Essential Insight Exam Guide

Mathematics Specialist

Year 12 WACE

Western Australian Curriculum

2025 Edition

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Essential Insight Exam Guide
Mathematics Specialist
Year 12 WACE

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Acknowledgements
School Curriculum and Standards Authority. (2019-2023). ATAR Examinations and Marking Keys. The School Curriculum and Standards Authority does not endorse this publication or product.

Contents

Unit 3

Unit 3.1 – Complex numbers

Section 1

2023 Section 1 Question 2 **Complex numbers**	$P(z) = z^5 + az^4 + bz^3 + cz^2 + dz + 14$ is a fifth order polynomial with real coefficients. It is known that $P(z) = (z - z_0)\,Q(z)$ where z_0 is real and $Q(z)$ is a fourth order polynomial. Two roots of $P(z)$ are $z_1 = 1 + i$ and $z_2 = 2 + \sqrt{3}i$. (a) Determine $Q(z)$ in expanded form. (3 marks) (b) Determine the values of the coefficients a, b, c and d. (2 marks)

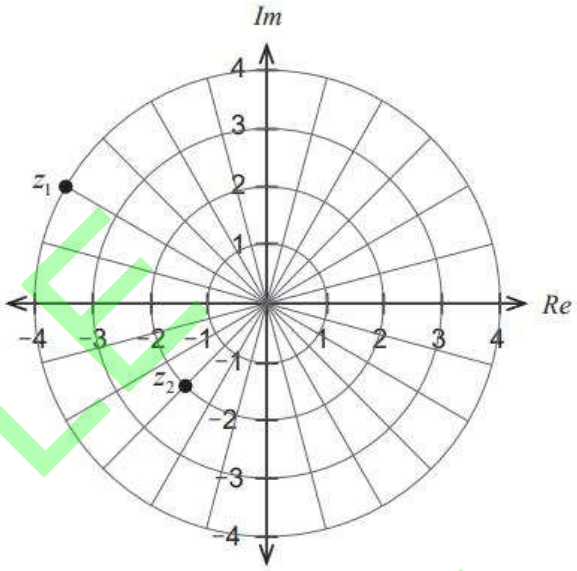
2023 Section 1 Question 6 **Complex numbers**	Solve the complex equation $z^4 = 2 - 2\sqrt{3}i$ giving solutions in the form $rcis\theta$ where $-\pi < \theta \leq \pi$. (5 marks)

2023 **Section 1** **Question 8** **Complex numbers**	In the following simultaneous equations, a and b are real numbers. $a^3 = 3ab^2 + 14$ $b^3 = 3a^2b + 2\sqrt{5}$ In order to determine the value of $a^2 + b^2$ from these equations, it is useful to consider the complex expansion for $(a + bi)^3$. Hence, or otherwise, determine the exact value of $a^2 + b^2$. (4 marks)

2022
Section 1
Question 6

Complex numbers

Two complex numbers $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ and z_2 are shown in the Argand plane below.



(a) Determine the exact polar form for z_2. (2 marks)

(b) Plot the complex number $w = z_1 \times (z_2)^{-1}$ on the Argand diagram above. (3 marks)

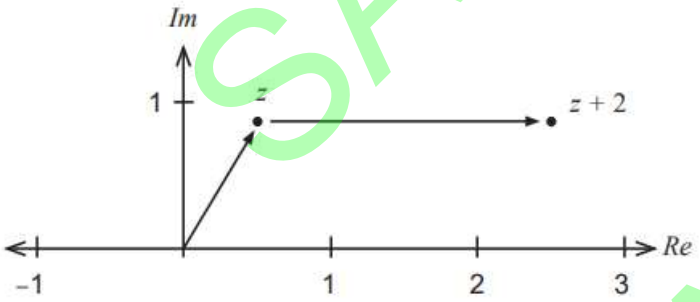
(c) If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution of the equation $z^n = r$ where r is a positive real number and n is a positive integer, determine the smallest possible value for r in the form 2^p.

Justify your answer. (3 marks)

2021 Section 1 Question 1

Complex numbers

The Argand diagram below shows the complex numbers z and $z+2$ where $z = cis\left(\frac{\pi}{3}\right)$.



Determine the exact value for:

(a) $Arg\,(-z)$. (1 mark)

	(b) $\lvert z + 2 \rvert$. (3 marks)

2021 Section 1 Question 6 **Complex numbers**	Consider the quartic polynomial $P(z) = z^4 - 6z^3 + 31z^2 - 52z + 60$. (a) Given that $P(2+4i) = 0$, determine a quadratic factor of $P(z)$. (2 marks) (b) Hence solve the equation $z^4 - 6z^3 + 31z^2 - 52z + 60 = 0$. (3 marks)

2021 Section 1 Question 7 **Complex numbers**	The number 2021 can be expressed as a product of two consecutive prime numbers: 43 × 47 = 2021. Consider the complex equation $z^{43} = 1$. (a) Write an expression for the roots of $z^{43} = 1$. (2 marks) Let w be any one of the roots of the equation $z^{43} = 1$. (b) How many of these roots will also be a solution of the equation $z^{47} = 1$? Justify your answer. (3 marks)

2020 Section 1 Question 8 **Complex numbers**	Consider the complex sum: $\sum_{n=1}^{2020} n i^n = 1i^1 + 2i^2 + 3i^3 + ... + 2020i^{2020}$ Express the value of this sum in the form $r\,cis\,\theta$ where $-\pi < \theta \leq \pi$. (3 marks)

2019 Section 1 Question 2 **Complex numbers**	Consider the function $P(z) = z^4 - 2z^3 + 14z^2 - 8z + 40$, defined over the complex numbers. (a) Show that $(z - 2i)$ is a factor of $P(z)$. (2 marks) (b) Hence or otherwise, solve the equation $P(z) = 0$, giving solutions in the form $a + bi$. (4 marks)

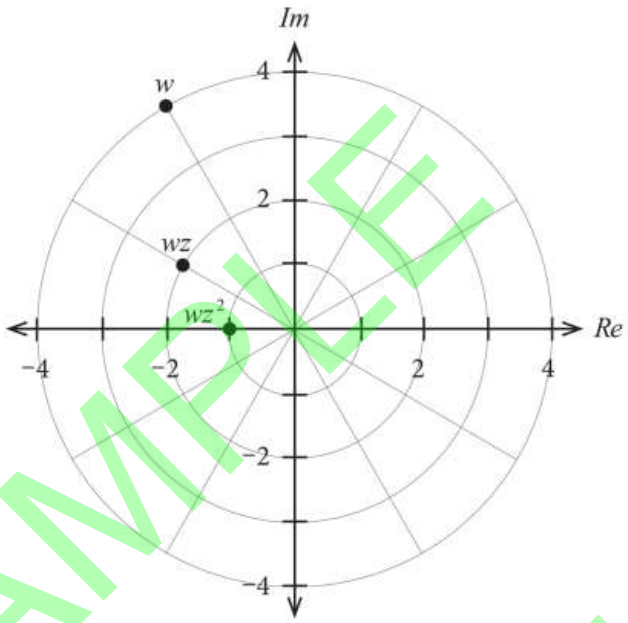
2019 **Section 1** **Question 9** **Complex numbers**	Consider the complex equation $z^n - 1 = 0$, where n is any positive integer $n \geq 3$. If the roots are designated as $z_0, z_1, z_2, \ldots, z_{n-1}$, then determine the exact value for the product of the roots $p = z_0 \times z_1 \times z_2 \times \ldots \times z_{n-1}$. (4 marks)

Section 2

2023 Section 2 Question 10

Complex numbers

The complex number $w = 4cis\left(\frac{2\pi}{3}\right)$ is shown in the Argand diagram, along with the complex numbers wz and wz^2.



(a) Express wz and wz^2 in exact polar form. (2 marks)

Consider the geometric transformation(s) applied to transform $w \rightarrow wz \rightarrow wz^2$ etc.

(b) Describe the geometric transformation(s) performed by successive multiplication by z. (2 marks)

(c) Determine z in exact polar form. (1 mark)

(d) Describe the geometric transformation(s) performed by successive multiplication by z^{-1}. (2 marks)

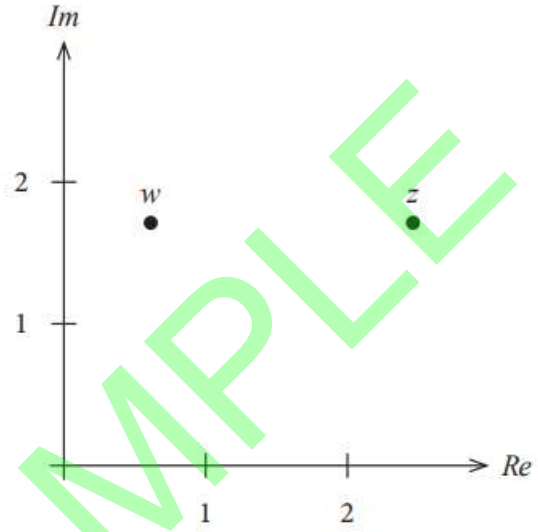
2023 Section 2 Question 12

Complex numbers

Complex numbers z and w are shown in the Argand diagram below. It is known that:

$|z| = 3,\ Arg\,(z) = \theta$ where $0 < \theta < \frac{\pi}{4}$

$w = z - k$ such that $Arg\,(w) = 2\theta$ where $Im\,(k) = 0,\ k > 0.$



(a) Represent the given information on the Argand diagram. (3 marks)

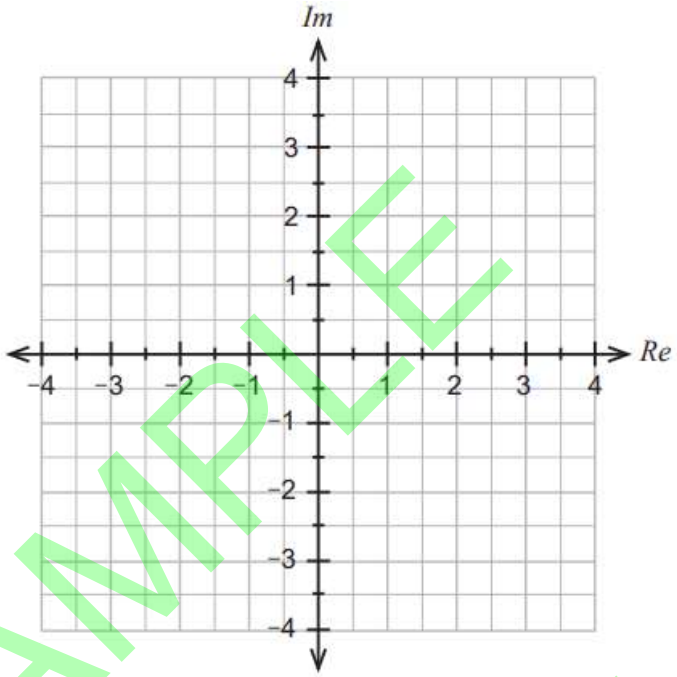
(b) Determine a simplified expression for k in terms of θ. Justify your answer. (3 marks)

2022 Section 2 Question 9

Complex numbers

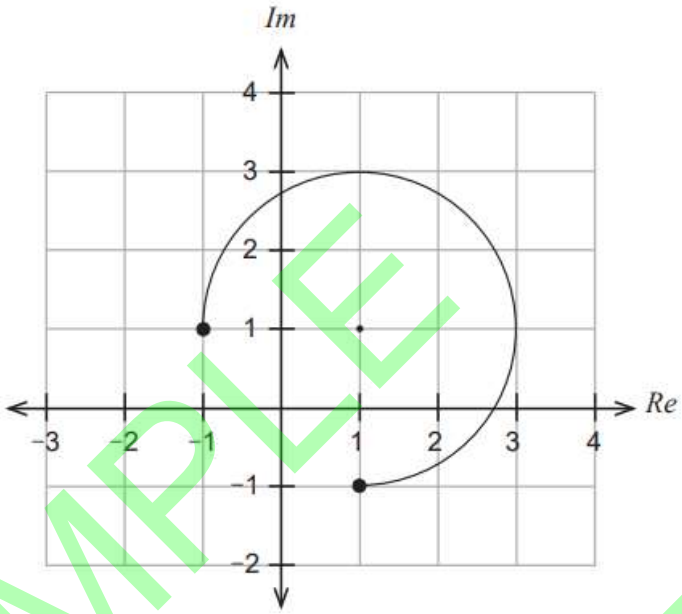
(a) Sketch the locus of a complex number z satisfying the condition:

$|z-2i|+|z-(3-2i)|=5$ (2 marks)

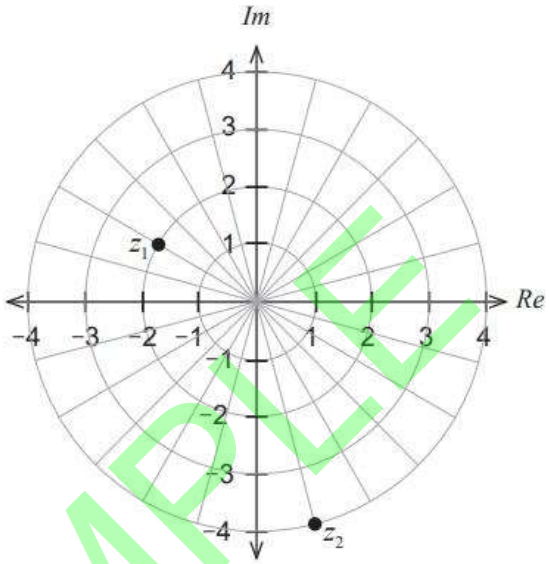


(b) Describe the locus of the equation $(z+i)(\overline{z+i})=2$. (3 marks)

