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JACARANDA

MATHS QUEST

FOR WESTERN AUSTRALIA

9

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jacaranda

A Wiley Brand

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The publisher acknowledges ongoing discussions related to gender-based population data. At the time of publishing, there was insufficient data available to allow for the meaningful analysis of trends and patterns to broaden our discussion of demographics beyond male and female gender identification.

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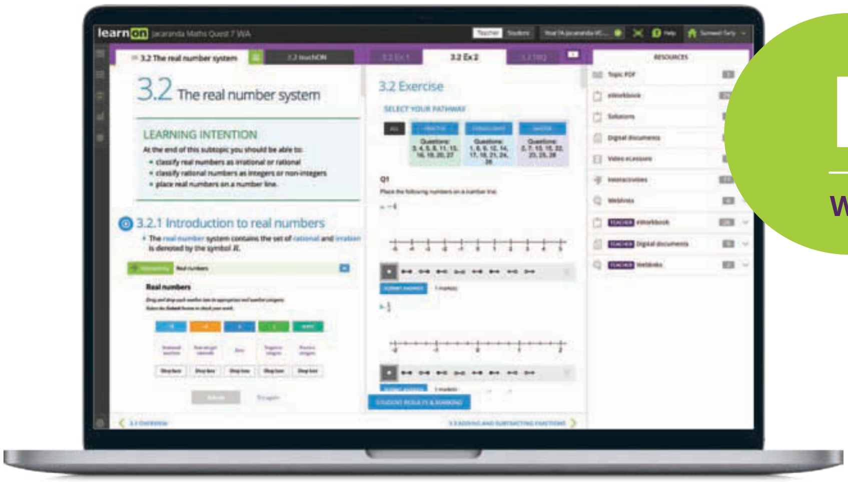

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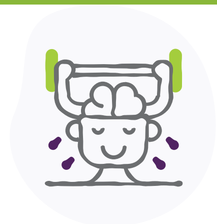

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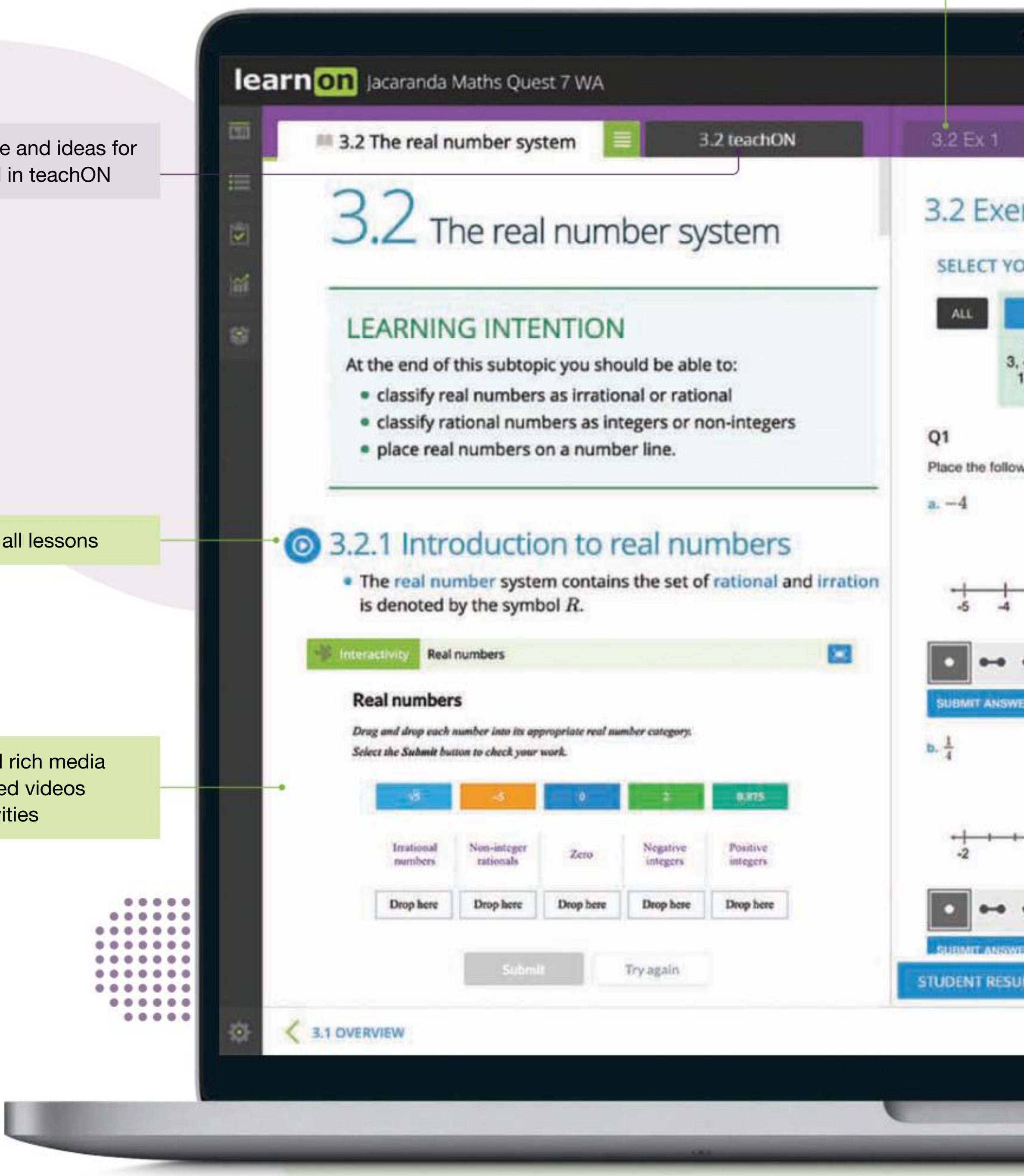

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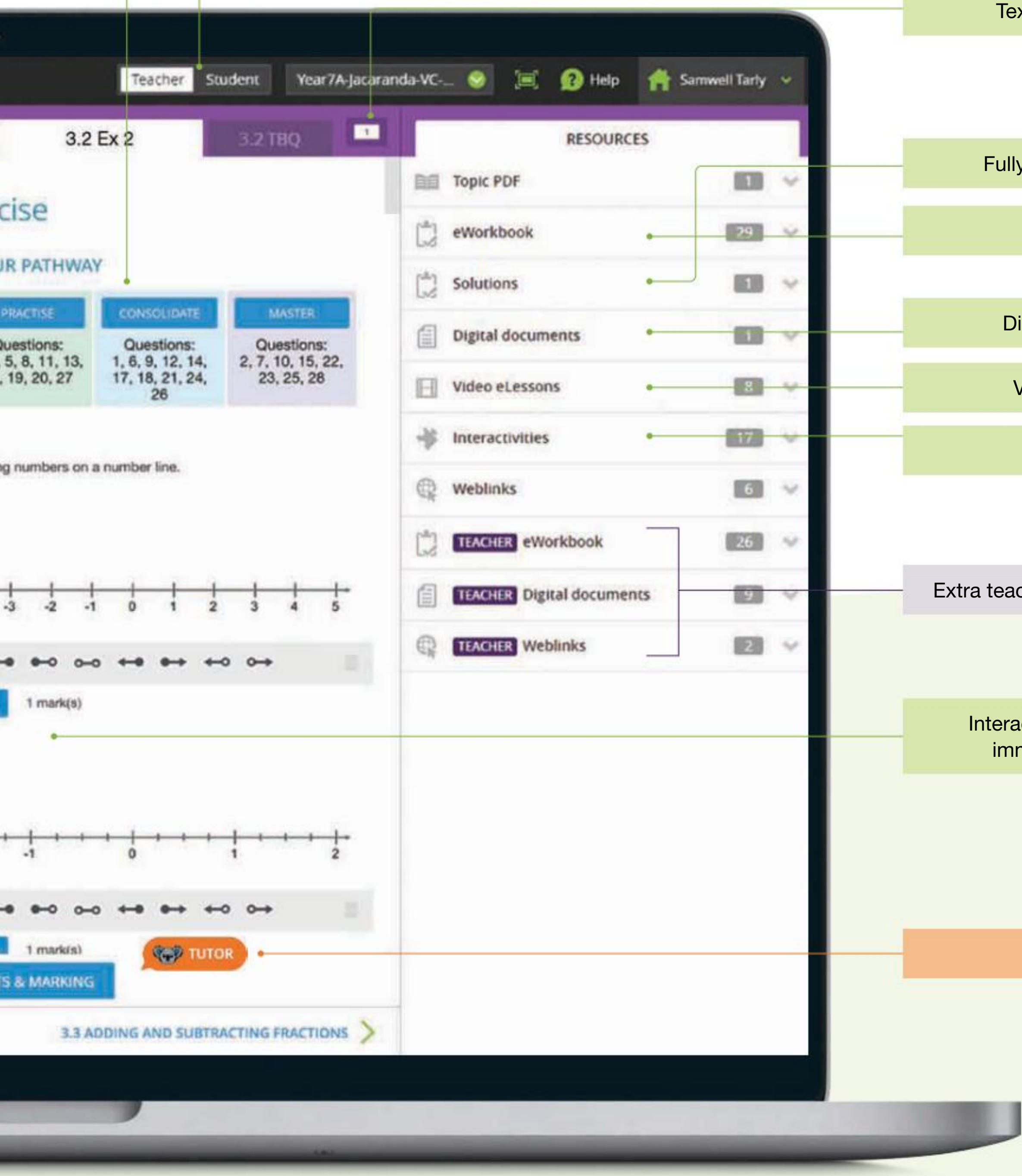

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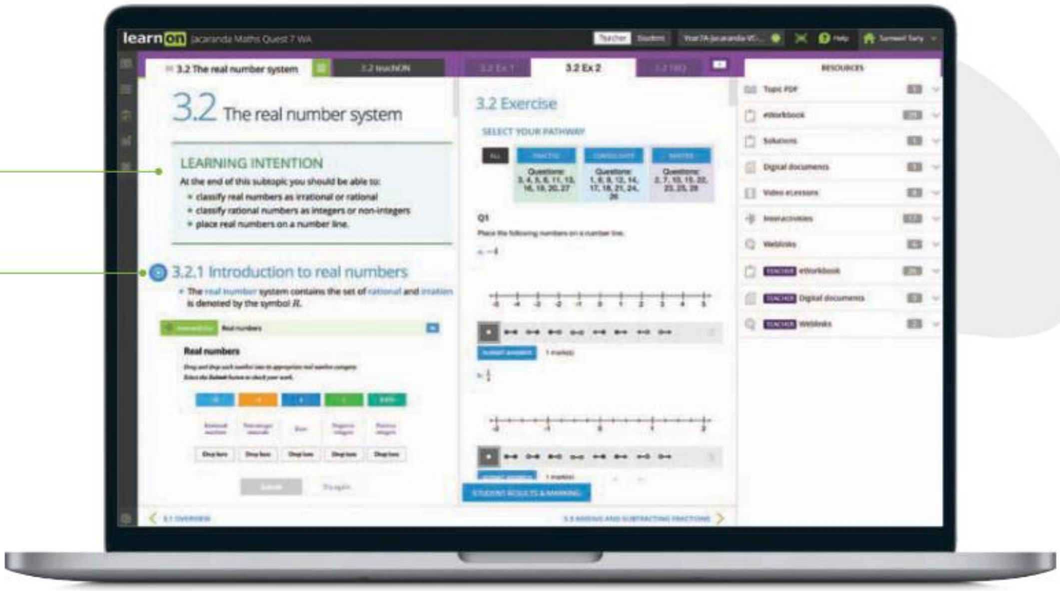

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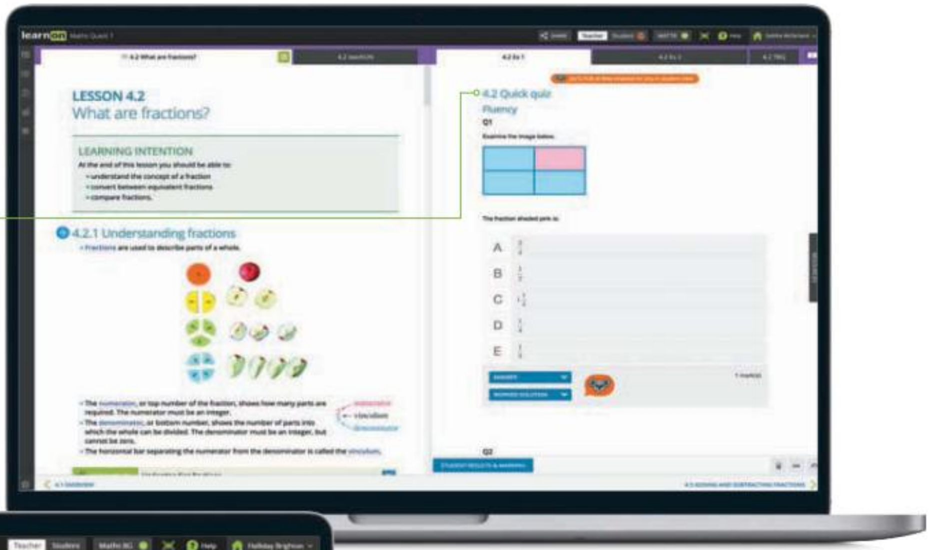

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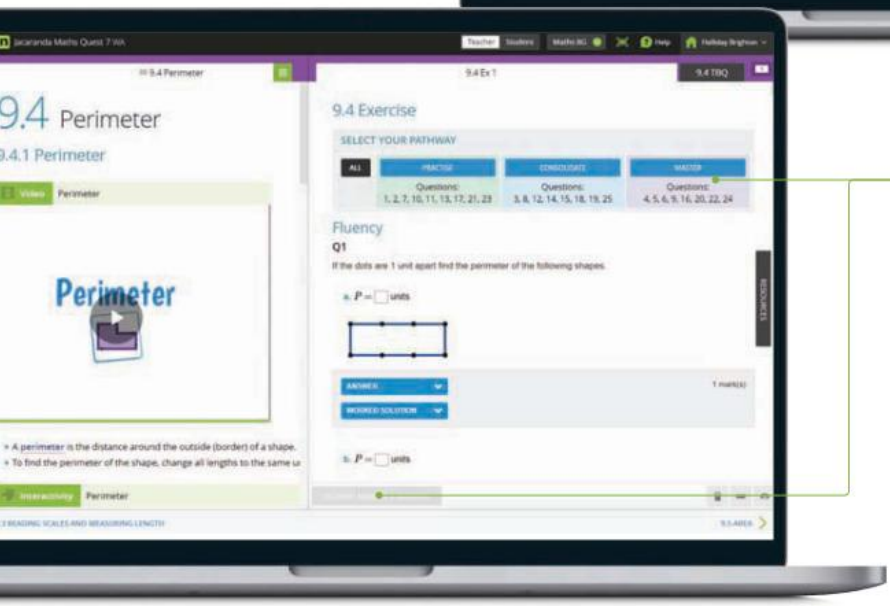

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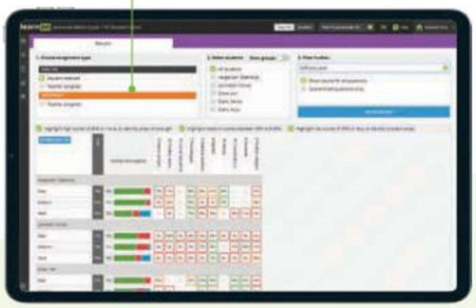

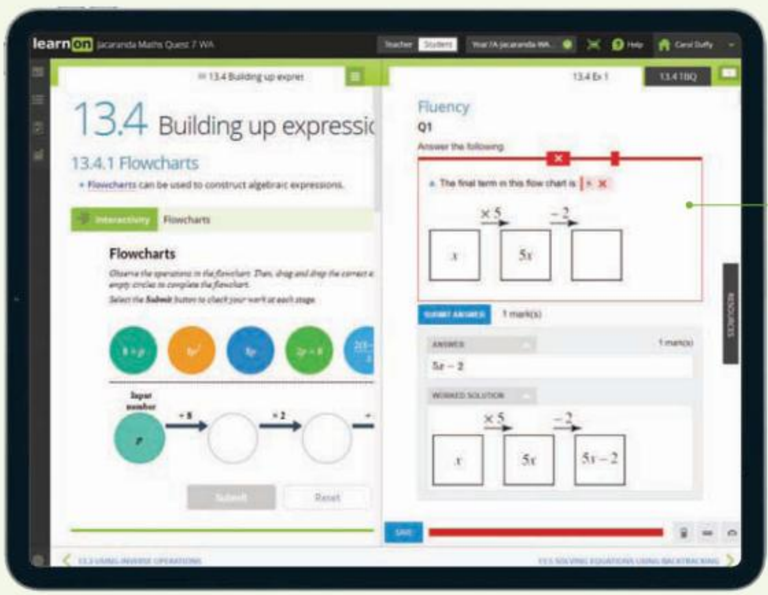

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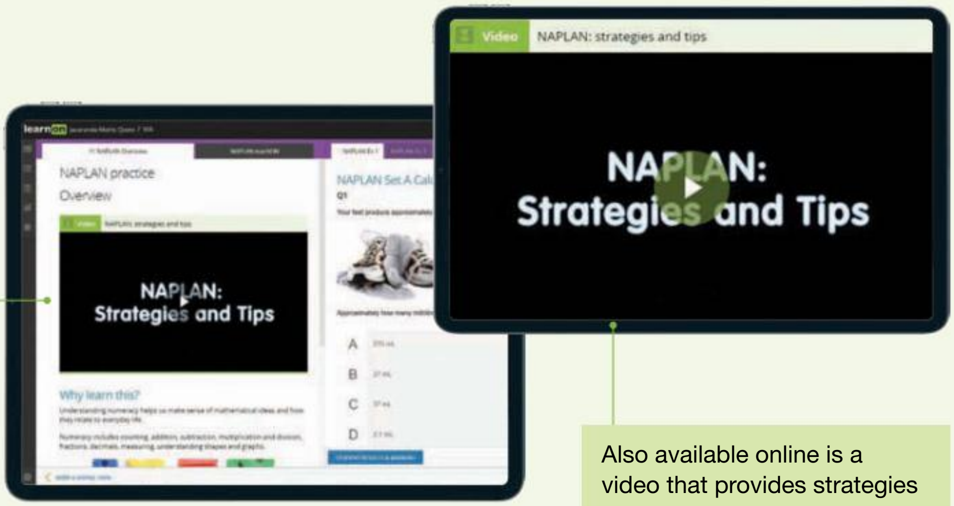

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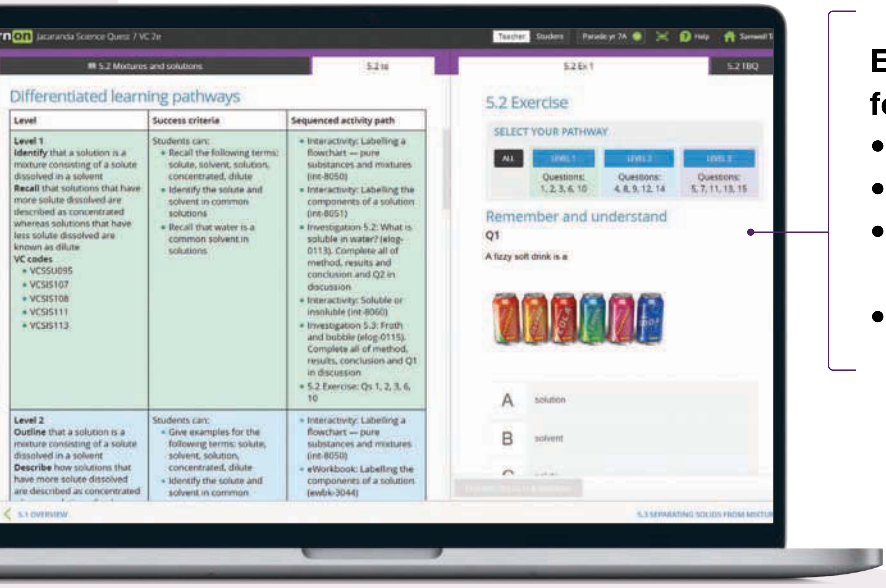

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- practical teaching advice
- three levels of differentiated teaching programs
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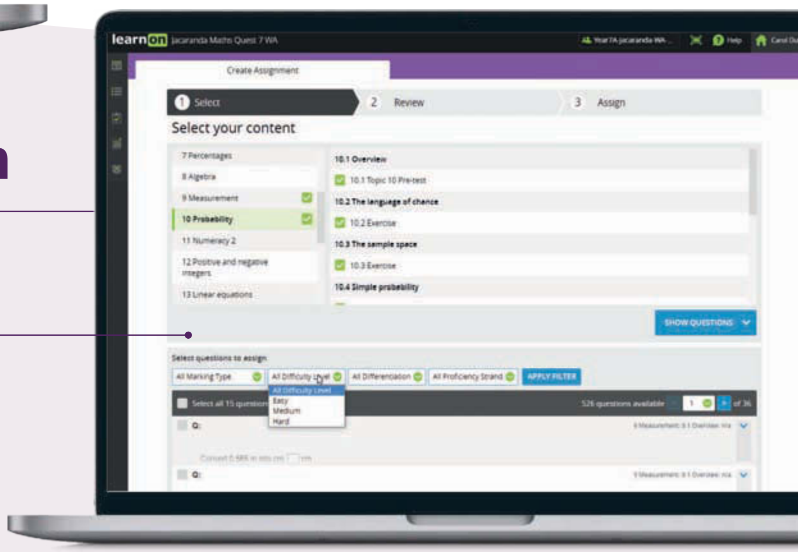

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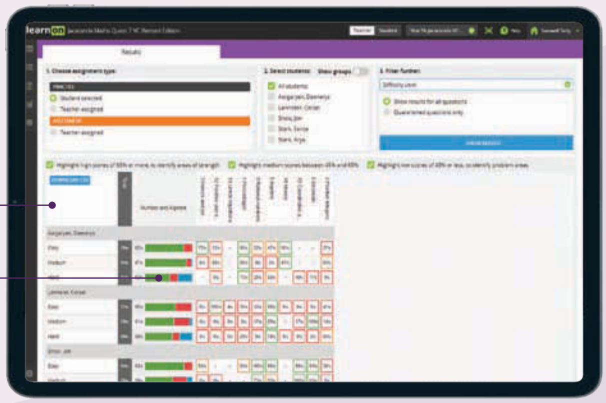

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A personal tutor for every student

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Combat anxiety

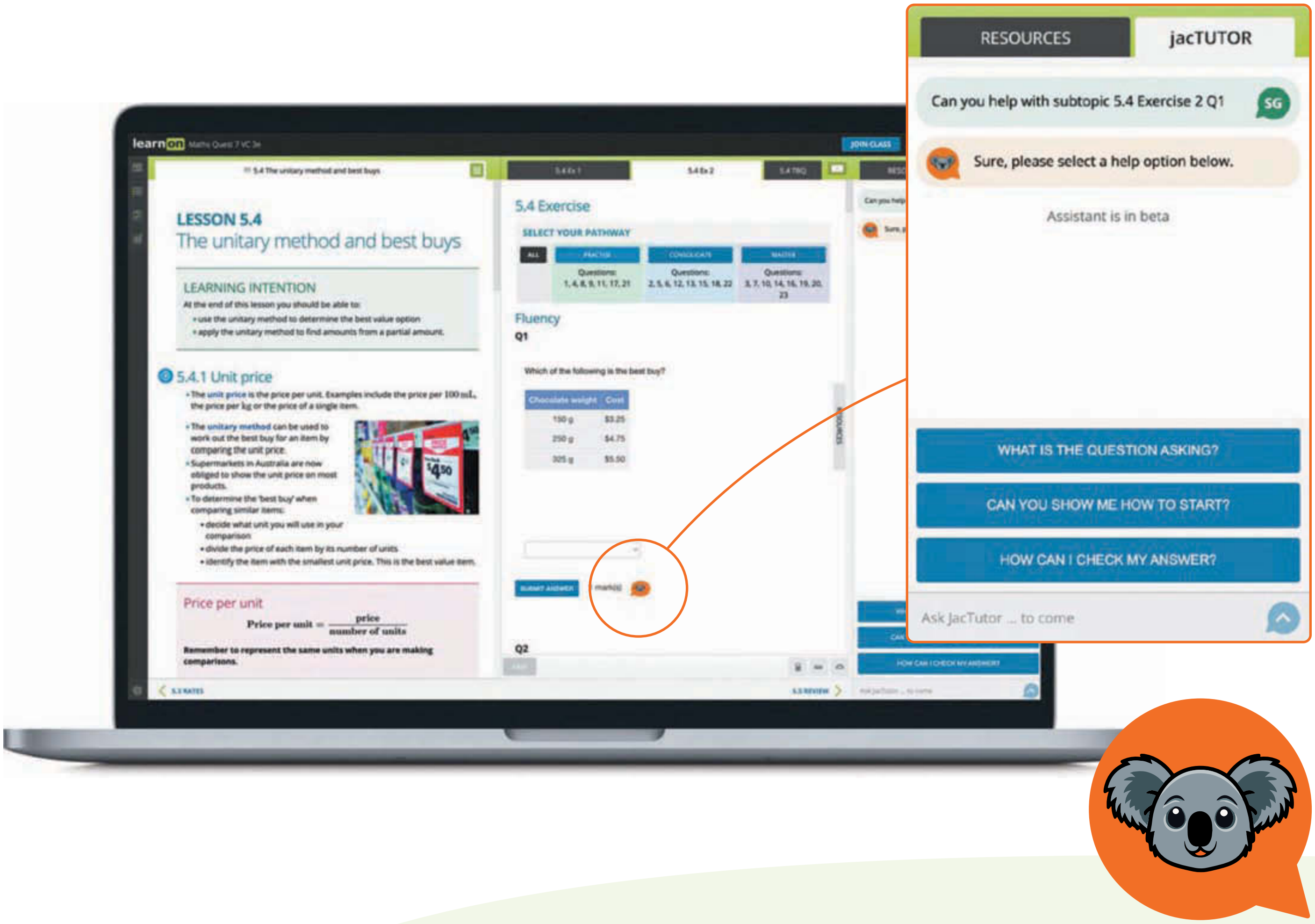
Afraid to raise a hand or ask more questions? With jacTUTOR, students will find a safe space to ask questions, get clarification and try again.

Consistency over confusion

There are so many ways to learn. To avoid confusion, jacTUTOR is built on the same trusted method used in the Jacaranda text.

A safe space

To keep students safe, any concerning or inappropriate comments are automatically flagged and sent to their teacher.



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SET A
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SET B
Non-calculator

SET C
Calculator allowed

SET D
Non-calculator

SET E
Calculator allowed

SET F
Non-calculator

STRAND: Number and algebra
SUB-STRANDS: Understanding number | Calculating with number | Algebraic techniques

1 Number skills and index laws

LESSON SEQUENCE

LESSON 1.1 Overview

1.1.1 Why learn this?

The earliest mathematics that humans used involved simple counting. Knowing that two goats could be swapped for six chickens was an important part of life twelve thousand years ago, so being able to count was a big deal.

Of course, being able to count is still a big deal today, but over thousands of years we have moved on from counting sheep. Ancient Egypt was the first recorded civilisation to think about fractions. Later on, ancient Greek mathematicians became the first people we know of to think about numbers that can't be expressed as fractions — these thoughts gave rise to the idea of irrational numbers. The value of π (pi) — which is the ratio of the circumference of a circle to its diameter — is one such irrational number. The value of π can only be approximated — if you were to try to say all of the digits in the value of π (that we presently know of) at a rate of 10 digits a second it would take you around 200 000 years to say them!

As new kinds of numbers have been discovered, new ways of writing them have been developed. These new ways of writing numbers can make it easier to work with numbers like the massively large ones that describe the distances between stars, or the incredibly small ones that describe the size of subatomic particles.

In this topic we will continue our mathematical journey by exploring a range of different types of numbers and the ways they can be written.

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Questions with immediate feedback, and fully worked solutions to help students get unstuck.

Exercise 1.1 Pre-test

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1. State whether the following statement is True or False.
Decimals that do not terminate and also do not recur are irrational numbers.

2. State the kind of number that $7.\overline{324}$ is an example of.

3. State whether the following statements are True or False.
   a. π is an irrational number.
   b. Every surd is an irrational number.

4. Calculate $\sqrt[3]{27}$.

5. **MC** Select the simplest form of $\sqrt{50}$ from the following.
   A. $5\sqrt{10}$
   B. $2\sqrt{5}$
   C. $5\sqrt{2}$
   D. $10\sqrt{5}$

6. **MC** Select the correct answer for the expression $\sqrt{5} - 2\sqrt{45} + 3\sqrt{20}$.
   A. $7\sqrt{5}$
   B. $\sqrt{5}$
   C. $2\sqrt{70}$
   D. $19\sqrt{5}$

7. The length of the hypotenuse of a right-angled triangle uses the formula $c^2 = a^2 + b^2$.
Calculate the length of the hypotenuse for a triangle with side lengths of $\sqrt{2}$ and $\sqrt{3}$. Leave your answer as a surd.

8. **MC** Select how many significant figures there are in 0.0305.
   A. 1
   B. 2
   C. 3
   D. 4

9. Round 7.43861 to 3 decimal places.

10. Round 7.43861 to 3 significant figures.

11. **MC** Select the correct scientific notation to 3 significant figures for 0.000 000 021 37.
   A. 2.137×10^8
   B. 2.14×10^8
   C. 2.14×10^{-8}
   D. 2.137×10^{-8}

12. **MC** Select the series of numbers that is correctly ordered from smallest to largest.
   A. $-\sqrt[3]{8}, -\sqrt[3]{27}, \sqrt[4]{81}$
   B. $\sqrt{3}, e, 5 \times 10^2$
   C. $2.1 \times 10^{-5}, 2.13 \times 10^2, 2.75 \times 10^{-1}$
   D. $\left(\frac{2}{5}\right)^2, \sqrt{0.16}, 0.\dot{3}$

13. An electron has a mass of $9.1093819 \times 10^{-31}$ kg, correct to 8 significant figures. A neutron is 1836 times the size of an electron. Evaluate the mass of a neutron correct to 4 significant figures.

14. Rationalise $\dfrac{4}{\sqrt{3}}$.

15. Simplify $\dfrac{5}{2}\sqrt{\dfrac{10}{5}}$.

LESSON 1.2 Real numbers

LEARNING INTENTIONS

At the end of this lesson you should be able to:

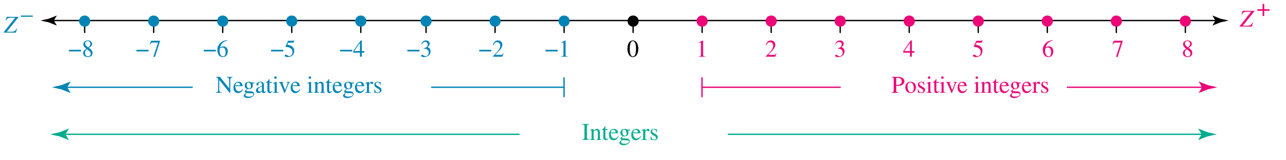
- distinguish between natural, integer, rational, irrational and real numbers
- remember and use the symbols used to represent natural, integer, rational, irrational and real numbers
- write integers and decimals as rational number fractions
- write recurring decimals using dot notation
- distinguish between recurring and terminating decimals.

1.2.1 Natural numbers

- The earliest use of numbers by humans was for counting things. The numbers we used were the positive whole numbers: 1, 2, 3, 4 ...
- These counting numbers are now known as the **natural numbers**.
- We use the symbol $\boldsymbol{N}$ to represent the set of natural numbers.

Integers

- **Integers** include the entire set of whole numbers. This means that the integer group contains:
  - the positive whole numbers (i.e. the natural numbers) 1, 2, 3, 4 ...
  - the negative whole numbers $-1, -2, -3, -4$ ...
  - 0 (which is neither positive nor negative).
- We use the symbol $\boldsymbol{Z}$ to represent the full set of integers. You may also see $\boldsymbol{Z}^-$ used to represent only the group of negative integers and $\boldsymbol{Z}^+$ used for only the group of positive integers. This means that $\boldsymbol{Z}^+$ and $\boldsymbol{N}$ represent the same group of numbers.



Rational numbers

- Integers, along with recurring decimals, are part of a larger group called the **rational numbers** represented by the symbol, $\boldsymbol{Q}$.
- Rational numbers are numbers that can be expressed as a fraction with whole numbers both above and below the dividing sign.

  Examples of rational numbers include $\frac{3}{7}$, -5 $\left(\text{which can be written as } \frac{-5}{1}\right)$, 3.2 $\left(\text{which can be written as } \frac{16}{5}\right)$ and -0.036 $\left(\text{which can be written as } \frac{-36}{1000}\right)$.

The denominator in a fraction

The denominator (bottom number) in a fraction cannot be 0. The quotient $9 \div 0$ has no answer. Because of this, numbers such as $\frac{9}{0}$ are said to be undefined. Because they do not exist, they are not rational numbers!

WORKED EXAMPLE 1 Writing rational numbers as fractions

By writing each of the following in fraction form, show that these numbers are rational.

a. **7** b. **−11** c. **0** d. $-4\frac{2}{5}$ e. **1.2**

THINK	WRITE
Each number must be written as a fraction using integers.	
a. The number 7 has to be written in fraction form. To write a number as a fraction, it must be written with a numerator and denominator. For a whole number like 7, the denominator is always 1.	a. $7 = \frac{7}{1}$. This number is rational.
b. The number −11 has to be written in fraction form. To write a number as a fraction, it must be written with a numerator and denominator. Because −11 is a whole number, the denominator is 1.	b. $-11 = \frac{-11}{1}$. This number is rational.
c. The number 0 has to be written in fraction form. To write a number as a fraction, it must be written with a numerator and a denominator. Because 0 is a whole number, the denominator is 1.	c. $0 = \frac{0}{1}$. This number is rational.
d. Write $-4\frac{2}{5}$ as an improper fraction.	d. $-4\frac{2}{5} = \frac{-22}{5}$. This number is rational.
e. The number 1.2 can be expressed as $1 + 0.2$. This can then be expressed as $1 + \frac{2}{10}$. Write this as an improper fraction.	e. $1.2 = \frac{12}{10}$. This number is rational.

1.2.2 Recurring decimals

- When a rational number is written in decimal form, there are two possibilities:
  1. The decimal will come to an end (or **terminate**).
     For example, $\frac{5}{4} = 1.25$, which is a decimal that terminates (ends) after two decimal places.
  2. The decimal will repeat itself endlessly. For example, $\frac{7}{6} = 1.16666...$
     The 6 in this number is a repeating digit, which means the decimal does not terminate – it has no end.
- Decimals that endlessly repeat digits (or blocks of digits) are called **recurring decimals**.
- Recurring decimals are written by placing dots or a line above the repeating digits.

Writing recurring decimals

Number of recurring digits	Example	How to write the decimal
1 recurring digit	1.615 555 555 … (Only the digit 5 repeats.)	$1.61\dot{5}$
2 recurring digits	1.615 151 515 … (The digits 1 and 5 repeat.)	$1.6\dot{1}\dot{5}$ or $1.6\overline{15}$
3 or more recurring digits	1.615 615 615 … (The digits 6, 1 and 5 repeat.)	$1.\dot{6}1\dot{5}$ or $1.\overline{615}$

WORKED EXAMPLE 2 Writing recurring decimals in extended form

Write the first 8 digits of each of the following recurring decimals.

a. $3.\dot{0}\dot{2}$ **b.** $47.\dot{1}$ **c.** $11.5\dot{4}\dot{9}$

THINK

a. In $3.\dot{0}\dot{2}$, the digits 0 and 2 repeat.

b. In $47.\dot{1}$, the digit 1 repeats.

c. In $11.5\dot{4}\dot{9}$, the digits 4 and 9 repeat.

WRITE

a. 3.020 202 0

b. 47.111 111

c. 11.549 494

WORKED EXAMPLE 3 Writing fractions as recurring decimals

Write each fraction as a recurring decimal using dot notation. Use a calculator for the division.

a. $\frac{5}{6}$ **b.** $\frac{57}{99}$

c. $\frac{25}{11}$ **d.** $\frac{4}{7}$

THINK

a. $5 \div 6 = 0.833\,333\,3$
3 repeats, so a dot goes above the 3.

b. $57 \div 99 = 0.575\,757\,575\,7$
5 and 7 repeat, so dots go above the 5 and the 7.

c. $25 \div 11 = 2.272\,727\,27$
2 and 7 repeat, so dots go above the 2 and the 7.

d. $4 \div 7 = 0.571\,428\,571$
It looks as though 571 428 will recur, so dots go above the 5 and the 8.

WRITE

a. $\frac{5}{6} = 0.8\dot{3}$

b. $\frac{57}{99} = 0.\dot{5}\dot{7}$

c. $\frac{25}{11} = 2.\dot{2}\dot{7}$

d. $\frac{4}{7} = 0.\dot{5}71\,42\dot{8}$

1.2.3 Irrational numbers

- A number is **irrational** if it cannot be written either as a fraction or as a terminating or recurring decimal.
- Irrational numbers may be expressed as decimals. In irrational numbers the digits do not repeat themselves in any particular order.
  For example, a calculator can be used to determine decimal expressions for the following:

$$\sqrt{18} = 4.242\ 640\ 687\ 12\ ...$$

$$\sqrt{0.03} = 0.173\ 205\ 080\ 757\ ...$$

- We use the symbol $\boldsymbol{I}$ to represent the set of irrational numbers.
- Surds are irrational numbers that cannot be expressed as the ratio of two integers. They are typically written in the form $\sqrt{n}$, where n is a positive integer that is not a perfect square.

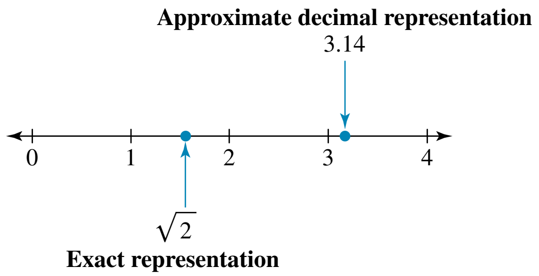
Identifying irrational numbers

To classify a number as either rational or irrational:

1. **Determine whether it can be expressed as a whole number, a fraction or a terminating or recurring decimal.**
2. **If the answer to step 1 is yes, then the number is rational. If the answer is no, then the number is irrational.**

- Exact representations of irrational numbers refer to representations that are precise numbers not in a decimal form.
  - They are expressed as roots, logarithms, or trigonometric functions.
  - Commonly used in theoretical or mathematical contexts where precision is crucial.
- Approximate decimal representations of irrational numbers are decimal numbers that are close to the actual value of the irrational number, but not exact.
  - These representations are obtained by truncating or rounding off the decimal expansion of the number to a certain number of digits.

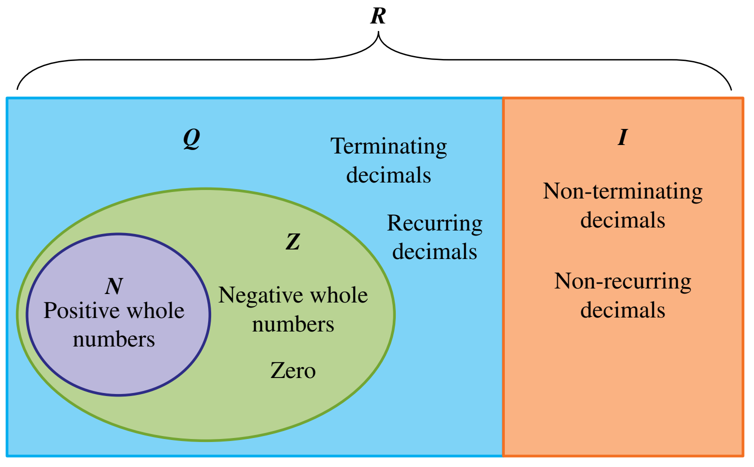
  For example, π is an irrational number that can be approximated to different degrees of accuracy, such as 3.14, 3.1416, or 3.14159265359. (These approximations are found using a calculator or computer.)



- The main difference between exact representations and approximate decimal representations of irrational numbers is the level of precision.

Real numbers

- The set of **real numbers** is made up of the group of rational numbers and the group of irrational numbers. These groups can, in turn, be broken up into subsets of other number groups.
- A real number is any number that lies on the number line.
- We use the symbol $\boldsymbol{R}$ to represent the set of real numbers.



DISCUSSION

Determine how many recurring digits there will be for $\frac{1}{13}$.

π (pi) — a special number

The symbol π (pi) is used for a particular number. That number is the circumference of a circle whose diameter length is 1 unit. Pi can be approximated as a decimal that is non-terminating and non-recurring. Therefore, π is classified as an irrational number. It is also called a transcendental number.

In decimal form, $\pi = 3.141\,592\,653\,589\,793\,23\ldots$ It has been calculated to 62.8 trillion decimal places with the aid of a computer.

1.2 Quick quiz on | 1.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 9, 11, 14	2, 5, 6, 8, 12, 15	3, 7, 10, 13, 16, 17

Fluency

1. WE1 Show that the following numbers are rational by writing each of them in fraction form.
   a. 15 b. -8 c. $2\frac{2}{3}$ d. $-5\frac{1}{8}$
2. Show that the following numbers are rational by writing each of them in fraction form.
   a. $\sqrt{16}$ b. $7\frac{3}{10}$ c. 0.002 d. 87.2
3. Show that the following numbers are rational by writing each of them in fraction form.
   a. 0 b. 1.56 c. 3.612 d. -0.08
4. WE2 Write the first 8 digits of each of the following recurring decimals.
   a. $0.\dot{5}$ b. $0.\dot{5}\dot{1}$ c. $0.5\dot{1}$ d. $6.0\dot{3}\dot{1}$ e. $5.\dot{1}8\dot{3}$
5. Write the first 8 digits of each of the following recurring decimals.
   a. $-7.02\dot{4}$ b. $8.\dot{9}12\dot{4}$ c. $5.12\dot{3}\dot{4}$ d. $5.\dot{1}23\dot{4}$ e. $3.0\dot{0}\dot{2}$
6. WE3 Write the following fractions as recurring decimals using dot notation. (Use a calculator.)
   a. $\frac{5}{9}$ b. $\frac{3}{11}$ c. $2\frac{2}{11}$ d. $\frac{3}{7}$ e. $\frac{-173}{99}$
7. Write the following fractions as recurring decimals using dot notation. (Use a calculator.)
   a. $\frac{73}{990}$ b. $-3\frac{5}{6}$ c. $\frac{7}{15}$ d. $\frac{46}{99}$ e. $\frac{46}{990}$

Understanding

8. From the following list of numbers:

$$-3, \frac{-3}{7}, 0, 2.\dot{3}, 2\frac{3}{5}, 15$$

   a. write down the natural numbers
   b. write down the integers
   c. write down the rational numbers
   d. write down the recurring decimals.
9. Write these numbers in order from smallest to largest.

$$2.\dot{1}, 2.\dot{1}\dot{2}, 2.\dot{1}2\dot{1}, 2.1\dot{2}\dot{1}, 2.1\dot{2}$$

10. State which of the following statements are true.
    a. All natural numbers are integers.
    b. All integers are rational numbers.
    c. All rational numbers have recurring decimals.
    d. Natural numbers can be negative integers.
    e. All numbers with decimals are rational numbers.
    f. Irrational numbers are not real numbers.

Reasoning

11. Explain why all integers are rational numbers.

12. a. If positive integers are represented by Z^+, explain what symbol we would use to represent the set of negative rational numbers.

b. Explain why we could not write N^-.

13. Are all fractions in which both the numerator and denominator are integers rational? Explain why or why not.

Problem solving

14. a. Using the fraction $\frac{a}{b}$, where a and b are natural numbers, write 3 recurring decimals in fractional form using the smallest natural numbers possible.

b. Determine the largest natural number you used.

15. Write 2 fractions that have the following number of repeating digits in their decimal forms.

a. 1 repeating digit
b. 2 repeating digits
c. 3 repeating digits
d. 4 repeating digits

16. If a \$2 coin weighs 6 g, a \$1 coin weighs 9 g, a 50 c coin weighs 15 g, a 20 c coin weighs 12g, a 10 c coin weighs 5 g and a 5 c coin weighs 3 g, determine what the maximum value of the coins would be if a bundle of them weighed exactly 10 kg.

17. A Year 9 student consumes two bottles of water every day, except on every fifth day when they only have one. Evaluate their annual bottled-water consumption.

LESSON 1.3 Rounding numbers and significant figures

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- round decimal numbers to a given number of decimal places
- identify the significant figures in a decimal number
- round decimal numbers to a given number of significant figures.

1.3.1 Rounding decimals

- If a number can't be written as an exact decimal, or if it has a lot of digits after the decimal point, we can use a shorter, approximate value.
- A number's approximate value can be found by using a calculator and then rounding it off to the desired level of accuracy.
- When we round numbers, we write them to either a certain number of decimal places or a certain number of significant figures.

Rounding to a given number of decimal places

1. **Decide how many decimal places you need to round the number to.**
2. **Identify the digit that is located at the position you need to round to. This digit is called the rounding digit.**
3. **Look at the digit that comes after the rounding digit.**
   - **If the digit's value is 0, 1, 2, 3 or 4, leave the rounding digit as it is.**
   - **If the digit's value is 5, 6, 7, 8 or 9, then increase the rounding digit by adding 1 to its value.**
4. **Remove all of the numbers after the rounding digit.**

WORKED EXAMPLE 4 Rounding a decimal number to three decimal places

Round 3.456 734 correct to three decimal places (3 d.p.).

THINK	WRITE
1. Identify the rounding digit located at the third decimal place. In this case it is 6.	3.456734
2. Look at the digit that comes after the rounding digit.	$\downarrow$ 3.456734
3. The number after the rounding digit is a 7, so increase the value of the rounding digit by adding 1. Next, remove the digits after it.	3.457

WORKED EXAMPLE 5 Writing a real number to three decimal places

Write these numbers correct to three decimal places (3 d.p.).

a. $\sqrt{3}$ b. π c. $5.1\dot{9}$ d. $\frac{2}{3}$ e. 7.123 456

THINK	WRITE
a. $\sqrt{3} = 1.7320\ldots$ The rounding digit is 2. The next digit is 0, so leave 2 as it is.	a. $\sqrt{3} \approx 1.732$
b. $\pi = 3.1415\ldots$ The rounding digit is 1. The next digit is 5, so add 1 to the rounding digit.	b. $\pi \approx 3.142$
c. $5.1\dot{9} = 5.1999\ldots$ The rounding digit is 9. The next digit is 9, so add 1 to the rounding digit.	c. $5.1\dot{9} \approx 5.200$
d. $\frac{2}{3} = 0.6666\ldots$ The rounding digit is 6. The next digit is 6, so add 1 to the rounding digit.	d. $\frac{2}{3} \approx 0.667$
e. 7.123 456. The rounding digit is 3. The next digit is 4, so leave 3 as is.	e. $7.1234 \approx 7.123$

1.3.2 Rounding to significant figures

- Another method of rounding decimals is to write them so that they are correct to a certain number of significant figures.
- When decimals are rounded, they don't always have the same number of decimal places as significant figures.

For example, the number 1.425 has three decimal places but four significant figures.

Counting significant figures in a decimal number

1. **Identify the first non-zero digit in the number. This digit may be located either before or after the decimal point. This digit is counted as the first significant figure.**
2. **Continue to count all digits in the number to the end of the number as it is written. This will give you the total number of significant figures.**

WORKED EXAMPLE 6 Counting significant figures

State the number of significant figures in each of the following numbers.

a. 25 b. 0.04 c. 3.02 d. 0.100

THINK	WRITE
a. In the number 25, the first significant figure is 2. There is 1 more significant figure after the first one.	a. 2 significant figures
b. In the number 0.04, the first significant figure is 4. There are no more significant figures after the first one.	b. 1 significant figure
c. In the number 3.02, the first significant figure is 3. There are 2 more significant figures after the first one.	c. 3 significant figures
d. In the number 0.100, the first significant figure is 1. There are 2 more significant figures after the first one.	d. 3 significant figures

WORKED EXAMPLE 7 Rounding real numbers to five significant figures

Round these numbers so that they are correct to five significant figures.

a. π b. $\sqrt{200}$ c. $0.\dot{0}\dot{3}$ d. 2530.166

THINK	WRITE
a. $\pi \approx 3.14159\ldots$ The first significant figure is 3. Starting with 3, write down the next 4 digits. The 6th digit is 9, so add 1 to the value of 5, the 5th digit.	a. 3.1416
b. $\sqrt{200} \approx 14.1421\ldots$ The first significant figure is 1. Starting with 1, write down the next 4 digits. The 6th digit is 1, so leave 2, the 5th digit, as it is.	b. 14.142
c. $0.\dot{0}\dot{3} = 0.030\,303\,0\ldots$ The first significant figure is 3. Starting with 3, write down the next 4 digits. The 6th digit is 0, so leave 3, the 5th digit, as it is.	c. 0.030 303
d. 2530.16 ... The first significant figure is 2. Starting with 2, write the next 4 digits. The 6th digit is 6, so add 1 to the value of 1, the 5th digit.	d. 2530.2

DISCUSSION

Is there an equal number of rational numbers and irrational numbers? Explain.

Exercise 1.3 Rounding numbers and significant figures

learnon

1.3 Quick quiz on | 1.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 7, 10, 12, 15, 18	2, 5, 8, 11, 13, 16, 19	3, 6, 9, 14, 17, 20

Fluency

1. **WE4&5** Write each of the following correct to 3 decimal places.
   a. $\frac{\pi}{2}$ b. $\sqrt{5}$ c. $\sqrt{15}$ d. 5.12×3.21 e. $5.\dot{1}$ f. $5.1\dot{5}$
2. Write each of the following correct to 3 decimal places.
   a. $5.\dot{1}\dot{5}$ b. 11.72^2 c. $\frac{3}{7}$ d. $\frac{1}{13}$ e. $2\frac{3}{7}$ f. 0.999 999
3. Write each of the following correct to three decimal places.
   a. 4.000 01 b. $2.79 \div 11$ c. 0.0254 d. 0.000 913 6 e. 5.000 01 f. 2342.156
4. **WE6** State how many significant figures there are in each of the following numbers.
   a. 36 b. 207 c. 1631 d. 5.04

5. State how many significant figures there are in each of the following numbers.

a. 176.2 b. 95.00 c. 0.21 d. 0.01

6. State how many significant figures there are in each of the following numbers.

a. 0.000 316 b. 0.1007 c. 0.010 d. 0.0512

7. WE7 Round each number so that it is correct to five significant figures.

a. $\frac{\pi}{2}$ b. $\sqrt{5}$ c. $\sqrt{15}$ d. 5.12×3.21 e. $5.\dot{1}$ f. $5.1\dot{5}$

8. Round each number so that it is correct to five significant figures.

a. $5.\dot{1}\dot{5}$ b. 11.72^2 c. $\frac{3}{7}$

d. $\frac{1}{13}$ e. $2\frac{3}{7}$ f. 0.999 999

9. Write each number so that it is correct to 5 significant figures.

a. 6.581 29 b. 4.000 01 c. $2.79 \div 11$ d. 0.0254 e. 0.000 913 6 f. 5.000 01

Understanding

10. Write the value of π correct to 4, 5, 6 and 7 decimal places.

11. State whether each of the following numbers has more significant figures than decimal places.

a. 17.26 b. 0.0032 c. 1.06 d. 0.010 005

12. State whether each statement is True or False.

a. Every surd is a rational number.
b. Every surd is an irrational number.
c. Every irrational number is a surd.
d. Every surd is a real number.

13. State whether each statement is True or False.

a. π is a rational number.
b. π is an irrational number.
c. π is a surd.
d. π is a real number.

14. State whether each statement is True or False.

a. $1.\dot{3}\dot{1}$ is a rational number.
b. $1.\dot{3}\dot{1}$ is an irrational number.
c. $1.\dot{3}\dot{1}$ is a surd.
d. $1.\dot{3}\dot{1}$ is a real number.

Reasoning

15. Write a decimal number that has 3 decimal places, but 4 significant figures. Show your working.

16. Explain why the number 12.995 412 3 becomes 13.00 when rounded to 2 decimal places.

17. The rational numbers 3.1416 and $\frac{22}{7}$ are both used as approximations to π.

a. Determine the largest number of decimal places these numbers can be rounded to in order to give the same value.
b. Explain which of these two numbers gives the best approximation to π.

Problem solving

18. The area of a circle is calculated using the formula $A = \pi \times r^2$, where r is the radius of the circle. Pi (π) is sometimes rounded to 2 decimal places to become 3.14. A particular circle has a radius of 7 cm.

a. Use $\pi = 3.14$ to calculate the area of the circle to 2 decimal places.
b. Use the π key on your calculator to calculate the area of the circle to 4 decimal places.
c. Round your answer for part **b** to 2 decimal places.
d. Are your answers for parts **a** and **c** different? Discuss why or why not.

19. The volume of a sphere (a ball shape) is calculated using the formula $v = \frac{4}{3} \times \pi \times r^3$, where r is the radius of the sphere. A beach ball with a radius of 25 cm is bouncing around the crowd at the MCG during the Boxing Day Test.

a. Calculate the volume of the beach ball to 4 decimal places.
b. Calculate the volume to 4 decimal places and determine how many significant figures the result has.
c. Explain whether the calculated volume is a rational number.

20. In a large sample of written English there are about 7 vowels for every 11 consonants. The letter e accounts for about one-third of the occurrence of vowels.
Explain how many times you would expect the letter e to occur in a passage of 100 000 letters. Round your answer to the nearest 100.

LESSON 1.4 Review of index laws

LEARNING INTENTIONS

At the end of this lesson you should be able to:

- identify the base and the exponent of a number written in index (exponent) form
- recall the first three index laws
- use the first three index laws to simplify expressions involving exponents.

1.4.1 Index notation

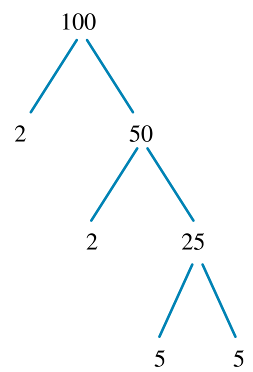
- The product of factors can be written in a shorter form called **index notation** (also known as exponent notation).

Index (exponent)

Base $\longrightarrow 6^4 = 6 \times 6 \times 6 \times 6 \longleftarrow$ Factor form

$= 1296$

- Any composite number can be written as a product of powers of prime factors using a factor tree, or by other methods, such as repeated division.



$$100 = 2 \times 2 \times 5 \times 5$$
$$= 2^2 \times 5^2$$

WORKED EXAMPLE 8 Using index notation to express a product of prime factors

Express 360 as a product of powers of prime factors using index notation.

THINK	WRITE
1. Express 360 as a product of a factor pair.	$360 = 6 \times 60$
2. Further factorise 6 and 60.	$= 2 \times 3 \times 4 \times 15$
3. Further factorise 4 and 15. They are both composite numbers.	$= 2 \times 3 \times 2 \times 2 \times 3 \times 5$
4. There are no more composite numbers.	$= 2 \times 2 \times 2 \times 3 \times 3 \times 5$
5. Write the answer using exponent notation. *Note:* The factors are generally expressed with their bases written in ascending order.	$360 = 2^3 \times 3^2 \times 5$

1.4.2 Multiplication using indices

- When multiplying **terms** with the same bases, add the indices. This is the **First Index Law**.
- When more than one base is involved, apply the First Index Law to each base separately.

The First Index Law

$$a^m \times a^n = a^{m+n}$$

For example,
$2x^2 \times x^3 = 2x^{2+3} = 2x^5$

WORKED EXAMPLE 9 Simplifying an expression using the First Index Law

Simplify $5e^{10} \times 2e^3$.

THINK	WRITE
1. The order is not important when multiplying, so place the coefficients first.	$5e^{10} \times 2e^3$ $= 5 \times 2 \times e^{10} \times e^3$
2. Simplify by multiplying the coefficients and then applying the First Index Law (add the indices). Write the answer.	$= 10e^{13}$

WORKED EXAMPLE 10 Simplifying using the First Index Law on multiple bases

Simplify $7m^3 \times 3n^5 \times 2m^8n^4$.

THINK	WRITE
1. The order is not important when multiplying, so place the coefficients first and group all similar pronumerals together.	$7m^3 \times 3n^5 \times 2m^8n^4$ $= 7 \times 3 \times 2 \times m^3 \times m^8 \times n^5 \times n^4$
2. Simplify by multiplying the coefficients and applying the First Index Law (add the exponents). Write the answer.	$= 42m^{11}n^9$

1.4.3 Division using indices

- When dividing terms with the same bases, subtract the indices. This is the **Second Index Law**.
- When the coefficients do not divide evenly, simplify by cancelling.

The Second Index Law

$$a^m \div a^n = a^{m-n} \text{ or}$$
$$\frac{a^m}{a^n} = a^{m-n}$$

For example,
$2x^5 \div x^3 = \dfrac{2x^5}{x^3} = 2x^{5-3} = 2x^2$

WORKED EXAMPLE 11 Simplifying using the Second Index Law

Simplify $\dfrac{25v^6 \times 8w^9}{10v^4 \times 4w^5}$.

THINK	WRITE
1. Simplify the numerator and the denominator by multiplying the coefficients.	$\dfrac{25v^6 \times 8w^9}{10v^4 \times 4w^5} = \dfrac{200v^6w^9}{40v^4w^5}$
2. Simplify further by dividing the coefficients and applying the Second Index Law (subtract the exponents). Write the answer.	$= \dfrac{{}^{5}\cancel{200}}{{}^{1}\cancel{40}} \times \dfrac{v^6}{v^4} \times \dfrac{w^9}{w^5}$ $= 5v^2w^4$

WORKED EXAMPLE 12 Simplifying using the Second Index Law with non-cancelling coefficients

Simplify $\dfrac{7t^3 \times 4t^8}{12t^4}$.

THINK	WRITE
1. Simplify the numerator by multiplying the coefficients.	$\dfrac{7t^3 \times 4t^8}{12t^4} = \dfrac{28t^{11}}{12t^4}$
2. Simplify further by dividing the coefficients by the highest common factor. Then apply the Second Index Law (subtract the exponents). Write the answer.	$= \dfrac{28}{12} \times \dfrac{t^{11}}{t^4}$ $= \dfrac{7t^7}{3}$

1.4.4 Zero exponent

- Any number divided by itself is 1.
- This means that $\dfrac{a^m}{a^m} = 1$. So, $a^{m-m} = a^0 = 1$.
- In general, any number (except zero) to the power of 0 is equal to 1. This is the **Third Index Law**.

The Third Index Law

$$a^0 = 1, \text{ where } a \neq 0$$

For example,
$2^0 = 1$

WORKED EXAMPLE 13 Evaluating using the Third Index Law

Evaluate the following.

a. t^0
b. $(xy)^0$
c. 17^0
d. $5x^0$
e. $(5x)^0 + 2$
f. $5^0 + 3^0$

THINK	WRITE
a. Apply the Third Index Law.	a. $t^0 = 1$
b. Apply the Third Index Law.	b. $(xy)^0 = 1$
c. Apply the Third Index Law.	c. $17^0 = 1$
d. Apply the Third Index Law.	d. $5x^0 = 5 \times x^0$ $= 5 \times 1$ $= 5$
e. Apply the Third Index Law.	e. $(5x)^0 + 2 = 1 + 2$ $= 3$
f. Apply the Third Index Law.	f. $5^0 + 3^0 = 1 + 1$ $= 2$

WORKED EXAMPLE 14 Simplifying using the First and Third Index laws

Simplify $\dfrac{9g^7 \times 4g^4}{6g^3 \times 2g^8}$.

THINK	WRITE
1. Simplify the numerator and the denominator by applying the First Index Law.	$\dfrac{9g^7 \times 4g^4}{6g^3 \times 2g^8}$
2. Simplify the fraction further by applying the Second Index Law.	$= \dfrac{36g^{11}}{12g^{11}}$ $= \dfrac{{}^{3}\cancel{36}g^{11}}{{}^{1}\cancel{12}g^{11}}$
3. Simplify by applying the Third Index Law.	$= 3g^0$ $= 3 \times 1$
4. Write the answer.	$= 3$

1.4.5 Cancelling fractions

- Consider the fraction $\dfrac{x^3}{x^7}$. This fraction can be cancelled by dividing the denominator and the numerator by the highest common factor (HCF), which is x^3, so $\dfrac{x^3}{x^7} = \dfrac{1}{x^4}$.

  Note: $\dfrac{x^3}{x^7} = x^{-4}$ when we apply the Second Index Law.

  We will study negative indices in a later section.

DISCUSSION

How do the First, Second and Third Index laws help with calculations?

WORKED EXAMPLE 15 Using the first three Index laws to simplify expressions

Simplify these fractions by cancelling.

a. $\dfrac{x^5}{x^7}$ b. $\dfrac{6x}{12x^8}$ c. $\dfrac{30x^5y^6}{10x^7y^3}$

THINK	WRITE
a. Divide the numerator and denominator by the highest common factor (HCF), which is x^5.	a. $\dfrac{x^5}{x^7}=\dfrac{1}{x^2}$
b. 1. Divide the numerator and denominator by the HCF, which is $6x$.	b. $\dfrac{6x}{12x^8}=\dfrac{6}{12}\times\dfrac{x}{x^8}$ $=\dfrac{1}{2}\times\dfrac{1}{x^7}$
2. Simplify and write the answer.	$=\dfrac{1}{2x^7}$
c. 1. Divide the numerator and denominator by the HCF, which is $10x^5y^3$.	c. $\dfrac{30x^5y^6}{10x^7y^3}=\dfrac{30}{10}\times\dfrac{x^5}{x^7}\times\dfrac{y^6}{y^3}$ $=\dfrac{3}{1}\times\dfrac{1}{x^2}\times\dfrac{y^3}{1}$
2. Simplify and write the answer.	$=\dfrac{3y^3}{x^2}$

Exercise 1.4 Review of index laws

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1.4 Quick quiz on | 1.4 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 6, 10, 13, 15, 19, 20, 21, 26	2, 4, 7, 8, 11, 18, 22, 23, 27, 28	5, 9, 12, 14, 16, 17, 24, 25, 29, 30

Fluency

1. WE8 Use index notation to express each of the following as a product of powers of prime factors.
   a. 12 b. 72 c. 75 d. 240 e. 640 f. 9800
2. WE9 Simplify each of the following.
   a. $4p^7\times 5p^4$ b. $2x^2\times 3x^6$ c. $8y^6\times 7y^4$ d. $3p\times 7p^7$ e. $12t^3\times t^2\times 7t$ f. $6q^2\times q^5\times 5q^8$

3. **WE10** Simplify each of the following.

a. $2a^2 \times 3a^4 \times e^3 \times e^4$ b. $4p^3 \times 2h^7 \times h^5 \times p^3$ c. $3k^2 \times 2k^{11} \times k$ d. $3^4 \times 3^4 \times 3^4$

4. Simplify each of the following.

a. $2m^3 \times 5m^2 \times 8m^4$ b. $2gh \times 3g^2 h^5$
c. $5p^4q^2 \times 6p^2q^7$ d. $8u^3w \times 3uw^2 \times 2u^5w^4$

5. Simplify each of the following.

a. $9y^8d \times y^5d^3 \times 3y^4d^7$
b. $7b^3c^2 \times 2b^6c^4 \times 3b^5c^3$
c. $4r^2s^2 \times 3r^6s^{12} \times 2r^8s^4$
d. $10h^{10}v^2 \times 2h^8v^6 \times 3h^{20}v^{12}$

6. **WE11** Simplify each of the following.

a. $\dfrac{15p^{12}}{5p^8}$ b. $\dfrac{18r^6}{3r^2}$ c. $\dfrac{45a^5}{5a^2}$ d. $\dfrac{60b^7}{20b}$

7. Simplify each of the following.

a. $\dfrac{100r^{10}}{5r^6}$ b. $\dfrac{9q^2}{q}$ c. $\dfrac{130d^3}{d^5}$ d. $\dfrac{21t^{11}}{3t^7}$

8. **WE12** Simplify each of the following.

a. $\dfrac{8p^6 \times 3p^4}{16p^5}$ b. $\dfrac{12b^5 \times 4b^2}{18b^2}$ c. $\dfrac{25m^{12} \times 4n^7}{15m^2 \times 8n}$ d. $\dfrac{27x^9y^3}{12xy^2}$

9. Simplify each of the following.

a. $\dfrac{12j^8 \times 6f^5}{8j^3 \times 3f^2}$ b. $\dfrac{8p^3 \times 7r^2 \times 2s}{6p \times 14r}$ c. $\dfrac{27a^9 \times 18b^5 \times 4c^2}{18a^4 \times 12b^2 \times 2c}$ d. $\dfrac{81f^{15} \times 25g^{12} \times 16h^{34}}{27f^9 \times 15g^{10} \times 12h^{30}}$

10. **WE13** Evaluate the following.

a. m^0 b. $6m^0$ c. $(6m)^0$ d. $(ab)^0$ e. $5(ab)^0$

11. Evaluate the following.

a. w^0x^0 b. 85^0 c. $85^0 + 15^0$ d. $x^0 + 1$ e. $5x^0 - 2$

12. Evaluate the following.

a. $\dfrac{x^0}{y^0}$ b. $x^0 - y^0$ c. $3x^0 + 11$ d. $3a^0 + 3b^0$ e. $3(a^0 + b^0)$

13. **WE14** Simplify each of the following.

a. $\dfrac{2a^3 \times 6a^2}{12a^5}$ b. $\dfrac{3c^6 \times 6c^3}{9c^9}$ c. $\dfrac{5b^7 \times 10b^5}{25b^{12}}$ d. $\dfrac{8f^3 \times 3f^7}{4f^5 \times 3f^5}$ e. $\dfrac{9k^{12} \times 4k^{10}}{18k^4 \times k^{18}}$

14. Simplify each of the following.

a. $\dfrac{2h^4 \times 5k^2}{20h^2 \times k^2}$ b. $\dfrac{p^3 \times q^4}{5p^3}$ c. $\dfrac{m^7 \times n^3}{5m^3 \times m^4}$ d. $\dfrac{8u^9 \times v^2}{2u^5 \times 4u^4}$ e. $\dfrac{9x^6 \times 2y^{12}}{3y^{10} \times 3y^2}$

Understanding

15. **WE15** Simplify each of the following by cancelling.

a. $\dfrac{x^7}{x^{10}}$ b. $\dfrac{m}{m^9}$ c. $\dfrac{m^3}{4m^9}$ d. $\dfrac{12x^6}{6x^8}$

16. Simplify each of the following by cancelling.

a. $\dfrac{12x^8}{6x^6}$ b. $\dfrac{24t^{10}}{t^4}$ c. $\dfrac{5y^5}{10y^{10}}$ d. $\dfrac{35x^2y^{10}}{20x^7y^7}$

17. Simplify each of the following by cancelling.

a. $\dfrac{12m^2n^4}{30m^5n^8}$ b. $\dfrac{16m^5n^{10}}{8m^5n^{12}}$ c. $\dfrac{20x^4y^5}{10x^5y^4}$ d. $\dfrac{a^2b^4c^6}{a^6b^4c^2}$

18. Calculate the value of each of the following expressions if $a = 3$.

a. $2a$ b. a^2 c. $2a^2$ d. $a^2 + 2$ e. $a^2 + 2a$ f. $a^2 - a$

Reasoning

19. Explain why x^2 and $2x$ are not the same number. Use an example to illustrate your reasoning.

20. MC

a. $12a^8b^2c^4(de)^0f$, when simplified, is equal to:

A. $12a^8b^2c^4$ B. $12a^8b^2c^4f$ C. $12a^8b^2f$ D. $12a^8b^2$

b. $\left(\dfrac{6}{11}a^2b^7\right)^0 \times -\ (3a^2b^{11})^0 + 7a^0b$, when simplified, is equal to:

A. $7b$ B. $1 + 7b$ C. $-1 + 7ab$ D. $-1 + 7b$

c. You are told that there is an error in the statement $3p^7q^3r^5s^6 = 3p^7s^6$.
Explain what the left-hand side should be for this statement to be correct.

A. $(3p^7q^3r^5s^6)^0$ B. $(3p^7)^0q^3r^5s^6$ C. $3p^7(q^3r^5s^6)^0$ D. $3p^7(q^3r^5)^0s^6$

21. MC

a. You are told that there is an error in the statement $\dfrac{8f^6g^7h^3}{6f^4g^2h} = \dfrac{8f^2}{g^2}$.

Explain what the left-hand side should be for this statement to be correct.

A. $\dfrac{8f^6(g^7h^3)^0}{(6)^0f^4g^2(h)^0}$ B. $\dfrac{8(f^6g^7h^3)^0}{(6f^4g^2h)^0}$ C. $\dfrac{8(f^6g^7)^0h^3}{(6f^4)^0g^2h}$ D. $\dfrac{8f^6g^7h^3}{(6f^4g^2h)^0}$

b. Select what $\dfrac{6k^7m^2n^8}{4k^7(m^6n)^0}$ is equal to.

A. $\dfrac{6}{4}$ B. $\dfrac{3}{2}$ C. $\dfrac{3n^8}{2}$ D. $\dfrac{3m^2n^8}{2}$

22. Explain why $5x^5 \times 3x^3$ is not equal to $15x^{15}$.

23. A multiple-choice question requires a student to multiply 5^6 by 5^3. The student is having trouble deciding which of these four answers is correct: 5^{18}, 5^9, 25^{18} or 25^9.

a. State the correct answer.
b. Explain your answer by using another example to explain the First Index Law.

24. A multiple-choice question requires a student to divide 5^{24} by 5^8. The student is having trouble deciding which of these four answers is correct: 5^{16}, 5^3, 1^{16} or 1^3.

a. Determine the correct answer.
b. Explain your answer by using another example to explain the Second Index Law.

25. a. Calculate the value of $\frac{5^7}{5^7}$.

b. Determine the value of any number divided by itself.

c. According to the Second Index Law, which deals with exponents and division, $\frac{5^7}{5^7}$ should equal 5 raised to what exponent?

d. Use an example to explain the Third Index Law.

Problem solving

26. Answer the following questions.

a. For $x^2 x^\Delta = x^{16}$ to be true, determine what number must replace the triangle.

b. For $x^\Delta x^\mathrm{O} x^\diamond = x^{12}$ to be true, there are 55 ways of assigning positive whole numbers to the triangle, circle, and diamond. Give at least four of these.

27. Answer the following questions.

a. Determine a pattern in the units digit for powers of 3.

b. The units digit of 3^6 is 9. Determine the units digit of 3^{2001}.

28. Answer the following questions.

a. Determine a pattern in the units digit for powers of 4.

b. Determine the units digit of 4^{105}.

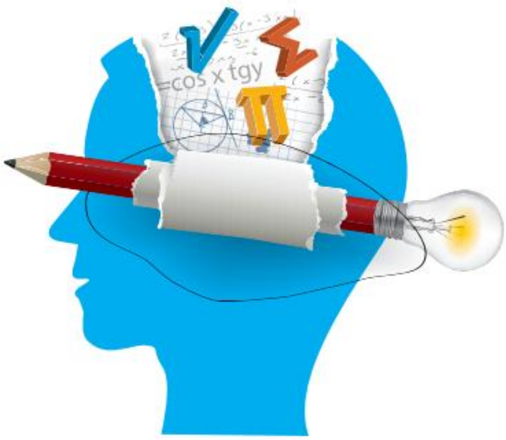
29. Answer the following questions.

a. Investigate the patterns in the units digit for powers of 2 to 9.

b. Predict the units digit for the following.

i. 2^{35} ii. 3^{16} iii. 8^{51}

30. Write $4^{n+1} + 4^{n+1}$ as a single power of 2.



LESSON 1.5 Raising powers

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- apply the Fourth, Fifth and Sixth index laws to raise a power to another power
- apply the first six index laws to simplify expressions involving indices.

1.5.1 More index laws

- When a power is raised to another power the indices are multiplied. This is the **Fourth Index Law**.
- The Fourth Index Law can then be applied to products and quotients of mixed bases to develop the Fifth and Sixth Index laws.

The Fourth Index Law

$$(a^m)^n = a^{m \times n}$$

For example, $(x^2)^3 = x^{2 \times 3} = x^6$

The Fifth Index Law

$$(a \times b)^m = a^m \times b^m$$

For example, $(2x)^3 = 2^3 \times x^3 = 8x^3$

The Sixth Index Law

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

For example, $\left(\frac{4x}{2}\right)^2 = \frac{4^2x^2}{2^2} = 4x^2$

WORKED EXAMPLE 16 Simplifying using the Fourth Index Law

Simplify the following.

a. $(7^4)^8$

b. $(3a^2b^5)^3$

THINK	WRITE
a. Simplify by applying the Fourth Index Law (multiply the indices).	**a.** $(7^4)^8$ $= 7^{4 \times 8}$ $= 7^{32}$
b. 1. Write the expression, including all indices.	**b.** $(3^1a^2b^5)^3$
2. Simplify by applying the Fourth Index Law (multiply the indices) for each term inside the brackets.	$= 3^{1 \times 3}a^{2 \times 3}b^{5 \times 3}$ $= 3^3a^6b^{15}$
3. Write the answer.	$= 27a^6b^{15}$

WORKED EXAMPLE 17 Simplifying using the Fifth Index Law

Simplify $(2b^5)^2 \times (5b)^3$.

THINK	WRITE
1. Write the expression, including all indices.	$(2^1 b^5)^2 \times (5^1\, b^1)^3$
2. Simplify by applying the Fifth Index Law.	$= 2^{1\times 2}\, b^{5\times 2} \times 5^{1\times 3}\, b^{1\times 3}$ $= 2^2 \times b^{10} \times 5^3 \times b^3$
3. Calculate the number values and bring them to the front. Then use the First Index Law to simplify the powers of b.	$= 4 \times 125 \times b^{10+3}$
4. Write the answer.	$= 500b^{13}$

WORKED EXAMPLE 18 Simplifying using the Sixth Index Law

Simplify $\left(\dfrac{2a^5}{d^2}\right)^3$.

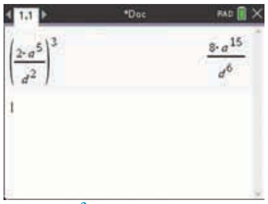
THINK	WRITE
1. Write the expression, including all indices.	$\left(\dfrac{2^1 a^5}{d^2}\right)^3$
2. Simplify by applying the Sixth Index Law for each term inside the brackets.	$= \dfrac{2^{1\times 3} a^{5\times 3}}{d^{2\times 3}}$ $= \dfrac{2^3\, a^{15}}{d^6}$
3. Simplify and then write the answer.	$= \dfrac{8\, a^{15}}{d^6}$

TI | THINK

In a new problem, on a Calculator page, complete the entry line as: $\left(\dfrac{2a^5}{d^2}\right)^3$

Press ENTER.

WRITE



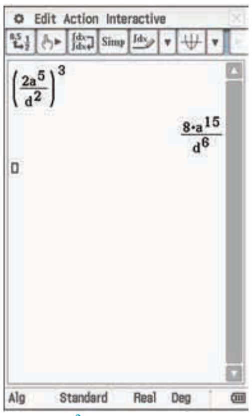
$\left(\dfrac{2a^5}{d^2}\right)^3 = \dfrac{8a^{15}}{d^6}$

CASIO | THINK

On the Main screen complete the entry line as: $\left(\dfrac{2a^5}{d^2}\right)^3$

Press EXE.

WRITE



$\left(\dfrac{2a^5}{d^2}\right)^3 = \dfrac{8a^{15}}{d^6}$

DISCUSSION

What difference, if any, is there between the operation of the index laws on numeric terms compared with similar operations on algebraic terms?

Exercise 1.5 Raising powers

learnon

1.5 Quick quiz on | 1.5 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 7, 9, 14, 15, 20	2, 5, 8, 10, 12, 16, 17, 21	4, 6, 11, 13, 18, 19, 22, 23

Fluency

1. WE16 Simplify each of the following.

   a. $(e^2)^3$ b. $(f^8)^{10}$ c. $(p^{25})^4$ d. $(r^{12})^{12}$ e. $(2a^2)^3$

2. Simplify each of the following.

   a. $(a^2b^3)^4$ b. $(pq^3)^5$ c. $(g^3h^2)^{10}$ d. $(3w^9q^2)^4$ e. $(7e^5r^2q^4)^2$

3. WE17 Simplify each of the following.

   a. $(p^4)^2 \times (q^3)^2$ b. $(r^5)^3 \times (w^3)^3$ c. $(b^5)^2 \times (n^3)^6$ d. $(j^6)^3 \times (g^4)^3$ e. $(2a)^3 \times (b^2)^2$

4. Simplify each of the following.

   a. $(q^2)^2 \times (r^4)^5$ b. $(h^3)^8 \times (j^2)^8$ c. $(f^4)^4 \times (a^7)^3$ d. $(t^5)^2 \times (u^4)^2$ e. $(i^3)^5 \times (j^2)^6$

5. WE18 Simplify each of the following.

   a. $\left(\frac{3b^4}{d^3}\right)^2$ b. $\left(\frac{5h^{10}}{2j^2}\right)^2$ c. $\left(\frac{2k^5}{3t^8}\right)^3$ d. $\left(\frac{7p^9}{8q^{22}}\right)^2$

6. Simplify each of the following.

   a. $\left(\frac{5y^7}{3z^{13}}\right)^3$ b. $\left(\frac{4a^3}{7c^5}\right)^4$ c. $\left(\frac{-4k^2}{7m^6}\right)^3$ d. $\left(\frac{-2g^7}{3h^{11}}\right)^4$

Understanding

7. Simplify each of the following.

   a. $(2^3)^4 \times (2^4)^2$ b. $(t^7)^3 \times (t^3)^4$ c. $(a^4)^0 \times (a^3)^7$ d. $(e^7)^8 \times (e^5)^2$

8. Simplify each of the following.

   a. $(g^7)^3 \times (g^9)^2$ b. $(3a^2) \times (2a^6)^2$ c. $(2d^7)^3 \times (3d^2)^3$ d. $(10r^2)^4 \times (2r^3)^2$

9. MC $(p^7)^2 \div p^2$ is equal to:

   A. p^7 B. p^{12} C. p^{16} D. $p^{4.5}$

10. MC $\dfrac{(w^5)^2 \times (p^7)^3}{(w^2)^2 \times (p^3)^5}$ is equal to:

A. w^2p^6 B. $(wp)^6$ C. $w^{14}p^{36}$ D. w^2p^2

11. MC $(r^6)^3 \div (r^4)^2$ is equal to:

A. r^3 B. r^4 C. r^8 D. r^{10}

12. Simplify each of the following.

a. $(a^3)^4 \div (a^2)^3$
b. $(m^8)^2 \div (m^3)^4$
c. $(n^5)^3 \div (n^6)^2$
d. $(b^4)^5 \div (b^6)^2$
e. $(f^7)^3 \div (f^2)^2$
f. $(g^8)^2 \div (g^5)^2$

13. Simplify each of the following.

a. $(p^9)^3 \div (p^6)^3$
b. $(y^4)^4 \div (y^7)^2$
c. $\dfrac{(c^6)^5}{(c^5)^2}$
d. $\dfrac{(f^5)^3}{(f^2)^4}$
e. $\dfrac{(k^3)^{10}}{(k^2)^8}$
f. $\dfrac{(p^{12})^3}{(p^{10})^2}$

Reasoning

14. a. Replace the triangle with the correct exponent for the equation $4^7 \times 4^7 \times 4^7 \times 4^7 \times 4^7 = (4^7)^{\Delta}$.
b. The expression $(p^5)^6$ means to write p^5 as a factor how many times?
c. If you rewrote the expression from part b without any exponents, in the format $p \times p \times p...$, determine how many factors you would need.
d. Explain the Fourth Index Law.

15. a. Simplify each of the following.

i. $(-1)^{10}$ ii. $(-1)^7$ iii. $(-1)^{15}$ iv. $(-1)^6$

b. Write a general rule for the result obtained when -1 is raised to a positive power. Explain your answer.

16. Jo and Danni are having an algebra argument. Jo is sure that $-x^2$ is equivalent to $(-x^2)$, but Danni thinks otherwise.
Explain who is correct and justify your answer.

17. A multiple-choice question requires a student to calculate $(5^4)^3$. The student is having trouble deciding which of these three answers is correct: 5^{64}, 5^{12} or 5^7.

a. Determine the correct answer.
b. Explain your answer by using another example to illustrate the Fourth Index Law.

18. a. Without using your calculator, simplify each side of the following equations to the same base and then solve each of them.

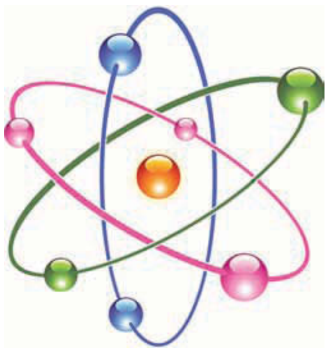
i. $8^x = 32$
ii. $27^x = 243$
iii. $1000^x = 100\,000$

b. Explain why all 3 equations have the same solution.

19. Consider the expression 4^{3^2}. Identify which is the correct answer, 4096 or 262 144. Justify your choice.

Problem solving

20. The diameter of a typical atom is so small that it would take about 10^8 atoms arranged in a line to reach a length of just 1 centimetre.
Estimate how many atoms are contained in a cubic centimetre. Write this number as a power of 10.



21. Writing a base as a power itself can be used as a way to simplify an expression. Copy and complete the following calculations.
   a. $16^{\frac{1}{2}} = (4^2)^{\frac{1}{2}} =$ ______
   b. $343^{\frac{2}{3}} = (7^3)^{\frac{2}{3}} =$ ______

22. Simplify the following using index laws.
   a. $8^{\frac{1}{3}}$
   b. $27^{\frac{4}{3}}$
   c. $125^{-\frac{2}{3}}$
   d. $512^{\frac{2}{9}}$
   e. $16^{-\frac{1}{2}}$
   f. $4^{-\frac{1}{2}}$
   g. $32^{-\frac{1}{5}}$
   h. $49^{-\frac{1}{2}}$

23. a. Use the index laws to simplify the following.
      i. $(3^2)^{\frac{1}{2}}$
      ii. $(4^2)^{\frac{1}{2}}$
      iii. $(8^2)^{\frac{1}{2}}$
      iv. $(11^2)^{\frac{1}{2}}$
   b. Use your answers from part **a** to calculate the value of the following.
      i. $9^{\frac{1}{2}}$
      ii. $16^{\frac{1}{2}}$
      iii. $64^{\frac{1}{2}}$
      iv. $121^{\frac{1}{2}}$
   c. Use your answers to parts **a** and **b** to write a sentence describing what happens when you raise a number to a power of one-half.

LESSON 1.6 Negative indices

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- apply the Seventh Index Law to evaluate expressions using negative indices
- apply the first seven index laws to simplify expressions involving indices.

1.6.1 The Seventh Index Law

- Negative indices occur when the power (or exponent) is a negative number.
For example, 3^{-2}.
- To explain the meaning or value of negative exponents it is useful to consider patterns of numbers written in exponent form.
For example, in the sequence $10^4 = 10\,000$, $10^3 = 1000$, $10^2 = 100$, $10^1 = 10$, $10^0 = 1$ each number is $\frac{1}{10}$ of the number before it.

Powers decrease by 1

10^4	10^3	10^2	10^1	10^0
10 000	1000	100	10	1

$\times\frac{1}{10}$ $\quad\times\frac{1}{10}$ $\quad\times\frac{1}{10}$ $\quad\times\frac{1}{10}$

- It is logical, then, that the next numbers in this sequence are:

$$10^{-1} = \frac{1}{10},\ 10^{-2} = \frac{1}{100},\ 10^{-3} = \frac{1}{1000},\ 10^{-4} = \frac{1}{10\,000}$$

Powers decrease by 1

10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10\,000}$
1	0.1	0.01	0.001	0.0001

$\times\frac{1}{3}$ $\quad\times\frac{1}{3}$ $\quad\times\frac{1}{3}$ $\quad\times\frac{1}{3}$

- When the base is 10, the negative indices show the relationship between the exponent and the decimal place value. For example, 10^{-2} means two place values to the right of the decimal.

The Seventh Index Law

$$a^{-n} = \frac{1}{a^n},\ \text{where } a \neq 0$$

For example, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

WORKED EXAMPLE 19 Evaluating using the Seventh Index Law

Evaluate the following.

a. 10^{-3}

b. 7^{-1}

c. $\left(\frac{3}{5}\right)^{-1}$

THINK	WRITE
a. 1. Apply the rule $a^{-n} = \frac{1}{a^n}$.	a. $10^{-3} = \frac{1}{10^3}$
2. Simplify and write the answer.	$= \frac{1}{1000}$ or 0.001
b. Apply the rule $a^{-n} = \frac{1}{a^n}$.	b. $7^{-1} = \frac{1}{7^1}$ $= \frac{1}{7}$
c. 1. Apply the Sixth Index Law, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	c. $\left(\frac{3^1}{5^1}\right) = \frac{3^{-1}}{5^{-1}}$
2. Apply the Seventh Index Law, $a^{-n} = \frac{1}{a^n}$, to the numerator and denominator.	$= \frac{1}{3} \div \frac{1}{5}$
3. Simplify and write the answer.	$= \frac{1}{3} \times \frac{5}{1}$ $= \frac{5}{3}$

WORKED EXAMPLE 20 Writing with positive indices

Write the following with positive indices.

a. x^{-3}

b. $5x^{-6}$

c. $\frac{x^{-3}}{y^{-2}}$

THINK	WRITE
a. Apply the Seventh Index Law, $a^{-n} = \frac{1}{a^n}$.	a. $x^{-3} = \frac{1}{x^3}$
b. 1. Write in expanded form and then apply the Seventh Index Law, $a^{-n} = \frac{1}{a^n}$.	b. $5x^{-6} = 5 \times x^{-6}$ $= 5 \times \frac{1}{x^6}$
2. Simplify and write the answer.	$= \frac{5}{x^6}$
c. 1. Write the fraction using division.	c. $\frac{x^{-3}}{y^{-2}} = x^{-3} \div y^{-2}$
2. Apply the Seventh Index Law, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{x^3} \div \frac{1}{y^2}$
3. Simplify and write the answer.	$= \frac{1}{x^3} \times \frac{y^2}{1}$ $= \frac{y^2}{x^3}$

WORKED EXAMPLE 21 Simplifying products of powers with positive and negative exponents

Simplify the following expressions, writing your answers with positive exponents.

a. $x^3 \times x^{-8}$

b. $3x^{-2}y^{-3} \times 5xy^{-4}$

THINK

a. 1. Apply the First Index Law $a^n \times a^m = a^{m+n}$.

$$x^3 \times x^{-8} = x^{3+-8} = x^{-5}$$

2. Write the answer with a positive exponent.

$$= \frac{1}{x^5}$$

b. 1. Write in expanded form and apply the First Index Law.

$$3x^{-2}y^{-3} \times 5xy^{-4} = 3 \times 5 \times x^{-2} \times x^1 \times y^{-3} \times y^{-4} = 15x^{-1}y^{-7}$$

2. Apply the rule $a^{-n} = \frac{1}{a^n}$.

$$= \frac{15}{1} \times \frac{1}{x} \times \frac{1}{y^7}$$

3. Simplify and write the answer.

$$= \frac{15}{xy^7}$$

(**WRITE** column shown as the equations above.)

WORKED EXAMPLE 22 Simplifying quotients of powers with positive and negative exponents

Simplify the following expressions, writing your answers with positive exponents.

a. $\frac{t^2}{t^{-5}}$

b. $\frac{15m^{-5}}{10m^{-2}}$

THINK / **WRITE**

a. Apply the Second Index Law, $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{t^2}{t^{-5}} = t^{2-(-5)} = t^{2+5} = t^7$$

b. 1. Apply the Second Index Law and simplify.

$$\frac{15m^{-5}}{10m^{-2}} = \frac{15}{10} \times \frac{m^{-5}}{m^{-2}} = \frac{3}{2} \times m^{-5-(-2)} = \frac{3}{2} \times m^{-3} = \frac{3}{2} \times \frac{1}{m^3}$$

2. Write the answer with positive indices.

$$= \frac{3}{2m^3}$$

DISCUSSION

What strategy will you use to remember the index laws?

Exercise 1.6 Negative indices

learnon

1.6 Quick quiz on | 1.6 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 5, 8, 11, 14, 17, 21	3, 6, 9, 12, 15, 18, 19, 22	4, 7, 10, 13, 16, 20, 23, 24

Fluency

1. Copy and complete the following patterns.

a. $3^5 = 243$
$3^4 = 81$
$3^3 = 27$
$3^2 =$
$3^1 =$
$3^0 =$
$3^{-1} = \frac{1}{3}$
$3^{-2} = \frac{1}{9}$
$3^{-3} =$
$3^{-4} =$
$3^{-5} =$

b. $5^4 = 625$
$5^3 =$
$5^2 =$
$5^1 =$
$5^0 =$
$5^{-1} =$
$5^{-2} =$
$5^{-3} =$
$5^{-4} =$

c. $10^4 = 10\ 000$
$10^3 =$
$10^2 =$
$10^1 =$
$10^0 =$
$10^{-1} =$
$10^{-2} =$
$10^{-3} =$
$10^{-4} =$

2. **WE19** Evaluate each of the following expressions.

a. 2^{-5} **b.** 3^{-3} **c.** 4^{-1} **d.** 10^{-2}

3. Evaluate each of the following expressions.

a. 5^{-3} **b.** $\left(\frac{1}{7}\right)^{-1}$ **c.** $\left(\frac{3}{4}\right)^{-1}$ **d.** $\left(\frac{3}{4}\right)^{-2}$

4. Evaluate each of the following expressions.

a. $\left(\frac{1}{3}\right)^{-3}$ **b.** $\left(\frac{3}{2}\right)^{-1}$ **c.** $\left(2\frac{1}{4}\right)^{-2}$ **d.** $\left(\frac{2}{7}\right)^{-2}$

5. **WE20** Write each expression with positive exponents.

a. x^{-4} **b.** y^{-5} **c.** z^{-1} **d.** a^2b^{-3} **e.** $m^{-2}n^{-3}$

6. Write each expression with positive indices.

a. $\left(m^2n^3\right)^{-1}$ **b.** $\frac{x^2}{y^{-2}}$ **c.** $\frac{5}{x^{-3}}$ **d.** $\frac{x^{-2}}{w^{-5}}$ **e.** $\frac{1}{x^{-2}y^{-2}}$

7. Write each expression with positive indices.

a. $a^2b^{-3}cd^{-4}$ **b.** $\frac{a^2b^{-2}}{c^2d^{-3}}$ **c.** $10x^{-2}y$ **d.** $3^{-1}x$ **e.** $\frac{m^{-3}}{x^2}$

Understanding

8. WE21 Simplify the following expressions, writing your answers with positive exponents.

a. $a^3 \times a^{-8}$ **b.** $m^7 \times m^{-2}$ **c.** $m^{-3} \times m^{-4}$ **d.** $2x^{-2} \times 7x$ **e.** $2b^{-2} \times 3b^2$ **f.** $ab^{-5} \times 5b^2$

9. Simplify the following expressions, writing your answers with positive indices.

a. $x^5 \times x^{-8}$ **b.** $3x^2y^{-4} \times 2x^{-7}y$ **c.** $10x^5 \times 5x^{-2}$
d. $x^5 \times x^{-5}$ **e.** $10a^2 \times 5a^{-7}$ **f.** $10a^{10} \times a^{-6}$

10. Simplify the following expressions, writing your answers with positive indices.

a. $16w^2 \times -2w^{-5}$ **b.** $4m^{-2} \times 4m^{-2}$ **c.** $\left(3m^2n^{-4}\right)^3$
d. $\left(a^2b^5\right)^{-3}$ **e.** $\left(a^{-1}b^{-3}\right)^{-2}$ **f.** $\left(5a^{-1}\right)^2$

11. WE22 Simplify the following expressions, writing your answers with positive exponents.

a. $\dfrac{x^3}{x^8}$ **b.** $\dfrac{x^{-3}}{x^8}$ **c.** $\dfrac{x^3}{x^{-8}}$ **d.** $\dfrac{x^{-3}}{x^{-8}}$ **e.** $\dfrac{6a^2c^5}{a^4c}$

12. Simplify the following expressions, writing your answers with positive indices.

a. $10a^2 \div 5a^8$ **b.** $5m^7 \div m^8$ **c.** $\dfrac{a^5b^6}{a^5b^7}$ **d.** $\dfrac{a^2b^8}{a^5b^{10}}$ **e.** $\dfrac{a^{-3}bc^3}{abc}$

13. Simplify the following expressions, writing your answers with positive exponents.

a. $\dfrac{4^{-2}ab}{a^2b}$ **b.** $\dfrac{m^{-3} \times m^{-5}}{m^{-5}}$ **c.** $\dfrac{2t^2 \times 3t^{-5}}{4t^6}$ **d.** $\dfrac{t^3 \times t^{-5}}{t^{-2} \times t^{-3}}$ **e.** $\dfrac{\left(m^2n^{-3}\right)^{-1}}{\left(m^{-2}n^3\right)^2}$

14. Write the following numbers as powers of 2.

a. 1 **b.** 8 **c.** 32 **d.** 64 **e.** $\dfrac{1}{8}$ **f.** $\dfrac{1}{32}$

15. Write the following numbers as powers of 4.

a. 1 **b.** 4 **c.** 64 **d.** $\dfrac{1}{4}$ **e.** $\dfrac{1}{16}$ **f.** $\dfrac{1}{64}$

16. Write the following numbers as powers of 10.

a. 1 **b.** 10 **c.** 10 000 **d.** 0.1 **e.** 0.01 **f.** 0.000 01

Reasoning

17. Answer the following questions.

a. The result of dividing 3^7 by 3^3 is 3^4. Determine the result of dividing 3^3 by 3^7.
b. Explain what it means to have a negative exponent.
c. Explain how you write a negative exponent as a positive exponent.

18. Indices are encountered in science, where they help to deal with very small and large numbers. The diameter of a proton is 0.000 000 000 000 3 cm.
Explain why it is logical to express this number in scientific notation as 3×10^{-13}.

19. Answer the following questions.

a. When asked to write an expression with positive indices that is equivalent to $x^3 + x^{-3}$, a student gave the answer x^0. Is this answer correct?
Explain why or why not.
b. When asked to write an expression with positive exponents that is equivalent to $\left(x^{-1} + y^{-1}\right)^{-2}$, a student gave the answer $x^2 + y^2$. Is this answer correct?
Explain why or why not.

20. Answer the following questions.

a. When asked to write an expression with positive indices that is equivalent to $x^8 - x^{-5}$, a student gave the answer x^3. Is this answer correct?
Explain why or why not.

b. Another student said that $\dfrac{x^2}{x^8 - x^5}$ is equivalent to $\dfrac{1}{x^6} - \dfrac{1}{x^3}$. Is this answer correct? Explain why or why not.

Problem solving

21. Write the following numbers as basic numerals.

a. 4.8×10^{-2} b. 7.6×10^3 c. 2.9×10^{-4} d. 8.1×10^0

22. Determine the value of n in the following expressions.

a. $4793 = 4.793 \times 10^n$ b. $0.631 = 6.31 \times 10^n$ c. $134 = 1.34 \times 10^n$ d. $0.000\,56 = 5.6 \times 10^n$

23. Evaluate the following.

a. Half of 2^{20} b. One-third of 3^{21}

24. Simplify the following expressions.

a. $\left(2^{-1} + 3^{-1}\right)^{-1}$ b. $\dfrac{3^{400}}{6^{200}}$

LESSON 1.7 Scientific notation

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify numbers that are written in scientific notation
- write numbers in scientific notation as decimal numbers
- write decimal numbers using scientific notation.

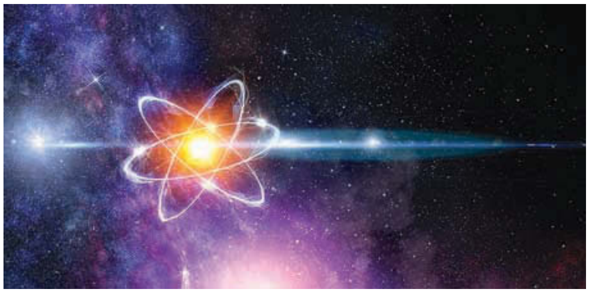
1.7.1 Using scientific notation

- Scientists often work with extremely large and extremely small numbers. These can range from numbers as large as 1 000 000 000 000 000 000 km (the approximate diameter of our galaxy) down to numbers as small as 0.000 000 000 06 mm (the diameter of an atom).
- Doing calculations with numbers written this way can be challenging — it can be easy to lose track of all of those zeros.
- It is more useful in these situations to use a notation system based on powers of 10. This system is called **scientific notation** (or **standard form**).
- A number written in scientific notation looks like this:

$$5.316 \times 10^2$$

A number between 1 and 10 × A power of 10

Using a calculator for scientific notation



- Numbers written in scientific notation can be entered into a calculator using its scientific notation buttons.
- Some calculators have different ways of displaying numbers written in scientific notation.
  For example, some calculators display 5.71×10^4 as 5.71E4.
- When you are writing down scientific notation yourself, you should always show this number as 5.71×10^4.

Writing any number in scientific notation

1. **Move the decimal point so that it is between the first and second significant figures.**
2. **Consider how many places the decimal point has moved — this number of places corresponds to the appropriate power of 10.**
3. **If the decimal point has moved to the left, the power of 10 will have a positive exponent or index (this happens with very large numbers).**
4. **If the decimal point has moved to the right, the power of 10 will have a negative exponent or index (this happens with very small numbers).**

WORKED EXAMPLE 23 Writing decimal numbers in scientific notation

Write each of these numbers in scientific notation.

a. **827.2** **b.** **51 920 000 000**

c. **0.0051** **d.** **0.000 000 007 648**

THINK	WRITE
a. 1. The numerical part must be written so it is a number between 1 and 10. This means we must move the decimal point 2 steps to the left.	**a.** 8.2 7 2 Decimal moves 2 steps left 8.272×10^2
2. Moving the decimal point 2 steps left corresponds to the power 10^2. Note that the number of steps moved is equal to the exponent.	
b. 1. Even though the decimal point is not written, we know that it lies after the final zero.	**b.** 5.192×10^{10}
2. To get a number between 1 and 10 we move the decimal 10 steps to the left. The corresponding power will be 10^{10}.	
c. To get a number between 1 and 10 we must move the decimal point 3 steps to the right. This corresponds to the power 10^{-3}. Note that moving to the right gives a negative exponent.	**c.** 5.1×10^{-3}
d. To get a number between 1 and 10 we must move the decimal point 9 steps to the right. This corresponds to the power 10^{-9}.	**d.** 7.648×10^{-9}

1.7.2 Converting scientific notation to decimal notation

- Converting scientific notation to decimal notation is simply the reverse of the process outlined in section 1.7.1.

WORKED EXAMPLE 24 Converting scientific notation with positive indices to decimals

The following numbers are written in scientific notation. Write them in decimal notation.

a. 7.136×10^2 b. 5.017×10^5 c. 8×10^6

THINK	WRITE
a. The exponent on the power of 10 is positive 2, therefore we move the decimal point 2 steps to the right.	a. $7.136 \times 10^2 = 713.6$
b. The exponent on the power of 10 is positive 5, therefore we move the decimal point 5 steps to the right. To do this we will need to add extra zeros.	b. $5.017 \times 10^5 = 501\,700$
c. The exponent on the power of 10 is positive 6, therefore we move the decimal point 6 steps to the right. While the decimal point has not been written as part of the scientific notation, we know it lies after the 8. We will need to add extra zeros.	c. $8 \times 10^6 = 8\,000\,000$

WORKED EXAMPLE 25 Converting scientific notation with negative indices to decimals

Write these numbers in decimal notation.

a. 9.12×10^{-1} b. 7.385×10^{-2} c. 6.32×10^{-7}

THINK	WRITE
a. The exponent on the power of 10 is negative 1, therefore we move the decimal point 1 step to the left.	a. $9.12 \times 10^{-1} = 0.912$
b. The exponent on the power of 10 is negative 2, therefore we move the decimal point 2 steps to the left. To do this we will need to add extra zeros.	b. $7.3857 \times 10^{-2} = 0.073\,857$
c. The exponent on the power of 10 is negative 7, therefore we move the decimal point 7 steps to the left. To do this we will need to add extra zeros.	c. $6.32 \times 10^{-7} = 0.000\,000\,632$

WORKED EXAMPLE 26 Representing decimals in exponential form

Write these numbers in exponential form.

a. **0.475** b. **0.03** c. **0.00023**

THINK	WRITE
a. 1. Write 0.475 in expanded fraction form.	a. $0.475 = \frac{4}{10} + \frac{7}{100} + \frac{5}{1000}$
2. Express the expanded from in the exponential form.	$= 4 \times 10^{-1} + 7 \times 10^{-2} + 5 \times 10^{-3}$

b. **1.** Write 0.03 in expanded fraction form.

2. Express the expanded fraction from in the exponential form.

b. $0.03 = \frac{0}{10} + \frac{3}{100}$
$= 3 \times 10^{-2}$

c. **1.** Write 0.00023 in expanded form.

2. Express the expanded from in the exponential form.

c. $0.00023 = \frac{0}{10} + \frac{0}{100} + \frac{0}{1000} + \frac{2}{10\,000} + \frac{3}{100\,000}$
$= 2 \times 10^{-4} + 3 \times 10^{-5}$

Metric prefixes for SI units

The following prefixes allow us to describe tiny numbers as well as incredibly large numbers of units succinctly.

Symbol	Prefix	Multiplication Factor	
T	tera	10^{12}	1 000 000 000 000
G	giga	10^{9}	1 000 000 000
M	mega	10^{6}	1 000 000
k	kilo	10^{3}	1 000
h	hecto	10^{2}	100
da	deka	10^{1}	10
d	deci	10^{-1}	0.1
c	centi	10^{-2}	0.01
m	milli	10^{-3}	0.001
μ	micro	10^{-6}	0.000 001
n	nano	10^{-9}	0.000 000 001
p	pico	10^{-12}	0.000 000 000 001

For example, 1 pico-gram = 10^{-12} gram and 1 terabyte = 10^{12} bytes

Exercise 1.7 Scientific notation

learnon

1.7 Quick quiz on | 1.7 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 6, 7, 10, 13, 14, 19, 23	2, 5, 8, 11, 15, 16, 20, 24	3, 9, 12, 17, 18, 21, 22, 25, 26

Fluency

1. WE23 Write these numbers in scientific notation.
   a. 5000 b. 431 c. 38 d. 350 000
2. Write these numbers in decimal notation.
   a. 1×10^9 b. $3.926\,73 \times 10^2$ c. 5.911×10^2 d. 5.1×10^3
3. Write these numbers in decimal notation.
   a. 7.34×10^5 b. 7.1414×10^6 c. 3.51×10 d. 8.05×10^4
4. WE24 Write these numbers in decimal notation.
   a. 6.14×10^2 b. 6.14×10^3 c. 6.14×10^4 d. 3.518×10^2
5. Write these numbers in scientific notation.
   a. 72.5 b. 725 c. 7250 d. 725 000 000
6. Write these numbers in scientific notation, correct to 4 significant figures.
   a. 43.792 b. 5317 c. 258.95 d. 110.11 e. 1 632 000 f. 1 million
7. WE25 Write these numbers in decimal notation.
   a. 2×10^{-1} b. 4×10^{-3} c. 7×10^{-4} d. 3×10^{-2}
8. Write these numbers in decimal notation.
   a. 8.273×10^{-2} b. 7.295×10^{-2} c. 2.9142×10^{-3} d. 3.753×10^{-5}
9. Write these numbers in decimal notation.
   a. 5.29×10^{-4} b. 3.3333×10^{-5} c. 2.625×10^{-9} d. 1.273×10^{-15}
10. Write these numbers in scientific notation.
    a. 0.7 b. 0.005 c. 0.000 000 3 d. 0.000 000 000 01
11. WE26 Write these numbers in exponential form.
    a. 0.231 b. 0.003 62 c. 0.000 7 d. 0.063
12. Write these numbers in scientific notation, correct to 3 significant figures.
    a. 0.006 731 b. 0.142 57 c. 0.000 068 3
    d. 0.000 000 005 12 e. 0.0509 f. 0.012 46

Understanding

13. Write each of the following sets of numbers in ascending order.
    a. 8.31×10^2, 3.27×10^3, 9.718×10^2, 5.27×10^2
    b. 7.95×10^2, 4.09×10^2, 7.943×10^2, 4.37×10^2
    c. 5.31×10^{-2}, 9.29×10^{-3}, 5.251×10^{-2}, 2.7×10^{-3}
    d. 8.31×10^2, 3.27×10^3, 7.13×10^{-2}, 2.7×10^{-3}

14. One carbon atom weighs 1.994×10^{-23} g.

a. Write this weight as a decimal.
b. Calculate how much one million carbon atoms will weigh.
c. Calculate how many carbon atoms there are in 10 g of carbon. Give your answer correct to 4 significant figures.

15. The distance from Earth to the Moon is approximately 3.844×10^5 km. If you could drive there at a constant speed of 100 km/h, calculate how long it would take. Determine how long that is in days, correct to 2 decimal places.

16. Earth weighs 5.97×10^{24} kg and the Sun weighs 1.99×10^{30} kg. Calculate how many Earths it would take to balance the Sun's weight. Give your answer correct to 2 decimal places.

17. Inside the nucleus of an atom, a proton weighs 1.6703×10^{-28} kg and a neutron weighs 1.6726×10^{-27} kg. Determine which one is heavier and by how much.

18. Earth's orbit has a radius of 7.5×10^7 km and the orbit of Venus has a radius of 5.4×10^7 km.

Calculate how far apart the planets are when:

a. they are closest to each other
b. they are furthest apart from each other.

Reasoning

19. A USB stick has 8 MB (megabytes) of storage.
1 MB = 1 048 576 bytes.

a. Determine the number of bytes in 8 MB of storage correct to the nearest 1000.
b. Write the answer to part **a** in scientific notation.

20. The basic unit of electric current is the ampere. It is defined as the constant current flowing in 2 parallel conductors 1 metre apart in a vacuum, which produces a force between the conductors of 2×10^7 newtons (N) per metre.
Complete the following statement. Write the answer as a decimal.
1 ampere $= 2 \times 10^7$ N/m = _____________ N/m

21. a. Without performing the calculation, state the power(s) of 10 that you believe the following equations will have when solved.

i. $5.36 \times 10^7 + 2.95 \times 10^3$ ii. $5.36 \times 10^7 - 2.95 \times 10^3$

b. Evaluate equations **i** and **ii**, correct to 3 significant figures.
c. Were your answers to part **a** correct? Why or why not?

22. Explain why $2.39 \times 10^{-3} + 8.75 \times 10^{-7} = 2.39 \times 10^{-3}$, correct to 3 significant figures.

Problem solving

23. Distance is equal to speed multiplied by time. If we travelled at 100 km/h it would take us approximately 0.44 years to reach the Moon, 89.6 years to reach Mars, 1460 years to reach Saturn and 6590 years to reach Pluto.

a. Assuming that there are 365 days in a year, calculate the distance (as a basic numeral) between Earth and:

i. the Moon **ii.** Mars **iii.** Saturn **iv.** Pluto.

b. Write your answers to part **a**, correct to 3 significant figures.

c. Write your answers to part **a** using scientific notation, correct to 3 significant figures.

24. A light-year is the distance that light travels in one year. Light travels at approximately 300 000 km/s.

a. i. Calculate the number of seconds in a year (assuming 1 year = 365 days).

ii. Write your answer to part **i** using scientific notation.

b. Calculate the distance travelled by light in one year. Express your answer:

i. as a basic numeral

ii. using scientific notation.

c. The closest star to Earth (other than the Sun) is in the star system Alpha Centauri, which is 4.3 light-years away.

i. Determine how far this is in kilometres, correct to 4 significant figures.

ii. If you were travelling at 100 km/h, determine how many years it would take to reach Alpha Centauri.

25. Scientists used Earth's gravitational pull on nearby celestial bodies (for example, the Moon) to calculate the mass of Earth. Their answer was that Earth weighs approximately 5.972 sextillion metric tonnes.

a. Write 5.972 sextillion using scientific notation.

b. Determine how many significant figures this number has.

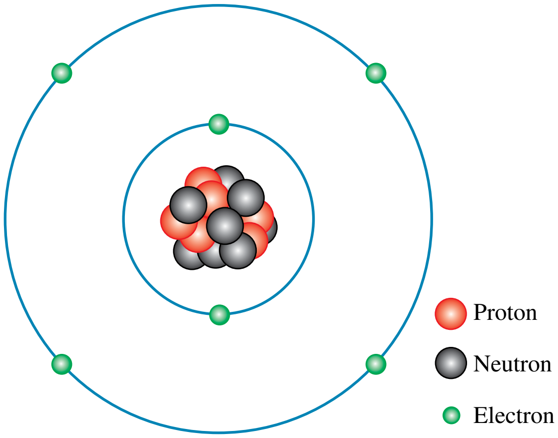
26. Atoms are made up of smaller particles called protons, neutrons and electrons. Electrons have a mass of $9.109\,381\,88 \times 10^{-31}$ kilograms, correct to 9 significant figures.

a. Write the mass of an electron correct to 5 significant figures.

b. Protons and neutrons are the same size. They are both 1836 times the size of an electron. Use the mass of an electron (correct to 9 significant figures) and your calculator to evaluate the mass of a proton correct to 5 significant figures.

c. Use the mass of an electron (correct to 3 significant figures) to calculate the mass of a proton, correct to 5 significant figures.

d. Explain why it is important to work with the original amounts and then round to the specified number of significant figures at the end of a calculation.



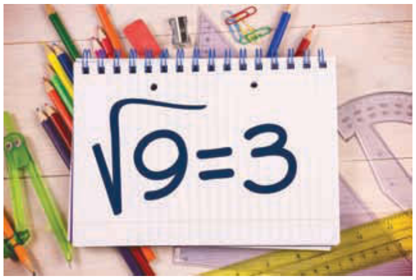
LESSON 1.8 Square roots and cube roots

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- write square and cube roots using the radical symbol
- use fractional indices to represent square roots and cube roots
- simplify expressions that use square roots and cube roots.

1.8.1 Square roots

- The symbol $\sqrt{}$ (the radical symbol) means '**square root**' — a number that multiplies by itself to give the original number.
- Every number actually has a positive and negative square root. For example, $(2)^2 = 4$ and $(-2)^2 = 4$.
- This means that the square root of 4 is +2 or –2. For this topic, however, assume that $\sqrt{}$ is positive unless otherwise indicated.
- A square root is the inverse of squaring (raising to the power 2).
- Because of this, a square root of a number is equivalent to raising that number to an exponent of $\frac{1}{2}$.
- In general, $\sqrt{a} = a^{\frac{1}{2}}$.



WORKED EXAMPLE 27 Evaluating square roots

Evaluate $\sqrt{16p^2}$.

THINK	WRITE
1. We need to work out the square root of both 16 and p^2.	$\sqrt{16p^2} = \sqrt{16} \times \sqrt{p^2}$
2. Remember that 4 is multiplied by itself to give 16, so the square root of 16 is 4. For the square root of the pronumeral, replace the square root sign with a power of $\frac{1}{2}$.	$= 4 \times \left(p^2\right)^{\frac{1}{2}}$
3. Use the Fourth Index Law.	$= 4 \times p^{2 \times \frac{1}{2}}$ $= 4 \times p^1$
4. Simplify and write the answer.	$= 4p$

1.8.2 Cube roots

- The symbol $\sqrt[3]{}$ means '**cube root**' — a number that multiplies by itself 3 times to give the original number.
- The cube root is the inverse of cubing (raising to the power 3).
- Because of this, a cube root is equivalent to raising a number to an exponent of $\frac{1}{3}$.
- In general, $\sqrt[3]{a} = a^{\frac{1}{3}}$.
- In more general terms, $a^{\frac{n}{m}} = \sqrt[m]{a^n}$.

WORKED EXAMPLE 28 Evaluating cube roots

Evaluate $\sqrt[3]{8j^6}$

THINK	WRITE
1. We need to work out the cube root of both 8 and j^6.	$\sqrt[3]{8j^6} = \sqrt[3]{8} \times \sqrt[3]{j^6}$
2. Remember that 2 multiplied by itself 3 times gives 8, so the cube root of 8 is 2. For the cube root of the pronumeral, replace the cube root sign with a power of $\frac{1}{3}$.	$= 2 \times \left(j^6\right)^{\frac{1}{3}}$
3. Use the Fourth Index Law.	$= 2 \times j^{6 \times \frac{1}{3}}$ $= 2 \times j^2$
4. Simplify and write the answer.	$= 2j^2$

DISCUSSION

Why does the cube root of a number always have the same sign (positive or negative) as the number itself?

Exercise 1.8 Square roots and cube roots

learnon

1.8 Quick quiz on | 1.8 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 7, 10, 14	2, 5, 8, 11, 15, 16	3, 6, 9, 12, 13, 17

Fluency

1. Write the following in exponent form.
   a. $\sqrt{15}$ b. $\sqrt{m}$ c. $\sqrt[3]{t}$ d. $\sqrt[3]{w^2}$
2. Evaluate the following.
   a. $49^{\frac{1}{2}}$ b. $4^{\frac{1}{2}}$ c. $27^{\frac{1}{3}}$ d. $125^{\frac{1}{3}}$
3. Evaluate the following.
   a. $1000^{\frac{1}{3}}$ b. $64^{\frac{1}{2}}$ c. $64^{\frac{1}{3}}$
   d. $1\,000\,000^{\frac{1}{2}}$ e. $1\,000\,000^{\frac{1}{3}}$ f. $\left(27^{\frac{1}{3}}\right)^2$

Understanding

4. WE27 Simplify the following expressions.
   a. $\sqrt{m^2}$ b. $\sqrt[3]{b^3}$ c. $\sqrt{36t^4}$ d. $\sqrt[3]{m^3n^6}$

5. Simplify the following expressions.

a. $\sqrt[3]{125t^6}$ b. $\sqrt[5]{x^5y^{10}}$ c. $\sqrt[4]{a^8m^{40}}$ d. $\sqrt[3]{216y^6}$

6. Simplify the following expressions.

a. $\sqrt[3]{64x^6y^6}$ b. $\sqrt{25a^2b^4c^6}$ c. $\sqrt[7]{b^{49}}$ d. $\sqrt[3]{b^3} \times \sqrt{b^4}$

7. WE28 MC $\sqrt[3]{8000m^6n^3p^3q^6}$ is equal to:

A. $2666.6m^2npq^2$ B. $20m^2npq^2$ C. $20m^3n^0p^0q^3$ D. $7997m^2npq^2$

8. MC $\sqrt[3]{3375a^9b^6c^3}$ is equal to:

A. $1125a^3b^2c$ B. $1125a^6b^3c^0$ C. $1123a^6b^3$ D. $15a^3b^2c$

9. MC $\sqrt[3]{15\,625f^3g^6h^9}$ is equal to:

A. $25fg^2h^3$ B. $25f^0g^3h^6$ C. $25g^3h^6$ D. $5208.3fg^2h^3$

Reasoning

10. Answer the following questions.

a. Using the First Index Law, explain how $3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3$.

b. Express $3^{\frac{1}{2}}$ in another way.

c. Write $\sqrt[n]{a}$ in exponent form.

d. Without a calculator, evaluate $8^{\frac{1}{3}}$.

11. Answer the following questions.

a. Explain why calculating $z^{2.5}$ is a square root problem.
b. Explain whether $z^{0.3}$ is a cube root problem.

12. Prithvi and Yasmin are having an algebra argument. Prithvi is sure that $\sqrt{x^2}$ is equivalent to x, but Yasmin thinks otherwise. State who is correct.
Explain how you would resolve this disagreement.

13. Verify that $(-8)^{\frac{1}{3}}$ can be evaluated and explain why $(-8)^{\frac{1}{4}}$ cannot be evaluated.

Problem solving

14. The British mathematician Augustus de Morgan enjoyed telling his friends that he was x years old in the year x^2. Determine the year of Augustus de Morgan's birth, given that he died in 1871.

15. Kepler's Third Law describes the relationship between the distance of planets from the Sun and their orbital periods. It is represented by the equation $d^{\frac{1}{2}} = t^{\frac{1}{3}}$. Solve for:

a. d in terms of t b. t in terms of d.

16. If $n^{\frac{3}{4}} = \frac{8}{27}$, evaluate the value of n.

17. An unknown number is multiplied by 4 and then has 5 subtracted from it. It is now equal to the square root of the original unknown number, squared.

a. Determine how many solutions to this problem are possible. Explain why.
b. Determine all possible values for the unknown number.

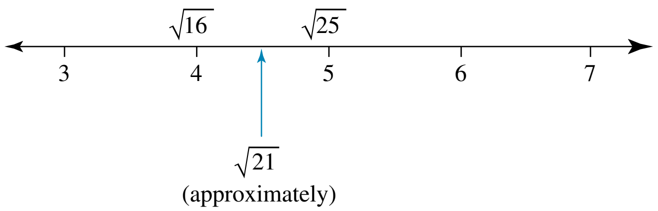
LESSON 1.9 Surds (extending)

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify when a square root forms a surd
- place surds in their approximate position on a number line
- simplify expressions containing surds.

1.9.1 Identifying surds

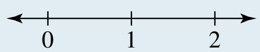
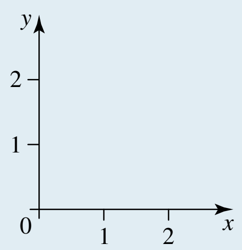
- In many cases the square root or cube root of a number results in a rational number.
  For example, $\sqrt{25} = \pm 5$ and $\sqrt[3]{1.728} = 1.2$.
- When the square root of a number is an irrational number, it is called a **surd**.
- For example, $\sqrt{10} \approx 3.162\,277\,660\,17\ldots$ Since $\sqrt{10}$ cannot be written as a fraction, a recurring decimal or a **terminating decimal**, it is irrational and therefore it is a surd.
- The value of a surd can be approximated using a number line.
  For example, we know that $\sqrt{21}$ lies between 4 and 5, because it lies between $\sqrt{16}$ (which equals 4) and $\sqrt{25}$ (which equals 5).
  We can show its approximate position on the number line like this:



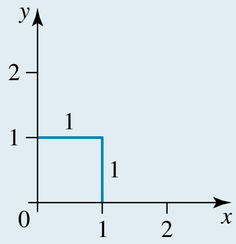
- Geometric constructions can be used to locate rational numbers and square root on a number line.
  For example, $\sqrt{2}$ is located at the intersection of an arc and the number line, where the radius of the arc is the length of the diagonal of a one-unit square.

WORKED EXAMPLE 29 Locating the square root on a number line

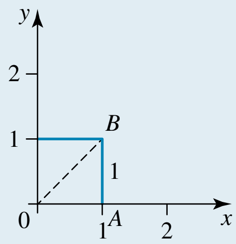
Represent $\sqrt{2}$ on the number line.

THINK	WRITE/ DRAW
1. Start with a horizontal number line from 0 to 2.	
2. Extend a vertical number line up from the origin.	

3. Draw in a unit square with its lower left corner at the origin. All sides of the unit square are 1 unit.



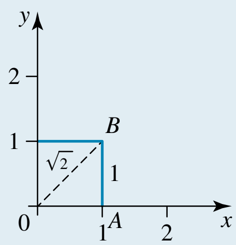
4. Draw a diagonal of the unit square from the origin, O to the opposite corner, B.



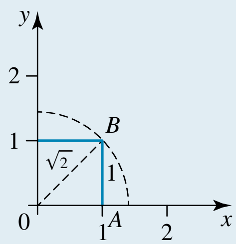
5. Apply the Pythagoras theorem to determine the length of the side OB:
$OB^2 = OA^2 + AB^2$

$$\begin{aligned} OB^2 &= OA^2 + AB^2 \\ &= 1^2 + 1^2 \\ &= 2 \\ OB &= \sqrt{2} \end{aligned}$$

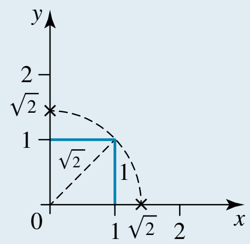
The length of the diagonal is $\sqrt{2}$.



6. Now to locate $\sqrt{2}$ accurately on the number line, draw an arc with its centre at the origin, O and its radius (diagonal) of $\sqrt{2}$ using a compass.



7. Label $\sqrt{2}$ at the intersections of the arc on both the horizontal and vertical number lines.



$\sqrt{2}$ is located at the intersection of an arc and the number line, where the radius of the arc is the length of the diagonal of a one-unit square.

WORKED EXAMPLE 30 Identifying surds

Identify which of the following are surds.

a. $\sqrt{0}$

b. $\sqrt{20}$

c. $-\sqrt{9}$

d. $\sqrt[3]{6}$

THINK

a. $\sqrt{0} = 0$. This is a rational number and therefore not a surd.

b. $\sqrt{20} \approx 4.472\,135\,955\ldots$ This is an irrational number and therefore a surd.

c. $-\sqrt{9} = -3$. This is a rational number and therefore not a surd.

d. $\sqrt[3]{6} \approx 1.817\,120\,592\,83\ldots$ This is an irrational number and therefore a surd.

WRITE

a. $\sqrt{0} = 0$. This is not a surd.

b. $\sqrt{20}$ is a surd.

c. $-\sqrt{9} = -3$. This is not a surd.

d. $\sqrt[3]{6}$ is a surd.

1.9.2 Multiplying and dividing surds

- Multiplication of surds is done by multiplying the numerical parts of each surd under a single radical sign.
- Division of surds is done by placing the quotient of the numerical parts of each surd under a single radical sign.

Multiplying surds

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

Dividing surds

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

WORKED EXAMPLE 31 Multiplying surds

Evaluate the following, leaving your answer in surd form.

a. $\sqrt{7} \times \sqrt{2}$

b. $5 \times \sqrt{3}$

c. $\sqrt{5} \times \sqrt{5}$

d. $-2\sqrt{3} \times 4\sqrt{5}$

THINK

a. Apply the rule $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

b. Only $\sqrt{3}$ is a surd. It is multiplied by 5, which is not a surd.

WRITE

a. $\sqrt{7} \times \sqrt{2} = \sqrt{14}$

b. $5 \times \sqrt{3} = 5\sqrt{3}$

c. Apply the rule $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

c. $\sqrt{5 \times 5} = \sqrt{25}$
$= 5$

d. Multiply the whole numbers. Then multiply the surds.

d. $-2\sqrt{3} \times 4\sqrt{5} = -2 \times 4 \times \sqrt{3} \times \sqrt{5}$
$= -8 \times \sqrt{15}$
$= -8\sqrt{15}$

WORKED EXAMPLE 32 Dividing surds

Evaluate the following, leaving your answer in surd form.

a. $\dfrac{\sqrt{10}}{\sqrt{5}}$ b. $\sqrt{\dfrac{10}{5}}$ c. $\dfrac{-6\sqrt{8}}{4\sqrt{4}}$ d. $\dfrac{\sqrt{20}}{\sqrt{5}}$ e. $\dfrac{5}{\sqrt{5}}$

THINK	WRITE
a. Apply the rule $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$.	a. $\frac{\sqrt{10}}{\sqrt{5}} = \sqrt{\frac{10}{5}}$ $= \sqrt{2}$
b. Simplify the fraction.	b. $\sqrt{\frac{10}{5}} = \sqrt{2}$
c. Simplify the whole numbers. Then apply the rule $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$.	c. $\frac{-6\sqrt{8}}{4\sqrt{4}} = \frac{-3\sqrt{2}}{2}$
d. Apply the rule $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$.	d. $\frac{\sqrt{20}}{\sqrt{5}} = \sqrt{\frac{20}{5}}$ $= \sqrt{4}$ $= 2$
e. Rewrite the numerator as the product of two surds and then simplify.	e. $\frac{5}{\sqrt{5}} = \frac{\sqrt{5} \times \sqrt{5}}{\sqrt{5}}$ $= \sqrt{5}$

1.9.3 Simplifying surds

- Just as a rational number can be written many different ways $\left(\text{e.g. } \frac{1}{2} = \frac{5}{10} = \frac{7}{14}\right)$, so can a surd. It is expected that surds should normally be written in their simplest form.
- A surd is in its simplest form when the number inside the radical sign has the smallest possible value.
- Note that $\sqrt{24}$ can be factorised several ways.
  For example:

$$\sqrt{24} = \sqrt{2} \times \sqrt{12}$$

$$\sqrt{24} = \sqrt{3} \times \sqrt{8}$$

$$\sqrt{24} = \sqrt{4} \times \sqrt{6}$$

In the last example, $\sqrt{4} = 2$, which means:

$$\begin{aligned}\sqrt{24} &= 2 \times \sqrt{6} \\ &= 2\sqrt{6}\end{aligned}$$

$2\sqrt{6}$ is equal to $\sqrt{24}$, and $2\sqrt{6}$ is $\sqrt{24}$ written in its simplest form.

- To simplify a surd you must determine a factor that is also a perfect square. For example, $4,\ 9,\ 16,\ 25,\ 36$ or 49.
- A surd like $\sqrt{22}$, for example, cannot be simplified because 22 has no perfect square factors.
- Surds can be simplified in more than one step.

$$\begin{aligned}\sqrt{72} &= \sqrt{4} \times \sqrt{18} \\ &= 2\sqrt{18} \\ &= 2 \times \sqrt{9} \times \sqrt{2} \\ &= 2 \times 3\sqrt{2} \\ &= 6\sqrt{2}\end{aligned}$$

WORKED EXAMPLE 33 Simplifying surds

Simplify the following surds.

a. $\sqrt{18}$ **b.** $6\sqrt{20}$

THINK	WRITE
a. 1. Rewrite 18 as the product of two numbers, one of which is a perfect square (9).	**a.** $\sqrt{18} = \sqrt{9} \times \sqrt{2}$
2. Simplify.	$= 3 \times \sqrt{2}$ $= 3\sqrt{2}$
b. 1. Rewrite 20 as the product of two numbers, one of which is square (4).	**b.** $6\sqrt{20} = 6 \times \sqrt{4} \times \sqrt{5}$
2. Simplify.	$= 6 \times 2 \times \sqrt{5}$ $= 12\sqrt{5}$

1.9.4 Entire surds

- The surd $\sqrt{45}$, when simplified, is written as $3\sqrt{5}$.
  The surd $\sqrt{45}$ is called an entire surd because it is written entirely inside the radical sign. The surd $3\sqrt{5}$, however, is not an entire surd.
- Writing a surd as an entire surd reverses the process of simplification.

WORKED EXAMPLE 34 Writing entire surds

Write $3\sqrt{7}$ as an entire surd.

THINK	WRITE
1. In order to place the 3 inside the radical sign it has to be written as $\sqrt{9}$.	$3\sqrt{7} = \sqrt{9} \times \sqrt{7}$
2. Apply the rule $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.	$= \sqrt{63}$

WORKED EXAMPLE 35 Comparing surds

Determine which number is larger, $3\sqrt{5}$ or $5\sqrt{3}$.

THINK	WRITE
1. Write $3\sqrt{5}$ as an entire surd.	$3\sqrt{5} = \sqrt{9} \times \sqrt{5}$ $= \sqrt{45}$
2. Write $5\sqrt{3}$ as an entire surd.	$5\sqrt{3} = \sqrt{25} \times \sqrt{3}$ $= \sqrt{75}$
3. Compare the values of each surd.	$\sqrt{75} > \sqrt{45}$
4. Write your answer.	$5\sqrt{3}$ is the larger number.

1.9.5 Addition and subtraction of surds

- Surds can be added or subtracted if they have like terms.
- Surds should be simplified before adding or subtracting like terms.

WORKED EXAMPLE 36 Adding and subtracting surds

Simplify each of the following.

a. $6\sqrt{3} + 2\sqrt{3} + 4\sqrt{5} - 5\sqrt{5}$ **b.** $3\sqrt{2} - 5 + 4\sqrt{2} + 9$

THINK	WRITE
a. Collect the like terms ($\sqrt{3}$ and $\sqrt{5}$).	a. $6\sqrt{3} + 2\sqrt{3} + 4\sqrt{5} - 5\sqrt{5}$ $= 8\sqrt{3} - \sqrt{5}$
b. Collect the like terms and simplify.	b. $3\sqrt{2} - 5 + 4\sqrt{2} + 9$ $= 3\sqrt{2} + 4\sqrt{2} - 5 + 9$ $= 7\sqrt{2} + 4$

WORKED EXAMPLE 37 Simplifying surd expressions

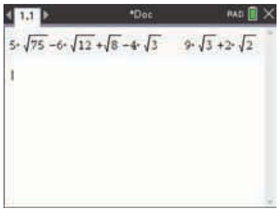
Simplify $5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$.

THINK	WRITE
1. Simplify $5\sqrt{75}$.	$5\sqrt{75} = 5 \times \sqrt{25} \times \sqrt{3}$ $= 5 \times 5 \times \sqrt{3}$ $= 25\sqrt{3}$
2. Next, simplify $6\sqrt{12}$.	$6\sqrt{12} = 6 \times \sqrt{4} \times \sqrt{3}$ $= 6 \times 2 \times \sqrt{3}$ $= 12\sqrt{3}$
3. Lastly, simplify $\sqrt{8}$.	$\sqrt{8} = \sqrt{4} \times \sqrt{2}$ $= 2\sqrt{2}$
4. Rewrite the original expression and simplify by adding like terms.	$5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$ $= 25\sqrt{3} - 12\sqrt{3} + 2\sqrt{2} - 4\sqrt{3}$ $= 9\sqrt{3} + 2\sqrt{2}$

TI | THINK

In a new problem, on a Calculator page, complete the entry line as: $5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$
Press ENTER.

WRITE

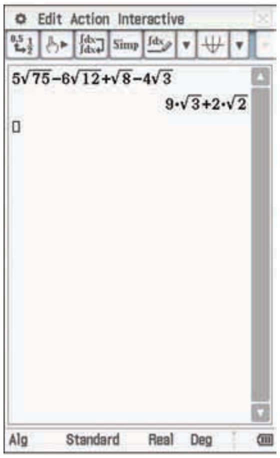


$5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$
$= 9\sqrt{3} + 2\sqrt{2}$

CASIO | THINK

Make sure the calculator is set to Standard mode.
On the Main screen, complete the entry line as:
$5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$
Press EXE.

WRITE



$5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$
$= 9\sqrt{3} + 2\sqrt{2}$

DISCUSSION

Are all square root numbers surds?

Exercise 1.9 Surds (extending)

1.9 Quick quiz on | 1.9 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 5, 7, 10, 13, 16, 19, 21, 23, 24, 27, 30, 35, 39	2, 4, 9, 11, 15, 17, 22, 25, 29, 31, 32, 36, 40	3, 6, 8, 12, 14, 18, 20, 26, 28, 33, 34, 37, 38, 41, 42

Fluency

1. Write down the square roots of each of the following.
   a. 1 b. 4 c. 0 d. $\frac{1}{9}$ e. $1\frac{9}{16}$
2. Write down the square roots of each of the following.
   a. 0.16 b. 400 c. 10 000 d. $\frac{4}{25}$ e. 1.44
3. Write down the square roots of each of the following.
   a. 20.25 b. 1 000 000 c. 0.0009 d. 256 e. $2\frac{23}{49}$
4. Write down the value of each of the following.
   a. $\sqrt{81}$ b. $-\sqrt{81}$ c. $\sqrt{121}$ d. $-\sqrt{441}$
5. Write down the value of each of the following.
   a. $\sqrt[3]{8}$ b. $\sqrt[3]{64}$ c. $\sqrt[3]{343}$ d. $\sqrt[4]{81}$
6. Write down the value of each of the following.
   a. $\sqrt[5]{1024}$ b. $\sqrt[3]{125}$ c. $-\sqrt{49}$ d. $\sqrt[3]{-27}$
7. WE29 Place the following on a number line.
   a. $\sqrt{3}$ b. $\sqrt{7}$ c. $\sqrt{12}$ d. $\sqrt{34}$
8. WE30 Identify which of the following are surds.
   a. $-\sqrt[3]{216}$ b. $\sqrt[3]{-216}$ c. $-\sqrt{2}$ d. $1+\sqrt{2}$
9. Identify which of the following are surds.
   a. 1.32 b. $1.\dot{3}\dot{2}$ c. $\sqrt[4]{64}$ d. 1.752 16
10. WE31 Simplify each of the following.
    a. $\sqrt{3}\times\sqrt{7}$ b. $-\sqrt{3}\times\sqrt{7}$ c. $2\times\sqrt{6}$
    d. $2\times3\sqrt{7}$ e. $2\sqrt{7}\times5\sqrt{2}$
11. Simplify each of the following.
    a. $3\sqrt{7}\times4$ b. $\sqrt{7}\times9$ c. $-2\sqrt{5}\times-11\sqrt{2}$
    d. $2\sqrt{3}\times11$ e. $\sqrt{3}\times\sqrt{3}$
12. Simplify each of the following.
    a. $-3\sqrt{2}\times-5\sqrt{5}$ b. $2\sqrt{3}\times4\sqrt{3}$ c. $\sqrt{6}\times\sqrt{6}$
    d. $\sqrt{11}\times\sqrt{11}$ e. $\sqrt{15}\times2\sqrt{15}$

13. WE32 Simplify each of the following.

a. $\sqrt{\frac{12}{4}}$
b. $-\sqrt{\frac{10}{5}}$
c. $\frac{\sqrt{18}}{\sqrt{3}}$
d. $\frac{-\sqrt{15}}{-\sqrt{3}}$

14. Simplify each of the following.

a. $\frac{15\sqrt{6}}{5\sqrt{2}}$
b. $\frac{15\sqrt{6}}{10}$
c. $\frac{15\sqrt{6}}{\sqrt{3}}$
d. $\frac{5}{3}\sqrt{\frac{15}{3}}$

15. Simplify each of the following.

a. $\frac{-10\sqrt{10}}{5\sqrt{2}}$
b. $\frac{\sqrt{9}}{\sqrt{3}}$
c. $\frac{3}{\sqrt{3}}$
d. $\frac{7}{\sqrt{7}}$

16. WE33 Simplify each of the following.

a. $\sqrt{20}$
b. $\sqrt{8}$
c. $\sqrt{18}$
d. $\sqrt{49}$

17. Simplify each of the following.

a. $\sqrt{30}$
b. $\sqrt{50}$
c. $\sqrt{28}$
d. $\sqrt{108}$

18. Simplify each of the following

a. $\sqrt{288}$
b. $\sqrt{48}$
c. $\sqrt{500}$
d. $\sqrt{162}$

19. Simplify each of the following.

a. $2\sqrt{8}$
b. $5\sqrt{27}$
c. $6\sqrt{64}$
d. $7\sqrt{50}$
e. $10\sqrt{24}$

20. Simplify each of the following.

a. $5\sqrt{12}$
b. $4\sqrt{42}$
c. $12\sqrt{72}$
d. $9\sqrt{45}$
e. $12\sqrt{242}$

21. WE34 Write each of the following in the form $\sqrt{a}$ (that is, as an entire surd).

a. $2\sqrt{3}$
b. $5\sqrt{7}$
c. $6\sqrt{3}$
d. $4\sqrt{5}$
e. $8\sqrt{6}$

22. Write each of the following in the form $\sqrt{a}$ (that is, as an entire surd).

a. $3\sqrt{10}$
b. $4\sqrt{2}$
c. $12\sqrt{5}$
d. $10\sqrt{6}$
e. $13\sqrt{2}$

23. WE35 Determine which number in each of the following pairs of numbers is larger.

a. $\sqrt{10}$ or $2\sqrt{3}$
b. $3\sqrt{5}$ or $5\sqrt{2}$
c. $10\sqrt{2}$ or $4\sqrt{5}$
d. $2\sqrt{10}$ or $\sqrt{20}$

24. WE36 Simplify each of the following.

a. $6\sqrt{2}+3\sqrt{2}-7\sqrt{2}$
b. $4\sqrt{5}-6\sqrt{5}-2\sqrt{5}$
c. $-3\sqrt{3}-7\sqrt{3}+4\sqrt{3}$
d. $-9\sqrt{6}+6\sqrt{6}+3\sqrt{6}$

25. Simplify each of the following.

a. $10\sqrt{11}-6\sqrt{11}+\sqrt{11}$
b. $\sqrt{7}+\sqrt{7}$
c. $4\sqrt{2}+6\sqrt{2}+5\sqrt{3}+2\sqrt{3}$
d. $10\sqrt{5}-2\sqrt{5}+8\sqrt{6}-7\sqrt{6}$

26. Simplify each of the following.

a. $5\sqrt{10}+2\sqrt{3}+3\sqrt{10}+5\sqrt{3}$

b. $12\sqrt{2}-3\sqrt{5}+4\sqrt{2}-8\sqrt{5}$

c. $6\sqrt{6}+\sqrt{2}-4\sqrt{6}-\sqrt{2}$

d. $16\sqrt{5}+8+7-11\sqrt{5}$

27. WE37 Simplify each of the following.

a. $\sqrt{8}+\sqrt{18}-\sqrt{32}$

b. $\sqrt{45}-\sqrt{80}+\sqrt{5}$

c. $-\sqrt{12}+\sqrt{75}-\sqrt{192}$

d. $\sqrt{7}+\sqrt{28}-\sqrt{343}$

e. $\sqrt{24}+\sqrt{180}+\sqrt{54}$

28. Simplify each of the following.

a. $\sqrt{12}+\sqrt{20}-\sqrt{125}$

b. $2\sqrt{24}+3\sqrt{20}-7\sqrt{8}$

c. $3\sqrt{45}+2\sqrt{12}+5\sqrt{80}+3\sqrt{108}$

d. $6\sqrt{44}+4\sqrt{120}-\sqrt{99}-3\sqrt{270}$

e. $2\sqrt{32}-5\sqrt{45}-4\sqrt{180}+10\sqrt{8}$

29. MC Choose the correct answer from the given options.

a. $\sqrt{2}+6\sqrt{3}-5\sqrt{2}-4\sqrt{3}$ is equal to:

A. $-5\sqrt{2}+2\sqrt{3}$ B. $-3\sqrt{2}+23$ C. $6\sqrt{2}+2\sqrt{3}$ D. $-4\sqrt{2}+2\sqrt{3}$

b. $6-5\sqrt{6}+4\sqrt{6}-8$ is equal to:

A. $-2-\sqrt{6}$ B. $14-\sqrt{6}$ C. $-2+\sqrt{6}$ D. $-2-9\sqrt{6}$

c. $4\sqrt{8}-6\sqrt{12}-7\sqrt{18}+2\sqrt{27}$ is equal to:

A. $-7\sqrt{5}$ B. $29\sqrt{2}-18\sqrt{3}$ C. $-13\sqrt{2}-6\sqrt{3}$ D. $-13\sqrt{2}+6\sqrt{3}$

d. $2\sqrt{20}+5\sqrt{24}-\sqrt{54}+5\sqrt{45}$ is equal to:

A. $19\sqrt{5}+7\sqrt{6}$ B. $9\sqrt{5}-7\sqrt{6}$ C. $-11\sqrt{5}+7\sqrt{6}$ D. $-11\sqrt{5}-7\sqrt{6}$

Understanding

30. MC Choose the correct answer from the given options.

a. $\sqrt{1000}$ is equal to:

A. 31.6228 B. $50\sqrt{2}$ C. $50\sqrt{10}$ D. $10\sqrt{10}$

b. $\sqrt{80}$ in simplest form is equal to:

A. $4\sqrt{5}$ B. $2\sqrt{20}$ C. $8\sqrt{10}$ D. $5\sqrt{16}$

c. Choose which of the following surds is in simplest form.

A. $\sqrt{60}$ B. $\sqrt{147}$ C. $\sqrt{105}$ D. $\sqrt{117}$

d. Choose which of the following surds is **not** in simplest form.

A. $\sqrt{102}$ B. $\sqrt{110}$ C. $\sqrt{116}$ D. $\sqrt{118}$

31. MC Choose the correct answer from the given options.

a. $6\sqrt{5}$ is equal to:

A. $\sqrt{900}$ B. $\sqrt{30}$ C. $\sqrt{150}$ D. $\sqrt{180}$

b. Choose the expression out of the following that is **not** equal to the rest.

A. $\sqrt{128}$ B. $2\sqrt{32}$ C. $8\sqrt{2}$ D. $64\sqrt{2}$

c. Choose the expression out of the following that is **not** equal to the rest.

A. $4\sqrt{4}$ B. $2\sqrt{16}$ C. 8 D. 16

d. $5\sqrt{48}$ is equal to:

A. $80\sqrt{3}$ B. $20\sqrt{3}$ C. $9\sqrt{3}$ D. $21\sqrt{3}$

32. Reduce each of the following to simplest form.

a. $\sqrt{675}$ b. $\sqrt{1805}$ c. $\sqrt{1792}$ d. $\sqrt{578}$

33. Reduce each of the following to its simplest form.

a. $\sqrt{a^2c}$ b. $\sqrt{bd^4}$ c. $\sqrt{h^2jk^2}$ d. $\sqrt{f^3}$

34. Write the following sequences of numbers in order from smallest to largest.

a. $6\sqrt{2}, 8, 2\sqrt{7}, 3\sqrt{6}, 4\sqrt{2}, \sqrt{60}$
b. $\sqrt{6}, 2\sqrt{2}, \sqrt{2}, 3, \sqrt{3}, 2, 2\sqrt{3}$

Reasoning

35. The formula for calculating the speed of a car before it brakes in an emergency is $v = \sqrt{20d}$, where v is the speed in m/s and d is the braking distance in metres.
Calculate the speed of a car before braking if the braking distance is 32.50 m.
Write your answer as a surd in its simplest form.

36. A gardener wants to divide their vegetable garden into 10 equal squares, each of area exactly 2 m^2.

a. Calculate the exact side length of each square. Explain your reasoning.
b. Determine the side length of each square, correct to 2 decimal places.
c. The gardener wishes to group the squares in their vegetable garden next to each other so as to have the smallest perimeter possible for the whole garden.
   i. Determine how they should arrange the squares.
   ii. Explain why the exact perimeter of the vegetable patch is $14\sqrt{2}$ m.
   iii. Calculate the perimeter, correct to 2 decimal places.

37. Explain why $\sqrt{a^3b^2}$ can be simplified to $ab\sqrt{a}$.

38. Determine the smallest values that a and b can have, given that they are both natural numbers, that $1 < a < b$, and also that $\sqrt{ab}$ is not a surd.

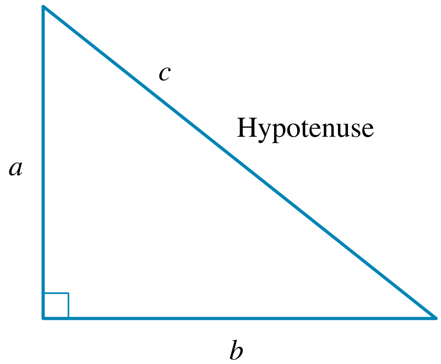
Problem solving

39. Kyle wanted a basketball court in his backyard, but he could not fit a full-sized court in his yard. He was able to get a rectangular court laid with a width of $6\sqrt{2}$ m and a length of $3\sqrt{10}$ m.
Calculate the area of the basketball court and represent it in its simplest surd form.

40. A netball team went on a pre-season training run. They completed 10 laps of a triangular course that had side lengths of $(200\sqrt{3}+50)$ m, $(50\sqrt{2}+75\sqrt{3})$ m and $(125\sqrt{2}-18)$ m.
Determine the distance they ran, in its simplest surd form.

41. The area of a square is $x\,\text{cm}^2$. Explain whether the side length of the square would be a rational number.

42. To calculate the length of the hypotenuse of a right-angled triangle, use the formula $c^2 = a^2 + b^2$.



a. Calculate the length of the hypotenuse for triangles with other side lengths as given in the following pairs of values.
   i. $\sqrt{5}, \sqrt{8}$
   ii. $\sqrt{2}, \sqrt{7}$
   iii. $\sqrt{15}, \sqrt{23}$

b. Describe any pattern in your answers to part a.

c. Calculate the length of the hypotenuse (without calculations) for triangles with the following side lengths.
   i. $\sqrt{1000}, \sqrt{500}$
   ii. $\sqrt{423}, \sqrt{33}$
   iii. $\sqrt{124}, \sqrt{63}$

d. Your friend wrote down the following explanation.

$$\sqrt{b} + \sqrt{c} = \sqrt{a}$$

Is this answer correct?
If not, explain what the correct answer should be.

LESSON 1.10 Review

1.10.1 Topic summary

NUMBER SKILLS AND INDEX LAWS

Natural numbers

- The natural numbers are the positive whole numbers, e.g. 1, 2, 3, 4 ... They were our first counting numbers.
- The set of natural numbers is represented by the symbol *N*.

Integers

- Integers are the whole number values on the number line. They include the positive and negative whole numbers as well as zero, e.g. –2, –1, 0, 1, 2.
- The set of integers is represented by the symbol *Z*.

Rational numbers

- Rational numbers are numbers that can be expressed as a whole number fraction, i.e. as $\frac{a}{b}$, where a and b are both integers and $b \neq 0$.
- The symbol *Q* is used for the set of rational numbers.
- The set of rational numbers includes the integers as well as terminating decimals and recurring decimals.

e.g. $\frac{3}{7}$, $-5\left(\frac{-5}{1}\right)$, $3.2\left(\frac{16}{5}\right)$, $-0.036\left(\frac{-36}{1000}\right)$

Real numbers

- Real numbers lie on the number line.
- The set of real numbers includes the irrational numbers and the rational numbers.
- The symbol *R* is used for real numbers.

Rounding to significant figures

- In a decimal number, the significant figures start at the first non-zero digit.
- To round a decimal number to significant figures:
  1. Start at the first non-zero digit.
  2. Write down the digits that follow until you have reached the required number of significant figures. (You may need to round off the final digit.)
  3. Leave out the digits that come after the last significant figure that is required.

  e.g. 1.425 has 3 decimal places but 4 significant figures.

Recurring decimals

- A recurring decimal is a decimal number with digits or groups of digits that repeat themselves endlessly.
- When writing recurring decimals, we place dots above the first and last repeating digits.

For example, 3.333333 … becomes $3.\dot{3}$ and 11.923 923 923 … becomes $11.\dot{9}2\dot{3}$.

Square and cube roots

- A square root is the inverse of squaring (raising to the power of 2).
- Every number has a positive and a negative square root.

  e.g. $2^2 = 4$ and $-2^2 = 4$
- The cube root is the inverse of cubing (raising to the power of 3).

Irrational numbers

- Irrational numbers are numbers that result in decimals that do not terminate and do not have repeated patterns of digits.

  e.g. $\sqrt{18} = 4.242\ 640\ 687\ 12...$
- The symbol *I* is used for the set of irrational numbers.

Rounding to decimal places

- Identify the rounding digit.
- If the digit after the rounding digit is 0, 1, 2, 3 or 4, leave the rounding digit as it is.
- If the digit after the rounding digit is 5, 6, 7, 8 or 9, increase the rounding digit by 1.
- Leave out all digits that come after the rounding digit.

Rounding

- Rounding is used to write approximate values of irrational numbers, recurring decimals or long terminating decimals.
- Numbers can be rounded to a set number of decimal places or to a set number of significant figures.

Transcendental numbers

- Transcendental numbers are irrational numbers that have special mathematical significance.
- Pi (π) is an example of a transcendental number.

Index laws

- There are a number of exponent laws:
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $a^0 = 1$, where $a \neq 0$
  - $(a^m)^n = a^{m \times n}$
  - $(a \times b)^m = a^m \times b^m$
  - $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
  - $a^{-n} = \frac{1}{a^n}$, where $a \neq 0$
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Scientific notation

- Scientific notation is used to write very large and very small numbers.
- A number written in scientific notation has the form: (number between 1 and 10) × (a power of 10).
- Large numbers have powers of 10 with positive exponents, while small numbers have powers of 10 with negative exponents.

For example:
- 8079 can be written as 8.079×10^3
- 0.000 15 can be written as 1.5×10^{-4}.

Surds (extending)

- Surds result when the root of a number is an irrational number.

  e.g. $\sqrt{10}$ $\approx 3.162\ 277\ 660\ 17$ ...
- They are written with a root (radical) sign in them.
- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

1.10.2 Success criteria

Tick the column to indicate that you have completed the lesson and how well you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
1.2	I can identify real numbers that are natural, integer, rational or irrational.			
	I can recall and use correct symbols for the sets of real, natural, integer, rational or irrational numbers.			
	I can write integers and decimals as rational number fractions.			
	I can write recurring decimals using dot notation.			
	I can explain the difference between recurring and terminating decimals.			
1.3	I can round decimals to a given number of decimal places.			
	I can identify how many significant figures a decimal number has.			
	I can round decimals to a given number of significant figures.			
1.4	I can identify the base and the exponent of a number written in index form.			
	I can recall the first three index laws.			
	I can use the first three index laws to simplify expressions involving indices.			
1.5	I can apply the Fourth, Fifth and Sixth index laws to raise a power to another power.			
	I can apply the first six index laws to simplify expressions involving indices.			
1.6	I can apply the Seventh Index Law to evaluate expressions using negative indices.			
	I can apply the first seven index laws to simplify expressions involving indices.			
1.7	I can identify numbers that are written in scientific notation.			
	I can write numbers in scientific notation as decimal numbers.			
	I can write decimal numbers using scientific notation.			
	I can represent decimals in exponential form.			

(continued)

(continued)

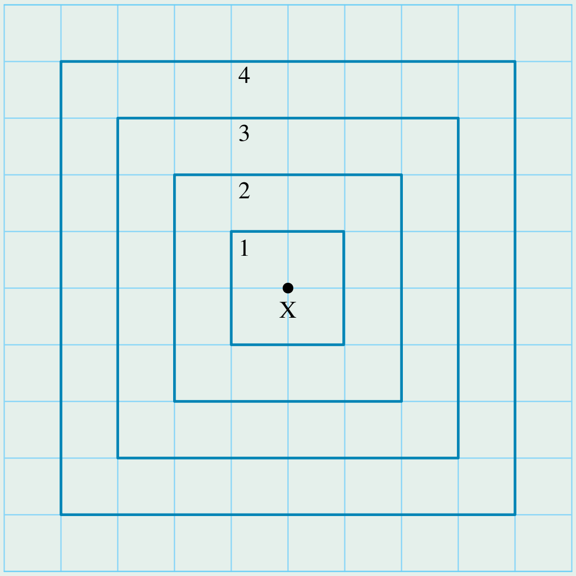
1.8	I can write square and cube roots using the radical symbol.			
	I can use fractional indices to represent square roots and cube roots.			
	I can simplify expressions that use square roots and cube roots.			
1.9	I can identify when a square root forms a surd.			
	I can place surds in their approximate position on a number line using geometric constructions.			
	I can simplify expressions containing surds.			

1.10.3 Project

Concentric squares

Consider a set of squares drawn around a central point on a grid, as shown.

These squares can be regarded as concentric squares, because the central point of each square is the point labelled **X**. The diagram shows four labelled squares drawn on one-centimetre grid paper.



1. Use the diagram above to complete the following table, leaving your answers in simplest surd form (if necessary). The first square has been completed for you.

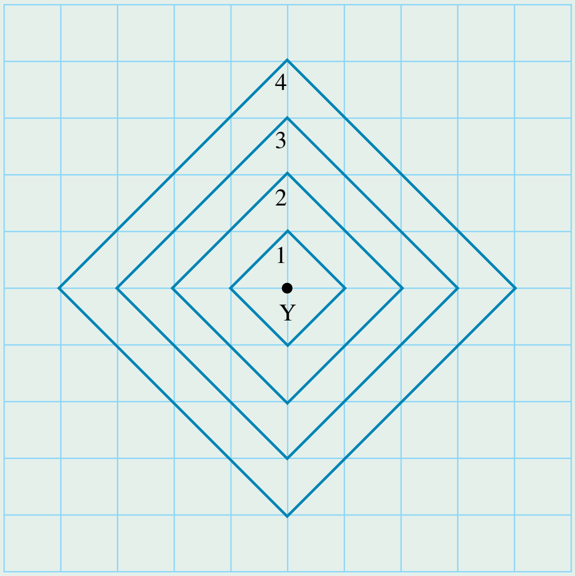
Square	**1**	**2**	**3**	**4**
Side length (cm)	2			
Diagonal length (cm)	$2\sqrt{2}$			
Perimeter (cm)	8			
Area (cm^2)	4			

Observe the patterns in the table and answer the following questions.

2. What would be the side length of the tenth square of this pattern?
3. What would be the length of the diagonal of this tenth square?

Consider a different arrangement of these squares on one-centimetre grid paper, as shown.

These squares all still have a central point, labelled **Y**.



4. Use the diagram to complete the following table for these squares, leaving your answers in simplest surd form if necessary. The first square has been done for you.

Square	1	2	3	4
Side length (cm)	$\sqrt{2}$			
Diagonal length (cm)	2			
Perimeter (cm)	$4\sqrt{2}$			
Area (cm^2)	2			

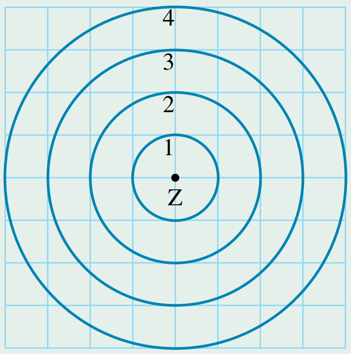
5. What would be the side length of the tenth square in this pattern of squares?
6. What would be the length of the diagonal of the tenth square?

Use the two diagrams above to answer the following questions.

7. In which of the two arrangements does the square have the greater side length?
8. Which of the two arrangements shows the squares with the greater diagonal length?
9. Compare the area of a square (your choice) in the first diagram with the area of the corresponding square in the second diagram.
10. Compare the perimeter of the same two corresponding squares in each of the two diagrams.
11. Examine the increase in area from one square to the next in these two diagrams. What increase in area would you expect from square 7 to square 8 in each of the two diagrams?

Consider a set of concentric circles around a centre labelled **Z**, drawn on one-centimetre grid paper, as shown.

12. Investigate the change in circumference of the circles, moving from one circle to the next (from smallest to largest). Write a general formula for this increase in circumference in terms of π.
13. Write a general formula in terms of π and r that can be used to calculate the increase in area from one circle to the next (from smallest to largest).



Exercise 1.10 Review questions

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Fluency

1. MC Choose which of the given numbers are rational: $\sqrt{\frac{6}{12}}$, $\sqrt{0.81}$, 5, -3.26, $0.\dot{5}$, $\frac{\pi}{5}$, $\sqrt{\frac{3}{12}}$.
   A. $\sqrt{0.81}$, 5, -3.26, $0.\dot{5}$ and $\sqrt{\frac{3}{12}}$
   B. $\sqrt{\frac{6}{12}}$ and $\frac{\pi}{5}$
   C. $\sqrt{\frac{6}{12}}$, $\sqrt{0.81}$ and $\sqrt{\frac{3}{12}}$
   D. 5, -3.26 and $\sqrt{\frac{6}{12}}$

2. For each of the following, state whether the number is rational or irrational and give the reason for your answer.
   a. $\sqrt{12}$
   b. $\sqrt{121}$
   c. $\frac{2}{9}$
   d. $0.\dot{6}$
   e. $\sqrt[3]{0.08}$

3. MC Choose which of the numbers in the set $\left\{3\sqrt{2}, 5\sqrt{7}, 9\sqrt{4}, 6\sqrt{10}, 7\sqrt{12}, 12\sqrt{64}\right\}$ are surds.
   A. $9\sqrt{4}, 12\sqrt{64}$
   B. $3\sqrt{2}$ and $7\sqrt{12}$ only
   C. $3\sqrt{2}, 5\sqrt{7}$ and $6\sqrt{10}$ only
   D. $3\sqrt{2}, 5\sqrt{7}, 6\sqrt{10}$ and $7\sqrt{12}$

4. State which of the following are not irrational numbers.
   a. $\sqrt{7}$
   b. $\sqrt{8}$
   c. $\sqrt{81}$
   d. $\sqrt{361}$

5. MC Choose which of the following represents the number 0.000 67 written using scientific notation.
   A. 67×10^{-5}
   B. 0.67×10^{-3}
   C. 6.7×10^{-4}
   D. 6.7×10^{-5}

6. MC Choose which of the following is a simplification of the expression $\sqrt{250}$.
   A. $25\sqrt{10}$
   B. $5\sqrt{10}$
   C. $10\sqrt{5}$
   D. $5\sqrt{50}$

7. Simplify each of the following.
   a. $b^7 \times b^3$
   b. $m^9 \times m^2$
   c. $k^3 \times k^5$
   d. $f^2 \times f^8 \times f^4$
   e. $h^4 \times h^5 \times h$

8. Simplify each of the following.

a. $2q^5 \times 3q^2 \times q^{10}$
b. $5w^3 \times 7w^{12} \times w^{14}$
c. $2e^2 p^3 \times 6e^3 p^5$
d. $5a^2 b^4 \times 3a^8 b^5 \times 7a^6 b^8$
e. $a^5 \div a^2$

9. Simplify each of the following.

a. $t^5 \div t$
b. $r^{19} \div r^{12}$
c. $p^8 \div p^5$
d. $\dfrac{f^{17}}{f^{12}}$
e. $\dfrac{y^{100}}{y^{10}}$

10. Simplify each of the following.

a. $\dfrac{m^{24}}{m^{14}}$
b. $\dfrac{g^4 \times g^5}{g^2}$
c. $\dfrac{x^6 \times x^2 \times x}{x^8}$
d. $\dfrac{d^6 \times d^7 \times d^2}{d^8}$
e. $\dfrac{t^7 \times t \times t^3}{t^2 \times t^4}$

11. Simplify each of the following.

a. $\dfrac{p^5 \times p^3 \times p \times p^4}{p^2 \times p^4 \times p^2}$
b. $\dfrac{16k^{13}}{21} \div \dfrac{8k^9}{42}$
c. $\dfrac{22b^{15}}{c} \div \dfrac{2b^8}{c^6}$
d. $\dfrac{9d^8}{16e^{10}} \div \dfrac{2d^{10}}{e^{16}}$
e. $\dfrac{2a^3 \times ab}{4 \times a \times b}$

12. Simplify each of the following.

a. 5^0
b. 12^0
c. 345^0
d. q^0
e. r^0

13. Simplify each of the following.

a. ab^0
b. $3w^0$
c. $5q^0 - 2q^0$
d. $100s^0 + 99t^0$
e. $a^7 b^0$

14. Simplify each of the following.

a. $v^{10} w^0$
b. prt^0
c. $a^9 b^4 c^0$
d. $j^8 k^0 m^3$
e. $4e^2 f^0 - 36\left(a^2 b^3\right)^0$

15. Simplify each of the following.

a. $2x^0 \times 3xy^2$
b. $-8\left(18x^2 y^4 z^6\right)^0$
c. $15 - 12x\left(\dfrac{3x}{8}\right)^0$
d. $-4p^0 \times 6\left(q^2 r^3\right)^0 \div 8\left(-12q^2\right)^0$
e. $3\left(6w^0\right)^2 \div 2\left(5w^5\right)^0$

16. Raise each of the following to the given power.

a. $\left(b^4\right)^2$
b. $\left(a^8\right)^3$
c. $\left(k^7\right)^{10}$
d. $\left(j^{100}\right)^2$

17. MC $\left(\dfrac{4b^4}{d^2}\right)^3$ is equal to:

A. $\dfrac{4b^3}{d^3}$
B. $\dfrac{12b^{12}}{d^6}$
C. $\dfrac{64b^{12}}{d^6}$
D. $\dfrac{64b^7}{d^5}$

18. Raise each of the following to the given power.

a. $\left(a^5 b^2\right)^3$
b. $\left(m^7 n^{12}\right)^2$
c. $\left(st^6\right)^3$
d. $\left(qp^{30}\right)^{10}$

19. Write each of the following with positive exponents.

a. a^{-1}
b. k^{-4}
c. $4m^3 \div 2m^7$
d. $7x^3y^{-4} \times 6x^{-3}y^{-1}$

20. Write each of the following using a negative exponent.

a. $\frac{1}{x}$
b. $\frac{2}{y^4}$
c. $z \div z^4$
d. $45p^2q^{-4} \times 3p^{-5}q$

21. Simplify each of the following.

a. $\sqrt{100}$
b. $\sqrt{36}$
c. $\sqrt{a^2}$
d. $\sqrt{b^2}$
e. $\sqrt{49f^4}$

22. Simplify each of the following.

a. $\sqrt[3]{27}$
b. $\sqrt[3]{1000}$
c. $\sqrt[3]{x^3}$
d. $\sqrt[3]{8d^3}$
e. $\sqrt[3]{64f^6g^3}$

Problem solving

23. If your body produces about 2.0×10^{11} red blood cells each day, determine how many red blood cells your body has produced after one week.

24. If your body produces about 1.0×10^{10} white blood cells each day, determine how many white blood cells your body produces each hour.

25. If Earth is approximately 1.496×10^8 km from the Sun, and Mercury is about 5.8×10^7 km from the Sun, evaluate the distance from Mercury to Earth.

26. Calculate the closest distance between Mars and Saturn if Mars is 2.279×10^8 km from the Sun and Saturn is 1.472×10^9 km from the Sun.

27. Consider the numbers 6×10^9, 3.4×10^7 and 1.5×10^9.

a. Place these numbers in order from smallest to largest.
b. If the population of Oceania is 3.4×10^7 and the world population is 6×10^9, determine the percentage of the world population that is represented by the population of Oceania. Give your answer to 2 decimal places.
c. Explain what you need to multiply 1.5×10^9 by to get 6×10^9.
d. Assume the surface area of Earth is approximately $1.5 \times 10^9 \text{km}^2$, and that the world's population is about 6×10^9. If we divided Earth up into equal portions, determine how much surface area each person in the world would get.

28. The total area of the state of Victoria is $2.27 \times 10^5 \text{ km}^2$. If the total area of Australia is $7.69 \times 10^6 \text{ km}^2$, calculate the percentage of the total area of Australia that is occupied by the state of Victoria. Write your answer correct to two decimal places.

29. A newly discovered colony of bees contains 2.05×10^8 bees. If 0.4% of these bees are estimated to be queen bees, calculate how many queen bees live in the colony.

30. Third-generation fibre optics that can carry up to 10 trillion (1×10^{12}) bits of data per second along a single strand of fibre have recently been tested.
Based on the current rate of improvement in this technology, the amount of data that one strand of fibre can carry is tripling every 6 months. This trend is predicted to continue for the next 20 years.
Calculate how many years it will take for the transmission speed along a single strand of fibre to exceed 6×10^{16} bits per second.

31. Mach speed refers to the speed of sound, where Mach 1 is the speed of sound (about 343 metres per second (m/s)). Mach 2 is double the speed of sound, Mach 3 is triple the speed of sound, and so on. Earth has a circumference of about 4.0×10^4km.

a. The fastest recorded aircraft speed is 11 270 kilometres per hour (km/h).
Express this as a Mach speed, correct to 1 decimal place.
b. Create and complete a table for three different speeds (Mach 1, 2 and 3) with columns labeled 'Mach number', 'Speed in m/s' and 'Speed in km/h'.
c. Calculate how long it would take something travelling at Mach 1, 2 and 3 to circle Earth. Give your answer in hours, correct to 2 decimal places.
d. Evaluate how long it would take for the world's fastest aircraft to circle Earth. Give your answer in hours, correct to 2 decimal places.
e. Evaluate how many times the world's fastest aircraft could circle Earth in the time it takes an aircraft travelling at Mach 1, 2 and 3 to circle Earth once. Give your answers correct to 2 decimal places.
f. Explain the relationship between your answers in part **e** and their corresponding Mach value.

on To test your understanding and knowledge of this topic, go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 1 Number skills and index laws

1.1 Pre-test

1. True
2. Recurring decimal
3. a. True b. True
4. 3
5. C
6. B
7. $\sqrt{5}$
8. C
9. 7.439
10. 7.44
11. C
12. B
13. 1.672×10^{-27}
14. $\dfrac{4\sqrt{3}}{3}$
15. $\dfrac{5\sqrt{2}}{2}$

1.2 Real numbers

1. a. $\dfrac{15}{1}$ b. $\dfrac{-8}{1}$ c. $\dfrac{8}{3}$ d. $\dfrac{-41}{8}$
2. a. $\dfrac{4}{1}$ b. $\dfrac{73}{10}$ c. $\dfrac{2}{1000}$ d. $\dfrac{872}{10}$
3. a. $\dfrac{0}{1}$ b. $\dfrac{156}{100}$ c. $\dfrac{3612}{1000}$ d. $\dfrac{-8}{100}$
4. a. 0.555 555 55 b. 0.515 151 51
   c. 0.511 111 11 d. 6.031 313 1
   e. 5.183 183 1
5. a. −7.024 444 4 b. 8.912 491 2
   c. 5.123 434 3 d. 5.123 412 3
   e. 3.002 020 2
6. a. $0.\dot{5}$ b. $0.\dot{2}\dot{7}$ c. $2.\dot{1}\dot{8}$
   d. $0.\dot{4}2857\dot{1}$ e. $-1.\dot{7}\dot{4}$
7. a. $0.0\dot{7}\dot{3}$ b. $-3.8\dot{3}$ c. $0.4\dot{6}$
   d. $0.\dot{4}\dot{6}$ e. $0.0\dot{4}\dot{6}$
8. a. 15
   b. −3, 0, 15
   c. $-3, \dfrac{-3}{7}, 0, 2.\dot{3}, 2\dfrac{3}{5}, 15$
   d. $2.\dot{3}$ and $-\dfrac{3}{7}$
9. $2.\dot{1}$, $2.\dot{1}\dot{2}\dot{1}$, $2.1\dot{2}\dot{1}$, $2.\dot{1}\dot{2}$, $2.1\dot{2}$
10. Statements **a** and **b** are true.
11. Every integer a can be written as $\dfrac{a}{1}$ and therefore every integer is rational.
12. a. Q^{-}
    b. N^{-} implies a set of negative natural numbers. As the natural numbers include only positive numbers, this set would not exist.
13. Yes, because the division of an integer by another integer will always result in a rational number.
14. a. $\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{1}{6}$ b. 6
15. Answers will vary. Sample responses are shown.
    a. $\dfrac{1}{9}$ b. $\dfrac{10}{99}$ c. $\dfrac{100}{999}$ d. $\dfrac{1000}{9999}$
16. \$3330.20
17. 657 bottles of water per year

1.3 Rounding numbers and significant figures

1. a. 1.571 b. 2.236 c. 3.873
   d. 16.435 e. 5.111 f. 5.156
2. a. 5.152 b. 137.358 c. 0.429
   d. 0.077 e. 2.429 f. 1.000
3. a. 4.000 b. 0.254 c. 0.025
   d. 0.001 e. 5.000 f. 2342.156
4. a. 2 b. 3 c. 4 d. 3
5. a. 4 b. 4 c. 2 d. 1
6. a. 3 b. 4 c. 2 d. 3
7. a. 1.5708 b. 2.2361 c. 3.8730
   d. 16.435 e. 5.1111 f. 5.1556
8. a. 5.1515 b. 137.36 c. 0.428 57
   d. 0.076 923 e. 2.4286 f. 1.0000
9. a. 6.5813 b. 4.0000 c. 0.253 64
   d. 0.025 400 e. 0.000 913 60 f. 5.0000
10. 3.1416, 3.14159, 3.141593, 3.1415927
11. a. Yes b. No c. Yes d. No
12. a. False b. True c. False d. True
13. a. False b. True c. False d. True
14. a. True b. False c. False d. True
15. A number of answers are possible. Example answer: 2.004.
16. The 5th digit (5) causes the 4th digit to round up from 9 to 10, which causes the 3rd digit to round up from 9 to 10, which causes the 2nd digit to round up from 2 to 3.
17. a. 2 decimal places b. 3.1416
18. a. 153.86 cm^2
    b. 153.9380 cm^2
    c. 153.94 cm^2
    d. Yes, because 3.14 is used as an estimate and is not the value of π as used on the calculator.
19. a. 65 449.8470 cm^3
    b. 9
    c. Yes, any number with a finite number of decimal places is a rational number.
20. 13 000

1.4 Review of index laws

1. a. $2^2 \times 3$ b. $2^3 \times 3^2$ c. 3×5^2
d. $2^4 \times 3 \times 5$ e. $2^7 \times 5$ f. $2^3 \times 5^2 \times 7^2$

2. a. $20p^{11}$ b. $6x^8$ c. $56y^{10}$
d. $21p^8$ e. $84t^6$ f. $30q^{15}$

3. a. $6a^6e^7$ b. $8p^6h^{12}$ c. $6k^{14}$ d. 3^{12}

4. a. $80m^9$ b. $6g^3h^6$ c. $30p^6q^9$ d. $48u^9w^7$

5. a. $27d^{11}y^{17}$ b. $42b^{14}c^9$ c. $24r^{16}s^{18}$ d. $60h^{38}v^{20}$

6. a. $3p^4$ b. $6r^4$ c. $9a^3$ d. $3b^6$

7. a. $20r^4$ b. $9q$ c. $\frac{130}{d^2}$ d. $7t^4$

8. a. $\frac{3p^5}{2}$ b. $\frac{8b^5}{3}$ c. $\frac{5m^{10}n^6}{6}$ d. $\frac{9x^8y}{4}$

9. a. $3j^5f^3$ b. $\frac{4p^2rs}{3}$ c. $\frac{9a^5b^3c}{2}$ d. $\frac{20f^6g^2h^4}{3}$

10. a. 1 b. 6 c. 1 d. 1 e. 5

11. a. 1 b. 1 c. 2 d. 2 e. 3

12. a. 1 b. 0 c. 14 d. 6 e. 6

13. a. 1 b. 2 c. 2 d. 2 e. 2

14. a. $\frac{h^2}{2}$ b. $\frac{q^4}{5}$ c. $\frac{n^3}{5}$ d. v^2 e. $2x^6$

15. a. $\frac{1}{x^3}$ b. $\frac{1}{m^8}$ c. $\frac{1}{4m^6}$ d. $\frac{2}{x^2}$

16. a. $2x^2$ b. $24t^6$ c. $\frac{1}{2y^5}$ d. $\frac{7y^3}{4x^5}$

17. a. $\frac{2}{5m^3n^4}$ b. $\frac{2}{n^2}$ c. $\frac{2y}{x}$ d. $\frac{c^4}{a^4}$

18. a. 6 b. 9 c. 18 d. 11 e. 15 f. 6

19. Sample responses can be found in the worked solutions in the online resources.

20. a. B b. D c. D

21. a. A b. D

22. Sample responses can be found in the worked solutions in the online resources.

23. a. 5^9
b. Sample responses can be found in the worked solutions in the online resources.

24. a. 5^{16}
b. Sample responses can be found in the worked solutions in the online resources.

25. a. 1
b. 1
c. 0
d. Sample responses can be found in the worked solutions in the online resources.

26. a. $\Delta = 14$
b. Answers will vary, but $\Delta + \text{O} + \Diamond$ must sum to 12. Possible answers include: $\Delta = 3, \text{O} = 2, \Diamond = 7$; $\Delta = 1, \text{O} = 3, \Diamond = 8$; $\Delta = 4, \text{O} = 4, \Diamond = 4$; $\Delta = 5, \text{O} = 1, \Diamond = 6$.

27. a. The repeating pattern is 1, 3, 9, 7.
b. 3

28. a. The repeating pattern is 4, 6.
b. 4

29. a. Sample responses can be found in the worked solutions in the online resources.
b. i. 8 ii. 1 iii. 2

30. 2^{2n+3}

1.5 Raising powers

1. a. e^6 b. f^{80} c. p^{100} d. r^{144} e. $8a^6$

2. a. a^8b^{12} b. p^5q^{15} c. $g^{30}h^{20}$
d. $81w^{36}q^8$ e. $49e^{10}r^4q^8$

3. a. p^8q^6 b. $r^{15}w^9$ c. $b^{10}n^{18}$
d. $j^{18}g^{12}$ e. $8a^3b^4$

4. a. q^4r^{20} b. $h^{24}j^{16}$ c. $a^{21}f^{16}$
d. $t^{10}u^8$ e. $i^{15}j^{12}$

5. a. $\frac{9b^8}{d^6}$ b. $\frac{25h^{20}}{4j^4}$ c. $\frac{8k^{15}}{27t^{24}}$ d. $\frac{49p^{18}}{64q^{44}}$

6. a. $\frac{125y^{21}}{27z^{39}}$ b. $\frac{256a^{12}}{2401c^{20}}$ c. $\frac{-64k^6}{343m^{18}}$ d. $\frac{16g^{28}}{81h^{44}}$

7. a. 2^{20} b. t^{33} c. a^{21} d. e^{66}

8. a. g^{39} b. $12a^{14}$ c. $216d^{27}$ d. $40\,000r^{14}$

9. B

10. B

11. D

12. a. a^6 b. m^4 c. n^3
d. b^8 e. f^{17} f. g^6

13. a. p^9 b. y^2 c. c^{20}
d. f^7 e. k^{14} f. p^{16}

14. a. 5
b. 6
c. 30
d. Sample responses can be found in the worked solutions in the online resources.

15. a. i. 1 ii. -1 iii. -1 iv. 1
b. $(-1)^{\text{even}} = 1(-1)^{\text{odd}} = -1$. Sample responses can be found in the worked solutions in the online resources.

16. Danni is correct. Explanations will vary but should involve $(-x)(-x) = (-x)^2 = x^2$ and $-x^2 = -1 \times x^2 = -x^2$.

17. a. 5^{12}
b. Sample responses can be found in the worked solutions in the online resources.

18. a. i. $x = \frac{5}{3}$ ii. $x = \frac{5}{3}$ iii. $x = \frac{5}{3}$
b. When equating the powers, $3x = 5$.

19. Answers will vary. Possible answers are 4096 and 262144.

20. $10^8 \times 10^8 \times 10^8 = (10^8)^3$ atoms

21. a. 4^1 b. 7^2

22. a. 2^1 b. 3^4 c. $\frac{1}{5^2}$ d. 2^2
e. $\frac{1}{2^2}$ f. $\frac{1}{2}$ g. $\frac{1}{2}$ h. $\frac{1}{7}$

23. a. i. 3 ii. 4 iii. 8 iv. 11
b. i. 3 ii. 4 iii. 8 iv. 11
c. Raising a number to a power of one-half is the same as calculating the square root of that number.

1.6 Negative indices

1. a. $3^5 = 243, 3^4 = 81, 3^3 = 27, 3^2 = 9, 3^1 = 3, 3^0 = 1,$
$3^{-1} = \frac{1}{3}, 3^{-2} = \frac{1}{9}, 3^{-3} = \frac{1}{27}, 3^{-4} = \frac{1}{81}, 3^{-5} = \frac{1}{243}$
b. $5^4 = 625, 5^3 = 125, 5^2 = 25, 5^1 = 5, 5^0 = 1, 5^{-1} = \frac{1}{5},$
$5^{-2} = \frac{1}{25}, 5^{-3} = \frac{1}{125}, 5^{-4} = \frac{1}{625}$
c. $10^4 = 10000, 10^3 = 1000, 10^2 = 100, 10^1 = 10, 10^0 = 1,$
$10^{-1} = \frac{1}{10}, 10^{-2} = \frac{1}{100}, 10^{-3} = \frac{1}{1000}, 10^{-4} = \frac{1}{10\,000}$

2. a. $\frac{1}{32}$ b. $\frac{1}{27}$ c. $\frac{1}{4}$ d. $\frac{1}{100}$

3. a. $\frac{1}{125}$ b. 7 c. $\frac{4}{3}$ d. $\frac{16}{9}$

4. a. 27 b. $\frac{2}{3}$ c. $\frac{16}{81}$ d. $\frac{49}{4}$

5. a. $\frac{1}{x^4}$ b. $\frac{1}{y^5}$ c. $\frac{1}{z}$
d. $\frac{a^2}{b^3}$ e. $\frac{1}{m^2n^3}$

6. a. $\frac{1}{m^2n^3}$ b. x^2y^2 c. $5x^3$
d. $\frac{w^5}{x^2}$ e. x^2y^2

7. a. $\frac{a^2c}{b^3d^4}$ b. $\frac{a^2d^3}{b^2c^2}$ c. $\frac{10y}{x^2}$
d. $\frac{x}{3}$ e. $\frac{1}{m^3x^2}$

8. a. $\frac{1}{a^5}$ b. m^5 c. $\frac{1}{m^7}$
d. $\frac{14}{x}$ e. 6 f. $\frac{5a}{b^3}$

9. a. $\frac{1}{x^3}$ b. $\frac{6}{x^5y^3}$ c. $50x^3$
d. 1 e. $\frac{50}{a^5}$ f. $10a^4$

10. a. $\frac{-32}{w^3}$ b. $\frac{16}{m^4}$ c. $\frac{27m^6}{n^{12}}$
d. $\frac{1}{a^6b^{15}}$ e. a^2b^6 f. $\frac{25}{a^2}$

11. a. $\frac{1}{x^5}$ b. $\frac{1}{x^{11}}$ c. x^{11}
d. x^5 e. $\frac{6c^4}{a^2}$

12. a. $\frac{2}{a^6}$ b. $\frac{5}{m}$ c. $\frac{1}{b}$
d. $\frac{1}{a^3b^2}$ e. $\frac{c^2}{a^4}$

13. a. $\frac{1}{16a}$ b. $\frac{1}{m^3}$ c. $\frac{3}{2t^9}$
d. t^3 e. $\frac{m^2}{n^3}$

14. a. 2^0 b. 2^3 c. 2^5 d. 2^6 e. 2^{-3} f. 2^{-5}

15. a. 4^0 b. 4^1 c. 4^3 d. 4^{-1} e. 4^{-2} f. 4^{-3}

16. a. 10^0 b. 10^1 c. 10^4 d. 10^{-1} e. 10^{-2} f. 10^{-5}

17. a. $3^{-4} = \frac{1}{3^4}$
b. Answers will vary, but should mention that if you are dividing, the power in the numerator is lower than that in the denominator. Sample responses can be found in the worked solutions in the online resources.
c. Sample responses can be found in the worked solutions in the online resources.

18. Answers will vary, but should mention that the negative 13 means the decimal point is moved 13 places to the left of 3. Using scientific notation allows the number to be expressed more concisely. Sample responses can be found in the worked solutions in the online resources.

19. a. No. The equivalent expression with positive indices is $\frac{x^6 + 1}{x^3}$.
b. No. The equivalent expression with positive indices is $\frac{(xy)^2}{(x+y)^2}$.

20. a. No. The equivalent expression with positive indices is $\frac{x^{13} - 1}{x^5}$.
b. No. The correct equivalent expression is $\frac{1}{x^6 - x^3}$.

21. a. 0.048 b. 7600 c. 0.000 29 d. 8.1

22. a. 3 b. −1 c. 2 d. −4

23. a. 2^{19} b. 3^{20}

24. a. $\frac{6}{5}$ b. $\left(\frac{3}{2}\right)^{200}$

1.7 Scientific notation

1. a. 5.00×10^3 b. 4.31×10^2
c. 3.8×10^1 d. 3.5×10^5

2. a. 1 000 000 000 b. 392.673
c. 591.1 d. 5100

3. a. 734 000 b. 7 141 400
c. 35.1 d. 80 500

4. a. 614 b. 6140
c. 61 400 d. 351.8

5. a. 7.25×10^1 b. 7.25×10^2
c. 7.25×10^3 d. 7.25×10^8

6. a. 4.379×10^1 b. 5.317×10^3 c. 2.590×10^2
d. 1.101×10^2 e. 1.632×10^6 f. 1.000×10^6

7. a. 0.2 b. 0.004 c. 0.0007 d. 0.03

8. a. 0.082 73 b. 0.072 95
c. 0.002 914 2 d. 0.000 037 53

9. a. 0.000 529 b. 0.000 033 333
c. 0.000 000 002 625 d. 0.000 000 000 000 001 273

10. a. 7×10^{-1} b. 5×10^{-3}
c. 3×10^{-7} d. 1×10^{-11}

11. a. $2 \times 10^{-1} + 3 \times 10^{-2} + 1 \times 10^{-3}$
b. $3 \times 10^{-3} + 6 \times 10^{-4} + 2 \times 10^{-5}$
c. 7×10^{-4}
d. $6 \times 10^{-2} + 3 \times 10^{-3}$

12. a. 6.73×10^{-3} b. 1.43×10^{-1} c. 6.83×10^{-5}
d. 5.12×10^{-9} e. 5.09×10^{-2} f. 1.25×10^{-2}

13. a. 5.27×10^2, 8.31×10^2, 9.718×10^2, 3.27×10^3
b. 4.09×10^2, 4.37×10^2, 7.943×10^2, 7.95×10^2
c. 2.7×10^{-3}, 9.29×10^{-3}, 5.251×10^{-2}, 5.31×10^{-2}
d. 2.7×10^{-3}, 7.13×10^{-2}, 8.31×10^2, 3.27×10^3

14. a. 0.000 000 000 000 000 000 000 019 94 g
b. 1.994×10^{-17} g
c. 5.015×10^{23} atoms

15. 3844 hours ≈ 160.17 days

16. 333 333.33

17. The neutron is heavier by 2.3×10^{-30} kg.

18. a. 2.1×10^7 b. 1.29×10^8

19. a. 8 389 000 bytes b. 8.389×10^6 bytes

20. 0.000 000 2 N/m

21. a. i. Sample responses can be found in the worked solutions in the online resources.
ii. Sample responses can be found in the worked solutions in the online resources.
b. i. 5.36×10^7 ii. 5.36×10^7
c. The answers are the same because 2.95×10^3 is very small compared with 5.36×10^7.

22. 8.75×10^{-7} is such a small amount (0.000 000 875) that, when it is added to 2.39×10^{-3}, it doesn't affect the value when given to 3 significant figures.

23. a. i. 385 440 km
ii. 78 489 600 km
iii. 1 278960 000 km
iv. 5 772 840 000 km
b. i. 385 000 km
ii. 78 500 000 km
iii. 1 280 000 000 km
iv. 5 770 000 000 km
c. i. 3.85×10^5 km
ii. 7.85×10^7 km
iii. 1.28×10^9 km
iv. 5.77×10^9 km

24. a. i. 31 536 000 s ii. 3.1536×10^7 s
b. i. 9 460 800 000 000 km ii. 9.4608×10^{12} km
c. i. 4.068×10^{13} km ii. 46 440 000 years

25. a. 5.972×10^{21} b. 4

26. a. 9.1094×10^{-31}
b. 1.6725×10^{-27}
c. 1.6726×10^{-27}
d. It is important to work with the original amounts and leave rounding until the end of a calculation so that the answer is accurate.

1.8 Square roots and cube roots

1. a. $15^{\frac{1}{2}}$ b. $m^{\frac{1}{2}}$ c. $t^{\frac{1}{3}}$ d. $(w^2)^{\frac{1}{3}}$

2. a. 7 b. 2 c. 3 d. 5

3. a. 10 b. 8 c. 4
d. 1000 e. 100 f. 9

4. a. m b. b c. $6t^2$ d. mn^2

5. a. $5t^2$ b. xy^2 c. a^2m^{10} d. $6y^2$

6. a. $4x^2y^2$ b. $5ab^2c^3$ c. b^7 d. b^3

7. B

8. D

9. A

10. a. $3^{\frac{1}{2}+\frac{1}{2}} = 3^1$ b. $\sqrt{3}$ c. $a^{\frac{1}{n}}$ d. 2

11. a. $z^{2.5} = z^{\frac{5}{2}} = \left(z^5\right)^{\frac{1}{2}} = \sqrt{z^5}$
b. No, it is the tenth root: $z^{0.3} = z^{\frac{3}{10}} = \left(z^3\right)^{\frac{1}{10}} = \sqrt[10]{z^3}$.

12. Prithvi is correct: $\sqrt{x^2} = x^{\frac{1}{2}\times 2} = x^1 = x$; x can be a positive or negative number.

13. $\left(-2^3\right)^{\frac{1}{3}} = -2$; answers will vary, but should include that we cannot take the fourth root of a negative number. Sample responses can be found in the worked solutions in the online resources.

14. $\sqrt{1871} \approx 43.25$
$42^2 = 1764$
$43^2 = 1849$
He was 43 years old in 1849. Therefore, he was born in $1849 - 43 = 1806$.

15. a. $d = t^{\frac{2}{3}}$ b. $t = d^{\frac{3}{2}}$

16. $\dfrac{16}{81}$

17. a. There are 2 possible solutions, because the square root of a number always has a positive and a negative answer.
b. $\dfrac{5}{3}$, 1

1.9 Surds (extending)

1. a. 1 b. 2 c. 0
   d. $\frac{1}{3}$ e. $\frac{5}{4}$

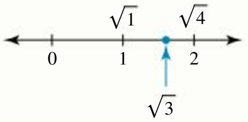
2. a. 0.4 b. 20 c. 100
   d. $\frac{2}{5}$ e. 1.2

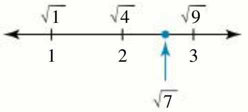
3. a. 4.5 b. 1000 c. 0.03
   d. 16 e. $\frac{11}{7}$

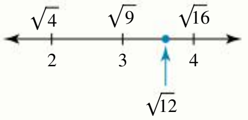
4. a. 9 b. -9 c. 11 d. -21

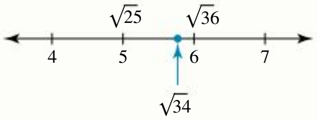
5. a. 2 b. 4 c. 7 d. 3

6. a. 4 b. 5 c. -7 d. -3

7. a.


   b.


   c.


   d.


8. c and d

9. c

10. a. $\sqrt{21}$ b. $-\sqrt{21}$ c. $2\sqrt{6}$
    d. $6\sqrt{7}$ e. $10\sqrt{14}$

11. a. $12\sqrt{7}$ b. $9\sqrt{7}$ c. $22\sqrt{10}$
    d. $22\sqrt{3}$ e. 3

12. a. $15\sqrt{10}$ b. 24 c. 6
    d. 11 e. 30

13. a. $\sqrt{3}$ b. $-\sqrt{2}$ c. $\sqrt{6}$ d. $\sqrt{5}$

14. a. $3\sqrt{3}$ b. $\frac{3\sqrt{6}}{2}$ c. $15\sqrt{2}$ d. $\frac{5\sqrt{5}}{3}$

15. a. $-2\sqrt{5}$ b. $\sqrt{3}$ c. $\sqrt{3}$ d. $\sqrt{7}$

16. a. $2\sqrt{5}$ b. $2\sqrt{2}$ c. $3\sqrt{2}$ d. 7

17. a. $\sqrt{30}$ b. $5\sqrt{2}$ c. $2\sqrt{7}$ d. $6\sqrt{3}$

18. a. $12\sqrt{2}$ b. $4\sqrt{3}$ c. $10\sqrt{5}$ d. $9\sqrt{2}$

19. a. $4\sqrt{2}$ b. $15\sqrt{3}$ c. 48 d. $35\sqrt{2}$ e. $20\sqrt{6}$

20. a. $10\sqrt{3}$ b. $4\sqrt{42}$ c. $72\sqrt{2}$ d. $27\sqrt{5}$ e. $132\sqrt{2}$

21. a. $\sqrt{12}$ b. $\sqrt{175}$ c. $\sqrt{108}$ d. $\sqrt{80}$ e. $\sqrt{384}$

22. a. $\sqrt{90}$ b. $\sqrt{32}$ c. $\sqrt{720}$ d. $\sqrt{600}$ e. $\sqrt{338}$

23. a. $2\sqrt{3}$ b. $5\sqrt{2}$ c. $10\sqrt{2}$ d. $2\sqrt{10}$

24. a. $2\sqrt{2}$ b. $-4\sqrt{5}$ c. $-6\sqrt{3}$ d. 0

25. a. $5\sqrt{11}$ b. $2\sqrt{7}$
    c. $10\sqrt{2}+7\sqrt{3}$ d. $8\sqrt{5}+\sqrt{6}$

26. a. $8\sqrt{10}+7\sqrt{3}$ b. $16\sqrt{2}-11\sqrt{5}$
    c. $2\sqrt{6}$ d. $5\sqrt{5}+15$

27. a. $\sqrt{2}$ b. 0 c. $-5\sqrt{3}$
    d. $-4\sqrt{7}$ e. $5\sqrt{6}+6\sqrt{5}$

28. a. $2\sqrt{3}-3\sqrt{5}$ b. $4\sqrt{6}+6\sqrt{5}-14\sqrt{2}$
    c. $29\sqrt{5}+22\sqrt{3}$ d. $9\sqrt{11}-\sqrt{30}$
    e. $28\sqrt{2}-39\sqrt{5}$

29. a. D b. A c. D d. A

30. a. D b. A c. C d. C

31. a. D b. D c. D d. B

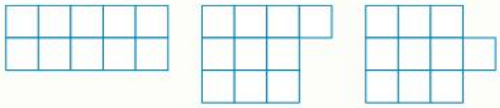
32. a. $15\sqrt{3}$ b. $19\sqrt{5}$ c. $16\sqrt{7}$ d. $17\sqrt{2}$

33. a. $a\sqrt{c}$ b. $d^2\sqrt{b}$ c. $hk\sqrt{j}$ d. $f\sqrt{f}$

34. a. $2\sqrt{7}, 4\sqrt{2}, 3\sqrt{6}, \sqrt{60}, 8, 6\sqrt{2}$
    b. $\sqrt{2}, \sqrt{3}, 2, \sqrt{6}, 2\sqrt{2}, 3, 2\sqrt{3}$

35. $5\sqrt{26}$

36. a. $\sqrt{2}$ m
    b. 1.41 m
    c. i. Any of the arrangements shown would work.



    ii. The length of each square is $\sqrt{2}$ m and there are 14 lengths around each shape.
    iii. 19.80 m

37.
$$\begin{aligned}\sqrt{a^3b^2} &= \sqrt{a^3}\times\sqrt{b^2}\\ &= \sqrt{a^2}\times\sqrt{a}\times\sqrt{b^2}\\ &= a\times\sqrt{a}\times b\\ &= ab\sqrt{a}\end{aligned}$$

38. $a=2$ and $b=8$.

39. $36\sqrt{5}\ \text{m}^2$

40. $\left(1750\sqrt{2}+2750\sqrt{3}+320\right)$ m

41. The side length will be rational if x is a perfect square. If x is not a perfect square, the side length will be a surd.

42. a. i. $\sqrt{13}$ ii. 3 iii. $\sqrt{38}$
    b. For side lengths $\sqrt{a}$ and $\sqrt{b}$, the length of the hypotenuse is $\sqrt{a+b}$.
    c. i. $\sqrt{1500}=10\sqrt{15}$
       ii. $2\sqrt{114}$
       iii. $\sqrt{187}$
    d. No. The correct answer is $\sqrt{b+a}=\sqrt{c}$.

Project

1.

Square	1	2	3	4
Side length (cm)	2	4	6	8
Diagonal length (cm)	$2\sqrt{2}$	$4\sqrt{2}$	$6\sqrt{2}$	$8\sqrt{2}$
Perimeter (cm)	8	16	24	32
Area (cm^2)	4	16	36	64

2. 20 cm
3. $20\sqrt{2}$ cm
4.

Square	1	2	3	4
Side length (cm)	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}$	$4\sqrt{2}$
Diagonal length (cm)	2	4	6	8
Perimeter (cm)	$4\sqrt{2}$	$8\sqrt{2}$	$12\sqrt{2}$	$16\sqrt{2}$
Area (cm^2)	2	8	18	32

5. $10\sqrt{2}$ cm
6. 20 cm
7. First arrangement
8. First arrangement
9. The squares in the first diagram are twice the area of each corresponding square in the second diagram.
10. The squares in the first diagram have a perimeter that is $\sqrt{2}$ times the perimeter of each corresponding square in the second diagram.
11. An increase of 60 cm^2 in the first diagram and an increase of 30 cm^2 for the second diagram.
12. 2π
13. $\pi(2r+1)$

1.10 Review questions

1. A
2. a. Irrational, because it is equal to a non-recurring and non-terminating decimal.
   b. Rational, because it can be expressed as a whole number.
   c. Rational, because it is given in a rational form.
   d. Rational, because it is a recurring decimal.
   e. Irrational, because it is equal to a non-recurring and non-terminating decimal.
3. D
4. c and d
5. C
6. B
7. a. b^{10} b. m^{11} c. k^8 d. f^{14} e. h^{10}
8. a. $6q^{17}$ b. $35w^{29}$ c. $12e^5p^8$ d. $105a^{16}b^{17}$ e. a^3
9. a. t^4 b. r^7 c. p^3 d. f^5 e. y^{90}
10. a. m^{10} b. g^7 c. x d. d^7 e. t^5
11. a. p^5 b. $4k^4$ c. $11b^7c^5$ d. $\dfrac{9e^6}{32d^2}$ e. $\dfrac{a^3}{2}$
12. a. 1 b. 1 c. 1 d. 1 e. 1
13. a. a b. 3 c. 3 d. 199 e. a^7
14. a. v^{10} b. pr c. a^9b^4 d. j^8m^3 e. $4e^2-36$
15. a. $6xy^2$ b. -8 c. $15-12x$ d. -3 e. 54
16. a. b^8 b. a^{24} c. k^{70} d. j^{200}
17. C
18. a. $a^{15}b^6$ b. $m^{14}n^{24}$ c. s^3t^{18} d. $q^{10}p^{300}$
19. a. $\dfrac{1}{a}$ b. $\dfrac{1}{k^4}$ c. $\dfrac{2}{m^4}$ d. $\dfrac{42}{y^5}$
20. a. x^{-1} b. $2y^{-4}$ c. z^{-3} d. $135p^{-3}q^{-3}$
21. a. 10 b. 6 c. a d. b e. $7f^2$
22. a. 3 b. 10 c. x d. $2d$ e. $4f^2g$
23. 1.4×10^{12}
24. 4.167×10^8
25. 9.16×10^7 km
26. 1.2441×10^9 km
27. a. $3.4\times10^7, 1.5\times10^9, 6\times10^9$
    b. 0.57%
    c. 4
    d. 0.25 km^2
28. 2.95%
29. 820 000 queen bees
30. At the beginning of the 5th year, or in 5.5 years.
31. a. Mach 9.1
    b.

Mach number	Speed in m/s	Speed in km/h
1	343	1234.8
2	686	2469.6
3	1029	3704.4

    c. At Mach 1 it would take 32.39 hours.
    At Mach 2 it would take 16.20 hours.
    At Mach 3 it would take 10.79 hours.
    d. 3.55 hours
    e. The fastest aircraft could circle Earth 9.12 times in the time an aircraft travelling at Mach 1 could circle Earth once.
    The fastest aircraft could circle Earth 4.56 times in the time an aircraft travelling at Mach 2 could circle Earth once.
    The fastest aircraft could circle Earth 3.04 times in the time an aircraft travelling at Mach 3 could circle Earth once.
    f. (Number of times to circle Earth) $\times$ (Mach value) ≈ 9.1

STRAND: Number and algebra
SUB-STRAND: Algebraic techniques

2 Algebra

LESSON SEQUENCE

LESSON 2.1 Overview

2.1.1 Why learn this?

Most people will tell you that if you want a career in fields like engineering, science, finance and software development, you need to study and do well in algebra at high school and university. Without algebra we wouldn't have landed on the Moon or be able to enjoy the technological marvels we do today, like our smartphones.

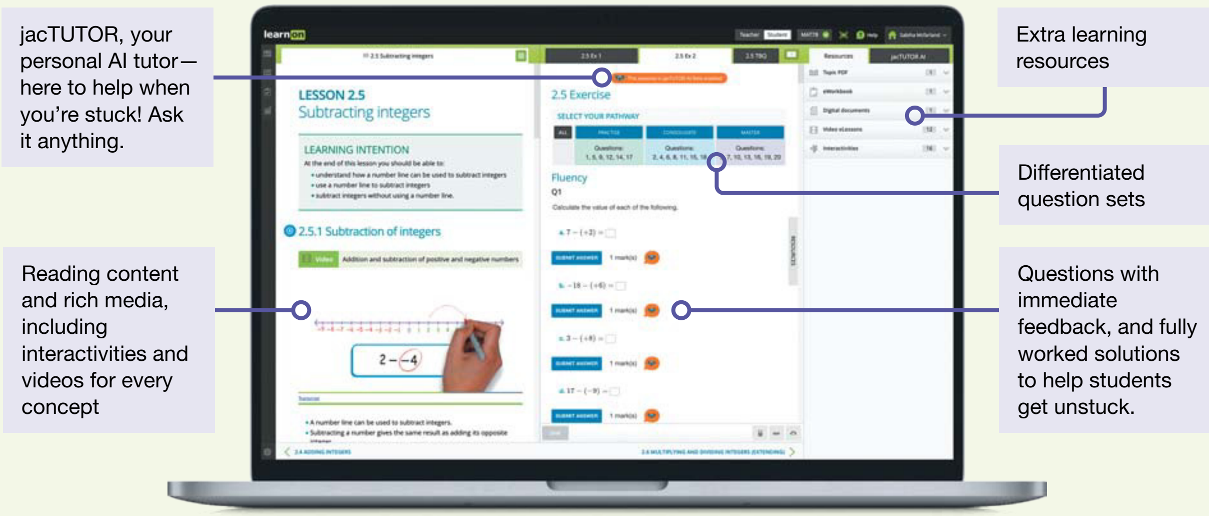
Learning algebra in school helps you to develop critical thinking skills. These skills can help with things like problem solving, logic, pattern recognition and reasoning.

Consider these situations:

- You want to work out the cost of a holiday to Japan at the current exchange rate.
- You're driving to a petrol station to refuel your car, but you only have \$30 in your pocket. You want to know how much petrol you can afford.
- You're trying to work out whether a new phone plan that charges \$5 per day over 24 months for the latest smartphone is worth it.

Working out the answers to these kinds of real-life questions is a direct application of algebraic thinking.

Understanding algebra can help you to make reasoned and well-considered financial and life decisions. It can help you make independent choices and assist you with many everyday tasks. Algebra is an important tool that helps us connect with — and make sense of — the world around us.



Exercise 2.1 Pre-test

learnon

1. MC Select the coefficient of x in the expression $\frac{-x^3}{3} + x^2 + 2ax + b$, where a and b are constants.

   A. 1 B. $-\frac{1}{3}$ C. 2 D. $2a$

2. Evaluate the following expression if $x = -3$, $y = 1$ and $z = -2$.

$$2x^2 - y^2 - xyz$$

3. Determine the value of m in the following equation if $E = 25$ and $v = 5$.

$$E = \frac{1}{2}mv^2$$

4. MC Choose the expression out of the following that is not equivalent.

   A. $x - y + z$ B. $z - y + x$ C. $x - (y - z)$ D. $z - (x + y)$

5. MC A bag of chips costs x cents. Select the correct expression for the cost of 6 bags of chips if each bag is discounted by 50 cents.

   A. $6(x - 50)$ B. $6(x - 0.5)$ C. $6x - 50$ D. $6x - 0.5$

6. Expand and simplify the following expressions.

   a. $4(m + 3n) - (2m - n)$

   b. $-x(y + 3) + y(5 - x)$

7. Expand and simplify the expression $(3q - r)(2q + r)$.

8. MC Students were asked to expand $-2(3x - 4)$. They gave the following answers.

   Student 1: $-6x - 8$
   Student 2: $-6x + 6$
   Student 3: $6x - 8$
   Student 4: $-6x + 8$
   Student 5: $-5x + 6$
   Choose the student who has the correct answer.

   A. Student 1 B. Student 2 C. Student 3 D. Student 4

9. State whether 23^2 can be expanded using the following method.

$$(20 + 3)^2 = 20^2 + 2 \times 20 \times 3 + 3^2$$

10. State whether the following statement is True or False.

    $(a + 3)^2$ expanded is equal to $a^2 + 9$

11. Expand and simplify the following expression.

$$(2b - 5)^2 - (b + 4)(b - 3)$$

12. Factorise the following expressions.

a. $8p^2 + 20p$

b. $25x^2 - 10y$

c. $6xy + 8x^2y - 3xy^2$

13. Factorise the following expressions.

a. $7a(b+3) - (b+3)$

b. $10xy + 5x - 4y - 2$

14. A box has the following dimensions: x, $2x - 1$ and $3x + 2$.
Determine the correct expression for calculating the total surface area of the box.

15. Factorise the expression $9mp - 3m + 6me$.

LESSON 2.2 Using variables

LEARNING INTENTIONS

At the end of this lesson you should be able to:

- identify the number of terms in an algebraic expression
- identify the variables and coefficients of the terms in an expression
- identify the constant terms in an expression
- evaluate an expression or formula by substituting values for each variable.

2.2.1 Expressions

- A **variable** is a letter or symbol that stands for a number. Algebra is the branch of mathematics in which we use and manipulate variables.
- In algebra an **expression** is a group of **terms** in which each term is separated by a + sign or a − sign.
- An expression with two terms is called a **binomial**. An expression with one term is called a **monomial**.
- Here are some important things to note about the expression $5x - 3y + 2$:
  - The expression has three terms and is called a **trinomial**.
  - The variables x and y may take different values and are sometimes called pronumerals.
  - The term 2 has only one value and is called a **constant**.
  - The **coefficient** (the number in front) of x is 5, and the coefficient of y is −3.

Algebraic expressions

An algebraic expression is a group of terms that are separated by + or − signs.

Consider the following expression: $5x^2 - 6y + 13x + 8 + 2y^3$.

- **This expression is made up of five terms.**
- **The variables/pronumerals used in this expression are x and y.**
- **The coefficients of the terms are (in order): 5, −6, 13 and 2.**
- **The coefficient of x is 13.**
- **The smallest coefficient is −6.**
- **The constant is 8.**

WORKED EXAMPLE 1 Identifying coefficients of terms in an expression

For the expression $6x-3xy+z+2+x^2z+\frac{y^2}{7}$, determine:

a. **the number of terms**
b. **the coefficient of the second term**
c. **the coefficient of the last term**
d. **the constant term**
e. **the term with the smallest coefficient**
f. **the coefficient of x^2z.**

THINK	WRITE
a. Count the number of terms.	**a.** There are six terms.
b. The second term is $-3xy$. The number part is the coefficient.	**b.** The coefficient of the second term is -3.
c. The last term is $\frac{y^2}{7}$. This can be rewritten as $\frac{1}{7}\times y^2$.	**c.** The coefficient of the last term is $\frac{1}{7}$.
d. The constant is the term with no variable.	**d.** The constant term is 2.
e. Identify the smallest coefficient and write the whole term to which it belongs.	**e.** The term with the smallest coefficient is $-3xy$.
f. x^2z can be written as $1x^2z$.	**f.** The coefficient of x^2z is 1.

2.2.2 Substitution

- We can evaluate (find the value of) an algebraic expression if we replace the variables with their known values.
- This process is called substitution.
- Consider the expression $4x + 3y$. If we substitute the values $x = 2$ and $y = 5$, then the expression can be evaluated like this:

$$\begin{aligned}4\times 2+3\times 5 &= 8+15\\ &= 23\end{aligned}$$

It is more common to use brackets for substitution instead of multiplication signs. For example:

$$\begin{aligned}4x+3y &= 4(2)+3(5)\\ &= 8+15\\ &= 23\end{aligned}$$

WORKED EXAMPLE 2 Substituting values into terms

Evaluate the following terms for the values $x=4$ and $x=-4$.

a. $3x^2$ **b.** $-x^2$ **c.** $\sqrt{5-x}$

THINK	WRITE
a. **1.** Substitute 4 for x.	**a.** If $x=43$ $\quad 3x^2 = 3(4)^2$
2. Only the x term is squared.	$= 3\times 16$
3. Evaluate.	$= 48$

4. Repeat for $x = -4$.

If $x = -4$: $3x^2 = 3(-4)^2$
$= 3 \times 16$
$= 48$

b. 1. Substitute 4 for x.

b. If $x = 4$: $-x^2 = -(4)^2$

2. Evaluate.

$= -16$

3. Repeat for $x = -4$.

If $x = -4$: $-x^2 = -(-4)^2$
$= -16$

c. 1. Substitute 4 for x.

c. If $x = 4$: $\sqrt{5 - x} = \sqrt{5 - 4}$

2. Evaluate.

$= \sqrt{1}$
$= 1$

3. Repeat for $x = -4$.

If $x = -4$ $\sqrt{5 - x} = \sqrt{5 - (-4)}$
$= \sqrt{5 + 4}$
$= \sqrt{9}$
$= 3$

WORKED EXAMPLE 3 Evaluating expressions using substitution

If $x = 3$ and $y = -2$, evaluate the following expressions.

a. $3x + 2y$
b. $5xy - 3x + 1$
c. $2x^2 + y^2$

THINK

a. 1. Write the expression.
2. Substitute $x = 3$ and $y = -2$.
3. Evaluate.

b. 1. Write the expression.
2. Substitute $x = 3$ and $y = -2$.
3. Evaluate.

c. 1. Write the expression.
2. Substitute $x = 3$ and $y = -2$.
3. Evaluate.

WRITE

a. $3x + 2y$
$= 3(3) + 2(-2)$
$= 9 - 4$
$= 5$

b. $5xy - 3x + 1$
$= 5(3)(-2) - 3(3) + 1$
$= -30 - 9 + 1$
$= -38$

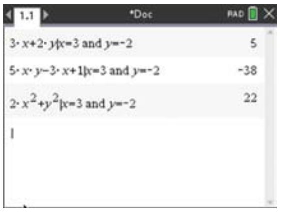
c. $2x^2 + y^2$
$= 2(3)^2 + (-2)^2$
$= 18 + 4$
$= 22$

TI | THINK

a–c. On a Calculator page, complete the entry lines as:
$3x + 2y|x = 3$ and $y = -2$
$5x \times y - 3x + 1|$ $x = 3$ and $y = -2$
$2x^2 + y^2|x = 3$ and $y = -2$

Press ENTER after each entry.

WRITE

a–c. 

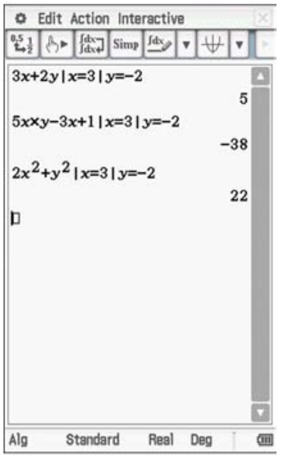
If $x = 3$ and $y = -2$, then
a. $3x + 2y = 5$
b. $5xy - 3x + 1 = -38$
c. $2x^2 + y^2 = 22$.

CASIO | THINK

a–c. On the Main screen, complete the entry lines as:
$3x + 2y|x = 3|$ $y = -2$
$5x \times y - 3x + 1|$ $x = 3|y = -2$
$2x^2 + y^2|x = 3|$ $y = -2$

Press EXE after each entry line.

WRITE

a–c. 

If $x = 3$ and $y = -2$, then
a. $3x + 2y = 5$
b. $5xy - 3x + 1 = -38$
c. $2x^2 + y^2 = 22$.

2.2.3 Substitution into formulas

- A **formula** is a mathematical rule. Formulas are usually written using variables.
- For example, the formula for the area of a rectangle is given by:

$$A = lw$$

where A represents the area of the rectangle, l represents the length and w represents the width.
- If a rectangular kitchen tile has length $l = 20$ cm and width $w = 15$ cm, we can substitute these values into the formula to find its area.

$$\begin{aligned} A &= lw \\ &= 20 \times 15 \\ &= 300\,\text{cm}^2 \end{aligned}$$

- If the area of a kitchen tile is 400 cm^2 and its width is 55 cm, then we can substitute these values into the formula to find its length.

$$\begin{aligned} A &= lw \\ 400 &= l \times 55 \\ l &= \frac{400}{55} \\ &= \frac{80}{11} \\ &\approx 7.3\,\text{cm (to 1 d.p.)} \end{aligned}$$

WORKED EXAMPLE 4 Substituting values into a formula

The formula for the voltage in an electrical circuit can be found using the formula known as Ohm's Law:

$$V = IR$$

where I = current (in amperes)
R = resistance (in ohms)
V = voltage (in volts).

a. Calculate V when:
i. $I = 2$ amperes, $R = 10$ ohms
ii. $I = 20$ amperes, $R = 10$ ohms.
b. Calculate I when $V = 300$ volts and $R = 600$ ohms.

THINK	WRITE
a. i. 1. Write the formula.	a. i. $V = IR$
2. Substitute $I = 2$ and $R = 10$.	$= (2)(10)$
3. Evaluate and write the answer using the correct units.	$= 20$ The voltage is 20 volts.
ii. 1. Write the formula.	ii. $V = IR$
2. Substitute $I = 20$ and $R = 10$.	$= (20)(10)$
3. Evaluate and write the answer using the correct units.	$= 200$ The voltage is 200 volts.
b. 1. Write the formula.	b. $V = IR$
2. Substitute $V = 300$ and $R = 600$.	$300 = I(600)$
3. Evaluate and write the answer using the correct units.	$I = \frac{300}{600}$ $= 0.5$ The current is 0.5 amperes.

Note: Even if we know nothing about volts, amperes and ohms, we can still do this calculation.

DISCUSSION

Which letters (variables) should you avoid using when writing algebraic expressions?

Exercise 2.2 Using variables

learnon

2.2 Quick quiz on | 2.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 7, 10, 13, 14, 17, 18, 25	2, 5, 8, 11, 15, 19, 20, 21, 26	3, 6, 9, 12, 16, 22, 23, 24, 27

Fluency

1. Determine the coefficient of each of the following terms.
   a. $3x$ b. $-2m$ c. $\frac{x}{3}$ d. $-\frac{at}{4}$
2. Evaluate the coefficient of each of the following terms.
   a. $5y$ b. $-2pq$ c. $\frac{x^2}{6}$ d. $-\frac{2mn^2}{7}$
3. Determine the coefficient of each of the following terms.
   a. $15xy$ b. $-\frac{2pq}{5}$ c. $\frac{7w^2z^4}{2}$ d. $-\frac{pq^3}{7}$
4. **WE1** For each expression below, determine:
   i. the number of terms
   ii. the coefficient of the first term
   iii. the constant term
   iv. the term with the smallest coefficient.
   a. $5x^2 + 7x + 8$
   b. $-9m^2 + 8m - 6$
   c. $5x^2y - 7x^2 + 8xy + 5$
   d. $9ab^2 - 8a - 9b^2 + 4$
5. For each expression below, determine:
   i. the number of terms
   ii. the coefficient of the first term
   iii. the constant term
   iv. the term with the smallest coefficient.
   a. $4a - 2 + 9a^2b^2 - 3ac$
   b. $5s + s^2t + 9 + 12t - 3u$
   c. $-m + 8 + 5n^2m + m^2 + 2n$
   d. $7c^2d + 5d^2 + 14 - 3cd^2 - 2e$
6. For each expression below, determine:
   i. the number of terms
   ii. the coefficient of the x term
   iii. the constant term
   iv. the term with the smallest coefficient.
   a. $-10x^2 + 3x - 7$
   b. $4y^2 + 6xy - 6x^2 + 7 - 3x$
   c. $10x^5 - 4x^{-3} + 2x^2 - 5x + 8$
   d. $6 - y^3 + 6x - 7y^5 + 10y - 15xy$

7. WE2 Evaluate the following expressions for $x = 0$, $x = 2$ and $x = -2$.

a. $5x + 2$ b. $x^2 - x$ c. $x^2 + 3x - 1$ d. $\frac{x}{2}$

8. Calculate the value of the following expressions for $x = 0$, $x = 3$ and $x = -3$.

a. $-5x + 4$ b. $x^2 - 2x + 1$ c. $x(x + 4)$ d. $\frac{8}{x + 1}$

9. Determine the value of the following expressions for $x = 0$, $x = 1$ and $x = -2$.

a. $\sqrt{x + 3}$ b. $\frac{2x^2 + 7x}{3}$ c. $(x + 3)(x - 4)$ d. $\frac{2x + 3}{5}$

10. MC Answer the following questions.

a. If $x = -3$, then state the value of $-5x - 3$.
A. -18 B. 12 C. 30 D. 18

b. Choose the expression that is a trinomial.
A. $3x^2$ B. $x + 5 + 2$ C. $x^2 - 7$ D. $x^2 + x + 2$

c. Choose the value of x for which the expression $\sqrt{(x - 6)}$ cannot be evaluated.
A. 2 B. 6 C. 7 D. 10

d. In the expression $5x^2 - 3xy + 0.5x - 0.3y - 5$, identify the smallest coefficient.
A. 5 B. -3 C. 0.5 D. -0.3

11. WE3 Determine the value of the following expressions if $x = 2$, $y = -1$ and $z = 3$.

a. $2x$ b. $3xy$ c. $2y^2z$
d. $\frac{14}{x}$ e. $6(2x + 3y - z)$ f. $x^2 - y^2 + xyz$

12. If $x = 4$ and $y = -3$, evaluate the following expressions.

a. $4x + 3y$ b. $3xy - 2x + 4$ c. $x^2 - y^2$

Understanding

13. WE4 The change in the voltage in an electrical circuit can be found using the formula known as Ohm's Law $V = IR$, where I = current (in amperes), R = resistance (in ohms) and V = voltage (in volts).

a. Calculate V when:
i. $I = 4$, $R = 8$ ii. $I = 25$, $R = 10$.

b. Calculate R when:
i. $V = 100$, $I = 25$ ii. $V = 90$, $I = 30$.