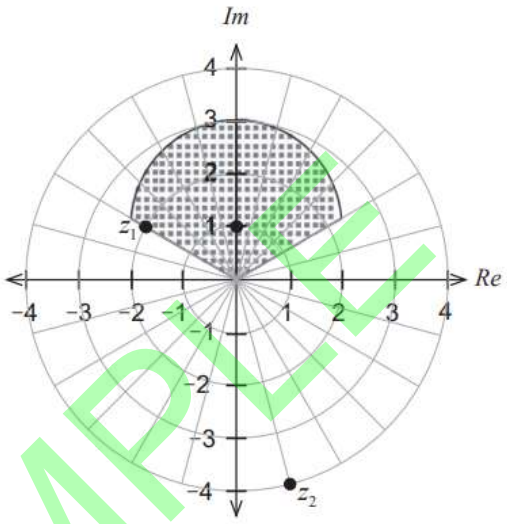
(c) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (3 marks)

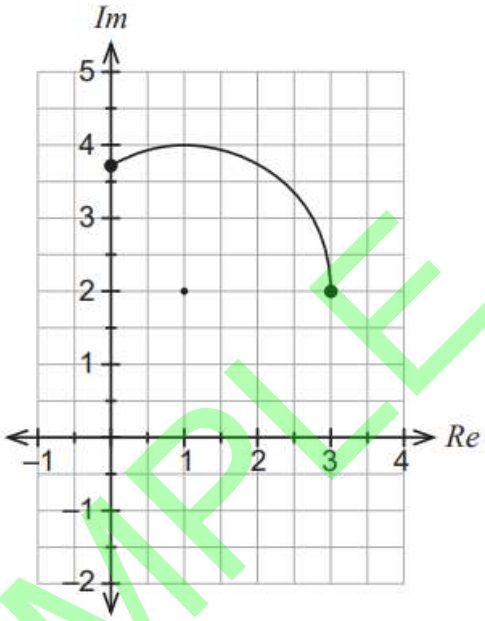
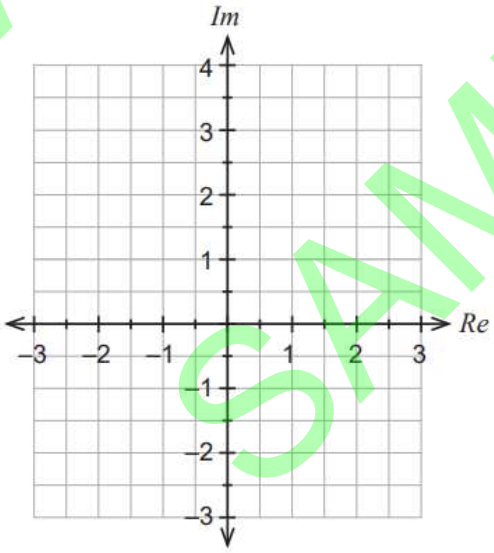


2022 Section 2 Question 18 **Complex numbers**	(a) Show that for all positive integers n and complex numbers z where $0 \le \theta \le \frac{\pi}{2}$, $(z^n - cis\,(\theta))(z^n + cis\,(-\theta)) = z^{2n} - (2i \sin\theta)\, z^n - 1$. (3 marks) (b) Hence, using the result from part (a), obtain all the solutions to the equation $z^6 - (i)z^3 - 1 = 0$ in exact polar form. (4 marks)

2021 Section 2 Question 11 **Complex numbers**	Two complex numbers z_1 and z_2 are shown in the Argand plane below.  (a) Write the expression for z_1 in exact polar form. (2 marks) (b) Write the expression for z_1 in exact Cartesian form. (1 mark) (c) Plot the complex number iz_1 on the Argand diagram above. (2 marks)

(d) A sketch of the locus of a complex number z is shown below. The upper boundary of the locus is part of a circle, centred at $z = i$. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (4 marks)



2020 Section 2 Question 10 **Complex numbers**	(a) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (4 marks)  (b) Sketch the locus of the equation $\|z+2\| = \|z-i\| + \sqrt{5}$ in the Argand diagram below. (3 marks) 

2020 Section 2 Question 11 **Complex numbers**	Let z, w and u be complex numbers where: $w = (1+i)\bar{z} \qquad Arg(w) = \frac{\pi}{3} \qquad |w| = 2$ $u = \frac{z}{2-2i}$ (a) Determine $Arg(u)$ exactly. (3 marks) (b) Determine $\|u\|$ exactly. (2 marks)

2020 Section 2 Question 13 **Complex numbers**	Solve the equation $z^4 = 8\sqrt{3} + 8i$ giving exact solutions in the form $r\,cis\,\theta$ where $-\pi < \theta \leq \pi$. (4 marks)

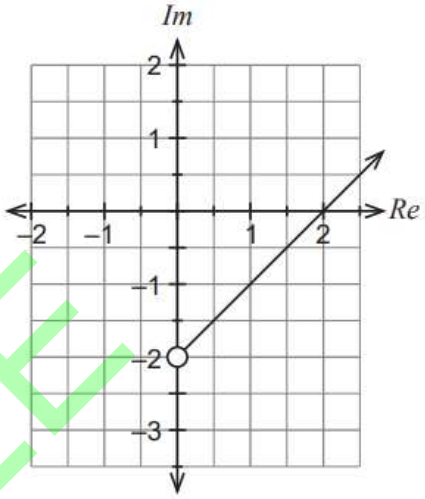
2020 Section 2 Question 15

Complex numbers

Let $z = r\,cis\,\theta$ be a complex number such that $\frac{\pi}{2} < \theta < \pi$.

(a) Express in terms of r and θ the complex number $\frac{\bar{z}}{-\sqrt{2}\,(i+1)}$. (3 marks)

(b) Express $\alpha = Arg(z - ri)$ in terms of θ where $0 < \alpha < 2\pi$. (3 marks)

2019 Section 2 Question 10 **Complex numbers**	The sketch of the locus of a complex number $z = x + iy$ is shown below.  (a) Given that the equation for the above locus is written as $Arg(z - z_0) = k\pi$, determine the value of the constants z_0 and k. (2 marks) (b) Determine the minimum value for $\|z - i\|$ as an exact value. (3 marks)

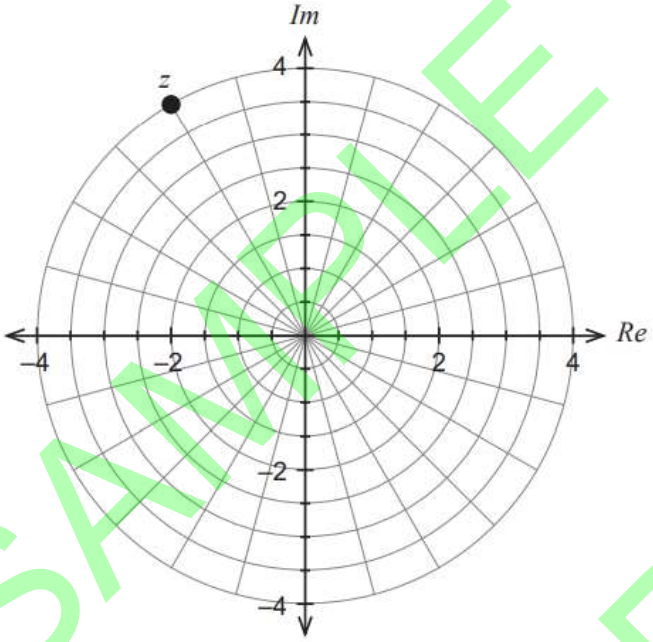
2019 Section 2 Question 12

Complex numbers

Let $w = \dfrac{1-i}{2\sqrt{2}}$.

(a) Express w in the form $w = r\,cis\theta$, where $-\pi < \theta \leq \pi$. (2 marks)

The complex number z is represented in the Argand diagram below.



(b) Express z exactly in the form $z = a + bi$. (2 marks)

(c) Determine the exact polar form for wz and $w^2 z$. (2 marks)

(d) On the Argand diagram on page 6, plot the position for wz and $w^2 z$. Ensure that each position is labelled clearly. (2 marks)

Consider the geometric transformation(s) applied to transform $z \rightarrow wz \rightarrow w^2 z \rightarrow w^3 z$ etc

(e) Describe the geometric transformation(s) performed by the successive multiplication by w. (2 marks)

2023 Section 1 Question 2

Complex numbers

$P(z) = z^5 + az^4 + bz^3 + cz^2 + dz + 14$ is a fifth order polynomial with real coefficients. It is known that $P(z) = (z - z_0)\,Q(z)$ where z_0 is real and $Q(z)$ is a fourth order polynomial. Two roots of $P(z)$ are $z_1 = 1 + i$ and $z_2 = 2 + \sqrt{3}i$.

(a) Determine $Q(z)$ in expanded form. (3 marks)

Solution

If two roots are $1+i$ and $2+\sqrt{3}i$, then so are the conjugates $1-i$ and $2-\sqrt{3}i$.

$$\begin{aligned}\therefore\ Q(z) &= \left(z-(1+i)\right)\left(z-(1-i)\right)\left(z-\left(2+\sqrt{3}i\right)\right)\left(z-\left(2-\sqrt{3}i\right)\right)\\ &= \left((z-1)-i\right)\left((z-1)+i\right)\left((z-2)-\sqrt{3}i\right)\left((z-2)+\sqrt{3}i\right)\\ &= \left((z-1)^2+1\right)\left((z-2)^2+3\right)\\ &= \left(z^2-2z+2\right)\left(z^2-4z+7\right) \quad \ldots (1)\\ &= z^4-6z^3+17z^2-22z+14 \quad \ldots (2)\end{aligned}$$

Specific behaviours

- ✓ states that are $1-i$ and $2-\sqrt{3}i$ are also roots of $P(z)$
- ✓ expresses $Q(z)$ as a product of four linear factors correctly
- ✓ determines $Q(z)$ as either expression (1) or (2)

(b) Determine the values of the coefficients a, b, c and d. (2 marks)

Solution

$$\begin{aligned}\text{Since } P(z) &= z^5+az^4+bz^3+cz^2+dz+14\\ &= (z-z_0)\left(z^4-6z^3+17z^2-22z+14\right)\end{aligned}$$

Then the constant term $14 = (-z_0)(14) \quad \therefore\ z_0 = -1$

$$\begin{aligned}\therefore\ P(z) &= (z+1)\left(z^4-6z^3+17z^2-22z+14\right)\\ &= z^5-5z^4+11z^3-5z^2-8z+14\end{aligned}$$

Hence $a = -5,\ b = 11,\ c = -5,\ d = -8$

Specific behaviours

- ✓ determines the value for z_0
- ✓ states the values for a, b, c and d correctly

2023 Section 1 Question 6

Complex numbers

Solve the complex equation $z^4 = 2 - 2\sqrt{3}i$ giving solutions in the form $rcis\theta$ where $-\pi < \theta \leq \pi$. (5 marks)

Solution

$$|z^4| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4 \qquad Arg(z^4) = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

$\therefore$ Solve $z^4 = 4cis\left(-\frac{\pi}{3}\right)$

$$\therefore \quad z = 4^{\frac{1}{4}} cis\left(\frac{-\pi}{12} + k\left(\frac{\pi}{2}\right)\right) \qquad k = 0,1,2,3 \qquad \ldots (1)$$

Roots are: $z_0 = \sqrt{2}\,cis\left(-\frac{\pi}{12}\right)$

$z_1 = \sqrt{2}\,cis\left(\frac{5\pi}{12}\right)$

$z_2 = \sqrt{2}\,cis\left(\frac{11\pi}{12}\right)$

$z_3 = \sqrt{2}\,cis\left(-\frac{7\pi}{12}\right)$ Note: $z_3 = \sqrt{2}\,cis\left(\frac{17\pi}{12}\right)$ does not satisfy $-\pi < \theta \leq \pi$.

Specific behaviours

- ✓ states the value for $|z^4|$ correctly
- ✓ states the value for $Arg(z^4)$ correctly
- ✓ states the principal solution $z_0 = \sqrt{2}\,cis\left(-\frac{\pi}{12}\right)$
- ✓ indicates a separation of $\frac{\pi}{2}$ between solution arguments
- ✓ states all solutions correctly using the condition $-\pi < \theta \leq \pi$

2023 Section 1 Question 8

Complex numbers

In the following simultaneous equations, a and b are real numbers.

$$a^3 = 3ab^2 + 14$$
$$b^3 = 3a^2b + 2\sqrt{5}$$

In order to determine the value of $a^2 + b^2$ from these equations, it is useful to consider the complex expansion for $(a + bi)^3$. Hence, or otherwise, determine the exact value of $a^2 + b^2$.
(4 marks)

Solution

$$\begin{aligned}(a+bi)^3 &= a^3 + 3a^2(bi) + 3a(bi)^2 + (bi)^3 \\ &= a^3 + (3a^2b)i - 3ab^2 - (b^3)i \\ &= (a^3 - 3ab^2) + (3a^2b - b^3)i\end{aligned}$$

From equation (1) we have: $a^3 - 3ab^2 = 14$

From equation (2) : $3a^2b - b^3 = -2\sqrt{5}$

Hence $(a+bi)^3 = 14 - 2\sqrt{5}\,i$.

$\therefore \left|(a+bi)^3\right| = \sqrt{14^2 + \left(2\sqrt{5}\right)^2} = \sqrt{196+20} = \sqrt{216}$

$\therefore \left|(a+bi)\right|^3 = \sqrt{216}$ since $\left|z^3\right| = |z|^3$

$\therefore \left|a+bi\right| = (216)^{\frac{1}{6}}$

$\therefore \left|a+bi\right|^2 = (216)^{\frac{1}{3}}$

i.e. $a^2 + b^2 = \sqrt[3]{216} = 6$ Note: accept $\sqrt[3]{216}$ as the final answer.

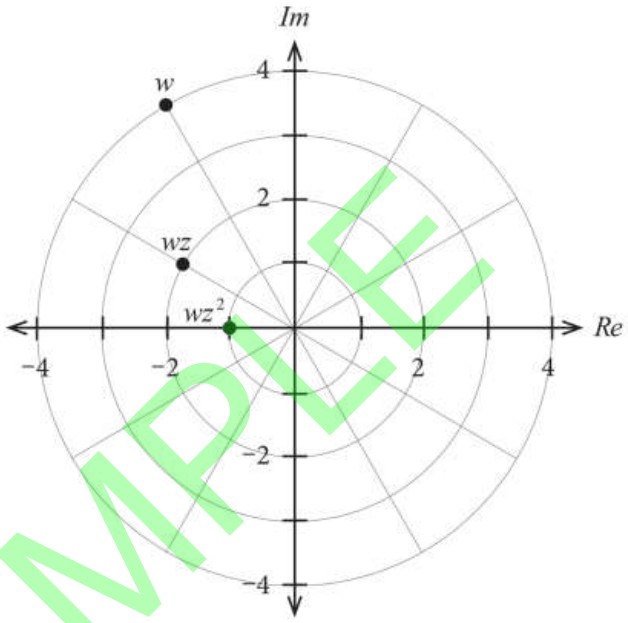
Specific behaviours

- ✓ obtains $(a+bi)^3$ correctly as $(a^3 - 3ab^2) + (3a^2b - b^3)i$ or its equivalent
- ✓ deduces $(a+bi)^3 = 14 - 2\sqrt{5}\,i$
- ✓ obtains the value for $\left|a+bi\right| = (216)^{\frac{1}{6}}$ or its equivalent
- ✓ deduces the value of $a^2 + b^2$

2023 Section 2 Question 10

Complex numbers

The complex number $w = 4cis\left(\frac{2\pi}{3}\right)$ is shown in the Argand diagram, along with the complex numbers wz and wz^2.



(a) Express wz and wz^2 in exact polar form. (2 marks)

Solution
$wz = 2cis\left(\frac{5\pi}{6}\right)$ $\quad wz^2 = cis(\pi)$
Specific behaviours
✓ writes the correct modulus for each complex number ✓ writes the correct argument for each complex number or ✓ writes the correct modulus and argument for one complex number ✓ writes the correct modulus and argument for the other complex number

Consider the geometric transformation(s) applied to transform $w \to wz \to wz^2$ etc.

(b) Describe the geometric transformation(s) performed by successive multiplication by z. (2 marks)

Solution
Successive multiplication by z results in the modulus changing by a factor of $\frac{1}{2}$ and the argument increasing by $\frac{\pi}{6}$ i.e. 30°. Geometric description: Each vector is scaled by a factor of 0.5. Rotation anti-clockwise (about origin) by 30°.
Specific behaviours
✓ describes the change in the modulus as a dilation by factor 0.5 ✓ describes the change in the argument as an anti-clockwise rotation by 30°

(c) Determine z in exact polar form. (1 mark)

Solution
$z = \frac{1}{2}cis\left(\frac{\pi}{6}\right)$
Specific behaviours
✓ determines the correct polar form for z

(d) Describe the geometric transformation(s) performed by successive multiplication by z^{-1}. (2 marks)

Solution
$z^{-1} = \left(\frac{1}{2}cis\left(\frac{\pi}{6}\right)\right)^{-1} = 2\,cis\left(-\frac{\pi}{6}\right)$ Successive multiplication by z^{-1} results in the modulus changing by a factor of 2 and the argument decreasing by $\frac{\pi}{6}$ i.e. 30°. Geometric description: Each vector is scaled by a factor of 2. Rotation clockwise (about origin) by 30°.
Specific behaviours
✓ describes the change in the modulus as an enlargement/dilation by factor 2 ✓ describes the change in the argument as a clockwise rotation by 30°

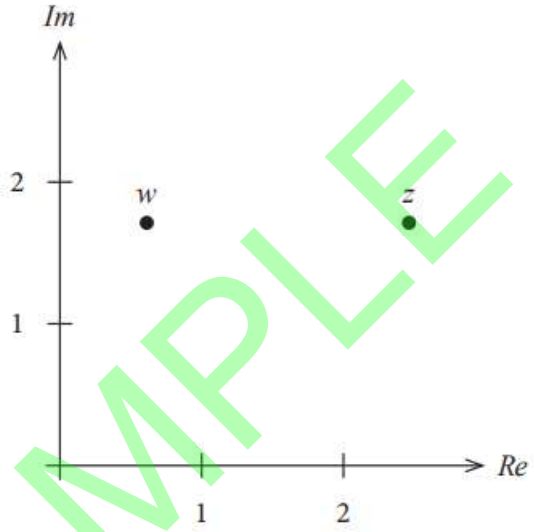
2023 Section 2 Question 12

Complex numbers

Complex numbers z and w are shown in the Argand diagram below. It is known that:

$|z| = 3,\ Arg\,(z) = \theta$ where $0 < \theta < \frac{\pi}{4}$

$w = z - k$ such that $Arg\,(w) = 2\theta$ where $Im\,(k) = 0,\ k > 0.$



(a) Represent the given information on the Argand diagram. (3 marks)

Solution
Shown on diagram.
Specific behaviours
✓ indicates $\|z\| = 3$ and $Arg\,(z) = \theta$ using line segments or vectors
✓ indicates horizontal length equal to k
✓ indicates $Arg\,(w) = 2\theta$

(b) Determine a simplified expression for k in terms of θ. Justify your answer. (3 marks)

Solution
Applying the Cosine rule in ΔOAB: $3^2 = k^2 + k^2 - 2(k)(k)\cos(\pi - 2\theta)$ i.e. $9 = 2k^2 - 2k^2\cos(-2\theta)$ i.e. $9 = 2k^2 + 2k^2\cos(2\theta)$ i.e. $9 = 2k^2(1 + \cos(2\theta))$ $\therefore\ 9 = 2k^2(2\cos^2\theta) = 4k^2\cos^2\theta$ This yields $k = \frac{3}{2\cos\theta}$.
Specific behaviours
✓ states that $\|w\| = k$ or refers to $s\angle OAB = \theta = s\angle AOB$
✓ forms an equation relating k, θ using appropriate trigonometry
✓ obtains the correct simplified expression for k in terms of θ

Alternative Solution

Applying the Sine rule in ΔOAB:

$$\frac{k}{\sin\theta} = \frac{3}{\sin(\pi - 2\theta)}$$

i.e. $k = \dfrac{3\sin\theta}{\sin(2\theta)} = \dfrac{3\sin\theta}{2\sin\theta\cos\theta}$

This yields $k = \dfrac{3}{2\cos\theta}$.

Specific behaviours

- ✓ states that $|w| = k$ or refers to $s\angle OAB = \theta = s\angle AOB$
- ✓ forms an equation relating k, θ using appropriate trigonometry
- ✓ obtains the correct simplified expression for k in terms of θ

Alternative Solution

Let $z = 3\cos\theta + (3\sin\theta)i$

$\therefore\ w = (3\cos\theta - k) + (3\sin\theta)i$

But we also have $w = k\cos 2\theta + (k\sin 2\theta)i$

Equating imaginary parts: $k\sin 2\theta = 3\sin\theta$

$$\therefore\ k = \frac{3\sin\theta}{\sin 2\theta}$$

This yields $k = \dfrac{3}{2\cos\theta}$.

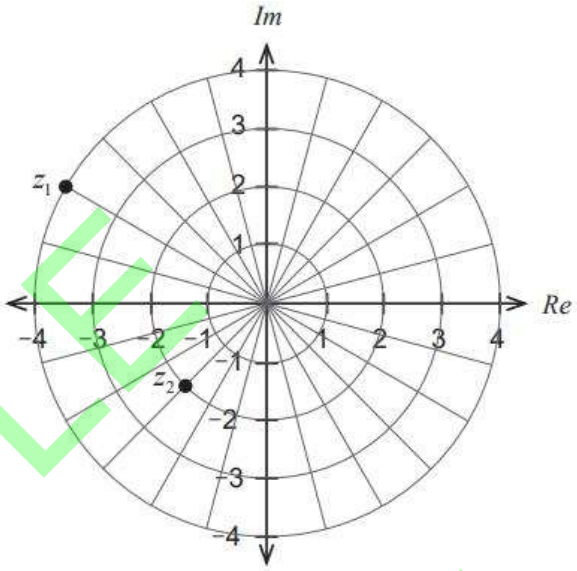
Specific behaviours

- ✓ states that $|w| = k$ or refers to $s\angle OAB = \theta = s\angle AOB$
- ✓ forms an equation relating k, θ using the real or imaginary parts
- ✓ obtains the correct simplified expression for k in terms of θ

2022
Section 1
Question 6

Complex numbers

Two complex numbers $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ and z_2 are shown in the Argand plane below.



(a) Determine the exact polar form for z_2. (2 marks)

Solution
$z_2 = 2cis\left(-\frac{3\pi}{4}\right)$ Accept also $2cis\left(\frac{5\pi}{4}\right)$.
Specific behaviours
✓ states the correct modulus ✓ states the correct argument

(b) Plot the complex number $w = z_1 \times (z_2)^{-1}$ on the Argand diagram above. (3 marks)

Solution
$w = \left(4cis\left(\frac{5\pi}{6}\right)\right) \times \left(2cis\left(-\frac{3\pi}{4}\right)\right)^{-1} = 4cis\left(\frac{5\pi}{6}\right) \times \frac{1}{2}cis\left(\frac{3\pi}{4}\right)$ $= 2cis\left(\frac{5\pi}{6} + \frac{3\pi}{4}\right)$ $= 2cis\left(\frac{19\pi}{12}\right)$ or $2cis\left(-\frac{5\pi}{12}\right)$ w shown on the Argand diagram above.
Specific behaviours
✓ applies DeMoivre's Theorem correctly to determine z_2^{-1} ✓ determines the correct polar form for w ✓ plots the correct position for w

(c) If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution of the equation $z^n = r$ where r is a positive real number and n is a positive integer, determine the smallest possible value for r in the form 2^p.

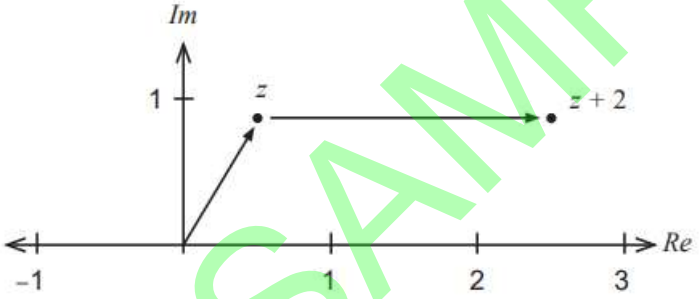
Justify your answer. (3 marks)

Solution
If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution then $\left(4cis\left(\frac{5\pi}{6}\right)\right)^n = r\,cis(2\pi k)$
i.e. $2^{2n}cis\left(\frac{5n\pi}{6}\right) = r\,cis(2\pi k)$ where $k = 0,1,2,\ \ldots\ , n-1$
i.e. $\frac{5n}{6} = 2k$ or $n = \frac{12k}{5}$
Hence the smallest possible value of $n = 12$ $(\text{when } k = 5)$ so that $n \in \mathbb{Z}^+$.
$\therefore\ r = 2^{2\times 12} = 2^{24}$ is the smallest value
Specific behaviours
✓ forms the equation that determines the relationship between n and integer k
✓ deduces the smallest value for n or k
✓ states the smallest value for r as a power of 2

2021 Section 1 Question 1

Complex numbers

The Argand diagram below shows the complex numbers z and $z+2$ where $z = cis\left(\frac{\pi}{3}\right)$.



Determine the exact value for:

(a) $Arg\,(-z)$. (1 mark)

Solution
$Arg(-z) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$.
Also accept $Arg(-z) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$
Specific behaviours
✓ states the correct value

(b) $|z+2|$. (3 marks)

Solution
Applying the cosine rule $\quad |z+2|^2 = 1^2+2^2-2(1)(2)\cos\left(\frac{2\pi}{3}\right)$ $= 1+4-4\left(-\frac{1}{2}\right) = 5+2 = 7$ $\therefore \quad |z+2| = \sqrt{7}$
Specific behaviours
✓ determines an angle of 120° between the vectors representing z and $z+2$ ✓ applies the cosine rule correctly ✓ determines the value for $\|z+2\|$ correctly

Alternative Solution
$z = \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \quad \therefore \quad z+2 = \left(\frac{5}{2}\right)+\frac{\sqrt{3}}{2}i$ $\|z+2\|^2 = \left(\frac{5}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{25}{4}+\frac{3}{4} = 7$ $\therefore \quad \|z+2\| = \sqrt{7}$
Specific behaviours
✓ determines $z+2$ in Cartesian form correctly ✓ forms the expression for $\|z+2\|^2$ correctly ✓ determines the value for $\|z+2\|$ correctly

2021 Section 1 Question 6

Complex numbers

Consider the quartic polynomial $P(z) = z^4 - 6z^3 + 31z^2 - 52z + 60$.