14. Evaluate each of the following by substituting the given values into each formula.

a. If $A = bh$, calculate the value of A when $b = 5$ and $h = 3$.

b. If $d = \frac{m}{v}$, determine the value of d when $m = 30$ and $v = 3$.

c. If $A = \frac{1}{2}xy$, calculate the value of A when $x = 18$ and $y = 2$.

d. If $A = \frac{1}{2}(a + b)h$, determine the value of A when $h = 10$, $a = 7$ and $b = 2$.

e. If $V = \frac{AH}{3}$, determine the value of V when $A = 9$ and $H = 10$.

15. Calculate each of the following by substituting the given values into each formula.

a. If $v = u + at$, calculate the value of v when $u = 4$, $a = 3.2$ and $t = 2.1$.
b. If $t = a + (n - 1)\,d$, determine the value of t when $a = 3$, $n = 10$ and $d = 2$.
c. If $A = \frac{1}{2}(x + y)\,h$, determine the value of A when $x = 5$, $y = 9$ and $h = 3.2$.
d. If $A = 2b^2$, calculate the value of A when $b = 5$.
e. If $y = 5x^2 - 9$, determine the value of y when $x = 6$.

16. Calculate each of the following by substituting the given values into each formula.

a. If $y = x^2 - 2x + 4$, determine the value of y when $x = 2$.
b. If $a = -3b^2 + 5b - 2$, calculate the value of a when $b = 4$.
c. If $s = ut + \frac{1}{2}at^2$, determine the value of s when $u = 0.8$, $t = 5$ and $a = 2.3$.
d. If $F = \frac{mp}{r^2}$, calculate the value of F correct to 2 decimal places, when $m = 6.9$, $p = 8$ and $r = 1.2$.
e. If $C = \pi d$, determine the value of C correct to 2 decimal places if $d = 11$.

Reasoning

17. The area of a triangle is given by the formula $A = \frac{1}{2}bh$, where b is the length of the base and h is the perpendicular height of the triangle.

a. Show that the area is 12 cm^2 when $b = 6$ cm and $h = 4$ cm.
b. Evaluate h if $A = 24$ cm^2 and $b = 4$ cm.

18. Using $E = F + V - 2$, where F is the number of faces on a prism, E is the number of edges and V is the number of vertices, calculate:

a. E if $F = 5$ and $V = 7$
b. F if $E = 10$ and $V = 2$.

19. The formula to convert degrees Fahrenheit (F) to degrees Celsius (C) is $C = \frac{5}{9}(F - 32)$.

a. Determine the value of C when $F = 59$.
b. Show that when Celsius (C) is 15, Fahrenheit (F) is 59.

20. The length of the hypotenuse of a right-angled triangle (c) can be found using the formula $c = \sqrt{a^2 + b^2}$, where a and b are the lengths of the other 2 sides.

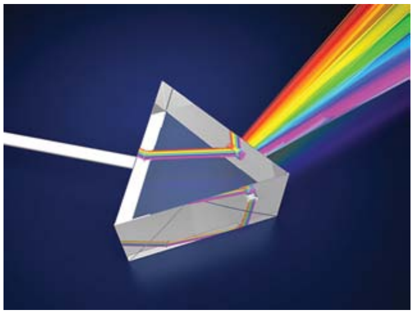
a. Calculate c when $a = 3$ and $b = 4$.
b. Show that when $a = 5$ and $c = 13$, $b = 12$.

21. The kinetic energy (E) of an object is found by using the formula $E = \frac{1}{2}mv^2$, where m is the mass and v is the velocity of the object.

a. Calculate E when $m = 3$ and $v = 3.6$.
b. Show that when $E = 25$ and $v = 5$, $m = 2$.

22. If the volume of a prism (V) is given by the formula $V = AH$, where A is the area of the cross-section and H is the height of the prism, calculate:

a. V when $A = 7\text{ cm}^2$ and $H = 9\text{ cm}$
b. H when $V = 120\text{ cm}^3$ and $A = 30\text{ cm}^2$.



23. The volume of a cylinder (v) is given by $v = \pi r^2 h$, where r is the radius in centimetres and h is the height of the cylinder in centimetres.

a. Determine v correct to 2 decimal places if $r = 7$ and $h = 3$.
b. Determine h correct to 2 decimal places if $v = 120$ and $r = 2$.

24. The surface area of a cylinder (S) is given by $S = 2\pi r(r + h)$, where r is the radius of the circular end and h is the height of the cylinder.

a. Calculate S (to 2 decimal places) if $r = 14$ and $h = 5$.
b. Show that for a cylinder of surface area 240 units^2 and radius 5 units, the height is 2.64 units, correct to 2 decimal places.

Problem solving

25. a. In a magic square, every row, column and diagonal adds to the same number. This number is called the magic number for that square. Complete the unfinished magic square by first finding an algebraic expression for the magic number.
b. Make three different magic squares by substituting different values for g and d in the completed magic square from part **a**.
c. Calculate the magic number for each magic square.

	$2d + 2g$	
	$d + g$	
	0	$d + 2g$

26. A flat, rectangular board is built by gluing together a number of square pieces of the same size. The board is m squares wide and n squares long. In terms of m and n, write expressions for:

a. the total number of squares
b. the number of completely surrounded squares
c. the perimeter of the figure, given that each square is 1 unit by 1 unit.

27. Determine which of the following eight expressions is **not** equivalent to the others.

$x - y + z$	$z - y + x$
$z - (y - x)$	$-y + x + z$
$z + x - y$	$x - (y - z)$
$y - (z - x)$	$x + z - y$

LESSON 2.3 Algebra in worded problems

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- assign variables to the unknown quantities in a worded problem
- convert the worded statements presented in a question into algebraic expressions.

2.3.1 Algebra in worded problems

- In order to make use of the skills you learn in algebra, it is useful to be able to take a real-life problem and convert it into an algebraic expression.
- The starting point for any worded problem is to define the unknown quantities in the question by assigning appropriate variables to those unknown quantities.

Key language used in worded problems

In order to be able to turn a worded problem into an algebraic expression, it helps to look out for the following kinds of words.
- **Words for addition: sum, altogether, add, more than, and, in total**
- **Words for subtraction: difference, less than, take away, take off, fewer than**
- **Words for multiplication: product, groups of, times, of, for each, double, triple**
- **Words for division: quotient, split into, halve, thirds**

WORKED EXAMPLE 5 Converting worded problems into algebraic expressions

Write an algebraic expression for each of the following, choosing an appropriate variable if necessary.

a. The number that is 6 more than Ben's age.
b. The total value of a bundle of $10 notes.
c. The total cost of 8 adults' and 3 children's train tickets.
d. The product of a and w.
e. w less than a.

THINK	WRITE
a. 1. Ben's age is the unknown quantity. *Note*: You must not say, 'Let Ben $= b$', because b is a number.	**a.** Let Ben's age $= b$ years.
2. Write an expression for Ben's age plus 6.	$b + 6$
b. 1. The number of notes is the unknown quantity.	**b.** Let $n =$ the number of $10 notes
2. The value is 10 times the number of notes. Write the expression for the value of the notes.	Total value $= 10n$ dollars
c. 1. The cost of tickets is the unknown quantity.	**c.** Let an adult's ticket cost a dollars
2. The adults' tickets cost $8 \times a$ and the children's Tickets cost $3 \times c$. Write the expression for the total cost of the tickets.	and a child's ticket cost c dollars. Total cost $= 8a + 3c$ dollars.
d. 'Product' means to multiply. Write the expression.	**d.** $a \times w$
e. In this expression, w is subtracted from a. Write the expression.	**e.** $a - w$

WORKED EXAMPLE 6 Solving a worded problem

Kot has a block of chocolate that is made up of 40 smaller pieces.

a. **If he breaks off x pieces to give to Akira, evaluate how much of the block remains.**

b. **Kot then eats $\frac{1}{5}$ of the remaining chocolate for dessert. Determine how much he ate.**

c. **Finally Kot gives a third of what remains to his sister Temar. Calculate how much Temar receives if the original amount Kot gave to Akira was 10 pieces.**

THINK	WRITE
a. **1.** The unknown quantity is the amount passed to Akira, which is x.	**a.** $x =$ number of pieces given to Akira
2. Write an expression for x less than 40.	Amount remaining $= 40 - x$
b. **1.** The unknown quantity is still x.	**b.** Amount eaten $= \frac{1}{5}$ of amount remaining
2. Using the answer to **a**, write an expression for $\frac{1}{5}$ of what is left. *Note:* 'of' is a word that means multiplication.	Amount eaten $= \frac{1}{5} \times (40 - x)$ Amount eaten $= \frac{1}{5}(40 - x)$
c. **1.** The unknown quantity is still x from part **a**.	**c.** Amount left $= 40 - x - \frac{1}{5}(40 - x)$
2. Use your answers to **a** and **b** to write an expression for the amount left over after Kot eats $\frac{1}{5}$ of the remaining chocolate.	Amount passed to Temar is $\frac{1}{3}$ of what is left over after Kot's dessert.
3. Multiply that expression by $\frac{1}{3}$.	Temar's amount $= \frac{1}{3}\left(40 - x - \frac{1}{5}(40 - x)\right)$
4. Substitute the value of $x = 10$ into the expression to find the final answer. Write the answer in words.	When $x = 10$ $= \frac{1}{3}\left(40 - 10 - \frac{1}{5}(40 - 10)\right)$ $= \frac{1}{3}\left(40 - 10 - \frac{1}{5}(30)\right)$ $= \frac{1}{3}\left(30 - \frac{1}{5}(30)\right)$ $= \frac{1}{3}(30 - 6)$ $= \frac{1}{3}(24)$ $= 8$ Temar receives 8 pieces of chocolate from Kot.

2.3 Quick quiz on | 2.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 5, 6, 10, 14	2, 7, 8, 11, 15, 16	3, 9, 12, 13, 17, 18

Fluency

1. Iam studies five more subjects than Xan. Determine how many subjects Iam studies if:
   a. Xan studies six subjects
   b. Xan studies x subjects
   c. Xan studies y subjects.

2. Eva and Juliette walk home from school together. Eva's home is 2 km further from school than Juliette's home. Calculate how far Eva walks if Juliette's home is:
   a. 1.5 km from school
   b. x km from school.
3. Lisa watched television for 2.5 hours today. Calculate how many hours she will watch tomorrow if she watches:
   a. 1.5 hours more than she watched today
   b. t hours more than she watched today
   c. y hours fewer than she watched today.
4. Samir ran d km today. Calculate the distance he will cover tomorrow if he runs:
   a. x km less than he ran today
   b. y km more than he ran today
   c. 4 km more than double the distance that he ran today.

Understanding

5. WE5 Write an algebraic expression for each of the following questions.
   a. If it takes 10 minutes to iron a single shirt, calculate how long it would take to iron all of Anthony's shirts.
   b. Ross has 30 dollars more than Nick. If Nick has N dollars, calculate how much money Ross has.
   c. In a game of Aussie Rules football Luciano kicked 4 more goals than he kicked behinds.
      i. Determine how many behinds Luciano scored, if g is the number of goals kicked.
      ii. Calculate the number of points Luciano scored.
      (*Note:* 1 goal scores 6 points, 1 behind scores 1 point.)
6. Jeff and Chris play Aussie Rules football for opposing teams. Jeff's team won when the two teams played each other.
   a. Calculate how many points Jeff's team scored if they kicked:
      i. 14 goals and 10 behinds
      ii. x goals and y behinds.
   b. Calculate how many points Chris's team scored if his team kicked:
      i. 10 goals and 6 behinds
      ii. p goals and q behinds.

c. Calculate how many points Jeff's team won by if:

i. Chris's team scored 10 goals and 6 behinds, and Jeff's team scored 14 goals and 10 behinds

ii. Chris's team scored p goals and q behinds, and Jeff's team scored x goals and y behinds.

(*Note:* 1 goal scores 6 points, 1 behind scores 1 point.)

7. Yvonne's mother gives her x dollars for each school subject she passes. If she passes y subjects, determine how much money Yvonne receives.

8. Rosya buys a bag containing x Smarties.

a. If they divide them equally among n people, calculate how many each person gets.

b. If they keep half of the Smarties for themselves and divide the remaining Smarties equally among n people, calculate how many Smarties each person gets.

9. A piece of licorice is 30 cm long.

a. If Ameer cuts d cm off, calculate how much licorice remains.

b. If Ameer cuts off $\frac{1}{4}$ of the remaining licorice, calculate how much licorice has been cut off.

c. Determine how much licorice remains after the two cuts that Ameer has made.

Reasoning

10. One-quarter of a class of x students plays tennis at the weekend. One-sixth of the class plays tennis and also swims at the weekend.

a. Write an expression to represent the number of students playing tennis at the weekend.

b. Write an expression to represent the number of students playing tennis and also swimming at the weekend.

c. Show that the number of students playing only tennis at the weekend is $\frac{x}{12}$.

11. During a 24-hour period, Vanessa uses her computer for c hours. Her brother Darren uses it for $\frac{1}{7}$ of the remaining time.

a. Determine the length of time for which Darren used the computer.

b. Show that the total number of hours that Vanessa and Darren use the computer during a 24-hour period can be expressed as $\frac{6c+24}{7}$.

12. Dizem had a birthday party last weekend and invited n friends. The table shown indicates the number of friends at Dizem's party at certain times during the evening. Everybody arrived by 8.30 pm and everyone left the party by 11 pm.

a. Show that 1 person arrived between 7.00 pm and 7.30 pm.

b. Determine the start and end times during which the most friends arrived.

c. Calculate how many friends were invited but did not arrive.

d. Show that 24 friends were invited in total.
e. Determine the start and end times during which the most friends were present at the party.

Time	Number of friends
7.00 pm	$n-24$
7.30 pm	$n-23$
8.00 pm	$n-8$
8.30 pm	$n-5$
9.00 pm	$n-5$
9.30 pm	$n-7$
10.00 pm	$n-12$
10.30 pm	$n-18$
11.00 pm	$n-24$

13. WE6 Ty earns \$$y$ per week working at his part-time job.
a. If he spends \$30 a week on his phone, calculate how much money he has left.
b. If he saves two-thirds of the remaining money for rent each week, calculate how much money is left.
c. If, after paying for his phone, saving for rent and spending \$100 on food, Ty still has \$200 left over, determine how much he earns in a week.

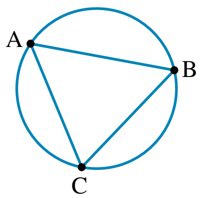
Problem solving

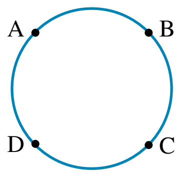
14. A child builds a pyramid out of building blocks using the pattern shown.
a. Determine a rule that gives the number of blocks on the bottom layer, b, for a tower that is h blocks high.
b. If the child wants the tower to be 10 blocks high, determine how many blocks they should begin with on the bottom layer.

15. A shop owner is ordering vases from a stock catalogue. Red vases cost \$4.20 and clear vases cost \$5.70. If she buys 25 vases for \$120, determine how many of each type of vase she bought.

16. A father is 4 times the age of his son. In 4 years time the father will be 3 times the age of his son. Determine how old the father is now.

17. If a circle has 3 equally spaced dots (A, B and C) on its circumference, only 1 triangle can be formed by joining the dots. Investigate to determine the number of triangles that could be formed by 4, 5 or more equally spaced dots on the circumference.





From your investigation, can you see a pattern in the number of triangles (t) formed for a circle with d equally spaced dots on its circumference?
Determine whether it is possible to express this as a formula.

18. Answer the following questions, showing full working.
a. Transpose the equation $T=\dfrac{x-3}{x-1}$ to make x the subject.
b. Evaluate T when $x=1$.
c. Evaluate x when $T=1$.

LESSON 2.4 Simplification of algebraic expressions

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify whether two terms are like terms
- simplify algebraic expressions by adding and subtracting like terms
- simplify algebraic expressions that involve multiplication and division of multiple algebraic terms
- add, subtract, multiply, and divide algebraic fractions.

2.4.1 Addition and subtraction of like terms

- Simplifying an expression involves writing it in a form with the least number of terms, without × or ÷ signs, and with any fractions expressed in their simplest form.
- **Like terms** have identical variable parts (including the power), but may have different coefficients. For example:
  - $5y$ and $10y$ are like terms, but $5y^2$ and $10y$ are not like terms.
  - $3mn^2$ and $-4mn^2$ and like terms, but $3m^2n$ and $-2mn^2$ are not like terms.
- If a term contains more than one variable, the convention is to write the variables in alphabetical order. This makes it easier to identify like terms.
  For example:
  - $3cb^2a$ and $-7b^2ca$ are like terms and should be written as $3ab^2c$ and $-7ab^2c$ to make this easier to identify.
- We can simplify like terms by adding (or subtracting) the coefficients to form a single term. We cannot simplify any terms that are not like terms.
  For example:
  - $10xy + 3xy - 7xy = 6xy$.

WORKED EXAMPLE 7 Simplifying algebraic expressions

Simplify the following expressions.

a. $x + 2x + 3x - 5$ **b.** $9a^2b + 5ba^2 + ab^2$ **c.** $-12 - 4c^2 + 10 + 5c^2$

THINK	WRITE
a. 1. Write the expression.	**a.** $x + 2x + 3x - 5$
2. Collect the like terms x, $2x$ and $3x$.	$= 6x - 5$
b. 1. Write the expression.	**b.** $9a^2b + 5ba^2 + ab^2$
2. Collect the like terms $9a^2b$ and $5ba^2$.	$= 14a^2b + ab^2$
c. 1. Write the expression.	**c.** $-12 - 4c^2 + 10 + 5c^2$
2. Collect the like terms $4c^2$ and $5c^2$.	$= -12 + 10 - 4c^2 + 5c^2$
3. Simplify the expression.	$= -2 + c^2$ or $c^2 - 2$

2.4.2 Multiplication of algebraic terms

- Any number of algebraic terms can be multiplied together to produce a single term.
- Numbers can be multiplied in any order, so it is easiest to multiply the coefficients first, then multiply the variables in alphabetical order.
- Remember that, when multiplying variables, we also add their indices together.

Note: $x = x^1$

$$
\begin{aligned}
4ab \times 3a^2b \times 2a^3 &= 4 \times a \times b \times 3 \times a^2 \times b \times 2 \times a^3 \\
&= 4 \times 3 \times 2 \times a \times a^2 \times a^3 \times b \times b \\
&= 24 \times a^6 \times b^2 \\
&= 24a^6b^2
\end{aligned}
$$

WORKED EXAMPLE 8 Multiplication of algebraic terms

Simplify the following expressions.

a. $\mathbf{4a \times 2b \times a}$

b. $\mathbf{7ax \times (-6bx) \times (-2abx)}$

THINK	WRITE
a. 1. Write the expression.	**a.** $4a \times 2b \times a$
2. Rearrange the expression, writing the coefficients first.	$= 4 \times 2 \times a \times a \times b$
3. Multiply the coefficients and variables separately.	$= 8a^2b$
b. 1. Write the expression.	**b.** $7ax \times (-6bx) \times (-2abx)$
2. Rearrange the expression, writing the coefficients first.	$= 7 \times (-6) \times (-2) \times a \times a \times b \times b \times x \times x \times x$
3. Multiply the coefficients and variables separately.	$= 84a^2b^2x^3$

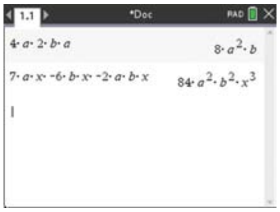
TI | THINK

a–b. In a new problem, on a Calculator page, complete the entry lines as:
$4a \times 2b \times a$
and
$7a \times x \times -6b \times x \times -2a \times b \times x$

Press ENTER after each entry line.

WRITE

a–b.



a. $4a \times 2b \times a = 8a^2b$

b. $7a \times x \times -6b \times x \times -2a \times b \times x = 84a^2b^2x^3$

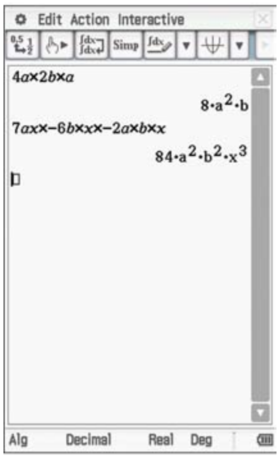
CASIO | THINK

a–b. On the Main screen, complete the entry lines as:
$4a \times 2b \times a$
and
$7a \times x \times -6b \times x \times -2a \times b \times x$

Press EXE after each entry line.

WRITE

a–b.



a. $4a \times 2b \times a = 8a^2b$

b. $7a \times x \times -6b \times x \times -2a \times b \times x = 84a^2b^2x^3$

2.4.3 Division of algebraic terms

- When dividing two algebraic terms, first rewrite the terms in fraction form.
- Simplify the fraction by dividing through by the highest common factor of the coefficients and cancelling down the variables.
- Remember that when dividing variables we subtract their indices.

$$36x^3yz^2 \div \left(40x^2y^2\right) = \frac{36x^3yz^2}{40x^2y^2}$$
$$= \frac{\cancel{36}^{9} \times x \times \cancel{x} \times \cancel{x} \times \cancel{y} \times z \times z}{{}_{10}\cancel{40} \times \cancel{x} \times \cancel{x} \times y \times \cancel{y}}$$
$$= \frac{9xz^2}{10y}$$

WORKED EXAMPLE 9 Dividing algebraic terms

Simplify the following terms.

a. $\frac{12xy}{4xz}$

b. $\left(-8ab^2\right) \div \left(-16a^2b\right)$

THINK	WRITE
a. 1. Write the term.	**a.** $\frac{12xy}{4xz}$
2. Cancel the common factors.	$= \frac{{}^{3}\cancel{12} \times \cancel{x} \times y}{{}_{1}\cancel{4} \times \cancel{x} \times z}$
3. Write the answer.	$= \frac{3y}{z}$
b. 1. Write the term as a fraction.	**b.** $\frac{-8ab^2}{-16a^2b}$
2. Cancel the common factors.	$= \frac{{}^{1}\cancel{-8} \times {}^{1}\cancel{a} \times \cancel{b^2}b}{{}_{2}\cancel{-16} \times {}_{a}\cancel{a^2} \times \cancel{b}_{1}}$
3. Simplify and write the answer.	$= \frac{1}{2} \times \frac{1}{a} \times \frac{b}{1}$ $= \frac{b}{2a}$

2.4.4 Adding and subtracting algebraic fractions (optional)

- Algebraic fractions contain pronumerals that may represent particular numbers or changing values.
- Simplifying algebraic fractions follows the same principles as those used for simplifying numerical fraction expressions such as $\frac{2}{5} + \frac{5}{7}$ or $\frac{3}{4} \div \frac{4}{5}$. That is, determine a common denominator for addition and subtraction and use reciprocals for division.
- To add or subtract algebraic fractions we perform the following steps.
  Step 1 Determine the lowest common denominator (LCD) by determining the lowest common multiple (LCM) of the denominators.
  Step 2 Rewrite each fraction as an equivalent fraction with this common denominator.

Step 3 Add (or subtract) the new numerators.
Step 4 Simplify.

WORKED EXAMPLE 10 Adding and subtracting algebraic fractions

Simplify each of the following expressions.

a. $\frac{x}{2}+\frac{x}{5}$

b. $\frac{y}{3}-\frac{2y}{7}$

c. $\frac{y+1}{5}+\frac{x+y}{6}$

THINK	WRITE
a. 1. Write the expression.	a. $\frac{x}{2}+\frac{x}{5}$
2. Determine the lowest common denominator (LCD). The lowest common multiple (LCM) of 2 and 5 is 10.	$=\frac{5x}{10}+\frac{2x}{10}$
3. Add the numerators.	$=\frac{7x}{10}$
b. 1. Write the expression.	b. $\frac{y}{3}-\frac{2y}{7}$
2. Determine the LCD. The LCM of 3 and 7 is 21.	$=\frac{7y}{21}-\frac{6y}{21}$
3. Subtract the numerators.	$=\frac{y}{21}$
c. 1. Write the expression.	c. $\frac{y+1}{5}+\frac{x+y}{6}$
2. Determine the LCD. The LCM of 5 and 6 is 30.	$=\frac{6(y+1)}{30}+\frac{5(x+y)}{30}$
3. Add the numerators.	$=\frac{6(y+1)+5(x+y)}{30}$
4. Expand the grouping symbols in the numerator.	$=\frac{6y+6+5x+5y}{30}$
5. Simplify the numerator by collecting like terms.	$=\frac{5x+11y+6}{30}$

2.4.5 Multiplying and dividing algebraic fractions (optional)

- The rules for multiplication and division are the same as for numerical fractions.
- When multiplying algebraic fractions, multiply the numerators and multiply the denominators, then cancel any common factors in the numerator and denominator.

WORKED EXAMPLE 11 Multiplying and dividing algebraic fractions

Simplify each of the following.

a. $\frac{x}{2}\times\frac{6}{y}$

b. $\frac{3xy}{2}\div\frac{4x}{9y}$

THINK	WRITE
a. 1. Write the algebraic fractions.	a. $\frac{x}{2}\times\frac{6}{y}$
2. Multiply the numerators and multiply the denominators.	$=\frac{6x}{2y}$

3. Check for common factors in the numerator and denominator and simplify. The numbers 6 and 2 have a common factor of 2.

$$= \frac{3x}{y}$$

b. 1. Write the algebraic fractions.

$$\text{b. } \frac{3xy}{2} \div \frac{4x}{9y}$$

2. Change the division sign to a multiplication sign and write the second fraction as its reciprocal.

$$= \frac{3xy}{2} \times \frac{9y}{4x}$$

3. Multiply the numerators and multiply the denominators.

$$= \frac{27xy^2}{8x}$$

4. Check for common factors in the numerator and denominator. Cancel x.

$$= \frac{27y^2}{8}$$

- When dividing algebraic fractions, change the division sign to a multiplication sign and write the following fraction as its reciprocal.

Exercise 2.4 Simplification of algebraic expressions

learnon

2.4 Quick quiz on	2.4 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 9, 10, 11, 14, 17, 21, 24, 27, 28	2, 4, 5, 7, 12, 15, 19, 22, 25, 29, 30	6, 8, 13, 16, 17, 20, 23, 26, 31, 32

Fluency

1. For each of the following terms, select the terms listed in brackets that are like terms.
   a. $6ab \quad (7a,\ 8b,\ 9ab,\ -ab,\ 4a^2b^2)$
   b. $-x \quad (3xy,\ -xy,\ 4x,\ 4y,\ -yx)$
   c. $3az \quad (3ay,\ -3za,\ -az,\ 3z^2a,\ 3a^2z)$
   d. $x^2 \quad (2x,\ 2x^2,\ 2x^3,\ -2x,\ -x^2)$
2. For each of the following terms, select the terms listed in brackets that are like terms.
   a. $2x^2y \quad (xy, -2xy, -2xy^2, -2x^2y, -2x^2y^2)$
   b. $3x^2y^5 \quad (3xy, 3x^5y^2, 3x^4y^3, -x^2y^5, -3x^2y^5)$
   c. $5x^2p^3w^5 \quad (-5x^3w^5p^3, p^3x^2w^5, 5xp^3w^5, -5x^2p^3w^5, w^5p^2x^3)$
   d. $-x^2y^5z^4 \quad (-xy^5, -y^2z^5x^4, -x+y+z,\ 4y^5z^4x^2, -2x^2z^4y^5)$
3. WE7 Simplify the following expressions.
   a. $5x + 2x$
   b. $3y + 8y$
   c. $7m + 12m$
   d. $13q - 2q$
   e. $17r - 9r$
   f. $-x + 4x$
4. Simplify the following expressions.
   a. $5a + 2a + a$
   b. $9y + 2y - 3y$
   c. $7x - 2x + 8x$
   d. $14p - 3p + 5p$
   e. $2q^2 + 7q^2$
   f. $5x^2 - 2x^2$
5. Simplify the following expressions.
   a. $6x^2 + 2x^2 - 3y$
   b. $3m^2 + 2n - m^2$
   c. $-2g^2 - 4g + 5g - 12$
   d. $-5m^2 + 5m - 4m + 15$
   e. $12a^2 + 3b + 4b^2 - 2b$
   f. $6m + 2n^2 - 3m + 5n^2$

6. Simplify the following expressions.

a. $3xy + 2y^2 + 9yx$ b. $3ab + 3a^2b + 2a^2b - ab$ c. $9x^2y - 3xy + 7yx^2$
d. $4m^2n + 3n - 3m^2n + 8n$ e. $-3x^2 - 4yx^2 - 4x^2 + 6x^2y$ f. $4 - 2a^2b - ba^2 + 5b - 9a^2$

7. MC Choose which of the following is a simplification of the expression $18p - 19p$.

A. p B. $-p$ C. p^2 D. -1

8. MC Choose which of the following is a simplification of the expression $5x^2 - 8x + 6x - 9$.

A. $3x - 9$ B. $3x^2 - 9$ C. $5x^2 + 2x - 9$ D. $5x^2 - 2x - 9$

9. MC Choose which of the following is a simplification of the expression $12a - a + 15b - 14b$.

A. $11a + b$ B. 12 C. $11a - b$ D. $13a + b$

10. MC Choose which of the following is a simplification of the expression $-7m^2n + 5m^2 + 3 - m^2 + 2m^2n$.

A. $-9m^2n + 4m^2 + 3$ B. $-9m^2n + 8$ C. $-5m^2n - 4m^2 + 3$ D. $-5m^2n + 4m^2 + 3$

11. WE8 Simplify the following.

a. $3m \times 2n$ b. $4x \times 5y$ c. $2p \times 4q$ d. $5x \times (-2y)$ e. $3y \times (-4x)$

12. Simplify the following.

a. $(-3m) \times (-5n)$ b. $5a \times 2a$ c. $3mn \times 2p$ d. $(-6ab) \times b$ e. $(-5m) \times (-2mn)$

13. Simplify the following.

a. $(-6a) \times 3ab$ b. $(-3xy) \times (-5xy) \times 2x$ c. $4pq \times (-p) \times 3q^2$
d. $4c \times (-7cd) \times 2c$ e. $\left(-3a^2\right) \times \left(-5ab^3\right) \times 2ab^4$

14. WE9 Simplify the following.

a. $\dfrac{6x}{2}$ b. $\dfrac{9m}{3}$ c. $\dfrac{12y}{6}$ d. $\dfrac{8m}{2}$ e. $12m \div 3$

15. Simplify the following.

a. $14x \div 7$ b. $-21x \div 3$ c. $-32m \div 8$ d. $\dfrac{4m}{8}$ e. $\dfrac{6x}{18}$

16. Simplify the following.

a. $\dfrac{8mn}{18n}$ b. $\dfrac{6ab}{12a^2b}$ c. $\dfrac{28xyz}{14x}$
d. $\dfrac{2x^2yz}{8xz}$ e. $-7xy^2z^2 \div (11xyz)$

17. WE10 Simplify each of the following expressions. Give your answers in simplified fraction form.

a. $\dfrac{x}{3} + \dfrac{x}{4}$ b. $\dfrac{y}{2} - \dfrac{y}{3}$ c. $\dfrac{m}{8} - \dfrac{m}{4}$ d. $\dfrac{x}{6} + \dfrac{x}{12}$ e. $\dfrac{m}{2} + \dfrac{m}{7}$

18. WE11 Simplify each of the following. Give your answer in simplified fraction form where appropriate.

a. $\dfrac{4}{3} \times \dfrac{m}{16}$ b. $\dfrac{n}{9} \times \dfrac{3}{2}$ c. $\dfrac{a}{5} \div \dfrac{a}{20}$ d. $\dfrac{3a}{14} \div \dfrac{a}{7}$

Understanding

19. Simplify the following.

a. $5x \times 4y \times 2xy$ b. $7xy \times 4ax \times 2y$ c. $\dfrac{6x^2y}{12y^2}$ d. $\dfrac{-15x^2ab}{12b^2x^2}$

20. Simplify the following.

a. $\dfrac{2p^3q^2}{p^3q^2}$

b. $(-4a)\times\left(-5ab^2\right)\times 2a$

c. $(-a)\times 4ab\times 2ba\times b$

d. $2a\times 2a\times 2a\times 2a$

21. Jim buys m pens at p cents each and n books at q dollars each.

a. Calculate how much Jim spends in:

i. dollars

ii. cents.

b. Calculate how much change Jim will have if he starts with \$20.

22. At a local discount clothing store 4 shirts and 3 pairs of shorts cost \$138 in total. If a pair of shorts costs 2.5 times as much as a shirt, calculate the cost of each kind of clothing.

23. Anthony and Jamila are taking their 3 children to see a movie at the cinema. The total cost for the 2 adults and 3 children is \$108. If an adult's ticket is 1.5 times the cost of a child's, calculate the cost of 1 child's ticket plus 1 adult's ticket.

Reasoning

24. Class 9A were given an algebra test. One of the questions is shown below.

Simplify the following expression: $\dfrac{3ab}{2}\times\dfrac{4ac}{6b}\times 7c$.

Sean, who is a student in class 9A, wrote his answer as $\dfrac{12aabc}{12b}\times 7c$. Explain why Sean's answer is incorrect, and write the correct answer.

25. Using the appropriate method to divide fractions, simplify the following expression.

$$\frac{15a^2b^2}{16c^4}\div\left(\frac{5ac^2}{4b^3}\times\frac{a}{2b^4}\right)$$

26. Answer the following questions.

a. Simplify the expression $\dfrac{n}{(n+1)}\times\dfrac{(n+1)}{(n+2)}$.

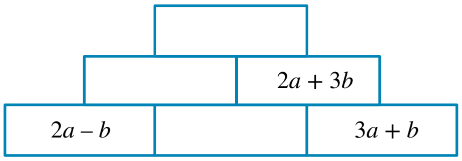
b. Simplify the expression $\dfrac{n}{(n+1)}\times\dfrac{(n+1)}{(n+2)}\times\dfrac{(n+2)}{(n+3)}$.

c. Use the results from parts **a** and **b** to evaluate $\dfrac{1}{2}\times\dfrac{2}{3}\times\dfrac{3}{4}\times\;\ldots\;\times\dfrac{99}{100}$.

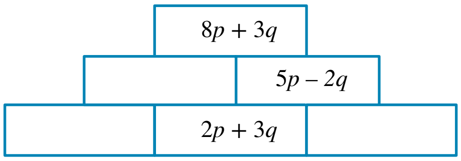
Problem solving

27. a. Fill in the empty bricks in the following pyramids. The expression in each brick is obtained by adding the expressions in the two bricks below it.

i.

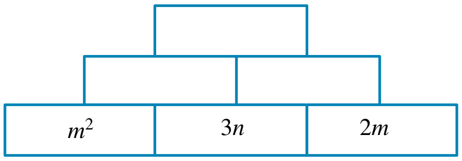


ii.

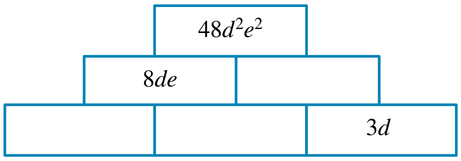


b. Fill in the empty bricks in the following pyramids. The expression in each brick is obtained by multiplying the expressions in the two bricks below it.

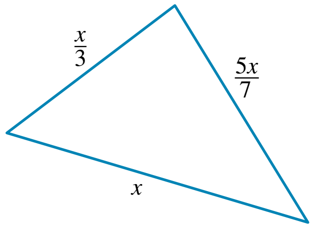
i.



ii.



28. For the triangle shown:

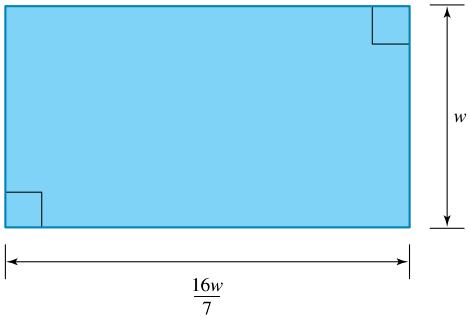


a. write an expression for the perimeter of the triangle

b. show that the triangle's perimeter can be simplified to $\frac{43}{21}x$

c. calculate the cost (to the nearest dollar) to frame the triangle using timber that costs \$4.50 per metre, if x equals 2 metres.

29. For the rectangle shown:



a. write an expression for the perimeter of the rectangle

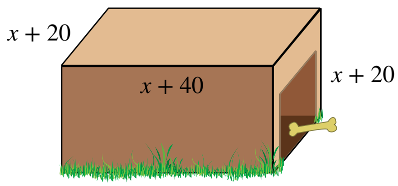
b. show that the expression for the perimeter can be simplified to $6\frac{4}{7}w$

c. calculate the cost (to the nearest cent) to create a wire frame for the rectangle using wire that costs \$1.57 per metre, if w equals seven metres.

30. A rectangular chocolate block has dimensions x and $(4x - 7)$.

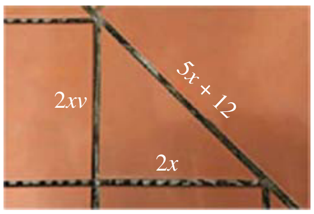
a. Write an expression for the perimeter (P) in the form $P = \square x + \square(4x - 7)$.
b. Expand and simplify this expression.
c. Calculate the perimeter of the chocolate block when $x = 3$.
d. Explain why x cannot equal 1.

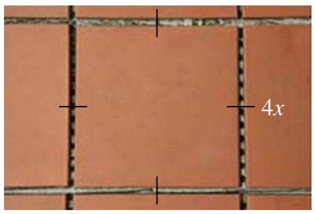
31. A doghouse in the shape of a rectangular prism is to be reinforced with steel edging along its outer edges including the base.
The cost of the steel edging is \$1.50 per metre. All measurements are in centimetres.



a. Write an expression for the total length of the straight edges of the frame.
b. Expand the expression.
c. Evaluate the cost of steel needed when $x = 80$ cm.

32. A tile manufacturer produces tiles that have the side lengths shown. All measurements are in centimetres.





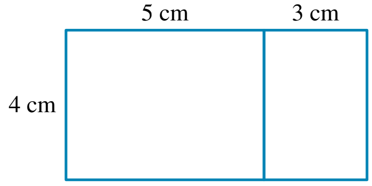
a. Write an expression for the perimeter of each shape.
b. Evaluate the value of x for which the perimeter of the triangular tile is the same as the perimeter of the square tile.

LESSON 2.5 Expanding brackets

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- apply the distributive law to expand and simplify an expression with a single set of brackets
- apply the distributive law to expand and simplify expressions containing two or more sets of brackets.

2.5.1 The Distributive Law



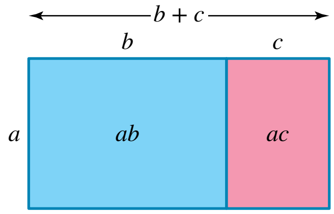
- There are two ways of calculating the area of the rectangle shown.
  1. The rectangle can be treated as a single shape with a length of 4 cm and a width of $5 + 3$ cm.

$$\begin{aligned} A &= l \times w \\ &= 4(5+3) \\ &= 4 \times 8 \\ &= 32\,\text{cm}^2 \end{aligned}$$

  2. The areas of the two smaller rectangles can be added together.

$$\begin{aligned} A &= 4 \times 5 + 4 \times 3 \\ &= 20 + 12 \\ &= 32\,\text{cm}^2 \end{aligned}$$

- The length of the rectangle shown is a and its width is $b + c$.



- This rectangle's area can be found in two different ways.
  1. $A = l \times w$
     $= a(b + c)$
     This expression is described as **factorised** because it shows one number (or factor) multiplied by another. The two factors are a and $(b + c)$.
  2. The areas of the two small rectangles can be added together.

$$A = ab + ac$$

  This expression is described as **expanded**, which means that it is written without brackets.
- $a(b + c) = ab + ac$ is called the **Distributive Law**.

> **The Distributive Law**
>
> **In order to expand a single set of brackets we apply the distributive law, which states that:**
>
> $$\boldsymbol{a(b + c) = a \times b + a \times c = ab + ac}$$

- A helpful way to expand brackets is to draw arrows between factors as you work out each multiplication.

$$x(x - y) = x^2 \ldots$$
$$x(x - y) = x^2 - xy$$

WORKED EXAMPLE 12 Expanding a single set of brackets

Use two different methods to calculate the value of the following.

a. $7(5 + 15)$ **b.** $10(9 - 1)$

THINK	WRITE
a. 1. Method 1: Work out the brackets first, then evaluate.	**a.** $7(5 + 15)$ $= 7 \times 20$ $= 140$
2. Method 2: Expand the brackets first, then evaluate.	$7(5 + 15)$ $= 7 \times 5 + 7 \times 15$ $= 35 + 105$ $= 140$
b. 1. Method 1: Work out the brackets first, then evaluate.	**b.** $10(9 - 1)$ $= 10 \times 8$ $= 80$
2. Method 2: Expand the brackets first, then evaluate	$10(9 - 1)$ $= 10 \times 9 + 10 \times -1$ $= 90 - 10$ $= 80$

WORKED EXAMPLE 13 Expanding using arrows

Expand the following expressions.

a. $5(x + 3)$ **b.** $-4y(2x - w)$ **c.** $3x(5 - 6x + 2y)$

THINK	WRITE
a. 1. Draw arrows to help with the expansion.	**a.** $5(x + 3)$
2. Simplify and write the answer.	$= 5 \times x + 5 \times 3$ $= 5x + 15$
b. 1. Draw arrows to help with the expansion.	**b.** $-4y(2x - w)$
2. Simplify and write the answer.	$= -4y \times 2x - 4y \times (-w)$ $= -8xy + 4wy$
c. 1. Draw arrows to help with the expansion.	**c.** $3x(5 - 6x + 2y)$
2. Simplify and write the answer.	$= 3x \times 5 + 3x \times (-6x) + 3x \times 2y$ $= 15x - 18x^2 + 6xy$

2.5.2 Expanding and simplifying

- When a problem involves **expanding** that is more complicated, it is likely that the like terms will need to be collected and simplified after you have expanded the brackets.
- When solving these more complicated problems, expand all sets of brackets first, then simplify any like terms that result from the expansion.

WORKED EXAMPLE 14 Expanding and simplifying

Expand and simplify the following expressions by collecting like terms.

a. $4(x-4)+5$ **b.** $x(y-2)+5x$ **c.** $-x(y-2)+5x$ **d.** $7x-6(y-2x)$

THINK	WRITE
a. 1. Expand the brackets.	**a.** $4(x-4)+5$
2. Simplify and write the answer.	$=4x-16+5$ $=4x-11$
b. 1. Expand the brackets.	**b.** $x(y-2)+5x$
2. Simplify and write the answer.	$=xy-2x+5x$ $=xy+3x$
c. 1. Expand the brackets.	**c.** $-x(y-z)+5x$
2. Simplify and write the answer. (*Note:* There are no like terms.)	$-xy+xz+5x$
d. 1. Expand the brackets.	**d.** $7x-6(y-2x)$
2. Simplify and write the answer.	$=7x-6y+12x$ $=19x-6y$

WORKED EXAMPLE 15 Expanding two sets of brackets and simplifying

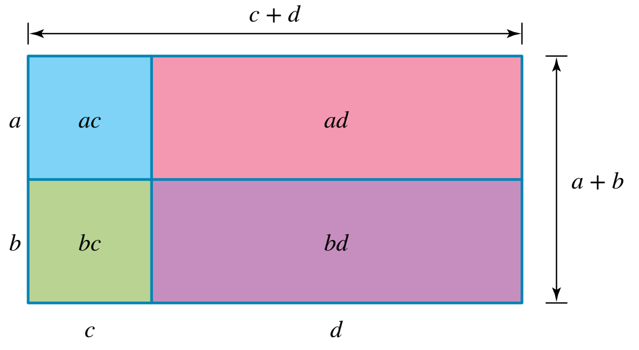
Expand and simplify the following expressions.

a. $5(x+2y)+6(x-3y)$ **b.** $5x(y-2)-y(x+3)$

THINK	WRITE
a. 1. Expand each set of brackets.	**a.** $5(x+2y)+6(x-3y)$
2. Simplify and write the answer.	$=5x+10y+6x-18y$ $=11x-8y$
b. 1. Expand each set of brackets.	**b.** $5x(y-2)-y(x+3)$
2. Simplify and write the answer.	$=5xy-10x-xy-3y$ $=4xy-10x-3y$

2.5.3 Expanding binomial factors

- Remember that a **binomial** is an expression containing two terms, for example $x + 3$ or $2y - z^2$. In this section we will look at how two binomials can be multiplied together.
- The rectangle shown has length $a + b$ and width $c + d$.



- There are two ways of finding the area of the large rectangle.
  1. $A = l \times w$
     $= (a + b) \times (c + d)$
     $= (a + b)(c + d)$
     This is a factorised expression in which the two factors are $(a + b)$ and $(c + d)$.
  2. The areas of the four small rectangles can be added together.
     $A = ac + ad + bc + bd$
     So $(a + b)(c + d) = ac + ad + bc + bd$

- There are several methods that can be helpful when remembering how to expand binomial factors.
  One commonly used method is **FOIL**.
  Each of the letters in FOIL stand for:
  First — multiply the first term in each bracket.
  Outer — multiply the two outer terms of each bracket.
  Inner — multiply the two inner terms of each bracket.
  Last — multiply the last term of each bracket.

FOIL – First Outer Inner Last

F O

$$(a + b)(c + d)$$

I L

$$(a + b)(c + d) = ac + ad + bc + bd$$

Expanding binomial factors

Using FOIL, the expanded product of two binomial factors is given by:

$$\boldsymbol{(a + b)(c + d) = a \times c + a \times d + b \times c + b \times d = ac + ad + bc + bd}$$

It is expected that the result will have 4 terms.

***Note:* It may be possible to simplify like terms after expanding.**

WORKED EXAMPLE 16 Expanding and simplifying two binomial factors

Expand and simplify each of the following expressions.

a. $(x-5)(x+3)$ **b.** $(x+2)(x+3)$ **c.** $(2x+2)(2x+3)$

THINK	WRITE
a. 1. Expand the brackets using FOIL.	**a.** $(x-5)(x+3)$
2. Simplify the expression by collecting like terms.	$= x^2 + 3x - 5x - 15$ $= x^2 - 2x - 15$
b. 1. Expand the brackets using FOIL.	**b.** $(x+2)(x+3)$
2. Simplify the expression by collecting like terms.	$= x^2 + 3x + 2x + 6$ $= x^2 + 5x + 6$
c. 1. Expand the brackets using FOIL.	**c.** $(2x+2)(2x+3)$
2. Simplify the expression by collecting like terms.	$= 4x^2 + 6x + 4x + 6$ $= 4x^2 + 10x + 6$

TI | THINK

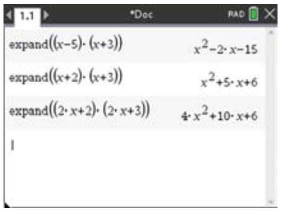
a–c. In a new problem, on a Calculator page, press:
- MENU
- 3: Algebra
- 3: Expand

Complete the entry lines as: expand $((x-5)\times(x+3))$ expand $((x+2)\times(x+3))$ expand $((2x+2)\times(2x+3))$

Press ENTER after each entry.
Note: Remember to include the multiplication sign between the brackets.

WRITE

a–c.



a. $(x-5)(x+3)$
$= x^2 - 2x - 15$

b. $(x+2)(x+3)$
$= x^2 + 5x + 6$

c. $(2x+2)(2x+3)$
$= 4x^2 + 10x + 6$

CASIO | THINK

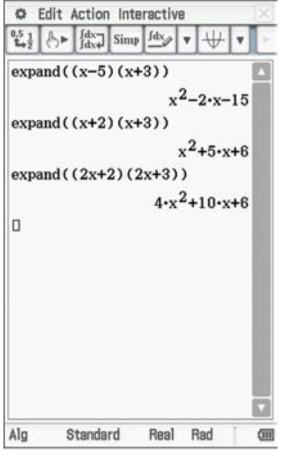
a–c. On the Main screen, press:
- Action
- Transformation
- Expand

Complete the entry lines as:
expand $((x-5)(x+3))$
expand $((x+2)(x+3))$
expand $((2x+2)(2x+3))$

Press EXE after each entry line.

WRITE

a–c.



a. $(x-5x+3) = x^2 - 2x - 15$

b. $(x+2x+3) = x^2 + 5x + 6$

c. $(2x+22x+3)$
$= 4x^2 + 10x + 6$

DISCUSSION

Explain why, when expanded, $(x+y)(2x+y)$ gives the same result as $(2x+y)(x+y)$.

Exercise 2.5 Expanding brackets

learn**on**

2.5 Quick quiz **on** | 2.5 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 7, 9, 12, 15, 18, 22, 24, 26, 30, 33	3, 4, 6, 10, 13, 17, 20, 21, 25, 27, 31, 34	5, 8, 11, 14, 16, 19, 23, 28, 29, 32, 35, 36

Fluency

1. **WE12** Use two different methods to calculate the value of the following expressions.
   a. $8(10-2)$ b. $11(99+1)$ c. $-5(3+1)$ d. $7(100-1)$
2. **WE13** Expand the following expressions. For the first two examples, draw a diagram to represent the expression.
   a. $3(x+2)$ b. $4(x+3)$ c. $4(x+1)$ d. $7(x-1)$
3. Expand the following expressions.
   a. $-3(p-2)$ b. $-(x-1)$ c. $3(2b-4)$ d. $8(3m-2)$
4. Expand each of the following. For the first two examples, draw a diagram to represent the expression.
   a. $x(x+2)$ b. $a(a+5)$ c. $x(4+x)$ d. $m(7-m)$
5. Expand each of the following.
   a. $2x(y+2)$ b. $-3y(x+4)$ c. $-b(3-a)$ d. $-6a(5-3a)$
6. **WE14** Expand and simplify the following expressions by collecting like terms.
   a. $2(p-3)+4$ b. $5(x-5)+8$ c. $-7(p+2)-3$
   d. $-4(3p-1)-1$ e. $6x(x-3)-2x$
7. Expand and simplify the following expressions by collecting like terms.
   a. $2m(m+5)-3m$ b. $3x(p+2)-5$ c. $4y(y-1)+7$
   d. $-4p(p-2)+5p$ e. $5(x-2y)-3y-x$
8. Expand and simplify the following expressions by collecting like terms.
   a. $2m(m-5)+2m-4$ b. $-3p(p-2q)+4pq-1$ c. $-7a(5-2b)+5a-4ab$
   d. $4c(2d-3c)-cd-5c$ e. $6p+3-4(2p+5)$
9. **WE15** Expand and simplify the following expressions.
   a. $2(x+2y)+3(2x-y)$ b. $4(2p+3q)+2(p-2q)$ c. $7(2a+3b)+4(a+2b)$
   d. $5(3c+4d)+2(2c+d)$ e. $-4(m+2n)+3(2m-n)$
10. Expand and simplify the following expressions.
    a. $-3(2x+y)+4(3x-2y)$ b. $-2(3x+2y)+3(5x+3y)$ c. $-5(4p+2q)+2(3p+q)$
    d. $6(a-2b)-5(2a-3b)$ e. $5(2x-y)-2(3x-2y)$
11. Expand and simplify the following expressions.
    a. $4(2p-4q)-3(p-2q)$ b. $2(c-3d)-5(2c-3d)$ c. $7(2x-3y)-(x-2y)$
    d. $-5(p-2q)-(2p-q)$ e. $-3(a-2b)-(2a+3b)$
12. Expand and simplify the following expressions.
    a. $a(b+2)+b(a-3)$ b. $x(y+4)+y(x-2)$ c. $c(d-2)+c(d+5)$
    d. $p(q-5)+p(q+3)$ e. $3c(d-2)+c(2d-5)$

13. Expand and simplify the following expressions.

a. $7a(b-3)-b(2a+3)$
b. $2m(n+3)-m(2n+1)$
c. $4c(d-5)+2c(d-8)$
d. $3m(2m+4)-2(3m+5)$
e. $5c(2d-1)-(3c+cd)$

14. Expand and simplify the following expressions.

a. $-3a(5a+b)+2b(b-3a)$
b. $-4c(2c-6d)+d(3d-2c)$
c. $6m(2m-3)-(2m+4)$
d. $7x(5-x)+6(x-1)$
e. $-2y(5y-1)-4(2y+3)$

15. **MC** Choose the expression that is the equivalent of $3(a+2b)+2(2a-b)$.

A. $5a+6b$ B. $7a+4b$ C. $5(3a+b)$ D. $7a+8b$

16. **MC** Choose the expression that is the equivalent of $-3(x-2y)-(x-5y)$.

A. $-4x+11y$ B. $-4x-11y$ C. $4x+11y$ D. $4x+7y$

17. **MC** Choose the expression that is the equivalent of $2m(n+4)+m(3n-2)$.

A. $3m+4n-8$ B. $5mn+4m$ C. $5mn+10m$ D. $5mn+6m$

18. **WE16** Expand and simplify each of the following expressions.

a. $(a+2)(a+3)$ b. $(x+4)(x+3)$ c. $(y+3)(y+2)$
d. $(m+4)(m+5)$ e. $(b+2)(b+1)$

19. Expand and simplify each of the following expressions.

a. $(p+1)(p+4)$ b. $(a-2)(a+3)$ c. $(x-4)(x+5)$
d. $(m+3)(m-4)$ e. $(y+5)(y-3)$

20. Expand and simplify each of the following expressions.

a. $(y-6)(y+2)$ b. $(x-3)(x+1)$ c. $(x-3)(x-4)$
d. $(p-2)(p-3)$ e. $(x-3)(x-1)$

21. Expand and simplify each of the following expressions.

a. $(2a+3)(a+2)$ b. $(c-6)(4c-7)$ c. $(7-2t)(5-t)$
d. $(2+3t)(5-2t)$ e. $(7-5x)(2-3x)$ f. $(5x-2)(5x-2)$

22. Expand and simplify each of the following expressions.

a. $(x+y)(z+1)$ b. $(2x+y)(z+4)$ c. $(3p+q)(r+1)$
d. $(a+2b)(a+b)$ e. $(2c+d)(c-3d)$ f. $(x+y)(2x-3y)$

23. Expand and simplify each of the following expressions.

a. $(4p-3q)(p+q)$ b. $(a+2b)(b+c)$ c. $(3p-2q)(1-3r)$
d. $(4x-y)(3x-y)$ e. $(p-q)(2p-r)$ f. $(5-2j)(3k-1)$

24. **MC** Choose the expression that is the equivalent of $(4-y)(7+y)$.

A. $28-y^2$ B. $28-3y+y^2$ C. $28-3y-y^2$ D. $11-2y$

25. **MC** Choose the expression that is the equivalent of $(2p+1)(p-5)$.

A. $2p^2-5$ B. $2p^2-11p-5$ C. $2p^2-9p-5$ D. $2p^2-6p-5$

Understanding

26. Expand the following expressions using FOIL, then simplify.

a. $(x+3)(x-3)$ b. $(x+5)(x-5)$ c. $(x+7)(x-7)$
d. $(x-1)(x+1)$ e. $(x-2)(x+2)$ f. $(2x-1)(2x+1)$

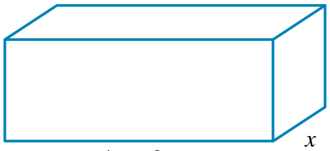
27. Expand the following expressions using FOIL, then simplify.

a. $(x+1)(x+1)$ b. $(x+2)(x+2)$ c. $(x+8)(x+8)$
d. $(x-3)(x-3)$ e. $(x-5)(x-5)$ f. $(x-9)(x-9)$

28. Simplify the following expressions.

a. $2.1x(3x+4.7y)-3.1y(1.4x+y)$
b. $(2.1x-3.2y)(2.1x+3.2y)$
c. $(3.4x+5.1y)^2$

29. For the box shown, calculate the following in expanded form:



a. the total surface area b. the volume.

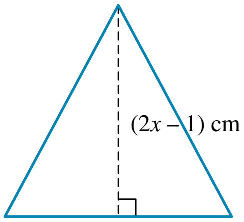
Reasoning

30. For each of the following shapes:

i. write down the area in factor form
ii. expand and simplify the expression
iii. discuss any limitations on the value of x.

a.

b.



31. Show that the following is true.

$$\begin{aligned}(a-x)(a+x)-2(a-x)(a-x)-2x(a-x)\\=-(a-x)^2\end{aligned}$$

32. A series of incorrect expansions of $(x+8)(x-3)$ are shown below. For each of these incorrect expansions, explain the mistake that has been made.

a. $x^2+11x+24$ b. $x^2+5x+24$ c. x^2-3x

Problem solving

33. Three students gave the following incorrect answers when expanding $-5(3x-20)$.

i. $-5x-20$ ii. $-8x+25$ iii. $15x-100$

a. Explain the errors made by each student.
b. Determine the correct answer.

34. In a test, a student expanded brackets and obtained the following answers. Identify and correct the student's errors and write the correct expansions.

a. $-2(a-5)=-2a-10$
b. $2b(3b-1)=6b^2-1$
c. $-2(c-4)=2c+8$

35. Three students' attempts at expanding $(3x+4)(2x+5)$ are shown.

Student A

$$(3x+4)(2x+5)$$
$$= 3x \times 2x + 3x \times 5 + 4 \times 2x + 4 \times 5$$
$$= 6x + 15x + 8x + 20$$
$$= 29x + 20$$
$$= 49x$$

Student B

$$(3x+4)(2x+5)$$
$$= 3x \times 2x + 4 \times 2x + 4 \times 5$$
$$= 6x^2 + 8x + 20$$

Student C

$$(3x+4)(2x+5)$$
$$= 3x \times 2x + 3x \times 5 + 4 \times 2x + 4 \times 5$$
$$= 6x^2 + 15x + 8x + 20$$
$$= 6x^2 + 23x + 20$$

a. State which student's work is correct.
b. Copy each of the incorrect answers into your workbook and correct the mistakes in each one as though you were their teacher.

36. Rectangular floor mats have an area of $(x^2+2x-15)\text{ cm}^2$.

a. The length of a mat is $(x+5)$ cm. Determine an expression for the width of this mat.
b. If the length of a mat is 70 cm, evaluate its width.
c. If the width of a mat is 1 m, evaluate its length.

LESSON 2.6 Difference of two squares and perfect squares

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- recognise and use the difference of two squares rule to quickly expand and simplify binomial products with the form $(a+b)(a-b)$
- recognise and use the perfect square rule to quickly expand and simplify binomial products with the form $(a \pm b)^2$.

2.6.1 Difference of two squares

- Consider the expansion of $(x+4)(x-4)$:

$$(x+4)(x-4) = x^2 - 4x + 4x - 16 = x^2 - 16$$

- Now consider the expansion of $(x-6)(x+6)$:

$$(x-6)(x+6) = x^2 + 6x - 6x - 36 = x^2 - 36$$

- In both cases the middle two terms cancel each other out, leaving two terms, both of which are perfect squares.
- The two terms that are left are the first value squared minus the second value squared. This is where the phrase **difference of two squares** originates.
- In both cases the binomial terms can be written in the form $(x+a)$ and $(x-a)$. If we can recognise expressions that have this form, we can use the pattern above to quickly expand those expressions. For example: $(x+12)(x-12) = x^2 - 12^2 = x^2 - 144$

Difference of two squares

The difference of two squares rule is used to quickly expand certain binomial products, as long as they are in the forms shown below:
- $(a+b)(a-b) = a^2 - b^2$
- $(a-b)(a+b) = a^2 - b^2$

Because the two binomial brackets are being multiplied, the order of the brackets does not affect the final result.

WORKED EXAMPLE 17 Expanding using difference of two squares

Use the difference of two squares rule to expand and simplify each of the following expressions.

a. $(x+8)(x-8)$ **b.** $(6-3)(6+x)$ **c.** $(2x-3)(2x+3)$ **d.** $(3x+5)(5-3x)$

THINK	WRITE
a. 1. Write the expression.	**a.** $(x+8)(x-8)$
2. This expression is in the form $(a+b)(a-b)$, so the difference of two squares rule can be used. Expand using the formula.	$= x^2 - 8^2$ $= x^2 - 64$

b. 1. Write the expression.

b. $(6-x)(6+x)$

2. This expression is in the form $(a+b)(a-b)$, so the difference of two squares rule can be used. Expand using the formula.
Note: $36-x^2$ is not the same as x^2-36.

$= 6^2 - x^2$
$= 36 - x^2$

c. 1. Write the expression.

c. $(2x-3)(2x+3)$

2. This expression is in the form $(a-b)(a+b)$, so the difference of two squares rule can be used. Expand using the formula.
Note: $(2x)^2$ and $2x^2$ are not the same. In this case $a = 2x$, so $a^2 = (2x)^2$.

$= (2x)^2 - 3^2$
$= 4x^2 - 9$

d. 1. Write the expression.

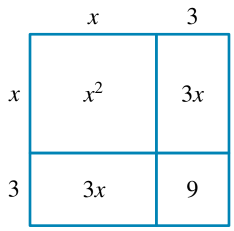
d. $(3x+5)(5-3x)$

2. The difference of two squares rule can be used if we rearrange the terms, since $3x+5 = 5+3x$. Expand using the formula.

$(5+3x)(5-3x)$
$= 5^2 - (3x)^2$
$= 25 - 9x^2$

2.6.2 Perfect squares

- A **perfect square** is the result of the square of a whole number. $1\times 1 = 1, 2\times 2 = 4$ and $3\times 3 = 9$, showing that $1, 4$ and 9 are all perfect squares.
- Similarly, $(x+3)(x+3) = (x+3)^2$ is a perfect square because it is the result of a binomial factor multiplied by itself.
- Consider the diagram illustrating $(x+3)^2$. What shape is it?
- The area is given by $x^2 + 3x + 3x + 9 = x^2 + 6x + 9$.
- We can see from the diagram that there are two squares produced $\left(x^2 \text{ and } 3^2 = 9\right)$ and two rectangles that are identical to each other $(2\times 3x)$.
- Compare this with the expansion of $(x+6)^2$:



$$(x+6)(x+6) = x^2 + 6x + 6x + 36 = x^2 + 12x + 36$$

- A pattern begins to emerge after comparing these two expansions. The square of a binomial equals the square of the first term, plus double the product of the two terms plus the square of the second term.
For example: $(x+10)(x+10) = x^2 + 2\times 10\times x + 10^2 = x^2 + 20x + 100$

Perfect squares

The rule for the expansion of the square of a binomial is given by:

- $\boldsymbol{(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2}$
- $\boldsymbol{(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2}$

We can use the above rules to quickly expand binomial products that are presented as squares.

WORKED EXAMPLE 18 Expanding perfect squares

Use the rules for expanding perfect binomial squares to expand and simplify the following.

a. $(x+1)(x+1)$ **b.** $(x-2)^2$ **c.** $(2x+5)^2$ **d.** $(4x-5y)^2$

THINK	WRITE
a. 1. This expression is the square of a binomial.	**a.** $(x+1)(x+1)$
2. Apply the formula for perfect squares: $(a+b)^2 = a^2+2ab+b^2$	$= x^2+2\times x\times 1+1^2$ $= x^2+2x+1$
b. 1. This expression is the square of a binomial.	**b.** $(x-2)^2$
2. Apply the formula for perfect squares: $(a-b)^2 = a^2-2ab+b^2$.	$=(x-2)(x-2)$ $= x^2-2\times x\times 2\times 2^2$ $= x^2-4x+4$
c. 1. This expression is the square of a binomial.	**c.** $(2x+5)^2$
2. Apply the formula for perfect squares: $(a+b)^2 = a^2+2ab+b^2$.	$=(2x)^2+2\times 2x\times 5+5^2$ $=4x^2+20x+25$
d. 1. This expression is the square of a binomial.	**d.** $(4x-5y)^2$
2. Apply the formula for perfect squares: $(a-b)^2 = a^2-2ab+b^2$.	$=(4x)^2-2\times 4x\times 5y+(5y)^2$ $=16x^2-40xy+25y^2$

DISCUSSION

How could you represent $(x-3)^2$ on a diagram?

Exercise 2.6 Difference of two squares and perfect squares

learnon

2.6 Quick quiz on | 2.6 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 8, 13, 16, 20, 21	2, 6, 9, 11, 14, 17, 22, 23	4, 7, 10, 12, 15, 18, 19, 24, 25

Fluency

1. WE17 Use the difference of two squares rule to expand and simplify each of the following.
   a. $(x+2)(x-2)$ **b.** $(y+3)(y-3)$ **c.** $(m+5)(m-5)$ **d.** $(a+7)(a-7)$
2. Use the difference of two squares rule to expand and simplify each of the following.
   a. $(x+6)(x-6)$ **b.** $(p-12)(p+12)$ **c.** $(a+10)(a-10)$ **d.** $(m-11)(m+11)$

3. Use the difference of two squares rule to expand and simplify each of the following.

a. $(2x+3)(2x-3)$
b. $(3y-1)(3y+1)$
c. $(5d-2)(5d+2)$
d. $(7c+3)(7c-3)$
e. $(2+3p)(2-3p)$

4. Use the difference of two squares rule to expand and simplify each of the following.

a. $(d-9x)(d+9x)$
b. $(5-12a)(5+12a)$
c. $(3x+10y)(3x-10y)$
d. $(2b-5c)(2b+5c)$
e. $(10-2x)(2x+10)$

5. **WE18** Use the rule for the expansion of the square of a binomial to expand and simplify each of the following.

a. $(x+2)(x+2)$
b. $(a+3)(a+3)$
c. $(b+7)(b+7)$
d. $(c+9)(c+9)$

6. Use the rule for the expansion of the square of a binomial to expand and simplify each of the following.

a. $(m+12)^2$
b. $(n+10)^2$
c. $(x-6)^2$
d. $(y-5)^2$

7. Use the rule for the expansion of the square of a binomial to expand and simplify each of the following.

a. $(9-c)^2$
b. $(8+e)^2$
c. $2(x+y)^2$
d. $(u-v)^2$

8. Use the rule for expanding perfect binomial squares rule to expand and simplify each of the following.

a. $(2a+3)^2$
b. $(3x+1)^2$
c. $(2m-5)^2$
d. $(4x-3)^2$

9. Use the rule for expanding perfect binomial squares to expand and simplify each of the following.

a. $(5a-1)^2$
b. $(7p+4)^2$
c. $(9x+2)^2$
d. $(4c-6)^2$

10. Use the rule for expanding perfect binomial squares to expand and simplify each of the following.

a. $(5+3p)^2$
b. $(2-5x)^2$
c. $(9x-4y)^2$
d. $(8x-3y)^2$

Understanding

11. Use the difference of two squares rule to expand and simplify each of the following.

a. $(x+3)(x-3)$
b. $(2x+3)(2x-3)$
c. $(7x-4)(7x+4)$
d. $(2x+7y)(2x-7y)$
e. $\left(x^2+y^2\right)\left(x^2-y^2\right)$

12. Expand and simplify the following perfect squares.

a. $(4x+5)^2$
b. $(7x-3y)^2$
c. $\left(5x^2-2y\right)^2$
d. $2(x-y)^2$
e. $\left(\frac{2}{x}+4x\right)^2$

13. A square has a perimeter of $4x+12$. Calculate its area.

14. Francis has fenced off a square in her paddock for spring lambs. The area of the paddock is $(9x^2+6x+1)\text{ m}^2$.
Using pattern recognition, determine the side length of the paddock in terms of x.

15. A square has an area of $x^2+18x+81$. Determine an expression for the perimeter of this square.

Reasoning

16. Show that $a^2-b^2=(a+b)(a-b)$ is true for each of the following.

a. $a=5, b=4$
b. $a=9, b=1$
c. $a=2,\ b=7$
d. $a=-10, b=-3$

17. Lin has a square bedroom. Her sister Tasneem has a room that is 1 m shorter in length than Lin's room, but 1 m wider.

a. Show that Lin has the larger bedroom.
b. Determine how much bigger Lin's bedroom is than Tasneem's bedroom.

18. Expand each of the following pairs of expressions.

a. i. $(x-4)(x+4)$ and $(4-x)(4+x)$

ii. $(x-11)(x+11)$ and $(11-x)(11+x)$

iii. $(2x-9)(2x+9)$ and $(9-2x)(9+2x)$

b. State what you notice about the answers to the pairs of expansions above.

c. Explain how this is possible.

19. Answer the following questions.

a. Expand $(10k+5)^2$.

b. Show that $(10k+5)^2 = 100k(k+1)+25$.

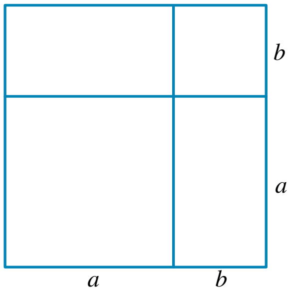
c. Using part **b**, evaluate 25^2 and 85^2.

Problem solving

20. A large square has been subdivided into 2 squares and 2 rectangles.

a. Write formulas for the areas of these 4 pieces, using the dimensions a and b marked on the diagram.

b. Write an equation that states that the area of the large square is equal to the combined area of its 4 pieces. Do you recognise this equation?



21. Expand each of the following pairs of expressions.

a. i. $(x-3)^2$ and $(3-x)^2$

ii. $(x-15)^2$ and $(15-x)^2$

iii. $(3x-7)^2$ and $(7-3x)^2$

b. State what you notice about the answers to the pairs of expansions above.

c. Explain how this is possible.

22. Use the perfect squares rule to quickly evaluate the following.

a. 27^2 b. 33^2 c. 39^2 d. 47^2

23. Allen is creating a square deck with a square pool installed in the middle of it. The side length of the deck is $(2x+3)$ m and the side length of the pool is $(x-2)$ m. Evaluate the area of the decking around the pool.

24. Ram wants to create a rectangular garden that is 10 m longer than it is wide. Write an expression for the area of the garden in terms of x, where x is the average length of the two sides.

25. The expansion of perfect squares $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$ can be used to simplify some arithmetic calculations. For example:

$$\begin{aligned} 97^2 &= (100-3)^2 \\ &= 100^2 - 2 \times 100 \times 3 + 3^2 \\ &= 9409 \end{aligned}$$

Use this method to evaluate the following.

a. 103^2 b. 62^2 c. 997^2 d. 1012^2 e. 53^2 f. 98^2

LESSON 2.7 Further expansions

LEARNING INTENTION

At the end of this lesson you should be able to:
- expand multiple sets of brackets and simplify the result.

2.7.1 Expanding multiple sets of brackets

- When expanding expressions with more than two sets of brackets, expand all brackets first before simplifying the like terms.
- It is important to be careful with signs, particularly when subtracting one expression from another. For example:

$$\begin{aligned}3x(x+6)-(x+5)(x-3) &= 3x^2+18x-\left(x^2+5x-3x-15\right)\\ &= 3x^2+18x-\left(x^2+2x-15\right)\\ &= 3x^2+18x-x^2-2x+15\\ &= 2x^2+16x+15\end{aligned}$$

- When expanding this, since the entire expression $(x+5)(x-3)$ is being subtracted from $3x(x+6)$, its expanded form needs to be kept inside its own set of brackets. This helps us to remember that, as each term is being subtracted, the sign of each term will switch when opening up the brackets.
- When expanding, it can help to treat the expression as though it has a -1 at the front, as shown.

$$\begin{aligned}-(x^2+2x-15) &= -1(x^2+2x-15)\\ &= -x^2-2x+15\end{aligned}$$

WORKED EXAMPLE 19 Expanding and simplifying multiple sets of brackets

Expand and simplify each of the following expressions.

a. $(x+3)(x+4)+4(x-2)$

b. $(x-2)(x+3)-(x-1)(x+2)$

THINK	WRITE
a. 1. Expand each set of brackets.	**a.** $(x+3)(x+4)+4(x-2)$
2. Simplify by collecting like terms.	$= x^2+4x+3x+12+4x-8$ $= x^2+11x+4$
b. 1. Expand and simplify each pair of brackets. Because the second expression is being subtracted, keep it in a separate set of brackets.	**b.** $(x-2)(x+3)-(x-1)(x+2)$ $= x^2+3x-2x-6-\left(x^2+2x-x-2\right)$
2. Subtract all of the second result from the first result. Remember that $-\left(x^2+x-2\right)=-1\left(x^2+x-2\right)$. Simplify by collecting like terms.	$= x^2+x-6-\left(x^2+x-2\right)$ $= x^2+x-6-x^2-x+2$ $= -4$

WORKED EXAMPLE 20 Further expanding with multiple sets of brackets

Consider the expression $(4x+3)(4x+3)-(2x-5)(2x-5)$.

a. Expand and simplify this expression.

b. Apply the difference of perfect squares rule to verify your answer.

THINK	WRITE
a. 1. Expand each set of brackets.	a. $(4x+3)(4x+3)-(2x-5)(2x-5)$
2. Simplify by collecting like terms.	$=16x^2+12x+12x+9-\left(4x^2-10x-10x+25\right)$ $=16x^2+24x+9-\left(4x^2-20x+25\right)$ $=16x^2+24x+9-4x^2+20x-25$ $=12x^2+44x-16$
b. 1. Rewrite as the difference of two squares.	b. $(4x+3)(4x+3)-(2x-5)(2x-5)$
2. Apply the difference of two squares rule, where $a=(4x+3)$ and $b=(2x-5)$.	$=(4x+3)^2-(2x-5)^2$ $=((4x+3)+(2x-5))((4x+3)-(2x-5))$
3 Simplify, then expand the new brackets.	$=(6x-2)(2x+8)$ $=12x^2+48x-4x-16$
4. Simplify by collecting like terms.	$=12x^2+44x-16$

DISCUSSION

On a diagram, how would you show $(m+2)(m+3)-(m+2)(m+4)$?

Exercise 2.7 Further expansions

learnon

2.7 Quick quiz on | 2.7 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 8, 11, 14	2, 5, 7, 9, 12, 15	3, 6, 10, 13, 16

Fluency

1. WE19 Expand and simplify each of the following expressions.

   a. $(x+3)(x+5)+(x+2)(x+3)$
   b. $(x+4)(x+2)+(x+3)(x+4)$
   c. $(x+5)(x+4)+(x+3)(x+2)$
   d. $(x+1)(x+3)+(x+2)(x+4)$

2. Expand and simplify each of the following expressions.

   a. $(p-3)(p+5)+(p+1)(p-6)$
   b. $(a+4)(a-2)+(a-3)(a-4)$
   c. $(p-2)(p+2)+(p+4)(p-5)$
   d. $(x-4)(x+4)+(x-1)(x+20)$

3. Expand and simplify each of the following expressions.

   a. $(y-1)(y+3)+(y-2)(y+2)$
   b. $(d+7)(d+1)+(d+3)(d-3)$
   c. $(x+2)(x+3)+(x-4)(x-1)$
   d. $(y+6)(y-1)+(y-2)(y-3)$

4. Expand and simplify each of the following expressions.

a. $(x+2)^2+(x-5)(x-3)$
b. $(y-1)^2+(y+2)(y-4)$
c. $(p+2)(p+7)+(p-3)^2$
d. $(m-6)(m-1)+(m+5)^2$

5. Expand and simplify each of the following expressions.

a. $(x+3)(x+5)-(x+2)(x+5)$
b. $(x+5)(x+2)-(x+1)(x+2)$
c. $(x+3)(x+2)-(x+4)(x+3)$
d. $(m-2)(m+3)-(m+2)(m-4)$

6. Expand and simplify each of the following expressions.

a. $(b+4)(b-6)-(b-1)(b+2)$
b. $(y-2)(y-5)-(y+2)(y+6)$
c. $(p-1)(p+4)-(p-2)(p-3)$
d. $(x+7)(x+2)-(x-3)(x-4)$

7. Expand and simplify each of the following expressions.

a. $(m+3)^2-(m+4)(m-2)$
b. $(a-6)^2-(a-2)(a-3)$
c. $(p-3)(p+1)-(p+2)^2$
d. $(x+5)(x-4)-(x-1)^2$

Understanding

8. **WE20** Consider the expression $(x+3)(x+3)-(x-5)(x-5)$.

a. Expand and simplify this expression.
b. Apply the difference of perfect squares rule to verify your answer.

9. Consider the expression $(4x-5)(4x-5)-(x+2)(x+2)$.

a. Expand and simplify this expression.
b. Apply the difference of perfect squares rule to verify your answer.

10. Consider the expression $(3x-2y)(3x-2y)-(y-x)(y-x)$.

a. Expand and simplify this expression.
b. Apply the difference of perfect squares rule to verify your answer.

Reasoning

11. Determine the value of x for which $(x+3)+(x+4)^2=(x+5)^2$ is true.

12. Show that $(p-1)(p+2)+(p-3)(p+1)=2p^2-p-5$.

13. Show that $(x+2)(x-3)-(x+1)2=-3x-7$.

Problem solving

14. Answer the following questions.

a. Show that $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$.
b. Using part a, write $(2^2+1^2)(3^2+4^2)$ as the sum of two squares and evaluate.

15. Answer the following questions.

a. Expand $\left(x^2+x-1\right)^2$.
b. Show that $\left(x^2+x-1\right)^2=(x-1)x(x+1)(x+2)+1$.
c. i. Evaluate $4\times3\times2\times1+1$.
   ii. Determine the value of x if $4\times3\times2\times1+1=(x-1)x(x+1)(x+2)+1$.

16. Answer the following questions.

a. Expand $(a+b)(d+e)$.
b. Expand $(a+b+c)(d+e+f)$. Draw a diagram to illustrate your answer.

LESSON 2.8 The highest common factor

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- determine the highest common factor for a group of terms
- factorise expressions by dividing out the highest common factor from each term.

2.8.1 Factorising expressions

- **Factorising** a number or term means writing it as the product of a pair of its factors. For example, $18ab$ could be factorised as $3a \times 6b$.
- Factorising an algebraic expression is the opposite process to expanding.
- $3(x+6)$ is the product of 3 and $(x+6)$ and is the factorised form of $3x+18$.
- In this section we will start with the expanded form of an expression, for example $6x+12$, and work to re-write it in factorised form, in this case $6(x+2)$.

WORKED EXAMPLE 21 Factorising terms

Complete the following factorisations.

a. $15x = 5x \times$ ________

b. $15x^2 = 3x \times$ ________

c. $-18ab^2 = 6a \times$ ________

THINK	WRITE
a. Divide $15x$ by $5x$.	**a.** $\frac{15x}{5x} = 3$ So, $15x = 5x \times 3$.
b. Divide $15x^2$ by $3x$.	**b.** $\frac{15x^2}{3x} = 5x$ So, $15x^2 = 3x \times 5x$.
c. Divide $-18ab^2$ by $6a$.	**c.** $\frac{-18ab^2}{6a} = -3b^2$ So, $-18ab^2 = 6a \times (-3b^2)$.

- The **highest common factor (HCF)** of two or more numbers is the largest factor that divides into all of them. The highest common factor of 18 and 27 is 9, since it is the biggest number that divides evenly into both 18 and 27.
- We can determine the HCF of two or more algebraic terms by determining the highest common factor of the coefficients, as well as any variables that are common to all terms. For example, the HCF of $30xyz$, $25x^2z$ and $15xz^2$ is $5xz$. It can help to write each term in expanded form so that you can determine all of the common variables.

WORKED EXAMPLE 22 Determining the highest common factor of two terms

Determine the highest common factor for each of the following pairs of terms.

a. $25a^2b$ and $10ab$

b. $3xy$ and $-3xz$

THINK	WRITE
a. 1. Write $25a^2b$ in expanded form.	**a.** $25a^2b = 5 \times 5 \times a \times a \times b$
2. Write $10ab$ in expanded form.	$10ab = 2 \times 5 \times a \times b$
3. Write the HCF.	$\text{HCF} = 5ab$
b. 1. Write $3xy$ in expanded form.	**b.** $3xy = 3 \times x \times y$
2. Write $-3xz$ in expanded form.	$-3xz = -1 \times 3 \times x \times z$
3. Write the HCF.	$\text{HCF} = 3x$

2.8.2 Factorising expressions by determining the highest common factor

- To factorise an expression such as $15x^2yz + 12xy^2$, determine the highest common factor of both terms. Writing each term in expanded form can help identify what is common to both terms.

$$\begin{aligned} 15x^2yz + 12xy^2 &= 3 \times 5 \times x \times x \times y \times z + 3 \times 4 \times x \times y \times y \\ &= 3 \times x \times y \times 5 \times x \times z + 3 \times x \times y \times 4 \times y \end{aligned}$$

- We can see that $3xy$ is common to both terms. To complete the factorisation, place $3xy$ on the outside of a set of brackets and simplify what is left inside the brackets.

$$\begin{aligned} &= 3 \times x \times y(5 \times x \times z + 4 \times y) \\ &= 3xy(5xz + 4y) \end{aligned}$$

- By removing the HCF from both terms, we have factorised the expression.
- The answer can always be checked by expanding out the brackets and making sure it produces the original expression.

WORKED EXAMPLE 23 Factorising by first determining the HCF

Factorise each expression by first determining the HCF.

a. $5x + 15y$

b. $-14xy - 7y$

c. $6x^2y + 9xy^2$

THINK	WRITE
a. 1. The HCF is 5.	**a.** $5x + 15y = 5(\quad)$
2. Divide each term by 5 to determine the binomial.	$= 5(x + 3y)$
3. Check the answer by expanding.	$5(x + 3y) = 5x + 15y$ (correct)

b. 1. The HCF is $7y$ or $-7y$, but $-7y$ makes things a little simpler.

2. Divide each term by $-7y$ to determine the binomial.

3. Check the answer by expanding.

b. $-14xy - 7y = -7y(\quad\quad)$

$= -7y(2x+1)$

$-7y(2x + 1) = -14xy - 7y$ (correct)

c. 1. The HCF is $3xy$.

2. Divide each term by $3xy$ to determine the binomial.

3. Check the answer by expanding.

c. $6x^2y + 9xy^2 = 3xy(\quad\quad)$

$= 3xy(2x+3y)$

$3xy(2x + 3y) = 6x^2y + 9xy^2$ (correct)

TI | THINK

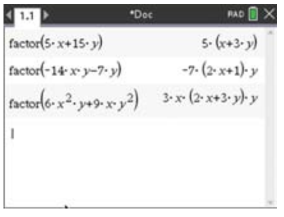
a–c. In a new problem, on a Calculator page, press:
- MENU
- 3: Algebra
- 2: Factor

Complete the entry lines as:
factor $(5x + 15y)$
factor $(-14x \times y - 7y)$
factor $(6x^2 \times y + 9x \times y^2)$

Press ENTER after each entry.
Note: Ensure you put a multiplication between each variable.

WRITE

a–c.



a. $5x + 15y = 5(x+3y)$
b. $-14xy - 7y = -7y(2x+1)$
c. $6x^2y + 9xy^2 = 3xy(2x+3y)$

CASIO | THINK

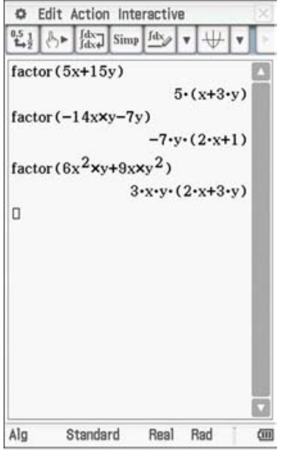
a–c. On the Main screen, press:
- Action
- Transformation
- Factor.

Complete the entry lines as:
factor $(5x + 15y)$
factor $(-14x \times y - 7y)$
factor $(6x^2 \times y + 9x \times y^2)$

Press EXE after each entry.

WRITE

a–c.



a. $(5x + 15y) = 5(x+3y)$
b. $(-14xy - 7y) = -7y(2x+1)$
c. $(6x^2y + 9xy^2) = 3xy(2x+3y)$

DISCUSSION

How do you find the factors of terms within algebraic expressions?

Exercise 2.8 The highest common factor

learnon

2.8 Quick quiz on | 2.8 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 7, 11, 13, 16, 21, 23, 26	2, 4, 6, 8, 10, 14, 17, 18, 22, 24, 27	5, 9, 12, 15, 19, 20, 25, 28

Fluency

1. **WE21** Complete each of the following factorisations by writing in the missing factor.
   a. $8a = 4 \times$ ______ b. $8a = 2a \times$ ______ c. $12x^2 = 4x \times$ ______
   d. $-12x^2 = 3x^2 \times$ ______ e. $3x^2 = x \times$ ______ f. $15a^2b = ab \times$ ______
2. Complete each of the following factorisations by writing in the missing factor.
   a. $12x = -4 \times$ ______ b. $10\,mn = 10\,n \times$ ______ c. $10\,mn = -10 \times$ ______
   d. $a^2b^2 = ab \times$ ______ e. $30x^2 = 10x \times$ ______ f. $-15\,mn^2 = -3\,m \times$ ______
3. **WE22** Determine the highest common factor (HCF) of each of the following.
   a. 4 and 12 b. 6 and 15 c. 10 and 25 d. 24 and 32 e. 12, 15 and 21
4. Determine the highest common factor (HCF) of each of the following.
   a. 25, 50 and 200 b. 17 and 23 c. $6a$ and $12ab$
   d. $14xy$ and $21xz$ e. $60pq$ and $30q$
5. Determine the highest common factor (HCF) of each of the following.
   a. $50cde$ and $70fgh$ b. $6x^2$ and $15x$ c. $6a$ and $9c$
   d. $5ab$ and 25 e. $3x^2y$ and $4x^2z$
6. **MC** Choose which of the following pairs has a highest common factor of $5m$.
   A. $2m$ and $5m$ B. $5m$ and m
   C. $25mn$ and $15lm$ D. $20m$ and $40m$
7. **WE23** Factorise each of the following expressions.
   a. $4x + 12y$ b. $5m + 15n$ c. $7a + 14b$ d. $7m - 21n$ e. $-8a - 24b$ f. $8x - 4y$
8. Factorise each of the following expressions.
   a. $-12p - 2q$ b. $6p + 12pq + 18q$ c. $32x + 8y + 16z$
   d. $16m - 4n + 24\,p$ e. $72x - 8y + 64\,pq$ f. $15x^2 - 3y$
9. Factorise each of the following expressions.
   a. $5p^2 - 20q$ b. $5x + 5$ c. $56q + 8p^2$
   d. $7p - 42x^2y$ e. $16p^2 + 20q + 4$ f. $12 + 36a^2b - 24b^2$
10. Factorise each of the following expressions.
    a. $9a + 21b$ b. $4c + 18d^2$ c. $12p^2 + 20q^2$ d. $35 - 14m^2n$ e. $25y^2 - 15x$
11. Factorise each of the following expressions.
    a. $16a^2 + 20b$ b. $42m^2 + 12n$ c. $63p^2 + 81 - 27y$
    d. $121a^2 - 55b + 110c$ e. $10 - 22x^2y^3 + 14xy$
12. Factorise each of the following expressions.
    a. $18a^2bc - 27ab - 90c$ b. $144p + 36q^2 - 84pq$ c. $63a^2b^2 - 49 + 56ab^2$
    d. $22 + 99p^3q^2 - 44p^2r$ e. $36 - 24ab^2 + 18b^2c$
13. Factorise each of the the following expressions.
    a. $-x + 5$ b. $-a + 7$ c. $-b + 9$ d. $-2m - 6$ e. $-6\,p - 12$ f. $-4a - 8$

14. Factorise each of the following expressions.

a. $-3n^2 + 15m$ b. $-7x^2y^2 + 21$ c. $-7y^2 - 49z$

d. $-12p^2 - 18q$ e. $-63m + 56$ f. $-12m^3 - 50x^3$

15. Factorise each of the following expressions.

a. $-9a^2b + 30$ b. $-15p - 12q$ c. $-18x^2 + 4y^2$

d. $-3ab + 18m - 21$ e. $-10 - 25p^2 - 45q$ f. $-90m^2 + 27n + 54p^3$

16. Factorise each of the following expressions.

a. $a^2 + 5a$ b. $14q - q^2$ c. $18m + 5m^2$

d. $6p + 7p^2$ e. $7n^2 - 2n$ f. $7p - p^2q + pq$

17. Factorise each of the following expressions.

a. $xy + 9y - 3y^2$ b. $5c + 3c^2d - cd$ c. $3ab + a^2b + 4ab^2$

d. $2x^2y + xy + 5xy^2$ e. $5p^2q^2 - 4pq + 3p^2q$ f. $6x^2y^2 - 5xy + x^2y$

18. Factorise each of the following expressions.

a. $5x^2 + 15x$ b. $24m^2 - 6m$ c. $32a^2 - 4a$

d. $-2m^2 + 8m$ e. $-5x^2 + 25x$ f. $-7y^2 + 14y$

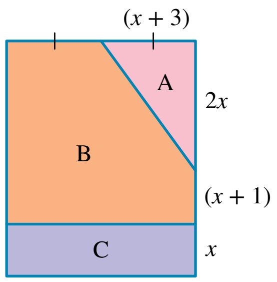
19. Factorise each of the following expressions.

a. $-3a^2 + 9a$ b. $-12p^2 - 2p$ c. $-26y^2 - 13y$

d. $4m - 18m^2$ e. $-6t + 36t^2$ f. $-8p - 24p^2$

Understanding

20. A large billboard display is in the shape of a rectangle as shown. The billboard has 3 regions (A, B, C) with dimensions in terms of x, as shown.



a. Calculate the total area of the billboard. Give your answer in factorised form.
b. Determine an expression for the area of each region of the billboard. Write the expression in its simplest form.

21. Consider the expression $3ab^2 + 18ab + 27a$.

a. Factorise this expression by taking out the highest common factor from each term.
b. Use the rule for perfect squares to fully factorise this expression.

22. Consider the expression $x^2(x + 4) + 8x(x + 4) + 16(x + 4)$.

a. Factorise this expression by taking out the highest common factor from each term.
b. Use the rule for perfect squares to fully factorise this expression.

Reasoning

23. A question on Marcia's recent Algebra test was, 'Using factorisation, simplify the following expression: $a^2(a - b) - b^2(a - b)$'. Marcia's answer was $(a + b)(a - b)^2$. If Marcia used the difference of two squares rule to get her solution, explain the steps she took to get that answer.

24. Prove that, as long as $a \neq 0$, then $(x + a)^2 \neq x^2 + a^2$.

25. Using the fact that any positive number, n, can be written as $\left(\sqrt{n}\right)^2$, factorise the following expressions using the difference of two squares rule.

a. $x^2 - 13$ b. $4x^2 - 17$ c. $(x + 3)^2 - 10$

Problem solving

26. Evaluate $\dfrac{x^2+2xy+y^2}{4x^2-4y^2} \div \dfrac{x^2+xy}{8xy-8x^2}$.

27. It has been said that, for any 2 numbers, the product of their LCM and HCF is the same as the product of the 2 numbers themselves. Show whether this is true.

28. **a.** Factorise $36x^2-100y^2$ by first taking out the common factor and then using the difference of two squares rule.
b. Factorise $36x^2-100y^2$ by first using the difference of two squares rule and then taking out the common factor.
c. Comment on whether you got the same answer for parts **a** and **b**.

LESSON 2.9 The highest common binomial factor (extending)

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- factorise expressions by taking out a common binomial factor
- factorise expressions by grouping terms.

2.9.1 The common binomial factor

- When factorising an expression, we look for the highest common factor(s) first.
- It is possible for the HCF to be a binomial expression.
- Consider the expression $7(a-b)+8x(a-b)$. The binomial expression $(a-b)$ is a common factor to both terms. Factorising this expression looks like this:

$$\begin{aligned}7(a-b)+8x(a-b) &= 7\times(a-b)+8x\times(a-b)\\ &= (a-b)(7+8x)\end{aligned}$$

WORKED EXAMPLE 24 Factorising by finding a binomial common factor

Factorise each of the following expressions.
a. $\mathbf{5(x+y)+6b(x+y)}$
b. $\mathbf{2b(a-3b)-(a-3b)}$
Note: **In both of these expressions the HCF is a binomial factor.**

THINK	WRITE
a. 1. The HCF is $(x+y)$.	**a.** $\dfrac{5(x+y)}{x+y}=5,\ \dfrac{6b(x+y)}{x+y}=6b$
2. Divide each term by $(x+y)$ to determine the binomial.	Therefore, $5(x+y)+6b(x+y)$ $=(x+y)(5+6b)$

b. 1. The HCF is $(a - 3b)$.

2. Divide each term by $(a - 3b)$ to determine the binomial.

b. $\dfrac{2b(a-3b)}{a-3b} = 2b, \dfrac{-1(a-3b)}{a-3b} = -1$

Therefore,

$$\begin{aligned} &2b(a-3b) - (a-3b) \\ &= 2b(a-3b) - 1(a-3b) \\ &= (a-3b)(2b-1) \end{aligned}$$

2.9.2 Factorising by grouping in pairs

- If an algebraic expression has four terms and no common factors in any of its terms, it may be possible to group the terms in pairs and find a common factor in each pair.
- Consider the expression $10x + 15 - 6ax - 9a$.
- We can attempt to factorise by grouping the first two terms and the last two terms:

$$\begin{aligned} 10x + 15 - 6ax - 9a &= 5 \times 2x + 5 \times 3 - 3a \times 2x - 3a \times 3 \\ &= 5(2x+3) - 3a(2x+3) \end{aligned}$$

- Once a common factor has been taken out from each pair of terms, a common binomial factor will appear. This common binomial factor can also be factorised out.

$$5(2x+3) - 3a(2x+3) = (2x+3)(5-3a)$$

- Thus the expression $10x + 15 - 6ax - 9a$ can be factorised to become $(2x+3)(5-3a)$.
- It is worth noting that it doesn't matter which terms are paired up first — the final result will still be the same.

$$\begin{aligned} 10x + 15 - 6ax - 9a &= 10x - 6ax + 15 - 9a \\ &= 2x \times 5 + 2x \times (-3a) + 3 \times 5 + 3 \times (-3a) \\ &= 2x(5-3a) + 3(5-3a) \\ &= (5-3a)(2x+3) \\ &= (2x+3)(5-3a) \end{aligned}$$

WORKED EXAMPLE 25 Factorising by grouping in pairs

Factorise each of the following expressions by grouping the terms in pairs.

a. $5a + 10b + ac + 2bc$ **b.** $x - 3y + ax - 3ay$ **c.** $5p + 6q + 15pq + 2$

THINK	WRITE
a. 1. Write the expression.	**a.** $5a + 10b + ac + 2bc$ $5a + 10b = 5(a+2b)$ $ac + 2bc = c(a+2b)$
2. Take out the common factor $a + 2b$.	$= 5(a+2b) + c(a+2b)$ $= (a+2b)(5+c)$
b. 1. Write the expression.	**b.** $x - 3y + ax - 3ay$ $x - 3y = 1(x-3y)$ $ax - 3ay = a(x-3y)$
2. Take out the common factor $x - 3y$.	$= 1(x-3y) + a(x-3y)$ $= (x-3y)(1+a)$

c. 1. Write the expression.

2. There are no simple common factors. Write the terms in a different order.

3. Take out the common factor $1+3q$. *Note:* $1+3q=3q+1$.

c. $5p+6q+15pq+2$

$= 5p+15pq+6q+2$

$5p+15pq=5p(1+3q)$

$6q+2=2(3q+1)$

$= 5p(1+3q)+2(3q+1)$

$= 5p(1+3q)+2(1+3q)$

$= (1+3q)(5p+2)$

DISCUSSION

How do you factorise expressions with four terms?

Exercise 2.9 The highest common binomial factor (extending)

learnon

2.9 Quick quiz on

2.9 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 6, 9, 12	2, 4, 7, 10, 13	5, 8, 11, 14

Fluency

1. WE24 Factorise each of the following expressions.

a. $2(a+b)+3c(a+b)$
b. $4(m+n)+p(m+n)$
c. $7x(2m+1)-y(2m+1)$
d. $4a(3b+2)-b(3b+2)$
e. $z(x+2y)-3(x+2y)$

2. Factorise each of the following expressions.

a. $12\,p(6-q)-5(6-q)$
b. $3p^2(x-y)+2q(x-y)$
c. $4a^2(b-3)+3b\,(b-3)$
d. $p^2(q+2p)-5(q+2p)$
e. $6(5m+1)+n^2(5m+1)$

3. WE25 Factorise each of the following expressions by grouping the terms in pairs.

a. $xy+2x+2y+4$
b. $ab+3a+3b+9$
c. $xy-4y+3x-12$
d. $2xy+x+6y+3$
e. $3ab+a+12b+4$
f. $ab-2a+5b-10$

4. Factorise each of the following expressions by grouping the terms in pairs.

a. $m-2n+am-2an$
b. $5+3p+15a+9ap$
c. $15mn-5n-6m+2$
d. $10pq-q-20p+2$
e. $6x-2-3xy+y$
f. $16p-4-12pq+3q$

5. Factorise each of the following expressions by grouping the terms in pairs.

a. $10xy+5x-4y-2$
b. $6ab+9b-4a-6$
c. $5ab-10ac-3b+6c$
d. $4x+12y-xz-3yz$
e. $5pr+10qr-3p-6q$
f. $ac-5bc-2a+10b$

Understanding

6. Simplify the following expressions using factorising.

a. $\dfrac{ax+2ay+3az}{bx+2by+3bz}$

b. $\dfrac{10(3x-4)+2y(3x-4)}{7a(10+2y)-5(10+2y)}$

7. Use factorising by grouping in pairs to simplify the following expressions.

a. $\dfrac{3x+6+xy+2y}{6+2y+18x+6xy}$

b. $\dfrac{5xy+10x+3ay+6a}{15bx-10x+9ab-6a}$

8. Use factorising by grouping in pairs to simplify the following expressions.

a. $\dfrac{6x^2+15xy-4x-10y}{6xy+4x+15y^2+10y}$

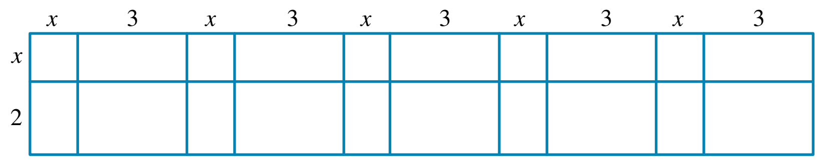
b. $\dfrac{mp+4mq-4np-16nq}{mp+4mq+4np+16nq}$

Reasoning

9. Using the method of rectangles to expand, show how $a(m+n)+3(m+n)$ equals $(a+3)(m+n)$.

10. Fully factorise $6x+4x^2+6x+9$ by grouping in pairs. Discuss what you noticed about this factorisation.

11. a. Write out the product $5(x+2)(x+3)$ and show that it also corresponds to the diagram shown.



b. Explain why $5(x+2)(x+3)$ is equivalent to $(5x+10)(x+3)$. Use bracket expansion and a labelled diagram to support your answer.

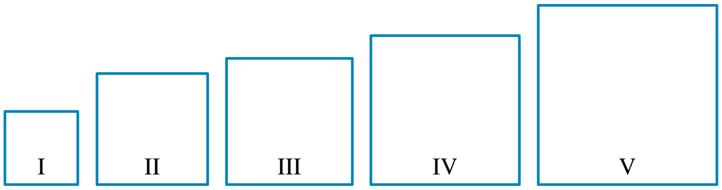
c. Explain why $5(x+2)(x+3)$ is equivalent to $(x+2)(5x+15)$. Use bracket expansion and a labelled diagram to support your answer.

Problem solving

12. A series of five squares of increasing size and a list of five area formulas are shown.

a. Use factorisation to calculate the side length that correlates to each area formula.

b. Using the area given and the side lengths found, match the squares below with the appropriate algebraic expression of their area.



$$A_a = x^2+6x+9$$
$$A_b = x^2+10x+25$$
$$A_c = x^2+16x+64$$
$$A_d = x^2-6x+9$$
$$A_e = x^2+12x+36$$

c. If $x=5$ cm, use the formula given to calculate the area of each square.

13. Fully factorise both sets and brackets in the expression $(-12xy + 27x + 8y - 18) - (-8xy + 18x + 12y - 27)$ using grouping in pairs, then factorise the result.

14. The area formulas shown relate to either squares or rectangles.

i. $9s^2 + 48s + 64$
ii. $25s^2 - 4$
iii. $s^2 + 4s + 3$
iv. $4s^2 - 28s - 32$

a. Without completing any algebraic operations, examine these formulas and work out which ones belong to squares and which ones belong to rectangles. Explain your answer.
b. Factorise each formula and classify it as a square or rectangle. Check your classifications against your answer to part a.

LESSON 2.10 Solving worded problems

LEARNING INTENTIONS

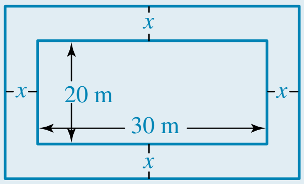
At the end of this lesson you should be able to:
- write an algebraic expression for worded problems
- apply the skills covered in this topic to solve worded problems.

2.10.1 Converting worded problems into algebraic expressions

- It is important to be able to convert worded problems (or 'real-world problems') into algebraic expressions. The key words that were identified in lesson 2.3 can help with this process.
- Drawing diagrams is a useful strategy for understanding and solving worded problems.
- If a worded problem uses units in its questions, make sure that any answers also use those units.

WORKED EXAMPLE 26 Converting worded problems

A rectangular swimming pool measures 30 m by 20 m. A path around the edge of the pool is x m wide on each side.
a. Determine the area of the pool.
b. Write an expression for the area of the pool plus the area of the path.
c. Write an expression for the area of the path.
d. If the path is 1.5 m wide, calculate the area of the path.

THINK	WRITE/DRAW
a. 1. Construct a drawing of the pool.	a. 
2. Calculate the area of the pool.	$\text{Area} = \text{length} \times \text{width}$ $= 20 \times 30$ $= 600\ \text{m}^2$

b. 1. Write an expression for the total length from one edge of the path to the other. Write another expression for the total width from one edge of the path to the other.

b. Length $= 30 + x + x$
$= 30 + 2x$
Width $= 20 + x + x$
$= 20 + 2x$

2. Area = length × width

Area $=$ length $\times$ width
$= (30 + 2x)(20 + 2x)$
$= 600 + 60x + 40x + 4x^2$
$= (600 + 100x + 4x^2)\text{ m}^2$

c. Determine an expression for the area of the path by subtracting the area of the pool from the total area of the pool and the path combined.

c. Area of path $=$ total area $-$ area of pool
$= 600 + 100x + 4x^2 - 600$
$= (100x + 4x^2)\text{ m}^2$

d. Substitute 1.5 for x in the expression you have found for the area of the path.

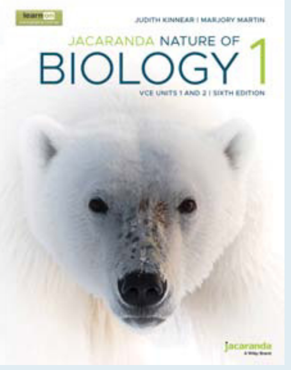
d. When $x = 1.5$,
Area of path $= 100(1.5) + 4(1.5)^2$
$= 159\text{ m}^2$

- The algebraic expression found in part **c** of Worked example 24 allows us to calculate the area of the path for any given width.

WORKED EXAMPLE 27 Writing expressions for worded problems

Suppose that the page of a typical textbook is 24 cm high by 16 cm wide. Each page has margins of x cm at the top and bottom, and margins of y cm on the left and right.

a. Write an expression for the height of the section of the page that lies inside the margins.
b. Write an expression for the width of the section of the page that lies inside the margins.
c. Write an expression for the area of the section of the page that lies inside the margins.
d. Show that, if the margins on the left and right are doubled, the area within the margins is reduced by $(48y - 4xy)$ cm.
e. If the margins at the top and the bottom of the page are 1.5 cm and the margins on the left and right of the page are 1 cm, calculate the size of the area that lies within the margins.

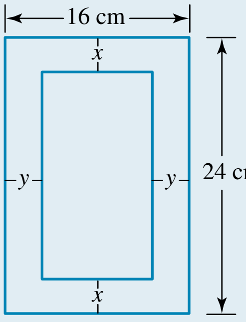


THINK

a. 1. Construct a drawing that shows the key dimensions of the page and its margins.

WRITE/DRAW

a.



2. The total height of the page (24 cm) is effectively reduced by x cm at the top and x cm at the bottom.

$$\text{Height} = 24 - x - x$$
$$= (24 - 2x)\text{ cm}$$

b. The total width of the page (16 cm) is reduced by y cm on the left and y cm on the right.

b. $$\text{Width} = 16 - y - y$$
$$= (16 - 2y)\text{ cm}$$

c. The area of the page that lies within the margins is the product of the width and height.

c. $$\text{Area}_1 = (24 - 2x)(16 - 2y)$$
$$= (384 - 48y - 32x + 4xy)\text{ cm}^2$$

d. 1. If the left and right margins are doubled they both become $2y$ cm. Determine the new expression for the new width of the page.

d. $$\text{Width} = 16 - 2y - 2y$$
$$= (16 - 4y)\text{ cm}$$

2. Determine the new expression for the reduced area.

$$\text{Area}_2 = (24 - 2x)(16 - 4y)$$
$$= 24 \times 16 + 24 \times -4y - 2x \times 16 - 2x \times -2y$$
$$= (384 - 96y - 32x + 8xy)\text{ cm}^2$$

3. Determine the difference in area by subtracting the reduced area obtained in part **d** from the original area obtained in part **c**.

$$\begin{aligned}\text{Difference in area} &= \text{Area}_1 - \text{Area}_2\\ &= 384 - 48y - 32x + 4xy\\ &\quad -(384 - 96y - 32x + 8xy)\\ &= 384 - 48y - 32x + 4xy\\ &\quad -384 + 96y + 32x - 8xy\\ &= 48y - 4xy\end{aligned}$$

So the amount by which the area is reduced is $(48y - 4xy)\text{ cm}^2$.

e. Using the area found in **c**, substitute 1.5 for x and 1 for y. Solve the expression.

e. $$\text{Area} = (384 - 48y - 32x + 4xy)\text{ cm}^2$$
$$= 384 - 48(1) - 32(1.5) + 4(1.5)(1)$$
$$294\text{ cm}^2$$

DISCUSSION

How can you use algebraic skills in real-life situations?

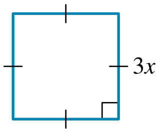
2.10 Quick quiz on | 2.10 Exercise

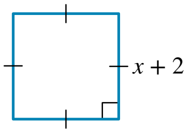
Individual pathways

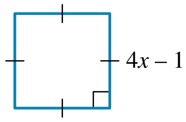
PRACTISE	CONSOLIDATE	MASTER
1, 2, 6, 9, 12	3, 5, 7, 10, 13	4, 8, 11, 14, 15

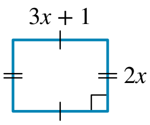
Fluency

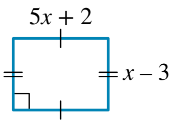
1. Answer the following for each shape shown.
   i. Determine an expression for the perimeter.
   ii. Determine the perimeter when $x = 5$.
   iii. Determine an expression for the area. If necessary, simplify the expression by expanding.
   iv. Determine the area when $x = 5$.

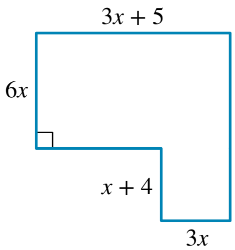
a.


b.


c.


d.


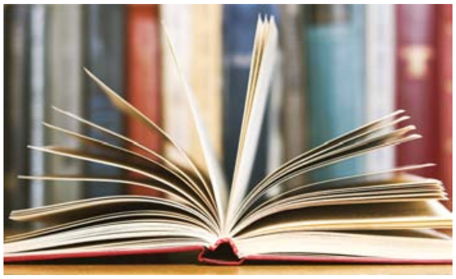
e.


f.


Understanding

2. WE26 A rectangular swimming pool measures 50 m by 25 m. A path around the edge of the pool is x m wide on each side.
   a. Determine the area of the pool.
   b. Write an expression for the area of the pool plus the area of the path.
   c. Write an expression for the area of the path.
   d. If the path is 2.3 m wide, calculate the area of the path.
   e. If the area of the path is 200 m^2, write an equation that can be solved to calculate the width of the path.

3. WE27 The pages of a book are 20 cm high by 15 cm wide. The pages have margins of x cm at the top and bottom and margins of y cm on the left and right.
   a. Write an expression for the height of the section of the pages that lie inside the margins.
   b. Write an expression for the width of the section of the pages that lie inside the margins.
   c. Write an expression for the area of the section of the pages that lie inside the margins.
   d. Show that, if the margins on the left and right are doubled, the area between the margins is reduced by $(40y - 4xy)$ cm.



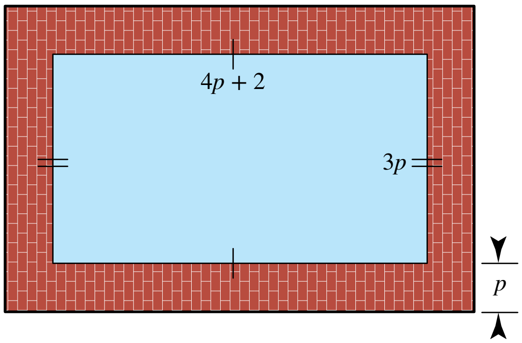
4. A rectangular book cover is 8 cm long and 5 cm wide.
 a. Calculate the area of the book cover.
 b. i. If the length of the book cover is increased by v cm, write an expression for its new length.
  ii. If the width of the book cover is increased by v cm, write an expression for its new width.
  iii. Write an expression for the new area of the book cover. Expand this expression.
  iv. Calculate the increased area of the book cover if $v = 2$ cm.
 c. i. If the length of the book cover is decreased by d cm, write an expression for its new length.
  ii. If the width of the book cover is decreased by d cm, write an expression for its new width.
  iii. Write an expression for the new area of the book cover. Expand this expression.
  iv. Calculate the new area of the book cover if $d = 2$ cm.
 d. i. If the length of the book cover is made x times longer, write an expression for its new length.
  ii. If the width of the book cover is increased by x cm, write an expression for its new width.
  iii. Write an expression for the new area of the book cover. Expand this expression.
  iv. Calculate the new area of the book cover if $x = 5$ cm.

5. A square has sides of length $5x$ m.
 a. Write an expression for its perimeter.
 b. Write an expression for its area.
 c. i. If the square's length is decreased by 2 m, write an expression for its new length.
  ii. If the square's width is decreased by 3 m, write an expression for its new width.
  iii. Write an expression for the square's new area. Expand this expression.
  iv. Calculate the square's area when $x = 6$ m.

6. A rectangular sign has a length of $2x$ cm and a width of x cm.
 a. Write an expression for the sign's perimeter.
 b. Write an expression for the sign's area.
 c. i. If the sign's length is increased by y cm, write an expression for its new length.
  ii. If the sign's width is decreased by y cm, write an expression for its new width.
  iii. Write an expression for the sign's new area and expand.
  iv. Calculate the sign's area when $x = 4$ cm and $y = 3$ cm using your expression.

7. A square has a side length of x cm.
 a. Write an expression for its perimeter.
 b. Write an expression for its area.
 c. i. If the square's side length is increased by y cm, write an expression for its new side length.
  ii. Write an expression for the square's new perimeter. Expand this expression.
  iii. Calculate the square's perimeter when $x = 5$ cm and $y = 9$ cm.
  iv. Write an expression for the square's new area and expand.
  v. Calculate the square's area when $x = 3.2$ cm and $y = 4.6$ cm.

8. A swimming pool with length $(4p + 2)$ m and width $3p$ m is surrounded by a path of width p m.
Write the following in expanded form.



 a. An expression for the perimeter of the pool.
 b. An expression for the area of the pool.
 c. An expression for the length of the pool and path.
 d. An expression for the width of the pool and path.
 e. An expression for the perimeter of the pool and path.
 f. An expression for the area of the pool and path.
 g. An expression for the area of the path.
 h. The area of the path when $p = 2$ m.

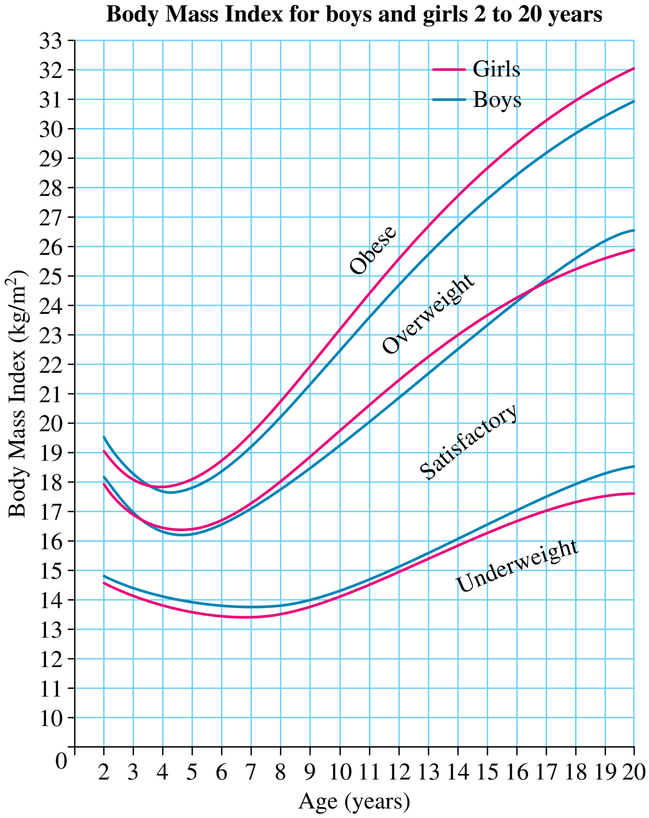
Reasoning

9. The Body Mass Index (B) is used as an indicator of whether or not a person is in a healthy weight range for their height. It can be calculated using the formula $B = \frac{m}{h^2}$, where m is the person's mass in kilograms and h is the person's height in metres. Use this formula to answer the following questions, correct to 1 decimal place.

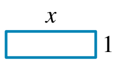
a. Calculate Samir's Body Mass Index if they weigh 85 kg and are 1.75 m tall.
b. A person is considered to be in a healthy weight range if their Body Mass Index is between 21 to 25 inclusive. Comment on Samir's weight for a person of their height.
c. Calculate the Body Mass Index, correct to 1 decimal place, for each of the following people:

i. Sammara, who is 1.65 m tall and has a mass of 52 kg
ii. Nimco, who is 1.78 m tall and has a mass of 79 kg
iii. Manuel, who is 1.72 m tall and has a mass of 65 kg.

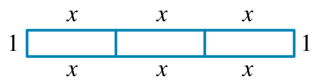
d. The healthy Body Mass Index range for children changes up until adulthood. These graphs show the Body Mass Index for boys and girls aged from 2 to 20 years of age.
Study these graphs carefully and decide on possible age ranges (between 2 and 20) for Sammara, Nimco and Manuel if their BMI was in the satisfactory range. Give the age ranges in half-year intervals.



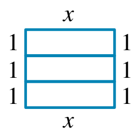
10. The figure shown is a rectangle. For the purpose of this activity we are going to call this rectangle an 'algebra rectangle'.
The length of each algebra rectangle is x cm and the width is 1 cm.
These algebra rectangles are put together to form larger rectangles in one of two ways.



Long algebra rectangle (3-long)



Tall algebra rectangle (3-tall)



To calculate the perimeter of each of these algebra rectangles:

- perimeter of 3-long algebra rectangle $P = (x + x + x + x + x + x) + (1 + 1) = 6x + 2$
- perimeter of 3-tall algebra rectangle $P = (x + x) + (1 + 1 + 1 + 1 + 1 + 1) = 2x + 6$.

a. Using the expressions for the perimeters of a 3-long algebra rectangle and a 3-tall algebra rectangle as a basis, find the perimeter of each of the following algebra rectangles and add them to the table shown.

Type of algebra rectangle	Perimeter	Type of algebra rectangle	Perimeter
1-long		1-tall	
2-long		2-tall	
3-long	$6x + 2$	3-tall	$2x + 6$
4-long		4-tall	
5-long		5-tall	

b. Can you see the pattern? Based on this pattern, calculate the perimeter of a 20-long algebra rectangle and a 20-tall algebra rectangle.

c. Determine the values of x that will result in a tall algebra rectangle that has a larger perimeter than a long algebra rectangle.

11. Can you evaluate 997^2 without a calculator in less than 90 seconds? It is possible to work out the answer using long multiplication, but it would take a fair amount of time and effort. Mathematicians are always looking for quick and simple ways of solving problems.
What if we consider the expanding formula that produces the difference of two squares?
$(a + b)(a - b) = a^2 - b^2$
Adding b^2 to both sides gives $(a + b)(a - b) + b^2 = a^2 - b^2 + b^2$.
Simplifying and swapping sides gives $a^2 = (a + b)(a - b) + b^2$.
We can use this formula, combined with the fact that multiplying by 1000 is an easy operation, to evaluate 997^2.

a. If $a = 997^2$, determine what the value of b should be so that $(a + b)$ equals 1000.

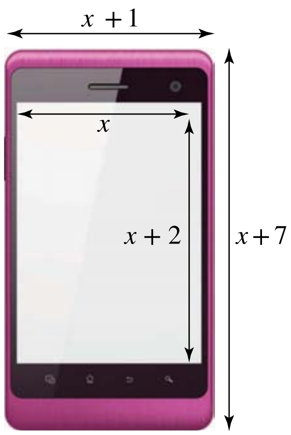
b. Substitute these a and b values into the formula to evaluate 997^2.

c. Try the above method to evaluate the following.

i. 995^2 ii. 990^2

Problem solving

12. In the picture of the phone shown, the dimensions are in cm.



a. Use the information in the picture to write an expression for the area of:

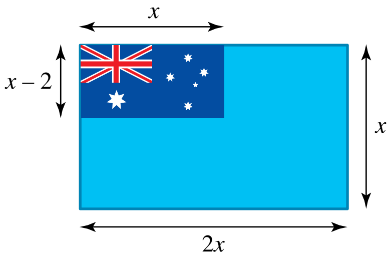
i. the viewing screen ii. the entire front face of the phone.

b. Your friend has a phone that is 4 cm longer than the one shown, but also 1 cm narrower. Write expressions in expanded form for:

i. the length and width of your friend's phone
ii. the area of your friend's phone.

c. If the phone in the picture is 5 cm wide, use your answer to part **b** to calculate the area of the front face of your friend's phone.

13. A proposed new flag for Australian schools will have the Australian flag in the top left-hand corner. The dimensions of this new flag are given in metres.



a. Write an expression in factorised form for the area of:
   i. the Australian flag section of the proposed flag
   ii. the whole area of the proposed flag.

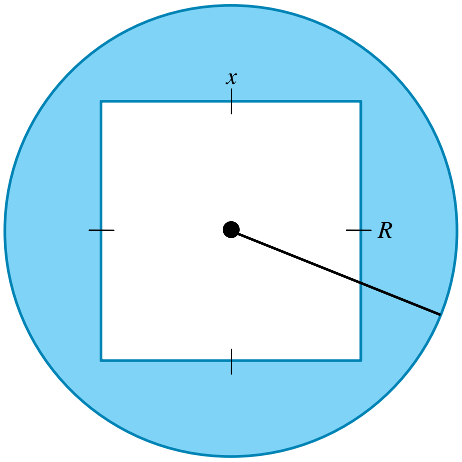
b. Use the answers from part **a** to write the area of the Australian flag as a fraction of the school flag. Simplify this fraction.

c. Use the fraction from part **b** to express the area of the Australian flag as a percentage of the proposed school flag.

d. Use the formula for the percentage of the area taken up by the Australian flag to find the percentages for the following suggested widths for the proposed school flag.
   i. 4 m
   ii. 4.5 m
   iii. 4.8 m

e. If the percentage of the school flag taken up by the Australian flag measures the importance a school places on Australia, determine what can be said about the three suggested flag widths.

14. A new game has been created by students for the school fair. To win the game you need to hit the shaded region of the target shown with 5 darts.



a. Write an expression to calculate the area of the shaded region of the game board.

b. Calculate the area of the shaded region of the game board if $R = 7.5$ cm and $x = 4$ cm.

c. Show that $R = \sqrt{\dfrac{A + x^2}{\pi}}$ by transposing the formula found in part **a**.

d. If $A = 80\text{ cm}^2$ and $x = 3$ cm, calculate the value of R.

e. The students found that the best size for the game board is when $R = 10$ and $x = 5$. For a board of this size, determine the percentage of the total board that is shaded.

15. Cubic expressions are the expanded form of expressions with three linear factors. The expansion process for three linear factors is called trinomial expansion.

a. Explain how the area model can be altered to show that binomial expansion can also be altered to become a model for trinomial expansion.

b. Investigate the powers in cubic expressions by expanding the following expressions.
   i. $(3x + 2)(-x + 1)(x + 1)$
   ii. $(x + 5)(2x - 2)(4x - 8)$
   iii. $(3 - x)(x + 8)(5 - x)$

c. Describe the patterns in the powers in a cubic expression.

LESSON 2.11 Review

2.11.1 Topic summary

Expressions

- An expression is a group of terms in which each term is separated by a + or a – sign.
- A variable is a letter that can take the place of a number. Variables are also referred to as pronumerals.
- Consider the expression $3x + 5y^2 - 8 - 9x^2z$.
  - This expression has 4 terms.
  - The constant term is –8.
  - The smallest coefficient is –9.
  - The coefficient of x is 3.
  - The variables used are x, y and z.

Substitution

- Evaluating an expression involves replacing variables with specific values.
- This process is called substitution.

e.g. Evaluate $3x^2 - 4y$ when $x = 2$ and $y = 5$:

$$\begin{aligned} 3x^2 - 4y &= 3(2^2) - 4(5) \\ &= 12 - 20 \\ &= -8 \end{aligned}$$

Formula

- A formula is a mathematical rule that represents a real-world relationship.
- The area of a triangle is given by the rule:

$$\text{Area} = \frac{1}{2} \times b \times h$$

e.g. The area of a triangle when $b = 10$ cm and $h = 15$ cm:

$$\text{Area} = \frac{1}{2} \times 10 \times 15 = 75 \text{ cm}^2$$

ALGEBRA

Simplifying algebraic expressions

- An algebraic expression can be simplified by adding or subtracting like terms.
- Two terms are considered like terms if they have exactly the same variable component.
  - $3x$ and $-10x$ are **like** terms.
  - $5x^2$ and $11x$ are **not like** terms.
  - $4abc$ and $-3cab$ are **like** terms.

e.g. The expression $3xy + 10xy - 5xy$ has 3 like terms and can be simplified to $8xy$.

The expression $4a \times 2b \times a$ can be simplified to $8a^2b$.

The expression $12xy \div (4xz)$ can be simplified to $\frac{3y}{z}$.

Setting up worded problems

- In order to turn a worded problem into an algebraic expression it is important to look out for the following words.
  - **Words that mean addition:** sum, altogether, add, more than, and, in total
  - **Words that mean subtraction:** difference, less than, take away, take off, fewer than,
  - **Words that mean multiplication:** product, groups of, times, of, for each, double, triple
  - **Words that mean division:** quotient, split into, halve, thirds.

Expanding brackets

- We can expand a single set of brackets using the **Distributive Law:**

$$a(b + c) = ab + ac$$

- The product of binomial factors can be expanded using **FOIL** (optional):

$$(a + b)(c + d) = ac + ad + bc + bd$$

e.g. $(x + 3)(y - 4) = xy - 4x + 3y - 12$

Factorising

- Factorising is the opposite process to expanding.
- An expression is factorised by determining the highest common factor of each term.

e.g. $$\begin{aligned} &9xy + 15xz - 21x^2 \\ &= 3x(3y + 5z - 7x) \end{aligned}$$

The highest common factor of each term is $3x$.

Solving application questions

- Drawing diagrams will help set up expressions to solve worded problems.
- Make sure you include units in your answers.

Special cases

- We can also expand if we recognise the following special cases.
  - Difference of two squares:

$$(a + b)(a - b) = a^2 - b^2$$

  - Perfect squares:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

e.g. $(x - 6)(x + 6) = x^2 - 36$

$(2x + 5)^2 = 4x^2 + 20x + 25$

$(x - 2)^2 = x^2 - 4x + 4$

Grouping in pairs

- When presented with an expression that has 4 terms with no common factor, we can factorise by grouping the terms in pairs.

e.g. $$\begin{aligned} &6xy + 8y - 12xz - 16z \\ &= 2y(3x + 4) - 4z(3x + 4) \\ &= (3x + 4)(2y - 4z) \end{aligned}$$

2.11.2 Success criteria

Tick the column to indicate that you have completed the lesson and how well you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
2.2	I can identify the number of terms in an algebraic expression.			
	I can identify the variables, constant terms and coefficients of the terms in an expression.			
	I can evaluate an expression or formula by substituting values for each variable.			
2.3	I can assign variables to the unknown quantities in a worded problem.			
	I can convert the worded statements presented in a question into algebraic expressions.			
2.4	I can identify like terms.			
	I can simplify algebraic expressions by adding and subtracting like terms.			
	I can simplify algebraic expressions that involve multiplication and division of multiple algebraic terms.			
	I can add, subtract, multiply, and divide algebraic fractions.			
2.5	I can use the distributive law to expand and simplify an expression with a single set of brackets.			
	I can use the distributive law to expand and simplify expressions containing two or more sets of brackets.			
2.6	I can quickly expand and simplify binomial products using the rule for difference of two squares.			
	I can quickly expand and simplify binomial products using the rule for perfect squares.			
2.7	I can expand multiple sets of brackets and simplify the result.			
2.8	I can determine the highest common factor for a group of terms.			
	I can factorise expressions by dividing out the highest common factor from each term.			
2.9	I can factorise an expression by taking out a common binomial factor			
	I can factorise expressions by grouping terms in pairs.			
2.10	I can write an algebraic expression for a worded problem and use it to solve that problem.			

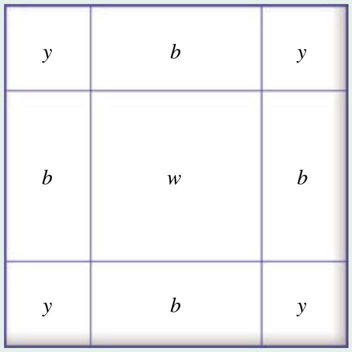
2.11.3 Project

Quilt squares

People all over the world are interested in quilt making, which involves stitching pieces of fabric together and inserting stuffing between layers of stitched-together fabric. When making a quilt, the fabric can be arranged and sewn in a variety of ways to create attractive geometric designs. Because of the potential for interesting designs in quilt-making, quilts are often used as decorative objects.

Medini is designing a quilt. She is sewing pieces of differently coloured fabric together to make a block, then copying the block and sewing the blocks together in a repeated pattern.

Making your own quilt



A scaled diagram of the basic block that Medini is using to make her quilt is shown. The letters indicate the colours of the fabric that make up the block: yellow, black and white. The yellow and white pieces are square, while the black pieces are rectangular. The finished blocks are sewn together in rows and columns.

Trace or copy the basic block shown onto a sheet of paper. Repeat this process until you have 9 blocks. Colour in each section and cut out all 9 blocks.

1. Place all 9 blocks together to form a 3×3 square. Draw and colour a scaled diagram of your result.
2. Describe the feature created by arranging the blocks in the manner described in question 1. Observe the shapes created by the different colours.

Medini sold her design to a company that now manufactures quilts made from 100 of these blocks. Each quilt covers an area of $1.44\,\text{m}^2$ Each row and column has the same number of blocks. Answer the following, ignoring any seam allowances.

3. Calculate the side length of each square block.
4. If the entire quilt has an area of $1.44\,\text{m}^2$, what is the area of each block?
5. Determine the dimensions of the yellow, black and white pieces of fabric of each block.
6. Calculate the area of the yellow, black and white pieces of fabric of each block.
7. Determine the total area of each of the three different colours required to construct this quilt.
8. Due to popular demand, the company that manufactures these quilts now makes them in different sizes. A customer can specify the approximate quilt area, the three colours they want and the number of blocks in the quilt, but the quilt must be either square or rectangular. Come up with a general formula that would let the company quickly work out the areas of the three coloured fabrics in each block. Give an example of this formula. Draw a diagram on a separate sheet of paper to illustrate your formula.

Fluency

1. **MC** Select the coefficient of the second term in the expression $-5xy^2 + 2y + 8y^2 + 6$.
 A. -5 B. 2 C. 8 D. 6

2. **MC** Jodie had \$55 in her purse. She spent \$$x$. Select the expression for the amount of money she has left.
 A. $\$(x - 55)$ B. $\$(x + 55)$ C. $\$55x$ D. $\$(55 - x)$

3. **MC** $\frac{3x}{5} - \frac{x}{4}$ can be simplified to:
 A. $\frac{7x}{20}$ B. $\frac{2x}{20}$ C. $\frac{2x}{1}$ D. $\frac{4x}{20}$

4. **MC** $\frac{3}{p} \div \frac{6}{p}$ is equal to:
 A. 2 B. $\frac{1}{2}$ C. $12p$ D. $12p^2$

5. **MC** The volume of a sphere, V, is given by the formula $V = \frac{4\pi r^3}{3}$, where r is the radius of the sphere. If $r = 3$, select the volume of the sphere from the following options.
 A. 25.12 B. 38.37 C. 64.05 D. 113.10

6. **MC** Choose the equivalent of $6 - 4x(x + 2) + 3x$.
 A. $6 - 4x^2 - 5x$ B. $6 + 4x^2 + 5x$ C. $6 - 4x^2 + 5x$ D. $6 + 4x^2 + 5x$

7. **MC** Choose the equivalent of $(3 - a)(3 + a)$.
 A. $9 + a^2$ B. $9 - a^2$ C. $3 + a^2$ D. $3 - a^2$

8. **MC** Choose the equivalent of $(2y + 5)^2$.
 A. $4y^2 + 20y + 25$ B. $4y^2 + 10y + 25$ C. $2y^2 + 20y + 25$ D. $2y^2 + 20y + 5$

9. **MC** Select what $6(a + 2b) - x(a + 2b)$ equals when it is factorised.
 A. $6 - x(a + 2b)$ B. $(6 - x)(a + 2b)$ C. $6(a + 2b - x)$ D. $(6 + x)(a - 2b)$

10. a. For the expression $-8xy^2 + 2x + 8y^2 - 5$:
 i. state the number of terms
 ii. state the coefficient of the first term
 iii. state the constant term
 iv. state the term with the lowest coefficient.
 b. Write expressions for the following, where x and y represent the following numbers.
 i. A number 8 more than y.
 ii. The difference between x and y.
 iii. The sum of x and y.
 iv. A number that is 7 times the product of x and y.
 v. The number given when 2 times x is subtracted from 5 times y.

11. a. Leo receives x dollars for each car he washes. If he washes y cars, calculate how much he earn.
b. A piece of rope is 24 m long.
i. If Rawiri cuts k m off, calculate how much is left.
ii. After Rawiri has cut k m off, he divides the rest of the rope into three pieces of equal length. Calculate the length of each piece.

12. Simplify the following expressions by collecting like terms.
a. $8p+9p$ b. $5y^2+2y-4y^2$ c. $9s^2t-12s^2t$
d. $11c^2d-2cd+5dc^2$ e. $n^2-p^2q-3p^2q+6$ f. $8ab+2a^2b^2-5a^2b^2+7ab$

13. Simplify the following expressions.
a. $6a\times 2b$ b. $2ab\times b$ c. $2xy\times 4yx$ d. $\frac{4x}{12}$ e. $18\div(4b)$

14. Expand the following expressions.
a. $5(x+3)$ b. $-(y+5)$ c. $-x(3-2x)$ d. $-4x(2m+1)$

15. Expand and simplify the following expressions by collecting like terms.
a. $3(x-2)+9$ b. $-2(5m-1)-3$
c. $4m(m-3)+3m-5$ d. $7p-2-(3p+4)$

16. Expand and simplify the following expressions.
a. $3(a+2b)+2(3a+b)$ b. $-4(2x+3y)+3(x-2y)$
c. $2m(n+6)-m(3n+1)$ d. $-2x(3-2x)-(4x-3)$

17. Expand and simplify the following expressions.
a. $(x+4)(x+5)$ b. $(m-2)(m+1)$ c. $(3m-2)(m-5)$ d. $(2a+b)(a-3b)$

18. Expand and simplify the following expressions.
a. $(x+4)(x-4)$ b. $(9-m)(9+m)$ c. $(x+y)(x-y)$ d. $(1-2a)(1+2a)$

19. Expand and simplify the following expressions.
a. $(x+5)^2$ b. $(m-3)^2$ c. $(4x+1)^2$ d. $(2-3y)^2$

20. Expand and simplify the following expressions.
a. $(x+2)(x+1)+(x+3)(x+2)$ b. $(m+7)(m-2)+(m+3)^2$
c. $(x+6)(x+2)-(x+3)(x-1)$ d. $(b-7)^2-(b-3)(b-4)$

21. Expand and simplify the following expressions.
a. $(x+2)^2+(x+3)^2$ b. $(x-2)^2-(x-3)^2$ c. $(x+4)^2-(x-4)^2$

22. If $y=5x^2+2x-1$, determine the value of y when:
a. $x=2$
b. $x=5$.

23. The volume (V) of each of the following paint tins is given by the formula $V=\pi r^2h$, where r is the radius and h is the height of the cylinder. Determine the value of V (to 2 decimal places) when:
a. $r=7$ cm and $h=2$ cm
b. $r=9.3$ cm and $h=19.8$ cm.

24. Factorise each of the following expressions by determining common factors.

a. $6x + 12$
b. $6x^2 + 12x^2y$
c. $8a^2 - 4b$
d. $16x^2 - 24xy$
e. $-2x - 4$
f. $b^2 - 3b + 4bc$

25. Simplify each of the following expressions using common factor techniques.

a. $5(x + y) - 4a(x + y)$
b. $7a(b + 5c) - 6c(b + 5c)$
c. $15x(d + 2e) + 25xy(d + 2e)$
d. $2x + 2y + ax + ay$
e. $6xy + 4x - 6y - 4$
f. $pq - r + p - rq$

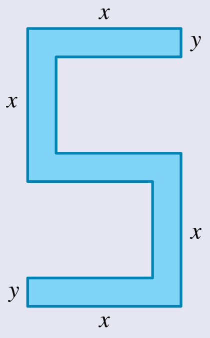
Problem solving

26. A rectangular rug has a length of $3x$ cm and a width of x cm.
   a. Write an expression for the rug's perimeter.
   b. Write an expression for the rug's area.
   c. The rug's side length is increased by y cm.
      i. Write an expression for its new side length.
      ii. Write an expression for the rug's new perimeter. Expand this expression.
      iii. Calculate the rug's perimeter when $x = 90$ cm and $y = 30$ cm.
      iv. Write an expression for the rug's new area. Expand this expression.
      v. Calculate the rug's area when $x = 90$ cm and $y = 30$ cm.

27. A rectangular garden bed has a length of 15 m and a width of 8 m. It is surrounded by a path with a width of p m.
   a. Write down the total area of the garden bed and path in factorised form.
   b. Expand the expression you wrote down in part a.
   c. Calculate the area of the path in terms of p.
   d. Write an equation that can be solved to find the width of the path if the area of the path is 200 m^2.

28. Calculate the circumference of a circle whose area is $\pi\left(\frac{x}{y}\right)$ cm^2. Give your answer in terms of π.

29. The large sign shown appears in a parking lot at the entrance to car park 5. It has a uniform width, with dimensions shown.



Write an algebraic expression for the area of the front of the sign.

30. At the end of a lesson involving boxes of Smarties, the Smarties left over are shared equally among the class. There are 6 boxes full of Smarties left over and 42 Smarties that are no longer in boxes.

a. Write an expression for the total number of Smarties left over.
(*Hint:* Let $n =$ the number of Smarties in a box.)

b. Write the expression from part **a** in factorised form.

c. If there are only three people in the class, write an expression for the number of Smarties each person receives.

d. If there are x people in the class, write an expression for the number of Smarties each person receives.

31. When entering numbers into an electronic device, or even writing numbers down, we can often make mistakes. A common type of mistake is called a transposition error. This is when two digits are written in the reverse order. For example, the number 2869 could be written as 8269, 2689 or 2896.

A common rule for checking if these mistakes have been made is to see if the difference between the correct number and the recorded number is a multiple of 9. If it is, a transposition error has occurred.

We can use algebraic expressions to check this rule. Let the digit in the thousands position be represented by a, the digit in the hundreds position by b, the digit in the tens position by c, and the digit in the ones position by d.

In this way the real number can be represented as $1000a + 100b + 10c + d$.

a. If the digits in the ones position and the tens position were written in the reverse order, the number would be $1000a + 100b + 10d + c$. The difference between the correct number and the incorrect one would then be $1000a + 100b + 10c + d - (1000a + 100b + 10d + c)$.

i. Simplify this expression.

ii. Is the expression a multiple of nine? Explain your answer.

b. If a transposition error had occurred with the tens and hundreds positions, the incorrect number would be $1000a + 100c + 10b + d$.
Perform the procedure shown in part **a** to work out whether the difference between the correct number and the incorrect one is a multiple of 9.

c. Consider a transposition error in the thousands and hundreds positions. Explain whether the difference between the two numbers is a multiple of 9.

d. Comment on this rule for transposition errors.

e. The price marked on the tag of a CD in a music store is x dollars and y cents. Its value is less than \$100. The person on the counter makes a mistake and enters a value of y dollars and x cents into the cash register.
Show whether the multiple of 9 rule for checking transposition errors applies in this case.

To test your understanding and knowledge of this topic, go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 2 Algebra

2.1 Pre-test

1. D
2. $2x^2 - y^2 - xyz = 11$
3. $m = 2$
4. D
5. A
6. a. $2m + 13n$ b. $-2xy - 3x + 5y$
7. $6q^2 + qr - r^2$
8. D
9. Yes
10. False
11. $3b^2 - 21b + 37$
12. a. $4p(2p+5)$ b. $5(5x^2 - 2y)$ c. $xy(6 + 8x - 3y)$
13. a. $(7a-1)(b+3)$ b. $(5x-2)\left(2y+1\right)$
14. $2x(2x-1) + 2x(3x+2) + 2(3x+2)(2x-1)$
15. $3m(3p - 1 + 2e)$

2.2 Using variables

1. a. 3 b. -2 c. $\frac{1}{3}$ d. $-\frac{1}{4}$
2. a. 5 b. -2 c. $\frac{1}{6}$ d. $-\frac{2}{7}$
3. a. 15 b. $-\frac{2}{5}$ c. $-\frac{7}{2}$ d. $-\frac{1}{7}$
4. a. i. 3 ii. 5 iii. 8 iv. $5x^2$
   b. i. 3 ii. -9 iii. -6 iv. $-9m^2$
   c. i. 4 ii. 5 iii. 5 iv. $-7x^2$
   d. i. 4 ii. 9 iii. 4 iv. $-9b^2$
5. a. i. 4 ii. 4 iii. -2 iv. $-3ac$
   b. i. 5 ii. 5 iii. 9 iv. $-3u$
   c. i. 5 ii. -1 iii. 8 iv. $-m$
   d. i. 5 ii. 7 iii. 14 iv. $-3cd^2$
6. a. i. 3 ii. 3 iii. -7 iv. $-10x^2$
   b. i. 5 ii. -3 iii. 7 iv. $-6x^2$
   c. i. 5 ii. -5 iii. 8 iv. $-5x$
   d. i. 6 ii. 6 iii. 6 iv. $-15xy$
7. a. $2, 12, -8$ b. $0, 2, 6$
   c. $-1, 9, -3$ d. $0, 1, -1$
8. a. $4, -11, 19$ b. $1, 4, 16$
   c. $0, 21, -3$ d. $8, 2, -4$
9. a. $\sqrt{3}, 2, 1$ b. $0, 3, -2$
   c. $-12, -12, -6$ d. $\frac{3}{5}, 1, -\frac{1}{5}$
10. a. B b. D c. A d. B
11. a. 4 b. -6 c. 6
    d. 7 e. -12 f. -3
12. a. 7 b. -40 c. 7
13. a. i. 32 ii. 250
    b. i. 4 ii. 3
14. a. 15 b. 10 c. 18
    d. 45 e. 30
15. a. 10.72 b. 21 c. 22.4
    d. 50 e. 171
16. a. 4 b. -30 c. 32.75
    d. 38.33 e. 34.56
17. a. Sample responses can be found in the worked solutions in the online resources.
    b. 12 cm
18. a. 10 b. 10
19. a. 15
    b. Sample responses can be found in the worked solutions in the online resources.
20. a. 5
    b. Sample responses can be found in the worked solutions in the online resources.
21. a. 19.44
    b. Sample responses can be found in the worked solutions in the online resources.
22. a. 63 cm^3 b. 4 cm
23. a. 461.81 b. 9.55
24. a. 1671.33 b. 2.64
25. a.

d	$2d + 2g$	g
$2g$	$d + g$	$2d$
$2d + g$	0	$d + 2g$

Magic number $= 3d + 3g$

b, c. Sample responses can be found in the worked solutions in the online resources.

26. a. mn
    b. $(m-2)(n-2)$
    c. $(2m + 2n)$ units
27. $y - (z - x)$

2.3 Algebra in worded problems

1. a. 11 b. $x + 5$ c. $y + 5$
2. a. 3.5 km b. $(x + 2)$ km
3. a. 4 b. $2.5 + t$ c. $2.5 - y$ hours
4. a. $d - x$ km b. $d + y$ km c. $2d + 4$ km
5. a. $10n$, where $n =$ number of shirts
   b. $N + 30$, where $N =$ the number of Nick's dollars
   c. i. $g - 4$ ii. $7g - 4$
6. a. i. 94 ii. $6x + y$
   b. i. 66 ii. $6p + q$
   c. i. 28 ii. $6x + y - 6p - q$
7. xy dollars
8. a. $\frac{x}{n}$ b. $\frac{x}{2n}$
9. a. $(30 - d)$ cm b. $\frac{30-d}{4}$ cm c. $\frac{3(30-d)}{4}$ cm
10. a. $\frac{x}{4}$ b. $\frac{x}{6}$ c. $\frac{x}{4} - \frac{x}{6} = \frac{x}{12}$

11. a. $\dfrac{24-c}{7}$ hours **b.** $\left(c+\dfrac{24-c}{7}\right)=\dfrac{6c+24}{7}$

12. a. 1 **b.** 7:30 pm and 8:00 pm
c. 5 **d.** 24
e. 8:30 pm and 9:00 pm

13. a. $\$\left(y-30\right)$
b. $\$\left(\dfrac{1}{3}(y-30)\right)$
c. \$930

14. a. $b=2h-1$ **b.** 19 blocks

15. 15 red vases and 10 clear vases

16. The father is 32 years old.

17. 4 dots = 4 triangles
5 dots = 10 triangles
6 dots = 20 triangles
7 dots = 35 triangles
$t=\dfrac{d!}{6(d-3)!}$ where $d!$ is the product of all positive integers less than or equal to d.

18. a. $x=\dfrac{T-3}{T-1}$ **b, c.** Answer is undefined.

2.4 Simplification of algebraic expressions

1. a. $9ab,\ -ab$ **b.** $4x$
c. $-3za, -az$ **d.** $2x^2,\ -x^2$

2. a. $-2x^2y$ **b.** $-x^2y^5,\ -3x^2y^5$
c. $p^3x^2w^5,\ -5x^2p^3w^5$ **d.** $4y^5z^4x^2,\ -2x^2z^4y^5$

3. a. $7x$ **b.** $11y$ **c.** $19m$ **d.** $11q$ **e.** $8r$ **f.** $3x$

4. a. $8a$ **b.** $8y$ **c.** $13x$ **d.** $16p$ **e.** $9q^2$ **f.** $3x^2$

5. a. $8x^2-3y$ **b.** $2m^2+2n$
c. $-2g^2+g-12$ **d.** $-5m^2+m+15$
e. $12a^2+b+4b^2$ **f.** $3m+7n^2$

6. a. $12xy+2y^2$ **b.** $2ab+5a^2b$
c. $16x^2y-3xy$ **d.** m^2n+11n
e. $-7x^2+2x^2y$ **f.** $-3a^2b-9a^2+5b+4$

7. B

8. D

9. A

10. D

11. a. $6mn$ **b.** $20xy$ **c.** $8pq$
d. $-10xy$ **e.** $-12xy$

12. a. $15mn$ **b.** $10a^2$ **c.** $6mnp$
d. $-6ab^2$ **e.** $10\,m^2n$

13. a. $-18a^2b$ **b.** $30x^3y^2$ **c.** $-12p^2q^3$
d. $-56c^3d$ **e.** $30a^4b^7$

14. a. $3x$ **b.** $3m$ **c.** $2y$ **d.** $4m$ **e.** $4m$

15. a. $2x$ **b.** $-7x$ **c.** $-4m$ **d.** $\dfrac{m}{2}$ **e.** $\dfrac{x}{3}$

16. a. $\dfrac{4m}{9}$ **b.** $\dfrac{1}{2a}$ **c.** $2yz$
d. $\dfrac{xy}{4}$ **e.** $\dfrac{-7yz}{11}$

17. a. $\dfrac{7x}{12}$ **b.** $\dfrac{y}{6}$ **c.** $-\dfrac{m}{8}$
d. $\dfrac{x}{4}$ **e.** $\dfrac{9m}{14}$

18. a. $\dfrac{1}{3}$ **b.** $\dfrac{n}{6}$ **c.** 4 **d.** $\dfrac{3}{2}$

19. a. $40x^2y^2$ **b.** $56ax^2y^2$
c. $\dfrac{x^2}{2y}$ **d.** $\dfrac{-5a}{4b}$

20. a. 2 **b.** $40a^3b^2$ **c.** $-8a^3b^3$ **d.** $16a^4$

21. a. i. $(0.01\,mp+nq)$ dollars **ii.** $(mp+100\,nq)$ cents
b. $20-(0.01mp+nq)$ dollars

22. Shirt = \$12 each
Shorts = \$30 each

23. \$45

24. The correct answer is $7a^2c^2$. Sample responses can be found in the worked solutions in the online resources.

25. $\dfrac{3b^9}{2c^6}$

26. a. $\dfrac{n}{(n+2)}$ **b.** $\dfrac{n}{(n+3)}$ **c.** $\dfrac{1}{100}$

27. a. i.

	$\mathbf{3a+4b}$	
$\mathbf{a+b}$	$2a+3b$	
$2a-b$	$\mathbf{-a+2b}$	$3a+b$

ii.

	$8p+3q$	
$\mathbf{3p+5q}$	$5p-2q$	
$\mathbf{p+2q}$	$2p+3q$	$\mathbf{3p-5q}$

b. i.

	$\mathbf{18m^3n^2}$	
$\mathbf{3m^2n}$	$\mathbf{6mn}$	
m^2	$3n$	$2m$

ii.

	$48d^2e^2$	
$8de$	$\mathbf{6de}$	
$\mathbf{4d}$	$\mathbf{2e}$	$3d$

28. a. $P=\dfrac{x}{3}+x+\dfrac{5x}{7}$
b. Sample responses can be found in the worked solutions in the online resources.
c. \$18

29. a. $P=2w+\dfrac{32w}{7}$
b. Sample responses can be found in the worked solutions in the online resources.
c. \$72.22

30. a. $P=2x+2(4x-7)$ **b.** $P=10x-14$
c. 16 units
d. If $x=1$ the rectangle will have a negative value for the perimeter, which is not possible.

31. a. $l=4(x+40)+8(x+20)$
b. $l=12x+320$
c. \$19.20

32. a. $P_1 = (10x + 12)$ cm
$P_2 = 16x$ cm
b. $x = 2$

2.5 Expanding brackets

1. a. 64 **b.** 1100
c. -20 **d.** 693

2. a. $3x + 6$

	x	2
3	$3x$	6

b. $4x + 12$

	x	3
4	$4x$	12

c. $4x + 4$ **d.** $7x - 7$

3. a. $-3p + 6$ **b.** $-x + 1$
c. $6b - 12$ **d.** $24m - 16$

4. a. $x^2 + 2x$

	x	2
x	x^2	$2x$

b. $a^2 + 5a$

	a	5
a	a^2	$5a$

c. $4x + x^2$ **d.** $7m - m^2$

5. a. $2xy + 4x$ **b.** $-3xy - 12y$
c. $-3b + ab$ **d.** $-30a + 18a^2$

6. a. $2p - 2$ **b.** $5x - 17$ **c.** $-7p - 17$
d. $-12p + 3$ **e.** $6x^2 - 20x$

7. a. $2m^2 + 7m$ **b.** $3px + 6x - 5$
c. $4y^2 - 4y + 7$ **d.** $-4p^2 + 13p$
e. $4x - 13y$

8. a. $2m^2 - 8m - 4$ **b.** $-3p^2 + 10pq - 1$
c. $-30a + 10ab$ **d.** $7cd - 12c^2 - 5c$
e. $-2p - 17$

9. a. $8x + y$ **b.** $10p + 8q$ **c.** $18a + 29b$
d. $19c + 22d$ **e.** $2m - 11n$

10. a. $6x - 11y$ **b.** $9x + 5y$ **c.** $-14p - 8q$
d. $-4a + 3b$ **e.** $4x - y$

11. a. $5p - 10q$ **b.** $-8c + 9d$ **c.** $13x - 19y$
d. $-7p + 11q$ **e.** $-5a + 3b$

12. a. $2ab + 2a - 3b$ **b.** $2xy + 4x - 2y$
c. $2cd + 3c$ **d.** $2pq - 2p$
e. $5cd - 11c$

13. a. $5ab - 21a - 3b$ **b.** $5m$
c. $6cd - 36c$ **d.** $6m^2 + 6m - 10$
e. $9cd - 8c$

14. a. $-15a^2 + 2b^2 - 9ab$
b. $-8c^2 + 3d^2 + 22cd$
c. $12m^2 - 20m - 4$
d. $-7x^2 + 41x - 6$
e. $-10y^2 - 6y - 12$

15. B

16. A

17. D

18. a. $a^2 + 5a + 6$ **b.** $x^2 + 7x + 12$
c. $y^2 + 5y + 6$ **d.** $m^2 + 9m + 20$
e. $b^2 + 3b + 2$

19. a. $p^2 + 5p + 4$ **b.** $a^2 + a - 6$
c. $x^2 + x - 20$ **d.** $m^2 - m - 12$
e. $y^2 + 2y - 15$

20. a. $y^2 - 4y - 12$ **b.** $x^2 - 2x - 3$
c. $x^2 - 7x + 12$ **d.** $p^2 - 5p + 6$
e. $x^2 - 4x + 3$

21. a. $2a^2 + 7a + 6$ **b.** $4c^2 - 31c + 42$
c. $2t^2 - 17t + 35$ **d.** $-6t^2 + 11t + 10$
e. $15x^2 - 31x + 14$ **f.** $25x^2 - 20x + 4$

22. a. $xz + x + yz + y$ **b.** $2xz + 8x + yz + 4y$
c. $3pr + 3p + qr + q$ **d.** $a^2 + 3ab + 2b^2$
e. $2c^2 - 5cd - 3d^2$ **f.** $2x^2 - xy - 3y^2$

23. a. $4p^2 + pq - 3q^2$ **b.** $ab + ac + 2b^2 + 2bc$
c. $3p - 9pr - 2q + 6qr$ **d.** $12x^2 - 7xy + y^2$
e. $2p^2 - pr - 2pq + qr$ **f.** $15k - 5 - 6jk + 2j$

24. C

25. C

26. a. $x^2 - 9$ **b.** $x^2 - 25$ **c.** $x^2 - 49$
d. $x^2 - 1$ **e.** $x^2 - 4$ **f.** $4x^2 - 1$

27. a. $x^2 + 2x + 1$ **b.** $x^2 + 4x + 4$
c. $x^2 + 16x + 64$ **d.** $x^2 - 6x + 9$
e. $x^2 - 10x + 25$ **f.** $x^2 - 18x + 81$

28. a. $6.3x^2 + 5.53xy - 3.1y^2$
b. $4.41x^2 - 10.24y^2$
c. $11.56x^2 + 34.68xy + 26.01y^2$

29. a. Surface area $= 38x^2 + 14x - 6$
b. Volume $= 12x^3 + 5x^2 - 3x$

30. a. i. $((x + 3y)(4x - y))$ m^2
ii. $(4x^2 + 11xy - 3y^2)$ cm^2
iii. Both brackets must be positive; therefore, $x > -3y$, $x > \dfrac{y}{4}$
b. i. $\dfrac{(2x - 1)(x + 5)}{2}$
ii. $\dfrac{2x^2 + 9x - 5}{2}$
iii. Sample responses can be found in the worked solutions in the online resources.

31. Sample responses can be found in the worked solutions in the online resources.

32. a. Negative sign ignored.
b. Negative sign ignored.
c. Distributive law not used.

33. a. i. The student did not multiply both terms.
ii. The student used addition instead of multiplication.
iii. The student did not change negative and positive signs.
b. $100 - 15x$

34. a. $-2(a - 5) = 2a - 10$ is incorrect because the student did not change the multiplied negative signs for $-2 \times (-5)$ to a positive sign. The correct answer is $-2a + 10$.

b. $2b(3b-1)=6b^2-1$ is incorrect because the student did not multiply -1 and $2b$ together. The correct answer is $6b^2-2b$.

c. $-2(c-4)=2c+8$ is incorrect because the student left out the negative sign when multiplying -2 and c. The correct answer is $-2c+8$.

35. a. Student C

b. Corrections to the students' answers are shown in bold.
Student A:
$(3x+4)(2x+5)$
$=3x\times 2x+3x\times 5+4\times 2x+4\times 5$
$=\mathbf{6x^2}+\mathbf{23x}+20$
(Also, $29x+20$ does not equal $49x$.)
Student B:
$(3x+4)(2x+5)$
$=3x\times 2x+\mathbf{3x\times 5}+4\times 2x+4\times 5$
$=6x^2+\mathbf{15}x+8x+20$
$=6x^2+\mathbf{23x}+20$

36. a. $w=(x-3)$ cm
b. $w=62$ cm
c. $l=108$ cm

2.6 Difference of two squares and perfect squares

1. a. x^2-4 b. y^2-9 c. m^2-25 d. a^2-49

2. a. x^2-36 b. p^2-144 c. a^2-100 d. m^2-121

3. a. $4x^2-9$ b. $9y^2-1$ c. $25d^2-4$ d. $49c^2-9$ e. $4-9p^2$

4. a. d^2-81x^2 b. $25-144a^2$ c. $9x^2-100y^2$ d. $4b^2-25c^2$ e. $100-4x^2$

5. a. x^2+4x+4 b. a^2+6a+9 c. $b^2+14b+49$ d. $c^2+18c+81$

6. a. $m^2+24m+144$ b. $n^2+20n+100$ c. $x^2-12x+36$ d. $y^2-10y+25$

7. a. $81-18c+c^2$ b. $64+16e+e^2$ c. $2x^2+4xy+2y^2$ d. $u^2-2uv+v^2$

8. a. $4a^2+12a+9$ b. $9x^2+6x+1$ c. $4m^2-20m+25$ d. $16x^2-24x+9$

9. a. $25a^2-10a+1$ b. $49p^2+56p+16$ c. $81x^2+36x+4$ d. $16c^2-48c+36$

10. a. $25+30p+9p^2$ b. $4-20x+25x^2$ c. $81x^2-72xy+16y^2$ d. $64x^2-48xy+9y^2$

11. a. x^2-9 b. $4x^2-9$ c. $49x^2-16$ d. $4x^2-49y^2$ e. x^4-y^4

12. a. $16x^2+40x+25$ b. $49x^2-42xy+9y^2$ c. $25x^4-20x^2y+4y^2$ d. $2x^2-4xy+2y^2$ e. $\frac{4}{x^2}+16+16x^2$

13. (x^2+6x+9) units^2

14. $(3x+1)$ m

15. Perimeter $=4x+36$

16. Sample responses can be found in the worked solutions in the online resources.

17. a. Sample responses can be found in the worked solutions in the online resources.
b. Lin's bedroom is larger by 1 m^2.

18. a. i. x^2-16 and $16-x^2$ ii. x^2-121 and $121-x^2$ iii. $4x^2-81$ and $81-4x^2$
b. The answers to the pairs of expansions are the same, except that the negative and positive signs are reversed.
c. This is possible because when a negative number is multiplied by a positive number, it becomes negative. When expanding a DOTS in which the expressions have different signs, the signs will be reversed.

19. a. $100k^2+100k+25$
b. $(10k+5)^2=100\times k\times k+100\times k+25$
$=100k(k+1)+25$
c. $25^2=(10\times 2+5)^2$
Let $k=2$.
$25^2=100k(k+1)+25$
$=100\times 2\times(2+1)+25$
$=625$
$85^2=(10\times 8+5)^2$
Let $k=8$.
$85^2=100k(k+1)+25$
$=100\times 8\times(8+1)+25$
$=7225$

20. a. $A_1=a^2$ units^2
$A_2=ab$ units^2
$A_3=ab$ units^2
$A_4=b^2$ units^2
b. $A=a^2+ab+ab+b^2$
$=a^2+2ab+b^2$
This is the equation for perfect squares.

21. a. i. x^2-6x+9 and $9-6x+x^2$
ii. $x^2-30x+225$ and $225-30x+x^2$
iii. $9x^2-42x+49$ and $49-42x+9x^2$
b. The answers to the pairs of expansions are the same.
c. This is possible because when a negative number is multiplied by itself, it becomes positive. When expanding a perfect square in which the two expressions are the same, the negative signs cancel out and result in the same answer.

22. a. 729 b. 1089 c. 1521 d. 2209

23. $(3x^2+16x+5)$ m^2

24. (x^2-25) m^2

25. a. 10609 b. 3844 c. 994 009 d. 1 024 144 e. 2809 f. 9604

2.7 Further expansions

1. a. $2x^2+13x+21$ b. $2x^2+13x+20$ c. $2x^2+14x+26$ d. $2x^2+10x+11$

2. a. $2p^2-3p-21$ b. $2a^2-5a+4$ c. $2p^2-p-24$ d. $2x^2+19x-36$

3. a. $2y^2+2y-7$ b. $2d^2+8d-2$ c. $2x^2+10$ d. $2y^2$

4. a. $2x^2 - 4x + 19$ b. $2y^2 - 4y - 7$
c. $2p^2 + 3p + 23$ d. $2m^2 + 3m + 31$

5. a. $x + 5$ b. $4x + 8$
c. $-2x - 6$ d. $3m + 2$

6. a. $-3b - 22$ b. $-15y - 2$
c. $8p - 10$ d. $16x + 2$

7. a. $4m + 17$ b. $-7a + 30$
c. $-6p - 7$ d. $3x - 21$

8. a. $16x - 16$
b. Sample responses can be found in the worked solutions in the online resources.

9. a. $15x^2 - 44x + 21$
b. Sample responses can be found in the worked solutions in the online resources.

10. a. $8x^2 - 10xy + 3y^2$
b. Sample responses can be found in the worked solutions in the online resources.

11. $x = -6$

12. Sample responses can be found in the worked solutions in the online resources.

13. Sample responses can be found in the worked solutions in the online resources.

14. a. Sample responses can be found in the worked solutions in the online resources.
b. $a = 2,\ b = 1,\ c = 3,\ d = 4$
$$\begin{aligned}(2^2 + 1^2)(3^2 + 4^2) &= (2 \times 3 - 1 \times 4)^2 + (2 \times 4 - 1 \times 3)^2 \\ &= 4 + 121 \\ &= 125\end{aligned}$$

15. a. $x^4 + 2x^3 - x^2 - 2x + 1$
b. Sample responses can be found in the worked solutions in the online resources.
c. i. 25
ii. $x = 2$

16. a. $ad + ae + bd + be$
b. $(a + b + c)(d + e + f) = ad + ae + bd + be + cd + ce + af + bf + cf$

	a	b	c
d	ad	bd	cd
e	ae	be	ce
f	af	bf	cf

2.8 The highest common factor

1. a. $2a$ b. 4 c. $3x$
d. -4 e. $3x$ f. $15a$

2. a. $-3x$ b. m c. $-mn$
d. ab e. $3x$ f. $5n^2$

3. a. 4 b. 3 c. 5 d. 8 e. 3

4. a. 25 b. 1 c. $6a$ d. $7x$ e. $30q$

5. a. 10 b. $3x$ c. 3 d. 5 e. x^2

6. C

7. a. $4(x + 3y)$ b. $5(m + 3n)$
c. $7(a + 2b)$ d. $7(m - 3n)$
e. $-8(a + 3b)$ f. $4(2x - y)$

8. a. $-2(6p + q)$ b. $6(p + 2pq + 3q)$
c. $8(4x + y + 2z)$ d. $4(4m - n + 6p)$
e. $8(9x - y + 8pq)$ f. $3(5x^2 - y)$

9. a. $5(p^2 - 4q)$ b. $5(x + 1)$
c. $8(7q + p^2)$ d. $7(p - 6x^2y)$
e. $4(4p^2 + 5q + 1)$ f. $12(1 + 3a^2b - 2b^2)$

10. a. $3(3a + 7b)$ b. $2(2c + 9d^2)$
c. $4(3p^2 + 5q^2)$ d. $7(5 - 2m^2n)$
e. $5(5y^2 - 3x)$

11. a. $4(4a^2 + 5b)$ b. $6(7m^2 + 2n)$
c. $9(7p^2 + 9 - 3y)$ d. $11(11a^2 - 5b + 10c)$
e. $2(5 - 11x^2y^3 + 7xy)$

12. a. $9(2a^2bc - 3ab - 10c)$ b. $12(12p + 3q^2 - 7pq)$
c. $7(9a^2b^2 - 7 + 8ab^2)$ d. $11(2 + 9p^3q^2 - 4p^2r)$
e. $6(6 - 4ab^2 + 3b^2c)$

13. a. $-(x - 5)$ b. $-(a - 7)$ c. $-(b - 9)$
d. $-2(m + 3)$ e. $-6(p + 2)$ f. $-4(a + 2)$

14. a. $-3(n^2 - 5m)$ b. $-7(x^2y^2 - 3)$ c. $-7(y^2 + 7z)$
d. $-6(2p^2 + 3q)$ e. $-7(9m - 8)$ f. $-2(6m^3 + 25x^3)$

15. a. $-3(3a^2b - 10)$ b. $-3(5p + 4q)$
c. $-2(9x^2 - 2y^2)$ d. $-3(ab - 6m + 7)$
e. $-5(2 + 5p^2 + 9q)$ f. $-9(10m^2 - 3n - 6p^3)$

16. a. $a(a + 5)$ b. $q(14 - q)$ c. $m(18 + 5m)$
d. $p(6 + 7p)$ e. $n(7n - 2)$ f. $p(7 - pq + q)$

17. a. $y(x + 9 - 3y)$ b. $c(5 + 3cd - d)$
c. $ab(3 + a + 4b)$ d. $xy(2x + 1 + 5y)$
e. $pq(5pq - 4 + 3p)$ f. $xy(6xy - 5 + x)$

18. a. $5x(x + 3)$ b. $6m(4m - 1)$ c. $4a(8a - 1)$
d. $-2m(m - 4)$ e. $-5x(x - 5)$ f. $-7y(y - 2)$

19. a. $-3a(a - 3)$ b. $-2p(6p + 1)$ c. $-13y(2y + 1)$
d. $2m(2 - 9m)$ e. $-6t(1 - 6t)$ f. $-8p(1 + 3p)$

20. a. $2(x + 3)(4x + 1)$
b. $\mathrm{A} = x(x + 3)$
$\mathrm{B} = (5x + 2)(x + 3)$
$\mathrm{C} = 2x(x + 3)$

21. a. $3a(b^2 + 6b + 9)$
b. $3a(b + 3)^2$

22. a. $(x + 4)(x^2 + 8x + 16)$
b. $(x + 4)^3$

23. Sample responses can be found in the worked solutions in the online resources.

24. Sample responses can be found in the worked solutions in the online resources.

25. a. $\left(x + \sqrt{13}\right)\left(x - \sqrt{13}\right)$
b. $\left(2x - \sqrt{17}\right)\left(2x + \sqrt{17}\right)$
c. $\left(x + 3 + \sqrt{10}\right)\left(x + 3 - \sqrt{10}\right)$

26. -2

27. True. Sample responses can be found in the worked solutions in the online resources.

28. a. $4(3x - 5y)(3x + 5y)$
b. $4(3x - 5y)(3x + 5y)$
c. Yes, the answers are the same.

2.9 The highest common binomial factor (extending)

1. a. $(a+b)(2+3c)$ b. $(m+n)(4+p)$
c. $(2m+1)(7x-y)$ d. $(3b+2)(4a-b)$
e. $(x+2y)(z-3)$

2. a. $(6-q)(12p-5)$ b. $(x-y)(3p^2+2q)$
c. $(b-3)(4a^2+3b)$ d. $(q+2p)(p^2-5)$
e. $(5m+1)(6+n^2)$

3. a. $(y+2)(x+2)$ b. $(b+3)(a+3)$
c. $(x-4)(y+3)$ d. $(2y+1)(x+3)$
e. $(3b+1)(a+4)$ f. $(b-2)(a+5)$

4. a. $(m-2n)(1+a)$ b. $(5+3p)(1+3a)$
c. $(3m-1)(5n-2)$ d. $(10p-1)(q-2)$
e. $(3x-1)(2-y)$ f. $(4p-1)(4-3q)$

5. a. $(2y+1)(5x-2)$ b. $(2a+3)(3b-2)$
c. $(b-2c)(5a-3)$ d. $(x+3y)(4-z)$
e. $(p+2q)(5r-3)$ f. $(a+5b)(c-2)$

6. a. $\dfrac{a}{b}$ b. $\dfrac{3x-4}{7a-5}$

7. a. $\dfrac{x+2}{2+6x}$ b. $\dfrac{y+2}{3b-2}$

8. a. $\dfrac{3x-2}{3y+2}$ b. $\dfrac{m-4n}{m+4n}$

9. Sample responses can be found in the worked solutions in the online resources.

10. $(2x+3)^2$, this is a perfect square.

11. a. $5(x+2)(x+3) = 5(x^2+5x+6) = 5x^2+25x+30$

	x	3	x	3	x	3	x	3	x	3
x	x^2	$3x$	x^2	$3x$	x^2	$3x$	x^2	$3x$	x^2	$3x$
2	$2x$	6	$2x$	6	$2x$	6	$2x$	6	$2x$	6

b. $5(x+2)(x+3) = (5\times x + 2\times 5)(x+3)$
$= (5x+10)(x+3)$

	x	2	x	2	x	2	x	2	x	2
x	x^2	$2x$	x^2	$2x$	x^2	$2x$	x^2	$2x$	x^2	$2x$
3	$3x$	6	$3x$	6	$3x$	6	$3x$	6	$3x$	6

c. $5(x+2)(x+3) = (5\times x + 3\times 5)(x+2)$
$= (5x+15)(x+2)$

	$5x$	15
x	$5x^2$	$15x$
2	$10x$	30

12. a. Side length$_a = x+3$
Side length$_b = x+5$
Side length$_c = x+8$
Side length$_d = x-3$
Side length$_e = x+6$

b. $A_a = \text{II} = (x+3)^2$
$A_b = \text{III} = (x+5)^2$
$A_c = \text{V} = (x+8)^2$
$A_d = \text{I} = (x-3)^2$
$A_e = \text{IV} = (x+6)^2$

c. $A_a = \text{II} = 64\text{ cm}^2$
$A_b = \text{III} = 100\text{ cm}^2$
$A_c = \text{V} = 169\text{ cm}^2$
$A_d = \text{I} = 4\text{ cm}^2$
$A_e = \text{IV} = 121\text{ cm}^2$

13. $-(x+1)(4y-9)$

14. a. i. Square, because it is a perfect square.
ii. Rectangle, because it is a DOTS.
iii. Rectangle, because it is a trinomial.
iv. Rectangle, because it is a trinomial.
b. i. $(3s+8)^2$ ii. $(5s+2)(5s-2)$
iii. $(s+1)(s+3)$ iv. $4(s-8)(s+1)$

2.10 Solving worded problems

1. a. i. $12x$ ii. 60 iii. $9x^2$ iv. 225
b. i. $4x+8$ ii. 28
iii. x^2+4x+4 iv. 49
c. i. $16x-4$ ii. 76
iii. $16x^2-8x+1$ iv. 361
d. i. $10x+2$ ii. 52
iii. $6x^2+2x$ iv. 160
e. i. $12x-2$ ii. 58
iii. $5x^2-13x-6$ iv. 54
f. i. $20x+18$ ii. 118
iii. $21x^2+42x$ iv. 735

2. a. 1250 m^2
b. $(1250+150x+4x^2)\text{ m}^2$
c. $(150x+4x^2)\text{ m}^2$
d. 366.16 m^2
e. $150x+4x^2=200$

3. a. $(20-2x)$ cm
b. $(15-2y)$ cm
c. $(300-40y-30x+4xy)\text{ cm}^2$
d. Sample responses can be found in the worked solutions in the online resources.

4. a. 40 cm^2
b. i. $(8+v)$ cm
ii. $(5+v)$ cm
iii. $(v^2+13v+40)\text{ cm}^2$
iv. 70 cm^2
c. i. $(8-d)$ cm
ii. $(5-d)$ cm
iii. $(40-13d+d^2)\text{ cm}^2$
iv. 18 cm^2
d. i. $8x\text{ cm}^2$ ii. $(5+x)$ cm
iii. $(8x^2+40x)\text{ cm}^2$ iv. 400 cm^2

5. a. $20x$ m b. $25x^2\text{m}^2$
c. i. $(5x-2)$ m
ii. $(5x-3)$ m
iii. $(25x^2-25x+6)\text{ m}^2$
iv. 756m^2

6. a. $6x$ cm
b. $2x^2\text{cm}^2$

c. i. $(2x + y)$ cm
ii. $(x - y)$ cm
iii. $(2x^2 - xy - y^2)$ cm^2
iv. 11 cm^2

7. a. $4x$ cm
b. x^2 cm^2
c. i. $(x + y)$ cm
ii. $(4x + 4y)$ cm
iii. 56 cm
iv. $(x^2 + 2xy + y^2)$ cm^2
v. 60.84 cm^2

8. a. $(14p + 4)$ m
b. $(12p^2 + 6p)$ m^2
c. $(6p + 2)$ m
d. $5p$ m
e. $(22p + 4)$ m
f. $(30p^2 + 10p)$ m^2
g. $(18p^2 + 4p)$ m^2
h. 80 m^2

9. a. 27.8
b. Samir is not within the healthy weight range.
c. i. 19.1 ii. 24.9 iii. 22.0
d. Sammara: 9 to 20 years of age; Nimco: 17.5 to 20 years of age; Manuel: 13.5 to 20 years of age

10. a.

Type of algebra	Perimeter	Type of algebra rectangle	Perimeter
1-long	$2x + 2$	1-tall	$2x + 2$
2-long	$4x + 2$	2-tall	$2x + 4$
3-long	$6x + 2$	3-tall	$2x + 6$
4-long	$8x + 2$	4-tall	$2x + 8$
5-long	$10x + 2$	5-tall	$2x + 10$

b. Perimeter: long $40x + 2$, tall $2x + 40$
c. $0 < x < 1$

11. a. 3
b. 994009
c. i. 990025 ii. 980100

12. a. i. $A_{\text{screen}} = x(x + 2) = x^2 + 2x$
ii. $A_{\text{phone 1}} = (x + 1)(x + 7) = x^2 + 8x + 7$
b. i. $l = x + 11,\ w = x$
ii. $A_{\text{phone 2}} = x(x + 11) = x^2 + 11x$
c. $x = 4$ cm, $A_{\text{phone2}} = 60$ cm^2

13. a. i. $x(x - 2)$ m^2 ii. $x(2x)$ m^2
b. $\dfrac{x-2}{2x}$
c. $\dfrac{50(x-2)}{x}\%$
d. i. 25% ii. 27.8% iii. 29.17%
e. The third flag places the most importance on Australia.

14. a. $A = \pi R^2 - x^2$
b. $A = 160.71$ cm^2
c. Sample responses can be found in the worked solutions in the online resources.
d. $R = 5.32$ cm
e. 92.0%

15. a. Sample responses can be found in the worked solutions in the online resources.
b. i. $-3x^3 - 2x^2 + 3x + 2$
ii. $8x^3 + 16x^2 - 104x + 80$
iii. $x^3 - 49x + 120$
c. In a cubic expression there are four terms with descending powers of x and ascending values of the variable.
For example, $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$, where the powers of x are descending and the values of a are ascending.

Project

1. Solution not required.
2. Students need to describe the feature created by arranging the blocks.
3. 0.12 m
4. 0.0144 m^2
5. Yellow: 0.03 m $\times$ 0.03 m
Black: 0.03 m $\times$ 0.06 m
White: 0.06 m $\times$ 0.06 m
6. Yellow: 0.0009 m^2
Black: 0.0018 m^2
White: 0.0036 m^2
7. Yellow: 0.0036 m^2
Black: 0.0072 m^2
White: 0.0036 m^2
8. Students are required to derive a general formula that would allow the company to determine the area of the three coloured fabrics in each block. They should give an example and illustrate it with the help of a diagram.

2.11 Review questions

1. B
2. D
3. A
4. B
5. D
6. A
7. B
8. A
9. B
10. a. i. 4
ii. -8
iii. -5
iv. $-8xy^2$
b. i. $y + 8$
ii. $x - y$
iii. $x + y$
iv. $7xy$
v. $5y - 2x$
11. a. $\$xy$
b. i. $(24 - k)$ m
ii. $\dfrac{24-k}{3}$ m

12. a. $17p$
b. $y^2 + 2y$
c. $-3s^2t$
d. $16c^2d - 2cd$
e. $n^2 - 4p^2q + 6$
f. $15ab - 3a^2b^2$

13. a. $12ab$
b. $2ab^2$
c. $8x^2y^2$
d. $\frac{x}{3}$
e. $\frac{9}{2b}$

14. a. $5x + 15$
b. $-y - 5$
c. $-3x + 2x^2$
d. $-8mx - 4x$

15. a. $3x + 3$
b. $-10m - 1$
c. $4m^2 - 9m - 5$
d. $4p - 6$

16. a. $9a + 8b$
b. $-5x - 18y$
c. $-mn + 11m$
d. $4x^2 - 10x + 3$

17. a. $x^2 + 9x + 20$
b. $m^2 - m - 2$
c. $3m^2 - 17m + 10$
d. $2a^2 - 5ab - 3b^2$

18. a. $x^2 - 16$
b. $81 - m^2$
c. $x^2 - y^2$
d. $1 - 4a^2$

19. a. $x^2 + 10x + 25$
b. $m^2 - 6m + 9$
c. $16x^2 + 8x + 1$
d. $4 - 12y + 9y^2$

20. a. $2x^2 + 8x + 8$
b. $2m^2 + 11m - 5$
c. $6x + 15$
d. $-7b + 37$

21. a. $2x^2 + 10x + 13$
b. $2x - 5$
c. $16x$

22. a. 23
b. 134

23. a. $307.88\,\text{cm}^3$
b. $5379.98\,\text{cm}^3$

24. a. $6(x + 2)$
b. $6x^2(1 + 2xy)$
c. $4(2a^2 - b)$
d. $8x(2x - 3y)$
e. $-2(x + 2)$
f. $b(b - 3 + 4c)$

25. a. $(5 - 4a)(x + y)$
b. $(7a - 6c)(b + 5c)$
c. $5x(3 + 5y)(d + 2e)$
d. $(2 + a)(x + y)$
e. $2(x - 1)(3y + 2)$
f. $(p - r)(q + 1)$

26. a. $8x\,\text{cm}$
b. $3x\,\text{cm}^2$
c. i. $(3x + y)\,\text{cm}$
ii. $(8x + 2y)\,\text{cm}$
iii. $780\,\text{cm}$
iv. $(3x^2 + xy)\,\text{cm}^2$
v. $27\,000\,\text{cm}^2$

27. a. $(8 + 2p)(15 + 2p)\,\text{m}^2$
b. $(4p^2 + 46p + 120)\,\text{m}^2$
c. $(4p^2 + 46p)\,\text{m}^2$
d. $4p^2 + 46p = 200$

28. $\frac{2\pi x}{y}\,\text{cm}$

29. $5xy - 4y^2$

30. a. $6n + 42$
b. $6(n + 7)$
c. $2(n + 7)$
d. $\frac{6(n + 7)}{x}$

31. a. i. $9(c - d)$
ii. Yes, this is a multiple of 9 because the number that multiplies the brackets is 9.
b. $90(b - c)$ –90 is a multiple of 9, so the difference between the correct and incorrect numbers is a multiple of 9.
c. $900(a - b)$ is a multiple of 9, so the difference between the correct and incorrect numbers is a multiple of 9.
d. If two adjacent digits are transposed, the difference between the correct number and the transposed number is a multiple of 9.
e. Let the correct price of the CD be $\$ab.cd$. This can be written as $10a + b + 0.1c + 0.01d$.
The cashier enters $\$cd.ab$ as the price. This can be written as $10c + d + 0.1$.
The difference is $10a + b + 0.1c + 0.01d - (10c + d + 0.1a + 0.01b)$.
This is equal to $9.9a + 0.99b - 9.9c - 0.99d$, which is a multiple of 9.
This means that the rule for checking transposition errors applies in this case.

STRAND: Number and algebra
SUB-STRAND: Linear and non-linear equations and inequalities

3 Linear equations

LESSON SEQUENCE

LESSON 3.1 Overview

3.1.1 Why learn this?

Mathematical equations are all around us. Learning how to solve equations lets us work out unknown values. Being able to manipulate and solve equations has many uses in many fields, including chemistry, medicine, economics and commerce, to name just a few.

Being able to solve linear equations is a useful skill for many everyday tasks. Linear equations can be used when converting temperature from Celsius to Fahrenheit, when working out how to balance chemical equations, or when converting between different currencies. Linear equations can also be used when working out fuel consumption for a road trip, deciding on the right dosage of medicine, or budgeting for a holiday.

Any calculation of rates will use some kind of linear equation. Linear equations can also be used to help predict future trends involving growth or decay. An understanding of the underlying principles for solving linear equations can also be applied to the solution of other mathematical models.

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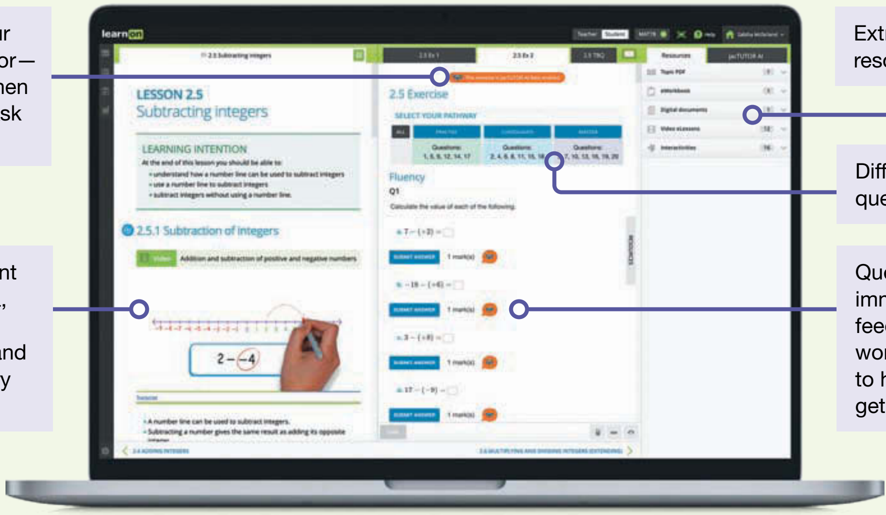
Extra learning resources

Differentiated question sets

Reading content and rich media, including interactivities and videos for every concept

Questions with immediate feedback, and fully worked solutions to help students get unstuck.



Exercise 3.1 Pre-test

learnon

1. Solve the linear equation $\frac{x}{3} - 2 = -1$.

2. Solve the linear equation $0.4x - 2.6 = 6.2$.

3. MC Select which of the following equations has the solution $x = 2$.
   A. $x + 3 = -5$ B. $3 - x = 5$ C. $3(x + 1) = 2x + 5$ D. $\frac{3}{x} = 6$

4. Solve the following linear equations.
   a. $\frac{3x + 1}{4} = -5$ b. $\frac{3 - z}{2} = -4$

5. Solve the equation $-3(2a - 4) = -12$.

6. Dylan is solving the equation $4(y + 3) = -10$. State whether it is true or false that Dylan will calculate the correct value of y if he subtracts 3 from both sides of the equation then divides both sides by 4.

7. MC Select the correct value for m in the equation $0.4(m - 8) = 0.6$.
   A. $m = -6.5$ B. $m = 6.5$ C. $m = 7.8$ D. $m = 9.5$

8. MC Select the first step to solve the equation $2x + 1 = 11 - 3x$.
   A. Add $2x$ to both sides. B. Add 1 to both sides.
   C. Add $3x$ to both sides. D. Add 11 to both sides.

9. Solve the following equations.
   a. $5(a + 2) = 4a + 12$ b. $0.2(b - 6) = 5.2 + b$

10. MC Choose the equation that matches the statement 'subtracting 6 from 5 multiplied by a certain number gives a result of -8'.
    A. $6 - 5x = -8$ B. $5x - 6 = -8$ C. $5 - 6x = -8$ D. $6x - 5 = -8$

11. The cost of renting a car is given by $c = 75d + 0.4k$, where $d =$ number of days rented and $k =$ number of kilometres driven. Toni has \$340 to spend on a car for three days.
    Calculate the total distance she can travel.

12. Solve the value of x in the statement 'x is multiplied by 5 and then 4 is subtracted. The result is the same as three times x minus 10'.

13. John is twice as old as his brother, and the sum of their ages is 30. How old is John and how old is his brother?

14. Sarah bought 3 packs of pencils and 2 packs of erasers. Each pack of pencils costs \$5 more than each pack of erasers. If she spent a total of \$29, how much did each pack of pencils and each pack of erasers cost?

15. MC Select the correct equation in which $x(y - 4) = y + 3$ is rearranged to make y the subject.
    A. $y = \frac{4x + 3}{x}$ B. $y = \frac{4x + 3}{x - 1}$ C. $y = 3x + 3$ D. $y = \frac{y + 3}{x} + 4$

LESSON 3.2 Solving multi-step linear equations

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- recognise inverse operations
- solve multi-step linear equations with variables on one side of the equation
- solve linear equations that involve simple algebraic fractions with numerical denominators (optional)

3.2.1 What is a linear equation?

- An **equation** is a mathematical statement that contains an equals sign (=).
- The **expression** on the left-hand side of the equals sign has the same value as the expression on the right-hand side of the equals sign.
- A **linear equation** is a mathematical statement where the highest power of the variable is 1, and it makes a straight line when you draw it on a graph.
- Solving a **linear equation** means determining a value for the variable that makes the statement true.
- 'Doing the same thing' to both sides of an equation (also known as applying the **inverse operation**) ensures that the expressions on either side of the equals sign remain equal, or **balanced**.

WORKED EXAMPLE 1 Using substitution to check equations

For each of the following equations, determine whether $x = 10$ is a solution.

a. $\dfrac{x+2}{3} = 6$ **b.** $2x + 3 = 3x - 7$ **c.** $x^2 - 2x = 9x - 10$

THINK	WRITE
a. 1. Substitute 10 for x in the left-hand side of the equation.	**a.** $\text{LHS} = \dfrac{x+2}{3}$ $= \dfrac{10+2}{3}$ $= \dfrac{12}{3}$ $= 4$
2. Write the right-hand side.	$\text{RHS} = 6$
3. Is the equation true? Does the left-hand side equal the right-hand side?	$\text{LHS} \neq \text{RHS}$
4. State whether $x = 10$ is a solution.	$x = 10$ is not a solution.
b. 1. Substitute 10 for x in the left-hand side of the equation.	**b.** $\text{LHS} = 2x + 3$ $= 2(10) + 3$ $= 23$
2. Substitute 10 for x in the right-hand side of the equation.	$\text{RHS} = 3x - 7$ $= 3(10) - 7$ $= 23$
3. Is the equation true? Does the left-hand side equal the right-hand side?	$\text{LHS} = \text{RHS}$
4. State whether $x = 10$ is a solution.	$x = 10$ is a solution.

c. 1. Substitute 10 for x in the left-hand side of the equation.

c. $\text{LHS} = x^2 - 2x$
$= 10^2 - 2(10)$
$= 100 - 20$
$= 80$

2. Substitute 10 for x in the right-hand side of the equation.

$\text{RHS} = 9x - 10$
$= 9(10) - 10$
$= 90 - 10$
$= 80$

3. Is the equation true? Does the left-hand side equal the right-hand side?

$\text{LHS} = \text{RHS}$

4. State whether $x = 10$ is a solution.

$x = 10$ is a solution.

3.2.2 Solving multi-step equations

- If an equation performs two operations on a variable, it is known as a **two-step equation**.
- If an equation performs more than two operations on a variable, it is known as a **multi-step equation**.
- To solve multi-step equations, first determine the order in which the operations were performed.
- Once the order of operations is determined, perform the inverse of those operations, in the reverse order, to both sides of the equation. This will keep the equation balanced.

Inverse operations

The inverse operation of an operation has the effect of undoing the original operation.

Operation	Inverse operation
+	−
−	+
×	÷
÷	×

- Each inverse operation must be performed one step at a time.
- These steps can be applied to any equation with two or more steps, as shown in the worked examples that follow.

WORKED EXAMPLE 2 Solving multi-step equations using inverse operations

Solve the following linear equations.

a. $3x + 1 = 11$ **b.** $6y - 32 = 16$ **c.** $15z + 4 - 7z = 48$

THINK

WRITE

a. 1. Subtract 1 from both sides of the equation.

a. $3x + 1 = 11$
$3x + 1 - 1 = 11 - 1$

2. Divide both sides by 3.

$\frac{3x}{3} = \frac{10}{3}$

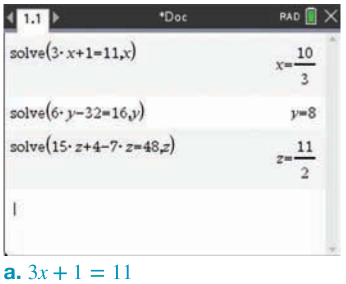
THINK	WRITE
3. Write the value for x. *Note:* It is preferable to write fractions in improper form rather than as mixed numbers.	$x=\frac{10}{3}$
4. Verify your solution by substituting it back into your original equation.	$3\left(\frac{10}{3}\right)+1=11$ $10+1=11$ $11=11$ ✓
b. 1. Add 32 to both sides of the equation	**b.** $6y-32=16$ $6y-32+32=16+32$ $6y=48$
2. Divide both sides by 6.	$\frac{6y}{6}=\frac{48}{6}$
3. Write the value of y.	$y=8$
4. Verify your solution by substituting it back into your original equation.	$6(8)-32=16$ $48-32=16$ $16=16$ ✓
c. 1. Collect like terms of z.	**c.** $15z+4-7z=48$
2. Subtract 4 from both sides.	$8z+4=48$ $8z+4-4=48-4$
3. Divide both sides by 8.	$8z=44$
4. Write the value of z.	$\frac{8z}{8}=\frac{44}{8}$ $z=5.5$
5. Verify your solution by substituting it back into your original equation.	$15(5.5)+4-7(5.5)=48$ $82.5+4-38.5=48$ $48=48$ ✓

TI | THINK

a-c. On a Calculator page, press:
- MENU
- 3: Algebra
- 1: Solve

Complete the entry lines as:
solve $(3x+1=11, x)$
solve $(6y-32=16, y)$
solve $(15z+4-7z=48, z)$
Press ENTER after each entry.

DISPLAY/WRITE



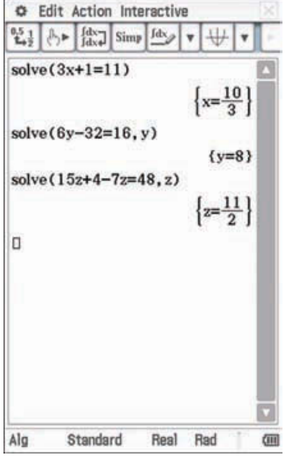
a. $3x+1=11$
$x=\frac{10}{3}$
b. $6y-32=16$
$y=8$
c. $15z+4-7z=48$
$z=\frac{11}{2}$ or 5.5
$6y-32=16$

CASIO | THINK

a-c. On a standard Main screen, press:
- Action
- Advanced
- Solve

Complete the entry lines as:
solve $(3x+1=11, x)$
solve $(16y-32=16, y)$
solve $(15z+4-7z=48, z)$
$(6y-32=16, y)$
Press EXE after each entry.

DISPLAY/WRITE



Note: If x is the only variable in each equation there is no need to specify 'solve for x', so the 'x' at the end of each entry line may be left out. This only applies if x is the variable.

a. $3x + 1 = 11$

$x = \frac{10}{3}$

b. $6y - 32 = 16$

$y = 8$

c. $15z + 4 - 7z = 48$

$z = \frac{11}{2}$ or 5.5

$6y - 32 = 16$

3.2.3 Algebraic fractions with numerical denominators (optional)

- To solve linear equations involving algebraic fractions, we need to eliminate the denominator through multiplication, simplify the expression, and isolate the variable.
- Check the solution by substitution to ensure the original equation is satisfied, confirming the accuracy of your answer.
- Consider the equation:

$$\frac{3x}{2} = 9$$

- Multiply both sides by 2 to undo the fraction:

$$\frac{3x}{2} \times 2 = 9 \times 2$$
$$3x = 18$$

- Divide both sides by 3 to isolate the variable:

$$\frac{3x}{3} = \frac{18}{3}$$
$$x = 6$$

WORKED EXAMPLE 3 Solving equations with algebraic fractions

Solve the following linear equations.

a. $1 - \frac{x}{3} = 11$ **b.** $\frac{2x}{3} + \frac{x}{4} = -11$ **c.** $\frac{-x - 0.3}{2} = -1.1$

THINK	WRITE
a. 1. Subtract 1 from both sides of the equation.	**a.** $1 - \frac{x}{3} = 11$ $1 - \frac{x}{3} - 1 = 11 - 1$
2. Multiply both sides by –3.	$\frac{-x}{3} \times (-3) = 10 \times (-3)$
3. Write the value for x.	$x = -30$

b. **1.** Subtract 1 from both sides.

2. Multiply both sides by 3.

3. Divide both sides by −1.

4. Write the value for *x*.

b.
$$1 - \frac{x}{3} = 11$$
$$1 - \frac{x}{3} - 1 = 11 - 1$$
$$\frac{-x}{3} \times 3 = 10 \times 3$$
$$\frac{-x}{-1} = \frac{30}{-1}$$
$$x = -30$$

c. **1.** Add like terms to simplify the left-hand side of the equation. To do this, you will need to determine the lowest common denominator.

2. Multiply both sides by 12.

3. Divide both sides by 11.

4. Write the value for *x*.

c.
$$\frac{2x}{3} + \frac{x}{4} = -11$$
$$\frac{2x}{3} \times \frac{4}{4} + \frac{x}{4} \times \frac{3}{3} = -11$$
$$\frac{8x + 3x}{12} = -11$$
$$\frac{11x}{12} \times 12 = -11 \times 12$$
$$\frac{11x}{11} = \frac{-132}{11}$$
$$x = -12$$

d. **1.** Multiply both sides of the equation by 2.

2. Add 0.3 to both sides.

3. Divide both sides by −1.

4. Write the value for *x*.

d.
$$\frac{-x - 0.3}{2} = -1.1$$
$$\frac{-x - 0.3}{2} \times 2 = -1.1 \times 2$$
$$-x - 0.3 + 0.3 = -2.2 + 0.3$$
$$\frac{-x}{-1} = \frac{-1.9}{-1}$$
$$x = 1.9$$

Exercise 3.2 Solving multi-step linear equations

learnon

3.2 Quick quiz on | 3.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 7, 11, 15, 17, 18, 21, 24	2, 4, 8, 9, 12, 13, 16, 19, 22, 25	6, 10, 14, 20, 23, 26, 27

Fluency

1. **WE1** For each of the following equations, determine whether $x = 6$ is a solution.
   a. $x + 3 = 7$
   b. $2x - 5 = 7$
   c. $x^2 - 2 = 38$
   d. $\frac{6}{x} + x = 7$
   e. $\frac{2(x+1)}{7} = 2$
   f. $3 - x = 9$
2. For each of the following equations, determine whether $x = 6$ is a solution.
   a. $x^2 + 3x = 39$
   b. $3(x + 2) = 5(x - 4)$
   c. $x^2 + 2x = 9x - 6$
   d. $x^2 = (x + 1)^2 - 14$
   e. $(x - 1)^2 = 4x + 1$
   f. $5x + 2 = x^2 + 4$
3. **WE2a** Solve each of the following linear equations. Check your answers by substitution.
   a. $x - 43 = 167$
   b. $x + 286 = 516$
   c. $58 + x = 81$
   d. $209 - x = 305$
   e. $5x = 185$
   f. $60x = 1200$
4. Solve each of the following linear equations. Check your answers by substitution.
   a. $5x = 250$
   b. $y - 16 = -31$
   c. $5.5 + y = 7.3$
   d. $y - 7.3 = 5.5$
   e. $6y = 14$
   f. $0.9y = -0.05$
5. **WE2** Solve each of the following linear equations.
   a. $2y - 3 = 7$
   b. $2y + 7 = 3$
   c. $5y - 1 = 0$
   d. $6y + 2 = 8$
   e. $7 + 3y = 10$
   f. $8 + 2y = 12$
6. Solve each of the following linear equations.
   a. $15 = 3y - 1$
   b. $-6 = 3y - 1$
   c. $6y - 7 = 140$
   d. $4.5y + 2.3 = 7.7$
   e. $0.4y - 2.7 = 6.2$
   f. $600y - 240 = 143$
7. Solve each of the following linear equations.
   a. $3 - 2x = 1$
   b. $-3x - 1 = 5$
   c. $-4x - 7 = -19$
   d. $1 - 3x = 19$
   e. $-5 - 7x = 2$
   f. $-8 - 2x = -9$
8. Solve each of the following linear equations.
   a. $9 - 6x = -1$
   b. $-5x - 4.2 = 7.4$
   c. $2 = 11 - 3x$
   d. $-3 = -6x - 8$
   e. $-1 = 4 - 4x$
   f. $35 - 13x = -5$
9. Solve each of the following linear equations.
   a. $7 - x = 8$
   b. $8 - x = 7$
   c. $5 - x = 5$
   d. $5 - x = 0$
   e. $15.3 = 6.7 - x$
   f. $5.1 = 4.2 - x$
10. Solve each of the following linear equations.
   a. $9 - x = 0.1$
   b. $140 - x = 121$
   c. $-30 - x = -4$
   d. $-5 = -6 - x$
   e. $-x + 1 = 2$
   f. $-2x - 1 = 0$

11. WE3 Solve each of the following linear equations.

a. $\frac{x}{4}+1=3$
b. $\frac{x}{3}-2=-1$
c. $\frac{x}{8}=\frac{1}{2}$
d. $-\frac{x}{3}=5$
e. $5-\frac{x}{2}=-8$
f. $4-\frac{x}{6}=11$

12. Solve each of the following linear equations.

a. $\frac{2x}{3}=6$
b. $\frac{5x}{2}=-3$
c. $-\frac{3x}{4}=-7$
d. $-\frac{8x}{3}=6$
e. $\frac{2x}{7}=-2$
f. $-\frac{3x}{10}=-\frac{1}{5}$

13. Solve each of the following linear equations.

a. $\frac{z-1}{3}=5$
b. $\frac{z+1}{4}=8$
c. $\frac{z-4}{2}=-4$
d. $\frac{6-z}{7}=0$
e. $\frac{3-z}{2}=6$
f. $\frac{-z-50}{22}=-2$

14. Solve each of the following linear equations.

a. $\frac{z-4.4}{2.1}=-3$
b. $\frac{z+2}{7.4}=1.2$
c. $\frac{140-z}{150}=0$
d. $\frac{-z-0.4}{2}=-0.5$
e. $\frac{z-6}{9}=-4.6$
f. $\frac{z+65}{73}=1$

15. Solve each of the following linear equations.

a. $\frac{5x+1}{3}=2$
b. $\frac{2x-5}{7}=3$
c. $\frac{3x+4}{2}=-1$
d. $\frac{4x-13}{9}=-5$
e. $\frac{4-3x}{2}=8$
f. $\frac{1-2x}{6}=-10$

16. Solve each of the following linear equations.

a. $\frac{-5x-3}{9}=3$
b. $\frac{-10x-4}{3}=1$
c. $\frac{4x+2.6}{5}=8.8$
d. $\frac{5x-0.7}{-0.3}=-3.1$
e. $\frac{1-0.5x}{4}=-2.5$
f. $\frac{-3x-8}{14}=\frac{1}{2}$

17. Solve each of the following linear equations.

a. $3a+7=4$
b. $5-b=-5$
c. $4c-4.4=44$
d. $\frac{d-4}{67}=0$
e. $5-3e=-10$
f. $\frac{2f}{3}=8$

Understanding

18. Write each of the following worded statements as a mathematical sentence and then solve for the unknown value.

a. Seven is added to the product of x and 3, which gives the result of 4.
b. Four is divided by x and this result is equivalent to $\frac{2}{3}$.
c. Three is subtracted from x and this result is divided by 12 to give 25.

19. Driving lessons are usually quite expensive, but a discount of \$15 per lesson is given if a family member belongs to the automobile club.
If 10 lessons cost \$760 after the discount, calculate the cost of each lesson before the discount.

20. Anton lives in Australia. His friend Utan lives in the USA. Anton's home town of Geraldton experienced one of the hottest days on record with a temperature of 46.7 °C.
Utan said that his home town had experienced a day hotter than that, with the temperature reaching 113 °F.
The formula for converting Celsius to Fahrenheit is $F = \frac{9}{5}C + 32$.
State whether Utan was correct. Show full working.

Reasoning

21. If the expression $\frac{12}{x-4}$ always results in a positive integer value, explain how you would determine the possible values for x.

22. Santo solved the linear equation $9 = 5 - x$. His second step was to divide both sides by -1. Trudy, his mathematics buddy, said she multiplied both sides by -1. Explain why they are both correct.

23. Determine the mistake in the following working and explain what is wrong.

$$\begin{aligned} \frac{x}{5} - 1 &= 2 \\ x - 1 &= 10 \\ x &= 11 \end{aligned}$$

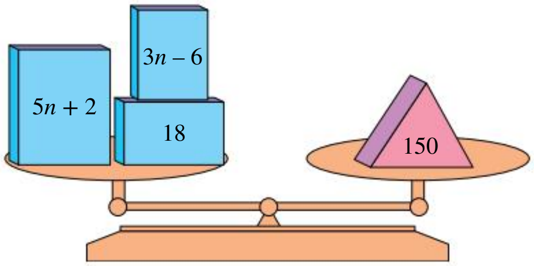
Problem solving

24. Sweet-tooth Sami goes to the corner store and buys an equal number of 25-cent and 30-cent lollies for a total of \$16.50. Determine the amount of lollies he bought.

25. In a cannery, cans are filled by two machines that together produce 16 000 cans during an 8-hour shift. If the newer machine of the two produces 340 more cans per hour than the older machine, evaluate the number of cans produced by each machine in an eight-hour shift.

26. General admission to a music festival is \$55 for an adult ticket, \$27 for a child and \$130 for a family of two adults and two children.
   a. Evaluate how much you would save by buying a family ticket instead of two adult and two child tickets.
   b. Determine if it is worthwhile buying a family ticket if a family has only one child.

27. A teacher asks her students to determine the value of n in the diagram shown. Use your knowledge of linear equations to solve this problem.



LESSON 3.3 Solving linear equations with brackets

LEARNING INTENTION

At the end of this lesson you should be able to:
- solve equations involving brackets.

3.3.1 Linear equations with brackets

- Consider the equation $3(x+5)=18$. There are two methods for solving this equation.

Method 1:

Start by dividing both sides by 3.

$$\frac{3(x+5)}{3}=\frac{18}{3}$$
$$x+5=6$$
$$x=1$$

Method 2:

Start by expanding the brackets.

$$3(x+5)=18$$
$$3x+15=18$$
$$3x=3$$
$$x=1$$

- In this case, method 1 works well because 3 divides exactly into 18.
- Now try the equation $7(x+2)=10$.

Method 1:

Start by dividing both sides by 7.

$$\frac{7(x+2)}{7}=\frac{10}{7}$$
$$x+2=\frac{10}{7}$$
$$x=-\frac{4}{7}$$

Method 2:

Start by expanding the brackets.

$$7(x+2)=10$$
$$7x+14=10$$
$$7x=-4$$
$$x=\frac{-4}{7}$$

- In this case, method 2 works well because it avoids the addition or subtraction of fractions.
- For each equation you need to solve, try both methods and choose the method that works best.

WORKED EXAMPLE 4 Solving equations with brackets

Solve each of the following linear equations.

a. $7(x-5)=28$ **b.** $6(x+3)=7$

THINK	WRITE
a. 1. 7 is a factor of 28, so it will be easier to divide both sides by 7.	**a.** $7(x-5)=28$ $\frac{7(x-5)}{7}=\frac{28}{7}$
2. Add 5 to both sides.	$x-5+5=4+5$
3. Write the value of x.	$x=9$

b. 1. 6 is not a factor of 7, so it will be easier to expand the brackets first.

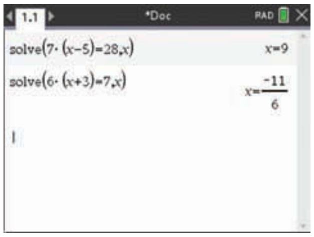
b.
$$6(x+3)=7$$
$$6x+18=7$$

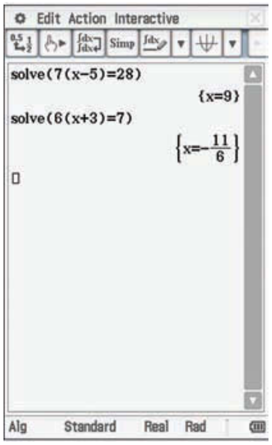
2. Subtract 18 from both sides.

$$6x+18-18=7-18$$
$$6x=-11$$

3. Divide both sides by 6.

$$x=-\frac{11}{6}$$

TI | THINK	DISPLAY/WRITE
a–b. In a new problem, on a Calculator page, press: • MENU • 3: Algebra • 1: Solve Complete the entry lines as: solve $(7(x-5)=28,\ x)$ solve $(6(x+3)=7,\ x)$ Press ENTER after each entry.	 **a.** $x=9$ **b.** $x=-\frac{11}{6}$

CASIO | THINK	DISPLAY/WRITE
a–b. On the Main screen, press: • Action • Advanced • Solve Complete the entry lines as: solve $(7(x-5)=28)$ solve $(6(x+3)=7)$ Press EXE after each entry.	 **a.** $x=9$ **b.** $x=-\frac{11}{6}$

Exercise 3.3 Solving linear equations with brackets

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3.3 Quick quiz on

3.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 11, 13, 16	2, 7, 9, 12, 14, 17	4, 6, 8, 10, 15, 18

Fluency

1. **WE4** Solve each of the following linear equations.

   a. $5(x-2)=20$ b. $4(x+5)=8$ c. $6(x+3)=18$

   d. $5(x-41)=75$ e. $8(x+2)=24$ f. $3(x+5)=15$

2. Solve each of the following linear equations.

   a. $5(x+4)=15$ b. $3(x-2)=-12$ c. $7(x-6)=0$

   d. $-6(x-2)=12$ e. $4(x+2)=4.8$ f. $16(x-3)=48$

3. Solve each of the following linear equations.

a. $6(b-1)=1$ b. $2(m-3)=3$ c. $2(a+5)=7$

d. $3(m+2)=2$ e. $5(p-2)=-7$ f. $6(m-4)=-8$

4. Solve each of the following linear equations.

a. $-10(a+1)=5$ b. $-12(p-2)=6$ c. $-9(a-3)=-3$

d. $-2(m+3)=-1$ e. $3(2a+1)=2$ f. $4(3m+2)=5$

5. Solve each of the following linear equations.

a. $9(x-7)=82$ b. $2(x+5)=14$ c. $7(a-1)=28$

d. $4(b-6)=4$ e. $3(y-7)=0$ f. $-3(x+1)=7$

6. Solve each of the following linear equations.

a. $-6(m+1)=-30$ b. $-4(y+2)=-12$ c. $-3(a-6)=3$

d. $-2(p+9)=-14$ e. $3(2m-7)=-3$ f. $2(4p+5)=18$

7. Solve each of the following linear equations. Round the answers to 3 decimal places where appropriate.

a. $2(y+4)=-7$ b. $0.3(y+8)=1$ c. $4(y+19)=-29$

d. $7(y-5)=25$ e. $6(y+3.4)=3$ f. $7(y-2)=8.7$

8. Solve each of the following linear equations. Round the answers to 3 decimal places where appropriate.

a. $1.5(y+3)=10$ b. $2.4(y-2)=1.8$ c. $1.7(y+2.2)=7.1$

d. $-7(y+2)=0$ e. $-6(y+5)=-11$ f. $-5(y-2.3)=1.6$

9. MC Select the best first step for solving the equation $7(x-6)=23$.

A. Add 6 to both sides. B. Subtract 7 from both sides.

C. Divide both sides by 23. D. Expand the brackets.

10. MC Select which one of the following is closest to the solution for the equation $84(x-21)=782$.

A. $x=9.31$ B. $x=9.56$

C. $x=30.31$ D. $x=-11.69$

Understanding

11. In 1994 a mother was 6 times as old as her daughter. If the mother turned 50 in the year 2020, calculate the year in which the mother's age was double her daughter's age.

12. New edging is to be placed around a rectangular children's playground. The width of the playground is x metres and the length is 7 metres longer than the width.
   a. Write down an expression for the perimeter of the playground. Write your answer in factorised form.
   b. If the amount of edging required is 54 m, determine the dimensions of the playground.

Reasoning

13. Explain the two possible methods for solving equations in factorised form.
14. Juanita is solving the equation $2(x-8)=10$. She performs the following operations to both sides of the equation in order: $+\ 8$, $\div\ 2$.
    Explain why Juanita will not be able to determine the correct value of x using her order of inverse operations, then solve the equation correctly.
15. As your first step to solve the equation $3(2x-7)=18$, you are given three options:
    - Expand the brackets on the left-hand side.
    - Add 7 to both sides.
    - Divide both sides by 3.

    Explain which of these options is your least preferred.

Problem solving

16. Five times the sum of four and an unknown number is equal to 35. Evaluate the value of the unknown number.
17. Oscar earns \$55 more than Josue each week, but Hector earns three times as much as Oscar. If Hector earns \$270 a week, determine how much Oscar and Josue earn each week.
18. A school wants to hire a bus to travel to a football game. The bus will take 28 passengers, and the school will contribute \$48 towards the cost of the trip.
    The price of each ticket is \$10. If the hiring of the bus is $\$300 + 10\%$ of the total cost of all the tickets, evaluate the cost per person.

LESSON 3.4 Solving linear equations with variables on both sides

LEARNING INTENTION

At the end of this lesson you should be able to:
- solve equations with variables on both sides by collecting variables on one side of the equation before solving.

3.4.1 Linear equations with variables on both sides

- If variables occur on both sides of an equation, the first step in solving the equation is to move all of the variables so that they appear on only one side of the equation.
- When solving equations, it is important to remember that whatever you do to one side of an equation you must also do to the other side.

WORKED EXAMPLE 5 Solving equations with variables on both sides

Solve each of the following linear equations.

a. $5y = 3y + 3$

b. $7x + 5 = 2 - 4x$

c. $3(x + 1) = 14 - 2x$

d. $2(x + 3) = 3(x + 7)$

THINK	WRITE
a. 1. $3y$ is smaller than $5y$. This means it is easiest to subtract $3y$ from both sides of the equation.	**a.** $5y = 3y + 3$ $5y - 3y = 3y + 3 - 3y$ $2y = 3$
2. Divide both sides by 2.	$\frac{2y}{2} = \frac{3}{2}$ $y = \frac{3}{2}$
b. 1. $-4x$ is smaller than $7x$. This means it is easiest to add $4x$ to both sides of the equation.	**b.** $7x + 5 = 2 - 4x$ $7x + 5 + 4x = 2 - 4x + 4x$ $11x + 5 = 2$
2. Subtract 5 from both sides.	$11x + 5 - 5 = 2 - 5$ $11x = -3$
3. Divide both sides by 11.	$\frac{11x}{11} = \frac{-3}{11}$ $x = \frac{-3}{11}$
c. 1. Expand the brackets.	**c.** $3(x + 1) = 14 - 2x$ $3x + 3 = 14 - 2x$
2. $-2x$ is smaller than $3x$. This means it is easiest to add $2x$ to both sides of the equation.	$3x + 3 + 2x = 14 - 2x + 2x$ $5x + 3 = 14$

3. Subtract 3 from both sides.

$$5x + 3 - 3 = 14 - 3$$
$$5x = 11$$

4. Divide both sides by 5.

$$\frac{5x}{5} = \frac{11}{5}$$
$$x = \frac{11}{5}$$

d. 1. Expand the brackets.

d.
$$2(x + 3) = 3(x + 7)$$
$$2x + 6 = 3x + 21$$

2. $2x$ is smaller than $3x$. This means it is easiest to subtract $2x$ from both sides of the equation.

$$2x + 6 - 2x = 3x + 21 - 2x$$
$$6 = x + 21$$

3. Subtract 21 from both sides.

$$6 - 21 = x + 21 - 21$$
$$-15 = x$$

4. Write the answer with the variable written on the left-hand side.

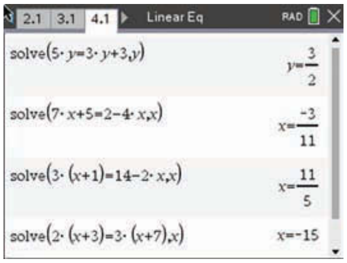
$$x = -15$$

TI | THINK

a-d. In a new problem, on a Calculator page, press:
- MENU
- 3: Algebra
- 1: Solve

Complete the entry lines as:
solve $(5y = 3y + 3, y)$
solve $(7x + 5 = 2 - 4x, x)$
solve $(3(x + 1) = 14 - 2x, x)$
solve $(2(x + 3) = 3(x + 7), x)$
Press ENTER after each entry.

DISPLAY/WRITE



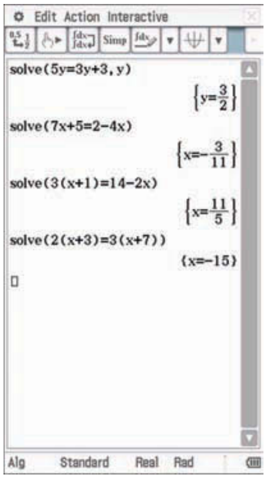
a. $y = \frac{3}{2}$
b. $x = \frac{-3}{11}$
c. $x = \frac{11}{5}$
d. $x = -15$

CASIO | THINK

a-d. On the Main screen, press:
- Action
- Advanced
- Solve

Complete the entry lines as:
solve $(5y = 3y + 3, y)$
solve $(7x + 5 = 2 - 4x, x)$
solve $(3(x + 1) = 14 - 2x, x)$
solve $(2(x + 3) = 3(x + 7), x)$
Press EXE after each entry.
Note: If the variable is *not* x, it must be specified.