(a) Given that $P(2+4i) = 0$, determine a quadratic factor of $P(z)$. (2 marks)

Solution
Since $P(2+4i) = 0$ then we also have $P(2-4i) = 0$ as all coefficients are real.
$Q(z) = (z-(2+4i))(z-(2-4i))$
$= (z^2 - 4z + 20)$
Specific behaviours
✓ states that $P(2-4i) = 0$ or states that $z-(2-4i)$ is a factor
✓ determines the quadratic factor $Q(z)$ correctly

(b) Hence solve the equation $z^4 - 6z^3 + 31z^2 - 52z + 60 = 0$. (3 marks)

Solution
$P(z) = (z^2 - 4z + 20)(z^2 - 2z + 3)$ i.e. $T(z) = z^2 - 2z + 3$
i.e. $P(z) = (z^2 - 4z + 20)((z-1)^2 + 2)$
$= (z-(2+4i))(z-(2-4i))(z-(1+\sqrt{2}i))(z-(1-\sqrt{2}i))$
Solving $T(z) = 0$ gives $z = 1 \pm \sqrt{2}i$
Solutions are $z = 2 \pm 4i,\ 1 \pm \sqrt{2}i$
Specific behaviours
✓ determines the quadratic factor $T(z)$ correctly
✓ states that $z = 1 + \sqrt{2}i$ is a solution
✓ states that $z = 1 - \sqrt{2}i$ is a solution

2021 Section 1 Question 7

Complex numbers

The number 2021 can be expressed as a product of two consecutive prime numbers: 43 × 47 = 2021.

Consider the complex equation $z^{43} = 1$.

(a) Write an expression for the roots of $z^{43} = 1$. (2 marks)

Solution
The equation $z^{43} = 1$ has 43 roots where any root is of the form given by:
$w = cis\left(\frac{2\pi k}{43}\right)$ where $k = 0, 1, 2, \ldots, 42$.
Specific behaviours
✓ writes the correct form $cis\left(\frac{2\pi k}{43}\right)$
✓ states that the integer $k = 0, 1, 2, \ldots, 42$.

Let w be any one of the roots of the equation $z^{43} = 1$.

(b) How many of these roots will also be a solution of the equation $z^{47} = 1$? Justify your answer. (3 marks)

Solution

If w is also a root of $z^{47} = 1$ then we must show that $w^{47} = 1$.

Examining the expression $w^{47} = \left(cis\left(\frac{2\pi k}{43}\right)\right)^{47} \qquad k = 0, 1, 2, ..., 42$

$$= cis\left(\frac{47 \times 2\pi k}{43}\right) \quad = \quad cis\left(\frac{47 \times k \times 2\pi}{43}\right)$$

This will be equal to ONE if and only if $\frac{47 \times k}{43}$ is an integer. If this occurs then the argument for w^{47} will be a multiple of 2π and hence $w^{47} = 1$.

Since 43 and 47 are both prime numbers, then 43 does not divide into 47 and that 43 will not divide into k when $k = 1, 2, \ldots, 42$.

Hence $\frac{47 \times k}{43}$ can never be an integer where $k = 1, 2, \ldots, 42$.

When $k = 0$, $w = 1$ is a solution of BOTH $z^{43} = 1$ and $z^{47} = 1$.

$\therefore$ *Only ONE* of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$.

Specific behaviours

- ✓ forms the expression for w^{47} correctly in terms of the integer k
- ✓ states that only ONE of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$
- ✓ justifies the answer using the fact that the argument for w^{47} is never an even multiple of π (for $k \neq 0$)

Alternative Solution

The equation $z^{43}=1$ has 43 roots where any root is of the form given by:

$w=cis\left(\frac{2\pi k}{43}\right)$ where $k=0,1,2,\ \ldots\ ,42.$

If w is also a root of $z^{47}=1$ then $w=cis\left(\frac{2\pi m}{47}\right)$ where $m=0,1,2,\ \ldots\ ,46.$

Hence we require : $w=cis\left(\frac{2\pi k}{43}\right)=cis\left(\frac{2\pi m}{47}\right)$

Hence $\left(\frac{2\pi k}{43}\right)=\left(\frac{2\pi m}{47}\right)$ where $k=0,1,2,\ \ldots\ ,42$ and $m=0,1,2,\ \ldots\ ,46.$

i.e. $\left(\frac{k}{43}\right)=\left(\frac{m}{47}\right)$

i.e. $m=\frac{47\times k}{43}$ must be an integer.

Since 43 and 47 are both prime numbers, then 43 does not divide into 47 and that 43 will not divide into k when $k=1,2,\ \ldots\ ,42$.

Hence $\frac{47\times k}{43}$ can never be an integer where $k=1,2,\ \ldots\ ,42$.

When $k=0,\ m=0$, then $w=1$ is a solution of BOTH $z^{43}=1$ and $z^{47}=1$.

$\therefore$ *Only ONE* of the roots $(w=1)$ of $z^{43}=1$ is also a root of $z^{47}=1$.

Specific behaviours

- ✓ forms the expression for the roots of $z^{47}=1$ correctly in terms of the integer m (a different parameter to k)
- ✓ states that only ONE of the roots $(w=1)$ of $z^{43}=1$ is also a root of $z^{47}=1$
- ✓ justifies the answer using the fact $m=\frac{47\times k}{43}$ cannot be an integer (unless both $k=0,\ m=0$)

2020 Section 1 Question 8

Complex numbers

Consider the complex sum: $\sum_{n=1}^{2020} n i^n = 1i^1 + 2i^2 + 3i^3 + \ldots + 2020i^{2020}$

Express the value of this sum in the form $r\,cis\,\theta$ where $-\pi < \theta \leq \pi$. (3 marks)

Solution

$$\sum_{n=1}^{4} n i^n = 1(i)^1 + 2(i)^2 + 3(i)^3 + 4(i)^4$$
$$= i - 2 - 3i + 4 = 2 - 2i$$

$$\sum_{n=5}^{8} n i^n = 5(i)^5 + 6(i)^6 + 7(i)^7 + 8(i)^8$$
$$= 5i - 6 - 7i + 8 = 2 - 2i$$

$$\sum_{n=9}^{12} n i^n = 9(i)^9 + 10(i)^{10} + 11(i)^{11} + 12(i)^{12}$$
$$= 9i - 10 - 11i + 12 = 2 - 2i$$

Hence $\sum_{n=1}^{2020} n i^n = (2-2i) + (2-2i) + (2-2i) + \ldots$ 505 terms

$$= 505(2-2i)$$
$$= 1010 - 1010i$$
$$= 1010\sqrt{2}\,cis\left(-\frac{\pi}{4}\right) \text{ or } \frac{2020}{\sqrt{2}}cis\left(-\frac{\pi}{4}\right)$$

Specific behaviours

- ✓ evaluates the sum of the first 4 terms correctly
- ✓ generalises that the sum of the first 4 terms repeats 505 times
- ✓ simplifies correctly in the form $r\,cis\,\theta$

2019 Section 1 Question 2

Complex numbers

Consider the function $P(z) = z^4 - 2z^3 + 14z^2 - 8z + 40$, defined over the complex numbers.

(a) Show that $(z - 2i)$ is a factor of $P(z)$. (2 marks)

Solution

$$P(2i) = (2i)^4 - 2(2i)^3 + 14(2i)^2 - 8(2i) + 40$$
$$= 16(1) - 16(-1)(i) + 14(4)(-1) - 16i + 40$$
$$= 16 + 16i - 56 - 16i + 40 \quad \ldots (1)$$
$$= 0$$

Hence $(z - 2i)$ is a factor of $P(z)$.

Specific behaviours

- ✓ substitutes $z = 2i$ correctly into $P(z)$
- ✓ obtains the 5 terms in expression (1) to deduce $P(2i) = 0$

(b) Hence or otherwise, solve the equation $P(z) = 0$, giving solutions in the form $a + bi$. (4 marks)

Solution
Since $(z-2i)$ is a factor then so is $(z+2i)$. Hence $(z+2i)(z-2i)=(z^2+4)$ is also a factor of $P(z)$. $\therefore\ P(z) = (z^2+4)Q(z)$ where $Q(z)=z^2-2z+10$ i.e. Solving $Q(z)=0$ $\quad z^2-2z+10 = 0$ $\quad$ OR $\quad \therefore\ (z^2+4)=0$ $\therefore\ (z-1)^2+9=0$ $\quad \therefore\ z = \pm 2i$ $\therefore\ (z-1)^2=-9$ *i.e.* $z=1\pm 3i$
Specific behaviours
✓ deduces $(z+2i)$ is a factor of $P(z)$ or states $z=-2i$ is a solution
✓ deduces (z^2+4) is a factor of $P(z)$
✓ factorises $P(z)$ as $(z^2+4)(z^2-2z+10)$
✓ states $z=1\pm 3i$ as solutions to $P(z)=0$

2019 Section 1 Question 9

Complex numbers

Consider the complex equation $z^n - 1 = 0$, where n is any positive integer $n \geq 3$.

If the roots are designated as $z_0, z_1, z_2, \ldots, z_{n-1}$, then determine the exact value for the product of the roots $p = z_0 \times z_1 \times z_2 \times \ldots \times z_{n-1}$.

(4 marks)

Solution

$z^n = 1 = cis(0) \quad \therefore \quad z = cis\left(\frac{0+2\pi k}{n}\right) = cis\left(\frac{2k\pi}{n}\right)$ where $k = 0, 1, 2, \ldots, n-1$

$\therefore \quad z_0 = cis(0) = 1, \; z_1 = cis\left(\frac{2\pi}{n}\right), \; z_2 = cis\left(\frac{4\pi}{n}\right), \; z_3 = cis\left(\frac{6\pi}{n}\right), \; z_4 = cis\left(\frac{8\pi}{n}\right)$

$z_{n-1} = cis\left(\frac{2(n-1)\pi}{n}\right)$

$p = cis(0)cis\left(\frac{2\pi}{n}\right)cis\left(\frac{4\pi}{n}\right)cis\left(\frac{6\pi}{n}\right) \ldots cis\left(\frac{2(n-1)\pi}{n}\right)$

$= cis\left(0 + \frac{2\pi}{n} + \frac{4\pi}{n} + \frac{6\pi}{n} + \ldots \frac{2(n-1)\pi}{n}\right)$

$= cis\left(\frac{2\pi}{n}(1+2+3+\ldots(n-1))\right)$

$= cis\left(\frac{2\pi}{n} \times \frac{(n-1)(n)}{2}\right)$

$= cis((n-1)\pi) = \cos(n-1)\pi + i\sin(n-1)\pi$

Since $\sin(n-1)\pi = 0$ for all integer values of n and $\cos(n-1)\pi = \pm 1$, then

Product $p = 1$ if n is ODD

$p = -1$ if n is EVEN.

Specific behaviours

✓ expresses the roots in the form $cis\left(\frac{2k\pi}{n}\right)$ where $k = 0, 1, 2, \ldots, n-1$

✓ forms the product $p = cis\left(\frac{2\pi}{n}\right)cis\left(\frac{4\pi}{n}\right)\ldots cis\left(\frac{2(n-1)\pi}{n}\right)$ correctly

✓ uses DeMoivres Theorem to obtain $cis((n-1)\pi)$ correctly

✓ states the two possible values correctly for n even and odd

Alternative Solution

Equation is $z^n - 1 = 0$

Given that the roots are: $z_0, z_1, z_2, \ldots, z_{n-1}$ means that the equation can be written in the form $(z-z_0)(z-z_1)(z-z_2)\ldots(z-z_{n-1}) = 0$

i.e. $(z-z_0)(z-z_1)(z-z_2)\ldots(z-z_{n-1}) = z^n - 1$

Hence the LHS constants $(-z_0)(-z_1)(-z_2)\ldots(-z_{n-1}) = -1$ (equating constants)

Since there are n factors :

IF n is EVEN then we have $(z_0)(z_1)(z_2)\ldots(z_{n-1}) = -1$ i.e. $p = -1$

IF n is ODD then we have $-(z_0)(z_1)(z_2)\ldots(z_{n-1}) = -1$ i.e. $p = 1$

Specific behaviours

✓ expresses the LHS in the form $(z-z_0)(z-z_1)(z-z_2)\ldots(z-z_{n-1})$

✓ states that the product of the constant terms $(-z_0)(-z_1)(-z_2)\ldots(-z_{n-1}) = -1$

✓ states that the product depends on whether n is even or odd

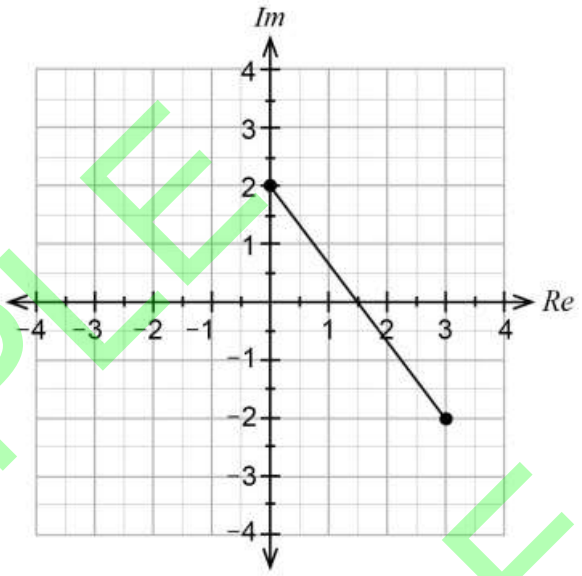
✓ states the correct value for the product for each case