DISPLAY/WRITE



a. $y = \frac{3}{2}$
b. $x = \frac{-3}{11}$
c. $x = \frac{11}{5}$
d. $x = -15$

DISCUSSION

Draw a diagram that could represent $2x + 4 = 3x + 1$.

Exercise 3.4 Solving linear equations with variables on both sides

learnon

3.4 Quick quiz on | 3.4 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 8, 12, 14, 17	2, 6, 10, 13, 15, 18	4, 7, 9, 11, 16, 19, 20

Fluency

1. WE5a Solve each of the following linear equations.
   a. $5y = 3y - 2$
   b. $6y = -y + 7$
   c. $10y = 5y - 15$
   d. $25 + 2y = -3y$
   e. $8y = 7y - 45$
   f. $15y - 8 = -12y$
2. Solve each of the following linear equations.
   a. $7y = -3y - 20$
   b. $23y = 13y + 200$
   c. $5y - 3 = 2y$
   d. $6 - 2y = -7y$
   e. $24 - y = 5y$
   f. $6y = 5y - 2$
3. MC Select the first step for solving the equation $3x + 5 = -4 - 2x$.
   A. Add $3x$ to both sides.
   B. Add 5 to both sides.
   C. Add $2x$ to both sides.
   D. Subtract $2x$ from both sides.
4. MC Select the first step for solving the equation $6x - 4 = 4x + 5$.
   A. Subtract $4x$ from both sides.
   B. Add $4x$ to both sides.
   C. Subtract 4 from both sides.
   D. Add 5 to both sides.
5. WE5b Solve each of the following linear equations.
   a. $2x + 3 = 8 - 3x$
   b. $4x + 11 = 1 - x$
   c. $x - 3 = 6 - 2x$
   d. $4x - 5 = 2x + 3$
   e. $3x - 2 = 2x + 7$
   f. $7x + 1 = 4x + 10$
6. Solve each of the following linear equations.
   a. $5x + 3 = x - 5$
   b. $6x + 2 = 3x + 14$
   c. $2x - 5 = x - 9$
   d. $10x - 1 = -2x + 5$
   e. $7x + 2 = -5x + 2$
   f. $15x + 3 = 7x - 3$
7. Solve each of the following linear equations.
   a. $x - 4 = 3x + 8$
   b. $2x + 9 = 7x - 1$
   c. $-2x + 7 = 4x + 19$
   d. $-3x + 2 = -2x - 11$
   e. $11 - 6x = 18 - 5x$
   f. $6 - 9x = 4 + 3x$
8. MC Determine the solution to $5x + 2 = 2x + 23$.
   A. $x = 3$
   B. $x = -3$
   C. $x = 5$
   D. $x = 7$
9. MC Determine the solution to $3x - 4 = 11 - 2x$.
   A. $x = 15$
   B. $x = 7$
   C. $x = 3$
   D. $x = 5$
10. WE5c,d Solve each of the following.
   a. $5(x - 2) = 2x + 5$
   b. $7(x + 1) = x - 11$
   c. $2(x - 8) = 4x$
   d. $3(x + 5) = x$
   e. $6(x - 3) = 14 - 2x$
   f. $9x - 4 = 2(3 - x)$
11. Solve each of the following.
   a. $4(x + 3) = 3(x - 2)$
   b. $5(x - 1) = 2(x + 3)$
   c. $8(x - 4) = 5(x - 6)$
   d. $3(x + 6) = 4(2 - x)$
   e. $2(x - 12) = 3(x - 8)$
   f. $4(x + 11) = 2(x + 7)$

Understanding

12. Aamir's teacher gave him the following algebra problem and told him to solve it.

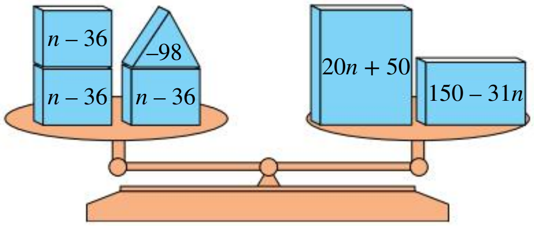
$$3x + 7 = x^2 + k = 7x + 15$$

Suggest how you can you help Aamir calculate the value of k.

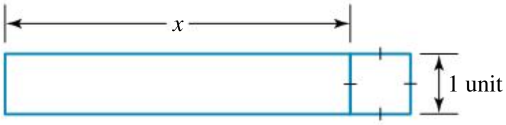
13. A classroom contained an equal number of boys and girls. Six girls left to play hockey, leaving twice as many boys as girls in the classroom.
Determine the original number of students present.

Reasoning

14. Express the information in the diagram shown as an equation, then show that $n = 29$ is the solution.



15. The block shown has a width of 1 unit and length of $(x + 1)$ units.



a. Draw two rectangles with different sizes, each using 3 of the blocks shown.

b. Show that the areas of both of the rectangles you have drawn using the blocks shown are the same. Explain how the areas of these rectangles relate to expanding brackets.

16. Explain what the difficulty is when trying to solve the equation $4(3x - 5) = 6(4x + 2)$ without expanding the brackets first.

Problem solving

17. This year Tom is 4 times as old as his daughter. In 5 years' time he will be only 3 times as old as his daughter. Determine the age of Tom and his daughter now.

18. If you multiply an unknown number by 6 and then add 5, the result is 7 less than the unknown number plus 1, multiplied by 3.
Evaluate the unknown number.

19. You are looking into getting a business card printed for your new game store. A local printing company charges $250 for the materials and an hourly rate for labour of $40.

a. If h is the number of hours of labour required to print the cards, construct an equation for the cost of the cards, C.
b. You have budgeted $1000 for this printing job. Determine the number of hours of labour that you can afford. Give your answer to the nearest minute.
c. The company estimates that it can print 1000 cards an hour. Determine the number of cards that you can afford to get printed with your current budget.
d. An alternative to printing is photocopying. The company charges 15 cents per side for the first 10 000 cards, then 10 cents per side for additional cards. Determine the cheaper option to get 18 750 single-sided cards made. Determine how much cheaper this is than the other option.

20. A local games arcade offers its regular customers the following deal. For a monthly fee of $40, players can play 25 $2 games of their choice. Extra games cost $2 each. After a player has played 50 games in a month, all further games are $1.

a. If Iman has $105 to spend in a month, determine the number of games she can play if she takes up the special deal.
b. Determine the amount of money Iman will save by taking up the special deal compared to playing the same number of games at $2 a game.

LESSON 3.5 Solving problems with linear equations

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- define a variable and write an equation to represent a problem
- solve a problem and answer the question (or questions) posed in the problem.

3.5.1 Solving problems with linear equations

- When variables are not provided in a problem, you need to introduce a variable to the problem.
- When translating a worded problem into an equation, make sure that the order of the operations is presented correctly.

WORKED EXAMPLE 6 Translating worded expressions into linear equations

Write linear equations for each of the following statements, using x to represent the unknown value. Solve the equations.

a. Three more than seven times a certain number is zero. Determine the number.

b. The sum of three consecutive integers is 102. Determine the three numbers.

c. The difference between two numbers is 11. Determine the numbers if, when eight is added to each of the two numbers, the larger result will be double the value of the smaller result.

THINK	WRITE
a. 1. Let x be the number.	**a.** $x =$ unknown number
2. 7 times the number is $7x$. Increasing the value of $7x$ by 3 gives $7x + 3$. This expression equals 0. Write the equation.	$7x + 3 = 0$
3. Solve the equation using inverse operations.	$7x + 3 - 3 = 0 - 3$ $\frac{7x}{7} = \frac{-3}{7}$
4. Write the value for x.	$x = \frac{-3}{7}$
b. 1. Let x be the first number.	**b.** $x =$ smallest number
2. The next two consecutive integers are $(x + 1)$ and $(x + 2)$. The sum of all three integers equals 102. Write the equation.	$x + (x + 1) + (x + 2) = 102$
3. Solve the equation using inverse operations.	$3x + 3 - 3 = 102 - 3$ $\frac{3x}{3} = \frac{99}{3}$ $x = 33$
4. Write the value for x and the next 2 integers.	The three numbers are 33, 34 and 35.
c. 1. Let x be the larger number.	**c.** $x =$ larger number
2. The difference between the 2 numbers is 11. This means the smaller number is 11 less than x. Write this as $(x - 11)$.	$(x - 11) =$ smaller number
3. Increasing both numbers by 8 makes the larger number $(x + 8)$ and the smaller number $(x - 11 + 8)$. The larger number becomes double the smaller number.	$(x + 8) = 2(x - 11 + 8)$ $x + 8 = 2(x - 3)$
4. Solve the equation using inverse operations.	$x + 8 - 8 = 2x - 6 - 8$ $x - x = 2x - 14 - x$ $0 + 14 = x - 14 + 14$ $x = 14$
5. Write the value of x and the other number.	The two numbers are 14 and 3.

WORKED EXAMPLE 7 Solving problems using linear equations

A taxi charges \$3.60 plus \$1.38 per kilometre for any trip in Perth. If Elena's taxi ride cost \$38.10, calculate the distance she travelled.

THINK	WRITE
1. The distance travelled by Elena has to be found. Define the variable.	Let $x =$ the distance travelled (in kilometres).
2. It costs 1.38 to travel 1 kilometre, so the cost to travel x kilometres $= 1.38x$. The fixed cost is \$3.60. Write an expression for the total cost.	Total cost $= 3.60 + 1.38x$
3. Let the total cost $= 38.10$.	$3.60 + 1.38x = 38.10$
4. Solve the equation.	$\begin{aligned} 3.60 + 1.38x &= 38.10 \\ 1.38x &= 34.50 \\ x &= \frac{34.50}{1.38} \\ &= 25 \end{aligned}$
5. Write the solution in words.	Elena travelled 25 kilometres.

WORKED EXAMPLE 8 Solving problems using linear equations

In a basketball game, Hao scored 5 more points than Seve. If they scored a total of 27 points between them, calculate how many points each of them scored.

THINK	WRITE
1. Define the variable.	Let Seve's score be x.
2. Hao scored 5 more than Seve. Write an expression for Hao's score.	Hao's score is $x + 5$.
3. Hao and Seve scored a total of 27 points. Write this as an equation.	$x + (x + 5) = 27$
4. Solve the equation.	$\begin{aligned} 2x + 5 &= 27 \\ 2x &= 22 \\ x &= 11 \end{aligned}$
5. Since $x = 11$, Seve's score is 11. Substitute this into the expression to work out Hao's score.	$\begin{aligned} \text{Hao's score} &= x + 5 \\ &= 11 + 5 \\ &= 16 \end{aligned}$
6. Write the answer in words.	Seve scored 11 points and Hao scored 16 points.

WORKED EXAMPLE 9 Solving problems using linear equations

A collection of 182 marbles is owned by four friends. Pat has twice the number of marbles that Quentin has, and Rachel has 20 fewer marbles than Pat. If Sam has two-thirds the number of marbles that Rachel has, determine who has the second-largest number of marbles.

THINK	WRITE
1. Let p represent the number of marbles Pat has.	$p =$ number of Pat's marbles
Pat has twice the number of marbles that Quentin has.	$\left(\frac{p}{2}\right) =$ number of Quentin's marbles
Rachel has 20 fewer marbles than Pat.	$p - 20 =$ number of Rachael's marbles
Sam has $\frac{2}{3}$ of the number of marbles that Rachel has.	$\frac{2}{3}(p - 20) =$ number of Sam's marbles
2. Let the sum of all marbles equal 182.	$p + \left(\frac{p}{2}\right) + (p - 20) + \frac{2}{3}(p - 20) = 182$
3. Simplify the left-hand side of the equation by adding like terms.	$p + \frac{p}{2} + p + \frac{2p}{3} - 20 - \frac{40}{3} = 182$ $\frac{6p}{6} + \frac{3p}{6} + \frac{6p}{6} + \frac{4p}{6} - \frac{120}{6} - \frac{80}{6} = 182$
4. Solve using inverse operations.	$\frac{19p - 200}{6} \times 6 = 182 \times 6$ $19p - 200 + 200 = 1092 + 200$ $\frac{19p}{19} = \frac{1292}{19}$ $p = 68$
5. Calculate the number of marbles that each friend has.	Pat has 68 marbles. Quentin has $\frac{68}{2} = 34$ marbles. Rachel has $68 - 20 = 48$ marbles. Sam has $\frac{2}{3} \times 48 = 32$ marbles.
6. Write down the second-highest number of marbles and the name of the person with that number of marbles.	Rachel has the second-highest number of marbles. She has 48 marbles.

DISCUSSION

Why is it important to define the variable when using a linear equation to solve a problem?

Exercise 3.5 Solving problems with linear equations

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3.5 Quick quiz on | 3.5 Exercise

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
1, 3, 7, 11, 12, 16	2, 4, 8, 9, 13, 17	5, 6, 10, 14, 15, 18

Fluency

1. **WE6** Write linear equations for each of the following statements, using x to represent the unknown, without solving the equations.
   - **a.** When 3 is added to a certain number, the answer is 5.
   - **b.** Subtracting 9 from a certain number gives a result of 7.
   - **c.** Multiplying a certain number by 7 gives 24.
   - **d.** A certain number divided by 5 gives a result of 11.
   - **e.** Dividing a certain number by 2 equals -9.

2. Write linear equations for each of the following statements, using x to represent the unknown, without solving the equations.
   - **a.** Subtracting 3 from 5 times a certain number gives a result of -7.
   - **b.** When a certain number is subtracted from 14 and the result is then multiplied by 2, the final result is -3.
   - **c.** When 5 is added to 3 times a certain number, the answer is 8.
   - **d.** When 12 is subtracted from 2 times a certain number, the result is 15.
   - **e.** The sum of 3 times a certain number and 4 is divided by 2. This gives a result of 5.

3. **MC** Select the equation that matches the following statement.
   A certain number, when divided by 2, gives a result of -12.
   - **A.** $x = \dfrac{-12}{2}$
   - **B.** $2x = -12$
   - **C.** $\dfrac{x}{2} = -12$
   - **D.** $\dfrac{x}{12} = -2$

4. **MC** Select the equation that matches the following statement.
   Dividing 7 times a certain number by -4 equals 9.
   - **A.** $\dfrac{x}{-4} = 9$
   - **B.** $\dfrac{-4x}{7} = 9$
   - **C.** $\dfrac{7+x}{-4} = 9$
   - **D.** $\dfrac{7x}{-4} = 9$

5. **MC** Select the equation that matches the following statement.
   Subtracting twice a certain number from 8 gives 12.
   - **A.** $2x - 8 = 12$
   - **B.** $8 - 2x = 12$
   - **C.** $2 - 8x = 12$
   - **D.** $8 - (x + 2) = 12$

6. **MC** Select the equation that matches the following statement.
   When 15 is added to a quarter of a number, the answer is 10.
   - **A.** $15 + 4x = 10$
   - **B.** $10 = \dfrac{x}{4} + 15$
   - **C.** $\dfrac{x + 15}{4} = 10$
   - **D.** $15 + \dfrac{4}{x} = 10$

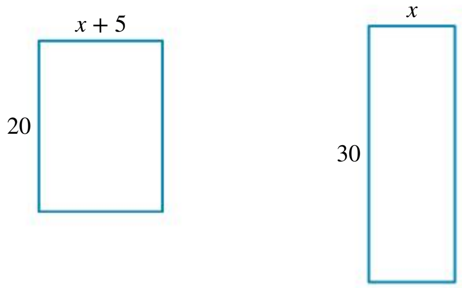
Understanding

7. When a certain number is added to 3 and the result is multiplied by 4, the answer is the same as when that same number is added to 4 and the result is multiplied by 3. Determine the number.

8. WE7 John is three times as old as his son Jack. The sum of their ages is 48. Calculate how old John is.

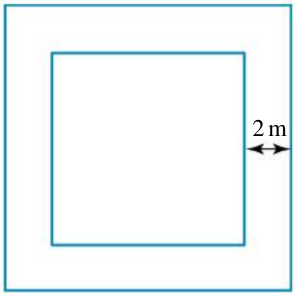
9. In one afternoon's shopping Seedevi spent half as much money as Georgia, but \$6 more than Amy. If the three of them spent a total of \$258, calculate how much Seedevi spent.

10. The rectangular blocks of land shown have the same area. Determine the dimensions of each block and use these dimensions to calculate the area.



Reasoning

11. WE8 A square pool is surrounded by a paved area that is 2 metres wide. If the area of the paving is 72 m^2, determine the length of the pool. Show all working.



12. Maria is paid \$11.50 per hour plus \$7 for each jacket that she sews. If she earned \$176 for a particular eight-hour shift, determine the number of jackets she sewed during that shift.

13. Mai hired a car for a fee of \$120 plus \$30 per day. Casey's rate for his car hire was \$180 plus \$26 per day. If their final cost and rental period is the same for both Mai and Casey, determine the rental period.

14. WE9 The cost of producing an album on CD is quoted as \$1200 plus \$0.95 per CD. If Maya has a budget of \$2100 for her debut album, determine the number of CDs she can make.

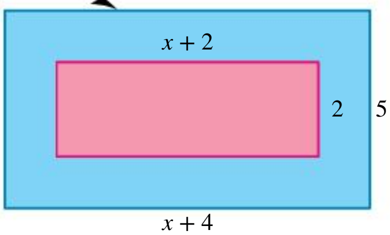
15. Joseph wants to have some flyers for his grocery business delivered. Post Quick quotes a price of \$200 plus 50 cents per flyer, and Fast Box quotes \$100 plus 80 cents per flyer.
    a. If Joseph needs to order 1000 flyers, determine the distributor that would be cheaper to use.
    b. Determine the number of flyers that Joseph need to get delivered for the cost to be the same for either distributor.

Problem solving

16. A certain number is multiplied by 8 and then 16 is subtracted. The result is the same as 4 times the original number minus 8. Evaluate the number.

17. Carmel sells three different types of healthy drinks: herbal, vegetable and citrus fizz. In an hour she sells 4 herbal, 3 vegetable and 6 citrus fizz drinks for \$60.50.
In the following hour she sells 2 herbal, 4 vegetable and 3 citrus fizz drinks. In the third hour she sells 1 herbal, 2 vegetable and 4 citrus.
The total amount in cash sales for the three hours is \$136.50.
Carmel made \$7 less in the third hour than she did in the second hour of sales.
Evaluate Carmel's sales in the fourth hour if she sells 2 herbal, 3 vegetable and 4 citrus fizz drinks.

18. A rectangular swimming pool is surrounded by a path that is enclosed by a pool fence. All measurements given are in metres and are not to scale in the diagram shown.



a. Write an expression for the area of the entire fenced-off section.
b. Write an expression for the area of the path surrounding the pool.
c. If the area of the path surrounding the pool is 34 m^2, evaluate the dimensions of the swimming pool.
d. Determine the fraction of the fenced-off area taken up by the pool.

LESSON 3.6 Rearranging formulas

LEARNING INTENTION

At the end of this lesson you should be able to:
- rearrange formulas with two or more variables to make only one of the variables the subject.

3.6.1 Rearranging formulas

- Formulas are equations that are used for specific purposes. They are generally written in terms of two or more variables (also known as 'pronumerals').
- One variable is usually written on the left of the equals sign. This variable is called the **subject**.
- Solving a linear equation requires determining a value for the variable (or variables), but rearranging formulas does not. For this reason formulas are defined as **literal equations**.

Inverse operations

When rearranging formulas, use inverse operations in reverse order, just as you do when you are solving linear equations.

For example, $C = 2\pi r$ is a formula describing the relationship of the circumference of a circle (C) to its radius (r). To make r the subject of this formula, the following steps are required:

Divide both sides by 2π.	$C = 2\pi r$ $\dfrac{C}{2\pi} = \dfrac{2\pi r}{2\pi}$ $\dfrac{C}{2\pi} = r$
Now r is the subject.	$r = \dfrac{C}{2\pi}$

DISCUSSION

In pairs, share some formulas that you know. Discuss each of the formulas by thinking about why we need them, how and when they are used in the real world, and why rearranging the variables in these formulas can be helpful.

WORKED EXAMPLE 10 Rearranging formulas

Rearrange each formula to make x the subject.

a. $y = kx + m$

b. $6(y + 1) = 7(x - 2)$

THINK	WRITE
a. 1. Subtract m from both sides.	**a.** $y = kx + m$ $y - m = kx$
2. Divide both sides by k.	$\dfrac{y-m}{k} = \dfrac{kx}{k}$ $\dfrac{y-m}{k} = x$
3. Rewrite the equation so that x is on the left-hand side.	$x = \dfrac{y-m}{k}$
b. 1. Expand the brackets.	**b.** $6(y + 1) = 7(x - 2)$ $6y + 6 = 7x - 14$
2. Add 14 to both sides.	$6y + 6 + 14 = 7x - 14 + 14$ $6y + 20 = 7x$
3. Divide both sides by 7.	$\dfrac{6y+20}{7} = \dfrac{7x}{7}$ $\dfrac{6y+20}{7} = x$

4. Rewrite the equation so that x is on the left-hand side.

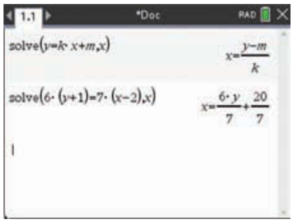
$$x = \frac{6y + 20}{7}$$

TI | THINK

a–b. In a new problem, on a Calculator page, press:
- MENU
- 3: Algebra
- 1: Solve

Complete the entry lines as:
solve $(y = k \times x + m, x)$
solve $(6(y + 1) = 7(x - 2), x)$
Press ENTER after each entry.

DISPLAY/WRITE



$x = \dfrac{y - m}{k}$

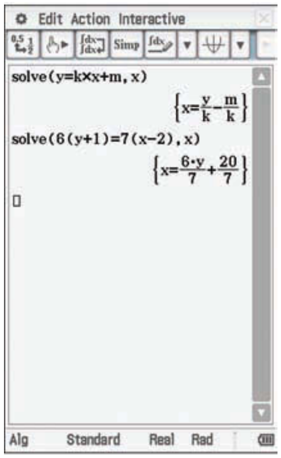
$x = \dfrac{6y}{7} + \dfrac{20}{7}$

CASIO | THINK

a–b. On the Main screen, press:
- Action
- Advanced
- Solve

Complete the entry lines as:
solve $(y = k \times x + m, x)$
solve $(6(y + 1) = 7(x - 2),\ x)$
Press EXE after each entry.

DISPLAY/WRITE



$x = \dfrac{y - m}{k}$

$x = \dfrac{6y}{7} + \dfrac{20}{7}$

WORKED EXAMPLE 11 Solving literal equations

a. If $g = 6d - 3$, solve for d.

b. Solve for v, given that $a = \dfrac{v - u}{t}$.

THINK	WRITE
a. 1. Add 3 to both sides.	a. $g = 6d - 3$ $g + 3 = 6d$
2. Divide both sides by 6.	$\dfrac{g+3}{6} = \dfrac{6d}{6}$ $\dfrac{g+3}{6} = d$
3. Rewrite the equation so that d is on the left-hand side.	$d = \dfrac{g+3}{6}$
b. 1. Multiply both sides by t.	b. $a = \dfrac{v-u}{t}$ $a \times t = \dfrac{v-u}{t} \times t$ $at = v - u$
2. Add u to both sides.	$at + u = v$
3. Rewrite the equation so that v is on the left-hand side.	$v = at + u$

Exercise 3.6 Rearranging formulas

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3.6 Quick quiz on | 3.6 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 6, 8	2, 4, 7, 9, 11	5, 10, 12

Fluency

1. **WE10** Rearrange each formula to make x the subject.
   a. $y = ax$
   b. $y = ax + b$
   c. $y = 2ax - b$
   d. $y + 4 = 2x - 3$
   e. $6(y + 2) = 5(4 - x)$
   f. $x(y - 2) = 1$
2. Rearrange each formula to make x the subject.
   a. $x(y - 2) = y + 1$
   b. $5x - 4y = 1$
   c. $6(x + 2) = 5(x - y)$
   d. $7(x - a) = 6x + 5a$
   e. $5(a - 2x) = 9(x + 1)$
   f. $8(9x - 2) + 3 = 7(2a - 3x)$
3. **WE11** For each of the following, make the variable shown in brackets the subject of the formula.
   a. $g = 4P - 3 \quad (P)$
   b. $f = \dfrac{9c}{5} \quad (c)$
   c. $f = \dfrac{9c}{5} + 32 \quad (c)$
   d. $v = u + at \quad (t)$
4. For each of the following, make the variable shown in brackets the subject of the formula.
   a. $d = b^2 - 4ac \quad (c)$
   b. $m = \dfrac{y - k}{h} \quad (y)$
   c. $m = \dfrac{y - a}{x - b} \quad (a)$
5. For each of the following, make the variable shown in brackets the subject of the formula.
   a. $f = ax + by \quad (x)$
   b. $s = ut + \dfrac{1}{2}at^2 \quad (a)$
   c. $F = \dfrac{GMm}{r^2} \quad (G)$

Understanding

6. The cost to rent a car is given by the formula $C = 50d + 0.2k$, where $d =$ the number of days rented and $k =$ the number of kilometres driven. Lin has \$300 to spend on car rental for her four-day holiday.

Calculate the distance that she can drive on her holiday.

7. A cyclist pumps up a bike tyre that has a slow leak.
The volume of air (in cm^3) after t minutes is given by the formula $V = 24\,000 - 300t$.
   a. Determine the volume of air in the tyre when it is first filled.
   b. Write out the equation for the tyre's volume and solve it to work out how long it would take the tyre to go completely flat.

Reasoning

8. The total surface area of a cylinder is given by the formula $T = 2\pi r^2 + 2\pi rh$, where r = radius and h = height. A car manufacturer wants its engines' cylinders to have a radius of 4 cm and a total surface area of 400 cm^2.
Show that the height of the cylinder is approximately 11.92 cm, correct to 2 decimal places. (*Hint:* Express h in terms of T and r.)

9. If $B = 3x - 6xy$, write x as the subject. Explain the process you followed by showing your working.

10. The volume, V, of a sphere can be calculated using the formula $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

Explain how you would work out the radius of a spherical ball that has the capacity to hold 5 litres of air.

Problem solving

11. Consider the formula $d = \sqrt{b^2 - 4ac}$.
Rearrange the formula to make a the subject.

12. Evaluate the values for a and b such that:
$$\frac{4}{x+1} - \frac{3}{x+2} = \frac{ax+b}{(x+1)(x+2)}.$$

LESSON 3.7 Review

3.7.1 Topic summary

LINEAR EQUATIONS

Inverse operations

- Solving a linear equation means determining a value for the variable that makes the statement true.
- Solving equations requires inverse operations.
- The inverse operation has the effect of undoing the original operation.

Operation	Inverse operation
$+$	$-$
$-$	$+$
$\times$	$\div$
$\div$	$\times$

- To keep an equation balanced, inverse operations are done on both sides of the equation.

e.g.

$$\begin{aligned} 3x + 1 &= 11 \\ 3x + 1 - \mathbf{1} &= 11 - \mathbf{1} \\ \frac{3x}{\mathbf{3}} &= \frac{10}{\mathbf{3}} \\ x &= \frac{10}{3} \end{aligned}$$

Equations with brackets

- The number in front of the brackets indicates the multiple of the expression inside the brackets.
- Brackets can be removed by dividing both sides by the multiple.

e.g.

$$\begin{aligned} 3(x+5) &= 18 \\ \frac{3(x+5)}{3} &= \frac{18}{3} \\ (x+5) &= 6 \end{aligned}$$

- Alternatively, brackets can be removed by expanding (multiplying) out the expression.

e.g.

$$\begin{aligned} 3(x+5) &= 18 \\ 3x + 15 &= 18 \end{aligned}$$

Equations with variables on both sides

- Variables must be moved so that they appear only
- on one side using inverse operations.

Once this is done, you can solve the equation.

e.g.

$$\begin{aligned} 5y &= 3y + 3 \\ 5y - 3y &= 3y + 3 - 3y \\ 2y &= 3 \\ y &= \frac{3}{2} \end{aligned}$$

Algebraic fractions (optional)

- Isolate the variable by multiplying both sides of the equation by the denominator first, then dividing by the coefficient of x.

e.g.

$$\begin{aligned} \frac{3x}{2} &= 9 \\ \frac{3x}{2} \times 2 &= 9 \times 2 \\ 3x &= 18 \\ \frac{3x}{3} &= \frac{18}{3} \\ x &= 6 \end{aligned}$$

Worded problems

- Worded problems should be translated into equations.
- Variables must be defined in the answer if this is not already done in the question.

e.g. The equation for the worded problem 'When 6 is subtracted from a certain number, the result is 15' is '$x - 6 = 15$'.

Rearranging formulas

- The subject of the formula is the variable that is written by itself.
- The difference between rearranging formulas and solving linear equations is that rearranging formulas does not require a value for the variable to be found.

e.g. Rearrange the following to make x the subject:

$$\begin{aligned} y &= kx + m \\ y - m &= kx \\ y - \frac{m}{k} &= \frac{kx}{k} \\ y - \frac{m}{k} &= x \end{aligned}$$

3.7.2 Success criteria

Tick the column to indicate that you have completed the lesson and how well you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
3.2	I can recognise inverse operations.			
	I can solve multi-step equations with variables on one side of the equation.			
	I can solve linear equations that involve simple algebraic fractions with numerical denominators (optional)			
3.3	I can solve equations involving brackets.			
3.4	I can solve equations with variables on both sides by collecting variables on one side of an equation before solving.			
3.5	I can define a variable and write an equation to represent the problem.			
	I can solve a problem and answer the question (or questions) posed in the problem.			
3.6	I can rearrange formulas with two or more variables to make only one of the variables the subject.			

3.7.3 Project

Forensic science

Scientific studies have been conducted on the relationship between a person's height and the measurements of a variety of body parts. One study has suggested that there is a general relationship between a person's height and the humerus bone in their upper arm, and that this relationship is slightly different for men and women.

According to this study, there is a general trend indicating that $h = 3.08l + 70.45$ for men, and $h = 3.36l + 57.97$ for women, where h represents body height in centimetres and l the length of the humerus in centimetres.

Imagine the following situation.

A decomposed body has been in the bushland outside your town. A team of forensic scientists suspects that the body could be the remains of either Alice Brown or James King, both of whom have been missing for several years.

From the descriptions provided by their Missing Persons files, Alice is 162 cm tall and James is 172 cm tall. The forensic scientists hope to identify the body based on the length of the body's humerus.

1. Based on the relationship suggested by the study previously mentioned, complete both of the following tables using the equations provided. Calculate body height to the nearest centimetre.

Body height (men)

Length of humerus, l (cm)	20	25	30	35	40
Body height, h (cm)					

Body height (women)

Length of humerus, l (cm)	20	25	30	35	40
Body height, h (cm)					

2. On a piece of graph paper, draw the first quadrant of a Cartesian plane. Since the length of the humerus is the independent variable, place it on the x-axis. Place the dependent variable, body height, on the y-axis.
3. Plot the points from the two tables onto the set of axes drawn in question 2. Join the points with straight lines, using different colours to represent men and women.

4. Describe the shape of the two graphs.
5. Measure the length of your humerus. Use your graph to predict your height. How accurate is the measurement?
6. The two lines of your graph will intersect if extended. At what point does this occur? Comment on this value.

The forensic scientists measured the length of the humerus of the decomposed body and found it to be 33 cm.

7. Using methods covered in this activity, is it more probable that the body is that of Alice or of James? Justify your decision with mathematical evidence.

Exercise 3.7 Review questions

learnon

Fluency

1. MC Select the linear equation represented by the sentence ‘When a certain number is multiplied by 3, the result is 5 times that certain number plus 7’.
   A. $3x + 7 = 5x$ B. $5(x + 7) = 3x$ C. $5x + 7 = 3x$ D. $5x = 3x + 7$
2. MC Calculate the solution to the equation $\frac{x}{3} = 5$.
   A. $x = -15$ B. $x = 15$ C. $x = 1\frac{2}{3}$ D. $x = 3$
3. MC Determine the solution to the equation $7 = 21 + x$.
   A. $x = 28$ B. $x = -28$ C. $x = -14$ D. $x = 14$
4. MC Calculate the solution to the equation $5x + 3 = 37$.
   A. $x = 8$ B. $x = -8$ C. $x = 6.8$ D. $x = 106$

5. MC Determine the solution to the equation $8 - 2x = 22$.

A. $x = 11$ B. $x = 15$ C. $x = -15$ D. $x = -7$

6. MC Determine the solution to the equation $4x + 3 = 7x - 33$.

A. $x = -12$ B. $x = 12$ C. $x = \frac{36}{11}$ D. $x = \frac{30}{11}$

7. MC Calculate the solution to the equation $7(x - 15) = 28$.

A. $x = 11$ B. $x = 19$ C. $x = 20$ D. $x = 6.14$

8. MC Select the correct rearrangement of $y = ax + b$ in terms of x.

A. $x = \frac{y-a}{b}$ B. $x = \frac{y-b}{a}$ C. $x = \frac{b-y}{a}$ D. $x = \frac{y+b}{a}$

9. Identify which of the following are linear equations.

a. $5x + y^2 = 0$ b. $2x + 3 = x - 2$ c. $\frac{x}{2} = 3$

d. $x^2 = 1$ e. $\frac{1}{x} + 1 = 3x$ f. $8 = 5x - 2$

g. $5(x + 2) = 0$ h. $x^2 + y = -9$ i. $r = 7 - 5(4 - r)$

10. Solve each of the following linear equations.

a. $3a = 8.4$ b. $a + 2.3 = 1.7$ c. $\frac{b}{21} = -0.12$

d. $b - 1.45 = 1.65$ e. $b + 3.45 = 0$ f. $7.53b = 5.64$

11. Solve each of the following linear equations.

a. $\frac{2x-3}{7} = 5$ b. $\frac{5-x}{2} = -4$ c. $\frac{-3x-4}{5} = 3$ d. $\frac{x+1.7}{2.3} = -4.1$

12. Solve each of the following linear equations.

a. $5(x - 2) = 6$ b. $7(x + 3) = 40$ c. $4(5 - x) = 15$

d. $6(2x + 3) = 1$ e. $4(x + 5) = 2x - 5$ f. $3(x - 2) = 7(x + 4)$

13. Liz has a packet of 45 lolly snakes. She saves 21 to eat tomorrow, but she rations the remainder so that she can eat 8 snakes every hour until today's share of snakes is gone.

a. Write a linear equation in terms of the number of hours, h, to represent this situation.

b. Determine the number of hours it will take Liz to eat today's share.

14. Solve each of the following linear equations.

a. $11x = 15x - 2$ b. $3x + 4 = 16 - x$ c. $5x + 2 = 3x + 8$

d. $8x - 9 = 7x - 4$ e. $2x + 5 = 8x - 7$ f. $3 - 4x = 6 - x$

15. Translate the following sentences into algebraic equations. Use x to represent the number in question.

a. Twice a certain number is equal to 3 minus that certain number.

b. When 8 is added to 3 times a certain number, the result is 19.

c. Multiplying a certain number by 6 equals 4.

d. Dividing 10 by a certain number equals one more than dividing that number by 6.

e. Multiply a certain number by 2, then add 5. Multiply this result by 7. This expression equals 0

f. Twice a certain distance travelled is 100 metres more than that certain distance travelled plus 50 metres.

16. Takanori decides to go on a holiday. He travels a certain distance on the first day, twice that distance on the second day, three times that distance on the third day and four times that distance on the fourth day. If Takanori's total journey is 2000 km, calculate the distance he travelled on the third day.

17. For each of the following, make the variable shown in brackets the subject of the formula.

a. $y = 6x - 4 \quad (x)$
b. $y = mx + c \quad (x)$
c. $q = 2(P - 1) + 2r \quad (P)$
d. $P = 2l + 2w \quad (w)$

18. For each of the following, make the variable shown in brackets the subject of the formula.

a. $v = u + at \quad (a)$
b. $v^2 = u^2 + 2as \quad (a)$
c. $2A = h(a + b) \quad (b)$

Problem solving

19. Saeed is comparing two car rental companies, Golden Ace Rental Company and Silver Diamond Rental Company. Golden Ace Rental Company charges a flat rate of \$38 plus \$0.20 per kilometre. The Silver Diamond Rental Company charges a flat rate of \$30 plus \$0.32 per kilometre.
Saeed plans to rent a car for three days.

a. Write an equation for the cost of renting a car for three days from the Golden Ace Rental Company, in terms of the number of kilometres travelled, k.
b. Write an algebraic equation for the cost of renting a car for three days from the Silver Diamond Rental Company, in terms of the number of kilometres travelled, k.
c. Evaluate the number of kilometres Saeed would have to travel so that the cost of hiring from each company is the same.

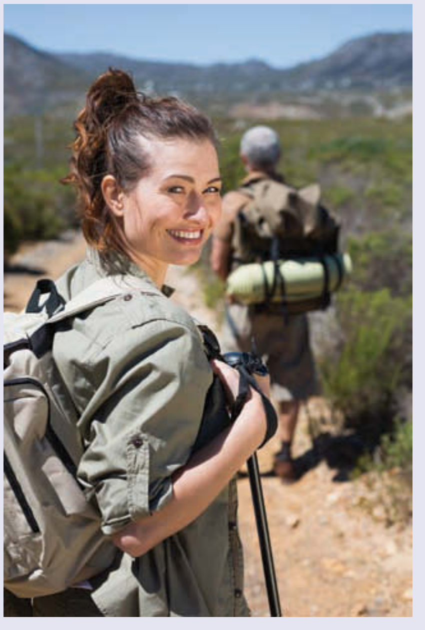
20. Frederika has \$24 000 saved to pay for a holiday. Her travel expenses are \$5400 and her daily expenses are \$260.

a. Write down an equation for the cost of her holiday if she stays for d days.
b. Determine the number of days Frederika can spend on holiday if she wants to be able to buy a new laptop for \$ 2500 when she gets back from her holidays.

21. A company that makes bottled orange juice buys their raw materials from two sources. The first source provides liquid with 6% orange juice, whereas the second source provides liquid with 3% orange juice. The company wants to make 1-litre bottles that have 5% orange juice.
Let x = the amount of liquid (in litres) that the company buys from the first source.

a. Write an expression for the amount of orange juice from the first supplier, given that x is the amount of liquid.
b. Write an expression for the amount of liquid from the second supplier, given that x is the amount of liquid used from the first supplier.
c. Write an expression for the amount of orange juice from the second supplier.
d. Write an equation for the total amount of orange juice in a mixture of liquids from the two suppliers, given that 1 litre of bottled orange juice will be mixed to contain 5% orange juice.
e. Determine the quantity of the first supplier's liquid that the company uses.

22. a. Jayani goes on a four-day bushwalk.
She travels a certain distance on the first day, half of that distance on the second day, a third of that distance on the third day and a quarter of that distance on the fourth day.
If Jayani's total journey is 50 km, evaluate the distance that she walked on the first day.
b. Svetlana goes on a five-day bushwalk, travelling the same relative distances as Jayani (a certain amount, then half that amount, then one third of that, then one quarter of that, and finally one fifth of that).
If Svetlana's journey is also 50 km, determine the distance that she travelled on the first day.



23. An online bookstore advertises its shipping cost to Australia as a flat rate of $20 for up to 10 books. Their major competitor offers a flat rate of $12 plus $1.60 per book. Determine the number of books you would have to buy (6, 7, 8, 9 or 10) for the first bookstore's shipping cost to be a better deal.

on To test your understanding and knowledge of this topic, go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 3 Linear equations

3.1 Pre-test

1. 3
2. 22
3. C
4. a. $x = -7$ b. $z = 11$
5. 4
6. False
7. D
8. C
9. a. $a = 2$ b. $b = -8$
10. B
11. 287.5km
12. -3
13. John's brother is 10 years old, and John is 20 years old.
14. The cost of each pack of erasers is \$2.80 and each pack of pencils is \$7.80.
15. D

3.2 Solving multi-step linear equations

1. a. No b. Yes c. No d. Yes e. Yes f. No
2. a. No b. No c. Yes d. No e. Yes f. No
3. a. $x = 210$ b. $x = 230$ c. $x = 23$ d. $x = -96$ e. $x = 37$ f. $x = 20$
4. a. $x = 50$ b. $y = -15$ c. $y = 1.8$ d. $y = 12.8$ e. $y = 2\frac{1}{3}$ f. $y = -\frac{1}{18}$
5. a. $y = 5$ b. $y = -2$ c. $y = 0.2$ d. $y = 1$ e. $y = 1$ f. $y = 2$
6. a. $y = 5\frac{1}{3}$ b. $y = -1\frac{2}{3}$ c. $y = 24.5$ d. $y = 1.2$ e. $y = 22.25$ f. $y = \frac{383}{600}$
7. a. $x = 1$ b. $x = -2$ c. $x = 3$ d. $x = -6$ e. $x = -1$ f. $x = \frac{1}{2}$
8. a. $x = 1\frac{2}{3}$ b. $x = -2.32$ c. $x = 3$ d. $x = -\frac{5}{6}$ e. $x = 1\frac{1}{4}$ f. $x = 3\frac{1}{13}$
9. a. $x = -1$ b. $x = 1$ c. $x = 0$ d. $x = 5$ e. $x = -8.6$ f. $x = -0.9$
10. a. $x = 8.9$ b. $x = 19$ c. $x = -26$ d. $x = -1$ e. $x = -1$ f. $x = -\frac{1}{2}$
11. a. $x = 8$ b. $x = 3$ c. $x = 4$ d. $x = -15$ e. $x = 26$ f. $x = -42$
12. a. $x = 9$ b. $x = -1\frac{1}{5}$ c. $x = 9\frac{1}{3}$ d. $x = -2\frac{1}{4}$ e. $x = -7$ f. $x = \frac{2}{3}$
13. a. $z = 16$ b. $z = 31$ c. $z = -4$ d. $z = 6$ e. $z = -9$ f. $z = -6$
14. a. $z = -1.9$ b. $z = 6.88$ c. $z = 140$ d. $z = 0.6$ e. $z = -35.4$ f. $z = 8$
15. a. $x = 1$ b. $x = 13$ c. $x = -2$ d. $x = -8$ e. $x = -4$ f. $x = 30\frac{1}{2}$
16. a. $x = -6$ b. $x = -\frac{7}{10}$ c. $x = 10.35$ d. $x = 0.326$ e. $x = 22$ f. $x = -5$
17. a. $a = -1$ b. $b = 10$ c. $c = 12.1$ d. $d = 4$ e. $e = 5$ f. $f = 12$
18. a. -1 b. 6 c. 303
19. \$91
20. No. 46.7 °C $\approx$ 116.1 °F
21. For a positive integer, $(x - 4)$ must be a factor of 12. $x = 5, 6, 7, 8, 10, 16$
22. Sample responses can be found in the worked solutions in the online resources.
23. The mistake is in the second line: the -1 should have been multiplied by 5.
24. 60 lollies
25. Old machine: 6640 cans; new machine: 9360 cans
26. a. \$34
    b. Yes, a saving of \$7
27. 17

3.3 Solving linear equations with brackets

1. a. $x = 6$ b. $x = -3$ c. $x = 0$ d. $x = 56$ e. $x = 1$ f. $x = 0$
2. a. $x = -1$ b. $x = -2$ c. $x = 6$ d. $x = 0$ e. $x = -0.8$ f. $x = 6$
3. a. $b = 1\frac{1}{6}$ b. $m = 4\frac{1}{2}$ c. $a = -1\frac{1}{2}$ d. $m = -1\frac{1}{3}$ e. $p = \frac{3}{5}$ f. $m = 2\frac{2}{3}$
4. a. $a = -1\frac{1}{2}$ b. $p = 1\frac{1}{2}$ c. $a = 3\frac{1}{3}$ d. $m = -2\frac{1}{2}$ e. $a = -\frac{1}{6}$ f. $m = -\frac{1}{4}$
5. a. $x = 16\frac{1}{9}$ b. $x = 2$ c. $a = 5$ d. $b = 7$ e. $y = 7$ f. $x = -3\frac{1}{3}$
6. a. $m = 4$ b. $y = 1$ c. $a = 5$ d. $p = -2$ e. $m = 3$ f. $p = 1$
7. a. $y = -7.5$ b. $y = -4.667$ c. $y = -26.25$ d. $y = 8.571$ e. $y = -2.9$ f. $y = 3.243$

8. a. $y = 3.667$ b. $y = 2.75$ c. $y = 1.976$
   d. $y = -2$ e. $y = -3.167$ f. $y = 1.98$

9. D

10. C

11. 2010

12. a. $[2(2x + 7)]$m
    b. Width 10 m, length 17 m

13. Expand out the brackets or divide by the factor.

14. $x = 13$. Sample responses can be found in the worked solutions in the online resources.

15. Adding 7 to both sides is the least preferred option, as it does not resolve the subtraction of 7 within the brackets.

16. 3

17. Oscar: \$90, Josue: \$35

18. \$20

3.4 Solving linear equations with variables on both sides

1. a. $y = -1$ b. $y = 1$ c. $y = -3$
   d. $y = -5$ e. $y = -45$ f. $y = \frac{8}{27}$

2. a. $y = -2$ b. $y = 20$ c. $y = 1$
   d. $y = -1\frac{1}{5}$ e. $y = 4$ f. $y = -2$

3. C

4. A

5. a. $x = 1$ b. $x = -2$ c. $x = 3$
   d. $x = 4$ e. $x = 9$ f. $x = 3$

6. a. $x = -2$ b. $x = 4$ c. $x = -4$
   d. $x = \frac{1}{2}$ e. $x = 0$ f. $x = -\frac{3}{4}$

7. a. $x = -6$ b. $x = 2$ c. $x = -2$
   d. $x = 13$ e. $x = -7$ f. $x = \frac{1}{6}$

8. D

9. C

10. a. $x = 5$ b. $x = -3$ c. $x = -8$
    d. $x = -7\frac{1}{2}$ e. $x = 4$ f. $x = \frac{10}{11}$

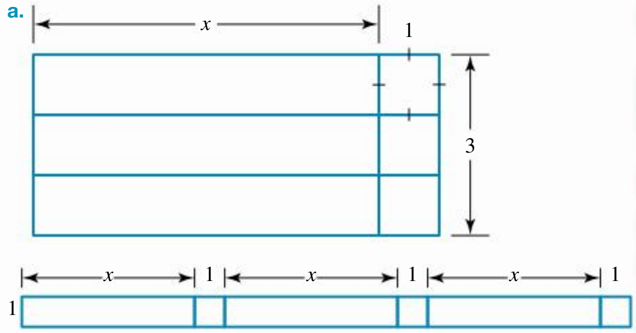
11. a. $x = -18$ b. $x = 3\frac{2}{3}$ c. $x = \frac{2}{3}$
    d. $x = -1\frac{3}{7}$ e. $x = 0$ f. $x = -15$

12. $k = -3$

13. 24

14. $3(n - 36) - 98 = -11n + 200$. Sample responses can be found in the worked solutions in the online resources.

15. a.



    b. Area of first rectangle $= (x + 1) \times 3$
    $= 3x + 3$
    Area of second rectangle $= (x + 1 + x + 1 + x + 1) \times 1$
    $= 3x + 3$
    Both rectangles have an area of $3x + 3$.

16. You cannot easily divide the left-hand side by 6 or the right-hand side by 4.

17. Daughter = 10 years, Tom = 40 years

18. The unknown number is -3.

19. a. $C = 40h + 250$
    b. 18 hours, 45 minutes
    c. 18 750
    d. The printing is cheaper by \$1375.

20. a. 65 games b. \$25

3.5 Solving problems with linear equations

1. a. $x + 3 = 5$ b. $x - 9 = 7$ c. $7x = 24$
   d. $\frac{x}{5} = 11$ e. $\frac{x}{2} = -9$

2. a. $5x - 3 = -7$ b. $2(14 - x) = -3$
   c. $3x + 5 = 8$ d. $2x - 12 = 15$
   e. $\frac{3x + 4}{2} = 5$

3. C

4. D

5. B

6. B

7. 0

8. 36 years old

9. \$66

10. 20×15; 30×10; area = 300 square units

11. 7 m

12. 12 jackets

13. 15 days

14. 947 CDs

15. a. Post Quick (cost = \$700)
    b. The cost is nearly the same for 333 flyers (\$366.50 and \$366.40).

16. 2

17. $42.50

18. a. $A_{\text{fenced}} = (5x + 20)\ \text{m}^2$ b. $A_{\text{path}} = (3x + 16)\ \text{m}^2$

c. $l = 8\ \text{m}, w = 2\ \text{m}$ d. $\frac{8}{25}$ m

3.6 Rearranging formulas

1. a. $x = \frac{y}{a}$ b. $x = \frac{y-b}{a}$

c. $x = \frac{y+b}{2a}$ d. $x = \frac{y+7}{2}$

e. $x = \frac{8-6y}{5}$ f. $x = \frac{1}{y-2}$

2. a. $x = \frac{y+1}{y-2}$ b. $x = \frac{4y+1}{5}$ c. $x = -5y - 12$

d. $x = 12a$ e. $x = \frac{5a-9}{19}$ f. $x = \frac{14a+13}{93}$

3. a. $P = \frac{g+3}{4}$ b. $c = \frac{5f}{9}$

c. $c = \frac{5(f-32)}{9}$ d. $t = \frac{v-u}{a}$

4. a. $c = \frac{b^2-d}{4a}$ b. $y = hm + k$

c. $a = y - m(x-b)$

5. a. $x = \frac{f-by}{a}$ b. $a = \frac{2(s-ut)}{t^2}$

c. $G = \frac{Fr^2}{Mm}$

6. 500 km

7. a. $24\,000\ \text{cm}^3$

b. $t = 80$ min $= 1$ h 20 min

8. Sample responses can be found in the worked solutions in the online resources.

9. $\frac{B}{3(1-2y)} = x$

10. radius $= 10.6$ cm

11. $a = \frac{b^2-d^2}{4c}$

12. $a = 1; b = 5$

Project

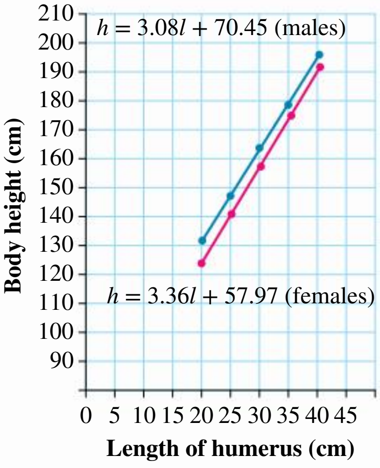
1. Body height (men)

Length of humerus, l (cm)	20	25	30	35	40
Body height, h (cm)	132	147	163	178	194

Body height (women)

Length of humerus, l (cm)	20	25	30	35	40
Body height, h (cm)	125	142	159	176	192

2. and 3.



3. Linear

4. Results will vary for each student.

5. (44.6, 207.8)

6. The height of the male body with $l = 33$ cm:
$h = 3.08 \times 33 + 70.45 = 172.09$ cm
The height of the female body with $l = 33$ cm:
$h = 3.36 \times 33 + 57.97 = 168.85$ cm
The estimated height for a male body is very close to the known height of James King; the estimated height for a female body is more different from the known height of Alice Brown. Therefore, the body is more likely to be that of James King.

3.7 Review questions

1. C
2. B
3. C
4. C
5. D
6. B
7. B
8. B
9. b, c, f, g, i

10. a. $a = 2.8$ b. $a = -0.6$

c. $b = -2.52$ d. $b = 3.1$

e. $b = -3.45$ f. $b = 0.749$

11. a. $x = 19$ b. $x = 13$

c. $x = -6\frac{1}{3}$ d. $x = -11.13$

12. a. $x = 3\frac{1}{5}$ b. $x = 2\frac{5}{7}$

c. $x = 1\frac{1}{4}$ d. $x = -1\frac{5}{12}$

e. $x = -12\frac{1}{2}$ f. $x = -8\frac{1}{2}$

13. a. $8h + 21 = 45$ b. 3 hours

14. a. $x = \frac{1}{2}$ b. $x = 3$ c. $x = 3$
d. $x = 5$ e. $x = 2$ f. $x = -1$

15. a. $2x = 3 - x$ b. $3x + 8 = 19$
c. $6x = 4$ d. $\frac{10}{x} - 1 = \frac{x}{6}$
e. $7(2x + 5) = 0$ f. $2x - 100 = x + 50$

16. 600 km

17. a. $x = \frac{y + 4}{6}$ b. $x = \frac{y - c}{m}$
c. $P = \frac{q - 2r}{2} + 1$ d. $w = \frac{P - 2l}{2}$

18. a. $a = \frac{v - u}{t}$ b. $a = \frac{v^2 - u^2}{2s}$
c. $b = \frac{2A - ah}{h}$

19. a. $C_G = 114 + 0.20k$
b. $C_S = 90 + 0.32k$
c. 200 km

20. a. $5400 + 260d = C_H$ b. 61 days

21. a. $0.06x$
b. $(1 - x)$
c. $0.03(1 - x)$
d. $0.06x + 0.03(1 - x) = 0.05$
e. 0.667 or 66.7%

22. a. 24 km
b. $21\frac{123}{137} \approx 21.9$ km

23. The online bookstore's fee is a better price than their major competitor's fee for all of the options given: 6, 7, 8, 9 and 10 books.

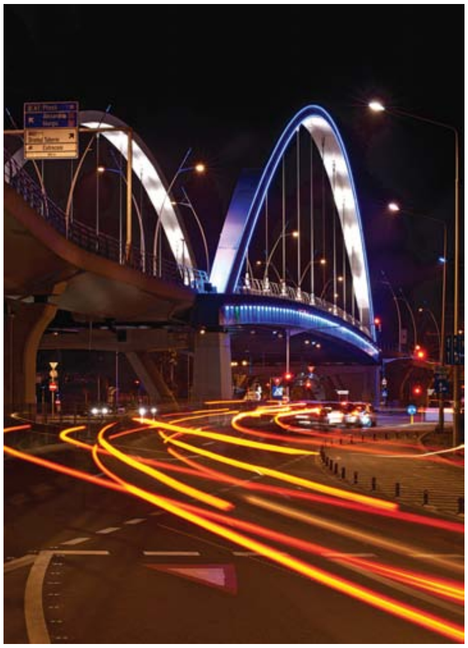
STRAND: Number and algebra
SUB-STRAND: Linear and non-linear patterns and relationships

4 Linear graphs

LESSON SEQUENCE

LESSON 4.1 Overview

4.1.1 Why learn this?



In many aspects of life, one quantity depends on another quantity, and the relationship between them can be described by equations. These equations can then be used in mathematical modelling to gain an understanding of the situation — we can draw graphs and use them to analyse, interpret and explain the relationship between the variables, and to make predictions about the future. Scientists, engineers, health professionals and financial analysts all rely heavily on mathematical equations to model real-life situations and solve problems.

Linear graphs are used extensively to represent trends, for example in the stock market or when considering the population growth of various countries. Some of the most common curves we see each day are in the arches of bridges. The arch shape is a parabolic function. Architects and structural engineers use arches extensively in buildings and other structures because its shape is very strong and stable. In space, satellites orbiting Earth follow an elliptical path, while the orbits of planets are almost circular.

Because of the many uses of linear and non-linear graphs, it is important to understand the basic concepts that you will study in this topic, such as gradient, how to calculate the distance between two points, and how to sketch linear and non-linear relationships.

Exercise 4.1 Pre-test

learnon

1. State the location of the point $(-3, 0)$.

2. Using the table shown, match each point in the left-hand column with the line in the right-hand column that passes through that point.

Point	Line
a. $(3, -1)$	**A.** $y = 3 - 2x$
b. $(0, 4)$	**B.** $2y = x$
c. $(2, 1)$	**C.** $x + y = 2$
d. $(-1, 5)$	**D.** $y = x + 4$

3. **MC** Select the rule that corresponds to the table of values shown.

x	-3	-2	-1	-0	1
y	2	1	0	-1	-2

A. $y = -x + 1$ **B.** $y = x + 1$ **C.** $y = -x - 1$ **D.** $y = x - 1$

4. Calculate the gradient of the line passing through the points $\left(-3, \frac{1}{2}\right)$ and $\left(5, -\frac{7}{2}\right)$.

5. **MC** Select the gradient of the line $3x + 4y = 12$.

A. $\frac{4}{3}$ **B.** 3 **C.** 4 **D.** $-\frac{3}{4}$

6. Match the gradients and y-intercepts to the rules given by $y = mx + c$.

Gradient and y-intercept	Rule
a. $m = \frac{1}{2}, c = 3$	**A.** $2y - 6x = 4$
b. $m = \frac{1}{4}, c = 0$	**B.** $y = \frac{1}{2}x + 3$
c. $m = 4, c = -1$	**C.** $4y = x$
d. $m = 3, c = 2$	**D.** $y = 4x - 1$

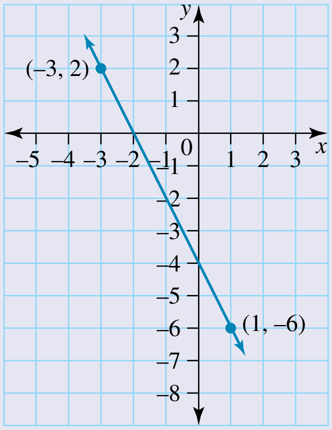
7. Determine the rule for a straight line that passes through the origin and point $(2.4, -0.6)$.

8. Determine the rule for a straight line that has an x-intercept $= -20$ and y-intercept $= 400$.

9. Determine the value of a so that the point $M\left(-\frac{1}{2}, -1\frac{1}{2}\right)$ is the midpoint of the segment joining points $A(3, 2)$ and $B(-4, a)$.

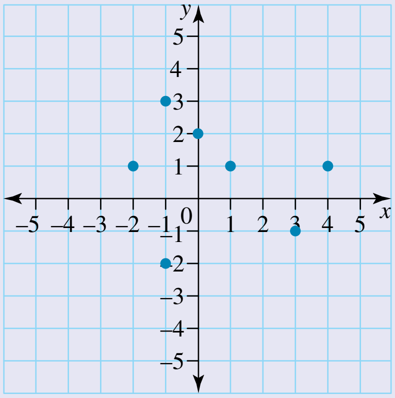
10. Calculate the exact distance between the points $(2, 5)$ and $(-3, 7)$.

11. MC Select the equation of the linear graph shown.



A. $y = 2x - 6$
B. $y = 2x - 8$
C. $y = -2x - 6$
D. $y = -2x - 4$

12. A student plays the game Blue Blobs. She has the following blobs on her screen, as shown in the diagram.
The student types in an equation for a line that will pass through four blobs.



a. Determine the equation of the straight line that passes through four blobs.
b. The student then types in the equation $x = -1$.
State the coordinate of the blob that both lines would hit.

13. An equation of a line is $2x + 4y = 10$. Determine the gradient and the coordinates of the y-intercept.

14. Determine the midpoint between the two points $A(-5, 4)$ and $B(3, -2)$.

15. The midpoint of a line segment is $M(2, -1)$. One endpoint of the segment is $A(5, 3)$. Determine the coordinates of the other endpoint B.

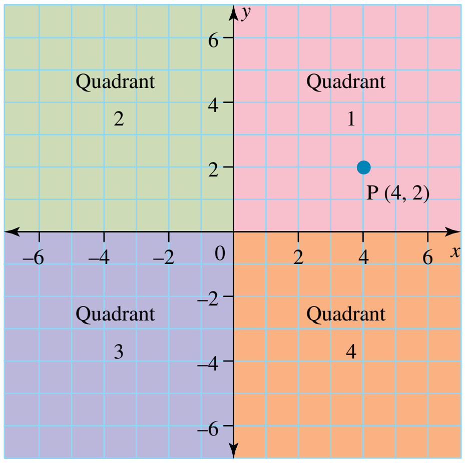
LESSON 4.2 Plotting linear graphs

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- plot points on the Cartesian plane
- plot a linear graph on the Cartesian plane.

4.2.1 Plotting linear graphs on the Cartesian plane

- The **Cartesian plane** is divided into four regions, known as quadrants, by the x- and y-axes, as shown.
- Every point in the plane is described exactly by a pair of coordinates (x, y), where x is the horizontal location and y is the vertical location of the point.
  For example, the point P (4, 2) is marked on the diagram.



Plotting linear graphs from a rule

- A straight-line graph is called a **linear graph**.
- The rule for a linear graph can always be written in the form $y = mx + c$.
  For example, $y = 4x - 5$ or $y = x + 2$.
- A minimum of two points are needed to plot a linear graph.

Plotting a linear graph from a rule on a Cartesian plane

1. **Draw a table of values for the x-values.**
2. **Substitute the x-values into the rule to find the y-values.**
3. **Plot the points on the Cartesian plane.**
4. **Join the points with a straight line and label the graph.**

WORKED EXAMPLE 1 Plotting a graph from a rule

Plot the graph specified by the rule $y = x + 2$ for the x-values $-3, -2, -1, 0, 1, 2, 3$.

THINK

1. Draw a table and write in the required x-values.

WRITE/DRAW

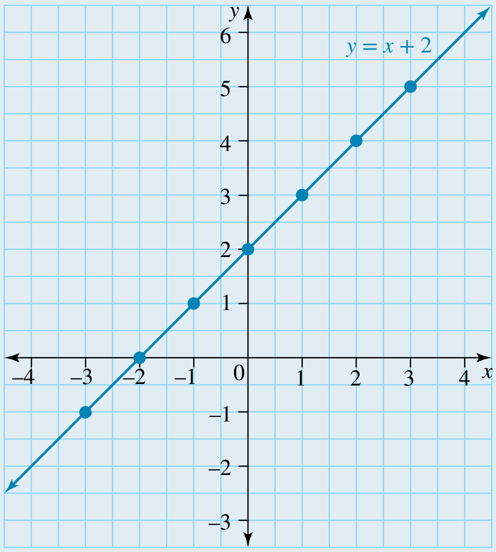
x	−3	−2	−1	0	1	2	3
y							

2. Substitute each x-value into the rule $y = x + 2$ to obtain the corresponding y-value.
   When $x = -3$, $y = -3 + 2 = -1$.
   When $x = -2$, $y = -2 + 2 = 0$ etc.
   Write the y-values into the table.

x	−3	−2	−1	0	1	2	3
y	−1	0	1	2	3	4	5

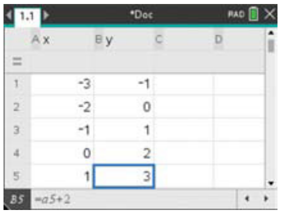
3. Plot the points from the table.

4. Join the points with a straight line and label the graph with its equation: $y = x + 2$.



TI | THINK — **WRITE/DISPLAY**

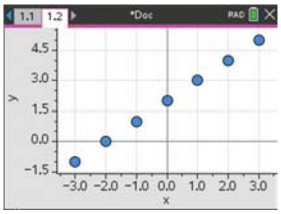
1. In a new document, on a Lists & Spreadsheet page, label column A as x and column B as y.
Enter the x-values from -3 to 3 into column A. Then in cell B1, complete the entry line as: $= a1 + 2$
Press ENTER. Hold down SHIFT and the down arrow key to fill down the y-values.



2. Open a Data & Statistics page.
Press TAB to locate the label of the horizontal axis and select the variable x-values.
Press TAB again to locate the label of the vertical axis and select the variable y-values. The points will be plotted.
To change the colour of the points, select any point and press:

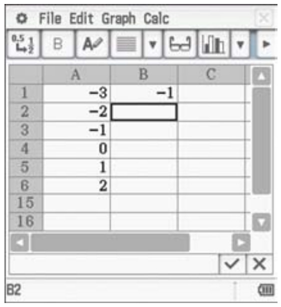
- CTRL
- MENU
- 3: Colour.

Choose a colour from the palette.



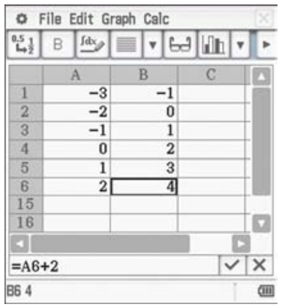
CASIO | THINK — **WRITE/DISPLAY**

1. On the Spreadsheet screen, enter the x-values into column A.
In cell B1, complete the entry line as: $=$ A1 $+$ 2
Press EXE.



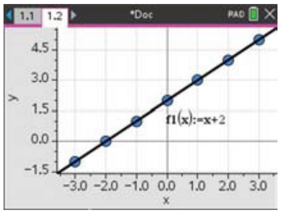
2. Highlight the required cells in column B.
Press:

- Edit
- Fill
- Fill Range

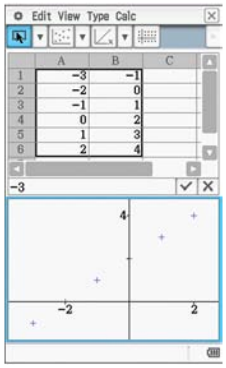


3. To join the points with a line, press:
   - MENU
   - 4: Analyze
   - 4: Plot Function.

   Complete the entry line as:
   $f1(x) = x + 2$

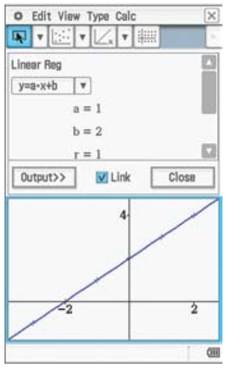


3. Highlight both columns and then press Graph. Select the scatterplot.



4. To determine the equation, press:
   - Calc
   - Regression
   - Linear Regression

   The equation is $y = x + 2$. The line will appear as continuous.



WORKED EXAMPLE 2 Drawing linear graphs by plotting two points

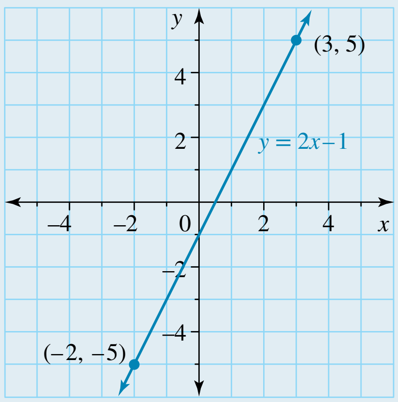
Plot two points and use them to draw the linear graph $y = 2x - 1$.

THINK

1. Choose any two x-values, for example $x = -2$ and $x = 3$.
2. Calculate y by substituting each x-value into $y = 2x - 1$.
   $y = 2 \times (-2) - 1 = -5$
   $x = 3: y = 2 \times 3 - 1 = 5$
3. Plot the points $(-2, -5)$ and $(3, 5)$.
4. Draw a line through the points and label the line.

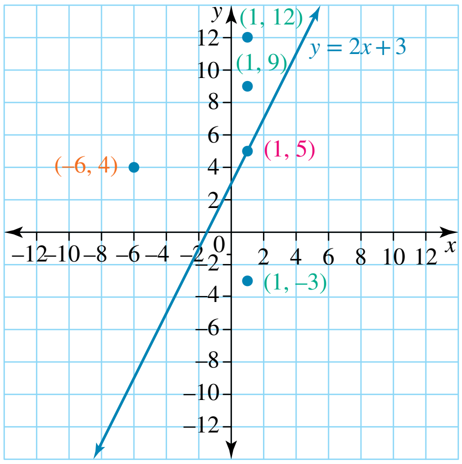
WRITE/DRAW

x	-2	3
y	-5	5



4.2.2 Points on a line

- Consider the line that has the rule $y = 2x + 3$ as shown in the graph.
  If $x = 1$, then $y = 2(1) + 3$
  $= 5$
  so the point $(1, 5)$ lies on the line $y = 2x + 3$.
- The points $(1, -3)$ $(1, 9)$, $(1, 12)$ … are not on the line, but lie above or below it.
- The point $(-6, 4)$ is in quadrant 2.
  If $x = -6$: $y = 2(-6) + 3$
  $y = -9$
  $y \neq 5$
  This shows that the point does not lie on the line.



WORKED EXAMPLE 3 Determining whether a point lies on a line

Determine whether the point (2, 4) lies on the line given by:

a. $y = 3x - 2$ **b.** $x + y = 5$.

THINK	WRITE
a. 1. Substitute $x = 2$ into the equation $y = 3x - 2$ and solve for y.	**a.** $y = 3x - 2$ $x = 2$: $y = 3(2) - 2$ $= 6 - 2$ $= 4$
2. When $x = 2$, $y = 4$, so the point $(2, 4)$ lies on the line. Write the answer.	The point $(2, 4)$ lies on the line $y = 3x - 2$.
b. 1. Substitute $x = 2$ into the equation $x + y = 5$ and solve for y.	**b.** $x + y = 5$ $x = 2$: $2 + y = 5$ $y = 3$
2. The point $(2, 3)$ lies on the line, but the point $(2, 4)$ does not. Write the answer.	The point $(2, 4)$ does not lie on the line $x + y = 5$.

DISCUSSION

In linear equations, what does the coefficient of x determine?

Exercise 4.2 Plotting linear graphs

4.2 Quick quiz on | 4.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 4, 9, 12	3, 5, 8, 11, 13	6, 7, 10, 14

Fluency

1. MC a. The point with coordinates $(-2, 3)$ is:
   - A. in quadrant 1
   - B. in quadrant 2
   - C. in quadrant 3
   - D. in quadrant 4

   b. The point with coordinates $(-1, -5)$ is:
   - A. in the first quadrant
   - B. in the second quadrant
   - C. in the third quadrant
   - D. in the fourth quadrant

   c. The point with coordinates $(0, -2)$ is:
   - A. in the third quadrant
   - B. in the fourth quadrant
   - C. on the x-axis
   - D. on the y-axis

2. WE1 For each of the following rules:
   i. complete the table
   ii. plot the linear graph.

x	-3	-2	-1	0	1	2	3
y							

   a. $y = x$
   b. $y = 2x + 2$
   c. $y = 3x - 1$
   d. $y = -2x$

Understanding

3. WE2 By first plotting two points, draw the linear graph given by each of the following.
   a. $y = -x$
   b. $y = \frac{1}{2}x + 4$
   c. $y = -2x + 3$
   d. $y = x - 3$

4. WE3 Determine whether these points lie on the graph of $y = 2x - 5$.
   a. $(3, 1)$
   b. $(-1, 3)$
   c. $(0, 5)$
   d. $(5, 5)$

5. Determine whether the given point lies on the given line.
   a. $y = -x - 7, (1, -8)$
   b. $y = 3x + 5, (0, 5)$
   c. $y = x + 6, (-1, 5)$
   d. $y = 5 - x, (8, 3)$
   e. $y = -2x + 11, (5, -1)$
   f. $y = x - 4, (-4, 0)$

6. MC Select the line that passes through the point $(2, -1)$.
   - A. $y = -2x + 5$
   - B. $y = 2x - 1$
   - C. $y = -2x + 1$
   - D. $x + y = 1$

7. Match each point with a line passing through that point.

Point	Line
a. $(1, 1)$	A. $x + y = 4$
b. $(1, 3)$	B. $2x - y = 1$
c. $(1, 6)$	C. $y = 3x - 7$
d. $(1, -4)$	D. $y = 7 - x$

Reasoning

8. The line through $(1, 3)$ and $(0, 4)$ passes through every quadrant except one. State the quadrant through which this line does not pass. Explain your answer.

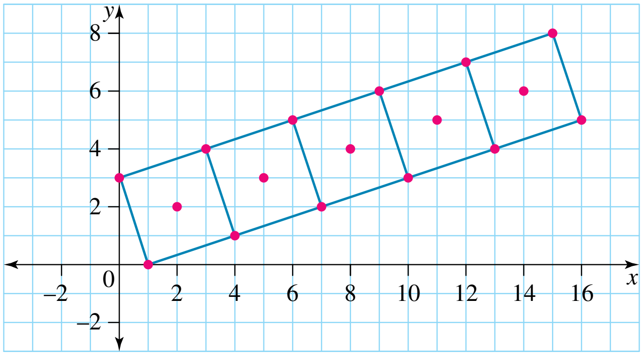
9. Answer the following questions.
   a. Determine which quadrant(s) the line $y = x + 1$ passes through.
   b. Show that the point $(1, 3)$ does not lie on the line $y = x + 1$.

10. Explain the process of how to check whether a point lies on a given line.

11. Using the coordinates $(-1, -3)$, $(0, -1)$ and $(2, 3)$, show that a rule for the linear graph is $y = 2x - 1$.

Problem solving

12. Consider the pattern of squares on the grid shown.
Determine the coordinates of the centre of the 20th square.

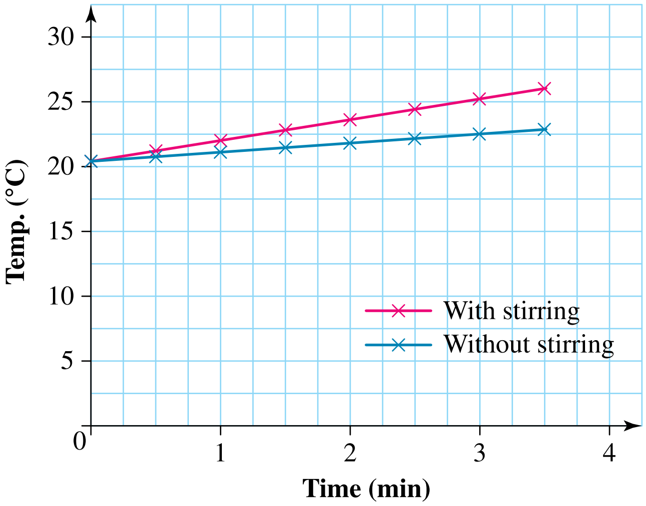


13. It is known that the mass of a certain kind of genetically modified tomato increases linearly over time. The following results were recorded.

Time, *t* (weeks)	1	4	6	9	16
Mass, *m* (grams)	6	21	31	46	81

   a. Plot the above points on a Cartesian plane.
   b. Determine the rule connecting mass with time.
   c. Show that the mass of one of these tomatoes is 101 grams after 20 weeks.

14. As a particular chemical reaction proceeds, the temperature increases at a constant rate. The graph represents the same chemical reaction with and without stirring. Interpret the graph and explain how stirring affects the reaction.



LESSON 4.3 Features of linear graphs

LEARNING INTENTIONS

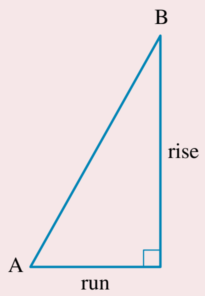
At the end of this lesson you should be able to:
- calculate the gradient of a line or line segment
- calculate the *y*-intercept of a linear graph.

4.3.1 The gradient (m)

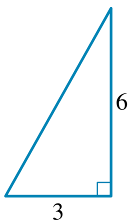
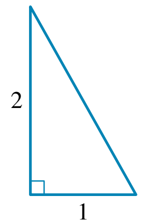
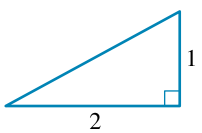
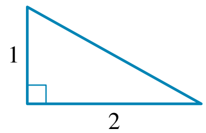
- The **gradient** of a line is a measure of the steepness of its slope.
- The symbol for the gradient is m.
- The gradient of an interval AB is defined as the rise (distance up) divided by the run (distance across).

Gradient formula

$$m = \frac{\text{rise}}{\text{run}}$$



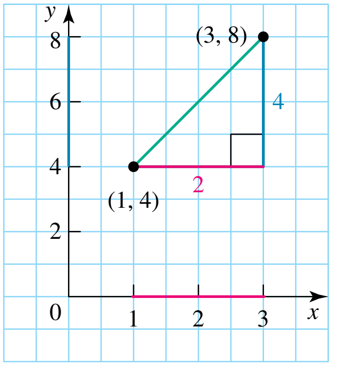
- **If the line is sloping upwards (from left to right), the gradient is positive.**
- **If the line is sloping downwards, the gradient is negative.**

Positive gradient (sloping upwards)	Negative gradient (sloping downwards)
 $m = 2$	 $m = -2$
 $m = \frac{1}{2}$	 $m = -\frac{1}{2}$

Determining the gradient of a line passing through two points

- Suppose a line passes through the points $(1, 4)$ and $(3, 8)$, as shown in the graph.
- By completing a right-angled triangle, it can be seen that:
  - the rise (difference in y-values): $8 - 4 = 4$
  - the run (difference in x-values): $3 - 1 = 2$
  - to determine the gradient:

$$\begin{aligned} m &= \frac{8-4}{3-1} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$



Gradient of a line

In general, if a line passes through the points (x_1, y_1) and (x_2, y_2), then:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

WORKED EXAMPLE 4 Determining the gradient of a line passing through two points

Calculate the gradient of the line passing through the points $(-2, 5)$ and $(1, 14)$.

THINK	WRITE
1. Let the two points be (x_1, y_1) and (x_2, y_2).	$(-2, 5) = (x_1, y_1), (1, 14) = (x_2, y_2)$
2. Write the formula for gradient.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
3. Substitute the coordinates of the given points into the formula and evaluate.	$m = \frac{14-5}{1--2}$ $= \frac{9}{1+2}$ $= \frac{9}{3}$ $= 3$
4. Write the answer.	The gradient of the line passing through $(-2, 5)$ and $(1, 14)$ is 3.

Note: If you were to switch the order of the points and let $(x_1, y_1) = (1, 14)$ and $(x_2, y_2) = (-2, 5)$, then the gradient could be calculated as shown and it can be seen that the result is the same.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 14}{-2 - 1} \\ &= \frac{-9}{-3} \\ &= 3 \end{aligned}$$

Rates of change

- A rate of change measures the change in the y-values with respect to the change in the x-values.
- The rate of change is equivalent to the gradient of the graph.

The gradient of horizontal and vertical lines

- The gradient of a horizontal line is zero, since it does not have any upwards or downwards slope.

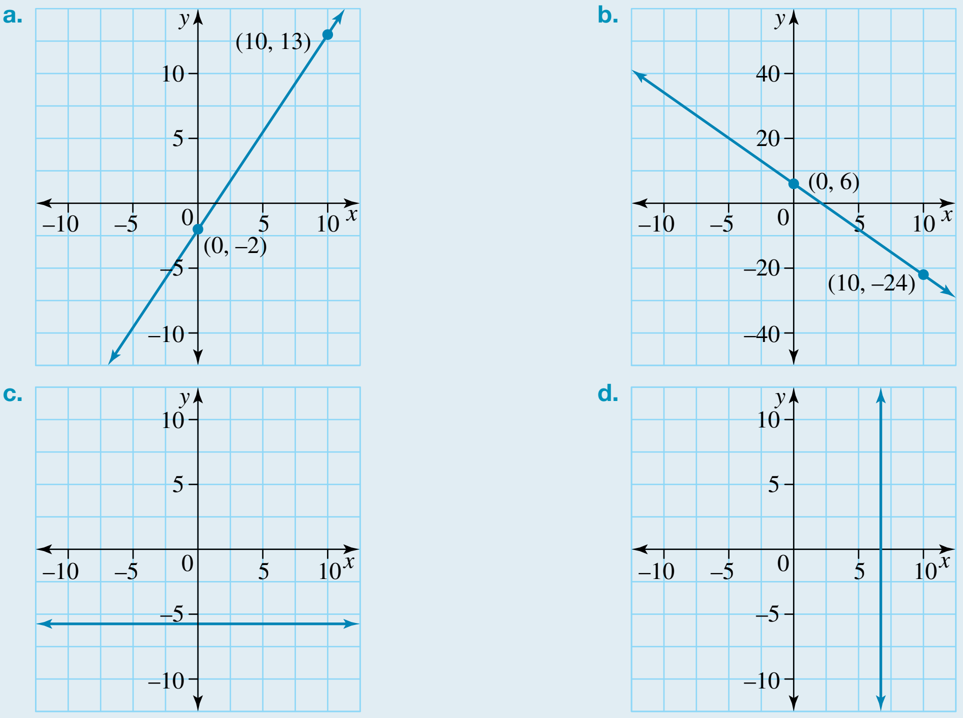
$$m = \frac{\text{rise}}{\text{run}} = \frac{0}{\text{run}} = 0$$

- The gradient of a vertical line is undefined, as the run is zero.

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{rise}}{0} = \text{undefined}$$

WORKED EXAMPLE 5 Calculating the gradient of a straight line

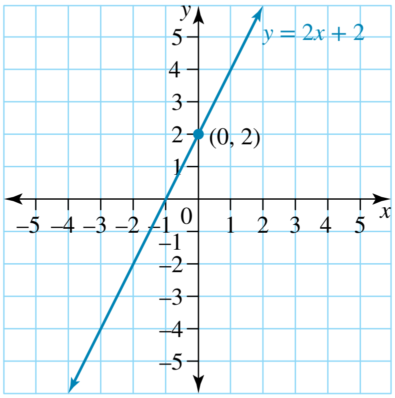
Calculate the gradients of the lines shown.



THINK	WRITE
a. 1. Write down two points that lie on the line.	**a.** Let $(x_1, y_1) = (0, -2)$ and $(x_2, y_2) = (10, 13)$. Rise $= y_2 - y_1 = 13 - -2 = 15$ Run $= x_2 - x_1 = 10 - 0 = 10$
2. Calculate the gradient by finding the ratio $\frac{\text{rise}}{\text{run}}$.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{15}{10}$ $= \frac{3}{2}$ or 1.5
b. 1. Write down two points that lie on the line.	**b.** Let $(x_1, y_1) = (0, 6)$ and $(x_2, y_2) = (10, -24)$. Rise $= y_2 - y_1 = -24 - 6 = -30$ Run $= x_2 - x_1 = 10 - 0 = 10$
2. Calculate the gradient.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{-30}{10}$ $= -3$
c. 1. Write down two points that lie on the line.	**c.** Let $(x_1, y_1) = (5, -6)$ and $(x_2, y_2) = (10, -6)$.
2. There is no rise between the two points.	Rise $= y_2 - y_1$ $= -6 - -6 = 0$ Run $= x_2 - x_1$ $= 10 - 5 = 5$
3. Calculate the gradient. *Note:* The gradient of a horizontal line is always zero. The line has no slope.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{0}{5}$ $= 0$
d. 1. Write down two points that lie on the line.	**d.** Let $(x_1, y_1) = (7, 10)$ and $(x_2, y_2) = (7, -3)$.
2. The vertical distance between the selected points is 13 units. There is no run between the two points.	Rise $= y_2 - y_1 = -3 - 10 = 13$ Run $= x_2 - x_1 = 7 - 7 = 0$
3. Calculate the gradient. *Note:* The gradient of a vertical line is always undefined.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{13}{0}$ $=$ undefined

4.3.2 The y-intercept

- The point where a line cuts the y-axis is called the **y-intercept**.
- In the graph shown the line cuts the y-axis at $y = 2$, so the y-intercept is $(0, 2)$.
- The y-intercept of any line is found by substituting 0 for x and calculating the y-value.



WORKED EXAMPLE 6 Determining the y-intercepts of lines

Determine the y-intercepts of the lines whose linear rules are given, and state the coordinates of the y-intercept.

a. $y = -4x + 7$ **b.** $5y + 2x = 10$ **c.** $y = 2x$ **d.** $y = -8$

THINK	WRITE
a. 1. To calculate the y-intercept, substitute $x = 0$ into the equation.	**a.** $y = -4x + 7$ $y = -4(0) + 7$
2. Solve for y.	$y = 7$
3. Write the coordinates of the y-intercept.	y-intercept: $(0, 7)$
b. 1. To calculate the y-intercept, substitute $x = 0$ into the equation.	**b.** $5y + 2x = 10$ $5y + 2(0) = 10$
2. Solve for y.	$5y = 10$ $y = 2$
3. Write the coordinates of the y-intercept.	y-intercept: $(0, 2)$
c. 1. To calculate the y-intercept, substitute $x = 0$ into the equation.	**c.** $y = 2x$ $y = 2(0)$ $y = 0$
2. Write the coordinates of the y-intercept.	y-intercept: $(0, 0)$
d. The value of y is -8 regardless of the x-value.	**d.** y-intercept: $(0, -8)$

DISCUSSION

Why is the y-intercept of a graph found by substituting $x = 0$ into the equation?

Exercise 4.3 Features of linear graphs

4.3 Quick quiz | 4.3 Exercise

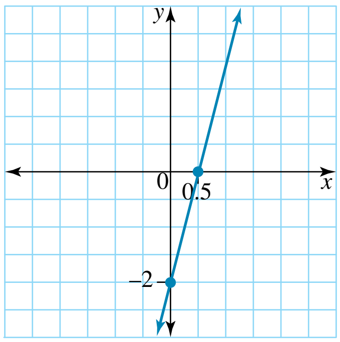
Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 8, 11, 14, 16	2, 4, 6, 9, 12, 15, 17, 18	7, 10, 13, 19, 20

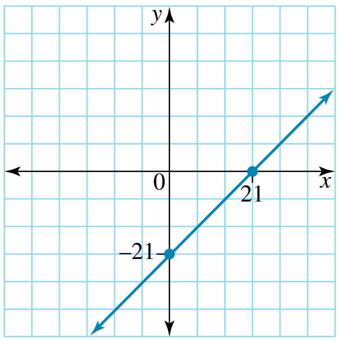
Fluency

1. WE4 Calculate the gradients of the lines passing through the following pairs of points.
   - a. $(2, 10)$ and $(4, 22)$
   - b. $(1, -2)$ and $(3, -10)$
   - c. $(-3, 0)$ and $(7, 0)$
   - d. $(-4, -7)$ and $(1, -1)$
   - e. $(0, 4)$ and $(4, -4.8)$
   - f. $(-2, 122)$ and $(1, -13)$
2. Calculate the gradients of the lines passing through the following pairs of points.
   - a. $(2, 3)$ and $(17, 3)$
   - b. $(-2, 2)$ and $(2, 2.4)$
   - c. $(1, -5)$ and $(5, -15.4)$
   - d. $(-12, -7)$ and $(8.4, -7)$
   - e. $(-2, -17.7)$ and $(0, 0.3)$
   - f. $(-3, 3.4)$ and $(5, 2.6)$
3. WE6 Determine the y-intercepts of the lines whose rules are given below.
   - a. $y = 5x + 23$
   - b. $y = 54 - 3x$
   - c. $y = 3(x - 2)$
   - d. $y = 70 - 2x$
4. Determine the y-intercepts of the lines whose rules are given below.
   - a. $y = \frac{1}{2}(x + 2)$
   - b. $y = \frac{x}{2} + 5.2$
   - c. $y = 100 - x$
   - d. $y = 100$
5. WE5 Calculate the gradients of the lines shown.

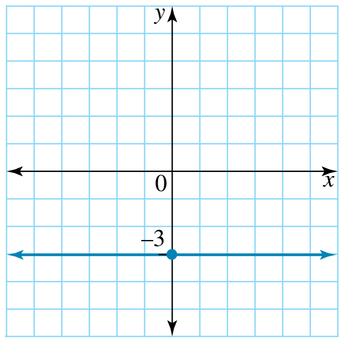
a.



b.

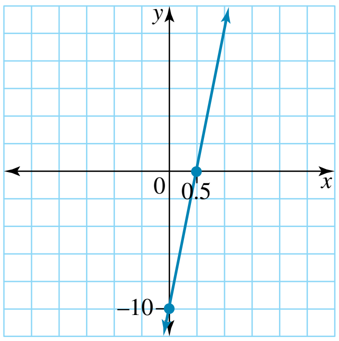


c.

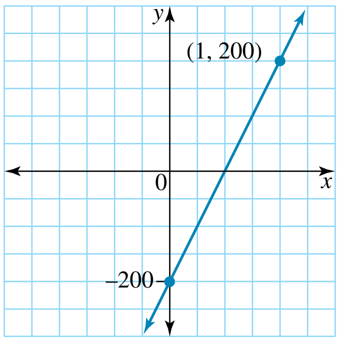


6. Calculate the gradients of the lines shown.

a.



b.



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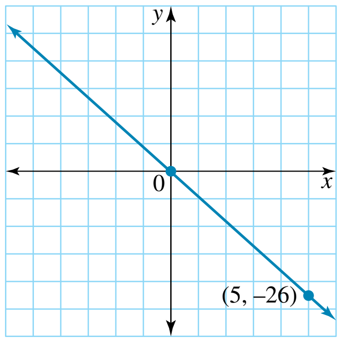
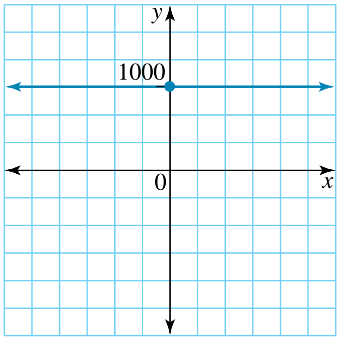
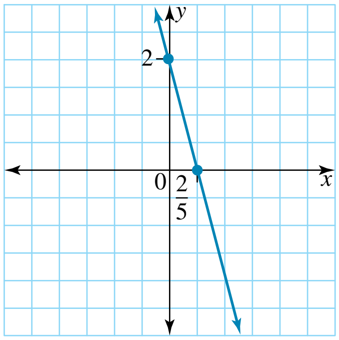


7. Calculate the gradients of the lines shown.

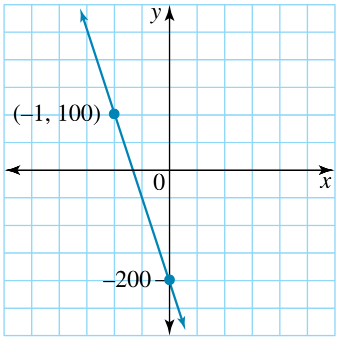
a.



b.



c.



Understanding

8. **MC** Select which of the following statements about linear graphs is false.
   - A. A gradient of zero means the graph is a horizontal line.
   - B. A gradient can be any real number.
   - C. A linear graph can have two y-intercepts.
   - D. In the form $y = mx + c$, the y-intercept equals c.

9. Identify the y-intercept of the line $y = mx + c$.

10. Determine the coordinates of the y-intercepts of the lines with the following rules.
    - a. $y = -6x - 10$
    - b. $3y + 3x = -12$
    - c. $7x - 5y + 15 = 0$
    - d. $y = 7$
    - e. $x = 9$

Reasoning

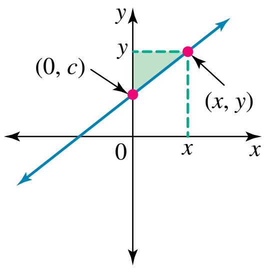
11. Explain why the gradient of a vertical line is undefined.

12. Explain why the gradient of a horizontal line is zero.

13. Show the gradient of the line passing through the points (a, b) and (c, d) is $\dfrac{d-b}{c-a}$.

14. When using the gradient to draw a line, does it matter if you rise before you run or run before you rise? Explain.

15. Consider the graph shown.
    - a. Determine a general formula for the gradient m in terms of x, y and c.
    - b. Transpose your formula to make y the subject.
      Discuss what you notice about this equation.

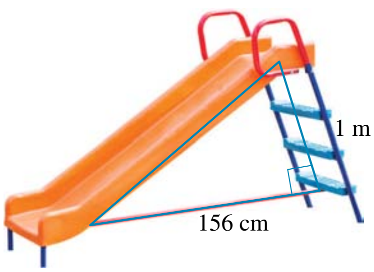


Problem solving

16. Calculate the gradient, correct to 2 decimal places, of the playground equipment in the diagrams given below.

a.

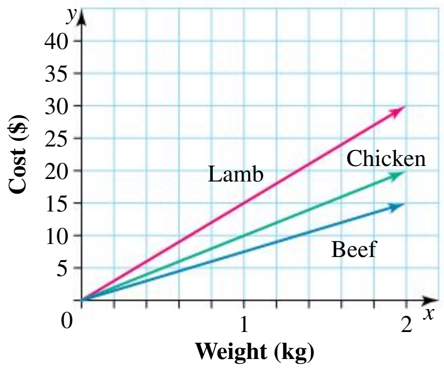
b.



17. Safety standards for escalators outline that the gradient of an escalator must not exceed 0.58. Determine if an escalator with a height of 18 metres that has a horizontal distance of 30 metres meets the required safety standards.

18. The maximum gradient for wheelchair ramps is $\frac{1}{14}$. If a ramp is being built across a horizontal distance of 4.2 metres determine the maximum height of the ramp.

19. The price per kilogram for three different types of meat is illustrated in the graph shown.



a. Calculate the gradient (using units) for each graph.
b. Determine the cost of 1 kg of each type of meat.
c. Evaluate the cost of purchasing:

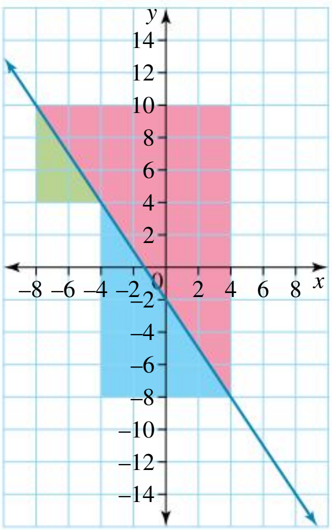
i. 1 kg of lamb **ii.** 0.5 kg of chicken **iii.** 2 kg of beef.

d. Calculate the total cost of the order in part **c**.
e. Complete the table below to confirm your answer from part **d**.

Meat type	Cost per kilogram (\$/kg)	Weight required (kg)	Cost = \$/kg × kg
Lamb		1	
Chicken		0.5	
Beef		2	
		Total cost	

20. Three right-angled triangles have been superimposed on the graph shown.

a. Use each of these to determine the gradient of the line.
b. Does it matter which points are chosen to determine the gradient of a line? Explain your answer.
c. Describe the shape of the graph.



LESSON 4.4 The equation of a straight line

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- determine the equation of a straight line given the gradient and the y-intercept
- determine the equation of a straight line given the gradient and one point
- determine the equation of a straight line given two points
- determine the equation of a straight line from a graph.

4.4.1 The general equation of a straight line

- The general equation of a straight line is

$$y = mx + c$$

where m is the gradient and c is the y-intercept of the line.
- Once the gradient and y-intercept are known the equation of the straight line can be stated.

WORKED EXAMPLE 7 Determining the equation from the gradient and y-intercept

Determine the equation of the line with a gradient of −2 and y-intercept of 3.

THINK	WRITE
1. Write the equation of a straight line.	$y = mx + c$
2. Substitute the values $m = -2$, $c = 3$ to write the equation.	$m = -2,\ c = 3$ $y = -2x + 3$

4.4.2 Determining the equation of a straight line

- If the gradient (m) and any single point on a straight line are known, then the y-intercept can be calculated algebraically.

WORKED EXAMPLE 8 Determining the equation of a line from the gradient and a point

Determine the equation of a straight line that goes through the point (1, − 3), if its gradient is −2.

THINK	WRITE
1. Write the equation of a straight line.	$y = mx + c$
2. Substitute the value $m = -2$.	$y = -2x + c$
3. Since the line passes through the point (1, −3), substitute $x = 1$ and $y = -3$ into $y = -2x + c$ to calculate the value of c.	When $x = 1$, $y = -3$. $-3 = -2 \times 1 + c$
4. Solve for c.	$-3 = -2 + c$ $-3 + 2 = c$ $c = -1$
5. Write the rule.	$y = -2x - 1$

- If two points on a straight line are known, then the gradient (m) can be calculated using the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, or $m = \dfrac{\text{rise}}{\text{run}}$.

WORKED EXAMPLE 9 Determining the equation of a line from two points

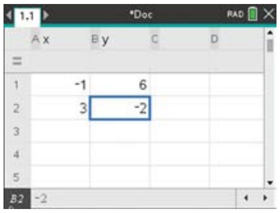
Determine the equation of the straight line passing through the points (−1, 6) and (3, −2).

THINK	WRITE
1. Write the equation of a straight line.	$y = mx + c$
2. Write the formula for calculating the gradient, m.	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$
3. Let $(x_1, y_1) = (-1, 6)$ and $(x_2, x_2) = (3, -2)$. Substitute the values into the formula and determine the value of m.	$m = \dfrac{-2-6}{3--1}$ $= \dfrac{-2-6}{3+1}$ $= \dfrac{-8}{4}$ $= -2$
4. Substitute the value of m into the general equation $y = mx + c$.	$y = -2x + c$
5. Select either of the two points, say $(3, -2)$, and substitute into $y = -2x + c$.	Point $(3, -2)$: $-2 = -2(3) + c$ $-2 = -6 + c$
6. Solve for c.	$c = 4$
7. Write the rule using the values $c = 4, m = -2$.	$y = -2x + 4$

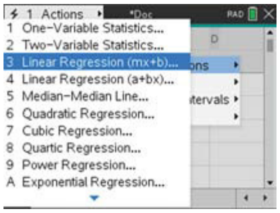
TI | THINK

1. On a Lists & Spreadsheet page, put your x-values of your coordinates in column A. Put your y-values in column B.

WRITE/DISPLAY



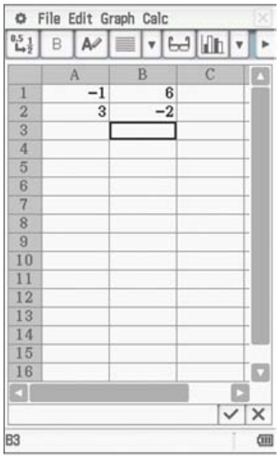
2. Press
- Menu
- 4: Statistics
- 1: Stat Calculations
- 3: Linear Regression ($mx + b$)



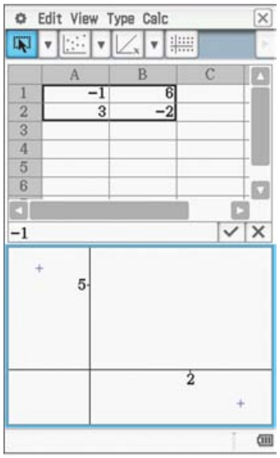
CASIO | THINK

1. On the Spreadsheet screen, put your x-values of your coordinates in column A. Put your y-values in column B.

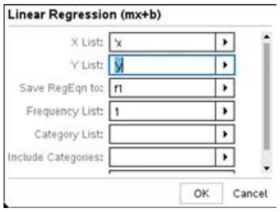
WRITE/DISPLAY



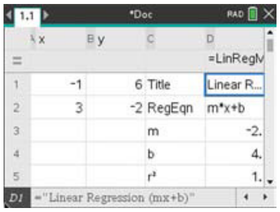
2. Highlight the data in both columns.
Press Graph and select the scatterplot.
The points will be plotted.



3. For X list, choose column A.
For Y list, choose column B.
And Press OK

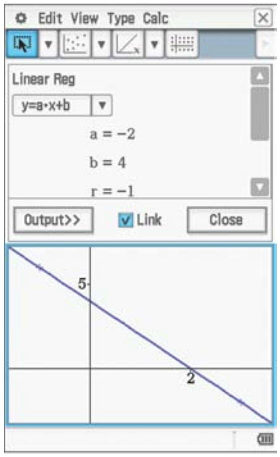


4. The values for the equation will appear on screen.
Note: The calculator uses $y = mx + b$ instead of $y = mx + c$.



$y = -2x + 4$

3. To get the equation of the line, press the Line icon. The equation will appear above the graph, and the points will be joined by a line.

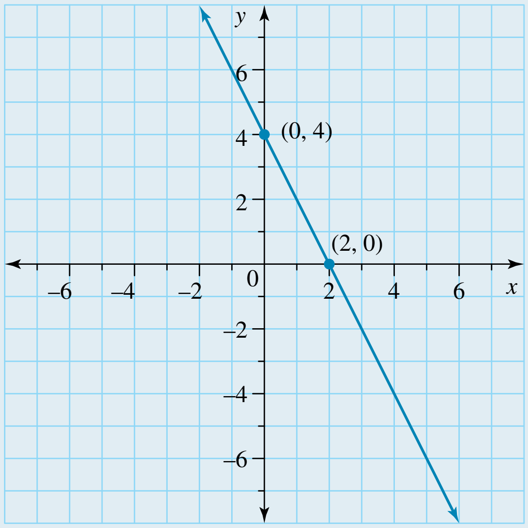


The equation is $y = -2x + 4$.

- The equation of a straight line can be determined from a graph by observing the y-intercept and calculating the gradient.

WORKED EXAMPLE 10 Determining the equation of a line from its graph

Determine the equation of the linear graph shown.



THINK

1. Read the important information from the graph. The y-intercept is 4 and the x-intercept is 2. Write the coordinates of each point.

WRITE

The graph passes through $(0, 4)$ and $(2, 0)$.

2. Calculate the gradient using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Let $(x_1, y_1) = (2,\ 0)$ and $(x_2, y_2) = (0,\ 4)$.	$m = \frac{4-0}{0-2}$ $= \frac{4}{-2}$ $= -2$ $y = -2x + c$
3. From the graph, the y-intercept is 4.	$c = 4$
4. Write the equation.	$y = -2x + 4$

Exercise 4.4 The equation of a straight line

learnon

4.4 Quick quiz on | 4.4 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 10, 15, 16, 19, 22, 25	2, 4, 6, 8, 11, 13, 17, 20, 23, 26, 27	7, 9, 12, 14, 18, 21, 24, 28, 29

Fluency

1. WE7 Determine the equations of the straight lines with the gradients and y-intercepts given.

a. Gradient $= 4$, y-intercept $= 2$
b. Gradient $= -4$, y-intercept $= 1$
c. Gradient $= 4$, y-intercept $= 8$

2. Determine the equations of the straight lines with the gradients and y-intercepts given.

a. Gradient $= 6$, y-intercept $= 7$
b. Gradient $= -2.5$, y-intercept $= 6$
c. Gradient $= 45$, y-intercept $= 135$

3. Determine the equation for each straight line passing through the origin and with the gradient given.

a. Gradient $= -2$ **b.** Gradient $= 4$ **c.** Gradient $= 10.5$

4. Determine the equation for each straight line passing through the origin and with the gradient.

a. Gradient $= -20$ **b.** Gradient $= 1.07$ **c.** Gradient $= 32$

5. WE8 Determine the equation of the straight lines with:

a. gradient $= 1$, point $= (3,\ 5)$
b. gradient $= -1$, point $= (3,\ 5)$
c. gradient $= -4$, point $= (-3,\ 4)$
d. gradient $= 2$, point $= (5, -3)$.

6. Determine the equation of the straight lines with:

a. gradient $= -5$, point $= (13,\ 5)$
b. gradient $= 2$, point $= (10, -3)$
c. gradient $= -6$, point $= (2, -1)$
d. gradient $= -1$, point $= (-2,\ 0.5)$.

7. Determine the equation of the straight lines with:

a. gradient $= 6$, point $= (-6, -6)$
b. gradient $= -3.5$, point $= (3,\ 5)$
c. gradient $= 1.2$, point $= (2.4, -1.2)$
d. gradient $= 0.2$, point $= (1.3, -1.5)$.

8. Determine the equations of the straight lines with:

a. gradient $= -4$, x-intercept $= -6$
b. gradient $= 2$, x-intercept $= 3$
c. gradient $= -2$, x-intercept $= 2$.

9. Determine the equation of the straight lines with:

a. gradient $= 5$, x-intercept $= -7$
b. gradient $= 1.5$, x-intercept $= 2.5$
c. gradient $= 0.4$, x-intercept $= 2.4$.

10. **WE9** Determine the equation for each straight line passing through the given points.

a. $(-6, 11)$ and $(6, 23)$
b. $(1, 2)$ and $(-5, 8)$
c. $(4, 11)$ and $(6, 11)$
d. $(3, 6.5)$ and $(6.5, 10)$

11. Determine the equation for each straight line passing through the given points.

a. $(1.5, 2)$ and $(6, -2.5)$
b. $(-7, 3)$ and $(2, 4)$
c. $(25, -60)$ and $(10, 30)$
d. $(5, 100)$ and $(25, 500)$

12. Determine the equation for each straight lines passing through the given points.

a. $(1, 3)$ and $(3, 1)$
b. $(2, 5)$ and $(-2, 6)$
c. $(9, -2)$ and $(2, -4)$
d. $(1, 4)$ and $(-0.5, 3)$

13. Determine the rules for the linear graphs that have the following x- and y-intercepts.

a. x-intercept $= -3$, y-intercept $= 3$
b. x-intercept $= 4$, y-intercept $= 5$
c. x-intercept $= 1$, y-intercept $= 6$
d. x-intercept $= -40$, y-intercept $= 35$

14. Determine the rules for the linear graphs that have the following x- and y-intercepts.

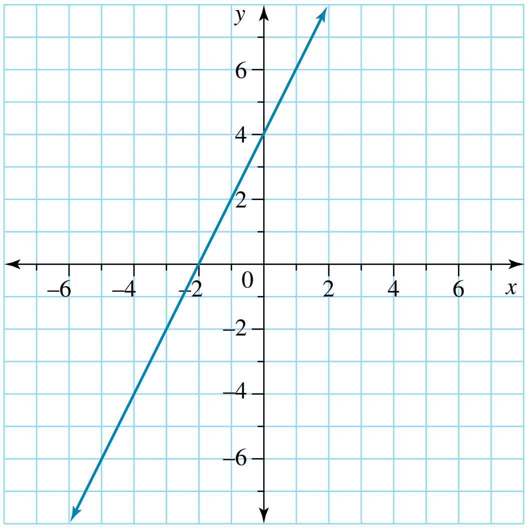
a. x-intercept $= -8$, y-intercept $= 8$
b. x-intercept $= 3$, y-intercept $= 6$
c. x-intercept $= -7$, y-intercept $= -3$
d. x-intercept $= -200$, y-intercept $= 50$

15. Determine the rule for each straight line passing through the origin and:

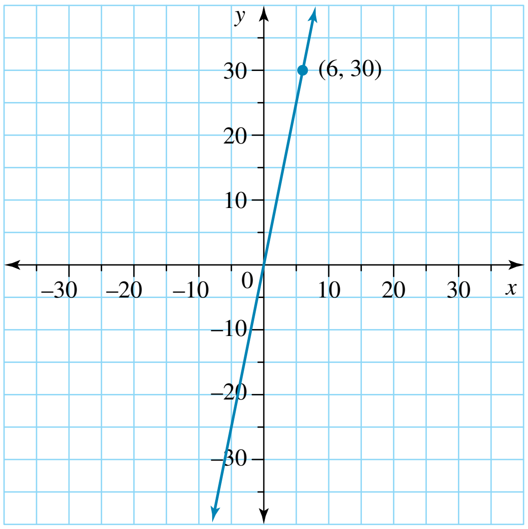
a. the point $(4, 7)$
b. the point $(5, 5)$
c. the point $(-4, 8)$
d. the point $(-1.2, 3.6)$
e. the point $(-22, 48)$
f. the point $(-105, 35)$.

16. **WE10** Determine the equation of the line shown on each of the following graphs.

a.

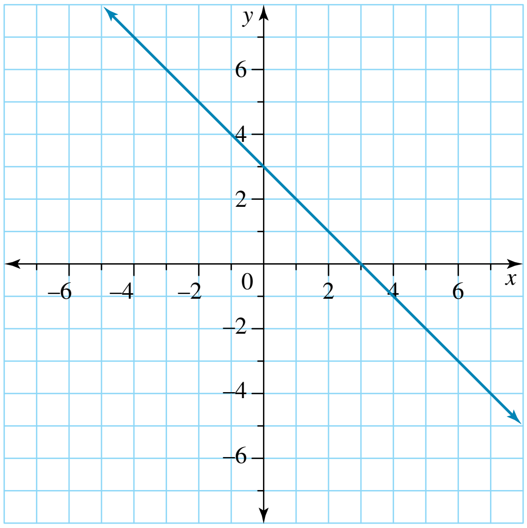


b.

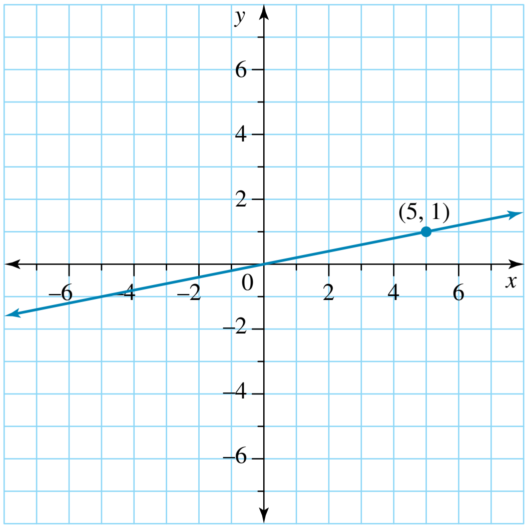


17. Determine the equation of the line shown on each of the following graphs.

a.

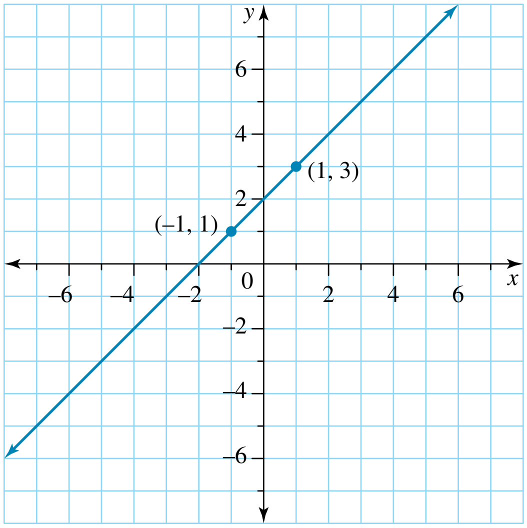


b.

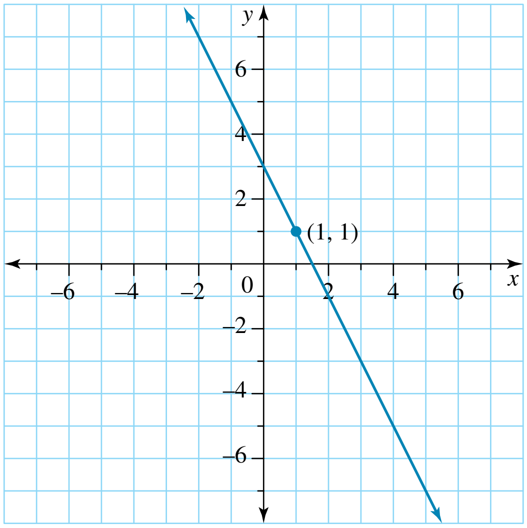


18. Determine the equation of the line shown on each of the following graphs.

a.



b.



Understanding

19. MC a. The gradient of the straight line that passes through (3, 5) and (5, 3) is:

A. -2 B. -1 C. 0 D. 1

b. A straight line with an x-intercept of 10 and a y-intercept of 20 has a gradient of:

A. -2 B. -1 C. -0.5 D. 0

c. The rule $2y - 3x = 20$ has an x-intercept at:

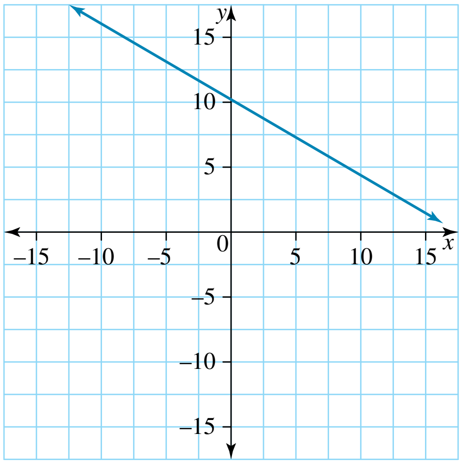
A. $-\frac{3}{2}$ B. $-\frac{2}{3}$ C. 0 D. None of these

20. Given that the x-intercept of a straight line graph is $(-5, 0)$ and the y-intercept is $(0, -12)$:

a. determine the equation of the straight line

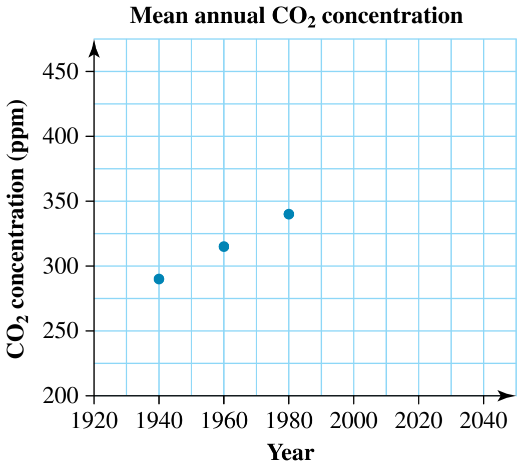
b. calculate the value of y when $x = 19.3$.

21. a. Determine the equation of the straight line shown in the graph. Use the fact that when $x = 5, y = 7$.
b. Determine the x-intercept.



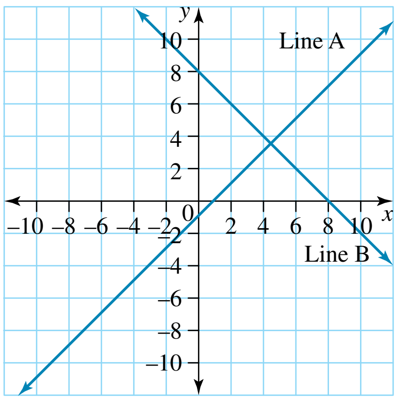
Reasoning

22. The graph shows the carbon dioxide (CO_2) concentration in the atmosphere, measured in parts per million (ppm).
a. If the trend follows a linear pattern, determine the equation for the line.
b. Explain why c cannot be read directly from the graph.
c. Infer the CO_2 concentration predicted for 2040. State the assumption made when determining this value.



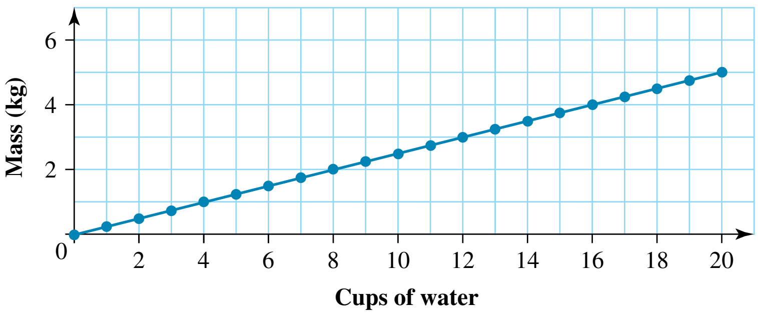
23. Show that the equation for the line that passes through the point (3, 6) parallel to the line through the points $(0, -7)$ and $(4, -15)$ is $y = -2x + 12$.

24. a. Determine the equations for line A and line B as shown in this graph.
b. Write the point of intersection between line A and line B and mark it on the Cartesian plane.
c. Show that the equation of the line that is perpendicular to line B and passes through the point $(-4, 6)$ is $y = x + 10$.

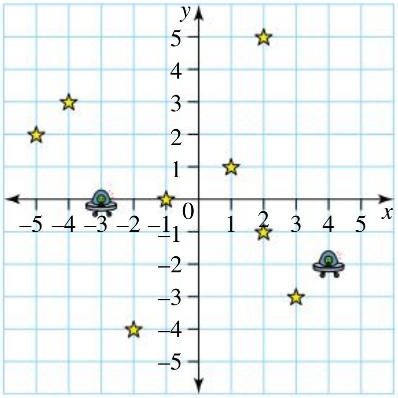


Problem solving

25. The graph shown describes the mass in kilograms of metric cups of water. Write a rule to describe the mass of water (m) relative to the number of cups (c).



26. Harpinder plays the game *Space Galaxy* on her phone. The stars and spaceships are displayed on her screen as shown.

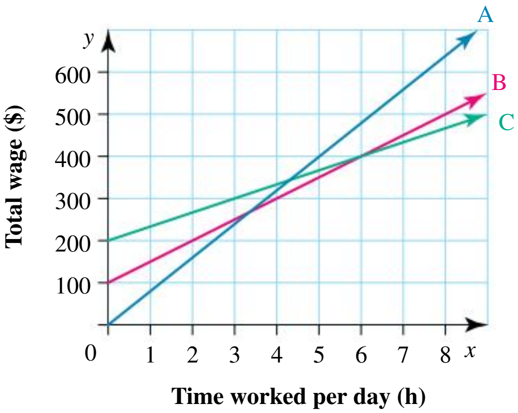


a. Copy the diagram. On the diagram, draw a straight line that will hit three stars.
b. Determine the equation of the straight line that will hit three stars.
c. Harpinder types in the equation $y = \frac{1}{2}x + \frac{1}{2}$ and manages to hit two stars. Draw the straight line on your diagram.
d. If Harpinder types in the equation from part **b** and the equation from part **c**, determine the coordinate of the star that both lines will hit.
e. If she types $y = 2$, state how many stars will she hit.
f. Give another equation of a straight line that will hit two stars.

27. The temperature of water in a kettle is 15 °C before the temperature increases at a constant rate for 20 seconds to reach boiling point (100 °C). Adel argues that $T = 5t + 15$ describes the water temperature, citing the starting temperature of 15 °C and that to reach 100 °C in 20 seconds an increase of 5 °C for every second is required.
Explain why Adel's equation is incorrect and devise another equation that correctly describes the temperature of the water.

28. A father wants to administer a children's liquid painkiller to his child. The recommended dosage is a range, 7.5 − 9 mL for an average weight of 12 − 14 kg. The child weighs 12.8 kg. The father uses a linear relationship to calculate an exact dosage.
Evaluate the dosage that the father calculates.

29. The graph shown displays the wages earned in three different workplaces.



a. Identify the set allowance for each workplace.
b. Determine the hourly rates for each workplace.
c. Using your answers from parts **a** and **b**, determine linear equations that describe the wages at each workplace.
d. Match each working lifestyle below to the most appropriate workplace.
   i. Working lifestyle 1: Earn the most money possible while working at most 4 hours in a day.
   ii. Working lifestyle 2: Earn the most money possible while working an 8-hour day.
e. If a person works an average of 8 hours a day, determine the advantage of workplace C.
f. Calculate how much money is earned at each workplace for a:
   i. 2-hour day
   ii. 6-hour day.

LESSON 4.5 Sketching linear graphs

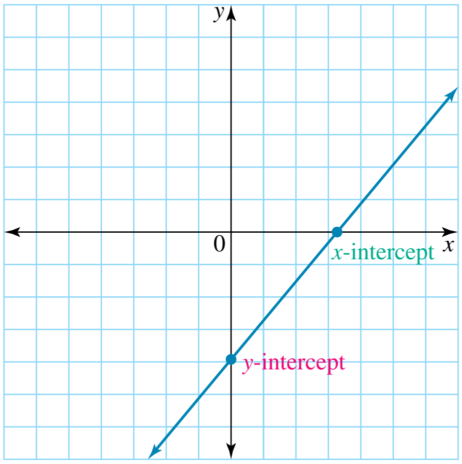
LEARNING INTENTIONS

At the end of this lesson you should be able to:

- sketch linear graphs using the x- and y-intercept method
- sketch linear graphs using the gradient–intercept method
- sketch linear graphs represented in both $y = mx + c$ and $ax + by = c$ form
- sketch vertical and horizontal lines.

4.5.1 The x- and y-intercept method

- To sketch a linear graph all you need is two points on the line. Once you have found two points, the line can be drawn through those points.
- One method to sketch a linear graph is to find both the ***x*-intercept** and the ***y*-intercept**.
- To determine the intercepts of a graph:
  - x-intercept: let $y = 0$ and solve for x
  - y-intercept: let $x = 0$ and solve for y.
- If both intercepts are zero (at the origin), another point is needed to sketch the line.



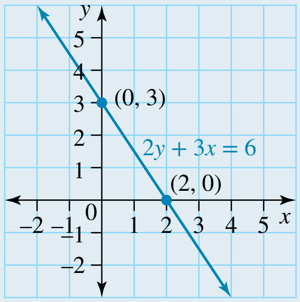
WORKED EXAMPLE 11 Sketching graphs using the x- and y-intercept method

Using the x- and y-intercept method, sketch the graphs of:

a. $2y + 3x = 6$ **b.** $y = \frac{4}{5}x + 5$ **c.** $y = 2x$.

THINK	WRITE/DRAW
a. 1. Write the rule.	**a.** $2y + 3x = 6$
2. To calculate the y-intercept, let $x = 0$. Write the coordinates of the y-intercept.	$x = 0$: $2y + 3 \times 0 = 6$ $2y = 6$ $y = 3$ y-intercept: $(0, 3)$
3. To calculate the x-intercept, let $y = 0$. Write the coordinates of the x-intercept.	$y = 0$: $2 \times 0 + 3x = 6$ $3x = 6$ $x = 2$ x-intercept: $(2, 0)$

4. Plot and label the x- and y-intercepts on a set of axes and rule a straight line through them. Label the graph.



b. 1. Write the rule.

b. $y = \frac{4}{5}x + 5$

2. The rule is in the form $y = mx + c$, so the y-intercept is the value of c.

$c = 5$

y-intercept: $(0, 5)$

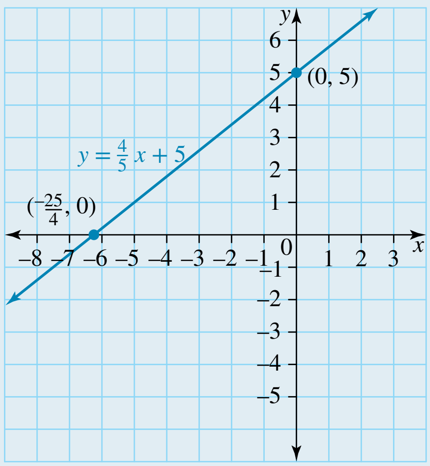
3. To calculate the x-intercept, let $y = 0$. Write the coordinates of the x-intercept.

$y = 0$:

$$\begin{aligned} y &= \frac{4}{5}x + 5 \\ 0 &= \frac{4}{5}x + 5 \\ -5 &= \frac{4}{5}x \\ x &= -\frac{25}{4}\left(= -6\frac{1}{4}\right) \end{aligned}$$

x-intercept: $\left(-\frac{25}{4}, 0\right)$

4. Plot and label the intercepts on a set of axes and rule a straight line through them. Label the graph.



c. 1. Write the rule.

c. $y = 2x$

2. To calculate the y-intercept, let $x = 0$.
Write the coordinates of the y-intercept.

$x = 0$: $\quad y = 2 \times 0$
$\quad\quad\quad\;\; = 0$

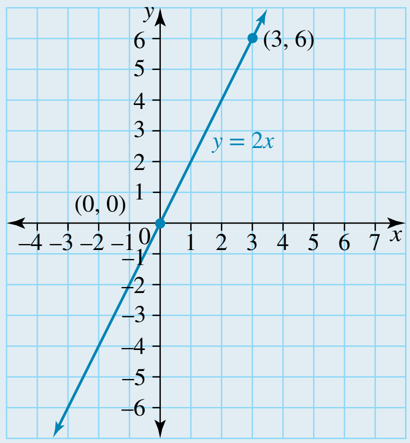
y-intercept: $(0, 0)$

3. The x- and y-intercepts are the same point, $(0, 0)$, so one more point is required.
Choose any value for x, such as $x = 3$.
Substitute and write the coordinates of the point.

$x = 3$: $\quad y = 2 \times 3$
$\quad\quad\quad\;\; = 6$

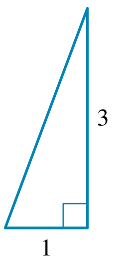
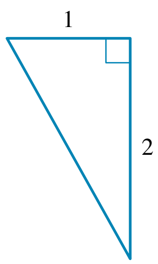
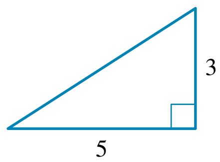
Another point: $(3, 6)$

4. Plot the points, then rule and label the graph.



4.5.2 The gradient–intercept method

- To use this method, the gradient and the y-intercept must be known.
- The line is drawn by plotting the y-intercept, then drawing a line with the correct gradient through that point.

A line interval of gradient $3\left(=\frac{3}{1}\right)$ can be drawn with a rise of 3 and a run of 1.	A line interval of gradient $-2\left(=\frac{-2}{1}\right)$ can be drawn with a rise downwards of 2 and a run of 1.	A line interval of gradient $\frac{3}{5}$ can be shown with a rise of 3 and a run of 5.
		

WORKED EXAMPLE 12 Sketching graphs using the gradient–intercept method

Using the gradient–intercept method, sketch the graphs of:

a. $y = \frac{3}{4}x + 2$

b. $4x + 2y = 3$.

THINK

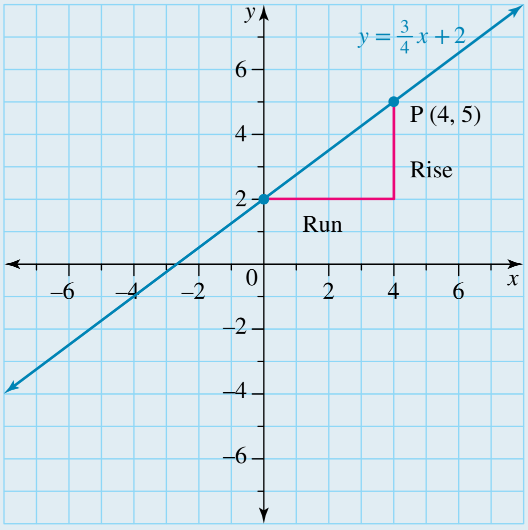
a. 1. From the equation, the y-intercept is 2. Plot the point $(0, 2)$.

2. From the equation, the gradient is $\frac{3}{4}$, so $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$.
From $(0, 2)$, run 4 units and rise 3 units. Mark the point P $(4, 5)$.

3. Draw a line through $(0, 2)$ and P$(4, 5)$. Label the graph.

WRITE

a.

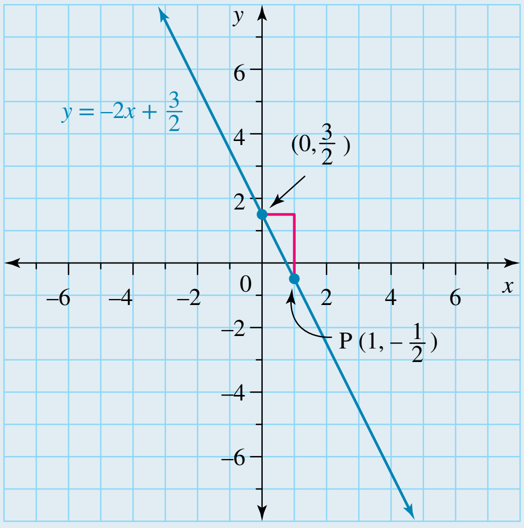


b. 1. Write the rule in gradient–intercept form: $y = mx + c$.
From the equation, $m = -2, c = \frac{3}{2}$.
Plot the point $\left(0, \frac{3}{2}\right)$.

b. $4x + 2y = 3$
$2y = 3 - 4x$
$y = \frac{3}{2} - 2x$
$y = -2x + \frac{3}{2}$

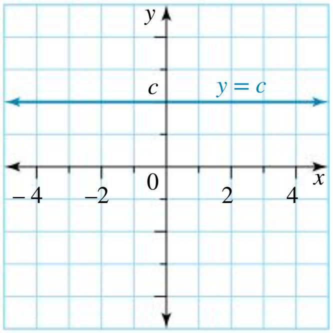
2. The gradient is -2, so $\frac{\text{rise}}{\text{run}} = \frac{-2}{1}$.
From $\left(0, \frac{3}{2}\right)$, run 1 units and rise -2 units (i.e. go down 2 units). Mark the point P $\left(1, -\frac{1}{2}\right)$.

3. Draw a line through $\left(0, \frac{3}{2}\right)$ and P $\left(1, -\frac{1}{2}\right)$.
Label the graph.



4.5.3 Horizontal and vertical lines

Horizontal lines ($y = c$)

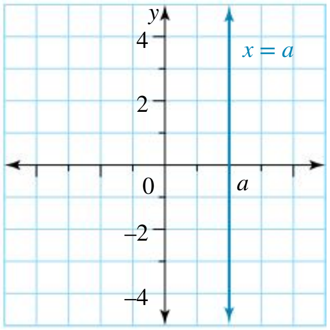


- Horizontal lines are expressed in the form $y = c$, where c is the y-intercept.
- In horizontal lines the y-value remains the same regardless of the x-value.
- This can be seen by looking at a table of values, like the one shown.

x	-2	0	2	4
y	c	c	c	c

- Plotting these points gives a horizontal line, as shown in the graph.
- Horizontal lines have a gradient of 0. They do not rise or fall.

Vertical lines ($x = a$)



- Vertical lines are expressed in the form $x = a$, where a is the x-intercept.
- In vertical lines the x-value remains the same regardless of the y-value.
- This can be seen by looking at a table of values, like the one shown.

x	a	a	a	a
y	-2	0	2	4

- Plotting these points gives a vertical line, as shown in the graph.
- The run of the graph is 0, so using the formula $m = \frac{\text{rise}}{\text{run}}$ involves dividing by zero, which cannot be done.
  The gradient is said to be **undefined**.

WORKED EXAMPLE 13 Sketching horizontal and vertical lines

On a pair of axes, sketch the graphs of the following and label the point of intersection of the two lines.

a. $x = -3$ **b.** $y = 4$

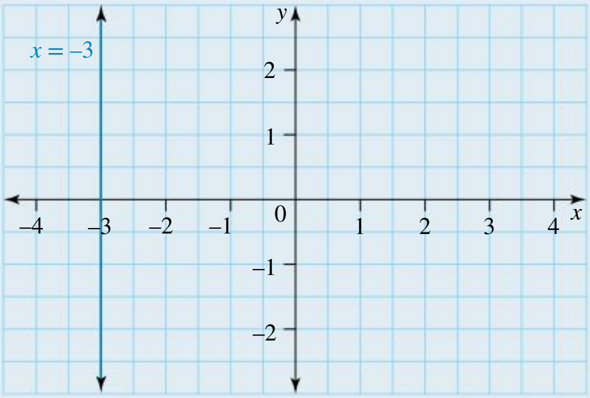
THINK

a. 1. The line $x = -3$ is in the form $x = a$.
This is a vertical line.

2. Rule the vertical line where $x = -3$.
Label the graph.

WRITE/DRAW

a.

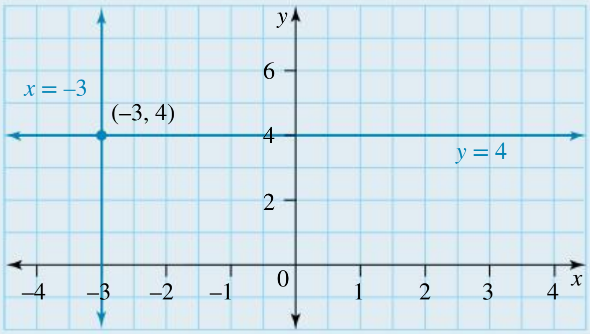


b. **1.** The line $y = 4$ is in the form $y = c$. This is a horizontal line.

2. Rule the horizontal line where $y = 4$.
Label the graph.

3. The lines intersect at $(-3, 4)$.

b.



Exercise 4.5 Sketching linear graphs

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4.5 Quick quiz on

4.5 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 7, 9, 11, 15, 20	4, 8, 12, 13, 16, 17, 21	2, 6, 10, 14, 18, 19, 22

Fluency

1. **WE11** Using the x- and y-intercept method, sketch the graphs of:

a. $5y - 4x = 20$
b. $y = x + 2$
c. $y = -3x + 6.$

2. Using the x- and y-intercept method, sketch the graphs of:

a. $3y + 4x = -12$
b. $x - y = 5$
c. $2y + 7x - 8 = 0.$

3. **WE12** Using the gradient–intercept method, sketch the graphs of:

a. $y = x - 7$
b. $y = 2x + 2$
c. $y = -2x + 2.$

4. Using the gradient–intercept method, sketch the graphs of:

a. $y = \frac{1}{2}x - 1$
b. $y = 4 - x$
c. $y = -x - 10.$

5. **WE13** On a pair of axes, sketch the graphs of the following.

a. $y = 4$
b. $y = -3$

6. On a pair of axes, sketch the graphs of the following.

a. $y = -12.5$

b. $y = \frac{4}{5}$

7. Sketch the graphs of the following.

a. $x = 2$

b. $x = -6$

8. Sketch the graphs of the following.

a. $x = -2.5$

b. $x = \frac{3}{4}$

9. Sketch the graphs of the following.

a. $y = 3x$

b. $y = -2x$

10. Sketch the graphs of the following.

a. $y = \frac{3}{4}x$

b. $y = -\frac{1}{3}x$

Understanding

11. **MC** Select which of the following statements about the rule $y = 4$ is false.

A. The gradient $m = 0$.
B. The y-intercept is at $(0,\ 4)$.
C. The graph is parallel to the x-axis.
D. The point $(4,\ 2)$ lies on this graph.

12. **MC** Select which of the following statements is not true about the rule $y = -\frac{3}{5}x$.

A. The graph passes through the origin.
B. The gradient $m = -\frac{3}{5}$.
C. The x-intercept is at $x = 0$.
D. The graph can be sketched using the x- and the y-intercept method.

13. $2x + 5y = 20$ is a linear equation in the form $ax + by = c$.

a. Rearrange this equation into the form $y = mx + c$.
b. State the gradient.
c. State the x- and y-intercepts.
d. Sketch this straight line.

14. A straight line has an x-intercept of -3 and a y-intercept of 5.

a. State the gradient.
b. Draw the graph.
c. Write the equation in the form:

i. $y = mx + c$

ii. $ax + by = c$.

Reasoning

15. Consider the relationship $4x - 3y = 24$.

a. Rewrite this relationship, making y the subject.
b. Show that the x- and y-intercepts are $(6, 0)$ and $(0, -8)$ respectively.
c. Sketch a graph of this relationship.

16. Consider the relationship $ax + by = c$.

a. Rewrite the relationship, making y the subject.
b. If a, b and c are positive integer values, explain how the gradient is negative.

17. Josie accidently spilled a drink on her work. Part of her calculations were smudged. The line $y = \frac{1}{2}x + \frac{3}{4}$ was written in the form $ax + 4y = 3$.
Show that the value of $a = -2$.

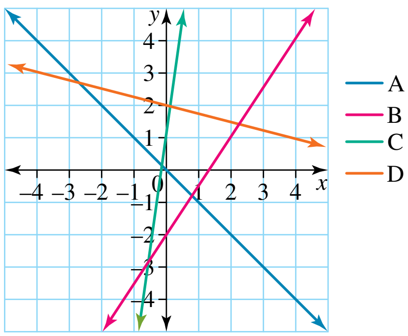
18. Explain why the descriptions 'right 3 up 2', 'right 6 up 4', 'left 3 down 2', 'right $\frac{3}{2}$ up 1' and 'left 1 down $\frac{2}{3}$' all describe the same gradient for a straight line.

19. A straight line passes through the points (3, 5) and (6, 11).

a. Determine the slope of the line.
b. Determine the equation of the line.
c. State the coordinates of another point that lies on the same line.

Problem solving

20. Consider the following graph.



a. Match the descriptions given below with their corresponding line.

i. Straight line with a y-intercept of $(0,\ 1)$ and a positive gradient.
ii. Straight line with a gradient of $1\frac{1}{2}$.
iii. Straight line with a gradient of -1.

b. Write a description for the unmatched graph.

21. a. Sketch the linear equation $y = -\frac{5}{7}x - \frac{3}{4}$:

i. using the y-intercept and the gradient
ii. using the x- and y-intercepts
iii. using two other points.

b. Compare and contrast the methods and generate a list of advantages and disadvantages for each method used in part **a**.
Explain which method you think is best. Give your reasons.

22. Consider these two linear graphs.

$$y - ax = b \text{ and } y - cx = d.$$

Show that if these two graphs intersect where both x and y are positive, then $a > c$ when $d > b$.

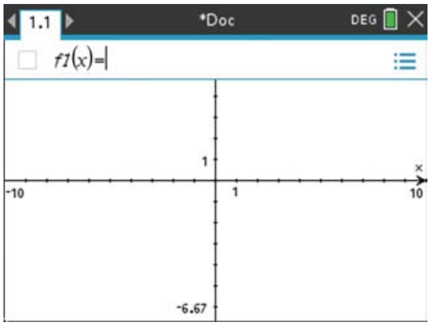
LESSON 4.6 Technology and linear graphs

LEARNING INTENTION

At the end of this lesson you should be able to:
- produce linear graphs using digital technologies.

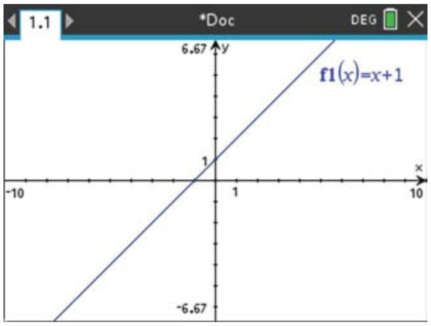
4.6.1 Graphing with technology

- There are many digital technologies that can be used to graph linear relationships.
- The Desmos Graphing Calculator is a free graphing tool that can be found online.
- Other commonly used digital technologies include Microsoft Excel and graphing calculators.
- Digital technologies can help identify important features and patterns in graphs.



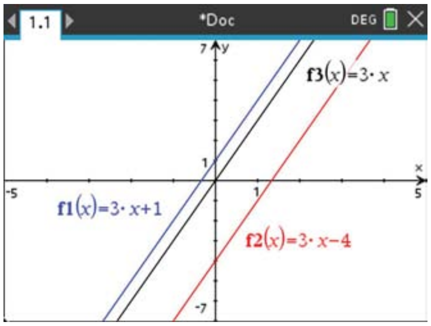
Sketching a linear graph using technology

- Depending on the choice of digital technology used, the steps involved to produce a linear graph may vary slightly.
- Most graphing calculators have an entry (or input) line to type in the equation of the line you wish to sketch.
- When using the Desmos Graphing Calculator you can simply type $y = x + 1$ into the input box to produce its graph.
- When using technologies such as a Texas Instruments or CASIO graphing calculator you will need to open a Graphs page first. The entry line for these types of technologies has a template that needs to be followed carefully. Both of these graphing calculators begin with $f1(x) =$ or $y1 =$, which is effectively saying $y =$.
- Thus, to draw the graph of $y = x + 1$, you would simply enter '$x + 1$'.
- The screen shows the graph of $y = x + 1$ sketched on a TI-Nspire CAS calculator.



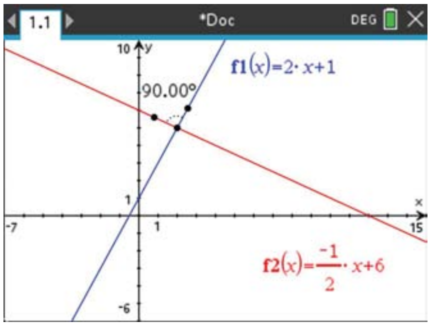
4.6.2 Graphing parallel lines

- Lines with the same gradient are called **parallel lines**. For a pair of parallel lines, $m_1 = m_2$.
- Digital technologies can be used to help visualise this concept. For example, $y = 3x + 1$, $y = 3x - 4$ and $y = 3x$ are all parallel lines, because $m = 3$.
- Select the digital technology of your choice and sketch these three parallel lines on the same set of axes.



4.6.3 Graphing perpendicular lines

- Lines that meet at right angles are called **perpendicular lines**.
- The product of the gradients of two perpendicular lines is equal to -1.
- For a pair of perpendicular lines, $m_1 \times m_2 = -1$.
- Digital technologies can be used to help visualise this concept. For example, $y = 2x + 1$ and $y = -\frac{1}{2}x + 6$ are perpendicular, because $2 \times -\frac{1}{2} = -1$.
- Select a digital technology of your choice and sketch these two lines on the same set of axes.



Exercise 4.6 Technology and linear graphs

learnon

4.6 Quick quiz on

4.6 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 4, 11, 15, 16	2, 5, 6, 7, 12, 14, 17	8, 9, 10, 13, 18

Understanding

Use technology wherever possible to answer the following questions.

1. Sketch the following graphs on the same Cartesian plane.

   a. $y = x$ b. $y = 2x$ c. $y = 3x$

   i. Describe what happens to the steepness of the graph (the gradient of the line) as the coefficient of x increases in value.
   ii. Identify where each graph cuts the x-axis (its x-intercept).
   iii. Identify where each graph cuts the y-axis (its y-intercept).

2. Sketch the following graphs on the same Cartesian plane.

   a. $y = x$ b. $y = \frac{1}{2}x$ c. $y = \frac{1}{3}x$

   i. Describe what happens to the steepness of the graph (the gradient of the line) as the coefficient of x decreases in value.
   ii. Identify where each graph cuts the x-axis (its x-intercept).
   iii. Identify where each graph cuts the y-axis (its y-intercept).

3. Sketch the following graphs on the same Cartesian plane.

a. $y = -x$ b. $y = -2x$ c. $y = -3x$

i. Describe what happens to the steepness of the graph as the magnitude of the coefficient of x decreases in value (becomes more negative).
ii. Identify where each graph cuts the x-axis (its x-intercept).
iii. Identify where each graph cuts the y-axis (its y-intercept).

4. Identify the correct word or words from the options given and rewrite the following sentences using the correct option.

a. If the coefficient of x is **(positive/negative)**, then the graph will have an upward slope to the right. That is, the gradient of the graph is **(positive/negative)**.
b. If the coefficient of x is negative, then the graph will have a **(downward/upward)** slope to the right. That is, the gradient of the graph is **(positive/negative)**.
c. The bigger the magnitude of the coefficient of x (more positive or more negative), the **(bigger/smaller)** the steepness of the graph.
d. If there is no constant term in the equation, the graph **(will/will not)** pass through the origin.

5. Sketch the following graphs on the same Cartesian plane and answer the following questions for each graph.

a. Is the coefficient of x the same for each graph? If so, state the coefficient.
b. State whether the steepness (gradient) of each graph differs.
c. Identify where each graph cuts the x-axis (its x-intercept).
d. Identify where each graph cuts the y-axis (its y-intercept).

i. $y = x$ ii. $y = x + 2$ iii. $y = x - 2$

6. Sketch the following graphs on the same Cartesian plane and answer the following questions for each graph.

a. Is the coefficient of x the same for each graph? If so, state the coefficient.
b. State whether the steepness (gradient) of each graph differs.
c. Identify where each graph cuts the x-axis (its x-intercept).
d. Identify where each graph cuts the y-axis (its y-intercept).

i. $y = -x$ ii. $y = -x + 2$ iii. $y = -x - 2$

7. Identify the correct word or words from the options given and rewrite the following sentences using the correct option.

a. For a given set of linear graphs, if the coefficient of x is **(the same/different)**, the graphs will be parallel.
b. The constant term in the equation is the **(y-intercept/x-intercept)** or where the graph cuts the **(y-axis/x-axis)**.
c. The **(y-intercept/x-intercept)** can be found by substituting $x = 0$ into the equation.
d. The **(y-intercept/x-intercept)** can be found by substituting $y = 0$ into the equation.

8. On the same Cartesian plane, sketch the following graphs.

a. $y = x + 5$ b. $y = -x + 5$

c. $y = 3x + 5$ d. $y = -\frac{2}{5}x + 5$

i. Is the coefficient of x the same for each graph? If so, state the coefficient.
ii. State whether the steepness (gradient) of each graph differ.
iii. Write down the gradient of each linear graph.
iv. Identify where each graph cuts the x-axis (its x-intercept).
v. Identify where each graph cuts the y-axis (its y-intercept).

9. Identify the correct word or words from the options given and rewrite the following sentences using the correct option.

 a. One of the general forms of the equation of a linear graph is $y = mx + c$, where m is the (**steepness/ *x*-coordinate**) of the graph. We call the steepness of the graph the *gradient*.
 b. The value of c is the (***x*-coordinate/*y*-coordinate**) where the graph cuts the (***x*-axis/*y*-axis**).
 c. All linear graphs with the (**same/different**) gradient are (**parallel/perpendicular**).
 d. All linear graphs that have the same y-intercept pass through (**the same/different**) point on the y-axis.

10. For each of the following lines, identify the gradient and the y-intercept.

 a. $y = 2x$
 b. $y = x + 1$
 c. $y = -3x + 5$
 d. $y = \frac{2}{3}x - 7$

Reasoning

11. Using a CAS calculator, graph the lines $y = 3(x - 1) + 5$, $y = 2(x - 1) + 5$ and $y = -\frac{1}{2}(x - 1) + 5$. Describe what they have in common and how they differ from each other.

12. Show on a CAS calculator how $y = \frac{1}{2}x - \frac{5}{3}$ can be written as $6y = 3x - 10$.

13. A phone company charges \$2.20 for international calls of 1 minute or less and \$0.55 for each additional minute. Using a CAS calculator, graph the cost for calls that last for whole numbers of minutes. Explain all the important values needed to sketch the graph.

14. Shirly walks dogs after school for extra pocket money. She finds that she can use the equation $P = -15 + 10N$ to calculate her profit (in dollars) each week.

 a. Explain the real-world meaning of the numbers -15 and 10 and the variable N.
 b. What is the minimum number of dogs that Shirly must walk in order to earn a profit?
 c. Using a CAS calculator, sketch the equation.

15. Graph $y = 0.2x + 3.71$ on a CAS calculator. Explain how to use the calculator to find an approximate value when $x = 70.3$.

Problem solving

16. Plot the points (6, 3.5) and (−1, −10.5) using a CAS calculator and:

 a. determine the equation of the line
 b. sketch the graph, showing x- and y-intercepts
 c. calculate the value of y when $x = 8$
 d. calculate the value of x when $y = 12$.

17. A school investigating the price of a site licence for their computer network found that it would cost \$1750 for 30 computers and \$2500 for 60 computers.

 a. Using a CAS calculator, find a linear equation that represents the cost of a site licence in terms of the number of computers in the school.
 b. Determine is the y-intercept of the linear equation and explain how it relates to the cost of a site licence.
 c. Calculate the cost for 200 computers.
 d. Evaluate how many computers you could connect for \$3000.

18. Dylan starts his exercise routine by jogging to the gym, which burns 325 calories. He then pedals a stationary bike burning 3.8 calories a minute.

 a. Graph this data using a CAS calculator.
 b. Evaluate how many calories Dylan has burnt after 15 minutes of pedalling.
 c. Evaluate how long it took Dylan to burn a total of 450 calories.

LESSON 4.7 Practical applications of linear graphs

LEARNING INTENTIONS

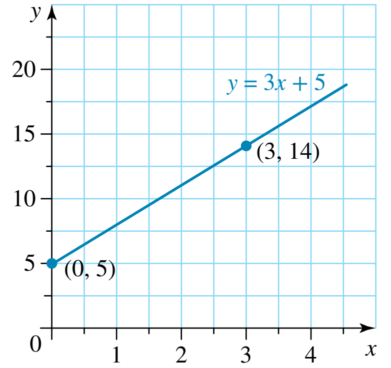
At the end of this lesson you should be able to:
- determine the linear rule from a table of values
- determine the linear rule that models a real-life problem
- interpret linear rules in real-life problems.

4.7.1 Determining a linear rule from a table of values

- If two **variables** are linked by a linear rule, then as one variable increases, the other increases, or decreases, at a steady rate.
- The table shown gives an example of a linear relationship.

x	0	1	2	3
y	5	8	11	14

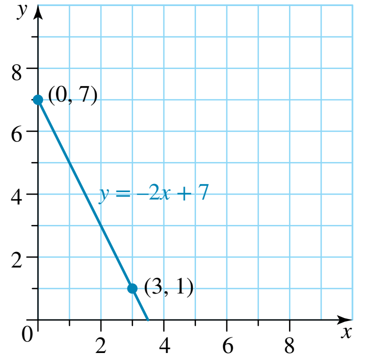
  - Each time x increases by 1, y increases by 3.
  - The linear rule connecting x and y is $y = 3x + 5$.



- Consider the relationship depicted in the table shown.

x	0	1	2	3
y	7	5	3	1

  - Each time x increases by 1, y decreases by 2.
  - The linear rule in this case is $y = -2x + 7$.



WORKED EXAMPLE 14 Determining a rule from a table of values

Determine the rule connecting x and y in each of the following tables.

a.

x	0	1	2	3
y	−3	2	7	12

b.

x	3	4	5	6
y	12	11	10	9

THINK	WRITE
a. 1. y increases at a steady rate, so this is a linear relationship. Write the rule.	**a.** $y = mx + c$
2. To calculate m: y increases by 5 each time x increases by 1. Write the value of the gradient.	$m = 5$
3. To determine c: From the table, when $x = 0$, $y = -3$. Write the value of the y-intercept.	$c = -3$
4. Write the rule.	$y = 5x - 3$

b. **1.** y decreases at a steady rate, so this is a linear relationship. Write the rule.

b. $y = mx + c$

2. To calculate m: y decreases by 1 each time x increases by 1.
Write the value of the gradient.

$m = -1$

3. To determine c: From the table, when $x = 3$, $y = 12$.
To calculate the y-intercept, substitute the x- and y-values of one of the points, and solve for c.

$$(3, 12): \quad \begin{aligned} y &= -x + c \\ 12 &= -(3) + c \\ 12 &= -3 + c \\ c &= 15 \end{aligned}$$

4. Write the rule.

$y = -x + 15$

4.7.2 Modelling linear relationships

- Relationships between real-life variables are often modelled by a mathematical equation. In other words, an equation or formula is used to link the two variables.
  For example:
  - $A = l^2$ represents the relationship between the area and the side length of a square
  - $C = \pi d$ represents the relationship between the circumference and the diameter of a circle.
- If one variable changes at a constant rate compared to the other, then the two variables have a linear relationship.
- Simple interest is another example of a relationship that can be modelled by a linear relationship. Simple interest will be studied in a later topic.

WORKED EXAMPLE 15 Determining the rule to model a relationship in a worded question

An online bookstore sells a certain textbook for \$21 and charges \$10 for delivery, regardless of the number of books being delivered.

a. **Determine the rule connecting the cost (\$$C$) with the number of copies of the textbook delivered (n).**

b. **Use the rule to calculate the cost of delivering 35 copies of the textbook.**

c. **Calculate how many copies of the textbook can be delivered for \$1000.**

THINK

WRITE

a. **1.** Set up a table.

$$\begin{aligned} \text{Cost for 1 copy} &= 21 + 10 \\ \text{Cost for 2 copies} &= 2(21) + 10 \\ &= 52 \end{aligned}$$

a.

n	1	2	3
C	31	52	73

2. The cost rises steadily, so there is a linear relationship. Write the rule.

$C = mn + c$

3. To calculate the gradient, use the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ with the points $(1, 31)$ and $(2, 52)$.

$$\begin{aligned} m &= \frac{52 - 31}{2 - 1} \\ &= 21 \end{aligned}$$

$C = 21n + c$

4. To determine the value of c, substitute $C = 31$ and $n = 1$.

$$(1, 31): \quad \begin{aligned} 31 &= 21(1) + c \\ c &= 10 \end{aligned}$$

5. Write the rule.

$C = 21n + 10$

b. 1. Substitute $n = 35$ and calculate the value of C.

b. $C = 21\,(35) + 10$
$= 735 + 10$
$= 745$

2. Write the answer.

The cost including delivery for 35 copies is \$745.

c. 1. Substitute $C = 1000$ and calculate the value of n.

c. $1000 = 21n + 10$
$21n = 990$
$n = 47.14$

2. You cannot buy 47.14 books, so round down. Write the answer.

For \$1000, 47 copies of the textbook can be bought and delivered.

- In Worked example 15, compare the rule $C = 21n + 10$ with the original question. It is clear that the $21n$ refers to the cost of the textbooks (a variable amount, depending on the number of copies) and that 10 refers to the fixed (constant) delivery charge.
- In this case C is called the **response variable** or the **dependent variable**, because it depends on the number of copies of the textbooks (n).
- The variable n is called the **explanatory variable** or the **independent variable** because it is the variable that may explain the changes in the response variable.
- When graphing numerical data, the response variable is plotted on the vertical axis and the explanatory variable is plotted on the horizontal axis.

Exercise 4.7 Practical applications of linear graphs

learnon

4.7 Quick quiz on | 4.7 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 6, 11, 16	2, 7, 9, 12, 13, 17	4, 8, 10, 14, 15, 18

Fluency

1. WE14 Determine the linear rule linking the variables in each of the following tables.

a.

x	0	1	2	3
y	−5	1	7	13

b.

x	0	1	2	3
y	8	5	2	−1

2. Determine the linear rule linking the variables in each of the following tables.

a.

x	0	1	2	3
y	4	6	8	10

b.

x	0	1	2	3
y	1.1	2.0	2.9	3.8

3. Determine the linear rule linking the variables in each of the following tables.

a.

x	2	3	4	5
y	7	10	13	16

b.

x	5	6	7	8
y	12	11	10	9

4. Determine the linear rule linking the variables in each of the following tables.

a.

t	3	4	5	6
v	18	15	12	9

b.

d	1	2	3	4
C	11	14	17	20

5. MC Sasha and Fiame hire a car. They are charged a fixed fee of \$150 for hiring the car and then \$25 per day. They hire the car for d days.
Select which one of the following rules describes the number of days the car is hired and the total cost, C, they would be charged for that number of days charged.

A. $C = 25d$
B. $C = 150d$
C. $C = 175d$
D. $C = 25d + 150$

Understanding

6. WE15 Fady's bank balance has increased in a linear manner since he started his part-time job. After 20 weeks of work his bank balance was at \$560 and after 21 weeks of work it was at \$585.

a. Determine the rule that relates the size of Fady's bank balance, A, and the time (in weeks) worked, t.
b. Use the rule to calculate the amount in Fady's account after 200 weeks.
c. Use the rule to identify the initial amount in Fady's account.

7. The cost of making a shoe increases as the size of the shoe increases. It costs \$5.30 to make a size 6 shoe, and \$6.40 to make a size 8 shoe. Assuming that a linear relationship exists:
a. determine the rule relating cost (C) to shoe size (s)
b. calculate much it costs to produce a size 12 shoe.

8. The number of books in a library (N) increases steadily with time (t). After 10 years there are 7200 publications in the library, and after 12 years there are 8000 publications.
a. Determine the rule predicting the number of books in the library.
b. Calculate how many books were there after 5.5 years.
c. Calculate how many books there will be after 25 years.

9. A skyscraper can be built at a rate of 4.5 storeys per month.

a. Calculate how many storeys will be built after 6 months.
b. Calculate how many storeys will be built after 24 months.

10. The Nguyens' water tank sprang a leak and has been losing water at a steady rate. Four days after the leak occurred, the tank contained 552 L of water, and ten days later it held only 312 L.
a. Determine the rule linking the amount of water in the tank (w) and the number of days (t) since the leak occurred.
b. Calculate the amount of water that was initially in the tank.
c. If water loss continues at the same rate, determine when the tank will be empty.

Reasoning

11. The pressure inside a boiler increases steadily as the temperature increases. For each 1 °C increase in temperature, the pressure increases by 10 units, and at a temperature of 100 °C the pressure is 1200 units. If the maximum pressure allowed is 2000 units, show that the temperature cannot exceed 180 °C.

12. After 11 pm a taxi company charges a \$3.50 flag fall plus \$2.57 for each kilometre travelled.
    a. Determine the linear rule connecting the cost, C, and the distance travelled, d.
    b. Calculate how much an 11.5 km trip will cost.
    c. If you have \$22 in your pocket, calculate how far you can afford to travel, correct to 1 decimal place.

13. A certain kind of eucalyptus tree grows at a linear rate for its first 2 years of growth. If the growth rate is 5 cm per month, show that the tree will be 1.07 m tall after 21.4 months.

14. A software company claims that its staff can fix 22 bugs each month. They are working on a project to fix a program that started out with 164 bugs.
    a. Determine the linear rule connecting the number of bugs left, N, and the time in months, t, from the beginning of the project.
    b. Calculate how many bugs will be left after 2 months.
    c. Determine how long it will be until there are only 54 bugs left.
    d. Determine how long it will take to eliminate all of the bugs. Justify your answer.

15. Michael produces and sells prints of his art at a local gallery. For each print run his profit (P) is given by the equation $P = 200n - 800$, where n is the number of prints sold.
    a. Sketch the graph of this rule.
    b. Identify the y-intercept. Determine what it represents in this situation.
    c. Identify the x-intercept. Determine what it represents in this example.
    d. Identify the gradient of the graph. Determine what this means in this situation.

Problem solving

16. The cost of a taxi ride is \$3.50 flag fall plus \$2.14 for each kilometre travelled.
    a. Determine the linear rule connecting the cost, C, and the distance travelled, d.
    b. Calculate how much an 11.5 km trip will cost.
    c. Calculate how much a 23.1 km trip will cost.

17. Theo is going on holiday to Japan. One yen (¥) buys 0.0127 Australian dollars (A\$).
    a. Write an equation that converts Australian dollars to Japanese yen (¥), where A represents amount of Australian dollars and Y represents amount of yen.
    b. Using the equation from part a, calculate how many yen Theo will receive in exchange if they have A\$2500.
    c. There is a commission to be paid on exchanging currency. Theo needs to pay 2.8% for each Australian dollar they exchange into yen. Write down an equation that calculates the total amount of yen Theo will receive. Write your equation in terms of Y_T (total amount of yen) and \$A (Australian dollars).

18. Burchill and Cody need to make a journey to the other branch of their store across town. The traffic is very busy at this time of the day, so Burchill catches a train that travels halfway, then walks the rest of the way. Cody travels by bike the whole way. The bike path travels along the train line and then follows the road to the other branch of their store.
    Cody's bike travels twice as fast as Burchill's walking speed, and the train travels 4 times faster than Cody's bike. Evaluate who arrives at the destination first.

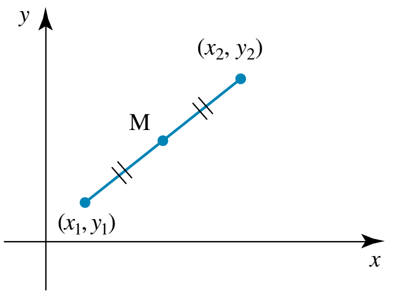
LESSON 4.8 Midpoint of a line segment and distance between two points

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- calculate the midpoint of a line segment
- calculate the distance between two points.

4.8.1 Calculating the midpoint of a line segment

- The x- and y-coordinates of the **midpoint** are halfway between those of the end points of a line segment.
- The coordinates of the midpoint, M, of a line can be found by averaging the x- and y-coordinates of the end points.



The midpoint of a line segment

The midpoint, M, of the line segment joining the points (x_1, y_1) and (x_2, y_2) is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

WORKED EXAMPLE 16 Calculating the midpoint of a line segment

Calculate the midpoint of the line segment joining $(5, 9)$ and $(-3, 11)$.

THINK	WRITE
1. Average the x-values: $\frac{x_1 + x_2}{2}$.	$x = \frac{5 - 3}{2}$ $= \frac{2}{2}$ $= 1$
2. Average the y-values: $\frac{y_1 + y_2}{2}$.	$y = \frac{9 + 11}{2}$ $= \frac{20}{2}$ $= 10$
3. Write the answer.	The midpoint is $(1, 10)$.

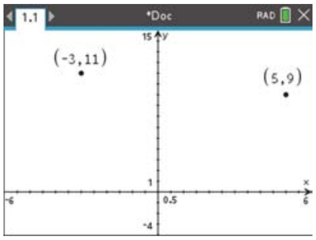
TI | THINK

WRITE

1. On a Graphs page, set an appropriate window as shown and press:
   - MENU
   - 8: Geometry
   - Points & Lines
   - 3: Point by Coordinates

   Type the coordinates of one point on the keypad, pressing ENTER after each number.

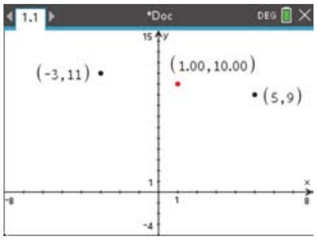
   Repeat the above step to plot the second point.



2. To determine the midpoint, press:
   - MENU
   - 8: Geometry
   - 4: Construction
   - 5: Midpoint.

   Move the cursor to one of the points and press ENTER.
   Move the cursor to the other point and press ENTER. This will place a dot on the screen representing the midpoint of the two plotted points.
   To display the midpoint's coordinates, press:
   - MENU
   - 1: Actions
   - 8: Coordinates and Equations

   Move the cursor to the midpoint and press ENTER twice.



3. State the coordinates of the midpoint.

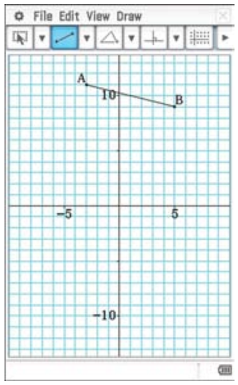
The coordinates of the midpoint are $(1, 10)$.

CASIO | THINK

WRITE

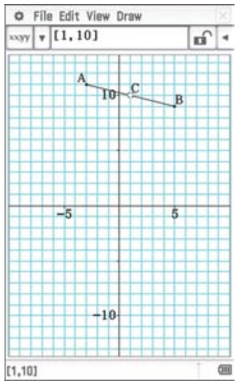
On a Geometry screen, select a pair of axes with scale and grid.
This is achieved by continuous clicking on the graph icon.
To get the desired window, select:
- View
- Zoom out

Using the Line segment tool, place the pointer at $(-3, 11)$ and hold to place the second point at $(5, 9)$.
The line segment will be automatically labelled.



To locate the midpoint:
- Click on the line segment.
- Click again for the point C to be placed in the centre or midpoint.

To determine the coordinates of the midpoint C:
- Click on the point C.
- Use the measuring tool at the top right to display the coordinates.



State the coordinates of the midpoint.

The coordinates of the midpoint are $(1, 10)$.

WORKED EXAMPLE 17 Determining the coordinates of a point given the midpoint and another point of an interval

M(7, 2) is the midpoint of the line segment AB. If the coordinates of A are (1, −4), determine the coordinates of B.

THINK

WRITE

1. Let B have the coordinates (x, y).

$A(1, -4), B(x, y), M(7, 2)$

2. The midpoint is $(7, 2)$, so the average of the x-values is 7. Solve for x.

$$\frac{1+x}{2} = 7$$

$$1 + x = 14$$

$$x = 13$$

3. The average of the y-values is 2. Solve for y.	$\frac{-4+y}{2}=2$ $-4+y=4$ $y=8$
4. Write the answer.	The coordinates of point B are $(13, 8)$.

4.8.2 Calculating the distance between two points

- The distance between two points on the Cartesian plane is calculated using Pythagoras' theorem applied to a right-angled triangle.
- Consider the formula for Pythagoras' Theorem:

$$c^2 = a^2 + b^2$$
$$c = \sqrt{a^2 + b^2}$$

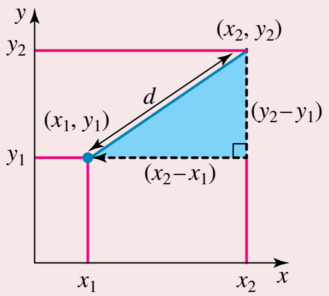
where c is the hypotenuse and a and b are the shorter sides.

- On a Cartesian plane with two points, the hypotenuse, c, would represent the length of the line.
- By drawing a triangle, the horizontal distance would be the difference between their x-values (a), and the vertical distance would be the difference between their y-values (b).

Distance between two points

The distance, d, of the line segment joining the points (x_1, y_1) and(x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



WORKED EXAMPLE 18 Calculating the distance between two points

Calculate the distance between the points $(-1, 3)$ and $(4, 5)$:

a. **in exact form**

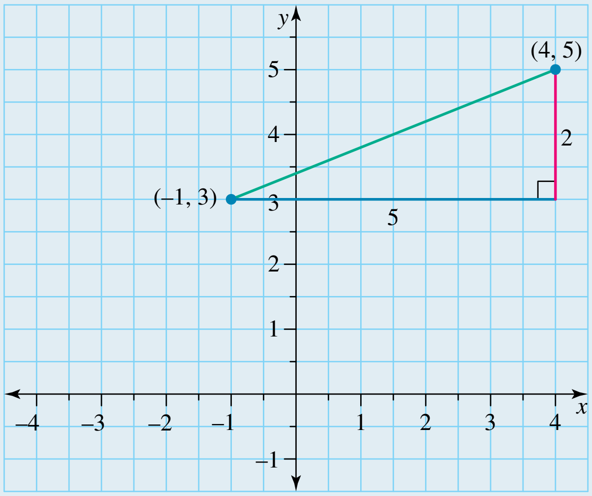
b. **correct to 3 decimal places.**

THINK

WRITE/DRAW

a. **1.** Draw a diagram showing the right-angled triangle (optional).

a.



2. Write the formula for the distance between two points.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

3. Let $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, 5)$. Substitute the x- and y-values into the equation.

$d = \sqrt{(4 - -1)^2 + (5 - 3)^2}$

4. Simplify.

$d = \sqrt{5^2 + 2^2}$

$d = \sqrt{25 + 4}$

$d = \sqrt{29}$

5. Write the answer.

The exact distance betwen points (–1, 3) and (4, 5) is $\sqrt{29}$.

b. **1.** Write $\sqrt{29}$ as a decimal to 5 decimal places.

b. $\sqrt{29} = 5.38516$

2. Round the answer correct to 3 decimal places.

$d \approx 5.385$

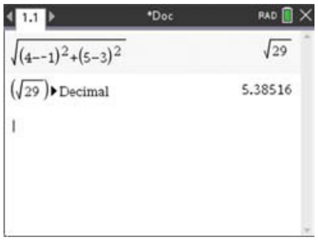
TI | THINK

a-b. On a Calculator page, complete the entry lines as:

$$\sqrt{(4--1)^2+(5-3)^2}$$

Press ENTER after each entry.

WRITE

a-b. 

b. Press CTRL ENTER to get a decimal approximation.

b. The distance is $\sqrt{29} \approx 5.385$ to 3 decimal places.

CASIO | THINK

a. Make sure the calculator is set to Standard mode. On the Main screen, complete the entry line as:

$$\sqrt{(4--1)^2+(5-3)^2}$$

Press EXE.

WRITE

a. $\sqrt{29}$

b. Convert the answer to a decimal by tapping Standard and changing it to Decimal.
Then press EXE.

b. 5.385

Exercise 4.8 Midpoint of a line segment and distance between two points

learnon

4.8 Quick quiz on | 4.8 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 8, 11, 13, 16, 20	2, 5, 9, 12, 14, 17, 21	3, 6, 7, 10, 15, 18, 19, 22

Fluency

1. WE16 Calculate the midpoint of the line segment joining each of the following pairs of points.

a. $(1,\ 3)$ and $(3,\ 5)$ **b.** $(6, 4)$ and $(4, -2)$ **c.** $(2,\ 3)$ and $(12,\ 1)$

2. Calculate the midpoint of the line segment joining each of the following pairs of points.

a. $(6,\ 3)$ and $(10,\ 15)$ **b.** $(4, 2)$ and $(-4, 8)$ **c.** $(0, -5)$ and $(-2,\ 9)$

3. Calculate the midpoint of the line segment joining each of the following pairs of points.

a. $(8,\ 2)$ and $(-18, -6)$ **b.** $(-3, -5)$ and $(7,\ 11)$ **c.** $(-8, -3)$ and $(8,\ 27)$

4. Calculate the midpoint of the segment joining each of the following pairs of points.

a. $(7, -2)$ and $(-4,\ 13)$ **b.** $(0, 22)$ and $(-6, -29)$
c. $(-15, 8)$ and $(-4,\ 11)$ **d.** $(-3, 40)$ and $(0, -27)$

5. WE17 Determine the value of a in each series of points so that the point M is the midpoint of the line segment joining points A and B.

a. $\mathrm{A}(-2, a), \mathrm{B}(-6, 5), \mathrm{M}(-4, 5)$ **b.** $\mathrm{A}(a, 0), \mathrm{B}(7, 3), \mathrm{M}\left(8, \frac{3}{2}\right)$

c. $\mathrm{A}(3, 3), \mathrm{B}(4, a), \mathrm{M}\left(3\frac{1}{2}, -6\frac{1}{2}\right)$ **d.** $\mathrm{A}(-4, 4), \mathrm{B}(a, 0), \mathrm{M}(-2, 2)$

6. M is the midpoint of the line interval AB. Determine the coordinates of B if:

a. $A = (0,\ 0)$ and $M = (2,\ 3)$
b. $A = (2,\ 3)$ and $M = (0,\ 0)$
c. $A = (-3,\ 2)$ and $M = (4,\ 2)$
d. $A = (3, -1)$ and $M = (-2, -2)$.

7. Determine the equation of a line that has a gradient of 5 and passes through the midpoint of the line segment joining $(-1, -7)$ and $(3,\ 3)$.

8. WE18 Calculate the distance between each of the following pairs of points.

a. $(4,\ 5)$ and $(1, 1)$
b. $(7, 14)$ and $(15, 8)$
c. $(2, 4)$ and $(2, 3)$

9. Calculate the distance between each of the following pairs of points.

a. $(12, 8)$ and $(10, 8)$
b. $(14, 9)$ and $(2, 14)$
c. $(5, -13)$ and $(-3, -7)$

10. Calculate the distance between each of the following pairs of points.

a. $(-14, -9)$ and $(-10, -6)$
b. $(0, 1)$ and $(-15, 9)$
c. $(-4, -8)$ and $(1, 4)$

11. Calculate the distance between the following pairs of points, correct to 3 decimal places.

a. $(-14, 10)$ and $(-8, 14)$
b. $(6, -7)$ and $(13, 6)$
c. $(-11, 1)$ and $(2, 2)$

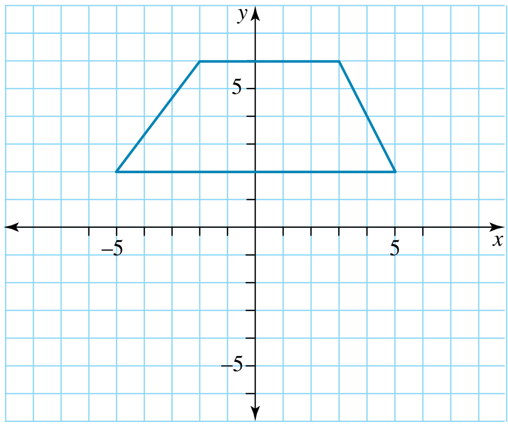
12. Calculate the distance between the following pairs of points, correct to 3 decimal places

a. $(9, 0)$ and $(5, -8)$
b. $(2, -7)$ and $(-2, 12)$
c. $(9, 4)$ and $(-10, 0)$

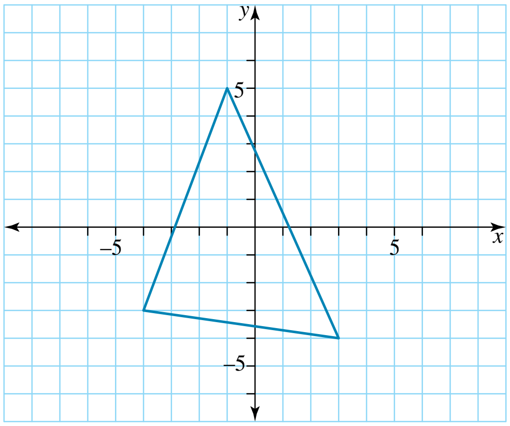
Understanding

13. Calculate the perimeter of each figure shown, giving your answers correct to 3 decimal places.

a.

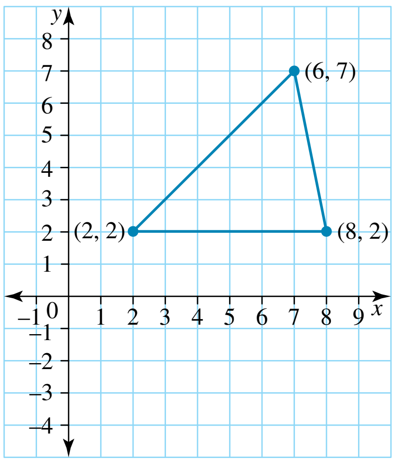


b.

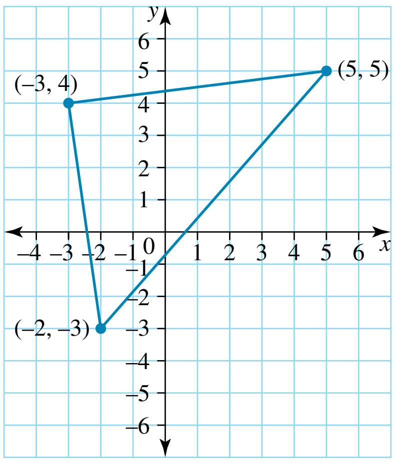


14. Calculate the perimeter of each triangle shown, giving your answers correct to 3 decimal places.

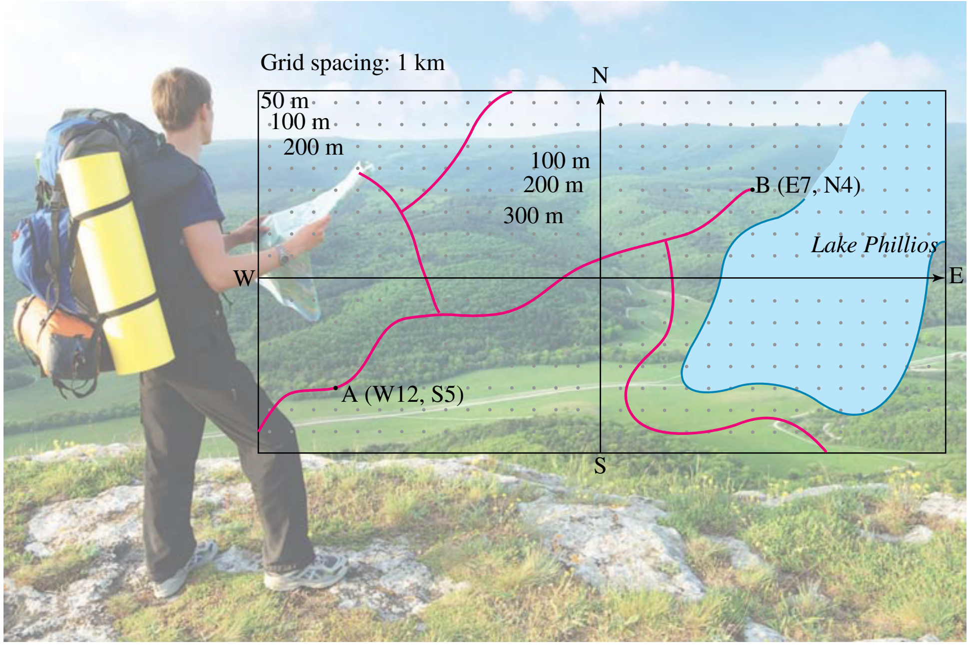
a.



b.



15. Two hikers are about to hike from A to B (shown on the map). Calculate the straight-line distance from A to B.



Reasoning

16. Show that the distance between the points A $(2, 2)$ and B $(6, -1)$ is 5.

17. The point M $(-2, -4)$ is the midpoint of the interval AB. Show that the point B is $(-9, -2)$, given A is $(5, -6)$.

18. Show that the point B $(6, -10)$ is equidistant from the points A $(15, 3)$ and C $(-7, -1)$.

19. Answer the following questions.
    a. Plot the following points on a Cartesian plane: A $(-1, -4)$, B $(2, 3)$, C $(-3, 8)$ and D $(4, -5)$.
    b. Show that the midpoint of the interval AC is $(-2, 2)$.
    c. Calculate the exact distance between the points A and C.
    d. If B is the midpoint of an interval CM, determine the coordinates of point M.
    e. Show that the gradient of the line segment AB is $\frac{7}{3}$.
    f. Determine the equation of the line that passes through points B and D.

Problem solving

20. Explain what type of triangle ΔABC is if it has vertices A $(-4, 1)$, B $(2, 3)$ and C $(0, -3)$.

21. Calculate the gradient of the line through the points $(-1, 3)$ and $(3 + 4t, 5 + 2t)$.

22. A map of a town drawn on a Cartesian plane shows the main street extending from $(-4, 5)$ to $(0, -7)$. There are five streetlights positioned in the street. There is one streetlight at either end, and three streetlights spaced evenly down the street.
    Give the position of the five lights in the street.

LESSON 4.9 Review

4.9.1 Topic summary

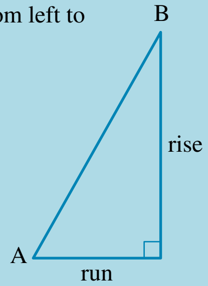
LINEAR GRAPHS

Gradient and axis intercepts

- The gradient of a straight line is given the label m. It can be determined by the following formula:

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

- If the line is sloping upwards (from left to right), the gradient is positive.
- If the line is sloping downwards, the gradient is negative.
- The y-intercept of a line can be determined by letting $x = 0$, then solving for y.
- The x-intercept of a line can be determined by letting $y = 0$, then solving for x.



Sketching linear graphs

- To sketch a linear graph, all you need are two points that the line passes through.
- The x- and y-intercept method involves finding both axis intercepts, then drawing the line through them.
- The gradient and y-intercept method involves plotting the y-intercept and then one other point (usually $x = 1$) using the gradient.

Equation of a straight line

- The equation of a straight line is $y = mx + c$ where:
  - m is the gradient
  - c is the y-intercept.

Horizontal and vertical lines

- Horizontal lines are of the form $y = c$. They have a gradient of zero ($m = 0$).
- Vertical lines are of the form $x = a$. They have an undefined gradient.

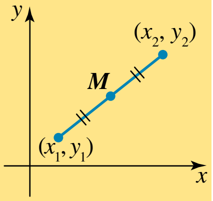
Midpoint and distance

- The midpoint of two points (x_1, y_1) and (x_2, y_2) is

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right).$$

- The distance between two points, (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



4.9.2 Success criteria

Tick the column to indicate that you have completed the lesson and how well you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
4.2	I can plot points on the Cartesian plane.			
	I can plot a linear graph on the Cartesian plane.			
4.3	I can calculate the gradient of a line or line segment.			
	I can calculate the y-intercept of a linear graph.			
4.4	I can determine the equation of a straight line given the gradient and the y-intercept.			
	I can determine the equation of a straight line given the gradient and a single point.			
	I can determine the equation of a straight line given two points.			
	I can determine the equation of a straight line from a graph.			
4.5	I can sketch linear graphs using the x- and y-intercept method.			
	I can sketch linear graphs using the gradient–intercept method.			
	I can sketch linear graphs represented in both $y = mx + c$ and $ax + by = c$ form.			
	I can sketch vertical and horizontal lines.			
4.6	I can produce linear graphs using digital technologies.			
4.7	I can determine the linear rule from a table of values.			
	I can determine the linear rule that models a real-life problem.			
	I can interpret linear rules in real-life problems.			
4.8	I can calculate the midpoint of a line segment.			
	I can calculate the distance between two points.			

4.9.3 Project

Path of a billiard ball

The path of a billiard ball can be mapped using mathematics. A billiard table can be represented by a rectangle, a ball by a point and its path by line segments. In this investigation, we will look at the trajectory of a single ball, unobstructed by other balls.

Note: A billiard table has a pocket at each of its corners and one in the middle of each of its long sides.

Consider the path of a ball that is hit on its side, from the lower left-hand corner of the table, so that it travels at a 45° angle from the corner of the table. Assume that the ball continues to move, rebounding from the sides and stopping only when it comes to a pocket.

Diagrams of the table drawn on grid paper are shown. Each grid square has a side length of 0.25 metres.

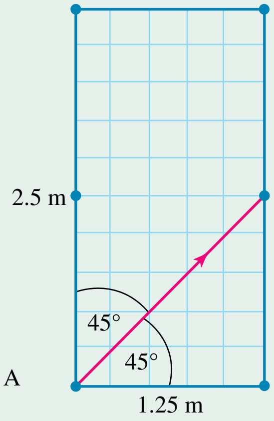


Diagram A

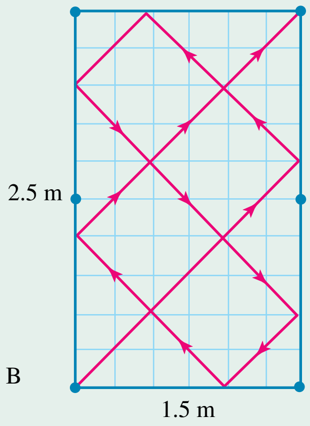


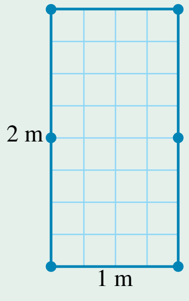
Diagram B

Diagram A shows the trajectory of a ball that has been hit on its side at a 45° angle, from the lower left-hand corner of a table that is 2.5 m long and 1.25 m wide.

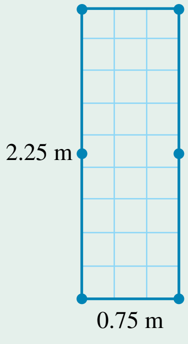
Note that, because the ball has been hit on its side at a 45° angle, it travels diagonally through each square in its path, from corner to corner. Diagram B shows the trajectory of a ball on a 2.5 m by 1.25 m table.

1. In Diagram B, how many times does the ball rebound off the sides before going into a pocket?
2. A series of eight tables of different sizes are drawn on grid paper in the following diagrams. For each table, determine the trajectory of a ball hit at 45° on its side, from the lower left-hand corner of the table. Draw the path each ball travels until it reaches a pocket.

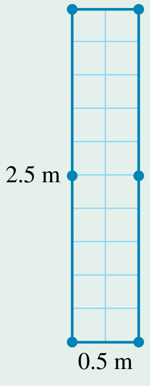
a.



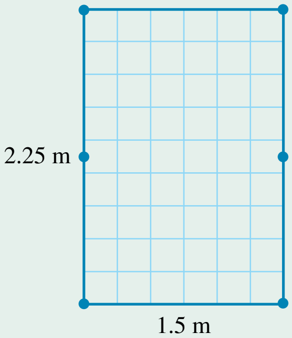
b.



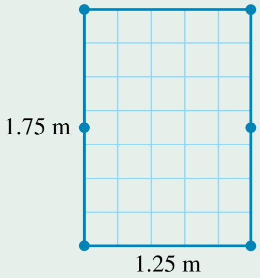
c.



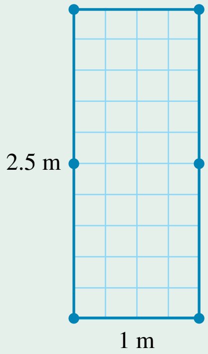
d.



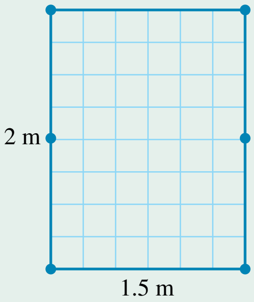
e.



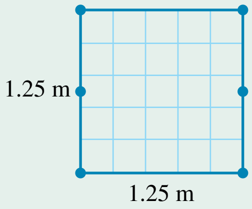
f.



g.



h.



3. Determine for which tables the ball will travel through the simplest path. What is special about the shape of these tables?

4. Determine for which table the ball will travel through the most complicated path. What is special about this path? Draw another table (and path of the ball) with the same feature.

5. Determine for which tables the ball will travel through a path that does not cross itself. Draw another table (and path of the ball) with the same feature.

6. Consider the variety of table shapes. Will a ball hit on its side from the lower left-hand corner of a table at 45° always end up in a pocket (assuming it does not run out of energy)?

Simplify matters a little and consider a billiard table with no pockets in the middle of the long sides. Use a systematic way to look for patterns for tables whose dimensions are related in a special way.

7. Draw a series of billiard tables of length 3 m. Increase the width of these tables from an initial value of 0.25 m in increments of 0.25 m. Investigate the final destination (the pocket the ball lands in) of a ball hit from the lower left-hand corner. Complete the table.

	Length of table (m)	Width of table (m)	Destination pocket
a.	3	0.25	
b.	3	0.5	
c.	3	0.75	
d.	3	1	
e.	3	1.25	
f.	3	1.5	
g.	3	1.75	
h.	3	2	

8. How can you predict (without drawing a diagram) the destination pocket of a ball hit from the lower left-hand corner of a table that is 3 m long? Provide an illustration to verify your prediction.

Exercise 4.9 Review questions

learnon

Fluency

1. MC For the rule $y = 3x - 1$, select the value of y when $x = 2$.
   A. -1 B. 1 C. 2 D. 5

2. MC Identify the gradient of the linear rule $y = 4 - 6x$.
   A. 6 B. -6 C. 4 D. -4

3. MC The graph with the rule $2y - x + 6 = 0$ has an x-intercept of:
   A. -2 B. 0 C. 2 D. 6

4. MC The graph with the rule $2y - x + 6 = 0$ has a y-intercept of:
   A. -6 B. -3 C. 0 D. 3

5. MC Consider a linear graph that goes through the points $(6, -1)$ and $(0, 5)$. The gradient of this line is:
   A. 5 B. -5 C. 1 D. -1

6. MC A straight line passes through the points $(2, 1)$ and $(5, 4)$. Its rule is:
   A. $y = x - 1$ B. $y = x + 1$ C. $y = 2x$ D. $4y = 5x$

7. MC The rule for a line whose gradient is -4 and y-intercept $= 8$ is:
   A. $y = -4x + 32$ B. $y = -4x + 8$ C. $y = 4x - 32$ D. $y = 4x - 8$

8. MC Identify which of the following linear rules will *not* intersect with the straight line defined by $y = 3x$.
   A. $y = 3x + 2$ B. $y = -3x + 1$ C. $y = -3x + 2$ D. $y = -2x + 1$

9. MC If $y = 2x + 1$, select from the following the point that could not be on the line.
   A. $(3, 7)$ B. $(-3, -5)$ C. $(0, 1)$ D. $(-3, 0)$

10. Write down the gradient and the y-intercept of the following linear graphs.
    a. $y = 8x - 3$
    b. $y = 5 - 9x$
    c. $2x + y - 6 = 0$
    d. $4x - 2y = 0$
    e. $y = \dfrac{2x - 1}{3}$

11. MC Determine which of the following lines are parallel to the line with the equation $y = 6 - x$. There could be more than one correct answer.
    A. $y + x = 4$
    B. $y = 13 - x$
    C. $2y - 2x = 1$
    D. $x + 2y - 4 = 0$

12. MC Determine which of the following rules will yield a linear graph. There could be more than one correct answer.

A. $3y = -5x - \frac{1}{2}$

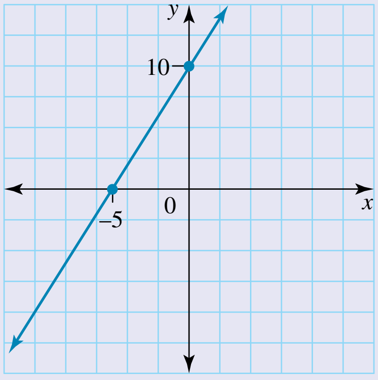
B. $y = 3x^2 - 2\frac{1}{2}$

C. $3x + 4y + 6 = 0$

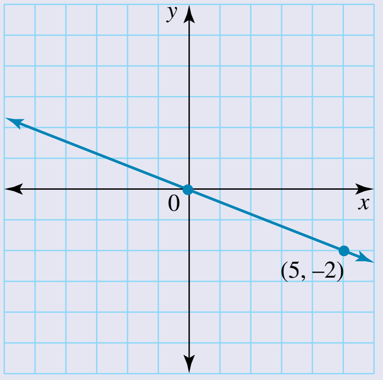
D. $2y = 4^2x + 9^2$

13. Determine the gradient of the lines shown.

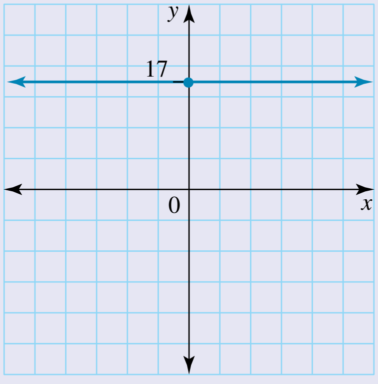
a.



b.



c.



14. Calculate the gradient of the line passing through the following pairs of points.

a. $(2, -3)$ and $(4, 1)$

b. $(0, -5)$ and $(4, 0)$.

15. For each of the following rules, state the gradient and the y-intercept.

a. $y = -3x + 7$

b. $2y - 3x = 6$

c. $y = -\frac{2}{5}x$

d. $y = 4$

16. For the following rules, use the gradient–intercept method to sketch linear graphs.

a. $y = -x + 5$

b. $y = 4x - 2.5$

c. $y = \frac{2}{3}x - 1$

d. $y = 3 - \frac{5}{4}x$

17. For the following rules, use the x- and y-intercept method to sketch linear graphs.

a. $y = -6x + 25$

b. $y = 20x + 45$

c. $2y + x = -5$

d. $4y + x - 2.5 = 0$

18. For the following rules, use an appropriate method to sketch linear graphs.

a. $y = -3x$

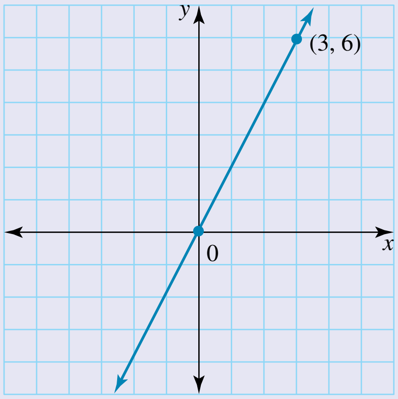
b. $y = \frac{1}{4}x$

c. $y = -2$

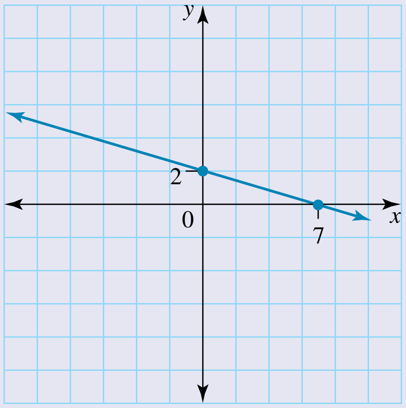
d. $x = 3$

19. Determine the rule of the lines shown.

a.



b.



20. Determine the linear rules given the following pieces of information.

a. Gradient $= 2$, y-intercept $= -7$

b. Gradient $= 2$, x-intercept $= 7$

c. Gradient $= 2$, passing through $(7, 9)$

d. Gradient $= -5$, passing through the origin

e. y-intercept $= -2$, passing through $(1, -3)$

f. Passing through $(1, 5)$ and $(5, -6)$

g. x-intercept $= 3$, y-intercept $= -3$

h. y-intercept $= 5$, passing through $(-4, 13)$

21. Determine the midpoint of the line interval joining the points $(-2, 3)$ and $(4, -1)$.

22. Calculate the distance between the points $(1, 1)$ and $(4, 5)$.

Problem solving

23. Louise owes her friend Sula \$400 and agrees to pay her back \$15 per week.

a. State a linear rule that demonstrates this debt reduction schedule and sketch the graph.

b. Calculate how many weeks it takes Louise to repay the debt.

c. Calculate how much she owes after 15 weeks.

d. Evaluate how many repayments Louise needs to make to reduce her debt to \$85.

24. A bushwalker is 40 km from their base camp when they decide to head back.
If they are able to walk 3.5 km each hour:

a. determine the linear rule that describes this situation and sketch its graph

b. calculate how long, correct to 1 decimal place, it will take them to reach base camp

c. calculate how far they will have walked in 6.5 hours.

25. Udaya is writing test questions. She has already written 25 questions and can write a further 5 questions per hour.

a. Represent this information as a linear equation where t hours is the time spent writing test questions and n is the number of questions written.

b. Predict the total number of questions written after a further 8 hours assuming the same linear rule.

c. Calculate how long, to the nearest minute, it will take Udaya to have written 53 questions.

26. Catherine earns a daily rate of \$200 for working in her mother's store. She receives \$5 for each necklace that she sells.

a. Write an equation to show how much money (m) Catherine earned for the day after selling (n) necklaces.
b. Graph the equation that you created in part **a**, showing the two intercepts.
c. Explain which part of the line applies to her earnings.
d. Explain which part of the line does not apply to her earnings.

27. Calculate the gradient of the line passing through the points $(2, 3)$ and $(6 + 4t, 5 + 2t)$. Write your answer in simplest form.

28. Determine the point on the line $y = 2x + 7$ that is also 5 units above the x-axis.

29. An experiment was conducted to collect data for two variables p and t. The data are presented in the table shown.

p	$-\frac{1}{2}$		$\frac{1}{2}$	3.6
t	$2\frac{1}{4}$	1.75		-5.95

It is known that the relationship between p and t is a linear one. Determine the two values missing from the table.

30. The distance from the origin to the y-intercept of a linear graph is three times the distance from the origin to the x-intercept. The area of the triangle formed by the line and the axes is 3.375 units^2. The line has a negative gradient and a negative y-intercept.
Determine the equation of the line.

on To test your understanding and knowledge of this topic go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 4 Linear graphs

4.1 Pre-test

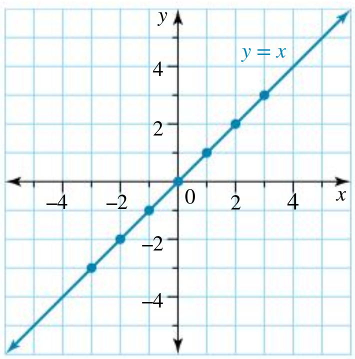
1. On the x-axis (because $y = 0$).
2. a. C b. D c. B d. A
3. C
4. –0.5
5. D
6. a. B b. C c. D d. A
7. $y = -\frac{1}{4}x$
8. $y = 20x + 400$
9. $a = -5$
10. $\sqrt{29}$
11. D
12. a. $y = -x + 2$ b. $(-1, 3)$
13. Gradient $= -\frac{1}{2}$
    y-intercept $= (0, 2.5)$
14. $(-1, 1)$
15. $(-1, -5)$

4.2 Plotting linear graphs

1. a. B b. C c. D
2. a. i. $y = x$

x	–3	–2	–1	0	1	2	3
y	–3	–2	–1	0	1	2	3

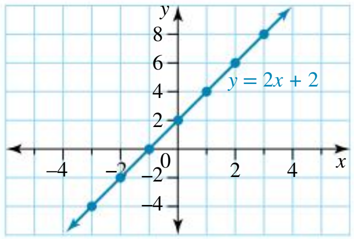
ii. $y = x$



b. i. $y = 2x + 2$

x	–3	–2	–1	0	1	2	3
y	–4	–2	0	2	4	6	8

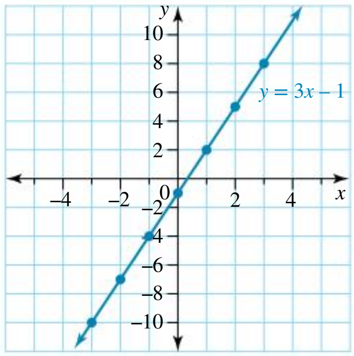
ii. $y = 2x + 2$



c. i. $y = 3x - 1$

x	–3	–2	–1	0	1	2	3
y	–10	–7	–4	–1	2	5	8

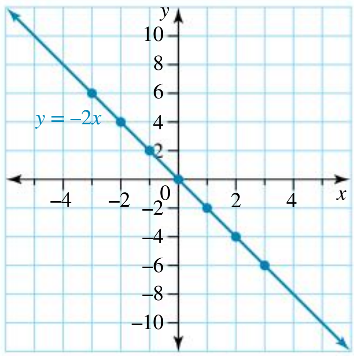
ii. $y = 3x - 1$



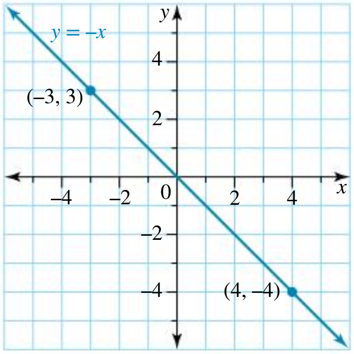
d. i. $y = -2x$

x	–3	–2	–1	0	1	2	3
y	6	4	2	0	–2	–4	–6

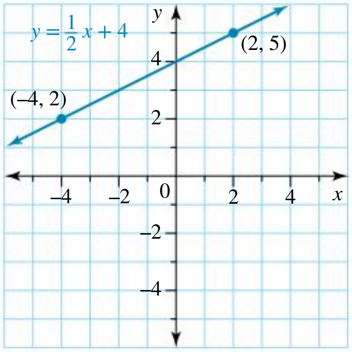
ii. $y = -2x$



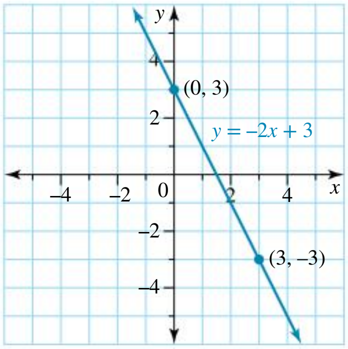
3. a. Sample points shown; $y = -x$



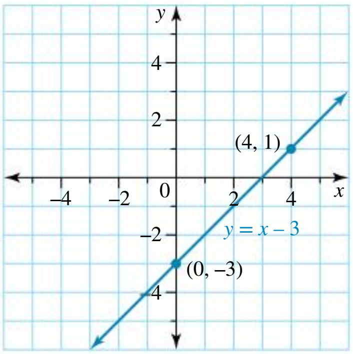
b. Sample points shown; $y = \frac{1}{2}x + 4$



c. Sample points shown; $y = -2x + 3$



d. Sample points shown; $y = x - 3$



4. a. Yes **b.** No **c.** No **d.** Yes

5. a. Yes **b.** Yes **c.** Yes
d. No **e.** No **f.** No

6. D

7. a. B **b.** A **c.** D **d.** C

8. By plotting the points on the Cartesian plane and joining them to make a line it can be seen that this line does not pass through the third quadrant.

9. a. The first, second and third quadrants.

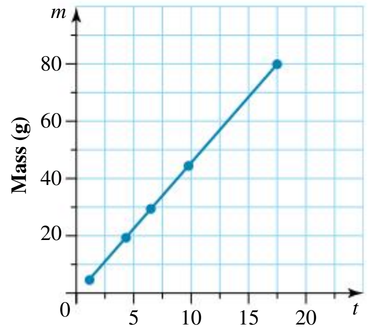
b. Answers will vary. Sample response: Substitute point $(1, 3)$ into line equation. $3 \neq 1 + 1$.

10. Answers will vary. Sample response: Substitute the point into the line equation. If the LHS equals the RHS of the equation, then the point lies on the line. Alternatively, graph the line and check the point on the Cartesian plane.

11. Answers will vary. Sample response: Plot the points on the Cartesian plane and draw a line through the points. Find the equation of the line. Check all points lie on the line by substituting into the line equation.

12. $(59, 21)$

13. a.



b. $m = 5t + 1$

c. Sample responses can be found in the worked solutions in the online resources.

14. Stirring increases the rate of reaction.

4.3 Features of linear graphs

1. a. 6 **b.** -4 **c.** 0
d. $\frac{6}{5}$ **e.** -2.2 **f.** -45

2. a. 0 **b.** 0.1 **c.** -2.6
d. 0 **e.** 9 **f.** -0.1

3. a. 23 **b.** 54 **c.** -6 **d.** 70

4. a. 1 **b.** 5.2 **c.** 100 **d.** 100

5. a. 4 **b.** 1 **c.** 0

6. a. 20 **b.** 400 **c.** -5.2

7. a. 0 **b.** -5 **c.** -300

8. C

9. y-intercept $= c$

10. a. $(0, -10)$ **b.** $(0, -4)$ **c.** $(0, 3)$
d. $(0, 7)$ **e.** No y-intercept

11. There is a rise and no run; $\frac{\text{rise}}{0} =$ undefined.

12. There is no rise and a run; $\frac{0}{\text{run}} = 0$

13. Sample responses can be found in the worked solutions in the online resources.

14. It does not matter if you rise before you run or run before you rise, as long as you take into account whether the rise or run is negative.

15. a. $m = \frac{y-c}{x}$ b. $y = mx + c$

16. a. 0.28 b. 0.64

17. $0.6 > 0.58$, therefore the safety standards have not been met.

18. 0.3 metres

19. a. Lamb: $m = \$15/\text{kg}$, Chicken: $m = \$10/\text{kg}$, Beef: $m = \$7.50/\text{kg}$

b. Lamb: \$15, Chicken: \$10, Beef: \$7.50

c. i. \$15 ii. \$5 iii. \$15

d. \$35

e.

Meat type	Cost per kilogram (\$/kg)	Weight required (kg)	Cost = \$/kg × kg
Lamb	15	1	15
Chicken	10	0.5	5
Beef	7.50	2	15
		Total cost	35

20. a. $m = -\frac{3}{2}$

b. It does not matter which points are chosen to determine the gradient of the graph because the gradient will always remain the same.

c. Straight line with a y-intercept of $(0, -2)$ and a slope of $-\frac{3}{2}$

4.4 The equation of a straight line

1. a. $y = 4x + 2$ b. $y = -4x + 1$ c. $y = 4x + 8$

2. a. $y = 6x + 7$ b. $y = -2.5x + 6$ c. $y = 45x + 135$

3. a. $y = -2x$ b. $y = 4x$ c. $y = 10.5x$

4. a. $y = -20x$ b. $y = 1.07x$ c. $y = 32x$

5. a. $y = x + 2$ b. $y = -x + 8$ c. $y = -4x - 8$ d. $y = 2x - 13$

6. a. $y = -5x + 70$ b. $y = 2x - 23$ c. $y = -6x + 11$ d. $y = -x - 1.5$

7. a. $y = 6x + 30$ b. $y = -3.5x + 15.5$ c. $y = 1.2x - 4.08$ d. $y = 0.2x - 1.76$

8. a. $y = -4x - 24$ b. $y = 2x - 6$ c. $y = -2x + 4$

9. a. $y = 5x + 35$ b. $y = 1.5x - 3.75$ c. $y = 0.4x - 0.96$

10. a. $y = x + 17$ b. $y = -x + 3$ c. $y = 11$ d. $y = x + 3.5$

11. a. $y = -x + 3.5$

b. $y = \frac{1}{9}x + \frac{34}{9}$ $(\text{or } 9y = x + 34)$

c. $y = -6x + 90$

d. $y = 20x$

12. a. $y = -x + 4$

b. $y = -0.25x + 5.5$

c. $y = \frac{2}{7}x - \frac{32}{7}$ $(\text{or } 7y = 2x - 32)$

d. $y = \frac{2}{3}x + \frac{10}{3}$ $(\text{or } 3y = 2x + 10)$

13. a. $y = x + 3$ b. $y = \frac{-5}{4}x + 5$ c. $y = -6x + 6$ d. $y = \frac{7}{8}x + 35$

14. a. $y = x + 8$ b. $y = -2x + 6$ c. $y = \frac{-3}{7}x - 3$ d. $y = \frac{1}{4}x + 50$

15. a. $y = \frac{7}{4}x$ b. $y = x$ c. $y = -2x$ d. $y = -3x$ e. $y = \frac{-24}{11}x$ f. $y = \frac{-x}{3}$

16. a. $y = 2x + 4$ b. $y = 5x$

17. a. $y = -x + 3$ b. $y = \frac{1}{5}x$

18. a. $y = x + 2$ b. $y = -2x + 3$

19. a. B b. A c. D

20. a. $y = -2.4x - 12$ b. $y = -58.32$

21. a. $y = -0.6x + 10$ b. $\left(16\frac{2}{3}, 0\right)$

22. a. The equation is $y = \frac{5}{4}x - 2135$

b. Values of c cannot be read directly from the graph because the graph doesn't contain the origin, and since c is found at $x = 0$, we need to use the equation.

c. In 2040 the concentration of CO_2 will be approximately 430 ppm. The assumption is that the concentration of CO_2 will continue to follow this linear pattern.

23. $m = \frac{-15 + 7}{4 - 0} = -2$

$y = -2x + c$
$6 = -6 + c$
$c = 12$

The equation is $y = -2x + 12$.

24. a. Line A $y = x - 1$, Line B: $y = -x + 8$

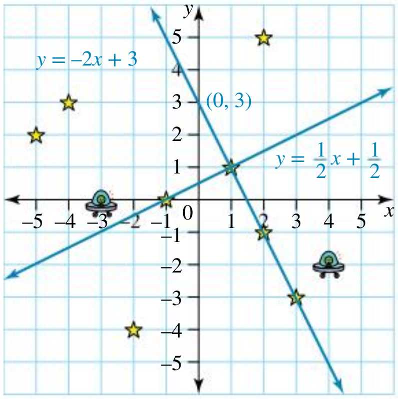
b. $(4.5, 3.5)$

c. Sample responses can be found in the worked solutions in the online resources. Possible methods include graphing on a Cartesian plane or using algebra.

Line B : $y = -x + 8, m = -1, m_{\perp} = 1$
$y = x + c$
$6 = -4 + c$
$c = 10$

25. $m = \frac{1}{4}C$

26. a, c



b. $y = -2x + 3$ **c.** $(1, 1)$ **d.** 1 star

e. Sample responses can be found in the worked solutions in the online resources. Possible responses could include the following.

$y = -x - 1, x = 2$

27. Sample responses can be found in the worked solutions in the online resources. Responses should include the correct equation: $T = 4.25t + 15$.

28. 8.1 mL

29. a. A: \$0, B: \$100, C: \$200

b. A: \$80, B: \$50 C: \$33.33

c. A: $w = 80\text{h}$, B: $w = 50\text{h} + 100$, C: $w = 33.3\text{h} + 200$

d. i. C **ii.** A

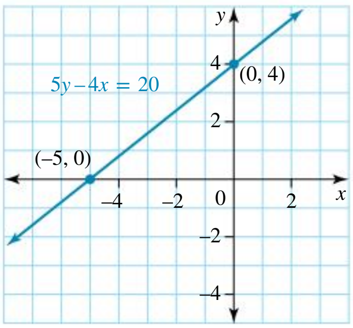
e. The advantage of location C is that it has the highest minimum pay, so you would be guaranteed at least \$200 per day, regardless of how many hours you work.

f. i. A: \$ 160, B: \$ 200, C: \$ 266.66

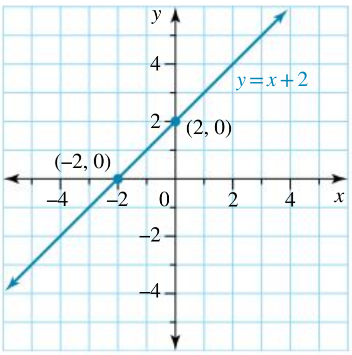
ii. A: \$ 480, B: \$ 400, C: \$ 400

4.5 Sketching linear graphs

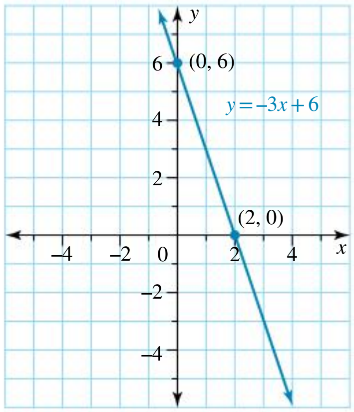
1. a. $5y - 4x = 20$



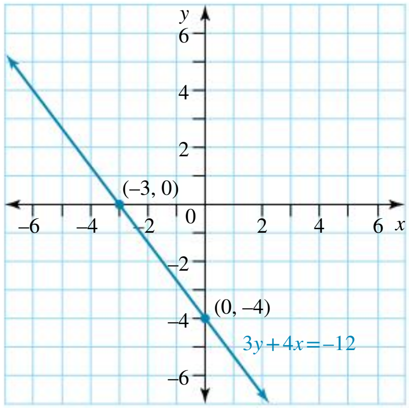
b. $y = x + 2$



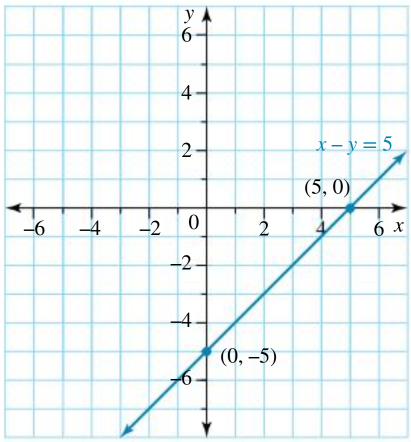
c. $y = -3x + 6$



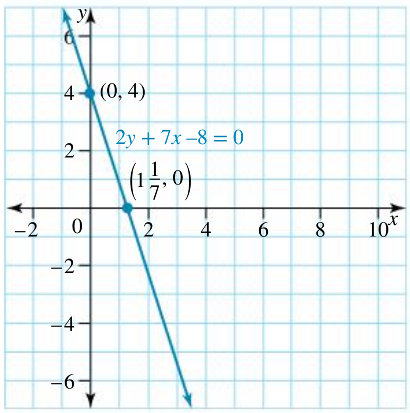
2. a. $3y + 4x = -12$



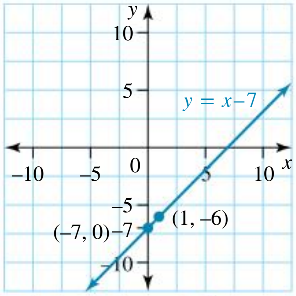
b. $x - y = 5$



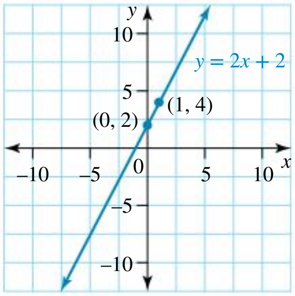
c. $2y + 7x - 8 = 0$



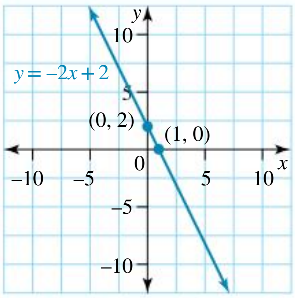
3. a.



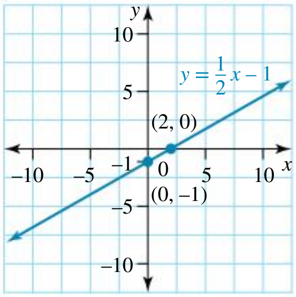
b.



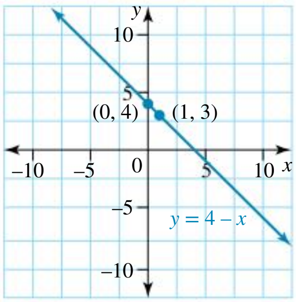
c.



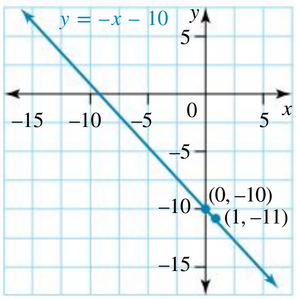
4. a.



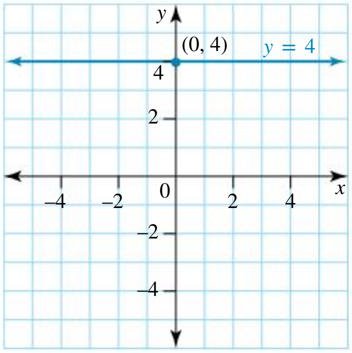
b.



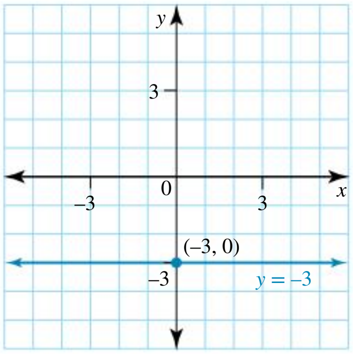
c.



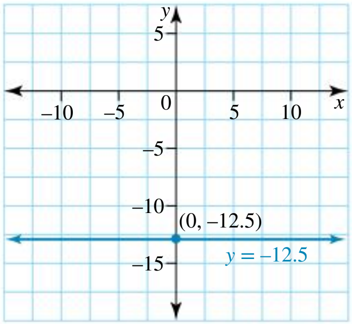
5. a.