2022 Section 2 Question 9

Complex numbers

(a) Sketch the locus of a complex number z satisfying the condition:

$|z-2i|+|z-(3-2i)|=5$ (2 marks)

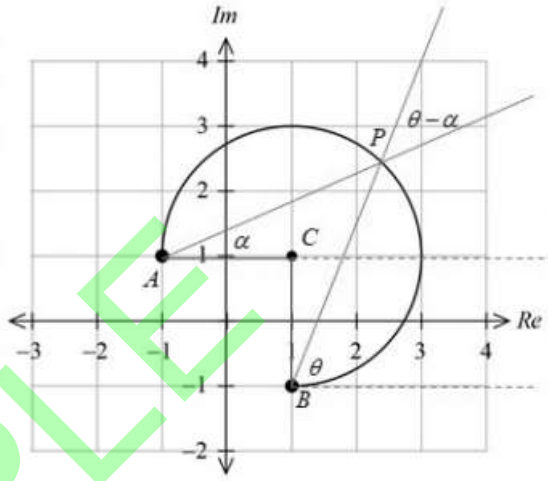


Solution
Locus can be interpreted as: "the distance of z from $2i$ added to the distance of z from $3-2i$ is equal to 5 units." Since the distance from $(0,2)$ and $(3,-2)$ is 5 units, then the locus is a line segment of the points connecting the end points $z=2i$ and $z=3-2i$.
Specific behaviours
✓ indicates a locus that contains $(0,2)$ and $(3,-2)$ ✓ indicates a line segment with correct end points

(b) Describe the locus of the equation $(z+i)(\overline{z+i})=2$. (3 marks)

Solution
$(z+i)(\overline{z+i})=2$ i.e. $\lvert z+i\rvert^2=2$ OR $x^2+(y+1)^2=2$ where $x=\mathrm{Re}(z)$, $y=\mathrm{Im}(z)$ i.e. $\lvert z-(-i)\rvert=\sqrt{2}$ Description: Locus is a CIRCLE with radius $\sqrt{2}$ and centre $z=-i$.
Specific behaviours
✓ uses appropriate complex number properties to re-write the equation correctly ✓ states that the locus is a CIRCLE with a radius of $\sqrt{2}$ units ✓ states that the centre is $z=-i$ or $(0,-1)$

(c) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (3 marks)



Solution
Locus is given by $\lvert z-(1+i)\rvert = 2$ and $-\frac{\pi}{2} \le Arg\left(z-(1+i)\right) \le \pi$
Specific behaviours
✓ states the equation $\lvert z-(1+i)\rvert = 2$ ✓ states an inequality about the argument from $1+i$ ✓ states the correct limits for the argument from $1+i$

Alternative Solution 1
Locus is given by $\lvert z-(1+i)\rvert = 2$ and $-\frac{\pi}{4} \le Arg(z) \le \frac{3\pi}{4}$
Specific behaviours
✓ states the equation $\lvert z-(1+i)\rvert = 2$ ✓ states an inequality about the argument from the origin ✓ states the correct limits for the argument from the origin

Alternative Solution 2
Locus is given by $\lvert z-(1+i)\rvert = 2$ Using the Central Angle Theorem: $s\angle ACB = \frac{\pi}{2} \quad \therefore\ s\angle APB = \frac{\pi}{4}$ *i.e.* $\theta - \alpha = \frac{\pi}{4} \quad \therefore\ Arg\left(z-(1-i)\right) - Arg\left(z-(-1+i)\right) = \frac{\pi}{4}$ $\therefore\ Arg\left(\frac{z-(1-i)}{z-(-1+i)}\right) = \frac{\pi}{4}$
Specific behaviours
✓ states the equation $\lvert z-(1+i)\rvert = 2$ ✓ writes a difference of arguments equal to $\frac{\pi}{4}$ OR its equivalent ✓ writes the correct expression for each of the two arguments

2022 Section 2 Question 18

Complex numbers

(a) Show that for all positive integers n and complex numbers z where $0 \le \theta \le \frac{\pi}{2}$,

$(z^n - cis(\theta))(z^n + cis(-\theta)) = z^{2n} - (2i \sin\theta) z^n - 1$. (3 marks)

Solution
$(z^n - cis(\theta))(z^n + cis(-\theta))$
$= z^{2n} - cis(\theta) z^n + cis(-\theta) z^n - cis(\theta) cis(-\theta)$
$= z^{2n} - (cis(\theta) - cis(-\theta)) z^n - cis(0)$
$= z^{2n} - (2i \sin\theta) z^n - 1$
Specific behaviours
✓ expands the binomial products correctly to obtain 4 terms
✓ uses the property $cis(\theta) cis(-\theta) = cis(0) = 1$
✓ uses the property $cis(\theta) - cis(-\theta) = 2i \sin\theta$

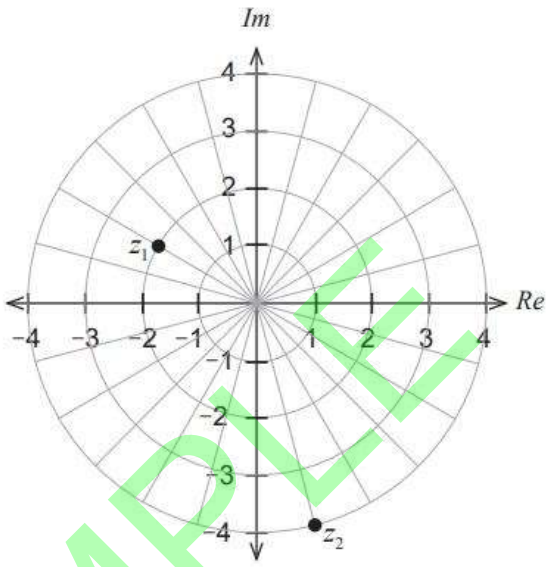
(b) Hence, using the result from part (a), obtain all the solutions to the equation $z^6 - (i) z^3 - 1 = 0$ in exact polar form. (4 marks)

Solution
To solve $z^6 - (i) z^3 - 1 = 0$
i.e. $(z^3)^2 - \left(2\left(\frac{1}{2}\right) i\right) z^3 - 1 = 0$
$\therefore\ n = 3$ and $\sin\theta = \frac{1}{2}$ i.e. $\theta = \frac{\pi}{6}$ using $0 \le \theta \le \frac{\pi}{2}$ from part (a).
Hence solve $\left(z^3 - cis\left(\frac{\pi}{6}\right)\right)\left(z^3 + cis\left(-\frac{\pi}{6}\right)\right) = 0$
$\therefore\ z^3 = cis\left(\frac{\pi}{6}\right)$ $\therefore\ z = cis\left(\frac{\pi}{18}\right), cis\left(\frac{13\pi}{18}\right), cis\left(\frac{25\pi}{18}\right)$ or $cis\left(-\frac{11\pi}{18}\right)$
OR $z^3 = -cis\left(-\frac{\pi}{6}\right) = cis(\pi) cis\left(-\frac{\pi}{6}\right) = cis\left(\frac{5\pi}{6}\right)$
$\therefore\ z = cis\left(\frac{5\pi}{18}\right), cis\left(\frac{17\pi}{18}\right), cis\left(\frac{29\pi}{18}\right)$ or $cis\left(-\frac{7\pi}{18}\right)$
Specific behaviours
✓ determines the value of n and θ correctly
✓ deduces that $z^3 = cis\left(\frac{\pi}{6}\right)$ and $z^3 = cis\left(\frac{5\pi}{6}\right)$
✓ gives ALL solutions for $z^3 = cis\left(\frac{\pi}{6}\right)$ correctly
✓ gives ALL solutions for $z^3 = cis\left(\frac{5\pi}{6}\right)$ correctly

2021 Section 2 Question 11

Complex numbers

Two complex numbers z_1 and z_2 are shown in the Argand plane below.



(a) Write the expression for z_1 in exact polar form. (2 marks)

Solution
$z_1 = 2cis\left(\frac{5\pi}{6}\right)$
Specific behaviours
✓ states the correct modulus ✓ states the correct argument

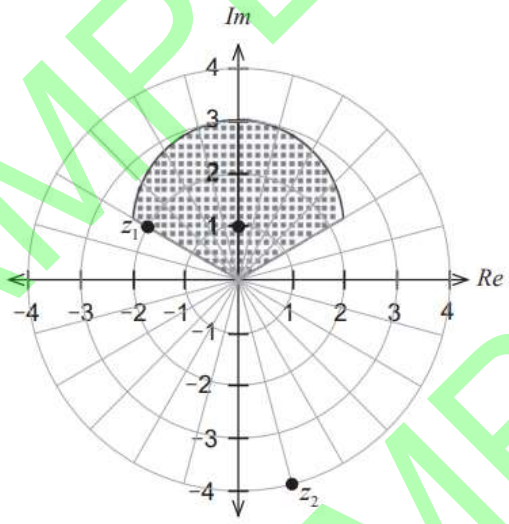
(b) Write the expression for z_1 in exact Cartesian form. (1 mark)

Solution
$z_1 = 2\cos\frac{5\pi}{6} + 2i\sin\frac{5\pi}{6} = 2\left(-\frac{\sqrt{3}}{2}\right) + 2i\left(\frac{1}{2}\right) = -\sqrt{3} + i$
Specific behaviours
✓ states the correct Cartesian form

(c) Plot the complex number iz_1 on the Argand diagram above. (2 marks)

Solution
$iz_2 = cis\left(\frac{\pi}{2}\right).4cis(a) = 4cis\left(\frac{\pi}{2}+a\right)$ i.e. multiplying by i is to rotate z_2 90° anti-clockwise about the origin
Specific behaviours
✓ determines the correct value for iz_2
✓ plots the correct position for iz_2

(d) A sketch of the locus of a complex number z is shown below. The upper boundary of the locus is part of a circle, centred at $z = i$. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (4 marks)

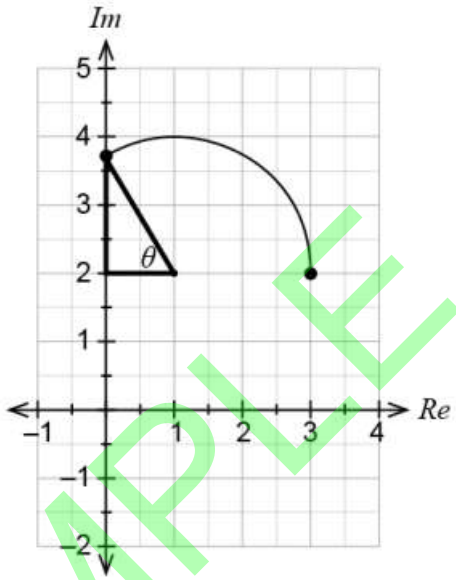


Solution
The locus is part of the interior of the circle with centre $z = i$ and radius 2. $\lvert z-i\rvert \le 2$ with $\frac{\pi}{6} \le Arg(z) \le \frac{5\pi}{6}$
Specific behaviours
✓ uses the form $\lvert z-c\rvert \le r$
✓ states $c = i$ and $r = 2$
✓ uses the form $\theta_1 \le Arg(z) \le \theta_2$
✓ states $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$

2020 Section 2 Question 10

Complex numbers

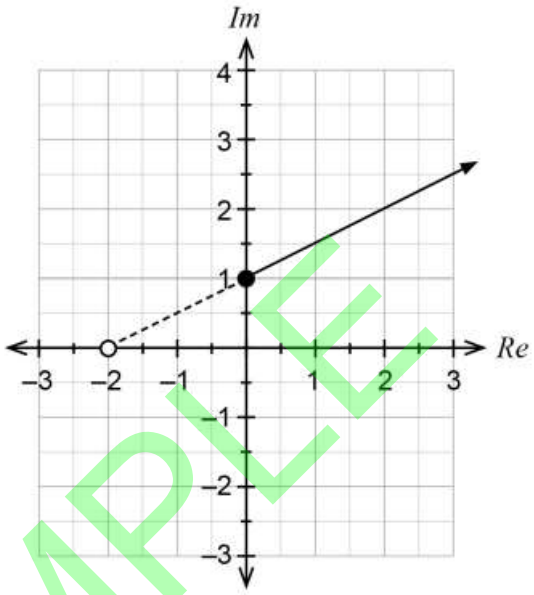
(a) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (4 marks)



Solution
Arc is part of the circle $\lvert z-(1+2i)\rvert = 2$ such that $0 \le Arg(z-(1+2i)) \le \pi-\theta$ where $\tan\theta=\sqrt{3}$ i.e. $\theta=\frac{\pi}{3}$ $\therefore$ Locus is $\lvert z-(1+2i)\rvert = 2$, $0 \le Arg(z-(1+2i)) \le \frac{2\pi}{3}$
Specific behaviours
✓ writes the equation of the form $\lvert z-(1+2i)\rvert = r$ correctly ✓ states $r=2$ ✓ writes the inequality of the form $c \le Arg(z-(1+2i)) \le k$ ✓ uses correct trigonometry to determine the limits c, k

(b) Sketch the locus of the equation $|z+2| = |z-i| + \sqrt{5}$ in the Argand diagram below.