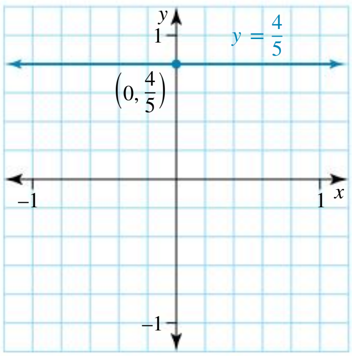
b.



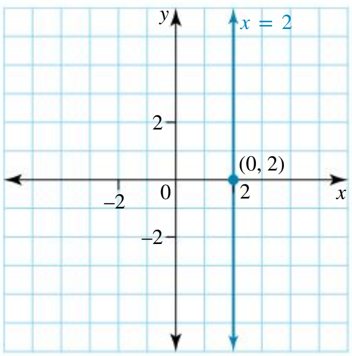
6. a.



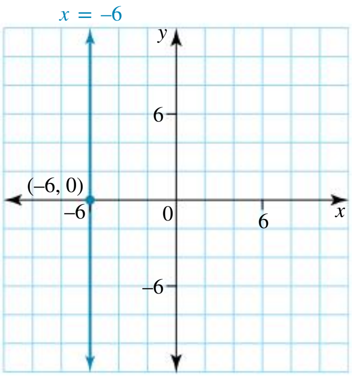
b.



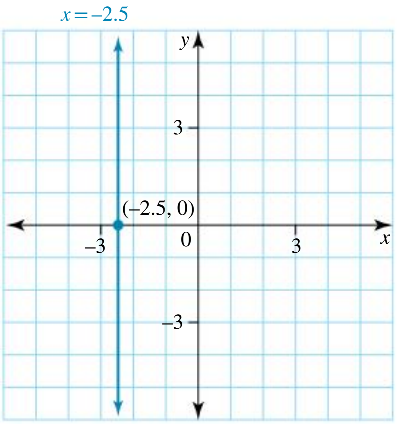
7. a.



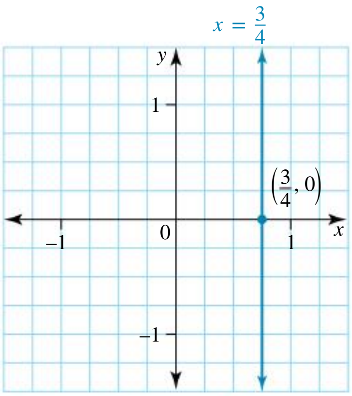
b.



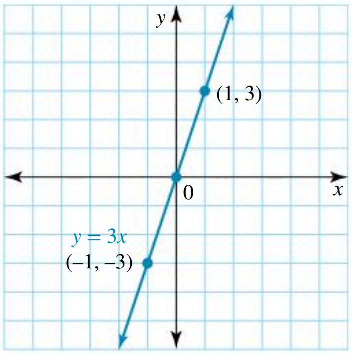
8. a.



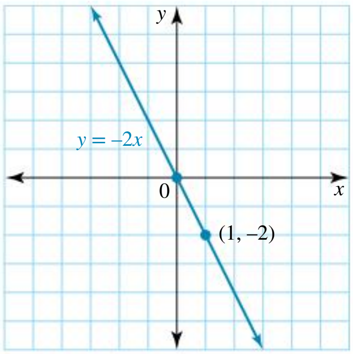
b.



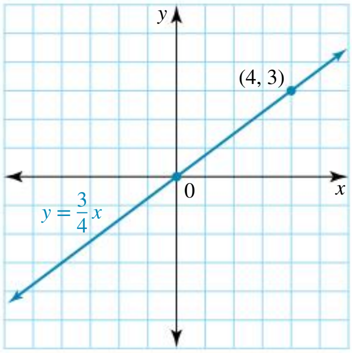
9. a.



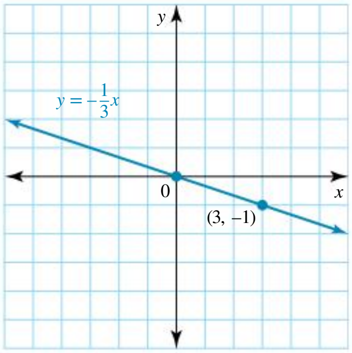
b.



10. a.



b.



11. D

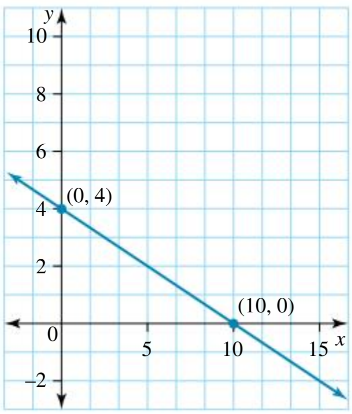
12. D

13. a. $y = -\dfrac{2x}{5} + 4$

b. $-\dfrac{2}{5}$

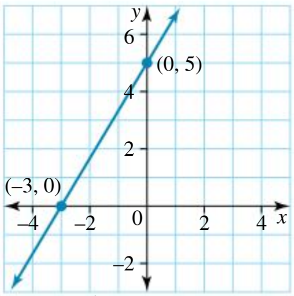
c. The x-intercept is 10, the y-intercept is 4.

d.



14. a. $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{5}{3}$

b.

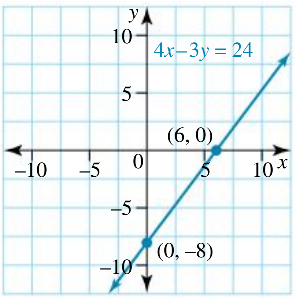


c. **i.** $y = \dfrac{5x}{3} + 5$ **ii.** $-5x + 3y = 15$

15. a. $y = \dfrac{4}{3}x - 8$

b. $y = 0, 4x = 24, x = 6$
$x = 0, -3y = 24, y = -8$

c.



16. a. $y = \dfrac{-ax + c}{b}$

b. Sample response: The gradient is $\dfrac{-a}{b}$ substituting positive values always results in a negative gradient.

17. Sample response:

$4y = \dfrac{4}{2}x + 3$

$4y = 2x + 3$

$-2x + 4y = 3$

18. Sample response: All descriptions use the idea that a gradient is equal to $\dfrac{\text{rise}}{\text{sun}}$, which equals $\dfrac{2}{3}$ in all of these cases.

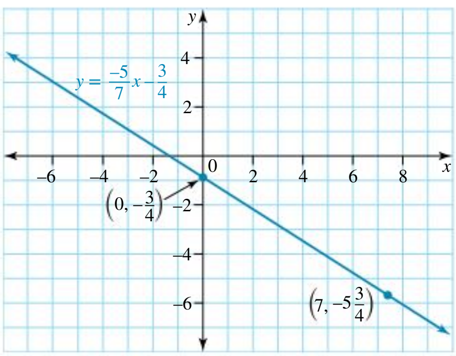
19. a. The gradient is 2. **b.** The equation of the line is $y = 2x - 1$.

c. One point is $(1, 1)$.

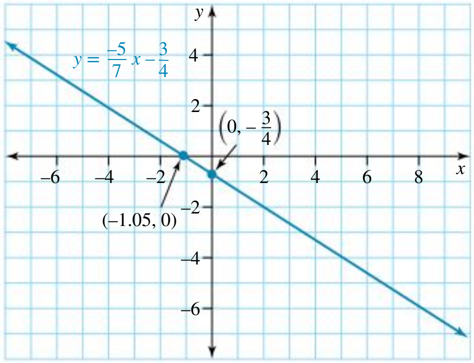
20. a. **i.** C **ii.** B **iii.** A

b. Straight line y-intercept of $(0, 2)$ with a slope of $-\dfrac{1}{4}$

21. a. **i.**



ii.



iii. Sample responses can be found in the worked solutions in the online resources.

b. Sample responses can be found in the worked solutions in the online resources.

22. Sample responses can be found in the worked solutions in the online resources.

4.6 Technology and linear graphs

1. i. As the size of the coefficient increases, the steepness of the graph increases.

ii. Each graph cuts the x-axis at $(0, 0)$.

iii. Each graph cuts the y-axis at $(0, 0)$.

2. i. As the size of the coefficient decreases, the steepness of the graph decreases.

ii. Each graph cuts the x-axis at $(0, 0)$

iii. Each graph cuts the y-axis at $(0, 0)$

3. i. As the magnitude of the coefficient decreases, the steepness of the graph increases.

ii. Each graph cuts the x-axis at $(0, 0)$.

iii. Each graph cuts the y-axis at $(0, 0)$.

4. a. positive, positive **b.** downward, negative

c. bigger **d.** will

5. a. Yes, 1

b. No, the lines are parallel.

c. **i.** $(0, 0), x = 0$

ii. $(-2, 0), x = -2$

iii. $(2, 0), x = 2$

d. **i.** $(0, 0), y = 0$

ii. $(0, 2), y = 2$

iii. $(0, -2), y = -2$

6. a. Yes, -1

b. No, the lines are parallel.

c. **i.** $(0, 0), x = 0$

ii. $(2, 0), x = 2$

iii. $(-2, 0), x = -2$

d. **i.** $(0, 0), y = 0$

ii. $(0, 2), y = 2$

iii. $(0, -2), y = -2$

7. a. same **b.** y-intercept, y-axis

c. y-intercept **d.** x-intercept

8. i. No

ii. Yes

iii. **a.** 1 **b.** -1 **c.** 3 **d.** $-\dfrac{2}{5}$

iv. a. $(-5, 0)$, i.e. $x = -5$

b. $(5, 0)$, i.e. $x = 5$

c. $\left(-\frac{5}{3}, 0\right)$, i.e. $x = -\frac{5}{3}$

d. $\left(\frac{25}{2}, 0\right)$, i.e. $x = \frac{25}{2}$

v. a. $(0, 5)$, i.e. $y = 5$

b. $(0, 5)$, i.e. $y = 5$

c. $(0, 5)$, i.e. $y = 5$

d. $(0, 5)$, i.e. $y = 5$

9. a. steepness
b. y-coordinate, y-axis
c. same, parallel
d. the same

10. a. Gradient = 2, y-intercept = (0, 0), $y = 0$
b. Gradient = 1, y-intercept = (0, 1), $y = 1$
c. Gradient = –3, y-intercept = (0, 5), $y = 5$
d. Gradient = $\frac{2}{3}$, y-intercept = (0, –7), $y = -7$

11. Sample response: They all have in common the point $(1, 5)$. Their gradients are all different.

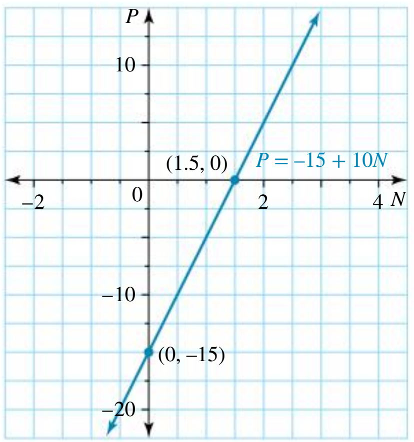
12. Sample responses can be found in the worked solutions in the online resources. Use solve on your CAS calculator.

13. Answers should show the y-intercept $(0, 2.2)$, point $(1, 2.75)$ and gradient 0.55.

14. a. N is the number of dogs walked; $-\$15$ is Shirly's starting cost and out-of-pocket expense before she walks a dog, and she earns \$10 for every dog walked.

b. She needs to walk at least two dogs a week before she can make a profit.

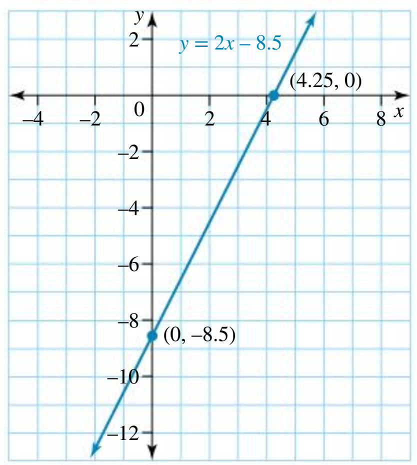
c.



15. Sample responses can be found in the worked solutions in the online resources.

16. a. $y = 2x - 8.5$

b.



c. $y = 7.5$

d. $x = 10.25$

17. a. $C = 25n + 1000$

b. $(0, 1000)$. This is the initial cost of the site licence.

c. \$6000

d. 80

18. a. Sample response: Possible methods include plotting the points $(0, 325)$ and $(1, 328.8)$ or using the equation $c = 3.8m + 325$.

b. 382 calories

c. 32.9 minutes

4.7 Practical applications of linear graphs

1. a. $y = 6x - 5$ b. $y = -3x + 8$

2. a. $y = 2x + 4$ b. $y = 0.9x + 1.1$

3. a. $y = 3x + 1$ b. $y = -x + 17$

4. a. $v = -3t + 27$ b. $C = 3d + 8$

5. D

6. a. $A = 25t + 60$ b. \$5060
c. \$60

7. a. $C = 0.55s + 2.0$ b. \$8.60

8. a. $N = 400t + 3200$ b. 5400
c. 13 200

9. a. 27 b. 108

10. a. $w = -40t + 712$ b. 712 L
c. 18 days

11. Sample responses can be found in the worked solutions in the online resources.

12. a. $C = 2.57d + 3.50$
 b. \$33.06
 c. 7.20 km

13. $5 \times 21.4 = 107\text{ cm} = 1.07\text{ m}$

14. a. $N = -22t + 164$ b. 120
 c. 5 d. 7.5 months

15. a. See the figure at the foot of the page.*
 b. $(0, -800)$, $y = -800$. Fixed costs are \$800.
 c. $(4, 0)$, $x = 4$, this is the break-even amount
 d. 200, this is the sale price per print that contributes to profit

16. a. $C = 2.14d + 3.50$
 b. \$28.11
 c. \$52.93

17. a. $Y = 78.7A$ b. ¥196 750 c. $Y_T = 76.5A$

18. Cody arrives first.

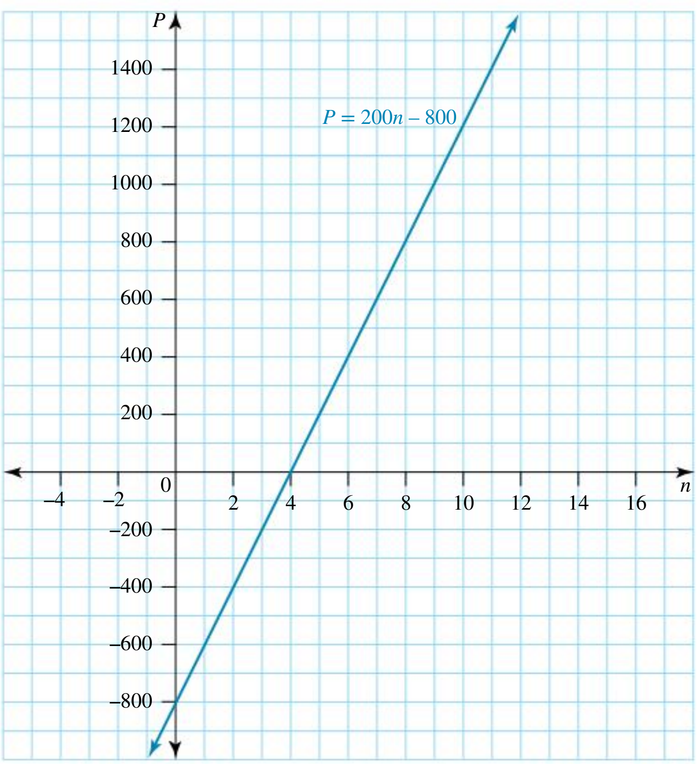
4.8 Midpoint of a line segment and distance between two points

1. a. $(2, 4)$ b. $(5, 1)$ c. $(7, 2)$
2. a. $(8, 9)$ b. $(0, 5)$ c. $(-1, 2)$
3. a. $(-5, -2)$ b. $(2, 3)$ c. $(0, 12)$
4. a. $\left(1\frac{1}{2}, 5\frac{1}{2}\right)$ b. $\left(-3, -3\frac{1}{2}\right)$
 c. $\left(-9\frac{1}{2}, 9\frac{1}{2}\right)$ d. $\left(-1\frac{1}{2}, 6\frac{1}{2}\right)$
5. a. 5 b. 9 c. -16 d. 0
6. a. $(4, 6)$ b. $(-2, -3)$
 c. $(11, 2)$ d. $(-7, -3)$
7. $y = 5x - 7$
8. a. 5 b. 10 c. 1
9. a. 2 b. 13 c. 10
10. a. 5 b. 17 c. 13
11. a. 7.211 b. 14.765 c. 13.038
12. a. 8.944 b. 19.416 c. 19.416
13. a. 24.472 b. 25.464
14. a. 17.788 b. 25.763
15. 21.024 km
16. $d = 5$
17. $-2 = \dfrac{x + 5}{2} \Rightarrow x = -9$

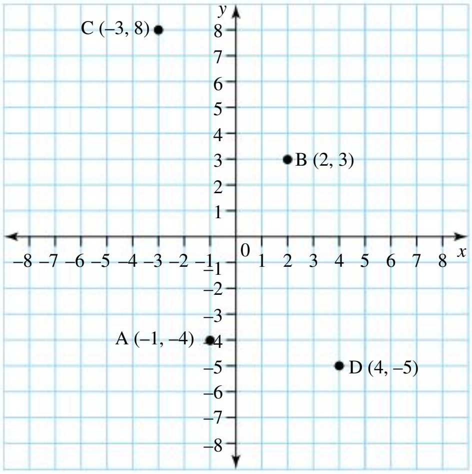
 $-4 = \dfrac{-6 + y}{2} \Rightarrow y = -2$

 The required point is $(-9, -2)$.
18. $d_{AB} = \sqrt[5]{10}$
 $d_{BC} = \sqrt[5]{10}$
 Therefore, B is equidistant from A and C.

*15. a.



19\. a.



b. $M = (-2, 2)$

c. $2\sqrt{37}$

d. $(7, -2)$

e. $\frac{3-(-4)}{2-(-1)} = \frac{7}{3}$

f. $y = -4x + 11$

20\. Sample responses can be found in the worked solutions in the online resources.

$d_{AB} = 2\sqrt{10}$

$d_{BC} = 2\sqrt{10}$

$d_{AC} = 4\sqrt{2}$

Side length AB is equal to side length BC but not equal to side length AC. Therefore, ΔABC is an isosceles triangle.

21\. $\frac{1}{2}$

22\. $(-4, 5), (-3, 2), (-2, -1), (-1, -4), (0, -7)$

Project

1. 6
2. Sample responses can be found in the worked solutions in the online resources.
3. a and h
4. e
5. a, b, c and h
6. Yes
7.

	Length of table (m)	Width of table (m)	Destination pocket
a.	3	0.25	Far left
b.	3	0.5	Far left
c.	3	0.75	Far left
d.	3	1	Far right
e.	3	1.25	Far left
f.	3	1.5	Far left
g.	3	1.75	Far right
h.	3	2	Close right

8\. Sample responses can be found in the worked solutions in the online resources.

4.9 Review questions

1. D
2. B
3. D
4. B
5. D
6. A
7. B
8. A
9. D

10\. a. Gradient = 8, y-intercept -3

b. Gradient = -9, y-intercept 5

c. Gradient = -2, y-intercept 6

d. Gradient = 2, y-intercept 0

e. Gradient = $\frac{2}{3}$, y-intercept $-\frac{1}{3}$

11\. A, B

12\. A, C, D

13\. a. 2 b. $-\frac{2}{5}$ c. 0

14\. a. 2 b. $\frac{5}{4}$

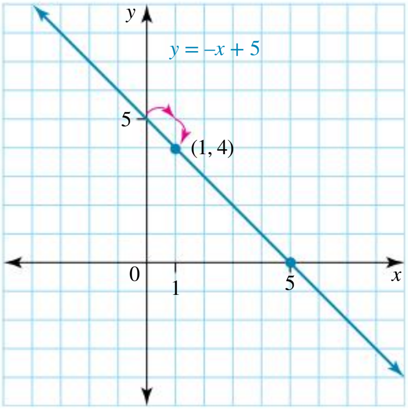
15\. a. $m = -3,\ c = 7$

b. $m = \frac{3}{2},\ c = 3$

c. $m = -\frac{2}{5},\ c = 0$

d. $m = 0,\ c = 4$

16\. a.



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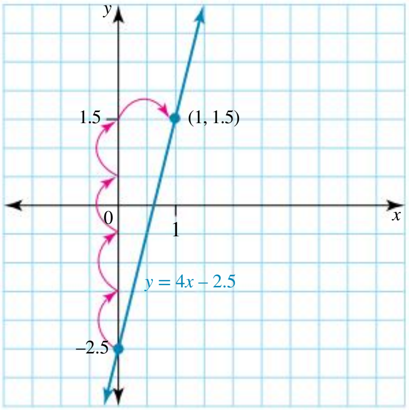
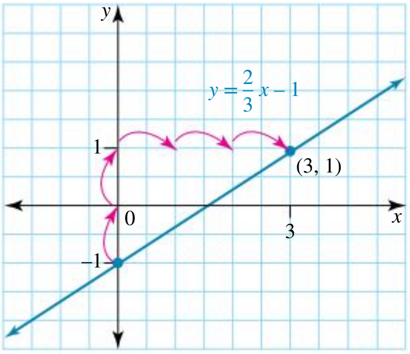


c.



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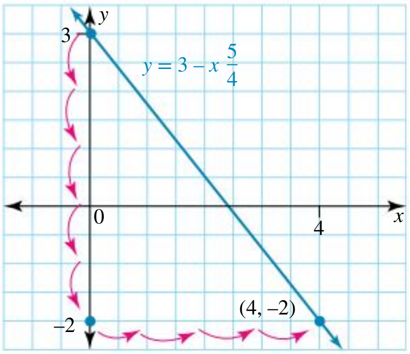
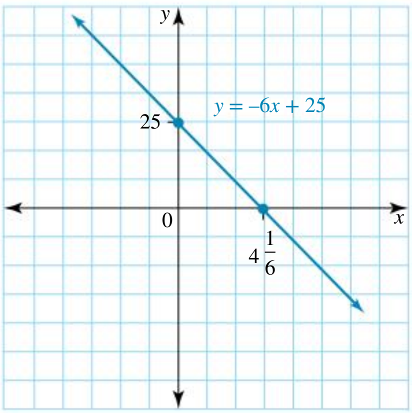


17. a.



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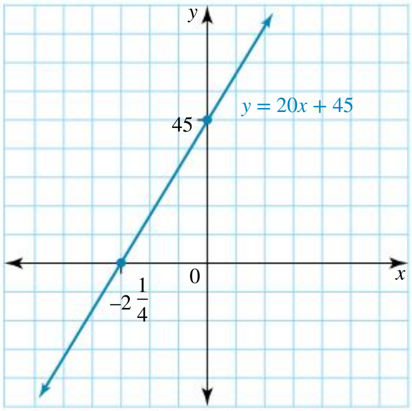

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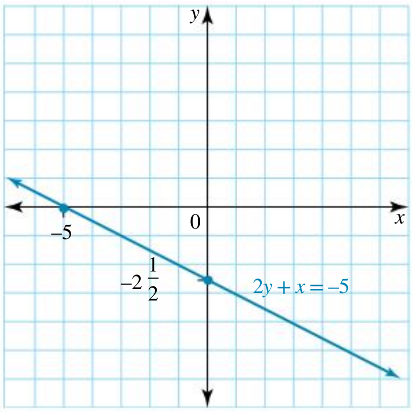
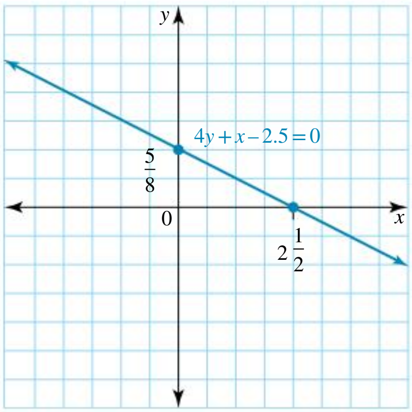
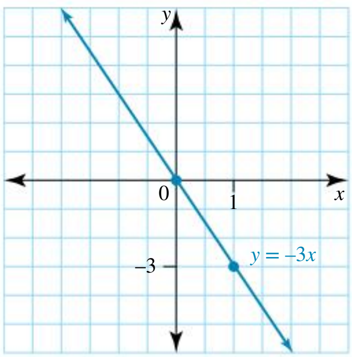


d.



18. a.



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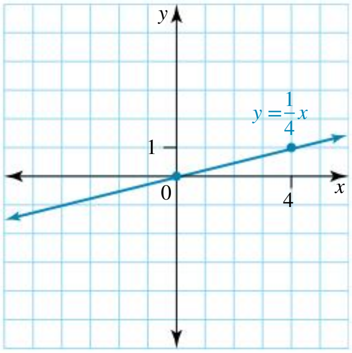
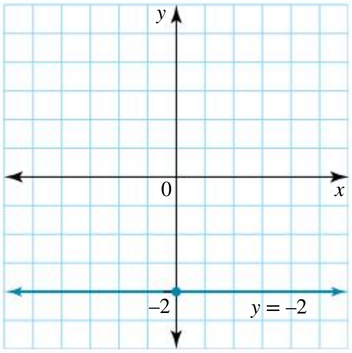


c.



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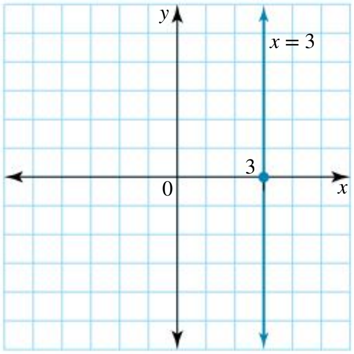


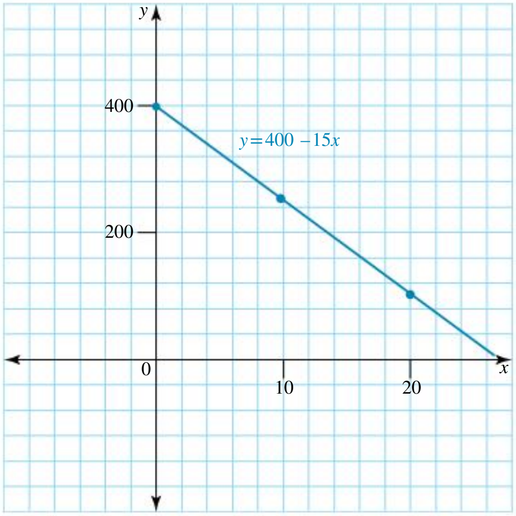
19. a. $y = 2x$ **b.** $y = 2 - \frac{2}{7}x$

20. a. $y = 2x - 7$ **b.** $y = 2x - 14$
c. $y = 2x - 5$ **d.** $y = -5x$
e. $y = -x - 2$ **f.** $y = -2.75x + 7.75$
g. $y = x - 3$ **h.** $y = -2x + 5$

21. (1.1)

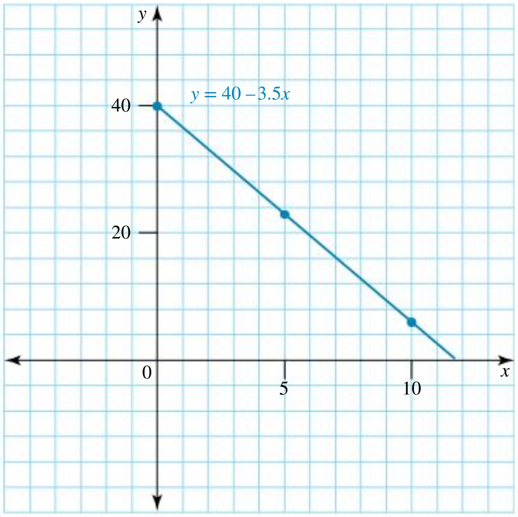
22. 5 units

23. a. $y = 400 - 15x$



b. 26.7 or 27 weeks
c. \$175
d. 21 repayments

24. a. $y = 40 - 3.5x$



b. 11.4 hours
c. 22.75 km

25. a. $n = 5t + 25$ **b.** 65 **c.** 336 minutes

26. a. $m = 5n + 200$

b. See the figure at the foot of the page.*

c. Everything to the right of the vertical axis and including the vertical axis applies to her earnings because she can only sell 0 or more necklaces.

d. Everything to the left of the vertical axis, because she cannot sell a negative number of necklaces and because she cannot make less than her base salary of $200.

27. $\frac{1}{2}$

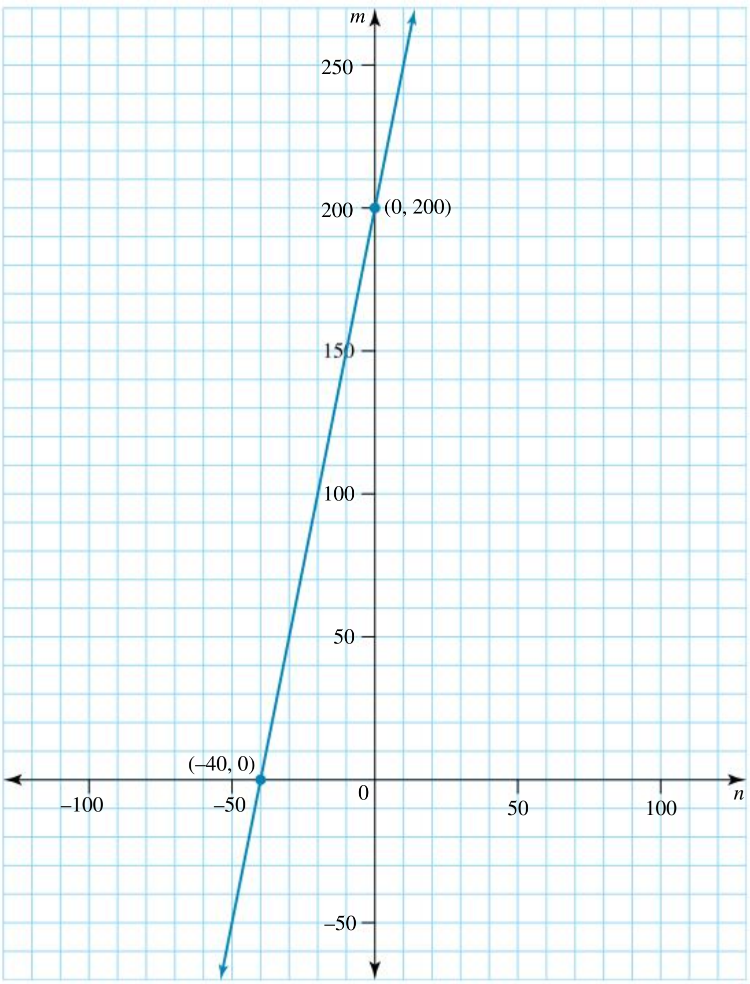
28. $(-1, 5)$

29. $p = -0.25, t = 0.25$

30. $y = -3x - 4.5$

---

*26. b.



STRAND: Number and algebra
SUB-STRAND: Linear and non-linear patterns and relationships

5 Proportions

LESSON SEQUENCE

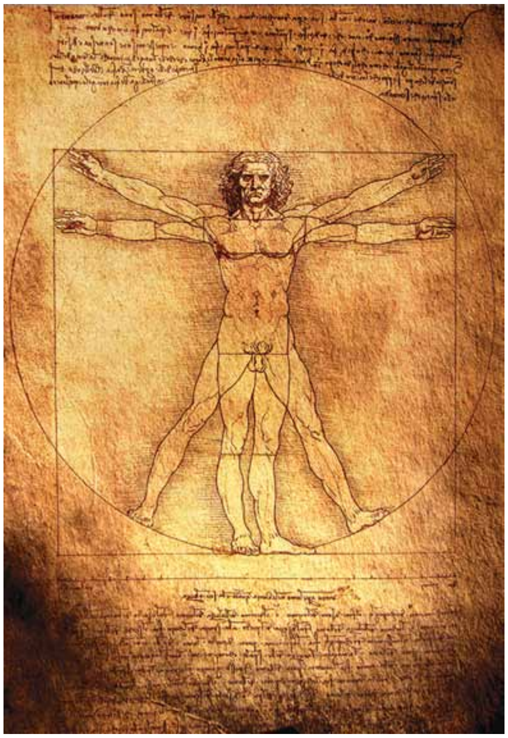
LESSON 5.1 Overview

5.1.1 Why learn this?

Proportional relationships appear in many areas of everyday life. When recipes are scaled up or down, ingredients are adjusted in fixed ratios. Builders and tradespeople use consistent proportions when mixing materials like concrete or paint to get the right consistency and strength.

In photography and graphic design, images are often resized while keeping their proportions the same, so they don't look stretched or distorted. Artists, architects, and engineers rely on proportionality when creating scale drawings or models that accurately represent real objects.

In geometry, the size of shapes may change, but their proportions remain consistent. For example, the circumference of a circle is proportional to its diameter. We say, simply, that the circumference, C, is pi (π) times its diameter, d; that is, $C = \pi d$. Calculations of π date back to the time of Archimedes, but the symbol became widely used in the 1700s. The Golden Ratio, approximately 1.618, is a proportion that is found in geometry, art and architecture and has been made famous in the illustrations of Leonardo da Vinci (1452–1519).



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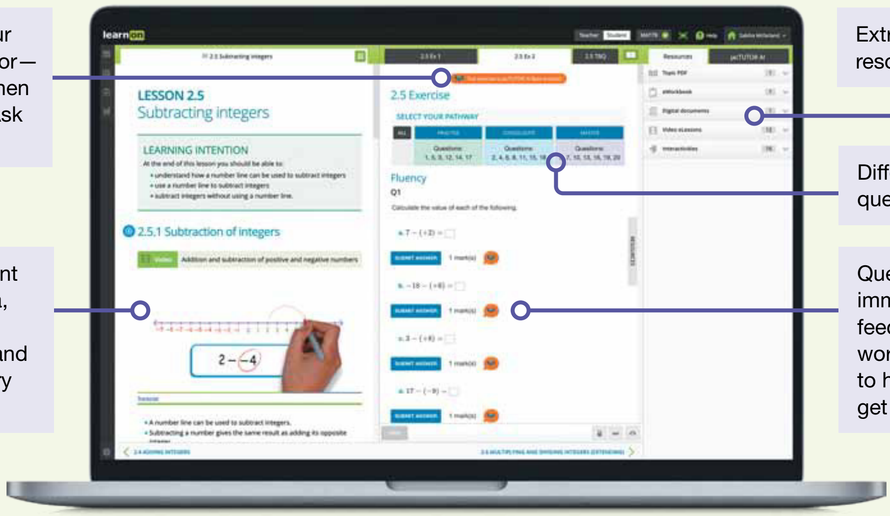

Extra learning resources

Differentiated question sets

Questions with immediate feedback, and fully worked solutions to help students get unstuck.

Exercise 5.1 Pre-test

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1. State whether this statement is True or False.
The expression $y \propto x$ means y is directly proportional to x.

2. **MC** Identify which of the following relationships is directly proportional.

A.

a	0	1	2	3
b	0	2	4	6

B.

a	0	1	2	3
b	0	1	4	9

C.

a	1	2	3	4
b	4	3	2	1

D.

a	0	1	2	3
b	1	3	5	7

3. Mobile phone calls are charged at \$1 per minute. State if the cost of a phone bill and the number of 1-minute time periods are directly proportional.

4. State whether this statement is True or False.
The perimeter of a square and its side length have a relationship that is directly proportional.

5. **MC** If $a \propto b$ and $a = 3$ when $b = 9$, select the rule that links a and b.
A. $a = \frac{1}{3}b$ **B.** $a = 3b$ **C.** $a = \frac{3}{b}$ **D.** $a = 9b$

6. The quantities x and y are related to each other by direct variation. If k is the constant of proportionality, complete the proportion statement $x =$ _____.

7. The voltage, v, that powers a mobile phone is directly proportional to the current, I amps.
A mobile phone with consistent resistance uses a current of two amps when powered by a voltage of 5.1 volts.
a. Determine the consistent resistance or constant of proportionality, k.
b. Calculate the voltage required to operate a mobile phone that would use a current of 1.2 amps.

8. State whether this statement is True or False.
The statement 'x varies directly with y' can be written as $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$.

9. **MC** If $y \propto x$ and $y = 12$ when $x = 3$, the proportionality constant is:
A. 12 **B.** 3 **C.** 36 **D.** 4

10. **MC** If $y \propto \frac{1}{x}$ and $y = 2$ when $x = 5$, the value of y when $x = 20$ is:
A. 2 **B.** 5 **C.** $\frac{1}{2}$ **D.** $\frac{1}{5}$

11. **MC** y is directly proportional to x.
If x is tripled, select what y is.
A. Doubled **B.** Tripled
C. Reduced to one-third **D.** Reduced to one-ninth

12. Given that $y \propto x$ and $\frac{y}{x} = 4$, determine the rule linking y and x.

13. A worker earns \$18 per hour. Write an equation to represent the relationship between hours worked, h, and total pay, P.

14. The number of cans in a vending machine is directly proportional to the number of rows. There are 72 cans in 6 rows. How many cans are there in 10 rows?

15. Is the following statement true or false? Explain your reasoning.
"If a graph is a straight line, then it shows a proportional relationship."

LESSON 5.2 Direct proportion

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- understand the concept of direct linear proportion
- recognise and write direct linear proportion statements.

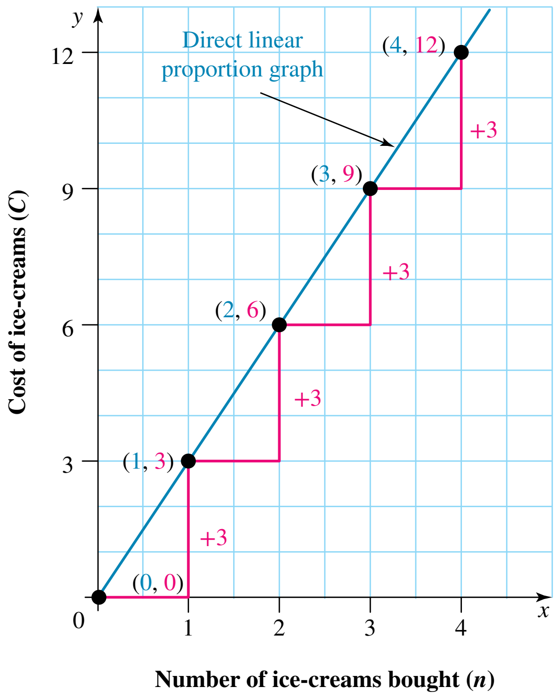
5.2.1 Direct linear proportion

- Proportion is described by two equal ratios.
- In **direct linear proportion**, the two variables change at the same rate.
- Suppose that ice-cream costs \$3 and that you are to buy some for your friends.
- There is a relationship between the cost of the ice-cream (C) and the number of ice-creams that you buy (n).
- The relationship between the two variables (number of ice-creams and cost of the ice-cream) can be illustrated in a table or a graph.

n	0	1	2	3	4
C (\$)	0	3	6	9	12

- This relationship is called direct linear proportion as it shows the following characteristics:
  - As n increases, so does C.
  - The graph of the relationship is a straight line passing through the origin as $n = 0$ and $C = 0$. (0 ice-creams costs \$0).

- We say that 'C is directly proportional to n' or 'C varies directly as n'.



Expressing direct linear proportion

If the y-values increase (or decrease) directly with the x-values, then this relationship can be expressed as:

$$y \propto x$$

where $\propto$ is the symbol for proportionality — this means that y is directly proportional to x.

WORKED EXAMPLE 1 Checking for direct linear proportion

For each of the following pairs of variables, state whether direct linear proportion exists.

a. The height of a stack of photocopy paper (h) and the number of sheets (n) in the stack.

b. Your Maths mark (m) and the number of hours of Maths homework you have completed (n).

THINK	WRITE
a. When n increases, so does h. When $n = 0$, $h = 0$. If graphed, the relationship would be linear.	a. $h \propto n$
b. As n increases, so does m. When $n = 0$, I may get a low mark but it is not necessarily zero, so $m \neq 0$.	b. m is not directly proportional to n.

WORKED EXAMPLE 2 Checking for direct linear proportion from a graph

For each of the following, determine whether direct linear proportion exists between the variables.

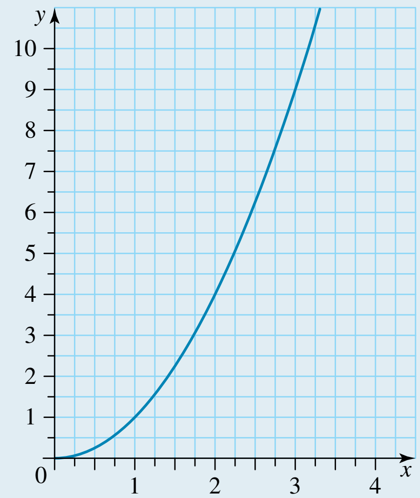
a.

t	0	1	2	3	4
y	0	1	3	7	15

b.

n	1	2	3	4
c	5	10	15	20

c.



THINK

a. From the table, when t increases, so does y.
When $t = 0$, $y = 0$. The t-values increase by a constant amount but the y-values do not, so the relationship is not linear.
When t is doubled, y is not.

b. From the table, as n increases, so does c.
Extending the pattern gives $n = 0$, $c = 0$. The n-values and C-values increase by constant amounts, so the relationship is linear.

c. When x increases, so does y.
When $x = 0$, $y = 0$.
The graph is not a straight line.

WRITE

a. y is not directly proportional to t.

b. $C \propto n$

c. y is not directly proportional to x.

Note: Not all direct proportion relationships are linear. Non-linear direct proportion will be studied in later years.

DISCUSSION

How do you know when two quantities are directly proportional?

Discuss the impact on image editing if proportion is not maintained.

Exercise 5.2 Direct proportion

learnon

5.2 Quick quiz on | 5.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 6, 9, 13	2, 5, 7, 10, 11, 14	4, 8, 12, 15, 16

Fluency

1. WE1 For each of the following pairs of variables, state whether direct linear proportion exists. If it does not exist, give a reason why.
   a. The distance (d) travelled in a car travelling at 60 km/h and the time taken (t).
   b. The speed of a swimmer (s) and the time the swimmer takes to complete one lap of the pool (t).
   c. The cost of a bus ticket (c) and the distance travelled (d).

2. For each of the following pairs of variables, state whether direct linear proportion exists. If it does not exist, give a reason why.
   a. The perimeter (p) of a square and the side length (l).
   b. The area of a square (A) and the side length (l).
   c. The total cost (C) and the number of boxes of pencils purchased (n).
   d. The length of a line in centimetres (c) and the length in millimetres (m).

3. For each of the following pairs of variables, state whether direct linear proportion exists. If it does not exist, give a reason why.
   a. The weight of an object in kilograms (k) and in pounds (p).
   b. The distance (d) travelled in a taxi and the cost (c).
   c. A person's height (h) and their age (a).
   d. The cost of an item in Australian dollars (A) and the cost of an item in Euros (E).

4. State whether the following statements are True or False.
   There is a direct linear relationship between:
   a. the total cost, $\$C$, of purchasing netballs and the number, n, purchased
   b. the circumference of a circle and its diameter
   c. the area of a semicircle and its radius.

5. MC For the table shown, if $y \propto x$, then the values of a and b would be:

x	1	2	4	8
y	3	6	a	b

A. $a = 8,\ b = 16$ B. $a = 8,\ b = 24$ C. $a = 9,\ b = 12$ D. $a = 12,\ b = 24$

Understanding

6. WE2 For each of the following, determine whether direct linear proportion exists between the variables. If it does not, explain why.

a.

x	0	1	2	3	4
y	0	1	3	8	15

b.

a	0	1	2	3	6
M	0	8	16	24	48

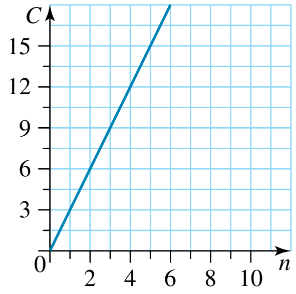
c.

t	0	1	2	4	8
d	0	3	6	9	12

d.

n	0	1	2	3	4
C	10	20	30	40	50

7. For each of the following, determine whether direct proportion exists between the variables. If it does not, explain why.

a.


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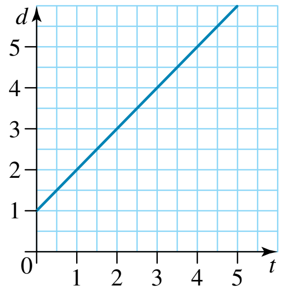

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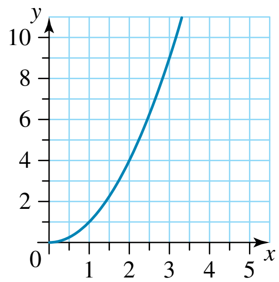
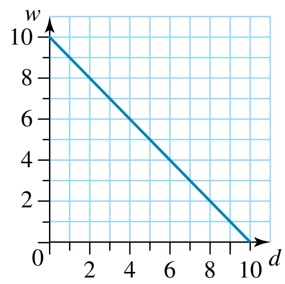


d.


8. List five pairs of real-life variables that exhibit direct linear proportion.

Reasoning

9. Explain which point must always exist in a table of values if the two variables exhibit direct linear proportionality.

10. If direct linear proportion exists between two variables m and n, fill out the table and explain your reasoning.

m	0	2	5
n	0		20

11. If the variables x and y in the following table are directly proportional, find the values of a, b and c. Explain your reasoning.

x	2	4	9	a
y	8	b	c	128

12. If $y \propto x$, explain what happens to:

a. y if x is doubled
b. y if x is halved
c. x if y is tripled.

Problem solving

13. Mobile phone calls are charged at 17 cents per 30 seconds.

a. Does direct linear proportion exist between the cost of a phone bill and the number of 30-second time periods? Justify your answer.
b. If a call went for 7.5 minutes, determine how much the call would cost.

14. An electrician charges $55 for every 30 minutes or part of 30 minutes for his labour on a building site.

a. Does direct linear proportion exist between the cost of hiring the electrician and the time spent at the building site? Justify your answer.
b. If the electrician worked for 8.5 hours, determine how much this would cost.

15. A one-litre can of paint covers five square metres of wall.
   a. Does a direct linear proportion exist between the number of litres purchased and the area of the walls to be painted? Justify your answer.
   b. Evaluate the number of cans needed to paint a wall 5 metres long and 2.9 metres high with two coats of paint.

16. Bruce is building a pergola and needs to buy treated pine timber. He wants 4.2-metre and 5.4-metre lengths of timber. If a 4.2-metre length costs $23.10 and a 5.4-metre length costs $29.16, determine if direct linear proportion exists between the cost of the timber and the length of the timber per metre.

LESSON 5.3 Proportionality

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- calculate the constant of proportionality
- express the constant of proportionality as a ratio of the two quantities.

5.3.1 Constant of proportionality

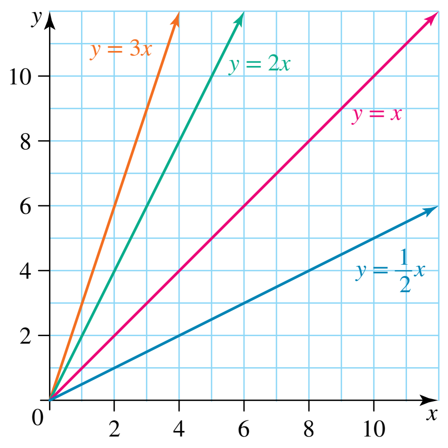
- Proportionality relationships ($y \propto x$) are shown by the graph with four straight lines:

$y = \frac{1}{2}x$

$y = x$

$y = 2x$

$y = 3x$



- For the equation $y = kx$:
  - k is a constant, called the **constant of proportionality**
  - y is directly proportional to x, or $y \propto x$
  - all the graphs start at $(0, 0)$
  - y is called the **dependent variable** and is normally placed on the vertical axis (y-axis)
  - x is called the **independent variable** and is normally placed on the horizontal axis (x-axis).

WORKED EXAMPLE 3 Calculating the constant of proportionality

Given that $y \propto x$ and $y = 12$ when $x = 3$, calculate the constant of proportionality and state the rule linking y and x.

THINK	WRITE
1. $y \propto x$, so write the linear rule.	$y = kx$
2. Substitute $y = 12$ and $x = 3$ into $y = kx$.	$12 = 3k$
3. Calculate the constant of proportionality by solving for k.	$k = 4$
4. Write the rule.	$y = 4x$

WORKED EXAMPLE 4 Determining the direct proportion relationship

The weight (W) of \$1 coins in a bag is directly proportional to the number of coins (n). Twenty coins weigh 180 g.

a. Determine the relationship between W and n.

b. Calculate how much 57 coins weigh.

c. Determine how many coins weigh 252 g.

THINK

Summarise the information given in a table.

WRITE

$W = kn$

n	20	57	
W	180		252

THINK	WRITE
a. 1. Substitute $n = 20$ and $W = 180$ into $W = kn$.	a. $180 = 20k$
2. Solve for k.	$k = 9$
3. Write the relationship between W and n.	$W = 9n$
b. 1. State the rule.	b. $W = 9n$
2. Substitute 57 for n to find W.	$W = 9 \times 57$ $= 513$
3. Write the answer in a sentence.	Fifty-seven coins weigh 513 g.
c. 1. State the rule.	c. $W = 9n$
2. Substitute 252 for W.	$252 = 9n$
3. Solve for n.	$n = \dfrac{252}{9}$ $= 28$
4. Write the answer in a sentence.	Twenty-eight coins weigh 252 g.

5.3.2 Ratio

- If $y \propto x$, then $y = kx$, where k is constant.
  Transposing this formula gives $\frac{y}{x} = k$.
- The constant of proportionality, k, is the ratio of any pair of values (x, y).
- For example, this table shows that $v \propto t$.

t	1	2	3	4
v	5	10	15	20

It is clear that $\frac{20}{4} = \frac{15}{3} = \frac{10}{2} = \frac{5}{1} = 5$, therefore, $k = 5$ and $v = 5t$.

WORKED EXAMPLE 5 Applying the constant of proportionality as a ratio

Sharon works part-time and is paid at a fixed rate per hour. If she earns \$135 for 6 hours' work, calculate how much she will earn for 11 hours.

THINK	WRITE
1. Sharon's payment is directly proportional to the number of hours worked. Write the rule.	$P = kn$
2. Summarise the information given. The value of x needs to be calculated.	see table below
3. Since $p = kn$, $\frac{P}{n}$ is constant.	$\frac{135}{6} = \frac{x}{11}$
Solve for x.	$x = \frac{135 \times 11}{6}$ $x = 247.5$
4. Write the answer in a sentence.	Sharon earns \$247.50 for 11 hours of work.

n	6	11
P	135	x

Exercise 5.3 Proportionality

learnon

5.3 Quick quiz on	5.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 6, 7, 12, 15	3, 4, 8, 9, 13, 16	5, 10, 11, 14, 17, 18

Fluency

1. **WE3** If a is directly proportional to b, and $a = 30$ when $b = 5$, calculate the constant of proportionality and state the rule linking a and b.
2. If $a \propto b$, and $a = 2.5$ when $b = 5$, determine the rule linking a and b.
3. If $C \propto t$, and $C = 100$ when $t = 8$, determine the rule linking C and t.
4. If $v \propto t$, and $t = 20$ when $v = 10$, determine the rule linking v and t.
5. If $F \propto a$, and $a = 40$ when $F = 100$, determine the rule linking a and F.

Understanding

6. **WE4** Springs are often used to weigh objects, because the extension of a spring (E) is directly proportional to the weight (W) of the object hanging from the spring. A 4-kg load stretches a spring by 2.5 cm.
   a. Determine the relationship between E and W.
   b. Calculate the load that will stretch the spring by 12 cm.
   c. Determine how far 7 kg will extend the spring.

7. Han finds that 40 shelled almonds weigh 52 g.
   a. Determine the relationship between the weight (W) and the number of almonds (n).
   b. Calculate how many almonds there would be in a 500 g bag.
   c. Calculate how much 250 almonds would weigh.
8. Petra knows that her bicycle wheel turns 40 times when she travels 100 m.
   a. Determine the relationship between the distance travelled (d) and the number of turns of the wheel (n).
   b. Calculate how far she goes if her wheel turns 807 times.
   c. Determine how many times her wheel turns if she travels 5 km.
9. Fiona, who operates a plant nursery, uses large quantities of potting mix. Last week she used 96 kg of potting mix to place 800 seedlings in medium-sized pots.
   a. Determine the relationship between the mass of potting mix (M) and the number of seedlings (n).
   b. Calculate how many seedlings she can pot with her remaining 54 kg of potting mix.
   c. Determine how much potting mix she will need to pot 3000 more seedlings.
10. **WE5** Tamara is paid at a fixed rate per hour. If she earns \$136 for five hours of work, calculate how much she will earn for eight hours of work.
11. It costs \$158 to buy 40 bags of birdseed. Calculate how much 55 bags will cost.

Reasoning

12. If 2.5 L of lawn fertiliser will cover an area of 150 m^2, determine how much fertiliser is needed to cover an area of 800 m^2 at the same rate. Justify your answer.
13. Paul paid \$68.13 for 45 L of fuel. At the same rate, determine how much he would pay for 70 L. Justify your answer.
14. Rose gold is an alloy of gold and copper that is used to make high-quality musical instruments. If it takes 45 g of gold to produce 60 g of rose gold, evaluate how much gold would be needed to make 500 g of rose gold.
    Justify your answer.

Problem solving

15. If Noah takes a group of friends to the movies for his birthday and it would cost \$62.50 for five tickets, determine how much it would cost if there were12 people (including Noah) in the group.
16. Sharyn enjoys quality chocolate, so she makes a trip to her favourite chocolate shop. She is able to select her favourite chocolates for \$7.50 per 150 grams. Since Sharyn loves her chocolate, she decides to purchase 675 grams.
    Evaluate how much she spent.
17. Anthony drives from Perth to Exmouth, covering an average of 75 kilometers in 45 minutes. The total distance from Perth to Exmouth is 1260 km.

    a. Determine the relationship between the distance he travelled in kilometres, D, and his driving time in hours, T.
    b. Evaluate how long it will take Anthony to complete his trip.
    c. If he stops at Geraldton, 410 kilometres from his starting position, determine how long it will take him to reach Geraldton.
    d. Determine how much longer it will take him to arrive at Exmouth after leaving Geraldton.
18. The volume of a bird's egg can be determined by the formula $V = kl^3$, where V is the volume in cm^3, l is the length of the egg in cm and k is a constant. A typical ostrich egg is 15 cm long and has a volume of 7425 cm^3.
    Evaluate the volume of a chicken egg that is 5 cm long.

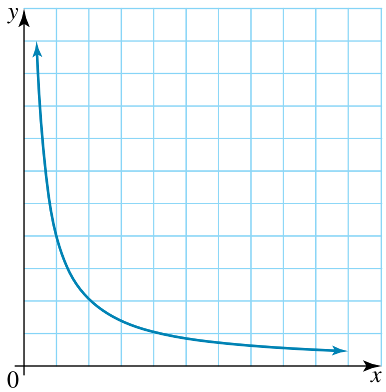
LESSON 5.4 Indirect proportion (optional)

LEARNING INTENTIONS

At the end of this lesson you should be able to:

- identify when two quantities indirect proportion
- create indirect proportion equations in the form $y = \frac{k}{x}$ to solve problems.

- If two quantities **vary indirectly**, then increasing one variable decreases the other.
- **Indirect proportion** produces the graph of a **hyperbola**.
- The statement 'quantity a varies indirectly with quantity b' can also be written as 'quantity a is indirectly proportional to quantity b'.
- If y varies indirectly to x, it is written as: $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$.
- As in direct variation, k is also called the *proportionality constant* or the *constant of proportionality*.



WORKED EXAMPLE 6 Determining indirect proportion equations

The table shows how the time that it takes to complete a task varies with the number of workers.

Number of workers	1	2	3	4	6
Time taken (hours)	12	6	4	3	2

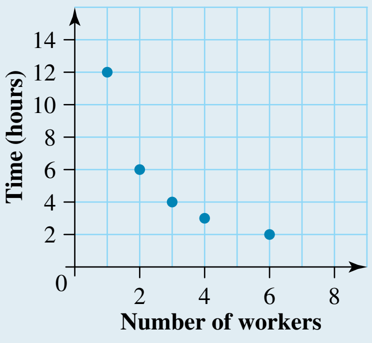
a. Draw a graph of the data.
b. Using the symbols t and n for the time and the number of workers respectively, and the variation symbol $\propto$, write a mathematical statement connecting the quantities.
c. Rewrite the statement from part b using an equals sign and the constant of variation k.
d. Choose one of the data points and calculate the value of k.
e. Restate the equation using the value of k.

THINK

a. The amount of time taken to complete the task depends on the number of workers; time is the dependent variable, so place it on the vertical axis.
The smallest number of hours is 2 and the highest is 12, so use a scale of 2 on the vertical axis.
The smallest number of workers is 1 and the highest is 6. Use a scale of 1 on the horizontal axis.

WRITE

a.



b. The graph looks like a hyperbola, so assume that the variables are indirectly proportional. Write this mathematically.

b. $t \propto \frac{1}{n}$

c. The proportion sign can be replaced with '$= k$'. Write the equation.

c. $t = \frac{k}{n}$

d. Choose one pair of coordinates to help calculate k — say (4, 3).
Substitute the values $t = 3$ and $n = 4$.
Solve for k by multiplying both sides by 4.

d. $3 = \frac{k}{4}$

$3 \times 4 = \frac{k}{4} \times \frac{4}{1}$

$12 = k$

e. Replace k with its value of 12 in the equation. Write the equation.

e. $t = \frac{12}{n}$

WORKED EXAMPLE 7 Indirect proportionality 2

The table shows the time taken (t) to fill a water tank using a different number of hoses (h) is indirectly proportional.

Number of hoses	1	2	3	4	6
Time to fill (mins)	60	30	20	15	10

a. **Write a mathematical statement connecting time taken (t) and number of hoses (h).**
b. **Determine how long it would take to fill the water tank if there were 9 hoses.**
c. **Determine the number of hoses required to fill the tank in 5 minutes.**

THINK | WRITE

a. 1. Write the statement for indirect proportionality.

a. $t \propto \frac{k}{h}$

2. The proportion sign can be replaced with '$= k$'. Write the equation.

$t = \frac{k}{h}$

3. Substitute a set of values for h and t such as $h = 6,\ t = 10$.

$10 = \frac{k}{6}$

4. Solve for k.

$k = 60$

5. Write the equation.

$t = \frac{60}{h}$

b. 1. To determine how long it would take, substitute $h = 9$ into the equation.

b. $t = \frac{60}{9}$

$t = 6\frac{2}{3}$

2. Convert the fractional answer into seconds.

$\frac{2}{3}$ of a minute would be 40 seconds.

3. Answer the question.

It would take 6 minutes and 40 seconds.

c. 1. To determine the number of hoses, substitute $t = 5$ into the equation

c. $5 = \frac{60}{h}$

$5h = 60$

2. Solve for h by multiplying both sides by h and then dividing by 5.

$h = \frac{60}{5}$

$h = 12$

3. Answer the question.

It would take 12 hoses.

Exercise 5.4 Indirect proportion (optional)

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5.4 Quick quiz on | 5.4 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 5, 10	2, 6, 8, 11, 13	3, 7, 9, 12, 14

Fluency

1. Explain, in your own words, some of the differences between direct and indirect proportion.

2. WE6 The table shows the time taken to travel 100 km at different speeds.

Speed (km/h)	100	50	25	10	5	1
Time (h)	1	2	4	10	20	100

a. Plot the information on a graph on paper or on a calculator.
b. Using the symbols s for speed and t for time, together with the variation sign, write a mathematical shorthand statement for the connection between the quantities.
c. Write the statement in part **b** as a rule with an equals sign.
d. Choose one of the data points and calculate the value of k.
e. Restate the equation using the value of k.

3. A packet of sweets contains 20 pieces. They are to be divided equally among several friends.

a. Copy and complete the table to show the number of sweets that each friend receives for various numbers of friends.

Number of friends	20	10		4	
Number of sweets each	1		4		20

b. Plot the information on a graph on paper or on a calculator.
c. If n is the number of friends and S is the number of sweets each friend receives, write a rule connecting the quantities, using an equals sign and a constant of variation, k.
d. Use this data to calculate the constant of variation, k.
e. Choose any of the other four data pairs and recalculate the value of k.
f. Write the rule connecting S and n using an equals sign.

4. WE7 The table shows the time taken (t) to paint a wall using different numbers of painters (p). The time and number of painters are indirectly proportional.

Number of painters	1	2	4	5	10
Time to paint (hrs)	20	10	5	4	2

a. Write a mathematical statement connecting time taken (t) and the number of painters (p).
b. Determine how long it would take to paint the wall if there were 8 painters.
c. Determine the number of painters required to paint the wall in 2.5 hours.

5. The table shows the time taken (t) to mow a field using different numbers of lawnmowers (m). The time and number of lawnmowers are indirectly proportional.

Number of lawnmowers	1	2	3	4	6
Time to mow (mins)	90	45	30	22.5	15

a. Write a mathematical statement connecting time taken (t) and number of lawnmowers (m).
b. Determine how long it would take to mow the field if there were 5 lawnmowers.
c. Determine the number of lawnmowers required to mow the field in 12 minutes.

Understanding

6. A company that makes wotzits determines that the number sold depends on the price. If the price is higher, fewer are sold. The market research gives the following expected sales results:

Price of the wotzit	\$1	\$5	\$20	\$50	\$100	\$200	\$400
Number sold (thousands)	400	80	20	8	4	2	1

a. Produce a graph of the number sold versus the price.
b. Using P for price and n for the number sold, state what equation connects the quantities.
c. Use the equation to predict the number sold if the price is \$25.
d. Use the equation to calculate the price if the required sales are 250 000 wotzits.

7. A process to sterilise surgical instruments is tested and its success is found to depend on the temperature used. The results are shown in the table.

Temperature of steriliser (°C)	10	20	40	50	80	90	100
Microbes remaining alive (%)	95	47.5	23.75	19	11.875	10.55	9.5

a. Produce a graph on paper or on a calculator of the percentage of microbes remaining versus the temperature of the steriliser.
b. Using M for the percentage of microbes remaining and T for the temperature, state what equation connects the quantities.
c. Use the equation to predict the percentage of microbes remaining if a temperature of 75 °C is used.
d. Use the equation to calculate the temperature required to ensure that no more than 2% of microbes remain.

8. The variable y is inversely proportional to the variable x. When $x = 4, y = 6$.

a. Write a variation statement using the $\propto$ sign for the relationship between y and x.
b. Calculate the constant of variation and hence write a rule for the relationship.
c. Use the rule to calculate the value of y when $x = 3$.
d. Calculate x when $y = 12$.

9. The time t (hours) it takes to complete a 100-km journey varies inversely with the speed v (m/s).

a. Write a proportionality statement connecting t and v.
b. Write a rule connecting t and v using a constant of variation, k.
c. Calculate the time needed to complete the 100 km at 50 km/h.
d. Calculate the constant of variation.
e. State the rule connecting t and v.

Reasoning

10. **MC** If $y \propto \frac{1}{x}$ and $y = 12$ when $x = 50$, when $x = 25$:

A. $y = 6$ **B.** $y = 18$ **C.** $y = 12\frac{1}{2}$ **D.** $y = 24$

11. **MC** When an amount of gas is enclosed in a container, the pressure is indirectly proportional to the volume. This is noticeable when using a bicycle pump. As the volume inside the pump is reduced by pushing the plunger, the pressure increases. Using P for pressure and V for volume, determine which of the following is *not* True.

A. As V increases, P decreases.

B. If V is halved, P is doubled.

C. $P \propto \frac{1}{V}$

D. $\frac{P}{V} = k$

12. If $y \propto \frac{1}{x}$ then $y = \frac{k}{x}$ and $yx = k$.

a. Copy and complete the following table.

x	48	24	16	12	8
y	2	4	6	8	12
yx					

b. State if it is true in this case that $y \propto \frac{1}{x}$.

Problem solving

13. A tech company uses drones to deliver packages across a city. The time it takes to complete all deliveries is indirectly proportional to the number of drones in operation. On a particular day, 5 drones completed the deliveries in 3 hours.

The next day, due to a software glitch, some drones were grounded. The deliveries took 5 hours to complete.

Determine how many drones were operational on the second day.

14. In a laboratory, a cooling system uses multiple identical fans to reduce the temperature of a heated metal block. The time it takes to cool the block is indirectly proportional to the number of fans running.

With 4 fans, the block cools in 18 minutes. During an experiment, the cooling time needed to be reduced to 12 minutes to prevent overheating.

Determine how many fans must be used to achieve this cooling time.

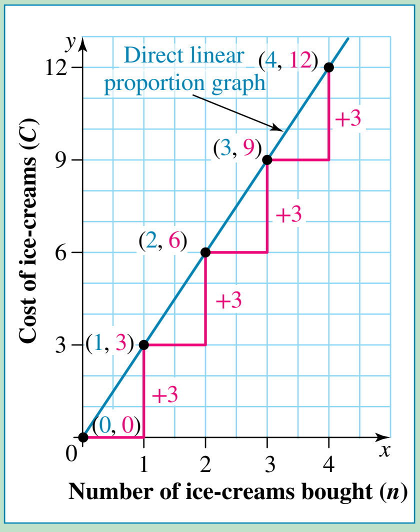
LESSON 5.5 Review

5.5.1 Topic summary

PROPORTIONS

Direct proportion

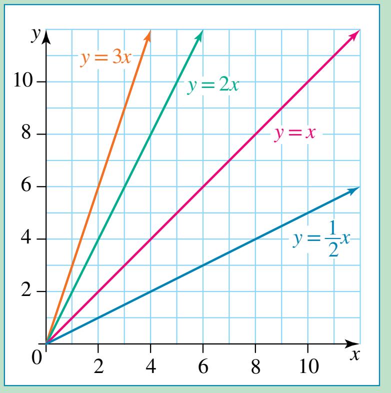
- If y is directly proportional to x, then:
  - y varies with the value of x
  - $y \propto x$
  - as x increases (or decreases), y increases (or decreases)
  - $x = 0$ and $y = 0$ results in a straight line.



- Not all direct proportion relationships are linear.

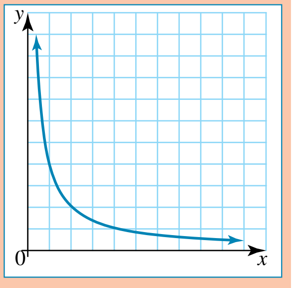
Constant of proportionality

- If y is directly proportional to x, then:
  - $y = kx$
  - k is a constant called the **constant of proportionality** — this is the ratio of any pair of values (x, y)
  - the gradient of the straight line is k
  - all graphs start at (0, 0).
- y is called the dependent variable and is normally placed on the vertical axis.
- x is called the independent variable and is normally placed on the horizontal axis.



Indirect proportion

- An indirect proportion produces a graph of a hyperbola.
- If two quantities vary indirectly, then increasing one variable decreases the other.
- If y varies indirectly to x, it is written as: $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$.



5.5.2 Success criteria

Tick a column to indicate that you have completed the lesson and how well you think you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
5.2	I can understand the concept of direct linear proportion.			
	I can recognise and write direct linear proportion statements.			
5.3	I can determine the constant of proportionality.			
	I can express the constant of proportionality as a ratio of the two quantities.			
5.4	I can identify when two quantities show an indirect proportion.			
	I can create indirect proportion equations in the form $y = \frac{k}{x}$ to solve problems.			

5.5.3 Project

Planning a Market Stall

You are running a stall at a weekend market selling homemade lemonade. You need to plan how much of each ingredient to bring, calculate costs and profits, and scale your recipe depending on expected sales.

The lemonade recipe you use makes 2 L of lemonade from the following ingredients:

- 4 lemons
- 200 g sugar
- 2 L water

1. You plan to make 24 L of lemonade for the market. How many lemons, grams of sugar, and litres of water will you need?
2. Lemons cost \$0.80 each, sugar costs \$2.00 per kg and water is free. What will your total ingredient cost be?
3. Each cup of lemonade is 300 mL. What is the maximum number of cups that you can sell?
4. If you plan to sell each cup for \$3.50, how much profit would you make if you sold the maximum number of cups?
5. On the day of the market, you realise you only brought 40 lemons. Using the original recipe ratio, how many litres of lemonade can you now make? What is the new maximum number of cups you can sell?
6. Your friend has an idea to make up for the reduced number of lemons. They say, "Why not dilute the lemonade a bit to make more cups?". You decide to keep the sugar and lemon quantities the same but increase the water.
   If you increase the water by 25%, how many litres of lemonade will you have? How many cups of lemonade is this?
7. The ratio of lemon to sugar is important for taste. You used 4 lemons for 200 g sugar in the original recipe. What is the constant of proportionality between lemons and sugar (in g per lemon)? Explain what this number means in context.

Exercise 5.5 Review questions

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Fluency

1. MC y is directly proportional to x and $y = 450$ when $x = 15$. The rule relating x and y is:
   A. $y = 0.033x$
   B. $y = 30x$
   C. $y = 60x$
   D. $y = 6750x$

2. MC If $y \propto x$ and $y = 10$ when $x = 50$, the constant of proportionality is:
   A. 10
   B. 5
   C. 1
   D. 0.2

3. MC If $y \propto x$ and $y = 10$ when $x = 50$, the value of x when $y = 12$ is:
   A. 6
   B. 60
   C. 40
   D. 2.4

4 MC If $y \propto x$ and $y = 10$ when $x = 5$, what is the value of x when $y = 4$?
   A. 12.5
   B. 2
   C. 8
   D. 5

5 MC If y is indirectly proportional to x, and $y = 6$ when $x = 3$, what is the value of x when $y = 9$?
   A. 1
   B. 2
   C. 4.5
   D. 6

6. In the table below, if $y \propto x$, state the rule for the relationship between x and y.

x	0	1	2	3
y	0	4	8	12

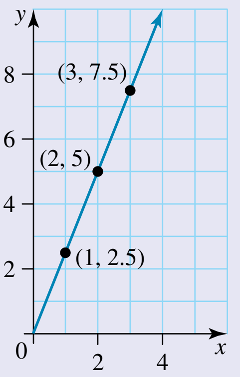
7. In the table below, if $q \propto p$, state the rule for the relationship between p and q.

p	0	3	6	9
q	0	1	2	3

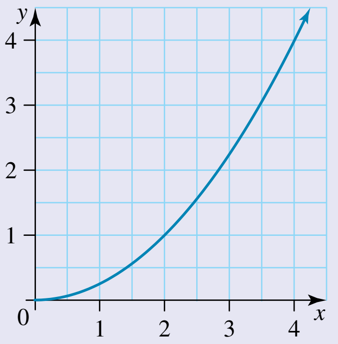
8. In the table below, if $q \propto \frac{1}{p}$, state the rule for the relationship between p and q.

p	**2**	**4**	**6**	**12**
q	6	3	2	1

9. In the graph below, y varies directly as x. State the rule for the relationship between x and y.



10. True or False: The following graph shows that $y \propto x$. Give your reason.



11. If $w \propto v$, and $w = 7.5$ when $v = 5$, calculate k, the constant of proportionality.

Problem solving

12. A recipe for 4 people uses 300 mL of cream.
   a. How much cream would be needed to serve 6 people?
   b. How much cream would be needed to serve 10 people?
   c. If you only have 225 mL of cream, how many people can you serve?

13. A print shop charges in proportion to the number of posters printed. Printing 50 posters costs \$62.50.
   a. How much would it cost to print 200 posters?
   b. How many posters can be printed with \$100?

14. A small photograph has a width of 6.4 cm and a height of 12 cm.
   It is enlarged so the proportions stay the same and the width becomes 20 cm.
   a. What scale factor was used to enlarge the photo?
   b. What is the height of the enlarged?

15. A model train travels along a track made of repeating metal segments. The train covers 24 segments in 36 seconds.
   Due to a design change, the new model track has longer segments, but the train travels at the same speed. Each new segment is 1.5 times the length of the original.
   a. How many new track segments will the train pass in 36 seconds?
   b. If the new track layout uses 100 segments, how long will the train take to travel the full track?

16. At a national park, bikes can be hired for a fixed amount of time. The cost is directly proportional to the number of bikes hired and the number of hours used.
A group hires 3 bikes for 4 hours and pays $72.
a. What is the constant of proportionality in this situation, and what does it represent in context?
b. How much would it cost to hire 5 bikes for 6 hours?
c. Another group paid $96 to hire 4 bikes. How long did they hire them for?

17. The pressure P of a gas is indirectly proportional to its volume V. When the volume is 8 litres, the pressure is 150 kPa.
a. Write a variation statement using the $\propto$ symbol for the relationship between P and V.
b. Calculate the constant of variation and hence write a rule for the relationship.
c. Use the rule to calculate the pressure when the volume is 10 litres.
d. Calculate the volume when the pressure is 200 kPa.

18. The time t (in hours) it takes to build a fence is indirectly proportional to the number of workers w.
a. Write a proportionality statement connecting t and w.
b. Write a rule connecting t and w using a constant of variation, k.
c. If 4 workers can build the fence in 15 hours, how long would it take 6 workers?
d. Calculate the constant of variation.
e. State the rule connecting t and w.

on To test your understanding and knowledge of this topic, go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 5 Proportions

5.1 Pre-test

1. True
2. A
3. Yes
4. True
5. A
6. $x = \frac{k}{y}$
7. **a.** $k = 2.55$ **b.** $v =$ 3.06 amps
8. False
9. D
10. C
11. B
12. $y = 4x$
13. $P = 18h$.
14. 120
15. False. A proportional relationship is a straight line that passes through the origin $(0, 0)$.

5.2 Direct proportion

1. **a.** Yes
   b. No; as speed increases, time decreases.
   c. No; doubling distance doesn't double the cost.
2. **a.** Yes
   b. No; doubling the side length doesn't double the area.
   c. Yes
   d. Yes
3. **a.** Yes
   b. No; doubling distance doesn't double the cost (due to the initial fee).
   c. No; doubling age doesn't double height.
   d. Yes
4. **a.** True: as the number, n, increases, so does the cost, $\$C$.
   b. True: $C = \pi D$, as C increases with D. π is a constant number.
   c. False: $A = \frac{1}{2}\pi r^2$, so it is not linear.
5. D
6. **a.** No; as the value of x increases, the value of y does not increase by a constant amount.
   b. Yes; as the value of a increases, the value of M increases by a constant amount. For every increase in a by 1, the value of M increases by 8.
   c. No; as the value of x increases, the value of y does not increase by a constant amount.
   d. No; when $n = 0$, C does not equal 0.
7. **a.** Yes
   b. No; when $t = 0$, d does not.
   c. No; as x doubles, y does not.
   d. No; when $d = 0$, w does not.
8. Sample responses can be found in the worked solutions in the online resources.
9. The point $(0, 0)$ must always exist in a table of values of two variables exhibiting direct proportionality.
10. The missing value is 8. The relationship between n and m can be calculated by the values 5 and 20 in the table $(n = 4m)$.
11. $a = 32, b = 16, c = 36$
12. **a.** y is doubled. **b.** y is halved.
    c. x is tripled.
13. **a.** Yes, direct proportion does exist.
    b. $2.55
14. **a.** Yes **b.** $935
15. **a.** Yes, direct linear proportion does exist.
    b. 6 cans of paint
16. Direct proportion does not exist. The price per metre for the 4.2-metre length is $5.50, and the price per metre for the 5.4-metre length is $5.40.

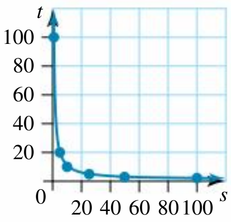
5.3 Proportionality

1. $k = 6$, $a = 6b$
2. $a = 0.5b$
3. $C = 12.5t$
4. $v = 0.5t$
5. $F = 2.5a$
6. **a.** $E = 0.625W$ **b.** 19.2 kg **c.** 4.375 cm
7. **a.** $W = 1.3n$
   b. $\approx$ 385 almonds
   c. 325 g
8. **a.** $d = 2.5n$ **b.** 2017.5 m **c.** 2000 turns
9. **a.** $M = 0.12n$ **b.** 450 seedlings **c.** 360 kg
10. $217.60
11. $217.25
12. 13.33 L
13. $105.98
14. 375 g
15. $150
16. $33.75
17. **a.** $D = 100T$ **c.** 2 hours 12 minutes
    b. 6 hours 6 minutes **d.** 3 hours 54 minutes
18. 275 cm^3

5.4 Indirect proportion (optional)

1.

Direct variation	Inverse variation
Increase in x means an increase in y.	Increase in x means a decrease in y.
Graph is a straight line through $(0, 0)$.	Graph is a hyperbola.
$y = kx$ $\Rightarrow k = \frac{y}{x}$	$y = \frac{k}{x}$ $\Rightarrow k = xy$

2. a.


b. $t \propto \frac{1}{s}$

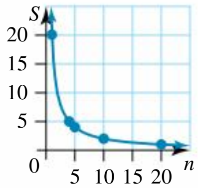
c. $t = \frac{k}{s}$

d. $k = 100$

e. $t = \frac{100}{s}$

3. a.

Number of friends	20	10	5	4	1
Number of sweets	1	2	4	5	20

b.


c. $S = \frac{k}{n}$

d. $k = 20$

e. $k = 20$

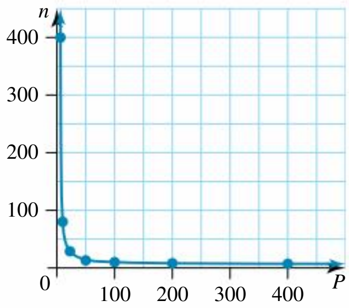
f. $S = \frac{20}{n}$

4. a. $t = \frac{20}{p}$ b. 2.5 hours c. 8 painters

5. a. $t = \frac{90}{m}$

b. 18 minutes

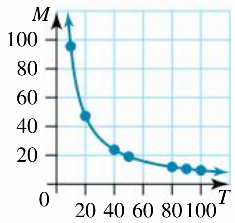
c. $7.5 \approx 8$ lawn mowers.

6. a.


b. $n = \frac{400}{P}$

c. $n = 16\,000$ sold

d. $P = \$1.60$

7. a.


b. $m = \frac{950}{T}$

c. 12.67%

d. More than 475 °C

8. a. $y \propto \frac{1}{x}$ b. $y = \frac{24}{x}$ c. 8

d. 2

9. a. $t \propto \frac{1}{v}$ b. $t = \frac{k}{v}$ c. 2 h

d. 100 e. $t = \frac{100}{v}$ f. 54.5 min

10. D

11. D

12. a.

x	48	24	16	12	8
y	2	4	6	8	12
yx	96	96	96	96	96

b. Yes

13. 3 drones

14. 6 fans

Project

1. 48 lemons, 2400 g of sugar, 24 L of water
2. \$43.20
3. 80 cups
4. \$236.80
5. 20 L of lemonade. 66 full cups
6. 25 L of diluted lemonade. 83 full cups
7. 50g per lemon. In context, this means that 50 g of sugar is used to sweeten the mixture for every lemon used.

5.5 Review questions

1. B
2. D
3. B
4. A
5. B
6. $y = 4x$
7. $q = \frac{1}{3}p$
8. $q = \frac{12}{p}$
9. $y = 2.5x$

10. False. The graph starts at $(0, 0)$ and increases, but it is not linear. Therefore, y does not vary directly with x.

11. 1.5

12. a. 450 mL b. 750 mL c. 3

13. a. \$250 b. 80

14. a. 3.125 b. 37.5 cm

15. a. 16 b. 225 seconds

16. a. Constant of proportionality $= 72 \div 12 =$ \$6 per bike-hour
b. \$180
c. 4 hours

17. a. $P \propto \frac{1}{V}$ b. $P = \frac{1200}{V}$ c. $P = 120$ kPa
d. $V = 6$ litres

18. a. $t \propto \frac{1}{w}$ b. $t = \frac{60}{w}$ c. $t = 10$ hours
d. $k = 60$ e. $t = \frac{60}{w}$

STRAND: Number and algebra
SUB-STRANDS: Linear and non-linear equations and inequalities | Linear and non-linear patterns and relationships

6 Quadratic equations and graphs

LESSON SEQUENCE

LESSON 6.1 Overview

6.1.1 Why learn this?

Galileo did a lot of important things in his lifetime. One of these was showing that objects launched into the air at an angle follow a parabolic path as they fall back to the ground. You can see this for yourself every time you throw a basketball, kick a football or hit a golf ball. As a ball moves through the air, gravity acts on it, causing it to follow a parabolic curved path. The parabolic curve that it follows can be described mathematically by quadratic equations. Every ball that is thrown, every arrow shot from a bow and every rocket launched into space has a quadratic equation that describes that object's position in space and also in time.

Quadratic equations also describe the parabolic shape of radio telescope dishes like the one at Parkes in New South Wales. The curved shape of these telescope dishes allows radio waves from space objects like quasars, galaxies and nebulae to be focused at a receiver to get a stronger signal. It was even used to relay transmissions between NASA and Apollo 11 during the first moon landing. On a much smaller scale, parabolic satellite dishes allow us to get clear internet and television. The bend in a banana, the loop of a rollercoaster, the curve of some suspension bridges or the path of a droplet of water from a fountain — these are all examples of the parabolas and quadratic equations that are all around us.

Hey students! Bring these pages to life online

Watch videos

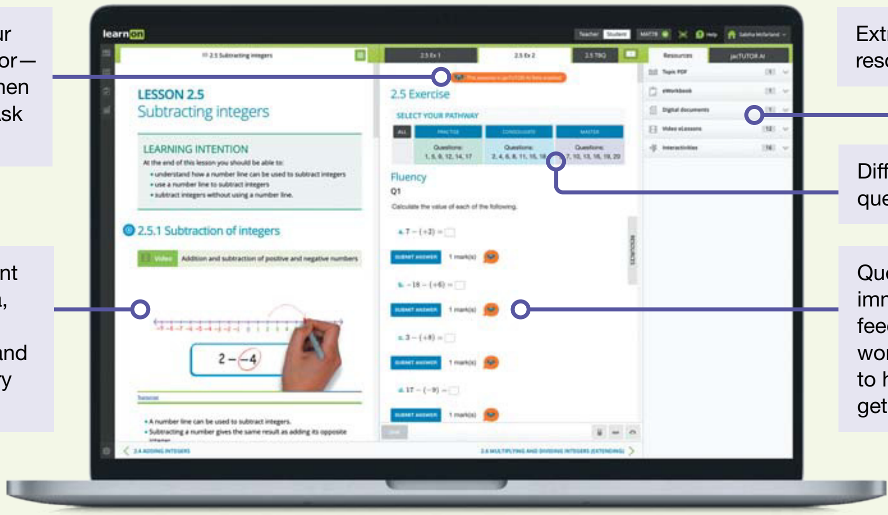
Engage with interactivities

Ask questions and check solutions

Find all this and MORE in jacPLUS

jacTUTOR, your personal AI tutor—here to help when you're stuck! Ask it anything.

Reading content and rich media, including interactivities and videos for every concept



Extra learning resources

Differentiated question sets

Questions with immediate feedback, and fully worked solutions to help students get unstuck.

Exercise 6.1 Pre-test

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1. Solve the equation $x^2 = 81$.

2. Solve the equation $2x^2 = 72$.

3. Verify whether $x = 5$ is a solution to the equation $-2x^2 = 50$.

4. Solve the equation $(x - 3)(x + 7) = 0$.

5. Solve the equation $5x(x - 8) = 0$.

6. The equation $6x^2 = k$ has one solution. Determine the value of k.

7. State whether the following is True or False.
   The quadratic equation $(x - 4)(x - 4) = 0$ has two solutions.

8. Solve the equation $(2x - 1)(3x - 5) = 0$.

9. State whether the following is True or False.
   A quadratic function of the form $y = ax^2$ crosses the y-axis at $(0, 0)$.

10. **MC** Which of the following is a point on the graph of the quadratic function $y = 3x^2$?
    A. $(1, 2)$ **B.** $(3, 1)$ **C.** $(0, 1)$ **D.** $(2, 12)$

11. Complete the following table of values for the equation $y = 5x^2$.

x	-2	-1	0	1	2
y					

12. Determine the value of y when $x = 6$ in the equation $y = 8x^2$.

13. **MC** The graph of $y = 2x^2$ is:
    A. wider than $y = x^2$
    b. the same as $y = x^2$
    c. narrower than $y = x^2$
    d. a reflection of $y = x^2$ in the x-axis.

14. **MC** Select the equation for a parabola that has a maximum turning point and is narrower than $y = x^2$.
    A. $y = 2x^2$
    B. $y = \frac{1}{2}x^2$
    C. $y = (x - 2)^2 + 1$
    D. $y = -2x^2$

15. **MC** Select the equation of a quadratic relation in the form $y = ax^2$ which passes through $(-2, 1)$.
    a. $y = 4x^2$
    b. $y = 2x^2$
    c. $y = -x^2$
    d. $y = \frac{1}{4}x^2$

LESSON 6.2 Quadratic equations

LEARNING INTENTIONS

At the end of this lesson you should be able to:

- identify equations that are quadratic
- rearrange quadratic equations into the general form $y = ax^2 + bx + c$
- solve quadratic equations of the form $ax^2 = k$.
- verify solutions to quadratic equations

6.2.1 Quadratic equations

- **Quadratic equations** are equations that contain an x^2 term. They may also contain the variable x and constants, but cannot contain any higher powers of x such as x^3.
- Examples of quadratic equations include $x^2 + 2x - 7 = 0$, $2x^2 = 18$ and $x^2 = 5x$.
- The general form of a quadratic equation is $y = ax^2 + bx + c$.
- Any quadratic equation can be expressed in general form by rearranging and combining like terms.

WORKED EXAMPLE 1 Rearranging equations into general quadratic form

Rearrange the following quadratic equations so that they are in general form and state the values of a, b and c.

a. $5x^2 - 2x + 3 = 2x^2 + 4x - 12$ **b.** $\frac{x^2}{2} - \frac{1}{6} = x\left(\frac{x}{3}\right) - 4$ **c.** $x(3 - 2x) = 4(x - 6)$

THINK	WRITE
a. 1. Write the equation.	**a.** $5x^2 - 2x + 3 = 2x^2 + 4x - 12$
2. Subtract $2x^2$ from both sides of the equation.	$3x^2 - 2x + 3 = 4x - 12$
3. Subtract $4x$ from both sides of the equation.	$3x - 6x + 3 = -12$
4. Add 12 to both sides of the equation. The equation is now in the general form $ax^2 + bx + c = 0$.	$3x^2 - 6x + 15 = 0$
5. Write the values of a, b and c.	$a = 3, b = -6, c = 15$
b. 1. Write the equation.	**b.** $\frac{x^2}{2} - \frac{1}{6} = x\left(\frac{x}{3}\right) - 4$
2. Expand the bracket. Lowest common denominator (LCD) = 6.	$\frac{x^2}{2} - \frac{1}{6} = \frac{x^2}{3} - 4$
3. Multiply through by 6.	$\frac{x^2}{2} \times \frac{6}{1} - \frac{1}{6} = \frac{x^2}{3} \times \frac{6}{1} - 4 \times 6$
4. Simplify each fraction.	$3x^2 - 1 = 2x^2 - 24$
5. Collect like terms on the left-hand side. The equation is now in the general form $ax^2 + bx + c = 0$.	$x^2 - 1 = -24$ $x^2 + 23 = 0$
6. Write the values of a, b and c.	$a = 1,\ b = 0,\ c = 23$

c. **1.** Write the equation. — **c.** $x(3-2x) = 4(x-6)$

2. Expand the brackets. — $3x - 2x^2 = 4x - 24$

3. To collect like terms on the left-hand side of the equation, subtract $4x$ from both sides. — $-2x^2 - x = -24$

4. Add 24 to both sides. — $-2x^2 - x + 24 = 0$

5. Multiply all terms by -1 to make the x^2 term positive. The equation is now in the general form $ax^2 + bx + c = 0$. — $2x^2 + x - 24 = 0$

6. Write the values of a, b and c. — $a = 2, b = 1, c = -24$

6.2.2 Solving equations of the form $ax^2 = k$

- Solving a quadratic equation means determining the values of x that satisfy the equation.
- Simple quadratic equations of the form $ax^2 = k$ can be solved by using square roots. To solve, rearrange the equation so that x^2 is isolated on one side of the equation, then take the square root of both sides.
- These equations can have 0, 1 or 2 solutions depending on the number that is being square rooted.
  - If the number under the square root is positive, there are 2 solutions.
  - If the number under the square root is zero, there is 1 solution.
  - If the number under the square root is negative, there are no real solutions.

WORKED EXAMPLE 2 Solving equations of the form $ax^2 = k$

Solve the following equations.

a. $2x^2 = 18$ **b.** $5x^2 = -20$ **c.** $3x^2 = 0$

THINK	WRITE
a. **1.** Divide both sides by 2 to isolate x^2.	**a.** $2x^2 = 18$ $x^2 = 9$
2. Take the square root of both sides.	$x = \pm\sqrt{9}$ $x = +3, -3$
3. State the conclusion.	The solutions are $x = 3$ and $x = -3$.
b. **1.** Divide both sides by 5 to isolate x^2.	**b.** $5x^2 = -20$ $x^2 = -4$
2. Take the square root of both sides.	$x = \pm\sqrt{-4}$ -4 has no square root
3. State the conclusion.	The equation $5x^2 = -20$ has no real solution.
c. **1.** Divide both sides by 3 to isolate x^2.	**c.** $3x^2 = 0$ $x^2 = 0$
2. Take the square root of both sides.	$x = \pm\sqrt{0}$ $x = 0$
3. State the conclusion.	The solution is $x = 0$.

Verifying solutions to quadratic equations

- Solutions to quadratic equations can be verified by substitution.
- Substitute the solution(s) into the equation, and if the RHS of the equation is equal to the LHS, the solutions are correct.

WORKED EXAMPLE 3 Verifying solutions to quadratic equations

Verify whether any of the following are solutions to the quadratic equation $4x^2 = 36$ using substitution.

a. $x = 1$ b. $x = 2$ c. $x = 3$

THINK	WRITE
a. 1. Substitute $x = 1$ into the equation.	a. $4x^2 = 36$ $4(1)^2 = 36$
2. Evaluate.	$4 \neq 36$
3. Answer the question.	Therefore, $x = 1$ is not a solution.
b. 1. Substitute $x = 2$ into the equation.	b. $4x^2 = 36$ $4(2)^2 = 36$
2. Evaluate.	$4 \times 4 = 36$ $16 \neq 36$
3. Answer the question.	Therefore, $x = 2$ is not a solution.
c. 1. Substitute $x = 3$ into the equation.	c. $4x^2 = 36$ $4(3)^2 = 36$
2. Evaluate.	$4 \times 12 = 36$ $36 = 36$
3. Answer the question.	Therefore, $x = 3$ is a solution.

Exercise 6.2 Quadratic equations

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6.2 Quick quiz on | 6.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 9, 14, 17	2, 4, 6, 10, 13, 15, 18	7, 8, 11, 12, 16, 19, 20

Fluency

1. WE1 Rearrange each of the following quadratic equations so that they are in the form $ax^2 + bx + c = 0$.

a. $3x - x^2 + 1 = 5x$ b. $5(x - 2) = x(4 - x)$ c. $x(5 - 2x) = 6\left(5 - x^2\right)$

2. Rearrange each of the following quadratic equations so that they are in the form $ax^2 + bx + c = 0$.
 a. $4x^2 - 5x + 2 = 6x - 4x^2$ b. $5(x - 12) = x(4 - 2x)$ c. $x^2 - 4x - 16 = 1 - 4x$

3. **MC** Select which of the following is a quadratic equation.
 A. $2x - 1 = 0$
 B. $2^x - 1 = 0$
 C. $x^2 - x = 1 + x^2$
 D. $x^2 - x = 1 - x^2$

4. **MC** Select which of the following is not a quadratic equation.
 A. $x(x - 1) = 2x - 1$
 B. $-3x^2 + 2x = 1$
 C. $3(x + 2) + 5(x + 3) = 2(x + 1)$
 D. $2(x - 1) + 3x = x(2x - 3)$

5. **WE2** Solve the following quadratic equations.
 a. $x^2 = 16$ b. $2x^2 = -18$ c. $5x^2 = 0$

6. Solve the following quadratic equations.
 a. $3x^2 = 27$ b. $2x^2 = 50$ c. $7x^2 = 112$

7. Solve the following quadratic equations, leaving your answer in exact form.
 a. $2x^2 = 14$ b. $8x^2 = 64$ c. $15x^2 = 45$

8. Solve the following quadratic equations.
 a. $-6x^2 = -6$ b. $x^2 = -4$ c. $-3x = -12$

Understanding

9. **WE3** Verify whether any of the following are solutions to the quadratic equation $2x^2 = 18$.
 a. $x = 2$ b. $x = 3$ c. $x = 4$

10. Verify whether any of the following are solutions to the quadratic equation $5x^2 = 80$.
 a. $x = 4$ b. $x = -4$ c. $x = 5$

11. Verify whether any of the following are solutions to the quadratic equation $3x^2 = 108$.
 a. $x = 6$ b. $x = -7$ c. $x = \sqrt{36}$

12. Determine whether $x = 0$ is a solution to the following equations.
 a. $x^2 = x + 12$ b. $x^2 = 3x + 1$ c. $x^2 = 4x$

13. Determine whether the following are solutions to the equation $5(x - 1)^2 + 7 = 27$.
 a. $x = -1$ b. $x = 1$ c. $x = 0$

Reasoning

14. Is $(x + 4)^2$ equal to $x^2 + 16$? Explain using a numerical example.

15. Explain why $x = -2$ is not a solution to $4x^2 = -16$.

16. Explain why $x^2 + 7x + 4 = 7x$ has no solutions.

Problem solving

17. The product of 4 and the square of a number is 100.

a. Determine the equation.

b. Solve the equation. Is there more than one possible number? If so, determine the other number and explain why there is more than one.

18. The sum of 5 and the square of a number is 41.

a. Determine the number.

b. Is there more than one possible number? If so, determine the other number and explain why there is more than one.

19. The product of 2 and the square of a number is 52.

a. Determine the equation.

b. Solve the equation. Is there more than one possible number? If so, determine the other number and explain why there is more than one.

20. Two less than a number is squared and its result is tripled. The difference between this and 7 is 41.

a. Write the algebraic equation.

b. Determine both possible answers that satisfy the criteria.

LESSON 6.3 The Null Factor Law (optional)

LEARNING INTENTIONS

At the end of this lesson you should be able to:

- recall the Null Factor Law
- solve quadratic equations using the Null Factor Law.

6.3.1 Using the Null Factor Law to solve equations

- The **Null Factor Law** states that if the product of two or more factors is zero, then at least one of the factors must be zero.

The Null Factor Law

If $a \times b = 0$ then:

$a = 0$ or $b = 0$ or both a and b equal 0.

- We can use the Null Factor Law to solve quadratic equations which are in factorised form.
- For example, if $(x-5)(x-2) = 0$ then

$$(x-5) = 0 \text{ or } (x-2) = 0 \text{ or both } x-5 = x-2 = 0.$$

$$\begin{array}{ll} \text{If } x-5 = 0 & \text{If } x-2 = 0 \\ x = 5 & x = 2 \end{array}$$

Both $x = 5$ and $x = 2$ make the equation true.

The solutions to the equation $(x-5)(x-2) = 0$ are therefore $x = 5$ and $x = 2$.

- These solutions can be verified by substituting the values into the original equation. For example:

If $x = 5$, then:	If $x = 2$, then:
$(x-5)(x-2) = (5-5)(5-2)$	$(x-5)(x-2) = (2-5)(2-2)$
$= 0 \times 3$	$= -3 \times 0$
$= 0$	$= 0$

Note: Quadratic equations can have a maximum of two solutions. This is known as the **Fundamental Theorem of Algebra**.

WORKED EXAMPLE 4 Using the Null Factor Law to solve quadratics

Solve each of the following quadratic equations.

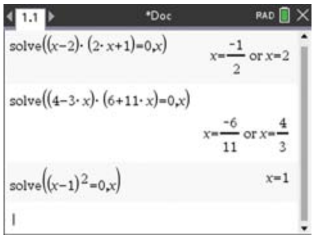
a. $(x-2)(2x+1) = 0$

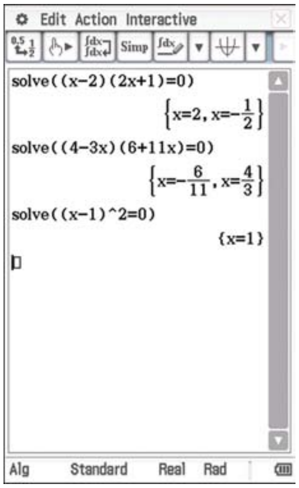
b. $(4-3x)(6+11x) = 0$

c. $x(x-3) = 0$

d. $(x-1)^2 = 0$

THINK	WRITE
a. 1. The product of 2 factors is 0, so apply the Null Factor Law.	**a.** $(x-2)(2x+1) = 0$
2. One of the factors must equal zero.	Either $x-2=0$ or $2x+1=0$
3. Solve the equations.	$x = 2$ $2x = -1$, $= -\frac{1}{2}$
4. Write the solutions.	$x = -\frac{1}{2}, x = 2$
b. 1. The product of two factors is 0, so apply the Null Factor Law.	**b.** $(4-3x)(6+11x) = 0$
2. One of the factors must equal zero.	Either $4-3x=0$ or $6+11x=0$
3. Solve the equations.	$-3x = -4$ $11x = -6$ $x = \frac{4}{3}$ $x = \frac{-6}{11}$
4. Write the solutions.	$x = \frac{4}{3}, -\frac{6}{11}$
c. 1. The product of two factors is 0, so apply the Null Factor Law.	**c.** $x(x-3) = 0$
2. One of the factors must equal zero.	Either $x=0$ or $x-3=0$
3. Solve the equations.	$x = 3$
4. Write the solutions.	$x = 0, 3$
d. 1. Write the equation.	**d.** $(x-1)^2 = 0$
2. Rewrite the squared factors as the product of two factors.	$(x-1)(x-1) = 0$
3. Apply the Null Factor Law.	Either $x-1=0$ or $x-1=0$
4. Solve the equations. The two solutions are the same.	$x = 1$ $x = 1$
5. Write the solution.	$x = 1$

TI | THINK	**DISPLAY/WRITE**
a–c. On a Calculator page, press: • Menu • 3. Algebra • 1. Solve Complete the entry lines as: solve $((x-2)(2x+1)=0, x)$ solve $((4-3x)(6+11x)=0, x)$ solve $\left((x-1)^2=0, x\right)$ Press ENTER after each entry.	 **a.** $x=-\frac{1}{2}$, $x=2$ **b.** $x=-\frac{6}{11}$, $x=\frac{4}{3}$ **c.** $x=1$

CASIO | THINK	**DISPLAY/WRITE**
a–c. On a Main screen, complete the entry lines as: solve $((x-2)(2x+1)=0)$ solve $((4-3x)(6+11x)=0)$ solve $\left((x-1)^2=0\right)$ Press EXE after each entry.	 **a.** $x=-\frac{1}{2}$, $x=2$ **b.** $x=-\frac{6}{11}$, $x=\frac{4}{3}$ **c.** $x=1$

Exercise 6.3 The Null Factor Law (optional)

learnon

6.3 Quick quiz on | 6.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 8, 12	3, 4, 6, 9, 10, 13, 14	5, 7, 11, 15

Fluency

1. **WE4** Solve each of the following quadratic equations.

a. $(x-2)(x+3)=0$ **b.** $(2x+4)(x-3)=0$ **c.** $(x+2)(x-3)=0$

d. $(2x+5)(4x+3)=0$ **e.** $(x+4)(2x+1)=0$

2. Solve each of the following quadratic equations.

a. $(x-1)(x+6)=0$

b. $(3x-2)(x+5)=0$

c. $(x-4)(x-7)=0$

d. $(2x-3)(5x+1)=0$

e. $(x+3)(3x-2)=0$

3. Solve each of the following quadratic equations.

a. $(x+1)(x-9)=0$

b. $(4x+5)(x-2)=0$

c. $(x-6)(x+2)=0$

d. $(3x+4)(2x-7)=0$

e. $(x-5)(x-1)=0$

4. Solve each of the following quadratic equations.

a. $(2x-1)(x+30)=0$

b. $(2x+1)(3-x)=0$

c. $(1-x)(3x-1)=0$

d. $x(x-2)=0$

e. $\left(x+\frac{1}{3}\right)\left(2x-\frac{1}{2}\right)=0$

5. Solve each of the following quadratic equations.

a. $(5x-1.5)(x+2.3)=0$

b. $\left(2x+\frac{1}{3}\right)\left(2x-\frac{1}{3}\right)=0$

c. $(x-2)^2=0$

d. $x(4x-15)=0$

e. $(x+4)^2=0$

6. The Null Factor Law can be extended to products of more than two factors. Use this to determine all the solutions to the following equations.

a. $(x-2)(x+2)(x+3)=0$

b. $(x+2)(x+2)(2x-5)=0$

c. $(x+2)(x+2)(x+4)=0$

d. $x(x+2)(3x+12)=0$

7. The Null Factor Law can be extended to products of more than two factors. Use this to determine all the solutions to the following equations.

a. $(2x-2.2)(x+2.4)(x+2.6)=0$

b. $(2x+6)\left(x+\frac{1}{2}\right)(9x-15)=0$

c. $3(x-3)^2=0$

d. $(x+1)(x-2)^2=0$

Understanding

8. MC Select the solutions to $(2x-4)(x+7)=0$.

A. $x=4,\ x=7$

B. $x=4, x=-7$

C. $x=2,\ x=7$

D. $x=2, x=-7$

9. MC The Null Factor Law cannot be applied to the equation $x\left(x+\frac{1}{2}\right)\left(\frac{x}{2}-1\right)=1$ because:

A. there are more than two factors

B. the right-hand side equals 1

C. the first factor is a simple x-term

D. the third term has x in a fraction

Reasoning

10. a. Explain why $x=2$ is not a solution of the equation $(2x-2)(x+2)=0$.

b. Determine the solutions to the equation.

11. Explain what is the maximum number of solutions a quadratic such as $(x+7)(x-8)=0$ can have.

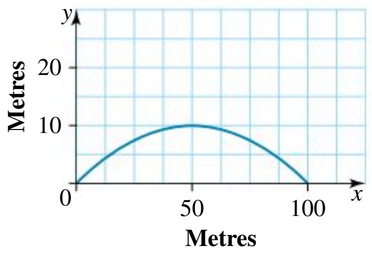
Problem solving

12. A bridge is constructed with a supporting structure in the shape of a parabola, as shown in the diagram. The origin $(0, 0)$ is at the left-hand edge of the bridge, which is 100 m long.

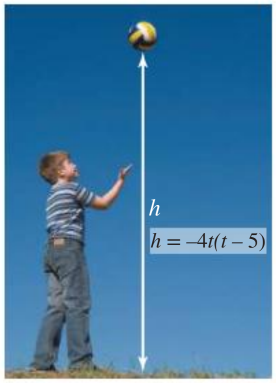
a. Identify the maximum height of the bridge support.

b. If the equation of the support is $y=ax(b-x)$, determine the values of a and b. (*Hint:* Let $y=0$.)

c. Calculate the height of the support when $x=62$.



13. Consider a ball thrown upwards so that it reaches a height h of metres after t seconds.
The expression $-4t(t-5)$ represents the height of the ball, in metres, after t seconds.
Calculate the height of the ball after:
a. 1 second
b. 5 seconds.



14. The product of 6 less than a number and 9 more than that same number is equal to zero.
Determine the possible values of the number.

15. The product of 3 more than a number and 5 more than twice that number is equal to zero.
Determine the possible values of the number.

LESSON 6.4 Graphing quadratic functions of the form $y = ax^2$

LEARNING INTENTIONS

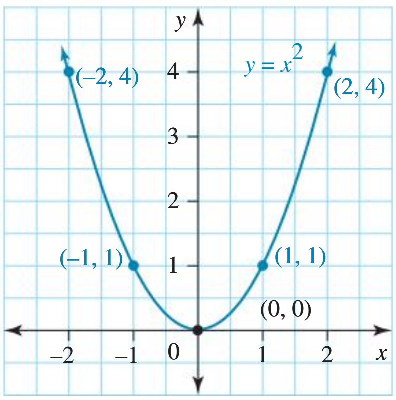
At the end of this lesson you should be able to:
- use a table of values to plot points and graph quadratic equations of the form $y = ax^2$
- describe the key features of quadratic functions of the form $y = ax^2$
- describe the effect of changing the value of a on quadratic functions of the form $y = ax^2$.

6.4.1 The graph of the quadratic function $y = x^2$

- The quadratic function $y = x^2$ can be graphed by using a table of values of x from -2 to 2 to substitute into the equation as shown below.

x	-2	-1	0	1	2
y	4	2	0	2	4

- By completing the table of values, 5 sets of (x, y) coordinates can be graphed on a Cartesian plane.

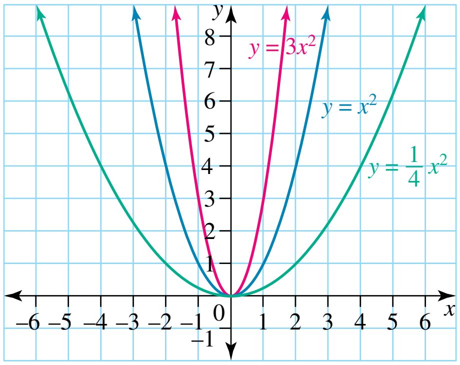


- Both the x- and y-axes are clearly indicated, along with their scales.
- The turning point $(0, 0)$ is indicated.
- The x- and y-intercepts are indicated. For this graph they are all $(0, 0)$.
- This is an example of a quadratic function that just touches (does not cross) the x-axis at $(0, 0)$.

6.4.2 Quadratic functions of the form $y = ax^2$, where $a > 0$

- A coefficient in front of the x^2 term affects the **dilation** of the graph, making it wider or narrower than the graph of $y = x^2$.
- If $a > 1$ then the graph becomes narrower, whereas if $0 < a < 1$, the graph becomes wider.
- The graph shows the effect of varying the coefficient a.

  Notice that $y = 3x^2$ is narrower than $y = x^2$ and $y = \frac{1}{4}x^2$ is wider than $y = x^2$.



Quadratic functions of the form $y = ax^2$, where $a > 0$

For all quadratic functions of the form $y = ax^2$, where $a > 0$:
- **the axis of symmetry is $x = 0$**
- **the turning point is $(0, 0)$**
- **the x-intercept is $(0, 0)$**
- **the y-intercept is $(0, 0)$**
- **the shape of the parabola is always upright, or a U shape (∪).**

WORKED EXAMPLE 5 Sketching dilation of quadratic functions with $a > 0$ using a table of values

On the same set of axes: a. sketch the graph of $y = x^2$ and $y = 3x^2$, marking the coordinates of the turning point and the intercepts; b. state which graph is narrower.

THINK	WRITE/DRAW
a. 1. Write the equation of the first graph.	a. $y = x^2$
2. Create a table of x values from −2 to 2 to substitute into the equation $y = x^2$.	(table below)
3. State its axis of symmetry.	The axis of symmetry is $x = 0$.
4. State the coordinates of the turning point.	The turning point is $(0, 0)$.
5. State the intercepts.	The x-intercept is (0, 0) and the y-intercept is also (0, 0).
6. Write the equation of the second graph.	$y = 3x^2$
7. Create a table of x values from −2 to 2 to substitute into the equation $y = 3x^2$.	(table below)
8. State its axis of symmetry.	The axis of symmetry is $x = 0$.
9. State the coordinates of the turning point.	The turning point is $(0, 0)$.
10. State the intercepts.	The x-intercept is 0 and the y-intercept is 0.

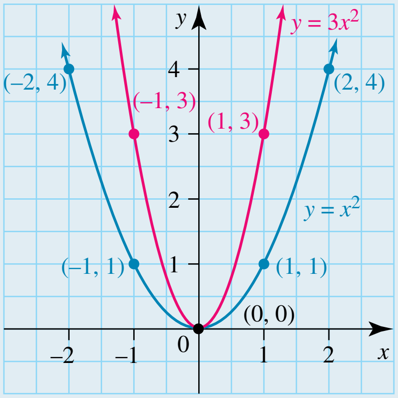
$y = x^2$:

x	−2	−1	0	1	2
y	4	1	0	1	4

$y = 3x^2$:

x	−2	−1	0	1	2
y	12	3	0	3	12

11. Sketch the graphs using the table of values as coordinates, labelling each point.



b. State the graph that is narrower.

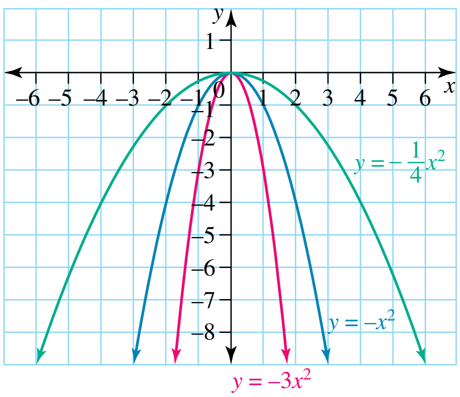
b. The graph of $y = 3x^2$ is narrower.

- Consider how the graphs of these quadratics relate to the number of solutions of equations of the form $ax^2 = k$.
- The solutions to $ax^2 = k$ will correspond to the points at which the y-value of the graph of $y = ax^2$ is equal to k.
- Note that when k is positive, the graph will have two points where the graph has a height of k (2 solutions). When k is 0, there is just the one point where the graph has a height of 0 (1 solution), and when k is negative, the graph never has a height of k (no solutions).

6.4.3 Quadratic functions of the form $y = ax^2$, where $a < 0$

- When $a < 0$, the graph is inverted; that is, it is ∩ shaped.
- A coefficient in front of the x^2 term affects the dilation of the graph, making it wider or narrower than the graph of $y = x^2$.
- If $-1 < a < 0$, the graph is wider than $y = -x^2$.
- If $a < -1$, the graph is narrower than $y = -x^2$.
  - The graph shows the effect of varying the coefficient a.

    Notice that $y = -3x^2$ is narrower than $y = -x^2$ and $y = -\frac{1}{4}x^2$ is wider than $y = -x^2$.



Quadratic functions of the form $y = ax^2$, where $a < 0$

For all quadratic functions of the form $y = ax^2$, where $a < 0$:
- **the axis of symmetry is $x = 0$**
- **the turning point is $(0, 0)$**
- **the x-intercept is $(0, 0)$**
- **the y-intercept is $(0, 0)$**
- **the shape of the parabola is always inverted or an upside-down U shape (∩).**

WORKED EXAMPLE 6 Sketching dilation of quadratic functions with $a < 0$, using a table of values

On the same set of axes: a. use a table of values to sketch the graphs of $y = -x^2$ and $y = -2x^2$, marking the coordinates of the turning point and the intercept; b. state which graph is narrower.

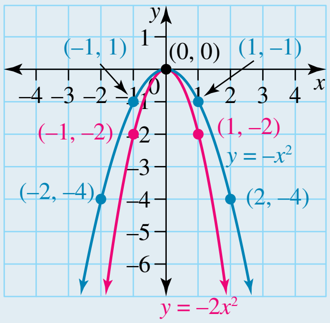
THINK	WRITE/DRAW
a. 1. Write the equation of the first graph.	**a.** $y = -x^2$
2. Create a table of x values from -2 to 2 to substitute into the equation $y = -x^2$.	(see table below)

x	-2	-1	0	1	2
y	-4	-1	0	-1	-4

THINK	WRITE/DRAW
3. State its axis of symmetry.	The axis of symmetry is $x = 0$.
4. State the coordinates of the turning point.	The turning point is $(0, 0)$.
5. State the intercepts.	The x-intercept is $(0, 0)$ and the y-intercept is also $(0, 0)$.
6. Write the equation of the second graph.	$y = -2x^2$
7. Create a table of x values from -2 to 2 to substitute into the equation $y = -2x^2$.	(see table below)

x	-2	-1	0	1	2
y	-8	-2	0	-2	-8

THINK	WRITE/DRAW
8. State its axis of symmetry.	The axis of symmetry is $x = 0$.
9. State the coordinates of the turning point.	The turning point is $(0, 0)$.
10. State the intercepts.	The x-intercept is $(0, 0)$ and the y-intercept is also $(0, 0)$.
11. Sketch the two graphs on a single set of axes, labelling the turning point (as well as intercepts and maximum).	(see graph below)



THINK	WRITE/DRAW
b. State which graph is narrower.	**b.** The graph of $y = -2x^2$ is narrower.

Exercise 6.4 Graphing quadratic functions of the form $y = ax^2$

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6.4 Quick quiz on	6.4 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 5, 6, 7, 9, 13, 16	2, 4, 10, 14, 17	3, 8, 11, 12, 15, 18

Fluency

1. **WE5** On the same set of axes: **a.** use a table of values to sketch the graph of $y = x^2$ and $y = 4x^2$, marking the coordinates of the turning point and the intercepts; **b.** state which graph is narrower.

2. On the same set of axes: **a.** use a table of values to sketch the graph of $y = x^2$ and $y = \frac{1}{2}x^2$, marking the coordinates of the turning point and the intercepts; **b.** state which graph is narrower.

3. Sketch the graph of the following table. State the equation of the graph.

x	−4	−3	−2	−1	0	1	2	3	4
y	4	2.25	1	0.25	0	0.25	1	2.25	4

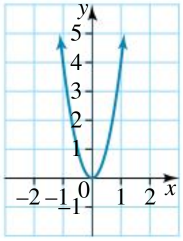
4. WE6 **a.** On the same set of axes, sketch the graphs of $y = -x^2$ and $y = -0.5x^2$, marking the coordinates of the turning point and the intercepts.
State which graph is narrower.
   b. Using the same set of axes, sketch the graph of $y = -5x^2$, marking the coordinates of the turning point and the intercepts. State which graph is the narrowest.

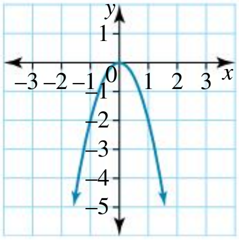
5. MC The graph of $y = -3x^2$ is:
   A. wider than $y = x^2$
   B. narrower than $y = x^2$
   C. the same width as $y = x^2$
   D. a reflection of $y = x^2$ in the x-axis.

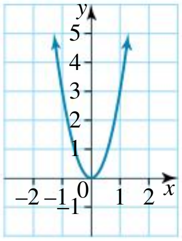
6. MC The graph of $y = \frac{1}{3}x^2$ is:
   A. wider than $y = x^2$
   B. narrower than $y = x^2$
   C. the same width as $y = x^2$
   D. a reflection of $y = x^2$ in the x-axis.

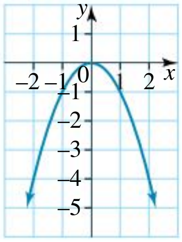
7. MC The graph of $y = \frac{1}{2}x^2$ is:
   A. wider than $y = \frac{1}{4}x^2$
   B. narrower than $y = \frac{1}{4}x^2$
   C. the same width as $y = \frac{1}{4}x^2$
   D. a reflection of $y = \frac{1}{4}x^2$ in the x-axis.

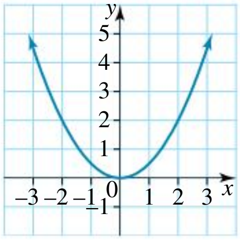
8. Match each of the following quadratic functions with the appropriate equation from the list.
   i. $y = 3x^2$
   ii. $y = -x^2$
   iii. $y = 4x^2$
   iv. $y = \frac{1}{2}x^2$
   v. $y = -4x^2$
   vi. $y = -2x^2$

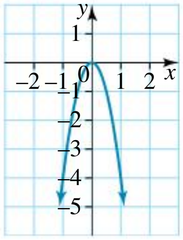
a.


b.


c.


d.


e.


f.


Understanding

9. Write an equation for a quadratic function that has a minimum turning point and is narrower than $y = x^2$.

10. Write an equation for a quadratic function that has a maximum turning point and is wider than $y = x^2$.

11. Determine the equation of a quadratic relation if it has an equation of the form $y = ax^2$ and passes through:

 a. $(1, 3)$
 b. $(-1, -1)$.

12. Consider the equation $y = -3.5x^2$. Calculate the values of y when x is:

 a. 10
 b. -10
 c. -3
 d. 1.5
 e. -2.2.

Reasoning

13. a. Sketch the following graphs on the same axes: $y = x^2$, $y = 2x^2$ and $y = -3x^2$. Shade the area between the two graphs above the x-axis and the area inside the graph below the x-axis.
 Describe the shape that has been shaded.

 b. Sketch the following graphs on the same axes: $y = x^2$, $y = \frac{1}{3}x^2$ and $y = -4x^2$. Shade the area inside the graphs of $y = x^2$ and $y = -4x^2$. Also shade the area between the graph of $y = \frac{1}{3}x^2$ and the x-axis.
 Describe the shape that you have drawn.

14. The photograph shows the parabolic shape of a skate ramp. The rule of the form $y = ax^2$ describes the shape of the ramp.
 If the top of the ramp has coordinates $(3, 6)$, determine a possible equation that describes the shape. Justify your answer.

15. The total sales of a fast food franchise vary as the square of the number of franchises in a given city. Let S be the total sales (in millions of dollars per month) and f be the number of franchises. If sales = \$25 million when $f = 4$, then:

 a. show that the equation relating S and f is $S = 1\,562\,500\,f^2$
 b. determine the number of franchises needed to (at least) double the sales from \$25 000 000.

Problem solving

16. The amount of power (watts) in an electric circuit varies as the square of the current (amperes). If the power is 100 watts when the current is 2 amperes, calculate:

 a. the power when the current is 4 amperes
 b. the power when the current is 5 amperes.

17. Xanthe and Carly were comparing parabolas on their CAS calculators. Xanthe graphed $y = 0.001x^2$ with window settings $-1000 \le x \le 1000$ and $0 \le y \le 1000$. Carly graphed $y = x^2$ with window settings $-k \le x \le k$ and $0 \le y \le k$.
 Except for the scale markings, the graphs looked exactly the same. What is the value of k?

18. The parabola $y = x^2$ is rotated 90° clockwise about the origin. Determine the equation.

LESSON 6.5 Review

6.5.1 Topic summary

QUADRATIC EQUATIONS AND GRAPHS

Quadratic equations

- Quadratic equations are equations in which the largest power of x is 2. They contain an x^2 term.
- The general form of a quadratic equation is $y = ax^2 + bx + c$, where $a \neq 0$.
- Quadratic equations for which $a = 1$ are called monic quadratic equations.
- The largest number of solutions a quadratic equation can have is 2.

Null Factor Law (optional)

- If $a \times b = 0$, then: $a = 0$ or $b = 0$ or $a = b = 0$

Axis intercepts

- All quadratic functions have a single y-intercept. The location of this y-intercept can be found by letting $x = 0$.
- Quadratic functions can have 0, 1 or 2 x-intercepts.
- x-intercepts can be found by letting $y = 0$ and solving for x. If the equation can't be solved, there is no x-intercept.

Axis of symmetry

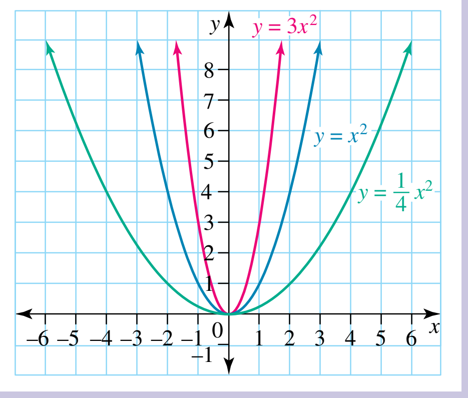
- Every quadratic functions has an axis of symmetry. This is the vertical line that passes through the turning point.

Turning point

- The turning point is the point at which a parabola changes direction.
- If $a > 0$ the turning point is the lowest point on the graph. This point is called the local minimum.
- If $a < 0$ the turning point is the highest point on the graph. This point is called the local maximum.

Quadratic functions of the form $y = ax^2$

- Quadratic functions of the form $y = ax^2$ have a turning point at the origin.
- If $a > 0$ the graph is ∪-shaped and the turning point is a local minimum.
- If $a < 0$ the graph is ∩-shaped and the turning point is a local maximum.
- Larger values of a create narrower graphs. Smaller values of a create wider graphs.



6.5.2 Success criteria

Tick a column to indicate that you have completed the lesson and how well you think you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
6.2	I can identify equations that are quadratic.			
	I can rearrange quadratic equations into the general form $y = ax^2 + bx + c$.			
	I can solve quadratic equations in the form $ax^2 + c = 0$.			
	I can verify solutions to quadratic equations.			
6.3	I can recall the Null Factor Law.			
	I can solve quadratic equations using the Null Factor Law.			
6.4	I can use a table of values to plot points and graph quadratic equations of the form $y = ax^2$.			
	I can describe the key features of quadratic functions of the form $y = ax^2$.			
	I can describe the effect of changing the value of a on quadratic functions of the form $y = ax^2$.			

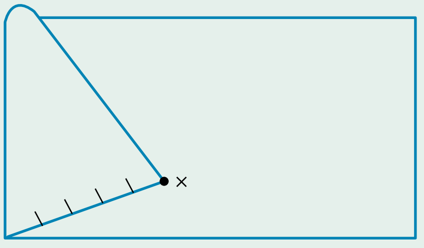
6.5.3 Project

Constructing a parabola

The word *parabola* comes from the Greek language and means 'thrown', because it is the path followed by a projectile in flight. Notice that the water streams shown in the photo are moving in the path of a parabola. This investigation explores the technique of folding paper to display the shape of a parabola. The instructions are given below. Take care with each step to ensure your finished product is a well-constructed parabola that can be used in later parts of this investigation.

Forming a parabola by folding paper

- Take a sheet of A4 paper. Cut it into two pieces by dividing the longer side into two. Only one of the halves is required for this investigation.
- Along one of the longer sides of your piece of paper, mark points that are equally spaced 1 cm apart. Start with the first point being on the very edge of the paper.
- Turn over the piece of paper and mark a point, X, 3 cm above the centre of the edge that has the markings on the reverse side.
- Fold the paper so that the first point you marked on the edge touches point X. Make a sharp crease and open the paper flat.
- Fold the paper again so that the second mark touches the point X. Crease and unfold again.
- Repeat this process until all the marks have been folded to touch point X.
- With the paper flat and the point X facing up, you should notice the shape of a parabola appearing in the creases.



1. Trace the curve with a pencil.
   The point X is called the focus of the parabola. Consider the parabola to represent a mirror. Rays of light from the focus would hit the mirror (parabola) and be reflected. The angle at which each ray hits the mirror is the same size as the angle at which it is reflected.
2. Using your curve traced from your folding activity, accurately draw a series of lines to represent rays of light from the point X to the parabola (mirror). Use a protractor to carefully measure the angle each line makes with the mirror and draw the path of these rays after reflection in the mirror.
3. Draw a diagram to describe your finding from question **2**. Provide a brief comment on your description.
4. Retrace your parabola onto another sheet of paper. Take a point other than the focus and repeat the process of reflection of rays of light from this point by the parabolic mirror.
5. Draw a diagram to describe your finding from question **4**. Provide a brief comment on your description.
6. Give examples of where these systems could be used in society.

Exercise 6.5 Review questions

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Fluency

1. MC Identify which of the following is *not* a quadratic equation.

   A. $x^2 - 1 = 0$ B. $x^2 - 1 + 2x = 0$ C. $x^2 - \frac{1}{2} = 0$ D. $x^2 - \frac{1}{x} = 0$

2. MC Identify which of the following is in general form.

   A. $x^2 + 1 = 0$ B. $x^2 - 1 = 2$ C. $2x^2 = x^2 + 3$ D. $x^2 = x^2 + 1$

3. MC The Null Factor Law cannot be applied to the equation $x(x+3)(x-2) = 1$ because:

   A. the first factor is a simple x term.
   B. there are more than two factors.
   C. the right-hand side equals 1.
   D. the second term is positive.

4. MC If the solutions to the quadratic equation $(x-3)(x-b) = 0$ are 3 and -5, then b is equal to:

   A. 5 B. -5 C. 3 D. -3

5. MC Select the solutions to $3x^2 = 27$.

   A. $x = 3$ and $x = -3$
   B. $x = 9$ and $x = -9$
   C. $x = 1$ and $x = -1$
   D. $x = 2$ and $x = -2$

6. Solve the following equations.

   a. $4x^2 = 400$
   b. $3x^2 = 75$

7. Solve the following equations, leaving your answer in exact form.

   a. $2x^2 = 16$
   b. $6x^2 = 120$

8. Verify if $x = 4$ is a solution to the quadratic equation $5x^2 = 80$.

9. Calculate the solution(s) to the quadratic equation $(x+4)(x-2) = 0$

10. Calculate the solutions to the quadratic equation $(x-3)(2x+8) = 0$.

11. Calculate the solutions to the quadratic equation $(4-x)(2x-7) = 0$.

12. MC The graph of $y = -4x^2$ is:

    A. wider than $y = x^2$.
    B. narrower than $y = x^2$.
    C. the same width as $y = x^2$.
    D. a reflection of $y = x^2$ in the x-axis.

13. MC The graph of $y = \frac{1}{2}x^2$ is:

    A. wider than $y = x^2$.
    B. narrower than $y = x^2$.
    C. the same width as $y = x^2$.
    D. a reflection of $y = x^2$ in the x-axis.

14. MC Compared to the graph of $y = x^2$, the graph of $y = -2x^2$ is:

    A. half as wide.
    B. twice as wide.
    C. moved 2 units to the right.
    D. moved 2 units up.

15. Use a table of values to sketch each of the following graphs, labelling the axis of symmetry, the turning point and the intercepts, and stating the type of turning point.

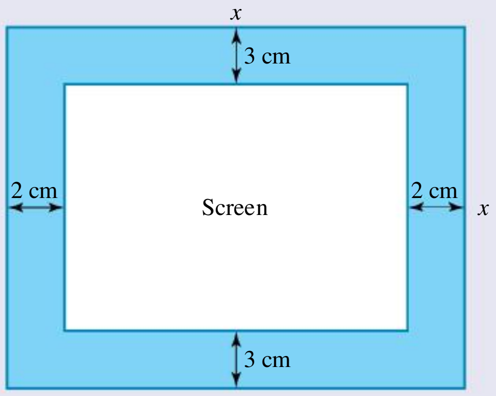
a. $y = 2x^2$
b. $y = \frac{1}{2}x^2$
c. $y = -4x^2$
d. $y = -\frac{1}{3}x^2$

Problem solving

16. The distance travelled by a motorbike is given by the formula $2t(t + 9)$, where t is the time in seconds and d is the distance in metres.
Calculate how long it would take the motorbike to travel a distance of 180 m.

17. The product of 5 and the square of a number is 180.
a. Determine the equation.
b. Solve the equation. Is there more than one possible number? If so, determine the other number and explain why there is more than one.

18. A new computer monitor is made up of a rectangular screen surrounded by a hard plastic frame in which speakers can be inserted.



a. Write expressions for the length and width of the screen in terms of x.
b. Write an expression for the area of the screen, using expanded form.
c. Calculate the area of the screen if $x = 30$ cm.

19. An astronaut needs to find a positive number such that twice the number plus its square gives 35.
a. Write this as an equation.
b. Solve the equation to find the answer for her.

on To test your understanding and knowledge of this topic, go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 6 Quadratic equations and graphs

6.1 Pre-test

1. $x = 9,\ x = -9$
2. $x = 6,\ x = -6$
3. $x = 5$ is not a solution
4. $x = 3,\ x = -7$
5. $x = 0,\ x = 8$
6. $k = 0$
7. False
8. $x = \frac{1}{2}$ or $x = \frac{5}{3}$
9. True
10. D
11.

x	-2	-1	0	1	2
y	20	5	0	5	20

12. 288
13. C
14. D
15. D

6.2 Quadratic equations

1. a. $a = 1, b = 2, c = -1$
   b. $a = 1, b = 1, c = -10$
   c. $a = 4, b = 5, c = -30$
2. a. $a = 8, b = -11, c = 2$
   b. $a = 2, b = 1, c = -60$
   c. $a = 1, b = 0, c = -17$
3. D
4. C
5. a. $x = 4, x = -4$
   b. There is no solution.
   c. $x = 0$
6. a. $x = \pm 3$ b. $x = \pm 5$ c. $x = \pm 4$
7. a. $x = \pm\sqrt{7}$ b. $x = \pm\sqrt{8}$ c. $x = \pm\sqrt{3}$
8. a. $x = 1, x = -1$
   b. There is no solution.
   c. $x = 2, x = -2$
9. a. ✗ Not a solution
   b. ✓ Is a solution
   c. ✗ Not a solution
10. a. ✓ Is a solution
    b. ✓ Is a solution
    c. ✗ Not a solution
11. a. ✓ Is a solution
    b. ✗ Not a solution
    c. ✓ Is a solution
12. a. N b. N c. Y
13. a. Y b. N
14. When $x = 2$:
$$\begin{aligned}(x+4)^2 &= x^2 + 8x + 16\\ &= 2^2 + 8\times 2 + 16\\ &= 4 + 16 + 16\\ &= 36\end{aligned}$$
$$\begin{aligned}x^2 + 16 &= 2^2 + 16\\ &= 4 + 16\\ &= 20\end{aligned}$$
The two expressions are not equal.
15. $16 \neq -16$
16. If $x^2 + 7x + 4 = 7x$, then $x^2 + 4 = 0$, which means that $x^2 = -4$ and, finally, $x = \sqrt{-4}$.
As the square root of a negative number is not a real number, x has no solutions in the real number range.
17. a. $4x^2 = 100$ b. 5 or -5
18. a. 6 or -6
    b. There are two possible numbers, as the square root of a positive number, in this case 36, has two possible solutions, one positive and one negative.
19. a. $2x^2 = 52$
    b. $\sqrt{26}$ or $-\sqrt{26}$
20. a. $3(x-2)^2 - 7 = 41$
    b. -2 and 6

6.3 The Null Factor Law (optional)

1. a. $x = -3, x = 2$
   b. $x = -2, x = 3$
   c. $x = -2, x = 3$
   d. $x = -2\frac{1}{2}, x = -\frac{3}{4}$
   e. $x = -4, x = -\frac{1}{2}$
2. a. $x = 1$ or $x = -6$
   b. $x = \frac{2}{3}$ or $x = -5$
   c. $x = 4$ or $x = 7$
   d. $x = \frac{3}{2}$ or $x = -\frac{1}{5}$
   e. $x = -3$ or $x = \frac{2}{3}$
3. a. $x = -1$ or $x = 9$
   b. $x = -\frac{5}{4}$ or $x = 2$
   c. $x = 6$ or $x = -2$
   d. $x = -\frac{4}{3}$ or $x = \frac{7}{2}$
   e. $x = 5$ or $x = 1$
4. a. $x = \frac{1}{2}, x = -30$
   b. $x = -\frac{1}{2}, x = 3$
   c. $x = 1, x = \frac{1}{3}$
   d. $x = 0, x = 2$

e. $x = -\frac{1}{3}, x = \frac{1}{4}$

5. a. $x = -2.3, x = 0.3$

b. $x = -\frac{1}{6}, x = \frac{1}{6}$

c. $x = 2$

d. $x = 0, x = \frac{15}{4}$

e. $x = -4$

6. a. $x = 2, x = -2, x = -3$

b. $x = -2, x = 2.5$

c. $x = -2, x = -4$

d. $x = 0, x = -2, x = -4$

7. a. $x = 1.1, x = -2.4, x = -2.6$

b. $x = -3, x = -\frac{1}{2}, x = 1\frac{2}{3}$

c. $x = 3$

d. $x = -1, x = 2$

8. D

9. B

10. a. When $x = 2$, the first bracket equals 2 and the second bracket equals 4; therefore, the product is 8.

b. -2 and 1

11. A quadratic can have a maximum of two solutions, because a quadratic can at most be factorised into two separate pairs of brackets, each of which represents one solution.

12. a. 10 metres

b. $y = \frac{1}{250}x(100 - x)$; $a = \frac{1}{250}$, $b = 100$

c. 9.424 m

13. a. 16 m

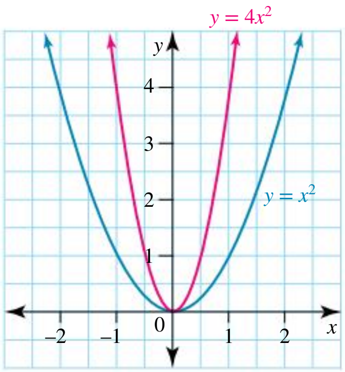
b. 0 m

14. 6 or -9

15. -3 or $-\frac{5}{2}$

6.4 Graphing quadratic functions of the form $y = ax^2$

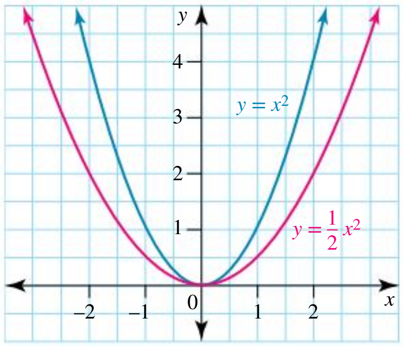
1. a.



Turning point for each is at (0, 0); x-int and y-int is 0 for both.

b. $y = 4x^2$ is narrower.

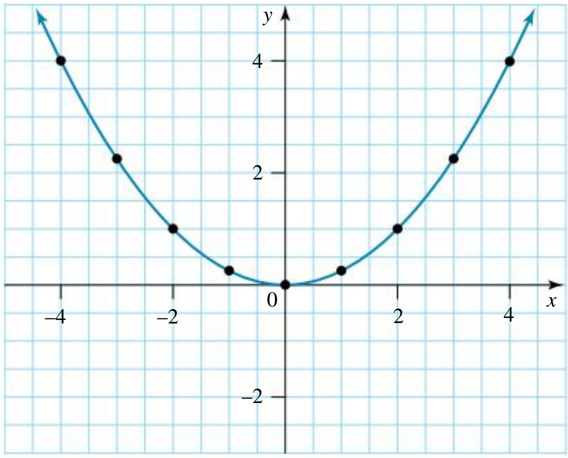
2. a.



Turning point for each is at (0, 0); x-int and y-int is 0 for both.

b. $y = x^2$ is narrower.

3.



$y = 0.25x^2$

4. a, b. See the figure at the foot of the page.*
Looking at the graphs, $y = -5x^2$ is the narrowest.

5. B

6. A

7. B

8. a. iii **b.** vi **c.** i
d. ii **e.** iv **f.** v

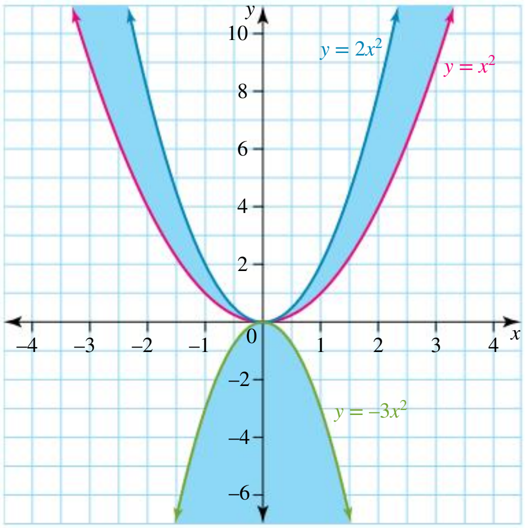
9. Sample responses can be found in the worked solutions in the online resources; for example, $y = 2x^2$.

10. Sample responses can be found in the worked solutions in the online resources; for example, $y = -0.5x^2$.

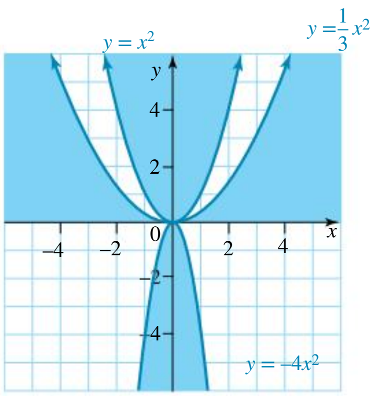
11. a. $y = 3x^2$ **b.** $y = -x^2$

12. a. -350 **b.** -350 **c.** -31.5
d. -7.875 **e.** -16.94

13. a.



b.



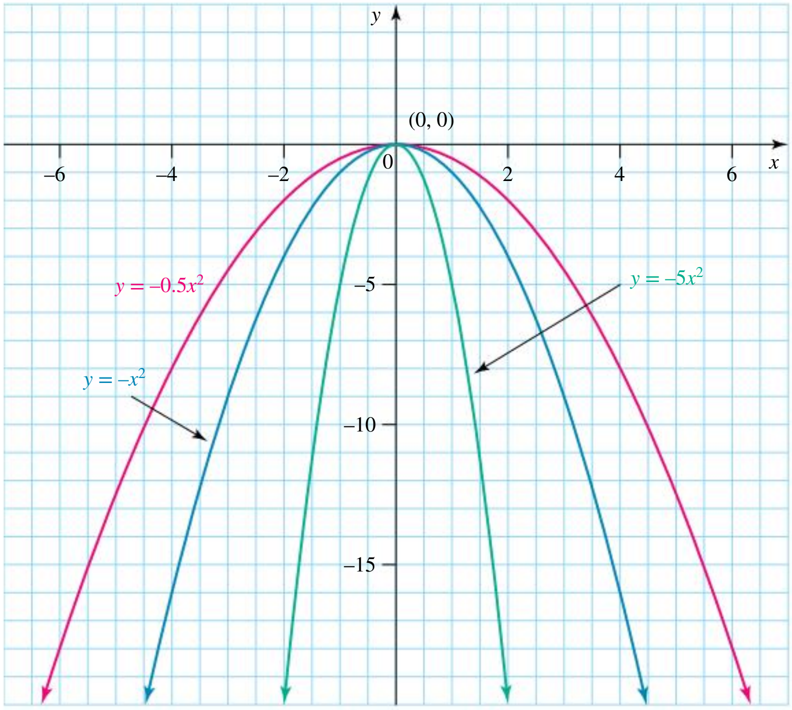
A bow

14. $y = \frac{2}{3}x^2$

15. a. Sample responses can be found in the worked solutions in the online resources.

b. 6

*4. a, b.



16. a. 400 watts b. 625 watts

17. $k = 1$

18. $x = y^2$

Project

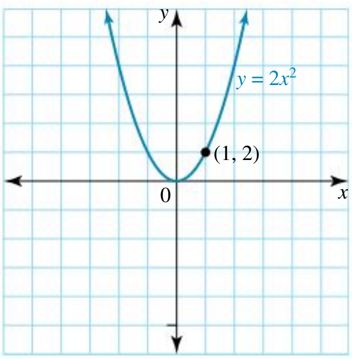
1–5. Students need to construct a parabola by folding paper with the directions given in this project. Draw incident and reflected rays of light and use a protractor to carefully measure the angle each line makes with the mirror. Also comment on the diagram.

6. Parabolas could be seen in real life in, for example, water shot by the fountain in a parabolic path, a ball thrown into the air, bridges, headlights, satellite dishes or telescopes.

6.5 Review questions

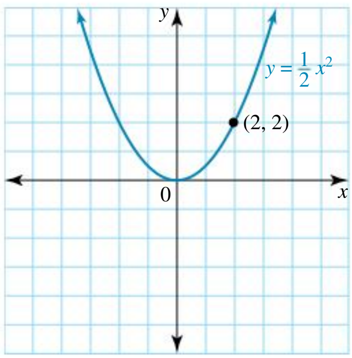
1. D
2. A
3. C
4. B
5. A
6. a. $x = 10$ or $x = -10$

   b. $x = 5$ or $x = -5$
7. a. $x = \pm\sqrt{8}$ b. $x = \pm\sqrt{20}$
8. $x = 4$ is a solution.
9. $x = -4$ or $x = 2$
10. $x = -4$, $x = 3$
11. $x = 3.5$, $x = 4$
12. B
13. A
14. A
15. a.



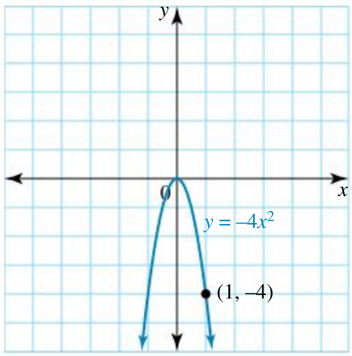
Minimum

b.



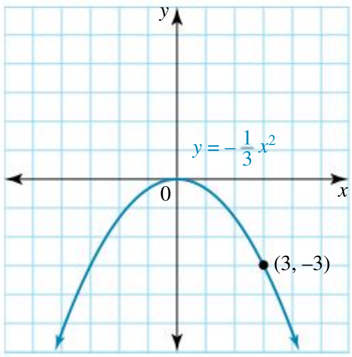
Minimum

c.



Maximum

d.



Maximum

16. 6 s
17. a. $5x^2 = 180$

    b. The solutions are:
    $x = \pm\sqrt{36} = \pm 6$
18. a. Length $= (x - 4)$ cm, width $= (x - 6)$ cm

    b. $\left(x^2 - 10x + 24\right)$ cm^2

    c. 624 cm^2
19. a. $x^2 + 2x = 35$ b. 5

STRAND: Number and algebra
SUB-STRAND: Financial mathematics

7 Financial mathematics

LESSON SEQUENCE

LESSON 7.1 Overview

7.1.1 Why learn this?

There are many famous sayings about money. You have probably heard that 'money doesn't grow on trees', 'money can't buy happiness' and 'a fool and his money are soon parted'. Most of us have to be conscious of where our money comes from and where it goes to. Understanding the basic principles of finance is very helpful for managing everyday life. Some of you will already be earning money from a part-time or casual job. Do you know what you spend your money on, or where it goes? Do you have your own savings account? Many of you may already be saving for a new games console, your first car or your dream holiday.

In this topic you will investigate different kinds of employment as well as different investment options for saving your money. Every branch of industry and business, whether large or small, international or domestic, will have to pay their employees either an annual salary or a wage based on an hourly rate. Financial investments are another way for businesses and individuals to make money — it is important to understand how investments work to be able to decide whether an investment is a good idea or not.

A sound knowledge of financial mathematics is essential in a range of careers, including financial consultancy, accountancy, business management and pay administration.

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Reading content and rich media, including interactivities and videos for every concept

Extra learning resources

Differentiated question sets

Questions with immediate feedback, and fully worked solutions to help students get unstuck.

Exercise 7.1 Pre-test

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1. Kira has an annual salary of $85 450.70 (without tax).
   Calculate how much she is paid:
   a. weekly
   b. fortnightly
   c. monthly.

2. Ishmael has a casual job at a department store. He is paid $11.50 per hour. He gets double time on a Sunday. Calculate his wage for a week in which he worked from 4 pm to 8 pm on Thursday and 12 pm to 6 pm on Sunday.

3. Jose is a tutor. He receives $975 for a week in which he works 30 hours. Calculate his hourly rate of pay.

4. MC This table shows a timesheet for Yumi, who works in an electronics store.

Day	Pay rate	Start time	Finish time
Monday	Normal	9 am	1 pm
Tuesday	Normal	9 am	5 pm
Wednesday	Normal	11 am	7 pm
Thursday	Normal	2 pm	7 pm
Friday	Normal	4 pm	8 pm
Saturday	Time-and-a-half	9 am	2 pm
Sunday	Double	12 pm	6 pm

   If Yumi's normal hourly rate is $15.30, the amount she earned for the week is:
   A. $612.00
   B. $734.00
   C. $818.55
   D. $742.05

5. Anita was offered a job with a salary of $65 000 per year, or $32.50 per hour. Assuming that she works a 40-hour week, compare the two rates to decide which is the better pay.

6. Ali earns $20.40 an hour. He needs to earn a minimum of $710 each week for his living expenses. Calculate the minimum number of hours he has to work each week.
   Give your answer to the nearest hour.

7. Mawluda is a casual worker in a café. She is paid time-and-a-half for working on Saturday and double time for working on Sunday. Her normal pay rate is $18.75 an hour.
   Over the weekend, she worked 8 hours on Saturday and 5 hours on Sunday.
   Calculate how much Mawluda earned over the weekend.

8. A bank has a special offer on a savings account of 3% p.a. simple interest.
Peter opens an account with $800 and leaves his money there for 5 months.
a. Calculate how much interest he earned over the 5 months.
b. Calculate the balance of Peter's account after the 5 months.

9. MC If an investment of $40 000 pays 8% simple interest per year, then the value of the investment at the end of 3 years is:
A. $3 200 B. $9 600 C. $43 200 D. $49 600

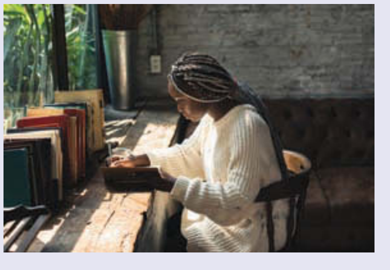
10. The Liang family wants to buy a family car at a cost of $22 700. They pay a deposit of $4500 and borrow the balance at an interest rate of 11.5% p.a.
The loan will be paid off with 48 equal monthly payments.
a. Calculate how much interest they will pay.
b. Calculate the total cost of the car, including the interest paid.
c. Calculate the amount of each repayment.

11. MC If $10 000 is invested at a simple interest rate of 6% per annum, select the correct formula for calculating the interest earned for 6 months.
A. $I = 10\,000 \times \frac{6}{100} \times 6$
B. $I = 10\,000 \times 0.6 \times 6$
C. $I = 10\,000 \times \frac{6}{100} \times \frac{1}{2}$
D. $I = 10\,000 \times (1.6)^6$

12. If the simple interest charged on a loan of $6400 over 30 months was $1450, calculate the percentage rate of interest charged.
Give your answer to the nearest whole percentage.

13. A worker earns $27.60 per hour and works 30 hours each week.
a. Calculate her total earnings for one week.
b. Calculate the amount of superannuation her employer pays on her behalf each week at a 12% rate.

14. MC Abena is an author who is paid a royalty of 8.5% on all sales. If a book she wrote sells copies to the value of $12 500 a month, what royalty will she earn in a year?
A. $1062.50
B. $12 750.00
C. $106 250.00
D. $127 500.00



15. A real estate agent receives 2.5% commission on the first $250 000 of a sale and 4.5% on the rest. What is the commission the real estate agent received on a sale of a $825 000 property?

LESSON 7.2 Salaries and wages

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- understand the difference between a salary and a wage
- calculate weekly, fortnightly or monthly pays from a given annual salary
- calculate the amount earned from the hours worked and the hourly rate of pay.

7.2.1 Employees' salaries and wages

- Employees may be paid for their work in a variety of ways — most receive either a **wage** or a **salary**.

WAYS OF PAYING EMPLOYEES

Salary	Wages
A fixed amount of money paid per year (annually), usually paid fortnightly or monthly regardless of the number of hours worked.	A fixed amount of money per hour worked (hours worked outside the normal work period are paid at a higher amount).
Examples of salaried jobs include: architect, company director, data analyst, teacher, doctor, accountant, federal or state government minister.	Examples of waged jobs include: waiter, kitchen hand, bar attendant, receptionist, technician, retail assistant, fruit picker or packer.

Key points

- **Normal working hours in Australia are 38 hours per week.**
- **There are 52 weeks in a year.**
- **There are 26 fortnights in a year (this value is slightly different for a leap year).**
- **There are 12 months in a year.**

WORKED EXAMPLE 1 Calculating pay from annual salary

Susan has an annual salary of $63 048.92. Calculate how much she is paid:

a. weekly **b. fortnightly** **c. monthly.**

THINK	WRITE
a. 1. Annual means per year, so divide the salary by 52 because there are 52 weeks in a year.	a. Weekly salary $= 63\,048.92 \div 52$ ≈ 1212.48
2. Write the answer in a sentence.	Susan's weekly salary is $1212.48.
b. 1. There are 26 fortnights in a year, so divide the salary by 26.	b. Fortnightly salary $= 63\,048.92 \div 26$ ≈ 2424.96
2. Write the answer in a sentence.	Susan's fortnightly salary is $2424.96.
c. 1. There are 12 months in a year, so divide the salary by 12.	c. Monthly salary $= 63\,048.92 \div 12$ ≈ 5254.08
2. Write the answer in a sentence.	Susan's monthly salary is $5254.08.

WORKED EXAMPLE 2 Calculating wage given hourly rate of pay

Frisco has casual work at a fast-food store. He is paid $12.27 per hour Monday to Saturday and $24.54 per hour on Sunday.
Calculate his wage for a week in which he worked from 5:00 pm to 10:00 pm on Friday and from 6:00 pm to 9:00 pm on Sunday.

THINK	WRITE
1. Work out the number of hours Frisco worked each day. He worked 5 hours on Friday and 3 hours on Sunday.	Friday: $5 \times 12.27 = 61.35$ Sunday: $3 \times 24.54 = 73.62$
2. Calculate the total amount earned by adding the wages earned on Friday and Sunday.	$61.35 + 73.62 = 134.97$
3. Write the answer in a sentence.	Frisco's wage was $134.97.

7.2.2 Superannuation

- **Superannuation** is money your employer contributes for you to use when you retire.
- Superannuation is usually additional to your salary or wages. Some companies include it in their salary package.
- From 1 July 2025, employers must put 12% of your salary into a super fund — this is called the **Super Guarantee**.
- You can also add extra money to your super if you want. Doing this can help you save more for the future, and you might pay less tax on that money.

WORKED EXAMPLE 3 Calculating superannuation contributions

An employee's hourly rate is $28.75 and they work 38 hours each week. They are paid fortnightly and will receive superannuation at a rate of 12%. Calculate the amount of superannuation the employer pays on their behalf each fortnight.

THINK	WRITE
1. Calculate the fortnightly wage.	$28.75 \times 38 \times 2 = \$2185$
2. Calculate 12% of the fortnightly wage. Remember to convert 12% to a decimal by dividing by 100.	$12\% \text{ of } 2185 = \frac{12}{100} \times 2185$ $= 262.20$
3. Write the answer in a sentence.	The employer must pay $262.20 into the superannuation fund each fortnight.

Exercise 7.2 Salaries and wages

learnon

7.2 Quick quiz on | 7.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 7, 10, 13, 14, 17, 20	2, 4, 8, 11, 15, 18, 21	5, 6, 9, 12, 16, 19, 22, 23, 24

Fluency

1. WE1 Johann has an annual salary of $57 482. Calculate how much he is paid:
   a. weekly b. fortnightly c. monthly.
2. Nigara earns $62 300 per annum. Calculate how much she earns:
   a. weekly b. fortnightly c. monthly.
3. Calculate the annual salary of workers with the following weekly incomes.
   a. $368 b. $892.50 c. $1320.85
4. Calculate how much is earned per annum by people who are paid fortnightly salaries of:
   a. $995 b. $1622.46 c. $3865.31.
5. Compare the following salaries, and determine which of each pair is the higher.
   a. $3890 per month or $45 700 per annum.
   b. $3200.58 per fortnight or $6700 per month.
6. Calculate the hourly rate for these workers.
   a. Rahni earns $98.75 for working 5 hours.
   b. Francisco is paid $54.75 for working $4\frac{1}{2}$ hours.
   c. Nhan earns $977.74 for working a 38-hour week.
   d. Jessica works $7\frac{1}{2}$ hours a day for 5 days to earn $1464.75.

Understanding

7. Henry is a second-year apprentice motor mechanic. He receives an award wage of \$12.08 per hour. Jenny, a fourth-year apprentice, earns \$17.65 per hour.
   a. Calculate how much Henry earns in a 38-hour week.
   b. Determine how much more Jenny earns in the same period of time.

8. WE2 Juan has casual work for which he is paid \$13.17 per hour Monday to Saturday and \$26.34 per hour on Sundays.
   Calculate his total pay for a week in which he worked from 11 am to 5 pm on Thursday and from 2 pm to 7 pm on Sunday.

9. Mimi worked the following hours in one week.

Wednesday	5 pm to 9 pm
Thursday	6 pm to 9 pm
Friday	7 pm to 11 pm

   If her pay is \$21.79 per hour up to 9 pm and \$32.69 per hour after that, calculate her total pay for that week.

10. Decide who earns more money each week: Rhonda, who receives \$38.55 an hour for 38 hours of work, or Rob, who receives \$41.87 an hour for 36 hours of work.

11. Glenn is a chef. He receives \$1076.92 for a week in which he works 35 hours.
    Calculate his hourly rate of pay.

12. Zack and Kaylah work in different department stores. Zack is paid \$981.77 per week. Kaylah is paid \$26.36 per hour. Calculate how many hours Kaylah must work to earn more money than Zack.

13. Calculate what pay each of the following salary earners will receive for each of the periods specified.
    a. Annual salary \$83 500, paid each week.
    b. Annual salary \$72 509, paid each fortnight.
    c. Annual salary \$57 200, paid each week.
    d. Annual salary \$105 240, paid each month.

14. WE3 A worker earns \$29.45 per hour and works 25.75 hours each week. She is paid fortnightly.
    a. Calculate her total earnings each fortnight.
    b. Calculate the amount of superannuation her employer pays on her behalf each fortnight at a 12% rate.

15. A school principal earns an annual salary of \$155 750.
    a. Determine how much they earn each month.
    b. Calculate how much their employer contributes to their superannuation fund annually at a rate of 12%.

16. An employee earns \$1850 per week.
    a. Determine their total annual salary.
    b. Calculate how much superannuation their employer contributes each year at a rate of 12%.

Reasoning

17. MC When Jack was successful in getting a job as a trainee journalist, he was offered the following choice of four salary packages.
Decide which salary package Jack should choose, and justify your answer.

A. \$456 per week
B. \$915 per fortnight
C. \$1980 per calendar month
D. \$23 700 per year

18. Julie is considering two job offers for work as a receptionist. Job A pays \$878.56 for a 38-hour working week. Job B pays \$812.16 for a 36-hour working week. Determine which job has the higher hourly rate of pay. Justify your answer.

19. In his job as a bookkeeper, Minh works 38 hours per week and is paid \$32.26 per hour. Michelle, who works 38 hours per week in a similar job, is paid a salary of \$55 280 per year.
Decide who has the higher-paying job. Justify your answer.

Problem solving

20. A lawyer is offered a choice between two jobs: one with a salary of \$74 000 per year and a second that pays \$40 per hour. Assuming that the lawyer will work 80 hours every fortnight, decide which job pays the most. Justify your answer.

21. Over the last four weeks Shahni has worked 35, 36, 34 and 41 hours. If she earns \$24.45 per hour, calculate how much she earned for each of the two fortnights.

22. Jackson works a 40-hour week (8 hours a day, Monday–Friday) and earns \$62 000 per annum.
   a. Calculate Jackson's hourly rate.
   b. If he works on average an extra half an hour per day from Monday to Friday and then another 4 hours over the weekend (for the same annual salary), explain how his hourly rate is affected.
   c. If Jackson was earning the hourly rate and was being paid for every hour worked, calculate his potential earnings for the year.

23. Mark saves \$10 per week. Phil saves 5 cents in the first week, 10 cents the second week, and doubles the amount each week from that time on.
Determine how many weeks it will take for Phil's savings to be worth more than Mark's.

24. An employer contributes an additional \$10 800 to an employee's superannuation fund in one year.
If the superannuation rate is 12%, determine the employee's annual salary before the contribution.

LESSON 7.3 Penalty rates

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- calculate total wages when overtime or penalty rates are included
- calculate hours paid when time-and-a-half or double time is involved.

7.3.1 Overtime and penalty rates

- **Overtime** is paid when a wage earner works more than their regular hours each week. These additional payments are often referred to as penalty rates.
- Penalty rates are usually paid for working on weekends, public holidays or at night.
- The extra hours are paid at a higher hourly rate, normally calculated at either time-and-a-half or double time.

Calculating overtime

The overtime hourly rate is usually a multiple of the regular hourly rate.

Some examples of overtime rates include:
- **1.5 × regular hourly wage (time-and-a-half)**
- **2 × regular hourly wage (double time)**
- **2.5 × regular hourly wage (double time-and-a-half).**

Regular hourly rate	→	Overtime hourly rate
$25.00		1.5 × regular hourly rate 1.5 × 25.00 = $37.50

WORKED EXAMPLE 4 Calculating wages with overtime

Ursula works in hospitality and earns $23.30 per hour. Last week she received the normal rate for 30 hours of work as well as time-and-a-half for 3 hours of overtime and double time for 5 hours of work on Sunday. Calculate her total pay.

THINK	WRITE
1. Calculate Ursula's normal pay for 30 hours.	Normal pay: $30 \times 23.30 = 699.00$
2. Calculate Ursula's pay for 3 hours at time-and-a-half (1.5 × regular hourly wage)	Overtime: $3 \times 1.5 \times 23.30 = 104.85$
3. Calculate Ursula's pay for 5 hours at double time (2 × regular hourly wage).	Sunday: $5 \times 2 \times 23.30 = 233.00$
4. Calculate the total amount by adding normal pay and overtime pay.	Total $= 699.00 + 104.85 + 233.00$ $= 1036.85$
5. Write the answer in a sentence.	Ursula's total pay was $1036.85.

7.3.2 Time sheets and pay slips

- Employers often use records to monitor the number of working hours of their employees.

FINANCIAL RECORD-KEEPING

Time sheets

Provide details of the number of hours worked by each employee, including overtime

Pay slips

Provide details of hours worked and the rate for each employee

WORKED EXAMPLE 5 Calculating pay slips showing overtime

Fiona works in a department store, and in the week before Christmas she works overtime. Her time sheet is shown. Fill in the details on her pay slip.

	Start	Finish	Normal hours	Over time 1.5
M	9:00	15:00	6	
T	9:00	17:00	8	
W	9:00	17:00	8	
T	9:00	19:00	8	2
F	9:00	19:00	8	2
S				

Pay slip for: Fiona Lee	Week ending December 21
Normal hours	
Normal rate	$17.95
Overtime hours	
Overtime rate	
Total wage	

THINK

1. Calculate the number of normal hours worked by adding the hours worked on each day of the week.

WRITE

Normal hours: $6 + 8 + 8 + 8 + 8 = 38$

2. Calculate the number of overtime hours worked by adding the overtime hours worked on Thursday and Friday.

Overtime hours: $2 + 2 = 4$

3. Calculate the overtime rate ($1.5 \times$ regular hourly wage).

$$\begin{aligned}\text{Overtime rate} &= 1.5 \times 17.95 \\ &= 26.93\end{aligned}$$

4. Calculate the total pay by multiplying the number of normal hours by the normal rate and adding the overtime amount (calculated by multiplying the number of overtime hours by the overtime rate).

$$\begin{aligned}\text{Total pay} &= 38 \times 17.95 + 4 \times 26.93 \\ &= 789.82\end{aligned}$$

5. Fill in the amounts on the pay slip.

Pay slip for: Fiona Lee	**Week ending December 21**
Normal hours	38
Normal rate	$17.95
Overtime hours	4
Overtime rate	$26.93
Total wage	$789.82

COLLABORATIVE TASK: Comparing pay rates and conditions

Working in small groups, collect several job advertisements from a mixture of print and digital sources. Compare the pay rates and conditions for the different positions and present a report to the class.

Exercise 7.3 Penalty rates

learnon

7.3 Quick quiz on | 7.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 6, 9, 12, 15	3, 4, 7, 10, 13, 16	5, 8, 11, 14, 17

Fluency

1. Calculate the following penalty rates:
   a. time-and-a-half when the hourly rate is $15.96
   b. double time when the hourly rate is $23.90
   c. double time-and-a-half when the hourly rate is $17.40.
2. Calculate the following total weekly wages:
   a. 38 hours at $22.10 per hour, plus 2 hours at time-and-a-half
   b. 40 hours at $17.85 per hour, plus 3 hours at time-and-a-half
   c. 37 hours at $18.32 per hour, plus 3 hours at time-and-a-half and 2 hours at double time.
3. Julio is paid $956.08 for a regular 38-hour week. Calculate:
   a. his hourly rate of pay
   b. how much he is paid for 3 hours of overtime at time-and-a-half rates
   c. his wage for a week in which he works 41 hours.

4. WE4 Geoff is a waiter in a cafe and works 8 hours most days. Calculate what he earns for 8 hours work on the following days:

a. a Monday, when he receives his standard rate of $21.30 per hour
b. a Sunday, when he is paid double time
c. a public holiday, when he is paid double time-and-a-half.

5. Albert is paid $870.58 for a 38-hour week. Determine his total wage for a week in which he works 5 extra hours on a public holiday with a double-time-and-a-half penalty rate.

Understanding

6. Jeleesa (aged 16) works at a supermarket on Thursday nights and weekends. The award rate for a 16-year-old is $7.55 per hour. Calculate what she would earn for working:

a. 4 hours on Thursday night
b. 6 hours on Saturday
c. 4 hours on Sunday at double time
d. the total of the 3 days described in parts **a**, **b** and **c**.

7. Jacob works in a pizza shop and is paid $13.17 per hour.

a. Jacob is paid double time-and-a-half for public holiday work. Calculate what he earns per hour on public holidays. Give your answer to the nearest cent.
b. Calculate Jacob's pay for working 6 hours on a public holiday.

8. If Bronte earns $7.80 on normal time, calculate how much she receives per hour:

a. at time-and-a-half
b. at double time
c. at double time-and-a-half.

9. WE5 a. Complete the time sheet shown. Calculate the number of hours Susan worked this week.

Day	Pay rate	Start time	Finish time	Hours worked
Monday	Normal	9:00 am	5:00 pm	
Tuesday	Normal	9:00 am	5:00 pm	
Wednesday	Normal	9:00 am	5:00 pm	
Thursday	Normal	9:00 am	5:00 pm	
Friday	Normal	9:00 am	3:00 pm	

b. Complete Susan's pay slip for this week.

Pay slip for: Susan Jones	Week ending 17 August
Normal hours	
Normal pay rate	$25.60
Overtime hours	0
Overtime pay rate	$38.40
Total pay	

10. a. Manu works in a department store. His time sheet is shown. Complete the table.

Day	Pay rate	Start time	Finish time	Hours worked
Monday	Normal	9:00 am	5:00 pm	
Tuesday	Normal	9:00 am	5:00 pm	
Wednesday	Normal	—		
Thursday	Normal	1:00 pm	9:00 pm	
Friday	Normal	—		
Saturday	Time-and-a-half	8:00 am	12:00 pm	

b. Complete Manu's pay slip for this week.

Pay slip for: Manu Taumata	Week ending 21 December
Normal hours	
Normal pay rate	$10.90
Overtime hours	
Overtime pay rate (time-and-a-half)	
Total pay	

11. a. Eleanor does shift work. Copy and complete their time sheet.

Day	Pay rate	Start time	Finish time	Hours worked
Monday	Normal	7:00 am	3:00 pm	
Tuesday	Normal	7:00 am	3:00 pm	
Wednesday				
Thursday				
Friday	Normal	11:00 pm	7:00 am	
Saturday	Time-and-a-half	11:00 pm	7:00 am	
Sunday	Double time	11:00 pm	7:00 am	

b. Copy and complete Eleanor's pay slip for the week.

Pay slip for: Eleanor Rigby	Week ending 15 September
Normal hours	
Normal pay rate	$16.80
Time-and-a-half hours	
Time-and-a-half pay rate	
Double time hours	
Double time pay rate	
Gross pay	

Reasoning

12. Calculate the following total weekly wages:

a. 38 hours at $18.40 per hour, plus 2 hours at time-and-a-half

b. 32 hours at $23.70 per hour plus 6 hours on a Sunday at double time.

13. Ruby earns $979.64 for her normal 38-hour week, but last week she also worked 6 hours of overtime at time-and-a-half rates.
 a. Calculate how much extra she earned and give a possible reason for her getting time-and-a-half rates.
 b. Calculate Ruby's total wage.

14. A standard working week is 38 hours and a worker puts in 3 hours overtime at time-and-a-half, plus 2 hours at double time.
Calculate the amount of standard work hours equivalent to the total time they have worked.

Problem solving

15. Joshua's basic wage is $22 per hour. His overtime during the week is paid at time-and-a-half. Over the weekend he is paid double time.
Calculate Joshua's gross wage in a week when he works his basic 40 hours, together with 1 hour overtime on Monday, 2 hours overtime on Wednesday and 4 hours overtime on Saturday.

16. Lin works 32 hours per week at $22 per hour and is paid overtime for any time worked over the 32 hours per week. In one week Lin worked 42 hours and was paid $814. Overtime is paid at 1.5 times the standard wage. Determine whether Lin was paid the correct amount. If not, then provide the correct amount.

17. The table shows the pay sheet for a small company. If a person works up to 36 hours, the regular pay is $14.50 per hour.
For hours over 36 and up to 40, the overtime is time-and-a-half.
For hours over 40, the overtime is double time.
Complete the table.

	Hours worked	Regular pay	Overtime pay	Total pay
a.	32			
b.	38.5			
c.	40.5			
d.	47.2			

LESSON 7.4 Piecework, commission and royalties

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- calculate the payment of piecework
- calculate commission on sales
- calculate royalties paid on copyright.

7.4.1 Non-wage earnings — piecework

- **Piecework** is a system of payment by which a worker is paid a fixed amount for each job or task they complete.

WORKED EXAMPLE 6 Calculating pay for a task with a fixed rate

Mitchell is washing cars for a fundraiser. He is paid $5.20 per car washed. Calculate the amount Mitchell earns in an afternoon when he washes 24 cars.

THINK	WRITE
1. Multiply the number of cars Mitchell washes by the amount paid for each car.	Amount earned $= 24 \times 5.20$ $= 124.80$
2. Write the answer in a sentence.	Mitchell earns $124.80.

- A person may also be paid on a sliding scale where the pay rate increases as the number of completed tasks increases.

WORKED EXAMPLE 7 Calculating pay for a task with a sliding rate

Angelica is a machinist in a clothing factory. Each week she is paid $4.28 per garment for the first 180 garments, and $5.35 per garment thereafter. What will she be paid if she produces 223 garments?

THINK	WRITE
1. Calculate the number of 'extra' garments Angelica makes.	Extra garments $= 223 - 180$ $= 43$
2. Calculate her total payment by adding the payment she receives for the first 180 garments to the payment she receives for the extra garments.	Payment $= 180 \times 4.28 + 43 \times 5.35$ $= 1000.45$
3. Write the answer in a sentence.	Angelica earns a total payment of $1000.45.

- In some cases, piecework is paid for multiple units rather than single units. For example, for letterbox deliveries you may be paid per 1000 deliveries made.

WORKED EXAMPLE 8 Calculating pay given a rate per 1000

Holly is delivering brochures to letterboxes in her local area. She is paid $43.00 per 1000 brochures delivered. Calculate the amount Holly will earn for a delivery of 3500 brochures.

THINK	WRITE
1. Calculate the number of thousands of brochures Holly will deliver.	$3500 \div 1000 = 3.5$ So Holly will deliver 3.5 thousand brochures.
2. Multiply the number of thousands of brochures delivered by 43 to calculate what Holly will earn.	$\text{Holly's pay} = 3.5 \times 43.00$ $= 150.50$
3. Write the answer in a sentence.	Holly will earn $150.50.

7.4.2 Non-wage earnings — commissions and royalties

- **Commission** is a method of payment used mainly for salespeople. The commission paid is usually calculated as a percentage of the value of goods sold.
- A **royalty** is a payment made to a person who owns a copyright. For example, a musician who writes a piece of music is paid a royalty on CD and online sales. An author who writes a book is also paid a royalty based on the number of books sold. Royalties are calculated as a percentage of sales.

WORKED EXAMPLE 9 Calculating royalties

Mohamad is a songwriter who is paid a royalty of 12% on all sales of his music. Calculate the royalty that Mohamad earns if a song he writes sells CDs to the value of $150 000.

THINK	WRITE
1. Determine the royalty by calculating 12% of $150 000.	$\text{Royalty} = 12\% \text{ of } 150\,000$ $= 0.12 \times 150\,000$ $= 18\,000$
2. Write the answer in a sentence.	Mohamad earns $18 000 in royalties.

- Sometimes a salesperson is paid a small wage, called a retainer, plus a percentage of the value of the goods sold.

WORKED EXAMPLE 10 Calculating total wage including commission

Gemma, a car salesperson, is paid a retainer of $350 per week, plus a commission of 8% of the profits made by the company on cars that she sells.
a. How much does Gemma earn in a week when no sales are made?
b. How much does she earn in a week when $5000 profit was generated by her sales?

THINK	WRITE
a. If no sales are made, only the retainer is paid.	a. Gemma earns $350.
b. 1. Determine the commission paid by calculating 8% of $5000.	b. $\text{Commission} = 8\% \text{ of } 5000$ $= 0.08 \times 5000$ $= \$400$

2. Determine the total amount paid by adding the retainer and the commission.

$$\text{Total earnings} = 350 + 400$$
$$= \$750$$

3. Write the answer in a sentence.

Gemma earns $750.

- Sometimes the commission is broken into several parts with differing rates.

WORKED EXAMPLE 11 Calculating commission including different rates

A real estate agency receives 2% commission on the first $300 000 of a sale and 3% on the remainder. How much commission is received on the sale of a $380 000 property?

THINK	WRITE
1. Calculate the difference between $380 000 and $300 000.	$380\,000 - 300\,000 = 80\,000$
2. Calculate 2% of $300 000.	2% of $300\,000 = 6000$
3. Calculate 3% of $80 000.	3% of $80\,000 = 2400$
4. Calculate the total commission by adding the commission earned on $300 000 and the commission earned on $80 000.	$6000 + 2400 = 8400$
5. Write the answer in a sentence.	The commission received is $8400.

Exercise 7.4 Piecework, commission and royalties

7.4 Quick quiz on | 7.4 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 4, 5, 8, 13, 17, 19, 27	3, 6, 9, 11, 14, 15, 18, 20, 22, 23, 26	7, 10, 12, 16, 21, 24, 25, 28, 29

Fluency

1. WE6 Hitani is paid 65 cents for each teacup she decorates. How much is she paid for decorating 150 teacups?
2. WE7 Jack makes leather belts. The piece rate is $1.25 each for the first 50 belts and $1.50 thereafter. What is his income for a day in which he produces 68 belts?
3. A production-line worker is paid $1.50 for each of the first 75 toasters assembled, then $1.80 per toaster thereafter. How much does she earn on a day in which she assembles 110 toasters?

4. WE8 Rudolf earns \$42.50 per 1000 leaflets delivered to letterboxes. Calculate what Rudolf will earn for a week in which he delivers 7500 leaflets.

5. WE9 Danyang is a writer who is paid a royalty of 10% on all sales. Calculate the royalty she earns in a year if a book she writes sells copies to the value of \$30 000.

6. A home improvement company pays commission at the rate of 16% on all sales. What would a person earn who had sales to the value of:

   a. \$8000 b. \$6972.50?

7. Linda is a car salesperson who is paid a 1.5% commission on her sales. Calculate the amount of money Linda earns in a week where her sales total \$95 000.

8. WE10 Gordon is paid a retainer of \$200 per week plus a commission of 6% of the profits made by the company on the goods that he sells.

   a. How much does Gordon earn in a week when no sales are made?
   b. How much does Gordon earn in a week during which a \$70 000 profit was generated by his sales?

9. Alfonso gets a retainer of \$235 per week plus a commission of $5\frac{1}{2}$% on sales. What are his total earnings in a week in which his sales are:

   a. \$1000 b. \$4500 c. \$17 384?

10. Bryce is an author. His publisher pays him a fixed allowance of \$500 per month, plus $4\frac{1}{2}$% royalty on sales. What would be his income for a month in which his book sales totalled:

    a. \$0
    b. \$2000
    c. \$15 000
    d. \$23 750?

Understanding

11. Dimitri earns \$7.20 for each box of fruit picked.

    a. How much does he make for picking 20 boxes?
    b. How many boxes must he pick to earn at least \$200?
    c. If he takes 4 hours to pick 12 boxes, what is his hourly rate of pay?

12. Pauline uses her home computer for word processing under contract with an agency. She is paid \$3 per page for the first 50 pages, \$4 per page from 51 to 100 pages, and \$5 per page thereafter. Calculate her total pay for a period in which she prepares:

    a. 48 pages b. 67 pages c. 123 pages.

13. WE11 A real estate agency receives 2% commission on the first \$250 000 of a sale and 4% on the rest. How much commission is received on the sale of a \$370 000 property?

14. At a second real estate agency, the commission rate is 5% on the first \$180 000 of sale price and 2% on the remainder. Calculate the commission on the sale of the \$370 000 property.

15. Ingrid's real estate agency pays her 1% commission on the first \$500 000 of sale price, then 4% thereafter. How much commission would she receive on the sale of a property worth:

a. \$480 000 b. \$510 000 c. \$735 000?

16. Yanu works for a boat broker who pays him 6% of the first \$50 000 of the sale price, then $3\frac{3}{4}$% on the rest. Calculate the commission he receives on the following sales.

a. \$40 000 b. \$70 000 c. \$395 000

17. Rani delivers bills to letterboxes and is paid \$43 per thousand.
   a. How much does she earn for delivering 2500 items?
   b. How many thousands must she deliver to earn at least \$1000?
   c. If she takes 6 hours to deliver each thousand on average, what is her hourly rate of pay?

18. Georgio delivers pizzas. He is paid \$3 per delivery from 5 pm to 9 pm and \$4 per delivery after 9 pm.
   a. How much does he earn on a night in which he makes 12 deliveries by 9 pm and 4 deliveries between 9 pm and 10.30 pm?
   b. What are his average earnings per hour if he has worked from 5 pm to 10.30 pm?

19. Veronica earns \$400 per week plus 4% on sales, whereas Francis earns 6% commission only.
   a. How much does each earn on sales of \$8400?
   b. What level of sales would yield each the same income?

20. Wolfgang, a car salesman, is paid a weekly retainer of \$550, plus 10% of the dealer's profit on each vehicle. Determine find his total income for weeks in which the dealer's profits on vehicles he sold were:

a. \$3500 b. \$5980 c. \$7036

Reasoning

21. What are the advantages and disadvantages of being paid by piecework?

22. What are the major advantages and disadvantages of getting paid by commission or royalties?

Problem solving

23. A shoemaker is paid \$5.95 for each pair of running shoes he can make.
   a. If the shoemaker made 235 pairs of shoes last week, what was the amount paid?
   b. The shoemaker is offered a bonus of 5% if he can make more than 250 pairs of shoes in a week. If he makes 251 pairs, what is the total amount earned, including the bonus?

24. A secretarial assistant gets paid \$12 per page that she types. If she manages to type more than 20 pages in a day, she gets a 10% bonus. If she typed 32 pages on Tuesday, how much did she earn?

25. There are both fixed and variable costs associated with some products. Consider the cost of importing a radio from China and selling it in Australia. The costs for a particular company are:
   - import of product \$12.50 per unit
   - transportation costs \$400 per 1000 units
   - warehouse rental space \$1 per unit per month
   - advertising costs \$2000 per month (fixed cost).

   a. If this company imports and sells 500 units per month, what is the total cost per month?
   b. At 500 units per month and a selling price of \$25.00, what is the total profit per month?

26. Using the commission table for house sales below, calculate the commission on each of the following sales.

Sale price	Commission	Plus
Between \$0 and \$80 000	2% of sale price	0
Between \$80 001 and \$140 000	1.5% of amount over \$80 000	\$1600 (2% of \$80 000)
\$140 000 and over	1.1% of amount over \$140 000	\$2500 (2% of \$80 000 + 1.5% of \$60 000)

a. \$76 000
b. \$122 500
c. \$145 000
d. \$600 000

27. Mr Hartney is a used car salesman. He receives a basic monthly salary of \$2400 together with 5% commission on all sales. Although his sales for the month amounted to \$48 300, he also had deductions for insurance (\$12.80), association fees (\$25.70) and income tax (\$1100). Calculate the amount, in dollars, he took home that month.

28. A rock musician makes a royalty on all record sales according to the following formula.

Sales from	Sales to	Royalty rate
0	\$100 000	3%
\$100 001	\$500 000	3.5% on amount over \$100 000
\$500 001	1 million	4% on amount over \$500 000
1 million	and above	5% on amount over 1 million

Calculate the musician's royalties for the following years.

a. 2007: sales = \$456 000
b. 2008: sales = \$1 234 500
c. 2009: sales = \$986 400
d. 2010: sales = \$2 656 000

29. Four years ago Inka became an employee of TrakRight Tourism, where her starting annual salary was \$55 600. After her first year she received a 2% pay rise. The next year she received a 3% pay rise. Last year she received an x% pay rise. If her annual salary is now \$61 042, determine the value of x, correct to 1 decimal place.

LESSON 7.5 Simple interest

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- calculate the simple interest on a loan or an investment
- apply the simple interest formula to determine the time, the rate or the principal.

7.5.1 Principal and interest

- When you put money in a financial institution such as a bank or credit union, the amount of money you start with is called the **principal**.
- **Interest** is the fee charged for the use of someone else's money.
- Investors are people who place money in a financial institution — they receive interest from the financial institution in return for the use of their money.
- Borrowers pay interest to financial institutions in return for being given money as a loan.
- The amount of interest is determined by the interest rate.
- Interest rates are quoted as a percentage over a given time period, usually a year.
- The total amount of money, which combines the principal (the initial amount) and the interest earned, is known as the value of the investment.

Value of an investment or loan

$$A = P + I$$

where:
- **A = the total amount of money at the end of the investment or loan period**
- **P = the principal (the initial amount invested or borrowed)**
- **I = the amount of interest earned.**

WORKED EXAMPLE 12 Calculating the total amount

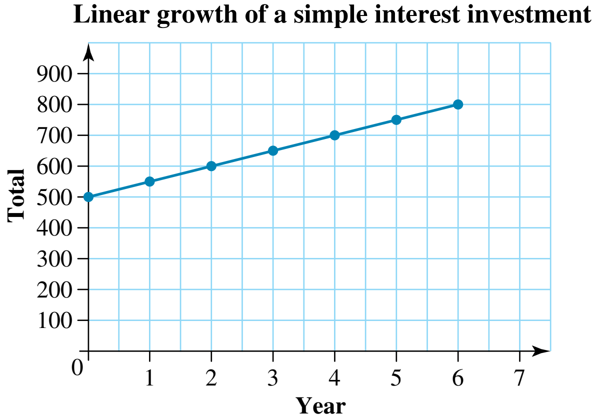
Sam borrowed \$450 000 from a bank for his business. After 30 years Sam had repaid his loan, with a total of \$500 000 paid to the bank.
Calculate how much interest Sam paid the bank.

THINK	WRITE
1. Write the known quantities: principal (P) and the total amount (A).	$P = \$450\,000$ $A = \$500\,000$
2. Write the formula to calculate interest paid.	$A = P + I$
3. Substitute the given values into the formula.	$500\,000 = 450\,000 + I$
4. Rearrange the formula to calculate the value of interest.	$I = 500\,000 - 450\,000$ $= 50\,000$
5. Write the answer in a sentence.	Sam paid \$50 000 in interest to the bank.

7.5.2 Simple interest

- **Simple interest** is the interest paid based on the principal of an investment.
- The principal remains constant and does not change from one period to the next.
- Since the amount of interest paid for each time period is based on the principal, the amount of interest paid is constant.
- Simple interest is usually used for short-term assets or loans.
  - For example, \$500 is placed in an account that earns 10% p.a. simple interest every year.
  - This means \$50 is earnt each year (10% of \$500 = \$50).
  - After 4 years the total interest earnt is $4 \times 50 = \$200$, so the value of the investment after 4 years is $500 + 200 = \$700$.
- Simple interest is an example of a linear relationship.
  - In the example above, a table could be constructed to show the addition of \$50 in interest added every year to the starting investment of \$500.
  - A graph can be drawn with the year as the explanatory variable and the total investment as the response variable.
  - The investment grows at a constant rate of \$50 per year.

Year	Investment
0	500
1	550
2	600
3	650
4	700
5	750



Formula for simple interest

$$I = Pin$$

where:

- I **= the amount of interest earned or paid**
- P **= the principal**
- i **= the interest rate as a percentage per annum (yearly), written as a decimal (e.g. 2% p.a. is equal to 0.02 p.a.)**
- n **= the duration of the investment in years.**

WORKED EXAMPLE 13 Calculating simple interest

Calculate the amount of simple interest earned over 4 years on an investment of \$35 000 that returns 2.1% p.a.

THINK	WRITE
1. Write the simple interest formula.	$I = Pin$
2. State the known values of the variables.	$P = \$35\,000$ $i = 0.021$ $n = 4$
3. Substitute the given values into the formula and evaluate the amount of interest.	$I = 35\,000 \times 0.021 \times 4$ $I = 2940$
4. Write the answer in a sentence.	The amount of simple interest over 4 years is \$2940.

- Note that the time, n, is in years. Months need to be expressed as a fraction of a year. For example, 27 months = 2 years and 3 months = 2.25 years.

WORKED EXAMPLE 14 Calculating the time of an investment

Karine invests \$2000 at a simple interest rate of 4% p.a. Calculate how long the money needs to be invested to earn \$120 in interest.

THINK	WRITE
1. Write the simple interest formula.	$I = Pin$
2. State the known values of the variables.	$P = \$2000$ $i = 0.04$ $n = \$120$
3. Substitute the given values into the formula and rearrange to calculate the value of the time.	$120 = 2000 \times 0.04 \times n$ $120 = 80n$ $\dfrac{120}{80} = \dfrac{80n}{80}$
4. Change the years into years and months.	$n = 1.5$ $n = 1$ year and 6 months
5. Write the answer in a sentence.	Karine will need to invest the \$2000 for 1 year and 6 months

WORKED EXAMPLE 15 Calculating the total amount of a loan

Zac borrows \$3000 for 2 years at 9% p.a. simple interest.
a. Calculate how much interest Zac is charged over the term of the loan.
b. Calculate the total amount Zac must repay.

THINK	WRITE
a. 1. Write the simple interest formula.	a. $I = Pin$

2. State the known values of the variables.

$P = \$3000$
$i = 0.09$
$n = 2$

3. Substitute the given values into the simple interest formula and evaluate the value of interest.

$I = 3000 \times 0.09 \times 2$
$I = 540$

4. Write the answer in a sentence.

Zac is charged $540 in interest.

b. 1. State the relationship for the value of a loan.

b. $A = P + I$
$A = 3000 + 540$

2. Substitute the values for P and I and calculate the total amount.

$A = 3540$

3. Write the answer in a sentence.

To repay the loan, Zac must pay $3540 in total.

WORKED EXAMPLE 16 Calculating simple interest on an item

The Carlon-Tozer family needs to buy a new refrigerator at a cost of $1679. They will pay a deposit of $200 and borrow the balance at an interest rate of 19.5% p.a.
The loan will be paid off in 24 equal monthly payments.

a. Calculate how much money the Carlon-Tozers need to borrow.
b. Calculate how much interest they will pay.
c. Determine the total cost of the refrigerator.
d. Calculate the amount of each payment.

THINK

WRITE

a. 1. Subtract the deposit from the cost to calculate the amount still owing.

a. $1679 - 200 = 1479$

2. Write the answer in a sentence.

They must borrow $1479.

b. 1. State the known values of the variables and write the simple formula.

b. $P = 1479$
$i = 0.195$
$n = 2$
$I = Pin$

2. Substitute the values into the formula to calculate interest (I), by multiplying the values.

$= 1479 \times 0.195 \times 2$
$= 576.81$

3. Write the answer in a sentence.

The interest will be $576.81.

c. 1. Add the interest to the initial cost to calculate the total cost of the refrigerator.

c. $1679.00 + 576.81 = 2255.81$

2. Write the answer in a sentence.

The total cost of the refrigerator will be $2255.81.

d. 1. Subtract the deposit from the total cost to calculate the amount to be repaid.

d. $2255.81 - 200 = 2055.81.$

2. Divide the total payment into 24 equal payments to calculate each repayment.

$2055.81 \div 24 = 85.66$

3. Write the answer in a sentence.

Each payment will be $85.66.

Technology and simple interest

- Digital technologies such as Microsoft Excel can be helpful in understanding the effects of changing P, i and n. Alternatively, you can investigate these effects using the interactivities in the Resources tab for this lesson.

COLLABORATIVE TASK: Interesting time periods

1. In groups, complete the table below for the pair of values $P = \$100$ and $R = 6\%$, with interest calculated and paid:
   a. each year
   b. every 6 months
   c. every 4 months
   d. every 3 months
   e. every 2 months
   f. every month.

Time period	2 years	3 years	5 years	7 years	10 years	15 years
Value of the investment						

2. As a class, discuss the effect of increasing the frequency of payments (for example, by paying interest monthly) at the same time as decreasing the interest rate (for example, by dividing it by 12). Would these changes make a difference to the final investment value? Why?

COLLABORATIVE TASK: Comparing financial options

Students will explore and compare different savings accounts, phone plans, and digital subscriptions based on interest rates, fees, withdrawal policies, price, quality, and terms and conditions.

Working in small groups, explore and compare a savings account, phone plan or a digital subscription. Each student should research a different account/plan/subscription.

Investigate and report on the following:
- **Savings accounts:** Interest rates, fees, withdrawal policies.
- **Phone plans:** Price, data limits, call/text limits, contract terms.
- **Digital subscriptions:** Price, content library, streaming quality, device compatibility.

Discuss the importance of comparing financial options. Reflect on what was learned and its real-life applications. Prepare and present findings to the class.

Exercise 7.5 Simple interest

learnon

7.5 Quick quiz on | 7.5 Exercise

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
1, 4, 6, 7, 10, 13, 18, 22	2, 5, 8, 11, 14, 17, 19, 20, 23, 24	3, 9, 12, 15, 16, 21, 25, 26

Fluency

1. **WE12** Alecia invests \$400. After 3 years, she withdraws the total value of her investment, which is \$496. Calculate the simple interest she earned per year.
2. **WE13** Calculate the amount of simple interest earned on an investment of \$4500:
   a. returning 3.5% p.a. for 6 years
   b. returning 2.75% p.a. for 4 years
   c. returning 7.5% p.a. for 2 years
   d. returning 2.5% p.a. for 8 years.
3. Calculate the amount of simple interest earned on an investment of:
   a. \$2000, returning 4.5% p.a. for 3 years and 6 months
   b. \$18 200, returning 3.6% p.a. for 6 months
   c. \$460, returning 2.15% p.a. for 2 years and 3 months
   d. \$6700, returning 3.2% p.a. for 7 years.
4. **WE14** Calculate how long it will take an investment of \$12 000 to earn the following:
   a. \$1260 interest at a simple interest rate of 3.5% p.a.
   b. \$2520 interest at a simple interest rate of 4.2% p.a.
   c. \$405 interest at a simple interest rate of 2.25% p.a.
   d. \$1485 interest at a simple interest rate of 4.5% p.a.
5. Calculate the simple interest rate per annum if:
   a. \$3000 earns \$270 in 3 years
   b. \$480 earns \$16.20 in 9 months
   c. \$5500 earns \$660 in 2.5 years
   d. \$2750 earns \$748 in 4 years.
6. **MC** If the total interest earned on a \$6000 investment is \$600 after 4 years, then the annual interest is:
   A. 10%
   B. 7.5%
   C. 5%
   D. 2.5%
7. **WE15** Monique borrows \$5000 for 3 years at 8% per annum simple interest.
   a. Calculate how much interest she is charged.
   b. Calculate the total amount she must repay.
8. Calculate the simple interest earned on an investment of \$15 000 at 5.2% p.a. over 30 months.
9. For each loan in the table, calculate:
   i. the simple interest
   ii. the amount repaid.

	Principal (\$)	Interest rate per annum	Time
a.	1000	5%	2 years
b.	4000	16%	3 years
c.	8000	4.5%	48 months
d.	2700	3.9%	2 years 6 months
e.	15 678	9.2%	42 months

10. Calculate the final value of each of the following investments.
    a. \$3000 for 2 years at 5% p.a.
    b. \$5000 for 3 years at 4.3% p.a.

11. Hasim borrows \$14 950 to buy a used car. The bank charges a 9.8% p.a. flat rate of interest over 60 months.

a. Calculate the total amount he must repay.

b. Calculate the monthly repayment.

12. Carla borrows \$5200 for an overseas trip at 8.9% p.a. simple interest over 30 months. If repayment is made in equal monthly instalments, calculate the amount of each instalment.

Understanding

13. Janan invested \$2000 at a simple interest rate of 4% p.a. Calculate how long he needs to invest it in order to earn \$200 in interest.

14. If Jodie can invest her money at 8% p.a., calculate how much she needs to invest to earn \$2000 in 2 years.

15. If the simple interest charged on a loan of \$9800 over 3 years is \$2352, determine the percentage rate of interest that was charged.

16. Calculate the missing quantity in each row of the table shown.

	Principal	Rate of interest p.a.	Time	Interest earned
a.	\$2000	6%		\$240.00
b.	\$3760	5.8%		\$545.20
c.		7%	3 years	\$126.00
d.		4.9%	1 year 9 months	\$385.88
e.	\$10 000		$1\frac{1}{2}$ years	\$1200.00
f.	\$8500		42 months	\$1041.25

17. WE16 Mika is buying a used car for \$19 998. He has a deposit of \$3000 and will pay the balance in equal monthly payments over 4 years. The simple interest rate will be 12.9% p.a.

a. Calculate how much money he needs to borrow.

b. Calculate how much interest he will pay.

c. Determine the total cost of the car.

d. Determine how many payments he will make.

e. Calculate the cost of each payment.

Reasoning

18. If a bank offers interest on its savings account of 4.2% p.a. and the investment is invested for 9 months, explain why 4.2 is not substituted into the simple interest formula as the interest rate.

19. Theresa invests \$4500 at 5.72% per annum in an investment that attracts simple interest for 6 months. Show that at the end of 6 months she should expect to have \$4628.70.

20. A \$269 000 business is purchased on an \$89 000 deposit with the balance payable over 5 years at an 8.95% p.a. flat rate.
   a. Calculate how much money is borrowed to purchase this business.
   b. Calculate how much interest is charged.
   c. Calculate the total amount that must be repaid.
   d. Determine the size of each of the equal monthly repayments.
   e. Explain two ways in which these payments could be reduced.
21. Nam has \$6273 in his bank account at a simple interest rate of 4.86% per annum. After 39 days he calculates that he will have \$6305.57 in his account. Did Nam calculate his interest correctly?
   Justify your answer by showing your calculations.

Problem solving

22. Giang is paid \$79.50 in interest for an original investment of \$500 for 3 years. Calculate the annual interest rate.
23. A loan is an investment in reverse: you borrow money from a bank and are charged interest. The value of a loan becomes its total cost.
   Jitto wishes to borrow \$10 000 from a bank, which charges 11.5% interest per year. If the loan is over 2 years:
   a. calculate the total interest paid
   b. calculate the total cost of the loan.
24. A new sound system costs \$3500, but it can be purchased for no deposit, followed by 48 equal monthly payments, at a simple interest rate of 16.2% p.a.

   a. Determine the total cost of the sound system.
   b. Under a 'no deposit, no payment for 2 years' scheme,48 payments are still required, but the first payment isn't made for 2 years. This will stretch the loan over 6 years. Calculate how much the system will cost using this scheme.
   c. Determine the monthly payments under each of these two schemes.
25. For the following questions, assume that the interest charged on a home loan is simple interest.

   a. Tex and Molly purchase their first home and arrange for a home loan of \$375 000. Their home loan interest rate rises by 0.25% per annum within the first 6 months of the loan.
      Determine the monthly increase of their repayments. Give your answer to the nearest cent.
   b. Brad and Angel's interest on their home loan is also increased by 0.25% per annum. Their monthly repayments increase by \$60.
      Determine the amount of their loan. Give your answer in whole dollars.
26. Juanita sells her car for \$10 984. She invests x% of the money in a bank account at a simple interest rate of 6.68% per annum for 1.5 years. She spends the remainder of the money. At the end of the investment she has exactly enough money to buy a new car for \$11 002.
   Determine the value of x, correct to 2 decimal places.

LESSON 7.6 Review

7.6.1 Topic summary

FINANCIAL MATHEMATICS

Salaries

- A salary is a fixed annual (yearly) payment, regardless of the number of hours worked.
- Salaries are generally paid in weekly, fortnightly or monthly amounts.
  - Normal working hours in Australia are **38 hours** per week.
  - There are **52 weeks** in a year.
  - There are **26 fortnights** in a year (this value is slightly different for a leap year).
  - There are **12 months** in a year.

  e.g. If an annual salary is $52 000:
  - Weekly salary = $52 000 ÷ 52 = $1000
  - Fortnightly salary = $52 000 ÷ 26 = $2000
  - Monthly salary = $52 000 ÷ 12 = $4333.33.

Wages

- A wage is paid at a fixed rate per hour.
- Hours worked outside of the usual 38 or 40 hours per week are usually paid at a higher rate per hour.
  e.g. If a wage's rate per hour is $24.50 and a person works 38 hours each week:
  Weekly pay = $24.50 × 38 = $931
- Superannuation is calculated at 12% of your total wage/salary.
  e.g. weekly pay × 12% = $931 × 0.12 = $111.72

Penalty rates

- Penalty rates are paid when employees work overtime or at weekends, on public holidays or at night.
  e.g. If the hourly pay is $24, then:
  time-and-a-half rate = $24 × 1.5
  = $36/hour
  double time rate = $24 × 2
  = $48/hour

Simple interest

- Interest is the amount earned from investing money with a bank.
- The formula to calculate the final value of a simple interest investment is $I = Pin$, where:
  - P = the principal, or the amount initially invested
  - i = the interest rate as a percentage per annum (yearly), written as a decimal (e.g. 2% p.a. is equal to 0.02 p.a.)
  - n = the duration of the investment in years.
- The formula to calculate the value of an investment or loan is $A = P + I$, where:
  - A = the total amount of money at the end of the investment or loan period
  - P = the principal
  - I = the amount of interest earned.

Piecework, commission and royalties

- Piecework is paid as a fixed amount for each job or task completed.
  e.g. Payment per car washed.
- Commission is paid as a percentage of the value of goods sold.
  e.g. A portion of each artwork sold.
- Royalties is a type of commission paid to the person who owns the copyright.

Simple interest investments

- For $8000 over 10 years with a 3% p.a. simple interest rate (also called a flat rate):

$$I = Pin = 8000 \times 0.03 \times 10 = 2400$$

- Final amount = $8000 + $2400
  = $10 400

7.6.2 Success criteria

Tick the column to indicate that you have completed the lesson and how well you think you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
7.2	I understand the difference between a salary and a wage.			
	I can calculate weekly, fortnightly or monthly pays from a given annual salary.			
	I can calculate the amount earned from the hours worked and the hourly rate of pay.			
7.3	I can calculate total wages when overtime or penalty rates are included.			
	I can calculate hours paid when time-and-a-half or double time is involved.			
7.4	I can calculate the payment of piecework.			
	I can calculate commission on sales.			
	I can calculate royalties paid on copyright.			
7.5	I can calculate the simple interest on a loan or an investment.			
	I can apply the simple interest formula to determine the time, the rate or the principal.			

7.6.3 Project

Australian currency

Since decimal currency was introduced in Australia in 1966, our notes and coins have undergone many changes. Only our 5c, 10c and 20c coins are still minted as they were back then. The 1c and 2c coins are no longer in circulation, the 50c coin is a different shape, the $1 and $2 notes have been replaced by coins, and our notes have changed from paper to a special type of plastic.

Coins have two sides: an obverse side and a reverse side. The obverse side of all Australian coins depicts the year in which the coin was minted and our reigning monarch at the time of minting. Most coins depict Queen Elizabeth II, however from 2023 onwards the coins will depict King Charles III. The reverse side depicts a typical Australian feature and sometimes a special commemorative event.

1. Describe what is depicted on the reverse side of each Australian coin.

The following table includes information on Australia's coins currently in circulation. Use the table to answer questions **2** to **4**.

Coin	Diameter (mm)	Mass (g)	Composition
5c	19.41	2.83	75% copper, 25% nickel
10c	23.60	5.65	75% copper, 25% nickel
20c	28.52	11.30	75% copper, 25% nickel
50c	31.51	15.55	75% copper, 25% nickel
$1	25.00	9.00	92% copper, 6% aluminium, 2% nickel
$2	20.50	6.60	92% copper 6% aluminium, 2% nickel

2. Identify the heaviest and lightest type of coin. List every type of coin in order from lightest to heaviest.
3. Identify which coin has the smaller diameter: the 5c coin or the $2 coin. Calculate the difference in size.

The following table displays information on the currency notes currently circulating in Australia. The column on the far right compares the average life of the previously used paper notes with that of the current plastic notes. Use the table to answer questions **4** to **8**.

			Average life of notes (months)	
Note	Date of issue	Size (mm)	Plastic	Paper
$5	07/07/1992 24/04/1995 01/01/2001 01/09/2016	130×65	40	6
$10	01/11/1993 20/09/2017	137×65	40	8
$20	31/10/1994 09/10/2019	144×65	50	10
$50	04/10/1995 18/10/2018	151×65	About 100	24
$100	15/05/1996 29/10/2020	158×65	About 450	104

4. List the denominations of the notes that are available in Australian currency.
5. State the date on which Australia's first plastic note was issued and the denomination of the note.
6. The $5 note has been issued 4 times, while all other notes have only been issued twice. Suggest a reason for this.
7. Suggest a reason why each note is a different size.
8. The table clearly shows that the plastic notes last about 5 times as long as the paper notes we once used. Suggest a reason why the $50 and $100 notes last longer than the $5 and $10 notes.

Exercise 7.6 Review questions

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Fluency

1. Jane earns an annual salary of $45 650. Calculate her fortnightly pay.

2. Express $638.96 per week as an annual salary.

3. Minh works as a casual shop assistant and is paid $8.20 an hour from 3:30 pm to 5:30 pm, Monday to Friday, and $9.50 an hour from 7 am to 12 noon on Saturday. Calculate his total pay for the week.

4. MC Below are the pay details for 5 people. Choose who receives the most money.
   A. Billy receives $18.50 per hour for a 40-hour working week.
   B. Jasmine is on an annual salary of $38 400.
   C. Omar receives $1476.90 per fortnight.
   D. Thuy receives $3205 per month.

5. Daniel earns $10 an hour for a regular 38-hour week. If he works overtime, he is paid double time. Calculate how much he would earn if he worked 42 hours in one week.

6. Bjorn earns $468.75 per week award wage for a 38-hour week.
   a. Calculate his standard hourly rate.
   b. If he is paid time-and-a-half for normal overtime, calculate his pay for a week in which he works 41 hours.

7. Phillipa is paid an annual salary of $48 800.
   Calculate how much she would be paid each time if she was paid:
   a. weekly
   b. monthly.

8. Deng works as a waiter and is paid $12.50 per hour. If he works 40 hours per week, calculate Deng's gross pay.

9. Xana makes gift cards as a hobby and is paid 55 cents for each card.
   a. How much would she earn if she made 50 cards?
   b. How many cards would she need to make to earn $44?

10. Kim sells cakes to the local shop. The shop pays for the ingredients. Kim is paid $3.20 for each cake that she makes and 50 cents for each slice. If Kim makes 5 cakes and 15 slices one week, how much does she earn?

11. A salesperson earns a $250 per week retainer, plus 2% commission on the first $10 000 sale and 1.5% on the remainder. What is their total income for a week in which sales to the value of $18 000 were made?

12. Jennifer is to start a new job selling mobile phones. She is paid commission only at the rate of 17.5% of sales. What value of sales must she make in order to receive commission of $600 in one week?

Problem solving

13. **a.** Calculate the simple interest on an investment of \$4000 for 9 months at 4.9% per annum.
b. Determine the final amount of the investment by the end of its term.

14. Calculate the monthly instalment needed to repay a loan of \$12 500 over 40 months at a 9.75% p.a. flat interest rate.

15. A simple interest loan of \$5000 over 4 years incurred interest of \$2300. Calculate the interest rate charged.

16. Calculate 3.5% of 900.

17. Milos works in a supermarket and earns \$11.70 per hour. If Milos works on Saturday he is paid at time-and-a-half. Determine the hourly rate for which Milos works on Saturday.

18. An employer contributes \$9600 to an employee's superannuation fund in one year.
If the superannuation rate is 12%, determine the employee's annual salary.

19. A car loan of \$18 000 is charged with 8.5% per annum simple interest over a 4-year period.
Determine how much interest will be charged for each month of the loan.

20. Savannah is charged \$600 in interest each month on a loan for home renovations over 5 years at a simple interest rate of 12% per annum.
Determine the original amount that Savannah borrowed.

on To test your understanding and knowledge of this topic, go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 7 Financial mathematics

7.1 Pre-test

1. a. $1643.28 b. $3286.57 c. $7120.89
2. $184
3. $32.50
4. D
5. $32.50 per hour
6. 35 hours
7. $412.50
8. a. $10 b. $810
9. D
10. a. $8372 b. $31 072 c. $553.58
11. C
12. 9% p.a
13. a. $828.00 b. $99.36
14. B
15. $32 120

7.2 Salaries and wages

1. a. $1105.42 b. $2210.85 c. $4790.17
2. a. $1198.08 b. $2396.15 c. $5191.67
3. a. $19 136 b. $46 410 c. $68 684.20
4. a. $25 870 b. $42 183.96 c. $100 498.06
5. a. $3890 per month b. $3200.58 per fortnight
6. a. $19.75/h b. $12.17/h
   c. $25.73/h d. $39.06/h
7. a. $459.04 b. Jenny earns $211.66 more.
8. $210.72
9. $261.49
10. Rob earns more.
11. $30.77
12. 38 hours
13. a. $1605.77 b. $2788.81
    c. $1100 d. $8770
14. a. $1515.59 b. $181.87
15. a. $12 979.17 b. $18 690.00
16. a. $96 200 b. $11 544.00
17. B, as the annual salary for option B is the highest.
18. Job A has a higher hourly rate.
19. Minh. Minh's annual salary is $63 745.76, which is higher than Michelle's annual salary.
20. $40 per hour. This is the option with the greatest pay.
21. $1735.95, $1833.75
22. a. $29.81 b. $25.64 c. $72 080.58
23. 12 weeks
24. Annual salary = $90 000

7.3 Penalty rates

1. a. $23.94 b. $47.80 c. $43.50
2. a. $906.10 b. $794.33 c. $833.56
3. a. $25.16 b. $113.22 c. $1069.30
4. a. $170.40 b. $340.80 c. $426.00
5. $1156.96
6. a. $30.20 b. $45.30
   c. $60.40 d. $135.90
7. a. $32.93 b. $197.55
8. a. $11.70 b. $15.60 c. $19.50
9. a. Mon: 8, Tue: 8, Wed: 8, Thu: 8, Fri: 6

   b.

Pay slip for Susan Jones	Week ending 17 August
Normal hours	38
Normal pay rate	$25.60
Overtime hours	0
Overtime pay rate	$38.40
Total pay	$972.80

10. a. Mon: 8, Tue: 8, Thu: 8, Sat: 4

    b.

Pay slip for Manu Taumata	Week ending 21 December
Normal hours	24
Normal pay rate	$10.90
Overtime hours	4
Overtime pay rate	$16.35
Total pay	$327.00

11. a. Mon: 8, Tue: 8, Fri: 8, Sat: 8, Sun: 8

    b.

Pay slip for Eleanor Rigby	Week ending 17 August
Normal hours	24
Normal pay rate	$16.80
Time-and-a half hours	8
Time-and-a half pay rate	$25.20
Double time hours	8
Double time pay rate	$33.60
Total pay	$873.60

12. a. $754.40 b. $1042.80
13. a. $232.02 b. $1211.66
14. $46\frac{1}{2}$
15. $1155
16. No. Lin should have been paid $1034.

17.

	Hours worked	Regular pay	Overtime pay	Total pay
a.	32	$464	$0	$464
b.	38.5	$522	$54.38	$576.38
c.	40.5	$522	$101.50	$623.50
d.	47.2	$522	$295.80	$817.80

7.4 Piecework, commission and royalties

1. $97.50
2. $89.50
3. $175.50
4. $318.75
5. $3000
6. a. $1280 b. $1115.60
7. $1425
8. a. $200 b. $4400
9. a. $290 b. $482.50 c. $1191.12
10. a. $500 b. $590 c. $1175 d. $1568.75
11. a. $144.00 b. 28 c. $21.60
12. a. $144 b. $218 c. $465
13. $9800
14. $12 800
15. a. $4800 b. $5400 c. $14 400
16. a. $2400 b. $3750 c. $15 937.50
17. a. $107.50 b. 24 c. $7.17
18. a. $52 b. $9.45
19. a. Veronica earns $736; Francis earns $504.
    b. $20 000
20. a. $900 b. $1148 c. $1253.60
21. Answers may vary. Advantages could include flexible hours or working from home; disadvantages could be that time per item is often at a very low hourly rate.
22. Answers may vary. One advantage is that hard work and a successful product or great selling skills will generate more money; one disadvantage is that other factors may influence the success of the product and your income may not reflect your work ethic.
23. a. $1398.25 b. $1568.12
24. $422.40
25. a. $8950 b. $3550
26. a. $1520 b. $2237.50 c. $2555 d. $7560
27. $3676.50
28. a. $15 460 b. $48 725 c. $36 456 d. $119 800
29. 4.5%

7.5 Simple interest

1. $32
2. a. $945 b. $495 c. $675 d. $900
3. a. $315 b. $327.60 c. $22.25 d. $1500.80
4. a. 3 years b. 5 years
   c. 1.5 years or 1 year and 6 months
   d. 2.75 years or 2 year and 9 months
5. a. 3% p.a. b. 4.5% p.a. c. 4.8% p.a. d. 6.8% p.a.
6. D
7. a. $1200 b. $6200
8. $1950
9. a. i. $100 ii. $1100
   b. i. $1920 ii. $5920
   c. i. $1440 ii. $9440
   d. i. $263.25 ii. $2963.25
   e. i. $5048.32 ii. $20 726.32
10. a. $3300 b. $5645
11. a. $22 275.50 b. $371.26
12. $211.90
13. 2.5 years
14. $12 500
15. 8%
16. a. 2 years b. 2.5 years c. $600 d. $4500.06 e. 8% f. 3.5%
17. a. $16 998 b. $8770.97 c. $28 768.97 d. 48 e. $536.85
18. The interest rate in the simple interest formula needs to be converted from a percentage into a decimal: 4.2% = 0.042.
19. Sample responses can be found in the worked solutions in the online resources.
20. a. $180 000
    b. $80 550
    c. $260 550
    d. $4342.50
    e. Increase the size of the deposit or increase the length of time over which the loan can be repaid
21. Yes
22. 5.3%
23. a. $2300 b. $12 300
24. a. $5768 b. $6902 c. $120.17, $143.79
25. a. $78.13 b. $288 000
26. 91.04

Project

1. 5c coin: echidna
   10c coin: lyrebird
   20c coin: platypus
   50c coin: coat of arms
   $1 coin: five kangaroos
   $2 coin: Walpiri-Anmatyerre Aboriginal Elder Gwoya Jungarai

2. 5c, 10c, \$2, \$1, 20c, 50c
3. The 5c coin has a smaller diameter. It is smaller by 1.09 mm.
4. \$5, \$10, \$20, \$50, \$100
5. 7 July 1992, \$5
6. Sample responses can be found in the worked solutions in the online resources.
7. The different sizes allow people with vision impairment to tell the difference between each note.
8. The \$50 and \$100 notes are used less frequently.

7.6 Review questions

1. \$1755.77
2. \$33 225.92
3. \$129.50
4. A
5. \$460
6. a. \$12.34 b. \$524.28
7. a. \$938.46 b. \$4066.67
8. \$500
9. a. \$27.50 b. 80 cards
10. \$23.50
11. \$570
12. \$3428.57
13. a. \$147 b. \$4147
14. \$414.06
15. 11.5%
16. 31.5
17. \$17.55
18. \$80 000
19. \$127.50
20. \$60 000

Modelling with number and algebra

Fully worked solutions for this topic are available online.

LESSON Modelling with number and algebra

Introduction

Mathematical modelling allows us to understand and solve many real-life situations. In number and algebra, you will apply your skills to manipulate relationships between two variables such as converting different currencies. Your understanding of financial mathematics is another important area to help you make wise decisions on your future.

Through applying your knowledge to different scenarios, you will make informed decisions and communicate solutions effectively. This process enhances your logical reasoning and deepens your understanding of how mathematics is in the world around us.

MODELLING TASK 1: Structured

To Alpha Centauri and back

Founded in 2016, Breakthrough Starshot is a research and engineering project to develop probes capable of making the journey to Alpha Centauri, the closest star to the Earth, other than the sun. Alpha Centauri is 4.3 light years away from Earth. Light travels at approximately 300 000 km/s. A light year is the distance light travels in one year.

How far is Alpha Centauri from Earth and, travelling at 100 km/h, how long in years would it take to reach Alpha Centauri?

a. **Analyse the situation:**
   - Give a brief overview of the star Alpha Centauri.
   - What information have you been given?
   - What are you asked to determine?

b. **Determine assumptions and constraints:**
   - What assumptions are you making? Assume 365 days in a year.
   - Are there any limitations or constraints you need to consider?
   - What is the best method to write these numbers?

c. **Represent the situation mathematically:**
   - Calculate the number of seconds in a year.
   - Calculate the distance travelled by light in one year.
   - Calculate the distance from Earth to Alpha Centauri, in kilometres.
   - How long would it take to reach Alpha Centauri?

d. **Interpret and communicate findings:**
   - What methods did you use to present your calculations?
   - What assumptions were you making?
   - Communicate your findings in a meaningful presentation.
   - Would travelling to Alpha Centauri seem possible or logical?

MODELLING TASK 2: Structured

The Lolly Shop

The local lolly shop has a 'Pick and Mix' section of sweets, where you pay by the total weight of your bag. You would like some of their strawberry twists and chocolate ripples. Each strawberry twists weighs 5 g and each chocolate ripple weighs 9 g.

You select some of each and fill your bag with 28 sweets. The bag weighs 188 g. Can you calculate how many sweets you bought of each type?

a. Analyse the situation:
- What information have you been given?
- What do you need to consider?
- Realise you will need to use algebra to solve the problem.

b. Determine assumptions and constraints:
- Have you made any assumptions?
- Are there any constraints or limitations you need to consider?
- How many sweets to you have?
- What is the weight of the different types of sweets?

c. Represent the situation mathematically:
- Write an expression for the number of one sweet in terms of the other.
- Write an equation using the weights of the sweets and the total weight of the lolly bag.
- Solve your linear equation.
- How many strawberry twists and chocolate ripples do you have?

d. Interpret and communicate findings:
- Present your calculations and explain your method of solving the problem.
- What assumptions did you made and could it have affected the outcome?
- Would you be able to swap 1 chocolate ripple for 2 strawberry twists without changing the cost since the cost is on the total weight of your lolly bag?

MODELLING TASK 3: Open ended

Hourly rate or salary?

A worker receives an hourly rate of \$28.40 to work a standard 38-hour week. They receive overtime of time-and-a-half for the first two hours worked above 38 hours and double time on any other hours overtime. On average, they work 42.5 hours each week. The worker is offered to go on a salary of \$68 000.

Will they be better off remaining on a wage or taking the salary offer?

Consider the following:
- What factors do you need to consider in advising the worker?
- What are the advantages/disadvantages of working on an hourly rate compared to a salary?
- How would you explain your recommendation to the worker?

MODELLING TASK 4: Open ended

Purchasing a new TV

Oliver is comparing purchasing plans for the latest 4K television. The recommended retail price of the television is $3500. He goes to three stores and they offer him the following buy now – pay later plans.

Store 1: $250 up-front + 12 monthly payments of $300
Store 2: 24 monthly payments of $165
Store 3: $500 up-front + 6 monthly payments of $540

Oliver is looking at both the best buy and the best rate of simple interest charged by these three stores.

Consider the following:
- What factors do you need to consider?
- How would you make the decision on which where to purchase the TV?
- How do you determine the rate of simple interest?
- Are there any other considerations that Oliver needs to consider?

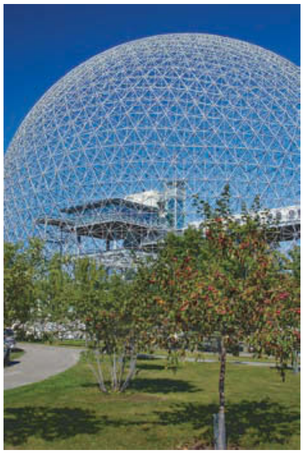
STRAND: Measurement and geometry
SUB-STRANDS: Two-dimensional space and structures

8 Pythagoras and trigonometry

LESSON SEQUENCE

LESSON 8.1 Overview

8.1.1 Why learn this?



Trigonometry is the branch of geometry that is concerned with triangles. The word 'trigonometry' was created in the sixteenth century from the Greek words *trigōnon* ('triangle') and *metron* ('measure'), but the study of the geometry of triangles goes back to at least the sixth century BCE, when the ancient Greek philosopher Pythagoras of Samos developed his famous theorem.

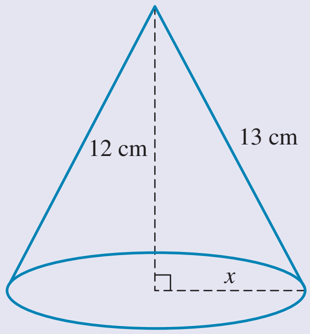
Pythagoras was particularly interested in right-angled triangles and the relationships between their sides. Later, Greek mathematicians used Pythagoras' theorem and the trigonometric ratios to calculate all kinds of distances, including Earth's circumference. Trigonometry is still the primary tool used by surveyors and geographers today when working out distances between points on Earth's surface.

None of the structures we build would be possible without our understanding of geometry and trigonometry. Engineers apply the principles of geometry and trigonometry regularly to make sure that buildings are strong, stable and capable of withstanding extreme conditions. Triangles are particularly useful to engineers and architects because they are the strongest shape. Any forces applied to a triangular frame will be distributed equally to all of its sides and joins. This fact has been known for thousands of years — triangular building frames were used as far back as the sixth century BCE.

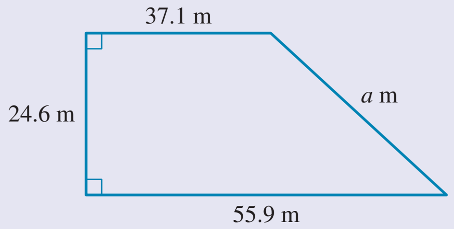
A truss is an example of a structure that relies on the strength of triangles. Trusses are often used to hold up the roofs of houses and to keep bridges from falling down. Triangular frames can even be applied to curved shapes. Geodesic domes, like the one shown here, are rounded structures that are made up of many small triangular frames connected together. This use of triangular frames makes geodesic domes very strong, but also very light and easy to build.

Exercise 8.1 Pre-test

1. Calculate the value of x in the diagram shown.



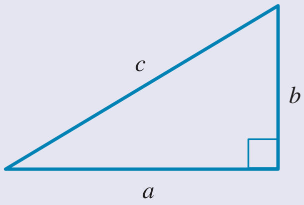
2. Calculate the value of a in the diagram shown, correct to 2 decimal places.



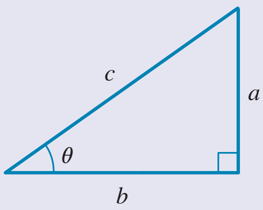
3. MC Identify which set of 3 numbers could represent the sides of a right-angled triangle.
   A. 2, 3, 4 B. 5, 12, 13 C. 1, 2, 3 D. 5, 6, 7

4. If a right-angled triangle has side lengths a, $a + b$ and $a - b$, where both a and b are greater than 0, state which one of the lengths is the hypotenuse.

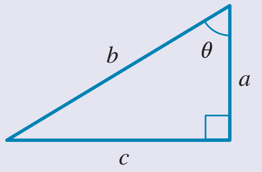
5. MC Identify the correct rule linking the sides of the right-angled triangle shown.
   A. $c^2 = (a + b)^2$ B. $c = a + b$
   C. $c^2 = a^2 + b^2$ D. $c = \dfrac{b}{a}$



6. MC Consider the right-angled triangle shown. Identify the correct option for angle θ.
   A. a is the adjacent side.
   B. b is the opposite side.
   C. a is the hypotenuse
   D. b is the adjacent side.



7. **MC** Identify the correct trigonometric ratio for the triangle shown.



A. $\tan(\theta) = \dfrac{a}{c}$

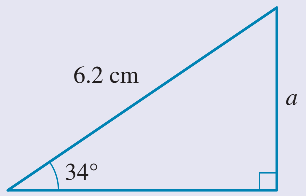
B. $\sin(\theta) = \dfrac{b}{a}$

C. $\cos(\theta) = \dfrac{c}{b}$

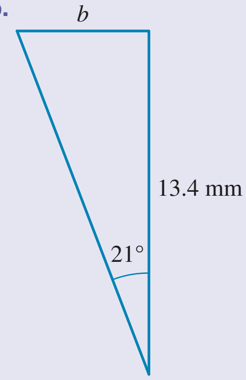
D. $\sin(\theta) = \dfrac{c}{b}$

8. Calculate the length of the unknown side of each of the following triangles, correct to 2 decimal places.

a.



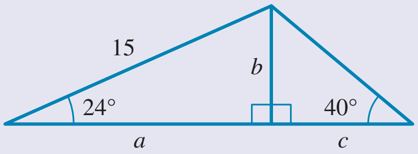
b.



9. Evaluate, correct to 4 decimal places.

a. $\sin(62.5°)$ b. $\cos(12.1°)$ c. $\tan(74.9°)$

10. **MC** Identify the lengths of the unknown sides in the triangle shown.



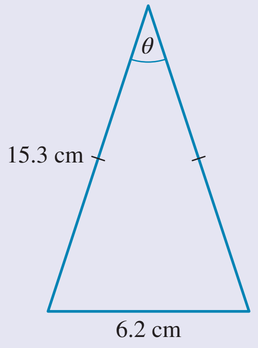
A. $a = 6.1,\ b = 13.7$ and $c = 7.3$

B. $a = 6.1,\ b = 13.7$ and $c = 5.1$

C. $a = 5.1,\ b = 6.1$ and $c = 13.7$

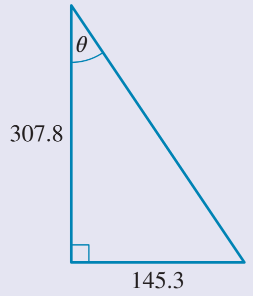
D. $a = 13.7,\ b = 6.1$ and $c = 7.3$

11. Calculate the value of θ in the triangle shown, correct to 1 decimal place.



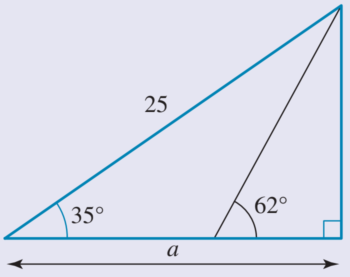
12. MC Identify the value of θ in the triangle shown, correct to 2 decimal places.

A. 25.27°
B. 28.16°
C. 28.17°
D. 61.83°

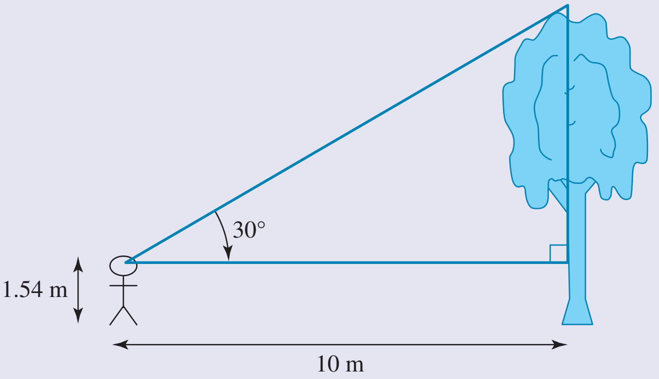


13. MC Determine the correct value of the side length a.

A. 7.62
B. 10.04
C. 14.34
D. 20.48



14. A person who is 1.54 m tall stands 10 m from the foot of a tree and records the angle to the top of the tree as 30° as shown. Evaluate the height of the tree, in metres correct to 2 decimal places.



15. Liam wants to reuse the 1.20 m long harwood desk top from his old desk to build a jumping ramp for his remote-control car. Looking around the house for options to use as a base on which to incline the wood plank, Liam picks up a big, discarded plastic box, of height 40 cm, and heads for the garden to install the ramp on a flat patch of turf.

After a few tests to ensure the plank does not slide from the box or tip under its own weight, and that the RC car can always clear the width of the box when jumping, Liam notes that plank must make contact with the top edge of the box between $\frac{2}{3}$ and $\frac{3}{4}$ of its length.

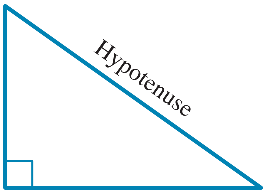
Calculate the corresponding minimum and maximum angles of inclination of the plank and the associated jumping heights. Give your answers to 2 decimal places where appropriate.

LESSON 8.2 Pythagoras' theorem

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify the hypotenuse in a right-angled triangle
- recall Pythagoras' theorem
- apply Pythagoras' theorem to calculate unknown side lengths
- identify Pythagorean triads
- apply the converse of Pythagoras' theorem to determine whether a triangle is right-angled.

8.2.1 Introducing Pythagoras' theorem



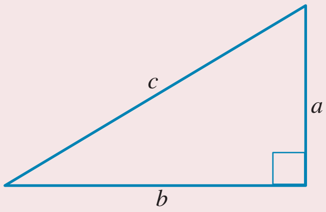
- A **right-angled triangle** is a triangle that contains a right angle (90°).
- The longest side of a right-angled triangle is always found opposite the right angle. This side is called the **hypotenuse**.
- Pythagoras (580–501 BCE) was a Greek mathematician who explored the relationship take between the lengths of the sides of right-angled triangles.
- The relationship he described is known as **Pythagoras' theorem**.
- If the lengths of two sides of a right-angled triangle are given, then Pythagoras' theorem lets us calculate the length of the unknown side.

Pythagoras' theorem

In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. This rule is written as follows.

$$c^2 = a^2 + b^2$$

In this rule, a and b represent the two shorter sides of a right-angled triangle, while c represents the hypotenuse.

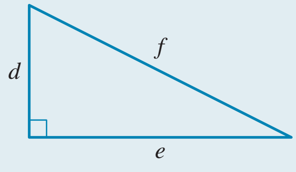


WORKED EXAMPLE 1 Assigning Pythagorean identities

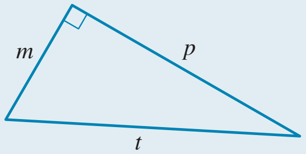
For the right-angled triangles shown below:

i. state which side is the hypotenuse **ii. write Pythagoras' theorem.**

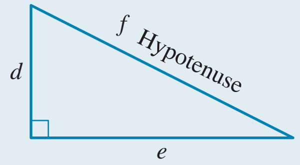
a.



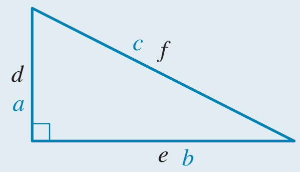
b.



THINK	WRITE
a. i. The hypotenuse is always opposite the right angle.	**a.** Side f is opposite the right angle. Therefore, side f is the hypotenuse.



ii. If the triangle is labelled as usual with a, b and c, as shown in blue, Pythagoras' theorem can be written and then the letters can be replaced with their new values (names).



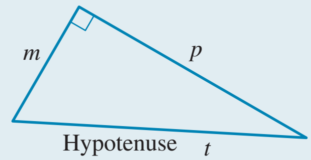
$c = f$; $a = d$; $b = e$

$c^2 = a^2 + b^2$

$f^2 = d^2 + e^2$

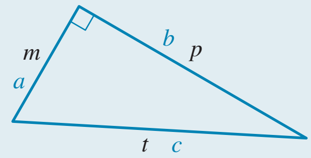
b. i. The hypotenuse is opposite the right angle.

b.



Side t is opposite the right angle. Therefore, side t is the hypotenuse.

ii. If the labels on the triangle are replaced with a, b and c, as shown in blue, Pythagoras' theorem can be written out, and then the letters a, b and c can be changed back to the original labels.



$c = t$; $b = p$; $a = m$

$c^2 = a^2 + b^2$

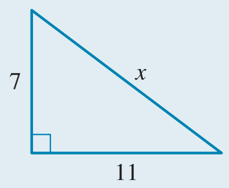
$t^2 = m^2 + p^2$

8.2.2 Calculating unknown side lengths using Pythagoras' theorem

- Pythagoras' theorem can be used to calculate the unknown side length of a right-angled triangle if the other two side lengths are known.
- To calculate an unknown side length, substitute the known values into the equation $c^2 = a^2 + b^2$ and solve for the remaining pronumeral.
- Remember that the hypotenuse always corresponds to the pronumeral c.

WORKED EXAMPLE 2 Calculating the length of the hypotenuse

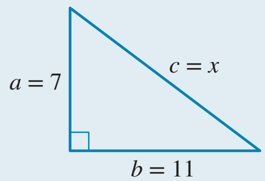
For the triangle shown, calculate the length of the hypotenuse, x, correct to 1 decimal place.



THINK

1. Copy the diagram and apply the labels a, b and c to the triangle, pairing them with the known values and the pronumeral. Remember to label the hypotenuse as c.

WRITE



2. Write out Pythagoras' theorem. $c^2 = a^2 + b^2$

3. Substitute the values of a, b and c into this rule and simplify.

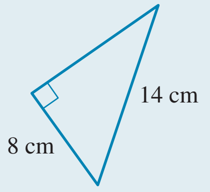
$$\begin{aligned} x^2 &= 7^2 + 11^2 \\ &= 49 + 121 \\ &= 170 \end{aligned}$$

4. Calculate x by taking the square root of 170. Round the answer to 1 decimal place.

$$x = \sqrt{170}$$
$$x = 13.0$$

WORKED EXAMPLE 3 Calculating the length of a shorter side

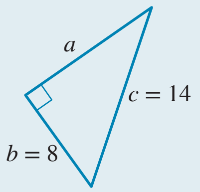
Calculate the length, as an exact value and correct to 1 decimal place, of the unmarked side of the triangle shown.



THINK

1. Copy the diagram and label the sides a, b and c. Remember to label the hypotenuse as c.

WRITE



2. Write out Pythagoras' theorem for a shorter side. $a^2 = c^2 - b^2$

3. Substitute the values of b and c into the equation and simplify.

$$\begin{aligned} a^2 &= 14^2 - 8^2 \\ &= 196 - 64 \\ &= 132 \end{aligned}$$

4. Calculate the value of a by taking the square root of 132. Round the answer to 1 decimal place.

$a = \sqrt{132}$ Exact answer.
$= 11.5$ cm 1 decimal place.

8.2.3 Applying Pythagoras' theorem

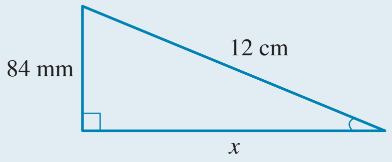
- If a problem involves a right-angled triangle, then Pythagoras' theorem can be applied to determine a solution.

Applying Pythagoras' theorem to a problem

1. **Read the question carefully and draw a right-angled triangle to represent the problem.**
2. **Identify the known values and place them on the diagram in their correct positions.**
3. **Identify the variable that you will need to determine.**
4. **Substitute the known values into Pythagoras' theorem. Make sure that they are all using the same units of length. You may need to convert some units.**
5. **Solve for the unknown value.**
6. **Use the result to write the answer as a complete sentence.**

WORKED EXAMPLE 4 Solving problems using Pythagoras' theorem

A wedge of cake has been cut so that it forms a right-angled triangle. The longest edge of the wedge is 12 cm long and its shortest edge is 84 mm long.
Calculate the length of the third side, in cm correct to 1 decimal place.

THINK	WRITE
1. Draw a right-angled triangle. Identify the longest side (this will always be the hypotenuse) and place its value on the diagram. Identify the shortest side and place its value on the diagram. Label the side with unknown length as x.	
2. Identify the length of the hypotenuse and the two shorter sides. Check that the units for all measurements are the same. Convert 84 mm to centimetres by dividing by 10.	$c = 12\text{ cm}$ $a = 84\text{ mm} = 8.4\text{ cm}$ $b = x$
3. Substitute the values into the equation and simplify.	$c^2 = a^2 + b^2$ $12^2 = (8.4)^2 + x^2$ $144 = 70.56 + x^2$
4. Solve the equation for x by: subtracting 70.56 from both sides and taking the square root of both sides.	$144 - 70.56 = 70.56 - 70.56 + x^2$ $73.44 = x^2$ $\sqrt{73.44} = \sqrt{x^2}$ $\pm\ 8.5697 \approx x$
5. It is not possible for a triangle to have a side with a negative length, therefore the solution must be the positive value. Round the answer to 1 decimal place and include the appropriate units.	$x = 8.6\text{ cm}$
6. Write the answer as a sentence.	The third side of the slice of cake is 8.6 cm long.

8.2.4 Pythagorean triads

- A **Pythagorean triad** is a group of any three whole numbers that satisfy Pythagoras' theorem.
  For example, $\{3, 4, 5\}$ is a Pythagorean triad because $3^2 = 9, 4^2 = 16$ and $5^2 = 25$, and $9 + 16 = 25$.
- Pythagorean triads are useful when solving problems using Pythagoras' theorem.
- If two known side lengths in a triangle belong to a triad, the length of the third side can be stated without performing any calculations.
- Some well known Pythagorean triads are $\{3, 4, 5\}, \{5, 12, 13\}, \{8, 15, 17\}$ and $\{7, 24, 25\}$.

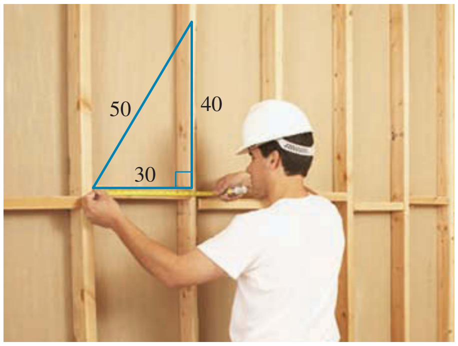
WORKED EXAMPLE 5 Identifying Pythagorean triads

Determine whether the following sets of numbers are Pythagorean triads.

a. **{9, 10, 14}** **b.** **{33, 65, 56}**

THINK	WRITE
a. 1. Pythagorean triads satisfy Pythagoras' theorem. Substitute the values into the equation $c^2 = a^2 + b^2$ and determine whether the equation is true. Remember that c is always the longest side.	**a.** $c^2 = a^2 + b^2$ LHS $= c^2 = 14^2 = 196$ RHS $= a^2 + b^2 = 9^2 + 10^2 = 81 + 100 = 181$
2. Write your conclusion.	Since LHS $\neq$ RHS, the set {9, 10, 14} is not a Pythagorean triad.
b. 1. Pythagorean triads satisfy Pythagoras' theorem. Substitute the values into the equation $c^2 = a^2 + b^2$ and determine whether the equation is true. Remember that c is always the longest side.	**b.** $c^2 = a^2 + b^2$ LHS $= 65^2 = 4225$ RHS $= 33^2 + 56^2 = 1089 + 3136 = 4225$
2. Write your conclusion.	Since LHS = RHS, the set {33, 65, 56} is a Pythagorean triad.

- If each term in a Pythagorean triad is multiplied by the same number, the result is also a triad.
  For example, if we multiply each number in {5, 12, 13} by 2, the result {10, 24, 26} is also a triad.
- Builders and gardeners use multiples of the Pythagorean triad {3, 4, 5} to make sure that walls and floors are at right angles to each other.



8.2.5 The converse of Pythagoras' theorem

- The converse of Pythagoras' theorem can be used to determine whether a given triangle is right-angled.
- If the three side lengths of a triangle are a Pythagorean triad, then that triangle is right-angled.

Applying the converse of Pythagoras' theorem

In any triangle, if the square of the longest side is equal to the sum of the squares of the other two sides, then that triangle is a right-angled triangle.

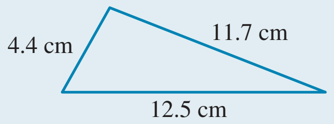
To determine whether a triangle is right-angled:

1. **Identify the longest side, c, among the three side lengths of a triangle. Make sure that they are all using the same units of length. You may need to convert some units.**
2. **Calculate the LHS, the square of the longest side. LHS $= c^2$**
3. **Calculate the RHS, the sum of the squares of the other two sides, a and b. RHS $= a^2 + b^2$**
4. **Determine whether LHS = RHS or LHS $\neq$ RHS.**
   If LHS = RHS, then the triangle is a right-angled triangle.
   If LHS $\neq$ RHS, then the triangle is not a right-angled triangle.
5. **Use the result to write a complete sentence on the nature of the triangle.**

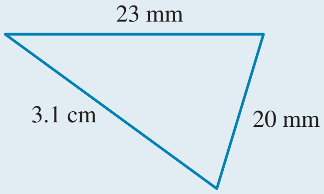
WORKED EXAMPLE 6 Determining whether a triangle is right-angled

For each of the triangles shown, determine whether they are right-angled.

a.



b.



THINK	WRITE
a. 1. Identify the longest side. Note that all the side lengths are in cm, so no conversion is required.	**a.** $c = 12.5$ cm
2. Calculate the LHS, the square of the longest side.	$\text{LHS} = c^2$ $= 12.5^2$ $= 156.25\ \text{cm}^2$
3. Calculate the RHS, the sum of the squares of the other two sides.	$\text{RHS} = a^2 + b^2$ $= 4.4^2 + 7.7^2$ $= 19.36 + 136.89$ $= 156.25\ \text{cm}^2$
4. Determine whether LHS = RHS or LHS ≠ RHS.	LHS = RHS
5. Write your conclusion.	The square of the longest side of the triangle is equal to the sum of the squares of the other two sides, thus the triangle is a right triangle.
b. 1. Identify the longest side. Note that one of the lengths is in cm, while the other two lengths are in mm, we need to use the same units, for example mm.	**b.** $c = 3.1$ cm $= 31$ mm
2. Calculate the LHS, the square of the longest side.	$\text{LHS} = c^2$ $= 31^2$ $= 961\ \text{mm}^2$
3. Calculate the RHS, the sum of the squares of the other two sides.	$\text{RHS} = a^2 + b^2$ $= 20^2 + 23^2$ $= 400 + 529$ $= 929\ \text{mm}^2$
4. Determine whether LHS = RHS or LHS ≠ RHS.	LHS ≠ RHS
5. Write your conclusion.	The square of the longest side of the triangle is not equal to the sum of the squares of the other two sides, thus the triangle is not a right triangle.

Exercise 8.2 Pythagoras' theorem

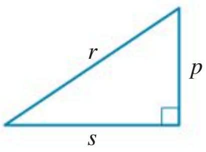
8.2 Quick quiz on | **8.2 Exercise**

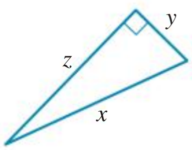
Individual pathways

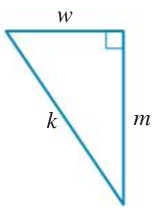
PRACTISE	CONSOLIDATE	MASTER
1, 5, 6, 7, 9, 13, 14, 18	2, 8, 10, 15, 17, 19	3, 4, 11, 12, 16, 20

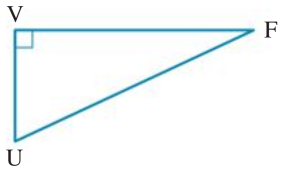
Fluency

1. **WE1** For the right-angled triangles shown below:
   i. state which side is the hypotenuse
   ii. write Pythagoras' theorem.

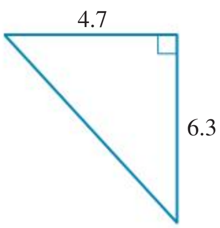
a.


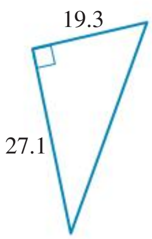
b.


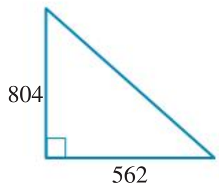
c.


d.


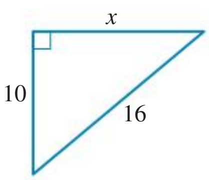
2. **WE2** For each of the following triangles, calculate the length of the hypotenuse, giving answers correct to 2 decimal places.

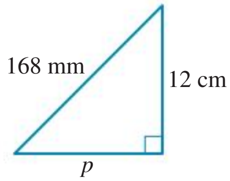
a.


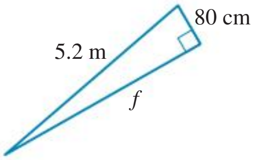
b.


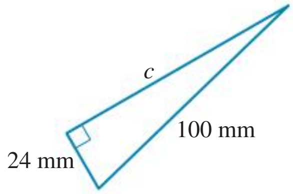
c.


3. **WE3** Calculate the value of the pronumeral in each of the following triangles. Give your answers correct to 2 decimal places.

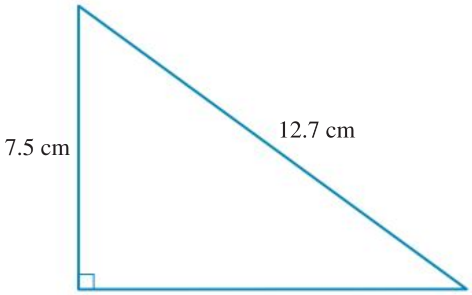
a.


b.


c.


d.


4. **WE4** Calculate the value of the missing side length. Give your answer correct to 1 decimal place.

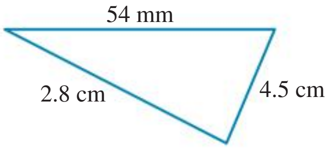


5. **WE5** Determine whether the following sets of numbers are Pythagorean triads.

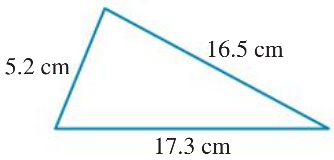
a. {2, 5, 6}
b. {7, 10, 12}
c. {18, 24, 30}
d. {72, 78, 30}
e. {13, 8, 15}
f. {50, 40, 30}

6. **WE6** For each of the triangles shown, determine whether they are right-angled.

a.



b.



Understanding

7. In a right-angled triangle, the two shortest sides are 4.2 cm and 3.8 cm.

a. Draw a sketch of the triangle.
b. Calculate the length of the hypotenuse correct to 2 decimal places.

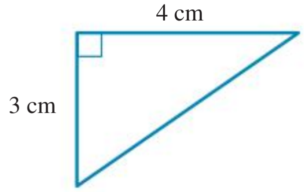
8. A right-angled triangle has a hypotenuse of 124 mm and another side of 8.5 cm.

a. Draw a diagram of the triangle.
b. Calculate the length of the third side of this triangle. Give your answer in millimetres, correct to 2 decimal places.

9. **MC** Identify which of the following sets is formed from the Pythagorean triad {21, 20, 29}.

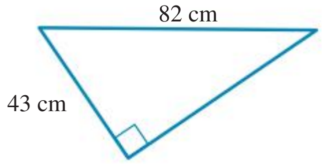
A. {95, 100, 125}
B. {105, 145, 100}
C. {84, 80, 87}
D. {105, 80, 87}

10. **MC** Identify the length of the hypotenuse in the triangle shown.



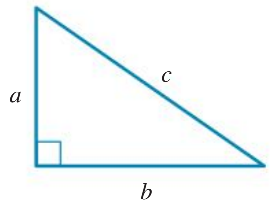
A. 25 cm
B. 50 cm
C. 50 mm
D. 500 mm

11. **MC** Identify the length of the third side in the triangle shown.



A. 48.75 cm
B. 0.698 m
C. 0.926 m
D. 92.6 cm

12. The lengths of two sides of a right-angled triangle are given below:



a. $a = 37$ cm, $c = 180$ cm; b is in cm.
b. $a = 856$ mm, $b = 1200$ mm; c is in cm.
c. $b = 4950$ m, $c = 5.6$ km; a is in km.
d. $a = 125\,600$ mm, $c = 450$ m; b is in m.

Calculate the value of the third side. Give your answer in the units specified (correct to 2 decimal places).
The diagram shown illustrates how each triangle is to be labelled.
Remember that c is always the hypotenuse.

13. Determine the exact perimeter of a right-angled triangle with base length of 7 cm and hypotenuse of 15 cm.

Reasoning

14. A ladder that is 7 m long leans up against a vertical wall. The top of the ladder reaches 6.5 m up the wall. Determine how far the foot of the ladder is from the wall.
Give your answer correct to 2 decimal places.

15. A rectangular park measures 260 m by 480 m. Danny usually trains by running five circuits around the edge of the park. After heavy rain, two of the adjacent sides are too muddy to run along, so Danny runs along a triangular path that traces the other two sides and the diagonal. Danny does five circuits of this path for training.
Show that Danny has run about 970 metres less than his usual training session.

16. A water park has hired Sharnee to build part of a ramp for a new water slide. She builds a 12-metre-long ramp that rises to a height of 250 cm. To meet safety regulations, the ramp can only have a gradient between 0.1 and 0.25.
Show that the ramp Sharnee has built is within the regulations.

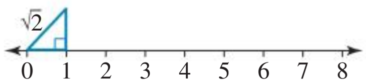
17. Determine the area of a rhombus with side length of 5 cm and one diagonal of 9 cm. Give your answer to 2 decimal places.

Problem solving

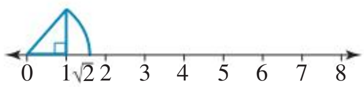
18. Spiridoula is trying to hang her newest painting on a hook in her hallway. She leans a ladder against the wall so that the distance between the foot of the ladder and the wall is 80 cm. The ladder is 1.2 m long.
  a. Draw a diagram showing the ladder leaning on the wall.
  b. Calculate how far up the wall the ladder reaches. Give your answer correct to 2 decimal places.
  c. Spiridoula climbs the ladder to check whether she can reach the hook from the step at the very top of the ladder. When she extends her arm, the distance from her feet to her fingertips is 1.7 m.
  If the hook is 2.5 m above the floor, determine whether she can reach it from the top step of the ladder.

19. a. The smallest numbers of four Pythagorean triads are given below. Determine the middle number of each triad, and use this answer to determine the third number of each triad.
  i. 9 ii. 11 iii. 13 iv. 29
  b. Comment about the triads formed in part a.

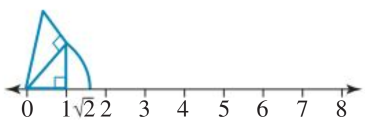
20. We know that it is possible to determine the exact square root of some numbers, but not others. For example, we can determine $\sqrt{4}$ exactly, but not $\sqrt{3}$ or $\sqrt{5}$. Our calculators can calculate decimal approximations of these square roots, but because they cannot be found exactly they are called 'irrational numbers'.
However, there is a method of showing the exact location of irrational numbers on a number line.
Using graph paper, draw a right-angled triangle with 2 equal sides of length 1 cm as shown below.



Using Pythagoras' theorem, we know that the longest side of this triangle is $\sqrt{2}$ units. Place the compass point at zero and make an arc that will show the location of $\sqrt{2}$ on the number line.



Draw another right-angled triangle using the longest side of the first triangle as one side, and make the other side 1 cm in length.



The longest side of this triangle will have a length of $\sqrt{3}$ units. Draw an arc to determine the location of $\sqrt{3}$ on the number line.
Repeat these steps to draw triangles that will have sides of length $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$ units, and so on.

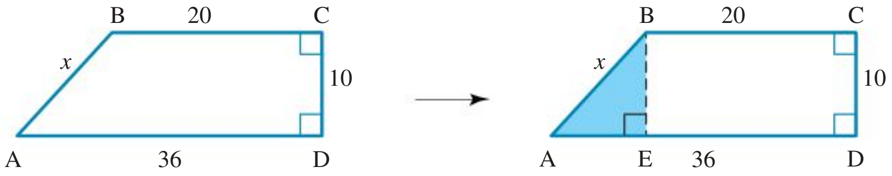
LESSON 8.3 Applications of Pythagoras' theorem

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- divide composite shapes into simple shapes
- calculate the perimeter and area of simple composite shapes
- calculate unknown side lengths in composite shapes and irregular triangles
- use Pythagoras' theorem to solve problems in 3D contexts.

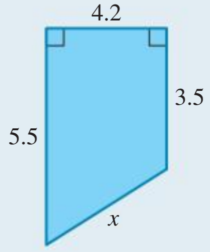
8.3.1 Using composite shapes to solve problems

- Complex or unusual shapes can be thought of as composites of simpler shapes with known properties.
- Dividing a shape into a series of these simpler shapes lets us use the known properties of those shapes to determine unknown values.
- For example, to calculate the value of x in the trapezium shown, a vertical line can be added to the trapezium to create two shapes: a right-angled triangle and a rectangle.
- Using Pythagoras' theorem, we can now calculate the length of x, which has become the hypotenuse of triangle ABE.



WORKED EXAMPLE 7 Calculating unknown side lengths in composite shapes

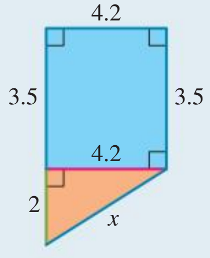
Calculate the value of x in the diagram shown, correct to 1 decimal place.



THINK

1. Add a horizontal line to divide the shape into 2 smaller shapes, one that is a right-angled triangle, as shown in orange. x is the hypotenuse of a right-angled triangle.
To use Pythagoras' theorem, the length of the side shown in green must be known. This length can be calculated as the difference between the long and short vertical edges of the trapezium: $5.5 - 3.5 = 2$.

WRITE



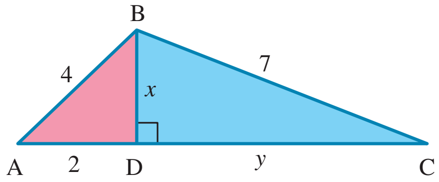
2. Write out Pythagoras' theorem and substitute the values of a, b and c.
$a = 2, b = 4.2, c = x$

$$c^2 = a^2 + b^2$$
$$x^2 = 2^2 + (4.2)^2$$
$$x^2 = 4 + 17.64$$
$$x^2 = 21.64$$
$$\sqrt{x^2} = \sqrt{21.64}$$
$$x \approx \pm 4.651\,88$$

3. It is not possible to have a side with a negative length. Therefore the solution is the positive value. Round the answer to 1 decimal place

$$x = 4.7$$

- To determine the value of y in the irregular triangle shown, the triangle can be split into two right-angled triangles: ΔABD (pink) and ΔBDC (blue). There is now enough information to calculate the missing side length from ΔABD. This newly calculated length can then be used to determine the value of y.



WORKED EXAMPLE 8 Applying Pythagoras' theorem to non-right-angled triangles

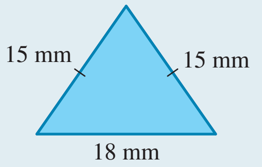
a. Calculate the perpendicular height of an isosceles triangle with equal sides that are both 15 mm long and a third side that is 18 mm long.
b. Calculate the area of this triangle.

THINK

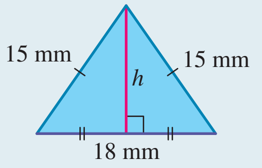
WRITE

a. 1. Draw the triangle and label all side lengths as described.

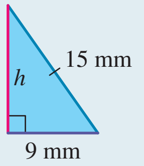
a.



2. Draw an additional line to represent the height of the triangle (shown in pink) and label it appropriately. Because the triangle is an isosceles triangle, h bisects the base of 18 mm (shown in purple) to create two right-angled triangles.



3. Consider one of these right-angled triangles, which both have a height of h. The base length of each of these triangles is $\frac{18}{2} = 9$ mm.



4. Substitute the values from this right-angled triangle into Pythagoras' theorem to calculate the value of h.

$$c^2 = a^2 + b^2$$
$$15^2 = h^2 + 9^2$$
$$225 = h^2 + 81$$
$$225 - 81 = h^2 + 81 - 81$$
$$144 = h^2$$
$$\sqrt{144} = \sqrt{h^2}$$
$$\pm 12 = h$$

5. It is not possible to have a side with a negative length. Therefore the solution is the positive value. Include the appropriate units in your answer.

$h = 12$ mm
The perpendicular height of the triangle is 12 mm.

b. 1. Write the formula for the area of a triangle. Using the answers from part **a**, the base is 18 mm and the height is 12 mm.

b. $$A_\Delta = \frac{1}{2}bh$$
$$= \frac{1}{2} \times 18 \times 12$$
$$= 108$$

2. Answer the question with appropriate units.

The area of the triangle is 108 mm^2.

WORKED EXAMPLE 9 Determining the perimeter and area of simple composite shapes

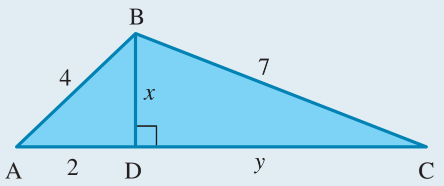
Consider the composite shape shown.

a. Calculate the exact value of x.

b. Calculate the exact value of y.

c. Determine the exact perimeter P of this shape.

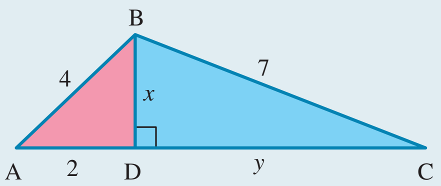
d. Determine the area A of this shape. Give your answer to 2 decimal places.



THINK | **WRITE**

a. 1. Consider the right-angled triangle ADB, with hypotenuse 4, and side lengths 2 and x.

a.



2. Apply Pythagoras' theorem to ΔADB and calculate x.

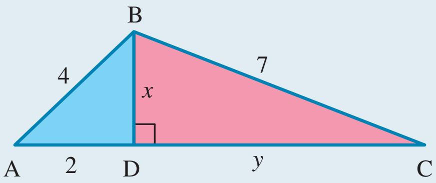
$$2^2 + x^2 = 4^2$$
$$4 + x^2 = 16$$
$$x^2 = 12$$
$$x = 2\sqrt{3}$$

3. Answer the question.

$x = 2\sqrt{3}$ units

b. 1. Consider the right-angled triangle CDB, with hypotenuse 7, and side lengths y and x.

b.



2. Apply Pythagoras' theorem to ΔCDB and calculate y.

$$y^2 + x^2 = 7^2$$
$$y^2 + 12 = 49$$
$$y^2 = 37$$
$$y = \sqrt{37}$$

3. Answer the question.

$y = \sqrt{37}$ units

c. 1. Calculate the perimeter P of the composite shape.

c. $P = 2 + y + 7 + 4$
$= 13 + \sqrt{37}$ units

2. Answer the question.

The perimeter of the composite shape is $13 + \sqrt{37}$ units.

d. 1. Calculate the area A_1 of ΔADB.

d. $$A_1 = \frac{2 \times x}{2}$$
$$= 2\sqrt{3} \text{ units}^2$$

2. Calculate the area A_2 of ΔCDB.

$$A_2 = \frac{y \times x}{2}$$
$$= \frac{\sqrt{37} \times 2\sqrt{3}}{2}$$
$$= \sqrt{111} \text{ units}^2$$

3. Calculate the exact area A of the composite shape and round the answer to 2 decimal places.

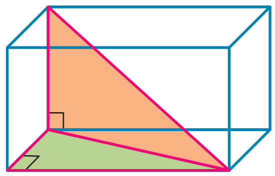
$$A = A_1 + A_2$$
$$= 2\sqrt{3} + \sqrt{111}$$
$$= 13.9997\ldots$$
$$\approx 14.00 \text{ units}^2$$

4. Answer the question.

The area of the composite shape is 14.00 units^2 to 2 decimal places.

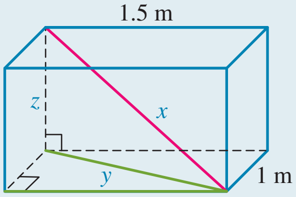
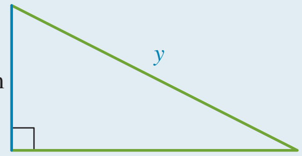
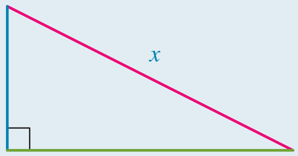
8.3.2 Pythagoras' theorem in 3D (optional)

- Pythagoras' theorem can be used to solve problems in three dimensions.
- 3D problems often involve multiple right-angled triangles.
- It is often helpful to redraw sections of a 3D diagram in two dimensions. This makes it easier to see
any right angles.



WORKED EXAMPLE 10 Using Pythagoras' theorem in 3D problems

Calculate the maximum length of a metal rod that would fit into a rectangular crate with dimensions 1 m × 1.5 m × 0.5 m.

THINK	WRITE
1. Draw a diagram of a rectangular box with a rod in it. Label the dimensions of this diagram. 2. Draw a right-angled triangle that has the metal rod as one of its sides (x, shown in pink). The lengths of x and y in this right-angled triangle are not known. Draw a second right-angled triangle (shown in green) to allow you to calculate the length of y.	
3. Calculate the length of y using Pythagoras' theorem.	 $c^2 = a^2 + b^2$ $y^2 = 1.5^2 + 1^2$ $y^2 = 3.25$ $y = \sqrt{3.25}$
4. Draw the right-angled triangle that has the rod as its hypotenuse. Use Pythagoras' theorem to calculate the length of x. z = height of the crate = 0.5 m	 $c^2 = a^2 + b^2$ $x^2 = \left(\sqrt{3.25}\right)^2 + 0.5^2$ $x^2 = 3.25 + 0.25$ $x^2 = 3.5$ $x = \sqrt{3.5}$ ≈ 1.87
5. Answer the question with appropriate units.	The maximum length of the metal rod is 1.87 m (correct to 2 decimal places).

Exercise 8.3 Applications of Pythagoras' theorem

learnon

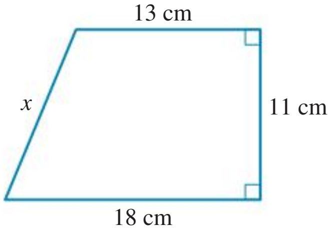
8.3 Quick quiz on | 8.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 4, 5, 10, 16, 20	3, 6, 8, 9, 11, 15, 17, 18, 21	7, 12, 13, 14, 19, 22

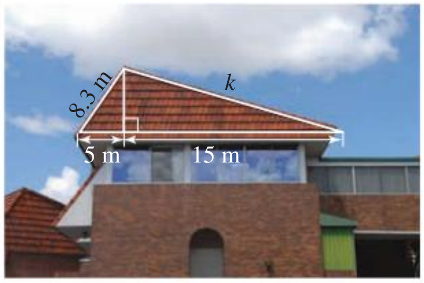
Fluency

1. WE7 Calculate the length of the side x in the figure shown, correct to 2 decimal places.

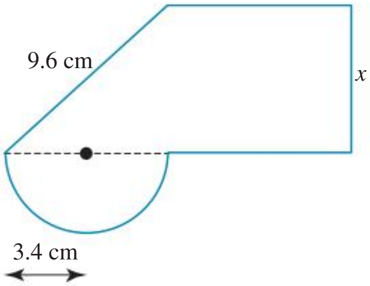


2. Calculate the values of the pronumerals in each of the following photographs and diagrams. Give your answers correct to 2 decimal places.

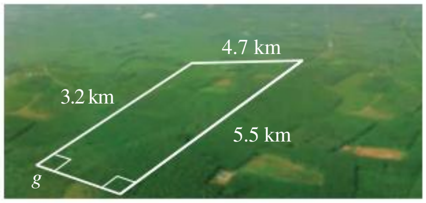
a.



b.

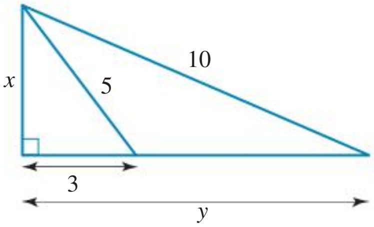


c.

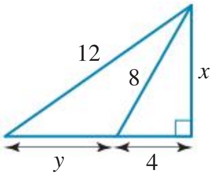


3. For each of the following diagrams, calculate the lengths of the sides marked x and y. Give your answer correct to 2 decimal places (where necessary).

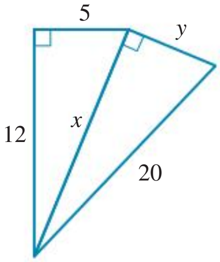
a.



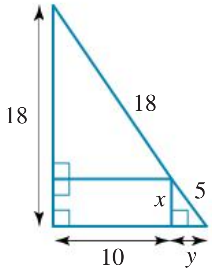
b.



c.



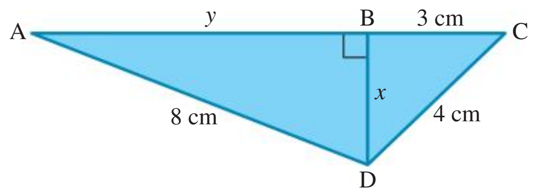
d.



4. WE8 The height of an isosceles triangle is 3.4 mm and its equal sides are twice as long.

   a. Sketch the triangle, showing all the information given.
   b. Calculate the length of the third side, correct to 2 decimal places.
   c. Calculate the area of the triangle, correct to 2 decimal places.

5. WE9 Consider the composite shape shown.

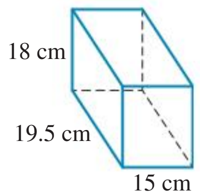


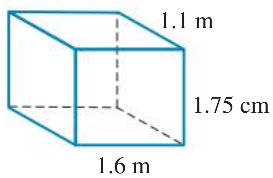
   a. Calculate the exact value of x.
   b. Calculate the exact value of y.
   c. Determine the exact perimeter P of this shape.
   d. Determine the area A of this shape. Give your answer to 2 decimal places.

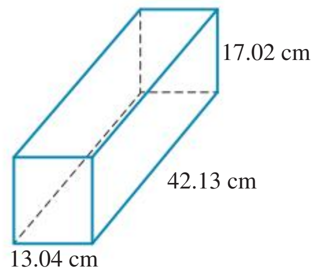
Understanding

6. Determine the area of a regular hexagon with side length 10 cm. Give your answer to 2 decimal places.

7. WE10 Calculate the length of the longest metal rod that could fit diagonally into each of the boxes shown below. Give your answers correct to 2 decimal places.

a.


b.


c.


8. The side length of an equilateral triangle is 1 m. Calculate:

   a. the height of the triangle in metres, correct to 2 decimal places
   b. the area of the triangle in m^2, correct to 3 decimal places.

9. MC Identify the longest length of an object that fits into a box with dimensions of 42 cm × 60 cm × 13 cm.

   A. 74.5 cm
   B. 60 cm
   C. 73.2 cm
   D. 74 cm

10. Priya wants to pack an umbrella into her suitcase, which measures 89 cm × 21 cm × 44 cm.

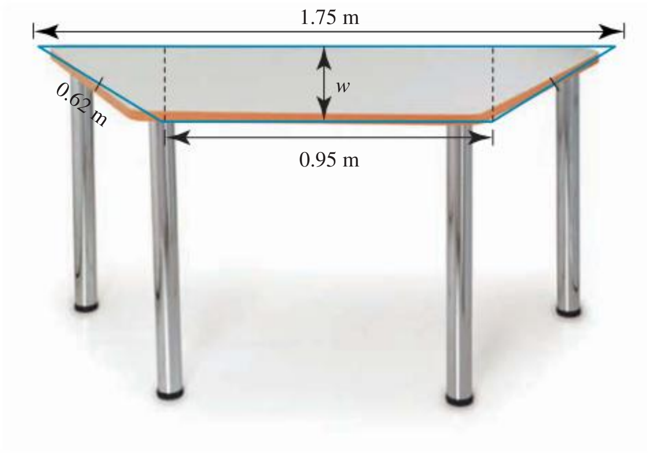
    a. State whether her 1 m umbrella will fit in.
    b. Give the length of the longest umbrella that Priya could fit in their suitcase, correct to 2 decimal places.

11. Giang is packing her lunch into her lunchbox, which measures 15 cm × 5 cm × 8 cm.

    a. State whether her 22 cm straw will fit in.
    b. Calculate the length of the longest object that will fit into Giang's lunchbox, correct to 2 decimal places.

12. A cylindrical pipe is 2.4 m long. It has an internal diameter of 30 cm. Calculate the size of largest object that could fit inside the pipe. Give your answer correct to the nearest cm.

13. A classroom contains 20 identical desks shaped like the one shown.



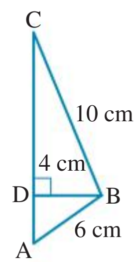
a. Calculate the width of each desktop (labelled *w*). Give your answer correct to 2 decimal places.
b. Calculate the area of each desktop, correct to 2 decimal places.
c. If the top surface of each desk needs to be painted, determine the total areaof the desktops that need to be painted. Give your answer correct to 2 decimal places.
d. Each desktop needs to be given 2 coats of paint. The paint being used is sold in 1-litre tins that cost $29.95 each.
Each of these tins holds enough paint to cover an area of 12 m^2. Using your answer to part **c**, calculate how much it will cost to paint all 20 desks.

14. A utility knife has a blade shaped like a right-angled trapezium. Its sloping edge is 2 cm long and its parallel sides are 32 mm and 48 mm long.
Calculate the width of the blade, then use your answer to calculate the blade's area.

15. A cylindrical pencil holder has an internal height of 16 cm. If its diameter is 8.5 cm and its width is 2 mm, calculate the length of the longest pen that could fit inside the holder. Give your answer correct to the nearest mm.

Reasoning

16. Consider the figure shown.

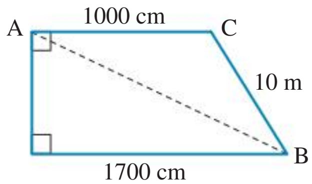


a. State whether you can calculate the length of AC without first calculating the lengths of AD and DC.
b. Calculate the lengths of the sides in part **a**. Give your answer correct to 2 decimal places.
c. Explain whether the triangle ABC is right-angled.

17. Explain in your own words how a 2D right-angled triangle can be seen in a 3D figure.

18. Katie goes on a hike. She walks 2.5 km north, then 3.1 km east. She then walks 1 km north and 2 km west. Finally, she walks in a straight line back to her starting point. Show that Katie has walked a total distance of 12.27 km.

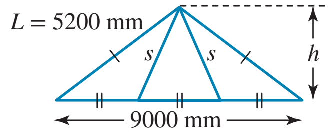
19. Consider the following diagram:



a. Show that the distance AB is 18.44 metres.
b. Show that the angle ∠ACB is not a right angle.

Problem solving

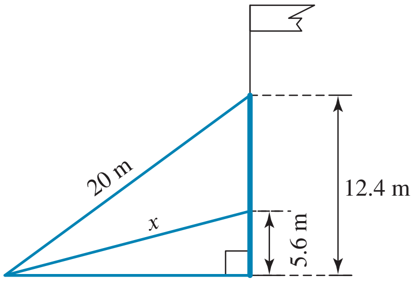
20. The diagram shown illustrates a cross-section of the roof of Gabriel's house.



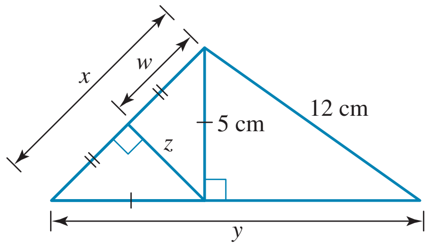
a. Calculate the height of the roof, h, to the nearest millimetre.

b. The roof's longer supports, labelled L, are 5200 mm long. Show that the shorter supports, labelled s, are 2193 mm shorter than the longer supports.

21. A flagpole is attached to the ground by two wires, as illustrated in the diagram shown. Use the information in the diagram to determine the length of the lower wire, x.
Give your answer correct to 1 decimal place.



22. Evaluate the values of the pronumerals w, x, y and z in the diagram shown.



LESSON 8.4 Trigonometric ratios

LEARNING INTENTIONS

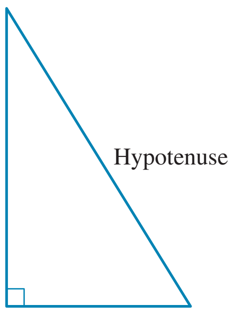
At the end of this lesson you should be able to:

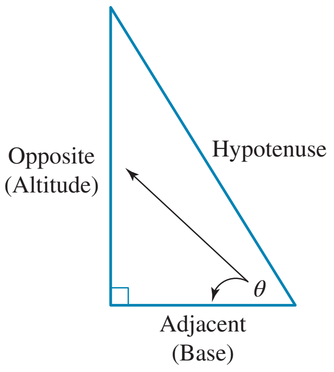
- identify the hypotenuse, opposite and adjacent sides of a right-angled triangle with respect to a given angle
- use the ratios of a triangle's sides to calculate the sine, cosine and tangent of an angle.

8.4.1 What is trigonometry?

- The word **trigonometry** is derived from the Greek words *trigonon* ('triangle') and *metron* ('measurement'). The literal translation of the word is 'to measure a triangle'.
- Trigonometry deals with the relationship between the sides and the angles of a triangle.
- As we have already seen, the longest side of a right-angled triangle is called the hypotenuse. It is always located opposite the right angle.
- In order to name the remaining two sides of a triangle, another angle, called the 'given angle', θ, must be labelled on the triangle. θ is pronounced 'theta'; it is the eighth letter of the Greek alphabet.
- The side that is across from the given angle is called the **opposite side** (or **altitude**). You can see that the opposite side does not touch the given angle at all.
- The remaining side is called the **adjacent side** (or **base**).
- Note that the given angle always sits between the hypotenuse and the adjacent side.

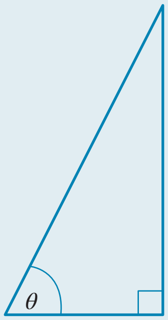
- When the hypotenuse equals one, $\sin(\theta) = $ Opposite (or altitude) and $\cos(\theta) = $ Adjacent (or base).





WORKED EXAMPLE 11 Labelling the sides of a right-angled triangle

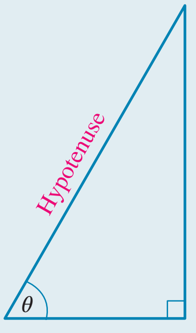
Label the sides of the right-angled triangle shown using the words 'hypotenuse', 'adjacent' and 'opposite'.



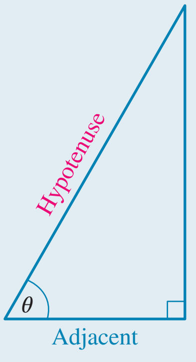
THINK

1. Identify the hypotenuse. Remember that the hypotenuse always lies opposite the right angle.

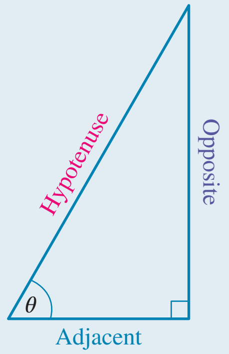
WRITE



2. Look at the position of the given angle. The given angle always sits between the hypotenuse and the adjacent side; therefore, the adjacent side is the bottom side of this triangle.



3. The opposite side does not touch the given angle; therefore, it is the vertical side on the right of this triangle.



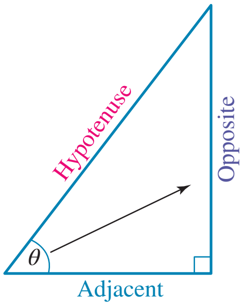
8.4.2 Trigonometric ratios

- Trigonometry is based upon the ratios between pairs of side lengths. Each of these ratios is given a special name.
- In any right-angled triangle:

$$\textbf{sine}(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\textbf{cosine}(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\textbf{tangent}(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$



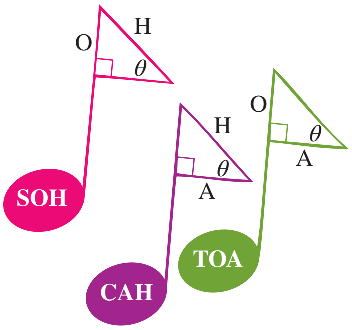
- These rules are often abbreviated as shown in the following.

Trigonometric ratios

$$\sin(\theta) = \frac{O}{H} \qquad \cos(\theta) = \frac{A}{H} \qquad \tan(\theta) = \frac{O}{A}$$

- The following mnemonic can be used to help remember the trigonometric ratios.

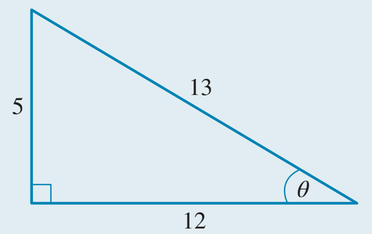
SOH – CAH – TOA



- In this mnemonic:
  - **SOH** refers to $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$
  - **CAH** refers to $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$
  - **TOA** refers to $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$.
- In right-angled triangles, for a given angle, the sine, cosine and tangent ratios are constant, independently of the size of the triangle.
- Similar right-angled triangles have the same trigonometric ratios for corresponding angles.

WORKED EXAMPLE 12 Identifying trigonometric ratios

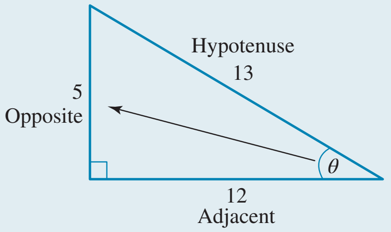
For the triangle shown, write the equations for the sine, cosine and tangent ratios of the given angle (θ).



THINK

1. Label the sides of the triangle.

WRITE



2. Write the trigonometric ratios.

$\sin(\theta) = \frac{\text{O}}{\text{H}}, \cos(\theta) = \frac{\text{A}}{\text{H}}, \tan(\theta) = \frac{\text{O}}{\text{A}}.$

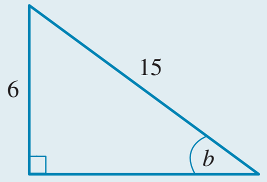
3. Substitute the values of A, O and H into each formula.

$\sin(\theta) = \frac{5}{13}, \cos(\theta) = \frac{12}{13}, \tan(\theta) = \frac{5}{12}$

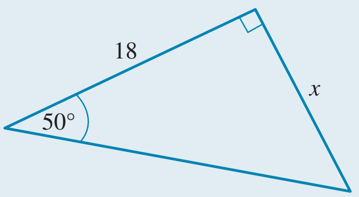
WORKED EXAMPLE 13 Relating sides to trigonometric ratios

For each of the following triangles, write the trigonometric ratio that relates the two given sides and the given angle.

a.



b.

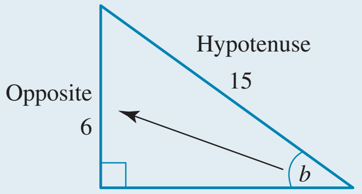


THINK

a. 1. Label the given sides.

WRITE

a.



2. We are given O and H. These are used in SOH. Write the ratio.

$\sin(\theta) = \frac{\text{O}}{\text{H}}$

3. Substitute the values of the pronumerals into the ratio.

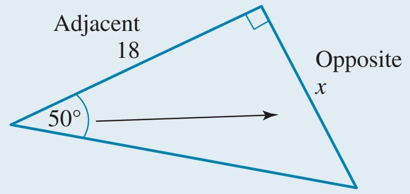
$\sin(b) = \frac{6}{15}$

4. Simplify the fraction.

$\sin(b) = \frac{2}{5}$

b. 1. Label the given sides.

b.



2. We are given A and O. These are used in TOA. Write the ratio.

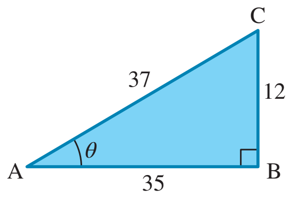
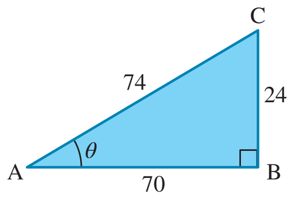
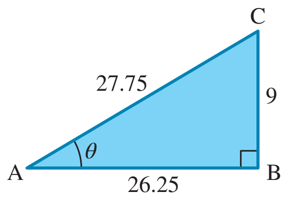
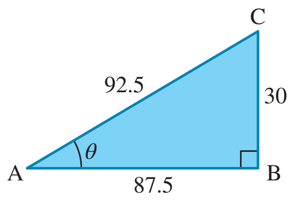
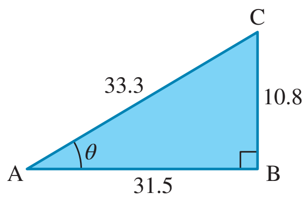
$$\tan(\theta) = \frac{O}{A}$$

3. Substitute the values of the angle and the pronumerals into the ratio.

$$\tan(50°) = \frac{x}{18}$$

COLLABORATIVE TASK

Working in pairs or small teams, calculate the trigonometric ratios of angle θ of the following right-angled triangles ABC and fill the table.

ΔABC	cos (θ)	sin (θ)	tan (θ)
			
			
			
			
			

What do you observe?

Exercise 8.4 Trigonometric ratios

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8.4 Quick quiz on | 8.4 Exercise

Individual pathways

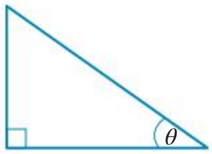
PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 8, 11, 13, 16, 19	2, 6, 9, 14, 17, 20	4, 7, 10, 12, 15, 18, 21

Fluency

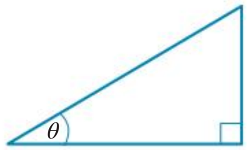
WE11 For questions **1** and **2**, label the sides of the following right-angled triangles using the words ‘hypotenuse’, ‘adjacent’ and ‘opposite’.

1. a.

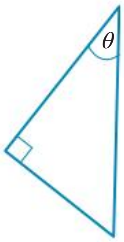
b.



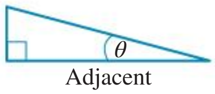
c.



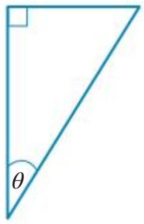
2. a.



b.

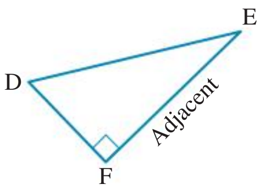


c.

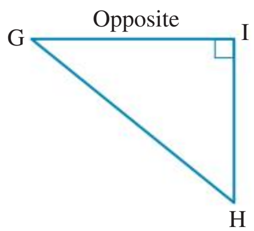


3. For each of the following right-angled triangles, label the given angle and, where appropriate, the hypotenuse, the adjacent side and the opposite side.

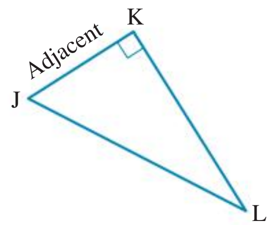
a.



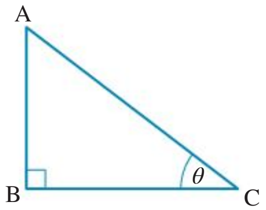
b.



c.



4. **MC** Identify which option correctly names the sides and angle of the triangle shown.



A. $\angle B = \theta$, AB = adjacent side, AC = hypotenuse, BC = opposite side

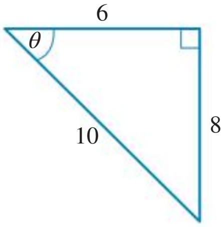
B. $\angle C = \theta$, AC = opposite side, BC = hypotenuse, AC = adjacent side

C. $\angle A = \theta$, AB = opposite side, BC = hypotenuse, AC = adjacent side

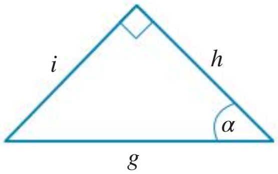
D. $\angle C = \theta$, AB = opposite side, AC = hypotenuse, BC = adjacent side

WE12 For questions **5–10**, based on the given angle in the triangles shown, write the expressions for the ratios of sine, cosine and tangent.

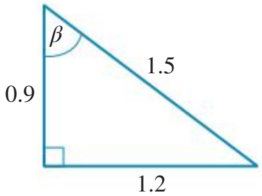
5.



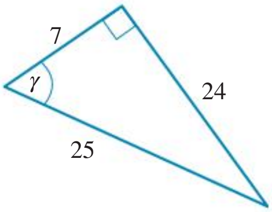
6.



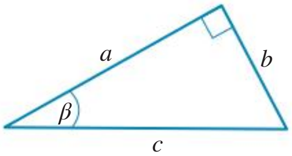
7.



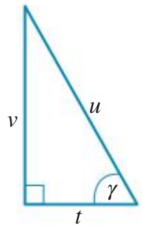
8.



9.

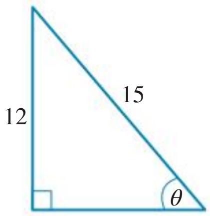


10.

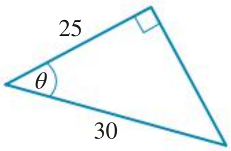


11. **WE13** Write the trigonometric ratio that relates the two given sides and the given angle in each of the following triangles.

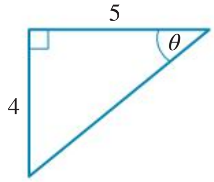
a.



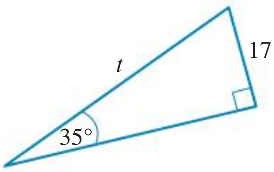
b.



c.

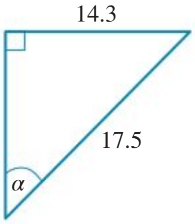


d.

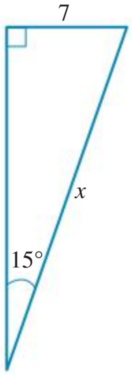


12. Write the trigonometric ratio that relates the two given sides and the given angle in each of the following triangles.

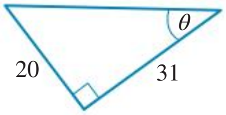
a.



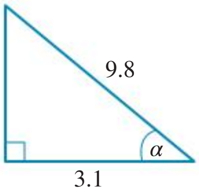
b.



c.

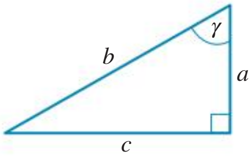


d.



Understanding

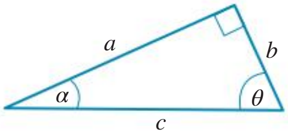
13. **MC** Identify the correct trigonometric ratio for the triangle shown.



A. $\tan(\gamma) = \dfrac{a}{c}$ **B.** $\sin(\gamma) = \dfrac{c}{a}$

C. $\cos(\gamma) = \dfrac{c}{b}$ **D.** $\sin(\gamma) = \dfrac{c}{b}$

14. **MC** Identify the incorrect trigonometric ratio for the triangle shown.



A. $\sin(\alpha) = \dfrac{b}{c}$ **B.** $\sin(\alpha) = \dfrac{a}{c}$

C. $\cos(\alpha) = \dfrac{a}{c}$ **D.** $\tan(\alpha) = \dfrac{b}{a}$

15. **MC** A right-angled triangle contains angles α and β as well as its right angle. Identify which of the following statements is always correct.

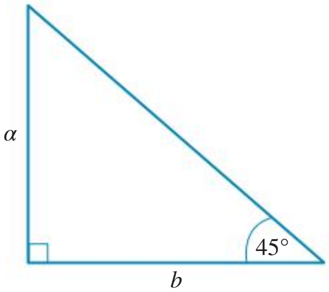
A. $\sin(\alpha) = \sin(\beta)$ **B.** $\tan(\alpha) = \tan(\beta)$

C. $\cos(\alpha) = 1 - \sin(\beta)$ **D.** $\tan(\alpha) = \dfrac{1}{\tan(\beta)}$

Reasoning

16. If a right-angled triangle has side lengths m, $(m + n)$ and $(m - n)$, explain which of the lengths is the hypotenuse.

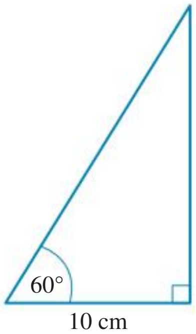
17. Given the triangle shown:



a. explain why $a = b$

b. determine the value of $\tan(45°)$.

18. Using a protractor and ruler, carefully measure and draw a right-angled triangle with a base 10 cm long and an angle of 60°, as shown in the diagram. Measure the length of the other two sides of this triangle and mark these lengths on the diagram as well. Give the lengths of these sides to the nearest millimetre.
Use your measurements to calculate the following three ratios, correct to 2 decimal places.



$$\frac{\text{opposite}}{\text{adjacent}} = \quad , \quad \frac{\text{opposite}}{\text{hypotenuse}} = \quad , \quad \frac{\text{adjacent}}{\text{hypotenuse}} =$$

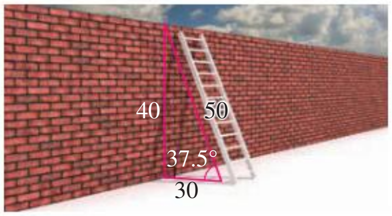
Draw another triangle, similar to the first. Make the base length any length you want, but make sure all angles are equal to the angles in the first triangle. Once this is done, measure the length of the remaining two sides.
Calculate the following three ratios correct to 2 decimal places.

$$\frac{\text{opposite}}{\text{adjacent}} = \quad , \quad \frac{\text{opposite}}{\text{hypotenuse}} = \quad , \quad \frac{\text{adjacent}}{\text{hypotenuse}} =$$

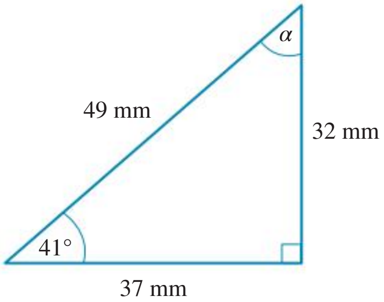
Comment on the conclusions you can draw from these two sets of ratios.

Problem solving

19. A ladder leans on a wall as shown. In relation to the angle given, identify the part of the image that represents:
    a. the adjacent side
    b. the hypotenuse
    c. the opposite side.



20. Consider the right-angled triangle shown.
    a. Label each of the sides using the letters O, A and H with respect to the 41° angle.
    b. Determine the value of each of the following trigonometric ratios, correct to 2 decimal places.
        i. $\sin(41°)$ ii. $\cos(41°)$ iii. $\tan(41°)$
    c. Calculate the value of the unknown angle, α.
    d. Determine the value of each of the following trigonometric ratios, correct to 2 decimal places. (*Hint:* Start by re-labelling the sides of the triangle with respect to angle α.)
        i. $\sin(\alpha)$ ii. $\cos(\alpha)$ iii. $\tan(\alpha)$
    e. Comment about the relationship between $\sin(41°)$ and $\cos(\alpha)$.
    f. Comment about the relationship between $\sin(\alpha)$ and $\cos(41°)$.
    g. Make a general statement about the two angles.



21. In relation to right-angled triangles, investigate the following.
    a. As the acute angle increases in size, determine what happens to the ratio of the length of the opposite side to the length of the hypotenuse in any right-angled triangle.
    b. As the acute angle increases in size, determine what happens to the ratio of the length of the adjacent side to the length of the hypotenuse. Determine what happens to the ratio of the length of the opposite side to the length of the adjacent side.
    c. Evaluate the largest possible value for the following.
        i. $\sin(\theta)$ ii. $\cos(\theta)$ iii. $\tan(\theta)$

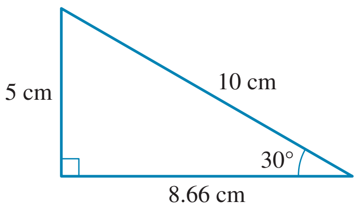
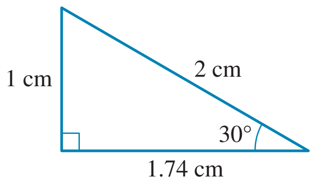
LESSON 8.5 Calculating unknown side lengths

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- use a calculator to calculate the trigonometric ratios for a given angle
- determine the lengths of unknown sides in a right-angled triangle given an angle and one known side length.

8.5.1 Values of trigonometric ratios

- The values of the trigonometric ratios are always the same for a given angle.
  For example, if a right-angled triangle has a given angle of 30°, then the ratio of the opposite side to the hypotenuse will always be equal to $\sin(30°) = 0.5$.
- The ratio of the adjacent side to the hypotenuse will always be equal to $\cos(30°) \approx 0.87$. This relationship is demonstrated in the table.

	$\sin(30°) = \frac{O}{H}$ $= \frac{5}{10}$ $= 0.5$	$\cos(30°) = \frac{A}{H}$ $= \frac{8.66}{10}$ $= 0.87$	$\tan(30°) = \frac{O}{A}$ $= \frac{5}{8.66}$ $= 0.58$
	$\sin(30°) = \frac{O}{H}$ $= \frac{1}{2}$ $= 0.5$	$\cos(30°) = \frac{A}{H}$ $= \frac{1.74}{2}$ $= 0.87$	$\tan(30°) = \frac{O}{A}$ $= \frac{1}{1.74}$ $= 0.58$

WORKED EXAMPLE 14 Evaluating trigonometric ratios using a calculator

Evaluate each of the following. Give your answers correct to 4 decimal places.

a. $\sin(53°)$ **b.** $\cos(31°)$ **c.** $\tan(79°)$

THINK	WRITE
a. 1. Set the calculator to degree mode. Make the calculation and then write out the first 5 decimal places.	**a.** $\sin(53°) = 0.798\,63$
2. Round the answer to 4 decimal places.	≈ 0.7986
b. 1. Set the calculator to degree mode. Make the calculation and then write out the first 5 decimal places.	**b.** $\cos(31°) = 0.857\,16$
2. Round the answer to 4 decimal places.	≈ 0.8572
c. 1. Set the calculator to degree mode. Make the calculation and then write out the first 5 decimal places.	**c.** $\tan(79°) = 5.144\,55$
2. Round the answer to 4 places.	≈ 5.1446

TI | THINK

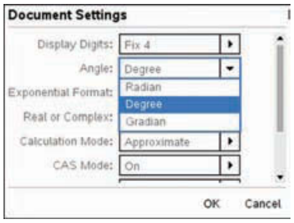
a–c. 1. Ensure your calculator is set to Degree and Approximate modes. To do this, press:

- HOME
- 5: settings
- 2: Document settings

In the Display Digits, select Fix 4, tab to Angle and select Degree, then tab to Calculation Mode and select Approximate.

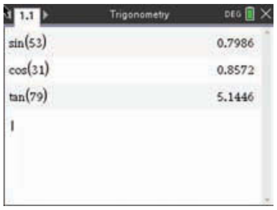
Tab to OK and press ENTER.

DISPLAY/WRITE



2. On a Calculator page, complete the entry lines as:
sin(53)
cos(31)
tan(79)

Press CTRL ENTER after each entry to get a decimal approximation.



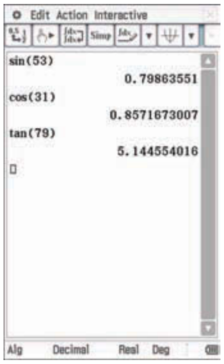
a. $\sin(53°) \approx 0.7986$
b. $\cos(31°) \approx 0.8572$
c. $\tan(79°) \approx 5.1446$

CASIO | THINK

a–c. Ensure the calculator is in Degree and Decimal modes. On the Main screen, complete the entry lines as:
sin(53)
cos(31)
tan(79)

Press EXE after each entry.

DISPLAY/WRITE



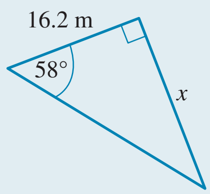
a. $\sin(53°) \approx 0.7986$
b. $\cos(31°) \approx 0.8572$
c. $\tan(79°) \approx 5.1446$

8.5.2 Determining side lengths using trigonometric ratios

- If a given angle and any side length of a right-angled triangle are known, it is possible to calculate the lengths of the other sides using trigonometry.

WORKED EXAMPLE 15 Calculating an unknown side length using the tangent ratio

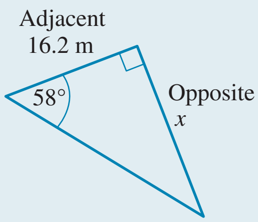
Use the appropriate trigonometric ratio to calculate the length of the unknown side in the triangle shown. Give your answer correct to 2 decimal places.



THINK

1. Label the given sides.

WRITE



2. These sides can be used in TOA. Write the ratio.

$$\tan(\theta) = \frac{O}{A}$$

3. Substitute the values of θ, O and A into the ratio.

$$\tan(58°) = \frac{x}{16.2}$$

4. Solve the equation for x. $16.2 \times \tan(58°) = x$

5. Calculate the value of x to 3 decimal places, then round the answer to 2 decimal places.

$$x = 16.2\ \tan(58°)$$
$$x = 25.925$$
$$x \approx 25.93\text{ m}$$

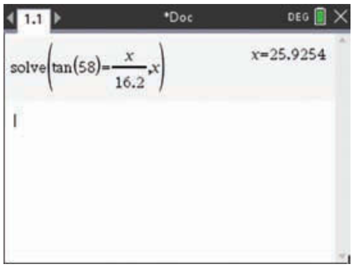
TI | THINK

Ensure your calculator is set to Degree mode. On a Calculator page, complete the entry line as:

$$\text{solve}\left(\tan(58) = \frac{x}{16.2}, x\right)$$

Then press ENTER.

DISPLAY/WRITE



$x = 25.93$ m (correct to 2 decimal places)

CASIO | THINK

Ensure your calculator is set to Degree mode. On the Main screen, complete the entry line as:

$$\text{solve}\left(\tan(58) = \frac{x}{16.2}, x\right)$$

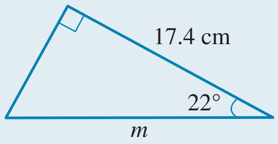
Then press EXE.

DISPLAY/WRITE

$x = 25.93$ m (correct to 2 decimal places)

WORKED EXAMPLE 16 Calculating an unknown side length using the cosine ratio

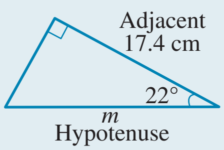
Use the appropriate trigonometric ratios to calculate the length of the side marked *m* in the triangle shown. Give your answer correct to 2 decimal places.



THINK

1. Label the given sides.

WRITE



2. These sides can be used in CAH. Write the ratio.

$$\cos(\theta) = \frac{A}{H}$$

3. Substitute the values of θ, A and H into the ratio.

$$\cos(22°) = \frac{17.4}{m}$$

4. Solve the equation for m.

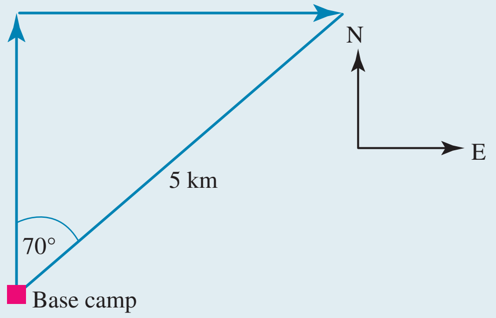
$$m\ \cos(22°) = 17.4$$
$$m = \frac{17.4}{\cos(22°)}$$

5. Calculate the value of m to 3 decimal places, then round the answer to 2 decimal places.

$$m = 18.766$$
$$m \approx 18.77\text{ cm}$$

WORKED EXAMPLE 17 Solving worded problems using trigonometric ratios

Bachar set out on a bushwalking expedition. Using a compass, he set off on a course N 70°E (or 070°T) and travelled a distance of 5 km from his base camp.



a. **Calculate how far east Bachar travelled.**

b. **Calculate how far north Bachar travelled from the base camp. Give answers in km correct to 2 decimal places.**

THINK

a. **1.** Label the eastern distance x.
Label the northern distance y.
Label the sides of the triangle as hypotenuse, opposite and adjacent.

2. To calculate the value of x, use the sides of the triangle: $x = \text{O}$, $\text{H} = 5$.
These are used in SOH. Write the ratio.

3. Substitute the values of the angle and the pronumerals into the sine ratio.

4. Make x the subject of the equation.

5. Solve for x using a calculator. Write out the answer to 3 decimal places.

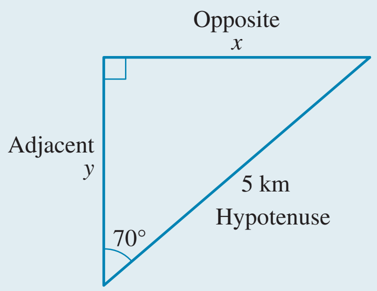
6. Round the answer to 2 decimal places.

7. Write the answer in sentence form.

b. **1.** To calculate the value of y, use the hypotenuse and the adjacent side: $y = \text{A}$, $\text{H} = 5$.
These are used in CAH. Write the ratio.

WRITE

a.



$\sin(\theta) = \dfrac{\text{O}}{\text{H}}$

$\sin(70°) = \dfrac{x}{5}$

$x = 5\sin(70°)$

$= 4.698$

$\approx 4.70\text{ km}$

Bachar has travelled 4.70 km east of the base camp.

b. $\cos(\theta) = \dfrac{\text{A}}{\text{H}}$

2. Substitute the values of the angle and the pronumerals into the cosine ratio. | $\cos(70°) = \dfrac{y}{5}$

3. Make y the subject of the equation. | $y = 5\cos(70°)$

4. Solve for y using a calculator. Write out the answer to 3 decimal places. | $= 1.710$

5. Round the answer to 2 decimal places. | ≈ 1.71 km

6. Write the answer in sentence form. | Bachar has travelled 1.71 km north of the base camp.

Exercise 8.5 Calculating unknown side lengths

learnon

8.5 Quick quiz on | 8.5 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 6, 9, 12, 13, 16, 19, 22	2, 5, 7, 10, 14, 17, 20, 23	3, 8, 11, 15, 18, 21, 24, 25

Fluency

1. a. WE14 Calculate the following. Give your answers correct to 4 decimal places.
   i. $\sin(55°)$
   ii. $\sin(11.6°)$

   b. Complete the table shown. Use your calculator to calculate each value of $\sin(\theta)$. Give your answers correct to 2 decimal places.

θ	0°	15°	30°	45°	60°	75°	90°
$\sin(\theta)$							

   c. Summarise the trend in these values.

2. a. Calculate the following. Give your answers correct to 4 decimal places.
   i. $\cos(38°)$
   ii. $\cos(53.71°)$

   b. Complete the table shown. Use your calculator to calculate each value of $\cos(\theta)$. Give your answers correct to 2 decimal places.

θ	0°	15°	30°	45°	60°	75°	90°
$\cos(\theta)$							

   c. Summarise the trend in these values.

3. a. Calculate the following. Give your answers correct to 4 decimal places.
   i. $\tan(18°)$
   ii. $\tan(51.9°)$

   b. Complete the table shown. Use your calculator to calculate each value of $\tan(\theta)$. Give your answers correct correct to 2 decimal places.

θ	0°	15°	30°	45°	60°	75°	90°
tan(θ)							

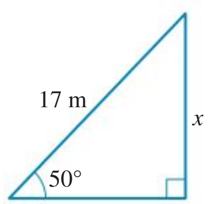
   c. Determine the value of $\tan(89°)$ and $\tan(89.9°)$.

   d. Comment on these results.

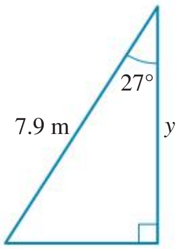
For questions 4–10, use the appropriate trigonometric ratios to calculate the length of the unknown side in each of the triangles shown. Give your answers correct to 2 decimal places.

4. WE15

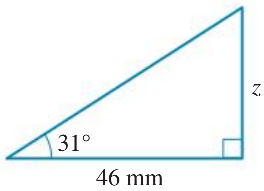
a.



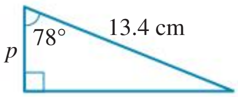
b.



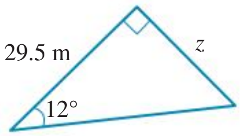
c.



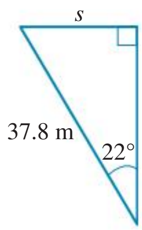
5. a.



b.

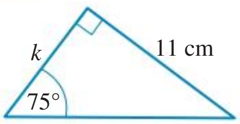


c.

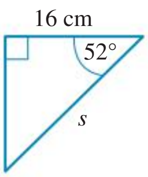


6. WE16

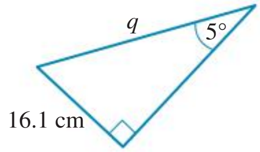
a.



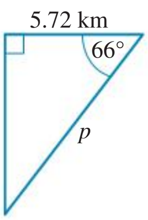
b.



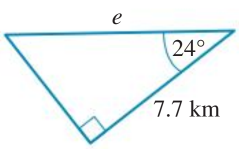
c.



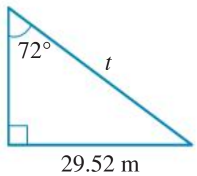
7. a.



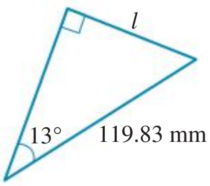
b.



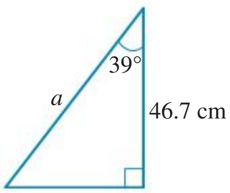
c.



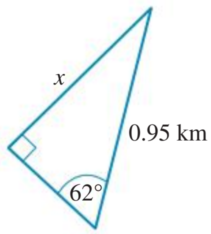
8. a.



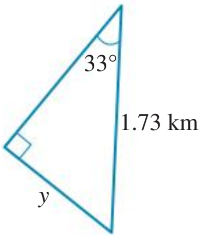
b.



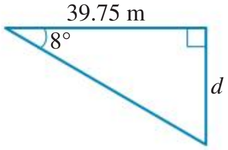
c.



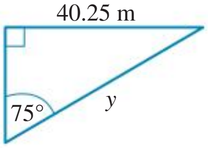
9. a.



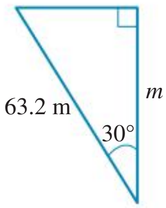
b.



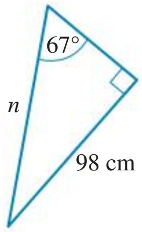
c.



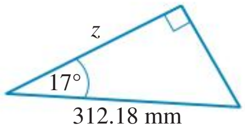
10. a.



b.

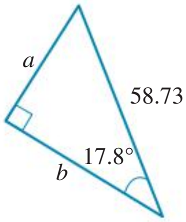


c.

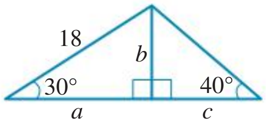


11. Calculate the lengths of the unknown sides in the triangles shown. Give your answers correct to 2 decimal places.

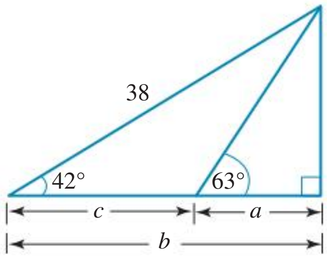
a.



b.

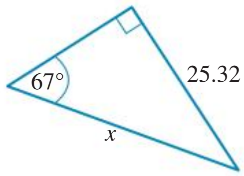


c.



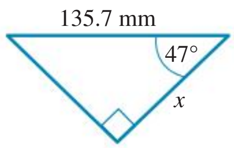
Understanding

12. MC Identify the value of x in the triangle shown, correct to 2 decimal places.



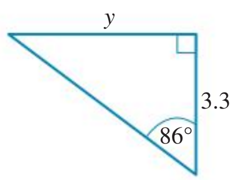
A. 59.65
B. 23.31
C. 64.80
D. 27.51

13. MC Identify the value of x in the triangle shown, correct to 2 decimal places.



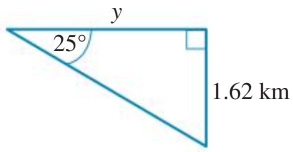
A. 99.24 mm
B. 92.55 mm
C. 185.55 mm
D. 198.97 mm

14. MC Identify y in the triangle shown, correct to 2 decimal places.



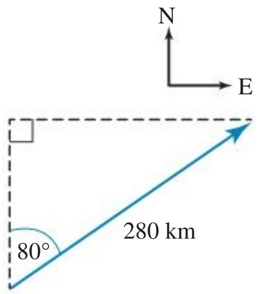
A. 47.19
B. 7.94
C. 1.37
D. 0.23

15. MC Identify the value of y in the triangle shown, correct to 2 decimal places.



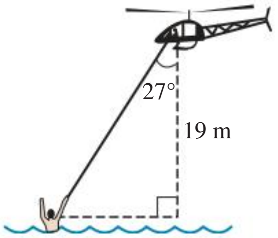
A. 0.76 km
B. 1.79 km
C. 3.83 km
D. 3.47 km

16. WE17 A ship that was supposed to travel due north veers off course and travels N 80°E (or 080°T) for a distance of 280 km instead. The ship's path is shown in the diagram.

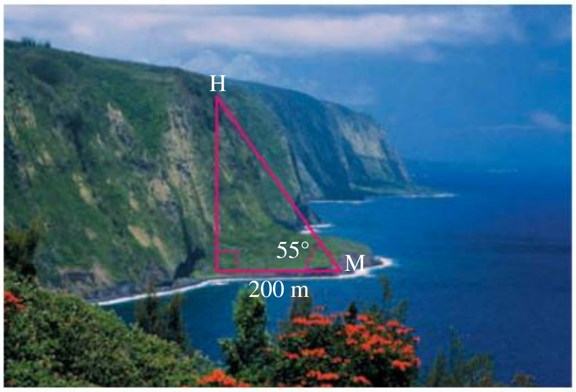


a. Calculate the distance east that the ship has travelled, in km correct to 2 decimal places.
b. Determine the distance north that the ship has travelled, in km correct to 2 decimal places.

17. A rescue helicopter spots a missing surfer drifting out to sea on their damaged board. The helicopter descends vertically to a height of 19 m above sea level and drops down an emergency rope, which the surfer grabs onto.
Due to the high winds, the rope swings at an angle of 27° to the vertical, as shown in the diagram.
Calculate the length of the rope, in metres correct to 2 decimal places.



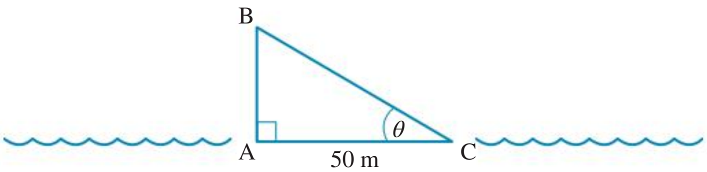
18. Walking along the coastline, Michelle (M) looks up at an angle of 55° and sees her friend Hella (H) at the lookout point on top of the cliff.
If Michelle is 200 m from the cliff's base, determine the height of the cliff, in metres correct to 2 decimal places (Assume Michelle and Hella are the same height.)



Reasoning

19. Using a diagram, explain why $\sin(70°) = \cos(20°)$ and $\cos(70°) = \sin(20°)$. Generally speaking, explain which cosine $\sin(\theta)$ will be equal to.

20. One method for determining the distance across a body of water is illustrated in the diagram shown.



The required distance is AB. A surveyor moves 50 m at a right angle to point C, then uses a tool called a transit to measure the angle θ (∠ACB).

a. If $\theta = 12.3°$, show that the distance from A to B is 10.90 m.
b. Show that a value of $\theta = 63.44°$ gives a distance from A to B of 100 m.
c. Determine a rule that can be used to calculate the distance from A to C.

21. Explain why $\cos(0°) = 1$ and $\sin(0°) = 0$, but $\cos(90°) = 0$ and $\sin(90°) = 1$

Problem solving

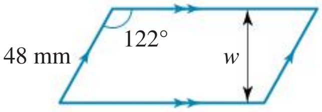
22. Calculate the value of the pronumeral in each of the triangles shown in the following photos.

a. 

b. 

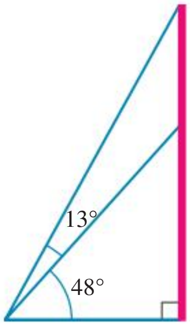
c. 

23. A tile in the shape of a parallelogram has the measurements shown. Determine the tile's width, w, to the nearest mm.

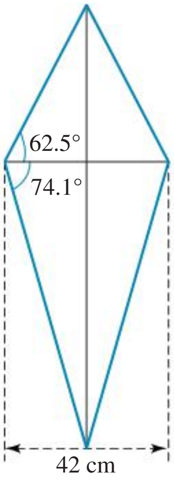


24. A pole is supported by two wires as shown. If the length of the lower wire is 4.3 m, evaluate the following to 1 decimal place.

a. The length of the top wire
b. The height of the pole



25. The frame of a kite is built from six wooden rods as shown. Evaluate the total length of wood used to make the frame of the kite. Give your answer to the nearest metre.



LESSON 8.6 Calculating unknown angles

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- use a calculator to identify an angle from a given trigonometric ratio
- determine the given angle in a right-angled triangle when given two side lengths.

8.6.1 Inverse trigonometric ratios

- When we are given an angle, the sine, cosine and tangent functions let us determine the ratio of side lengths, but what if we only know the side lengths and we want to determine the angle in a right-angled triangle?
- The inverse trigonometric functions allow us to calculate angles using sine, cosine or tangent ratios.
- We have seen that $\sin(30°) = 0.5$. This means that 30° is the inverse sine of 0.5. This is written as $\sin^{-1}(0.5) = 30°$.

Inverse trigonometic ratios

Sine function	Inverse sine function
$\sin(30°) = 0.5$	$\sin^{-1}(0.5) = 30°$

- When reading the expression $\sin^{-1}(x)$ out loud, we say, 'the inverse sine of x'.
- When reading the expression $\cos^{-1}(x)$ out loud, we say, 'the inverse cosine of x'.
- When reading the expression $\tan^{-1}(x)$ out loud, we say, 'the inverse tangent of x'.
- You can calculate inverse trigonometric ratios using the SIN^{-1}, COS^{-1} and TAN^{-1} buttons on your calculator.

Digital technology

1. Use your calculator to determine $\sin(30°)$, then determine the inverse sine ($\sin^{-1}$) of the answer. Choose another angle and do the same thing.
2. Now determine $\cos(30°)$ and then determine the inverse cosine ($\cos^{-1}$) of the answer. Choose another angle and do the same thing.
3. Lastly, determine $\tan(45°)$ and then determine the inverse tangent ($\tan^{-1}$) of the answer. Choose another angle and do the same thing.

The fact that sin and $\sin^{-1}$ cancel each other out is useful when solving equations like the following.
Consider: $\sin(\theta) = 0.3$.
Take the inverse sine of both sides

$$\sin^{-1}(\sin(\theta)) = \sin^{-1}(0.3)$$
$$\theta = \sin^{-1}(0.3)$$

Consider: $\sin^{-1}(x) = 15°$.

Take the sine of both sides.

$$\sin(\sin^{-1}(x)) = \sin(15°)$$
$$x = \sin(15°)$$
Similarly, $\cos(\theta) = 0.522$ means that
$$\theta = \cos^{-1}(0.522)$$
and $\tan(\theta) = 1.25$ means that
$$\theta = \tan^{-1}(1.25).$$

WORKED EXAMPLE 18 Evaluating inverse cosine values

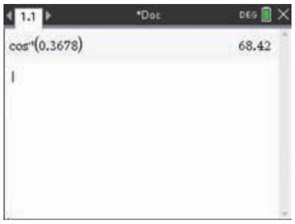
Evaluate $\cos^{-1}(0.3678)$, correct to the nearest degree.

THINK	WRITE
1. Set your calculator to degree mode and make the calculation.	$\cos^{-1}(0.3678) = 68.4$
2. Round the answer to the nearest whole number and include the degree symbol in your answer.	$\approx 68°$

TI | THINK

Ensure your calculator is set to Degree mode. On a Calculator page, complete the entry line as:
$\cos^{-1}(0.3678)$
Then press ENTER.

DISPLAY/WRITE

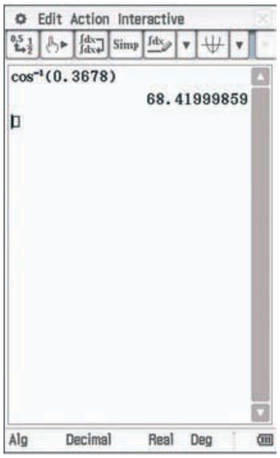


$\cos^{-1}(0.3678) = 68°$ (to the nearest degree)

CASIO | THINK

Ensure your calculator is set to Degree mode. On the Main screen, complete the entry line as:
$\cos^{-1}(0.3678)$
Then press EXE.

DISPLAY/WRITE



$\cos^{-1}(0.3678) = 68°$ (to the nearest degree)

WORKED EXAMPLE 19 Determining angles using inverse trigonometric ratios

Determine the size of angle θ in each of the following. Give answers correct to the nearest degree.

a. $\sin(\theta) = 0.6543$ **b.** $\tan(\theta) = 1.745$

THINK	WRITE
a. 1. θ is the inverse sine of 0.6543.	**a.** $\sin(\theta) = 0.6543$ $\theta = \sin^{-1}(0.6543)$
2. Make the calculation using your calculator's inverse sine function and record the answer.	$= 40.8$
3. Round the answer to the nearest degree and include the degree symbol in your answer.	$\approx 41°$

b. **1.** θ is the inverse tangent of 1.745.

2. Make the calculation using your calculator's inverse tangent function and record the answer.

3. Round the answer to the nearest degree and include the degree symbol in your answer.

b. $\tan(\theta) = 1.745$

$\theta = \tan^{-1}(1.745)$

$= 60.18$

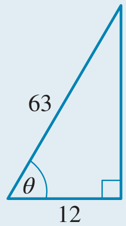
$\approx 60°$

8.6.2 Determining the angle when two sides are known

- If the lengths of any two sides of a right-angled triangle are known, it is possible to determine an angle using inverse sine, inverse cosine or inverse tangent.

WORKED EXAMPLE 20 Determining an unknown angle when two sides are known

Determine the value of θ in the triangle shown. Give your answer correct to the nearest degree.



THINK

1. Label the given sides. These are used in CAH. Write out the ratio.

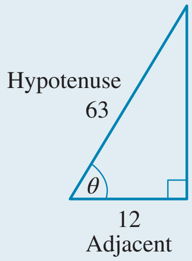
2. Substitute the given values into the cosine ratio.

3. θ is the inverse cosine of $\frac{12}{63}$.

4. Evaluate.

5. Round the answer to the nearest degree.

WRITE



$\cos(\theta) = \frac{A}{H}$

$\cos(\theta) = \frac{A}{H}$

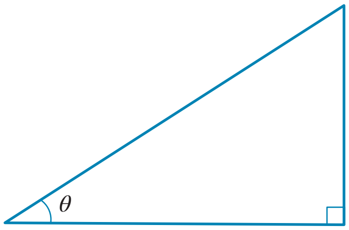
$\theta = \cos^{-1}\left(\frac{12}{63}\right)$

$= 79.0$

$\approx 79°$

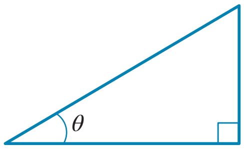
8.6.3 Relating the tangent of the angle to the gradient

- $\tan(\theta) = \dfrac{O}{A}$



- The gradient is given as:

$$\text{Gradient } (m) = \frac{\text{Rise}}{\text{Run}}$$



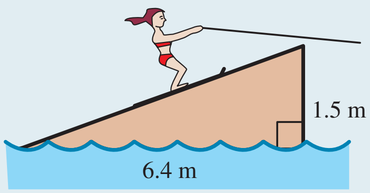
- Comparing the two right-angled triangles, we can see:
  Opposite (Altitude) = Rise
  and
  Adjacent (Base) = Run
- Therefore, we can say that:

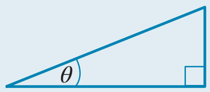
$$\tan(\theta) = \frac{O}{A} = \frac{\text{Rise}}{\text{Run}} = m$$

Therefore, $\tan(\theta) = m$
Where m is the gradient.

WORKED EXAMPLE 21 Solving word problems using the inverse tangent

Roberta goes water skiing on the Hawkesbury River. She is going to try out a new ramp. The ramp rises 1.5 m above the water level and has a length of 6.4 m.
Determine the magnitude (size) of the angle that the ramp makes with the water's surface. Give your answer to the nearest degree. Hence determine the gradient of the ramp to 2 decimal places.



THINK	WRITE
1. Draw a simple diagram, showing the known lengths and the angle to be found.	

2. Label the given sides. These are used in TOA. Write out the ratio. $\tan(\theta) = \dfrac{\text{O}}{\text{A}}$

3. Substitute the values of the pronumerals into the tangent ratio. $\tan(\theta) = \dfrac{1.5}{6.4}$

4. θ is the inverse inverse tangent of $\dfrac{1.5}{6.4}$. $\theta = \tan^{-1}\left(\dfrac{1.5}{6.4}\right)$

5. Evaluate. $= 13.19$

6. Round the answer to the nearest degree. $\approx 13°$

7. Write the answer in words. The ramp makes an angle of 13° with the water's surface.

8. Since $m = \tan(\theta)$ and $\theta = 13.19°$. Substitute the angle into the equation to determine the gradient.
$$\begin{aligned} m &= \tan(\theta) \\ &= \tan(13.19°) \\ &= 0.23 \end{aligned}$$

9. Write the answer. The gradient of the ramp is 0.23.

Exercise 8.6 Calculating unknown angles

learnon

8.6 Quick quiz on	8.6 Exercise

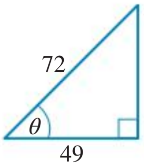
Individual pathways

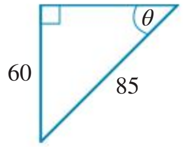
PRACTISE	CONSOLIDATE	MASTER
1, 4, 7, 10, 11, 14, 18, 21	2, 5, 8, 12, 15, 17, 19, 22	3, 6, 9, 13, 16, 20, 23, 24

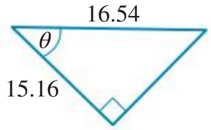
Fluency

1. WE18 Calculate each of the following, rounding your answers to the nearest degree.
   a. $\sin^{-1}(0.6294)$ b. $\cos^{-1}(0.3110)$ c. $\tan^{-1}(0.7409)$
2. Calculate each of the following, rounding your answers to the nearest degree.
   a. $\tan^{-1}(1.3061)$ b. $\sin^{-1}(0.9357)$ c. $\cos^{-1}(0.3275)$
3. Calculate each of the following, rounding your answers to the nearest degree.
   a. $\cos^{-1}(0.1928)$ b. $\tan^{-1}(4.1966)$ c. $\sin^{-1}(0.2554)$
4. WE19 Determine the size of the angle in each of the following, rounding your answers to the nearest degree.
   a. $\sin(\theta) = 0.3214$ b. $\sin(\theta) = 0.6752$ c. $\sin(\beta) = 0.8235$ d. $\cos(\beta) = 0.9351$
5. Determine the size of the angle in each of the following, rounding your answers to the nearest degree.
   a. $\cos(\alpha) = 0.6529$ b. $\cos(\alpha) = 0.1722$ c. $\tan(\theta) = 0.7065$ d. $\tan(a) = 1$
6. Determine the size of the angle in each of the following, rounding your answers to the nearest degree.
   a. $\tan(b) = 0.876$ b. $\sin(c) = 0.3936$ c. $\cos(\theta) = 0.5241$ d. $\tan(\alpha) = 5.6214$

7. **WE20** Determine the value of θ in each of the following triangles, rounding your answers to the nearest degree.

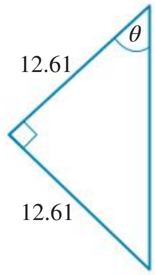
a. 

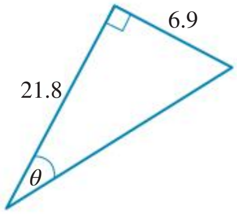
b. 

c. 

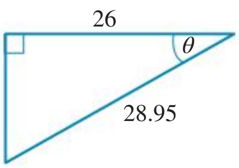
8. Determine the value of θ in each of the following triangles, rounding your answers to the nearest degree.

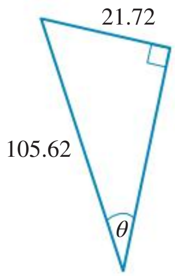
a. 

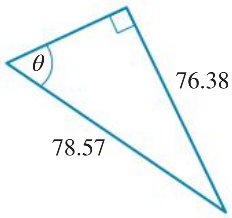
b. 

c. 

9. Determine the value of θ in each of the following triangles, rounding your answers to the nearest degree.

a. 

b. 

c. 

10. **MC** If $\cos(\theta) = 0.8752$, identify the value of θ, correct to 2 decimal places.

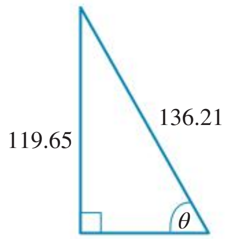
A. 61.07°
B. 41.19°
C. 25.84°
D. 28.93°

11. **MC** If $\sin(\theta) = 0.5530$, identify the value of θ, correct to 2 decimal places.

A. 56.43°
B. 33.57°
C. 28.94°
D. 36.87°

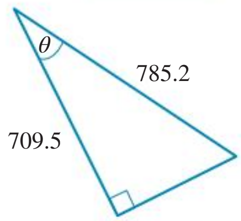
12. **MC** Identify the value of θ in the triangle shown, correct to 2 decimal places.

A. 41.30°
B. 28.55°
C. 48.70°
D. 61.45°



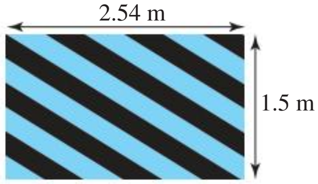
13. **MC** Identify the value of θ in the triangle shown, correct to 2 decimal places.

A. 42.10°
B. 64.63°
C. 25.37°
D. 47.90°

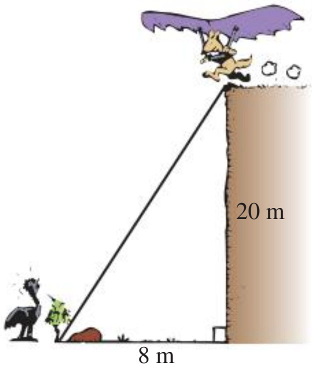


Understanding

14. A piece of fabric measuring 2.54 m by 1.5 m has been printed with parallel diagonal stripes. Determine the angle each diagonal makes with the length of the fabric.
Give your answer in degrees correct to 2 decimal places.



15. WE21 Danny Dingo is perched on top of a 20-metre-high cliff. They are watching Erwin Emu, who is feeding on a bush that stands 8 m away from the base of the cliff. Danny has purchased a flying contraption that they hope will help them capture Erwin.
Calculate the angle to the cliff that Danny should follow downwards to catch their prey.
Give your answer in degrees correct to 2 decimal places.



16. a. Complete the table shown.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y = \cos^{-1}(x)$	90°					60°					0°

b. Plot the table from part **a** on graph paper. Alternatively you can use a spreadsheet program or a suitable calculator.

17. A zipline runs from an observation platform 400 m above sea level to a landing platform on the ground 1200 metres away.
Calculate the angle that the zipline makes with the ground at the landing platform and hence determine the gradient of the zipline.
Give your answers correct to 2 decimal places.

Reasoning

18. Safety guidelines for wheelchair access ramps used to state that the gradient of a ramp had to be in the ratio 1 : 20.

a. Using this ratio, show that the angle that any ramp needs to make with the horizontal is closest to 3°.

b. New regulations have changed the guidelines about the ratio of the gradient of a ramp. Now the angle the ramp makes with the horizontal must be closest to 6°.
Explain why, using this new angle size, the new ratio could be 1 : 9.5.

19. Jayani and Lee are camping with their friends Awer and Susie. Both couples have tents that are 2 m high. The top of each 2 m tent pole has to be tied with a piece of rope so that the pole stays upright. To make sure this rope doesn't trip anyone, Jayani and Lee decide that the angle between the rope and the ground should be 60°.

a. Determine the length of the rope that Jayani and Lee need to run from the top of their tent pole to the ground. Give your answer in metres correct to 2 decimal places.

b. Awer and Susie set up their tent further into the camping ground. They want to secure their tent pole using a piece of rope that they know is somewhere between 2 and 3 metres long.

i. Explain why Awer and Susie's rope will have to be longer than 2 m.

ii. Show that the minimum angle Awer and Susie's rope will make with the ground will be 41.8°.

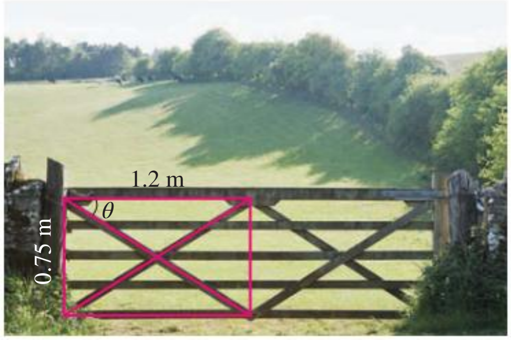
20. Use the formulas $\sin(\theta) = \dfrac{\text{O}}{\text{H}}$ and $\cos(\theta) = \dfrac{\text{A}}{\text{H}}$ to prove that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

Problem solving

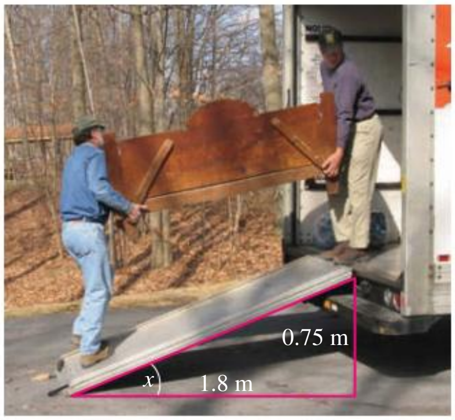
21. Calculate the value of the pronumeral in each of the following, in degrees correct to 2 decimal places.

a.

b.



c.

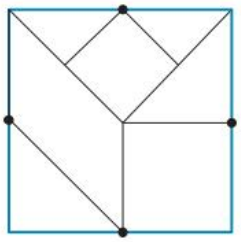


22. A family is building a patio extension for their house. One section of the new patio will have a gable roof. A similar structure is shown, with the planned post heights and the span of the roof given.
To allow more light in, the family wants the peak (the highest point) of the roof to be at least 5 m above ground level.
According to building regulations, the slope of the roof (the angle that the sloping edge makes with the horizontal) must be 22°.

a. Use trigonometry to calculate whether the roof would be high enough if the angle was 22°.
b. Using trigonometry, determine the size of the obtuse angle formed at the peak of the roof.

23. The height of a square-based prism is twice its base length. Calculate the angle the diagonal of the prism makes with the diagonal of the base.

24. A series of seven shapes are marked out as shown inside a square with a side length of 10 cm. If the dots marked on the diagram represent the midpoints of the square's sides, determine the dimensions of each of the seven smaller shapes.



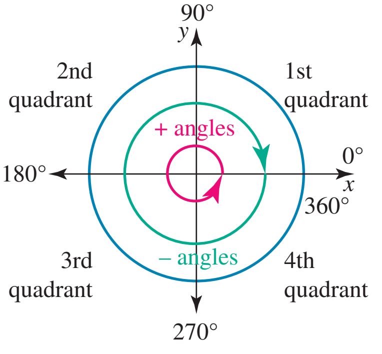
LESSON 8.7 The unit circle (optional)

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- determine in which quadrant an angle lies
- use and interpret the relationship between a point on the unit circle and the angle made with the positive x-axis
- use the unit circle to determine approximate trigonometric ratios for angles greater than 90°
- use and interpret the relationships between supplementary and complementary angles.

8.7.1 The unit circle

- A **unit circle** is a circle with a radius of 1 unit.
- The unit circle is divided into 4 quadrants, numbered in an anticlockwise direction, as shown in the diagram.
- *Positive angles* are measured anticlockwise from 0°.
- *Negative angles* are measured clockwise from 0°.



WORKED EXAMPLE 22 Identifying where an angle lies on the unit circle

State the quadrant of the unit circle in which each of the following angles is found.

a. **145°**

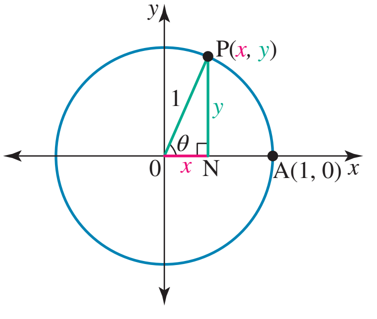
b. **282°**

THINK	WRITE
a. The given angle is between 90° and 180°. State the appropriate quadrant.	**a.** 145° is in quadrant 2.
b. The given angle is between 270° and 360°. State the appropriate quadrant.	**b.** 282° is in quadrant 4.

- Consider the unit circle with point P(x, y) making the right-angled triangle OPN as shown in the diagram.
- Using the trigonometric ratios:

$$\frac{x}{1} = \cos(\theta),\ \frac{y}{1} = \sin(\theta),\ \frac{y}{x} = \tan(\theta)$$

where θ is measured anticlockwise from the positive x-axis.



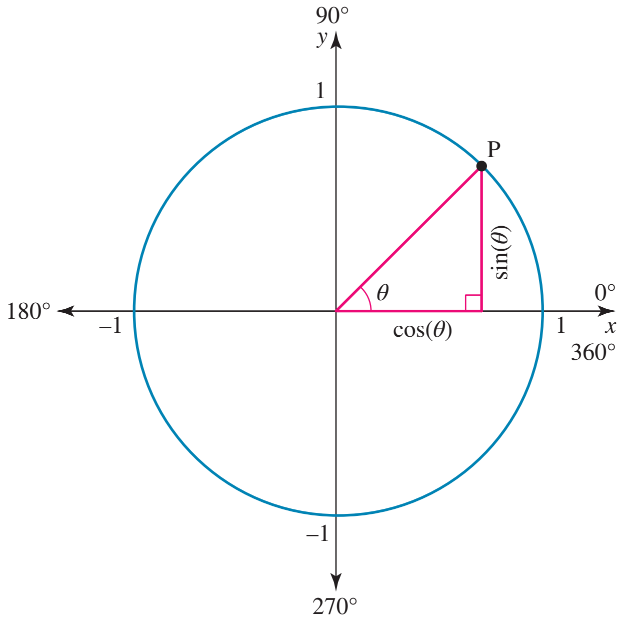
Calculating values of sine, cosine and tangent

To calculate the value of sine, cosine or tangent of any angle θ from the unit circle:

$$\cos(\theta) = x$$
$$\sin(\theta) = y$$
$$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$$

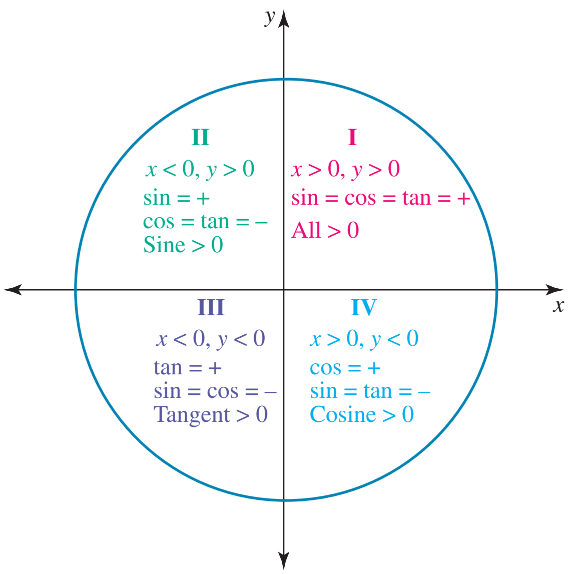
8.7.2 The four quadrants of the unit circle

- Approximate values for sine, cosine and tangent of an angle can be found from the unit circle using the following steps, as shown in the diagram.
  Step 1: Draw a unit circle and label the x- and y-axes.
  Step 2: Mark the angles 0°, 90°, 180°, 270° and 360°.
  Step 3: Draw the given angle θ.
  Step 4: Mark $x = \cos(\theta)$, $y = \sin(\theta)$.
  Step 5: Approximate the values of x and y and equate to give the values of $\cos(\theta)$ and $\sin(\theta)$.
- Where the angle lies in the unit circle determines whether the trigonometric ratio is positive or negative.



Sign of the trigonometric functions

- Consider the following.
  - In the first quadrant $x > 0$, $y > 0$; therefore, **All** trig ratios are positive.
  - In the second quadrant $x < 0$, $y > 0$; therefore, **Sine** (the y-value) is positive.
  - In the third quadrant $x < 0$, $y < 0$; therefore, **Tangent** $\left(\frac{y}{x} \text{ values}\right)$ is positive.
  - In the fourth quadrant $x > 0$, $y < 0$; therefore, **Cosine** (the x-value) is positive.
  - There are several mnemonics for remembering the sign of the trigonometric functions.
    A common saying is '**A**ll **S**tations **T**o **C**entral'.



WORKED EXAMPLE 23 Using the unit circle to approximate the trigonometric ratios of an angle

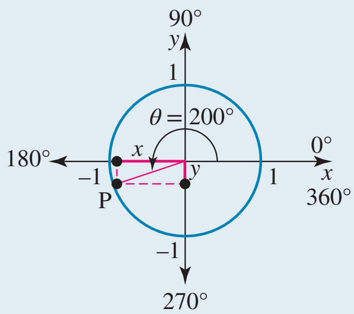
Determine the approximate value of each of the following using the unit circle.

a. **sin(200°)** **b.** **cos(200°)** **c.** **tan(200°)**

THINK

WRITE/DRAW

Draw a unit circle and construct an angle of 200°. Label the point corresponding to the angle of 200° on the circle P. Highlight the lengths representing the *x*- and *y*-coordinates of point P.



a. The sine of the angle is given by the *y*-coordinate of P. Determine the *y*-coordinate of P by measuring the distance along the *y*-axis. State the value of sin(200°). (*Note:* The sine value will be negative as the *y*-coordinate is negative.)

a. $\sin(200°) = -0.3$

b. The cosine of the angle is given by the *x*-coordinate of P. Determine the *x*-coordinate of P by measuring the distance along the *x*-axis. State the value of cos(200°). (*Note:* Cosine is also negative in quadrant 3, as the *x*-coordinate is negative.)

b. $\cos(200°) = -0.9$

c. $\tan(200°) = \dfrac{\sin(200°)}{\cos(200°)}$

c. $\dfrac{-0.3}{-0.9} = \dfrac{1}{3} = 0.3333$

- The approximate results obtained in Worked example 23 can be verified with the aid of a calculator:

$$\sin(200°) = -0.342\,020\,143,\ \cos(200°) = -0.939\,692\,62 \text{ and } \tan(200°) = 0.3640.$$

Rounding these values to 1 decimal place would give −0.3, −0.9 and 0.4 respectively, which match the values obtained from the unit circle.

8.7.3 The relationship between supplementary and complementary angles

Supplementary angles

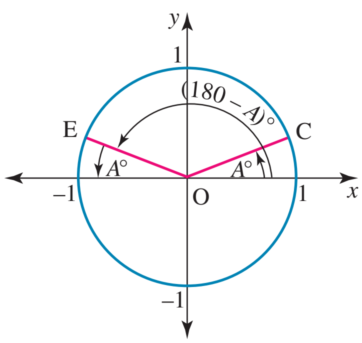
- Consider the special relationship between the sine, cosine and tangent of supplementary angles, say $A°$ and $(180 - A)°$.

  In the diagram, the *y*-axis is an axis of symmetry.
  - The *y*-values of points C and E are the same.
    That is, $\sin(A°) = \sin(180 - A)°$.
  - The *x*-values of points C and E are opposites in value.
    That is, $\cos(A°) = -\cos(180 - A)°$

  Thus: $\sin(180 - A)° = \sin(A°)$

  $\cos(180 - A)° = -\cos(A°)$

  $\tan(180 - A)° = \dfrac{\sin(180 - A)°}{\cos(180 - A)°} = \dfrac{\sin(A°)}{-\cos(A°)} = -\tan(A°)$.



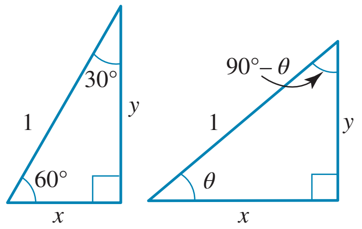
Supplementary angles $A°$ and $(180° - A)$ where $0° \le A \le 90°$

$$\sin(180° - A) = \sin(A)$$

$$\cos(180° - A) = -\cos(A)$$

$$\tan(180° - A) = -\tan(A)$$

Complementary angles



- Complementary angles add to 90°. Therefore, 30° and 60° are complementary angles, and θ and $(90° - \theta)$ are also complementary angles.
- The sine of an angle is equal to the cosine of its complement. Therefore, $\sin(60°) = \cos(30°)$.
- We say that sine and cosine are complementary functions.

$$\sin A = \cos(90° - A)$$
$$\cos A = \sin(90° - A)$$

Complementary angles where $0° \le A \le 90°$

$$\sin A = \cos(90° - A)$$

$$\cos A = \sin(90° - A)$$

Exercise 8.7 The unit circle

learnon

8.7 Quick quiz on | 8.7 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 6, 8, 12, 17, 20	3, 4, 7, 9, 11, 13, 18, 21	5, 10, 14, 15, 16, 19, 22

Where appropriate in this exercise, give answers correct to 2 decimal places.

Fluency

1. WE22 State which quadrant of the unit circle each of the following angles is in.
   a. 60° b. 130° c. 310° d. 260° e. 100° f. 185°
2. MC If $\theta = 43°$, the triangle drawn to show this would be in:
   A. quadrant 1 B. quadrant 2 C. quadrant 3 D. quadrant 4
3. MC If $\theta = 295°$, the triangle drawn to show this would be in:
   A. quadrant 1 B. quadrant 2 C. quadrant 3 D. quadrant 4

4. **WE23** Determine the approximate value of each of the following using the unit circle.

a. $\sin(20°)$ b. $\cos(20°)$ c. $\cos(100°)$ d. $\sin(100°)$

5. Determine the approximate value of each of the following using the unit circle.

a. $\sin(320°)$ b. $\cos(320°)$ c. $\sin(215°)$ d. $\cos(215°)$

6. Use the unit circle to determine the approximate value of each of the following.

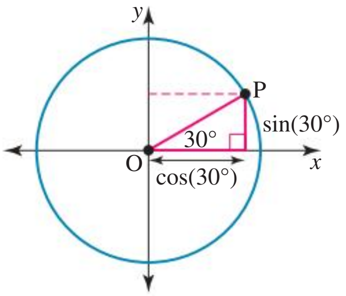
a. $\sin(90°)$ b. $\cos(90°)$ c. $\sin(180°)$ d. $\cos(180°)$

7. Use the unit circle to determine the approximate value of each of the following.

a. $\sin(270°)$ b. $\cos(270°)$ c. $\sin(360°)$ d. $\cos(360°)$

Understanding

8. On the unit circle, use a protractor to measure an angle of 30° from the positive x-axis. Mark the point P on the circle. Use this point to construct a triangle in quadrant 1 as shown.

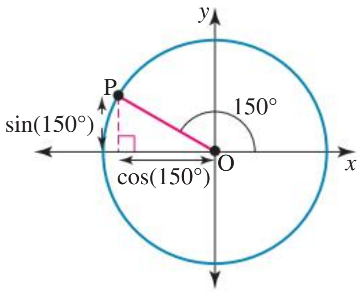


a. Calculate the value of $\cos(30°)$.
(Remember that the length of the adjacent side of the triangle is $\cos(30°)$.)

b. Calculate the value of $\sin(30°)$.
(This is the length of the opposite side of the triangle.)

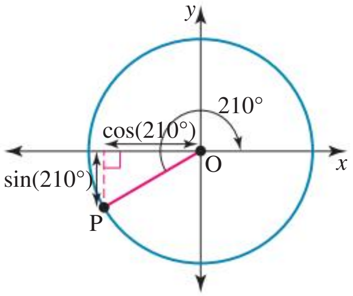
Check your answers in **a** and **b** by calculating these values with a calculator.

9. Using a graph of the unit circle, measure 150° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 2 as shown.



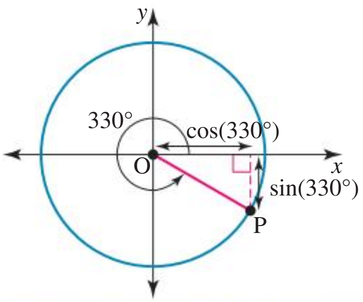
a. Determine the angle the radius OP makes with the negative x-axis.
b. Remembering that $x = \cos(\theta)$, use your circle to determine the value of $\cos(150°)$.
c. Comment on how $\cos(150°)$ compares to $\cos(30°)$.
d. Remembering that $y = \sin(\theta)$, use your circle to determine the value of $\sin(150°)$.
e. Comment on how $\sin(150°)$ compares with $\sin(30°)$.

10. On a unit circle, measure 210° with a protractor and mark the point P on the circle.
Use this point to draw a triangle in quadrant 3 as shown.



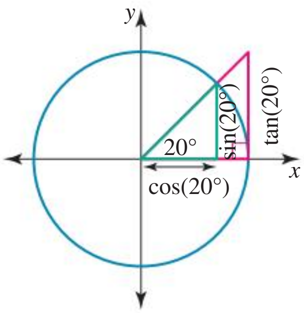
a. Determine the angle the radius OP makes with the negative x-axis.
b. Use your circle to determine the value of $\cos(210°)$.
c. Comment on how $\cos(210°)$ compares to $\cos(30°)$.
d. Use your circle to determine the value of $\sin(210°)$.
e. Comment on how $\sin(210°)$ compares with $\sin(30°)$.

11. On a unit circle, measure 330° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 4 as shown.



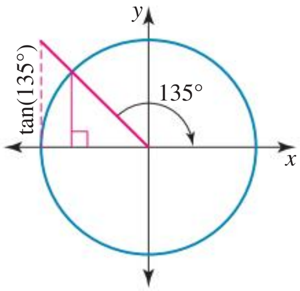
a. Determine the angle the radius OP makes with the positive x-axis.
b. Use your circle to determine the value of $\cos(330°)$.
c. Comment on how $\cos(330°)$ compares to $\cos(30°)$.
d. Use your circle to determine the value of $\sin(330°)$.
e. Comment on how $\sin(330°)$ compares with $\sin(30°)$.

12. On a unit circle, draw an appropriate triangle for the angle of 20° in quadrant 1.



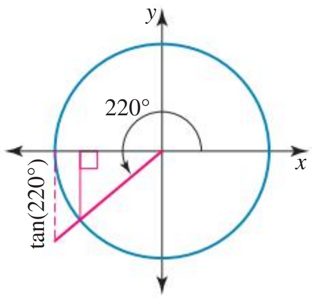
a. Determine the value of sin(20°).
b. Determine the value of cos(20°).
c. Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.
Accurately measure the length of the tangent between the *x*-axis and the point where it meets the hypotenuse, and hence state the value of tan(20°).
d. Determine the value of $\frac{\sin(20°)}{\cos(20°)}$.
e. Comment on how tan(20°) compares with $\frac{\sin(20°)}{\cos(20°)}$.

13. On a unit circle, draw an appropriate triangle for the angle of 135° in quadrant 2.



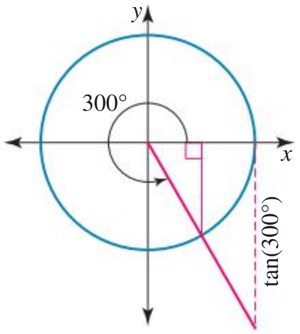
a. Determine the value of sin(135°) using sin(45°).
b. Determine the value of cos(135°) using cos(45°).
c. Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.
Accurately measure the length of the tangent to where it meets the hypotenuse to calculate the value of tan(135°).
d. Determine the value of $\frac{\sin(135°)}{\cos(135°)}$.
e. Comment on how tan(135°) compares with $\frac{\sin(135°)}{\cos(135°)}$.
f. Comment on how tan(135°) compares with tan(45°).

14. On a unit circle, draw an appropriate triangle for the angle of 220° in quadrant 3.



a. Determine the value of sin(220°).
b. Determine the value of cos(220°).
c. Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.
Determine the value of tan(220°) by accurately measuring the length of the tangent to where it meets the hypotenuse.
d. Determine the value of $\frac{\sin(220°)}{\cos(220°)}$.
e. Comment on how tan(220°) compares with $\frac{\sin(220°)}{\cos(220°)}$.
f. Comment on how tan(220°) compares with tan(40°). (Use a calculator.)

15. On a unit circle, draw an appropriate triangle for the angle of 300° in quadrant 4.



a. Determine the value of sin(300°).
b. Determine the value of cos(300°).
c. Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown. Determine the value of tan(300°) by accurately measuring the length of the tangent to where it meets the hypotenuse.
d. Determine the value of the value of $\frac{\sin(300°)}{\cos(300°)}$.
e. Comment on how tan(300°) compares with $\frac{\sin(300°)}{\cos(300°)}$.
f. Comment on how tan(300°) compares with tan(60°). (Use a calculator.)

16. **MC** In a unit circle, the length of the radius is equal to:

A. $\sin(\theta)$
B. $\cos(\theta)$
C. $\tan(\theta)$
D. 1

Reasoning

17. Show that $\sin^2(\alpha°) + \cos^2(\alpha°) = 1$.

18. Show that $1 - \sin^2(180 - \alpha)° = \cos^2(180 - \alpha)°$.

19. Show that $1 + \tan^2(\alpha°) = \sec^2(\alpha°)$, where $\sec(\alpha°) = \frac{1}{\cos(\alpha°)}$.

Problem solving

20. If $\sin(x°) = p$, $0 \leq x \leq 90°$, write each of the following in terms of p.

a. $\cos(x°)$
b. $\sin(180 - x)°$
c. $\cos(180 - x)°$
d. $\cos(90 - x)°$

21. Simplify $\sin(180 - x)° - \sin(x°)$.

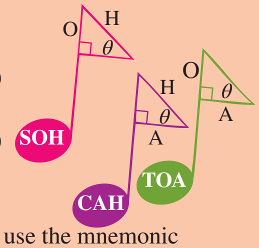
22. Simplify $\cos(180 - x)° + \cos(x°)$, where $0 < x° < 90°$.

LESSON 8.8 Review

8.8.1 Topic summary

Trigonometric ratios

- For a given angle, θ, in a right-angled triangle, the trigonometric ratios sine, cosine and tangent are constant.
  - $\sin(\theta) = \dfrac{\text{opposite (O)}}{\text{hypotenuse (H)}}$
  - $\cos(\theta) = \dfrac{\text{adjacent (A)}}{\text{hypotenuse (H)}}$
  - $\tan(\theta) = \dfrac{\text{opposite (O)}}{\text{adjacent (A)}}$



- To remember these ratios, use the mnemonic **SOH CAH TOA** (**S**ine **O**pposite **H**ypotenuse, **C**os **A**djacent **H**ypotenuse, **T**an **O**pposite **A**djacent).

Inverse trigonometric ratios

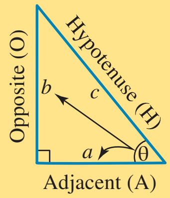
- These ratios can be used to determine the size of a given angle when given two side lengths.
  - $\sin^{-1}\left(\dfrac{\text{O}}{\text{H}}\right) = \theta$
  - $\cos^{-1}\left(\dfrac{\text{A}}{\text{H}}\right) = \theta$
  - $\tan^{-1}\left(\dfrac{\text{O}}{\text{A}}\right) = \theta$

PYTHAGORAS AND TRIGONOMETRY

Pythagoras' theorem

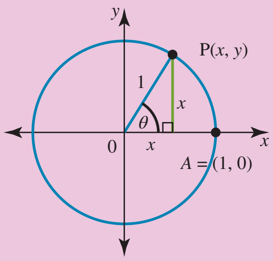
Pythagoras' theorem states the relationship between the lengths of the sides of right angled triangles. Mathematically it is written as:

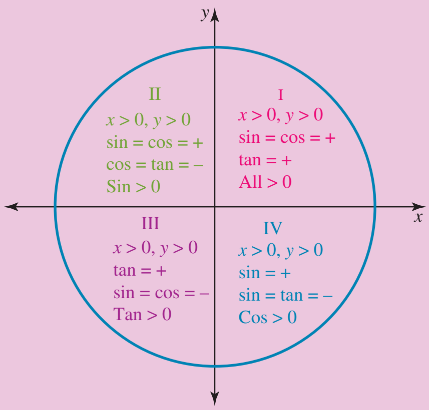
- $a^2 + b^2 = c^2$, where c is the length of the hypotenuse (the longest side) and a and b are the lengths of the other sides.
- This rule can be applied to any right-angled triangle.
- The hypotenuse is always located opposite the right angle.
- A Pythagorean triad is a group of any three whole numbers that satisfy Pythagoras' theorem. e.g. {3, 4, 5} is a Pythagorean triad because $3^2 = 9$, $4^2 = 16$ and $5^2 = 25$, and $9 + 16 = 25$.
- The converse of Pythagoras' theorem can be used to determine whether a triangle is right-angled.



Unit circle (optional)

- Equation of the unit circle: $x^2 + y^2 = 1$
- Radius of length 1 unit.
- For any point on circumference:
  $x = \cos\theta$
  $y = \sin\theta$
- $\tan\theta = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$
- $\sin(180 - A)° = -\sin A°$
  $\cos(180 - A)° = -\cos A°$
  $\tan(180 - A)° = -\tan A°$
- $\sin A = -\cos(90° - A)$
  $\cos A = -\sin(90° - A)$
- Quadrants are positive for:





8.8.2 Success criteria

Tick the column to indicate that you have completed the lesson and how well you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

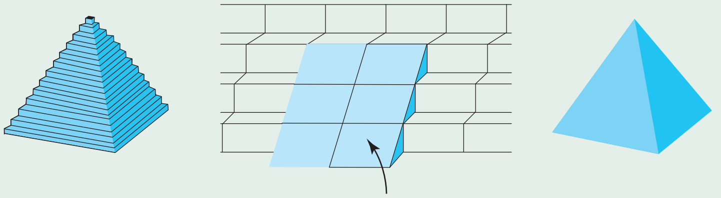
Lesson	Success criteria	Green	Yellow	Red
8.2	I can identify the hypotenuse in a right-angled triangle.			
	I can recall Pythagoras' theorem.			
	I can identify Pythagorean triads.			
	I can apply the converse of Pythagoras' theorem to determine whether a triangle is right-angled.			
	I can apply Pythagoras' theorem to calculate unknown side lengths.			
8.3	I can divide composite shapes into simple shapes.			
	I can calculate the perimeter and area of simple composite shapes.			
	I can calculate unknown side lengths in composite shapes and irregular triangles.			
	I can use Pythagoras' theorem to solve problems in 3D contexts.			
8.4	I can identify the hypotenuse, opposite and adjacent sides of a right-angled triangle with respect to a given angle.			
	I can use the ratios of a triangle's sides to calculate the sine, cosine and tangent of an angle.			
8.5	I can use a calculator to calculate the trigonometric ratios for a given angle.			
	I can determine the lengths of unknown sides in a right-angled triangle given an angle and one known side length.			
8.6	I can use a calculator to identify an angle from a given trigonometric ratio.			
	I can determine the given angle in a right-angled triangle when given two side lengths.			
8.7	I can determine in which quadrant an angle lies.			
	I can use and interpret the relationship between a point on the unit circle and the angle made with the positive x-axis.			
	I can use the unit circle to determine approximate trigonometric ratios for angles greater than $90°$.			
	I can use and interpret the relationships between supplementary and complementary angles.			

8.8.3 Project

The Great Pyramid of Giza

The Great Pyramid of Giza, in Egypt, was built over 4500 years ago. It was constructed using approximately 2 300 000 rectangular granite blocks. This construction took over 20 years to complete. When it was finished, one side of its square base was 230 m long and its vertical height was 146.5 m.

1. Each side of the Great Pyramid of Giza has a triangular face. Use Pythagoras' theorem and the dimensions provided to calculate the height of each triangular face. Give your answer correct to 2 decimal places.
2. Special finishing blocks were added to the ends of each row of stones that were used to make the pyramid. These finishing blocks gave each triangular face a smooth, flat finish. Calculate the area of each face of the pyramid.

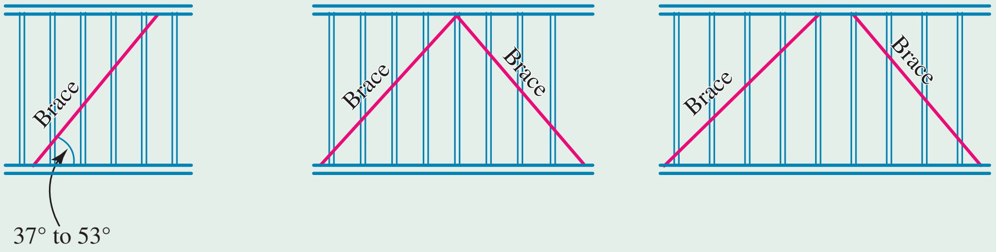


3. The edges of the pyramid join two of its faces and run all the way from the ground to the pyramid's tip. Use Pythagoras' theorem to calculate the length of these edges.

Wall braces

Builders use braces to strengthen wall frames. These braces typically run between the top and bottom horizontal sections of the wall frame.

Building industry standards require that the acute angle that a brace makes with the horizontal sections needs to be somewhere between 37° and 53°. Sometimes more than one brace may be required if the frame is particularly long, as shown in the diagram.



1. Cut out some thin strips of cardboard and arrange them in the shape of a rectangle to represent a wall frame. Place pins at the corners of the rectangle to hold the strips together.
   Notice that the frame can easily be moved out of shape. Attach a brace to this frame according to the building industry standards discussed.
   Write a brief comment to describe what effect this brace has on the frame.
2. Investigate what happens to the length of the brace as the acute angle it creates with the base is increased from 37° to 53°.
3. Use your findings from question **2** to calculate the angle that requires the shortest brace and the angle that requires the longest brace.

Most modern houses are constructed with a ceiling height (the height of the walls from floor to ceiling) of 2.4 m. Use this information to help with your calculations for the following questions.

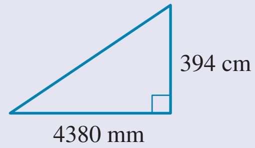
4. Assume you are working with a section of wall that is 3.5 m long. Calculate the length of the longest possible brace.
   Draw a diagram and show your working to support your answer.
5. Calculate the minimum wall length for which two braces are required. Draw a diagram and show your working to support your answer.
6. Some older houses have ceilings that are over 2.4. m high. Answer the questions in **4** and **5** for a house with a ceiling that is 3 m high.
   Draw diagrams and show your workings to support your answers.
7. Take the measurements of a wall with no windows at your school or at home. Draw a scale drawing of the frame of this wall and show where the brace or braces for this wall might lie.
   Calculate the length and angle of each brace.

Exercise 8.8 Review questions

learnon

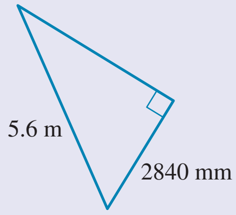
Fluency

1. MC Identify the length of the third side in the triangle shown.



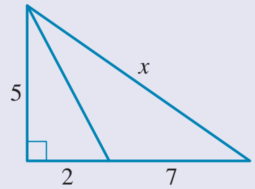
A. 34.71 m **B.** 2.96 m
C. 5.89 m **D.** 1722 cm

2. MC Select the most accurate measure for the length of the third side in the triangle shown.



A. 4.83 m **B.** 23.3 cm
C. 3.94 m **D.** 4826 mm

3. MC Select the value of x in this figure.

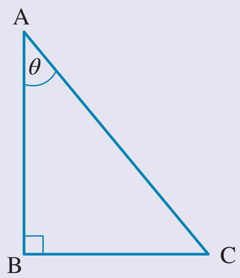


A. 5.4 **B.** 7.5
C. 10.1 **D.** 10.3

4. MC Identify which of the following is not a Pythagorean triad.

A. 3, 4, 5 **B.** 6, 8, 10
C. 5, 12, 13 **D.** 2, 3, 4

5. MC Identify which of the following correctly names the sides and angle of the triangle shown.

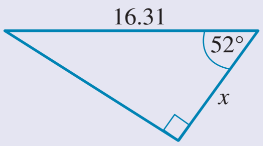


A. $\angle C = \theta$, AB = adjacent side, AC = hypotenuse, BC = opposite side
B. $\angle C = \theta$, AB = opposite side, BC = hypotenuse, AC = adjacent side
C. $\angle A = \theta$, AB = opposite side, AC = hypotenuse, BC = adjacent side
D. $\angle A = \theta$, AB = adjacent side, AC = hypotenuse, BC = opposite side

6. MC Select which of the following statements is correct.

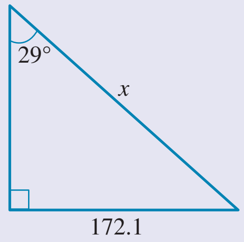
A. $\sin(60°) = \cos(60°)$ **B.** $\cos(25°) = \cos(65°)$
C. $\cos(60°) = \sin(30°)$ **D.** $\sin(70°) = \cos(70°)$

7. MC Identify the value of x in the triangle shown, correct to 2 decimal places.



A. 26.49 **B.** 10.04 **C.** 12.85 **D.** 20.70

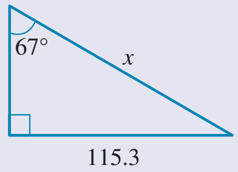
8. **MC** Identify which of the following could be used to calculate the value of x in the triangle shown.



A. $x = \dfrac{172.1}{\cos(29°)}$ **B.** $x = \dfrac{172.1}{\sin(29°)}$

C. $x = 172.1 \times \sin(29°)$ **D.** $x = 172.1 \times \cos(29°)$

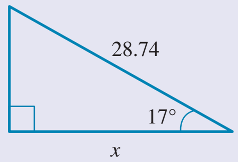
9. **MC** Identify which of the following could be used to calculate the value of x in the triangle shown.



A. $x = \dfrac{115.3}{\sin(23°)}$ **B.** $x = \dfrac{115.3}{\cos(67°)}$

C. $x = 115.3° \times \sin(67°)$ **D.** $x = \dfrac{115.3}{\cos(23°)}$

10. **MC** Identify which of the following could be used to calculate the value of x in the triangle shown.



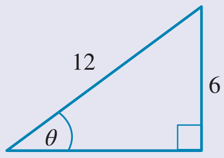
A. $x = \dfrac{28.74}{\cos(17°)}$ **B.** $x = 28.74 \times \sin(17°)$

C. $x = 28.74 \times \cos(17°)$ **D.** $x = 28.74 \times \cos(73°)$

11. **MC** Select which of the following is closest to the value of $\tan^{-1}(1.8931)$.

A. 62° **B.** 0.0331° **C.** 1.08° **D.** 69°

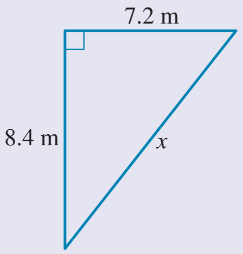
12. **MC** Select the value of θ in the triangle shown, correct to 2 decimal places.



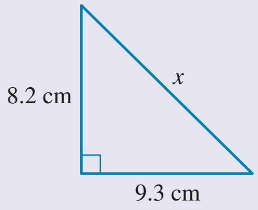
A. 40.89° **B.** 60° **C.** 35.27° **D.** 30°

13. Calculate x, correct to 2 decimal places.

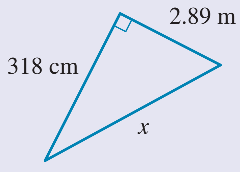
a.



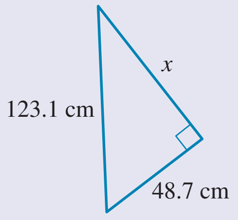
b.

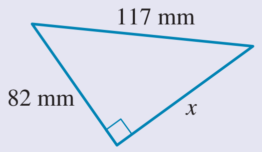


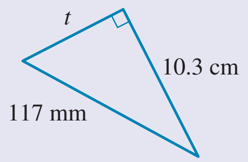
c.



14. Calculate x, correct to 2 decimal places.

a. 

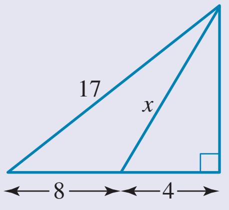
b. 

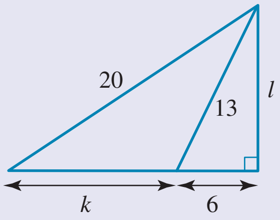
c. 

15. The top of a kitchen table measures 160 cm by 90 cm. A beetle walks diagonally across the table from one corner to the other. Calculate how far it walks. Give your answer in cm correct to 2 decimal places.

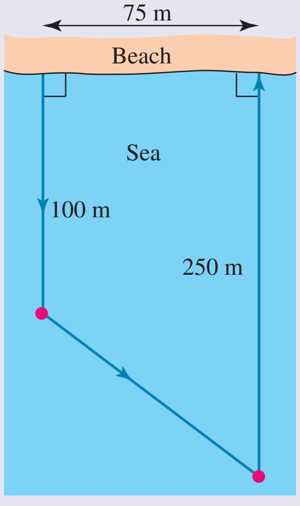
16. A broom leans against a wall. The broom is 1.5 m long and reaches 1.2 m up the wall. Calculate how far the bottom of the broom is from the base of the wall.
Write your answer in metres correct to 1 decimal place.

17. Calculate the unknown values in the figures shown. Give your answers correct to 2 decimal places.

a. 

b. 

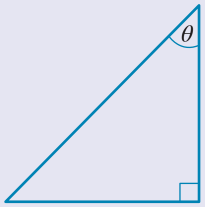
18. A beach athletics event involves three swimming legs and a run along the beach back to the start of the first swimming leg. Some of the distances of the race course are shown.
Calculate the total distance covered in the race, correct to the nearest metre.

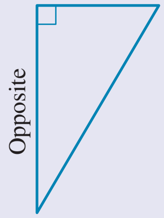


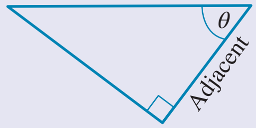
19. True or False? The following are Pythagorean triads.

a. 15, 36, 39
b. 50, 51, 10
c. 50, 48, 14

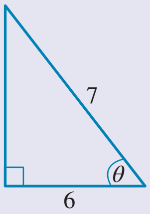
20. Label the unlabelled sides of the following right-angled triangles using the symbol θ and the words 'hypotenuse', 'adjacent' and 'opposite' where appropriate.

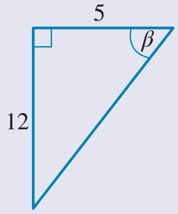
a.


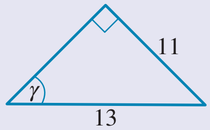
b.


c.


21. Write a trigonometric ratio, in fraction form, that connects the lengths of the given sides and the size of the given angle in the following triangles.
Use your ratio to then calculate the size of the angle, correct to the nearest degree.

a.


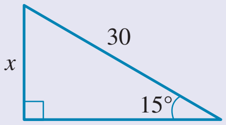
b.


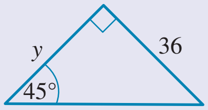
c.


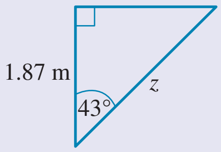
22. Use a calculator to evaluate the following trigonometric ratios, correct to 4 decimal places.

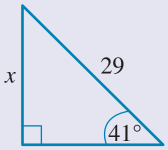
a. $\sin(54°)$
b. $\cos(39°)$
c. $\tan(12°)$

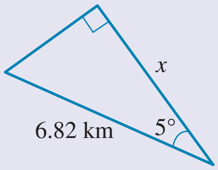
23. Calculate the values of the pronumerals in each of the following triangles. Give your answers correct to 2 decimal places.

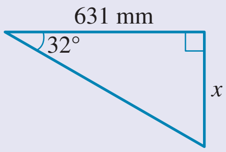
a.


b.


c.


d.


e.


f.


24. Evaluate each of the following, correct to the nearest degree.

a. $\sin^{-1}(0.1572)$
b. $\cos^{-1}(0.8361)$
c. $\tan^{-1}(0.5237)$

25. Calculate the size of the angle in each of the following. Give your answers correct to the nearest degree.

a. $\sin(\theta) = 0.5321$
b. $\cos(\theta) = 0.7071$
c. $\tan(\theta) = 0.8235$
d. $\cos(\alpha) = 0.3729$
e. $\tan(\alpha) = 0.5774$
f. $\sin(\beta) = 0.8660$
g. $\cos(\beta) = 0.5050$
h. $\tan(\beta) = 8.3791$

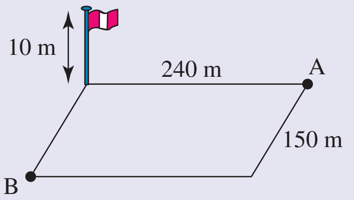
26. If an angle of $\theta = 290°$ was represented on the unit circle, state which quadrant the triangle to show this would be drawn in.

27. On the unit circle, draw an appropriate triangle for the angle 110° in quadrant 2.

a. Determine the value of $\sin(110°)$ and $\cos(110°)$, correct to 2 decimal places.
b. Determine the value of $\tan(110°)$, correct to 2 decimal places.

Problem solving

28. A 10-metre-high flagpole stands in the corner of a rectangular park that measures 240 m by 150 m.



a. Determine the following, in metres correct to 2 decimal places.

  i. The length of the diagonal of the park

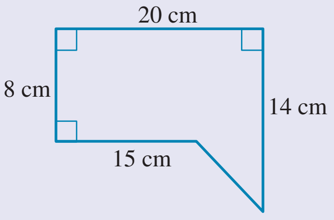
  ii. The distance from A to the top of the pole

  iii. The distance from B to the top of the pole

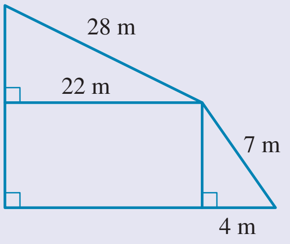
b. A bird flies from the top of the pole to the middle of the park. Determine far the bird has to fly to reach this point, in metres correct to 2 decimal places.

29. Calculate the perimeters of the shapes shown, round your answers to 2 decimal places.

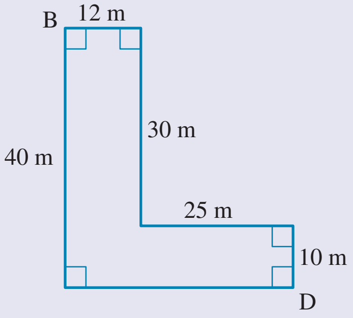
a.



b.



30. Evaluate the length of the shortest distance between points B and D inside the figure shown. Write your answer in metres correct to 2 decimal places.



31. Calculate the angles of a triangle whose sides can be described using the Pythagorean triad 3, 4, 5. Round your answers to the nearest degree.

32. A stack of chairs is 2 m tall. The stack needs to fit through a doorway that is 1.8 m high. The maximum angle that the stack of chairs can be safely tilted is 25° to the vertical.
Based on this information, determine if it is safe to try to move the stack of chairs through the doorway.

33. Evaluate the side length of the largest square that can be drawn within a circle that has a radius of r. Give your answer as a surd.

on To test your understanding and knowledge of this topic go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

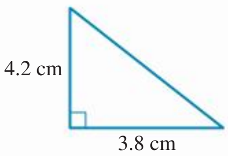
Topic 8 Pythagoras and trigonometry

8.1 Pre-test

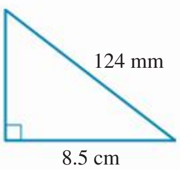
1. 5 cm
2. 30.96 m
3. B
4. $a + b$
5. C
6. D
7. D
8. a. 3.47 cm b. 5.14 mm
9. a. 0.8870 b. 0.9778 c. 3.7062
10. D
11. 23.4°
12. A
13. D
14. 7.31 m
15. Minimum angle of inclination: 26.39°, corresponding jumping height: 53.33 cm
    Maximum angle of inclination: 30°, corresponding jumping height: 60 cm

8.2 Pythagoras' theorem

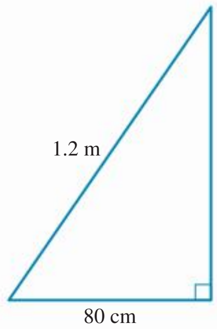
1. a. i. r
      ii. $r^2 = p^2 + s^2$
   b. i. x
      ii. $x = y^2 + z^2$
   c. i. k
      ii. $k^2 = m^2 + w^2$
   d. i. FU
      ii. $(\text{FU})^2 = (\text{VU})^2 + (\text{VF})^2$
2. a. 7.86 b. 33.27 c. 980.95
3. a. $x = 12.49$ b. $p = 11.76$ cm
   c. $f = 5.14$ m d. $c = 97.08$ mm
4. 10.2 cm
5. a. No b. No c. Yes
   d. Yes e. No f. Yes
6. a. No b. Yes
7. a.



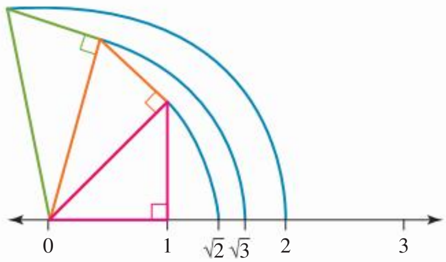
   b. 5.66 cm
8. a.



   b. 90.28 mm
9. B
10. C
11. B
12. a. 176.16 cm b. 147.40 cm
    c. 2.62 km d. 432.12 m
13. $22 + 4\sqrt{11}$ cm
14. 2.60 m
15. The length of the diagonal can be calculated using Pythagoras' theorem.
    $c^2 = a^2 + b^2$
    $c^2 = 260^2 + 480^2$
    $c^2 = 298\,000$
    $c = \sqrt{298\,000} = 546$ m
    The length of the triangular circuit is
    260 m + 480 m + 546 m = 1286 m.
    5 laps of this circuit is $5 \times 1286 = 6430$ m.
    5 laps of the regular circuit is
    $5 \times (260 + 260 + 480 + 480) = 7400$ m.
    7400 m − 6430 m = 970 m.
16. The horizontal distance is 11.74 m, so the gradient is 0.21, which is within the limits.
17. 4.90 cm^2
18. a.



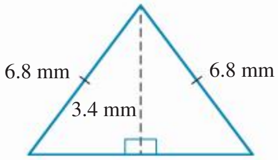
    b. 89.44 cm
    c. Yes, she can reach the hook from the top step.
19. a. i. {9, 40, 41} ii. {11, 60, 61}
       iii. {13, 84, 85} iv. {29, 420, 421}
    b. The middle number and the large number are one number apart.
20. The following figure shows the first three triangles, with the values of $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{4}$.



8.3 Applications of Pythagoras' theorem

1. 12.08 cm
2. a. $k = 16.40$ m b. $x = 6.78$ cm c. $g = 4.10$ km
3. a. $x = 4$, $y = 9.17$ b. $x = 6.93$, $y = 5.80$
   c. $x = 13$, $y = 15.20$ d. $x = 2.52$, $y = 4.32$

4. a.



b. 11.78 mm

c. $20.02\ \text{mm}^2$

5. a. $x = \sqrt{7}$ cm b. $y = \sqrt{57}$ cm

c. $P = 15 + \sqrt{57}$ cm d. $A = 13.96\ \text{cm}^2$

6. $259.81\ \text{cm}^2$

7. a. 30.48 cm b. 2.61 cm c. 47.27 cm

8. a. 0.87 m b. $0.433\ \text{m}^2$

9. D

10. a. Yes.

b. The longest umbrella that can fit in the suitcase is an umbrella that is 101.48 cm long.

11. a. No.

b. The longest straw that can fit in the lunchbox is a straw that is 17.72 cm long.

12. 242 cm

13. a. $w = 0.47$ m b. $0.64\ \text{m}^2$

c. $12.79\ \text{m}^2$ d. \$89.85

14. Width = 12 mm, area = $480\ \text{mm}^2$

15. 179 mm

16. a. No, AC is calculated by summing the lengths of AD and DC, so these lengths have to be calculated first.

b. AD = 4.47 cm, DC = 9.17 cm, AC = 13.64 cm

c. The triangle ABC is not right-angled because $(AB)^2 + (BC)^2 \neq (AC)^2$.

17. Even though a problem may be represented in 3D, right-angled triangles in 2D can often be found within the problem. This can be done by drawing a cross-section of the shape or by looking at individual faces of the shape.

18. Sample responses can be found in the worked solutions in the online resources.

19. a, b. Sample responses can be found in the worked solutions in the online resources.

20. a. 2606 mm

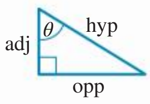
b. Sample responses can be found in the worked solutions in the online resources.

21. 16.7 m

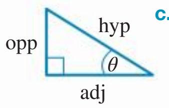
22. $w = 3.536$ m, $x = 7.071$ cm, $y = 15.909$ cm, $z = 3.536$ cm

8.4 Trigonometric ratios

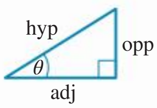
1. a.



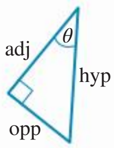
b.



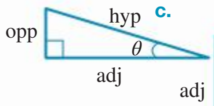
c.



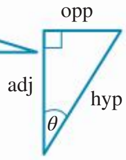
2. a.



b.



c.



3. a. DE = hypotenuse DF = opposite $\angle E = \theta$

b. GH = hypotenuse IH = adjacent $\angle H = \theta$

c. JL = hypotenuse KL = opposite $\angle J = \theta$

4. D

5. $\sin(\theta) = \frac{4}{5}$, $\cos(\theta) = \frac{3}{5}$, $\tan(\theta) = \frac{4}{3}$

6. $\sin(\alpha) = \frac{i}{g}$, $\cos(\alpha) = \frac{h}{g}$, $\tan(\alpha) = \frac{i}{h}$

7. $\sin(\beta) = 0.8$, $\cos(\beta) = 0.6$, $\tan(\beta) = 1.3$

8. $\sin(\gamma) = \frac{24}{25}$, $\cos(\gamma) = \frac{7}{25}$, $\tan(\gamma) = \frac{24}{7}$

9. $\sin(\beta) = \frac{b}{c}$, $\cos(\beta) = \frac{a}{c}$, $\tan(\beta) = \frac{b}{a}$

10. $\sin(\gamma) = \frac{v}{u}$, $\cos(\gamma) = \frac{t}{u}$, $\tan(\gamma) = \frac{v}{t}$

11. a. $\sin(\theta) = \frac{12}{15} = \frac{4}{5}$ b. $\cos(\theta) = \frac{25}{30} = \frac{5}{6}$

c. $\tan(\theta) = \frac{4}{5}$ d. $\sin(35°) = \frac{17}{t}$

12. a. $\sin(\alpha) = \frac{14.3}{17.5}$ b. $\sin(15°) = \frac{7}{x}$

c. $\tan(\theta) = \frac{20}{31}$ d. $\cos(\alpha) = \frac{3.1}{9.8}$

13. D

14. B

15. D

16. Provided n is a positive value, $(m + n)$ would be the hypotenuse, because it has a greater value than both m and $(m - n)$.

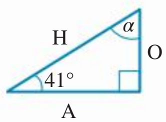
17. a. Sample responses can be found in the worked solutions in the online resources.

b. 1

18. The ratios remain constant when the angle is unchanged. The size of the triangle has no effect.

19. a. Ground b. Ladder c. Brick wall

20. a.



b. i. $\sin(41°) = 0.65$ ii. $\cos(41°) = 0.76$

iii. $\tan(41°) = 0.86$

c. $\alpha = 49°$

d. i. $\sin(49°) = 0.76$ ii. $\cos(49°) = 0.65$

iii. $\tan(49°) = 1.16$

e. They are equal.

f. They are equal.

g. The sine of an angle is equal to the cosine of its complement.

21. a. The ratio of the length of the opposite side to the length of the hypotenuse will increase.

b. The ratio of the length of the adjacent side will decrease, and the ratio of the opposite side to the adjacent will increase.

c. i. 1 ii. 1 iii. ∞

8.5 Calculating unknown side lengths

1. a. i. 0.8192
   ii. 0.2011

   b.

θ	0°	15°	30°	45°	60°	75°	90°
$\sin(\theta)$	0	0.26	0.50	0.71	0.87	0.97	1.00

   c. As θ increases, so does $\sin(\theta)$, starting at 0 and increasing to 1.

2. a. i. 0.7880
   ii. 0.5919

   b.

θ	0°	15°	30°	45°	60°	75°	90°
$\cos(\theta)$	1.00	0.97	0.87	0.71	0.50	0.26	0

   c. As θ increases, $\cos(\theta)$ decreases, starting at 1 and decreasing to 0.

3. a. i. 0.3249
   ii. 1.2753

   b.

θ	0°	15°	30°	45°	60°	75°	90°
$\tan(\theta)$	0	0.27	0.58	1.00	1.73	3.73	Undefined

   c. $\tan(89°) = 57.29$, $\tan(89.9°) = 572.96$
   d. As θ increases, $\tan(\theta)$ increases, starting at 0 and becoming very large. There is no value for $\tan(90°)$.

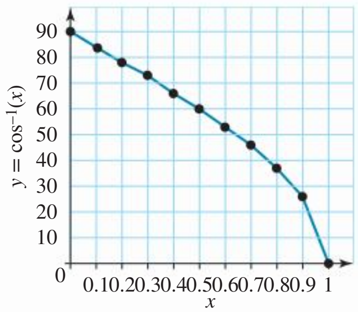
4. a. 13.02 m b. 7.04 m c. 27.64 mm
5. a. 2.79 cm b. 6.27 m c. 14.16 m
6. a. 2.95 cm b. 25.99 cm c. 184.73 cm
7. a. 14.06 km b. 8.43 km c. 31.04 m
8. a. 26.96 mm b. 60.09 cm c. 0.84 km
9. a. 0.94 km b. 5.59 m c. 41.67 m
10. a. 54.73 m b. 106.46 cm c. 298.54 mm
11. a. $a = 17.95$, $b = 55.92$
    b. $a = 15.59$, $b = 9.00$, $c = 10.73$
    c. $a = 12.96$, $b = 28.24$, $c = 15.28$
12. D
13. B
14. A
15. D
16. a. 275.75 km b. 48.62 km
17. 21.32 m
18. 285.63 m
19. Sample responses can be found in the worked solutions in the online resources.
20. a. Sample responses can be found in the worked solutions in the online resources.
    b. Sample responses can be found in the worked solutions in the online resources.
    c. $AC = \dfrac{AB}{\tan(\theta)}$
21. Sample responses can be found in the worked solutions in the online resources.
22. a. $x = 12.87$ m b. $h = 3.01$ m c. $x = 2.60$ m
23. $w = 41$ mm
24. a. 5.9 m b. 5.2 m
25. 4 m

8.6 Calculating unknown angles

1. a. 39° b. 72° c. 37°
2. a. 53° b. 69° c. 71°
3. a. 79° b. 77° c. 15°
4. a. 19° b. 42° c. 55° d. 21°
5. a. 49° b. 80° c. 35° d. 45°
6. a. 41° b. 23° c. 58° d. 80°
7. a. 47° b. 45° c. 24°
8. a. 43° b. 45° c. 18°
9. a. 26° b. 12° c. 76°
10. D
11. B
12. D
13. C
14. 30.56°
15. 21.80°
16. a.

x	$y = \cos^{-1}(x)$
0.0	90°
0.1	84°
0.2	78°
0.3	73°
0.4	66°
0.5	60°
0.6	53°
0.7	46°
0.8	37°
0.9	26°
1.0	0°

    b.



17. 18.43° and $m = 0.33$.
18. a. Sample responses can be found in the worked solutions in the online resources.
    b. Sample responses can be found in the worked solutions in the online resources.

19. a. 2.31 m

b. Sample responses can be found in the worked solutions in the online resources.

20. Sample responses can be found in the worked solutions in the online resources.

21. a. $\theta = 26.01°$ b. $\theta = 32.01°$ c. $x = 22.62°$

22. a. The roof would not be high enough.

b. 136°

23. 54.74°

24. Large square dimensions: length $= 5$ cm, width $= 5$ cm
Large triangle dimensions: base $= 5$ cm, height $= 5$ cm

Small square dimensions: length $= \dfrac{5\sqrt{2}}{2}$ cm,

width $\dfrac{5\sqrt{2}}{2}$ cm

Small triangle dimensions: base $= \dfrac{5\sqrt{2}}{2}$ cm, height $\dfrac{5\sqrt{2}}{2}$ cm

Parallelogram dimensions: height $= 5$ cm, length $5\sqrt{2}$ cm

8.7 The unit circle (optional)

1. a. 1st b. 2nd c. 4th d. 3rd e. 2nd f. 3rd

2. A

3. D

4. a. 0.34 b. 0.94 c. −0.17 d. 0.98

5. a. −0.64 b. 0.77 c. −0.57 d. −0.82

6. a. 1 b. 0 c. 0 d. −1

7. a. −1 b. 0 c. 0 d. 1

8. a. 0.87 b. 0.50

9. a. 30° b. −0.87
c. $\cos(150°) = -\cos(30°)$ d. 0.5
e. $\sin(150°) = \sin(30°)$

10. a. 30° b. −0.87
c. $\cos(210°) = -\cos(30°)$ d. −0.50
e. $\sin(210°) = -\sin(30°)$

11. a. 30° b. 0.87
c. $\cos(330°) = -\cos(30°)$ d. −0.50
e. $\sin(330°) = -\sin(30°)$

12. a. 0.34 b. 0.94
c. 0.36 d. 0.36
e. They are equal.

13. a. 0.71 b. −0.71
c. −1 d. −1
e. They are equal. f. $\tan(135°) = -\tan(45°)$

14. a. −0.64 b. −0.77
c. 0.84 d. 0.83
e. They are approximately equal.
f. $\tan(220°) = \tan(40°)$

15. a. −0.87 b. 0.5
c. −1.73 d. −1.74
e. They are approximately equal.
f. $\tan(300°) = -\tan(60°)$

16. D

17, 18, 19. Sample responses can be found in the worked solutions in the online resources.

20. a. $\sqrt{1-p^2}$ b. p
c. $-\sqrt{1-p^2}$ d. p

21. 0

22. 0

Project

The Great Pyramid of Giza

1. 186.25 m

2. 21 418.75 m^2

3. 218.89 m

Wall braces

1. Sample responses can be found in the worked solutions in the online resources.

2. Sample responses can be found in the worked solutions in the online resources.

3. 53° requires the shortest brace and 37° requires the longest brace.

4. 4.24 m

5. 3.62 m

6. 4.61 m, 4.52 m

7. Sample responses can be found in the worked solutions in the online resources.

8.8 Review questions

1. C

2. D

3. D

4. D

5. D

6. C

7. B

8. B

9. D

10. C

11. A

12. D

13. a. 11.06 m
b. 12.40 cm
c. 429.70 cm or 4.30 m

14. a. 113.06 cm
b. 83.46 mm
c. 55.50 mm or 5.55 cm

15. 183.58 cm

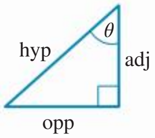
16. 0.9 m

17. a. $x = 12.69$
b. $l = 11.53$, $k = 10.34$

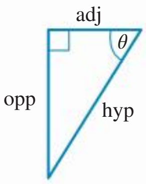
18. 593 m

19. a. True b. False c. True

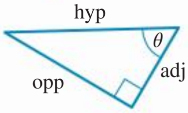
20. a.



b.



c.



21. a. $\cos(\theta) = \frac{6}{7}$

$\theta = 31°$

b. $\tan(\beta) = \frac{12}{5}$

$\beta = 67°$

c. $\sin(\gamma) = \frac{11}{13}$

$\gamma = 58°$

22. a. 0.8090 b. 0.7771 c. 0.2126

23. a. 7.76 b. 36.00 c. 2.56 m
d. 19.03 e. 6.79 km f. 394.29 mm

24. a. 9° b. 33° c. 28°

25. a. 32° b. 45° c. 39°
d. 68° e. 30° f. 60°
g. 60° h. 83°

26. 4^{th} quadrant

27. a. 0.94, −0.34 b. −2.75

28. a. i. 283.02 m
ii. 240.21 m
iii. 150.33 m
b. 141.86 m

29. a. 64.81 cm b. 84.06 m

30. 59.24 m

31. 90°, 53° and 37°

32. No. The stack of chairs must be tilted by 25.84° to fit through the doorway, which is more than the safe angle of 25°.

33. $\frac{2r}{\sqrt{2}} = \sqrt{2}r$

STRAND: Measurement and geometry
SUB-STRANDS: Two-dimensional space and structures | Three-dimensional space and structures

9 Measurement

LESSON SEQUENCE

LESSON 9.1 Overview

9.1.1 Why learn this?

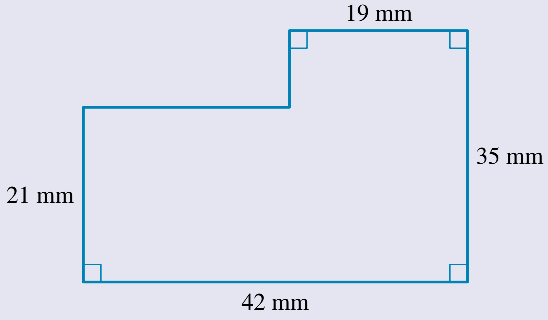
We live in a world surrounded by shapes and objects. Often, we ask questions such as ‘how long?’, ‘how far?’ or ‘how big?’. These questions are all answered using measurement. Measurements may be in one, two or three dimensions. Examples of measurements in one dimension would be length and perimeter, temperature and time. Measurements in two dimensions include areas of shapes including the areas of curved surfaces of objects such as drink cans. Measurements in three dimensions include volumes of objects and capacity, such as the number of millilitres in a drink can.

Since objects are all around us, measurement skills are used in many real-world situations. Carpenters, builders, concreters, landscape gardeners and construction workers are just a few of the many tradespeople who need to understand various aspects of measurement when ordering or working onsite and following plans. Designers, interior decorators and architects use measurement in their drawings and calculations. Chefs measure ingredients in their cooking. Nurses and health professionals follow instructions regarding the amount of a drug to administer to a patient. To maximise profits, manufacturers need to minimise the amount of raw materials used in production. This means knowing the measurements of various parts. Understanding the basic concepts involved in measurement is beneficial in many real-world situations.

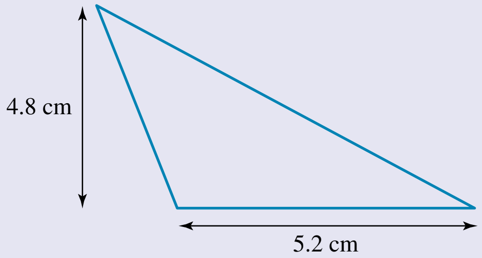
Exercise 9.1 Pre-test

learnon

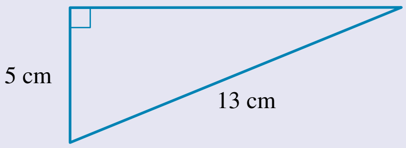
1. Calculate the perimeter of this figure, giving your answer in centimetres.



2. Calculate the area of the triangle, correct to 2 decimal places.

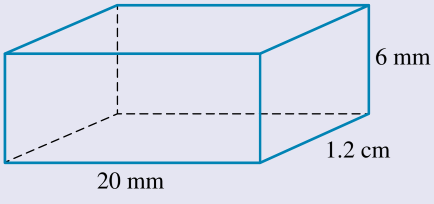


3. Calculate the perimeter of the triangle shown.



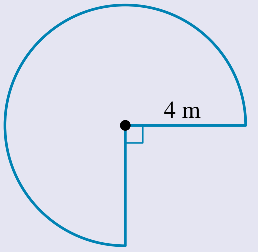
4. MC $0.005\ \text{m}^2$ is equivalent to:
   A. $0.000\,05\ \text{cm}^2$
   B. $0.5\ \text{cm}^2$
   C. $50\ \text{cm}^2$
   D. $0.005\ \text{cm}^2$

5. Calculate the volume of the cuboid shown in mm^3.



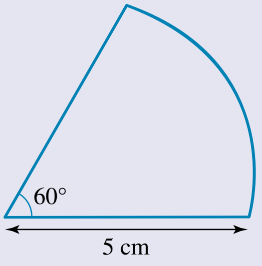
6. **MC** Select the correct formula to calculate the perimeter of the shape shown.

A. $P = \pi \times 4^2$
B. $P = 2 \times \pi \times 4^2$
C. $P = 8 \times \pi$
D. $P = 6 \times \pi + 8$



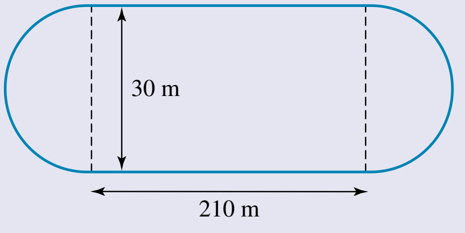
7. **MC** Select the correct formula to calculate the area of the sector shown.

A. $A = \dfrac{60}{360} \times \pi \times 5$
B. $A = \dfrac{60}{360} \times \pi \times 25$
C. $A = \dfrac{60}{300} \times \pi \times 25$
D. $A = \dfrac{60}{360} \times 2 \times \pi \times 5$



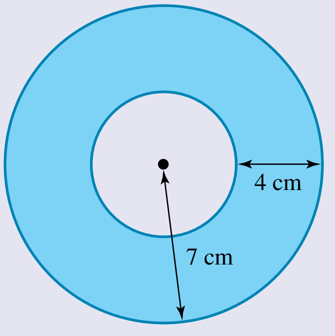
8. **MC** Select the perimeter of the running track shown, correct to 2 decimal places.

A. 94.25 m
B. 304.24 m
C. 467.12 m
D. 514.25 m

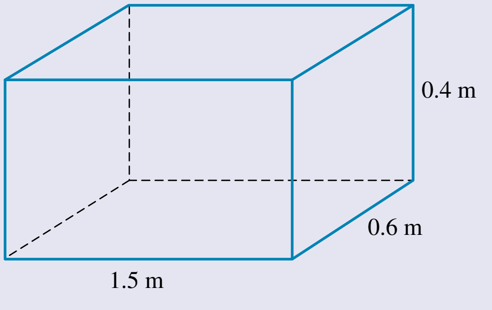


9. **MC** Select the correct formula to calculate the area of the annulus.

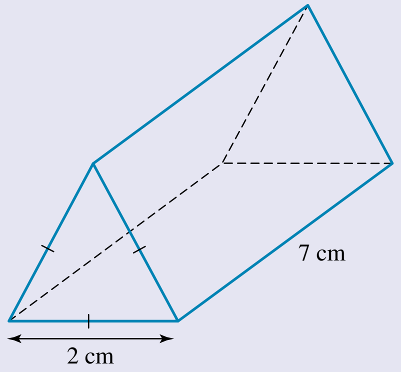
A. $A = \pi(7^2 - 4^2)$
B. $A = \pi(7^2 - 3^2)$
C. $A = \pi \times 4^2$
D. $A = 2 \times \pi \times (7 - 3)$



10. Calculate the surface area (SA) of the rectangular prism, giving your answer correct to 2 decimal places.



11. **MC** Select the correct formula to calculate the surface area of the triangular prism.



A. $SA = \frac{1}{2} \times 2 \times 2 \times 7$

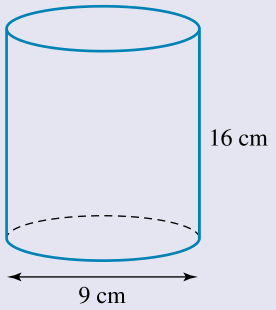
B. $SA = \frac{1}{2} \times 2 \times 2 \times 7 \times 3$

C. $SA = \frac{1}{2} \times 2 \times 2 \times 2 + 3 \times 7 \times 2$

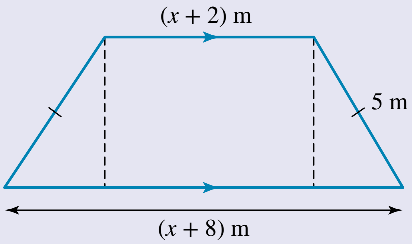
D. $SA = 2 \times \sqrt{3} + 3 \times 7 \times 2$

12. Convert 3.1 L into cm^3.

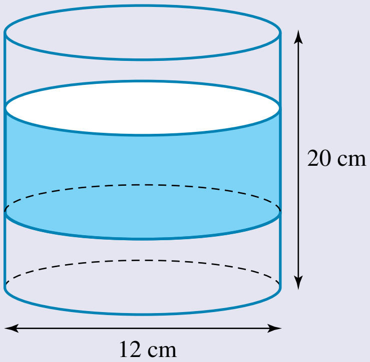
13. Calculate the capacity (in mL) of two-thirds of the soup can shown, correct to 1 decimal place.



14. A garden bed is designed in the shape of a trapezium with the dimensions shown. If the area of the garden bed is $30\,m^2$, calculate its perimeter.



15. A label is made for a tin as shown in the diagram. The label is symmetrically positioned so that it is 3 cm from the top and bottom of the tin. Determine the area of paper required to make the label, correct to 1 decimal place.



LESSON 9.2 Perimeter and circumference

LEARNING INTENTIONS

At the end of this lesson you should be able to:

- convert between different units of length
- calculate the perimeter of a given shape
- calculate the circumference of a circle.

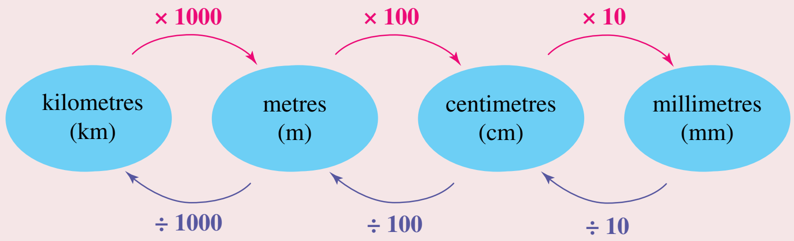
9.2.1 Length

- In the **metric system**, units of length are based on the metre. The following units are commonly used.

millimetre	(mm)	one-thousandth of a metre
centimetre	(cm)	one-hundredth of a metre
metre	(m)	one metre
kilometre	(km)	one thousand metres

Converting units of length

The following chart is useful when converting from one unit of length to another.



For example, $36\,000 \text{ mm} = 36\,000 \div 10 \div 100 \text{ m}$

$= 36 \text{ m}$

WORKED EXAMPLE 1 Converting units of length

Convert the following lengths into cm.

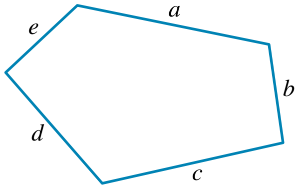
a. **37 mm**

b. **2.54 km**

THINK	WRITE
a. There will be fewer cm, so divide by 10.	**a.** $37 \div 10 = 3.7 \text{ cm}$
b. There will be more cm, so multiply. km → m: $\times 1000$ m → cm: $\times 100$	**b.** $2.54 \times 1000 = 2540 \text{ m}$ $2540 \times 100 = 254\,000 \text{ cm}$

9.2.2 Perimeter

- The **perimeter** of a plane (flat) figure is the distance around the outside of the figure.
- If the figure has straight edges, then the perimeter can be found by simply adding all the side lengths.
- Ensure that all lengths are in the same unit.



Perimeter $= a + b + c + d + e$

Circumference

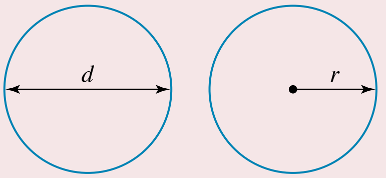
- **Circumference** is a special name given to the perimeter of a circle.
- Circumference is calculated using the formula:

Circumference of a circle

$C = \pi d$, where d is the diameter of the circle

or

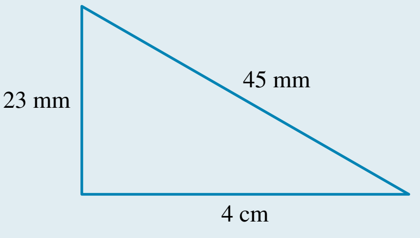
$C = 2\pi r$, where r is the radius of the circle.



- Use the exact value of π unless otherwise directed.

WORKED EXAMPLE 2 Calculating the perimeter of a shape

Calculate the perimeter of this figure. Give your answer in centimetres.

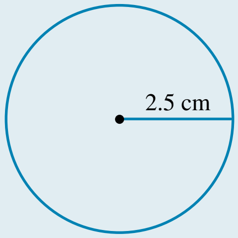


THINK	WRITE
1. State the lengths of the sides in the same unit of length. Convert cm to mm: $4\text{ cm} = 4 \times 10 = 40\text{ mm}$	23 mm, 45 mm, 40 mm
2. Add the side lengths together.	$P = 45 + 40 + 23$ $= 108\text{ mm}$
3. As the answer is required in cm, convert mm to cm by dividing 118 by 10.	$P = \frac{108}{10}$ $= 10.8\text{ cm}$
4. Write the answer in the correct unit.	The perimeter is 10.8 cm.

WORKED EXAMPLE 3 Calculating the circumference of a circle

Calculate the circumference of the circle. Give your answer correct to 2 decimal places.

a. Give your answer as an exact value.
b. Give your answer correct to 2 decimal places.

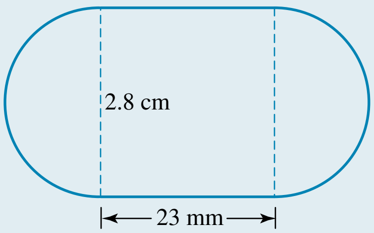


THINK	WRITE
1. The radius is known, so apply the formula $C = 2\pi r$.	$C = 2\pi r$
2. Substitute $r = 2.5$ into $C = 2\pi r$.	$C = 2 \times \pi \times 2.5$
3. Write your answer as an exact value.	$C = 5\pi$
4. Calculate the circumference to 4 decimal places and then round correct to 2 decimal places.	$C \approx 15.7079$ $C \approx 15.71$ cm

- Some shapes will contain sectors of circles. To calculate the perimeter of these shapes, determine what proportion of a circle is involved, and multiply this by the perimeter of the circle.

WORKED EXAMPLE 4 Determining the perimeter of a given shape

Determine the perimeter of the shape shown. Give your answer in cm correct to 2 decimal places.



THINK	WRITE
1. There are two straight sections and two semicircles. The two semicircles make up a full circle, the diameter of which is known, so apply the formula $C = \pi d$.	$C = \pi d$
2. Substitute $d = 2.8$ into $C = \pi d$ and give the answer to three decimal places.	$= \pi \times 2.8$ ≈ 8.796 cm
3. Convert 23 mm to cm (23 mm = 2.3 cm). The perimeter is the sum of all the outside lengths.	$P = 8.796 + 2 \times 2.3$ $= 13.396$
4. Round to 2 decimal places.	$P \approx 13.40$ cm

Exercise 9.2 Perimeter and circumference

learnon

9.2 Quick quiz on | 9.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 6, 7, 9, 12, 14, 20, 24	2, 4, 10, 13, 15, 17, 21, 25	5, 8, 11, 16, 18, 19, 22, 23, 26

Fluency

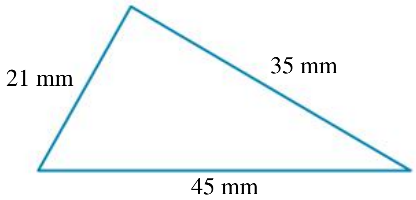
1. WE1 Convert the following lengths to the units shown.

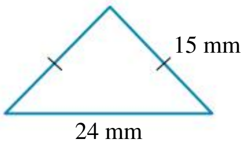
a. 5 cm = ______________ mm
b. 1.52 m = ______________ cm
c. 12.5 mm = ______________ m
d. 0.0322 m = ______________ mm

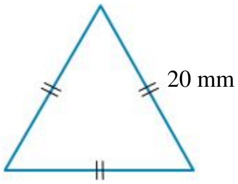
2. Convert the following lengths to the units shown.

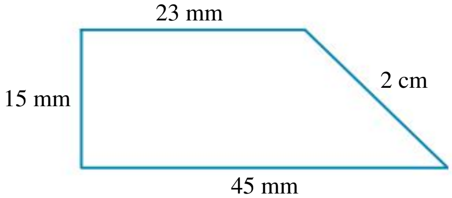
a. 6.57 m = ________________ km
b. 64 cm = ________________ km
c. 0.000 014 35 km = ________________ mm
d. 18.35 cm = ________________ km

3. WE2 Calculate the perimeter of the following figures, in millimetres.

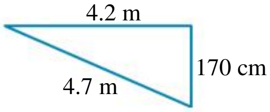
a.


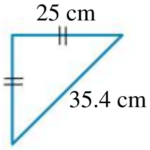
b.


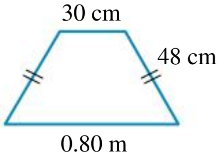
c.


d.


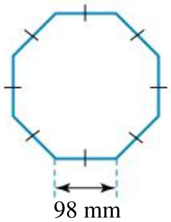
4. Calculate the perimeter of each of the following figures. Give your answers in centimetres.

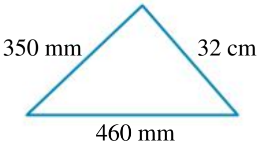
a.


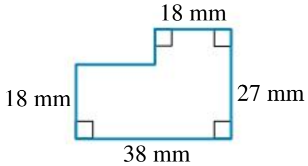
b.


c.


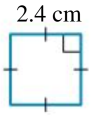
5. Calculate the perimeter of the following figures, in millimetres.

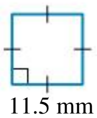
a.


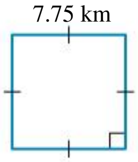
b.


c.


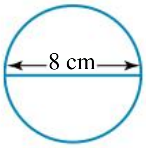
6. Calculate the perimeter of each of the squares.

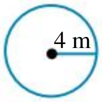
a.


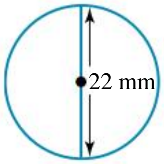
b.


c.


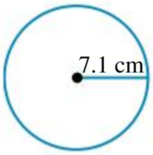
7. WE3 Calculate the circumference of the circles. Give your answer correct to 2 decimal places.

a.


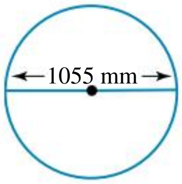
b.


c.


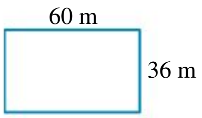
8. Calculate the circumference of the circles. Give your answer correct to 2 decimal places.

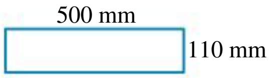
a.


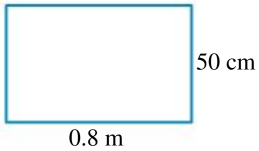
b.


c.


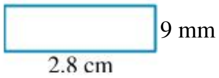
9. Calculate the perimeter of the rectangles shown.

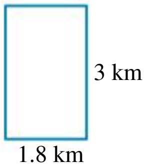
a.


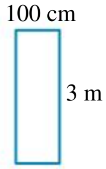
b.


c.


10. Calculate the perimeter of the rectangles shown.

a.


b.


c.


11. MC A circle has a radius of 34 cm. Its circumference, to the nearest centimetre, is:

A. 102 cm B. 214 cm C. 107 cm D. 3630 cm

Understanding

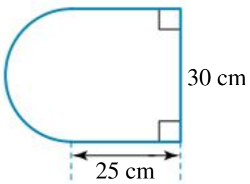
12. Timber is sold in standard lengths, which increase in 300-mm intervals from the smallest available length of 900 mm. (The next two standard lengths available are therefore 1200 mm and 1500 mm.)

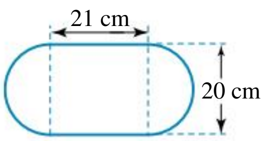
a. Write the next four standard lengths (after 1500 mm) in mm, cm and m.
b. Calculate the number of pieces 600 mm in length that could be cut from a length of timber that is 2.4 m long.
c. If I need to cut eight pieces of timber, each 41 cm long, determine the smallest standard length I should buy.
Note: Ignore any timber lost due to the cuts.

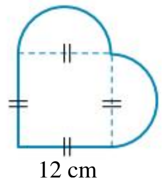
13. The world's longest bridge is the Akashi–Kaikyo Bridge, which links the islands of Honshu and Shikoku in Japan. Its central span covers 1.990 km.

a. State the length of the central span, in metres.
b. Calculate how much longer the span of the Akashi–Kaikyo Bridge is than the span of the Sydney Harbour Bridge, which is 1149 m.

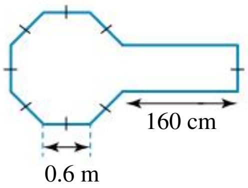
14. WE4 Determine the perimeter of the shapes shown. Give your answer correct to 2 decimal places.

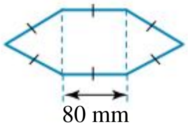
a.


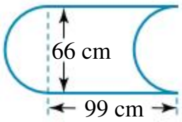
b.


c.


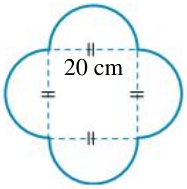
15. Determine the perimeter of the shapes shown. Give your answer correct to 2 decimal places.

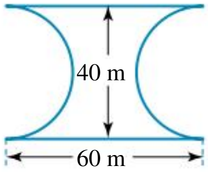
a. 

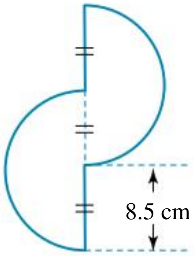
b. 

c. 

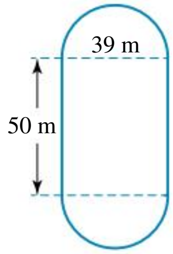
16. Determine the perimeter of the shapes shown. Give your answer correct to 2 decimal places.

a. 

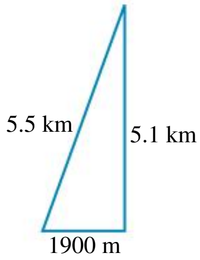
b. 

c. 

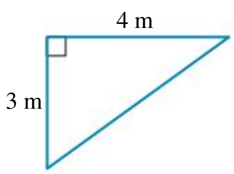
17. Determine the perimeter of the racetrack shown in the plan.



18. Yacht races are often run over a triangular course, as shown. Determine the distance covered by the yachts if they completed 3 laps of the course.



19. Use Pythagoras' theorem to calculate the length of the missing side and, hence, the perimeter of the triangular frame shown.



Reasoning

20. The Hubble Space Telescope is over 13 m in length. It orbits the Earth at a height of 559 km, where it can take extremely sharp images outside the distortion of the Earth's atmosphere.

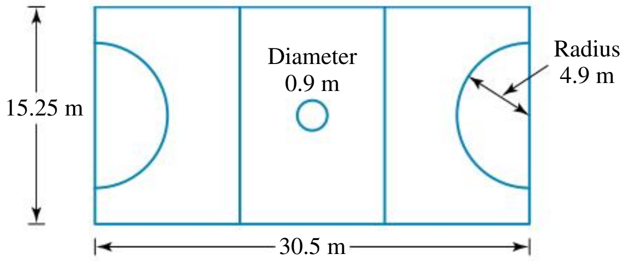
a. If the radius of the Earth is 6371 km, show that the distance travelled by the Hubble Space Telescope in one orbit, to the nearest km, is 43 542 km.

b. If the telescope completes one orbit in 96 minutes, show that its speed is approximately 7559 m/s.

21. A bullet can travel in air at 500 m/s.

a. Show how the bullet travels 50 000 cm in 1 second.
b. Calculate how long it takes for the bullet to travel 1 centimetre.
c. If a super-slow-motion camera can take 100 000 pictures each second, determine how many shots would be taken by this camera to show the bullet travelling 1 cm.

22. Edward is repainting all the lines of a netball court at the local sports stadium. The dimensions of the netball court are shown.



a. Calculate the total length of lines that need to be repainted.

Edward starts painting at 8 pm when the centre is closing, and it takes him $1\frac{1}{2}$ minutes on average to paint each metre of line.

b. Show that it will take him 233 minutes to complete the job.

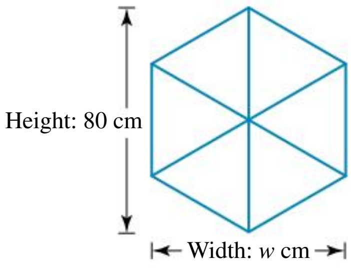
23. The radius of the Earth is accepted to be roughly 6400 km.

a. Calculate how far, to the nearest km, you travel in one complete rotation of the Earth.
b. As the Earth spins on its axis once every 24 hours, calculate the speed you are moving at.
c. If the Earth is 150 000 000 km from the Sun, and it takes 365.25 days to circle around the Sun, show that the speed of the Earth's orbit around the Sun is 107 515 km/h.
Give your answer to the nearest whole number.

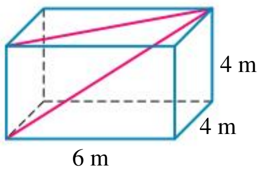
Problem solving

24. One-fifth of an 80 cm length of jewellery wire is cut off. A further 22 cm length is then removed. Evaluate whether there is enough wire remaining to make a 40 cm necklace.

25. A church needs to repair one of its hexagonal stained glass windows. Use the information given in the diagram to calculate the width of the window.



26. A spider is sitting in one top corner of a room that has dimensions 6 m by 4 m by 4 m. It needs to get to the corner of the floor that is diagonally opposite. The spider must crawl along the ceiling, then down a wall, until it reaches its destination.



a. If the spider crawls first to the diagonally opposite corner of the ceiling, then down the wall to its destination, calculate the distance it would crawl.
b. Determine the shortest distance from the top back corner to the lower left corner.

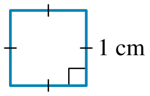
LESSON 9.3 Area

LEARNING INTENTIONS

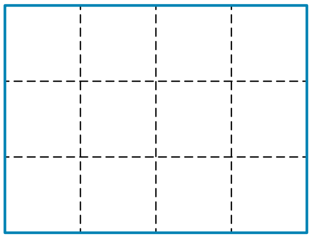
At the end of this lesson you should be able to:
- convert between units of area
- calculate areas of triangles, quadrilaterals and circles using formulas
- determine areas of composite shapes.

9.3.1 Area

- The diagram shows a square of side length 1 cm.
  By definition it has an area of 1 cm^2 (1 square centimetre).
  Note: This is a 'square centimetre', not a 'centimetre squared'.



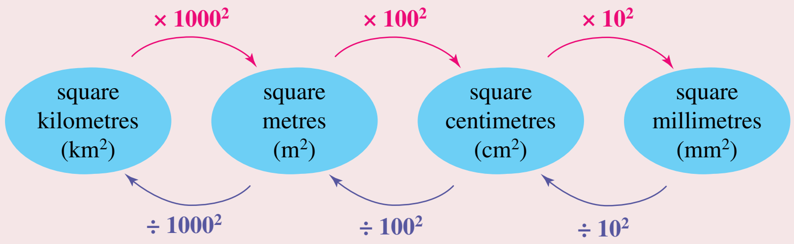
- Area tells us how many squares it takes to cover a figure, so the area of the rectangle shown is 12 cm^2.



- Area is commonly measured in square millimetres (mm^2), square centimetres (cm^2), square metres (m^2), or square kilometres (km^2).

Converting units of area

The following chart is useful when converting between units of area.



$$\textbf{For example, } 54\,\text{km}^2 = 54 \times 1000^2 \times 100^2$$
$$= 540\,000\,000\,000\,\text{cm}^2$$

- Another common unit is the **hectare** (ha), a 100 m × 100 m square equal to 10 000 m^2, which is used to measure small areas of land.

The hectare

$$\textbf{1 hectare (ha)} = 100\,\text{m} \times 100\,\text{m}$$
$$= 10\,000\,\text{m}^2$$

WORKED EXAMPLE 5 Converting units of area

Convert $1.3\,km^2$ into:

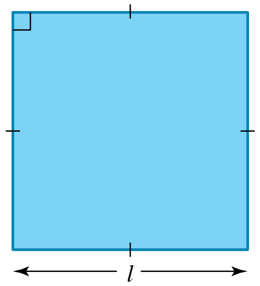
a. **square metres** **b.** **hectares.**

THINK	WRITE
a. There will be more m^2, so multiply by 1000^2.	**a.** $1.3 \times 1000^2 = 1\,300\,000\,m^2$
b. Divide the result of part **a** by 10 000, as $1\,ha = 10\,000\,m^2$.	**b.** $1\,300\,000 \div 10\,000 = 130\,ha$

9.3.2 Using formulas to calculate area

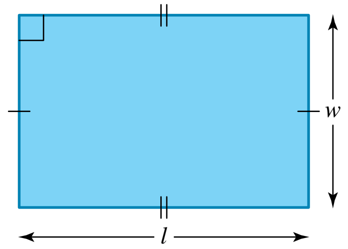
- There are many useful formulas that can be used to calculate the areas of simple shapes. Some common formulas are summarised here.

Square



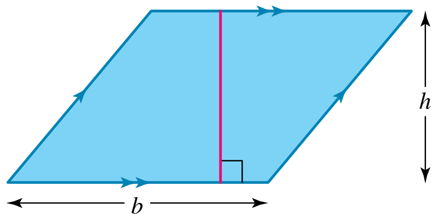
$A = l^2$

Rectangle



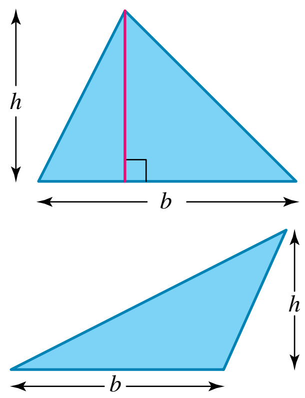
$A = lw$

Parallelogram



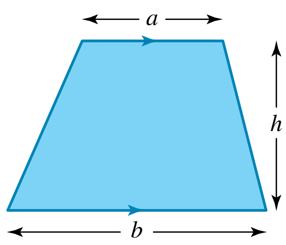
$A = bh$

Triangle



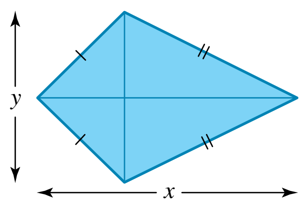
$A = \frac{1}{2}bh$

Trapezium

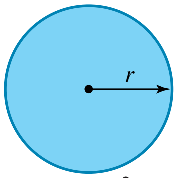
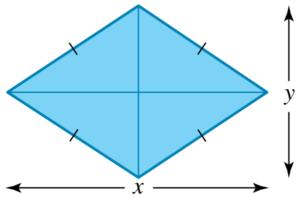


$A = \frac{1}{2}(a + b)h$

Kite



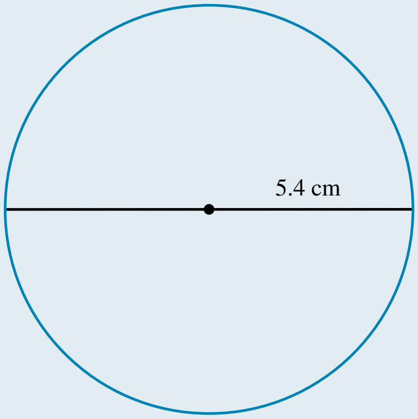
$A = \frac{1}{2}xy$

Circle	Rhombus
 $A = \pi r^2$	 $A = \frac{1}{2}xy$

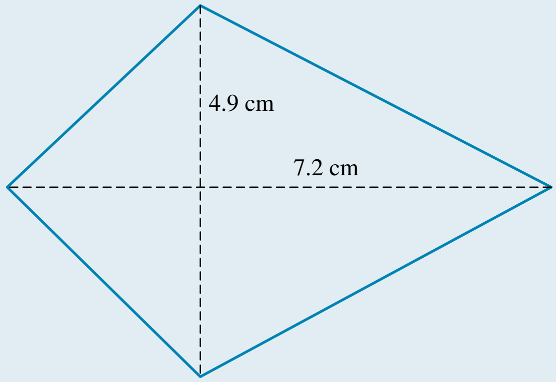
WORKED EXAMPLE 6 Calculating the areas of figures

Calculate the area of each of the following figures in cm², correct to 1 decimal place.

a.



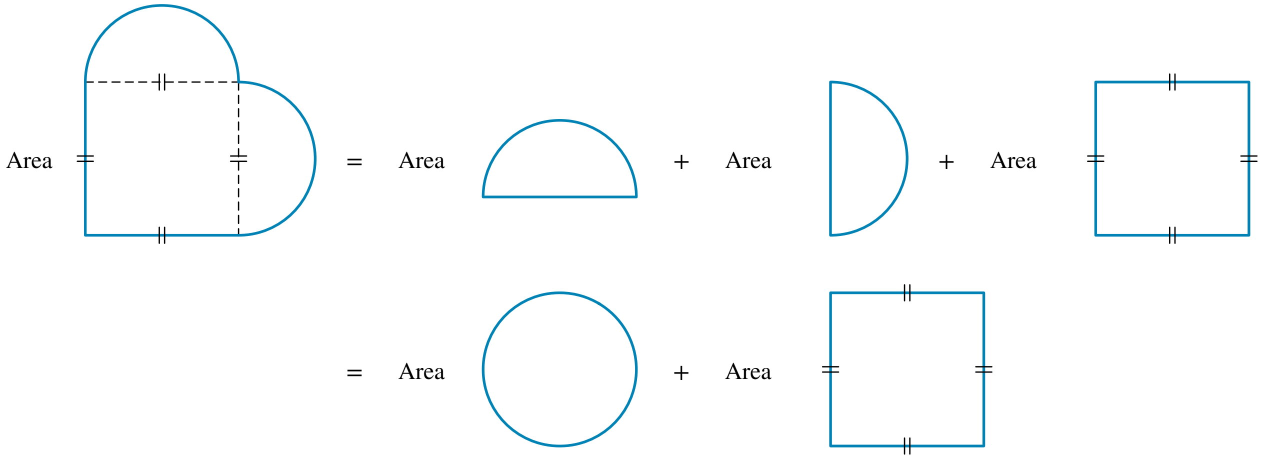
b.

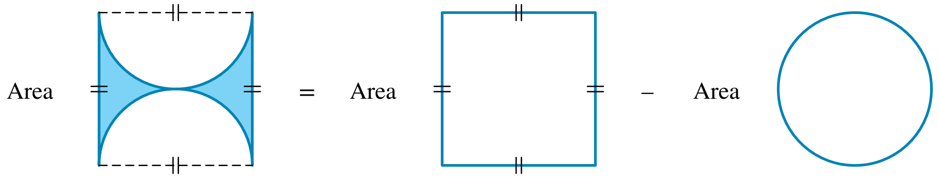


THINK	WRITE
a. 1. The figure is a circle. State the radius.	a. $r = 2.7$ cm
2. Apply the formula for area of a circle: $A = \pi r^2$.	$A = \pi r^2$ $= \pi \times 2.7^2$ ≈ 22.90
3. Round the answer to 1 decimal place.	$A \approx 22.9\text{ cm}^2$
b. 1. The figure is a kite. State the lengths of the diagonals.	b. $x = 4.9$ cm, $y = 7.2$ cm
2. Apply the formula for area of a kite: $A = \frac{1}{2}xy$.	$A = \frac{1}{2}xy$ $= \frac{1}{2} \times 4.9 \times 7.2$ $= 17.64$
3. Round the answer to 1 decimal place.	$A \approx 17.6\text{ cm}^2$

9.3.3 Composite shapes

- A composite shape is made up of smaller, simpler shapes. Here are two examples.

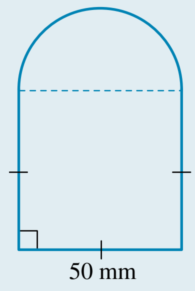




- In the second example, the two semicircles are subtracted from the square to obtain the shaded area on the left.

WORKED EXAMPLE 7 Calculating the area of composite shapes

Calculate the area of the figure shown, giving your answer correct to 1 decimal place.

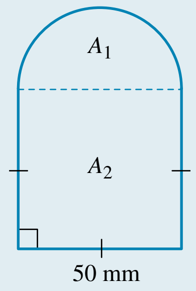


THINK

1. Draw the diagram, divided into basic shapes.

2. A_1 is a semicircle of radius $\frac{50}{2} = 25$ mm.
   The area of a semicircle is half the area of a complete circle.

WRITE/DRAW



$A_1 = \frac{1}{2}\pi r^2$

3. Substitute $r = 25$ into the formula and evaluate, correct to 4 decimal places.	$A_1 = \frac{1}{2} \times \pi \times 25^2$ $\approx 981.7477\ \text{mm}^2$
4. A_2 is a square of side length 50 mm. Write the formula.	$A_2 = l^2$
5. Substitute $l = 50$ into the formula and evaluate A_2.	$= 50^2$ $= 2500\ \text{mm}^2$
6. Sum to calculate the total area.	Total area $= A_1 + A_2$ $= 981.7477 + 2500$ $= 3481.7477\ \text{mm}^2$
7. Round the final answer correct to 1 decimal place.	$\approx 3481.7\ \text{mm}^2$

Exercise 9.3 Area

learnon

9.3 Quick quiz on | 9.3 Exercise

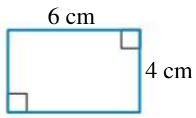
Individual pathways

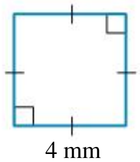
PRACTISE	CONSOLIDATE	MASTER
1, 2, 4, 7, 9, 13, 17, 19, 22, 27	3, 5, 8, 10, 14, 15, 18, 20, 23, 24, 28	6, 11, 12, 16, 21, 25, 26, 29, 30

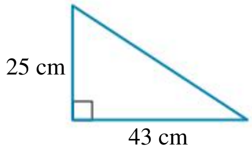
Fluency

1. **MC** To convert an area measurement from square kilometres to square metres:
   - **A.** divide by 1000
   - **B.** multiply by 1000
   - **C.** divide by 1 000 000
   - **D.** multiply by 1 000 000
2. **WE5** Convert the following to the units shown.
   - **a.** $13\,400\ \text{m}^2 =$ ________________ km^2
   - **b.** $0.04\ \text{cm}^2 =$ ________________ mm^2
   - **c.** $3\,500\,000\ \text{cm}^2 =$ ________________ m^2
   - **d.** $0.005\ \text{m}^2 =$ ________________ cm^2
3. Convert the following to the units shown.
   - **a.** $0.043\ \text{km}^2 =$ ________________ m^2
   - **b.** $200\ \text{mm}^2 =$ ________________ cm^2
   - **c.** $1.41\ \text{km}^2 =$ ________________ ha
   - **d.** $3800\ \text{m}^2 =$ ________________ ha

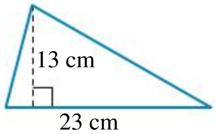
4. WE6 Calculate the area of each of the following shapes. Where appropriate, give your answer correct to 2 decimal places.

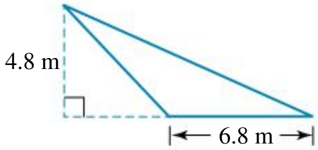
a. 

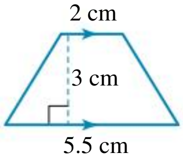
b. 

c. 

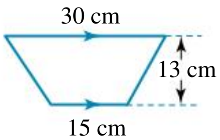
5. Calculate the area of each of the following shapes. Where appropriate, give your answer correct to 2 decimal places.

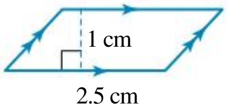
a. 

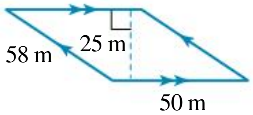
b. 

c. 

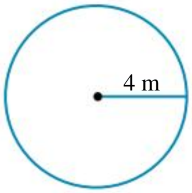
6. Calculate the area of each of the following shapes. Where appropriate, give your answer correct to 2 decimal places.

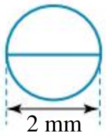
a. 

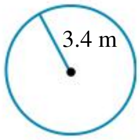
b. 

c. 

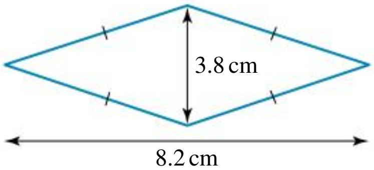
7. Calculate the area of each of the following shapes. Where appropriate, give your answer correct to 2 decimal places.

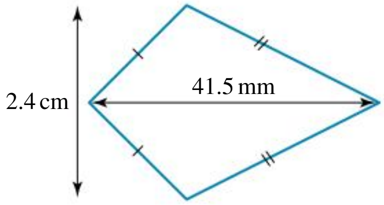
a. 

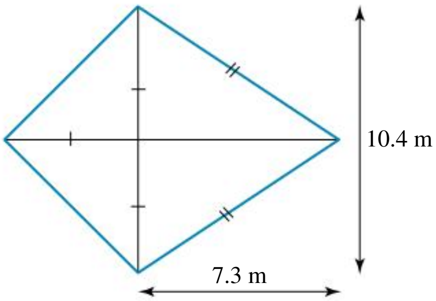
b. 

c. 

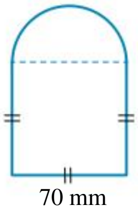
8. Calculate the area of each of the following shapes. Where appropriate, give your answer correct to 2 decimal places.

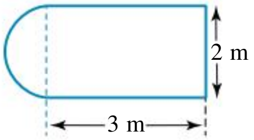
a. 

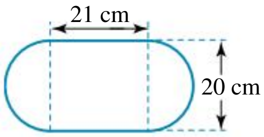
b. 

c. 

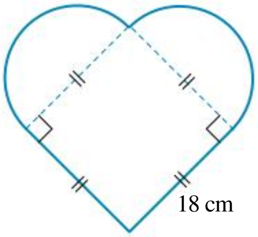
9. WE7 Calculate the areas of the composite shapes shown. Where appropriate, express your answers correct to 1 decimal place.

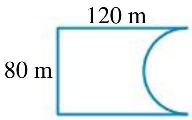
a. 

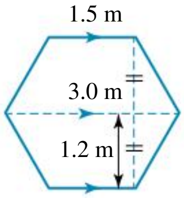
b. 

c. 

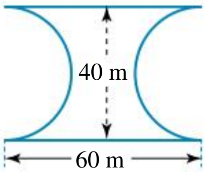
10. Calculate the areas of the composite shapes shown. Where appropriate, express your answers correct to 1 decimal place.

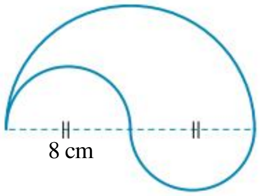
a. 

b. 

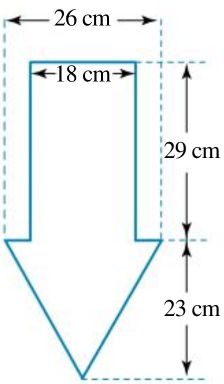
c. 

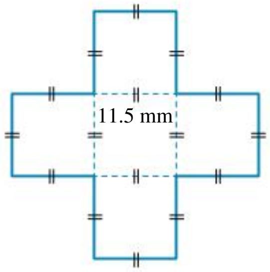
11. Calculate the areas of the composite shapes shown. Where appropriate, express your answers correct to 1 decimal place.

a. 

b. 

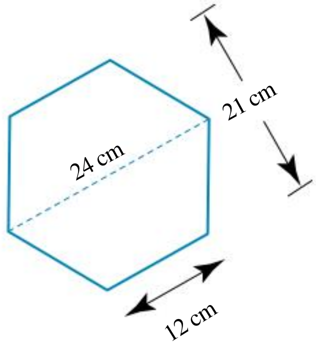
12. Calculate the areas of the composite shapes shown. Where appropriate, express your answers correct to 1 decimal place.

a. 