(3 marks)



Solution
Shown above. This equation can be interpreted as 'the distance from $z=-2$ is equal to $\sqrt{5}$ more than the distance from $z=i$'.
Specific behaviours
✓ indicates the locus as a ray (part of a line) ✓ indicates $(0,1)$ i.e. from $z=i$, is an element of the locus ✓ indicates the correct ray (correct slope from $z=-2$ to $z=i$)

2020 Section 2 Question 11

Complex numbers

Let z, w and u be complex numbers where:

$$w = (1+i)\bar{z} \qquad Arg(w) = \frac{\pi}{3} \qquad |w| = 2$$

$$u = \frac{z}{2-2i}$$

(a) Determine $Arg(u)$ exactly. (3 marks)

Solution
$Arg(w) = Arg(1+i) + Arg(\bar{z})$ $\frac{\pi}{3} = \frac{\pi}{4} - Arg(z)$ $\therefore \quad Arg(z) = -\frac{\pi}{12}$ $Arg(u) = Arg(z) - Arg(2-2i)$ $= -\frac{\pi}{12} - \left(-\frac{\pi}{4}\right)$ $= \frac{\pi}{6}$
Specific behaviours
✓ expresses relationships between arguments correctly ✓ determines $Arg(z)$ correctly ✓ determines $Arg(u)$ correctly

(b) Determine $|u|$ exactly. (2 marks)

Solution
$\|w\| = \|1+i\| \times \|\bar{z}\|$ $2 = \sqrt{2} \times \|z\|$ $\therefore \quad \|z\| = \sqrt{2}$ $\|u\| = \frac{\|z\|}{\|2-2i\|} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$
Specific behaviours
✓ expresses relationship between the modulus of numbers correctly ✓ determines $\|u\|$ correctly

2020 Section 2 Question 13

Complex numbers

Solve the equation $z^4 = 8\sqrt{3} + 8i$ giving exact solutions in the form $r\,cis\,\theta$ where $-\pi < \theta \leq \pi$.
(4 marks)

Solution

$z^4 = 8\sqrt{3} + 8i \qquad r^2 = \left(8\sqrt{3}\right)^2 + 8^2$

$= 8^2(3) + 8^2 = 4\left(8^2\right) \quad \therefore \; r = 2(8) = 16$

$\tan\theta = \dfrac{8}{8\sqrt{3}} \quad \therefore \; \theta = \dfrac{\pi}{6}$

Hence solve $z^4 = 16cis\left(\dfrac{\pi}{6}\right)$

Solutions are given by $z = 16^{\frac{1}{4}}cis\left(\dfrac{\frac{\pi}{6} + 2\pi k}{4}\right)$ where $k = 0,1,2,3$.

Solutions are: $z_0 = 2cis\left(\dfrac{\pi}{24}\right)$

$z_1 = 2cis\left(\dfrac{13\pi}{24}\right)$

$z_2 = 2cis\left(\dfrac{13\pi}{24} + \dfrac{12\pi}{24}\right) = 2cis\left(-\dfrac{23\pi}{24}\right)$

$z_3 = 2cis\left(-\dfrac{11\pi}{24}\right)$

Specific behaviours

- ✓ determines the modulus correctly for $8\sqrt{3} + 8i$
- ✓ determines the argument correctly for $8\sqrt{3} + 8i$
- ✓ states one solution as $z = 2cis\left(\dfrac{\pi}{24}\right)$
- ✓ states the correct arguments for the other 3 solutions (using $-\pi < \theta \leq \pi$)

2020 Section 2 Question 15

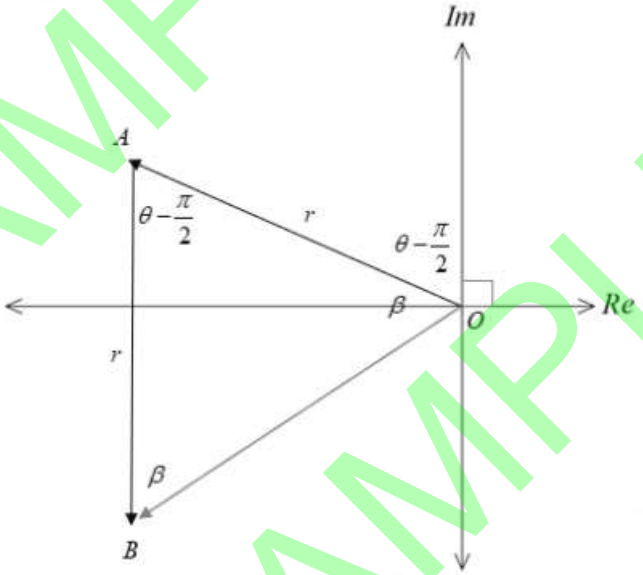
Complex numbers

Let $z = r\,cis\,\theta$ be a complex number such that $\frac{\pi}{2} < \theta < \pi$.

(a) Express in terms of r and θ the complex number $\frac{\overline{z}}{-\sqrt{2}(i+1)}$. (3 marks)

Solution
$\frac{\overline{z}}{-\sqrt{2}(i+1)} = \frac{rcis(-\theta)}{2cis\left(-\frac{3\pi}{4}\right)} = \frac{r}{2}cis\left(\frac{3\pi}{4}-\theta\right)$
Specific behaviours
✓ writes the correct expression for $\overline{z}$
✓ converts $-\sqrt{2}(i+1)$ into polar form correctly
✓ simplifies expression correctly in polar form

(b) Express $\alpha = Arg(z - ri)$ in terms of θ where $0 < \alpha < 2\pi$. (3 marks)

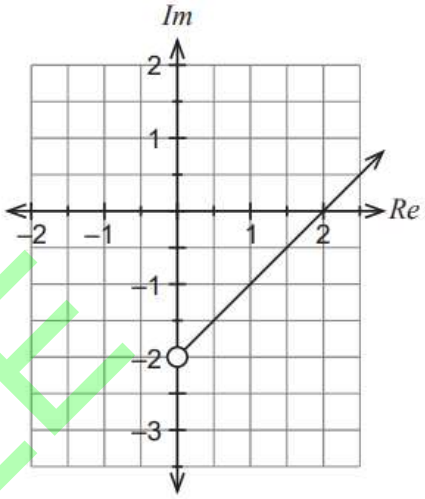


Solution
Vector $\overrightarrow{OB}$ represents $z - ri$. $s\angle OAB = \theta - \frac{\pi}{2}$
In isosceles ΔOAB, $OA = AB$ Hence $s\angle AOB = s\angle ABO = \beta$
We have $2\beta + \left(\theta - \frac{\pi}{2}\right) = \pi$ (Angle sum in ΔOAB)
$\beta = \frac{3\pi}{4} - \frac{\theta}{2}$ Hence $\alpha = Arg(z - ri) = \theta + \beta = \frac{3\pi}{4} + \frac{\theta}{2}$
Specific behaviours
✓ draws the correct representation for $z - ri$
✓ uses that $OA = AB$ to deduce congruent angles in ΔOAB
✓ deduces $\alpha = \frac{3\pi}{4} + \frac{\theta}{2}$

2019 Section 2 Question 10

Complex numbers

The sketch of the locus of a complex number $z = x + iy$ is shown below.



(a) Given that the equation for the above locus is written as $Arg(z - z_0) = k\pi$, determine the value of the constants z_0 and k. (2 marks)

Solution
The equation can be read as the argument of z from z_0 is equal to $k\pi$. i.e. $Arg\left(z-(-2i)\right) = \frac{\pi}{4}$ i.e. $z_0 = -2i,\ k = \frac{1}{4}$
Specific behaviours
✓ states the correct value for z_0 ✓ states the correct value for k

(b) Determine the minimum value for $|z - i|$ as an exact value. (3 marks)

Solution
We require the minimum distance of a point in the locus from $z = i$ (point A). This will be the perpendicular distance AB to the locus. Point B will be the point $(1.5, -0.5i)$. Hence $AB = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{18}{4}} = \frac{3\sqrt{2}}{2}$ Hence the minimum value for $\lvert z-i\rvert = \frac{3\sqrt{2}}{2}$.
Specific behaviours
✓ indicates how the minimum value $\lvert z-i\rvert$ is found ✓ determines coordinates for point B correctly ✓ determines the minimum value $\lvert z-i\rvert$ correctly

2019 Section 2 Question 12

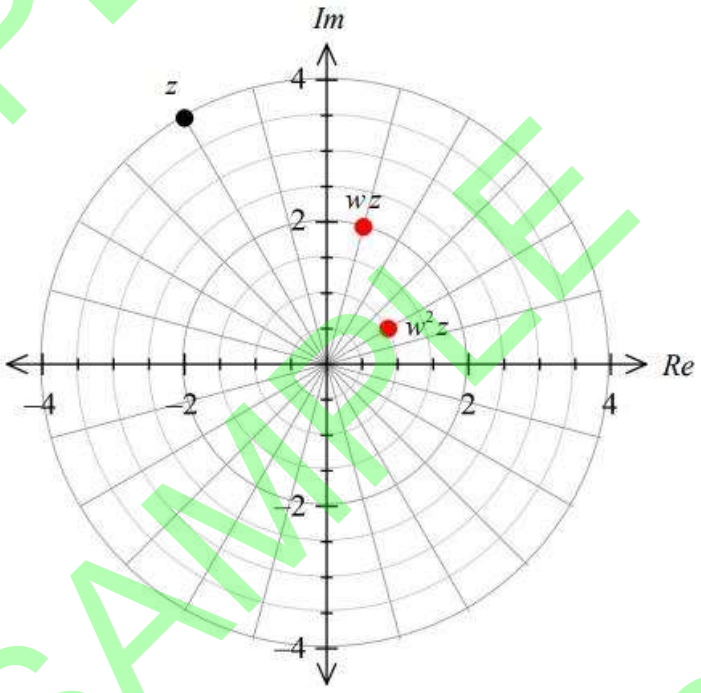
Complex numbers

Let $w = \dfrac{1-i}{2\sqrt{2}}$.

(a) Express w in the form $w = r\,cis\theta$, where $-\pi < \theta \leq \pi$. (2 marks)

Solution
$w = \dfrac{1-i}{2\sqrt{2}} = \dfrac{\sqrt{2}cis\left(-\dfrac{\pi}{4}\right)}{2\sqrt{2}} = \dfrac{1}{2}cis\left(-\dfrac{\pi}{4}\right)$
Specific behaviours
✓ determines the correct modulus r ✓ determines the correct argument θ

The complex number z is represented in the Argand diagram below.



(b) Express z exactly in the form $z = a + bi$. (2 marks)

Solution
From the Argand diagram $z = 4\,cis\left(\dfrac{2\pi}{3}\right)$ Hence $z = 4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right) = 4\left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) = -2 + 2\sqrt{3}i$
Specific behaviours
✓ determines the polar form for z correctly (interprets the Argand diagram) ✓ determines the correct exact values for a, b

(c) Determine the exact polar form for wz and $w^2 z$. (2 marks)

Solution
Given $w = \frac{1}{2}cis\left(-\frac{\pi}{4}\right)$ and $z = 4cis\left(\frac{2\pi}{3}\right)$
Then $wz = \frac{1}{2}\times 4cis\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right) = 2cis\left(\frac{5\pi}{12}\right)$
Also $w^2 z = \left(\frac{1}{2}\right)^2 \times 4 \times cis\left(-\frac{\pi}{2}+\frac{2\pi}{3}\right) = 1cis\left(\frac{\pi}{6}\right)$
Specific behaviours
✓ determines the correct modulus for both wz and $w^2 z$
✓ determines the correct argument for both wz and $w^2 z$

(d) On the Argand diagram on page 6, plot the position for wz and $w^2 z$. Ensure that each position is labelled clearly. (2 marks)

Solution
Indicated on the Argand diagram.
Specific behaviours
✓ indicates the correct modulus for both wz and $w^2 z$ (distance from origin)
✓ indicates the correct argument for both wz and $w^2 z$ (angle to real axis)

Consider the geometric transformation(s) applied to transform $z \rightarrow wz \rightarrow w^2 z \rightarrow w^3 z$ etc

(e) Describe the geometric transformation(s) performed by the successive multiplication by w. (2 marks)

Solution
Successive multilplication by w results in the modulus changing by a factor of $\frac{1}{2}$ (successive points becoming twice as close to the origin) and the argument decreasing by 45° or $\frac{\pi}{4}$. Geometric description: Each vector is REDUCED by a factor of 0.5. Each vector is ROTATED clockwise (about origin) by 45°
Specific behaviours
✓ describes the change in the modulus a dilation by factor 0.5
✓ describes the change in the argument as a clockwise rotation by 45° or $\frac{\pi}{4}$

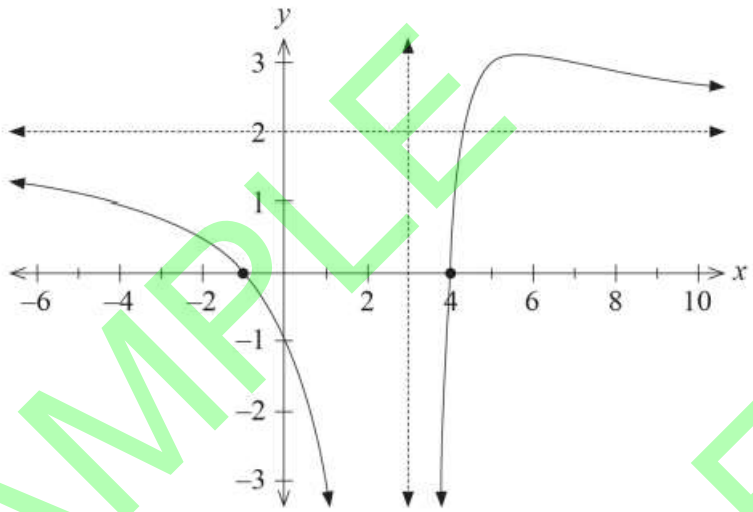
Unit 3.2 – Functions and sketching graphs

Section 1

2023 Section 1 Question 1

Functions and sketching graphs

The graph of the function $f(x) = \dfrac{k(x+a)(x-b)}{(x-c)^2}$ is shown below. The constants a, b, c and k are positive.