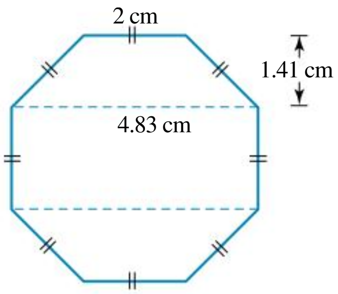
b. 

Understanding

13. Calculate the area of the regular hexagon shown by dividing it into two trapeziums.

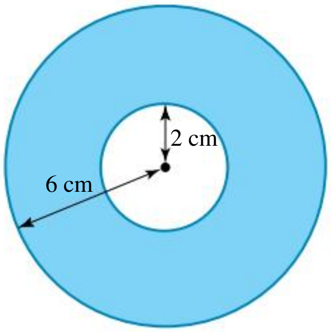


14. Calculate the area of the regular octagon by dividing it into two trapeziums and a rectangle, as shown in the figure. Express your answer correct to 1 decimal place.

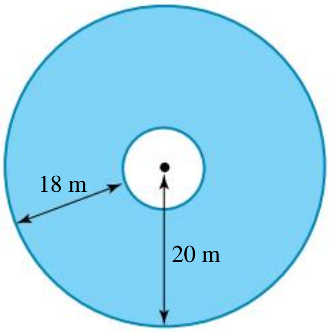


15. An annulus is a shape formed by two concentric circles (two circles with a common centre). Calculate the area of each of the annuli shown by subtracting the area of the smaller circle from the area of the larger circle. Give answers correct to 2 decimal places.

a.

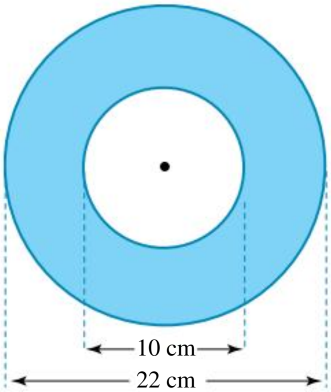


b.

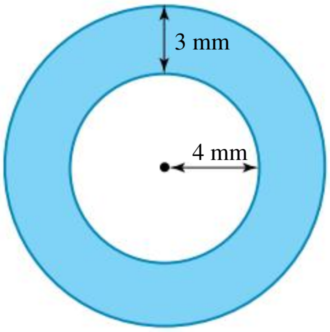


16. Calculate the area of each of the annuli shown (the shaded area). Give answers correct to 2 decimal places.

a.



b.

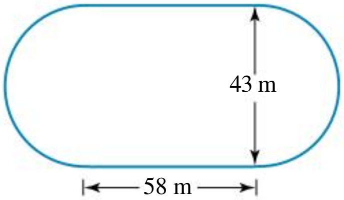


17. MC A pizza has a diameter of 30 cm. If your sister eats one-quarter, determine what area of pizza remains.

A. $168.8\ \text{cm}^2$ B. $530.1\ \text{cm}^2$ C. $706.9\ \text{cm}^2$ D. $176.7\ \text{cm}^2$

18. A circle has an area of $4500\ \text{cm}^2$. Calculate its diameter to the nearest mm.

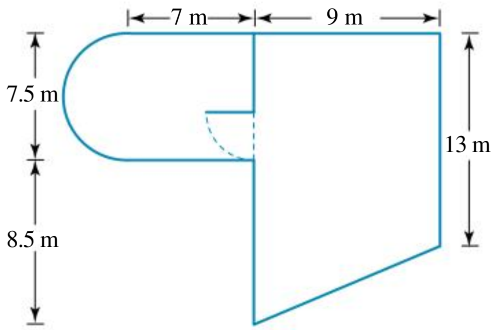
19. Determine the cost of covering the sportsground shown in the figure with turf, if the turf costs \$7.50 per square metre.



20. The Murray–Darling River Basin is Australia's largest catchment. Irrigation of farms in the Murray–Darling Basin has caused soil degradation due to rising salt levels. Studies indicate that about 500 000 hectares of the basin could be affected in the next 50 years.

a. Convert the possible affected area to square kilometres. ($1\ \text{km}^2 = 100$ hectares)
b. The total area of the Murray–Darling Basin is about 1 million square kilometres, about one-seventh of the continent. Calculate what percentage of this total area may be affected by salinity.

21. The plan shows two rooms, which are to be refloored. Calculate the cost if the flooring costs \$45 per square metre. Allow 10% more for wastage and round to the nearest \$10.



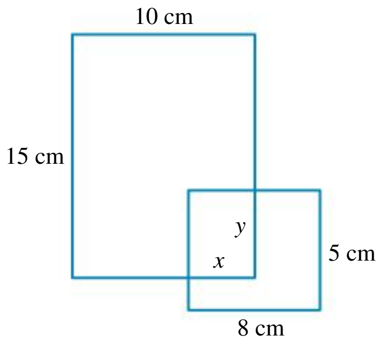
Reasoning

22. A sheet of paper measures 29.5 cm by 21.0 cm.

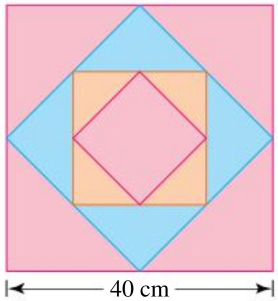
a. Calculate the area of the sheet of paper.
b. Determine the radius of the largest circle that could be drawn on this sheet.
c. Calculate the area of this circle.
d. If the interior of the circle is shaded red, show that 56% of the paper is red.

23. A chessboard is made up of 8 rows and 8 columns of squares. Each square is $42\ \text{cm}^2$ in area. Show that the shortest distance from the upper right corner to the lower left corner of the chessboard is 73.32 cm.

24. Two rectangles of sides 15 cm by 10 cm and 8 cm by 5 cm overlap as shown. Show that the difference in area between the two non-overlapping sections of the rectangles is $110\ \text{cm}^2$.



25. Answer the following questions using the information in the figure.



a. Calculate the area of a square with side lengths 40 cm.
b. If the midpoints of each side of the previous square are joined by straight lines to make another square, calculate the area of the smaller square.
c. Now the midpoints of the previous square are also joined with straight lines to make another square. Calculate the area of this even smaller square.
d. This process is repeated again to make an even smaller square. Calculate the area of this smallest square.
e. Explain any patterns you observe.
f. Determine the percentage of the original square's area taken up by the smallest square.
g. Show that the area of the combined figure that is coloured pink is 1000 cm^2.

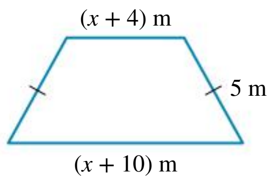
26. Show that a square of perimeter $4x + 20$ has an area of $x^2 + 10x + 25$.

Problem solving

27. The area of a children's square playground is 50 m^2.

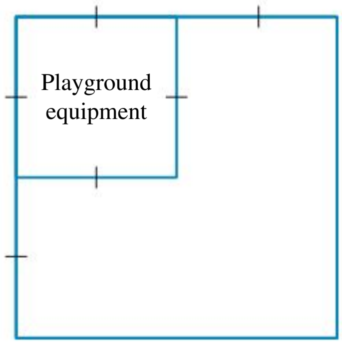
a. Calculate the exact length of the playground.
b. Pine logs 3 m long are to be laid around the playground. Determine how many logs need to be bought. (The logs can be cut into smaller pieces if required.)

28. A sandpit is designed in the shape of a trapezium, with the dimensions shown. If the area of the sandpit is 14 m^2, determine its perimeter.



29. A rectangular classroom has a perimeter of 28 m and its length is 4 m shorter than its width. Determine the area of the classroom.

30. The playground equipment is half the length and half the width of a square kindergarten yard, as shown.



a. Identify the fraction of the kindergarten yard that is occupied by the play equipment.
b. During a working bee, the playground equipment area is extended 2 m in length and 1 m in width. If x represents the length of the kindergarten yard, write an expression for the area of the play equipment.
c. Write an expression for the area of the kindergarten yard *not* taken up by the playground equipment.
d. The kindergarten yard that is not taken up by the playground equipment is divided into 3 equal-sized sections:
   - A grassed section
   - A sandpit
   - A concrete section

   i. Write an expression for the area of the concrete section.
   ii. The children usually spend their time on the play equipment or in the sandpit. Write a simplified expression for the area of the yard where the children usually play.

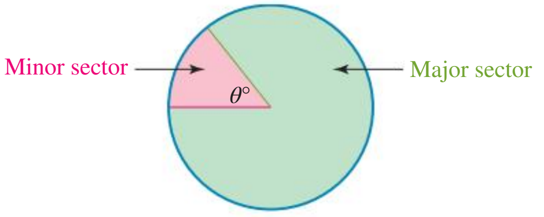
LESSON 9.4 Area and perimeter of a sector

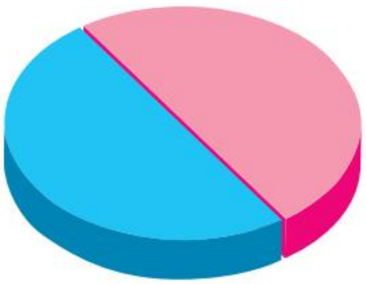
LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify sectors, semicircles and quadrants
- calculate the area of a sector
- calculate the length of the arc and the perimeter of a sector.

9.4.1 Sectors

- A **sector** is the shape created when a circle is cut by two radii. The smaller area is called the minor sector and the larger area is called the major sector.
- A circle, like a pizza, can be cut into many sectors.
- Two important sectors that have special names are the **semicircle** (half-circle) and the **quadrant** (quarter-circle).



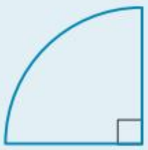


Semicircle

Quadrant

WORKED EXAMPLE 8 Calculating the area of a quadrant

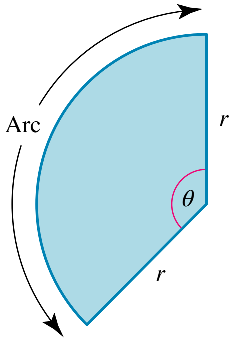
Calculate the area enclosed by the figure, correct to 1 decimal place.



THINK	WRITE
1. The figure is a quadrant or quarter-circle. Write the formula for its area.	$A = \frac{1}{4} \times \pi r^2$
2. Substitute $r = 11$ into the formula.	$A = \frac{1}{4} \times \pi \times 11^2$
3. Evaluate and round the answer correct to 1 decimal place. Include the units.	≈ 95.03 $\approx 95.0\,\text{cm}^2$

9.4.2 Area of a sector

- Sectors are specified by the angle (θ) between the two radii.
- To calculate the area, we first determine the angle of the sector (out of 360°) and then multiply it by the area of a circle (πr^2).



Area of a sector

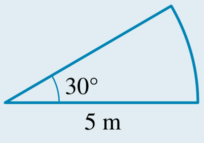
For any value of θ, the area of the sector is given by:

$$A_{\text{sector}} = \frac{\theta}{360} \times \pi r^2$$

where r is the radius of the sector.

WORKED EXAMPLE 9 Calculating the area of a sector

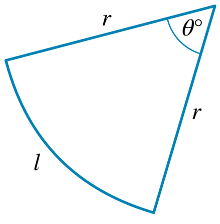
Calculate the area of the sector shown, correct to 1 decimal place.



THINK	WRITE
1. This sector is $\frac{30}{360}$ of a circle. Write the formula for its area.	$A = \frac{30}{360}\pi r^2$
2. Substitute $r = 5$ into the formula.	$A = \frac{30}{360} \times \pi \times 5^2$ ≈ 6.54
3. Evaluate and round the answer correct to 1 decimal place. Include the units.	$\approx 6.5\ \text{m}^2$

9.4.3 Perimeter of a sector

- The perimeter, P, of a sector is the sum of the 2 radii and the curved section, which is called an arc of a circle.
- The length of the arc, l, will be $\frac{\theta}{360}$ of the circumference of the circle.



Arc length of a sector

For any value of θ, the arc length, l, of the sector is given by:

$$l = \frac{\theta}{360} \times 2\pi r$$

where r is the radius of the sector.

- The perimeter of a sector can therefore be calculated using the following formula.

Perimeter of a sector

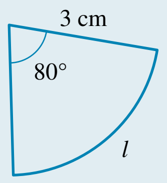
The perimeter, P, of a sector is given by:

$$P = 2r + l$$

where r is the radius of the sector and l is the arc length.

WORKED EXAMPLE 10 Calculating the perimeter of a sector

Calculate the perimeter of the sector shown, correct to 1 decimal place.



THINK	WRITE
1. The sector is $\frac{80}{360}$ of a circle. Write the formula for the length of the curved side.	$l = \frac{80}{360} \times 2\pi r$
2. Substitute $r = 3$ and evaluate l. Don't round off until the end.	$l = \frac{80}{360} \times 2 \times \pi \times 3$ $\approx 4.189\text{ cm}$
3. Add all the sides together to calculate the perimeter.	$P = 4.189 + 3 + 3$ $= 10.189\text{ cm}$
4. Round the answer to 1 decimal place.	$P \approx 10.2\text{ cm}$

learnon

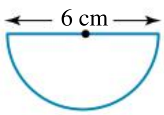
9.4 Quick quiz on	9.4 Exercise

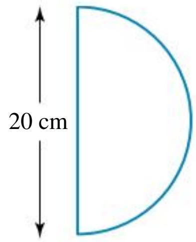
Individual pathways

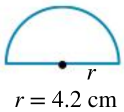
PRACTISE	CONSOLIDATE	MASTER
1, 2, 7, 9, 13, 15, 20, 23	3, 4, 8, 10, 14, 16, 19, 21, 24	5, 6, 11, 12, 17, 18, 22, 25

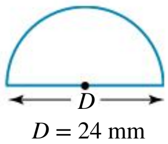
Fluency

1. Calculate the area of the semicircles shown, correct to 2 decimal places.

a.


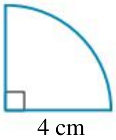
b.


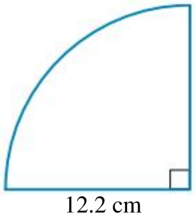
c.


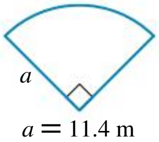
d.


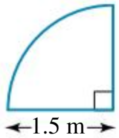
WE8 For each of the quadrants in questions **2**–**5**, calculate to 1 decimal place:

a. the perimeter **b.** the area enclosed.

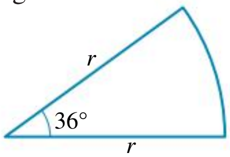
2.


3.


4.


5.


6. **MC** Select the correct formula for calculating the area of the sector shown.



A. $A = \frac{1}{4}\pi r^2$

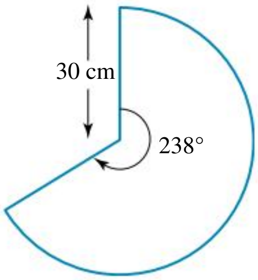
B. $A = \frac{3}{4}\pi r^2$

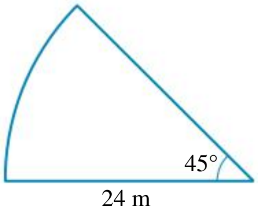
C. $A = \frac{1}{2}\pi r^2$

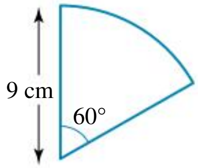
D. $A = \frac{1}{10}\pi r^2$

WE9&10 For each of the sectors in questions **7**–**12**, calculate to 1 decimal place:

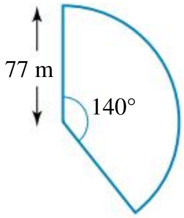
a. the perimeter **b.** the area.

7.


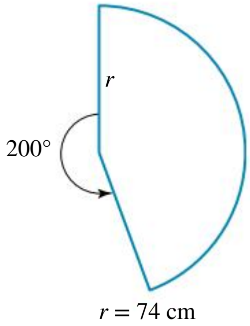
8.


9.


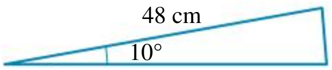
10.



11.

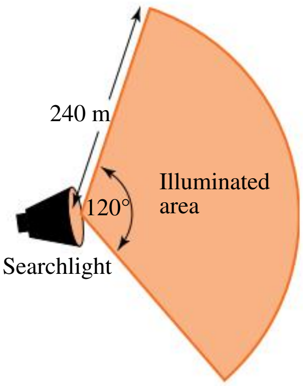


12.

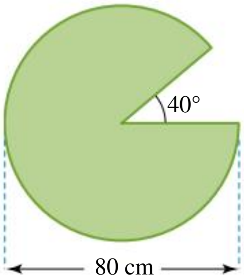


Understanding

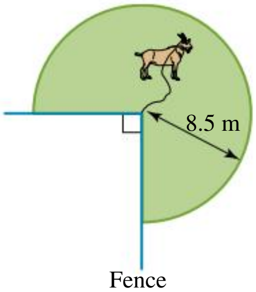
13. A searchlight lights up the ground to a distance of 240 m. Calculate the area the searchlight illuminates if it can swing through an angle of 120°, as shown in the diagram. (Give your answer correct to 1 decimal place.)



14. Calculate the perimeter, correct to 1 decimal place, of the figure shown.



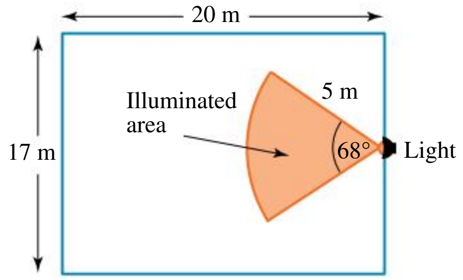
15. A goat is tethered by an 8.5 m rope to the outside of a corner post in a paddock, as shown in the diagram.



Calculate the area of grass (shaded) on which the goat is able to graze. (Give your answer correct to 1 decimal place.)

16. A sector has an angle of 80° and a radius of 8 cm; another sector has an angle of 160° and a radius of 4 cm. Determine the ratio of the area of the first sector to the area of the second sector.

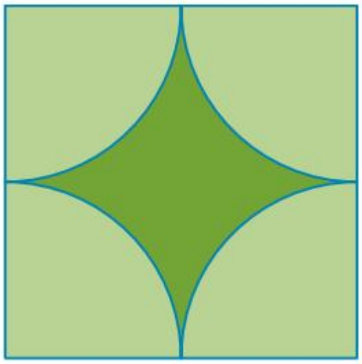
17. A beam of light is projected onto a theatre stage as shown in the diagram.



a. Calculate the illuminated area (correct to 1 decimal place) by first evaluating the area of the sector.
b. Calculate the percentage of the total stage area that is illuminated by the light beam.

18. The minute hand on a vintage clock is 20 centimetres long and the hour hand is 12 centimetres long.
a. Calculate the distance travelled by the minute hand from midday until 2 pm.
b. During this same time, calculate how far the hour hand travelled. Give your answers correct to the nearest centimetre.

19. Four baseball fields are to be constructed inside a rectangular piece of land. Each field is in the shape of a sector of a circle, as shown in light green. The radius of each sector is 80 m.



a. Calculate the area of one baseball field, correct to the nearest whole number.
b. Calculate the percentage, correct to 1 decimal place, of the total area occupied by the four fields.
c. The cost of the land is $24 000 per hectare. Calculate the total purchase price of the land.

Reasoning

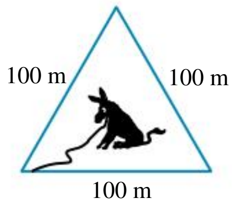
20. John and Jim are twins, and on their birthday they have identical birthday cakes, each with a diameter of 30 cm. Grandma Maureen cuts John's cake into 8 equal sectors. Grandma Mary cuts Jim's cake with a circle in the centre and then 6 equal portions from the rest.

a. Show that each sector of John's cake makes an angle of 45° at the centre of the cake.
b. Calculate the area of one slice of John's cake, correct to 1 decimal place.
c. Calculate the area of the small central circular part of Jim's cake, correct to 1 decimal place.
d. Calculate the area of one of the larger portions of Jim's cake, correct to 1 decimal place.
e. If each boy eats one slice of the largest part of his own cake, does John eat more cake? Explain your answer.

21. A lighthouse has a light beam in the shape of a sector of a circle that rotates at 10 revolutions per minute and covers an angle of 40°. A person stands 200 m from the lighthouse and observes the beam.
Show that the time between the end of one flash and the start of the next is approximately 5.33 seconds.

22. Answer the following questions. Where appropriate, give all answers to the nearest whole number.

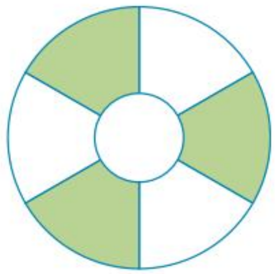
a. A donkey inside a square enclosure is tethered to a post at one of the corners. Show that a rope of length 120 m is required to prevent the donkey from eating more than half the grass in the enclosure.
b. Suppose two donkeys are tethered at opposite corners of the square region shown. Calculate how long each rope should be so that the donkeys together can graze half of the area.
c. This time four donkeys are tethered, one at each corner of the square region. Calculate how long each rope should be so that all the donkeys can graze only half of the area.



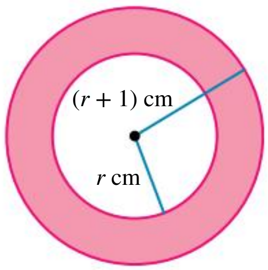
d. Another donkey is tethered to a post inside an enclosure in the shape of an equilateral triangle. The post is at one of the vertices.
Show that a rope of length 64 m is required so that the donkey eats only half of the grass in the enclosure.
e. This time the donkey is tethered halfway along one side of the equilateral triangular region shown in the diagram.
Calculate how long the rope should be so that the donkey can graze half of the area.

Problem solving

23. A new logo, divided into 6 equal sectors, has been designed based on two circles with the same centre, as shown in the diagram. The radius of the larger circle is 60 cm and the radius of the smaller circle is 20 cm.
The shaded sections are to be coated with a special paint costing \$145 a square metre. The edge of the shaded sections is made from a special material costing \$36 a metre.
Calculate the cost to make the shaded sections. Give your answer to the nearest dollar.



24. The metal washer shown has an inner radius of r cm and an outer radius of $(r + 1)$ cm.



a. State, in terms of r, the area of the circular piece of metal that was cut out of the washer.
b. State, in terms of r, the area of the larger circle.
c. Show that the area of the metal washer in terms of r is $\pi(2r + 1)\,\text{cm}^2$.
d. If r is 2 m, what is the exact area of the washer?
e. If the area of the washer is $15\pi\ \text{cm}^2$, show that the radius would be 7 cm.

25. The area of a sector of a circle is $\pi\ \text{cm}^2$, and the length of its arc is 2 cm. Determine the radius of the circle (in terms of π).

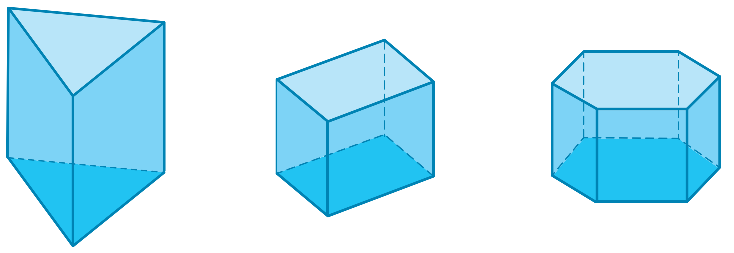
LESSON 9.5 Surface area of rectangular and triangular prisms

LEARNING INTENTION

At the end of this lesson you should be able to:
- calculate the surface area of rectangular and triangular prisms.

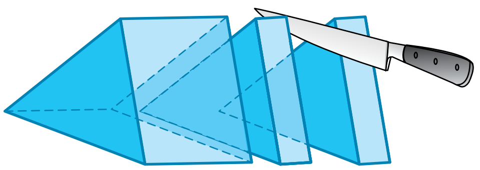
9.5.1 Prisms and surface areas of prisms

- A **prism** is a three-dimensional figure with a uniform (unchanging) **cross-section** and all faces flat.
- An oblique prism has two opposite faces identical and the remaining faces are parallelograms.
- A right prism has two opposite faces identical and the remaining faces are rectangles.
  Note: In this topic, we will only work with right prisms.



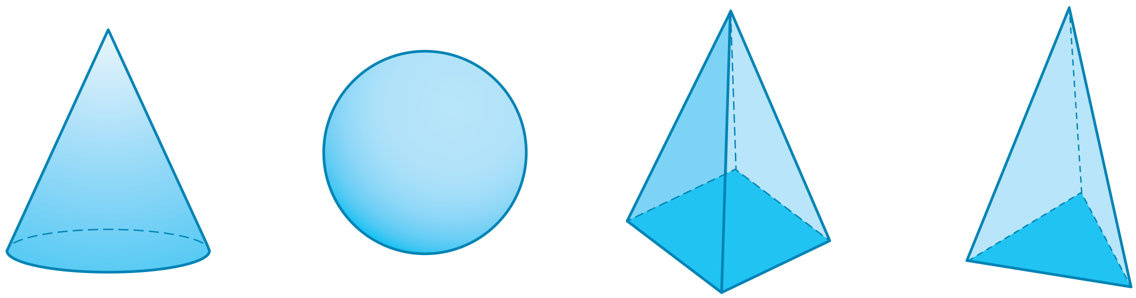
Triangular prism   Rectangular prism (cuboid)   Hexagonal prism

- A prism can be sliced (cross-sectioned) in such a way that each 'slice' has an identical base.



'Slicing' a prism into pieces produces congruent cross-sections.

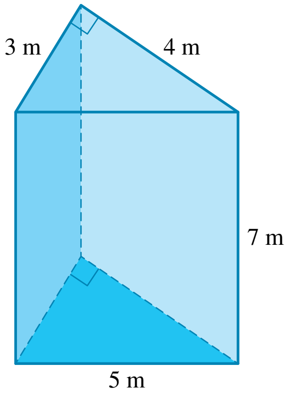
- The following objects are not prisms because they do not have uniform cross-sections.

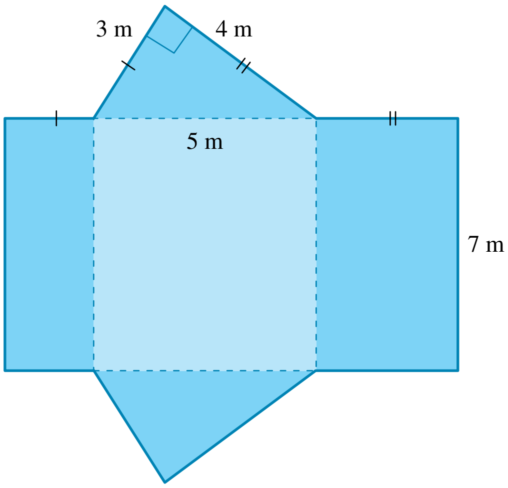


Cone   Sphere   Square pyramid   Triangular pyramid

Surface area of a prism

- Consider the **triangular prism** shown.
- When unfolded as a 2D **net**, it has 5 faces: 2 bases, which are right-angled triangles, and 3 rectangular sides.

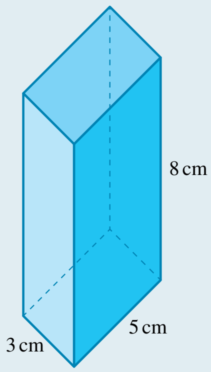


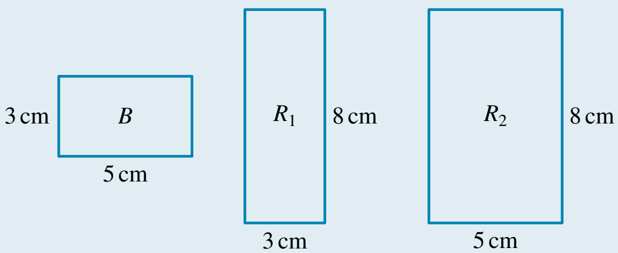


- The area of the net is the same as the total surface area (*SA*) of the prism.

WORKED EXAMPLE 11 Calculating the surface area of a rectangular prism

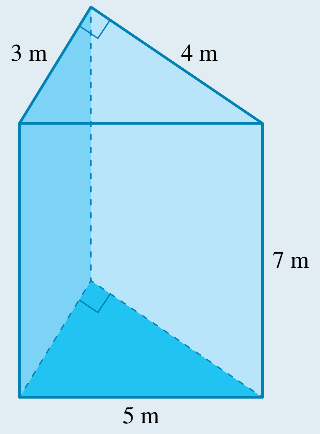
Calculate the surface area of the rectangular prism (cuboid) shown.



THINK	WRITE/DRAW
1. There are 6 faces: 2 rectangular bases and 4 rectangular sides. Draw diagrams for each pair of faces and label each region.	
2. Calculate the area of each rectangle by applying the formula $A = lw$, where l is the length and w is the width of the rectangle.	$B = 3 \times 5 = 15$ $R_1 = 3 \times 8 = 24$ $R_2 = 5 \times 8 = 40$
3. The total surface area (*SA*) is the sum of the area of 2 of each shape. Write the answer.	$SA = 2 \times B + 2 \times R_1 + 2 \times R_2 = 30 + 48 + 80 = 158\ \text{cm}^2$

WORKED EXAMPLE 12 Calculating the surface area of a triangular prism

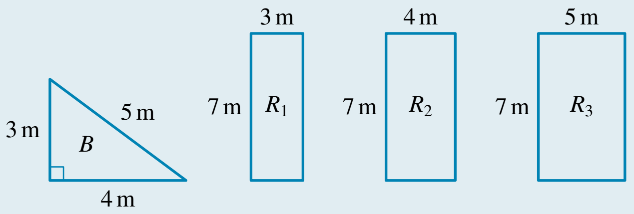
Calculate the surface area of the right-angled triangular prism shown.



THINK

WRITE/DRAW

1. There are 5 faces: 2 triangles and 3 rectangles. Draw diagrams for each face and label each region.



2. Calculate the area of the triangular base by applying the formula $A = \frac{1}{2}bh$.

$$B = \frac{1}{2} \times 3 \times 4$$
$$= 6\text{ m}^2$$

3. Calculate the area of each rectangular face by applying the formula $A = lw$.

$$R_1 = 3 \times 7 = 21 \qquad R_2 = 4 \times 7 = 28 \qquad R_3 = 5 \times 7 = 35$$

4. The total surface area is the sum of all the areas of the faces, including 2 bases.

$$SA = 2 \times B + R_1 + R_2 + R_3$$
$$= 12 + 21 + 28 + 35$$
$$= 96\text{ m}^2$$

Exercise 9.5 Surface area of rectangular and triangular prisms

learnon

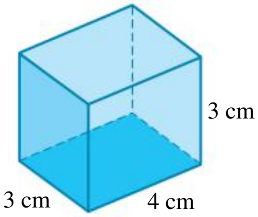
9.5 Quick quiz on	9.5 Exercise

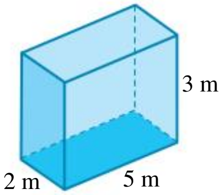
Individual pathways

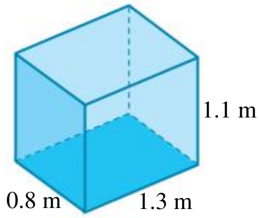
■ PRACTISE	■ CONSOLIDATE	■ MASTER
1, 3, 7, 9, 12, 14, 17	2, 4, 6, 8, 10, 15, 18	5, 11, 13, 16, 19

Fluency

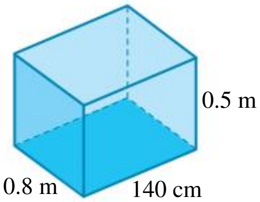
1. **WE11** Calculate the surface area of the following rectangular prisms (cuboids).

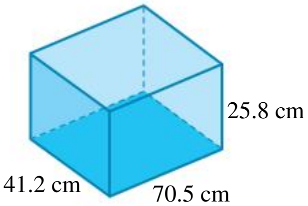
a.


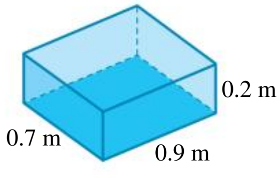
b.


c.


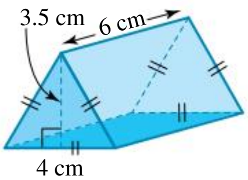
2. Calculate the surface area of the following rectangular prisms (cuboids).

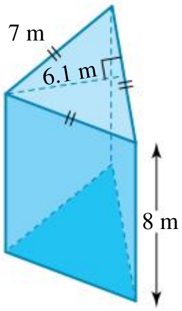
a.


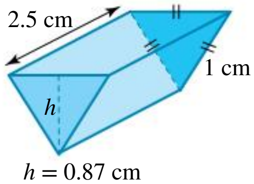
b.


c.


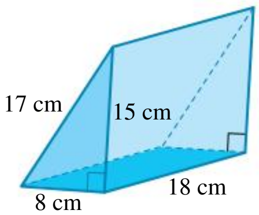
3. **WE12** Calculate the surface area of each of the following triangular prisms.

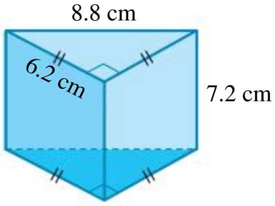
a.


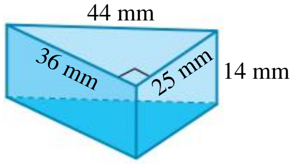
b.


c.


4. Calculate the surface area of each of the triangular prisms shown.

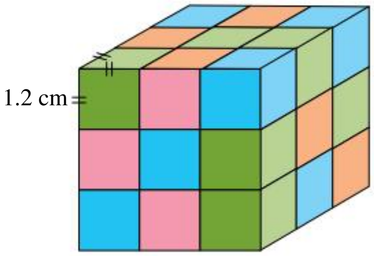
a.


b.


c.


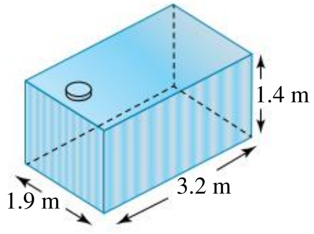
5. A shipping company is planning to buy and paint the outside surface of one of these shipping containers. Determine how many cans of paint the company should buy if the base of the container is not painted and each can of paint covers about 40 m^2.

6. The aim of the Rubik's cube puzzle is to make each face of the cube one colour. Calculate the surface area of the Rubik's cube if each small coloured square is 1.2 cm in length.
Assume that there are no gaps between the squares.



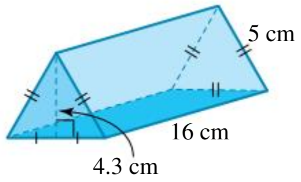
7. Determine how many square metres of iron sheet are needed to construct the water tank shown.

8. Calculate the surface area of the tank in question 7 if no top is made.

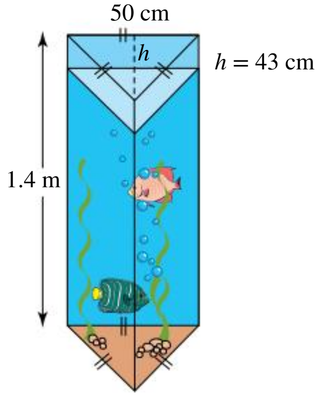


Understanding

9. Calculate the area of cardboard that would be needed to construct a box to pack this prism, assuming that no overlap occurs.

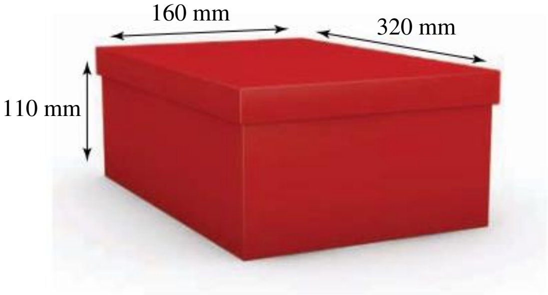


10. An aquarium is a triangular prism with the dimensions shown. The top of the tank is open. Calculate the area of glass that was required to construct the tank. Give your answer correct to 2 decimal places.



11. A tent is constructed as shown. Calculate the area of canvas needed to make the tent, including the floor.

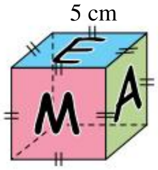
12. a. Determine how many square centimetres of cardboard are needed to construct the shoebox shown, ignoring the overlap on the top.
    b. Draw a sketch of a net that could be used to make the box.



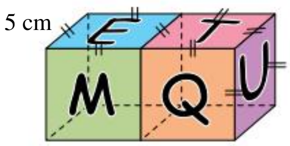
13. Calculate the surface area of a square-based prism of height 4 cm, given that the side length of its base is 6 cm.

Reasoning

14. a. Calculate the surface area of the toy block shown.



    b. If two of the blocks are placed together as shown, calculate the surface area of the prism that is formed.



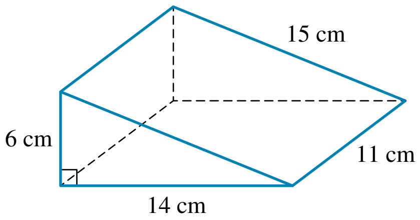
    c. Calculate the surface area of the prism formed by three blocks.
    d. Use the pattern to determine the surface area of a prism formed by eight blocks arranged in a line. Explain your reasoning.

15. A cube has a side length of 2 cm. Show that the least surface area of a solid formed by joining eight such cubes is 96 cm^2.

16. A prism has an equilateral triangular base with a perimeter of 12 cm. If the total surface area of the prism is 302 cm^2, show that the length of the prism is approximately 24 cm.

Problem solving

17. A wedge in the shape of a triangular prism, as shown in the diagram, is to be painted.



a. Draw a net of the wedge so that it is easier to calculate the area to be painted.

b. Calculate the area to be painted. (Do not include the base.)

18. Ken wants to paint his son's bedroom walls blue and the ceiling white. The room measures 3 m by 4 m, with a ceiling height of 2.6 m. There is one door that measures 1 m by 2 m and one window that measures 1.8 m by 0.9 m.
Each surface takes two coats of paint and 1 L of paint covers 16 m^2 on the walls and 12 m^2 on the ceiling. Cans of wall paint cost $33.95 for 1 L, $63.90 for 4 L, $147 for 10 L and $174 for 15 L. Ceiling paint costs $24 for 1 L and $60 for 4 L.
Determine the cheapest options for Ken to paint the room.

19. A swimming pool has a length of 50 m and a width of 28 m. The shallow end of the pool has a depth of 0.80 m, which increases steadily to 3.8 m at the deep end.

a. Calculate how much paint would be needed to paint the floor of the pool.

b. If the pool is to be filled to the top, calculate how much water will be needed.

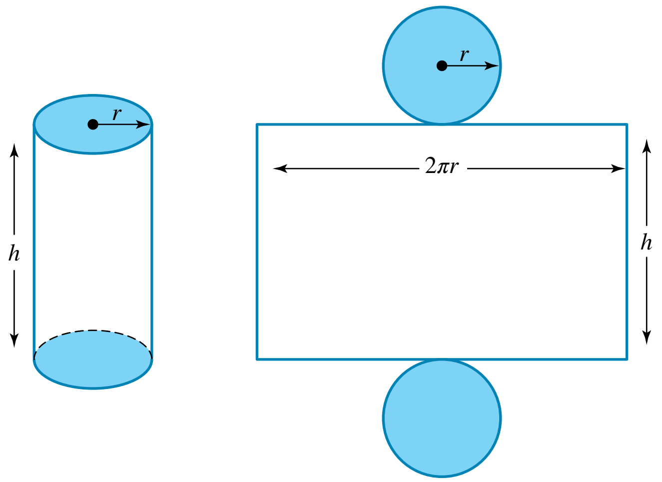
LESSON 9.6 Surface area of a cylinder

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- draw the net of a cylinder
- calculate the curved surface area of a cylinder
- calculate the total surface area of a cylinder.

9.6.1 Surface area of a cylinder

- A **cylinder** is a solid object with two identical flat circular ends and one curved side. A cylinder has a uniform cross-section.
- The net of a cylinder has two circular bases and one rectangular face. The rectangular face is the curved surface of the cylinder.
- Because the rectangle wraps around the circular base, the width of the rectangle is the same as the circumference of the circle. Therefore, the width is equal to $2\pi r$.
- The area of each base is πr^2, and the area of the rectangle is $2\pi rh$.



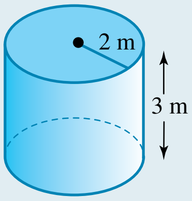
Surface area of a cylinder

$$SA = 2\pi r^2 + 2\pi rh$$
$$= 2\pi r(r + h)$$

WORKED EXAMPLE 13 Calculating the surface area of a cylinder

a. **Use the formula $A = 2\pi rh$ to calculate the area of the curved surface of the cylinder, correct to 1 decimal place.**

b. **Use the formula $SA = 2\pi rh + 2\pi r^2$ to calculate the surface area of the cylinder, correct to 1 decimal place.**



THINK	WRITE
a. 1. Write the formula for the curved surface area.	**a.** $A = 2\pi rh$
2. Identify the values of the pronumerals.	$r = 2,\ h = 3$
3. Substitute $r = 2$ and $h = 3$.	$A = 2 \times \pi \times 2 \times 3$
4. Evaluate, round to 1 decimal place and include units.	≈ 37.69 $\approx 37.7\ \text{m}^2$
b. 1. Write the formula for the surface area of a cylinder.	**b.** $SA = 2\pi rh + 2\pi r^2$
2. Identify the values of the pronumerals.	$r = 2,\ h = 3$
3. Substitute $r = 2$ and $h = 3$.	$SA = (2 \times \pi \times 2 \times 3) + \left(2 \times \pi \times 2^2\right)$
4. Evaluate, round to 1 decimal place and include units.	≈ 62.83 $\approx 62.8\ \text{m}^2$

Exercise 9.6 Surface area of a cylinder

learnon

9.6 Quick quiz on | 9.6 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 7, 9, 13, 15, 18	3, 4, 8, 10, 14, 16, 19	5, 6, 11, 12, 17, 20, 21

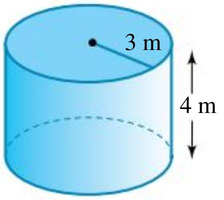
Fluency

WE13 For each of the cylinders in questions 1–6, answer the following questions.

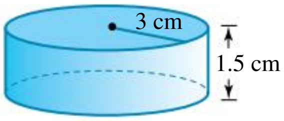
a. Use the formula $A = 2\pi rh$ to calculate the area of the curved surface of the cylinder.

b. Use the formula $SA = 2\pi rh + 2\pi r^2$ to calculate the total surface area of the cylinder.

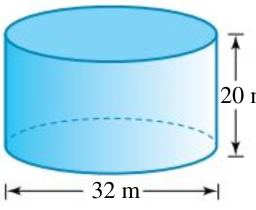
1.



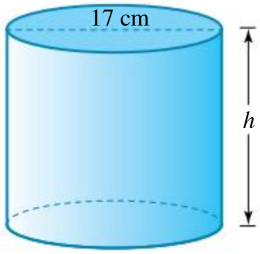
2.



3.

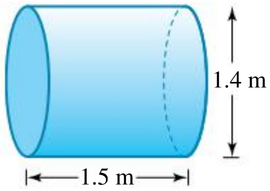


4.

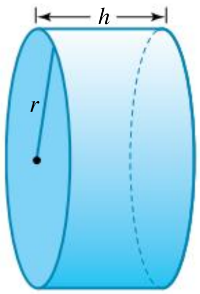


$h = 21$ cm

5.



6.



$r = 2.4$ m
$h = 1.7$ m

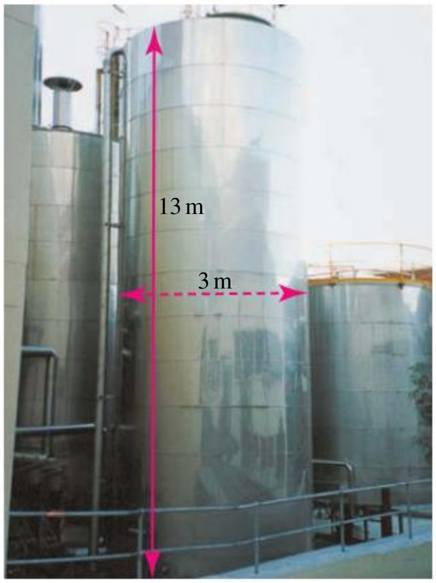
7. A can of energy drink has a height of 130 mm and a radius of 24 mm.

a. Draw a net of the can.

b. Calculate the surface area of the can, to the nearest cm^2.

8. A cylinder has a radius of 15 cm and a height of 45 mm.
Determine its surface area correct to 1 decimal place.

Understanding



9. A cylinder has a surface area of 2000 cm^2 and a radius of 8 cm. Determine the cylinder's height correct to 2 decimal places.

10. A can of asparagus is 137 mm tall and has a diameter of 66 mm; a can of tomatoes is 102 mm tall and has a diameter of 71 mm; and a can of beetroot is 47 mm tall with a diameter of 84 mm.

a. Determine which can has the largest surface area and which has the smallest surface area.

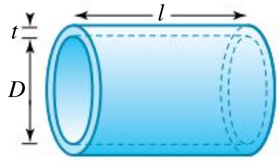
b. Calculate the difference between the largest and smallest surface areas, correct to the nearest cm^2.

11. A 13 m-high storage tank was constructed from stainless steel (including the lid and the base). The diameter is 3 metres, as shown.

a. Calculate the surface area of the tank. Give your answer correct to 2 decimal places.

b. Determine the cost of the steel for the side of the tank if it comes in sheets 1 m wide that cost \$60 a metre.

12. The concrete pipe shown in the diagram has the following measurements: $t = 30$ mm, $D = 18$ cm, $l = 27$ cm.
Answer the following, giving your answers correct to 2 decimal places.



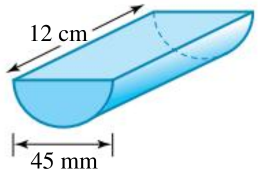
a. Calculate the outer curved surface area.

b. Calculate the inner curved surface area.

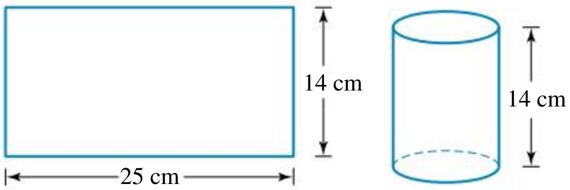
c. Calculate the total surface area of both ends.

d. Hence calculate the surface area for the entire shape.

13. Wooden mouldings are made by cutting cylindrical dowels in half, as shown. Calculate the surface area of the moulding.



14. Kiara has a rectangular sheet of cardboard with dimensions 25 cm by 14 cm. She rolls the cardboard to form a cylinder so that the shorter side, 14 cm, is its height, and glues the edges together with a 1 cm overlap.



a. Determine the radius of the circle Kiara needs to construct to put at the top of her cylinder. Give your answer correct to 2 decimal places.

b. Determine the total surface area of her cylinder if she also makes the top and bottom of her cylinder out of cardboard. Give your answer correct to 1 decimal place.

Reasoning

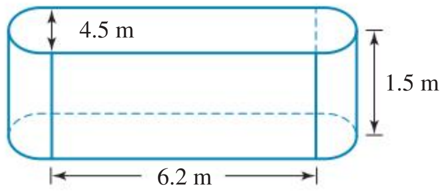
15. If the radius of a cylinder is twice its height, write a formula for the surface area in terms of its height only.

16. Cylinder A has a 10% greater radius and a 10% greater height than Cylinder B. Show that the ratio of their surface areas is 121 : 100.

17. Two identical cylinders (with height h cm and radius r cm) are modified slightly.
The first cylinder's radius is increased by 10% and the second cylinder's height is increased by 20%.
After these modifications have been made, the surface area of the second cylinder is greater than that of the first cylinder. Show that $h > 2.1r$.

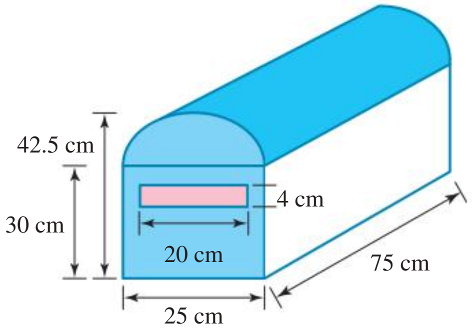
Problem solving

18. An above-ground swimming pool has the following shape, with semicircular ends.
Calculate how much plastic would be needed to line the base and sides of the pool.

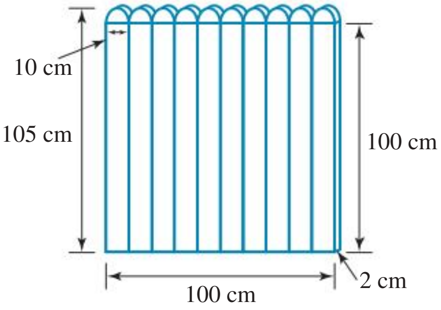


19. A over-sized solid wooden die is constructed for a children's playground. The side dimensions of the die are 50 cm. The number on each side of the die will be represented by cylindrical holes that will be drilled out of each side.
Each hole will have a diameter of 10 cm and depth of 2 cm. All surfaces on the die will be painted (including the die holes).
Show that the total area required to be painted is 1.63 m^2.

20. The following letterbox is to be spray-painted on the outside. Calculate the total area to be spray-painted.
Assume that the end of the letterbox is a semicircle above a rectangle.
The letter slot is open and does not require painting.



21. A timber fence is designed as shown. Evaluate how many square metres of paint are required to completely paint the fence front, back, sides and top with 2 coats of paint.
Assume each paling is 2 cm in thickness and that the top of each paling is a semicircle.



LESSON 9.7 Volume of prisms and cylinders

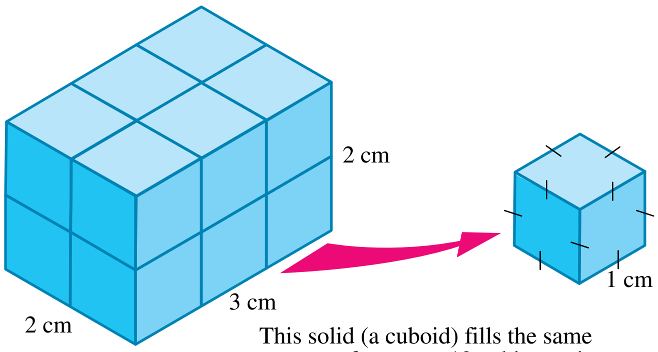
LEARNING INTENTIONS

At the end of this lesson you should be able to:
- calculate the volume of prisms and cylinders
- convert between units of volume
- convert between units of capacity and units of volume.

9.7.1 Volume and capacity

Volume

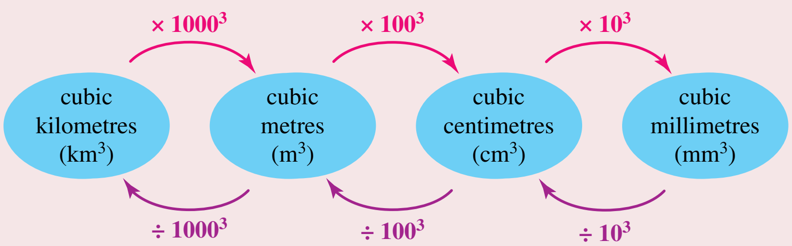
- The volume of a solid is the amount of space it fills or occupies.
- The diagram at far right shows a single cube of side length 1 cm. By definition, the cube has a **volume** of 1 cm^3 (1 cubic centimetre). *Note:* This is a 'cubic centimetre', not a 'centimetre cubed'.
- The volume of some solids can be found by dividing them into cubes with 1 cm sides.
- Volume is commonly measured in cubic millimetres (mm^3), cubic centimetres (cm^3), cubic metres (m^3) or cubic kilometres (km^3).



This solid (a cuboid) fills the same amount of space as 12 cubic centimetres.
The volume of this cuboid is 12 cm^3.

Converting units of volume

The following chart is useful when converting between units of volume.



For example, $3\text{ m}^3 = 3 \times 100^3 \times 10^3\text{ mm}^3$
$= 3\,000\,000\,000\text{ mm}^3$

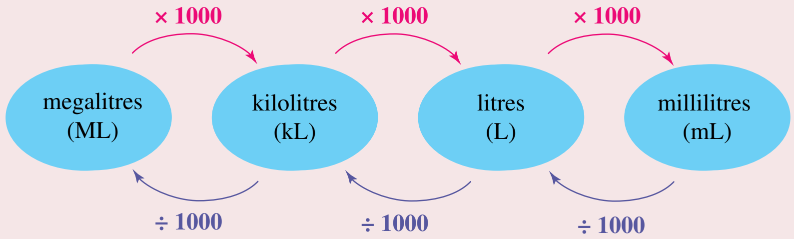
Capacity

- **Capacity** is another term for volume, which is usually applied to the measurement of liquids and containers.
- The capacity of a container is the volume of liquid that it can hold.
- The standard measurement for capacity is the litre (L).
- Other common units are the millilitre (mL), kilolitre (kL), and megalitre (ML), where:

$$1\text{ L} = 1000\text{ mL}$$
$$1\text{ kL} = 1000\text{ L}$$
$$1\text{ ML} = 1\,000\,000\text{ L}$$

Converting units of capacity

The following chart is useful when converting between units of capacity.



- The units of capacity and volume are related as follows:

Converting between units of volume and capacity

$$\mathbf{1\text{ cm}^3 = 1\text{ mL}}$$
$$\mathbf{1000\text{ cm}^3 = 1\text{ L}}$$
$$\mathbf{1\text{ m}^3 = 1000\text{ L} = 1\text{ kL}}$$

WORKED EXAMPLE 14 Converting units of volume and capacity

Convert:

a. **13.2 L into cm^3**
b. **3.1 m^3 into litres**
c. **0.13 cm^3 into mm^3**
d. **3.8 kL into m^3.**

THINK	WRITE
a. 1. 1 L = 1000 mL, so multiply by 1000.	**a.** $13.2 \times 1000 = 13\,200\text{ mL}$
2. 1 mL = 1 cm^3. Convert to cm^3.	$= 13\,200\text{ cm}^3$
b. 1 m^3 = 1000 L, so multiply by 1000.	**b.** $3.1 \times 1000 = 3100\text{ L}$
c. There will be more mm^3, so multiply by 10^3.	**c.** $0.13 \times 1000 = 130\text{ mm}^3$
d. 1 kL = 1 m^3. Convert to m^3.	**d.** $3.8\text{ kL} = 3.8\text{ m}^3$

9.7.2 Volume of a prism

- The volume of a prism can be found by multiplying its cross-sectional area (A) by its height (h).

Volume of a prism

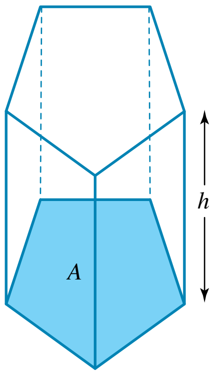
$$V_{\text{prism}} = A \times h$$

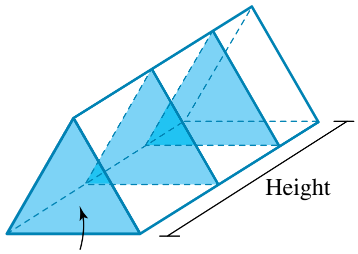
where:

A is the area of the base or cross-section of the prism

h is the perpendicular height of the prism.

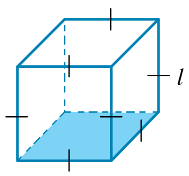
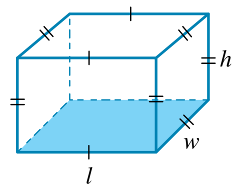
- The cross-section (A) of a prism is often referred to as the base, even if it is not at the bottom of the prism.
- The height (h) is always measured perpendicular to the base, as shown in the following diagrams.





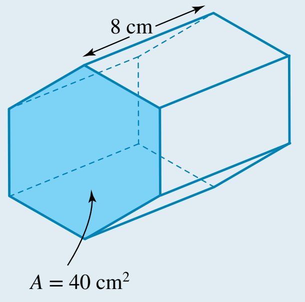
Volume of a cube or a cuboid

- A specific formula can be developed for the volumes of cubes and cuboids.

Cube	**Cuboid (rectangular prism)**
 $\text{Volume} = \text{base area} \times \text{height}$ $= l^2 \times l$ $= l^3$	 $\text{Volume} = \text{base area} \times \text{height}$ $= lw \times h$ $= lwh$

WORKED EXAMPLE 15 Calculating the volume of a prism

Calculate the volume of the hexagonal prism.



THINK

1. Write the formula for the volume of a prism.
2. Identify the values of the pronumerals.
3. Substitute $A = 40$ and $h = 8$ into the formula and evaluate. Include the units.

WRITE

$V = A \times h$

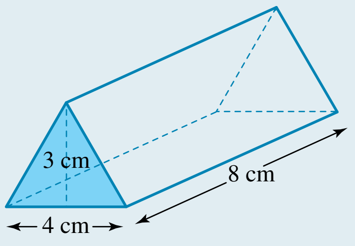
$A = 40,\ h = 8$

$V = 40 \times 8$

$= 320\ \text{cm}^3$

WORKED EXAMPLE 16 Calculating the volume of a prism

Calculate the volume of the prism.



THINK

1. The base of the prism is a triangle. Write the formula for the area of the triangle.
2. Substitute $b = 4$, $h = 3$ into the formula and evaluate.
3. Write the formula for the volume of a prism.
4. State the values of A and h.
5. Substitute $A = 6$ and $h = 8$ into the formula and evaluate. Include the units.

WRITE

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 4 \times 3$

$= 6\ \text{cm}^3$

$V = A \times h$

$A = 6,\ h = 8$

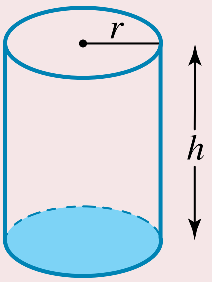
$V = 6 \times 8$

$= 48\ \text{cm}^3$

9.7.3 Volume of a cylinder

- A cylinder has a circular base and uniform cross-section.
- A formula for the volume of a cylinder is shown.

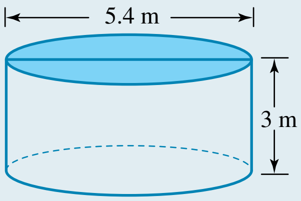
Volume of a cylinder



$$\begin{aligned}\textbf{Volume} &= \textbf{base area} \times \textbf{height}\\ &= \textbf{area of circle} \times \textbf{height}\\ &= \pi r^2 \times h\\ &= \pi r^2 h\end{aligned}$$

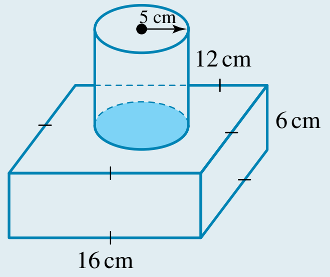
WORKED EXAMPLE 17 Calculating the capacity of a cylinder

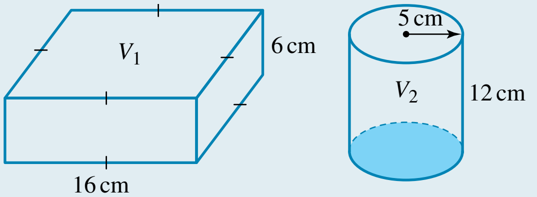
Calculate the capacity, in litres, of a cylindrical water tank that has a diameter of 5.4 m and a height of 3 m. Give your answer correct to the nearest litre.

THINK	WRITE/DRAW
1. Draw a labelled diagram of the tank.	
2. The base is a circle, so $A = \pi r^2$. Write the formula for the volume of a cylinder.	$V = \pi r^2 h$
3. Recognise the values of r and h. Recall that the radius, r, is half the diameter. $r = \frac{5.4}{2} = 2.7$ m	$r = 2.7,\ h = 3$
4. Substitute $r = 2.7$, $h = 3$ into the formula and calculate the volume.	$V = \pi \times (2.7)^2 \times 3$ $\approx 68.706\,631\ \text{m}^3$
5. Convert this volume to litres (multiply by 1000).	$V = 68.706\,631 \times 1000$ $= 68\,706.631$ $\approx 68\,707$ L

WORKED EXAMPLE 18 Calculating the volume of a solid

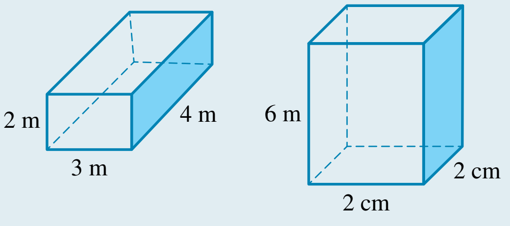
Calculate the total volume of this solid.



THINK	WRITE/DRAW
1. The solid is made from two objects. Draw and label each object. Let $V_1 =$ volume of the square prism. Let $V_2 =$ volume of the cylinder.	
2. Calculate V_1 (square prism).	$V_1 = Ah$ $= 16^2 \times 6$ $= 1536\text{ cm}^3$
3. Calculate V_2 (cylinder).	$V_2 = \pi r^2 h$ $= \pi \times 5^2 \times 12$ $\approx 942.478\text{ cm}^3$
4. To calculate the total volume, add the volumes found above.	$V = V_1 + V_2$ $= 1536 + 942.478$ $= 2478.478$ $\approx 2478\text{ cm}^3$

WORKED EXAMPLE 19 Comparing volume and surface area

Calculate the volume and surface area of the following rectangular prisms, compare and comment on your results.



THINK	WRITE
1. Use the volume formula to calculate the volume of the first shape.	$V = A \times h$ $= 3 \times 2 \times 4$ $= 24\text{ m}^3$

2. Use the volume formula to calculate the volume of the second shape.

$$V = A \times h$$
$$= 6 \times 2 \times 2$$
$$= 24 \text{ m}^3$$

3. Use the surface area formula to calculate the surface area of the first shape.

$$SA = 2(wh + lh + lw)$$
$$= 2(3 \times 2 + 4 \times 2 + 4 \times 3)$$
$$= 52 \text{ m}^2$$

4. Use the surface area formula to calculate the surface area of the second shape.

$$SA = 2(wh + lh + lw)$$
$$= 2(2 \times 6 + 2 \times 6 + 2 \times 2)$$
$$= 56 \text{ m}^2$$

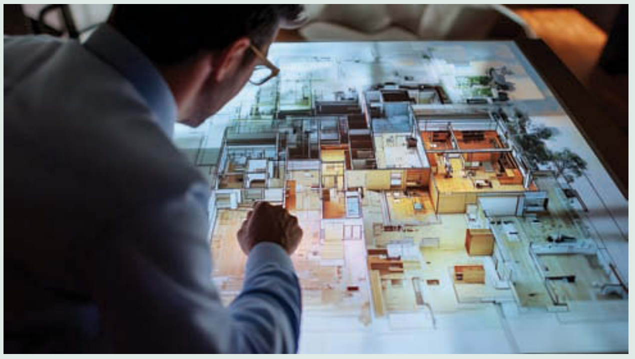
5. Compare the volume of each shape.

Both shapes are rectangular prisms but with different dimensions. Even though they have different dimensions the have the same volume of 24 m^3.

6. Compare the surface area of each shape.

Even though these two shapes have the same volume, they do not have the same surface area. For these shapes, their surface areas are 52 m^2 and 56 m^2.

COLLABORATIVE TASK: Designing 3D objects



In this task, you'll work in small groups to explore how two-dimensional shapes can be transformed into three-dimensional objects using dynamic geometry software.

Start by choosing a familiar object you can model in 3D. This could be something simple like a cube, a pyramid, a house, or even something creative like a trophy or a piece of playground equipment. The key is to start with two-dimensional shapes.

Use a dynamic geometry software like GeoGebra, Desmos or another similar tool to:

- Construct a two-dimensional shape or set of shapes.
- Apply transformations to turn these shapes into three-dimensional structures.
- Experiment with changing views and rotating your model to see it from different angles.

Once your model is complete, prepare a short presentation. Share your object, explain the transformations you used, and reflect on what you learned about how two-dimensional geometry builds the foundation for three-dimensional design.

Think about these questions:

- How did each transformation contribute to the final 3D object?
- What challenges did you face using the software or visualising depth?
- Why might this be useful in the real world, especially in design, architecture or engineering?

Exercise 9.7 Volume of prisms and cylinders

9.7 Quick quiz | 9.7 Exercise

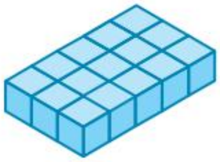
Individual pathways

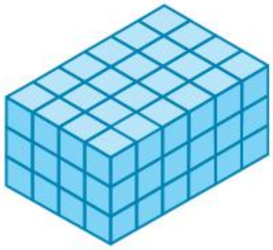
PRACTISE	CONSOLIDATE	MASTER
1, 4, 5, 8, 9, 12, 22, 25	2, 6, 10, 13, 14, 16, 17, 20, 23, 26	3, 7, 11, 15, 18, 19, 21, 24, 27, 28

Fluency

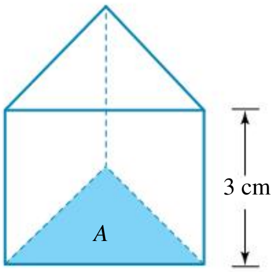
1. **WE14** Convert the following units into mL.
   a. 325 cm^3 b. 2.6 m^3 c. 5.1 L d. 0.63 kL
2. Convert the following units into cm^3.
   a. 5.8 mL b. 6.1 L c. 3.2 m^3 d. 59.3 mm^3
3. Convert the following units into kL.
   a. 358 L b. 55.8 m^3 c. 8752 L d. 5.3 ML
4. Calculate the volumes of the cuboids shown. Assume that each small cube has sides of 1 cm.

   a. 

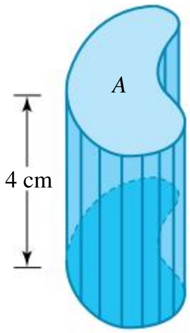
   b. 

   c. 

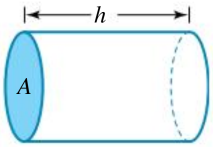
5. **WE15** Calculate the volumes of these objects.

   a. 

   $A = 4\text{ cm}^2$

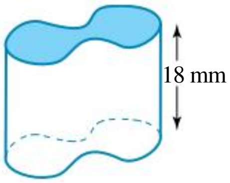
   b. 

   $A = 17\text{ cm}^2$

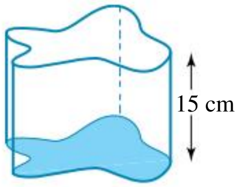
   c. 

   $A = 3.2\text{ m}^2$
   $h = 3.0\text{ m}$

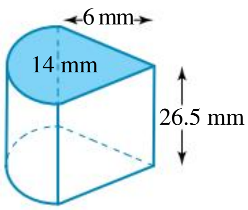
6. Calculate the volumes of these objects, correct to 1 decimal place where necessary.

   a. 

   Base area = 35 mm^2

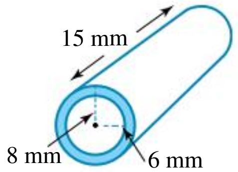
   b. 

   Base area = 28 cm^2

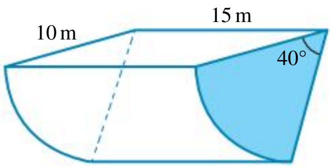
   c. 

7. Calculate the volumes of these objects, correct to 1 decimal place.

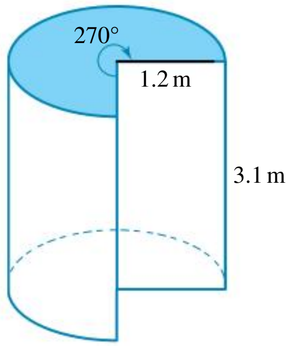
a.



b.

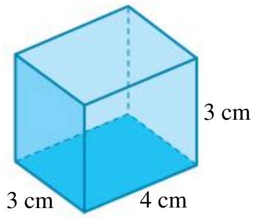


c.

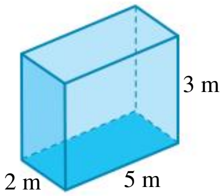


8. Calculate the volumes of these rectangular prisms.

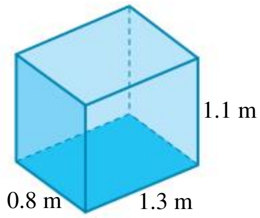
a.



b.

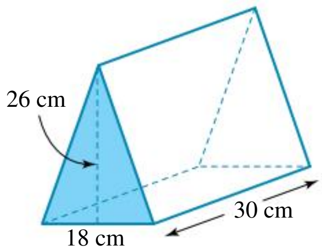


c.

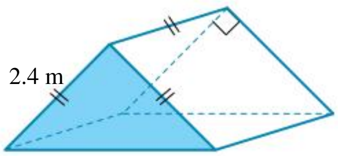


9. WE16 Calculate the volumes of the prisms shown. Give your answers correct to 2 decimal places where necessary.

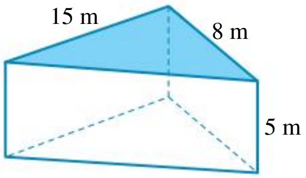
a.



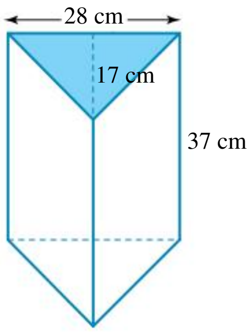
b.



c.

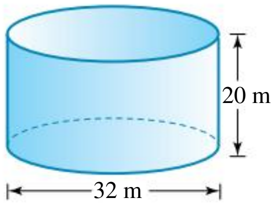


d.

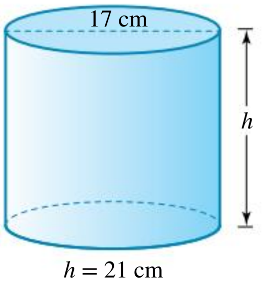


10. Calculate the volume of the following cylinders. Give your answers correct to 1 decimal place.

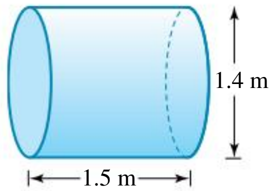
a.



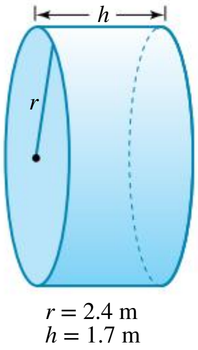
b.

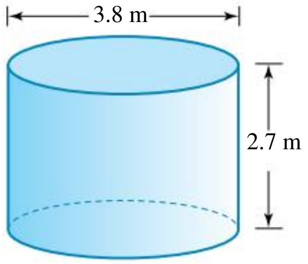


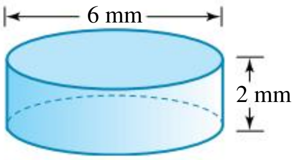
c.



11. Calculate the volume of the following cylinders. Give your answers correct to 1 decimal place.

a. 

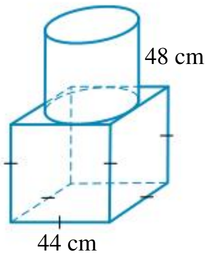
b. 

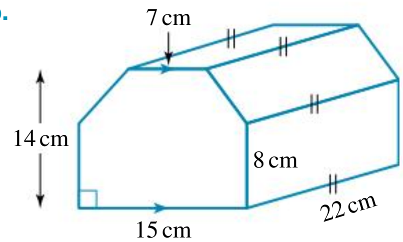
c. 

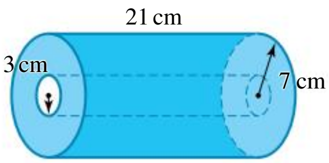
12. WE17 Calculate the volume of water, in litres (L), that can fill a cylindrical water tank that has a diameter of 3.2 m and a height of 1.8 m.

13. Calculate the capacity of the Esky shown, in litres.

14. WE18 Calculate the volume of each solid to the nearest cm^3.

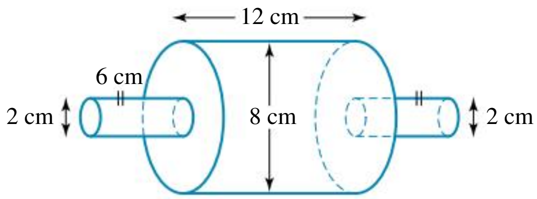
a. 

b. 

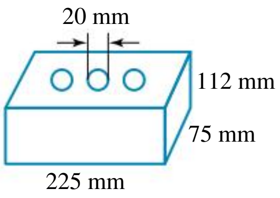
c. 

15. Calculate the volume of each solid to the nearest cm^3.

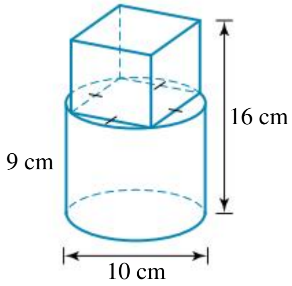
a.



b.



c.



Understanding

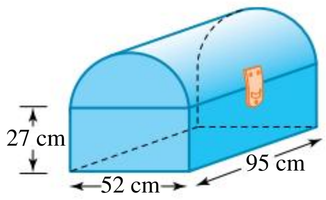
16. Calculate the capacity, in litres, of the cylindrical storage tank shown. Give your answer correct to 1 decimal place.

17. Calculate the capacity (in mL) of this cylindrical coffee plunger when it is filled to the level shown.
Give your answer correct to 1 decimal place.

18. Sudhira is installing a rectangular pond in a garden. The pond is 1.5 m wide, 2.2 m long and has a uniform depth of 1.5 m.
   a. Calculate the volume of soil $\left(\text{in } m^3\right)$ that Sudhira must remove to make the hole before installing the pond.
   b. Calculate the capacity of the pond in litres.
   (Ignore the thickness of the walls in this calculation.)

19. Until its closure in 2001, Fresh Kills on Staten Island outside New York City was one of the world's biggest landfill garbage dumps. (New Yorkers throw out about 100 000 tonnes of refuse weekly.)
Calculate the approximate volume $\left(\text{in } m^3\right)$ of the Fresh Kills landfill if it covers an area of 1215 hectares and is about 240 m high.
(*Note:* 1 hectare = 10 000 m^2.)

20. Calculate the internal volume (in cm^3) of the wooden chest shown, correct to 2 decimal places. (Ignore the thickness of the walls.)



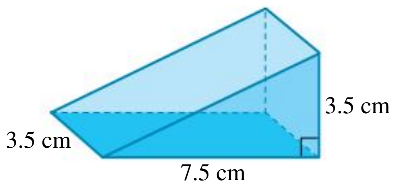
21. Answer the following questions using the given information.

a. Calculate the volume, in m^3, of the refrigerator shown.

b. Determine the capacity of the refrigerator if the walls are all 5 cm thick.

Reasoning

22. a. Calculate the volume of plastic needed to make the door wedge shown. Give your answer correct to 2 decimal places.



b. The wedges can be packed snugly into cartons with dimensions 45 cm × 70 cm × 35 cm. Determine how many wedges fit into each carton.

23. A cylindrical glass is designed to hold 1.25 L. Show that its height is 132 mm if it has a diameter of 110 mm.

24. Mark is responsible for the maintenance of the Olympic (50 m) pool at an aquatic centre. The figure shows the dimensions of an Olympic pool.

a. Identify the shape of the pool.
b. Draw the cross-section of the prism and calculate its area.
c. Show that the capacity of the pool is 1 650 000 L.
d. Mark needs to replace the water in the pool every 6 months. If the pool is drained at 45 000 L per hour and refilled at 35 000 L per hour, calculate how long it will take to:
   i. drain
   ii. refill (in hours and minutes).

Problem solving

25. A cylindrical container of water has a diameter of 16 cm and is 40 cm tall.
Determine how many full cylindrical glasses can be filled from the container if the glasses have a diameter of 6 cm and are 12 cm high.

26. An internal combustion engine consists of 4 cylinders. In each cylinder a piston moves up and down. The diameter of each cylinder is called the bore and the height that the piston moves up and down within the cylinder is called the stroke (stroke = height).
a. If the bore of a cylinder is 84 mm and the stroke is 72 mm, calculate the volume (in litres) of 4 such cylinders, correct to 2 decimal places.
b. When an engine gets old, the cylinders have to be 're-bored'; that is, the bore is increased by a small amount (and new pistons put in them).
If the re-boring increases the diameter by 1.1 mm, determine the increase in volume of the four cylinders (in litres).
Give your answer correct to 3 decimal places.

27. Answer the following questions.
a. A square sheet of metal with dimensions 15 cm by 15 cm has a 1 cm square cut out of each corner. The remainder of the square is folded to form an open box.
Calculate the volume of the box.
b. Write a general formula for calculating the volume of a box created from a metal sheet of any size with any size square cut out of the corners.

28. Identify two prisms that have the same volume but different surface areas.
Show your working.

LESSON 9.8 Review

9.8.1 Topic summary

MEASUREMENT

Units of measurement

- Length:

$$\text{kilometres (km)} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \text{metres (m)} \underset{\div 100}{\overset{\times 100}{\rightleftarrows}} \text{centimetres (cm)} \underset{\div 10}{\overset{\times 10}{\rightleftarrows}} \text{millimetres (mm)}$$

- Area:

$$\text{square kilometres (km}^2\text{)} \underset{\div 1000^2}{\overset{\times 1000^2}{\rightleftarrows}} \text{square metres (m}^2\text{)} \underset{\div 100^2}{\overset{\times 100^2}{\rightleftarrows}} \text{square centimetres (cm}^2\text{)} \underset{\div 10^2}{\overset{\times 10^2}{\rightleftarrows}} \text{square millimetres (mm}^2\text{)}$$

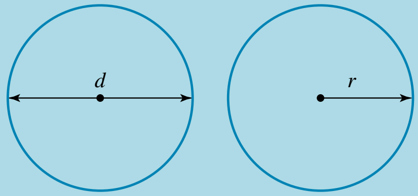
- Volume:

$$\text{cubic kilometres (km}^3\text{)} \underset{\div 1000^3}{\overset{\times 1000^3}{\rightleftarrows}} \text{cubic metres (m}^3\text{)} \underset{\div 100^3}{\overset{\times 100^3}{\rightleftarrows}} \text{cubic centimetres (cm}^3\text{)} \underset{\div 10^3}{\overset{\times 10^3}{\rightleftarrows}} \text{cubic millimetres (mm}^3\text{)}$$

- When converting between units of capacity and volume:
  - $1 \text{ cm}^3 = 1 \text{ mL}$
  - $1000 \text{ cm}^3 = 1 \text{ L}$
  - $1 \text{ m}^3 = 1000 \text{ L} = 1 \text{ kL}$

Perimeter and circumference

- The **perimeter** is the distance around a shape.
- The perimeter of a circle is called the **circumference**.
- The circumference of a circle can be calculated using either of the following formulas, where r is the radius of the circle and d is the diameter of the circle:
  - $C = 2\pi r$
  - $C = \pi d$

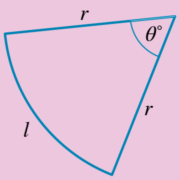


Area

- The area of a shape is the amount of space it takes up.
- The areas of common shapes can be calculated using the following formulas:
  - Square: $A = l^2$
  - Rectangle: $A = lw$
  - Parallelogram: $A = bh$
  - Triangle: $A = \frac{1}{2}bh$
  - Trapezium: $A = \frac{1}{2}(a + b)h$
  - Kite: $A = \frac{1}{2}xy$
  - Circle: $A = \pi r^2$
  - Rhombus: $A = \frac{1}{2}xy$

Area and perimeter of a sector

- The curved part of a sector is often labeled l. Its length can be calculated by the formula $l = \frac{\theta}{360} \times 2\pi r$, where r is the radius and θ is the angle between the two straight sides.
- The perimeter of a sector is: $P = 2r + l$
- The area of a sector is: $A = \frac{\theta}{360} \times \pi r^2$



Surface area

- The surface area of an object can be calculated by adding the areas of all its faces.
- The formula for the surface area of a cylinder is: $SA = 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$

Volume

- The volume of a prism is equal to its cross-sectional area multiplied by its height: $V_{\text{prism}} = A \times h$
- The volume of a cube is: $V_{\text{cube}} = l^3$
- The volume of a cuboid is: $V_{\text{cuboid}} = lwh$
- The volume of a cylinder is: $V_{\text{cylinder}} = \pi r^2 h$
- Capacity is another term for volume, and is usually applied to the measurement of liquids and containers.

9.8.2 Success criteria

Tick a column to indicate that you have completed the lesson and how well you think you have understood it using the traffic light system.
(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

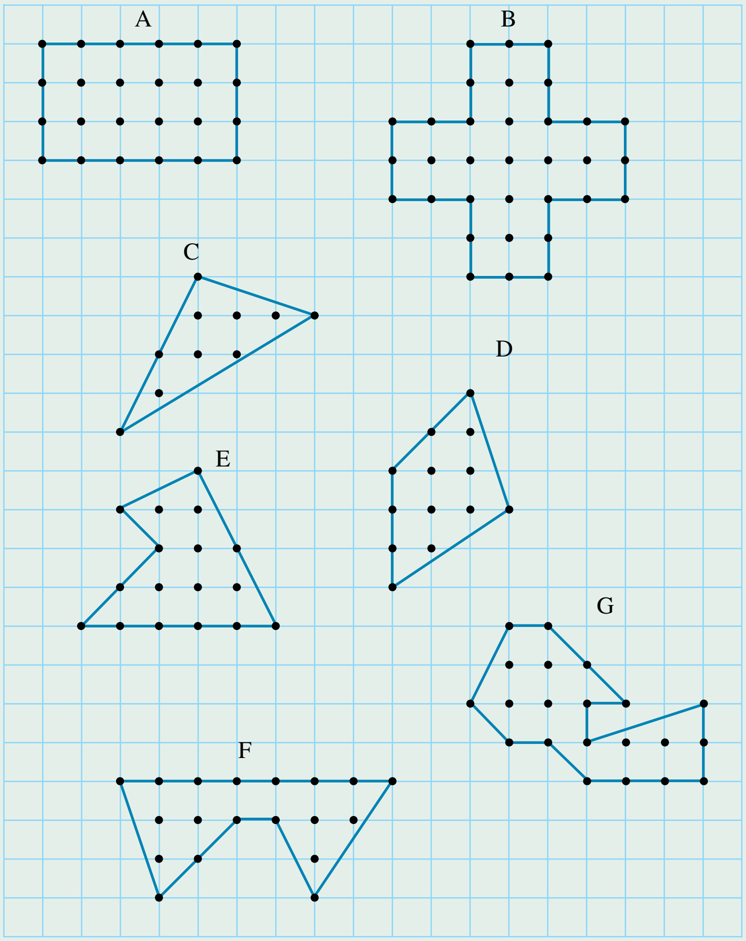
Lesson	Success criteria	Green	Yellow	Red
9.2	I can convert between different units of length.			
	I can calculate the perimeter of a given shape.			
	I can calculate the circumference of a circle.			
9.3	I can convert between units of area.			
	I can calculate areas of triangles, quadrilaterals and circles using formulas.			
	I can determine areas of composite shapes.			
9.4	I can identify sectors, semicircles and quadrants.			
	I can calculate the area of a sector.			
	I can calculate the length of the arc and the perimeter of a sector.			
9.5	I can calculate the surface area of rectangular and triangular prisms.			
9.6	I can draw the net of a cylinder.			
	I can calculate the curved surface area of a cylinder.			
	I can calculate the total surface area of a cylinder.			
9.7	I can calculate the volume of prisms and cylinders.			
	I can convert between units of volume.			
	I can convert between units of capacity and units of volume.			

9.8.3 Project

Areas of polygons

The area of plane figures can be found using a formula. For example, if the figure is a rectangle, its area is found by multiplying its length by its width. If the figure is a triangle, the area is found by multiplying half the base length by its perpendicular height. If the figures are complex, break them down into simple shapes and determine the area of each of these shapes. The area of a complex shape is the sum of the area of its simple shapes. Drawing polygons onto grid paper is one method that can be used to determine their area.

Consider the following seven polygons drawn on 1-cm grid paper.



1. By counting the squares or half-squares, determine the area of each polygon in cm^2. In some cases, it may be necessary to divide some sections into half-rectangles in order to determine the exact area.

Formulas define a relationship between dimensions of figures. In order to search for a formula to calculate the area of polygons drawn on the grid paper, consider the next question.

2. For each of the polygons provided above, complete the following table. Count the number of dots on the perimeter of each polygon and count the number of dots that are within the perimeter of each polygon.

Polygon	Dots on perimeter (b)	Dots within perimeter (i)	Area of polygon (A)
A			
B			
C			
D			
E			
F			
G			

3. Choose a pronumeral to represent the headings in the table. Investigate and determine a relationship between the area of each polygon and the dots on and within the perimeter. Test that the relationship determined works for each polygon. Write the relationship as a formula.
4. Draw some polygons on grid paper. Use your formula to determine the area of each shape. Confirm that your formula works by counting the squares in each polygon. The results of both methods (formula and method) should be the same.

Exercise 9.8 Review questions

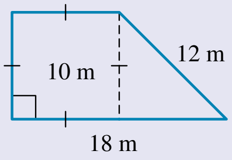
learnon

Fluency

1. MC Identify which of the following is true.
   A. 5 cm is 100 times as big as 5 mm.
   B. 5 metres is 100 times as big as 5 cm.
   C. 5 km is 1000 times as small as 5 metres.
   D. 5 mm is 100 times as small as 5 metres.

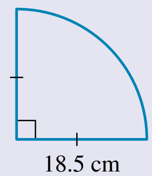
2. MC The circumference of a circle with a diameter of 12.25 cm is:
   A. 471.44 cm B. 384.85 mm C. 76.97 cm D. 117.86 cm

3. MC The area of the given shape is:
   A. 216 m^2 B. 140 m^2 C. 150 m^2 D. 90 m^2



4. MC The area of a circle with diameter 7.5 cm is:
   A. 176.7 cm^2 B. 47.1 cm^2 C. 23.6 cm^2 D. 44.2 cm^2

5. MC The perimeter of the shape shown is:
   A. 1075.2 cm B. 55.5 cm C. 153.2 cm D. 66.1 cm



6. MC The surface area and volume of a cube with side length 7 m are respectively:
   A. 294 m^2, 343 m^3 B. 49 m^2, 343 m^3 C. 147 m^2, 49 m^3 D. 28 m^2, 84 m^3

7. MC The surface area of a rectangular box with dimensions 7 m, 3 m, 2 m is:
   A. 41 m^2 B. 42 m^2 C. 72 m^2 D. 82 m^2

8. MC The surface area and volume of a cylinder with radius 35 cm and height 40 cm are:
   A. 16 493.36 cm^2 and 153 938 cm^3
   B. 8796.5 cm^2 and 11 246.5 cm^3
   C. 153 938 cm^2 and 11 246.5 cm^3
   D. 8796.5 cm^2 and 153 938 cm^3

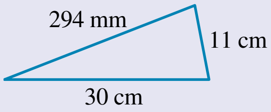
9. Convert the following lengths into the required units.
   a. 26 mm = ________ cm
   b. 1385 mm = ________ cm
   c. 1.63 cm = ________ mm
   d. 1.5 km = ________ m
   e. 0.077 km = ________ m
   f. 2850 m = ________ km

10. Calculate the cost of 1.785 km of cable, if the cable costs $4.20 per metre.

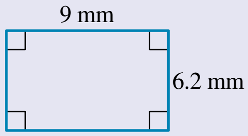
11. Calculate the circumference of circles with the following dimensions (correct to 1 decimal place).
    a. Radius 4 cm b. Radius 5.6 m c. Diameter 12 cm

12. Calculate the perimeters of the following shapes (correct to 1 decimal place).

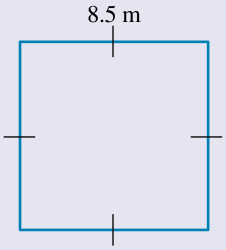
a.



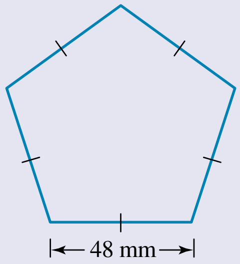
b.



c.

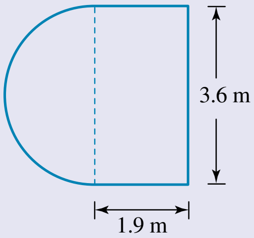


d.

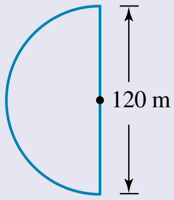


13. Calculate the perimeter of the following shapes (correct to 1 decimal place).

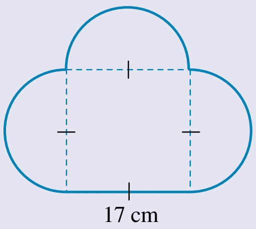
a.



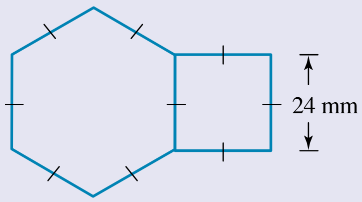
b.



c.

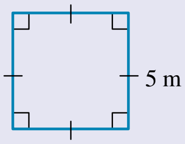


d.

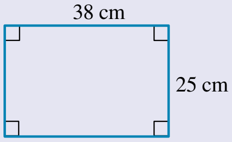


14. Calculate the area of the following shapes.

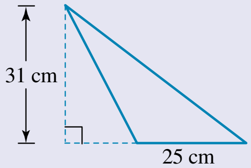
a.



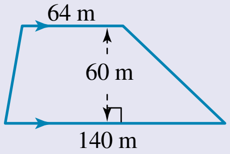
b.



c.

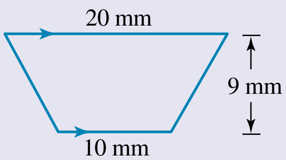


d.

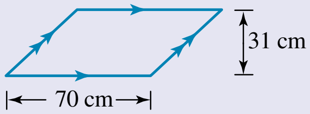


15. Calculate the area of the following shapes, correct to 1 decimal place where necessary.

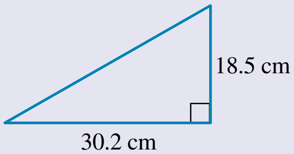
a.



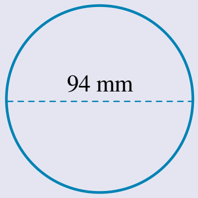
b.



c.

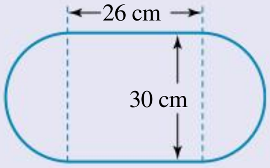


d.

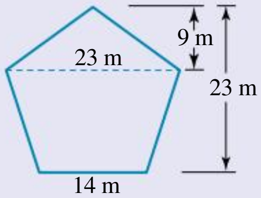


16. Calculate the area of the following 2-dimensional shapes by dividing them into simpler shapes. (Where necessary, express your answer correct to 1 decimal place.)

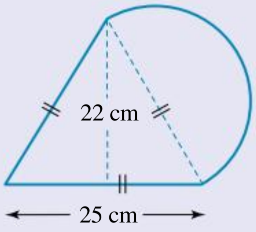
a.



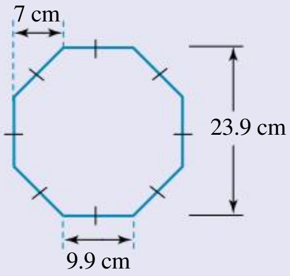
b.



c.

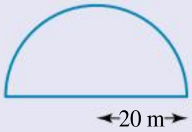


d.

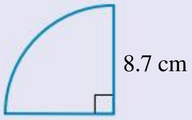


17. Calculate the area of the sectors shown (correct to 2 decimal places).

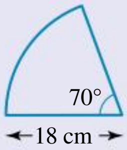
a.



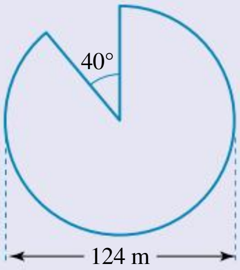
b.



c.



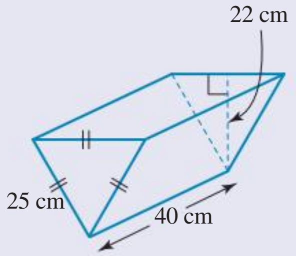
d.



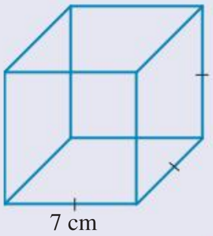
18. Calculate the perimeter of the sectors in question 14, correct to 2 decimal places.

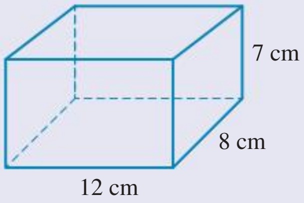
19. Calculate the inner surface area of the grape-collecting vat using the dimensions for length, width and depth shown.

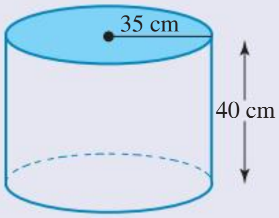
20. Calculate the surface area of the triangular prism shown.

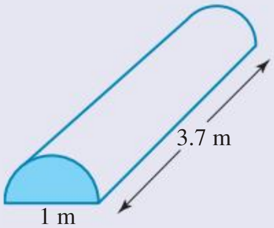


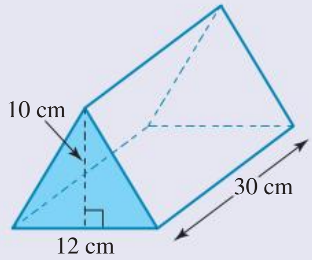
21. Calculate the volume of each of the following, giving your answers to 2 decimal places where necessary.

a. 

b. 

c. 

d. 

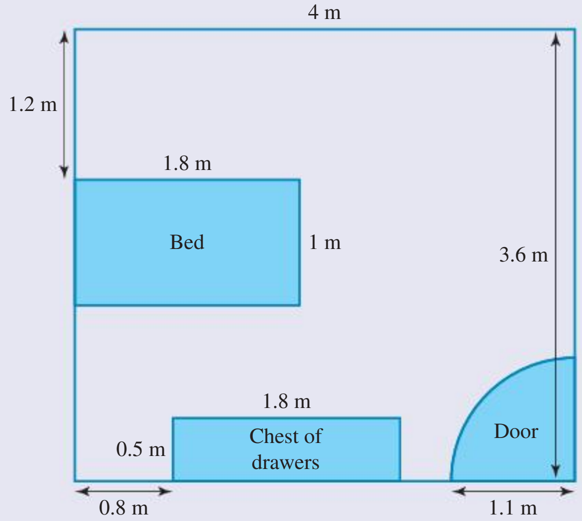
e. 

Problem solving

22. Calculate the cost of painting an outer surface (including the lid) of a cylindrical water tank that has a radius of 2.2 m and a height of 1.6 m, if the paint costs \$1.90 per square metre.

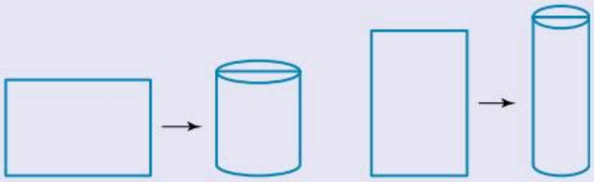
23. A harvester travels at 11 km/h. The comb (harvesting section) is 8.7 m wide. Determine how many hectares per hour can be harvested with this comb.

24. Nina decides to invite some friends for a sleepover. She plans to have her guests sleep on inflatable mattresses with dimensions 60 cm by 170 cm. A plan of Nina's bedroom is shown.

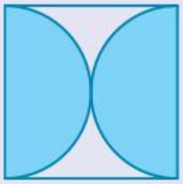


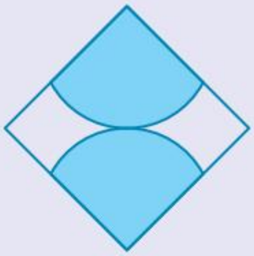
a. Calculate the area of floor space available.
b. The area of an inflatable mattress is 1.02 m^2. Nina's best friend suggests that, by dividing the available floor space by the area of one inflatable mattress, the number of mattresses that can fit into the bedroom can be calculated. Explain whether the friend is correct.
c. Determine how many self-inflatable mattresses can fit in the room as it is.
d. Explain how your answer to c would change if the bed could be moved within the bedroom.

25. An A4 sheet of paper has dimensions $210\,\text{mm} \times 297\,\text{mm}$ and can be rolled two different ways (by rotating the paper) to make baseless cylinders, as illustrated.
Compare the volumes to decide which shape has the greatest volume.



26. These congruent squares have shaded parts of circles inside them.

a.


b.


Compare the shaded area in figure **a** with that in figure **b**.

27. The widespread manufacture of soft drinks in cans only really started in the 1960s. Today cans of soft drink are very common. They can be purchased in a variety of sizes. You can also purchase them as a single can or bulk-buy them in larger quantities.
Does the arrangement of the cans in a multi-pack affect the amount of packaging used to wrap them?
For this investigation we will look at the packaging of 375 mL cans. This size of can has a radius of approximately 3.2 cm and a height of approximately 12 cm.
We will consider packaging these cans in cardboard in the shape of a prism, ignoring any overlap for glueing.
   a. A 6-can pack could be packed in a single row of cans.
      i. Draw a diagram to show what this pack would look like.
      ii. Calculate the amount of cardboard required for the package.
   b. The cans could also be packaged in a 3×2 arrangement.
      i. Show the shape of this pack.
      ii. Calculate the amount of cardboard required for this pack.
   c. Evaluate which of the two 6-can pack arrangements is the more economical in terms of packaging.
   d. These cans can be packaged as a single layer of 4×3. Calculate how much cardboard this would require.
   e. They could also have a two-level arrangement of 3×2 each layer. Determine what packaging this would require.
   f. It is also possible to have a three-level arrangement of 2×2 each layer. Determine how much packaging would be needed in this case.
   g. Comment on the best way to package a 12-can pack so that it uses the minimum amount of wrapping.
   Provide diagrams and mathematical evidence to support your conclusion.

on To test your understanding and knowledge of this topic, go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 9 Measurement

9.1 Pre-test

1. 15.4 cm
2. 12.48 cm^2
3. 30 cm
4. C
5. 1440 mm^3
6. D
7. B
8. D
9. B
10. 3.48 m^2
11. D
12. 3100 cm^3
13. 678.6 mL
14. 25 m
15. 527.8 cm^2

9.2 Perimeter and circumference

1. a. 50 mm b. 152 cm c. 0.0125 m d. 32.2 mm
2. a. 0.006 57 km b. 0.000 64 km c. 14.35 mm d. 0.000 183 5 km
3. a. 101 mm b. 54 mm c. 60 mm d. 103 mm
4. a. 1060 cm b. 85.4 cm c. 206 cm
5. a. 78.4 cm b. 113 cm c. 13 cm
6. a. 9.6 cm b. 46 mm c. 31 km
7. a. 25.13 cm b. 25.13 m c. 69.12 mm
8. a. 44.61 cm b. 19 741.77 km c. 3314.38 mm
9. a. 192 m b. 1220 mm c. 260 cm
10. a. 74 mm b. 9.6 km c. 8 m
11. B
12. a. 1800 mm, 2100 mm, 2400 mm, 2700 mm, 180 cm, 210 cm, 240 cm, 270 cm, 1.8 m, 2.1 m, 2.4 m, 2.7 m
    b. 4
    c. 3300 mm
13. a. 1990 m b. 841 m
14. a. 127.12 cm b. 104.83 cm c. 61.70 cm
15. a. 8 m b. 480 mm c. 405.35 cm
16. a. 125.66 cm b. 245.66 m c. 70.41cm
17. 222.5 m
18. 37.5 km
19. 12 m
20. a. $r = 6371 + 559$
    $= 6930$ km
    $C = 2\pi r$
    $= 2 \times \pi \times 6930$
    $= 13\,860\pi$
    $= 43\,542.474$
    $\approx 43\,542$ km
    b. 96 minutes $= 96 \times 60$
    $= 5760$ seconds
    43 542 km $=$ 43 542 000 m
    Speed $= 43\,542\,000 \div 5760$
    $= 7559.375\ldots$
    ≈ 7559 m/s
21. a. 500 m/s $=$ (500 $\times$ 100) cm/s
    $=$ 50 000 cm/s
    b. 0.000 02 seconds
    c. 2
22. a. 155.62 m
    b. 155.62 m $\times$ 1.5 minutes/metre $=$ 233.43 minutes
23. a. 40 212 km
    b. 1676 km/h
    c. Sample responses can be found in the worked solutions in the online resources.
24. Yes, 42 cm of wire remains.
25. 69.3 cm
26. a. $\left(2\sqrt{13} + 4\right)$ m b. $2\sqrt{17}$ m

9.3 Area

1. D
2. a. 13 400 m^2 $=$ 0.0134 km^2
   b. 0.04 cm^2 $=$ 4 mm^2
   c. 3 500 000 cm^2 $=$ 350 m^2
   d. 0.005 m^2 $=$ 50 cm^2
3. a. 0.043 km^2 $=$ 43 000 m^2
   b. 200 mm^2 $=$ 2 cm^2
   c. 1.41 km^2 $=$ 141 ha
   d. 3800 m^2 $=$ 0.38 ha
4. a. 24 cm^2 b. 16 mm^2 c. 537.5 cm^2
5. a. 149.5 cm^2 b. 16.32 m^2 c. 11.25 cm^2
6. a. 292.5 cm^2 b. 2.5 cm^2 c. 1250 m^2
7. a. 50.27 m^2 b. 3.14 mm^2 c. 36.32 m^2
8. a. 15.58 cm^2 b. 4.98 cm^2 c. 65 m^2
9. a. 6824.2 mm^2 b. 7.6 m^2 c. 734.2 cm^2
10. a. 578.5 cm^2 b. 7086.7 m^2 c. 5.4 m^2
11. a. 1143.4 m^2 b. 100.5 cm^2
12. a. 821 cm^2 b. 661.3 mm^2
13. 378 cm^2
14. 19.3 cm^2
15. a. 100.53 cm^2 b. 1244.07 m^2
16. a. 301.59 cm^2 b. 103.67 mm^2
17. B
18. 75.7 cm or 757 mm
19. $29 596.51
20. a. 5000 km^2 b. 0.5%
21. $10 150
22. a. 619.5 cm^2
    b. 10.5 cm
    c. 346.36 cm^2
    d. Sample responses can be found in the worked solutions in the online resources.

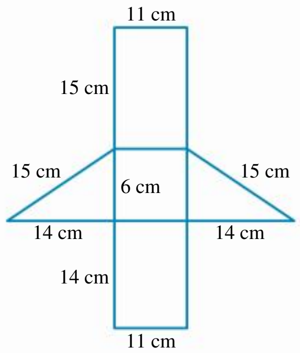
23. Sample responses can be found in the worked solutions in the online resources.
24. Sample responses can be found in the worked solutions in the online resources.
25. a. 1600 cm^2
b. 800 cm^2
c. 400 cm^2
d. 200 cm^2
e. The area halves each time.
f. 12.5%
g. Sample responses can be found in the worked solutions in the online resources.
26. Sample responses can be found in the worked solutions in the online resources.
27. a. $\sqrt{50} = 5\sqrt{2}$ m b. 10 logs
28. 17 m
29. 45 m^2
30. a. $\frac{1}{4}$ b. $\frac{1}{4}\left(x^2 + 6x + 8\right)$
c. $\frac{1}{4}\left(3x^2 - 6x - 8\right)$
d. i. $\frac{1}{12}\left(3x^2 - 6x - 8\right)$
ii. $\frac{1}{6}\left(3x^2 + 6x + 8\right)$

9.4 Area and perimeter of a sector

1. a. 14.14 cm^2 b. 157.08 cm^2
c. 27.71 cm^2 d. 226.19 mm^2
2. a. 14.3 cm b. 12.6 cm^2
3. a. 43.6 cm b. 116.9 cm^2
4. a. 40.7 m b. 102.1 m^2
5. a. 5.4 m b. 1.8 m^2
6. D
7. a. 184.6 cm b. 1869.2 cm^2
8. a. 66.8 m b. 226.2 m^2
9. a. 27.4 cm b. 42.4 cm^2
10. a. 342.1 m b. 7243.6 m^2
11. a. 354.6 cm b. 7645.9 cm^2
12. a. 104.4 cm b. 201.1 cm^2
13. 60 318.6 m^2
14. 303.4 cm
15. 170.2 m^2
16. 2 : 1
17. a. 14.8 m^2 b. 4.4%
18. a. 251 cm b. 13 cm
19. a. 5027 m^2 b. 78.5% c. \$61 440
20. a. Sample responses can be found in the worked solutions in the online resources.
b. 88.4 cm^2
c. 50.3 cm^2
d. 109.4 cm^2
e. No, Jim eats more cake
21. Sample responses can be found in the worked solutions in the online resources.
22. a. Sample responses can be found in the worked solutions in the online resources.
b. 85 m each
c. 60 m each
d. Sample responses can be found in the worked solutions in the online resources.
e. 37 m
23. \$250
24. a. πr^2 cm^2
b. $\pi(r + 1)^2$ cm^2
c. Sample responses can be found in the worked solutions in the online resources.
d. 5π cm^2
e. Sample responses can be found in the worked solutions in the online resources.
25. Radius $= \pi$ cm

9.5 Surface area of rectangular and triangular prisms

1. a. 66 cm^2 b. 62 m^2 c. 6.7 m^2
2. a. 4.44 m^2 b. 11 572.92 cm^2 c. 1.9 m^2
3. a. 86 cm^2 b. 210.7 m^2 c. 8.37 cm^2
4. a. 840 cm^2 b. 191.08 cm^2 c. 2370 mm^2
5. 2 cans of paint
6. 77.76 cm^2
7. 26.44 m^2
8. 20.36 m^2
9. 261.5 cm^2
10. 2.21 m^2
11. 15.2 m^2
12. a. 2080 cm^2
b. Sample responses can be found in the worked solutions in the online resources.
13. 168 cm^2
14. a. 150 cm^2 b. 250 cm^2 c. 350 cm^2 d. 850 cm^2
15. The solid formed is a cube with side length 4 cm.
16. Sample responses can be found in the worked solutions in the online resources.
17. a.



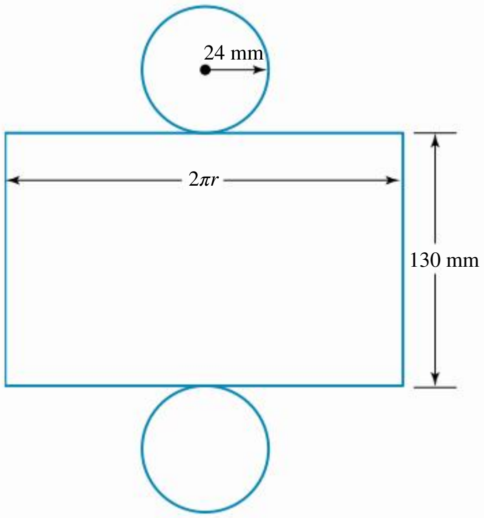
b. 315 cm^2

18. \$145.85 (2 L of ceiling paint, 1 L + 4 L for walls)

19. a. 1402.52 m^2 b. 3220 m^3

9.6 Surface area of a cylinder

1. a. 75.4 m^2 b. 131.9 m^2
2. a. 28.3 cm^2 b. 84.8 cm^2
3. a. 2010.6 m^2 b. 3619.1 m^2
4. a. 1121.5 cm^2 b. 1575.5 cm^2
5. a. 6.6 m^2 b. 9.7 m^2
6. a. 25.6 m^2 b. 61.8 m^2
7. a.



b. 232 cm^2

8. 1837.8 cm^2
9. 31.79 cm
10. a. Asparagus is largest; beetroot is smallest.
    b. 118 cm^2
11. a. 136.66 m^2 b. \$7351.33
12. a. 2035.75 cm^2 b. 1526.81 cm^2
    c. 395.84 cm^2 d. 3958.41 cm^2
13. 154.73 cm^2
14. a. 3.82 cm b. 427.7 cm^2
15. $12\pi h^2$
16. Sample responses can be found in the worked solutions in the online resources.
17. Sample responses can be found in the worked solutions in the online resources.
18. 83.6 m^2
19. The area to be painted is 1.63 m^2.
20. 11 231.12 cm^2
21. 4.34 m^2

9.7 Volume of prisms and cylinders

1. a. 325 mL b. 2 600 000 mL
   c. 5100 mL d. 630 000 mL
2. a. 5.8 cm^3 b. 6100 cm^3
   c. 3 200 000 cm^3 d. 0.0593 cm^3
3. a. 0.358 kL b. 55.8 kL
   c. 8.752 kL d. 5300 kL
4. a. 36 cm^3 b. 15 cm^3 c. 72 cm^3
5. a. 12 cm^3 b. 68 cm^3 c. 9.6 m^3
6. a. 630 mm^3 b. 420 cm^3 c. 3152.7 mm^3
7. a. 1319.5 mm^3 b. 523.6 m^3 c. 10.5 m^3
8. a. 36 cm^3 b. 30 m^3 c. 1.144 m^3
9. a. 7020 cm^3 b. 6.91 m^3 c. 300 m^3 d. 8806 cm^3
10. a. 16 085.0 m^3 b. 4766.6 cm^3 c. 2.3 m^3
11. a. 30.8 m^3 b. 30.6 m^3 c. 56.5 mm^3
12. 14 476.5 L
13. 176.4 L
14. a. 158 169 cm^3 b. 4092 cm^3 c. 2639 cm^3
15. a. 641 cm^3 b. 1784 cm^3 c. 1057 cm^3
16. 75 322.8 L
17. 1145.1 mL
18. a. 4.95 m^3 b. 4950 L
19. 2.916×10^9 m^3
20. 234 256.54 cm^3
21. a. 1.1475 m^3 b. 840 L
22. a. 45.94 cm^3
    b. 2400 (to the nearest whole number)
23. Sample responses can be found in the worked solutions in the online resources.
24. a. Prism
    b. 75 m^2
    c. Sample responses can be found in the worked solutions in the online resources.
    d. i. 36 h 40 min ii. 47 h 9 min
25. 23
26. a. 1.60 L b. 0.042 L
27. a. 169 cm^3
    b. $V = x(l - 2x)(w - 2x)$
28. There are many possible answers. An example of one possible solution.
    Shape 1: Length = 6 m, Width = 4 m, Height = 2 m
    Shape 2: Length = 8 m, Width = 4.5 m, Height = 2 m

Project

1. A. 15 cm^2 B. 20 cm^2 C. 6 cm^2 D. 8.5 cm^2
   E. 10.5 cm^2 F. 11.5 cm^2 G. 12.5 cm^2

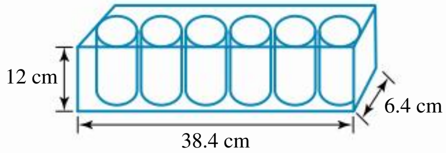
2.

Polygon	Dots on perimeter (b)	Dots within perimeter (i)	Area of polygon (A)
A	16	8	15
B	24	9	20
C	4	6	6
D	7	6	8.5
E	11	6	10.5
F	13	6	11.5
G	15	6	12.5

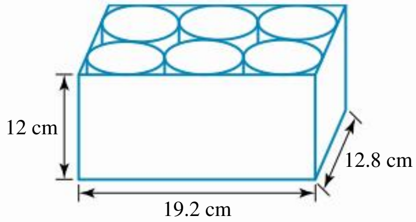
3. Sample responses can be found in the worked solutions in the online resources.
4. Sample responses can be found in the worked solutions in the online resources.

9.8 Review questions

1. B
2. B
3. B
4. D
5. D
6. A
7. D
8. A
9. a. 26 mm = 2.6 cm
   b. 1385 mm = 138.5 cm
   c. 1.63 cm = 16.3 mm
   d. 1.5 km = 1500 m
   e. 0.077 km = 77 m
   f. 2850 m = 2.85 km
10. $7497
11. a. 25.1 cm b. 35.2 m c. 37.7 cm
12. a. 70.4 cm b. 30.4 mm
    c. 34 m d. 240 mm or 24 cm
13. a. 13.1 m b. 308.5 m
    c. 97.1 cm d. 192.0 mm or 19.2 cm
14. a. 25 m^2 b. 950 cm^2
    c. 387.5 cm^2 d. 6120 m^2
15. a. 135 mm^2 b. 2170 cm^2
    c. 279.4 cm^2 d. 6939.8 mm^2
16. a. 1486.9 cm^2 b. 362.5 m^2
    c. 520.4 cm^2 d. 473.2 cm^2
17. a. 628.32 m^2 b. 59.45 cm^2
    c. 197.92 cm^2 d. 10 734.47 m^2
18. a. 102.83 m b. 31.07 cm
    c. 57.99 cm d. 470.27 m
19. 48 920 cm^2 or 4.892 m^2
20. 3550 cm^2
21. a. 343 cm^3 b. 672 cm^3
    c. 153 938.04 cm^3 d. 1.45 m^3
    e. 1800 cm^3
22. $99.80
23. 9.57 hectares
24. a. 10.75 m^2
    b. No
    c. 8
    d. 9 mattresses could fit in the room.
25. The cylinder of height 210 mm has the greater volume.
26. The shaded parts have the same area.
27. a. i.



    ii. 1566.72 cm^2
    b. i.



    ii. 1259.52 cm^2
    c. 3×2 arrangement
    d. 2058.24 cm^2
    e. 2027.52 cm^2
    f. Yes, 2170.88 cm^2
    g. The smallest amount of packaging required is 3×2, with two levels of packaging.

STRAND: Measurement and geometry
SUB-STRAND: Two-dimensional space and structures

10 Congruence and similarity

LESSON SEQUENCE

LESSON 10.1 Overview

10.1.1 Why learn this?

Geometry allows us to explore our world in very precise ways. It is also one of the oldest areas of mathematics. It involves the study of points, lines and angles and how they can be combined to make different shapes. Similarity and congruence are two important concepts in geometry. When trying to determine whether two shapes are exactly the same, or if they are enlargements of each other, the answers can be found by considering the sides and angles of those shapes.

When you take a photo on your phone it is very small. If you save it to your computer you can make it larger. The larger photo is an enlargement of the original photo — this is an example of how similar figures work in the everyday world.

The principle of similar triangles can be used to work out the height of tall objects by calculating the length of their shadows. This technique was extremely important in early engineering and architecture. Today, architects and designers still prepare scale diagrams before starting the building process.

In manufacturing, the products that come off an assembly line all have exactly the same shape and size. These products can be described as congruent. Designers, engineers and surveyors all use the concepts of congruence and similarity in their daily work.

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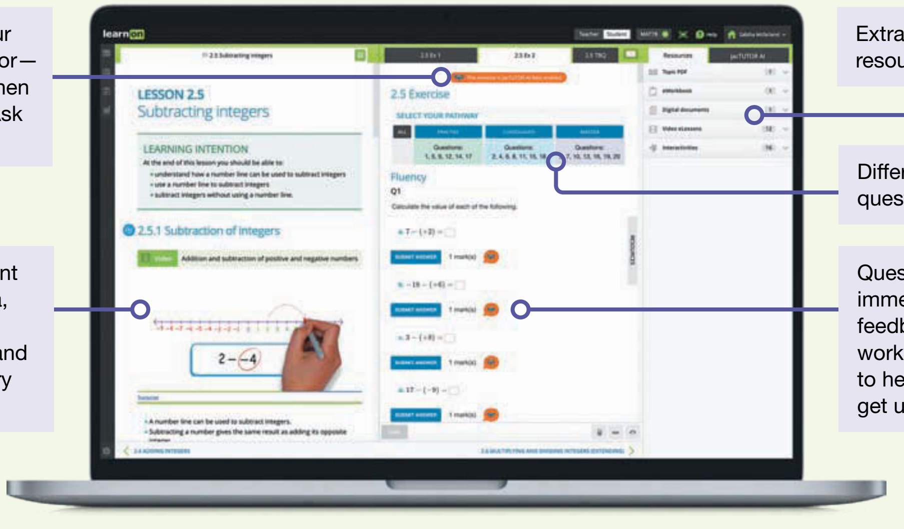


Extra learning resources

Differentiated question sets

Questions with immediate feedback, and fully worked solutions to help students get unstuck.

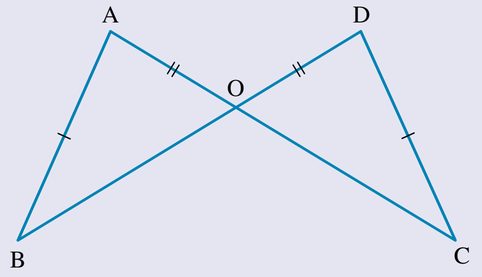
1. Express the ratio $1\frac{2}{3} : 2\frac{3}{5}$ in simplest form.

$$1\frac{2}{3} : 2\frac{3}{5} = \boxed{} : \boxed{}$$

2. Determine the value of x in the proportion $2 : 3 = 9 : x$

3. **MC** Select the correct symbol for congruence.
   A. $=$
   B. $\approx$
   C. $\sim$
   D. $\cong$

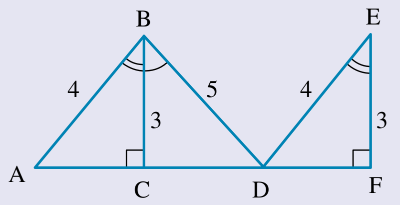
4. **MC** If OB = OC, choose which of the following congruency tests can be used to prove $\Delta AOB \cong \Delta DOC$.
   A. AAA
   B. RHS
   C. SSS
   D. AAS



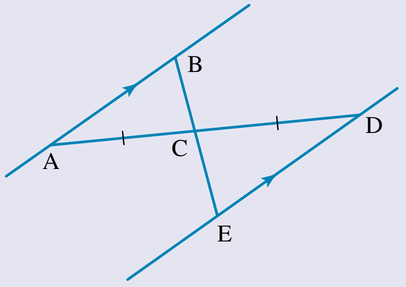
5. Determine the simplest ratio of $y : z$ if $3x = 2y$ and $4x = 3z$.

6. The ratio 1.4 : 0.2 in its simplest form is 14 : 2. State whether this statement is True or False.

7. **MC** Select the only pair of congruent triangles from the following.
   A. $\Delta ABC \cong \Delta BCD$
   B. $\Delta ABD \cong \Delta BCD$
   C. $\Delta BCD \cong \Delta DEF$
   D. $\Delta ABC \cong \Delta DEF$

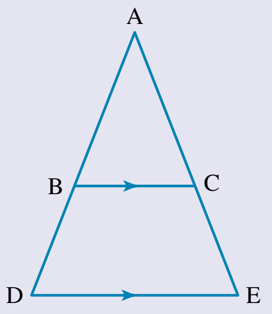


8. For the diagram given:
   a. name the angle that is equal to angle ABC
   b. state the congruency test that can be used to prove $\Delta ABC \cong \Delta CED$.



9. MC Select which similarity test can be used to prove $\Delta ABC \sim \Delta ADE$.

A. AAA
B. RHS
C. SSS
D. AAS



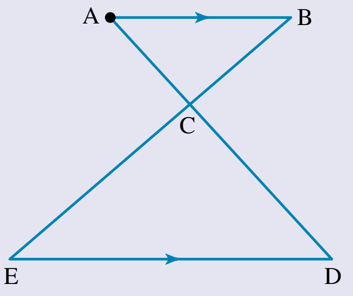
10. MC ΔABC and ΔCDE are similar.
Choose which of the following statements is true.

A. $\dfrac{AB}{ED} = \dfrac{BC}{CD}$

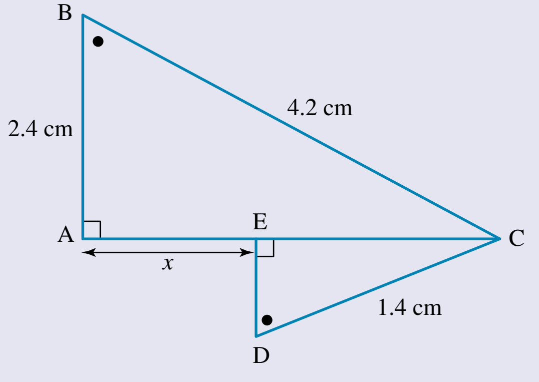
B. $\dfrac{AB}{ED} = \dfrac{AC}{CD}$

C. $\dfrac{AC}{EC} = \dfrac{BC}{CD}$

D. $\dfrac{AB}{ED} = \dfrac{BC}{CE}$

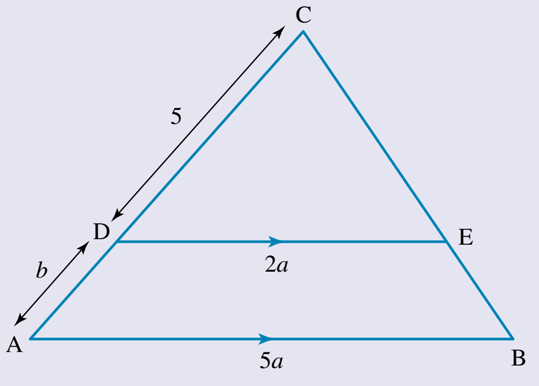


11. ΔABC and ΔCDE are similar.



Determine the value of the length x correct to 1 decimal place.

12. ΔABC and ΔCDE are similar. Determine the value of b.



13. A pair of cones are similar. The ratio of their volumes is 27 : 125.

a. The perpendicular height ratio of the two cones is $\square : \square$.

b. The areas of the base ratio of the two cones is $\square : \square$.

14. A pair of rectangles are similar. If the width of the first rectangle is 3 times the width of the other, the ratio of their areas is:
larger area : smaller area = $\square : \square$.

15. **MC** A rectangular box has a volume of 60 cm^3 and a surface area of 94 cm^2. Select the volume and surface area of a similar box that has side lengths that are twice the size of the original.

A. Volume = 480 cm^3 and surface area = 752 cm^2
B. Volume = 480 cm^3 and surface area = 376 cm^2
C. Volume = 120 cm^3 and surface area = 188 cm^2
D. Volume = 240 cm^3 and surface area = 376 cm^2

LESSON 10.2 Ratio and scale

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- compare two quantities of the same type using ratios
- simplify ratios
- enlarge or reduce a figure by applying a scale factor
- calculate and use a scale factor, including plans.

10.2.1 Ratio

- **Ratios** are used to compare quantities of the same kind, measured in the same unit.
- The ratio '1 is to 4' can be written in two ways: as 1 : 4 or as $\frac{1}{4}$.
- The order of the numbers in a ratio is important.
- In simplest form, a ratio is written using the smallest whole numbers possible.

WORKED EXAMPLE 1 Expressing ratios in simplest form

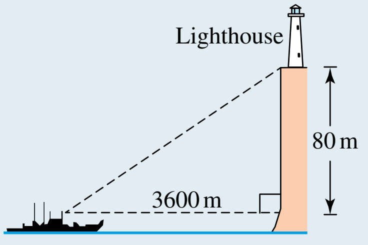
A lighthouse is positioned on a cliff that is 80 m high. A ship at sea is 3600 m from the base of the cliff.

a. Write the following ratios in simplest form.

i. The height of the cliff to the distance of the ship from shore.

ii. The distance of the ship from shore to the height of the cliff.

b. Compare the distance of the ship from shore with the height of the cliff.



THINK | **WRITE**

a. i. 1. The height and distance are in the same units (m). Write the height first and the distance second.

a. i. Height of cliff : distance of ship from shore $= 80 : 3600$

2. Simplify the ratio by dividing both terms by the highest common factor (80).

$= 1 : 45$

ii. 1. Write the distance first and the height second.

ii. Distance of ship from shore : height of cliff $= \dfrac{3600}{80}$

2. Simplify by dividing both terms by the highest common factor (80).

$= \dfrac{45}{1}$

Note: Do not write $\dfrac{45}{1}$ as 45, because a ratio is a comparison between two numbers.

b. 1. Write the ratio 'distance of the ship from shore to height of the cliff'.

b. $45 : 1$

2. Write the answer.

The distance of the ship from shore is 45 times the height of the cliff.

WORKED EXAMPLE 2 Simplifying ratios into simplest form

Express each of the following ratios in simplest form.

a. $\mathbf{24 : 8}$ **b.** $\mathbf{3.6 : 8.4}$ **c.** $\mathbf{1\frac{4}{9} : 1\frac{2}{3}}$

THINK | **WRITE**

a. Divide both terms by the highest common factor (8).

a. $24 : 8$
$= 3 : 1$

b. 1. Multiply both terms by 10 to obtain whole numbers.

b. $3.6 : 8.4$
$= 36 : 84$

2. Divide both terms by the highest common factor (12).

$= 3 : 7$

c. 1. Change both mixed numbers into improper fractions.

c. $1\frac{4}{9} : 1\frac{2}{3}$
$= \frac{13}{9} : \frac{5}{3}$

2. Multiply both terms by the lowest common denominator (9) to obtain whole numbers.

$= 13 : 15$

- A **proportion** is a statement that indicates that two ratios are equal.
- A proportion can be written in two ways, for example in the format used for $4 : 7 = x : 15$ or in the format used for $\dfrac{4}{7} = \dfrac{x}{15}$.

WORKED EXAMPLE 3 Calculating a value in a proportion

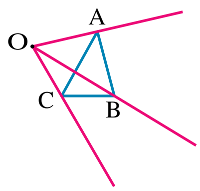
Determine the value of x in the proportion $4:9=7:x$.

THINK	WRITE
1. Write the ratios as equal fractions.	$\frac{4}{9}=\frac{7}{x}$
2. Multiply both sides by x.	$\frac{4x}{9}=7$
3. Solve the equation to obtain the value of x.	$4x=63$
4. Write the answer.	$x=15.75$

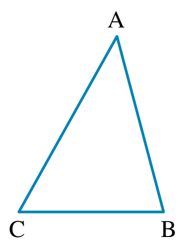
10.2.2 Scale

- Ratios are used when creating scale drawings or maps.
- Consider the situation in which we want to enlarge a triangle ABC (the **object**) by a **scale factor** of 2 (this means we want to make it twice its size). The following is one method that we can use.

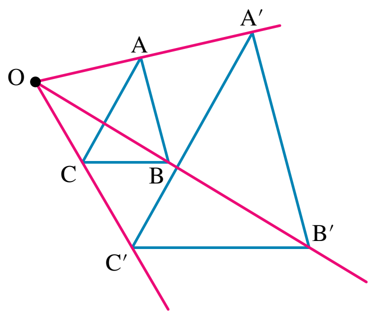
1. Mark a point O somewhere outside the triangle and draw the lines OA, OB and OC as shown.



2. Measure the length of OA and mark in the point A′ (this is called the **image** of A) so that the distance OA′ is twice the distance of OA.



3. In the same way, mark in points B′ and C′. $(OB'=2\times OB,\text{ and } OC'=2\times OC.)$



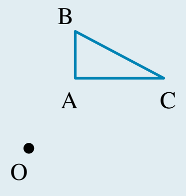
4. Joining $A'B'C'$ gives a triangle that has side lengths double those of ΔABC. $\Delta A'B'C'$ is called the image of ΔABC.

Scale factor

$$\text{Scale factor} = \frac{\text{image length}}{\text{object length}}$$

WORKED EXAMPLE 4 Enlarging a figure

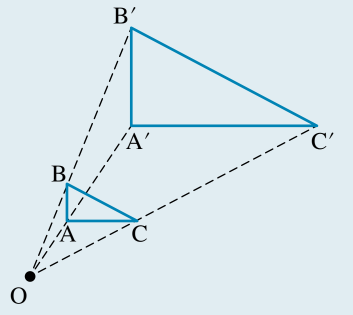
Enlarge triangle ABC by a scale factor of 3, with the centre of enlargement at point O.



THINK

1. Join each vertex of the triangle to the centre of enlargement (O) with straight lines, then extend them.
2. Locate points A', B' and C' along the lines so that $OA' = 3OA$, $OB' = 3OB$ and $OC' = 3OC$.
3. Join points A', B' and C' to complete the image.

DRAW

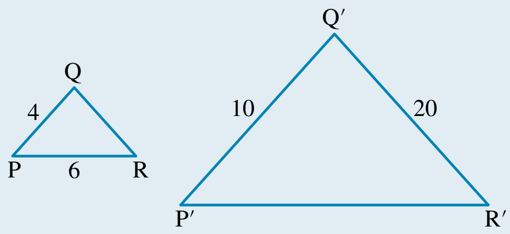


- Enlargements have the following properties.
  - The corresponding side lengths of the enlarged figure are changed in a fixed ratio (they have same ratio).
  - The corresponding angles are the same.
  - A scale factor greater than 1 produces an enlarged figure.
  - If the scale factor is a positive number less than 1, the image is smaller than the object (this means that a reduction has taken place).
  - the original figure and the enlarged figure are called similar figures as their angles are the same and their corresponding side lengths are in the same ratio.

WORKED EXAMPLE 5 Calculating the scale factor

A triangle PQR has been enlarged to create triangle $P'Q'R'$. $PQ = 4$ cm, $PR = 6$ cm, $P'Q' = 10$ cm and $Q'R' = 20$ cm. Calculate:

a. **the scale factor for the enlargement**
b. **the length of $P'R'$**
c. **the length of QR.**

THINK	WRITE/DRAW
a. 1. Draw a diagram.	**a.** 
2. Identify two corresponding sides. $P'Q'$ corresponds to PQ.	$\text{Scale factor} = \dfrac{\text{image length}}{\text{object length}}$ $= \dfrac{P'Q'}{PQ}$ $= \dfrac{10}{4}$ $= 2.5$
b. 1. Apply the scale factor. $P'R' = 2.5 \times PR$	**b.** $P'R' = 2.5 \times PR$ $= 2.5 \times 6$ $= 15$
2. Write the answer.	$P'R'$ is 15 cm long.
c. 1. Apply the scale factor. $Q'R' = 2.5 \times QR$	**c.** $\dfrac{Q'R'}{QR} = \dfrac{20\text{ cm}}{x\text{ cm}} = 2.5$ $Q'R' = 2.5 \times QR$ $20 = 2.5 \times QR$ $QR = \dfrac{20}{2.5}$ $= 8$
2. Write the answer.	QR is 8 cm long.

10.2.3 Scales in photographs, plans and drawings

- A scale on a photograph, plan or drawing describes the ratio between the distance or size on the photograph, plan or drawing and the actual distance or size.
- A scale may be written in several ways.
  - A ratio such as 1 : 100 000 means that 1 cm on the map or plan represents a distance of 100 000 cm on the Earth, 1 mm on the map or plan represents 100 000 mm on the Earth, etc.
  - A statement such as 1 cm ⇔ 500 km means that 1 cm on the map or plan represents a distance of 500 km on the Earth.
  - A graphical bar scale, such as the one shown in Worked example 6, can be used. In this case, we need to measure the length of the bar to determine the scale ratio.

WORKED EXAMPLE 6 Writing scales

a. **The scale on a house plan is 1 : 2000. How many metres does a length of 1 cm on the map represent?**
b. **Write the scale 1 cm ⇔ 5000 m as a ratio scale.**
c. **A map shows a scale of 1 cm ⇔ 1500 m. What distance is represented by 18 mm on the map?**
d. **Use the following graphical scale to determine the distance that 1 cm on the map represents.**

THINK	WRITE
a. 1. Use the ratio to add the same units to both sides. Since the answer asks for cm, use this unit.	**a.** 1 : 2000 is the same as: 1 cm ⇔ 2000 cm
2. Bring the length to an appropriate unit. In this case, the answer is required in metres.	2000 cm = (2000 ÷ 100) m = 20 m So, 1 cm ⇔ 20 m.
3. Write the answer.	1 cm on the map represents a length of 20 m in reality.
b. 1. Convert the ground units to the same as those of the map.	**b.** 5000 m = (5000 × 100) cm = 500 000 cm
2. Write the scale as a ratio.	So, 1 cm ⇔ 5000 m is the same as 1 cm ⇔ 500 000 cm. Expressed as a ratio, this is 1 : 500 000.
c. 1. Write a statement explaining this scale.	**c.** 1 cm ⇔ 1500 m means that 1 cm on the map represents a distance of 1500 m.
2. Change the question distance to the same units as those for the map.	18 mm = (18 ÷ 10) cm = 1.8 cm
3. Multiply by this scale factor.	1.8 cm represents (1.8 × 1500) m, which is 2700 m or 2.7 km.
d. 1. Measure the length of the bars with a ruler.	**d.** The bars measure 3.5 cm.
2. Relate this measurement to the length shown in the scale.	This represents a length of 15 km.
3. Write as a scale.	3.5 cm ⇔ 15 km
4. Calculate a 1 cm map distance and write the answer.	So, 1 cm ⇔ $\frac{15}{35}$ km. 1 cm on the map represents a distance of 4.3 km.

- Sometimes it is necessary to do the reverse, and convert from a real length to a plan measurement.

WORKED EXAMPLE 7 Calculating length from a scale

The length of a wall for a particular house is 12 m. If the scale of a house plan is 1 : 200, what length, on the plan, would represent the actual length of the wall?

THINK	WRITE
1. Write the scale as a ratio using units of cm.	1 : 200 means that 1 cm on the plan represents an actual length of 200 cm. 1 cm ⇔ 200 cm

2. Convert the actual length from cm to m units.	$200\text{ cm} = (200 \div 100)\text{ m}$ $= 2\text{ m}$ So 1 cm on the plan represents an actual length of 2 m.
3. Divide the actual length by 2 to find how many lots of 2 m there are in 12 m.	$\dfrac{12\text{ m}}{2\text{ m}} = 6$
4. Multiply both sides of the scale by this scale factor, then simplify.	$1\text{ cm} \times 6 \Leftrightarrow 2\text{ m} \times 6$ $6\text{ cm} \Leftrightarrow 12\text{ m}$
5. Answer the question.	The length of the wall on the house plan would be 6 cm.

10.2.4 House plans

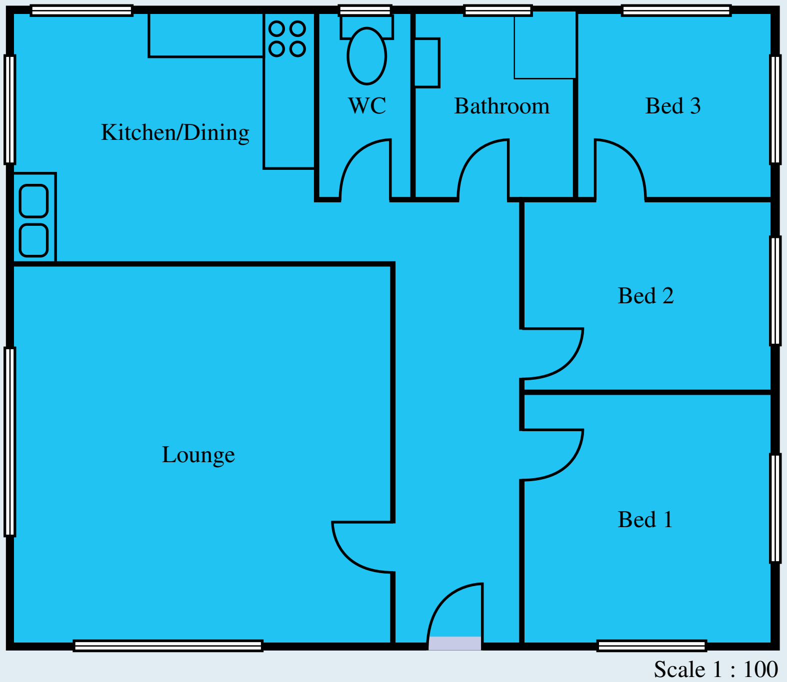
- Whenever a house is being constructed, floor plans are supplied to the builder so that the footings, walls, roof and other parts of the house are all placed in the correct positions and are the correct length.
- The scale of the plan is generally given as a ratio. This enables us to calculate all dimensions within the house.

WORKED EXAMPLE 8 Using a house plan to calculate dimensions within a house

A plan for a house is shown.

a. Calculate the dimensions of the house.

b. Calculate the area of the lounge room.



THINK	WRITE
a. **1.** Measure the length and width of the house on the plan.	**a.** Length of house on plan = 12 cm Width of house on plan = 10 cm
2. Multiply each of these measurements by 100.	$\text{Actual length of house} = 12\text{ cm} \times 100$ $= 1200\text{ cm}$ $= 12\text{ m}$ $\text{Actual width} = 10\text{ cm} \times 100$ $= 1000\text{ cm}$ $= 10\text{ m}$
3. Write your answer.	The dimensions of the house are 12 m by 10 m.
b. **1.** Measure the length and width of the lounge room on the plan.	**b.** Length of lounge room on plan = 6 cm Width of lounge room on plan = 6 m
2. Multiply each of these measurements by 100.	$\text{Actual length of lounge room} = 6\text{ cm} \times 100$ $= 600\text{ cm}$ $= 6\text{ m}$ The actual width of the lounge room is also 6 m.
3. Calculate the area of the lounge room.	$\text{Area} = 6^2$ $= 36\text{ m}^2$
4. Write your answer.	The area of the lounge room is 36 m^2.

Exercise 10.2 Ratio and scale

learnon

10.2 Quick quiz on

10.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 10, 13, 16, 17, 18, 20, 23, 27	2, 4, 6, 8, 11, 15, 19, 22, 24, 28, 29	7, 9, 12, 14, 21, 25, 26, 30, 31

Fluency

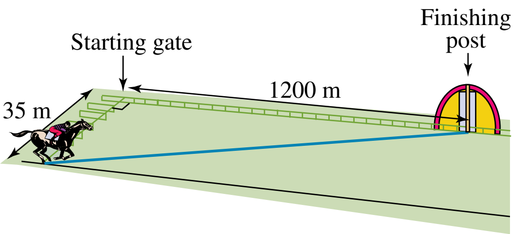
1. WE1 The horse track shown is 1200 m long and 35 m wide.

   a. Write the following ratios in simplest form.

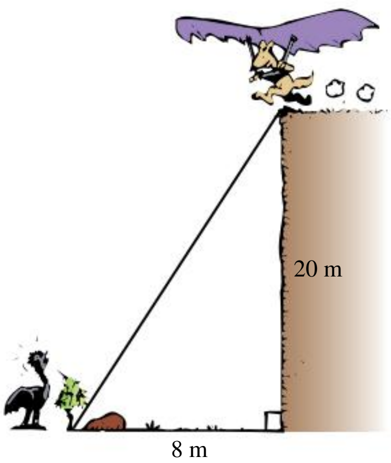
      i. Track length to track width.

      ii. Track width to track length.

   b. Compare the distance of the length of the track with the width of the track.



2. A dingo perched on top of a cliff spots an emu on the ground below, as represented by the diagram to the right.



a. Write the following ratios in simplest form.

i. Cliff height to distance between cliff base and emu.

ii. Distance between emu and cliff base to cliff height.

b. Compare the height of the cliff with the horizontal distance between the base of the cliff and the emu.

3. **WE2a** Express each of the following ratios in simplest form.

a. 12 : 18 b. 8 : 56

c. 9 : 27 d. 14 : 35

4. Express each of the following ratios in simplest form.

a. 16 : 60 b. 200 : 155

c. 32 : 100 d. 800 : 264

5. **WE2b** Express each of the following ratios in simplest form.

a. 1.2 : 0.2 b. 3.9 : 4.5 c. 9.6 : 2.4 d. 18 : 3.6

6. Express each of the following ratios in simplest form.

a. 1.8 : 3.6 b. 4.4 : 0.66 c. 0.9 : 5.4 d. 0.35 : 0.21

7. Express each of the following ratios in simplest form.

a. 6 : 1.2 b. 12.1 : 5.5 c. 8.6 : 4 d. 0.07 : 14

8. **WE2c** Write each of the following ratios in the simplest form.

a. $1\frac{1}{2} : 2$ b. $2 : 1\frac{3}{4}$ c. $1\frac{1}{3} : 2$ d. $1\frac{2}{5} : 1\frac{1}{4}$

9. Write each of the following ratios in the simplest form.

a. $\frac{4}{7} : 2$ b. $5 : 1\frac{1}{2}$ c. $2\frac{3}{4} : 1\frac{1}{3}$ d. $3\frac{5}{6} : 2\frac{1}{2}$

10. **WE3** Determine the value of the pronumeral in each of the following proportions.

a. $a : 15 = 3 : 5$ b. $b : 18 = 4 : 3$ c. $24 : c = 3 : 4$

11. Determine the value of the pronumeral in each of the following proportions.

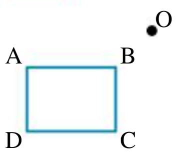
a. $e : 33 = 5 : 44$ b. $6 : f = 5 : 12$ c. $3 : 4 = g : 5$

12. Determine the value of the pronumeral in each of the following proportions.

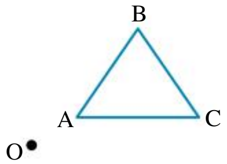
a. $11 : 3 = i : 8$ b. $7 : 20 = 3 : j$ c. $15 : 13 = 12 : k$

For questions **13** to **15**, enlarge the figures shown by the given scale factor and the centre of enlargement marked with O. Show the image of each of these figures.

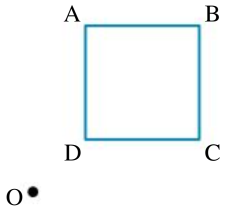
13. **WE4** Scale factor = 2



14. Scale factor = 1.5



15. Scale factor = $\frac{1}{2}$



16. WE5 A quadrilateral ABCD is enlarged to $A'B'C'D'$. AB = 7 cm, AD = 4 cm, $A'B'$ = 21 cm, $B'C'$ = 10.5 cm. Determine:

a. the scale factor for enlargement b. $A'D'$ c. BC.

17. WE6&7 If a scale of 1 : 50 is used to make each of the following drawings:

a. what would be the width of the drawing of a house that is 25 m wide
b. what would be the height of the drawing of a man who is 2 m tall
c. what would be the dimensions of the drawing of a rectangular pool 50 m by 20 m
d. what would be the dimensions of the drawing of a table 4 m by 1.5 m?

Understanding

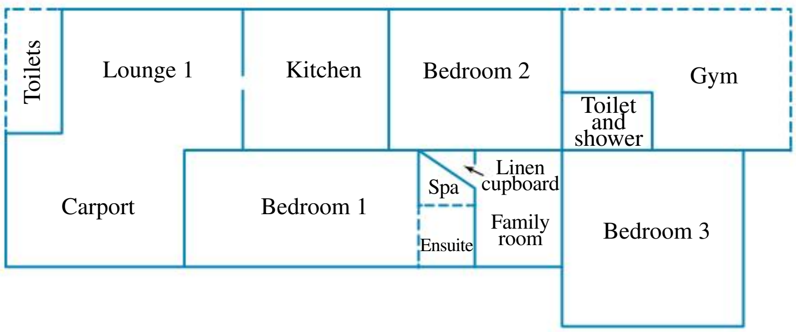
18. The estimated volume of Earth's salt water is 1 285 600 000 cubic kilometres. The estimated volume of fresh water is about 35 000 000 cubic kilometres.

a. Determine the ratio of fresh water to salt water (in simplest form).
b. Determine the value of x, to the nearest whole number, when the ratio found in part a. is expressed in the form 1 : x.

19. Super strength glue comes in two tubes that contain 'Part A' and 'Part B' pastes. These pastes have to be mixed in the ratio 1 : 4 for maximum strength. Determine how many mL of Part A should be mixed with 10 mL of Part B.

20. A recipe for a tasty cake says the butter and flour needs to be combined in the ratio 2 : 7. Determine the amount of butter that should be mixed with 3.5 kg of flour.

21. WE8 The diagram shown lays out the floor plan of a house. The actual size of bedroom 1 is 8 m × 4 m.



a. If the dimensions of bedroom 1 as it appears on the ground plan are 4 cm by 2 cm, calculate the scale factor when the actual house (object) is built from the plan (image).
b. Determine the real-life dimensions of bedroom 3 if the dimensions as shown on the ground plan are $3\text{ cm} \times 3\text{ cm}$.
c. Determine the real-life dimensions of the kitchen if the dimensions as shown on the ground plan are $2.5\text{ cm} \times 2.5\text{ cm}$.

Reasoning

22. Pure gold is classed as 24-carat gold. This kind of gold is too soft to use for making jewellery, so it gets combined with other metals to form an alloy. The ratio of gold to other metals in 18-carat gold is 18 : 6.
The composition of 18-carat rose gold is 75% gold, 22.25% copper and 2.75% silver.

a. Show that the mass of silver in a 2.5-gram rose gold bracelet is 0.07 g.
b. Determine the composition of metals in a rose gold bracelet that contains 0.5 g of copper.

23. The angles of a triangle have the ratio 3 : 4 : 5. Show that the sizes of the three angles are 45°, 60° and 75°.

24. The dimensions of a rectangular box have the ratio 2 : 3 : 5. The box's volume is $21\ 870\text{ cm}^3$. Show that the dimensions of the box are 18, 27 and 45 cm.

25. Tyler, Dylan and Ari invested money in the ratio 11 : 9 : 4. If their profits are shared in the ratio 17 : 13 : 6, is this fair to each person? Explain your answer.

26. It costs the same amount to buy either 5 pens or 2 pens and 6 pencils. This is also the cost of 6 sharpeners and a pencil. Show a relationship between the cost of each kind of item.

Problem solving

27. Sharnee, a tourist at Kakadu National Park, takes a picture of a 2-metre-long crocodile beside a cliff.
When they develops the pictures, they can see that on the photo the crocodile is 2.5 cm long and the cliff is 8.5 cm high.
Determine the actual height of the cliff in cm.

28. Evaluate the ratio of $y : z$ if $2x = 3y$ and $3x = 4z$.

29. The quantities P and Q are in the ratio 2 : 3. If P is reduced by 1, the ratio becomes $\frac{1}{2}$. Determine the values of P and Q.

30. The ratio of boys to girls among the students who signed up for a basketball competition is 4 : 3.
If 3 boys drop out of the competition and 4 girls join, there will be the same number of boys and girls.
Evaluate the number of students who have signed up for the basketball competition.

31. In a group of students who voted in a Year 9 class president election, the ratio of girls to boys is 2 : 3.
If 10 more girls and 5 more boys had voted, the ratio would have been 3 : 4.
Evaluate the number of students who voted altogether.

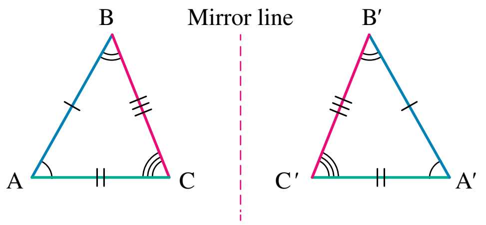
LESSON 10.3 Congruent figures

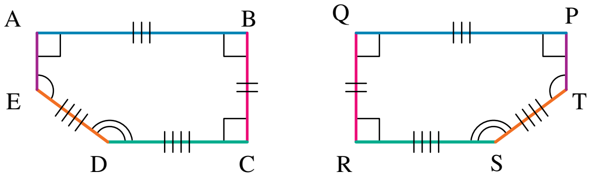
LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify congruent figures
- show that two triangles are congruent using the appropriate congruency test
- Use proofs to determine congruent triangles (optional).

10.3.1 Congruent figures

- **Congruent figures** are identical figures. They have exactly the same shape and size.
- Congruent figures can be superimposed exactly on top of each other using reflection, rotation or translation (or a combination of some or all of these actions).

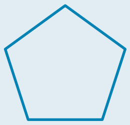


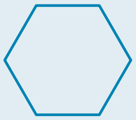


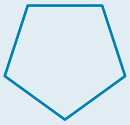
- The symbol for congruence is ≅. When reading this symbol out loud we say, 'is congruent to'.
- For the diagrams shown, ABC ≅ A′B′C′ and ABCDE ≅ PQRST.
- When writing congruence statements, the vertices of each figure are named in corresponding order.

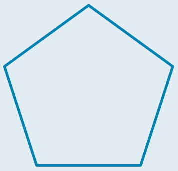
WORKED EXAMPLE 9 Identifying congruent shapes

Identify a pair of congruent shapes from the following set.

a.


b.


c.


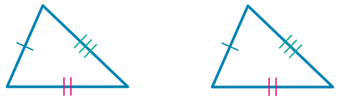
d.


THINK	WRITE
Figures **a** and **c** are identical in shape and size. They just have different orientations.	Shape **a** ≅ shape **c**

10.3.2 Testing triangles for congruence

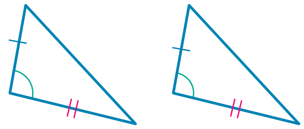
- To determine whether two triangles are congruent, it is not necessary to know that all three sides and all three angles of a each triangle are equal to the corresponding sides and angles of the other triangle.
- Certain minimum conditions can guarantee that two triangles are congruent.

Side-side-side condition for congruence (SSS)



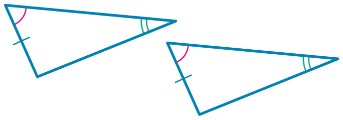
- If all of the corresponding sides of two triangles are equal, the angles opposite those corresponding sides will also be equal. This means that the two triangles are congruent.
- This is known as the *side-side-side* (*SSS*) condition for congruence.

Side-angle-side condition for congruence (SAS)



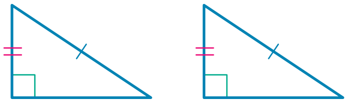
- If two triangles have two corresponding sides that are equal, and the angles between those corresponding sides are equal, then the two triangles are congruent.
- This is known as the *side–angle–side* (*SAS*) condition for congruence.

Angle-angle-side condition for congruence (AAS)



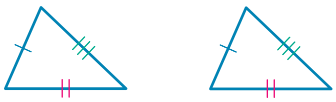
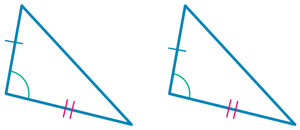
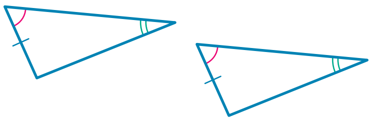
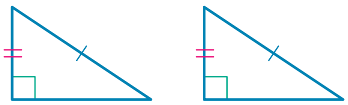
- If two triangles have two corresponding angles that are equal, and 1 side that is also equal, then the two triangles are congruent. (*Note:* The third corresponding angle will also be equal.)
- This is known as the *angle-angle-side* (*AAS*) condition for congruence.

Right angle-hypotenuse-side condition for congruence (RHS)



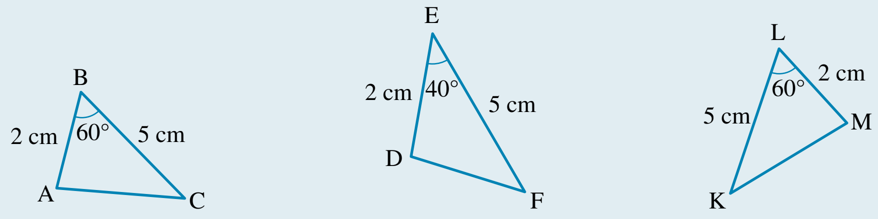
- If the hypotenuse and one other side of two right-angled triangles are equal, then the two triangles are congruent.
- This is known as the *right angle–hypotenuse–side* (*RHS*) condition for congruence.

Summary of congruence tests

Congruence test	Description	Abbreviation
	• All corresponding sides are equal in length.	SSS (side–side–side)
	• Two corresponding sides are equal in length. • The corresponding angles between them are equal in size.	SAS (side–angle–side)
	• Two corresponding angles are equal in size. • One corresponding side is equal in length.	AAS (angle-angle-side)
	• The hypotenuse and one other corresponding side of two right-angled triangles are equal in length.	RHS (right angle–hypotenuse–side)

WORKED EXAMPLE 10 Identifying congruent triangles

Identify which of the following triangles are congruent. Give reasons for your answer.



THINK

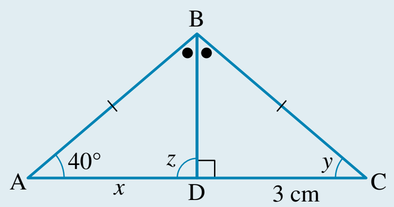
In all three triangles two given sides are of equal length (2 cm and 5 cm). The included angles in triangles ABC and KLM are also equal (60°). B corresponds to L, and A corresponds to M.

WRITE

$\Delta ABC \cong \Delta MLK$ (SAS)

WORKED EXAMPLE 11 Using congruent triangles to determine values of the pronumerals

Given that $\Delta ABD \cong \Delta CBD$, determine the values of the pronumerals in the figure shown.



THINK

1. Congruent triangles have corresponding sides that are equal in length. Side AD (marked x) corresponds to side CD.

WRITE

$\Delta ABD \cong \Delta CBD$
$AD = CD$
$x = 3$ cm

THINK

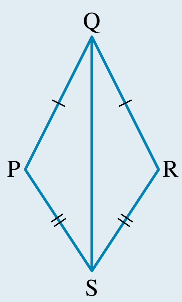
2. Since these triangles are congruent, the corresponding angles are equal.

WRITE

$\angle A = \angle C$
$y = 40°$
$\angle BDA = \angle BDC$
$z = 90°$

WORKED EXAMPLE 12 Using proofs to determine that two triangles are congruent (optional)

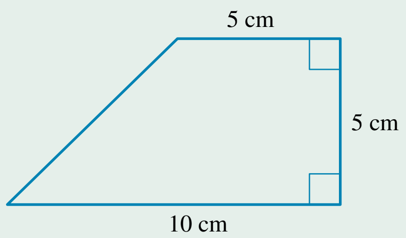
Prove that ΔPQS is congruent to ΔRQS.



THINK	WRITE
1. Study the diagram and state which sides and/or angles are equal.	QP = QR (given) PS = RS (given) QS is common.
2. This fits the SSS condition, which proves congruence.	$\Delta PQS \cong \Delta RQS$ (SSS)

COLLABORATIVE TASK

On a piece of paper, draw the trapezium shown as per the dimensions given in the diagram. Working in pairs or small teams, try to divide the trapezium into four congruent trapeziums that are similar in shape to the original trapezium.



Exercise 10.3 Congruent figures

learnon

10.3 Quick quiz on

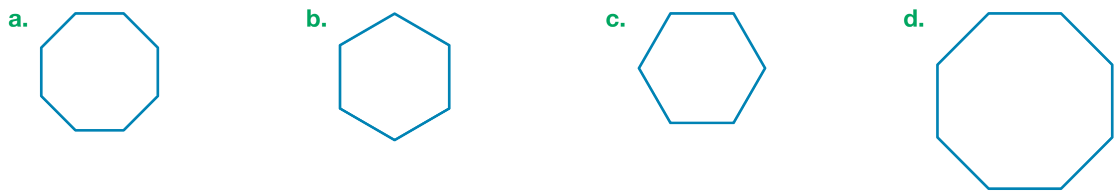
10.3 Exercise

Individual pathways

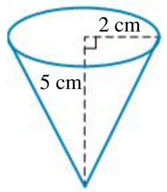
PRACTISE	CONSOLIDATE	MASTER
1, 4, 5, 8, 11, 12, 16, 17, 22	2, 6, 9, 13, 14, 18, 19, 23	3, 7, 10, 15, 20, 21, 24, 25

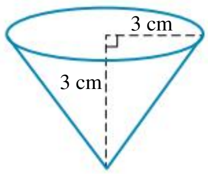
Fluency

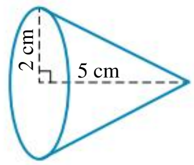
1. WE6 Select a pair of congruent shapes from the figures shown.

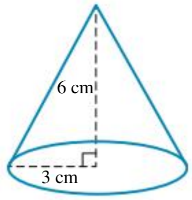


2. Select a pair of congruent shapes from the figures shown.

a.


b.


c.


d.


3. Select a pair of congruent shapes from the figures shown.

a.

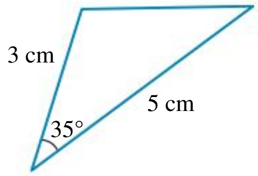

b.

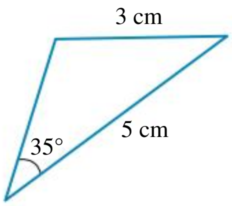

c.

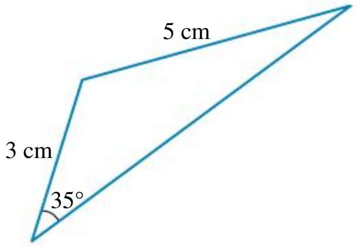

d.

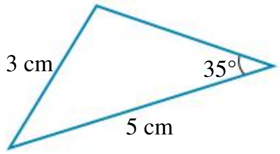

Understanding

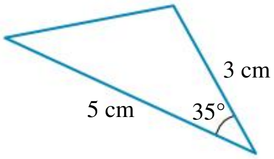
4. MC Select which of the following is congruent to the triangle shown.



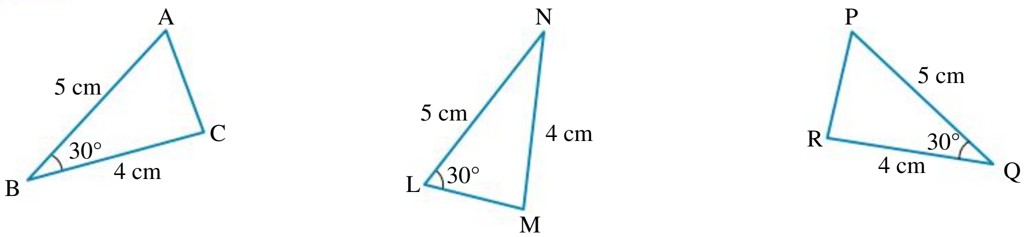
A.


B.


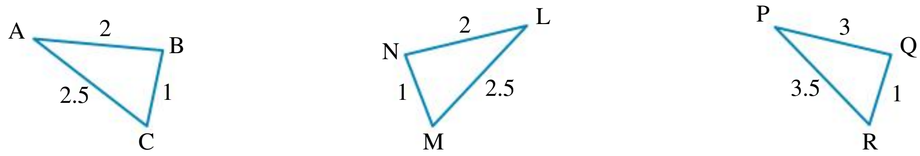
C.


D.


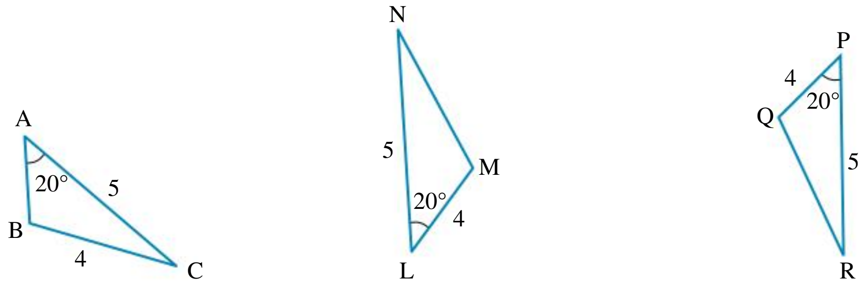
5. WE7 Identify which of the following triangles are congruent. Give a reason for your answer.



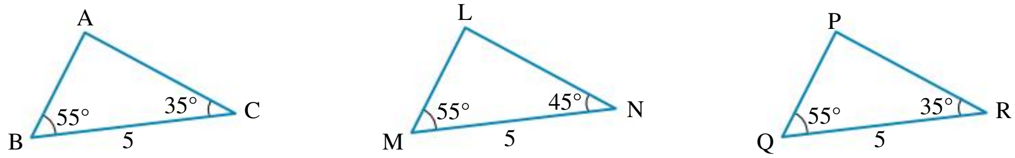
6. Identify which of the following triangles are congruent. Give a reason for your answer.



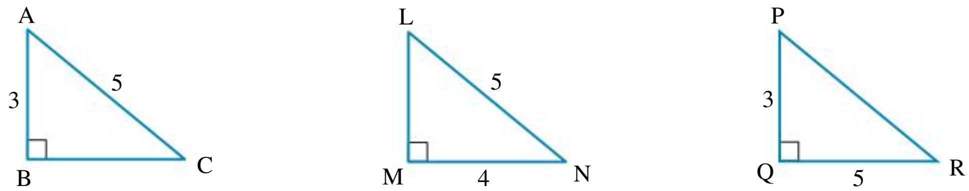
7. Identify which of the triangles are congruent. Give a reason for your answer.



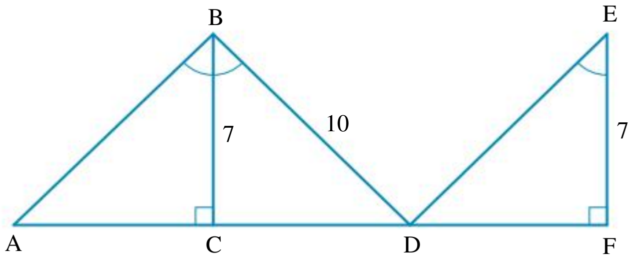
8. Identify which of the following triangles are congruent. Give a reason for your answer.



9. Identify which of the following triangles are congruent. Give a reason for your answer.



10. Identify which of the following triangles are congruent. Give a reason for your answer.



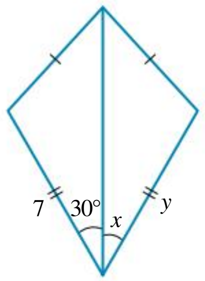
11. **WE8** Determine the value of the pronumeral in the following pair of congruent triangles. All side lengths are in centimetres.

12. Determine the value of the pronumeral in the pair of congruent triangles shown. All side lengths are in centimetres.

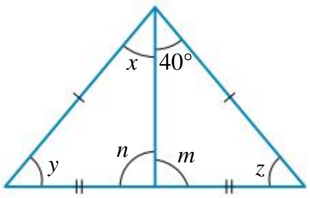
13. Determine the value of the pronumerals in the pair of congruent triangles shown. All side lengths are in centimetres.

14. Determine the value of the pronumerals in the pair of congruent triangles shown. All side lengths are in centimetres.

a.

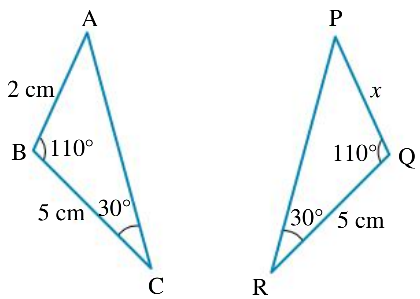


b.

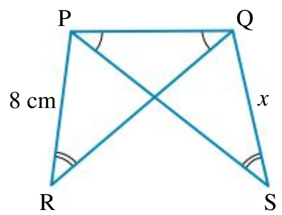


15. Calculate the length of the side marked with the pronumeral using the following congruent triangles.

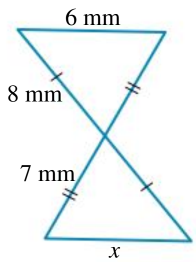
a.



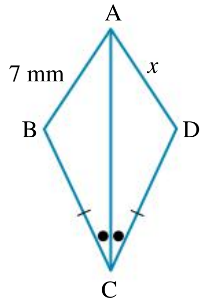
b.



c.



d.



Reasoning

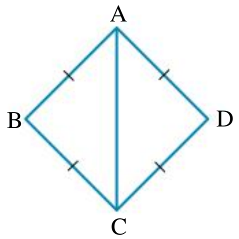
16. Do congruent figures have the same area? Explain your answer.

17. If two congruent triangles have a right angle, is the reason for their congruence always due to the RHS condition of congruence? Justify your answer.

18. Give an example to show that triangles with two angles of equal size and a pair of non-corresponding sides of equal length may not be congruent.

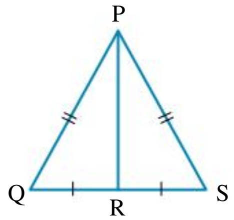
19. WE9 For each of the following, prove that the statement accompanying the diagram is true.

a.



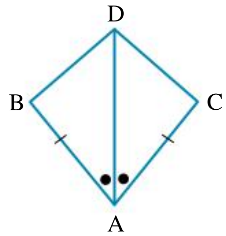
$\Delta ABC \cong \Delta ADC$

b.



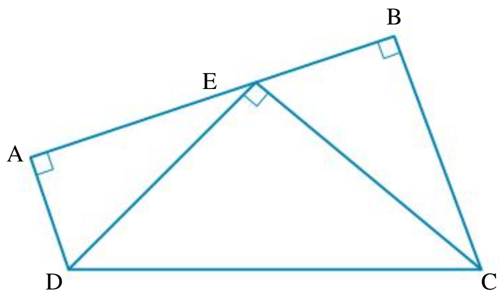
$\Delta PQR \cong \Delta PSR$

c.



$\Delta DBA \cong \Delta DCA$

20. ABCD is a trapezium with both AD and BC perpendicular to AB. If a right-angled triangle DEC is constructed with an angle $\angle ECD$ equal to 45°, prove that $\Delta EDA \cong \Delta ECB$.



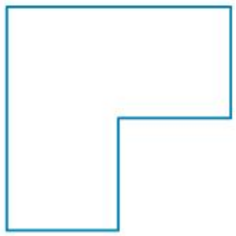
21. A teacher asked their class to each draw a triangle with side lengths of 5 cm and 4 cm, and an angle of 45° that is not formed at the point joining the 5 cm and 4 cm side.
Explain why the triangles drawn by every member of the class would be congruent.

Problem solving

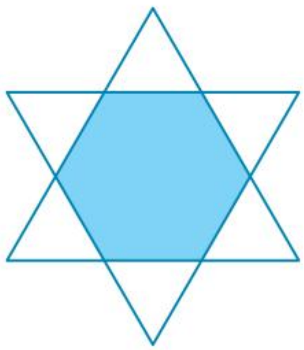
22. Construct five congruent triangles from nine matchsticks.

23. Construct seven congruent triangles from nine matchsticks.

24. Demonstrate how the figure shown can be cut into four congruent pieces.



25. Determine the ratio of the outer (unshaded) area to the inner (shaded) area of the six-pointed star shown.



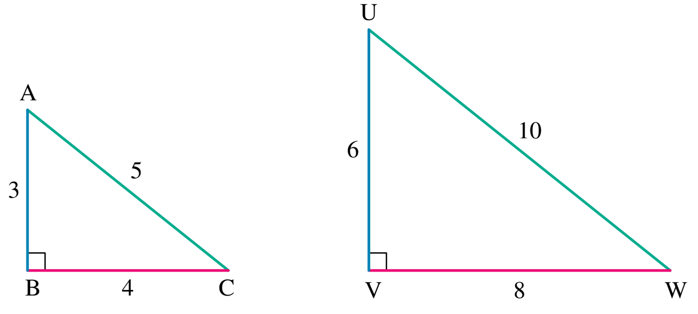
LESSON 10.4 Similar figures

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify similar figures
- determine the scale factor between similar figures
- show that two triangles are similar using the appropriate similarity test.

10.4.1 Similar figures and similarity condition

- **Similar figures** are identical in shape, but different in size.
- The corresponding angles in similar figures are equal in size and the corresponding sides are in the same ratio.
- The symbol used to denote similarity is ~. When reading this symbol out loud we say, 'is similar to'.
- In the triangles shown ΔABC is similar to ΔUVW. That is, ΔABC~ΔUVW.
- The ratio of side lengths is known as the scale factor.



Scale factor

$$\text{Scale factor} = \frac{\text{length of image}}{\text{length of object}}$$

- **If the scale factor is > 1, an enlargement has occurred.**
- **If the scale factor is < 1, a reduction has occurred.**

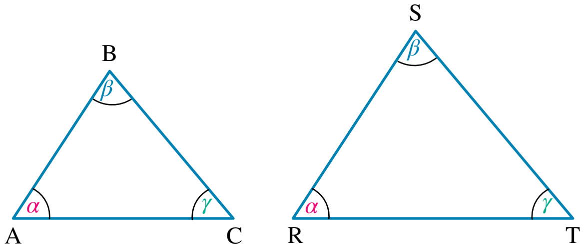
- The scale factor for the triangles shown is 2. Each side in UVW is twice the length of the corresponding side in ABC.
- Enlargements and reductions are transformations that create similar figures.
- The method for creating enlarged figures that is explained in lesson 10.2 can also be used to create similar figures.

10.4.2 Testing triangles for similarity

- As with congruent triangles, to determine whether two triangles are similar, it is not necessary to know that all pairs of corresponding sides are in the same ratio and that all corresponding angles are equal.
- Certain minimum conditions can guarantee that two triangles are similar.

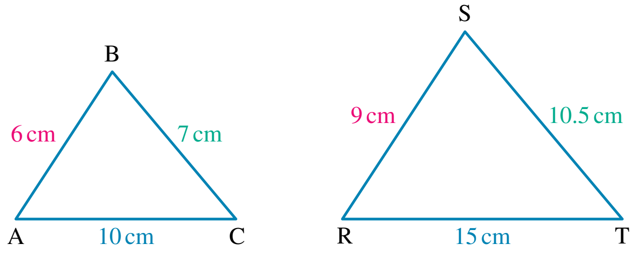
Angle–angle–angle condition of similarity (AAA)

- If all of the angles of two triangles are equal, then the triangles are similar.
- This is known as the angle–angle–angle (AAA) condition for similarity.
- In the diagram shown, $\Delta ABC \sim \Delta RST$ (AAA).



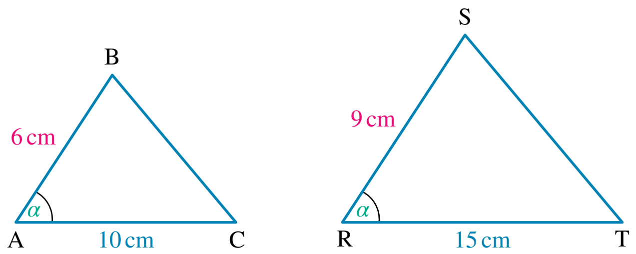
Side–side–side condition for similarity (SSS)

- If two triangles have a constant ratio for all corresponding side lengths, then the two triangles are similar.
- This is known as the side–side–side (SSS) condition for similarity.
- In the diagram shown, the ratios of all corresponding side lengths are equal $\left(\frac{9}{6} = \frac{15}{10} = \frac{10.5}{7} = 1.5\right)$, therefore $\Delta ABC \sim \Delta RST$ (SSS).



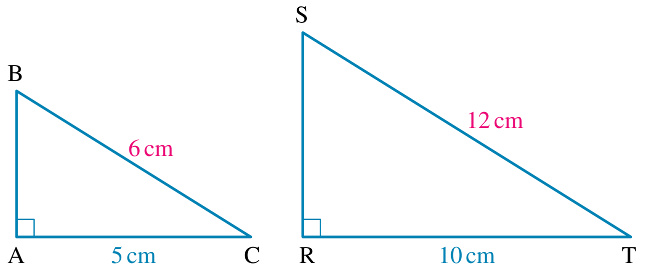
Side–angle–side condition for similarity (SAS)

- If two triangles have two corresponding sides in the same ratio and the included angles of those sides are equal, then the two triangles are similar.
- This is known as the side–angle–side (SAS) condition for similarity.
- In the diagram shown, the ratio of the triangles' two corresponding side lengths are equal $\left(\frac{9}{6} = \frac{15}{10} = 1.5\right)$ and the included angles are also the same, therefore $\Delta ABC \sim \Delta RST$ (SAS).



Right angle–hypotenuse–side condition for similarity (RHS)

- If the hypotenuse and one other corresponding side of two right-angled triangles are in the same ratio, then the two triangles are similar.
- This is known as the the right angle–hypotenuse–side (RHS) condition for similarity.
- In the diagram shown, the ratio of the hypotenuses and one other pair of corresponding sides are equal $\left(\frac{12}{6} = \frac{10}{5} = 2\right)$, therefore $\Delta ABC \sim \Delta RST$ (RHS).



Summary of similarity tests

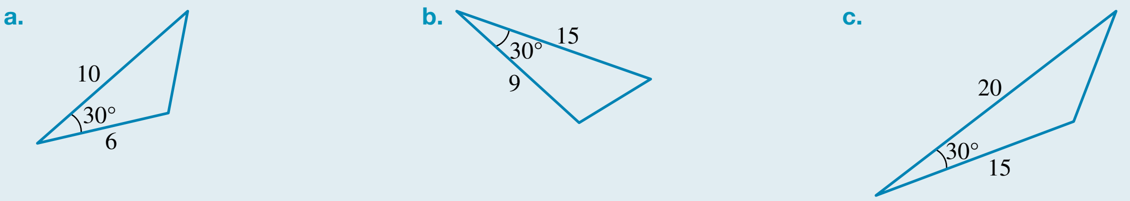
- Triangles can be checked for similarity using one of the tests described in the table shown.

Test description	Abbreviation
All corresponding angles are equal in size	AAA (angle–angle–angle)
All corresponding sides are in the same ratio	SSS (side–side–side)
Two corresponding sides are in the same ratio and the included angles are equal in size	SAS (side–angle–side)
The hypotenuse and one other corresponding side of two right-angled triangles are in the same ratio	RHS (right angle–hypotenuse–side)

Note: When using the AAA test, it is sufficient to show that two corresponding angles are equal. Since the sum of the interior angles in any triangle is 180°, the third corresponding angle will automatically be equal.

WORKED EXAMPLE 13 Identifying similar triangles

Identify a pair of similar triangles from the triangles shown. Give a reason for your answer.



THINK

1. In each triangle we know the size of two sides and the included angle, so the SAS test can be applied. Since all included angles are equal (30°), we need to determine the ratios of the corresponding sides, looking at two triangles at a time.

2. Write the answer.

WRITE

For triangles **a** and **b**:

$\frac{15}{10} = \frac{9}{6} = 1.5$

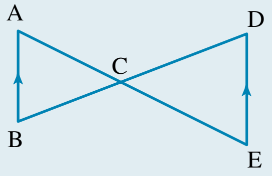
For triangles **a** and **c**:

$\frac{20}{10} = 2, \frac{15}{6} = 2.5$

Triangle **a** ~ triangle **b** (SAS)

WORKED EXAMPLE 14 Proving that two triangles are similar

Prove that ΔABC is similar to ΔEDC.

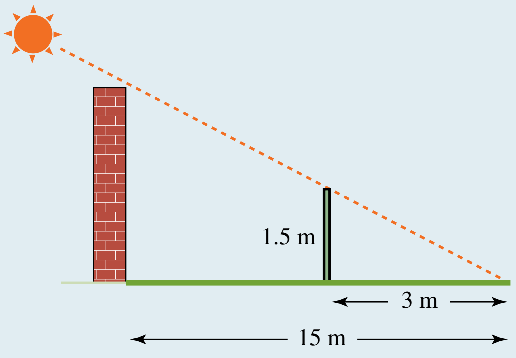


THINK	WRITE
1. AB is parallel to DE. ∠ABC and ∠EDC are alternate angles.	∠ABC = ∠EDC (alternate angles)
2. ∠BAC and ∠DEC are alternate angles.	∠BAC = ∠DEC (alternate angles)
3. The third pair of angles must be equal.	∠BCA = ∠DCE (vertically opposite angles)
4. This proves that the triangles are similar.	ΔABC ~ ΔEDC (AAA)

- The ratio of the corresponding sides in similar figures can be used to calculate missing side lengths or angles in those figures.

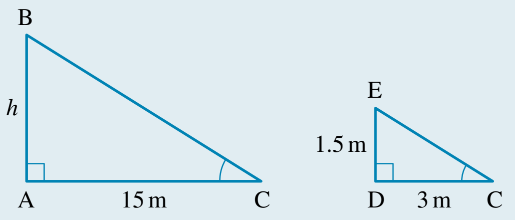
WORKED EXAMPLE 15 Solving worded problems using similar triangles

A pole 1.5 metres high casts a shadow 3 metres long, as shown. Calculate the height of a building that casts a shadow 15 metres long at the same time of the day.



THINK | **WRITE/DRAW**

1. Represent the given information on a diagram.
∠BAC = ∠EDC; ∠BCA = ∠ECD



2. Triangles ABC and DEC are similar. This means the ratios of corresponding sides are the same. Write the ratios.

ΔABC ~ ΔDEC (AAA)

$$\frac{h}{1.5} = \frac{15}{3}$$

3. Solve the equation for h.

$$h = \frac{15 \times 1.5}{3}$$
$$= 7.5$$

4. Write the answer in words, including units.

The building is 7.5 metres high.

Digital tools to develop similar shapes

We have looked at similar shapes that get bigger with a scale factor greater than one and a similar shape that reduces in size with a scale factor greater than zero and less than one.

When a shape in enlarged or reduced:

- Angles at the vertex stay the same.
- The side lengths increase or decrease.

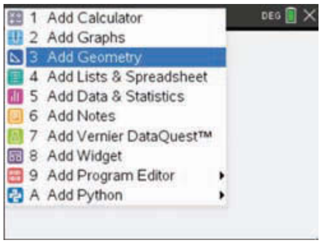
These similar shapes can be produced by using CAS technology (as shown below) or using dynamic geometry software.

Digital technology

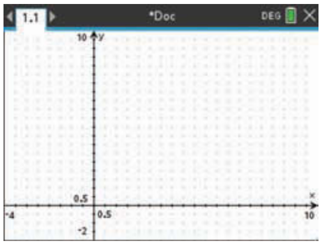
As an example, we will draw a square and enlarge it with a scale factor of 3 and then reduce it with a scale factor of 0.5.

TI | THINK — **DISPLAY/WRITE**

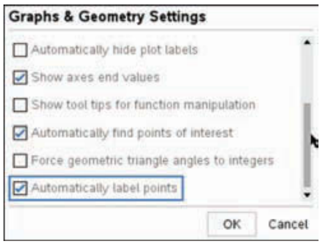
1. On a Calculator page, press MENU and select: 3 Add Geometry



2. Select 2 View and click on 1 Graphing. Then select 6 Grid and click on 2 Dot Grid.

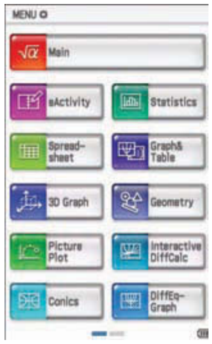


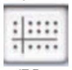
3. Click on  and select: 9 Settings and tick Automatically label points then click OK.

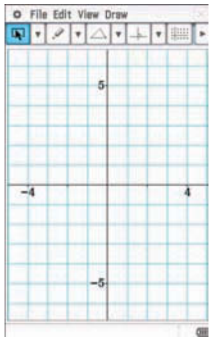


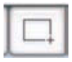
CASIO | THINK — **DISPLAY/WRITE**

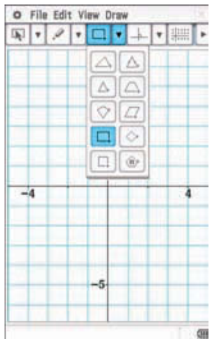
1. On the Main screen tap:
   - Geometry



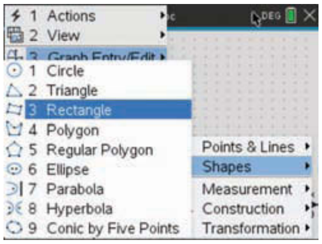
2. After selecting the Geometry screen:
   - Tap  to create a grid. (You might need to tap this multiple times to get the grid).



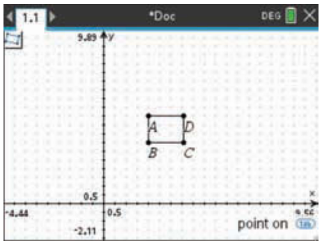
3. Tap:  to select a rectangle, or the arrow next to it to choose other shapes.



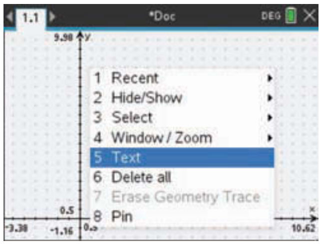
4. To draw a shape, press MENU and select:
8 Geometry
2 Shapes
To select a rectangle, select
3 Rectangle



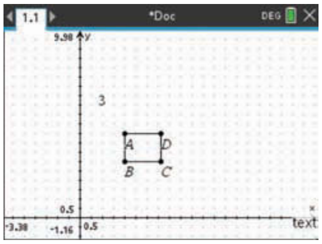
5. Tap on the screen where you want one of the corners to be, then drag to the size of the shape you require.



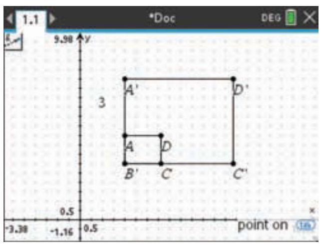
6. From the Menu, click on
8 Geometry,
then select
1 Points & Lines and click on 1 Point.
Right click on the point and select5 Text.



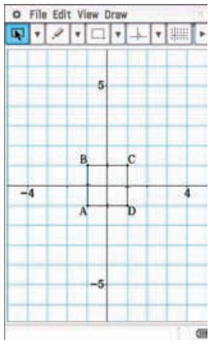
7. Add text to the screen. Text being a numerical quantity '3' in this case.



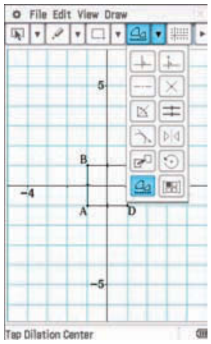
8. To enlarge the rectangle by a factor of 3. Select from the Menu:
8 Geometry
5 Transformation
5 Dilation
Dilation of the rectangle by a factor of 3 about a point A. The diagram shows the new shape increased by a scale factor of 3.



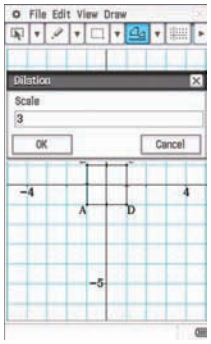
4. Tap on the screen where you want one of your corners to be, then drag to the size of the shape you require.



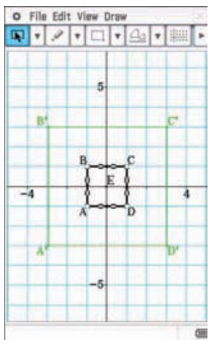
5. Tap:
-  on the arrow as shown.
-  to enlarge/reduce the size of the shape.



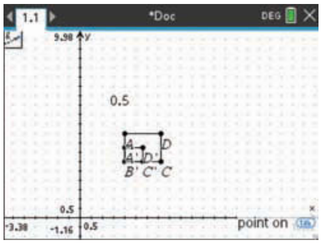
6. Tap the centre of the shape you want to enlarge/reduce.
Then type in 3 for the scale factor and then OK.



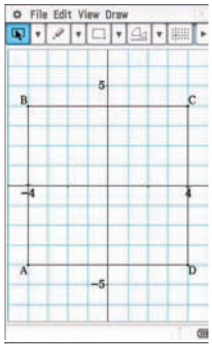
8. The diagram shows the new shape increased by a scale factor of 3.
This shows that the angle at the vertices stay the same (90°), but the side lengths are three times longer.

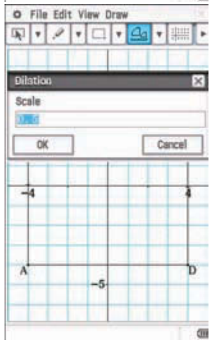


9. Completing the same process but reducing the size of the shape by using a scale factor of 0.5.



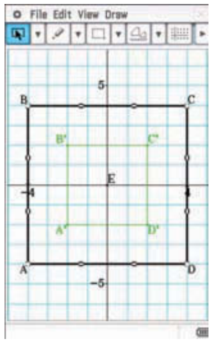
9. Starting with the shape shown, complete the same process but reducing the size of the shape by using a scale factor of 0.5.
Use the same process to input a scale factor of 0.5.





10. Tap:
- OK

The shape again has the same angles at the vertices (90°), but the side lengths are half their original length to produce the similar shape.



Exercise 10.4 Similar figures

learnon

10.4 Quick quiz on	10.4 Exercise

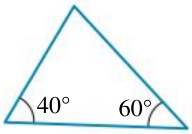
Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
1, 2, 6, 7, 11, 14, 15, 19, 20	3, 4, 8, 12, 16, 21	5, 9, 10, 13, 17, 18, 22, 23, 24

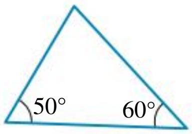
Fluency

WE10 For questions **1** to **5**, Identify the pair of similar triangles among those shown. Give reasons for your answers.

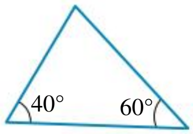
1. a.



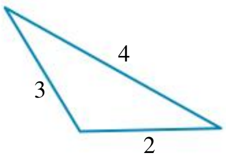
b.



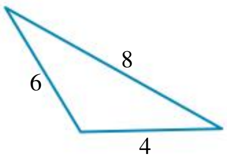
c.



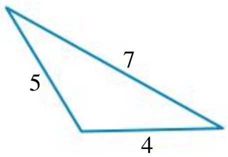
2. a.



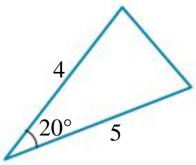
b.



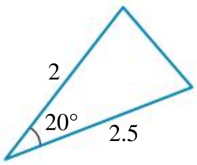
c.



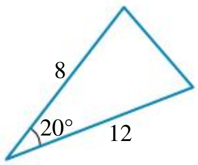
3. a.



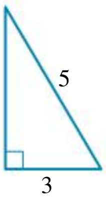
b.



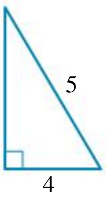
c.



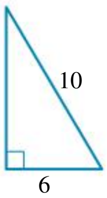
4. a.



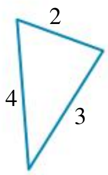
b.



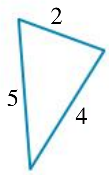
c.



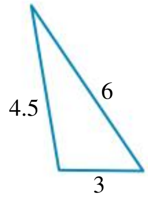
5. a.



b.

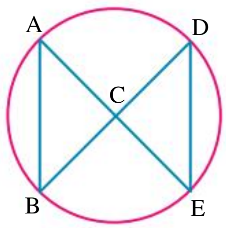


c.

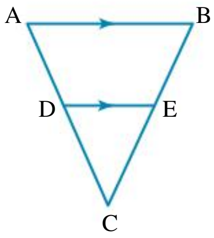


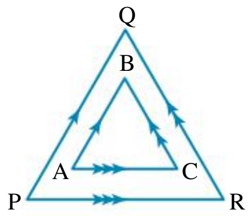
Understanding

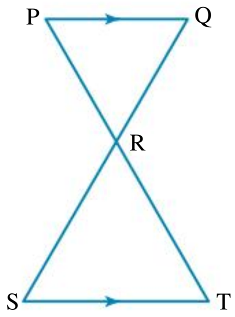
6. In the diagram shown, C is the centre of the circle. Complete this statement: ΔABC is similar to …

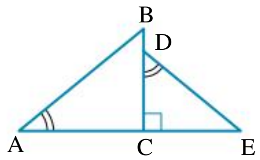


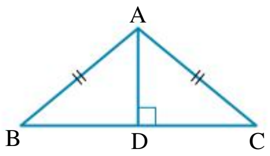
7. Name two similar triangles in each of the following figures, ensuring that vertices are listed in the correct order.

a. 

b. 

c. 

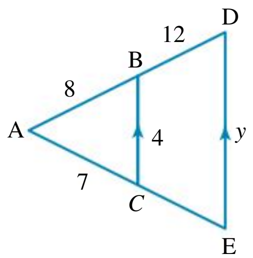
d. 

e. 

8. For the diagram shown:

a. complete the statement: $\dfrac{AB}{AD} = \dfrac{BC}{\square} = \dfrac{\square}{AE}$

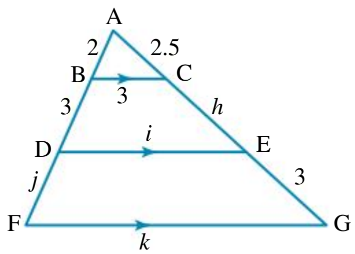
b. determine the value of the pronumerals.



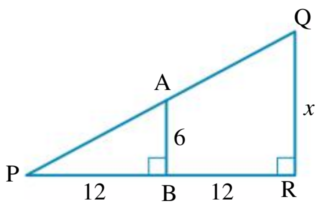
9. Using the diagram shown:

a. determine the values of h and i

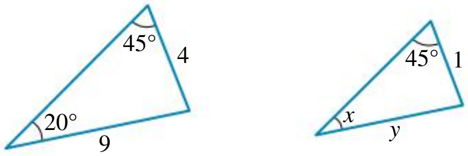
b. determine the values of j and k.



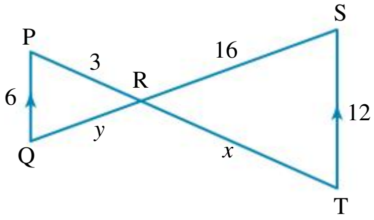
10. Determine the value of the pronumeral in the diagram shown.



11. If the two triangles shown are similar, determine the values of the pronumerals x and y.



12. Determine the values of the pronumerals x and y in the diagram shown.



13. Determine the value of each pronumeral in the following triangles. Show how you arrived at your answers.

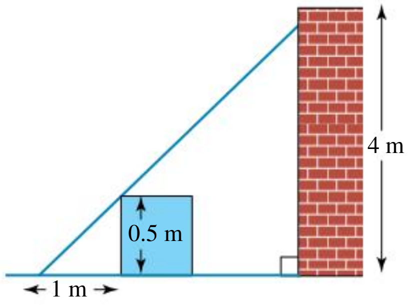
a.

b.

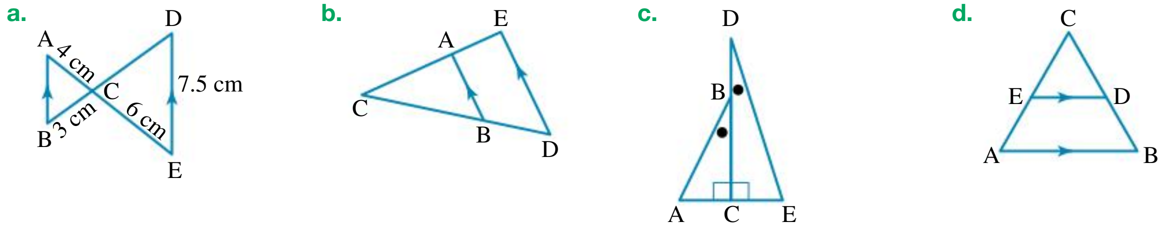
c.

Reasoning

14. WE12 A ladder just touches a bench and also leans on a wall that is 4 metres high, as shown in the diagram.
If the bench is 50 centimetres high and 1 metre from the base of the ladder, show that the base of the ladder is 8 metres from the wall.



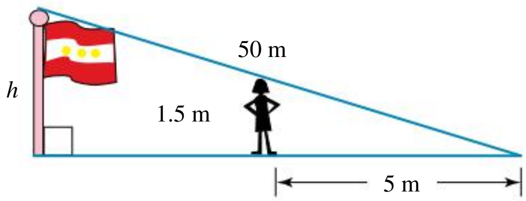
15. **WE11** Prove that ΔABC is similar to ΔEDC in each of the following.



16. Natalie is 1.5 metres tall. They cast a shadow 2 metres long at a certain time of the day.

If Alex is 1.8 metres tall, show that their shadow would be 2.4 m long at the same time of day.

17. A string 50 metres long is pegged to the ground and tied to the top of a flagpole. It just touches Maz on the top of her head. If Maz is 1.5 metres tall and 5 metres away from the point where the string is held to the ground, show that the height, h, of the flagpole is 14.37 m.



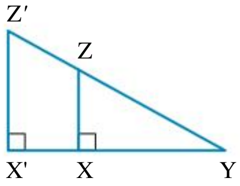
18. Using diagrams or other methods, explain whether the following statements are True or False.
   a. All equilateral triangles are similar.
   b. All isosceles triangles are similar.
   c. All right-angled triangles are similar.
   d. All right-angled isosceles triangles are similar.

Problem solving

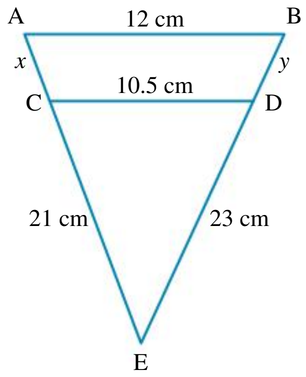
19. Paw and Thuy play tennis at night under floodlights. When Paw stands 2.5 m from the base of the floodlight, her shadow is 60 cm long.
   a. If Paw is 1.3 m tall, evaluate the height of the floodlight in metres, correct to 2 decimal places.
   b. If Thuy, who is 1.6 m tall, stands in the same place, calculate her shadow length in cm.

20. To determine the height of a flagpole, Jenna and Mia decide to measure the shadow cast by the flagpole. They place a 1 m ruler at a distance of 3 m from the base of the flagpole and measure the shadows that both the ruler and flagpole cast. Both shadows finish at the same point.
After measuring the shadow of the flagpole, Jenna and Mia calculate that the height of the flagpole is 5 m. Determine the length of the shadow cast by the flagpole, as measured by Jenna and Mia. Give your answer in metres.

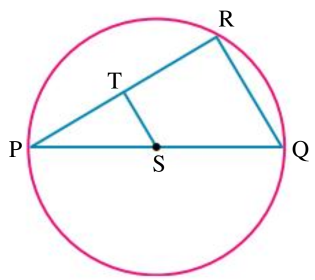
21. Use the diagram shown to determine the value of a if XZ = 8 cm, X′Z′ = 12 cm, X′X = a cm and XY = $(a + 1)$ cm.



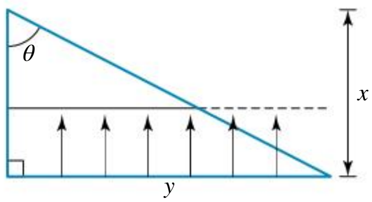
22. AB and CD are parallel lines in the figure shown.
 a. State the similar triangles.
 b. Determine the values of x and y.



23. PQ is the diameter of the circle shown. The circle's centre is located at S. R is any point on the circumference. T is the midpoint of PR.
 a. Write down everything you know about this figure.
 b. Explain why ΔPTS is similar to ΔPRQ.
 c. Determine the length of TS if RQ is 8 cm.
 d. Determine the length of every other side, given that PT is 3 cm and the angle PRQ is a right angle.



24. For the diagram shown, show that, if the base of the triangle is raised to half of the height of the triangle, the length of the base of the newly formed triangle will be half of its original length.



LESSON 10.5 Area and volume of similar figures (extending)

LEARNING INTENTIONS

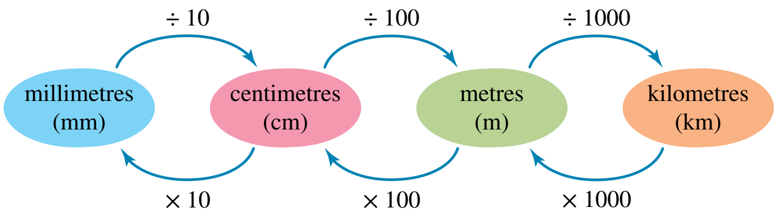
At the end of this lesson you should be able to:

- convert between different units of length, area and volume
- determine the area of similar figures, given a scale factor of n^2
- determine the volume of similar figures, given a scale factor of n^3.

10.5.1 Converting between units of length, area and volume

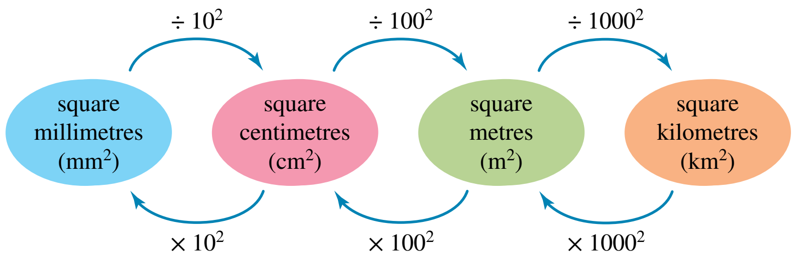
Units of length

- Metric units of length include millimetres (mm), centimetres (cm), metres (m) and kilometres (km).
- Length units can be converted using the chart shown.



Units of area

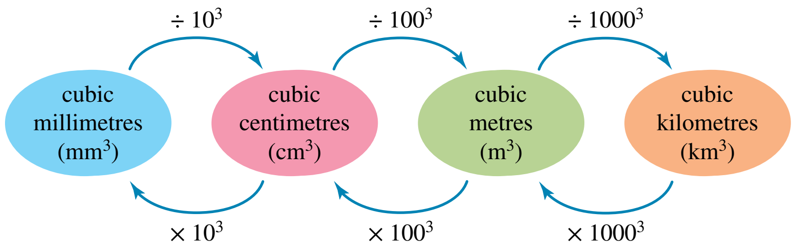
- **Area** is measured in square units, such as square millimetres $\left(\text{mm}^2\right)$, square centimetres $\left(\text{cm}^2\right)$, square metres $\left(\text{m}^2\right)$ and square kilometres $\left(\text{km}^2\right)$.
- Area units can be converted using the chart shown. Note, $10^2 = 100$; $100^2 = 10\,000$; $1000^2 = 1\,000\,000$



- Area units are the squares of the corresponding length units.

Units of volume

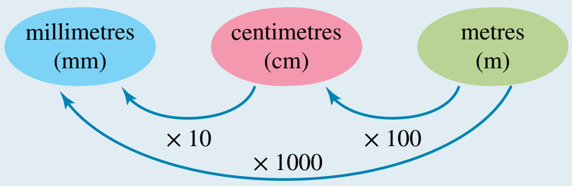
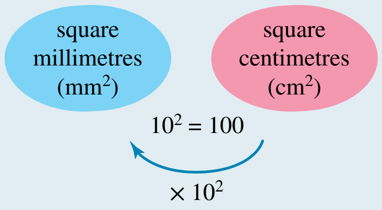
- **Volume** is measured in cubic units, such as cubic millimetres $\left(mm^3\right)$, cubic centimetres $\left(cm^3\right)$ and cubic metres $\left(m^3\right)$.
- Volume units can be converted using the chart shown. Note, $10^3 = 1000$; $100^3 = 1\,000\,000$; $1000^3 = 1\,000\,000\,000$



- Volume units are the cubes of the corresponding length units.

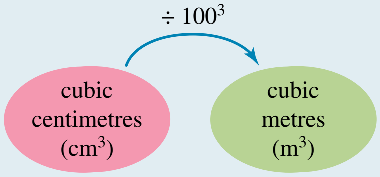
WORKED EXAMPLE 16 Converting between units of measurements

a. Convert 9 m into mm.
b. Convert 150 cm^2 into mm^2.
c. A cube has a side length of 8 cm. Calculate the volume of the cube in m^3.

THINK	WRITE
a. **1.** To convert m to mm: 	**a.** 1 m = 1000 mm
2. To convert 9 m to mm, multiply 9 with 1000.	$9\ m = 9 \times 1000\ mm$
3. Simplify and write the answer.	9 m is 9000 mm.
b. **1.** To convert cm^2 into mm^2: 	**b.** $1\ cm^2 = 100\ mm^2$
2. To convert 150 cm^2 into mm^2, multiply 150 by 10^2 (100).	$150\ cm^2 = 150 \times 100\ mm^2$
3. Simplify and write the answer.	150 cm^2 is 15 000 mm^2.
c. **1.** Write the formula for the volume of a cube.	**c.** Volume of a cube $(V) = l \times l \times l$, where l is the side length.
2. Substitute the value of the side length (l) in the volume formula.	$V = 8 \times 8 \times 8$ $= 512\ cm^3$

3. The question states that the answer should be given in m^3. To convert cm^3 into m^3:

$1\ \text{cm}^3 = \dfrac{1}{100^3}\ \text{m}^3$



4. To convert $512\ \text{cm}^3$ into m^3, multiply 512 by $\dfrac{1}{100^3}\ \text{m}^3$.

$512\ \text{cm}^3 = \dfrac{512}{100^3}\ \text{m}^3$

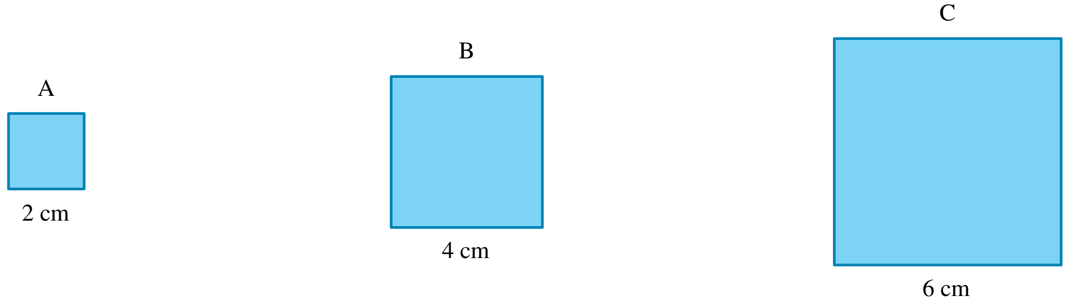
5. Simplify and write the answer.

The volume of the cube is $0.000\,512\ \text{m}^3$.

10.5.2 Area and volume of similar figures

Area of similar figures

- If the side lengths of any figure are increased by a scale factor of n, then the area of similar figures increases by a scale factor of n^2.
  For example, consider the squares shown.



$$\text{Area A} = 2 \times 2 = 4\ \text{cm}^2$$

$$\text{Area B} = 4 \times 4 = 16\ \text{cm}^2$$

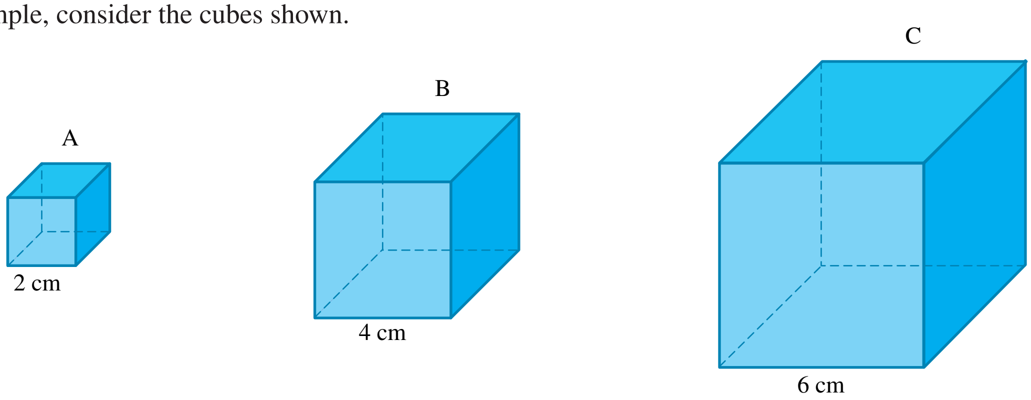
$$\text{Area C} = 6 \times 6 = 36\ \text{cm}^2$$

- The scale factors for the side lengths and the scale factors for the areas are calculated in the table shown.

Squares	Scale factor for side length	Scale factor for area
A and B	$\frac{4}{2} = 2$	$\frac{16}{4} = 4 = 2^2$
A and C	$\frac{6}{2} = 3$	$\frac{36}{4} = 9 = 3^2$
B and C	$\frac{6}{4} = \frac{3}{2}$	$\frac{36}{16} = \frac{9}{4} = \left(\frac{3}{2}\right)^2$

Surface area of similar figures

- The surface area of a 3D object also increases by the square of the length scale factor.
  For example, consider the cubes shown.



Surface area A $= 6 \times 4$
$= 24\ \text{cm}^2$

Surface area B $= 6 \times 16$
$= 96\ \text{cm}^2$

Surface area C $= 6 \times 36$
$= 216\ \text{cm}^2$

- The scale factors for the side lengths and the scale factors for the surface areas are calculated in the table shown.

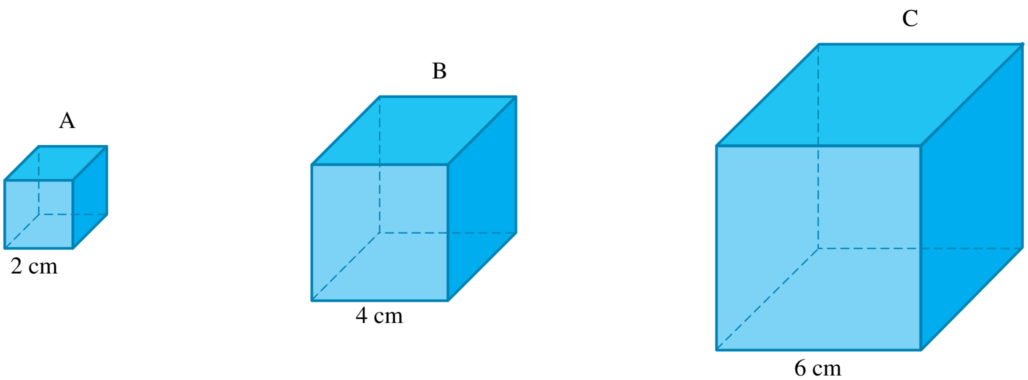
Cubes	Scale factor for side length	Scale factor for surface area
A and B	$\frac{4}{2} = 2$	$\frac{96}{24} = 4 = 2^2$
A and C	$\frac{6}{2} = 3$	$\frac{216}{24} = 9 = 3^2$
B and C	$\frac{6}{4} = \frac{3}{2}$	$\frac{216}{96} = \frac{9}{4} = \left(\frac{3}{2}\right)^2$

Areas of similar figures

When side lengths are increased by a factor of n, the area increases by a factor of n^2.

Volume of similar figures

- If the side lengths of any solid are increased by a scale factor of n, then the volume of similar solids increases by a scale factor of n^3.
  For example, consider the cubes shown.



Volume A $= 2 \times 2 \times 2$
$= 8\ \text{cm}^3$

Volume B $= 4 \times 4 \times 4$
$= 64\ \text{cm}^3$

Volume C $= 6 \times 6 \times 6$
$= 216\ \text{cm}^3$

- The scale factors for the side lengths and the scale factors for the volumes are calculated in the table shown.

Cubes	Scale factor for side length	Scale factor for volume
A and B	$\frac{4}{2}=2$	$\frac{64}{8}=8=2^3$
A and C	$\frac{6}{2}=3$	$\frac{216}{8}=27=3^3$
B and C	$\frac{6}{4}=\frac{3}{2}$	$\frac{216}{64}=\frac{27}{8}=\left(\frac{3}{2}\right)^3$

Volumes of similar figures

When side lengths are increased by a factor of n, the volume increases by a factor of n^3.

WORKED EXAMPLE 17 Calculating area and volume of similar figures

The side lengths of a box have been increased by a factor of 3.

a. Calculate the surface area of the new box if the original surface area was 94 cm^2.

b. Determine the volume of the new box if the original volume was 60 cm^3.

THINK	WRITE
a. 1. State the scale factor for side length used to increase the size of the original box.	**a.** Scale factor for side length $= 3$
2. The scale factor for surface area is the square of the scale factor for length.	Scale factor for surface area $= 3^2$ $= 9$
3. Calculate the surface area of the new box.	Surface area of new box $= 94 \times 9$ $= 846\text{ cm}^2$
b. 1. The scale factor for volume is the cube of the scale factor for length.	**b.** Scale factor for volume $= 3^3$ $= 27$
2. Calculate the volume of the new box.	Volume of new box $= 60 \times 27$ $= 1620\text{ cm}^3$

WORKED EXAMPLE 18 Using technology to calculate volume for similar figures

A small cone has a radius of 5 cm and a height of 15 cm.

a. **If the scale factor is 4, determine the dimensions of a larger similar cone.**

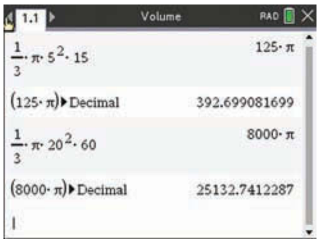
b. **Giving your answers both in exact form and correct to 3 decimal places, calculate the volume of:**

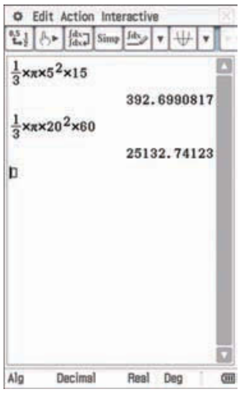
  i. **the smaller cone**

  ii. **the larger cone.**

c. **Show that the volumes of the two cones are in the ratio 1 : 64.**

THINK	WRITE
a. The scale factor is 4, so multiply the radius and height by 4. Write the answers.	**a.** For the larger cone: radius $= 5\text{ cm} \times 4 = 20\text{ cm}$ height $= 15\text{ cm} \times 4 = 60\text{ cm}$
b. i. Use your calculator to work out the volume of the smaller cone by substituting the following into the formula: $r = 5$ $h = 15$.	**b. i.** $V_s = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi \times 5^2 \times 15$ $= 125\pi\text{ cm}^3$ $\approx 392.699\text{ cm}^3$ (to 3 d.p)
ii. Use your calculator to work out the volume of the larger cone (V_L) by substituting the following into the formula: $r = 20$ $h = 60$.	**ii.** $V_L = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi \times 20^2 \times 60$ $= 8000\pi$ $\approx 25\,132.741\text{ cm}^3$ (to 3 d.p)
c. Use your calculator to evaluate the ratio of the volumes of the smaller cone and the larger cone. $\frac{V_s}{V_L} = \frac{125\pi}{8000\pi}$	**c.** $\frac{V_s}{V_L} = \frac{1}{64}$ The volumes are in the ratio 1 : 64.

TI | THINK	DISPLAY/WRITE
a-b. In a new problem, on a Calculator page, complete the entry line as: $\frac{1}{3} \times \pi \times 5^2 \times 15$ Then press ENTER. $\frac{1}{3} \times \pi \times 20^2 \times 60$ Then press ENTER. To convert the answer to decimal: • Menu • 2: Number • 1: Convert to Decimal Then press ENTER.	**a-b.**  $V_s = 125\pi\text{ cm}^3$ $V_s \approx 392.699\text{ cm}^3$ (to 3 d.p.) $V_L = 8000\pi\text{ cm}^3$ $V_L \approx 25\,132.741\text{ cm}^3$ (to 3 d.p.)

CASIO | THINK	DISPLAY/WRITE
a-b. On the Main screen, complete the entry line as: $\frac{1}{3} \times \pi \times 5^2 \times 15$ Then press EXE. $\frac{1}{3} \times \pi \times 20^2 \times 60$ Then press EXE. *Note:* Change Standard to Decimal	**a-b.**  $V_s = 125\pi\text{ cm}^3$ $V_s \approx 392.699\text{ cm}^3$ (to 3 d.p.) $V_L = 8000\pi\text{ cm}^3$ $V_L \approx 25\,132.741\text{ cm}^3$ (to 3 d.p.)

Exercise 10.5 Area and volume of similar figures (extending) learnon

10.5 Quick quiz on | 10.5 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 4, 8, 9, 13	2, 5, 7, 10, 14, 15	6, 11, 12, 16, 17, 18

Fluency

1. WE14a The side lengths of the following shapes have all been increased by a factor of 3.
Copy and complete the table shown.

Original surface area	Enlarged surface area
100 cm^2	**a.**
7.5 cm^2	**b.**
95 mm^2	**c.**
d.	918 cm^2
e.	45 m^2
f.	225 mm^2

2. A rectangular box has a surface area of 96 cm^2 and volume of 36 cm^3. Calculate the volume and surface area of a similar box that has side lengths that are double the size of the original box.

3. WE14b The side lengths of the following shapes have all been increased by a factor of 3.
Copy and complete the table shown.

Original volume	Enlarged volume
200 cm^3	**a.**
12.5 cm^3	**b.**
67 mm^3	**c.**
d.	2700 cm^3
e.	67.5 m^3
f.	27 mm^3

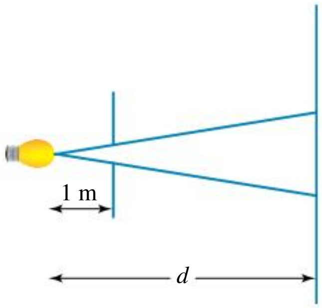
Understanding

4. The area of a bathroom as drawn on a house plan is 5 cm^2. Calculate the area of the actual bathroom if the map has a scale of 1 : 100.