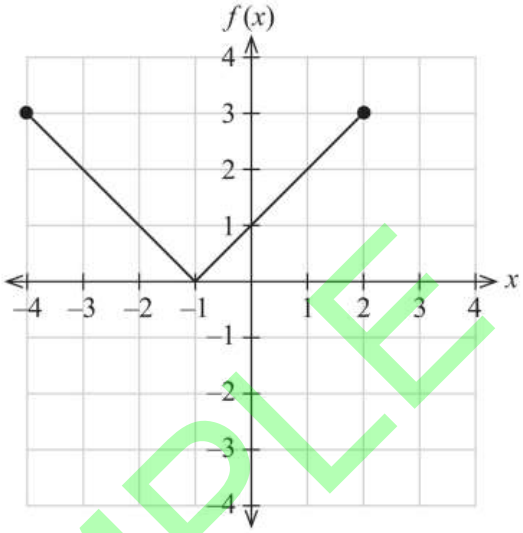
Complete the table below by determining the values for a, b, c and k.

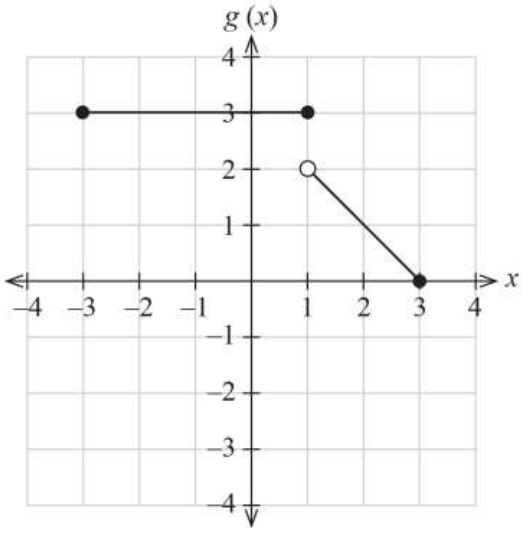
a	b	c	k

2023 Section 1 Question 4

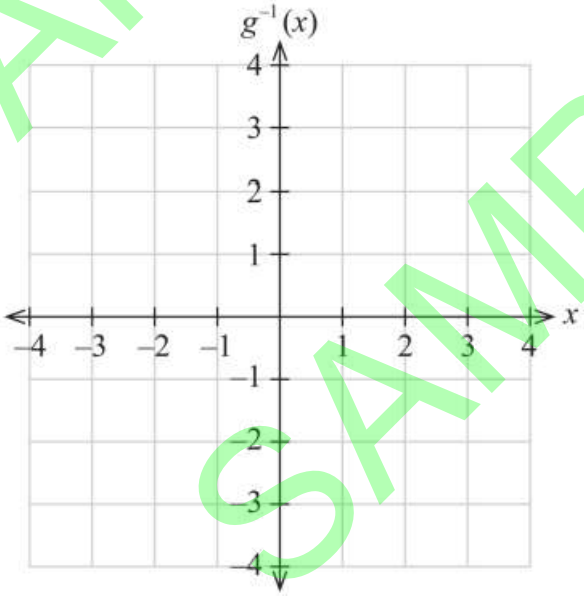
Functions and sketching graphs

The graphs of functions $f(x)$ and $g(x)$ are shown.





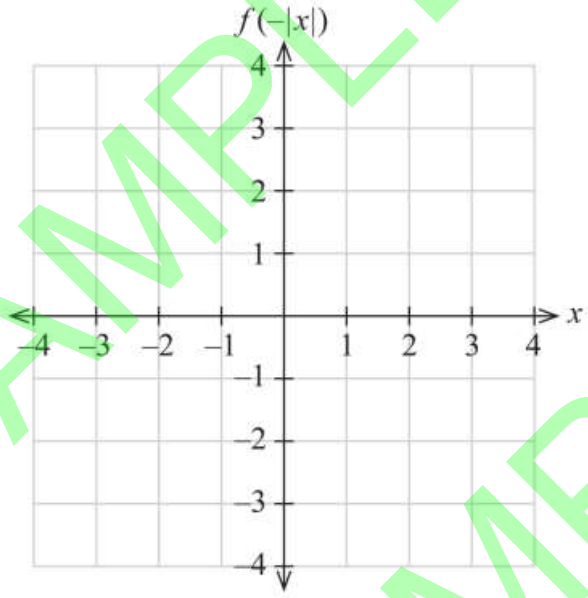
(a) Sketch the graph of $y = g^{-1}(x)$ on the axes below. (2 marks)



(b) State the value for $g\left(f^{-1}(0)\right)$. (2 marks)

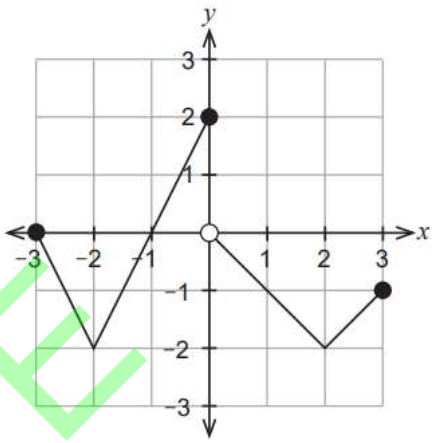
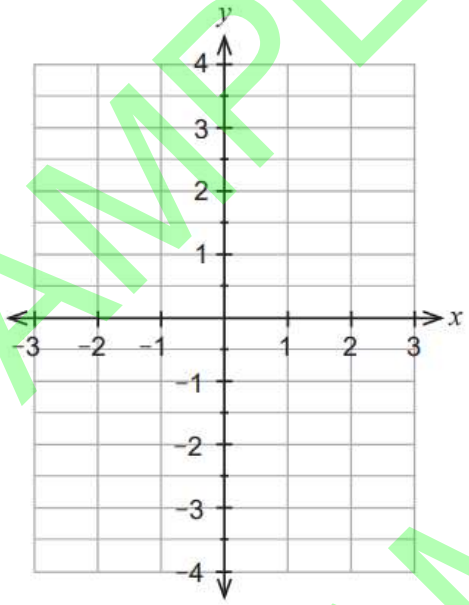
(c) Determine the set of values of x such that $f(g(x))$ is defined. (2 marks)

(d) Sketch the graph of $y = f(-|x|)$ on the axes below. (2 marks)



(e) The equation $|x+1| = k - |x+a|$ has an infinite number of solutions, with the solution set being $-3 \leq x \leq -1$. Determine the values of the constants a and k. (3 marks)

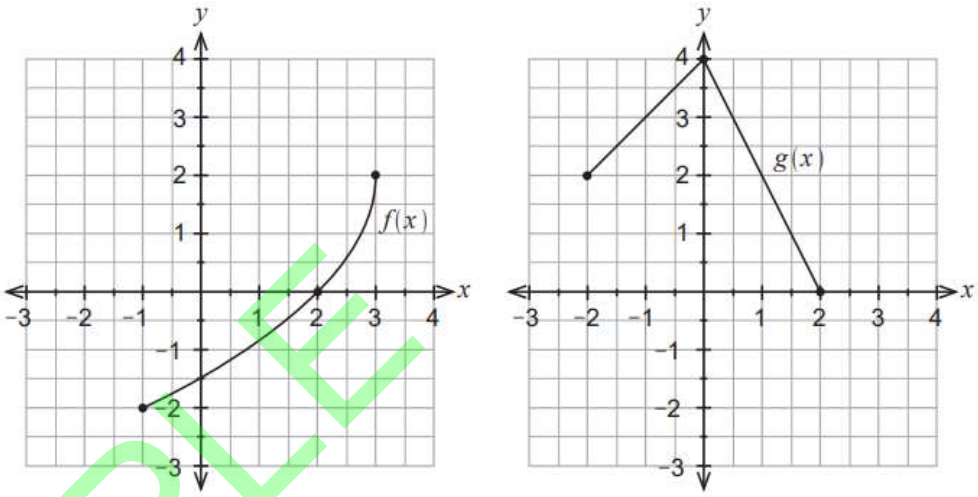
2022 Section 1 Question 1 **Functions and sketching graphs**	Consider functions $f(x) = \sqrt{4-x}$ and $g(x) = \frac{1}{x^2}$. (a) Determine the exact value of $g(f(-5))$. (2 marks) (b) Determine the domain for $f(g(x))$. (3 marks) (c) Explain why function g is not a one-to-one function. (1 mark)

2022 Section 1 Question 2 **Functions and sketching graphs**	The graph of $y = f(x)$ is shown below.  (a) Solve the equation $\lvert f(x) \rvert = x$. (2 marks) (b) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below. (5 marks) 

2021 Section 1 Question 2

Functions and sketching graphs

The graphs of functions f and g are shown below.



(a) Sketch the graph of function f^{-1} on the same axes used for function f. (2 marks)

(b) Explain why the inverse of g is not a function. (1 mark)

The defining rule for function f is $f(x) = 2 - 2\sqrt{3-x}$ where $-1 \leq x \leq 3$.

(c) Determine the rule for $y = f^{-1}(x)$. (3 marks)

(d) Determine the exact value for $g(f(0))$. (2 marks)

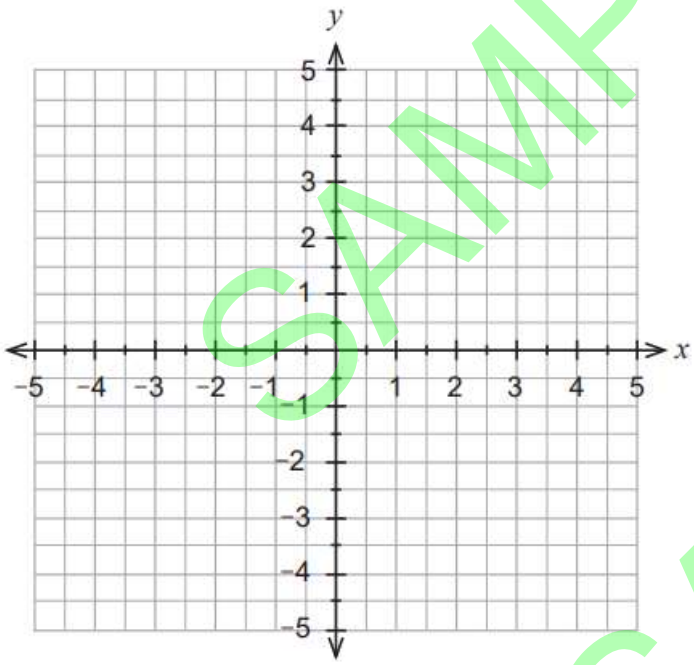
(e) Determine the domain for the function $y = f(g(x))$. Justify your answer. (3 marks)

2021 Section 1 Question 4

Functions and sketching graphs

Consider the function $f(x) = \dfrac{x^2-4}{x+1} = x - 1 - \dfrac{3}{x+1}$.

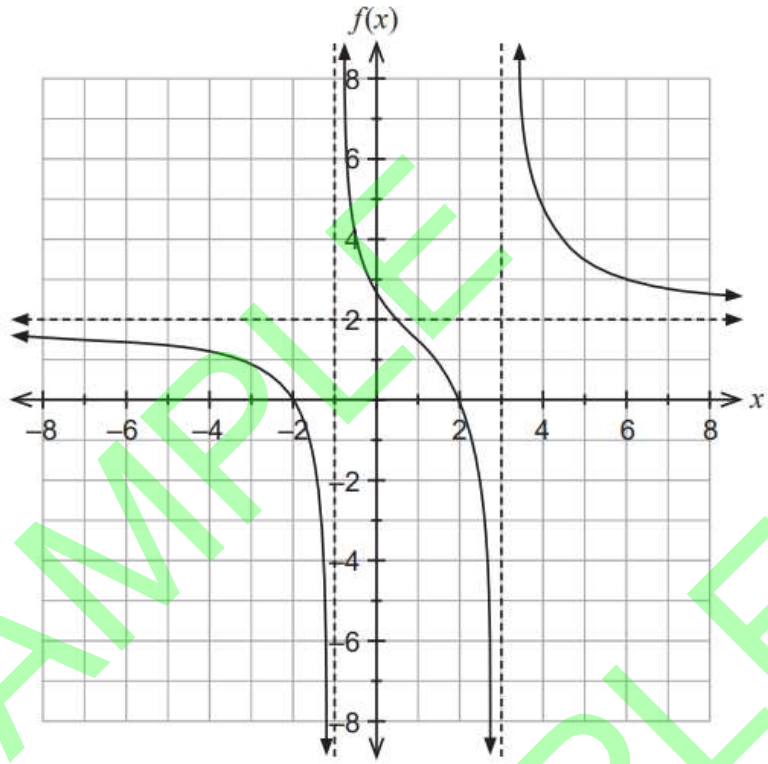
Sketch the graph of the function $y = f(x)$ on the axes below. Indicate clearly the x and y intercepts and any asymptotes.