5. The area of a kitchen is $25\,m^2$.
   a. WE13 Convert $25\,m^2$ to cm^2.
   b. Calculate the area of the kitchen as drawn on a plan if the scale of the plan is 1 : 120. (Give your answer correct to 1 decimal place.)
6. The volume of a swimming pool as it appears on its construction plan is $20\,cm^3$. Determine the actual volume of the pool if the plan has a scale of 1 : 75.
7. The total surface area of an aeroplane's wings is $120\,m^3$.
   a. Convert $120\,m^2$ to cm^2.
   b. Calculate the total surface area of the wings of a scale model of the aeroplane if the model is built using the scale 1 : 80.

Reasoning

8. A triangle ABC maps to triangle $A'B'C'$ under an enlargement.
   $AB = 7\,cm$, $AC = 5\,cm$, $A'B' = 21\,cm$, $B'C' = 30\,cm$.
   a. Show that the scale factor for enlargement is 3.
   b. Determine BC.
   c. Determine $A'C'$.
   d. If the area of ΔABC is $9\,cm^2$, show that the area of $\Delta A'B'C'$ is $81\,cm^2$.
9. A pentagon has an area of $20\,cm^2$. If all the side lengths are doubled, show that the area of the enlarged pentagon is $80\,cm^2$.
10. Two rectangles are similar. If the width of the first rectangle is twice of width of the other, prove that the ratio of their areas is 4 : 1.
11. A cube has a surface area of $253.5\,cm^2$. (Give answers correct to 1 decimal place where appropriate.)
    a. Show that the side length of the cube is 6.5 cm.
    b. Show that the volume of the cube is $274.625\,cm^3$.
    c. Determine the volume of a similar cube that has side lengths twice as long.
    d. Determine the volume of a similar cube that has side lengths half as long.
    e. Determine the surface area of a similar cube that has side lengths one third as long.
12. In the diagram shown, a light is shining through a hole, resulting in a circular bright spot with a radius of 5 cm on the screen.



The hole is 10 mm wide. If the light is 1 m behind the hole, show that the light is 10 m from the screen.

Problem solving

13. The areas of two similar trapeziums are 9 and 25. Determine the ratio of one pair of these trapeziums' corresponding side lengths.

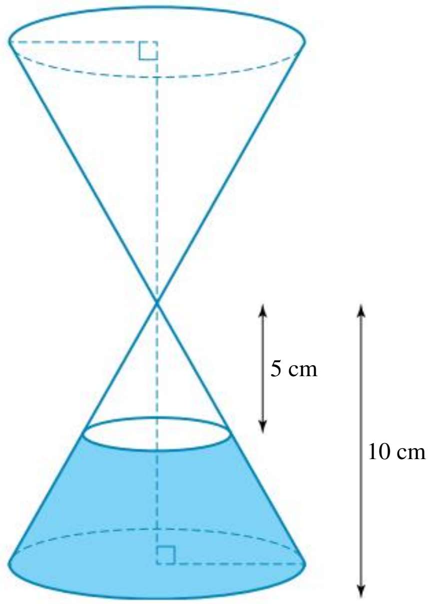
14. Answer the following questions.
    a. Calculate the areas of squares with sides 2 cm, 5 cm, 10 cm and 20 cm.
    b. State in words how the ratio of the areas of these squares is related to the ratio of their side lengths.

15. Two cones are similar. The ratio of these cones' volumes is 27 : 64.
    Determine the ratio of:
    a. the perpendicular heights of the cones
    b. the areas of the bases of the cones.

16. Rectangle A has dimensions 5 by 4 units, rectangle B has the dimensions 4 by 3 units, and rectangle C has the dimensions 3 by 2.4 units.
    a. Determine which of these rectangles are similar. Explain your answer.
    b. Evaluate the area scale factor for the similar rectangles that you have identified.

17. A balloon in the shape of a sphere has an initial volume of 840 cm^3. Its volume is then increased to 430 080 cm^3.
    Determine the increase in the radius of the balloon.

18. WE15 The bottom half of an egg timer, which is shaped like two cones connected at their apexes, has sand poured into it as shown by the blue section of this diagram.



Using the measurements given, evaluate the ratio of the volume of sand in the bottom half of the egg timer to the volume of empty space that is left in the bottom half of the egg timer.
You could also use technology to evaluate the answer.

LESSON 10.6 Review

10.6.1 Topic summary

Units of measurement

When converting units of **length**:

millimetres (mm)		centimetres (cm)		metres (m)		kilometres (km)
	$\div 10$ →		$\div 100$ →		$\div 1000$ →	
	← $\times 10$		← $\times 100$		← $\times 1000$	

When converting units of **area**:

millimetres (mm^2)		centimetres (cm^2)		metres (m^2)		kilometres (km^2)
	$\div 10^2$ →		$\div 100^2$ →		$\div 1000^2$ →	
	← $\times 10^2$		← $\times 100^2$		← $\times 1000^2$	

When converting units of **volume**:

millimetres (mm^3)		centimetres (cm^3)		metres (m^3)		kilometres (km^3)
	$\div 10^3$ →		$\div 100^3$ →		$\div 1000^3$ →	
	← $\times 10^3$		← $\times 100^3$		← $\times 1000^3$	

Ratio and scale factors

- Ratios compare quantities of the same type.
- Always simplify ratios. For example: 8 : 24 = 1 : 3
- Scale factor = $\dfrac{\text{image length}}{\text{object length}}$.
- For a scale factor of n:
  - if $n > 1$ the image is larger than the object
  - if $n < 1$ the image is smaller than the object.

Similar figures

- Similar figures have the same shape, but different sizes.
- The symbol for similarity is ~.

If the scale factor of similar figures' sides is n, then:

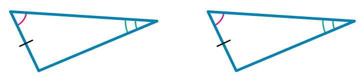
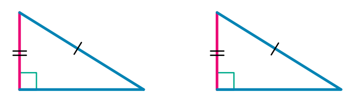
- the scale factor of their areas is n^2
- the scale factor of their volumes is n^3.

CONGRUENCE AND SIMILARITY

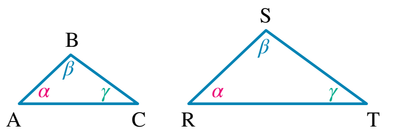
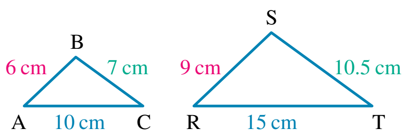
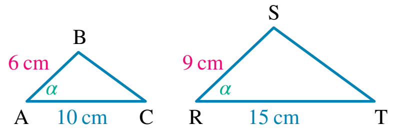
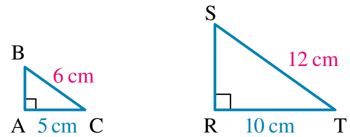
Congruent figures

- Congruent figures have exactly the same shape and size.
- The symbol for congruence is ≅.

Tests for congruent triangles

	SSS
	SAS
	ASA
	RHS

Tests for similar triangles

	AAA
	SSS
	SAS
	RHS

- *Note:* Sides of similar triangles are not equal. They are proportional, or have the same scale factor.

10.6.2 Success criteria

Tick the column to indicate that you have completed the lesson and how well you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
10.2	I can compare two quantities of the same type using ratios.			
	I can simplify ratios.			
	I can enlarge or reduce a figure by applying a scale factor.			
	I can calculate and use a scale factor, including plans.			
10.3	I can identify congruent figures.			
	I can show that two triangles are congruent using the appropriate congruency test.			
	Use proofs to determine congruent triangles (optional).			
10.4	I can identify similar figures.			
	I can determine the scale factor between similar figures.			
	I can show that two triangles are similar using the appropriate similarity test.			
10.5	I can convert between different units of length, area and volume.			
	I can, for a given scale factor of n^2, determine the area of similar figures.			
	I can, for a given scale factor of n^3, determine the volume of similar figures.			

10.6.3 Project

What's this object?

When using geometrical tools to construct shapes, we have to make sure that our measurements are precise. Even a small error in one single step of the measuring process can result in an incorrect shape.

The object you will be making in this task is a very interesting one. This object is made by combining three congruent shapes.

The instructions for making the congruent shape that you will use to then make the final object are given below.

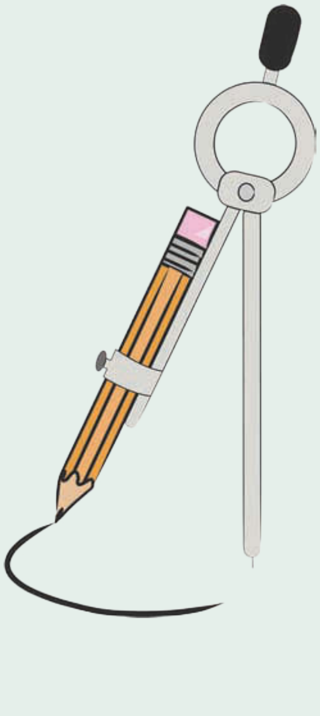
Part 1: Making the congruent shape

1. Using a ruler, a protractor, a pencil and two compasses, draw a shape in your workbook by following the instructions below.
   - Measure a horizontal line (AB) that is 9.5 cm long. To make sure there is enough space for the whole of the final object, draw this first line close to the bottom of the page you are working on.
   - At point B, measure an angle of 120° above the line AB. Draw a line that follows this angle for 2 cm. Mark the end of this line as point C, creating the line BC.
   - At point C, measure an angle of 60° on the same side of line BC as point A. Draw a line that follows this angle for 7.5 cm. Mark the end of this line as point D, creating the line CD.
   - At point D, measure an angle of 60°above the line CD. Draw a line that follows this angle for 3.5 cm. Mark the end of this line as point E, creating the line DE.
   - At point E, measure an angle of 120° above the line DE. Draw a line that follows this angle for 2 cm. Mark the end of this line as point F, creating the line EF.
   - Join point F to point A.
     Colour or shade this shape using any colour you wish.
2. Determine the length of the line joining point A to point F.
3. Describe what you notice about the size of the angles FAB and AFE.

Part 2: Making the final object

You have now constructed the shape that will be used three times to make the final object. To make this object, follow the instructions below.

- Trace the original shape onto a piece of tracing paper twice. Cut around the edges of both of the shapes you have just drawn. Label the shapes with the same letters you used to mark the points A–F when making the original shape.
- Place line AF of the first traced copy of the original shape so that it is covering up line CD of the shape you first drew. Use the tracing paper to transfer this shape onto the original shape. Colour or shade the traced shape using a colour that is different to the colour you used for the first shape.
- Place the line DC of your second traced copy so that it is covering up line FA of the shape you first drew. Use the tracing paper to transfer this shape onto the original shape. Colour or shade this third section with a third colour.



4. Describe the object you have drawn.
5. Using the internet, your school library or other references, investigate other ‘impossible objects’ that can be drawn as 2-dimensional shapes. Recreate these shapes on a separate sheet of paper. Beside these shapes, briefly write some reasons for these shapes being referred to as ‘impossible’.

Exercise 10.6 Review questions

learnon

Fluency

1. Express each of the following ratios in simplest form.
   a. 8 : 16
   b. 24 : 16
   c. 27 : 18

2. Express each of the following ratios in simplest form.
   a. 56 : 80
   b. 8 : 20
   c. 49 : 35

3. **MC** There are 9 girls and 17 boys in a Year 9 maths class. Select the ratio of boys to girls.
   A. 9 : 17
   B. 17 : 9
   C. 17 : 26
   D. 9 : 26

4. Jan raised \$15 for a charity fundraising event, while her friend Lara raised \$25. Calculate the ratio of:
   a. the amount Jan raised to the amount Lara raised
   b. the amount Jan raised to the total amount raised by the pair
   c. the amount the pair raised to the amount Lara raised.

5. A pigeon breeder has 45 pigeons. These include 15 white, 21 speckled pigeons, and the rest grey. Calculate the following ratios.
   a. White pigeons to grey pigeons.
   b. Grey pigeons to speckled pigeons.
   c. White pigeons to the total number of pigeons.

6. Express each of the following ratios in simplest form.
   a. $4\frac{1}{2} : 1$
   b. $6\frac{1}{8} : 9$
   c. $7\frac{1}{4} : 3\frac{1}{2}$

7. Express each of the following ratios in simplest form.
   a. 8.4 : 7.2
   b. 0.2 : 2.48
   c. 6.6 : 0.22

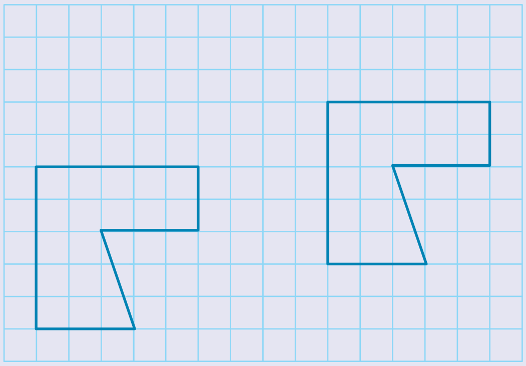
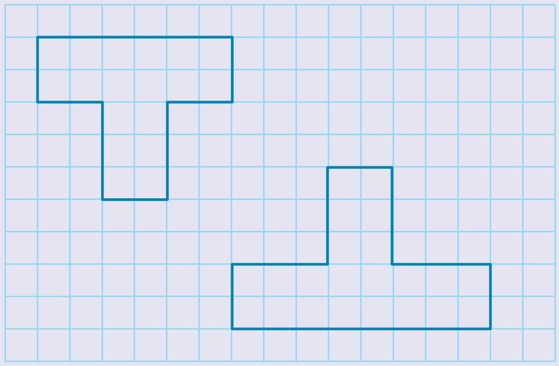
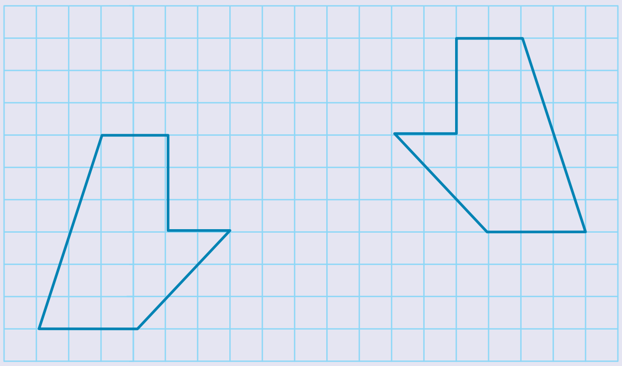
8. **MC** Jack has completed 2.5 km of a 4.5 km race. Choose the ratio of the distance Jack has completed to the remaining distance he has left to run.
   A. 9 : 5
   B. 4 : 5
   C. 4 : 9
   D. 5 : 4

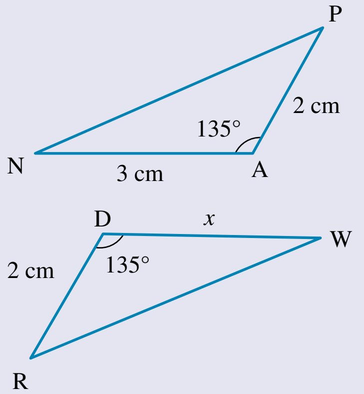
9. **MC** If $x : 16 = 5 : 4$, select the value of the pronumeral x.
   A. 1
   B. 4
   C. 16
   D. 20

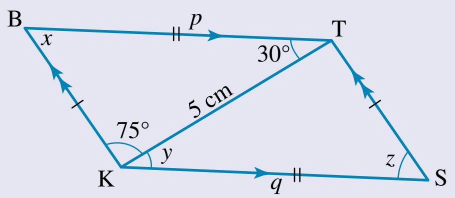
10. **MC** If $40 : 9 = 800 : y$, select the value of the pronumeral y.
    A. 20
    B. 40
    C. 180
    D. 450

11. Determine the value of the pronumeral in each of the following.
    a. $a : 15 = 2 : 5$
    b. $b : 20 = 5 : 8$
    c. $9 : 10 = 12 : c$

12. $\Delta PQR \sim \Delta DEF$. Determine the length of the missing side in each of the following combinations.
   a. Calculate DF if PQ = 10 cm, DE = 5 cm and PR = 6 cm
   b. Calculate EF if PQ = 4 cm, DE = 12 cm and QR = 5 cm
   c. Calculate QR if DE = 4 cm, PQ = 6 cm and EF = 8 cm
   d. Calculate PQ if DF = 5 cm, PR = 8 cm and DE = 6 cm
   e. Calculate DE if QR = 16 cm, EF = 6 cm and PQ = 12 cm

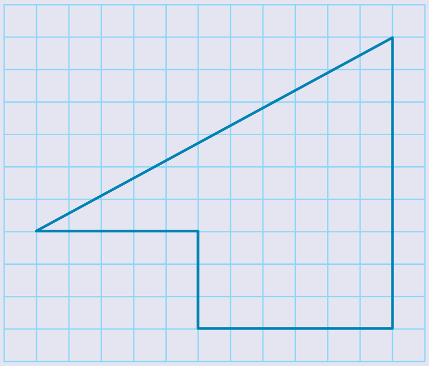
13. Identify which of the following pairs of shapes are congruent.
   a.

   b.

   c.


14. Name the congruent triangles in these figures and determine the value of the pronumerals in each case.
   a.


   b.


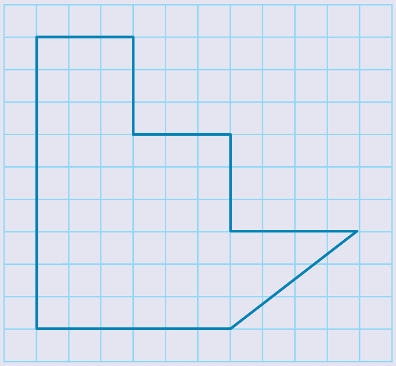
15. Copy each of the following shapes and enlarge (or reduce) them by the given factor.

a.



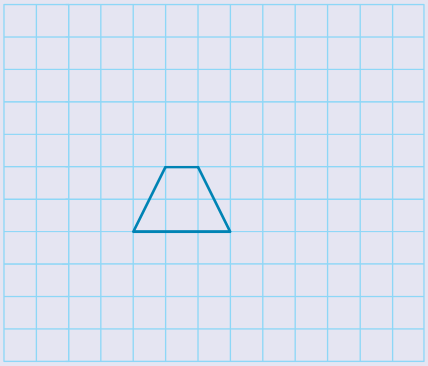
Enlarge by a factor of 2.

b.



Reduce by a factor of 3.

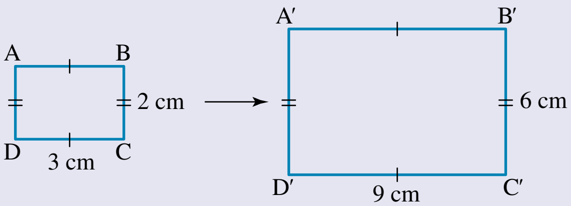
c.



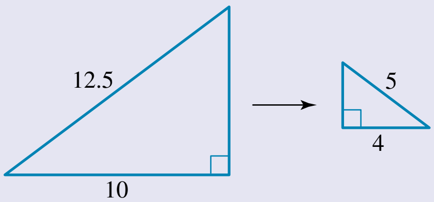
Enlarge by a factor of 4.

16. Determine the enlargement factors that have been used on the following shapes.

a.



b.



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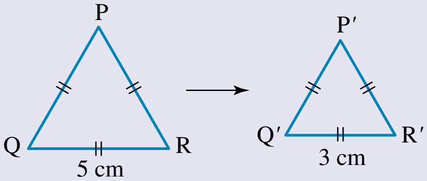
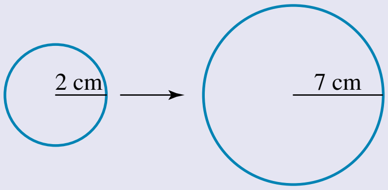
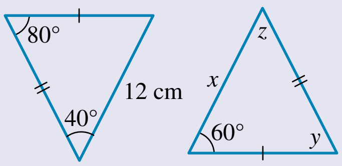


d.



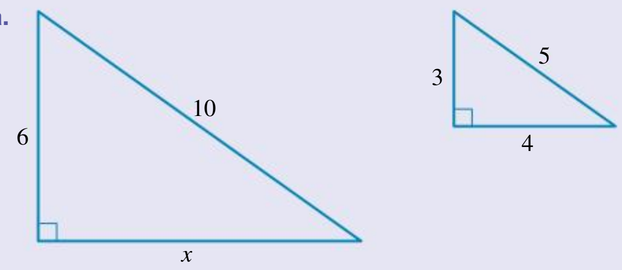
17. Evaluate the value of the pronumerals in the pair of congruent triangles shown.



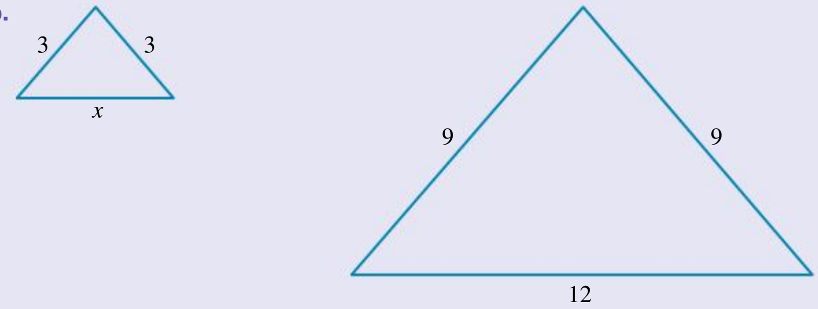
18. The area of a family room is 16 m^2 and the length of the room is 6.4 m. Calculate the area of the room if it is drawn on a plan that uses a scale of 1 : 20.

19. Each of the diagrams shown shows a pair of similar triangles. Calculate the value of x in each case.

a.



b.



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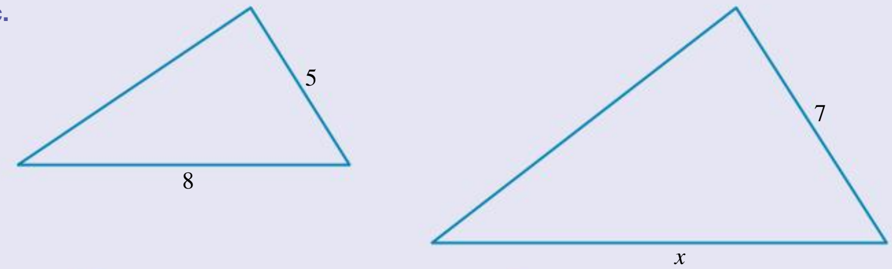

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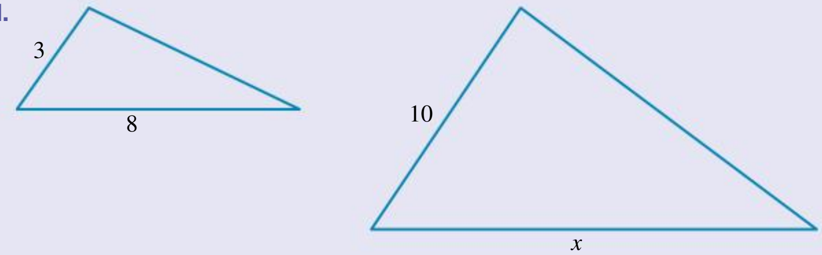
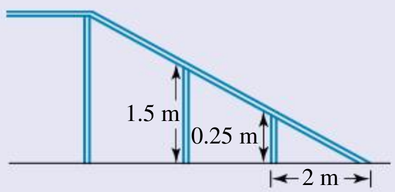


Problem solving

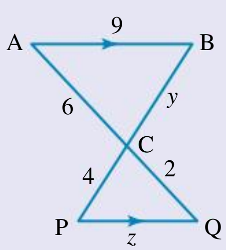
20. The diagram shows a ramp made by Jinghua for her automotive class. The first post has a height of 0.25 m and is placed 2 m from the end of the ramp.
If the second post is 1.5 m high, determine the distance it should be placed from the first post.



21. Consider the figure shown.

a. Write a formal proof for a similarity relationship between two triangles in this figure. Give a reason for the similarity.

b. Determine the values of the pronumerals.

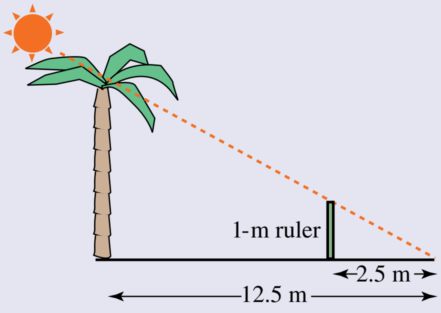


22. *Slocum* is a yacht with a length of 12 m and a beam (width) of 2.5 m. A model of *Slocum* is constructed for a museum. If the length and width of the model are one fifth of the yacht's actual length and width, determine how the volume of the yacht and the volume of the model compare.

23. Before the start of a yacht race, judges must certify that all of the yachts have the same size of sail. Without removing the triangular sails from their masts, state the congruency rule that the official could most efficiently use to work out if the sails on each of the boats are exactly the same.
Explain why this is the most appropriate rule.

24. Calculate the height of the top of the ladder in the photo shown by using similar triangles. Give your answer correct to 1 decimal place.

25. Poh is given a 1 m ruler and asked to estimate the height of a palm tree. She places the ruler vertically so that its shadow ends at exactly the same point as the shadow of the palm tree. The ruler's shadow is 2.5 m long and the palm tree's shadow is 12.5 m long.

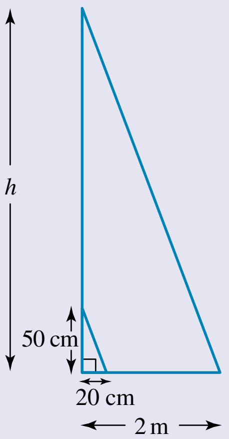


Poh performed some calculations using similar triangles and calculated the height of the palm tree to be 4 m. Her friend Mikalya said that she thought Poh's calculations were incorrect, and that the answer should be 5 m.

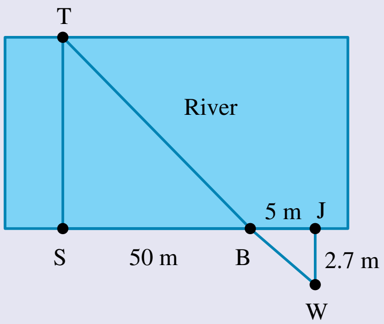
a. State the correct answer.

b. Explain the error that was made by the person with the incorrect answer.

26. A flagpole casts a shadow 2 m long. If a 50 cm ruler is placed upright at the base of the flagpole it casts a shadow 20 cm long. Evaluate the height of the flagpole.



27. A student (S) uses a tape measure, backpack (B), water bottle (W), tree (T) and jacket (J) to help calculate the width of a river.



a. Copy and complete the following:
  i. ∠____ and ∠ _____ are both ____.
  ii. ∠SBT and ∠ _____ are equal because _____.
  iii. ∠STB and ∠ _____ are equal because _____.

b. Rewrite the following statement, selecting the correct alternatives from within each set of brackets.
∠STB and ∠JWB are (similar / congruent) (SSS / SAS / AAA / ASA / RHS)

c. Copy and complete the following equation.

$$\frac{\overline{ST}}{\overline{JW}} = \frac{\square}{\square}$$

d. Use your answer to part **c** to evaluate the width of the river.

on To test your understanding and knowledge of this topic go to your learnON title at www.jacplus.com.au and complete the **post-test**.

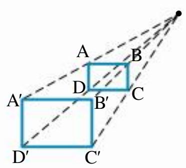
Answers

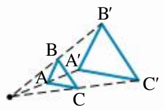
Topic 10 Congruence and similarity

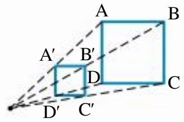
10.1 Pre-test

1. 25 : 39
2. $x = 13.5$
3. D
4. C
5. $y : z = 9 : 8$
6. False
7. D
8. a. $\angle CED$ b. ASA
9. A
10. B, D
11. $x = 2.3$ cm
12. $b = 7.5$
13. a. 3 : 5 b. 9 : 25
14. 9 : 1
15. B

10.2 Ratio and scale

1. a. i. 240 : 7 ii. 7 : 240
   b. The track is $34\frac{2}{7}$ times as long as it is wide.
2. a. i. 5 : 2 ii. 2 : 5
   b. The cliff is 2.5 times as high as the distance from the base of the cliff to the emu.
3. a. 2 : 3 b. 1 : 7 c. 1 : 3 d. 2 : 5
4. a. 4 : 15 b. 40 : 31 c. 8 : 25 d. 100 : 33
5. a. 6 : 1 b. 13 : 15 c. 4 : 1 d. 5 : 1
6. a. 1 : 2 b. 20 : 3 c. 1 : 6 d. 5 : 3
7. a. 5 : 1 b. 11 : 5 c. 43 : 20 d. 1 : 200
8. a. 3 : 4 b. 8 : 7 c. 2 : 3 d. 28 : 25
9. a. 2 : 7 b. 10 : 3 c. 33 : 16 d. 23 : 15
10. a. $a = 9$ b. $b = 24$ c. $c = 32$
11. a. $e = 3\frac{3}{4}$ b. $f = 14\frac{2}{5}$ c. $g = 3\frac{3}{4}$
12. a. $i = 29\frac{1}{3}$ b. $j = 8\frac{4}{7}$ c. $k = 10\frac{2}{5}$
13. 

14. 

15. 

16. a. 3 b. 12 cm c. 3.5 cm
17. a. 50 cm b. 4 cm
    c. 100 cm × 40 cm d. 8 cm × 3 cm
18. a. 175 : 6428 b. 37
19. 2.5 mL
20. 1000 g
21. a. 200 b. 6 m × 6 m c. 5 m × 5 m
22. a. Sample responses can be found in the worked solutions in the online resources.
    b. 1.69 g gold, 0.5 g copper, 0.06 g silver
23. Sample responses can be found in the worked solutions in the online resources.
24. $2k \times 3k \times 5k = 30k^3$

$$30k^3 = 21870$$
$$k^3 = 729$$
$$k = 9$$

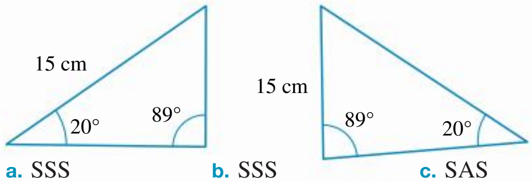
Substituting k into the ratio $2k : 3k : 5k$, the dimensions are 18 cm, 27 cm and 45 cm.

25. The profits aren't shared in a fair ratio. Tyler gets more profit than his share and Dylan gets less profit than his share. Only Ari gets the correct share of the profit.
26. Sample responses can be found in the worked solutions in the online resources.
27. 680 cm
28. $y : z = 9 : 8$
29. $P = 4$, $Q = 6$
30. 49 students
31. 125 students

10.3 Congruent figures

1. **b** and **c**
2. **a** and **c**
3. **a** and **b**
4. D
5. ΔABC and ΔPQR, SAS
6. ΔABC and ΔLNM, SSS
7. ΔLMN and ΔPQR, SAS
8. ΔABC and ΔPQR, ASA
9. ΔABC and ΔLMN, RHS
10. ΔABC and ΔDEF, ASA
11. $x = 3$ cm
12. $x = 85°$
13. $x = 80°$, $y = 30°$, $z = 70°$
14. a. $x = 30°$, $y = 7$
    b. $x = 40°$, $y = 50°$, $z = 50°$, $n = 90°$, $m = 90°$
15. a. 2 cm b. 8 cm c. 6 mm d. 7 mm
16. Yes, because they are identical.
17. No. The reason for congruence in this situation could alternatively be ASA.

18.

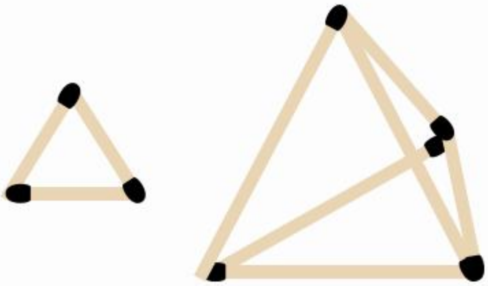


19. a. SSS b. SSS c. SAS

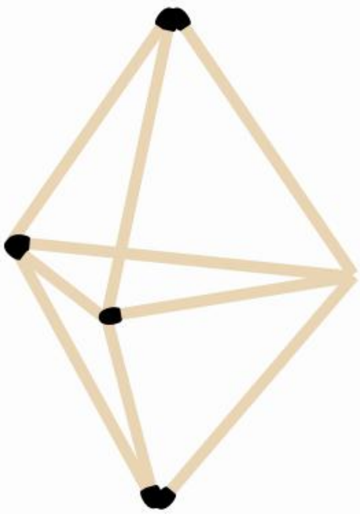
20. Sample responses can be found in the worked solutions in the online resources.

21. Because the angle is not between the two given sides, the general shape of the triangle is not set. This means that it is possible to draw many different shapes that fit these conditions.

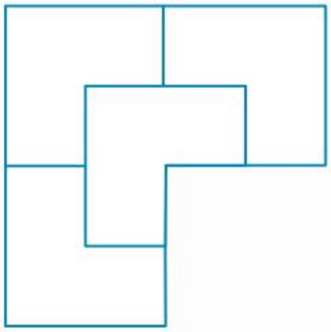
22. This can be done by making an equilateral triangle and a regular tetrahedron.



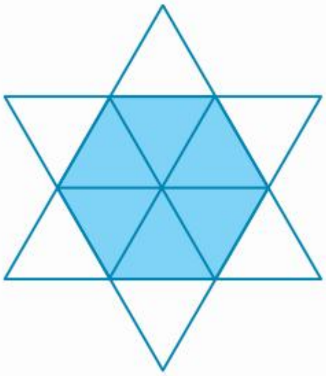
23. This can be done by making a a double regular tetrahedron.



24. Each piece shown in this solution is similar to the original shape.



25. 1 : 1



10.4 Similar figures

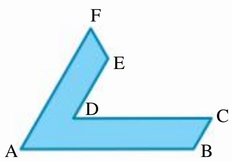
1. a and c, AAA
2. a and b, SSS
3. a and b, SAS
4. a and c, RHS
5. a and c, SSS
6. ΔEDC
7. a. ΔABC and ΔDEC
   b. ΔPQR and ΔABC
   c. ΔPQR and ΔTSR
   d. ΔABC and ΔDEC
   e. ΔADB and ΔADC
8. a. $\dfrac{AB}{AD} = \dfrac{BC}{DE} = \dfrac{AC}{AE}$
   b. $f = 9$, $g = 8$
9. a. $h = 3.75$, $i = 7.5$
   b. $j = 2.4$, $k = 11.1$
10. $x = 4$
11. $x = 20°$, $y = 2\frac{1}{4}$
12. $x = 3$, $y = 4$
13. a. $x = 7.1$ b. $x = 3.1$ c. $x = 7.5$, $y = 7.7$
14. Sample responses can be found in the worked solutions in the online resources.
15. Sample responses can be found in the worked solutions in the online resources.
16. Sample responses can be found in the worked solutions in the online resources.
17. Sample responses can be found in the worked solutions in the online resources.
18. a. True b. False c. False d. True
19. a. 6.72 m b. 78 cm
20. 3.75 m
21. $a = 1$ cm
22. a. ΔEDC and ΔEBA
    b. $x = 3$ cm, $y \approx 3.29$ cm
23. a. Sample response can be found in the worked solutions in the online resources.
    b. SAS
    c. 4 cm
    d. PR 6 cm, PS 5 cm and PQ 10 cm
24. The triangles are similar (AAA). $l = \dfrac{y}{2}$.

10.5 Area and volume of similar figures (extending)

1. a. 900 cm^2 b. 67.5 cm^2 c. 855 mm^2
   d. 102 cm^2 e. 5 m^2 f. 25 mm^2
2. $\text{SA} = 384\text{ cm}^2$, $\text{V} = 288\text{ cm}^3$
3. a. 5400 cm^3 b. 337.5 cm^3 c. 1809 mm^3
   d. 100 cm^3 e. 2.5 m^3 f. 1 mm^3
4. $50\,000\text{ cm}^2$
5. a. $250\,000\text{ cm}^2$ b. 17.4 cm^2
6. $8\,437\,500\text{ cm}^3$
7. a. $1\,200\,000\text{ cm}^2$ b. 187.5 cm^2
8. a. Sample responses can be found in the worked solutions in the online resources.
   b. 10 cm
   c. 15 cm
   d. Sample responses can be found in the worked solutions in the online resources.
9. Sample responses can be found in the worked solutions in the online resources.
10. Sample responses can be found in the worked solutions in the online resources.
11. a. Sample responses can be found in the worked solutions in the online resources.
    b. Sample responses can be found in the worked solutions in the online resources.
    c. 2197 cm^3
    d. 34.3 cm^3
    e. 28.2 cm^2
12. Sample responses can be found in the worked solutions in the online resources.
13. 3 : 5
14. a. 4 cm^2, 25 cm^2, 100 cm^2, 400 cm^2
    b. The ratio of the areas is equal to the square of the ratio of the side lengths.
15. a. 3 : 4 b. 9 : 16
16. a. A and C are similar rectangles by the ratio 5 : 3 or scale factor $\frac{5}{3}$.
    b. $\frac{25}{9}$
17. The new radius is 8 times the old radius.
18. The ratio is 7 : 1.

Project

1.

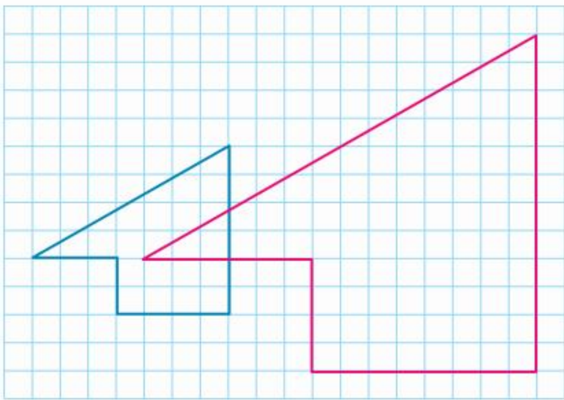


2. 7.5 cm
3. $\angle FAB = 60°$, $\angle AFE = 60°$
4. The impossible triangle

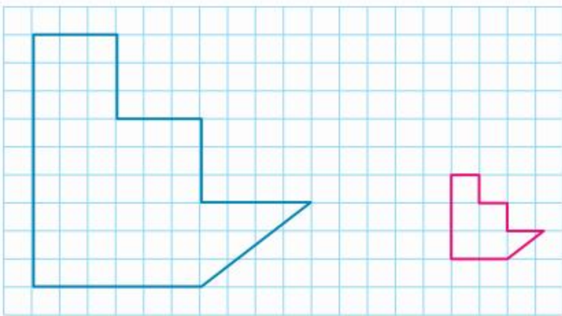
5. Students are required to research and investigate other 'impossible' objects drawn as two-dimensional shapes. These need to be created on a separate sheet of paper with reasons as to why they are termed 'impossible'.

10.6 Review questions

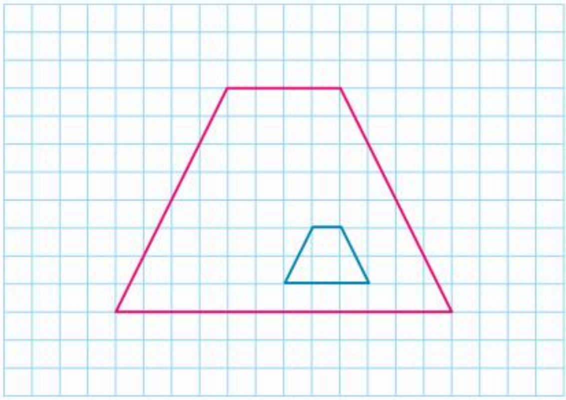
1. a. 1 : 2 b. 3 : 2 c. 3 : 2
2. a. 7 : 10 b. 2 : 5 c. 7 : 5
3. B
4. a. 3 : 5 b. 3 : 8 c. 8 : 5
5. a. 5 : 3 b. 3 : 7 c. 1 : 3
6. a. 9 : 2 b. 49 : 72 c. 29 : 14
7. a. 7 : 6 b. 5 : 62 c. 30 : 1
8. D
9. D
10. C
11. a. $a = 6$ b. $b = 12\frac{1}{2}$ c. $c = 13\frac{1}{3}$
12. a. 3 cm b. 15 cm c. 12 cm
    d. 9.6 cm e. 4.5 cm
13. a. Congruent b. Not congruent c. Congruent
14. a. $\Delta ANP \cong \Delta DWR$, $x = 3\text{ cm}$
    b. $\Delta TKB \cong \Delta KTS$ or $\Delta TKB \cong \Delta KST$, $x = 75°$, $y = 30°$, $z = 75°$, $p = 5\text{ cm}$, $q = 5\text{ cm}$
15. a.



b.



c.



16. a. 3 b. 0.4 c. $\frac{3}{5}$ d. 3.5

17. $x = 12$ cm, $y = 80°$, $z = 40°$

18. 400 cm^2

19. a. $x = 8$ b. $x = 4$ c. $x = 11\frac{1}{5}$ d. $x = 26\frac{2}{3}$

20. 10 m

21. a. $\Delta ABC \sim \Delta QPC$, AAA

b. $y = 12$, $z = 3$

22. The model is $\frac{1}{125}$ the volume of the yacht.

23. The easiest way to check for congruency is to measure the parts of each sail that are easiest to reach. This would be the bottom of each sail and the angles at the bottom of each of those sails. In this situation ASA would be the most appropriate rule to use.

24. 2.1 m

25. a. 5 m

b. Mikalya had the correct answer. Poh used the distance of 10 m (from the ruler to the tree) in her calculations instead of 12.5 m (the whole length of the tree's shadow).

26. 5 m

27. a. i. $\angle TSB$ and $\angle WJB$ are both right angles.

ii. $\angle SBT$ and $\angle JBW$ are equal because they are vertically opposite.

iii. $\angle STB$ and $\angle JWB$ are equal because they are alternate.

b. $\angle SBT$ and $\angle BJW$ are similar (AAA).

c. $\frac{\overline{ST}}{\overline{JW}} = \frac{50}{5}$

d. 27 m

Modelling with measurement and geometry

Fully worked solutions for this topic are available online.

LESSON Modelling with measurement and geometry

Introduction

We all use measurement every day, often without even realising that we are doing so. Understanding measurement allows us to understand the world around us. The knowledge of calculating the perimeter, area and volume of shapes is applied in several fields such as mapping, architecture and surveying. Calculating surface area of shapes is useful when estimating, for example, the amount of paint you need to repaint your room. None of the structures we build would be possible without our understanding of geometry and trigonometry.

MODELLING TASK 1: Structured

Beach huts

The outside walls of the beach hut shown are to be painted. A litre of paint covers 20 m^2 and the new paint colour will have to be applied twice to the whole area. As a paint contractor, your need to calculate the cost of the paint.

a. Analyse the situation:
- The front and back faces are the same shape.
- What do you need to determine?
- Do you need to consider the size of the cans of paint?

b. Determine assumptions and constraints:
- Identify any assumptions you need to make.
- A litre of paint covers 20 m^2.
- Cans of paint cost \$47 for a 1 L can, \$79 for a 4 L can and \$204 for a 15 L can.

c. Represent the situation mathematically:
- Calculate the surface area of the sides of the beach hut.
- Calculate the amount of paint to be used.
- The paint comes in various sized cans. Which would be the most economical?

d. Interpret and communicate findings:
- Present your calculations for the surface area and explain your method.
- Communicate your answer to the number of cans of paint, referring to the various sizes.
- What assumptions did you make when deciding on the number of cans of paint?

MODELLING TASK 2: Structured

The Louvre Museum Pyramid

The glass pyramid in the courtyard of the Louvre Museum in Paris has a height of 21.6 m and a square base of 35.4 m with metal edging. The glass needs cleaning and a specialised cleaning remote-controlled robot is used for this. The person controlling the robot needs to know the area the robot has to cover each time it is cleaning the glass pyramid. They also check the metal edging.

a. **Analyse the situation:**
- Describe the glass structure in the courtyard and its edges.
- What does the person controlling the robot need to know?

b. **Determine assumptions and constraints:**
- Are there any constraints or limitations you need to consider?
- What are you assuming?

c. **Represent the situation mathematically:**
- Calculate the perpendicular height of the triangular sides.
- Calculate the area of the triangular sides.
- State the area the robot has to clean.
- Calculate the slant edge of the glass pyramid.
- Calculate the total length of metal edging.

d. **Interpret and communicate findings:**
- Explain how you calculated the surface area.
- Discuss which edges you included in the metal edging and why.
- Communicate all of your findings.

MODELLING TASK 3: Open ended

New flag

Your sports association is running a competition for a new flag to use at their sporting events. Flags normally are in the ratio of 1: 2. The rules for the competition state the flag must include at least three and must include at least one pair of congruent shapes. Since they are offering three prizes, you are keen to enter and hopefully win one of the prizes. To enter, you need to submit a scale drawing by the due date. Your proposal needs to include all of the details to satisfy the judges.

Consider the following:
- What factors do you need to consider when planning this?
- What size is your flag?
- Have you included a pair of congruent shapes?
- Is your flag in the correct ratio?

MODELLING TASK 4: Open ended

Flying the flag

Your sporting field is rectangular in shape, measureing 90 m by 45 m. One corner has a flagpole, 10 m in height. You would like to see your winning flag flying if you are standing somewhere on the field. To do this, you calculate the distances and angles of elevation from different positions on the field. It has been suggested to see the flag clearly, the angle of elevation should be at least 20º.

How far do you think you could be? What angle would that be? You are advised to include diagrams in making your decisions.

Consider the following:

- What do you need to calculate?
- How are you going to use your mathematical skills to determine this?
- What rules of mathematics will you use?
- Are your answers realistic?

STRAND: Probability and statistics
SUB-STRAND: Probability and statistics

11 Probability

LESSON SEQUENCE

LESSON 11.1 Overview

11.1.1 Why learn this?

New technologies which predict human behaviour are all based on probability. On 11 May 1997, IBM's supercomputer Deep Blue made history by defeating chess grandmaster Gary Kasparov in a six-game match under standard time controls. This was the first time that a computer had defeated the highest ranked chess player in the world. Deep Blue won by evaluating millions of possible positions each second and determining the probability of victory from each possible choice. This application of probability allowed artificial intelligence to defeat a human who had spent his life mastering the game.

Combining advanced computing with probability is no longer used only to play games. From the daily convenience of predictive text, through to assisting doctors with cancer diagnosis, companies spend vast amounts of money on developing probability-based software. Self-driving cars use probability functions to predict the behaviour of other cars on the road and pedestrians. The more accurate the predictive functions, the safer self-driving cars can become. Politicians use campaign data analysis to develop models that produce predictions about individual citizens' likelihood of supporting specific candidates and issues, and the likelihood of these citizens changing their support if they're targeted with various campaign interventions. Even ChatGPT uses probability to create the responses it gives.

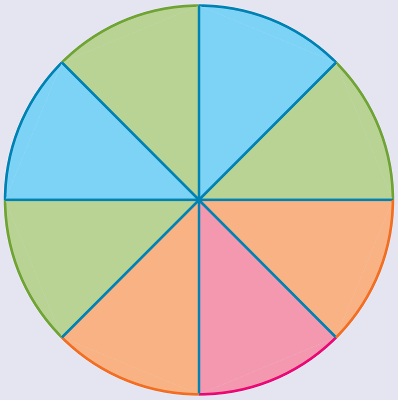
As technology improves, so will its predictive power in determining the likelihood of certain outcomes occurring. It is important that we study and understand probability so we know how technology is being used and the impact it will have on our day-to-day life.

Exercise 11.1 Pre-test

learnon

1. **MC** A six-sided die is rolled, and the number uppermost is noted. The event of rolling an even number is:
   A. $\{1, 2, 3, 4, 5, 6\}$
   B. $\{0, 1, 2, 3, 4, 5, 6\}$
   C. $\{2, 4, 6\}$
   D. $\{1, 3, 5\}$

2. The coloured spinner shown is spun once and the colour noted. Written in its simplest form, state the probability of spinning:
   a. an orange and a blue
   b. an orange or a pink.



3. A coin is tossed in an experiment and the outcomes recorded.

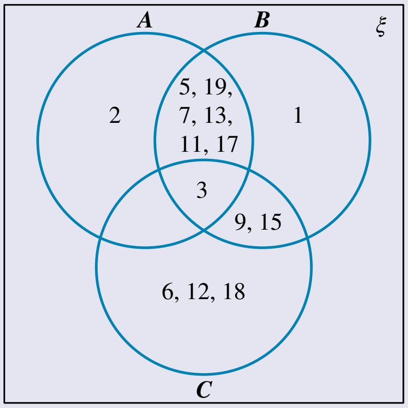
Outcome	Heads	Tails
Frequency	72	28

   a. Identify how many trials there were.
   b. Calculate the experimental probability for tossing a Tail, giving your answer in simplest form.
   c. Calculate the theoretical probability for tossing a Tail with a fair coin, giving your answer in simplest form.
   d. In simplest form, calculate the difference in this experiment between the theoretical and experimental probabilities for tossing a Tail.

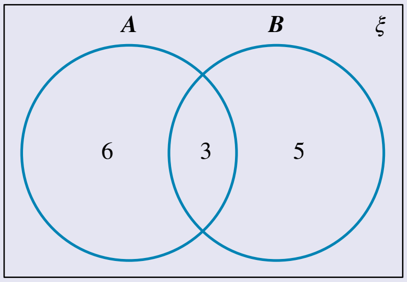
4. A random number is selected from:
   $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$
   Calculate the exact probability of selecting a number that is a multiple of 3, giving your answer in simplest form.

5. Identify whether the following statement is True or False.
   If a large number of trials is conducted in an experiment, the relative (experimental) frequency of each outcome will be very close to its theoretical probability.

6. **MC** Consider the Venn diagram shown.
   If $\xi = \{$numbers between 1 and 20 inclusive$\}$, identify which of the following set B is equal to.
   A. $B = \{$multiples of 3 numbers between 1 and 20$\}$
   B. $B = \{$number 1$\}$
   C. $B = \{$even numbers between 1 and 20$\}$
   D. $B = \{$odd numbers between 1 and 20$\}$



7. **MC** The information presented in the Venn diagram can be shown on a two-way frequency table as:



A.

	A	A'	Total
B	5	3	8
B'	0	6	6
Total	5	9	14

B.

	A	A'	Total
B	3	6	9
B'	5	0	5
Total	8	6	14

C.

	A	A'	Total
B	3	5	8
B'	6	0	6
Total	9	5	14

D.

	A	A'	Total
B	3	2	5
B'	3	3	6
Total	6	5	11

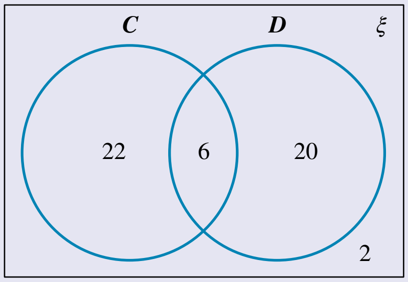
8. **MC** If $\xi = \{\text{numbers between 1 and 20 inclusive}\}$, identify the complement of the event $A = \{\text{multiples of 5 and prime numbers}\}$.

A. $\{1, 4, 6, 8, 9, 12, 14, 16, 18\}$
B. $\{5, 10, 15, 20\}$
C. $\{2, 3, 5, 7, 11, 13, 17, 19\}$
D. $\{2, 3, 5, 7, 10, 11, 13, 15, 17, 19, 20\}$

9. The jacks and aces from a deck of cards are shuffled, then two cards are drawn. Calculate the exact probability, in simplest form, that two aces are chosen:

a. if the first card is replaced
b. if the first card is not replaced.

10. The Venn diagram shows the results of a survey where students were asked to indicate whether they prefer drama (D) or comedy (C) movies.
In simplest form, determine the probability that a student selected at random prefers drama movies but does not like comedy.

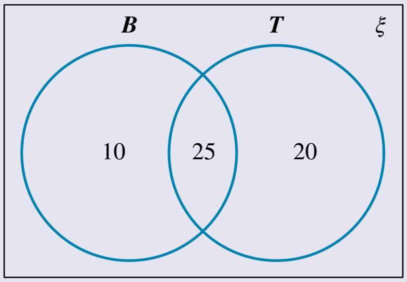


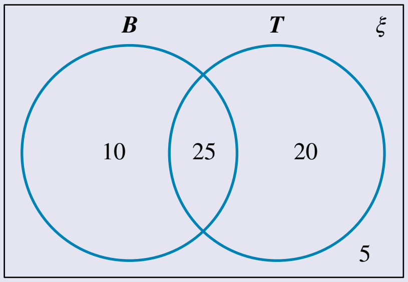
11. From a bag of mixed lollies, students can choose from three different types of lollies: red frogs, milk bottles or jelly babies. The bag contains five of each type of lolly.
If Mahsa chooses two lollies without looking into the bag, calculate the probability that she will choose two different types of lollies.

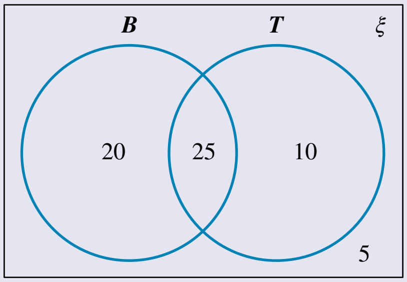
12. The students in a class were asked about their sport preferences — whether they played basketball or tennis or neither.
The information was recorded in a two-way frequency table.

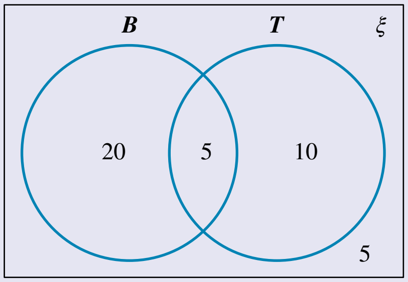
	Basketball (B)	**No basketball (B')**
Tennis (T)	25	20
No tennis (T')	10	5

a. MC The student sport preferences recorded in the two-way frequency table is represented on a Venn diagram as:

A.


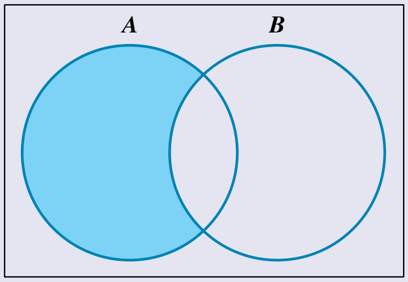
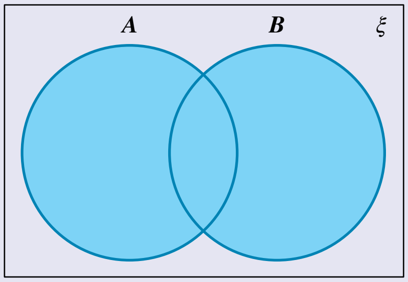
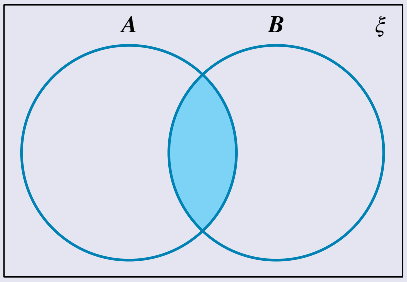
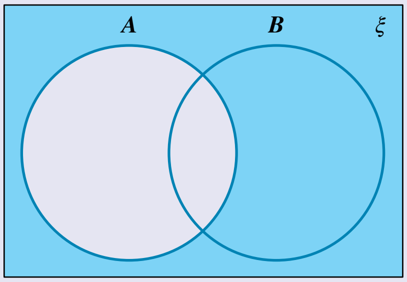
B.


C.


D.


b. If one student is selected at random, calculate the probability that the student plays tennis only, correct to 2 decimal places.

13. Match the following Venn diagrams to the correct set notation for the shaded regions.

Venn diagram	Set notation
a. 	**A.** $A \cap B$
b. 	**B.** A'
c. 	**C.** $A \cap B'$
d. 	**D.** $A \cup B$

14. A survey of Year 8, 9 and 10 students asked them to choose dinner options for their school camp.

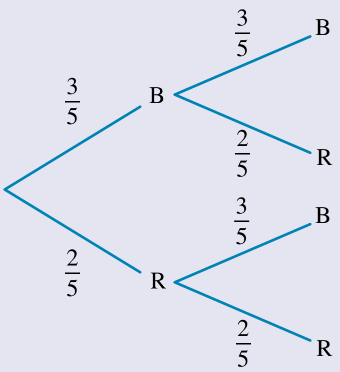
Year level	Lasagna	Stir-fry
8	82	75
9	67	90
10	89	45
Total	240	210

Calculate the following probabilities.

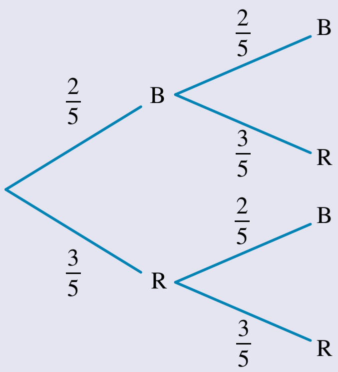
a. The probability that a randomly selected student chose lasagne.
b. The probability that a randomly selected student was in Year 10.
c. The probability that a randomly selected student was in Year 8 and chose stir-fry.
d. The probability that a randomly selected student who chose lasagna was in Year 9.

15. **MC** A bag contains three blue balls and two red balls. A ball is taken at random from the bag and its colour noted. Then a second ball is drawn, without replacing the first one.
Identify the tree diagram that best represents this sample space.

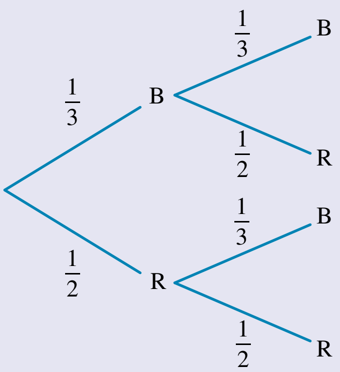
A.



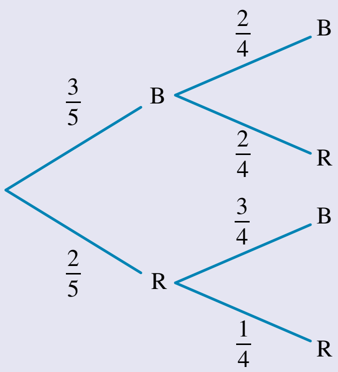
B.



C.



D.



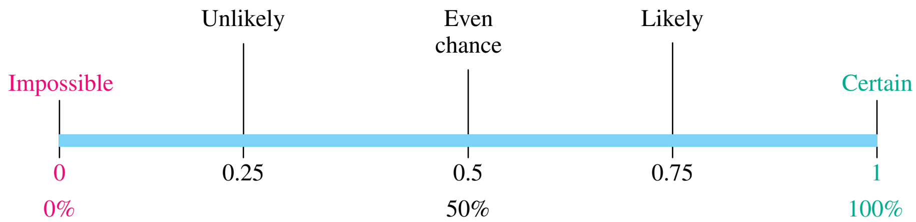
LESSON 11.2 Theoretical probability

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- estimate whether events are impossible, unlikely, likely or certain
- use key probability terminologies such as trials, outcomes, sample space and events
- calculate the theoretical probability of an event.

11.2.1 The language of probability

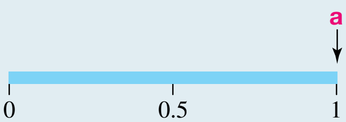
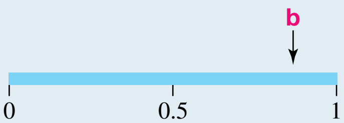
- The **probability** of an event is a measure of the likelihood that the event will take place.
- If an event is certain to occur, then it has a probability of 1.
- If an event is impossible, then it has a probability of 0.
- The probability of any other event taking place is given by a number between 0 and 1.
- An event is likely to occur if it has a probability between 0.5 and 1.
- An event is unlikely to occur if it has a probability between 0 and 0.5.



WORKED EXAMPLE 1 Placing events on a probability scale

On the probability scale given, insert each of the following events at appropriate points.

a. If today is Saturday, then tomorrow is Sunday.
b. You will come to school the next Monday during a school term.
c. It will reach 35 degrees Celsius in Perth in summer.

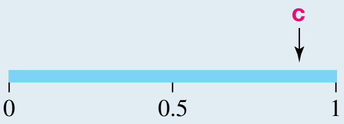
THINK	WRITE/DRAW
a. 1. Carefully read the given statement and label its position on the probability scale.	a. 
2. Write the answer and provide reasoning.	It is certain that if the current day is Saturday, then the next day will be Sunday.
b. 1. Carefully read the given statement and label its position on the probability scale.	b. 

2. Write the answer and provide reasoning.

It is very likely but not certain that I will come to school on a Monday during term. Circumstances such as illness or public holidays may prevent me from coming to school on a specific Monday during a school term.

c. 1. Carefully read the given statement and label its position on the probability scale.

c.



2. Write the answer and provide reasoning.

It is highly likely but not certain that it will reach 35 degrees Celsius in Perth during the summer.

11.2.2 Key terms of probability

- The study of probability uses many special terms.
- **Chance experiment**: A chance experiment is a process that can be repeated many times. For example, rolling a die.
- **Trial**: A trial is one performance of an experiment to get a result. For example, each roll of the die is called a trial.
- **Outcome**: The outcome is the result obtained when the experiment is conducted. For example, when a normal six-sided die is rolled the outcome can be 1, 2, 3, 4, 5 or 6.
- **Sample space**: The set of all possible outcomes is called the sample space and is given by the symbol ξ. For the example of rolling a die, $\xi = \{1, 2, 3, 4, 5, 6\}$.
- **Event**: An event is the favourable outcome of a trial and is often represented by a capital letter. For example, when a die is rolled, A could be the event of getting an even number; $A = \{2, 4, 6\}$.
- **Favourable outcome**: A favourable outcome for an event is any outcome that belongs to the event. For event A above (rolling an even number), the favourable outcomes are 2, 4 and 6.

WORKED EXAMPLE 2 Identifying sample space, events and outcomes

For the chance experiment of rolling a die:

a. list the sample space

b. list the events for:

i. rolling a 4

ii. rolling an even number

iii. rolling at least 5

iv. rolling at most 2

c. list the favourable outcomes for:

i. $\{4, 5, 6\}$

ii. not rolling 5

iii. rolling 3 or 4

iv. rolling 3 and 4.

THINK

a. The outcomes are the numbers 1 to 6.

b. i. This describes only one outcome.
ii. The possible even numbers are 2, 4 and 6.
iii. 'At least 5' means 5 is the smallest.
iv. 'At most 2' means 2 is the largest.

c. i. The outcomes are shown inside the brackets.
ii. 'Not 5' means everything except 5.
iii. The event is $\{3, 4\}$.
iv. There is no number that is both 3 and 4.

WRITE

a. $\xi = \{1, 2, 3, 4, 5, 6\}$

b. i. $\{4\}$
ii. $\{2, 4, 6\}$
iii. $\{5, 6\}$
iv. $\{1, 2\}$

c. i. 4, 5, 6
ii. 1, 2, 3, 4, 6
iii. 3, 4
iv. There are no favourable outcomes.

11.2.3 Theoretical probability

- When a coin is tossed, there are two possible outcomes, Heads or Tails. That is, $\xi = \{\text{H}, \text{T}\}$.
- We assume the coin is unbiased, meaning each outcome is **equally likely** to occur.
- Since each outcome is equally likely, they must have the same probability.
- The total of all probabilities is 1, so the probability of an outcome is given by:
  - $\text{P(outcome)} = \dfrac{1}{\text{number of outcomes}}$
  - $\text{P(Heads)} = \dfrac{1}{2}$ and $\text{P(Tails)} = \dfrac{1}{2}$
- The probability of an event A is found by adding up all the probabilities of the favourable outcomes in event A and dividing by the total number of outcomes.

Theoretical probability

When determining probabilities of equally likely outcomes, use the following:

$$\textbf{P(outcome)} = \frac{\textbf{1}}{\textbf{total number of outcomes}}$$

$$\textbf{P(event A)} = \frac{\textbf{number of favourable outcomes}}{\textbf{total number of outcomes}}$$

WORKED EXAMPLE 3 Calculating theoretical probability

A die is rolled and the number uppermost is noted. Determine the probability of each of the following events.

a. $A = \{1\}$ b. $B = \{\text{odd numbers}\}$ c. $C = \{4 \text{ or } 6\}$

THINK

There are six possible outcomes.

WRITE

a. *A* has one favourable outcome.

a. $P(A) = \frac{1}{6}$

b. *B* has three favourable outcomes: 1, 3 and 5.

b. $P(B) = \frac{3}{6}$

$= \frac{1}{2}$

c. *C* has two favourable outcomes.

c. $P(C) = \frac{2}{6}$

$= \frac{1}{3}$

Exercise 11.2 Theoretical probability

learnon

11.2 Quick quiz on | 11.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 5, 8, 10, 14, 15, 19, 22	2, 4, 6, 9, 11, 16, 20, 23	7, 12, 13, 17, 18, 21, 24, 25

Fluency

1. WE1 On the given probability scale, insert each of the following events at appropriate points.
Indicate the chance of each event using one of the following terms: certain, likely, unlikely, impossible.

a. The school will have a lunch break on Friday.
b. Australia will host two consecutive Olympic Games.

2. On the given probability scale, insert each of the following events at appropriate points.
Indicate the chance of each event using one of the following terms: certain, likely, unlikely, impossible.

a. At least one student in a particular class will obtain an A for Mathematics.
b. Australia will have a swimming team in the Commonwealth Games.

3. On the given probability scale, insert each of the following events at appropriate points.
Indicate the chance of each event using one of the following terms: certain, likely, unlikely, impossible.

a. Mathematics will be taught in secondary schools.
b. In the future most cars will run without LPG or petrol.

4. On the given probability scale, insert each of the following events at appropriate points.
Indicate the chance of each event using one of the following terms: certain, likely, unlikely, impossible.

a. Winter will be cold.
b. Bean seeds, when sown, will germinate.

5. WE2a For each chance experiment below, list the sample space.

a. Rolling a die.
b. Tossing a coin.
c. Testing a light bulb to see whether it is defective or not.
d. Choosing a card from a normal deck and noting its colour.
e. Choosing a card from a normal deck and noting its suit.

6. WE2b A normal six-sided die is rolled. List each of the following events.

a. Rolling a number less than or equal to 3.
b. Rolling an odd number.
c. Rolling an even number or 1.
d. Not rolling a 1 or 2.
e. Rolling at most a 4.
f. Rolling at least a 5.

7. WE2c A normal six-sided die is rolled. List the favourable outcomes for each of the following events.

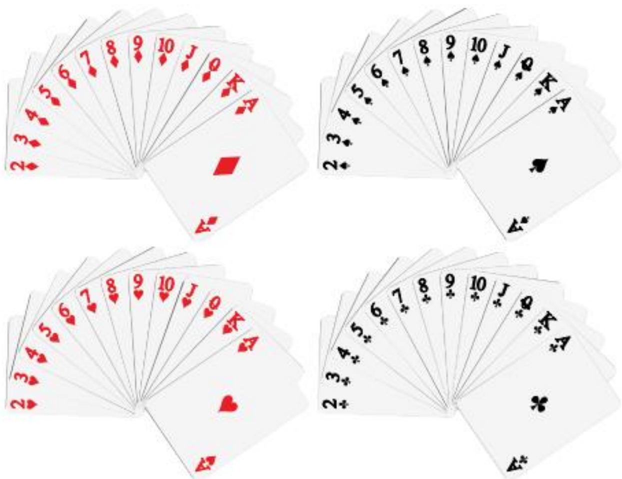
a. $A = \{3, 5\}$
b. $B = \{1, 2\}$
c. C = 'rolling a number greater than 5'
d. D = 'not rolling a 3 or a 4'
e. E = 'rolling an odd number or a 2'
f. F = 'rolling an odd number and a 2'
g. G = 'rolling an odd number and a 3'

8. A card is selected from a normal deck of 52 cards and its suit is noted.

a. List the sample space.
b. List each of the following events.
   i. Drawing a black card.
   ii. Drawing a red card.
   iii. Not drawing a heart.
   iv. Drawing a black or a red card.

9. Determine the number of outcomes there are for:
a. rolling a die
b. tossing a coin
c. drawing a card from a standard deck
d. drawing a card and noting its suit
e. noting the remainder when a number is divided by 5.

10. A card is drawn at random from a standard deck of 52 cards.
Note: 'At random' means that every card has the same chance of being selected.
Calculate the probability of selecting:



a. an ace
b. a king
c. the 2 of spades
d. a diamond.

11. WE3 A card is drawn at random from a deck of 52. Determine the probability of each event below.

a. A = {5 of clubs}
b. B = {black card}
c. C = {5 of clubs or queen of diamonds}
d. D = {hearts}
e. E = {hearts or clubs}

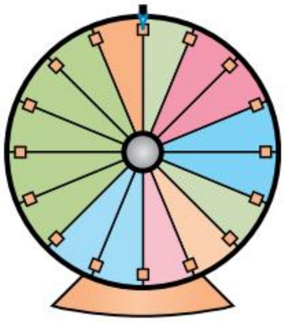
12. A card is drawn at random from a deck of 52. Determine the probability of each event below.

a. $F = \{\text{hearts and } 5\}$
b. $G = \{\text{hearts or } 5\}$
c. $H = \{\text{aces or kings}\}$
d. $I = \{\text{aces and kings}\}$
e. $J = \{\text{not a } 7\}$

13. A letter is chosen at random from the letters in the word PROBABILITY. Determine the probability that the letter is:

a. B
b. not B
c. a vowel
d. not a vowel.

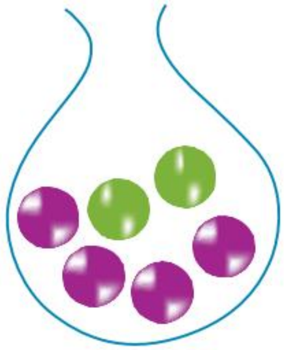
14. The following coloured spinner is spun and the colour is noted. Determine the probability of each of the events given below.



a. $A = \{\text{blue}\}$
b. $B = \{\text{orange}\}$
c. $C = \{\text{orange or pink}\}$
d. $D = \{\text{orange and pink}\}$
e. $E = \{\text{not blue}\}$

Understanding

15. A bag contains four purple balls and two green balls.



a. If a ball is drawn at random, then calculate the probability that it will be:
   i. purple
   ii. green.
b. Design an experiment like the one in part **a** but where the probability of drawing a purple ball is 3 times that of drawing a green ball.

16. Design spinners (see question **14**) using red, white and blue sections so that:

a. each colour has the same probability of being spun
b. red is twice as likely to be spun as either of the other two colours
c. red is twice as likely to be spun as white and three times as likely to be spun as blue.

17. A bag contains red, green and blue marbles. Calculate how many marbles there must be in the bag for the following to be true when a single marble is selected at random from the bag.

a. Each colour is equally likely to be selected and there are at least six red marbles in the bag.
b. Blue is twice as likely to be selected as the other colours and there are at least five green marbles in the bag.
c. There is a $\frac{1}{2}$ chance of selecting red, a $\frac{1}{3}$ chance of selecting green and a $\frac{1}{6}$ chance of selecting blue from the bag when there is between 30 and 40 marbles in the bag.

18. a. A bag contains seven gold and three silver coins. If a coin is drawn at random from the bag, calculate the probability that it will be:
   i. gold
   ii. silver.

b. After a gold coin is taken out of the bag, a second coin is then selected at random. Assuming the first coin was not returned to the bag, calculate the probability that the second coin will be:
   i. gold
   ii. silver.

Reasoning

19. Do you think that the probability of tossing Heads is the same as the probability of tossing Tails if your friend tosses the coin? Suggest some reasons that it might not be.

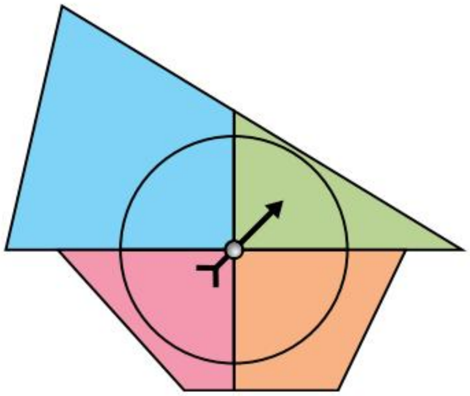
20. If the following four probabilities were given to you, explain which two you would say were not correct.

$$0.725, -0.5, 0.005, 1.05$$

21. A coin is going to be tossed five times in a row. During the first four flips the coin comes up Heads each time. What is the probability that the coin will come up Heads again on the fifth flip? Justify your answer.

Problem solving

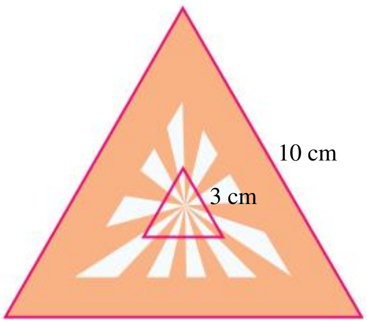
22. Consider the spinner shown. Discuss whether the spinner has an equal chance of falling on each of the colours.



23. A box contains two coins. One is a double-headed coin, and the other is a normal coin with Heads on one side and Tails on the other. You draw one of the coins from a box and look at one of the sides. It is Heads. Determine the probability that the other side also shows Heads.

24. 'Unders and Overs' is a game played with two normal six-sided dice. The two dice are rolled, and the numbers uppermost added to give a total. Players bet on the outcome being 'under 7', 'equal to 7' or 'over 7'.
If you had to choose one of these outcomes, which would you choose? Explain why.

25. Justine and Mary have designed a new darts game for their Year 9 Fete Day. Instead of a circular dart board, their dart board is in the shape of two equilateral triangles. The inner triangle (bullseye) has a side length of 3 cm, while the outer triangle has side length 10 cm.
Given that a player's dart falls in one of the triangles, determine the probability that it lands in the bullseye. Write your answer correct to 2 decimal places.



LESSON 11.3 Experimental probability

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- determine the relative frequency of an outcome
- understand the difference between theoretical probability and experimental probability.

11.3.1 Relative frequency

- A die is rolled 12 times and the outcomes are recorded in the table shown.

Outcome	1	2	3	4	5	6
Frequency	3	1	1	2	2	3

In this chance experiment there were 12 trials.
The table shows that the number 1 was rolled 3 times out of 12.

- So the **relative frequency** of 1 is 3 out of 12, or $\frac{3}{12} = \frac{1}{4}$.

As a decimal, the relative frequency of 1 is equal to 0.25.

Relative frequency

The relative frequency of an outcome is given by:

$$\textbf{Relative frequency} = \frac{\textbf{frequency of an outcome}}{\textbf{total number of trials}}$$

- As the number of trials becomes larger, the relative frequency of each outcome will become very close to the theoretical probability.

WORKED EXAMPLE 4 Calculating relative frequency

For the chance experiment of rolling a die, the following outcomes were noted.

Outcome	1	2	3	4	5	6
Frequency	3	1	4	6	3	3

a. Calculate the number of trials.
b. Identify how many threes were rolled.
c. Calculate the relative frequency for each number written as a decimal.

THINK	WRITE
a. Adding the frequencies will give the number of trials.	a. $1 + 3 + 4 + 6 + 3 + 3 = 20$ trials
b. The frequency of 3 is 4.	b. 4 threes were rolled.

c. Add a relative frequency row to the table and complete it. The relative frequency is calculated by dividing the frequency of the outcome by the total number of trials.

c.

Outcome	1	2	3	4	5	6
Frequency	3	1	4	6	3	3
Relative frequency	$\frac{3}{20} =$ 0.15	$\frac{1}{20} =$ 0.05	$\frac{4}{20} =$ 0.2	$\frac{6}{20} =$ 0.3	$\frac{3}{20} =$ 0.15	$\frac{3}{20} =$ 0.15

11.3.2 Experimental probability

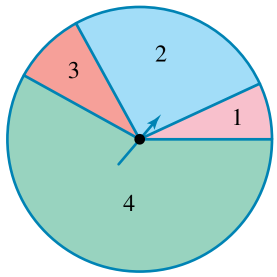
- When it is not possible to calculate the theoretical probability of an outcome, carrying out simulations involving repeated trials can be used to determine the **experimental probability**.
- The relative frequency of an outcome is the experimental probability.

Experimental probability

The experimental probability of an outcome is given by:

$$\textbf{Experimental probability} = \frac{\textbf{frequency of an outcome}}{\textbf{total number of trials}}$$

- The spinner shown is not symmetrical, and the probability of each outcome cannot be determined theoretically.
- The experimental probability of each outcome can be found by using the spinner many times and recording the outcomes. As more trials are conducted, the experimental probability will become more accurate and closer to the true probability of each section.



WORKED EXAMPLE 5 Calculating experimental probabilities

The spinner shown above was spun 100 times and the following results were achieved.

Outcome	1	2	3	4
Frequency	**7**	**26**	**9**	**58**

a. **Calculate the number of trials.**
b. **Calculate the experimental probability of each outcome.**
c. **Calculate the sum of the four probabilities.**

THINK	WRITE
a. Adding the frequencies will determine the number of trials.	a. $7 + 26 + 9 + 58 = 100$ trials

b. The experimental probability equals the relative frequency. This is calculated by dividing the frequency of the outcome by the total number of trials.

b. $P(1) = \frac{7}{100}$
$= 0.07$

$P(2) = \frac{26}{100}$
$= 0.26$

$P(3) = \frac{9}{100}$
$= 0.09$

$P(4) = \frac{58}{100}$
$= 0.58$

c. Add the probabilities (they should equal 1).

c. $0.07 + 0.26 + 0.09 + 0.58 = 1$

11.3.3 Expected number of results

- If we tossed a coin 100 times, we would expect there to be 50 Heads, since $P(\text{Heads}) = \frac{1}{2}$.

Expected number of results

The expected number of favourable outcomes from a series of trials is found from:

$$\textbf{Expected number} = \textbf{probability of outcome} \times \textbf{number of trials}$$

- The probability of an outcome can be the theoretical probability or an experimental probability.

WORKED EXAMPLE 6 Expected number of outcomes

Calculate the expected number of results in the following situations.

a. The number of Tails after flipping a coin 250 times.

b. The number of times a one comes up after rolling a dice 120 times.

c. The number of times a royal card is picked from a deck that is reshuffled with the card replaced 650 times.

THINK

a. The probability of getting a Tail is $P(\text{Tails}) = \frac{1}{2}$.

b. The probability of getting a one is $P(\text{one}) = \frac{1}{6}$.

WRITE

a. $\text{Number of Tails} = P(\text{Tails}) \times 250$
$= \frac{1}{2} \times 250$
$= 125 \text{ times}$

b. $\text{Number of ones} = P(\text{one}) \times 120$
$= \frac{1}{6} \times 120$
$= 20 \text{ times}$

c. A deck of cards has 52 cards and 12 of them are royal cards. This means the probability of getting a royal is $P(\text{royal}) = \frac{12}{52} = \frac{3}{13}$.

c. $\text{Number of royals} = P(\text{royal}) \times 650$

$$= \frac{3}{13} \times 650$$

$$= 150 \text{ times}$$

Digital technology

There are digital tools available to conduct probability simulations, including calculator technology.

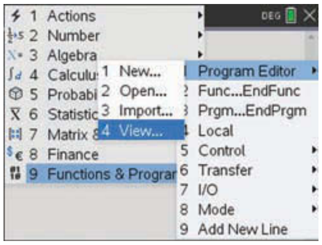
TI | THINK

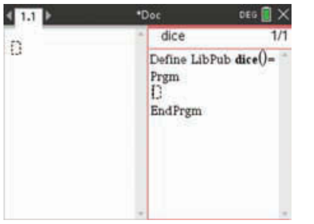
In a new document on a Calculator page, select:

- Menu
- 9 Functions & Programs
- 1 Program Editor
- 1 New

Call the program 'Dice' and select LibPub (Show in Catalog) for Library Access. Press OK.

WRITE/DISPLAY

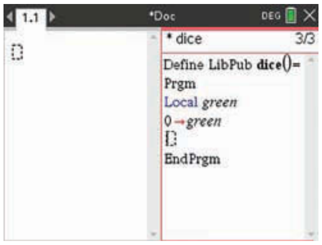


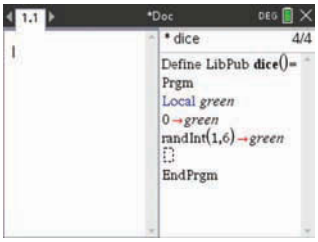


Localize a die, namely the green die. This is to make the variables used only in this program. More dice can be added by naming them with different names.

Initialise them, to make sure no other values are already in a variable. The arrow is found by sto-> (ctl+var).

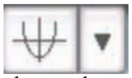
Store a randomly generated integer from 1 to 6 into the die. Use int(6*rand+1) or randint(1,6).





CASIO | THINK

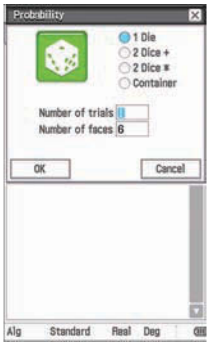
In a Main screen select the drop-down box in the



menu and choose the probability simulator by selecting

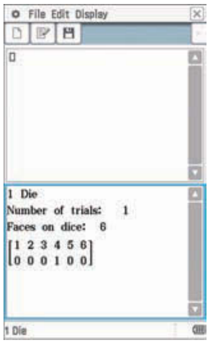
The following menu will appear as shown.

WRITE/DISPLAY



To simulate rolling a six sided die once, select OK. This simulation resulted in a 4.

The number of trials and number of faces can be adjusted to suit and there is also an option to change the experiment to simulate picking items out of a container.



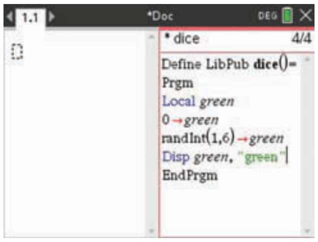
TI | THINK	WRITE/DISPLAY	CASIO | THINK	WRITE/DISPLAY

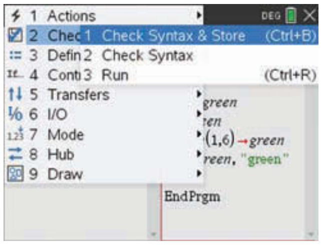
TI | THINK

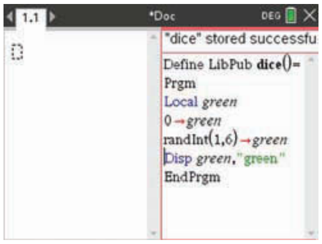
To display results, enter the following:
Disp green, “green”
To check syntax and store, go to Menu and select:

- 2 Check Syntax & Store
- 1 Check Syntax & Store

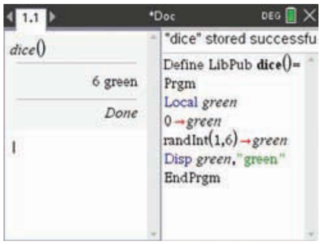
WRITE/DISPLAY

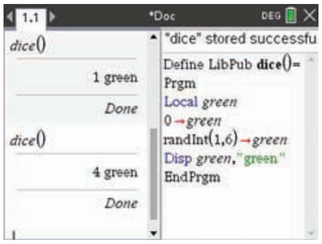






Enter the following on the left side of the document page:
dice ()
Press Enter.
This simulation will generate values by pressing Enter and the program will keep on generating the values.
The number of trials and number of faces can be adjusted to suit.





CASIO | THINK

WRITE/DISPLAY

COLLABORATIVE TASK

Construct an irregular spinner using cardboard and a toothpick. By carrying out a number of trials, estimate the probability of each outcome.

Exercise 11.3 Experimental probability

learnon

11.3 Quick quiz on | 11.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 8, 10, 11, 15, 17, 19, 22	2, 4, 6, 9, 12, 18, 20, 23, 24	5, 7, 13, 14, 16, 21, 25, 26, 27

Fluency

1. WE4 Each of the two tables shown contains the results of a chance experiment (rolling a die). For each table, calculate:
   i. the number of trials held
   ii. the number of fives rolled
   iii. the relative frequency for each outcome, correct to 2 decimal places
   iv. the sum of the relative frequencies.

   a.

Number	1	2	3	4	5	6
Frequency	3	1	5	2	4	1

   b.

Number	1	2	3	4	5	6
Frequency	52	38	45	49	40	46

2. A coin is tossed in two chance experiments. The outcomes are recorded in the tables shown. For each experiment, calculate:
   i. the relative frequency of both outcomes
   ii. the sum of the relative frequencies.

   a.

Outcome	H	T
Frequency	22	28

   b.

Outcome	H	T
Frequency	31	19

3. WE5 An unbalanced die was rolled 200 times and the following outcomes were recorded.

Number	1	2	3	4	5	6
Frequency	18	32	25	29	23	73

   Using these results, calculate:
   a. P(6)
   b. P(odd number)
   c. P(at most 2)
   d. P(not 3).

4. A box of matches claims on its cover to contain 100 matches.
A survey of 200 boxes established the following results.

Number of matches	95	96	97	98	99	100	101	102	103	104
Frequency	1	13	14	17	27	55	30	16	13	14

If you were to purchase a box of these matches, calculate the probability that:

a. the box would contain 100 matches
b. the box would contain at least 100 matches
c. the box would contain more than 100 matches
d. the box would contain no more than 100 matches.

5. A packet of chips is labelled as weighing 170 grams. This is not always the case and there will be some variation in the weight of each packet.
A packet of chips is considered underweight if its weight is below 168 grams. Chips are made in batches of 1000 at a time. A sample of the weights from a particular batch are shown below.

Weight (grams)	166	167	168	169	170	171	172	173	174
Frequency	2	3	13	17	27	18	9	5	1

Calculate the probability that:

a. a packet of chips is its advertised weight
b. a packet of chips is above its advertised weight
c. a packet of chips is underweight.

A batch of chips is rejected if more than 50 packets in a batch are classed as underweight.

d. Explain if the batch that this sample of chips is taken from should be rejected.

Understanding

6. Here is a series of statements based on experimental probability. If a statement is not reasonable, give a reason why.

a. I tossed a coin five times and there were four Heads, so P(Heads) = 0.8.
b. The West Coast Eagles have won 473 matches out of the 891 that they have played, so P(West Coast will win their next game) = 0.53.
c. P(the sun will rise tomorrow) = 1
d. At a factory, a test of 10 000 light globes showed that 7 were faulty. Therefore, P(faulty light globe) = 0.0007.
e. In Perth it rains an average of 109 days each year, so P(it will rain in Perth on the 17th of next month) = 0.30.

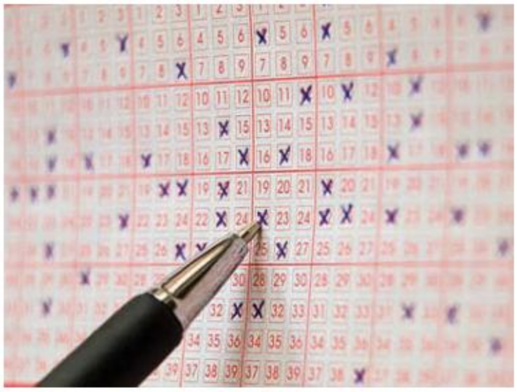
7. At a birthday party, some cans of soft drink were put in a container of ice. There were 16 cans of Coke, 20 cans of Sprite, 13 cans of Fanta, 8 cans of Sunkist and 15 cans of Pepsi.
If a can was picked at random, calculate the probability that it was:

a. a can of Pepsi
b. not a can of Fanta.

8. **WE6** Calculate the expected number of Tails if a fair coin is tossed 400 times.

9. Calculate the expected number of threes if a fair die is rolled 120 times.

10. MC In Tattslotto, six numbers are drawn from the numbers 1, 2, 3, ... 45. The number of different combinations of six numbers is 8 145 060. If you buy one ticket, select from the following the probability that you will win the draw.



A. $\frac{1}{8\,145\,060}$ B. $\frac{1}{45}$

C. $\frac{45}{8\,145\,060}$ D. $\frac{1}{6}$

11. MC A survey of high school students asked 'Should Saturday be a normal school day?' 350 students voted yes, and 450 voted no. From the following, select the probability that a student chosen at random said no.

A. $\frac{7}{16}$ B. $\frac{9}{16}$ C. $\frac{7}{9}$ D. $\frac{9}{14}$

12. In a poll of 200 people, 110 supported party M, 60 supported party N and 30 were undecided. If a person is chosen at random from this group of people, calculate the probability that he or she:

a. supports party M
b. supports party N
c. supports a party
d. is not sure what party to support.

13. A random number is picked from $N = \{1, 2, 3, \ldots 100\}$. Calculate the probability of picking a number that is:

a. a multiple of 3
b. a multiple of 4 or 5
c. a multiple of 5 and 6.

14. The numbers 3, 5 and 6 are combined to form a three-digit number such that no digit may be repeated.

a. i. Determine how many numbers can be formed.
ii. List them.
b. Determine P(the number is odd).
c. Determine P(the number is even).
d. Determine P(the number is a multiple of 5).

15. Gorri was a disc-bowling game played in Central Australia which involved the throwing of a spear at a moving target. If a person had success hitting the target 2 out of every 10 times, calculate the number of times they are expected to hit the target in 25 games.

16. Svetlana, Sarah, Leonie and Trang are volleyball players. The probabilities that they will score a point on serve are 0.6, 0.4, 0.3 and 0.2 respectively. Calculate how many points on serve are expected from each player if they serve 10 times each.

17. MC A survey of the favourite leisure activity of 200 Year 9 students produced the following results.

Activity	Playing sport	Fishing	Watching TV	Video games	Surfing
Number of students	58	26	28	38	50

The probability (given as a percentage) that a student selected at random from this group will have surfing as their favourite leisure activity is:

A. 50% B. 100% C. 25% D. 0%

18. The numbers 1, 2 and 5 are combined to form a three-digit number, allowing for any digit to be repeated up to three times.
   a. Determine how many different numbers can be formed.
   b. List the numbers.
   c. Determine P(the number is even).
   d. Determine P(the number is odd).
   e. Determine P(the number is a multiple of 3).

Reasoning

19. John has a 12-sided die numbered 1 to 12 and Lisa has a 20-sided die numbered 1 to 20. They are playing a game where the first person to get the number 10 wins.
They are rolling their dice individually.
   a. Calculate P(John gets a 10).
   b. Calculate P(Lisa gets a 10).
   c. Explain whether the game is fair.
20. At a supermarket checkout, the scanners have temporarily broken down and the cashiers must enter in the bar codes manually. One particular cashier overcharged 7 of the last 10 customers she served by entering the incorrect bar code.
   a. Based on the cashier's record, determine the probability of making a mistake with the next customer.
   b. Explain if another customer should have any objections with being served by this cashier.
21. If you flip a coin six times, determine how many of the possible outcomes could include a Tail on the second toss.

Problem solving

22. In a jar, there are 600 red balls, 400 green balls, and an unknown number of yellow balls. If the probability of selecting a green ball is $\frac{1}{5}$, determine how many yellow balls are in the jar.
23. In a jar there are an unknown number of balls, N, with 20 of them green. The other colours contained in the jar are red, yellow and blue, with P(red or yellow) $= \frac{1}{2}$, P(red or green) $= \frac{1}{4}$ and P(blue) $= \frac{1}{3}$.
Determine the number of red, yellow and blue balls in the jar.
24. The biological sex of babies in a set of triplets is simulated by flipping three coins. If a coin lands Tails up, the baby is male. If a coin lands Heads up, the baby is female. In the simulation, the trial is repeated 40 times.
The following results show the number of Heads obtained in each trial:
0, 3, 2, 1, 1, 0, 1, 2, 1, 0, 1, 0, 2, 0, 1, 0, 1, 2, 3, 2, 1, 3, 0, 2, 1, 2, 0, 3, 1, 3, 0, 1, 0, 1, 3, 2, 2, 1, 2, 1
   a. Calculate the probability that exactly one of the babies in a set of triplets is female.
   b. Calculate the probability that more than one of the babies in the set of triplets is female.
25. Use your calculator to generate three random sets of numbers between 1 and 6 to simulate rolling a dice. The number of trials in each set will be:
   a. 10
   b. 25
   c. 50

   Calculate the relative frequency of each outcome (1 to 6) and comment on what you notice about experimental and theoretical probability.

26. A survey of the favourite foods of Year 9 students is recorded, with the following results.

Meal	Tally
Hamburger	45
Fish and chips	31
Macaroni and cheese	30
Lamb souvlaki	25
BBQ pork ribs	21
Cornflakes	17
T-bone steak	14
Banana split	12
Corn-on-the-cob	9
Hot dogs	8
Garden salad	8
Veggie burger	7
Smoked salmon	6
Muesli	5
Fruit salad	3

a. Estimate the probability that macaroni and cheese is the favourite food of a randomly selected Year 9 student.
b. Estimate the probability that a vegetarian dish is a randomly selected student's favourite food.
c. Estimate the probability that a beef dish is a randomly selected student's favourite food.

27. A spinner has six sections of different sizes. Steven conducts an experiment and finds the following results:

Section	1	2	3	4	5	6
Frequency	6	24	15	30	48	12

Determine the angle size of each section of the spinner using the results above.

LESSON 11.4 Venn diagrams and two-way tables

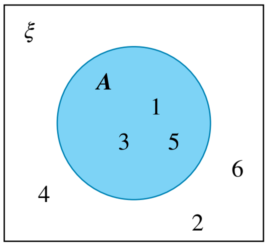
LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify the complement of an event
- create and interpret Venn diagrams and two-way tables
- use a Venn diagram or two-way table to determine $A \cap B$
- use a Venn diagram or two-way table to determine $A \cup B$.

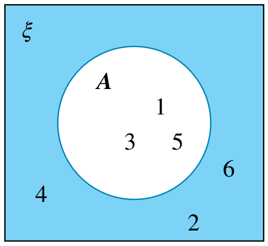
11.4.1 The complement of an event

- Suppose that a die is rolled. The sample space is given by: $\xi = \{1, 2, 3, 4, 5, 6\}$.
- If A is the event 'rolling an odd number', then $A = \{1, 3, 5\}$.

- There is another event called 'the **complement** of A', or not A. This event contains all the outcomes that do not belong to A. It is given by the symbol A'.
- In this case $A' = \{2, 4, 6\}$.
- A and A' can be shown on a **Venn diagram**.



A is shaded.



A' (not A) is shaded.

Complementary events

- **For the event A, the complement is denoted A', and the two are related by the following:**

$$\mathbf{P}(A) + \mathbf{P}(A') = 1$$

WORKED EXAMPLE 7 Determining the complement

For the sample space $\xi = \{1, 2, 3, 4, 5\}$, list the complement of each of the following events.

a. $A = \{\text{multiples of } 3\}$

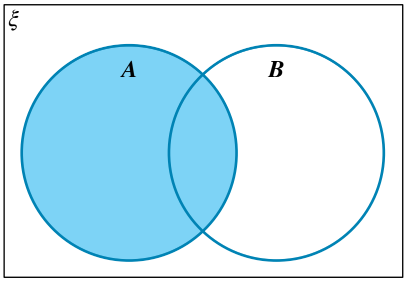
b. $B = \{\text{square numbers}\}$

c. $C = \{1, 2, 3, 5\}$

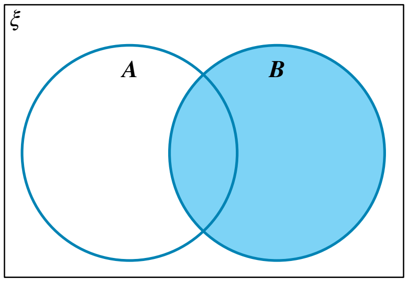
THINK	WRITE
a. The only multiple of 3 in the set is 3. Therefore, $A = \{3\}$. A' is every other element of the set.	**a.** $A' = \{1, 2, 4, 5\}$
b. The only square numbers are 1 and 4. Therefore, $B = \{1, 4\}$. B' is every other element of the set.	**b.** $B' = \{2, 3, 5\}$
c. $C = \{1, 2, 3, 5\}$. C' is every other element of the set.	**c.** $C' = \{4\}$

11.4.2 Venn diagrams: the intersection of events

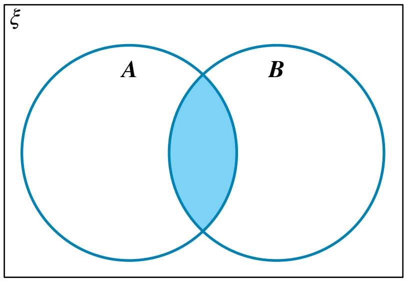
- A Venn diagram consists of a rectangle and one or more circles.
- A Venn diagram contains all possible outcomes in the sample space and will have the ξ symbol in the top left corner.
- A Venn diagram is used to illustrate the relationship between sets of objects or numbers.
- All outcomes for a given event will be contained within a specific circle.
- Outcomes that belong to multiple events will be found in the overlapping region of two or more circles.
- The overlapping region of two circles is called the intersection of the two events and is represented using the $\cap$ symbol.



The circle on the left contains all outcomes in event A.

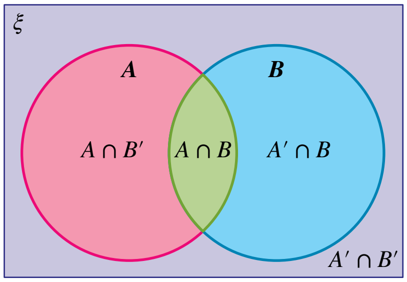


The circle on the right contains all outcomes in event B.



The overlap or intersection of the two circles contains the outcomes that are in event A 'and' in event B. This is denoted by $A \cap B$.

- A Venn diagram for two events A and B has four distinct regions.
- $A \cap B'$ contains the outcomes in event A and not in event B.
- $A \cap B$ contains the outcomes in event A and in event B.
- $A' \cap B$ contains the outcomes not in event A and in event B.
- $A' \cap B'$ contains the outcomes not in event A and not in event B.



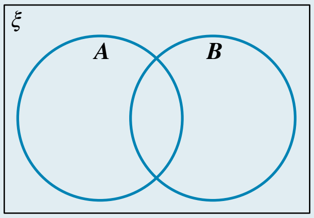
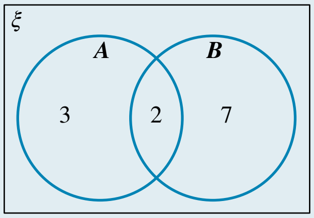
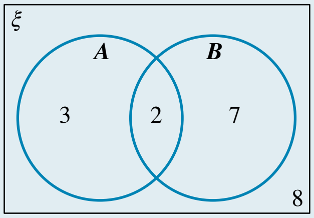
WORKED EXAMPLE 8 Setting up a Venn diagram

In a class of 20 students, 5 study Art, 9 study Biology and 2 students study both.
Let A = {students who study Art} and B = {students who study Biology}.

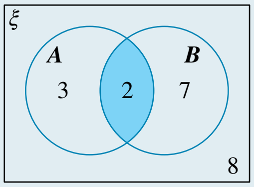
a. **Create a Venn diagram to represent this information.**

b. **Identify the number of students represented by the following and state what these regions represent:**

i. $A \cap B$ **ii.** $A \cap B'$ **iii.** $A' \cap B$ **iv.** $A' \cap B'$

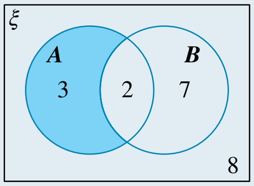
THINK	WRITE/DRAW
a. Draw a sample space with events A and B.	**a.** 
Place a 2 in the intersection of both circles since we know 2 students take both subjects. Since 5 study Art and there are 2 already in the middle, place a 3 in the remaining section of circle A. Since 9 study Biology and there are 2 already in the middle, place a seven in the remaining section of circle B.	
The total number inside the three circles is $3 + 2 + 7 = 12$. This means there must be $20 - 12 = 8$ outside of the two circles. Place an 8 outside the circles, within the rectangle.	

b. **i.** From the Venn diagram, $A \cap B = 2$. These are the students in both A and B.



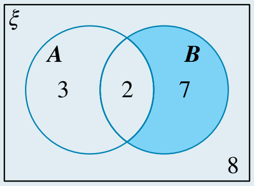
b. **i.** There are 2 students who study Art and Biology.

ii. From the Venn diagram, $A \cap B' = 3$. These are the students in A and not in B.



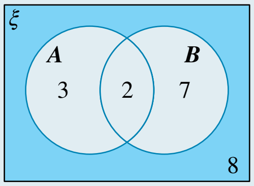
ii. There are 3 students who study Art and not Biology, i.e. 3 students study Art only.

iii. From the Venn diagram, $A' \cap B = 7$. These are the students not in A and in B.



iii. There are 7 students who do not study Art and do study Biology, i.e. 7 students study Biology only.

iv. From the Venn diagram, $A' \cap B' = 8$. These are the students not in A and not in B.

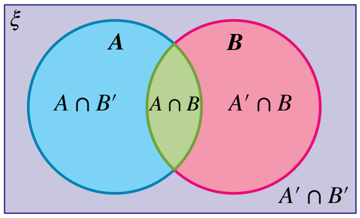


iv. There are 8 students who do not study Art and do not study Biology.

11.4.3 Two-way tables

- The information in a Venn diagram can also be represented using a **two-way table**. The relationship between the two is shown below.

	Event B	**Event B'**	**Total**
Event A	$A \cap B$	$A \cap B'$	A
Event A'	$A' \cap B$	$A' \cap B'$	A'
Total	B	B'	



WORKED EXAMPLE 9 Creating a two-way table

In a class of 20 students, 5 study Art, 9 study Biology and 2 students study both. Create a two-way table to represent this information.

THINK

1. Create an empty two-way table.

WRITE/DRAW

	Biology	Not Biology	Total
Art			
Not Art			
Total			

2. Fill the table in with the information provided in the question.
   - 2 students study both subjects
   - 5 in total take Art
   - 9 in total take Biology
   - 20 students in the class

	Biology	Not Biology	Total
Art	2		5
Not Art			
Total	9		20

3. Use the totals of the rows and columns to fill in the gaps in the table.
   - $9-2=7$ Biology and not Art
   - $5-2=3$ Art and not Biology
   - $20-9=11$ not Biology total
   - $20-5=15$ not Art total

	Biology	Not Biology	Total
Art	2	3	5
Not Art	7	8	15
Total	9	11	20

4. The last value, not Biology and not Art, can be found from either of the following:
   - $11-3=8$
   - $15-7=8$

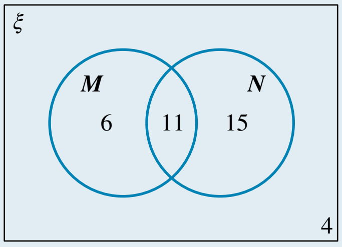
11.4.4 Number of outcomes

- If event A contains seven outcomes or members, this is written as $n(A)=7$.
- So $n(A \cap B')=3$ means that there are three outcomes that are in event A and not in event B.

WORKED EXAMPLE 10 Determining the number of outcomes in an event

For the Venn diagram shown, write down the number of outcomes in each of the following.

a. M **b.** M' **c.** $M \cap N$ **d.** $M \cap N'$ **e.** $M' \cap N'$

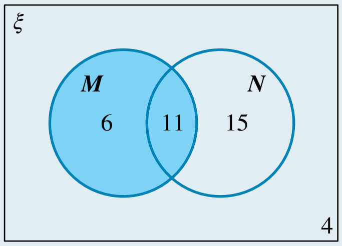


THINK

WRITE/DRAW

a. Identify the regions showing M and add the outcomes.

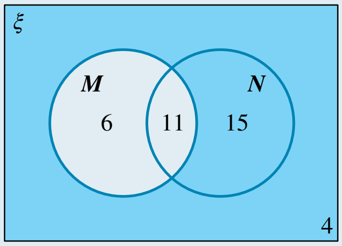
a.



$n(M) = 6 + 11 = 17$

b. Identify the regions showing M' and add the outcomes.

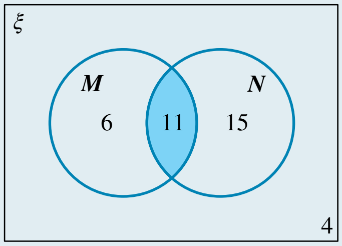
b.



$n(M') = 4 + 15 = 19$

c. $M \cap N$ means 'M and N'. Identify the region.

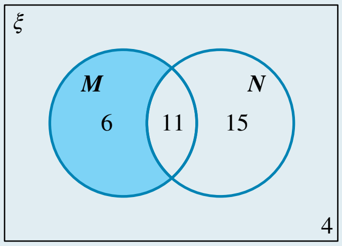
c.



$n(M \cap N) = 11$

d. $M \cap N'$ means 'M and not N'. Identify the region.

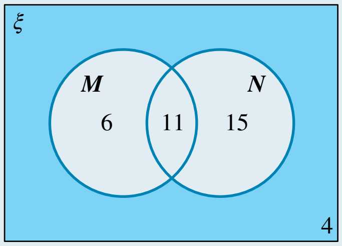
d.



$n(M \cap N') = 6$

e. $M' \cap N'$ means 'not M and not N'. Identify the region.

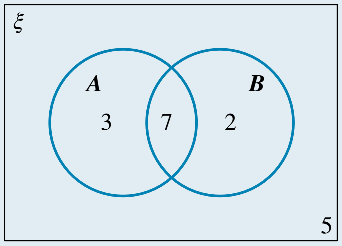
e.



$n(M' \cap N') = 4$

WORKED EXAMPLE 11 Using a Venn diagram to create a two-way table

Show the information from the Venn diagram on a two-way table.



THINK

1. Draw a 2×2 table and add the labels A, A', B and B'.

WRITE/DRAW

	A	A'
B		
B'		

2. There are 7 elements in A and B.
There are 3 elements in A and 'not B'.
There are 2 elements in 'not A' and B.
There are 5 elements in 'not A' and 'not B'.

	A	A'
B	7	2
B'	3	5

3. Add in a column and a row to show the totals.

	A	A'	Total
B	7	2	9
B'	3	5	8
Total	10	7	17

WORKED EXAMPLE 12 Using a two-way table to create a Venn diagram

Show the information from the two-way table on a Venn diagram.

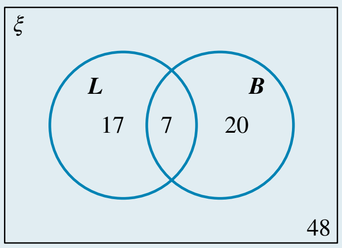
	Left-handed	Right-handed
Blue eyes	**7**	**20**
Not blue eyes	**17**	**48**

THINK

Draw a Venn diagram that includes a sample space and events L for left-handedness and B for blue eyes. (Right-handedness $= L'$.)

$n(L \cap B) = 7$
$n(L \cap B') = 17$
$n(L' \cap B) = 20$
$n(L' \cap B') = 48$

WRITE/DRAW



WORKED EXAMPLE 13 Probability from a two-way table

In a class of 30 students, 15 swim for exercise and 20 run for exercise and 5 participate in neither activity.

a. Create a two-way table to represent this information.

b. Calculate the probability that a randomly chosen student from this class does running and swimming for exercise.

THINK

a. 1. Create an empty two-way table.

WRITE/DRAW

a.

	Swim	Not swim	Total
Run			
Not run			
Total			

2. Fill the table in with the information provided in the question.
- 5 students do neither activity.
- 15 in total swim.
- 20 in total run.
- 30 students are in the class.

	Swim	Not swim	Total
Run			20
Not run		5	
Total	15		30

3. Use the totals of the rows and columns to fill in the gaps in the table.

	Swim	Not swim	Total
Run	10	10	20
Not run	5	5	10
Total	15	15	30

b. The probability a student runs and swims is given by:

$$P(\text{run} \cap \text{swim}) = \frac{\text{number in run} \cap \text{swim}}{\text{total in the class}}$$

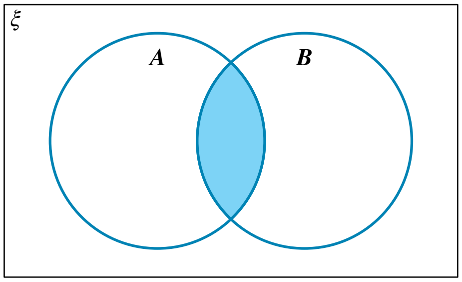
b. $P(\text{run} \cap \text{swim}) = \dfrac{\text{number in run} \cap \text{swim}}{\text{total in the class}}$

$= \dfrac{10}{30}$

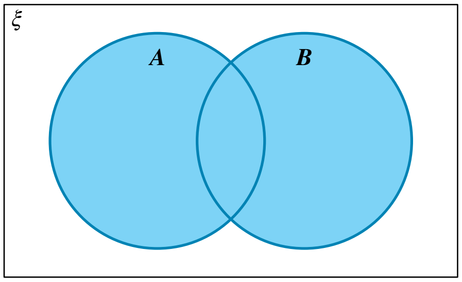
$= \dfrac{1}{3}$

11.4.5 Venn diagrams: the union of events

- The **intersection** of two events ($A \cap B$) is all outcomes in event A 'and' in event B.
- The **union** of two events ($A \cup B$) is all outcomes in events A 'or' in event B.



The intersection of the two circles contains the outcomes that are in event A 'and' in event B. This is denoted by $A \cap B$.

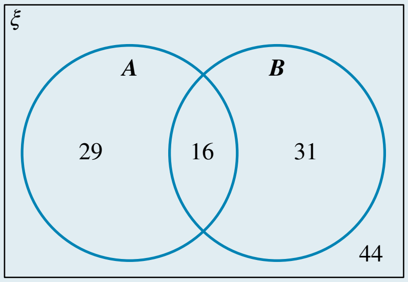


Everything contained within the two circles is an outcome that is in event A 'or' in event B. This is denoted by $A \cup B$.

WORKED EXAMPLE 14 Calculating the union of two events

Use the Venn diagram shown to calculate the value of the following:

a. $n(A)$
b. $\mathrm{P}(B)$
c. $n(A \cup B)$
d. $\mathrm{P}(A \cap B)$



THINK	WRITE
a. Identify the number of outcomes in the A circle.	**a.** $n(A) = 29 + 16 = 45$
b. Identify the number of outcomes in the B circle. $\mathrm{P}(B) = \dfrac{\text{number of favourable outcomes in } B}{\text{total number of outcomes}}$	**b.** $\mathrm{P}(B) = \dfrac{\text{favourable outcomes in } B}{\text{total number of outcomes}} = \dfrac{n(B)}{\text{total number of outcomes}} = \dfrac{16+31}{29+16+31+44} = \dfrac{47}{120}$
c. Identify the number of outcomes in either of the two circles.	**c.** $n(A \cup B) = 29 + 16 + 31 = 76$
d. Identify the number of outcomes in the intersection of the two circles. $\mathrm{P}(A \cap B) = \dfrac{\text{favourable outcomes in } A \cap B}{\text{total number of outcomes}}$	**d.** $\mathrm{P}(A \cap B) = \dfrac{\text{number of favourable outcomes in } A \cap B}{\text{total number of outcomes}} = \dfrac{n(A \cap B)}{\text{total number of outcomes}} = \dfrac{16}{29+16+31+44} = \dfrac{16}{120} = \dfrac{2}{15}$

WORKED EXAMPLE 15 Completing a Venn diagram

a. Place the elements of the following sets of numbers in their correct position in a single Venn diagram.

$$\xi = \{\text{Number 1 to 20 inclusive}\}$$
$$A = \{\text{Multiples of } 3 \text{ from 1 to 20 inclusive}\}$$
$$B = \{\text{Multiples of } 2 \text{ from 1 to 20 inclusive}\}$$

b. Use this Venn diagram to determine the following:

i. $A \cap B$ **ii.** $A \cup B$ **iii.** $A \cap B'$ **iv.** $A' \cup B'$

THINK

a. Write out the numbers in each event A and B:

$A = \{3, 6, 9, 12, 15, 18\}$

$A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$

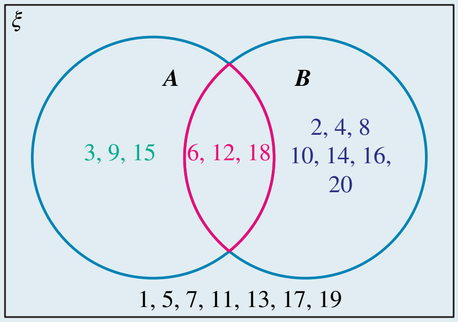
Identify the numbers that appear in both sets. In this case it is 6, 12 and 18. These numbers will be placed in the overlap of the two circles for A and B.

All numbers not in A or B are placed outside the two circles.

After placing the numbers in the Venn diagram, check that all numbers from 1 to 20 are written down.

WRITE/DRAW

a.



b. i. $A \cap B$ are the numbers in A 'and' in B.

ii. $A \cup B$ are the numbers in A 'or' in B.

iii. $A \cap B'$ are the number in A and not in B. Refer to the four sections of the Venn diagram to locate this region.

iv. $A' \cup B'$ is any number that is not in A 'or' not in B. This ends up being any number not in $A \cap B$.

b. i. $A \cap B = \{6, 12, 18\}$

ii. $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$

iii. $A \cap B' = \{3, 9, 15\}$

iv. $A' \cup B' = \left\{\begin{matrix} 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, \\ 14, 15, 16, 17, 19, 20 \end{matrix}\right\}$

WORKED EXAMPLE 16 Calculating the probability of the union of two events

In a class of 24 students, 11 students play basketball, 7 play tennis, and 4 play both sports.

a. Show the information on a Venn diagram.

b. If one student is selected at random, then calculate the probability that:

i. the student plays basketball

ii. the student plays tennis or basketball

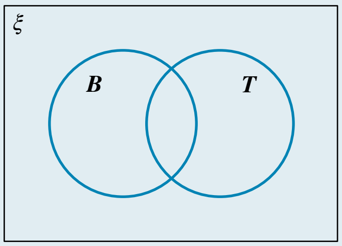
iii. the student plays tennis or basketball but not both.

THINK

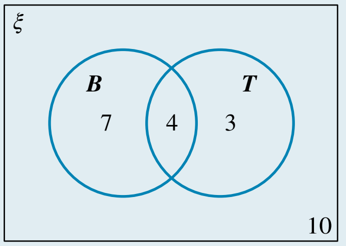
a. 1. Draw a sample space with events B and T.

WRITE/DRAW

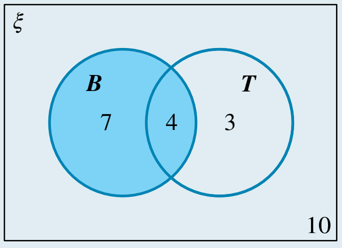
a.



2. $n(B \cap T) = 4$
$n(B \cap T') = 11 - 4 = 7$
$n(T \cap B') = 7 - 4 = 3$
So far, 14 students out of 24 have been placed.
$n(B' \cap T') = 24 - 14 = 10$

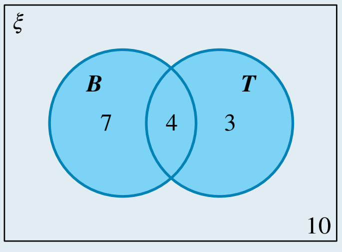


b. **i.** Identify the number of students who play basketball.

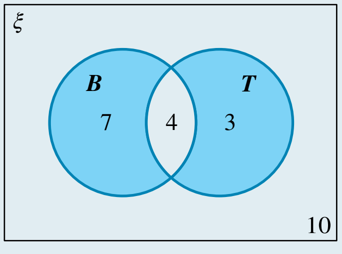


$\text{P}(B) = \dfrac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$

ii. Identify the number of students who play tennis or basketball.



iii. Identify the number of students who play tennis or basketball but not both.



b. **i.** $\text{P}(B) = \dfrac{\text{number of students who play basketball}}{\text{total number of students}}$

$= \dfrac{n(B)}{24}$

$= \dfrac{11}{24}$

ii. $\text{P}(T \cup B) = \dfrac{n(T \cup B)}{24}$

$= \dfrac{14}{24}$

$= \dfrac{7}{12}$

iii. $n(B \cap T') + n(B' \cap T) = 3 + 7$
$= 10$

P(tennis or basketball but not both)

$= \dfrac{10}{24}$

$= \dfrac{5}{12}$

DISCUSSION

How will you remember the difference between when one event **and** another occurs and when one event **or** another occurs?

Exercise 11.4 Venn diagrams and two-way tables

learnon

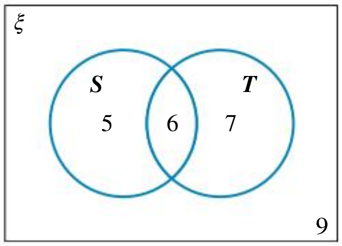
11.4 Quick quiz on | 11.4 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 5, 10, 13, 16, 17, 20, 24	2, 6, 7, 9, 11, 14, 18, 21, 25	3, 8, 12, 15, 19, 22, 23, 26, 27

Fluency

1. **WE7** For the sample space $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, list the complement of each of the following events.
   a. $A = \{\text{evens}\}$
   b. $B = \{\text{multiples of } 5\}$
   c. $C = \{\text{squares}\}$
   d. $D = \{\text{numbers less than } 8\}$
2. If $\xi = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$, list the complement of each of the following events.
   a. $A = \{\text{multiples of } 3\}$
   b. $B = \{\text{numbers less than } 20\}$
   c. $C = \{\text{prime numbers}\}$
   d. $D = \{\text{odd numbers or numbers greater than } 16\}$
3. If $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$, list the complement of each of the following events.
   a. $A = \{\text{multiples of } 4\}$
   b. $B = \{\text{primes}\}$
   c. $C = \{\text{even and less than } 13\}$
   d. $D = \{\text{even or greater than } 13\}$
4. **WE10** For the Venn diagram shown, write down the number of outcomes in:

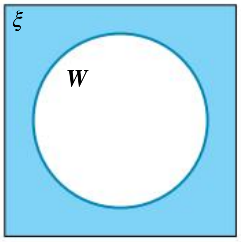


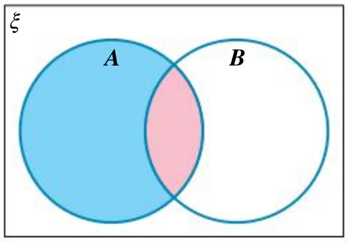
   a. ξ
   b. S
   c. T
   d. $T \cap S$
   e. $T \cap S'$
   f. $S' \cap T'$
5. **WE11** Show the information from question 4 on a two-way table.
6. **WE12** Show the information from this two-way table on a Venn diagram.

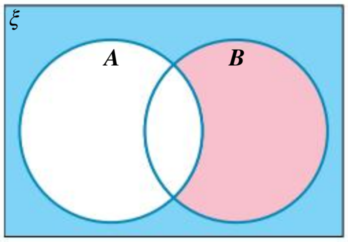
	S	S'
V	21	7
V'	2	10

7. For each of the following Venn diagrams, use set notation to write the name of the region coloured in:

i. blue ii. pink.

a. 

b. 

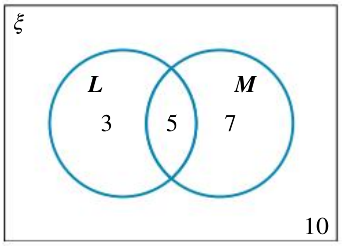
c. 

8. WE9&13 The membership of a tennis club consists of 55 men and 45 women. There are 27 left-handed people, including 15 men.

a. Show the information on a two-way table.
b. Show the information on a Venn diagram.
c. If one member is chosen at random, calculate the probability that the person is:
   i. right-handed
   ii. a right-handed man
   iii. a left-handed woman.

9. WE14 Using the information given in the Venn diagram, if one outcome is chosen at random, determine:

a. $P(L)$ b. $P(L')$ c. $P(L \cap M)$ d. $P(L \cap M')$



10. WE15 Place the elements of the following sets of numbers in their correct position in a single Venn diagram.
ξ = {numbers between 1 to 10 inclusive}
A = {odd numbers from 1 to 10}
B = {squared numbers between 1 to 10 inclusive}

11. Place the elements of the following sets of numbers in their correct position in a single Venn diagram.
ξ = {numbers between 1 to 25 inclusive}
A = {multiples of 3 from 1 to 25}
B = {numbers that are odd or over 17 from 1 to 25 inclusive}

12. Place the elements of the following sets of numbers in their correct position in a single Venn diagram.
$A = \{\text{prime numbers from 1 to 20}\}$
$B = \{\text{even numbers from 1 to 20}\}$
$C = \{\text{multiples of 5 from 1 to 20}\}$
$\xi = \{\text{numbers between 1 and 20 inclusive}\}$

13. Using the information given in the table, if one family is chosen at random, calculate the probability that they own:

Pets owned by families		
	Cat	**No cat**
Dog	4	11
No dog	16	9

a. a cat
b. a cat and a dog
c. a cat or a dog or both
d. a cat or a dog but not both
e. neither a cat nor a dog.

14. Using the information given in the table, if a person is chosen at random, calculate the probability that for exercise, this person:

Type of exercise		
	Cycling	**No cycling**
Running	12	19
No running	13	6

a. cycles
b. cycles and runs
c. cycles or runs
d. cycles or runs but not both.

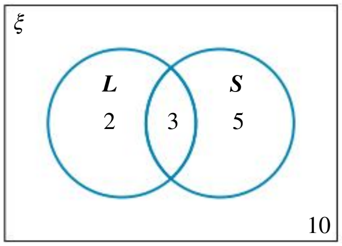
15. A barista decides to record what the first 45 customers order on a particular morning. The information is partially filled out in the table shown.

	Croissant	No croissant	Total
Coffee	27		
No coffee		7	
Total	36		

a. Fill in the missing information in the table.
b. Using the table, calculate the probability that the next customer:
   i. orders a coffee and a croissant
   ii. order a coffee or a croissant
   iii. orders a coffee or a croissant but not both.
c. The barista serves an average of 315 customers a day.
   If the café is open six days a week, calculate how many coffees the barista will make each week.

Understanding

16. A group of athletes was surveyed and the results were shown on a Venn diagram. $S = \{\text{sprinters}\}$ and $L = \{\text{long jumpers}\}$.



a. Write down how many athletes were included in the survey.
b. If one of the athletes is chosen at random, calculate the probability that the athlete competes in:
   i. long jump
   ii. long jump and sprints
   iii. long jump or sprints
   iv. long jump or sprints but not both.

17. WE8a,16 In a class of 40 students, 26 take a train to school, 19 take a bus and 8 take neither of these. Let $T = \{\text{takes the train}\}$ and $B = \{\text{takes the bus}\}$.
a. Show the information on a Venn diagram.
b. If one student is selected at random, calculate the probability that:
   i. the student takes the bus
   ii. the student takes the train or the bus
   iii. the student takes the train or the bus but not both.

18. WE8b If $\xi = \{\text{children}\}$, $S = \{\text{swimmers}\}$ and $R = \{\text{runners}\}$, describe in words each of the following.

a. S' b. $S \cap R$ c. $R' \cap S'$ d. $R \cup S$

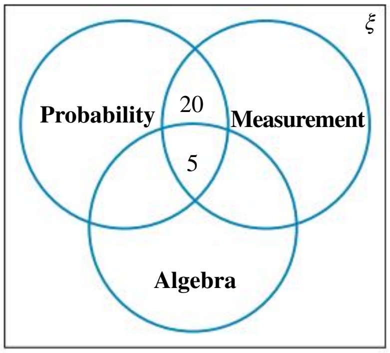
19. A group of 12 students was asked whether they liked hip hop (H) and whether they liked classical music (C). The results are shown in the table.

	C	H
Ali	✓	✓
Anu		
Chris		✓
George		✓
Imogen		✓
Jen	✓	✓
Luke	✓	✓
Pam	✓	
Petra		
Roger	✓	
Seedevi		✓
Tomas		

a. Show the results on:
   i. a Venn diagram
   ii. a two-way table.
b. If one student is selected at random, calculate:
   i. $P(H)$
   ii. $P(H \cup C)$
   iii. $P(H \cap C)$
   iv. P(student likes classical or hip hop but not both)

Reasoning

20. One hundred Year 9 Maths students were asked to indicate their favourite topic in mathematics. Sixty chose Probability, 50 chose Measurement and 43 chose Algebra. Some students chose two topics: 15 chose Probability and Algebra, 18 chose Measurement and Algebra, and 25 chose Probability and Measurement. Five students chose all three topics.



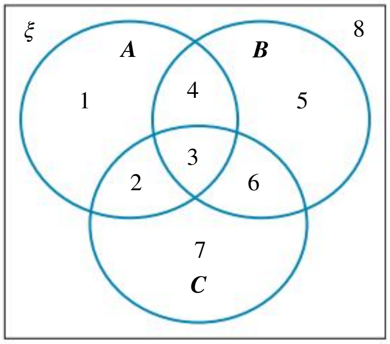
a. Copy and complete the Venn diagram shown.
b. Calculate how many students chose Probability only.
c. Calculate how many students chose Algebra only.
d. Calculate how many students chose Measurement only.
e. Calculate how many students chose any two of the three topics.

A student is selected at random from this group. Calculate the probability that this student has chosen:

f. Probability
g. Algebra
h. Algebra and Measurement
i. Algebra and Measurement but not Probability
j. all of the topics.

21. Create a Venn diagram using two circles to accurately describe the relationships between the following quadrilaterals: rectangle, square and rhombus.

22. Use the Venn diagram shown to write the numbers of the correct regions for each of the following problems.



a. $A' \cup (B' \cap C)$
b. $A \cap (B \cap C')$
c. $A' \cap (B' \cap C')$
d. $A \cup (B \cap C)'$

23. A recent survey taken at a cinema asked 90 teenagers what they thought about three different movies. In total, 47 liked 'Hairy Potter', 25 liked 'Stuporman' and 52 liked 'There's Something About Fred'.
16 liked 'Hairy Potter' only.
4 liked 'Stuporman' only.
27 liked 'There's Something About Fred' only.
There were 11 who liked all three films and 10 who liked none of them.

a. Construct a Venn diagram showing the results of the survey.
b. Calculate the probability that a teenager chosen at random liked 'Hairy Potter' and 'Stuporman' but not 'There's Something About Fred'.

Problem solving

24. 120 children attended a school holiday program during September. They were asked to select their favourite board game from Cluedo, Monopoly and Scrabble. They all selected at least one game, and four children chose all three games.
In total, 70 chose Monopoly and 55 chose Scrabble.
Some children selected exactly two games — 12 chose Cluedo and Scrabble, 15 chose Monopoly and Scrabble, and 20 chose Cluedo and Monopoly.

a. Draw a Venn diagram to represent the children's selections.
b. Calculate the probability that a child selected at random did not choose Cluedo as a favourite game.

25. Valleyview High School offers three sports at Year 9: baseball, volleyball and soccer. There are 65 students in Year 9.
2 have been given permission not to play sport due to injuries and medical conditions.
30 students play soccer.
9 students play both soccer and volleyball but not baseball.
9 students play both baseball and soccer (including those who do and don't play volleyball).
4 students play all three sports.
12 students play both baseball and volleyball (including those who do and don't play soccer).
The total number of players who play baseball is one more than the total of students who play volleyball.

a. Determine the number of students who play volleyball.
b. If a student was selected at random, calculate the probability that this student plays soccer and baseball only.

26. A Venn diagram consists of overlapping ovals which are used to show the relationships between sets. Consider the numbers 156 and 520.
Show how a Venn diagram could be used to determine their:

a. HCF
b. LCM.

27. A group of 200 shoppers was asked which type of fruit they had bought in the last week. The results are shown in the table.

Fruit	Number of shoppers
Apples (A) only	45
Bananas (B) only	34
Cherries (C) only	12
A and B	32
A and C	15
B and C	26
A and B and C	11

a. Display this information in a Venn diagram.
b. Calculate $n(A \cap B' \cap C)$.
c. Determine how many shoppers purchased apples and bananas but not cherries.
d. Calculate the relative frequency of shoppers who purchased:
  i. apples
  ii. bananas or cherries.
e. Estimate the probability that a shopper purchased cherries only.

LESSON 11.5 Compound events

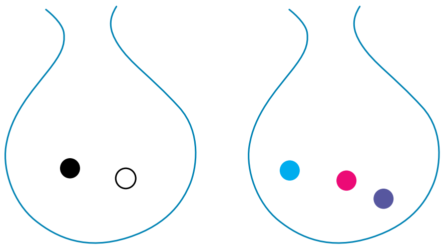
LEARNING INTENTIONS

At the end of this lesson you should be able to:

- create the sample space for compound events using arrays and tree diagrams
- solve problems using tree diagrams for experiments with and without replacement.

11.5.1 The sample space of compound events

- A two-step experiment involves two actions. It may be the same action repeated (tossing a coin twice) or two separate actions (tossing a coin and rolling a dice).
- Imagine two bags that contain coloured counters. The first bag has a mixture of black and white counters, and the second bag holds blue, magenta (pink) and purple counters. In a probability experiment, one counter is to be selected at random from each bag and its colour noted.



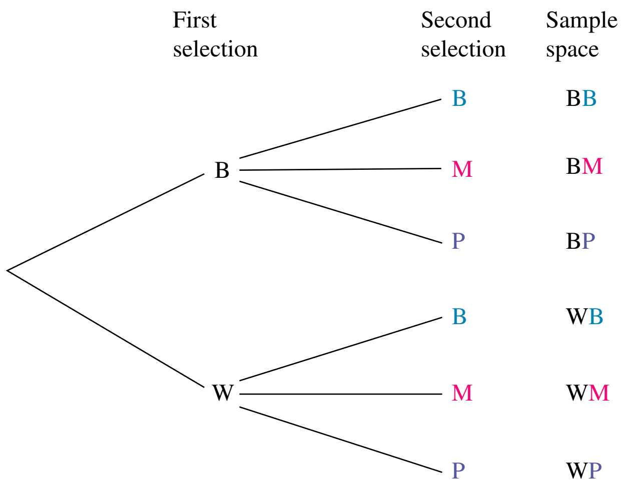
Bag 1 **Bag 2**

- The sample space for this experiment can be found using a table called an **array** that systematically displays all the outcomes.

			Second action		
			Bag 2		
			B	**M**	**P**
First action	**Bag 1**	**B**	B B	B M	B P
		W	W B	W M	W P

The sample space, $\xi = \{BB, BM, BP, WB, WM, WP\}$.

- The sample space can also be found using a **tree diagram**.



WORKED EXAMPLE 17 Using an array to create a sample space

Two dice are rolled and the numbers uppermost are noted.
List the sample space in an array.

a. Determine how many outcomes there are.
b. Calculate how many outcomes contain at least one 5.
c. Calculate P(at least one 5).

THINK

Draw an array (table) showing all the possible outcomes.

WRITE/DRAW

		Second die					
		1	2	3	4	5	6
First die	1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

a. The table shows 36 outcomes.

a. There are 36 outcomes.

b. Count the outcomes that contain 5. The cells are shaded in the table.

b. Eleven outcomes include 5.

c. There are 11 favourable outcomes and 36 in total.

c. $\text{P(at least one 5)} = \frac{11}{36}$

WORKED EXAMPLE 18 Using a tree diagram for two step experiments

Two coins are tossed and the outcomes are noted.
Show the sample space on a tree diagram.

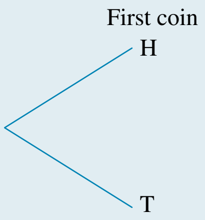
a. Determine how many outcomes there are.
b. Calculate the probability of tossing at least one Head.

THINK

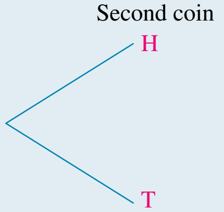
a. 1. Draw a tree representing the outcomes for the toss of the first coin.

WRITE/DRAW

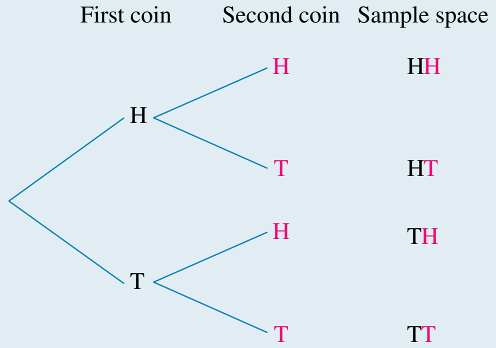
a.



2. For the second coin the tree looks like this:



Add this tree to both ends of the first tree.



3. List the outcomes in the sample space column.
4. Count the outcomes in the sample space. — There are four outcomes (HH, HT, TH, TT).

b. Three outcomes have at least one Head. — **b.** P(at least one Head) $= \frac{3}{4}$

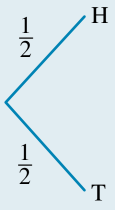
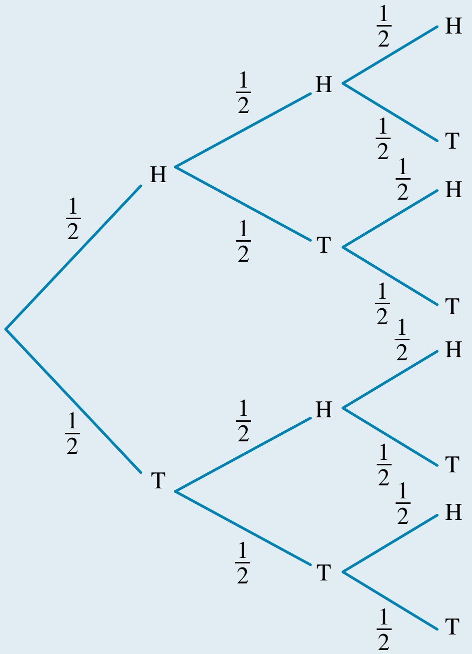
WORKED EXAMPLE 19 Using a tree diagram for three step experiments (Extending)

Three coins are tossed and the outcomes are noted.

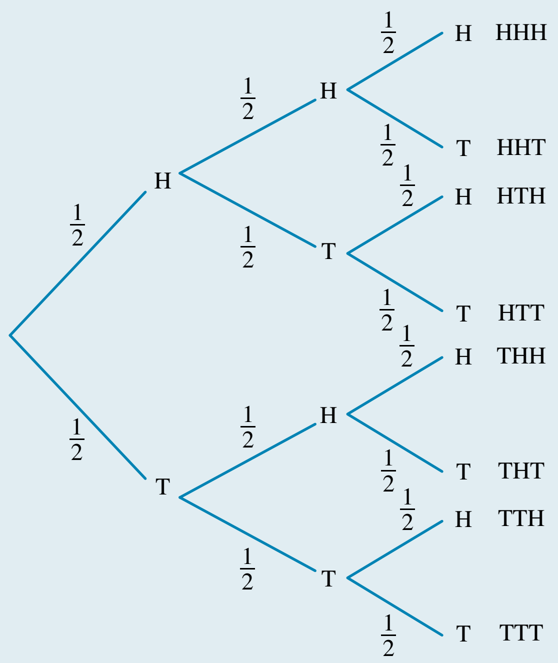
a. Show the outcomes on a tree diagram.

b. Calculate the probability of getting three tails.

c. Calculate the probability of getting at least one head.

THINK	WRITE/DRAW
Draw a tree representing the outcomes for the toss of the first coin. Write the probability on the branch.	
a. 1. For the second trial and third trials the tree is the same. Add this tree to both ends of the first tree, and all ends of the second tree.	**a.** 

2. List all the outcomes.



b. Count the outcomes. There is one outcome that has three tails out of the total 8 outcomes.

b. $P(TTT) = \frac{1}{8}$

c. Count the outcomes. There are seven outcomes that have at least one head.

c. $P(\text{at least one Head}) = \frac{7}{8}$

11.5.2 Two step probabilities

- When a coin is tossed, $P(H) = \frac{1}{2}$, and when a die is rolled, $P(3) = \frac{1}{6}$.

- If a coin is tossed and a die is rolled, what is the probability of getting a Head *and* a 3?
- Consider the sample space.

	1	**2**	**3**	**4**	**5**	**6**
H	H, 1	H, 2	**H, 3**	H, 4	H, 5	H, 6
T	T, 1	T, 2	T, 3	T, 4	T, 5	T, 6

There are 12 outcomes, and $P(\text{Head and } 3) = \frac{1}{12}$.

- In this case, $P(\text{Head and } 3)$ can be calculated as $P(H) \times P(3)$; that is, $\frac{1}{12} = \frac{1}{2} \times \frac{1}{6}$.

Two-step probabilities

If A is the outcome of the first action and B is the outcome of the second action in a two-step experiment, then:

- **$A \cap B$ is the outcome of A followed by B**
- **$P(A \cap B) = P(A) \times P(B)$.**

Note: **This formula is only applied when the two events are independent (if the occurrence of one event does not affect the chances of the occurrence of the other event).**

WORKED EXAMPLE 20 Calculating two-step probabilities

In one cupboard Joe has two black T-shirts and one yellow one. In his drawer there are three pairs of white socks and one black pair. If he selects his clothes at random, calculate the probability that his socks and T-shirt will be the same colour.

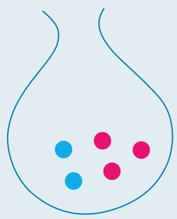
THINK	WRITE
If they are the same colour then they must be black.	$P(B_t \cap B_s) = P(B_t) \times P(B_s)$
$P(\text{black T-shirt}) = P(B_t) = \frac{2}{3}$	$= \frac{2}{3} \times \frac{1}{4}$
$P(\text{black socks}) = P(B_s) = \frac{1}{4}$	$= \frac{1}{6}$

11.5.3 Experiments with replacement

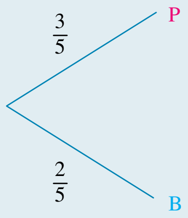
- If an experiment requires an object to be selected, say from a bag, the person doing the selecting has two options after the first selection.
  - Place the object bag back in the bag, meaning the experiment is being carried out **with replacement**. In this case, the number of objects in the bag remains constant.
  - Permanently remove the object from the bag, meaning the experiment is being carried out **without replacement**. In this situation, the number of the objects in the bag is reduced by 1 after every selection.

WORKED EXAMPLE 21 Calculating two-step experiments with replacement

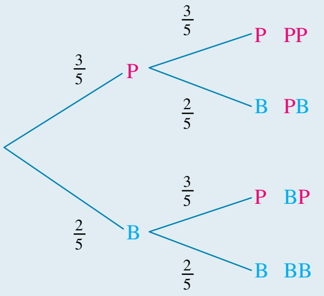
A bag contains three pink and two blue counters. A counter is taken at random from the bag, its colour is noted, then it is returned to the bag and a second counter is chosen.



a. **Show the outcomes on a tree diagram.**
b. **Calculate the probability of each outcome.**
c. **Calculate the sum of the probabilities.**

THINK	WRITE/DRAW
a. 1. Draw a tree for the first trial. Write the probability on the branch. *Note:* The probabilities should sum to 1.	**a.** 

2. For the second trial the tree is the same. Add this tree to both ends of the first tree, then list the outcomes.



b. For both draws $P(P) = \frac{3}{5}$ and $P(B) = \frac{2}{5}$.

Use the rule $P(A \cap B) = P(A) \times P(B)$ to determine the probabilities.

b. $P(P \cap P) = P(P) \times P(P)$

$$= \frac{3}{5} \times \frac{3}{5}$$

$$= \frac{9}{25}$$

$P(P \cap B) = P(P) \times P(B)$

$$= \frac{3}{5} \times \frac{2}{5}$$

$$= \frac{6}{25}$$

$P(B \cap P) = P(B) \times P(P)$

$$= \frac{2}{5} \times \frac{3}{5}$$

$$= \frac{6}{25}$$

$P(B \cap B) = P(B) \times P(B)$

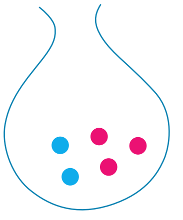
$$= \frac{2}{5} \times \frac{2}{5}$$

$$= \frac{4}{25}$$

c. Add the probabilities.

c. $\frac{9}{25} + \frac{6}{25} + \frac{6}{25} + \frac{4}{25} = 1$

11.5.4 Experiments without replacement

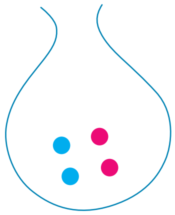


- Let us consider again the situation described in Worked example 21, and consider what happens if the first counter is not replaced.
- Initially the bag contains three pink and two blue counters, and either a pink counter or a blue counter will be chosen.
- In Worked example 21, $P(P) = \frac{3}{5}$ and $P(B) = \frac{2}{5}$ for both trials.

  This would not be true if a counter is selected *but not replaced.*

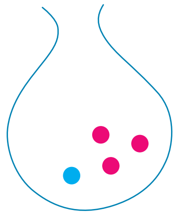
- If the counter is not replaced, then the sample space is affected as follows:

If the first counter randomly selected is pink, then the sample space for the second draw looks like this:



So $\text{P}(P) = \frac{2}{4}$ and $\text{P}(B) = \frac{2}{4}$.

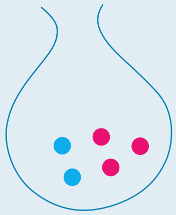
If the first counter randomly selected is blue, then the sample space for the second draw looks like this:



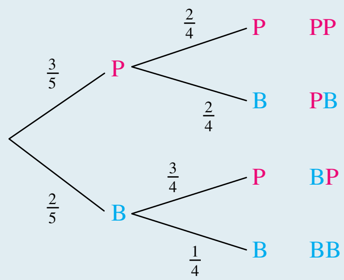
So $\text{P}(P) = \frac{3}{4}$ and $\text{P}(B) = \frac{1}{4}$.

WORKED EXAMPLE 22 Calculating two-step experiments without replacement

A bag contains three pink and two blue counters. A counter is taken at random from the bag and its colour is noted, then a second counter is drawn, without replacing the first one.



a. Show the outcomes on a tree diagram.
b. Calculate the probability of each outcome.
c. Calculate the sum of the probabilities.

THINK	WRITE/DRAW
a. Draw a tree diagram, listing the probabilities.	**a.** 
b. Use the rule $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$ to determine the probabilities.	**b.** $\text{P}(P \cap P) = \text{P}(P) \times \text{P}(P)$ $= \frac{3}{5} \times \frac{2}{4}$ $= \frac{6}{20}$ $= \frac{3}{10}$ $\text{P}(P \cap B) = \text{P}(P) \times \text{P}(B)$ $= \frac{3}{5} \times \frac{2}{4}$ $= \frac{6}{20}$ $= \frac{3}{10}$

$$P(B \cap P) = P(B) \times P(P)$$
$$= \frac{2}{5} \times \frac{3}{4}$$
$$= \frac{6}{20}$$
$$= \frac{3}{10}$$
$$P(B \cap B) = P(B) \times P(B)$$
$$= \frac{2}{5} \times \frac{1}{4}$$
$$= \frac{2}{20}$$
$$= \frac{1}{10}$$

c. Add the probabilities.

c. $\frac{3}{10} + \frac{3}{10} + \frac{3}{10} + \frac{1}{10} = 1$

WORKED EXAMPLE 23 Language of two-step experiments

Consider the situation presented in Worked example 22. Use the tree diagram to calculate the following probabilities:

a. **P(a pink counter and a blue counter)**
b. **P(a pink counter then a blue counter)**
c. **P(a matching pair)**
d. **P(different colours)**

THINK	WRITE
a. Think of the outcomes that have a pink and a blue counter. There are two: PB and BP. Add their probabilities together to find the answer.	a. P(a pink counter and a blue counter) $= P(PB) + P(BP)$ $= \frac{3}{10} + \frac{3}{10}$ $= \frac{3}{5}$
b. In this case, we have to take the order the counters are selected into account. The only outcome that has a pink then a blue is PB.	b. P(a pink counter then a blue counter) $= P(PB)$ $= \frac{3}{10}$
c. A matching pair implies both counters are the same colour. These outcomes are BB and PP.	c. P(a matching pair) $= P(BB) + P(PP)$ $= \frac{1}{10} + \frac{3}{10}$ $= \frac{2}{5}$

d. One way to approach this is to think that if the colours are different, this is all outcomes not counted in part **c**. Thus, we can find the answer by subtracting the answer for part **c** from 1.

d. P(different colours)

$$= 1 - \text{P(a matching pair)}$$
$$= 1 - \frac{2}{5}$$
$$= \frac{3}{5}$$

COLLABORATIVE TASK: Simulating compound events

In this task, you'll work in pairs to explore the differences between multi-step chance experiments with and without replacement.

Chance experiment:

You and a friend are drawing different coloured marbles from a bag at random. There are 30 marbles in the bag, 15 are green, 8 are red, 5 are yellow and 2 are pink.

Use a spreadsheet software (such as Excel or Google sheets) to simulate drawing 20 marbles out of the bag. Repeat this simulation 10 times and determine the average number of marbles of each colour that are obtained.

One person from the pair should simulate the chance experiment with replacement of the marble after each trial and their partner should simulate the chance experiment without replacement of marbles between trials.

Once you have both simulated the experiment 10 times and calculated the average number of marbles of each colour that are drawn work together to think about and answer the following questions.

1. Is there a difference between this experiment with and without replacement? If so, describe the differences.
2. Given your results, can you calculate the probability of drawing a particular colour out of the bag? Are these probabilities different when comparing with and without replacement?
3. Which colours were most/least impacted by the replacement or lack of replacement?
4. Are your results reliable? Would you get similar results if you were to do the experiment again?
5. How could you make your results more reliable?

Hints for setting up the simulation:

- Use integers from 1 to 30 to label the marbles, grouping by colour (for example, the red marbles could be labelled 1 to 15).
- Generate random integers between 1 and 30 to simulate each individual trial using the RANDBETWEEN() function.

11.5 Quick quiz on | 11.5 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 4, 5, 9, 10, 13, 16, 17, 20, 23, 24	3, 7, 8, 11, 14, 18, 21, 25, 26	6, 12, 15, 19, 22, 27, 28, 29

Fluency

1. WE17 If two dice are rolled and their *sum* is noted, complete the array below to show the sample space.

		Die 1					
		1	2	3	4	5	6
Die 2	1	2					
	2					7	
	3						
	4						
	5						
	6			9			

   a. Calculate P(rolling a total of 5).
   b. Calculate P(rolling a total of 1).
   c. Determine the most probable outcome.

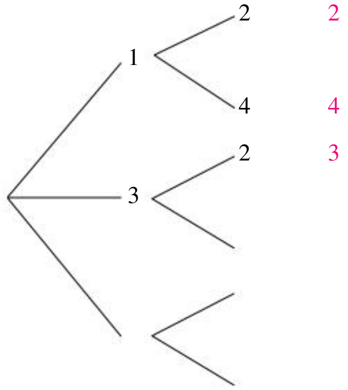
2. In her cupboard Rosa has three scarves (red, blue and pink) and two beanies (brown and purple). If she randomly chooses one scarf and one beanie, show the sample space in an array.

3. A ten-sided die is rolled and then a coin is flipped.
   a. Use an array to determine all the outcomes.
   b. Calculate:
      i. P(even and a Head)
      ii. P(even or a Head)
      iii. P(a Tail and a number greater than 7)
      iv. P(a Tail or a number greater than 7)

4. One box contains red and blue pencils, and a second box contains red, blue and green pencils. If one pencil is chosen at random from each box and the colours are noted, draw a tree diagram to show the sample space.

5. WE18 A bag contains three discs labelled 1, 3 and 5, and another bag contains two discs, labelled 2 and 4, as shown below. A disc is taken from each bag and the larger number is recorded.

a. Complete the tree diagram below to list the sample space.



b. Calculate:
   i. P(2)
   ii. P(1)
   iii. P(odd number)

6. Two dice are rolled and the difference between the two numbers is found.
   a. Use an array to find all the outcomes.
   b. Calculate:
      i. P(odd number)
      ii. P(0)
      iii. P(a number more than 2)
      iv. P(a number no more than 2)

7. WE19 Two coins and one die are tossed.
   a. List all possible outcomes.
   b. Calculate the probability of getting a head, then a tail, then rolling a 2.
   c. Calculate the probability of getting no heads.
   d. Calculate the probability of getting no heads and an even number.

8. WE20 A die is rolled twice. Calculate the probability of rolling:
   a. a 6 on the first roll
   b. a double 6
   c. an even number on both dice
   d. a total of 12.

9. A coin is tossed twice.
   a. Show the outcomes on a tree diagram.
   b. Calculate:
      i. P(2 Tails)
      ii. P(at least 1 Tail)

10. WE21 A bag contains three red counters and one blue counter. A counter is chosen at random. A second counter is drawn with replacement.
   a. Show the outcomes and probabilities on a tree diagram.
   b. Calculate the probability of choosing:
      i. a red counter then a blue counter
      ii. two blue counters.

11. A bag contains five red, six green and four blue counters. A counter is chosen at random. A second counter is drawn with replacement. Use a tree diagram to calculate the probability of choosing:

a. a red and a green counter
b. a red counter then a green counter
c. a red or green counter
d. a matching pair of the same colour.

12. A bag contains eight black, nine white and three red counters. A counter is chosen at random. A second counter is drawn with replacement. Use a tree diagram to calculate the probability of choosing:

a. at least one black counter
b. a black counter and a white counter
c. a black counter or a white counter
d. a matching pair of the same colour
e. different colours.

13. WE22 A bag contains three black balls and two red balls. If two balls are selected, randomly, without replacement:

a. show the outcomes and their probabilities on a tree diagram
b. calculate P(2 red balls).

14. WE23 A bag contains five red, three blue and two green balls. If two balls are randomly selected without replacement, calculate:

a. P(two red balls)
b. P(a red ball then a green ball)
c. P(a red ball and a green ball)
d. P(the same colour selected twice)

15. A bag contains five black, one white and six red balls. If two balls are randomly selected without replacement, calculate:

a. P(two red balls)
b. P(a red ball then a white ball)
c. P(a red ball and a black ball)
d. P(the same colour selected twice)

Understanding

16. The kings and queens from a deck of cards are shuffled, then two cards are chosen. Calculate the probability that two kings are chosen:

a. if the first card is replaced
b. if the first card is not replaced.

17. A coin is flipped and two dice are tossed and their values summed. Calculate the probability that:

a. there is a Head and two 1s
b. the dice sum to 12
c. there is a Tail.

18. The 12 royal cards from a deck of cards are shuffled together, then two cards are chosen. Determine the probability that:

a. two queens are chosen if the first card is replaced
b. a matching pair (same value, not suit) is chosen if the first card is not replaced
c. two cards from different suits are chosen if the first card is not replaced.

19. Each week John and Paul play two sets of tennis against each other. They each have an equal chance of winning the first set.
When John wins the first set, his probability of winning the second set rises to 0.6, but if he loses the first set, he has only a 0.3 chance of winning the second set.

a. Show the possible outcomes on a tree diagram.
b. Calculate:
   i. P(John wins both sets)
   ii. P(Paul wins both sets)
   iii. P(they win one set each)

Reasoning

20. A bag contains four red and six yellow balls. If the first ball drawn is yellow, explain the difference in the probability of drawing the second ball if the first ball was replaced compared to not being replaced.

21. James has six different ties, five different shirts and three different suits that he can choose from when getting ready for work.
    a. Determine how many days he can go without repeating an item of clothing.
    b. Determine how many possible combinations of clothing there are.
    c. James receives a new shirt and tie for his birthday. Determine how many more combinations of clothing are now possible.

22. Three dice are tossed and the total is recorded.
    a. State the smallest and largest possible totals.
    b. Calculate the probabilities for all possible totals.

Problem solving

23. You draw two cards, one after the other without replacement, from a deck of 52 cards. Calculate the probability of:
    a. drawing two aces
    b. drawing two face cards (J, Q, K)
    c. getting a 'pair' (22, 33, 44 ... QQ, KK, AA).

24. A chance experiment involves flipping a coin and rolling two dice. Determine the probability of obtaining Tails and two numbers whose sum is greater than 4.

25. In a jar there are 10 red balls and 6 green balls. Jacob takes out two balls, one at a time, without replacing them. Calculate the probability that both balls are the same colour.

26. A coin is being tossed repeatedly. Determine how many possible outcomes there will be if it is tossed:
    a. 3 times b. 5 times c. 4 times d. x times.

27. A box of chocolates contains milk chocolates and dark chocolates. The probability of selecting a dark chocolate from a full box is $\frac{1}{6}$. Once a dark chocolate has been taken from the box, the chance of selecting a second dark chocolate drops to $\frac{1}{7}$.
    a. Calculate how many chocolates are in the box altogether.
    b. If two chocolates were randomly selected from the box, calculate the probability of getting two of the same type of chocolate.

28. Claire's maths teacher decided to surprise the class with a four-question multiple-choice quiz. If each question has four possible options, calculate the probability the Claire passes the test given she guesses every question.

29. In the game of 'Texas Hold'Em' poker, five cards are progressively placed face up in the centre of the table for all players to use.
    At one point in the game there are three face-up cards (two hearts and one diamond). You have two diamonds in your hand for a total of three diamonds. Five diamonds make a flush.
    Given that there are 47 cards left, determine the probability that the next two face-up cards are both diamonds.

LESSON 11.6 Review

11.6.1 Topic summary

PROBABILITY

Probability of an outcome

- The probability of any outcome always falls between 0 and 1.
- An outcome that is **certain** has a probability of 1.
- An outcome that is **impossible** has a probability of 0.
- An outcome that is **likely** has a probability between 0.5 and 1.
- An outcome that is **unlikely** has a probability between 0 and 0.5.

Key terms

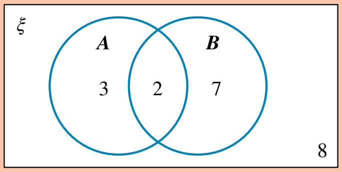
- **Trial:** A single performance of an experiment to produce a result, such as rolling a die.
- **Sample space:** The set of all possible outcomes, represented by the symbol ξ. When rolling a 6-sided die, $\xi = \{1, 2, 3, 4, 5, 6\}$.
- **Event:** A set of favourable outcomes
  For example, A can represent the event of rolling an even number on a 6-sided die: $A = \{2, 4, 6\}$.
- **Complement:** All outcomes that are not part of an event. The complement of event A above is denoted by A'. $A' = \{1, 3, 5\}$.

Experimental probability

- **Experimental probability** is used when it is difficult or impossible to determine the theoretical probability.
- **Experimental probability** = Relative frequency
- **Relative frequency** = $\dfrac{\text{frequency of outcome}}{\text{total number of trials}}$
- The more trials conducted, the more accurate the experimental probability will be.
- **Expected number** = probability of outcome × number of trials

Venn diagrams and two-way tables

- These are two different methods that can be used to represent a sample space and visualise the interaction of different events.
- For example, in a class of 20 students, 5 study Art (event A), 9 study Biology (event B) and 2 study both subjects.
  - Venn diagram:



  - Two-way table:

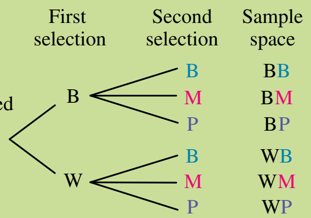
	B	B'	Total
A	2	3	5
A'	7	8	15
Total	9	11	20

- Using these we can see:
  - $n(A' \cap B) = 7$
  - $P(A \cap B') = \dfrac{3}{20}$

Two-step experiments

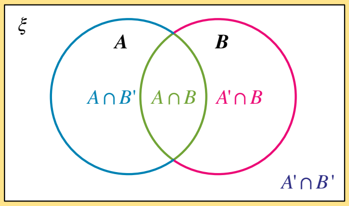
- Two-step experiments involve two separate actions performed one after the other.
- The sample space of a two-step experiment can be represented with an array or a tree diagram.
- An experiment can be conducted with replacement or without replacement.
- If an experiment is performed without replacement, the probabilities will change for the second action.
- $P(A \text{ and } B) = P(A) \times P(B)$

		Bag 2		
		B	M	P
Bag 1	B	BB	BM	BP
	W	WB	WM	WP



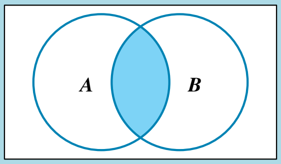
Sections of a Venn diagram

- A Venn diagram can be split into four distinct sections, as shown.

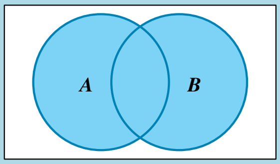


Intersection and union

- The intersection of two events A and B is written $A \cap B$. These are the outcomes that are both in A and in B.



- The union of two events A and B is written $A \cup B$. These are the outcomes that are either in A or in B.



11.6.2 Success criteria

Tick a column to indicate that you have completed the lesson and how well you think you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

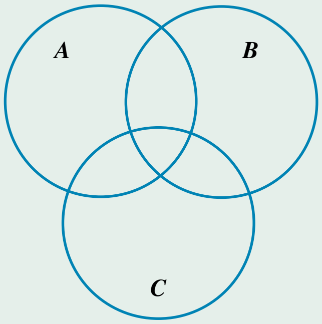
Lesson	Success criteria	Green	Yellow	Red
11.2	I can determine whether an event is likely or unlikely.			
	I am familiar with key terms such as *trials*, *outcomes*, *events* and *sample space*.			
	I can calculate theoretical probability of an event.			
11.3	I can calculate the relative frequency and experimental probability of an outcome.			
	I can explain the difference between theoretical and experimental probability.			
11.4	I can identify the complement of an event A.			
	I can create and interpret a Venn diagram.			
	I can create and interpret a two-way table.			
	I can use a Venn diagram or two-way table to find $A \cap B$.			
	I can use a Venn diagram or two-way table to find $A \cup B$.			
11.5	I can display the sample space for a two-step experiment using an array and a tree diagram.			
	I can solve two-step probabilities for experiments with replacement.			
	I can solve two-step probabilities for experiments without replacement.			

11.6.3 Project

Sand-rings

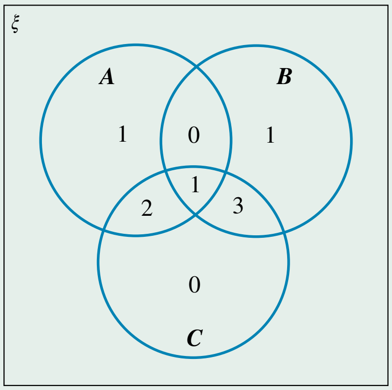
A class of students and their teacher spent a day at the beach as part of a school excursion.

Part of the day was devoted to activities involving puzzles in the sand. One of the popular — and most challenging — puzzles was 'sand-rings'. Sand-rings involves drawing three rings in the sand, as shown in the diagram.



The first sand-rings puzzle requires eight shells to be arranged inside the circles, so that four shells appear inside circle A, five shells appear inside circle B and six shells appear inside circle C. The overlapping of the circles shows that the shells can be counted in two or three circles. One possible arrangement is shown below. Use this diagram to answer questions **1** to **4**.

1. How many shells appear inside circle A, but not circle B?
2. How many shells appear in circles B and C, but not circle A?



A shell is selected at random from the sand.

3. Calculate the probability it came from circle A.
4. Calculate the probability it was not in circle C.
5. The class was challenged to find the rest of the arrangements of the eight shells. (*Remember:* Four shells need to appear in circle A, five in circle B and six in circle C.)

After completing the first puzzle, the students are given new rules. The number of shells to be arranged in the circles is reduced from eight to six. However, the number of shells to be in each circle remains the same; that is, four shells in circle A, five shells in circle B and six shells in circle C.

6. Using six shells, calculate how many ways the shells can be arranged so that there are four, five and six shells in the three circles.
7. Explain the system or method you used to determine your answer to question 6 above. Draw diagrams to show the different arrangements.
8. Using seven shells, calculate how many ways the shells can be arranged so that there are four, five and six shells in the three circles.
9. Again, explain the system or method you used to determine your answer to question 8. Draw diagrams to show the different arrangements.
10. Determine the minimum number of shells required to play sand-rings, so that there are four, five and six shells in the three circles.
11. Modify the rules of this game so that different totals are required for the three circles. Challenge your classmates to find all possible solutions to your modified game.

Exercise 11.6 Review questions

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Fluency

1. MC In a trial, it was found that a drug cures $\frac{2}{5}$ of those treated by it. If 700 sufferers are treated with the drug, calculate how many of them are not expected to be cured.

   A. 280 B. 420 C. 140 D. 350

2. MC If a die is rolled and a coin tossed, determine the probability of a 6–Heads result.

   A. $\frac{1}{6}$ B. $\frac{1}{2}$ C. $\frac{1}{8}$ D. $\frac{1}{12}$

3. MC Twelve nuts are taken from a jar containing macadamias and cashews. If three macadamias are obtained, the experimental probability of obtaining a cashew is:

   A. $\frac{1}{12}$ B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{3}{4}$

4. MC From a normal pack of 52 playing cards, one card is randomly drawn and replaced. If this is done 208 times, the number of red or picture cards expected to turn up is:

   A. 150 B. 130 C. 128 D. 120

5. MC A cubic die with faces numbered 2, 3, 4, 5, 6 and 6 is rolled. The probability of rolling an even number is:

A. $\frac{1}{3}$ B. $\frac{2}{3}$ C. $\frac{1}{6}$ D. $\frac{1}{2}$

6. MC The probability of rolling an odd number or a multiple of 2 using the die in question 5 is:

A. 1 B. $\frac{1}{3}$ C. $\frac{1}{4}$ D. $\frac{3}{4}$

The following information should be used to answer questions 7 and 8.

Students in a Year 9 class chose the following activities for a recreation day.

Activity	Tennis	Fishing	Golf	Bushwalking
Number of students	8	15	5	7

7. MC If a student is selected at random from the class, the probability that the student chose fishing is:

A. $\frac{1}{7}$ B. $\frac{2}{7}$ C. $\frac{3}{7}$ D. $\frac{4}{7}$

8. MC If a student is selected at random, the probability that the student did not choose bushwalking is:

A. $\frac{1}{35}$ B. $\frac{2}{5}$ C. $\frac{3}{5}$ D. $\frac{4}{5}$

9. The mass of 40 students in a Year 9 Maths class was recorded in a table.

Mass (kg)	Less than 50	$50- <55$	$55- <60$	$60- <65$	65 and over
Number of students	4	6	10	15	5

Calculate the experimental probability of selecting a student who has:

a. a mass of 55 kg or more, but less than 60 kg
b. a mass less than 50 kg
c. a mass of 65 kg or greater.

10. Calculate the following expected values.

a. The number of Heads in 80 tosses of a coin.
b. The number of sixes in 200 rolls of a die.
c. The number of hearts if a card is picked from a reshuffled pack and replaced 100 times.

11. A normal six-sided die is rolled. Calculate the probability of getting an odd number or a multiple of 4.

12. A card is drawn from a pack of 52 cards. Calculate the probability that the card is a heart or a club.

13. A jar contains 2 red balls and 3 blue balls. Two balls are selected with replacement.

a. Show the outcomes and their probabilities on a tree diagram.
b. Calculate the probability of selecting two red balls.

14. In a video game, a player opens a loot box that can contain one of the following items:

- 5 common items
- 3 rare items
- 2 legendary items.

The player opens two loot boxes, and each box is reset after opening (i.e., with replacement).

a. Draw a tree diagram showing all possible outcomes and their probabilities.

b. Determine the probability of getting two legendary items in a row.

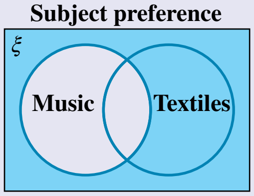
15. At a music festival, there are 4 VIP passes, 3 standard passes and 3 backstage passes in a prize draw box. Two winners are drawn without replacement.

a. Show the outcomes and their probabilities using a tree diagram.

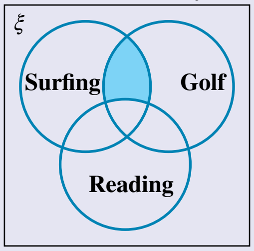
b. Determine the probability that both winners get VIP passes.

16. Indicate the set that each of the shaded regions represents.

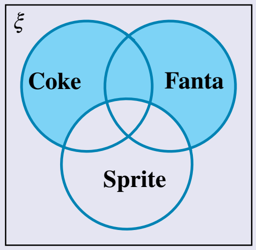
a. **Subject preference**



b. **Leisure activity**



c. **Favourite drinks**



17. An equal number of red (R), black (B) and yellow (Y) counters makes up a total of 30 in a bag.

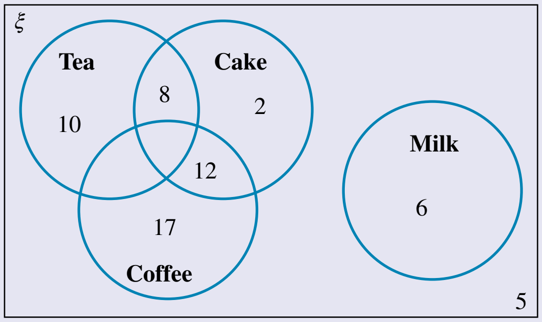
a. List the sample space for picking a counter from the bag.

b. Event A is ‘draw a yellow counter, then randomly draw another counter from the bag’. List the sample space of event A.

c. Explain whether ‘choosing a green counter’ is an outcome.

Problem solving

18. Teachers at a school opted for the choice of morning recess refreshments shown in the Venn diagram.



a. State how many teachers are in the set ‘cake ∩ coffee’.

b. State the total number of teachers surveyed.

c. If a teacher is selected at random, calculate the probability that the teacher:

i. chose tea

ii. chose coffee only

iii. chose milk

iv. did not choose tea, coffee, cake or milk

v. did not choose coffee.

19. Thirty-two students ordered fried rice, chicken wings and dim sims for lunch. Four students ordered all three, two ordered fried rice and chicken wings only, three ordered fried rice and dim sims only, and three ordered chicken wings and dim sims only.

When the waiter organised the orders, he found that 16 students ordered fried rice and 18 students ordered dim sims.

a. Show this information on a Venn diagram.

b. Calculate how many students ordered chicken wings only.

c. If a student is picked at random, calculate the probability that the student has:

i. ordered chicken wings and dim sims

ii. ordered fried rice

iii. not ordered dim sims.

20. The following are options for dorm rooms at a university. You are required to choose one option from each of the four categories.
    - Upstairs or downstairs rooms
    - Single or double rooms
    - Male or female roommates
    - A choice of 10 different locations on campus

    Calculate how many different combinations of rooms there are to choose from.

21. A future king is the oldest male child. The future king of Mainland has two siblings. Determine the probability that he has an older sister.

22. A witness described a getaway car as having a WA registration plate starting with 1IBK. The witness could not remember the three digits that followed, but recalled that all three digits were different. Calculate how many cars in WA could have a registration plate with these letters and numbers.

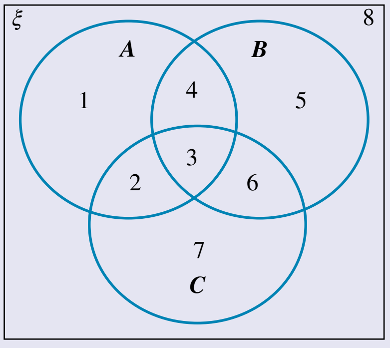
23. There are 12 people trying out for a badminton team. Five of them are girls. Calculate the probability that a team chosen at random to play is a mixed doubles team.

24. An ace is chosen from a deck of standard cards and not replaced. A king is then chosen from the deck. Calculate the probability of choosing an ace and a king in this order.

25. If you randomly select one number from 1 to 500 (inclusive), calculate the probability that the selected number will have at least one 4 in the digits.

26. If you flip a coin eight times, calculate how many of the possible outcomes would you expect to have a Head on the second toss.

27. Use the Venn diagram to calculate each of the following.



a. $A' \cup (B' \cap C')$

b. $A' \cap (B \cap C')$

c. $A \cap (B' \cup C')$

d. $(A \cup B \cup C)' \cap (A \cup B \cup C)$

on To test your understanding and knowledge of this topic go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 11 Probability

11.1 Pre-test

1. C
2. a. Zero b. $\frac{3}{8}$ or 0.375
3. a. 100 b. $\frac{7}{25}$ c. $\frac{1}{2}$ d. $\frac{11}{50}$
4. $\frac{1}{3}$
5. True
6. D
7. C
8. A
9. a. $\frac{1}{4}$ b. $\frac{3}{14}$
10. P(drama not comedy) $= \frac{2}{5}$
11. $\frac{5}{7}$
12. a. B b. 0.33
13. a. C b. D c. A d. B
14. a. $\frac{8}{15}$ b. $\frac{67}{225}$ c. $\frac{1}{6}$ d. $\frac{67}{240}$
15. D

11.2 Theoretical probability

1.


a. Certain b. Unlikely

2.


a. Likely b. Certain

3.


a. Certain b. Likely

4.


a. Likely b. Likely

5. a. {1, 2, 3, 4, 5, 6}
   b. {H, T}
   c. {defective, not defective}
   d. {red, black}
   e. {hearts, clubs, diamonds, spades}
6. a. {1, 2, 3} b. {1, 3, 5}
   c. {1, 2, 4, 6} d. {3, 4, 5, 6}
   e. {1, 2, 3, 4} f. {5, 6}
7. a. 3, 5
   b. 1, 2
   c. 6
   d. 1, 2, 5, 6
   e. 1, 2, 3, 5
   f. No favourable outcomes
   g. 3
8. a. {hearts, clubs, diamonds, spades}
   b. i. {clubs, spades}
      ii. {hearts, diamonds}
      iii. {clubs, diamonds, spades}
      iv. {hearts, clubs, diamonds, spades}
9. a. 6 b. 2 c. 52
   d. 4 e. 5
10. a. $\frac{1}{13}$ b. $\frac{1}{13}$ c. $\frac{1}{52}$ d. $\frac{1}{4}$
11. a. $\frac{1}{52}$ b. $\frac{1}{2}$ c. $\frac{1}{26}$ d. $\frac{1}{4}$ e. $\frac{1}{2}$
12. a. $\frac{1}{52}$ b. $\frac{4}{13}$ c. $\frac{2}{13}$ d. 0 e. $\frac{12}{13}$
13. a. $\frac{2}{11}$ b. $\frac{9}{11}$ c. $\frac{4}{11}$ d. $\frac{7}{11}$
14. a. $\frac{1}{4}$ b. $\frac{1}{8}$ c. $\frac{5}{16}$ d. 0 e. $\frac{3}{4}$
15. a. i. $\frac{2}{3}$ ii. $\frac{1}{3}$
    b. Sample responses can be found in the worked solutions in the online resources.
16. Sample responses can be found in the worked solutions in the online resources.
17. a. 18 b. 20 c. 36
18. a. i. $\frac{7}{10}$ ii. $\frac{3}{10}$
    b. i. $\frac{2}{3}$ ii. $\frac{1}{3}$
19. Sample responses can be found in the worked solutions in the online resources.
20. Probabilities must be between 0 and 1, so −0.5 and 1.05 can't be probabilities.
21. $\frac{1}{2}$
22. The coloured portions outside the arc of the spinner shown are of no consequence. The four colours within the arc of the spinner are of equal area (each $\frac{1}{4}$ circle), so there is equal chance of falling on each of the colours.
23. $\frac{2}{3}$
24. There are 36 outcomes, 15 under 7, 6 equal to 7 and 15 over 7. So, you would have a greater chance of winning if you chose 'under 7' or 'over 7' rather than 'equal to 7'.
25. 0.09

11.3 Experimental probability

1. a. i. 16

ii. 4

iii.

Outcome	1	2	3	4	5	6
Relative frequency	0.19	0.06	0.31	0.13	0.25	0.06

iv. 1

b. i. 270

ii. 40

iii.

Outcome	1	2	3	4	5	6
Relative frequency	0.19	0.14	0.17	0.18	0.15	0.17

iv. 1

2. a. i. r.f. (H) $= 0.44$, r.f. (T) $= 0.56$

ii. 1

b. i. r.f. (H) $= 0.62$, r.f. (T) $= 0.38$

ii. 1

3. a. 0.365 b. 0.33 c. 0.25 d. 0.875

4. a. 0.275 b. 0.64 c. 0.365 d. 0.635

5. a. $\frac{27}{95}$ b. $\frac{33}{95}$

c. $\frac{1}{19}$ d. Yes, reject this batch.

6. a. Not reasonable; not enough trials were held.

b. Not reasonable; the conditions are different under each trial.

c. Reasonable; the Sun rises every morning, regardless of the weather or season.

d. Reasonable; enough trials were performed under the same conditions.

e. Not reasonable; monthly rainfall in Perth is not consistent throughout the year.

7. a. $\frac{5}{24}$ b. $\frac{59}{72}$

8. 200

9. 20

10. A

11. B

12. a. $\frac{11}{20}$ b. $\frac{3}{10}$ c. $\frac{17}{20}$ d. $\frac{3}{20}$

13. a. $\frac{33}{100}$ b. $\frac{40}{100} = \frac{2}{5}$ c. $\frac{3}{100}$

14. a. i. 6

ii. $\{356, 365, 536, 563, 635, 653\}$

b. $\frac{2}{3}$

c. $\frac{1}{3}$

d. $\frac{1}{3}$

15. 5

16. Svetlana 6, Sarah 4, Leonie 3, Trang 2

17. C

18. a. 27

b. $\left\{\begin{matrix} 111, 112, 115, 121, 122, 125, 151, 152, 155, 211, 212, \\ 215, 221, 222, 225, 251, 252, 255, 511, 512, 515, 521, \\ 522, 525, 551, 552, 555 \end{matrix}\right\}$

c. $\frac{1}{3}$

d. $\frac{2}{3}$

e. $\frac{1}{3}$

19. a. $\frac{1}{12}$

b. $\frac{1}{20}$

c. No, because John has a higher probability of winning.

20. a. $\frac{7}{10}$

b. Yes, far too many mistakes

21. 32

22. 1000 balls

23. Red $= 10$, yellow $= 50$, blue $= 40$

24. a. $\frac{7}{20}$ b. $\frac{2}{5}$

25. Sample responses can be found in the worked solutions in the online resources.

26. a. $\frac{30}{241}$ b. $\frac{91}{241}$ c. $\frac{59}{241}$

27. 16, 64, 40, 80, 128 and 32 degrees.

11.4 Venn diagrams and two-way tables

1. a. $A' = \{1, 3, 5, 7, 9\}$

b. $B' = \{1, 2, 3, 4, 6, 7, 8, 9\}$

c. $C' = \{2, 3, 5, 6, 7, 8, 10\}$

d. $D' = \{8, 9, 10\}$

2. a. $A' = \{11, 13, 14, 16, 17, 19, 20\}$

b. $B' = \{20\}$

c. $C' = \{12, 14, 15, 16, 18, 20\}$

d. $D' = \{12, 14, 16\}$

3. a. $A' = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19\}$

b. $B' = \{1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$

c. $C' = \{1, 3, 5, 7, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20\}$

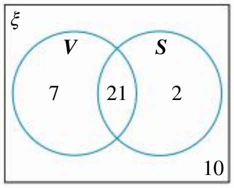
d. $D' = \{1, 3, 5, 7, 9, 11, 13\}$

4. a. 27 b. 11 c. 13 d. 6

e. 7 f. 9

5.

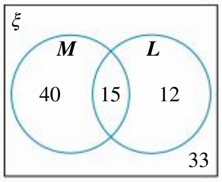
	T	T'
S	6	5
S'	7	9

6. 

7. a. i. W' ii. None
b. i. $A \cap B'$ ii. $A \cap B$
c. i. $A' \cap B'$ ii. $B \cap A'$

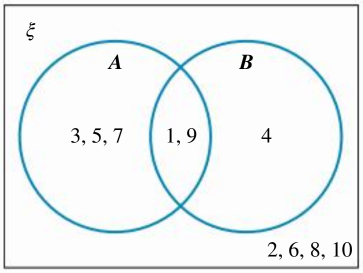
8. a.

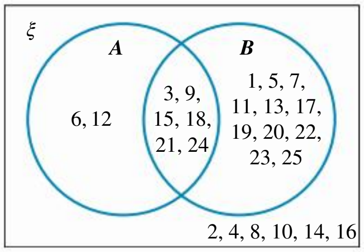
	Left-handed	Right-handed
Male	15	40
Female	12	33

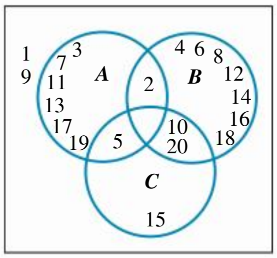
b. 

c. i. $\frac{73}{100}$ ii. $\frac{2}{5}$ iii. $\frac{3}{25}$

9. a. $\frac{8}{25}$ b. $\frac{17}{25}$ c. $\frac{1}{5}$ d. $\frac{3}{25}$

10. 

11. 

12. 

13. a. $\frac{1}{2}$ b. $\frac{1}{10}$ c. $\frac{31}{40}$
d. $\frac{27}{40}$ e. $\frac{9}{40}$

14. a. $\frac{1}{2}$ b. $\frac{6}{25}$
c. $\frac{22}{25}$ d. $\frac{16}{25}$

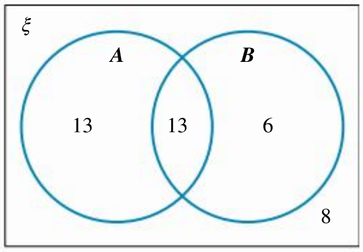
15. a.

	Croissant	No croissant	Total
Coffee	27	2	**29**
No coffee	9	7	**16**
Total	**36**	**9**	**45**

b. i. $\frac{3}{5}$ ii. $\frac{38}{45}$ iii. $\frac{11}{45}$

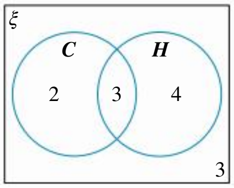
c. 1218

16. a. 16
b. i. $\frac{5}{16}$ ii. $\frac{3}{16}$ iii. $\frac{5}{8}$ iv. $\frac{7}{16}$

17. a. 

b. i. $\frac{29}{40}$ ii. $\frac{4}{5}$ iii. $\frac{19}{40}$

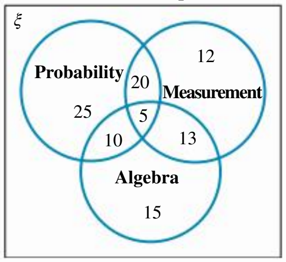
18. a. Children who are not swimmers
b. Children who are swimmers and runners
c. Children who neither swim nor run
d. Children who swim or run or both

19. a. i. 

ii.

	H	H'
C	3	2
C'	4	3

b. i. $\frac{7}{12}$ ii. $\frac{3}{4}$ iii. $\frac{1}{4}$ iv. $\frac{1}{2}$

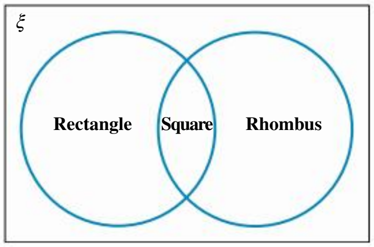
20. a. Favourite topic


b. 25 c. 15 d. 12

e. 43 **f.** $\frac{3}{5}$ **g.** $\frac{43}{100}$

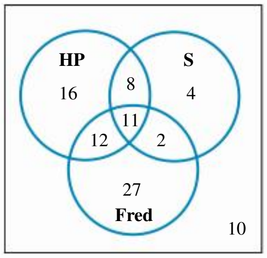
h. $\frac{9}{50}$ **i.** $\frac{13}{100}$ **j.** $\frac{1}{20}$

21. Quadrilaterals



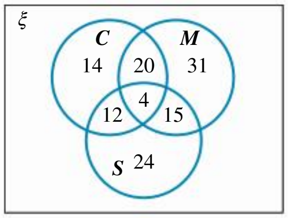
22. a. 2, 5, 6, 7, 8 **b.** 4
c. 8 **d.** 1, 2, 3, 4, 5, 7, 8

23. a.



b. $\frac{8}{90} = \frac{4}{45}$

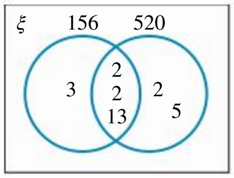
24. a.



b. $\frac{7}{12}$

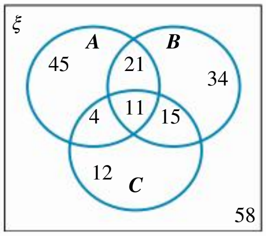
25. a. 31 students **b.** $\frac{1}{13}$

26.



a. HCF $= 2 \times 2 \times 13 = 52$
b. LCM $= 3 \times 2 \times 2 \times 13 \times 2 \times 5 = 1560$

27. a.



b. 4
c. 21
d. i. $\frac{81}{200}$ **ii.** $\frac{97}{200}$
e. $\frac{3}{50}$

11.5 Compound events

1. See table at the foot of the page.*
a. $\frac{1}{9}$ **b.** 0 **c.** A total of 7

2.

		Scarves		
		R	***Bl***	***Pi***
Beanies	***Br***	Br, R	Br, Bl	Br, Pi
	Pu	Pu, R	Pu, Bl	Pu, Pi

3. a. See table at the foot of the page.*
b. i. $\frac{1}{4}$ **ii.** $\frac{3}{4}$ **iii.** $\frac{3}{20}$ **iv.** $\frac{13}{20}$

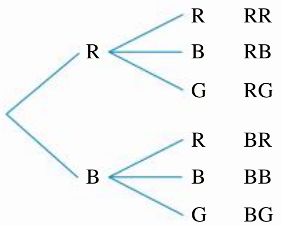
*__1.__

		Die 1					
		1	**2**	**3**	**4**	**5**	**6**
Die 2	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

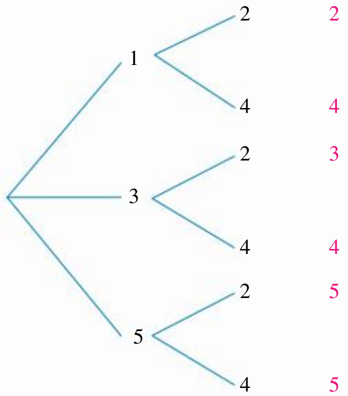
*__3. a.__

		Dice roll									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Coin toss	**H**	1, H	2, H	3, H	4, H	5, H	6, H	7, H	8, H	9, H	10, H
	T	1, T	2, T	3, T	4, T	5, T	6, T	7, T	8, T	9, T	10, T

4.



5. a.

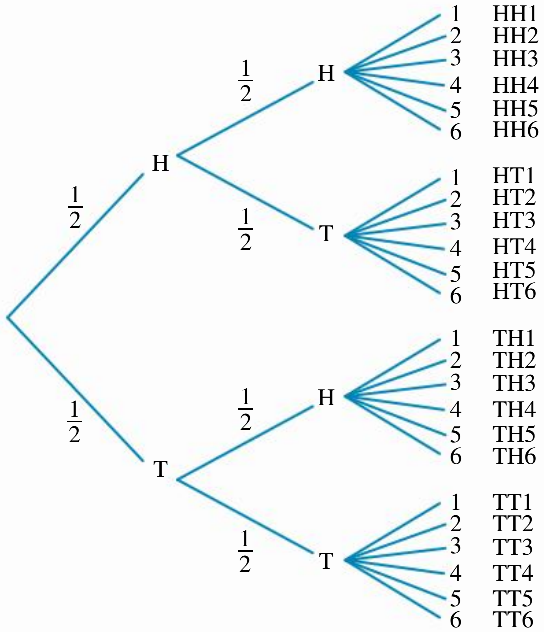


b. i. $\frac{1}{6}$ ii. 0 iii. $\frac{1}{2}$

6. a. See table at the foot of the page.*

b. i. $\frac{1}{2}$ ii. $\frac{1}{6}$ iii. $\frac{1}{3}$ iv. $\frac{2}{3}$

7. a.

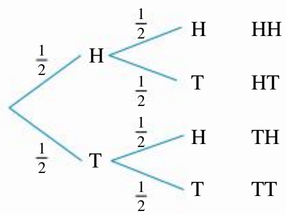


b. P(HT2) = $\frac{1}{24}$ c. P(no heads) = $\frac{1}{4}$

d. P(no heads) = $\frac{1}{8}$

8. a. $\frac{1}{6}$ b. $\frac{1}{36}$ c. $\frac{1}{4}$ d. $\frac{1}{36}$

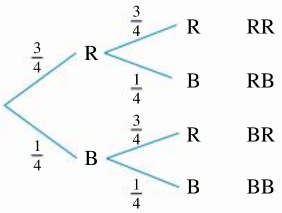
9. a.



b. i. $\frac{1}{4}$ ii. $\frac{3}{4}$

*6. a.

		Die 1					
		1	**2**	**3**	**4**	**5**	**6**
Die 2	**1**	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

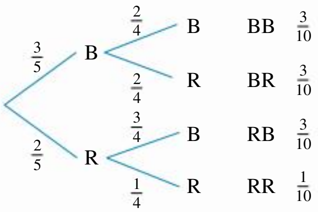
10. a.


b. i. $\frac{3}{16}$ ii. $\frac{1}{16}$

11. a. $\frac{4}{15}$ b. $\frac{2}{15}$ c. $\frac{209}{225}$ d. $\frac{77}{225}$

12. a. $\frac{16}{25}$ b. $\frac{9}{25}$ c. $\frac{391}{400}$

d. $\frac{77}{200}$ e. $\frac{123}{200}$

13. a.


b. $\frac{1}{10}$

14. a. $\frac{2}{9}$ b. $\frac{1}{9}$ c. $\frac{2}{9}$ d. $\frac{14}{45}$

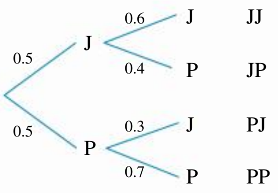
15. a. $\frac{5}{22}$ b. $\frac{1}{22}$ c. $\frac{5}{11}$ d. $\frac{25}{66}$

16. a. $\frac{1}{4}$ b. $\frac{3}{14}$

17. a. $P(H11) = \frac{1}{72}$ b. $P(\text{dice sum is } 12) = \frac{1}{36}$

c. $P(\text{Tail}) = \frac{1}{2}$

18. a. $\frac{1}{9}$ b. $\frac{3}{11}$ c. $\frac{8}{11}$

19. a.


b. i. 0.3 ii. 0.35 iii. 0.35

20. If the first ball is replaced, the probability of drawing a yellow ball stays the same on the second draw, i.e. $\left(\frac{3}{5}\right)$.
If the first ball isn't replaced, the probability of drawing a yellow ball on the second draw decreases, i.e. $\left(\frac{5}{9}\right)$.

21. a. 3 days b. 90 combinations
c. 36 new combinations

22. a. Smallest total: 3, largest total: 18

b.

Total	Probability
3	$\frac{1}{216}$
4	$\frac{3}{216}$
5	$\frac{6}{216}$
6	$\frac{10}{216}$
7	$\frac{15}{216}$
8	$\frac{21}{216}$
9	$\frac{25}{216}$
10	$\frac{27}{216}$
11	$\frac{27}{216}$
12	$\frac{25}{216}$
13	$\frac{21}{216}$
14	$\frac{15}{216}$
15	$\frac{10}{216}$
16	$\frac{6}{216}$
17	$\frac{3}{216}$
18	$\frac{1}{216}$

23. a. $\frac{1}{221}$ b. $\frac{11}{221}$ c. $\frac{1}{17}$

24. $\frac{5}{12}$

25. $\frac{1}{2}$

26. a. 8 b. 32 c. 16 d. 2^x

27. a. 36 b. $\frac{5}{7}$

28. $\frac{67}{256}$

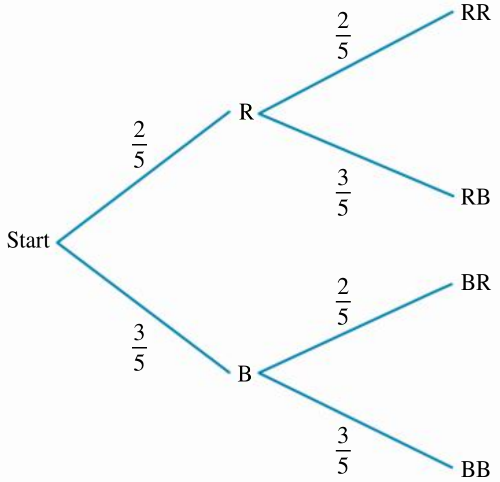
29. $\frac{45}{1081}$

Project

1. 3
2. 4
3. $\frac{1}{2}$
4. $\frac{1}{4}$
5. 18
6. 2
7. Sample responses can be found in the worked solutions in the online resources.
8. 8
9. Sample responses can be found in the worked solutions in the online resources.
10. 6
11. Sample responses can be found in the worked solutions in the online resources.

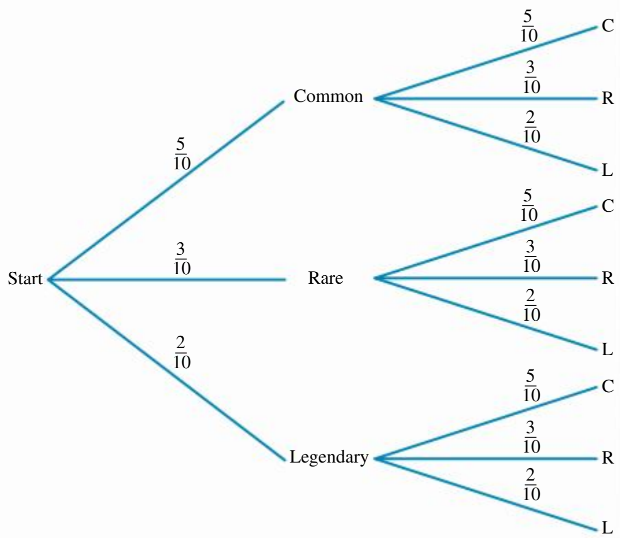
11.6 Review questions

1. B
2. D
3. D
4. C
5. B
6. A
7. C
8. D
9. a. $\frac{1}{4}$ b. $\frac{1}{10}$ c. $\frac{1}{8}$
10. a. 40 b. 33 c. 25
11. $\frac{2}{3}$
12. $\frac{1}{2}$
13. a.



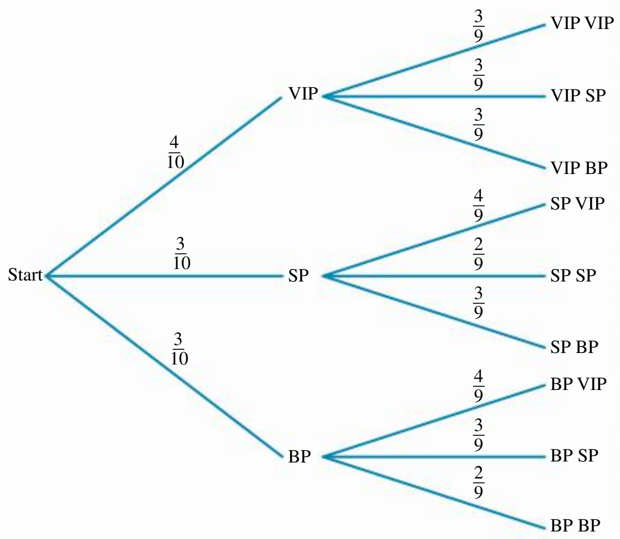
b. $\frac{4}{25}$

14. a.



b. $\frac{1}{25}$

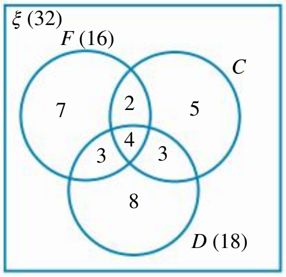
15. a.



b. $\frac{2}{15}$

16. a. Students who do not like music.
    b. People who like surfing and golf as leisure activities, but not reading.
    c. People who like Coke or Fanta or both, but not Sprite.
17. a. {R, B, Y}
    b. {YR, YY, YB}
    c. No, there is no green counter.
18. a. 12
    b. 60
    c. i. $\frac{3}{10}$ ii. $\frac{17}{60}$ iii. $\frac{1}{10}$
    iv. $\frac{1}{12}$ v. $\frac{31}{60}$

19. a.



b. 5

c. **i.** $\frac{7}{32}$ **ii.** $\frac{1}{2}$ **iii.** $\frac{7}{16}$

20. 80

21. $\frac{3}{7}$

22. 720

23. $\frac{35}{66}$

24. $\frac{4}{663}$

25. $\frac{176}{500}$

26. 128

27. a. $\{1, 5, 6, 7, 8\}$ **b.** $\{5\}$

c. $\{1, 2, 4\}$ **d.** ϕ

STRAND: Probability and statistics
SUB-STRAND: Probability and statistics

12 Statistics

LESSON SEQUENCE

LESSON 12.1 Overview

12.1.1 Why learn this?

The science of statistics is concerned with understanding the world around us. It presents a way of processing data about people, places and things in order to make that data useful and easier to understand.

One example of statistics is the data that social media companies collect about how you use their platforms. This data is statistically analysed and used to adapt these platforms to the way you use them (and to help the company maximise profits). Another example of statistics is the Australian Bureau of Meteorology — they collect data about the weather, analyse the statistics from that data and then use that knowledge to help predict what the weather will be like tomorrow, next week and even next year.

A good understanding of statistics can help you develop critical analysis skills so that you are not tricked by bad or misleading statistics. It is also important because statistics are a valuable tool used in a broad range of careers including product design, science, sports coaching and finance. Next time you are reading an article or watching a news report, keep your eye out for statistics and you'll start to notice that they are everywhere!

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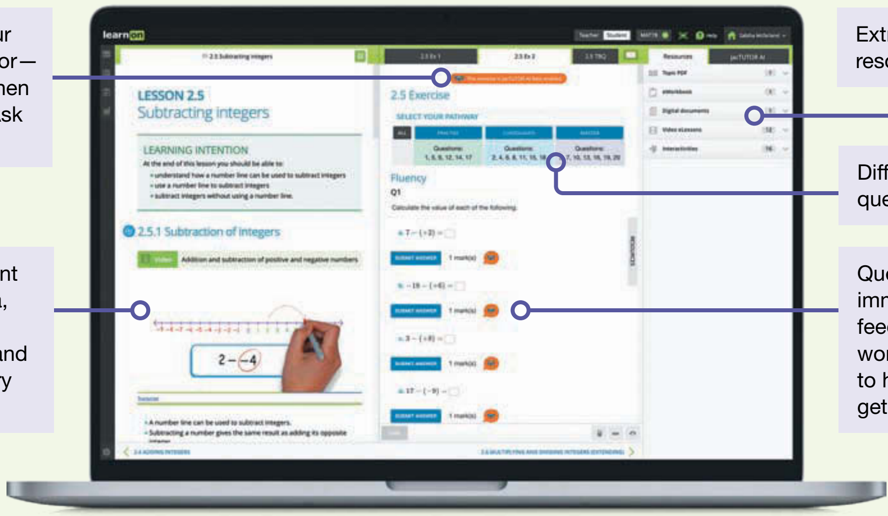


Extra learning resources

Differentiated question sets

Questions with immediate feedback, and fully worked solutions to help students get unstuck.

Exercise 12.1 Pre-test

1. Match and classify the following different types of data.

Example	Data type
a. Attendance at a concert	**i.** nominal
b. Mass of bag of lollies	**ii.** ordinal
c. Eye colour	**iii.** discrete
d. The quality of food at a restaurant	**iv.** continuous

2. Thirty Year 9 students from a particular school were surveyed on the number of hours they spend each week using social media on their phones.
State whether this type of data is a population or a sample.

3. Fifty members of a tennis club were selected to participate in a survey. The club has 80 male adult, 70 female adult, 55 male adolescent and 45 female adolescent members.
State how many members are required from each of these four groups to perform a stratified sample survey.

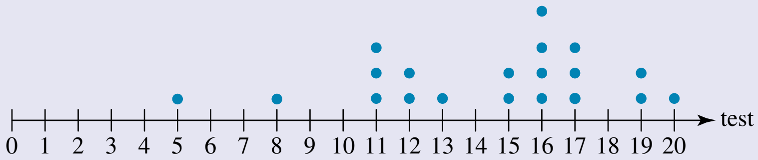
4. The stem and leaf plot shown represents the speeds of cars recorded by a roadside camera outside a school where the speed limit is: 40 km/h.

Key: 1 | 9 = 19 km/h

Stem	*Leaf*
1	9
2	5 7 7 9 9
3	0 0 4 5 6 7 8 8
4	0 0 0 0 2 4 5 5
5	2 3
6	0

a. State how many cars had their speed recorded.
b. Determine how many cars were travelling at a speed over 40 km/h.

5. The dot plot shown displays the results of a maths test for a Year 9 class.
The results show the students' marks out of 20.

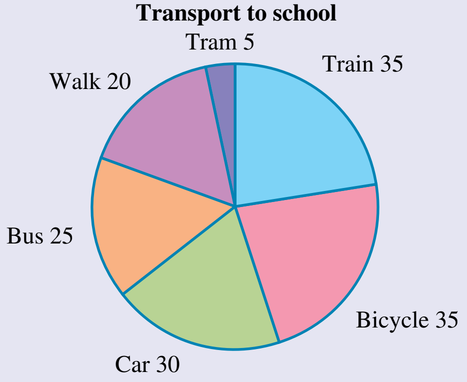


a. State how many students are in the class.
b. Determine how many students scored more than 15 out of 20 for the test.

6. Match the different types of data shown in the table with the most appropriate type of graph.

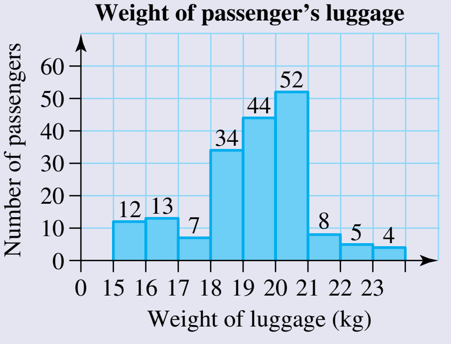
Data type	Type of graph
a. Categorical data	i. Box plot
b. Numerical continuous data	ii. Stem plot
c. Numerical discrete data	iii. Bar chart

7. MC Based on the pie chart shown, the percentage of students who catch public transport to school is approximately:



A. 40% B. 43% C. 52% D. 60%

8. The histogram shows the weights of passengers' luggage for a flight to Perth.
a. State the type of data shown the histogram.
b. State how many passengers exceeded the maximum limit of 20 kg.
c. Explain whether the data distribution is skewed or symmetrical.



9. MC Identify the correct meaning of the spreadsheet function =RANDBETWEEN(1, 5).
A. Randomly select a real number between 1 and 5.
B. Randomly select an integer between 1 and 5.
C. Randomly select a number 5 times.
D. Randomly select 5 numbers.

10. MC Megan wanted to select a random number in the range 10 to 20. Select the command she used to generate the number in her spreadsheet.
A. = range (10, 20) B. = ran (10, 20)
C. = randbetween (10, 20) D. = rand (10, 20)

11. Calculate the median for the frequency table shown.

Score	Frequency
0	3
1	8
2	10
3	14
4	5
5	2

12. Examine the grouped frequency table shown and answer the following questions.

Score	Frequency
$0-<20$	3
$20-<40$	4
$40-<60$	8
$60-<80$	15
$80-<100$	7

a. Calculate the mean for the grouped frequency table, rounded to 1 decimal place.
b. MC Select the modal class for the grouped frequency table.
A. $0-<20$ B. $20-<40$ C. $40-<60$ D. $60-<80$

13. Determine the missing value from the set of numbers $\{8, 8, 10, 12, 15\}$ if the mean is 10, the mode is 8 and the median is 9.

14. MC A high school wants to gather student opinions on a new school uniform policy. The principal is considering selecting every 10th student from an alphabetical list of all students.
This is an example of:
A. Systematic Sampling
B. Cluster Sampling
C. Stratified Sampling
D. Simple Random Sampling

15. MC Select the correct statement to describe symmetrical distributions.
A. The data on the right have more low values.
B. Distribution has two peaks.
C. Mean has a smaller value than median.
D. Mean, median and mode have similar values.

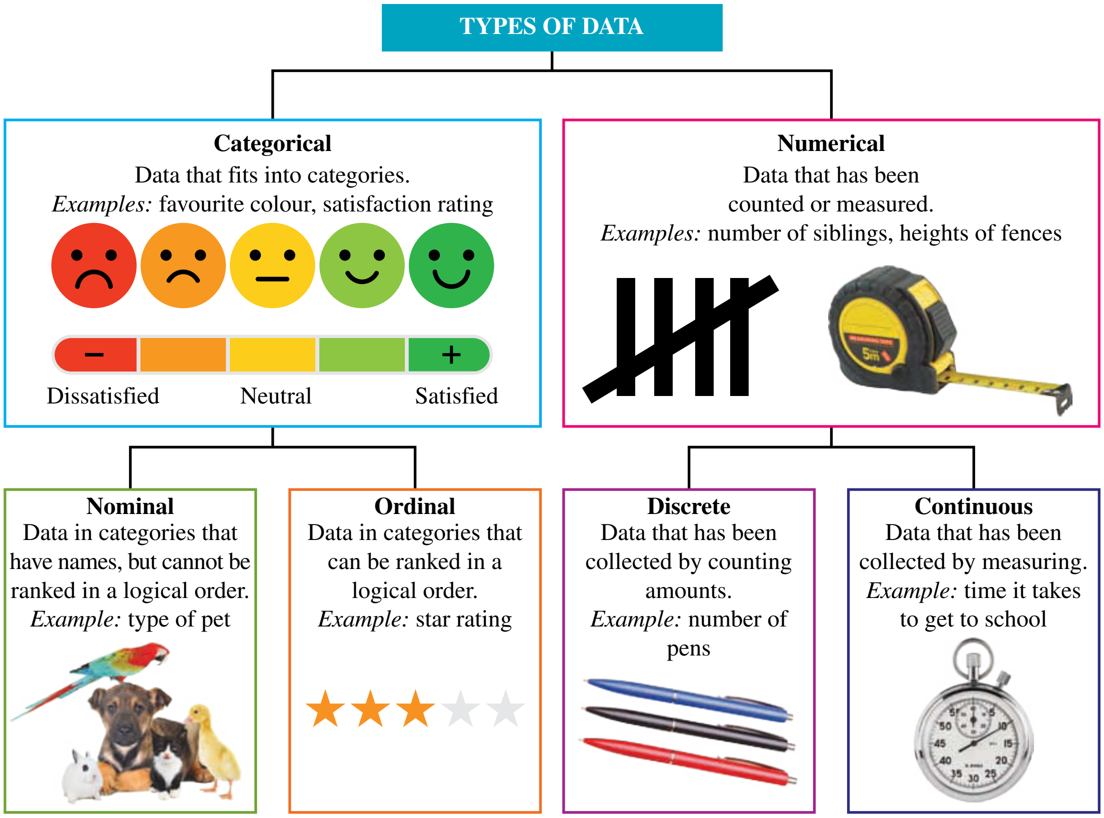
LESSON 12.2 Collecting data

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- identify different types of data
- determine if a data set is a sample or a population
- generate random numbers.

12.2.1 Types of data

- Data can be **categorical** or **numerical**.
- There are two types of categorical data:
  - **nominal data**
  - **ordinal data**.
- There are two types of numerical data:
  - **discrete data**
  - **continuous data**.
- The information in the flowchart shown can be used to determine the type of data being considered.



WORKED EXAMPLE 1 Classifying data as categorical or numerical

Classify each of the following type of data as categorical data or numerical data.

a. **Favourite sport watched on TV.**
b. **Quantity of books carried in a school bag by 20 Year 9 students.**
c. **Cars passing your home each hour in a day.**
d. **Eye colour of 100 preschool children.**

THINK	WRITE
a. Decide how the data would initially be recorded, as either numbers or categories. Examples of sports watched on TV could include football, netball, soccer or tennis.	**a.** The sports watched on TV are collected into categories so this is categorical data.
b. Decide how the data would be recorded, as either numbers or categories. The quantity of books carried in a school bag by 20 Year 9 students will be collected by counting the number of books.	**b.** Because the recorded data would be collected by counting the number of books in a school bag, this is numerical data.
c. Decide how the data would be recorded, as numbers or categories. Data for cars passing your home each hour of a day would be collected by counting the number of cars.	**c.** Because the recorded data would be collected by counting the number of cars, this is numerical data.
d. Decide how the data would be recorded, as numbers or categories. Eye colour of 100 preschool children could include blue, brown, green or hazel.	**d.** Because the eye colour of 100 preschool children would be collected into categories, this is categorical data.

WORKED EXAMPLE 2 Classifying types of data

Classify each of the following types of data into the correct groups, stating whether it is categorical or numerical. If it is categorical, state whether it is nominal or ordinal.
If it is numerical, state whether it is discrete or continuous.

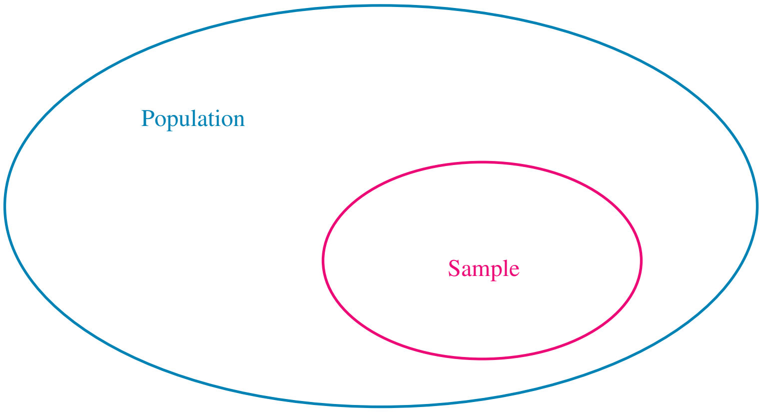
a. **Type of transport used to go to work**
b. **Daily attendance at a zoo**
c. **Hair colour of 100 Year 9 students**
d. **Mass of an individual Easter egg**
e. **A student's behavioural report**

THINK	WRITE
a. **1.** Types of transport used to go to work could include car, bicycle, tram, bus or train.	**a.** The possible transport modes are categories; therefore, they are categorical data.
2. For categorical data, decide whether the categories can be ranked in a logical order (categorical ordinal) or not (categorical nominal).	Because types of transport cannot be logically ranked, the data can be classified as categorical nominal data.

b. 1. Attendance at a zoo each day would be counted and recorded as a whole number.	**b.** The recorded data would be the number of visitors at a zoo; therefore, this is numerical data.
2. For numerical data, decide whether it is collected by counting (numerical discrete data) or by measuring (numerical continuous data).	The visitors would be counted; therefore, it is classified as numerical discrete data.
c. 1. Hair colour of 100 Year 9 students could include black, brown, blonde, brunette or redhead.	**c.** The hair colours are categories; therefore, this is categorical data.
2. For categorical data, decide if the categories can be ranked in a logical order (categorical ordinal) or not (categorical nominal).	Because hair colours cannot be logically ranked, the data can be classified as categorical nominal data.
d. 1. Mass of an individual Easter egg would be measured on a scale.	**d.** Because the recorded data would be the mass of an individual egg, this is numerical data.
2. For numerical data, decide whether it is collected by counting (numerical discrete data) or by measuring (numerical continuous data).	The mass of each egg would be measured. Therefore, it is numerical continuous data.
e. 1. A student's behavioural report could be described as excellent, good, satisfactory, improvement needed or unsatisfactory.	**e.** A student's behavioural report uses categories; therefore, it is categorical data.
2. For categorical data, decide whether the categories can be ranked in a logical order (categorical ordinal) or not (categorical nominal).	Because the student's behavioural report can be logically ranked, it can be classified as categorical ordinal data.

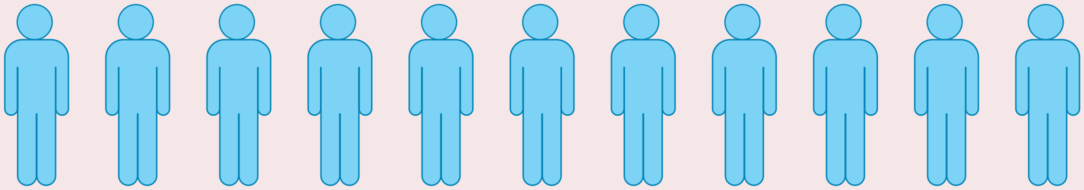
12.2.2 Collecting data

- Data is collected to obtain information and draw conclusions.
- Data can be collected using:
  - a census of the entire **population**, or
  - a survey of a **sample** of the population



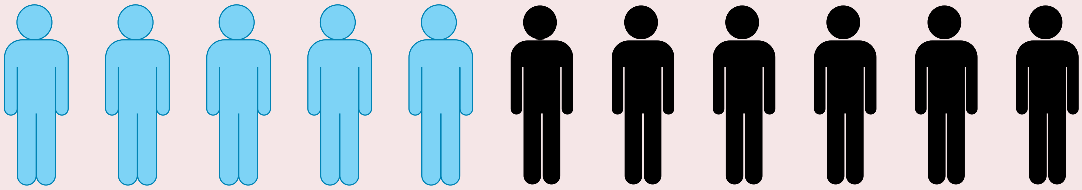
Samples and populations

- **Data is classified as a *population* if data is collected from an entire group.**
  Examples of population data include surveying all students at a school or measuring the size of all frogs in a pond.



In this example, 11 out of 11 people in the group were surveyed.

- **Data is classified as a *sample* if data is collected from parts of a group.**
  Examples of sample data include surveying only some students at a school or measuring only some of the frogs in a pond.



In this example, 5 out of 11 people in the group were surveyed.

WORKED EXAMPLE 3 Classifying data collection methods

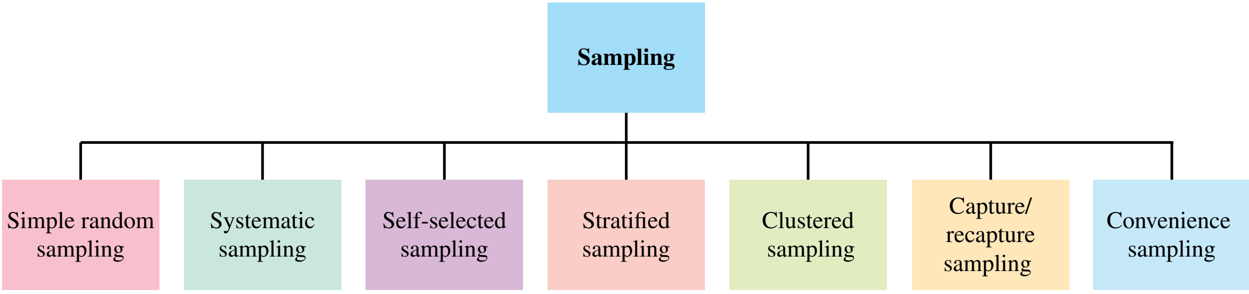
For the following data collection activities, define whether the data collected is a sample or a population.

a. **Sixty Year 7 students collected data on the amount of money they donated to a charity, then used this data to calculate the average donation from the 1523 students at their school.**
b. **A data set of the force (measured in Newtons) needed to break wooden ice-block sticks.**
c. **The average age of the members of your own family.**

THINK	WRITE
a. Only 60 Year 7 students were surveyed out of a total of 1523.	**a.** This is a sample, because only 60 of the 1523 students were surveyed.
b. An unknown number of wooden ice-block sticks are required. It is impossible to collect data on the entire population (which would mean every wooden ice-block stick in the world).	**b.** This is a sample, because there is an unknown number of wooden ice-block sticks.
c. Every family member is surveyed to find the average age of the group.	**c.** This is a population, because every member of the group has been surveyed.

12.2.3 Sampling

- Sampling is the process of selecting a sample of a population to provide an estimate of the entire population.
- When a sample of data is collected, it is important to ensure that the sample is indicative of the population and is selected at random. If the sample is not large enough and is not selected at random, it may provide data that is biased towards one particular group of people.



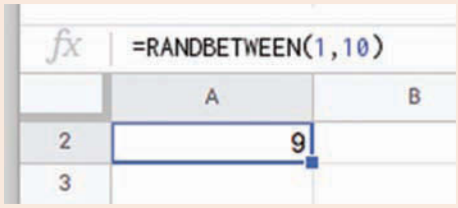
Simple random sampling

- **Simple random sampling** is the basic method of selecting a sample. This type of sampling ensures that each individual in a population has an equal chance of being selected for the sample.

Digital technology

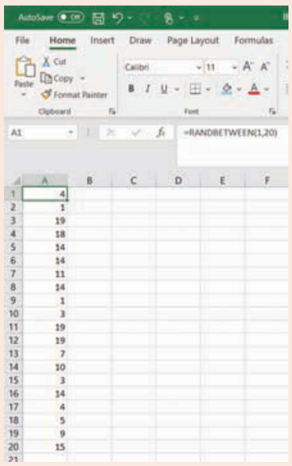
Digital technology often features random number generators that can be used to create simple random samples.

- In a spreadsheet, click on a cell and type =**RANDBETWEEN(**<lowest number>,<highest number>).
- Press **ENTER**.
- A random number will appear in the cell.
- If you want a list of random numbers, use **Fill Down** to get more numbers.



For example, in the screenshot shown, a random number in a sample of 20 is generated by:

- clicking on a cell
- typing =**RANDBETWEEN**(1, 20)
- pressing **ENTER**.

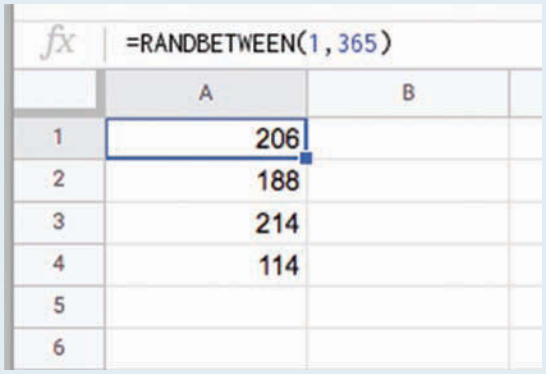
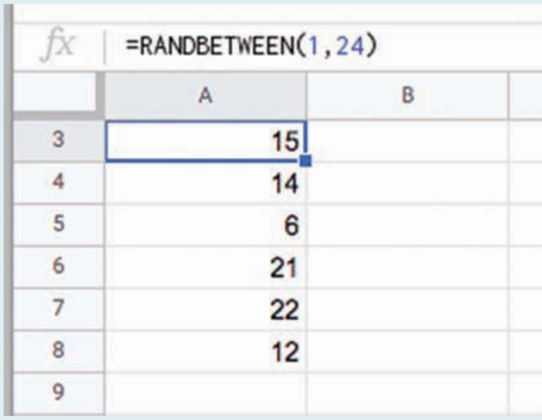


WORKED EXAMPLE 4 Selecting a sample using random number generator

Use a random number generator on a spreadsheet to select a sample for the following surveys.

a. Randomly select 4 days from an entire year.

b. Randomly select 6 months that fall between January 2022 and December 2023

THINK	WRITE
a. 1. Use the Julian date system, which associates each day of the year with the numbers 1 to 365.	**a.**
2. Click on a cell. Type =RANDBETWEEN(1, 365). Click on the bottom right corner of the cell. Fill down by dragging your cursor down four cells. *Note:* If any numbers are repeated, fill down as required to generate another number to replace the repeated number.	
3. Use the Julian calendar to convert each number to a calendar date.	Day 206 is 25 July Day 188 is 7 July Day 214 is 2 August Day 114 is 24 April
b. 1. Convert the months to numbers by creating a numbered list.	**b.** Label the months numerically as: 1. January 2022 2. February 2022 . . . 23. November 2023 24. December 2023.
2. Click on a cell. Type =RANDBETWEEN (1, 24) Press ENTER. Click on the bottom right corner of the cell. Fill down by dragging your cursor down six cells. *Note:* If any numbers are repeated, fill down an extra cell to replace the repeated numbers.	
3. Convert the numbers back to months to write the answer.	15 is March 2023 14 is February 2023 6 is June 2022 21 is September 2023 22 is October 2023 12 is December 2022

Spreadsheet (a): fx =RANDBETWEEN(1,365)

	A	B
1	206	
2	188	
3	214	
4	114	
5		
6		

Spreadsheet (b): fx =RANDBETWEEN(1,24)

	A	B
3	15	
4	14	
5	6	
6	21	
7	22	
8	12	
9		

TI | THINK

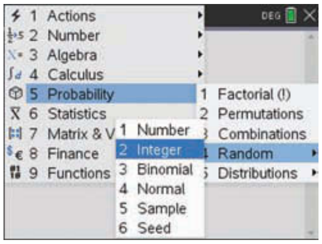
a. On a Calculator page, press:
- Menu
- 5. Probability
- 4. Random
- 2. Integers

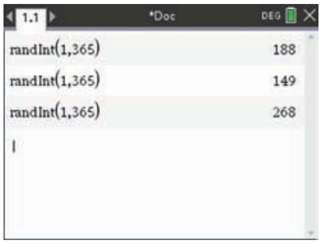
Complete the entry lines as:
randInt(1,365)
Press Enter.

Random numbers are generated every time the entry line is completed.

DISPLAY/WRITE

a.



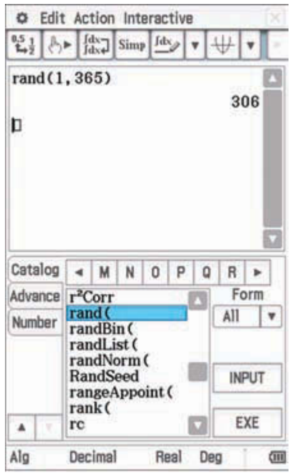


CASIO | THINK

a. A random number in a sample of 365 is generated by:
- selecting **rand(** from the Catalog tab in the Keyboard
- typing 1, 365 to indicate the lowest number and highest number
- pressing **EXE.**

DISPLAY/WRITE

a.



Systematic sampling

- **Systematic sampling** picks people at regular intervals from a population.
- Then select every nth person, where n = *population size ÷ sample size.*
- For example, a population of 30 students and a sample of 5:
  - Divide: $30 \div 5 = 6$ → choose every 6th student.
  - Pick a random number from 1 to 6 to start.
  - Continue selecting every 6th student from that point.

WORKED EXAMPLE 5 Determining sample using systematic sampling

Using a systematic random sampling method, a researcher wants to choose 8 people from a population of 40 people. Determine how the researcher would go about this.

THINK	WRITE
1. Determine the interval of people to sample.	$\dfrac{40}{8} = 5$ The researcher should choose every 5th person. The starting point should be between the 1st and 5th person (inclusive). Lets choose the 5th person as the starting number.
2. Determine which people would be sampled.	The sample will be the 5th, 10th, 15th, 20th, 25th, 30th, 35th, 40th person

Stratified sampling

- **Stratified sampling** is a method used to ensure that different subgroups (or “strata”) within a population are fairly represented in a sample.
- Stratified sampling works by:
  - Dividing the population into strata (groups based on shared traits eg. Age, gender, income level)
  - Determining how many to sample from each group (Usually proportional to each group’s size in the population)
  - Randomly select individuals from each group to ensure fair representation.

Calculating sample size for subgroups

$$\text{sample size for each subgroup} = \frac{\text{sample size}}{\text{population size}} \times \text{subgroup size}$$

WORKED EXAMPLE 6 Determining sample size using stratified sampling

A school wants to survey their Year 10, 11 and 12 students regarding the lunch options at their school canteen. There are:

- **120 students in Year 10**
- **80 students in Year 11**
- **50 students in Year 12**

If they want a stratified sample of 25, determine how many students from each group are to be surveyed.

THINK	WRITE
1. State the formula for determining the sample size.	$\text{sample size for each group} = \dfrac{\text{sample size}}{\text{population size}} \times \text{subgroup size}$
2. Calculate the sample size for each subgroup.	$\text{sample size of Year 10} = \dfrac{25}{250} \times 120 = 12$ $\text{sample size of Year 11} = \dfrac{25}{250} \times 80 = 8$ $\text{sample size of Year 12} = \dfrac{25}{250} \times 50 = 5$
3. Write the answer.	The sample should contain 12 Year 10 students, 8 Year 11 students, and 5 Year 12 students.

Clustered sampling

- Cluster sampling is a method used when a population is naturally divided into groups (or "clusters"), and it's easier or more practical to sample entire groups rather than individuals.
- For example, A city has 100 schools, and you want to survey students. To survey them:
  - Divide the schools into clusters (each school is a cluster).
  - Randomly select 10 schools.
  - Survey all students in those 10 schools
- This method is useful when the population is spread out geographically and is costly/difficult to sample individuals directly.

Capture/recapture sampling

- Capture/recapture sampling is a method used to estimate the size of a population, especially when it's difficult to count every individual (like animals in the wild, bacteria in a sample, estimating homeless populations).
- The population can be estimated by the following formula.

Capture/recapture population estimation

$$\text{Estimated population} = \frac{\text{Number in first sample} \times \text{Number in second sample}}{\text{Number of tagged individuals in second sample}}$$

WORKED EXAMPLE 7 Estimating populations using capture/recapture sampling.

A fish farm is trying to estimate the population of one of their ponds using the capture/recapture method. In the first capture, 40 fish were caught, tagged and realised. In the second capture, 50 fish were caught, 10 of them tagged from the first capture.
Estimate the population of the pond.

THINK	WRITE
1. State the formula for determining the estimated population.	$\text{Estimated population} = \dfrac{\text{Number in first sample} \times \text{Number in second sample}}{\text{Number of tagged individuals in second sample.}}$
2. Substitute the values and estimate the population.	$= \dfrac{40 \times 50}{10}$ $= 200$
3. Write your answer.	The estimated population of the pond is 200 fish.

Convenience sampling

- Convenience sampling is a non-random sampling method where individuals are selected based on how easy they are to reach or access.
- For example, a dance instructor wanted quick feedback on a lesson by asking the first 5 students who walk into the classroom.

COLLABORATIVE TASK: Designing a survey

1. Working in pairs, think of a topic on which you would like to gather your year groups' opinions. Some suggestions for things to ask them about include pets, favourite movies, favourite social platform or school subjects.
2. Think of five questions you would like to ask your classmates about this topic, ensuring that you collect both categorical and numerical data.
3. When coming up with your questions, you should consider what sorts of answers you want. Do you want yes and no answers, or would you like questions that require people to make a judgement on a scale? Your questions should be easy to understand and must be relevant to your topic.
4. Write down how you intend to collect the data, the type of data you expect to collect, and how you think it is best to represent this data. Think about the possible responses you may get. It is important to consider the methods used to collect data, because they could possibly bias the results of the analysis, which would lead to incorrect conclusions.
5. As a class, listen to and offer constructive advice on the questions that each pair has developed. Your feedback might include comments about the type of responses you might expect, how you could collect the data, whether the questions make sense (or if they could give a biased result), and whether they are relevant to the topic.

6. Once you have received your feedback, rewrite some of your questions and rethink the way you will collect the data.
7. Collect your data and keep it ready to use again during this topic.

Exercise 12.2 Collecting data

learnon

12.2 Quick quiz on | 12.2 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 7, 8, 10, 14, 17, 20, 23	2, 5, 11, 12, 15, 16, 18, 21, 24, 25	3, 6, 9, 13, 19, 22, 26, 27, 28

Fluency

1. WE1 Classify each of the following data sets according to whether they are numerical or categorical data.
   a. Favourite TV programs of each student in your year level.
   b. Shoe size of the top ten international models.
2. Classify each of the following data sets according to whether they are numerical or categorical data.
   a. Hours of games played at home each week by each student in your class.
   b. Birth weight of each baby in a mothers' group.
3. WE2 Classify each of the following numerical data sets according to whether they are discrete or continuous.
   a. Length of a stride of each golfer in a local golf club.
   b. Distance travelled on one full tank from the top ten most popular vehicles.
4. Classify each of the following numerical data sets according to whether they are discrete or continuous.
   a. Rainfall recorded for each day during spring.
   b. Cars owned by each family of the students in your class.
5. Classify each of the following categorical data sets as nominal or ordinal.
   a. Satisfaction levels of customers (Satisfied, Indifferent, Dissatisfied).
   b. Favourite magazine of each student in your year level.
6. Classify each of the following categorical data sets as nominal or ordinal.
   a. Sales ranking of the top ten magazines in Australia.
   b. Age group of the readers of the top ten magazines in Australia.
7. Classify each of the following categorical data sets as nominal or ordinal.
   a. Types of fish caught while fishing off a pier.
   b. Genres of movies on a streaming service.
8. WE3 Classify each of the following data collections according to whether they are samples or populations.
   a. Favourite TV programs of secondary school students, determined by surveying students in Year 8.
   b. Size of shoes to be stocked in a department store, determined by the shoe sizes of the top ten international models.
   c. Number of hours your classmates spend playing games at home, determined by surveying each student in your class.

9. Classify each of the following data collections according to whether they are samples or populations.
   a. Survey of first 40 customers at the opening of a new store to gauge customer satisfaction.
   b. The most popular duco colours for cars sold in the last month in Western Australia taken from the registration board database of all 2310 cars registered last month.

10. MC A researcher wants to survey students from different year levels in a school. She divides the students into Year 7, Year 8, and Year 9 groups, then randomly selects 10 students from each group.
This type of sampling is called:
   A. Cluster sampling
   B. Stratified sampling
   C. Systematic sampling
   D. Convenience sampling
11. MC A student conducting a survey for a class project interviews people sitting near them in the library.
This type of sampling is called:
   A. Stratified sampling
   B. Convenience sampling
   C. Systematic sampling
   D. Capture/recapture
12. MC Marine biologists tag 100 dolphins and release them. A week later, they capture 80 dolphins and find that 20 are tagged. They use this data to estimate the total dolphin population.
This type of sampling is called:
   A. Cluster sampling
   B. Systematic sampling
   C. Capture/recapture
   D. Stratified sampling
13. MC A health department wants to survey households in a city. They randomly select 10 neighbourhoods and then survey every household within those neighbourhoods.
This type of sampling is called:
   A. Cluster sampling
   B. Stratified sampling
   C. Systematic sampling
   D. Convenience sampling

Understanding

14. WE5 Using a systematic random sampling method, a company wants to survey 10 people from their workforce of 400 people. Determine how the researcher would go about this.
15. WE6 A local council wants to survey residents from three suburbs about public transport. There are:
   - 240 residents in Suburb A
   - 160 residents in Suburb B
   - 100 residents in Suburb C

   If they want a stratified sample of 50, determine how many residents from each suburb should be surveyed.
16. WE7 A wildlife reserve is trying to estimate the population of kangaroos in a fenced area using the capture/recapture method. In the first capture, 60 kangaroos were caught, tagged, and released. In the second capture, 80 kangaroos were caught, and 20 of them were already tagged. Estimate the total kangaroo population in the area.
17. Identify the missing word from each of the following sentences.
   a. When writing questions for a survey, it is important to consider which type of ____________ will be useful for you to collect.
   b. If you __________ an entire population, this data can be used for the purpose of a ____________.
   c. If the data you collect is grouped numerical data, it is best to use a __________ to visually represent the data.

18. WE4 Complete the following.

a. Use a random number generator to select six days from the month of March.
b. Use a random number generator to select five students from your mathematics class.
c. State whether your selections from **a** and **b** are samples or populations.

19. a. Randomly select five days of a non-leap year.
b. State whether this data is a sample or population.

Reasoning

20. The following are 2020 statistics for motorcycle and car fatalities in Australia.

Motorcycle fatalities: 190
Car fatalities: 732
Registered motorcycles: 880 881
Registered cars: 14 679 249
Kilometres travelled by motorcycles (millions): 1683
Kilometres travelled by cars (millions): 162 983
Population of Australia:25 693 059

a. State whether motorcycles or cars are the safer mode of transport, according to the data. Explain your answer.
b. Calculate the ratios or percentages that could be extracted from this data to reinforce your conclusion.
c. State the information that you would include if you wanted to argue for the safety of riding a motorcycle. State the information that you would leave out in order to make this argument.
d. Write a sentence to convince your readers that a motorcycle is a safer mode of transport.
e. Write a sentence to convince your readers that a car is a safer mode of transport.

21. Explain how data is classified into different types, giving examples for each type.

22. Identify and justify two sets of information a person might want to collect from each of the following sources.

a. A maternity hospital.
b. An airport.
c. Shoppers at a supermarket.

Problem solving

23. Questions are being written for a survey about students' views on sports. Select the questions from the following list that you would use.
Explain your choices.

a. What do you think about sport?
b. List these sports in order from your most favourite to least favourite: football; tennis; netball; golf.
c. How often do you watch sports on television?
Choose from these options: once a week; once a day; once a month; once a year; never.
d. Why do you like sports?
e. How many live sporting events do you attend each year?
f. Can you play a sport?
g. Which sports do you watch on the television?
Choose from these options: football; tennis; netball; golf; all of these; none of these.

24. Businesses collect and analyse all sorts of data.

a. Suggest some types of data that businesses might want to collect.
b. Suggest some ways in which businesses might collect data.
c. Provide some examples of how businesses might use the data that they collect.

25. The local government wishes to survey people about expanding a large windfarm in the South West region of Western Australia.

a. Explain if a census or survey would be most appropriate.
b. Explain how survey respondents should be selected to participate.

26. A marine biologist estimates that there are 360 fish in a reef using the capture/recapture method. In the second capture, 90 fish were caught, and 18 of them were tagged from the first capture.
Based on this information, determine how many fish were tagged in the first capture.

27. A company interested in constructing a desalination plant south of Perth in Mandurah wishes to gather some data on public opinion about the presence of the desalination plant in the area and its possible effects on the environment.
The company has decided to conduct a survey and has asked you to plan and implement the survey and then analyse the data and interpret the results.
   a. Suggest three questions you could include in the survey.
   b. Explain what you are trying to find out by asking these questions.
   c. State the types of data you would collect. (Choose at least one categorical and one numerical variable.)
   d. Suggest what the company could use this data for.
   e. The company conducts the survey on residents of Mandurah.
   Discuss the appropriateness of this sample.

28. A large supermarket surveyed shoppers in their store between 7 pm and 9 pm each day to ask their thoughts on the ban on single-use plastics and found that most shoppers did not support the ban.
Discuss the reasonableness of this inference.

LESSON 12.3 Organising data

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- organise data in frequency tables
- display data in dot plots
- display data in stem-and-leaf plots.

12.3.1 Frequency tables

- Data needs to be recorded and organised so that graphs and tables can be drawn and analysis conducted.
- A **frequency table** can be used to record and organise data.
- Data in frequency tables can be numerical or categorical data, and can even include two sets of data.
- A frequency table looks like this:

The tally is recorded from the data.

This is the value/category.

Person's age	Tally	Number of people (f)
0–29	𝍸 |	5
30–59	𝍸 𝍸	10
60–89	|||	3
		$\Sigma f = 18$

This is the frequency. It is the total number given by the tally. It is labelled f.

This is the sum of values in this column.
Note: Σf means 'sum of f'.

WORKED EXAMPLE 8 Drawing frequency tables with values

The following is a result of a survey of 30 students showing the number of people in their family living at home.
Draw a frequency table to summarise the data.

2, 4, 5, 6, 4, 3, 4, 3, 2, 5, 6, 6, 5, 4, 5,
3, 3, 4, 5, 8, 7, 4, 3, 4, 2, 5, 4, 6, 4, 5

THINK

1. Identify the data as numerical discrete data.
2. There are only seven different scores so the data can be left ungrouped.
3. Set up a frequency table. The first column is for the possible scores and can be labelled 'Number of members in the family'.
The last column is for the frequency (f) and can be labelled 'Number of families'. An optional column between these two can be used to keep a tally for each category.
4. Add up the frequency column $\left(\sum f\right)$ to confirm that 30 results have been recorded.

WRITE/DRAW

Number of members in the family	Tally	Number of families (f)
2	|||	3
3	~~||||~~	5
4	~~||||~~ ||||	9
5	~~||||~~ ||	7
6	||||	4
7	|	1
8	|	1
		$\sum f = 30$

WORKED EXAMPLE 9 Drawing a frequency table with categories

The final scores for 30 golfers were recorded as follows. Summarise the data as a frequency table, grouping the scores in intervals of 2, starting at 62.

73, 69, 75, 79, 68, 85, 78, 76, 72, 73,
71, 70, 81, 73, 74, 78, 68, 75, 76, 72,
63, 72, 74, 71, 70, 75, 66, 82, 87, 78

THINK

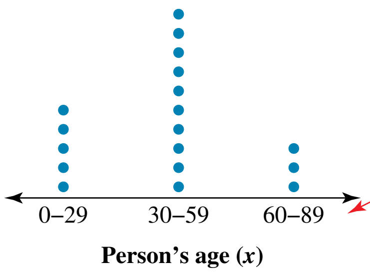
1. Identify the data as numerical discrete data.
2. The scores range from 63 to 87. This is a range of 24 shots.
3. For this example the data, starting at 62, will be grouped in intervals of 2.
4. Add up the frequency column to check that all 30 scores have been recorded.

WRITE/DRAW

Golf scores (x)	Tally	Number of players (f)
62–63	$\vert$	1
64–65		0
66–67	$\vert$	1
68–69	$\vert\vert\vert$	3
70–71	$\vert\vert\vert\vert$	4
72–73	$\cancel{\vert\vert\vert\vert}\ \vert$	6
74–75	$\cancel{\vert\vert\vert\vert}$	5
76–77	$\vert\vert$	2
78–79	$\vert\vert\vert$	4
80–81	$\vert$	1
82–83	$\vert$	1
84–85	$\vert$	1
86–87	$\vert$	1
		$\sum f = 30$

12.3.2 Dot plots

- A **dot plot** is another way of displaying data.
- Data in a dot plot can be categorical or numerical.
- A dot plot looks like this:



Person's age (x)	Tally	Number of people (f)
0–29	$\cancel{\vert\vert\vert\vert}\ \vert$	5
30–59	$\cancel{\vert\vert\vert\vert}\quad\cancel{\vert\vert\vert\vert}$	10
60–89	$\vert\vert\vert$	3
		$\Sigma f = 18$

The x column categories are listed along the base of the plot.

The f column values indicate the number of dots.

WORKED EXAMPLE 10 Drawing a dot plot

A survey was conducted, asking students about the type of vehicle driven by their parents. Their responses were collected and recorded as shown.
Display the data as a frequency table and as a dot plot.

Sedan	**4WD**	**Sedan**	**Sedan**
4WD	**SUV**	**SUV**	**Station wagon**
SUV	**Convertible**	**Station wagon**	**4WD**
Sedan	**Sports car**	**Convertible**	**Station wagon**
Sedan	**Station wagon**	**SUV**	**Station wagon**
SUV	**SUV**	**Sedan**	**Sedan**

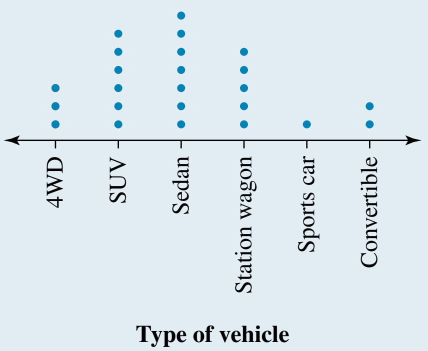
THINK

WRITE/DRAW

1. Identify this data as categorical ordinal data.
2. There are six different scores (or categorical labels).
3. Use a frequency table of two (or three) columns.
   The first column is for the possible scores (x) and can be labelled as 'Type of vehicle'.
   The last column is for the frequency (f) and can be labelled 'Number of vehicles'.
   An optional middle column can be used to keep a tally for each category.

Type of vehicle (x)	Tally	Number of vehicles (f)
4WD	|||	3
SUV	~~||||~~ |	6
Sedan	~~||||~~ ||	7
Station wagon	~~||||~~	5
Sports car	|	1
Convertible	||	2
		$\sum f = 24$

4. Add up the frequency column $\left(\sum f\right)$ to confirm that 24 scores have been recorded.
5. Place a dot for each vehicle type in the appropriate column.



12.3.3 Stem-and-leaf plots

- A **stem-and-leaf plot**, or stem plot, is a graph that displays numerical data.
- Stem plots have two columns:
  - the stem shows the leading part of the value
  - the leaf shows the trailing part of the value.

- The following stem plot shows the sizes of 20 students' book collections.

Book collection size for 20 students

Key: 2 | 5 = 25 books

Stem	*Leaf*
0	4
1	4 6
2	2 3 5 8
3	0 4 4 6 7 7 9
4	1 3 5 9
5	4 6

- From the stem plot shown, the sizes collected were 4, 14, 16, 22, 23, 25, 28, 30, 34, 34, 36, 37, 37, 39, 41, 43, 45, 49, 54, 56.

WORKED EXAMPLE 11 Summarising data in a stem plot

The following scores represent the distance (in km) travelled by a group of people over a particular weekend.
Summarise the data in a stem plot.

48, 67, 87, 2, 34, 105, 34, 45, 63, 98, 12, 23, 35,
54, 65, 41, 34, 23, 12, 38, 18, 58, 53, 44, 39, 29

THINK

1. Write a title and key and set up a stem plot.
2. The scores range from 2 to 105. Grouping them in ranges of 10 will give 11 categories.
   The stem will hold the tens place value and the leaf will hold the unit/ones place value (the last digit) for each score.
3. Transfer each score to the stem plot by recording its last digit in the leaf column alongside the row of the stem column that corresponds to its tens place digit.
4. Rearrange each row in the leaf column so that the digits are in ascending order from left to right.

WRITE/DRAW

Distance travelled over one weekend

Key: 4 | 7 = 47 km

Stem	*Leaf*
0	2
1	2 2 8
2	3 3 9
3	0 4 4 5 8 9
4	1 4 5 8
5	3 4 8
6	3 5 7
7	
8	7
9	8
10	5

- Sometimes the leaves of a stem plot can become too long. This can be fixed by breaking the stems into smaller intervals. The second part of the interval is marked with an asterisk (*).

WORKED EXAMPLE 12 Representing data in a stem plot with smaller stem intervals

The heights of 30 students are measured (in cm) and recorded as follows.

125, 143, 119, 136, 127, 131, 139, 122, 140, 118,
120, 123, 132, 134, 127, 129, 124, 131, 138, 133,
122, 128, 130, 135, 141, 139, 121, 138, 131, 126

Represent the data in a stem plot.

THINK	WRITE/DRAW
1. Write a title and key, then draw up the stem plot with the numbers in each row of the leaf column arranged in ascending order from the stem.	Heights of 30 students (in cm) Key: 11 | 8 = 118 cm
2. The leaves of the two middle stem values are long. They would be easier to interpret with the stem broken into smaller intervals, for example intervals of 5. The stem of 12 would then include the numbers from 120 to 124 while the stem of 12* would include the numbers from 125 to 129. Reconstruct the stem plot according to this new set of intervals.	Heights of 30 students (in cm) Key: 12* | 5 = 125 cm

Step 1 stem plot:

Stem	*Leaf*
11	8 9
12	0 1 2 2 3 4 5 6 7 7 8 9
13	0 1 1 1 2 3 4 5 6 8 8 9 9
14	0 1 3

Step 2 stem plot:

Stem	*Leaf*
11	8 9
12	0 1 2 2 3 4
12*	5 6 7 7 8 9
13	0 1 1 1 2 3 4
13*	5 6 8 8 9 9
14	0 1 3

Exercise 12.3 Organising data

learnon

12.3 Quick quiz on | 12.3 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 4, 8, 9, 12, 15	2, 5, 6, 10, 13, 16	3, 7, 11, 14, 17, 18

Fluency

1. WE8&10 The following set of data represents the scores achieved in a multiple choice Maths test.

   4, 6, 8, 3, 6, 9, 1, 3, 5, 6, 4, 7, 5, 9, 3, 2, 7, 8, 9, 6, 5, 4, 6, 5, 3, 5, 7, 6, 7, 7

   a. Present the data in a frequency table.
   b. Present the data as a dot plot.

2. The following set of data shows the types of pets kept by 18 people.

cat, dog, fish, cat, cat, cat, dog, dog, bird, turtle, dog, cat, turtle, cat, dog, dog, turtle, bird

a. Present the data in a frequency table.
b. Present the data as a dot plot.
c. Explain whether this data can be presented in a stem plot.

3. a. WE9 Present the data shown in the table as a frequency table by grouping the data in intervals of 10 (0–9, 10–19, ...).

b. WE11 Present the data as a stem plot.

	Death by motor vehicle accidents: Males aged 15–24 years (rates per 100 000 people)			
State	**2021**	**2022**	**2023**	**2024**
NSW	14	13	18	9
Vic.	19	25	14	12
Qld	25	17	26	15
SA	23	34	22	26
WA	22	24	36	27
Tas.	42	18	36	42
NT	43	60	71	45
ACT	0	11	14	21

4. A sample of students surveyed by the Australian Bureau of Statistics were asked the colour of their eyes. The results were as follows:

Brown	Hazel	Blue	Brown	Brown	Green	Green	Brown	Brown	Hazel
Brown	Brown	Brown	Brown	Blue	Brown	Green	Blue	Blue	Blue
Brown	Brown	Brown	Brown	Hazel	Blue	Hazel	Blue	Brown	Blue
Brown	Blue	Green	Hazel	Hazel	Hazel	Green	Blue	Blue	Brown

a. Set up a frequency table for the data.
b. Construct a dot plot for the data.
c. Comment on the proportion of people who have brown eyes.

5. Draw a stem-and-leaf plot for each of the following sets of data. Comment on each distribution.

a. 18, 22, 20, 19, 20, 21, 19, 20, 21
b. 24, 19, 31, 43, 20, 36, 26, 19, 27, 24, 31, 42, 29, 25, 38
c. 346, 353, 349, 368, 371, 336, 346, 350, 359, 362

6. WE12 The test scores of 30 students in a mathematics exam are recorded as follows:
125, 143, 119, 136, 127, 131, 139, 122, 140, 118,
120, 123, 132, 134, 127, 129, 124, 131, 138, 133,
122, 128, 130, 135, 141, 139, 121, 138, 131, 126
Represent the data in a stem-and-leaf plot using intervals of 5.

7. The following times (to the nearest minute) were achieved by 40 students during a running race.

23	45	25	48	21	56	33	34	63	43	42	41	26	44
45	41	40	39	37	53	26	55	48	39	29	52	57	33
31	32	71	60	49	52	32	28	47	42	37	33		

a. Construct a stem plot with intervals of 5 minutes.
b. Construct a frequency table. Use intervals of 5 minutes.

c. If the top 10 runners were chosen for the representative team, identify the qualifying time as given by:
   i. the stem plot
   ii. the frequency table.

d. Identify the number of times that the finish time of 33 minutes was recorded. Explain which summary (stem plot or frequency table) was able to give this information.

8. The ages of participants in a weights class at a gym are listed below:

17, 21, 36, 38, 23, 45, 32, 53, 18, 25, 14, 29, 42, 26, 18, 27, 37, 19, 34, 20, 35

Display the data as a stem plot using intervals of 5.

9. **MC** Consider the stem plot shown.

Key 7|6 = 7.6

Stem	*Leaf*
7	8
8	0 8 9
9	1 6 7 8
10	3 5 8
11	2

Identify which of the following data sets matches the stem plot.

A. 78, 80, 88, 89, 91, 96, 97, 98, 103, 105, 108, 112
B. 8, 0, 8, 9, 1, 6, 7, 8, 3, 5, 8, 2
C. 7.8, 8.0, 8.8, 8.9, 9.1, 9.6, 9.7, 9.8, 1.03, 1.05, 1.08, 1.12
D. 7.8, 8.0, 8.8, 8.9, 9.1, 9.6, 9.7, 9.8, 10.3, 10.5, 10.8, 11.2

Understanding

10. The following data represent the life expectancy (in years) of Australians from 40 different age groups.

83.7	84.5	84.7	85.2	84.6	84.5	85.9	84.9	88.3	86.5
84.1	84.5	84.8	84.4	84.7	84.5	86.0	85.0	88.3	86.6
84.1	84.5	84.8	84.4	84.7	85.7	87.3	85.0	88.8	86.6
84.1	84.5	84.9	84.5	84.7	85.8	87.6	85.1	88.8	86.9

a. Present the data as a frequency table.
b. Present the data as a stem plot.

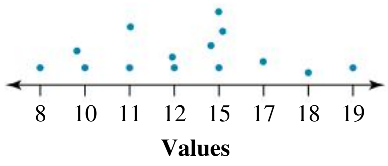
11. Choose the most appropriate technique for summarising the data in the table. Justify your choice.

Median age of Australian females and males from 2017 to 2024		
Year	**Females**	**Males**
2017	39.6	38.2
2018	39.9	38.3
2019	40.1	38.5
2020	40.3	38.7
2021	40.4	38.9
2022	40.6	39.1
2023	40.6	39.1
2024	40.7	39.1

Reasoning

12. The dot plot shown is a Year 9 student's attempt to display the values:

8, 10, 10, 11, 11, 12, 12, 15, 15, 15, 15, 17, 18, 19



a. List the mistakes that the student made in drawing the dot plot.
b. Redraw the dot plot so that it correctly displays all of the data given.

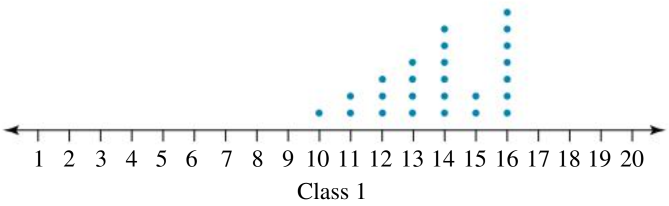
13. a. Sometimes you are asked to represent two sets of data on separate stem-and-leaf plots. Suggest how you could represent two sets of data on a single stem plot while keeping them separate.
Demonstrate your method using the data sets below that show the heights (in cm) of 12 boys and 12 girls in a Year 8 class:

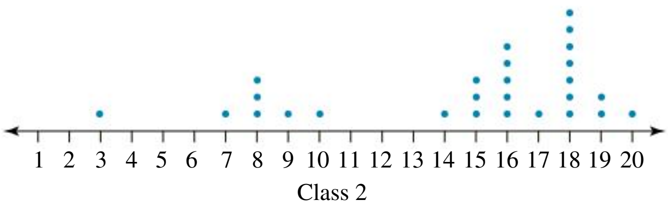
Boys	154	156	156	158	159	160	162	164	168	171	172	180
Girls	145	148	152	152	155	156	160	161	161	163	165	168

b. Consider the stem-and-leaf plot below. Explain how you could make this plot easier to read and use.

Stem	*Leaf*
1	0 2 2 2 5 7 8 9
2	0 0 0 0 0 1 1 2 2 2 2 2 4 4 7 7 7 7 8 8 8 8 9 9
3	0 2 2 3 3 3 5 5 7 8 9

14. The dot plots below display Maths test results for two Year 9 classes. The results show the marks out of 20.





a. Determine the number of students in each class.
b. For each class, determine the number of students who scored 15 out of 20 on the test.
c. For each class, determine the number of students who scored more than 10 on the test.
d. Use the dot plots to describe the performance of each class on the test.

Problem solving

15. The following data shows the speeds of 30 cars recorded by a roadside camera outside a school, where the speed limit is 40 km/h.

20, 27, 30, 36, 45, 39, 15, 22, 29, 30, 30, 38, 40, 40, 40,
40, 42, 44, 20, 45, 29, 30, 34, 37, 40, 45, 60, 38, 35, 32

a. Present the data as an ordered stem-and-leaf plot.
b. Write a paragraph for the school newsletter about the speed of cars outside the school.

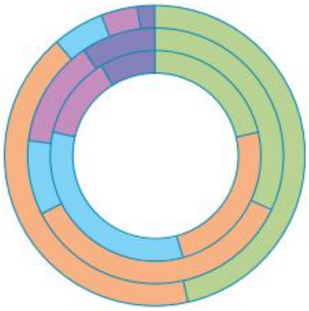
16. A Year 9 Science class recorderd the masses of mice (measured in grams) as shown below.

25.1, 24.8, 25.1, 27.3, 25.3, 29.5, 24.5, 26.7, 24.0, 26.3, 25.4,
26.3, 23.9, 25.8, 25.4, 24.6, 25.1, 23.9, 33.2, 24.5, 28.1, 27.3

a. Draw an ordered stem-and-leaf plot for this data.
b. Calculate the range of this data.
c. Draw an ordered stem-and-leaf plot for the masses of the mice rounded to the nearest gram. Discuss whether a stem-and-leaf plot is suitable for the rounded data.
Compare the stem-and-leaf plot with the original data. Explain your conclusions.

17. A doughnut graph can be used to compare two or more sets of data using the basic principles of a pie graph. A doughnut graph comparing students' methods of travel to school on Monday, Tuesday and Wednesday is shown.

Method of travel to school
Monday (inner ring)
Tuesday (middle ring)
Wednesday (outer ring)



a. List the similarities and differences between a doughnut graph and a pie graph.
b. Write a series of instructions for drawing a doughnut graph.
c. Test your instructions on the following data by creating a doughnut graph to represent it.

Method of travel	Monday	Tuesday	Wednesday
Walking	5	7	25
Cycling	6	8	23
Car	8	2	3
Bus	3	3	2
Train	2	2	1

d. Compare your doughnut chart with the one shown.
e. Suggest some advantages and disadvantages of using a doughnut graph instead of a multiple column graph to compare two or more sets of data.

18. Analysis of written English prose reveals the proportion of vowels and consonants to be roughly the same in every large sample — approximately 7:11. The letter *e* accounts for one-third of the occurrence of vowels. In a passage consisting of 10 000 letters, determine approximately how many times you would expect the letter *e* to occur.

LESSON 12.4 Displaying data

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- display data in bar charts, pie charts, histograms and frequency polygons
- present data as back-to-back stem-and-leaf plots.

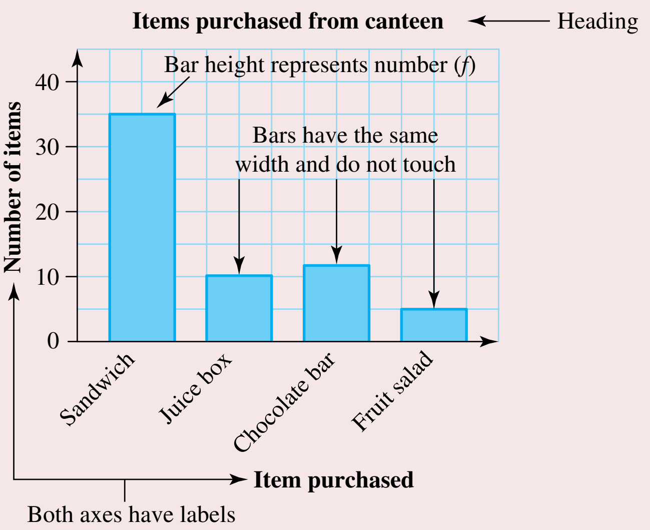
12.4.1 Bar charts and pie charts

- **Bar charts** and **pie charts** are used to display categorical data.

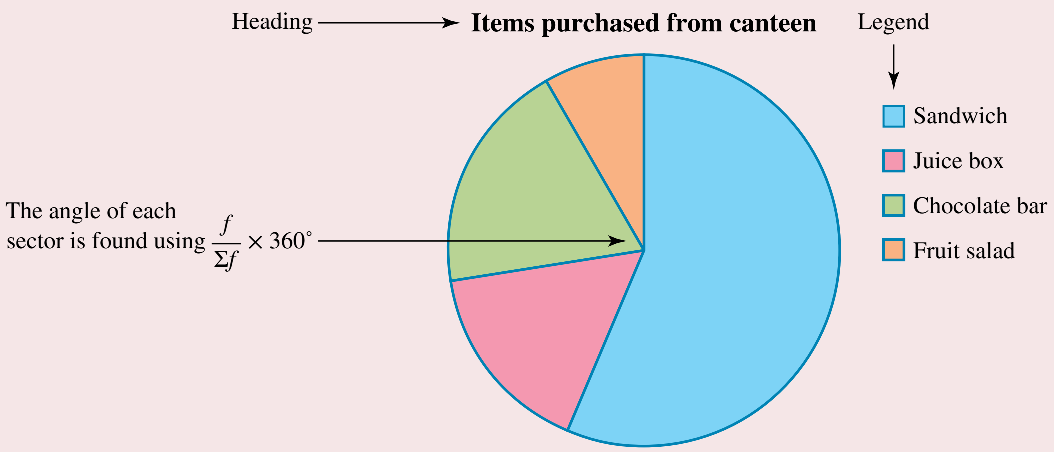
Displaying data in bar charts and pie charts

- **A bar chart for the following data is shown as an example.**

Items purchased from canteen	
Item purchased	**Number (f)**
Sandwich	**35**
Juice box	**10**
Chocolate bar	**12**
Fruit salad	**5**



- **A pie chart for the same data is shown as an example.**



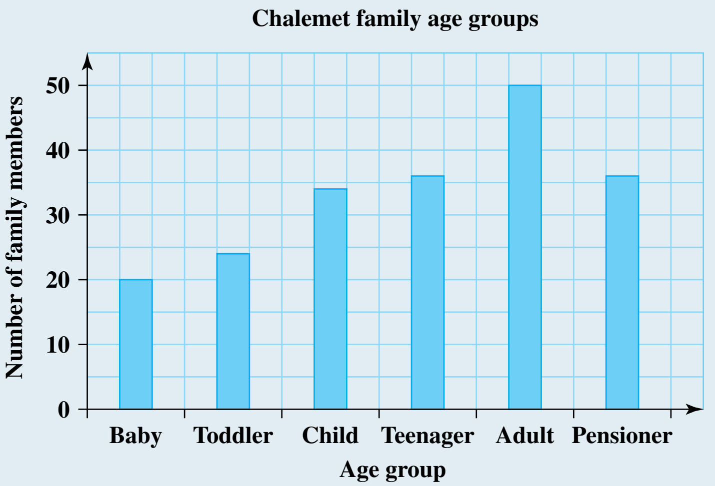
WORKED EXAMPLE 13 Displaying a frequency table as a bar chart and a pie chart

The Chalemet family is having a major reunion. The age groups of the family members who are attending have been recorded in the table below.

Age group	Number
Baby	20
Toddler	24
Child	34
Teenager	36
Adult	50
Pensioner	36

Display the frequency table as a:

a. **bar chart** **b.** **pie chart.**

THINK	WRITE/DRAW
a. 1. Give the bar chart a suitable title. **2.** Identify the category names and use them to label the horizontal axis. It is the convention to leave a space between each bar on a bar chart. **3.** Label the vertical axis with a suitable scale and title. **4.** Draw each bar so that its height matches the frequency of the category it represents.	**a.** 

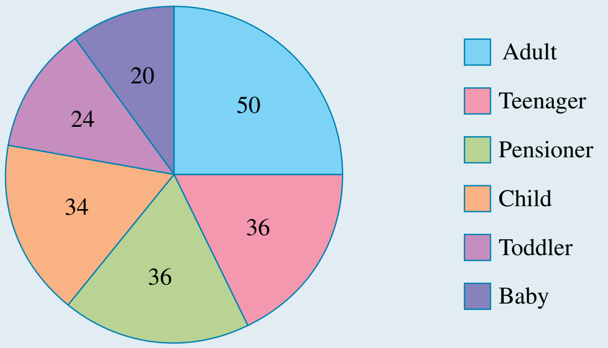
b. 1. Add one extra column to the frequency table to calculate the size of the sector. The size of each sector is determined by its angle. To calculate this angle, use the use the rule angle $= \dfrac{f}{\sum f} \times 360°$.

b.

Age group	Number (f)	Size of sector
Baby	20	$\frac{20}{200} \times 360° = 36°$
Toddler	24	$\frac{24}{200} \times 360° = 43.2° \approx 43°$
Child	34	$\frac{34}{200} \times 360° = 61.2° \approx 61°$
Teenager	36	$\frac{36}{200} \times 360° = 64.8° \approx 65°$
Adult	50	$\frac{50}{200} \times 360° = 90°$
Pensioner	36	$\frac{36}{200} \times 360° = 64.8° \approx 65°$
$\sum$	$\sum f = 200$	360°

2. Give the pie chart a suitable title.
3. Create a legend with a suitable colour for each category.
4. Draw a circle.
5. Using a protractor, measure and draw the correct angle of each sector.
6. Colour the sectors using the legend.

Chalemet family age groups

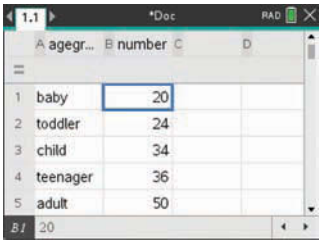


TI | THINK

a. 1. Open a Lists & Spreadsheet page. Label column A as agegroup and column B as number. Enter the data as shown in the table.

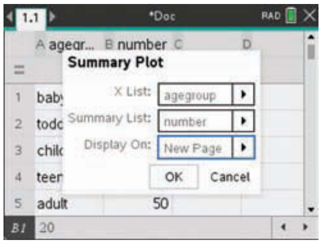
WRITE

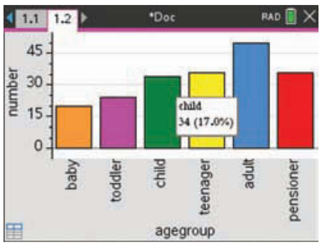
a.



2. To create a summary plot, press:
- MENU
- 3: Data
- 8: Summary Plot.

Set the X List (category) as agegroup and the Summary List as number. You have the option of displaying the summary plot in a split screen or in a new page. Choose New Page for Display On. Use TAB to move to OK. Press ENTER. Hover the pointer over each bar to see the percentage frequency.



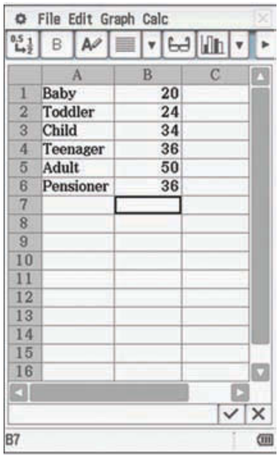


CASIO | THINK

a. 1. Open a new Spreadsheet screen. Enter the data into columns A and B.

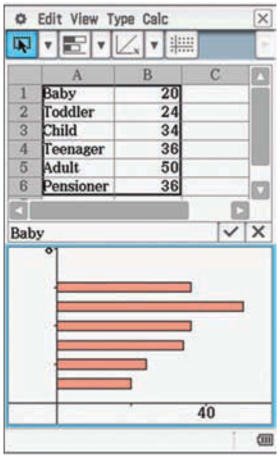
WRITE

a.



2. Highlight columns A and B. Press:
- Graph
- Bar
- Stacked.

Note: This is a frequency bar graph. To draw a percentage frequency bar graph, change the entered data to percentages.

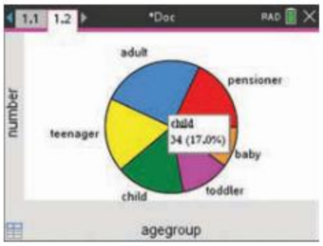


TI | THINK

b. To draw the pie chart, press:
- MENU
- 1: Plot Type
- 9: Pie Chart.

Hover the pointer over each bar to see the percentage frequency.

WRITE

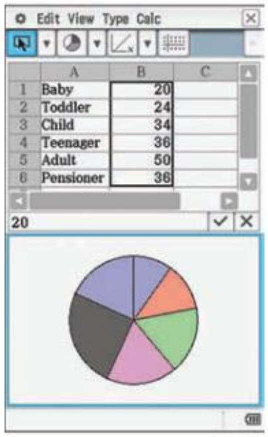
b. 

CASIO | THINK

b. To draw the pie chart, highlight the numerical data only. Then press:
- Graph
- Pie.

Hover the pointer over each sector to see the percentage frequency.

WRITE

b. 

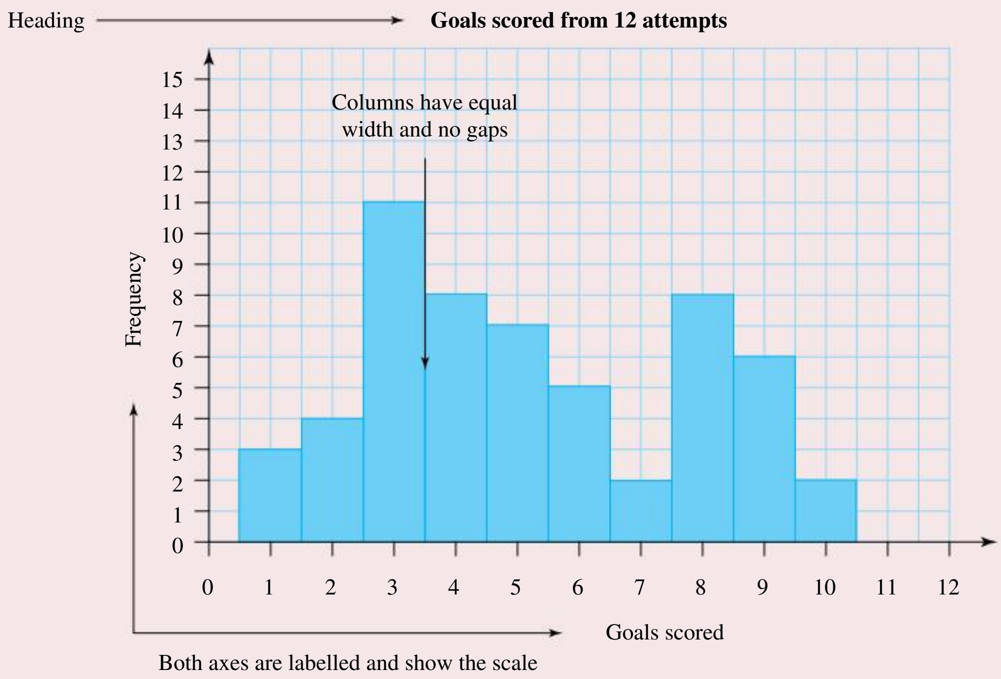
12.4.2 Histograms

- **Histograms** are used to display numerical data.
- They are used most commonly to display frequency data.
- The horizontal axis of a histogram can be labelled with:
  - values
  - intervals
  - mid points of intervals.

Displaying data in histograms

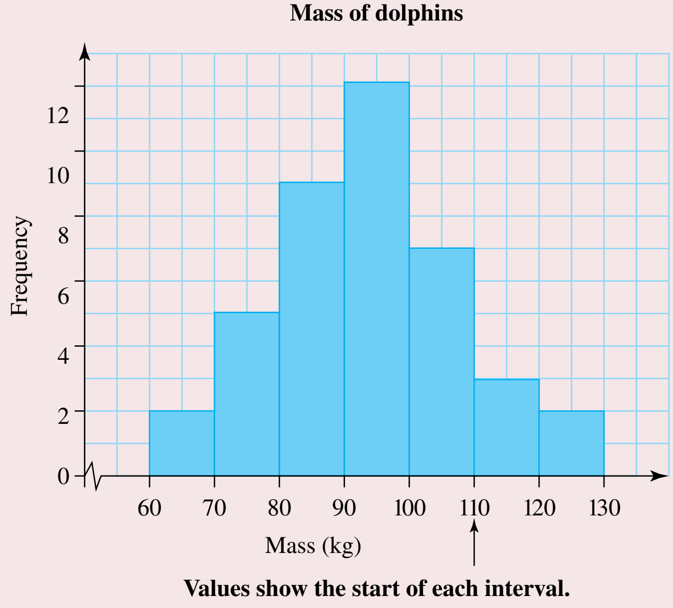
Histograms with values along the horizontal axis

- **The *x*-axis of this type of histogram shows the values or results of the data collected.**
- **The *y*-axis of this type of histogram shows the frequency of each result.**



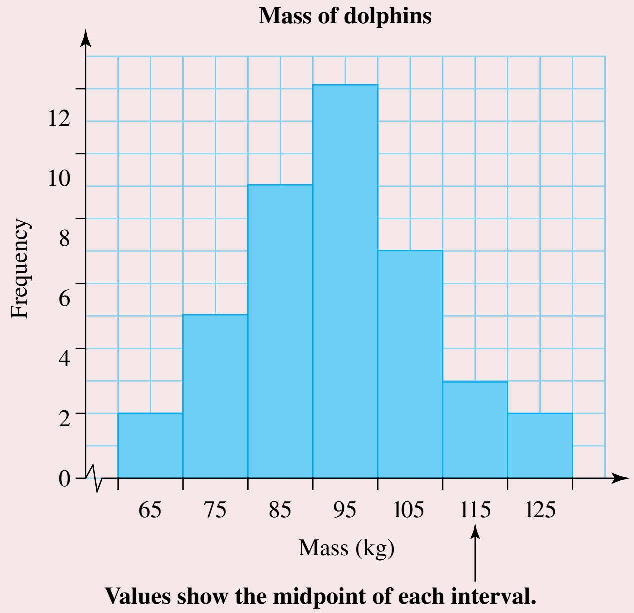
Histograms with intervals along the horizontal axis

- **The information along the horizontal axis are intervals (or groups) of values.**



Displaying data with midpoints along the horizontal axis

- **The midpoint of a class interval is calculated by taking the average of the two extreme values of that class interval.**



- **For example, the midpoint of the 60–70 class interval is $\frac{60 + 70}{2} = 65$.**

WORKED EXAMPLE 14 Constructing a histogram to represent data

A sample of 40 people was surveyed regarding the number of hours per week they spent on their phones. The results are listed below.

12.9,	**18.3,**	**9.9,**	**17.1,**	**20,**	**7.8,**	**24.2,**	**16.7,**	**9.1,**	**27,**
7.2,	**16,**	**26.5,**	**15,**	**7.4,**	**28,**	**11.4,**	**20,**	**9,**	**11,**
23.7,	**19.8,**	**29,**	**12.6,**	**19,**	**12.5,**	**16,**	**21,**	**8.3,**	**5,**
16.4,	**20.1,**	**17.5,**	**10,**	**24,**	**21,**	**5.9,**	**13.8,**	**29,**	**25**

a. **Calculate the range for the data.**
b. **Organise the data into five class intervals and create a frequency distribution table that displays class intervals, midpoints and frequencies.**
c. **Construct a histogram to represent the data, using midpoints.**

THINK

a. Determine the range of the number of hours spent on phones.

b. **1.** Determine the size of the class intervals. Class intervals have been recorded as 5–<10, 10–<15 and so on, to accommodate the continuous data.

2. Rule a table with three columns, headed 'Number of hours on phone each week' (class interval), 'Midpoint' (class centre) and 'Frequency'.

3. Tally the scores and enter the information into the frequency column.

4. Calculate the total of the frequency column.

c. **1.** Rule a set of axes on graph paper. Label the horizontal axis 'Number of hours on phone' and the vertical axis 'Frequency'.

2. Draw the first column so that it starts and finishes halfway between class intervals and reaches a vertical height of 9 people. Make sure you leave space between the y-axis and the first column.

3. Draw the columns for each of the other scores.

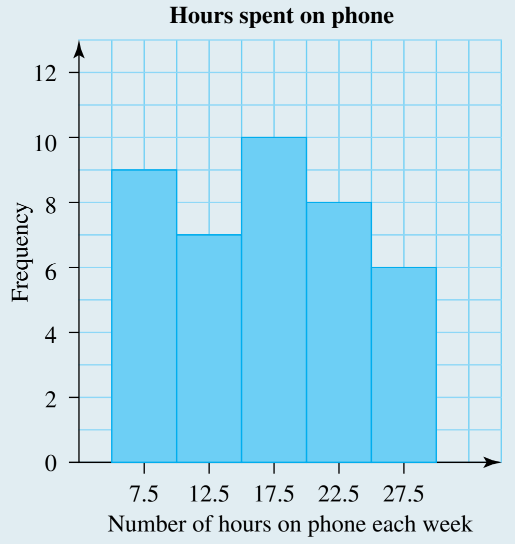
WRITE/DRAW

a. Range = largest value − smallest value
= 29 − 5
= 24

b. Class intervals of 5 hours will create 5 groups, and cover a range of 25 numbers.

Number of hours on phone each week	Midpoint	Frequency (f)
5– < 10	7.5	9
10– < 15	12.5	7
15– < 20	17.5	10
20– < 25	22.5	8
25– < 30	27.5	6
	Total	40

c.



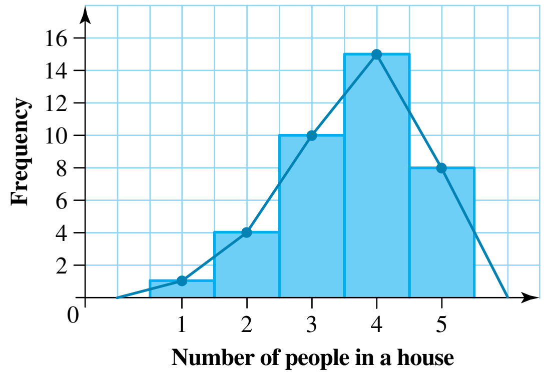
12.4.3 Frequency polygons (optional)

A **frequency polygon** is a line graph that can be drawn by joining the centres of the tops of each column of the histogram.

Frequency polygons are a graphical device for understanding the shapes of distributions. They serve the same purpose as histograms, but are especially helpful for comparing sets of data.

The **polygon** starts and finishes on the horizontal axis a half-column-width space from the group boundary of the first and last groups.

The figure shows the frequency polygon drawn on top of the histogram from Worked example 15.



It is common practice to draw the histogram and the polygon on the same set of axes.

WORKED EXAMPLE 15 Creating a frequency histogram with a polygon from a table

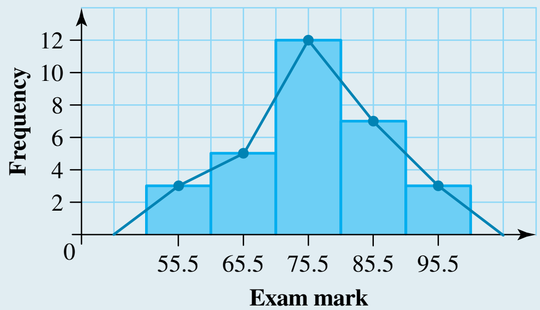
The frequency table below shows a class set of marks on an exam. Draw a frequency histogram and polygon on the same set of axes.

Mark	Class centre	Frequency
51–60	55.5	3
61–70	65.5	5
71–80	75.5	12
81–90	85.5	7
91–100	95.5	3

THINK

1. Draw a set of axes with the exam mark on the horizontal axis and frequency on the vertical axis. Show the class centres for the exam marks.
2. Draw the columns, leaving a half-column-width space before the first column.
3. Draw a line graph to the centre of each column.
4. Make sure the line graph begins and ends on the horizontal axis.

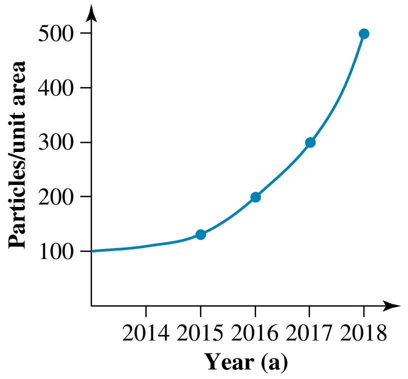
WRITE

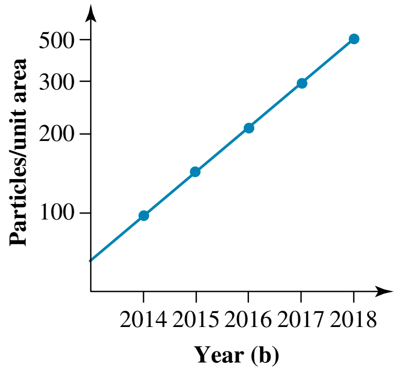


12.4.4 Misleading graphs

- Some graphs misrepresent data, which can lead to incorrect conclusions.
- Data can be misrepresented by:
  - adjusting scales
  - omitting values
  - shortening or lengthening axes
  - distorting images.

Consider the following two graphs depicting the rate of growth of pollutants.



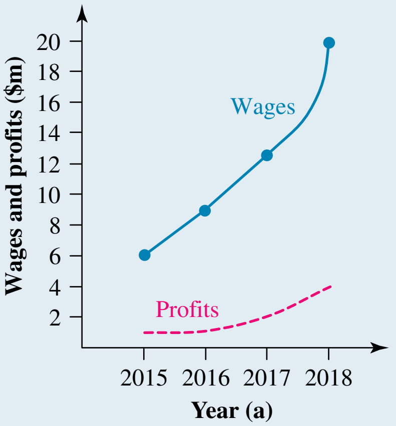


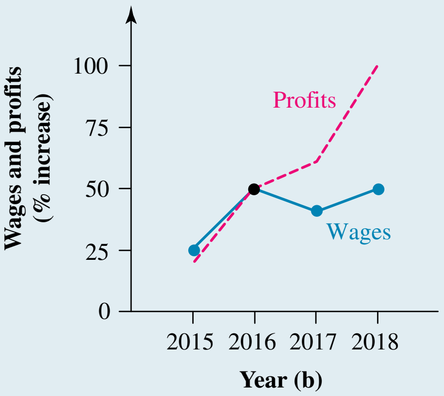
Both of these graphs show the same numerical information, but graph (a) has a linear scale on the vertical axis and graph (b) does not. Graph (a) emphasises the ever-increasing rate of growth of pollutants, whereas graph (b) suggests a slower, linear growth.

WORKED EXAMPLE 16 Determining whether a graph is misleading

The following data give wages and profits for a certain company. All figures are in millions of dollars.

Year	2015	2016	2017	2018
Wages	6	9	13	20
% increase in wages	25	50	44	54
Profits	1	1.5	2.5	5
% increase in profits	20	50	66	100





a. **State whether the graphs accurately reflect the data.**
b. **Suggest which graph would you rather have published if you were:**
  i. **an employer dealing with employees requesting pay increases**
  ii. **an employee negotiating with an employer for a pay increase.**

THINK

a. 1. Look at the scales on both axes. Both scales are linear.

WRITE

a. The graphs do represent data accurately. However, quite a different picture of wage and profit increases is painted by graphing with different units on the y-axis.

2. Look at the units on both axes. Graph (a) has its y-axis in \$, whereas graph (b) has its y-axis in %.

b. **i.** 1. Compare wage increases with profit increases.

2. The employer wants a graph that shows wages are already increasing exponentially to argue against pay increases.

b. **i.** The employer would prefer graph (a) because he/she could argue that employees' wages were increasing at a greater rate than profits.

ii. 1. Consider again the increases in wages and profits.

2. The employee doesn't like to see profits increasing at a much greater rate than wages.

ii. The employee would choose graph (b), arguing that profits were increasing at a great rate while wage increases clearly lagged behind.

12.4.5 Back-to-back stem-and-leaf plots

- **Back-to-back stem-and-leaf plots** are used to display two related numerical data sets.
- The leaf values start in the centre and count outwards.

WORKED EXAMPLE 17 Displaying data as a back-to-back stem-and-leaf plot

The ages of experienced and inexperienced bowlers using a ten-pin bowling centre are listed.

Experienced: 65, 15, 50, 15, 54, 16, 57, 16, 16, 21, 17, 28, 17, 27, 17, 35, 18, 19, 22, 30, 34, 22, 22, 31, 43, 23, 48, 23, 46, 25, 30, 21.

Inexperienced: 16, 60, 16, 52, 17, 38, 38, 43, 20, 17, 45, 18, 45, 36, 21, 34, 19, 32, 29, 21, 23, 32, 23, 22, 23, 31, 25, 28.

Display the data as a back-to-back stem-and-leaf plot and comment on the distribution.

THINK

1. Rule three columns, headed Leaf (Inexperienced), Stem and Leaf (Experienced).
2. Make a note of the smallest and largest values of both sets of data (15 and 65). List the stems in ascending order in the middle column.
3. Beginning with the experienced bowlers, work through the given data and enter the leaf (unit component) of each value in a row beside the appropriate stem.

WRITE/DRAW

Leaf (Inexperienced)	*Stem*	*Leaf (Experienced)*
9 8 7 7 6 6	1	5 5 6 6 6 7 7 7 8 9
8 5 3 2 3 3 1 9 1 0	2	1 8 7 2 2 2 3 3 5 1
1 2 2 4 6 8 8	3	5 0 4 1 0
5 5 3	4	3 8 6
2	5	0 4 7
0	6	5

4. Repeat step **3** for the set of data for inexperienced bowlers.
5. Include a key to the plot that informs the reader of the meaning of each entry.
6. Redraw the stem-and-leaf plot so that the numbers in each row of the leaf columns are in ascending order.
   Note: The smallest values should be closest to the stem column and increase as they move away from the stem.

Key: $1 \mid 5 = 15$

Leaf (Inexperienced)	*Stem*	*Leaf (Inexperienced)*
9 8 7 7 6 6	1	5 5 6 6 6 7 7 7 8 9
9 8 5 3 3 3 2 1 1 0	2	1 1 2 2 2 3 3 5 7 8
8 8 6 4 2 2 1	3	0 0 1 4 5
5 5 3	4	3 6 8
2	5	0 4 7
0	6	5

7. Comment on any interesting features.

The youngest experienced bowler attending the ten-pin bowling centre is 15 and the oldest is 65. The youngest and oldest inexperienced bowlers attending the ten-pin bowling centre are 16 and 60 respectively. There were an equal amount of experienced and inexperienced bowlers in their 20s.
The next largest number of experienced bowlers were in their teens, while the next largest number of inexperienced bowlers were in their 30s.

Exercise 12.4 Displaying data

learnon

12.4 Quick quiz on	12.4 Exercise

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
1, 2, 5, 8, 12, 15, 18	3, 6, 7, 9, 11, 13, 16, 19	4, 10, 14, 17, 20

Fluency

1. **WE13** The number of participants in various classes at the local gym is given as:

   Aerobics — 120 members   Pilates — 60 members   Aqua aerobics — 30 members
   Boxing — 15 members   Spin — 45 members   Step — 90 members.

   Represent this information as a:
   a. bar chart
   b. pie chart.

2. WE14 In a survey, 40 people were asked about the number of hours a week they spent on their phone. The results are listed below.

10.3, 13, 7.1, 12.4, 16, 11, 6, 14, 6, 11.1, 5.8, 14, 12.9, 8.5, 27.5, 17.3, 13.5, 8, 14, 10.2, 13, 7.9, 15, 10.7, 16, 8.4, 18.4, 14, 21, 28, 9, 12, 11.5, 13, 9, 13, 29, 5, 24, 11

a. State whether this data is continuous or discrete.
b. Determine the range of the data.
c. Organise the data into class intervals and create a frequency distribution table that displays the class intervals, midpoints and frequencies.
d. Construct a histogram.

3. The number of phone calls made per week in a sample of 56 people is listed below.

21, 50, 8, 64, 33, 58, 35, 61, 3, 51, 5, 62, 16, 44, 56, 17, 59, 23, 34, 57, 49, 2, 24, 50, 27, 33, 55, 7, 26, 52, 17, 54, 78, 69, 53, 2, 42, 52, 25, 48, 63, 12, 72, 36, 66, 15, 28, 67, 13, 23, 10, 72, 72, 89, 80, 62

a. Organise the data into a grouped frequency distribution table using a suitable class interval.
b. Display the data as a combined histogram.

4. For the following data, construct:

a. a frequency distribution table
b. a histogram.

Class sizes at Jacaranda Secondary College									
38	24	20	23	27	27	22	17	30	26
25	16	29	26	15	26	19	22	13	25
21	19	23	18	30	20	23	16	24	18
12	26	22	25	14	21	25	21	31	25

5. The table below shows the length of 71 fish caught in a competition. Show this information in a frequency histogram and polygon.

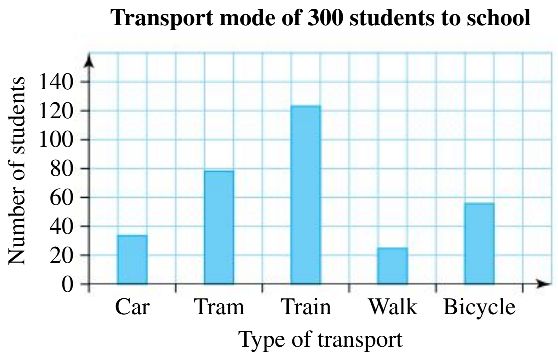
Length of fish (mm)	Class centre	Frequency
300–309	304.5	9
310–319	314.5	15
320–329	324.5	20
330–339	334.5	12
340–349	344.5	8
350–359	354.5	7

6. The distances 300 students have to travel to attend their school are summarised in the table below.

Distance km	Number of students
$0-<2$	112
$2-<4$	65
$4-<6$	56
$6-<8$	44
$8-<10$	15
$10-<12$	8

Represent this information a histogram.

7. MC Examine the bar chart shown and state the number of students who catch public transport to school.



A. 80 B. 100 C. 120 D. 200

Understanding

8. A block of houses in a suburb was surveyed to find the size of each house (in m^2). These are results.

Size of house (m^2)	Frequency
$100-<150$	13
$150-<200$	18
$200-<250$	19
$250-<300$	17
$300-<350$	14
$350-<400$	11
Total	92

Draw a combined histogram and frequency polygon for this data.

9. A drama teacher is organising his DVD collection into categories based on the genre of the movie or TV show. He recorded the following for the first 20 DVDs he categorised.

musical, musical, drama, comedy, horror,
musical, comedy, action, drama, comedy,
drama, drama, musical, drama, comedy,
romance, drama, musical, romance, drama

a. Show the data set as a bar chart.

He continued with the task, and the classification for his collection of 200 DVDs is summarised in the frequency table shown below.

Genre	Number of DVDs	Size of sector for a pie chart
Musical	52	
Drama	98	
Comedy	30	
Horror	10	
Action	5	
Romance	5	
Total	200	360°

b. Copy and complete the frequency table.
c. Construct a pie chart for the whole 200 DVD collection.

10. The height, in centimetres, of 32 students in Year 9 was recorded as follows.

167, 162.9, 157, 166.7, 146.5, 163, 156.1, 168, 159.4, 170.1,
152.1, 152.1, 174.5, 156, 163, 157.8, 161, 178, 151, 148, 166.7,
154.6, 150.3, 166, 160, 155.9, 164, 157, 171.1, 168, 158, 162

a. State whether the data is continuous or discrete.
b. Organise the data into seven classes, starting at 145 cm, and create a frequency distribution table.
c. Display the data as a histogram.
d. Explain whether it is possible to obtain the range of heights from the frequency distribution table without being given any other information.
e. MC Determine how many students stood at least 160 cm tall.

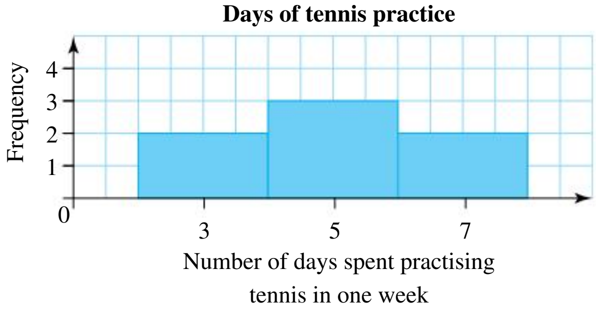
A. 15 B. 16 C. 17 D. 7

f. MC Calculate what percentage of students were under the minimum height of 165 cm required to qualify for the basketball team.

A. 21% B. 30% C. 87% D. 70%

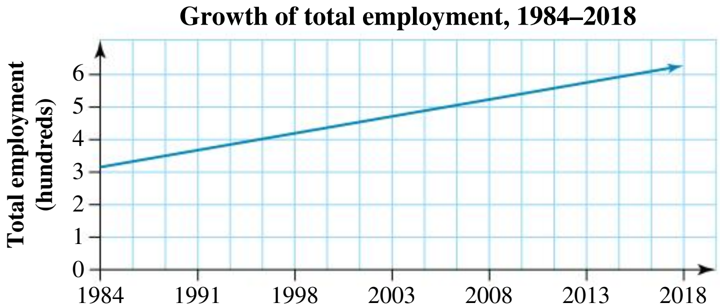
g. Reorganise the data into class intervals of 4 cm (for example, 145–148 cm).
h. Draw a new histogram and compare it to your histogram from part c. Discuss any advantages or disadvantages of having a smaller class interval.

11. The following histogram illustrates the number of days that members of a tennis team practised in a week.



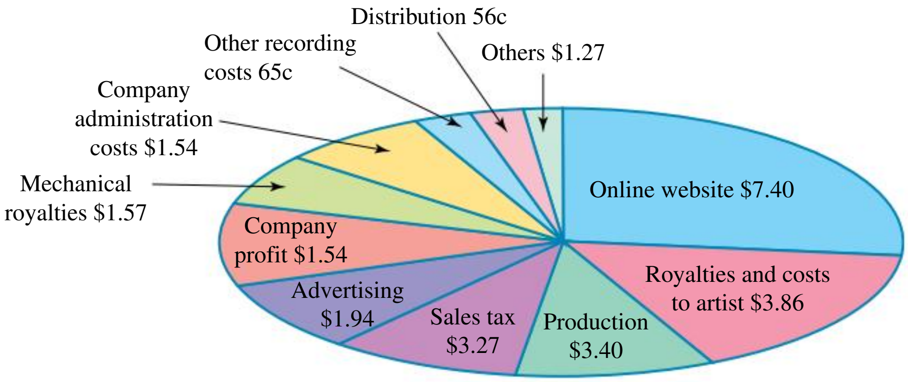
a. Describe what the frequency in this problem represents.
b. Calculate the average number of days that the team members practised in a week.

12. WE16 Examine this graph of employment growth in a company.



Explain why this graph is misleading.

13. This graph shows how the $27 that a buyer pays for a new album by their favourite singer-songwriter is distributed among the departments of a major recording company involved in its production and marketing.



You are required to find out whether or not the graph is misleading, to explain fully your reasoning, and to support any statements that you make.

a. Comment on the shape of the graph and how it could be obtained.
b. Does your visual impression of the graph support the figures? Explain your response.

14. The back-to-back stem-and-leaf plot displays the heights of a group of Year 9 students.

Key: 13 | 7 = 137 cm

Leaf (males)	*Stem*	*Leaf (females)*
9 8	13	7 8
9 8 8 7 6	14	3 5 6
9 8 8	15	1 2 3 7
7 6 6 5	16	3 5 6
8 7 6	17	1

a. **MC** The total number of Year 9 students is:
A. 13 B. 17 C. 30 D. 36

b. **MC** The tallest male and shortest female heights respectively are:
A. 186 cm and 137 cm
B. 171 cm and 148 cm
C. 137 cm and 188 cm
D. 178 cm and 137 cm

c. Compare the heights of boys and girls.

Reasoning

15. **WE17** The number of goals scored in football matches by Mitch and Yani were recorded as follows:

Mitch	0	3	1	0	1	2	1	0	0	1
Yani	1	2	0	1	0	1	2	2	1	1

a. Display the data as a back-to-back stem-and-leaf plot and comment on the distribution.
b. Calculate how many times Mitch and Yani scored more than a single goal.
c. Determine who scored the most goals in any single match.
d. Determine who scored the greatest number of goals overall.
e. State who is the most consistent performer.
Explain your answer.

16. Percentages in a Mathematics exam for two classes were as follows:

9A	32	65	60	54	85	73	67	65	49	96	57	68
9B	46	74	62	78	55	73	60	75	73	77	68	81

a. Construct a back-to-back stem-and-leaf plot of the data.
b. Calculate the percentage of each group that scored above 50.
Give your answers to the nearest whole number.
c. Identify the group with more scores over 80.
d. Compare the clustering for each group.
e. Comment on extreme values.
f. Calculate the average percentage for each group.

17. Suggest two reasons stem plots rather than frequency tables are preferred by statisticians.

Problem solving

18. When reading the menu at the local Chinese restaurant, you notice that the dishes are divided into sections. The sections are labelled chicken, beef, duck, vegetarian and seafood.

a. Identify the type of data that this menu presents.
b. Suggest the best way to represent this data.

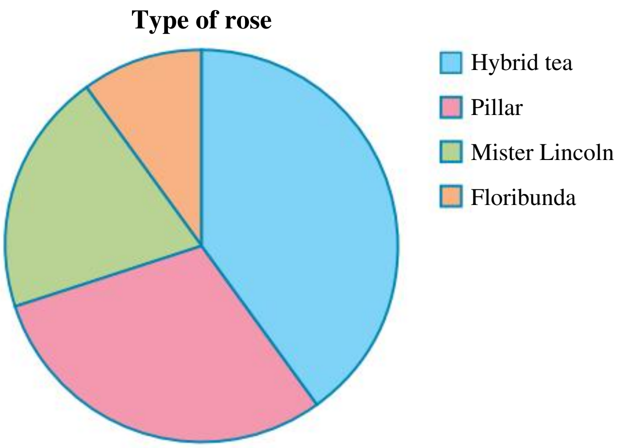
19. You are working in the marketing department of a company that makes sunscreen and your boss wants to know if there are any unusual results from the market research you completed recently.

You completed a survey of people who use your sunscreen, asking them how many times a week they used it. Their answers are given in the data set shown.

$$7, 2, 5, 4, 7, 5, 7, 2, 5, 4, 3, 5, 7, 7, 4, 5, 21, 5, 5, 2$$

Analyse the data set to work out what you will say to your boss, and try to explain any unusual variables.

20. The pie chart shown displays data collected about the types of roses in the local botanical gardens.



a. Carefully measure the angle of each section and record the results in a table.
b. Determine the smallest possible total number of rose bushes in the original data.
c. Create a bar chart displaying the data, assuming the total number of bushes is your answer from b.

LESSON 12.5 Measures of centre

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- calculate the mean, median and mode of raw data
- calculate the mean, median and mode of data in tables and plots
- calculate the mean, median and mode of grouped data
- determine whether a distribution is symmetrical, skewed or bimodal.

12.5.1 Mean, median and mode

- There are three values commonly used to measure the centre of data. These are the **mean**, **median** and **mode**.
  - The mean is also known as the average.
  - The median is the middle of the data.
  - The mode is the most common value in the data.

Calculating the mean

- **For raw data, calculate the mean using the formula:**

$$\bar{x} = \frac{\sum x}{n} = \frac{\text{sum of all data values}}{\text{number of all data values}}$$

For example, for the data 2, 2, 2, 3, 3, 7, 12, 13, 19:

$$\text{sum of data values} = 1 + 2 + 2 + 2 + 3 + 3 + 7 + 13 + 13 = 63$$
$$\text{number of data values} = 9$$
$$\text{mean} = \frac{63}{9} = 7$$

- **For data in frequency tables, calculate the mean using the formula:**

$$\bar{x} = \frac{\sum xf}{\sum f} = \frac{\text{sum of (data values} \times \text{frequency)}}{\text{sum of frequencies}}$$

Calculating the median

- **Start with data in order from smallest to largest.**
- **The median is the middle of the data.**
- **The middle value will be the $\frac{n+1}{2}$th score, where n is the number of scores or pieces of data.**

For example, for the data 2, 2, 2, 3, 3, 7, 12, 13, 19 there are 9 scores, so the middle value will be the 5th score (because $\frac{9+1}{2} = 5$).

$$2, 2, 2, 3, 3, 7, 12, 13, 19$$

$$\text{median} = 3$$

***Note:* If n is even, find the median by averaging the two middle scores.**

Calculating the mode

To calculate the mode, determine the most common value in the data.

For example, for the data 2, 2, 2, 3, 3, 7, 12, 13, 19:

the most common score is 2

mode = 2

***Note:* It is possible to have more than one mode.**

WORKED EXAMPLE 18 Calculating the median, mean and mode

Consider the data set shown.

$$3, 7, 4, 5, 6, 2, 8, 3, 3, 5$$

Calculate the:

a. **median** **b.** **mean** **c.** **mode.**

THINK	WRITE
a. 1. The median is the middle of the data. Organise the data from smallest to largest.	**a.** 2 3 3 3 4 5 5 6 7 8
2. The median, or middle score, is found by first counting how many scores there are. If there is an even number of scores, then the median is the average of the two middle scores. Use the rule: median score $= \frac{n+1}{2}$th score where n is the number of scores.	Median score $= \frac{n+1}{2}$th score $= \frac{10+1}{2}$ $= 5.5$th score The median score is between the 5th and 6th scores, which is between 4 and 5. $\therefore$ Median $= 4.5$
b. 1. The mean is the average of the data values Add up all the data values.	**b.** Mean $= \bar{x}$ $= \frac{\sum x}{n} = \frac{\text{sum of all data values}}{\text{number of data values}}$
2. Divide the result by the total number of scores.	$\bar{x} = \frac{3+7+4+5+6+2+8+3+3+5}{10}$ $\bar{x} = \frac{46}{10}$ $= 4.6$
c. The mode is the score that occurs the most frequently in the list.	**c.** Mode $= 3$

WORKED EXAMPLE 19 Calculating median, mean and mode from a frequency table

The combined number of goals scored in each of the 48 first round games of this year's World Cup is summarised in the frequency table below.

Total combined goals scored	Number of matches
0	6
1	13
2	12
3	9
4	5
5	2
6	0
7	1
$\sum f$	48

Calculate the:

a. **median number of goals scored**
b. **mean number of goals scored**
c. **modal number of goals scored.**

THINK

a. 1. The median is the middle value of the data set.
The sum of the frequencies is the number of scores, n. If there is an even number of scores, then the median is the average of the 2 middle scores.

2. Determine where the 24th and 25th scores lie.

b. 1. The mean is the average value.
Set up a third column on the frequency table and multiply the score and its frequency.

2. Add up all of the products of xf.

WRITE

a. $$\text{Median score} = \frac{n+1}{2}\text{th match}$$
$$= \frac{48+1}{2}$$
$$= 24.5\text{th match}$$

This score is between the 24th match and the 25th match.

The 24th score is 2 combined goals.
The 25th score is 2 combined goals.
Median = 2 combined goals

b.

Combined goals (x)	Number of matches (f)	$x \times f$
0	6	0
1	13	13
2	12	24
3	9	27
4	5	20
5	2	10
6	0	0
7	1	7
$\sum f$	48	$\sum xf = 101$

3. Divide the result by the sum of the frequencies, Σf.

$$\bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{\text{sum of (scores} \times \text{frequency)}}{\text{sum of frequencies}}$$

$$= \frac{101}{48}$$

$$= 2.1$$

c. The mode is the score with the highest frequency. **c.** Mode = 1 combined goal scored

- Mean, median and mode can be calculated from plots.

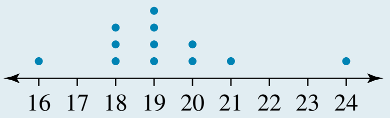
WORKED EXAMPLE 20 Calculating the median from a plot

Calculate the median of each data set.

a. **Key : 1|8 = 18**

Stem	*Leaf*
1	**8 9**
2	**2 2 5 7 7 8**
3	**0 1 4 6 7**
4	**0 5**

b.



THINK

WRITE

a. 1. Check that the scores are arranged in ascending order.

a. The scores are arranged in ascending order.

2. Locate the position of the median using the rule $\frac{n+1}{2}$, where $n = 15$.

Note: The position of the median is $\frac{15+1}{2} = 8$ (the eighth score).

Key : 1|8 = 18

Stem	*Leaf*
1	8 9
2	2 2 5 7 7 **8**
3	0 1 4 6 7
4	0 5

3. Write the answer in a sentence.

The median of the set of data is 18.

b. 1. Locate the position of the median using the rule $\frac{n+1}{2}$, where $n = 12$.

Note: The position of the median is $\frac{12+1}{2} = 6.5$. This means it is between the sixth and seventh scores.

b. The 6th score is 19.
The 7th score is 19.

$$\text{Median} = \frac{19+19}{2}$$

3. Calculate the average of the two middle scores.

$$= \frac{38}{2}$$

$$= 19$$

4. Write the answer in a sentence.

The median of the set of data is 19.

12.5.2 Grouped data (extending)

- **Grouped data** is given when a frequency table shows intervals instead of a value.
- To calculate the mean of a grouped data set, multiply the midpoint of each class interval by the corresponding frequency, and then divide the sum of these products by the sum of the frequencies.
- The median of a grouped data set is the middle value when the values are in order.
- The mode of a grouped data set is the most common class interval. This is known as the **modal class**.
  For example, let us calculate the mean, median and mode for the following data.

Interval	Frequency (f)
10– < 15	3
15– < 20	8
20– < 25	6
$\sum f$	17

- To calculate the mean, add a column to the data table showing midpoints, and another column showing the product of frequency and midpoint.

Interval	Frequency (f)	Midpoint (x_{mid})	$f \times x_{mid}$
10 – < 15	3	12.5	$3 \times 12.5 = 37.5$
15 – < 20	8	17.5	$8 \times 17.5 = 140$
20 – < 25	6	22.5	$6 \times 22.5 = 135$
Σf	**17**	$\Sigma f \times x_{min}$	$37.5 + 140 + 135 =$ **312.5**

- Mean $= \dfrac{312.5}{17} = 18.38$ (rounded to 2 decimal places)

- To calculate the median: since there are 17 data values, the middle value will be the 9th value, because $\dfrac{17+1}{2} = 9$.

Interval	Frequency (f)	
10 – < 15	3	⟵ First 3 values — need 6 more to get to 9
15 – < 20	8	⟵ The median is the 6th value in this group of 8
20 – < 25	6	
Σf	**17**	

- To calculate the median, look at the interval group that contains the median and use the rule:
  - median $= 15 + \dfrac{6}{8} \times 5 = 18.75$
- To calculate the modal class, determine the interval with the highest frequency.
  - Modal class $= 15- < 20$.

WORKED EXAMPLE 21 Calculating the median, mean and mode from grouped data

The actual diagonal lengths of 45 televisions in a store are presented as a frequency table. Calculate the:

a. **median (to the nearest inch)** **b.** **mean (to the nearest inch)** **c.** **modal class.**

Diagonal length (inches)	Number of TVs (f)
10− < 20	1
20− < 30	3
30− < 40	8
40− < 50	19
50− < 60	9
60− < 70	3
70− < 80	2
Σf	45

THINK

a. 1. The median is the middle value. The sum of the frequencies is the number of values, n. If there is an odd number of values, then the median is the middle value.

2. Determine where the 23rd value lies in the table by looking at the cumulative frequencies.
< 20 inches $= 1$ TV
< 30 inches $= 4$ TVs
< 40 inches $= 12$ TVs
< 50 inches $= 31$ TVs
The 23rd value is in the 40− < 50 inches group. It is the 11th value in that group of 19 TVs.

b. 1. The mean is the average value. Set up a third and fourth column in the frequency table.

2. Record the midpoint as the middle value of each interval, for example $\frac{10+20}{2} = 15$.

3. Multiply the midpoint of each interval, x_{mid} and the interval's frequency, f.

4. Add up all the products of $x_{mid} \times f$.

WRITE/DISPLAY

a. Median value $= \frac{n+1}{2}$th value

$= \frac{45+1}{2}$

$= 23$rd value

The 23rd value is in the 40− < 50 inches class.
It is the 11th value in a group of 19 values.

$\frac{11}{19} \times 10 \text{ inches} = 5.79 \text{ inches}$

Median $= 40 + 5.79$ inches
$= 45.79$ inches
≈ 46 inches

b.

Diagonal length (inches)	Number of TVs (f)	Class midpoint x_{mid} (inches)	Product of x_{mid} and f
10− < 20	1	15	$1 \times 15 = 15$
20− < 30	3	25	$3 \times 25 = 75$
30− < 40	8	35	$8 \times 35 = 280$
40− < 50	19	45	$19 \times 45 = 855$
50− < 60	9	55	$9 \times 55 = 495$
60− < 70	3	65	$3 \times 65 = 195$
70− < 80	2	75	$2 \times 75 = 150$
Σf	$= 45$	$\Sigma x_{mid} \times f$	$= 2065$

5. Divide $x_{mid} \times f$ by the sum of the frequencies, Σf.

$$\bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{\text{data values (score} \times \text{frequency)}}{\text{sum of frequencies}}$$

$$= \frac{2065}{45}$$

$$= 45.89 \approx 46 \text{ inches}$$

6. Write the answer in a sentence.

The mean TV size is 46 inches.

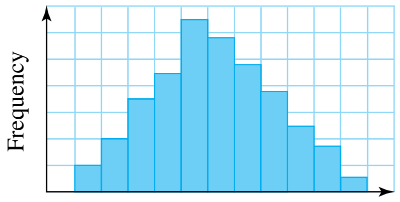
c. The modal class is the class interval with the highest frequency.

c. The modal class is 40 to less than 50 inches.

12.5.3 Types of distributions

- The type of distribution describes the shape of the data.
- **Symmetrical distributions** have a peak in the centre and evenly balanced data above and below the centre.
  - In symmetrical distributions, the mean, median and mode have similar values.

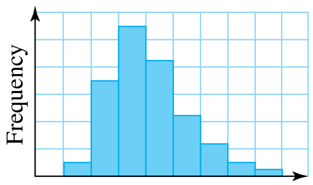
Symmetrical distribution



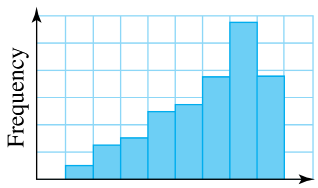
Stem	*Leaf*
0	7
1	2 3
2	2 4 5 7 9
3	0 2 3 6 8 8
4	4 7 8 9 9
5	2 7 8
6	1 3

- **Skewed distributions** are not symmetrical.
  - **Positively skewed distributions** (or right-skewed distributions) have a tail towards the positive direction, or right-hand side of the histogram.
    - In positively skewed distributions, the mean has a larger value than the median.
  - **Negatively skewed distributions** (or left-skewed distributions) have a tail towards the negative direction, or left-hand side of the histogram.
    - In negatively skewed distributions, the mean has a smaller value than the median.

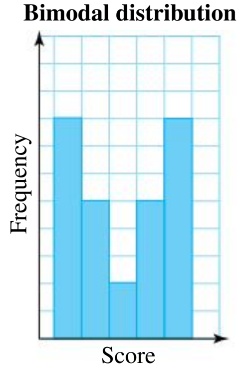
Positively skewed distribution



Negatively skewed distribution



- **Bimodal distributions** have two peaks (the prefix *bi-* means *two*).



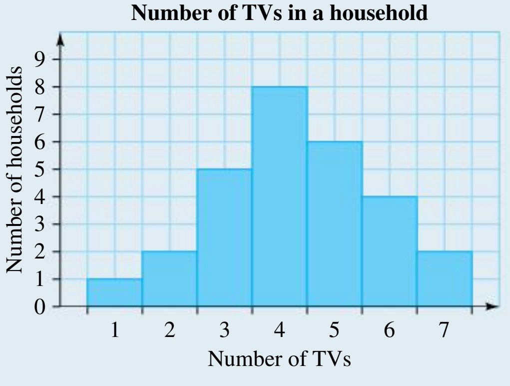
Types of distribution and measures of centre

Type of distribution	Most appropriate measure of centre
Symmetrical	Mean or Median
Skewed	Median

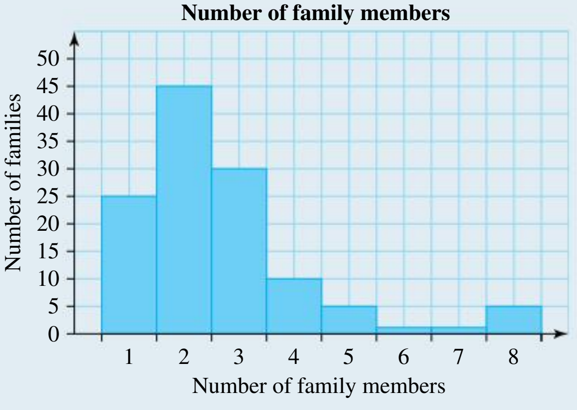
WORKED EXAMPLE 22 Identifying distribution type and calculating measure of centre

For each of the distributions shown, describe the type of distribution, determining its measure of centre using the most appropriate measures.

a.



b.



THINK	WRITE
a. 1. Describe the shape of the distribution.	**a.** The graph is roughly symmetrical.
2. Select an appropriate measure of centre.	Measure of centre: mean
3. Determine the mean.	$\text{Mean} = \bar{x} = \frac{\sum x}{n} = \frac{\text{sum of all data values}}{\text{number of data values}}$ $= \frac{120}{28}$ $= 4.3$
4. Write the answer in a sentence.	The mean number of TVs per household is 4.3.

b. **1.** Describe the shape of the distribution.

2. Select the appropriate measures of centre.

3. Determine the median by first determining the number of data values.
$25 + 45 + 30 + 10 + 5 + 1 + 1 + 5 = 121$

4. Write the answer in a sentence.

b. The graph is positively skewed.

Measure of centre: median

$$\text{Median} = \frac{n+1}{2}\text{th data value}$$
$$= \frac{122+1}{2}$$
$$= 61.5$$

The median is the average of the 61st and 62nd data values.

$$\text{Median} = \frac{2+2}{2}$$
$$= 2$$

The median number of members in a family is 2.

WORKED EXAMPLE 23 Determining the measures of centre from a stem plot

Determine the best measures of centre for the heights of 25 people shown in the stem plot. Give your answer to the nearest centimetre.

Key: 17 | 8 = 178 cm

Stem	*Leaf*
14	8
15	3 6
16	1 4 7
17	0 0 7 8 9
18	1 2 2 4 6 8 8
19	3 4 7 7
20	0 2
21	2

THINK

1. Describe the distribution.
2. State the most appropriate measure of centre.
3. Calculate the mean.
4. Write the answer in a sentence.

WRITE

The distribution is symmetrical.

The most appropriate measure of centre is the mean.

$$\text{Mean} = \bar{x} = \frac{\sum x}{n} = \frac{\text{sum of all data values}}{\text{number of data values}}$$
$$= \frac{4509}{25}$$
$$= 180.36$$

The mean height of the 25 people is 180 cm.

Outliers and measures of centre

The median is less affected by extremely large or extremely small data values (outliers) than the mean and is usually the preferred measure of centre when the distribution is not symmetrical.

DISCUSSION

It is important to understand what the numbers on an exam report mean, as these are used to make decisions about student learning. A student in a Year 9 exam receives a mean, median and range of scores. In pairs, discuss and write down what each of these measures suggests about the score of an individual student in comparison to the class.

Exercise 12.5 Measures of centre

learnon

12.5 Quick quiz on | 12.5 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 2, 3, 6, 10, 11, 14, 16, 19, 20, 25	4, 7, 8, 12, 15, 17, 21, 22, 26	5, 9, 13, 18, 23, 24, 27, 28

Fluency

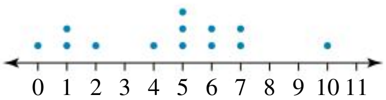
1. WE18 Calculate the median, mean and mode for the data sets given. Confirm your answers for the median and mean using a graphics calculator. Where necessary, give your answers to 2 decimal places.
   a. 27, 32, 45, 48, 53, 55, 55, 57, 59, 61, 75, 81
   b. 8, 8, 2, 8, 8, 8, 8, 8, 9, 9, 1, 9, 5, 9, 7, 10, 0

2. WE19 Calculate the median, mean (to 2 decimal places) and mode for the ungrouped frequency table shown.

Score	Frequency
0	2
1	5
2	12
3	10
4	7
5	2

3. WE20 Calculate the median of each of the following data sets.
   a. Key: $1|0 = 10$

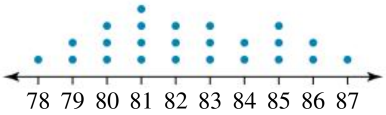
Stem	*Leaf*
1	0 2
2	1 3 3 5
3	
4	4

   b. 

4. Calculate the median of each of the following data sets.

a. Key: $10|0 = 100$

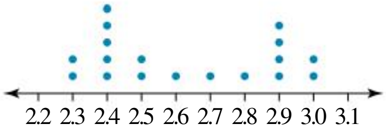
Stem	*Leaf*
10	0
11	0 2 2 2
12	0 4 6 6
13	3

b. 

5. Calculate the median of each of the following data sets.

a. Key: $6.1|8 = 6.18$

Stem	*Leaf*
6.1	8 8 9
6.2	0 5 6 8
6.3	0 1 2 4 4 4

b. 

6. For the stem plot given below, calculate:

a. the median

b. the mean (to 2 decimal places)

c. the mode.

Key: $3^*|7 = 37$

Stem	*Leaf*
1	3 4
1*	5 6 7 9
2	0 2 3 3 4
2*	5 5 5 6 6 7 8
3	1 2 3 4
3*	5 7
4	1

7. **WE21** For the following grouped frequency tables, calculate the:

i. median ii. mean iii. modal class.

Where necessary, give your answers to 1 decimal place.

a.

Score x	Frequency f
120–129	2
130–139	8
140–149	10
150–159	5
160–169	1

b.

Score	Frequency
120–129	2
130–139	8
140–149	10
150–159	16
160–169	5
170–179	1

Hint: The midpoint of the first class interval is $\frac{120 + 129}{2} = 124.5$.

8. **MC** Select the correct mean, median and mode for the following set of data.

1, 5, 2, 8, 7, 3, 5, 7, 2, 8, 7, 6, 4, 5, 5

A. 5, 6, 5 B. 6, 3, 5 C. 5, 5, 5 D. 5, 5, 7

9. MC Select the median number of members per family surveyed as presented in this frequency table.

Number of family members	Frequency
3	19
4	53
5	21
6	4
7	2
8	1

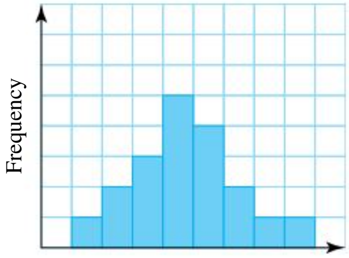
A. 3 B. 4 C. 4.5 D. 4.6

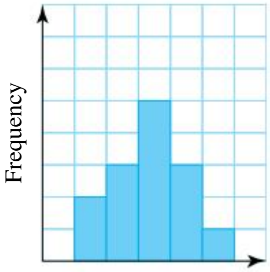
10. MC Select the best estimate of the mean score for the grouped frequency table shown.

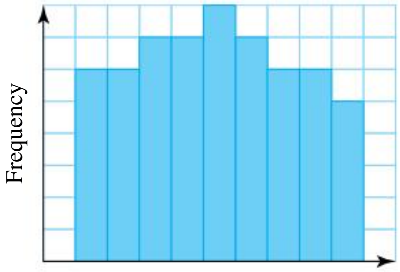
Score	Frequency
$0-<50$	1
$50-<100$	4
$100-<150$	10
$150-<200$	3
$200-<250$	1
$250-<300$	1

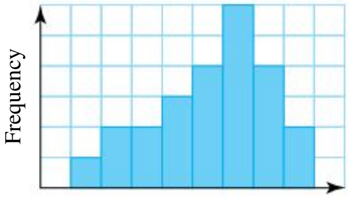
A. 100 B. 150 C. 125 D. 130

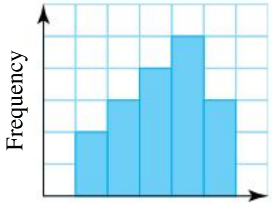
11. WE22 For each of the following histograms, state whether the distribution is symmetrical, positively skewed or negatively skewed.

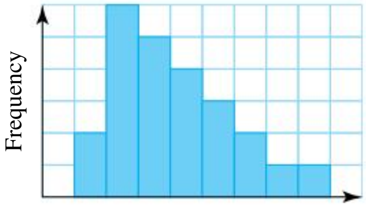
a.


b.


c.


d.


e.


f.


12. State the type of distribution shown by each of the following stem plots.

a. Key: $14 \mid 6 = 14.6$

Stem	*Leaf*
12	0
13	8 9
14	0 8 9 9
15	1 5 6 6 6 7 8
16	3 5 7 8
17	0 2
18	3 5

b. Key: $14 \mid 6 = 146$

Stem	*Leaf*
13	1 2 4 6 7 8 9
14	0 4 5 6 8 9 9
15	1 5 6 6 7 8
16	3 7 8
17	0 2
18	3

c. Key: $14 \mid 6 = 1.46$

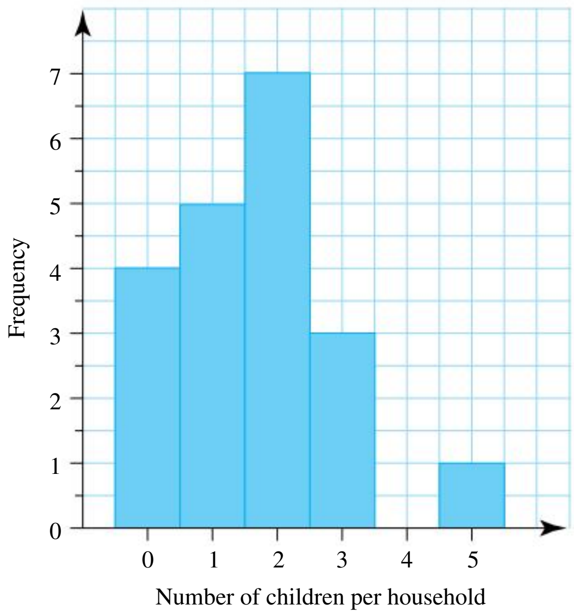
Stem	*Leaf*
12	7 8
13	6 7 8 9
14	0 9
15	1 5 6
16	3 7 8 9 9
17	0 2
18	3

d. Key: $14 \mid 6 = 1.46$

Stem	*Leaf*
1	1
2	
3	6 9
4	0 4 9
5	1 3 5 6 8
6	2 3 7 8 9 9
7	0 4 6 7 7 8

13. **WE22** This histogram represents the results of data collected from households in a suburban street. Describe the type of distribution, determining its centre using the most appropriate measure.

Children per household



14. WE23 The number of customers served at a cafe each hour for the past 24-hour period is summarised in the stem plot shown.

a. State the type of data and type of distribution.
b. Choose the most appropriate measure of centre.
c. Calculate the centre chosen in b.

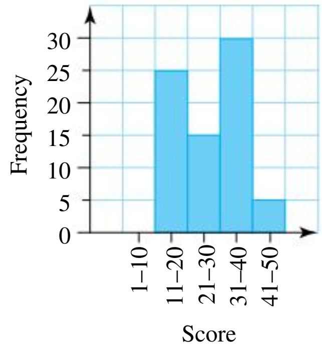
Key: 2 | 4 = 24 customers

Stem	*Leaf*
0	0 0 1 4
0*	5 5 8 8 9 9 9
1	1 2 3 4 4 4
1*	5 5 5 7
2	0 2
2*	
3	4

Understanding

15. Consider the distribution shown.

a. State whether the data is symmetrical.
b. State the modal class.
c. Explain whether the mean and median can be seen from the graph. If they can, state their values.



16. The grouped frequency table shown is a summary of 40 golf scores recorded at the local 9-hole golf course. For the data given, determine the:

a. type of data
b. median
c. mean
d. modal class.

Golf score	Number of games
30–39	4
40–49	32
50–59	2
60–69	1
70–79	1

17. The frequency table shown is a summary of 100 long-jump attempts by a group of primary school students during a recent school sports carnival. For the data given, determine the:

a. type of data
b. median
c. mean
d. mode.

Where necessary, round your answers to the nearest whole number.

Long jump distance (cm)	Number of jumps
120 – < 140	22
140 – < 160	44
160 – < 180	17
180 – < 200	14
200 – < 220	2
220 – < 240	0
240 – < 260	0
260 – < 280	1

18. A survey of the number of people living in each house on a residential street produced the following data:

$$2, 5, 1, 6, 2, 3, 2, 1, 4, 3, 4, 3, 1, 2, 2, 0, 2, 4$$

a. Prepare a frequency distribution table with an $f \times x$ column and use it to calculate the average (mean) number of people per household. Give your answer to 1 decimal place.
b. Draw a dot plot of the data and use it to find the median number of people per household.
c. Calculate the modal number of people per household.
d. Explain which of the measures would be most useful to:
   i. a real-estate agent renting out houses
   ii. a government population survey
   iii. an ice-cream truck owner.

Reasoning

19. A Geography teacher with a class of 20 students commented that test results for their class out of a score of 50 were negatively skewed. Show and explain a data set for the possible results obtained by the 20 students.

20. If the mean $= 11$, mode $= 4$, and the median$= 9$ and the missing values are positive integers:
   a. determine the missing two values in the following set of numbers:
   4, ___, 19, ____, 21, 4
   b. calculate two different values that meet the stated criteria
   c. explain whether you can substitute 9 for both missing values.

21. A class of 26 students had a median mark of 54 in Mathematics. In reality, though, no-one actually obtained this result.
   a. Explain how this is possible.
   b. Explain how many students must have scored below 54.

22. A small business pays the following wages (in thousands of dollars) to its employees:

18, 18, 18, 18, 26, 26, 26, 35, 80 (boss)

   a. MC Select the wage earned by most workers at this business.
   A. \$29 400 B. \$18 000 C. \$29 000 D. \$26 000
   b. Calculate the average wage at this business.
   c. MC Select the median of this distribution of wages.
   A. \$26 000 B. \$26 500 C. \$18 000 D. \$18 500
   d. Explain which measure might be used in wage negotiations by:
      i. the union, representing the employees (other than the boss)
      ii. the boss.

23. a. Create a data set of 8 numbers that meet the following criteria.
   - Data values are positive integers
   - Mean $= 9.5$
   - Median $= 7$
   - Mode $= 6$
   - Maximum value in data set $= 21$
   - Minimum value in data set $= 1$
   b. Explain whether there is only one data set that can meet this criteria

24. A tyre manufacturer selects 48 tyres at random from the production line for testing. The total distance travelled during the safe life of each tyre is shown in the table.

Distance in km ('000)	82	78	56	52	50	46
Number of tyres	2	4	10	16	12	4

a. Calculate the mean, median and mode.
b. Explain which measure best describes average tyre life.
c. Recalculate the mean with the 6 longest-lasting tyres removed. By how much is it lowered?
   Give your answer to the nearest km.
d. Determine what tyre life has the highest probability of being selected if you selected a tyre at random.
e. In a production run of 10 000 tyres, determine how many could be expected to last for a maximum of 50 000 km.
f. As the manufacturer, state the distance for which you would be prepared to guarantee your tyres.
   Explain your answer.

Problem solving

25. Answer the following questions.
   a. Create a set of five values with a mean of 4 and a median of 5.
      Demonstrate that your data set meets the given criteria.
   b. Change only one of the numbers in your set so that the mean is now 5 and the median remains 5.
      Demonstrate that your modified set meets the new criteria.

26. Seven students completed a challenging Science test. The test was scored out of 15.
    Six of the students got the following marks: 4, 4, 5, 5, 5, 6.
    Determine the mark that the 7th student obtained if:
   a. the mean score is the same as the median score
   b. the mean score is greater than the median score
   c. the mean score is equal to the modal score
   d. the mean score is greater than 6.

27. Calculate the mean for this data set.

Score	4	8	12	16	20	24
Frequency	$4a$	$3a$	$2a$	a	$\frac{a}{2}$	$\frac{a}{4}$

28. On a piece of paper is a list of six 2-digit prime numbers. They have a mean and median of 39. Their mode is 31 and the smallest number is 13. What are the six numbers?
    Explain how you got your answer.

LESSON 12.6 Comparing multiple data sets

LEARNING INTENTIONS

At the end of this lesson you should be able to:
- Compare back-to-back stem plots and histograms for different datasets

12.6.1 Back-to-back stem plots

- We have seen how to construct a stem plot for a single set of data.
- We can extend a stem plot to display two different sets of data in order to be able to make a comparison. We shall limit ourselves in this section to categorical variables with just two categories; for example, passing or failing an exam, voting yes or no.
- The two categories are used to provide two back-to-back leaves of a stem plot.

Back-to-back stem plots

A back-to-back stem plot is used to display two sets of data involving a numerical variable and a categorical variable with 2 categories.

WORKED EXAMPLE 24 Creating a back-to-back stem plot

Two classes submitted projects on the Olympic Games. The marks they obtained out of 20 are shown.

Class A's marks	**16**	**17**	**19**	**15**	**12**	**16**	**17**	**19**	**19**	**16**
Class B's marks	**14**	**15**	**16**	**13**	**12**	**13**	**14**	**13**	**15**	**14**

Display the data in the form of a back-to-back stem plot.

THINK

1. Identify the highest and lowest scores in order to decide on the stems.

WRITE

Highest score = 19
Lowest score = 12
Use a stem of 1, and divide into five class intervals. (Values of 12 and 13 will be included in the first class interval, values of 14 and 15 will be included in the next class interval, etc.)

THINK

2. Create an unordered stem plot first. Put Class B's scores on the left, and Class A's scores on the right.

WRITE

Key: 1|2 = 12

Leaf: Class B	Stem	Leaf: Class A
3 2 3 3	1	2
4 5 4 5 4	1	5
6	1	6 7 6 7 6
	1	9 9 9

3. Now order the stem plot. The scores on the left should increase in value from right to left, while the scores on the right should increase in value from left to right.

Key: 1|2 = 12

Leaf: Class B	Stem	Leaf: Class A
3 3 3 2	1	2
5 5 4 4 4	1	5
6	1	6 6 6 7 7
	1	9 9 9

12.6.2 Comparing back-to-back stem plots

- The back-to-back stem plot allows us to make some visual comparisons of two distributions for easy comparison to be made visually between the two distributions.
- In Worked example 5, the centre of the distribution for Class A is higher than the centre of the distribution for Class B. The spread of each of the distributions seems to be about the same.
- The marks for Class B are grouped between 12 and 15, and Class A marks are grouped between 16 and 19.
- Overall, we can conclude that Class A obtained better marks on average than Class B did.
- A more accurate way to compare each distribution is by the measures of centre (mean, median and mode).
- If the distribution of either category is skewed and/or contains outliers, then the median is a better measure of centre than the mean.

WORKED EXAMPLE 25 Comparing back-to-back stem plots

The number of 'how-to-vote' cards handed out by various Australian Labor Party and Liberal Party volunteers during the course of a polling day is shown below.

Labor	**180**	**233**	**246**	**252**	**263**	**270**	**229**	**238**	**226**	**211**
	193	**202**	**210**	**222**	**257**	**247**	**234**	**226**	**214**	**204**
Liberal	**204**	**215**	**226**	**253**	**263**	**272**	**285**	**245**	**267**	**275**
	287	**273**	**266**	**233**	**244**	**250**	**261**	**272**	**280**	**279**

a. Construct a back-to-back stem plot using the data.

b. Using the stem plot, together with the mean, median and mode, compare the distributions of the number of cards handed out by the Labor and Liberal volunteers.

THINK

a. 1. Identify the highest and lowest scores in order to decide on the stems.

2. Construct the stem plot.

WRITE

a. The highest score is 287, and the lowest score is 180, so use stems 18 to 28.

Key: 18|0 = 180

Leaf: Labor	Stem	Leaf: Liberal
0	18	
3	19	
4 2	20	4
4 1 0	21	5
9 6 6 2	22	6
8 4 3	23	3
7 6	24	4 5
7 2	25	0 3
3	26	1 3 6 7
0	27	2 2 3 5 9
	28	0 5 7

b. 1. Calculate the mean,median and mode for both Labor and Liberal.

Labor:

$$\text{Mean} = \frac{\sum x}{n}$$
$$= \frac{4557}{20}$$
$$= 227.85$$

$$\text{Median} = \frac{n+1}{2}$$
$$= 10.5 \text{ number (between the } 10^{\text{th}} \text{ and } 11^{\text{th}} \text{ score)}$$
$$= 227.5$$
$$\text{Mode} = 226$$

Liberal:

$$\text{Mean} = \frac{\sum x}{n}$$
$$= \frac{5150}{20}$$
$$= 256.5$$

$$\text{Median} = \frac{n+1}{2}$$
$$= 10.5 \text{ number (between the } 10^{\text{th}} \text{ and } 11^{\text{th}} \text{ score)}$$
$$= 264.5$$
$$\text{Mode} = 272$$

2. Comment on the relationship.

From the stem plot, we see that the Labor distribution is symmetric and therefore the mean and the median are very close, whereas the Liberal distribution is negatively skewed.

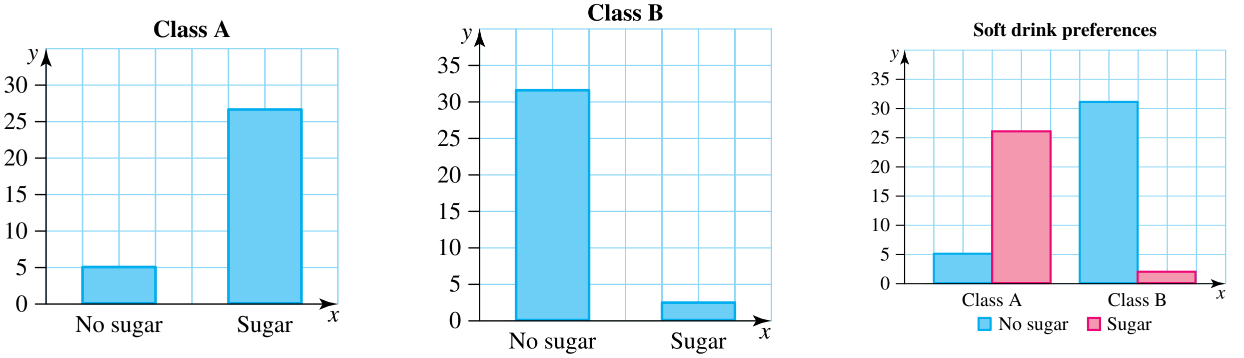
Since the distribution is skewed, the median is a better indicator of the centre of the distribution than the mean.

The median number of cards handed out by Labor (228 cards) was lower than that of Liberals (265 cards).

Overall, the Liberal Party volunteers handed out more ‘how-to-vote’ cards than the Labor Party volunteers.

12.6.3 Comparing histograms

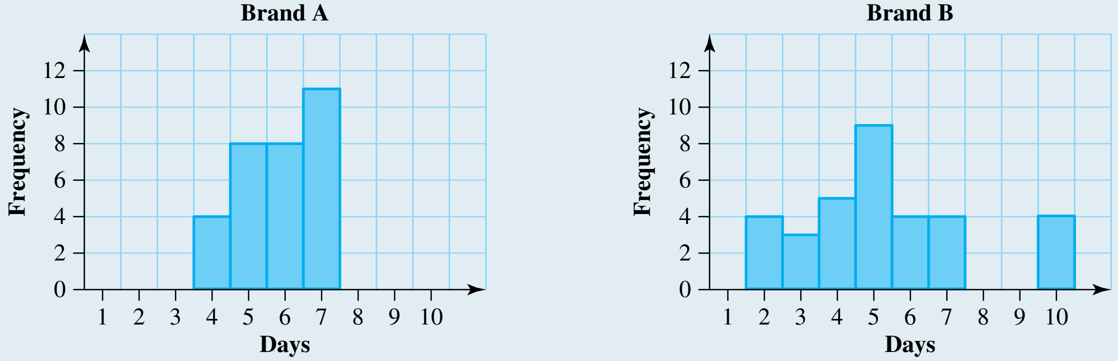
- Histograms can be drawn as side-by-side bar or column graphs or as separate graphs.
- For example, note the two separate graphs indicating preference for sugar or sugar-free soft drinks for two separate classes. The same information can also be drawn side-by-side on the right.



- Either representation allows easy comparison of two or more sets of data. In this instance, it is clear that Class B preferred sugar-free while Class A preferred sugar.
- When comparing histograms, it is important include a comparison of the key characteristics of each dataset:
  - Shape (symmetry or skew)
  - Distribution and spread
  - Outliers
  - Modality
- Comparisons made should always be in context.

WORKED EXAMPLE 26 Comparing histograms

Two power tool brands have measured their battery lifespan (in days) before it needs to be recharged. Compare these two histograms to determine which brand has more consistent performance.



THINK

1. Determine the shapes and key characteristics of each histogram.

WRITE

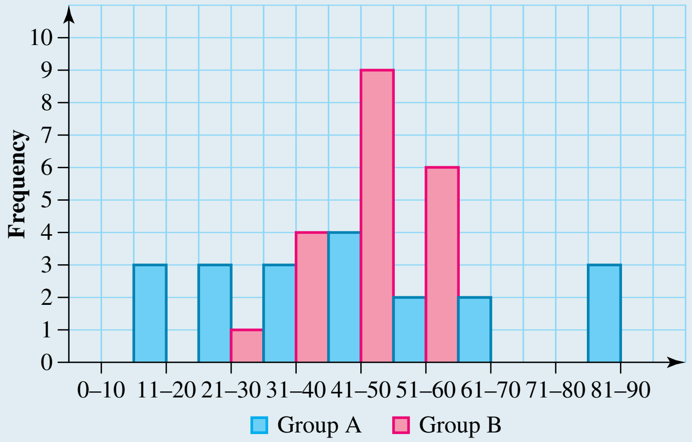
Brand A is clustered between 4 and 7 days with a modal number of 7. It appears to be negatively skewed.
Brand B is much more spread out (or dispersed) between 2 and 10 days, with a modal number of 5. It appears to be symmetrical around 5, with two outliers at 10 days.

2. In context, determine which battery performs more consistently based on the characteristics listed.

Brand A appears to be more consistent, as all batteries lasted between 4 and 7 days, with the majority lasting a full work week (5 to 7 days) without needing to be recharged.
Brand B performed less consistently, with a greater spread of lifespans. It did have two batteries lasting well over a week — these were likely outliers. It also had several batteries lasting only 2 or 3 days before needing to be recharged.
Overall, Brand A seems to be more consistent.

WORKED EXAMPLE 27 Comparing side-by-side histograms

Below is a side-by-side histogram representing the scores of two groups of teenagers in a popular online game.



By comparing these two datasets, determine which group you would consider more consistent.

THINK

1. Determine the shape and distribution of each dataset.

WRITE

Both groups A and B appear to have a fairly symmetrical spread. Group A is more spread out, with 3 outliers in the 81–90 score range. Group B has a smaller spread and shows more consistency, with most of the data in the 30–60 range.

2. Determine which group you prefer to join.

Although Group A got some higher scores, Group B is more consistent and therefore the better group to go with.

Exercise 12.6 Comparing multiple data sets

learnon

12.6 Quick quiz on | 12.6 Exercise

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
1, 3, 4, 8, 11, 14, 17, 20, 23	2, 5, 6, 9, 12, 15, 18, 21, 24	7, 10, 13, 16, 19, 22, 25

Fluency

1. WE24 The marks (out of 50) obtained for the end-of-term test by the students in German and French classes are given below. Display the data in the form of a back-to-back stem plot.

German	20	38	45	21	30	39	41	22	27	33	30	21	25	32	37	42	26	31	25	37
French	23	25	36	46	44	39	38	24	25	42	38	34	28	31	44	30	35	48	43	34

2. The birth masses of 10 boys and 10 girls (in kilograms, to the nearest 100 grams) are recorded in the table.

Boys	3.4	5.0	4.2	3.7	4.9	3.4	3.8	4.8	3.6	4.3
Girls	3.0	2.7	3.7	3.3	4.0	3.1	2.6	3.2	3.6	3.1

Display the data using a back-to-back stem plot.

3. WE25 A nursery plants seedlings in two locations, one of which gets full sun and the other partial sun. The heights of the seedlings after 1 week are recorded in the stem plot below. Use this stem plot to identify which growing location is better.

Key $1|0 = 10$ cm

Full sun	Stem	Partial sun
9 8 8 7 6 4 4 0 0	1	9
3 2 1 0 0	2	5 5 7 8 9
6	3	0 1 3 4 4 6 7 7 8

4. The following data shows the heights of students from two high school classes, in centimetres.

Class A	150	157	160	165	170	175	177	168	181	187	159	162
Class B	135	138	149	162	147	131	148	136	125	142	155	143

a. Construct a back-to-back stem plot of this data.
b. Based on this plot, determine which class would be the older class.

5. Two local cafes record the ages of their customers in the morning. Their results are recorded in the stem plot below.

Key $1|6 = 16$

Cafe A	Stem	Cafe B
8 8 6	1	5 5 7 7 9 9 9
4 0	2	0 1 1 3 4 4 6 9
9 7 6	3	1 3 3 4 5 5 5 8 9
6 3 3 0	4	0 1 2 6
8 5 5 0	5	0 5 7
8 8 4 3 2	6	2
9 7 7 5 4 3 1 1	7	
1 1 0	8	

a. Determine which cafe has a larger spread of data.
b. Looking at the distribution of the data, determine which cafe attracts an older population. Justify your answer.

6. A class of 30 students sat for a Mathematics test. Their results out of 100 are shown below.

68, 72, 58, 45, 69, 92, 38, 51, 70, 65, 69, 73, 52, 76, 48,
69, 73, 41, 42, 73, 80, 50, 60, 49, 65, 94, 88, 85, 53, 60

a. Use these results to copy and complete the frequency table.

Score	Tally	Frequency
30–39		
40–49		
50–59		
60–69		
70–79		
80–89		
90–99		

b. Construct a histogram to display the data.

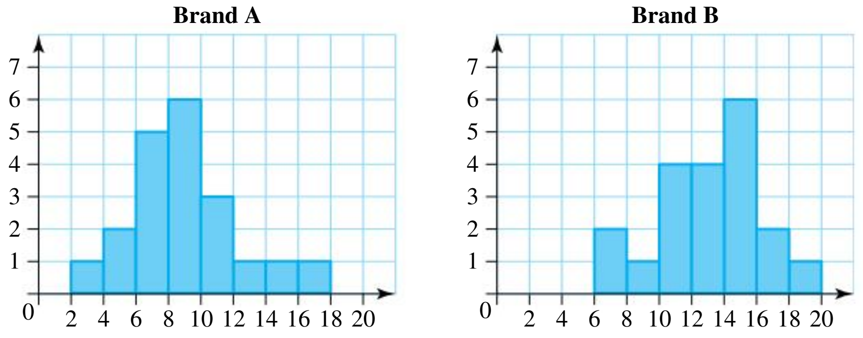
7. The following data give the times (in seconds) taken for athletes to complete a 100-m sprint.

12.2, 12.0, 11.9, 12.0, 12.6, 11.7,
11.4, 11.0, 10.9, 11.7, 11.2, 11.8,
12.2, 12.0, 12.7, 12.9, 11.3, 11.2,
12.8, 12.4, 11.7, 10.8, 13.3, 11.7,
11.6, 11.7, 12.2, 12.7, 13.0, 12.2

a. Construct a frequency distribution table for the data. Use class intervals of 0.5 seconds.
b. Construct a histogram to display the data.

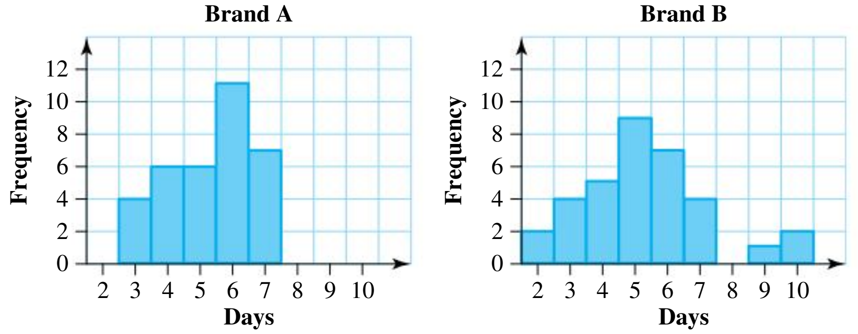
Understanding

8. WE26 Two groups of students measured the battery life of their phones over a day.

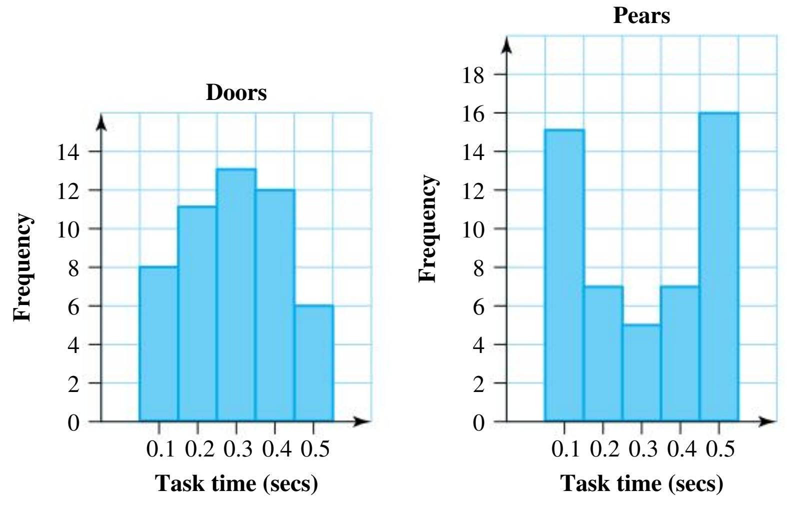


Compare these two histograms to determine which brand has more consistent performance.

9. Two electric scooter companies have measured their battery lifespan (in days) before needing to be recharged. Compare these two histograms to determine which brand has more consistent performance. Comment on any other factors that may affect battery life.



10. A computer programmer is comparing the time to complete video-editing tasks across two different computer platforms. The results are presented as histograms. Compare these two histograms to determine the best computer brand for completing video-editing tasks.



11. The number of delivery trucks making deliveries to a supermarket each day over a 2-week period was recorded for two neighbouring supermarkets — Supermarket A and Supermarket B. The data are as follows.

A	11	15	20	25	12	16	21	27	16	17	17	22	23	24
B	10	15	20	25	30	35	16	31	32	21	23	26	28	29

a. Construct a back-to-back stem plot using the data.
b. Use the stem plot, together with the mean, median and mode, to compare the distributions of the number of trucks delivering to supermarkets A and B.

12. The marks out of 20 for Class A and Class B on a Science test are as follows.

Class A	12	13	14	14	15	15	16	17
Class B	10	12	13	14	14	15	17	19

a. Construct a back-to-back stem plot using the data.
b. Use the stem plot, together with summary statistics, to compare the distributions of the marks of Class A and Class B.

13. The end-of-year marks of 10 students in an English class were compared over 2 years. The marks from the first year and for the same students in the second year are shown.

First year	30	31	35	37	39	41	41	42	43	46
Second year	22	26	27	28	30	31	31	33	34	36

a. Construct a back-to-back stem plot using the data.
b. Use the stem plot, together with summary statistics, to compare the distributions of the marks obtained by the students over the 2 years.

14. The ages of participants in two different sports teams were recorded as follows:

Football	23	24	25	26	27	28	30	31
Soccer	22	25	30	31	36	37	42	46

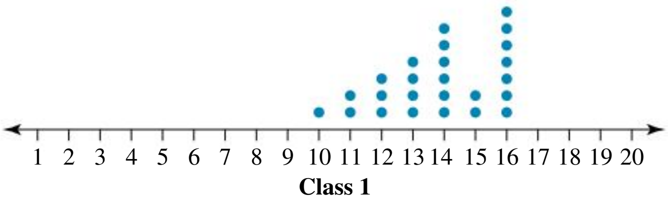
a. Construct a back-to-back stem plot using the data.
b. Use the stem plot, together with summary statistics, to compare the distributions of the ages of the football and soccer players.

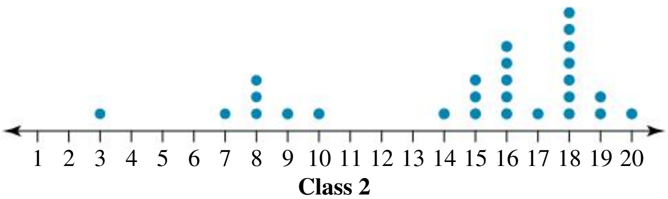
15. The scores on a board game are recorded for a group of kindergarten children and for a group of children in a preparatory school.

Kindergarten	3	13	14	25	28	32	36	41	47	50
Prep. school	5	12	17	25	27	32	35	44	46	52

a. Construct a back-to-back stem plot using the data.
b. Use the stem plot, together with summary statistics, to compare the distributions of the scores of the kindergarten children compared to the preparatory school children.

16. The two dot plots shown display the latest Maths test results for two Year 11 classes. The results show the marks out of 20.





a. Determine the number of students are in each class.
b. For each class, determine how many students scored 15 out of 20 on the test.
c. For each class, determine how many students scored more than 10 on the test.
d. Use the dot plots to describe the performance of each class on the test.

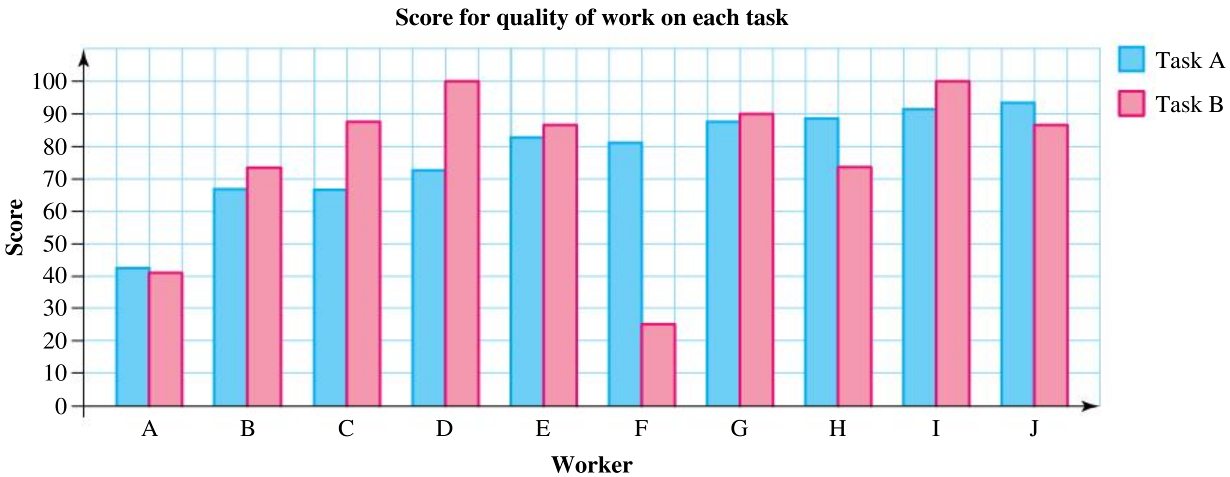
17. The following data show the ages of male and female players at a ten-pin bowling centre.

Males	20	36	16	38	32	18	19	21	25	45	29	60	31	21		
	16	38	52	43	17	28	23	23	43	17	22	23	32	34		
Females	21	23	30	16	31	46	15	17	22	17	50	34	65	25	27	19
	15	43	22	17	22	16	48	57	54	23	16	30	18	21	28	35

Draw a back-to-back stem plot of the data and compare the ages for each group.

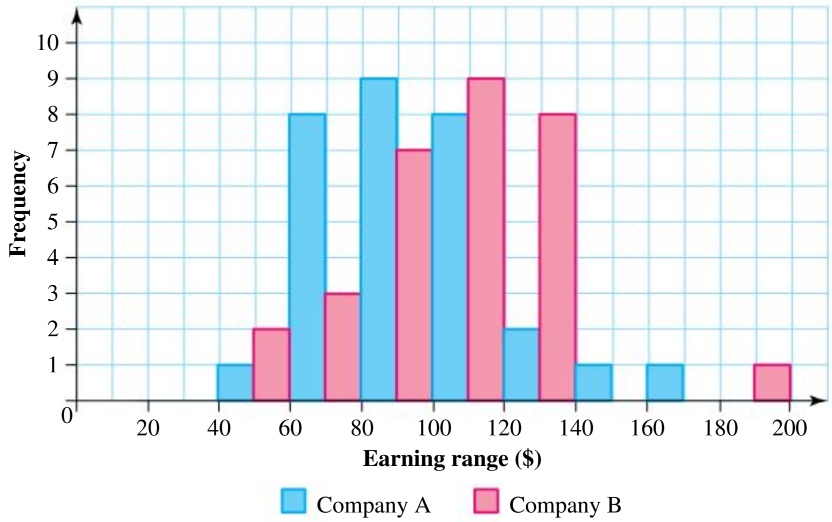
Reasoning

18. Ten workers were required to complete two tasks. Their supervisor observed them and gave them a score for the quality of their work on each task, where higher scores indicated better-quality work. The results are indicated in the following side-by-side bar chart.



a. Identify which worker had the largest difference between scores for the two tasks.
b. Determine how many workers received a lower score for task B than task A.

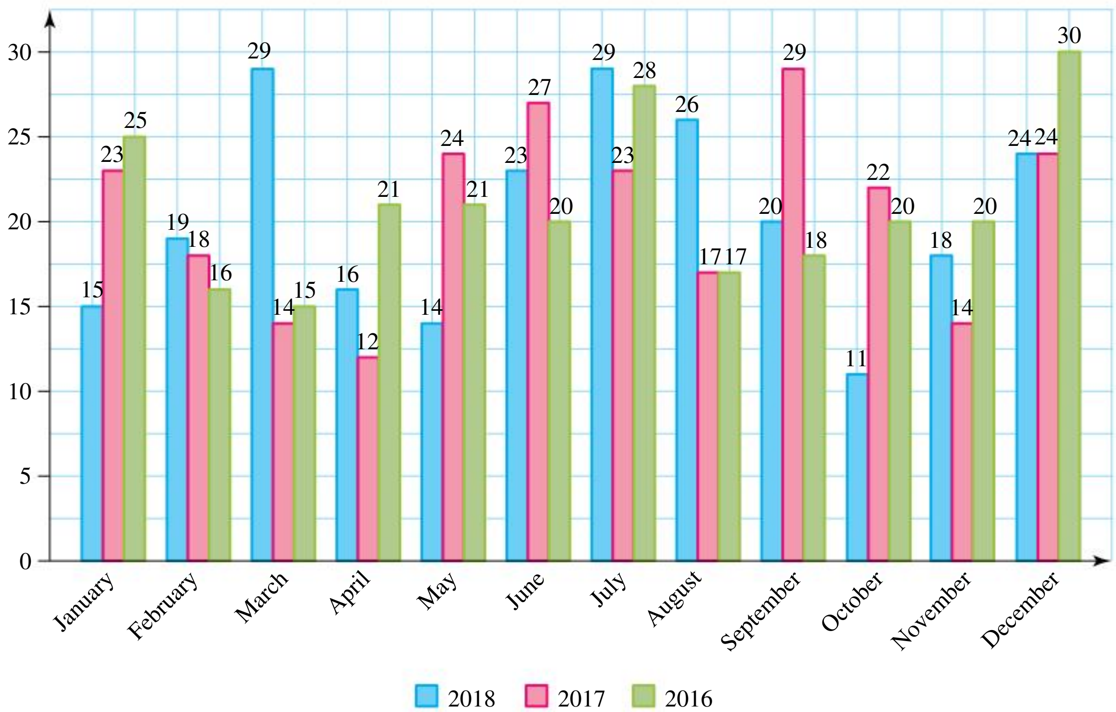
19. WE27 Employees' salaries (in $1000s) at two companies are compared using the following side-by-side histogram.



a. Determine which company has the highest paid employee.
b. Compare the distributions of Company A and Company B.
c. Determine which company you would prefer to work for. Justify your answer using mathematical reasoning.

20. The side-by-side bar chart shows a monthly comparison of road fatalities in Queensland.

Monthly comparison of Queensland road fatalities



Source: Australian Government, Department of Infrastructure, Regional Development and Cities, Australian Road Deaths Database

a. Identify the year that recorded the highest number of fatalities on Queensland roads.
b. Determine the year that had the highest number of fatalities from January to March.
c. Suggest a possible reason for the high number of fatalities in December 2016.

21. The comparisons between the battery lives of two mobile phone brands are shown in the back-to-back stem plot below.

Key: 6|1 = 61 hours

Brand A	Stem	Brand B
8 8 7 5	0	7
9 7 4 1 0	1	0 5 5 5 7 9
2 2 2 1	2	0 2 2 6 7
8 6 4 2 0	3	0 2 4 6 8
	4	
	5	6
1	6	
	7	5

Determine which mobile phone brand has the better battery life. Explain.

Problem solving

22. Two internet providers were compared for their internet speeds (in Mbps). The results were recorded in the back-to-back stem plot below.

Key: 4|5 = 45 Mbps

Provider A	Stem	Provider B
5 2	4	
3	5	
8 4	6	8 9
	7	0 2 2 5 5 8
6	8	1 2
7 1	9	
8	10	
	11	
	12	
0	13	

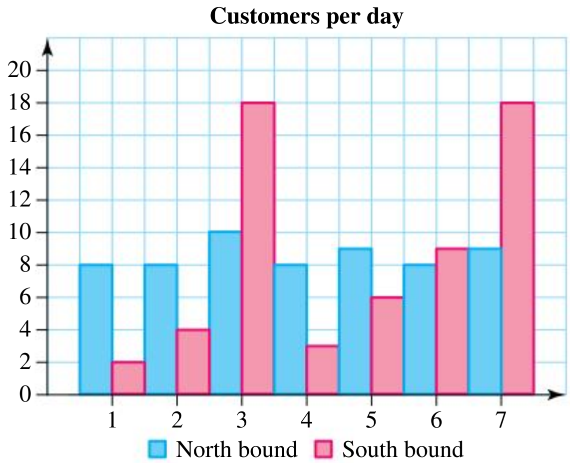
Determine which company you should choose as your internet provider. Justify your answer using mathematical reasoning.

23. The following datasets show the rental price (in $) of one-bedroom apartments in two different suburbs of Townsville.

Suburb A	215	225	211	235	244	210	215	210	256	207
	200	200	242	225	231	205	240	205	235	200
Suburb B	235	245	231	232	240	280	280	270	255	275
	275	285	245	265	270	255	260	258	251	285

Your friend tells you that it doesn't matter where you find a house, the price is pretty much the same everywhere. Provide a mathematically justified response to your friend whether you support their claim or not.

24. Below is a side-by-side column graph representing the number of customers at a service station on each side of the highway for the past week.
By comparing these two datasets, determine which side of the highway would be better to open a diner.



25. A coffee chain serves either skim, reduced-fat or whole milk in coffees. The coffees sold on a particular day are shown in the table below, sorted by the type of milk and the two different cafes.

	Cafes	
Type of milk	**Cafe A**	**Cafe B**
Skim	87	124
Reduced fat	55	73
Whole	112	49

a. Determine the total number of coffees sold on this day.
b. Construct a horizontal side-by-side bar chart to represent these data.
c. Calculate the percentage of Cafe A's customers who wanted skim milk in their coffees, rounding your answer to the nearest whole number.
d. Calculate the percentage of coffees sold that contained reduced-fat milk, rounding your answer to the nearest whole number.
e. Predict the percentage of the coffee chain's customers that would be from Cafe B if this sales trend continued daily, rounding your answer to the nearest whole number.

COLLABORATIVE TASK: Surveys in the media

At times, reports of surveys in the media can be used to lead people to draw conclusions by representing graphs and statistics in a way that supports a certain point of view. Find two statistical report in the media and analyse by linking claims made in the report to displays, statistics and sampling methods. Choose one report featuring categorical data and one report featuring numerical data.

LESSON 12.7 Review

12.7.1 Topic summary

STATISTICS

Types of data

- Categorical data is expressed using labelled categories.
  - Nominal data cannot be ranked in a logical order; e.g. favourite colours, pet types.
  - Ordinal data can be ranked in a logical order; e.g. AFL players' salaries.
- Numerical data is expressed using numbers.
  - Discrete data is collected by counting; e.g. number of pens, number of days.
  - Continuous data is collected by measuring; e.g. the mass of eggs, the distance between school and each student's home.

Size of data set

- Population: Data collected from an entire group; e.g. when all members of a family are surveyed.
- Sample: Data collected from only parts of a group; e.g. when only Year 9 students are surveyed at a school.

Random number generator

- To create random data on a spreadsheet, use the command =RANDBETWEEN()
- Then enter the lowest and highest desired value within the brackets, separated by a comma.

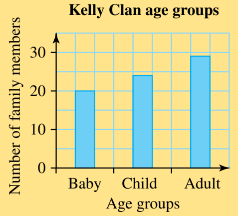
Measure of centre

- The **mean** is also known as the average.
  - For raw data:
    $$\bar{x} = \frac{\Sigma x}{n} = \frac{\text{sum of all data values}}{\text{number of data values}}$$
  - For data in a frequency table:
    $$\bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{\text{sum of (data values} \times \text{frequency)}}{\text{sum of frequencies}}$$
- The **median** is the middle value of the data.
  $\dfrac{n+1}{2}$ th data value, where n is the number of pieces of data or scores.
- The **mode** is the most common value in the data.

Displaying data

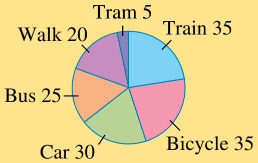
- Methods of displaying data include:
  - bar charts

Kelly Clan age groups



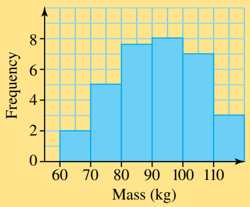
  - pie charts

Transport to school

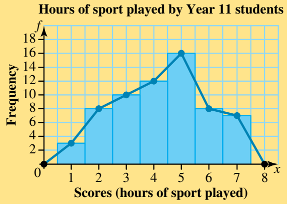


  - histograms

Mass of dolphin



Hours of sport played by Year 11 students



  - back-to-back stem-and-leaf plots.

Leaf (female)	Stem	Leaf (male)
987766	1	55667789
53231910	2	18722351
1224688	3	50410
553	4	386
2	5	047

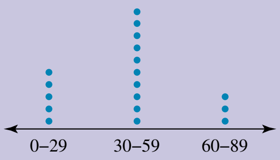
Collecting data

- Methods of collecting data include:
  - frequency tables

Person's age (x)	Tally	Number of people (f)
0–29	卌 I	5
30–59	卌 卌	10
60–89	III	3
		$\Sigma f = 18$

  - dot plots

Person's age (x)



  - stem-and-leaf plots.

Heights of 30 students (in cm)
Key: 11 | 8 = 118 cm

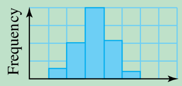
Stem	*Leaf*
11	8 9
12	0 1 2 2 3 4 5 6 7 7 8 9
13	0 1 1 1 2 3 4 5 6 8 8 9 9
14	0 1 3

  - Types of sampling include simple random, systematic, self-selected, convenience, stratified, clustered and capture/recapture.

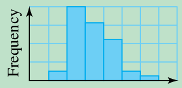
Types of distribution

Types of distribution include:

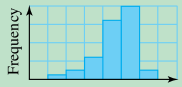
- symmetrical distribution, which has a peak in the centre and evenly balanced data above and below the centre.



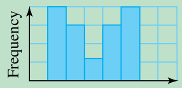
- skewed data, which is not evenly balanced.
  - For positively skewed data, the mean has a larger value than the median.



  - For negatively skewed data, the mean has a smaller value than the median.



- bimodal data, which has a distribution pattern with two peaks.



12.7.2 Success criteria

Tick the column to indicate that you have completed the lesson and how well you think you have understood it using the traffic light system.

(**Green:** I understand; **Yellow:** I can do it with help; **Red:** I do not understand)

Lesson	Success criteria	Green	Yellow	Red
12.2	I can identify different types of data.			
	I can determine if a data set is a sample or a population.			
	I can generate random numbers.			
12.3	I can organise data in frequency tables.			
	I can display data in dot plots.			
	I can display data in stem-and-leaf plots.			
12.4	I can display data in bar charts, pie charts, histograms and frequency polygons			
	I can present data as a back-to-back stem-and-leaf plots.			
12.5	I can calculate the mean, median and mode of raw data.			
	I can calculate the mean, median and mode of data in tables and plots.			
	I can calculate the mean, median and mode of grouped data.			
12.6	I can compare back-to-back stem and leaf plots and histograms for different data sets.			

12.7.3 Project

Graphical displays of data

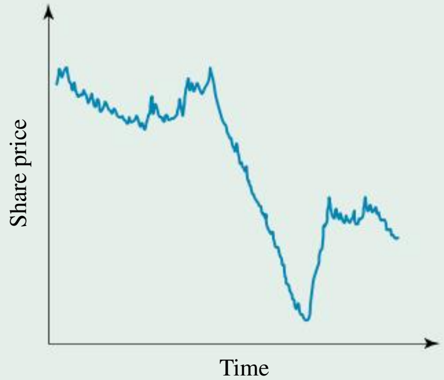
Statistical data can be communicated using words, tables or graphs.

When data is presented in words, it can be difficult to identify and compare trends.

When data is presented in table form, additional calculations are often needed to gain a complete understanding.

When data is presented as a graph it has an immediate visual impact, but the impression that graphs give can often be misleading, particularly if the graph is not looked at carefully. It is easy to manipulate graphs to produce a desired effect.

Consider the two graphs shown.



These graphs both display the same data (the value of a company's shares over the same period of time), but they look quite different. If the objective of the graph is to create the impression that the value of the shares has dropped quite dramatically, the first graph would serve this purpose. On the other hand, if we are to be convinced that the shares have not changed very much in value, the second graph reflects this point of view.

1. What technique is used to create two quite opposing impressions in the two graphs?

The following table shows the times for the men's and women's 100-m sprints in the Summer Olympic Games. A sports journalist wants to convince readers that, since 1928, women have been more successful in reducing their time than men.

Year	100 m men	Men's time (seconds)	100 m women	Women's time (seconds)
1928	Percy Williams, CAN	10.80	Betty Robinson, USA	12.20
1932	Eddie Tolan, USA	10.30	Stella Walsh, POL	11.90
1936	Jesse Owens, USA	10.30	Helen Stephens, USA	11.50
1948	Harrison Dillard, USA	10.30	Fanny Blankers-Koen, HOL	11.90
1952	Lindy Remigino, USA	10.40	Marjorie Jackson, AUS	11.50
1956	Bobby Morrow, USA	10.50	Betty Cuthbert, AUS	11.50
1960	Armin Hary, GER	10.20	Wilma Rudolph, USA	11.00
1964	Bob Hayes, USA	10.00	Wyomia Tyus, USA	11.40
1968	Jim Hines, USA	9.95	Wyomia Tyus, USA	11.08
1972	Valery Borzov, URS	10.14	Renate Stecher, GDR	11.07
1976	Hasely Crawford, TRI	10.06	Annegret Richter, FRG	11.08
1980	Allan Wells, GBR	10.25	Lyudmila Kondratyeva, URS	11.06
1984	Carl Lewis, USA	9.99	Evelyn Ashford, USA	10.97
1988	Carl Lewis, USA	9.92	Florence Griffith Joyner, USA	10.54
1992	Linford Christie, GBR	9.96	Gail Devers, USA	10.82
1996	Donovan Bailey, CAN	9.84	Gail Devers, USA	10.94
2000	Maurice Greene, USA	9.87	Marion Jones, USA	10.75
2004	Justin Gatlin, USA	9.85	Yuliya Nesterenko, BLR	10.93
2008	Usain Bolt, JAM	9.69	Shelley-Ann Fraser, JAM	10.78
2012	Usain Bolt, JAM	9.63	Shelly-Ann Fraser-Pryce, JAM	10.75
2016	Usain Bolt, JAM	9.81	Elaine Thompson, JAM	10.71

Note: The Summer Olympic Games were not held in 1940 and 1944.

2. Using graph paper, draw two graphs to support the sports journalist's view.
3. Describe the features of your graphs that helped you to present the journalist's view effectively.
4. Imagine you are the sports journalist and you are about to publish the graphs in the local newspaper. Write a short article to accompany the graphs.

Exercise 12.7 Review questions

learnon

Fluency

1. **MC** Recognise which one of the following is an example of numerical discrete data.
   A. Your favourite weeknight television show
   B. The speed of a car recorded on a speed camera
   C. The number of home runs by a baseball player
   D. Placing in a field of eight swimmers of a 100 m freestyle race

The following information relates to questions 2 and 3.

Number of bedrooms	1	2	3	4	5	6
Number of homes	4	9	32	36	7	2

2. **MC** For the data given, select the most likely type of distribution.
   A. A negatively skewed distribution
   B. A positively skewed distribution
   C. Skewed to the left
   D. A symmetrical distribution

3. **MC** Select the mode for the data collected.
   A. 36 **B.** 5 **C.** 4 **D.** 3

4. **MC** The results of a Science test marked out of 60 are represented by the stem plot shown. The median, mean and mode, respectively, are:
   A. 60, 42, 52
   B. 40, 45, 60
   C. 45, 60, 45
   D. 45, 44, 60

Key: $2 \mid 3 = 23$

Stem	*Leaf*
0	8
1	8
2	3 5
3	0 3 6
4	2 4 5 5 5 8
5	4 7 8 8 9
6	0 0 0 0

5. **MC** Select the best estimate of the mean score for the grouped frequency table shown.

Score	Frequency
$0-<50$	12
$50-<100$	24
$100-<150$	23
$150-<200$	4

   A. 63 **B.** 90 **C.** 92 **D.** 100

6. A school wants to survey students about their opinions on a new cafeteria menu. Determine which of the following is an example of cluster sampling.
   A. The school randomly selects 10 students from each grade level to participate in the survey.
   B. The school selects every 5th student from an alphabetical list of all students.
   C. The school randomly selects 3 entire classrooms and surveys all students in those classrooms.
   D. The school uses a random number generator to select 50 students from the entire student body.

7. For the following surveys or studies, decide what type of data would be collected and how it should be displayed.
   a. The blood type of all workers at a shoe factory.
   b. A survey of 500 Australians' views on bringing back corporal punishment (strongly agree, agree, disagree, strongly disagree).
   c. Number of phones in the households of the students in your class.
   d. The top 20 fastest recorded times for the women's 100 metres butterfly.

8. Classify each of the following data collections according to whether they are samples or populations.
   a. A survey of 1000 travellers on the quality of train service provided.
   b. A survey of all the students at a secondary college about the state government's rules on underage alcohol laws.
   c. Performance in Mathematics exams by this year's Year 9 cohort.

9. The following set of data shows the types of phones owned by the first 30 shoppers surveyed.

   iPhone, Samsung, Motorola, iPhone, iPhone, iPhone, Samsung, Samsung, Nokia, Motorola, Samsung, iPhone, Motorola, iPhone, Samsung, Samsung, Motorola, iPhone, Samsung, Nokia, Samsung, Motorola, iPhone, iPhone, iPhone, iPhone, Samsung, Samsung, Nokia, Motorola

   a. Suggest two appropriate ways to display this data.
   b. Present this data in the ways suggested in part a.

10. The following data represent the number of songs on people's favourite Spotify playlists. An initial analysis of 25 playlists resulted in the following data being collected.

$$47, 103, 45, 111, 67, 54, 59, 44, 31, 89, 34, 2, 43, 96, 70, 90, 66, 48, 67, 82, 74, 56, 35, 9, 77$$

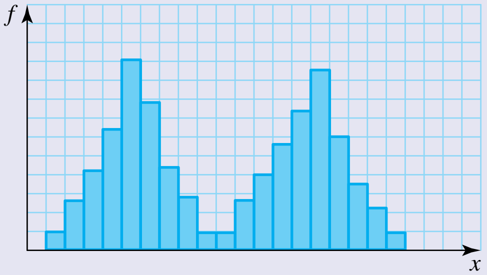
   a. Present the data as a stem-and-leaf plot.
   b. Comment on the shape of distribution.
   c. Present the data as a frequency table grouped in intervals of 20.
   d. State the type of data and type of distribution, and give the most appropriate measure of centre.

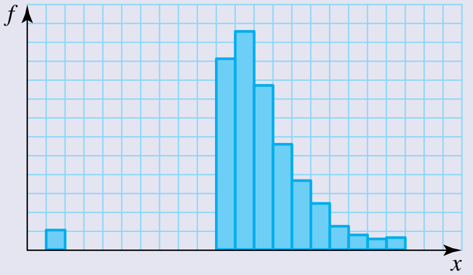
11. The following table shows the salaries for AFL players in 2019.

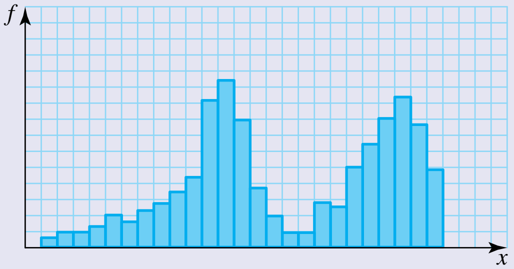
AFL players' salaries	Number of players
0–\$100 000	146
\$100 001–\$200 000	188
\$200 001–\$300 000	107
\$300 001–\$400 000	57
\$400 001–\$500 000	24
\$500 001–\$600 000	12
\$600 001–\$700 000	4
\$700 001–\$800 000	0
\$800 001–\$900 000	4
Total number of players	542

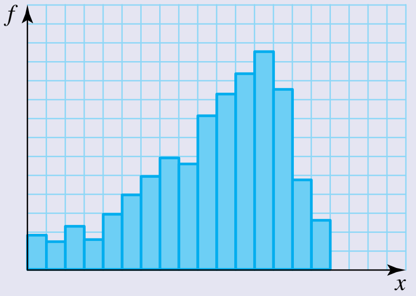
   a. Draw a histogram to display the data.
   b. Describe the type of distribution the histogram displays.
   c. Calculate the median salary of these football players (to the nearest thousand).
   d. Calculate the modal salary for AFL footballers in 2019.

12. Identify the modality and shape of each of the following graphs.

a. 

b. 

c. 

d. 

13. The number of hours of counselling received by a group of 9 full-time firefighters and 9 volunteer firefighters after a serious bushfire is given in the table.

Full-time	2	4	3	5	2	4	6	1	3
Volunteer	8	10	11	11	12	13	13	14	15

a. Construct a back-to-back stem plot to display the data.
b. Comment on the distributions of the number of hours of counselling of the full-time firefighters and the volunteers.

Problem solving

14. The workers in an office are trying to obtain a salary rise. In the previous year, all ten people who work in the office each received a 2% salary rise, while the company CEO received a 42% salary rise.
a. Calculate the mean salary rise received last year. Give your answer to 1 decimal place.
b. Calculate the median percentage salary rise received last year.
c. Calculate the modal percentage salary rise received last year.
d. The company is trying to avoid paying the salary rise. Explain the statistics you think they would quote about last year's rise. Give your reason for using these statistics.
e. Explain the statistics that you think the trade union would quote about salary rises. Give your reason for using these statistics.
f. Identify which statistic you think is the most 'honest' reflection of last year's salary rises. Explain your reasoning.

15. A data set with the values 8, ___, 25, ___, 21, 8 has a mean of 15, mode of 8 and a median of 14. If the missing values are positive integers:
a. calculate two values for the set
b. calculate two different values that meet the given criteria.

16. Create a data set of 9 numbers that meet the following criteria.
- Data values are positive integers
- Mean = 8
- Median = 7
- Mode = 10
- Maximum value in data set = 14
- Minimum value in data set = 3

17. The following datasets show the rental prices (in \$) of one-bedroom apartments in two different suburbs of Sydney.

Suburb A	275	275	281	285	284	310	315	310	296	307
	300	300	242	295	281	305	290	305	295	300
Suburb B	235	225	231	232	240	280	300	310	295	275
	275	285	245	305	270	255	270	228	241	285

a. Draw a back-to-back stem plot to compare the datasets.
b. Compare and contrast the rental prices in the two suburbs.
c. The rental prices in a third suburb, Suburb C, were also analysed.

Suburb C	335	325	351	335	340	360	300	390	395	285
	357	385	354	305	375	345	270	358	340	365

Compare the rents in the third suburb with the rents in the other two suburbs.

18. The average maximum temperature (in °C) in a small town in the Darling Downs region for two 20-year time periods is shown in the following tables.
Time period (2005–2024):

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Temp.	22.3	22.6	21.8	22.1	22.4	22.7	22.1	22.7	23.1	23.1

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011	2024
Temp.	22.7	23.4	23.4	23.1	22.7	22.1	22.9	22.6	22.6.	22.7

Time period (1893–1912):

Year	1893	1894	1895	1896	1897	1898	1899	1900	1901	1902
Temp.	20.5	20.9	21.0	20.9	21.5	21.3	21.0	20.9	21.0	21.0

Year	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912
Temp.	20.7	20.9	20.9	21.5	21.4	21.3	21.1	21.4	21.3	21.4

Display each data set as a histogram and analyse the differences between the two data sets.

on To test your understanding and knowledge of this topic go to your learnON title at www.jacplus.com.au and complete the **post-test**.

Answers

Topic 12 Statistics

12.1 Pre-test

1. a. iii b. iv c. i d. ii
2. Sample
3. 16 male adult, 14 female adult, 11 male adolescent, 9 female adolescent
4. a. 25 b. 7
5. a. 20 b. 10
6. a. ii b. iii c. i
7. B
8. a. Discrete b. 17 c. Skewed
9. B
10. C
11. 2.5
12. a. 60.3 b. D
13. 7
14. A
15. D

12.2 Collecting data

1. a. Categorical b. Numerical
2. a. Numerical b. Numerical
3. a. Continuous b. Continuous
4. a. Continuous b. Categorical
5. a. Ordinal b. Nominal
6. a. Ordinal b. Ordinal
7. a. Nominal b. Nominal
8. a. Sample b. Sample c. Population
9. a. Sample b. Population
10. B
11. B
12. C
13. A
14. The researcher would use systematic random sampling by selecting every 40th person from a randomly chosen starting point between 1 and 40.
15. Suburb A: 20 residents
Suburb B: 13 residents
Suburb C: 17 residents
16. 240
17. a. data b. survey, census c. graph
18. a. RANDBETWEEN (1, 31)
b. RANDBETWEEN (1, class size)
c. Samples
19. a. RANDBETWEEN (1, 365)
b. Sample
20. a. Car
b. Sample responses can be found in the worked solutions in the online resources.
c. Sample responses can be found in the worked solutions in the online resources. Answers could include the lower number of motor cycle fatalities as compared to the number of car fatalities.
d. According to the data, there were 190 motorcycle fatalities in 2020, compared to 732 car fatalities.
e. Sample responses can be found in the worked solutions in the online resources.
21. Sample responses can be found in the worked solutions in the online resources. Answers will vary but should include the following groups.
Categorical data can be:
   - nominal — arranged in categories, for example by colour or size
   - ordinal — arranged in categories that have an order, for example, karate ability, which ranges from white belt (beginner) to black belt (accomplished).

   Numerical data can be:
   - discrete — data can have only particular values, for example amounts of money or shoe sizes
   - continuous — data can have any value within a range, for example the time it takes a person to run around an oval.
22. a. Sample responses can be found in the worked solutions in the online resources. Answers could include:
Data that might be collected from a maternity hospital:
   - birth weights of babies
   - length of time in labour.

   b. Sample responses can be found in the worked solutions in the online resources. Answers could include:
Data that might be collected from an airport:
   - number of delays
   - time spent waiting for flights.

   c. Sample responses can be found in the worked solutions in the online resources. Answers could include:
Data that might be collected from shoppers in a supermarket:
   - shoppers' favourite brands of detergent
   - the amount of money spent on an average weekly shop.
23. a. Sample responses can be found in the worked solutions in the online resources. Answers could include:
'What do you think of sport?' — This question should not be used because it is too broad.
b. Sample responses can be found in the worked solutions in the online resources. Answers could include:
'List these sports in order from your most favourite to your least favourite: football, tennis, netball, golf.'
c. Sample responses can be found in the worked solutions in the online resources.
d. Sample responses can be found in the worked solutions in the online resources. Answers could include:
'Why do you like sports?' — This question should not be used because it is too broad.
e. Sample responses can be found in the worked solutions in the online resources.
f. Sample responses can be found in the worked solutions in the online resources. Answers could include

'Can you play a sport?' — This question should not be used because it is too broad. It would be better to compile a list of sports and ask about those specifically.

g. Sample responses can be found in the worked solutions in the online resources.
'Which of these sports do you watch on the television? Football, tennis, netball, golf, all of these, none of these.' — This is a good question, though it must be noted that results may vary depending on the selection of the sample population.

24. a. Sample responses can be found in the worked solutions in the online resources.

b. Sample responses can be found in the worked solutions in the online resources. Answers could include:
Data can be collected manually, or by using technology to capture data as specific events occur. For example, a computer system might be set up to automatically record the number of sales made per month.

c. Sample responses can be found in the worked solutions in the online resources. Answers could include:
Data is used for forward planning and budgeting purposes. For example, it can be used to identify the most or least popular items that the company sells; this data can then be used to see which items require updating or which items should no longer be sold.

25. a. A survey would be most appropriate as it would require fewer participants, take less time and cost less to survey.

b. Survey respondents should be selected by simple random sampling, which could include digital technologies to assist in random number generation.

26. 72 fish

27. a. Sample responses can be found in the worked solutions in the online resources.

b. Sample response: We are trying to find out whether the population approves or disapproves of the desalination plant and we are also trying to capture data about their reasons.

c. Sample response: You are collecting categorical data in both instances.

d. Sample responses can be found in the worked solutions in the online resources.

e. Sample responses can be found in the worked solutions in the online resources.

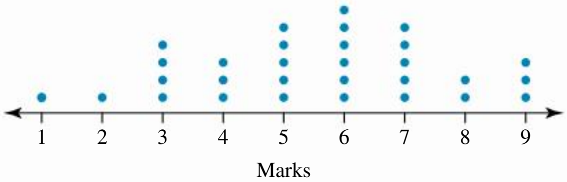
28. This inference is not reasonable as the survey was not a simple random sample and only included shoppers who were at the supermarket in the evenings. This could have created biased results.

12.3 Organising data

1. a. **Frequency of Maths test results**

Score x	Frequency f
1	1
2	1
3	4
4	3
5	5
6	6
7	5
8	2
9	3

b. **Multiple choice Maths test results**

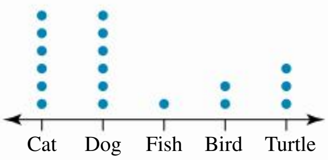


2. **Frequency of type of pet**

a.

Type of pet x	Frequency f
Cat	6
Dog	6
Fish	1
Bird	2
Turtle	3

b. **Type of pet**



c. No, only numeric data can be presented in a stem plot.

3. a.

Deaths by motor vehicle accidents (rates per 100 000 people)	Frequency f
$0- < 10$	2
$10- < 20$	11
$20- < 30$	10
$30- < 40$	3
$40- < 50$	4
$50- < 60$	0
$60- < 70$	1
$70- < 80$	1
	$\Sigma f = 32$

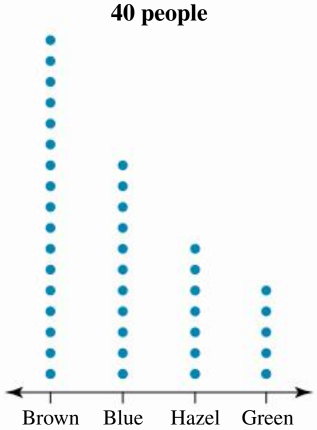
b. Deaths by motor vehicle accidents (rates per 100 000 people)
Key: 7 | 3=73 deaths per 100 000 people

Stem	Leaf
0	0 9
1	1 2 3 4 4 4 5 7 8 8 9
2	1 2 2 3 4 5 5 6 6 7
3	4 6 6
4	2 2 3 5
5	
6	0
7	1

4. a. Frequency of eye colour in a sample of 40 people.

Score x	Frequency f
Brown	17
Blue	11
Hazel	7
Green	5
	$f = 40$

b. **Eye colour in a sample of 40 people**



c. 17 out of 40 people (or 42.5% of the people) have brown eyes.

5. a. Key: 1 | 8 = 18

Stem	Leaf
1	8 9 9
2	0 0 0 1 1 2

b. Key: 1 | 9 = 19

Stem	Leaf
1	9 9
2	0 4 4 5 6 7 9
3	1 1 6 8
4	2 3

c. Key: 33 | 6 = 336

Stem	Leaf
33	6
34	6 6 9
35	0 3 9
36	2 8
37	1

6. Key: 12* | 5 = 125

Stem	Leaf
11	8 9
12	0 1 2 2 3 4
12	5 6 7 7 8 9
13	0 1 1 1 2 3 4
13	5 6 8 8 9 9
14	0 1 3

7. a. Running times for 40 students

Key: 4 | 3 = 43 minutes, 4* | 8 = 48 minutes

Stem	Leaf
2	1 3
2*	5 6 6 8 9
3	1 2 2 3 3 3 4
3*	7 7 9 9
4	0 1 1 2 2 3 4
4*	5 5 7 8 8 9
5	2 2 3
5*	5 6 7
6	0 3
6*	
7	1

b.

Run times (minutes) x	Number of runners f
20– < 25	2
25– < 30	5
30– < 35	7
35– < 40	4
40– < 45	7
45– < 50	6
50– < 55	3
55– < 60	3
60– < 65	2
65– < 70	0
70– < 75	1
$\Sigma f =$	40

c. i. 32 min
ii. 30 min – < 35 min

d. 3, stem plot

8. Key: $1 | 4 = 14$ $1^* | 7 = 17$

Stem	*Leaf*
1	4
1^*	7 8 8 9
2	0 1 3
2^*	5 6 7 9
3	2 4
3^*	5 6 7 8
4	2
4^*	5
5	3

9. D

10. a.

Life expectancy (years) x	**Number of age groups f**
$83- < 84$	1
$84- < 85$	21
$85- < 86$	7
$86- < 87$	5
$87- < 88$	2
$88- < 89$	4
$89- < 90$	0
$\Sigma f =$	40

b. Life expectancy of Australians across 40 different age groups.

Key: $86 | 1 = 86.1$ years

Stem	*Leaf*
83	7
84	1 1 1 4 4 5 5 5 5 5 5 5 6 7 7 7 7 8 8 9 9
85	0 0 1 2 7 8 9
86	0 5 6 6 9
87	3 6
88	3 3 8 8
89	

11. Median age of Australian females and males from 2017 to 2024

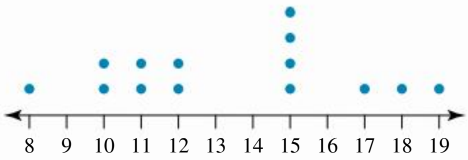
Key: $39 | 1 = 39.1$ years

Stem	*Leaf*
38	2 3
38^*	5 7 9
39	1 1 1
39^*	6 9
40	1 3 4
40^*	6 6 7

Note: The alternative method — showing categories for 38, 39 and 40 years — is not appropriate because it only allows for 3 groups, which is below the 5 minimum groups allowed.

Score x	**Frequency f**
$38- < 39$	5
$39- < 40$	5
$40- < 41$	6
	16

12. a. Dots are not aligned vertically. The horizontal scale is not equally spaced.

b. 

13. a. Key: $15 | 4 = 154$ cm

Leaf (boys)	*Stem*	*Leaf (girls)*
	14	5 8
9 8 6 6 4	15	2 2 5 6
8 4 2 0	16	0 1 1 3 5 8
2 1	17	
0	18	

b. The data could be grouped into smaller intervals, such as 5, instead of 10. For example, the stem value of 2 could be broken into two groups, encompassing values 20–24 and 25–29.

14. a. Class 1 : 25, class 2 : 27

b. Class 1 : 2, class 2 : 3

c. Class 1 : 24, class 2 : 20

d. A total of 11 students in class 2 scored better than class 1, but 6 students scored worse than class 1. Class 2's results are more widely spread than class 1's results.

15. a. Key: $2 | 1 = 21$ km/h

Stem	*Leaf*
1	5
2	0 0 2 7 9 9
3	0 0 0 0 2 4 5 6 7 8 8 9
4	0 0 0 0 0 2 4 5 5 5
5	
6	0

b. Sample responses can be found in the worked solutions in the online resources. A possible answer could include that there were a high number of speeds ending in 0 (possibly indicating a machine malfunction), and most of the speeds (24 out of 30) were within the legal speed limit.

16. a. Key: $25 | 1 = 25.1$ g

Stem	*Leaf*
23	9 9
24	0 5 5 6 8
25	1 1 1 3 4 4 8
26	3 3 7
27	3 3
28	1
29	5
30	
31	
32	
33	2

b. The range is $33.2 - 23.9 = 9.3$.

c. Sample responses can be found in the worked solutions in the online resources. Answers could mention that the plot for rounded values provides less detail.

Key: $2 \mid 4 = 24$ g

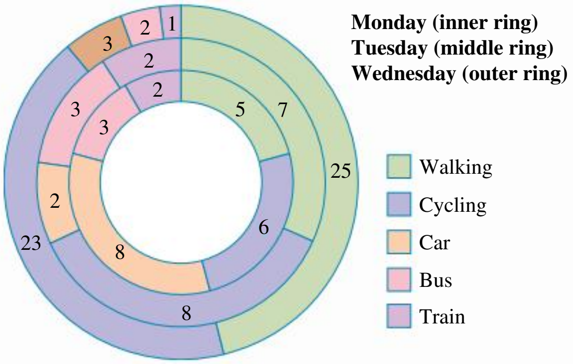
$2^* \mid 6 = 26$ g

Stem	*Leaf*
2	4 4 4
2^*	5 5 5 5 5 5 5 5 5 5 6 6 6 7 7 7 8
3	0 3

17. a. Sample responses can be found in the worked solutions in the online resources. Answers could mention that a doughnut graph and a pie graph are both circular, and both use colour to emphasise different categories.

b. Draw circles of different size with the same centre. Then use the same instructions as for creating a pie chart, but mark only the lines between the two circles.

c. **Methods of travel to school**



d. They are the same graph.

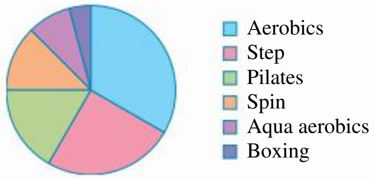
e. Sample responses can be found in the worked solutions in the online resources. Answers could include that a disadvantage of using a doughnut graph could be that you are restricted in the number of circles you can use, so this type of graph would not be suitable for large data sets.

18. 1296

12.4 Displaying data

1. a. See the figure at the foot of the page.*

b. **Pie chart of gym class participation**



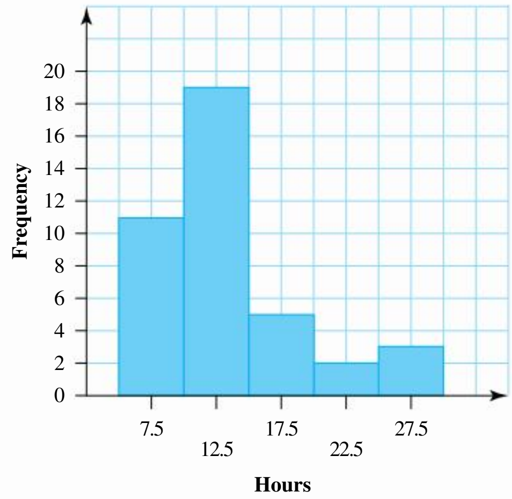
2. a. Continuous

b. 24

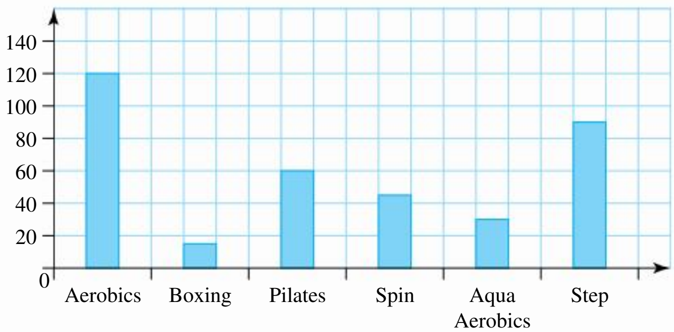
c.

Class interval	Midpoint (class centre)	Tally	Frequency
5– < 10	7.5	~~||||~~ ~~||||~~ |	11
10– < 15	12.5	~~||||~~ ~~||||~~ ~~||||~~ ||||	19
15– < 20	17.5	~~||||~~	5
20– < 25	22.5	||	2
25– < 30	27.5	|||	3
		Total	40

d. **Hours of phone use per week**



***1. a.** **Gym class participation**



3. a.

Class interval	Midpoint (class centre)	Tally	Frequency
0 – 9	4.5	~~||||~~ |	6
10 – 19	14.5	~~||||~~ ||	7
20 – 29	24.5	~~||||~~ |||	8
30 – 39	34.5	~~||||~~	5
40 – 49	44.5	||||	4
50 – 59	54.5	~~||||~~ ~~||||~~ ||	12
60 – 69	64.5	~~||||~~ |||	8
70 – 79	74.5	||||	4
80 – 89	84.5	||	2
		Total	56

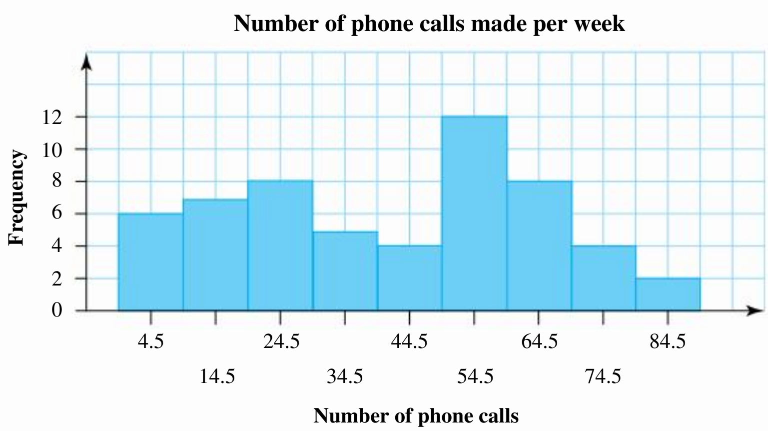
b. See the figure at the foot of the page.*

4. a.

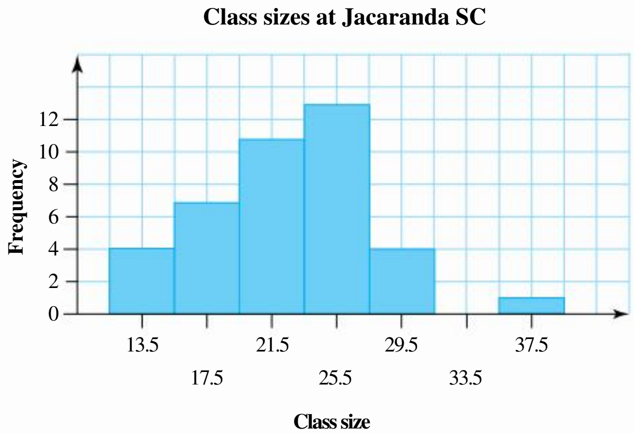
Class interval	Midpoint (class centre)	Tally	Frequency
12 – 15	13.5	||||	4
16 – 19	17.5	~~||||~~ ||	7
20 – 23	21.5	~~||||~~ ~~||||~~ |	11
24 – 27	25.5	~~||||~~ ~~||||~~ |||	13
28 – 31	29.5	||||	4
32 – 35	33.5	____	0
36 – 39	37.5	|	1
		Total	40

b. See the figure at the foot of the page.*

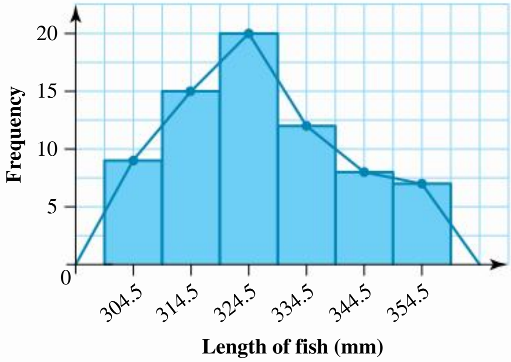
***3. b.**



***4. b.**

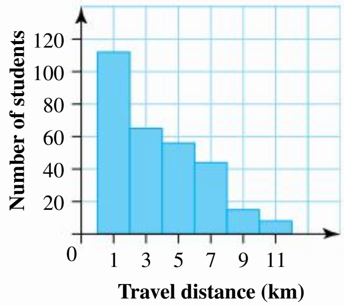


5.



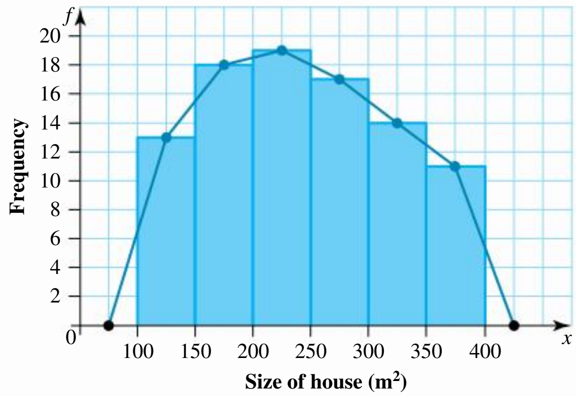
6.

Distances travelled



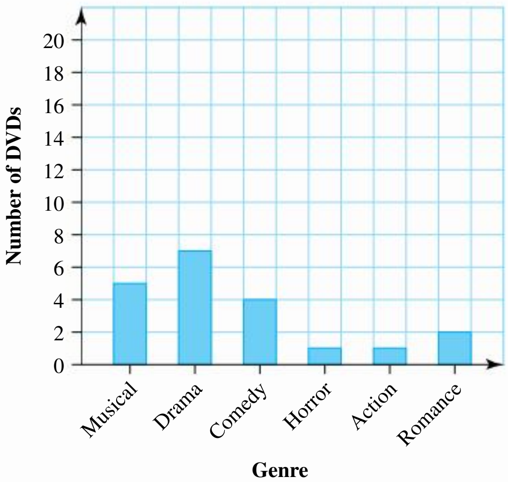
7. D

8.



9. a.

Bar chart of DVD genres (20 DVDs)

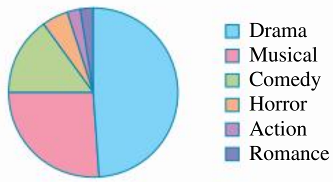


b.

Genre	Number of DVDs	Size of sector for a pie chart
Musical	52	93.6°
Drama	98	176.4°
Comedy	30	54°
Horror	10	18°
Action	5	9°
Romance	5	9°
Total	200	360°

c.

DVD genres
(sample size 200)



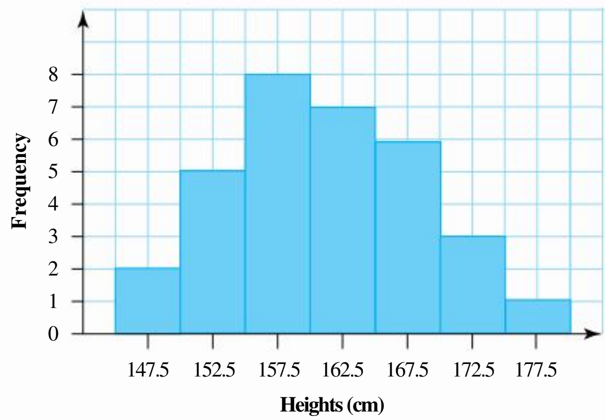
10. a. Continuous

b.

Class interval	Midpoint (class centre)	Tally	Frequency
145– < 150	147.5	||	2
150– < 155	152.5	~~||||~~	5
155– < 160	157.5	~~||||~~ |||	8
160– < 165	162.5	~~||||~~ ||	7
165– < 170	167.5	~~||||~~ |	6
170– < 175	172.5	|||	3
175– < 180	177.5	|	1
		Total	32

c.

Heights of Year 9 students



d. No. Individual scores are lost when data are grouped into class intervals.

e. C

f. D

g.

Class interval	Midpoint (class centre)	Tally	Frequency
145– < 148	146.5	|	1
148– < 151	149.5	||	2
151– < 154	152.5	|||	3
154– < 157	155.5	||||	4
157– < 160	158.5	~~||||~~	5
160– < 163	161.5	||||	4
163– < 166	164.5	|||	3
166– < 169	167.5	~~||||~~ |	6
169– < 172	170.5	||	2
172– < 175	173.5	|	1
175– < 178	176.5	____	0
178– < 181	179.5	|	1
		Total	32

h. See the figure at the foot of the page.*
Advantages: resolution is better — more details gained.
Disadvantages: requires more space to display.

11. a. Number of players b. 5

12. Horizontal axis uses same division for 5 and 7 year periods.

13. a. It is a circle viewed on an angle to produce an ellipse.
b. No, because it causes some angles to be larger and others to be smaller.

14. a. C
b. D
c. In general the boys are slightly taller than the girls, with the tallest 3 students all being boys.

15. a. Key: $0 \mid 0 = 0$

Leaf (Mitch)	*Stem*	*Leaf (Yani)*
3 2 1 1 1 1 0 0 0 0	0	0 0 1 1 1 1 1 2 2 2

Mitch scored between 0 and 3 goals inclusive.
Yani scored between 0 and 2 goals inclusive.

b. Mitch: 2, Yani: 3
c. Mitch
d. Yani: 11
e. Yani, because he scored goals in more games.

16. a. Key: $3 \mid 2 = 32$

Leaf (9A)	*Stem*	*Leaf (9B)*
2	3	
9	4	6
7 4	5	5
8 7 5 5 0	6	0 2 8
3	7	3 3 4 5 7 8
5	8	1
6	9	

b. 9A: 83%, 9B: 92%
c. 9A: 2
d. Sample responses can be found in the worked solutions in the online resources.
e. Sample responses can be found in the worked solutions in the online resources.
f. 9A: 64.25%, 9B: 68.5%

17. Sample responses can be found in the worked solutions in the online resources.

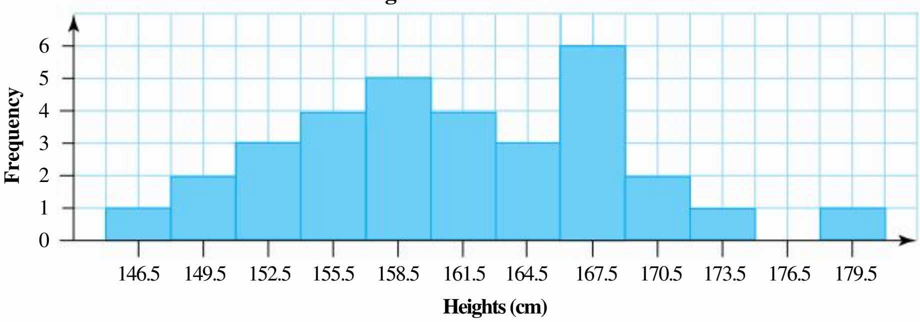
18. a. Categorical
b. A column graph or a pie chart

19. The average amount of times per week people used the sunscreen was 5.6. The minimum amount of times was 2 and the maximum amount was 21, although this value was much higher than the rest of the data. 75% of the people surveyed used the sunscreen between 4 and 7 times per week.

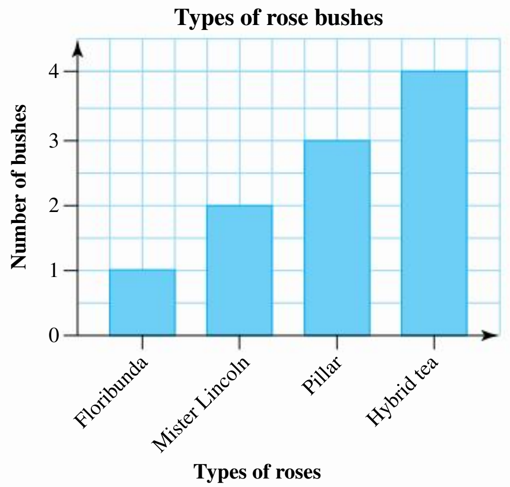
20. a. Floribunda: 36°, Mister Lincon: 72°, Pillar: 108°, Hybrid tea: 144°
b. 10

*10. h.

Heights of Year 9 students



c.



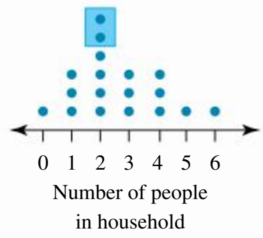
12.5 Measures of centre

1. a. 55, 54, 55 b. 8, 6.88, 8
2. 2.5, 2.55, 2
3. a. 23 b. 5
4. a. 116 b. 82
5. a. 6.28 b. 2.55
6. a. 25 b. 25.24 c. 25
7. a. 143.5, 142.6, 140–149
   b. 150.9, 148.5, 150–159
8. C
9. B
10. D
11. a. Symmetrical b. Symmetrical
    c. Symmetrical d. Negatively skewed
    e. Negatively skewed f. Positively skewed
12. a. Symmetrical b. Positively skewed
    c. Bimodal d. Negatively skewed with an outlier
13. The data is positively skewed with an outlier. This indicates that the mean will not be a suitable measure of centre. The median, with a value of 2, is an appropriate measure of centre because it is not affected by the presence of an outlier.
14. a. Numerical discrete data with a positively skewed distribution
    b. Median is the best measure of centre for a skewed distribution.
    c. Median = 11.5 customers
15. a. No
    b. 31 – 40
    c. The mean and median cannot be seen directly from the graph itself. You would need to calculate the mean and the median using a frequency distribution table, which can be constructed from the distribution shown.
16. a. Numerical discrete
    b. 45.2
    c. 45.25
    d. 40–49
17. a. Numerical continuous
    b. 153
    c. 157
    d. 140– < 160
18. a.

Score (x)	Frequency (f)	Freq × score ($f \times x$)
0	1	0
1	3	3
2	6	12
3	3	9
4	3	12
5	1	5
6	1	6
	$n = 18$	$\Sigma fx = 47$

$\bar{x} = 2.6$

b.



c. 2
d. i. Median ii. Mean iii. Mode

19. Possible answers could include that for data to be negatively skewed, the data needs to be a set with few low scores spread over a large range to the left. The most common scores are at the right. Assuming that the horizontal axis has test results in increasing order along the horizontal axis, it could be assumed that many of his students achieved good results.
20. a. Sample responses can be found in the worked solutions in the online resources. Possible answers include 7, 11 and 8, 10.
    b. Sample responses can be found in the worked solutions in the online resources. Possible answers include 7, 11 and 8, 10.
    c. No, 4 is the mode.
21. a. The median was calculated by taking the average of the 2 middle scores.
    b. 13
22. a. B
    b. $29 444
    c. A
    d. Sample responses can be found in the worked solutions in the online resources.
23. a. Sample responses can be found in the worked solutions in the online resources.
    b. No
24. a. 55 250 km, 52 000 km, 52 000 km
    b. Discuss in class.
    c. 51 810 km (it is reduced by 3440 km)
    d. 52 000 km

e. 3333

f. 50 000 km (92% of tyres last that distance or more)

25. Sample responses can be found in the worked solutions in the online resources. Possible answers include:

a. 1, 2, 5, 6, 6 $\dfrac{1+2+5+6+6}{5}=\dfrac{20}{5}=4$

b. 1, 2, 5, 6, 11 $\dfrac{1+2+5+6+11}{5}=\dfrac{25}{5}=5$

26. a. 6 b. > 6 c. 6 d. > 13

27. 8.93

28. 13, 31, 31, 47, 53 and 59

12.6 Comparing multiple data sets

1. Key: $2|3 = 23$

$2^*|5 = 25$

German	Stem	French
2 1 1 0	2	3 4
7 6 5 5	2^*	5 5 8
3 2 1 0 0	3	0 1 4 4
9 8 7 7	3^*	5 6 8 8 9
2 1	4	2 3 4 4
5	4^*	6 8

2. Key: $2|3 = 2.3$ kg

$2^*|7 = 2.7$ kg

Boys	Stem	Girls
	2^*	6 7
4 4	3	0 1 1 2 3
8 7 6	3^*	6 7
3 2	4	0
9 8	4^*	
0	5	

3. Partial sun is the better growing location.

4. a. Key: $15|0 = 150$ cm

Class A	Stem	Class B
	12	5
	13	1 5 6 8
	14	2 3 7 8 9
9 7 0	15	5
8 5 2 0	16	2
7 5 0	17	
7 1	18	

b. Students in Class A are older because they are taller.

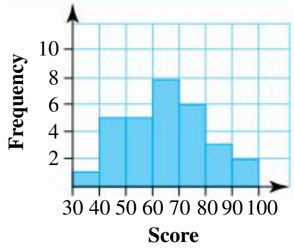
5. a. Cafe A has the larger spread of data, with a range of 65 years.

b. Cafe A: it has a greater spread of data and more customers from an older age group.

6. a.

Score	Tally	Frequency
30–39	𝍷	1
40–49	𝍸	5
50–59	𝍸	5
60–69	𝍸𝍷𝍷𝍷	8
70–79	𝍸𝍷	6
80–89	𝍷𝍷𝍷	3
90–99	𝍷𝍷	2

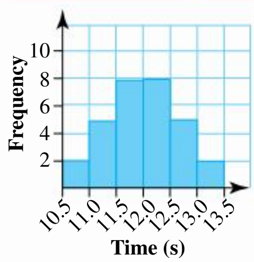
b.



7. a.

Class	Class centre	Frequency
10.5–<11	10.7	2
11–<11.5	11.2	5
11.5–<12	11.7	8
12–<12.5	12.2	8
12.5–<13	12.7	5
13–<13.5	13.2	2

b.



8. Brand B performed consistently higher than Brand A.

9. Brand A data appears to be more consistent in spread, with most batteries lasting 6 to 7 days. Brand B data appears symmetrical around 5 days, with three potential outliers at 9 and 10 days. Factors affecting charge would include use; for example, if a scooter was used sparingly and for short distances, it would require charging less frequently.

10. The Doors computer has a symmetrical spread centred around 0.3 seconds. The Pears computer has the opposite spread, with a dip at 0.3 and peaks at 0.1 and 0.5 seconds. The Doors computer would be recommended, as it seems to perform consistently within the 0.2–0.4 second range, with most tasks taking less than 0.5 seconds to complete.

11. a. Key: $2|3 = 23$ trucks

$2^*|5 = 25$ trucks

A	Stem	B
2 1	1	0
7 7 6 6 5	1^*	5 6
4 3 2 1 0	2	0 1 3
7 5	2^*	5 6 8 9
	3	0 1 2
	3^*	5

b. Statistical analysis

Supermarket A	Supermarket B
Mean = 19 $s_x = 4.9$	Mean = 24.4 $s_x = 7.2$
Min = 11 $Q_1 = 16$	Min = 10 $Q_1 = 20$
Median = 18.5 $Q_3 = 23$	Median = 25.5 $Q_3 = 30$
Max = 27 IQR = 7	Max = 35 IQR = 10

For Supermarket A, the mean is 19, the median is 18.5, the standard deviation is 4.9 and the interquartile range is 7. The distribution is symmetric.
For Supermarket B, the mean is 24.4, the median is 25.5, the standard deviation is 7.2 and the interquartile range is 10. The distribution is slightly negatively skewed. The centre and spread of the distribution of supermarket B is higher than that of supermarket A. There is greater variation in the number of trucks arriving at supermarket B.

12. a. Key: $1|7 = 17$ marks

Class A	Stem	Class B
	1	0
3 2	1	2 3
5 5 4 4	1	4 4 5
7 6	1	7
	1	9

b. Statistical analysis

Class A	Class B
Mean = 14.5 $s_x = 1.6$	Mean = 14.25 $s_x = 2.8$
Min = 12 $Q_1 = 13.5$	Min = 10 $Q_1 = 12.5$
Median = 14.5 $Q_3 = 15.5$	Median = 14 $Q_3 = 16$
Max = 17 IQR = 2	Max = 19 IQR = 3.5

For the marks of the Class A, the mean is 14.5, the median is 14.5, the standard deviation is 1.6 and the interquartile range is 2.
The distribution is symmetric.
For the marks of the Class B, the mean is 14.25, the median is 14, the standard deviation is 2.8 and the interquartile range is 3.5.
The distribution is symmetric.
The centre of each distribution is about the same.

13. a. Key: $2|4 = 24$ marks

$2^*|6 = 25$ marks

First year	Stem	Second year
	2	2
	2^*	6 7 8
1 0	3	0 1 1 3 4
9 7 5	3^*	6
3 2 1 1	4	
6	4^*	

b. Statistical analysis

First year	Second year
Mean = 38.5 $s_x = 5.2$	Mean = 29.8 $s_x = 4.2$
Min = 30 $Q_1 = 35$	Min = 22 $Q_1 = 27$
Median = 40 $Q_3 = 42$	Median = 30.5 $Q_3 = 33$
Max = 46 IQR = 7	Max = 36 IQR = 6

The distributions of marks for the first year and for the second year are each symmetric.
For the first-year marks, the mean is 38.5, the median is 40, the standard deviation is 5.2 and the interquartile range is 7. The distribution is symmetric.
For the second-year marks, the mean is 29.8, the median is 30.5, the standard deviation is 4.2 and the interquartile range is 6.
The spread of each of the distributions is much the same, but the centre of each distribution is quite different, with the centre of the second-year distribution quite a lot lower. The work may have become a lot harder!

14. a. Key: $2|3 = 23$ years old

$2^*|5 = 25$ years old

Football	Stem	Soccer
4 3	2	2
8 7 6 5	2^*	5
1 0	3	0 1
	3^*	6 7
	4	2
	4^*	6

b. Statistical analysis

Football	Soccer
Mean = 26.75 s_x = 2.8	Mean = 33.6 s_x = 8.2
Min = 23 Q_1 = 24.5	Min = 22 Q_1 = 27.5
Median = 26.5 Q_3 = 29	Median = 33.5 Q_3 = 39.5
Max = 31 IQR = 4.5	Max = 46 IQR = 12

For the distribution of the football players, the mean is 26.75, the median is 26.5, the standard deviation is 2.8 and the interquartile range is 4.5.
For the distribution of the soccer players, the mean is 33.6, the median is 33.5, the standard deviation is 8.2 and the interquartile range is 12.
The centre of the distributions is very different: it is much higher for the soccer players. The spread of the ages of the football players who attend the fitness class is very small but very large for soccer players. Older soccer players are more likely to the attend fitness classes than older football players.

15. a. Key: 3|2 = 32

Kindergarten	Stem	Prep. school
3	0	5
4 3	1	2 7
8 5	2	5 7
6 2	3	2 5
7 1	4	4 6
0	5	2

b. Statistical analysis

Kindergarten	Prep. school
Mean = 28.9 s_x = 15.4	Mean = 29.5 s_x = 15.3
Min = 3 Q_1 = 14	Min = 5 Q_1 = 17
Median = 30 Q_3 = 41	Median = 29.5 Q_3 = 44
Max = 50 IQR = 27	Max = 52 IQR = 27

For the distribution of scores of the kindergarten children, the mean is 28.9, the median is 30, the standard deviation is 15.4 and the interquartile range is 27.
For the distribution of scores for the prep. school children, the mean is 29.5, the median is 29.5, the standard deviation is 15.3 and the interquartile range is 27.
The distributions are very similar. There is not a lot of difference between the way the kindergarten children and the prep. school children scored.

16. a. Class 1:25; Class 2:27

b. Class 1:2; Class 2:3

c. Class 1:24; Class 2:20

d. Class 1's results are clustered tightly, with a range of 6 from 10 to 16 marks out of 20. Class 2's results are more widely spread, with a range of 17 from 3 to 20. 11 of 27 students scored higher than any member of Class 1.

17. Key: 1|6 = 16 years

Male	Stem	Female
987766	1	5566677789
9853332110	2	1122233578
8864221	3	00145
533	4	368
2	5	047
0	6	5

18. a. Worker F

b. 4

19. a. Company B

b. Company A has a positive skew, whereas Company B has a negative skew.

c. Company B

20. a. 2016

b. 2018

c. Answers will vary but could include that more people are on the roads during the Christmas break and school holidays.

21. Both Brand A and B's battery life is clustered around 10–38 hours. Brand A had more batteries with a life of less than 10 hours (4 batteries) in comparison to Brand B (1 battery). The median value of Brand A was 21 hours in comparison to Brand B, which was 22 hours. From this comparison, Brand B has the better battery life span.

22. Provider B would be the better provider, because it offers a more stable connection. The consistent performance, as evidenced by the small range of speeds, means you can expect a reliable connection every time.

23. Key: 23|1 = \$231

Suburb A	Stem	Suburb B
7 5 5 0 0 0	20	
5 5 1 0 0	21	
5 5	22	
5 5 1	23	1 2 5
4 2 0	24	0 5 5
6	25	1 5 5 8
	26	0 5
	27	0 0 5 5
	28	0 0 5 5

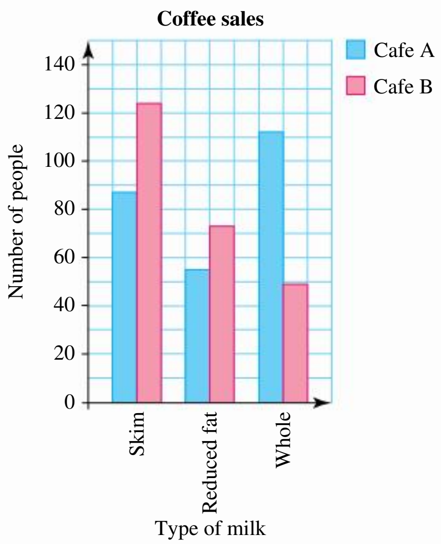
Suburb A has a lower median rent (\$215) compared to suburb B (\$259). Suburb A has a positively skewed distribution for median rental price whereas Suburb B has

an even distribution. The range of rental price for Suburb A is \$200–\$256, compared to Suburb B, which is \$231–\$285. More than half of the proprieties in Suburb A are cheaper than the cheapest house available in Suburb B.
Overall, despite some overlap in prices, your friend's claim is incorrect — Suburb A is cheaper.

24. Both stations had the same number of customers in the week. However, northbound was more evenly distributed and consistent. Southbound had bimodal peaks on day 3 and day 7, with more than half of customers coming in these two days. The best location for a diner would be northbound, as there would be a consistent flow of customers every day.

25. a. 500

b.



c. 34%

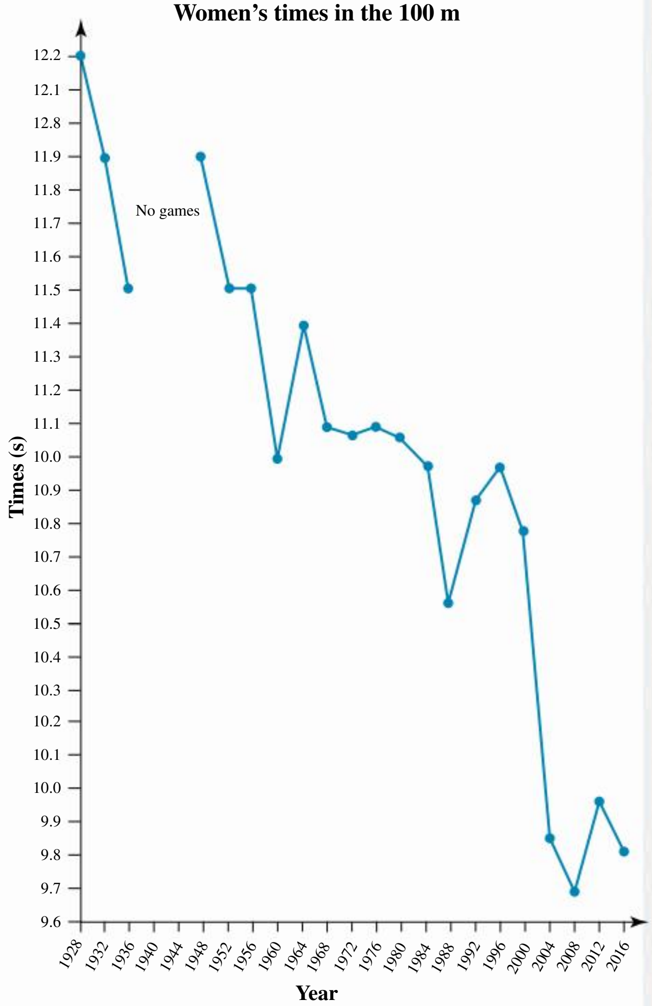
d. 26%

e. 49%

Project

1. Stretching the axes vertically gives the impression of a fall in the value of the shares. Stretching the axes horizontally gives the impression the value of the shares has not changed very much.

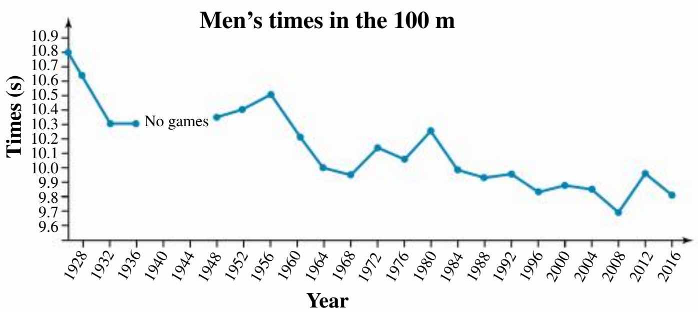
2. See figure at the foot of the page.*



3. For the women's graph, stretching the vertical axis and reducing the distance between the values on the horizontal axis gives the impression women's times have fallen dramatically. To show that the men's times have not fallen greatly, you can reduce the distance between the values on the vertical axis and stretch the values on the horizontal axis.

4. Sample responses can be found in the worked solutions in the online resources.

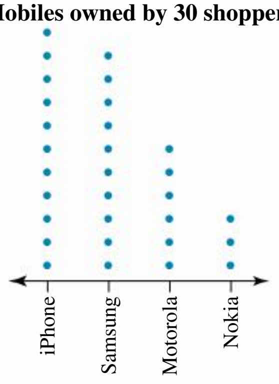
***2.**



12.7 Review questions

1. C
2. D
3. C
4. D
5. B
6. C
7. a. Categorical nominal; frequency table, dot plot, bar chart or pie chart
   b. Categorical ordinal; frequency table, dot plot, bar chart or pie chart
   c. Numerical discrete; stem plot
   d. Numerical continuous: box plot
8. a. Sample b. Sample c. Population
9. a. This is categorical nominal data. It could be displayed as a frequency table, dot plot, bar chart or pie chart.
   b. Examples of a dot plot and frequency table are shown. Pie or bar charts are also acceptable.

Mobiles owned by 30 shoppers



Mobiles owned by 30 shoppers	
Brand of mobile	**Number of mobiles**
iPhone	11
Samsung	10
Motorola	6
Nokia	3

10. a. Key: $1 \mid 2 = 12$ songs

Stem	*Leaf*
0	2 9
1	
2	
3	1 4 5
4	3 4 5 7 8
5	4 6 9
6	6 7 7
7	0 4 7
8	2 9
9	0 6
10	3
11	1

   b. Positively skewed with outliers
   c. **Number of songs on favourite Spotify playlist**

Number of songs	Number of playlists
0-19	2
20-39	3
40-59	8
60-79	6
80-99	4
100-119	2

   d. Numerical discrete, symmetrical distribution, mean $= 60$ (rounded to the nearest whole number) and standard deviation $= 27.3$
11. a. See figure at the foot of the page.*
   b. Positively skewed
   c. The median is \$166 755, which is \$167 000 to the nearest thousand.
   d. The modal class is \$100 001–\$200 000

*11. a.

2019 AFL player salaries

12. a. Bimodal, symmetric
b. Unimodal, positively skewed with one outlier
c. Bimodal, negatively skewed
d. Unimodal, negatively skewed

13. a. Key: 1|3 = 13 hours

Full-time	Stem	Volunteer
1	0	
3 3 2 2	0	
5 4 4	0	
6	0	
	0	8
	1	0 1 1
	1	2 3 3
	1	4 5
	1	
	1	

b. The full-time firefighters distribution is slightly positively skewed and the volunteers distribution is approximately symmetric.
The centre of the volunteers, distribution is much higher than that of the full-time firefighters distribution.

14. a. 5.6%
b. 2%
c. 2%
d. In order to give the impression that the pay rise is generous, the company would quote the mean.
e. The trade union would quote the median or the mode, because they would be arguing that the wage rise is too low.
f. The mode is probably the most 'honest' reflection of last year's wage rise, because it represents the situation of most of the workers.

15. a. 8, 20 b. 9, 19

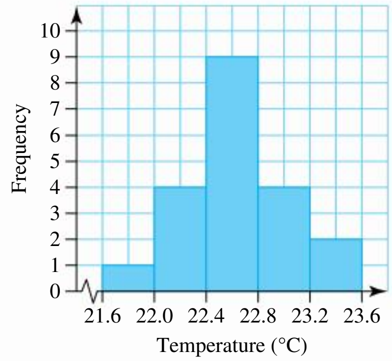
16. Sample response: 3, 4, 5, 6, 7, 10, 10, 13, 14

17. a. Key: 22 | 5 = 225

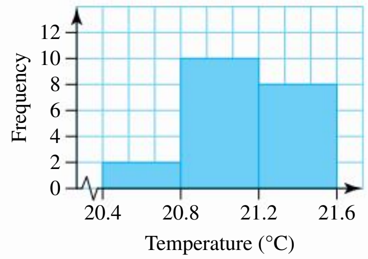
Suburb A	Stem	Suburb B
	22	5 8
	23	1 2 5
2	24	0 1 5
	25	5
	26	
5 5	27	0 0 5 5
5 4 1 1	28	0 0 5
6 5 5 0	29	5
7 5 5 0 0 0	30	0 5
5 0 0	31	0

b. Answers will vary. Suburb A has higher rent than Suburb B.
c. Answers will vary. Suburb C is higher than Suburb A, which is higher than Suburb B. The most expensive suburb to rent in is Suburb C. The least expensive suburb to rent a one-bedroom unit in is Suburb B.

18. a.



b.



c. The distribution for the interval 2005–2024 is approximately symmetrical. The distribution for the interval 1893–1912 is negatively skewed, with a smaller overall range. Both distributions are unimodal. The average temperature of 2005–2024 has more variance than the temperature of 1893–1912.

Modelling with probability and statistics

Fully worked solutions for this topic are available online.

LESSON Modelling with probability and statistics

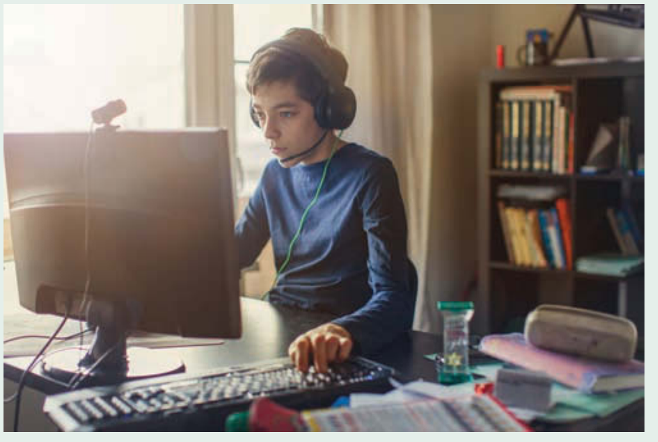
Introduction

The world we live in produces an ever-increasing amount of data. Being able to understand and interpret these large sets of data is a valuable skill. The Australian Bureau of Statistics runs a census night every five years to collect data about the population. This data is analysed so future decisions can be made by the government. Probability is widely used to describe the chance of an event happening. By engaging with these topics, you will develop a better understanding of logical reasoning and making informed decisions.

MODELLING TASK 1: Structured

Time spent on computer games

Henry and Oliver love playing their computer games. They decide to record the time they spend playing a particular game for 15 days. As they play AFL in winter and cricket in summer, they try to limit their time playing computer games. In school, they have learned about back-to-back stem plots. The table below shows their times.



Key: 1 | 8 = 18 minutes

Henry Leaf	**Stem**	**Oliver** Leaf
	1	8 9
9 9 8 5 5 4	2	2 5 5 7
8 6 5 5 2	3	0 0 2 5 5 8
2 0 0	4	0 3 6
	5	
9	6	

a. Analyse the situation:
- Give a brief overview of what information is given.
- What statistics can be determined and are appropriate for their analysis?
- What type of data is displayed?

b. Determine assumptions and constraints:
- Identify any assumptions that you may need to make.
- What are the constraints/limitations on the data?
- Is the data reliable?

c. Represent the situation mathematically:
- Calculate the mean, mode, median and range of the data for Henry.
- Calculate the mean, mode, median and range of the data for Oliver.
- Describe the shape of the data for both Henry and Oliver.

d. Interpret and communicate findings:
- Identify any outliers and discuss what effect this has on the statistical results.
- Recalculate and compare the results when the outlier is removed.
- Compare Henry and Oliver's results.

MODELLING TASK 2: Structured

Deck of cards

Two bundles of cards from a normal set of playing cards have been made to play a game. One bundle has the spades from 1 to 9. The other bundle has the hearts from 1 to 9. Two cards are drawn consecutively from each bundle to make a two-digit number. From the bundle of hearts, the card will be replaced before the second card is drawn. From the bundle of spades, the card will not be replaced before the second card is chosen. Would the probability that the number formed is even be the same for both the bundle of hearts and the bundle of spades.

a. Analyse the situation:
- What is the difference between the two experiments?
- Should the outcomes from the bundles be the same?
- How will you determine the two-digit numbers?

b. Determine assumptions and constraints:
- Are you making any assumptions in this experiment?
- What are the limitations you may need to consider?

c. Represent the situation mathematically:
- Generate a data set for drawing two cards from the bundle of hearts.
- Determine the experimental probability that the two-digit number formed from the bundle of hearts is even.
- Generate a data set for drawing two cards from the bundle of spades.
- Determine the experimental probability that the two-digit number formed from the bundle of spades is even.
- Determine the theoretical probability of forming two-digit numbers from 1 to 9 with and without repeats.

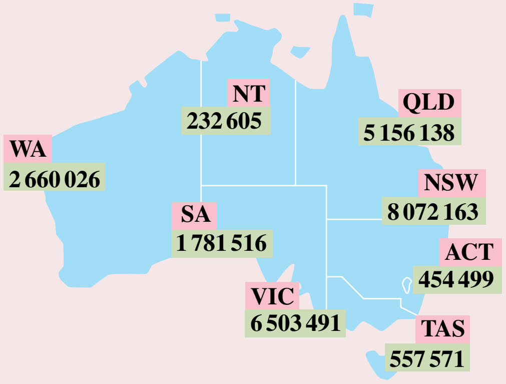
d. Interpret and communicate findings:
- Present your calculations.
- Discuss any assumptions made and how these may affect the outcomes.
- Compare your experimental and theoretical probabilities when randomly making two-digit numbers with or without replacement.

MODELLING TASK 3: Open ended

The 2021 Census

The Australian Bureau of Statistics undertakes a census every five years. The following table comes from information released by the Bureau in 2022.

State or Territory	2016	2021
New South Wales	7 480 228	8 072 163
Victoria	5 926 624	6 503 491
Queensland	4 703 193	5 156 138
South Australia	1 676 653	1 781 516
Western Australia	2 474 410	2 660 026
Tasmania	509 965	557 571
Northern Territory	228 833	232 605
Australian Capital Territory	397 397	454 499
Australia	**23 401 892**	**25 422 788**



What does the 2021 Census tell you about Australia?

Consider the following:

- Where do the majority of people live?
- Where was the fastest/slowest growth rate?
- What else can you infer from this data?
- What other information could be helpful in looking at the population of Australia?
- How could you present this data graphically?

MODELLING TASK 4: Open ended

Australian Bureau of Statistics Population Clock Pyramid

The Australian Bureau of Statistics is used to make many predictions to assist government in making decisions for the country. Population projections help assess future needs for resources like food, water and energy, as well as services like eduction and healthcare. The Australian Bureau of Statistics Population Clock Pyramid is a projection based on births, deaths, immigration with an overall total population increase of one person every 57 seconds. The Population Clock Pyramid shows populations from 1981 until years in the future and is accessible online.

What does it tell you about Australia from the past, the present and into the future?

Consider the following:

- What facts does each Population Clock Pyramid give?
- Does it give any specific data along with the Pyramid?
- Choose three years, one in the past, the present and the future, describing the shape, spread and centre in terms of the age structure of males and females in Australia.
- Are there any noticeable trends?