2020 Section 1 Question 3

Functions and sketching graphs

The graph of $y = f(x)$ is shown on the axes below. The defining rule is given by $f(x) = \dfrac{a(x^2 - b)}{(x + c)(x - d)}$ where a, b, c and d are positive constants.



Determine the value of the constants a, b, c and d. Justify your answers. (6 marks)

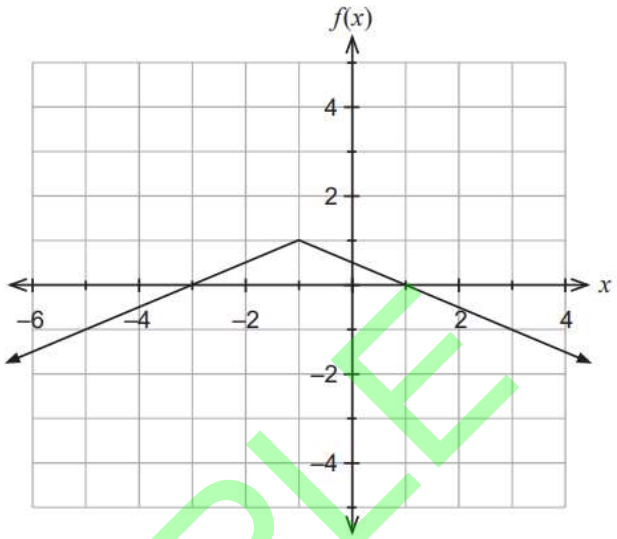
a	b	c	d

2020 Section 1

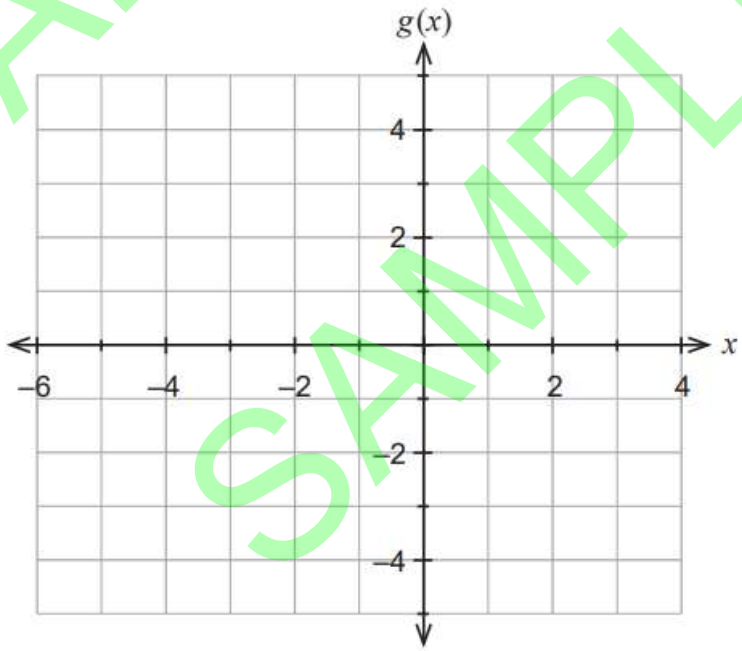
The graph of $f(x) = 1 - \dfrac{|x + 1|}{2}$ is shown below.

Question 5

Functions and sketching graphs



(a) Sketch the graph of $g(x) = \frac{1}{f(x)}$ on the axes below. (4 marks)



(b) Hence give the domain and range for $h(x) = \dfrac{4}{2 - |x + 1|}$. (3 marks)

2020 Section 1 Question 6

Functions and sketching graphs

Consider $f(x) = 2\tan(x)$ where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

Let $g(x) = f^{-1}(x)$ be the inverse of function f.

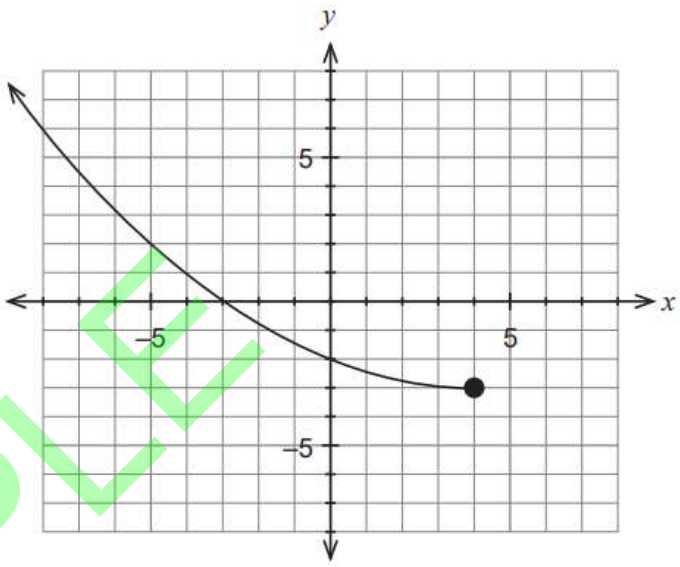
(a) Determine the defining rule for $y = g(x)$. (2 marks)

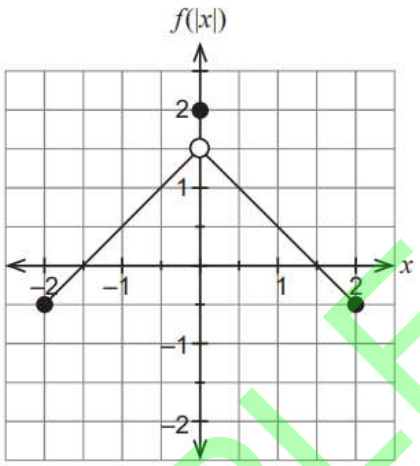
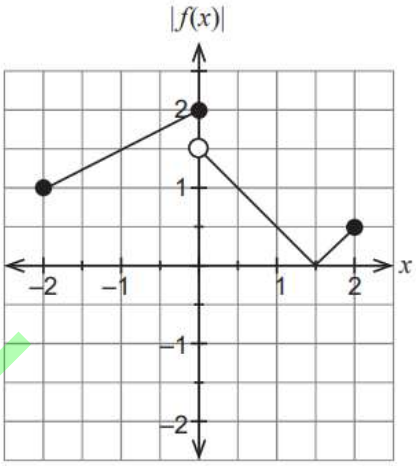
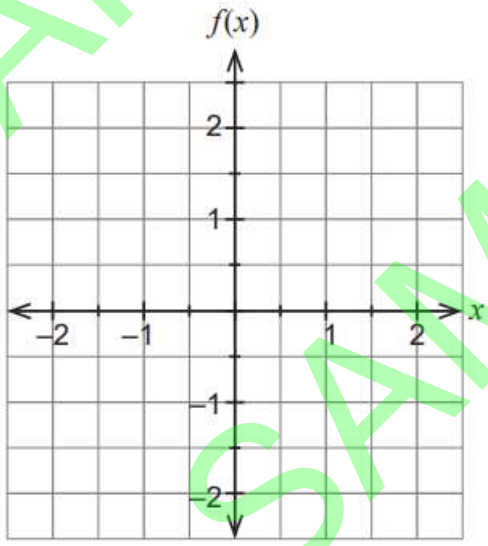
(b) By using implicit differentiation show that $g'(x)$ can be written in the form $\dfrac{a}{x^2 + b}$. (4 marks)

(c) Show that $\frac{3x^2+2x+6}{(x^2+4)(x-3)}$ can be expressed as $\frac{q}{x^2+4}+\frac{r}{x-3}$ and hence determine the values for q and r. (3 marks)

(d) Hence determine $\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx$. (4 marks)

2019 Section 1 Question 4 **Functions and sketching graphs**	Functions f, g and h are defined such that: $f(x)=\frac{1}{x-1}$, $g(x)=x^2$, $h(x)=\sqrt{x}$. (a) Determine the defining rule for $f(h(x))$. (1 mark) (b) Determine the domain for $f(h(x))$. (2 marks) (c) Determine the range for $f(h(x))$. (2 marks) (d) Is it true that $f(h(g(x)))=\frac{1}{x-1}=f(x)$? Justify your answer. (2 marks)

2019 Section 1 Question 5 **Functions and sketching graphs**	The graph of $y = g(x)$ is shown below.  (a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. (3 marks) (b) Given that $g(x) = \frac{1}{16}(x-4)^2 - 3$ where $x \leq 4$, determine the defining rule for $y = g^{-1}(x)$. (3 marks)

2019 Section 1 Question 7 **Functions and sketching graphs**	The graphs of $y = f(\lvert x \rvert)$ and $y = \lvert f(x) \rvert$ are shown below.   Given that $y = f^{-1}(x)$ is also a function, sketch a possible graph for $y = f(x)$ on the axes below. Justify your answer considering $y = f^{-1}(x)$. 

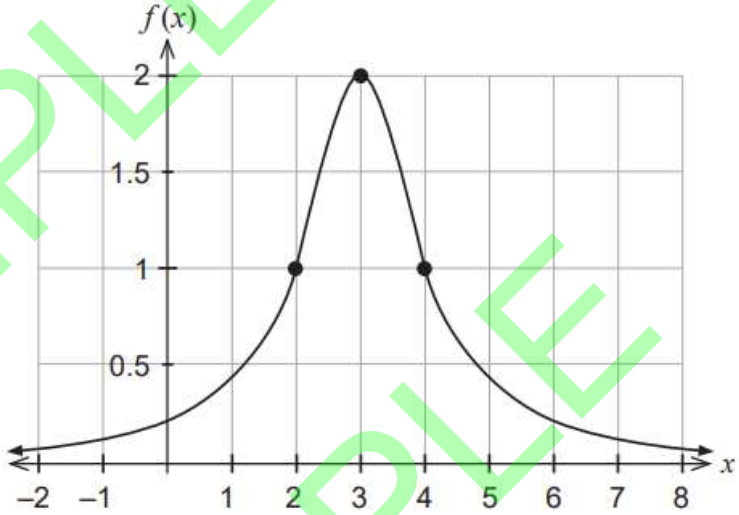
Section 2

2022 Section 2 Question 15

Functions and sketching graphs

The graph of a rational function f is shown below. Function f has the form $f(x) = \dfrac{k}{q(x)}$ with the following properties:

- f has no x intercepts or vertical asymptotes
- $f(x) \to 0$ for $|x| \to \infty$
- f is symmetric about $x = 3$
- function q is quadratic
- k is a constant.

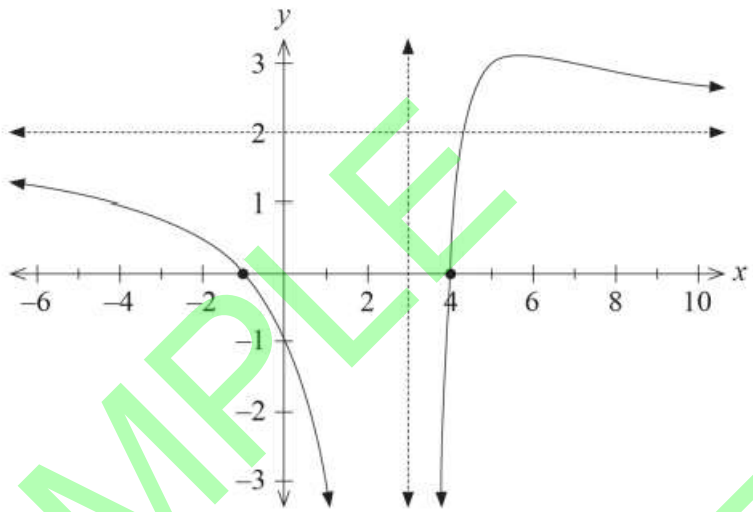


Determine the defining rule for f. (4 marks)

2023 Section 1 Question 1

Functions and sketching graphs

The graph of the function $f(x) = \dfrac{k(x+a)(x-b)}{(x-c)^2}$ is shown below. The constants a, b, c and k are positive.



Complete the table below by determining the values for a, b, c and k.

a	b	c	k

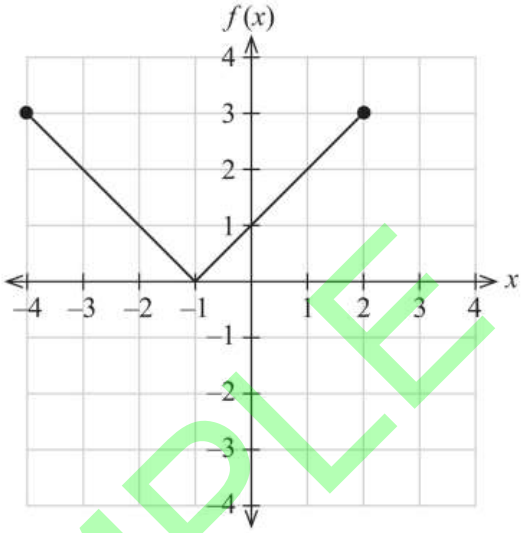
a	b	c	k
1	**4**	**3**	**2**

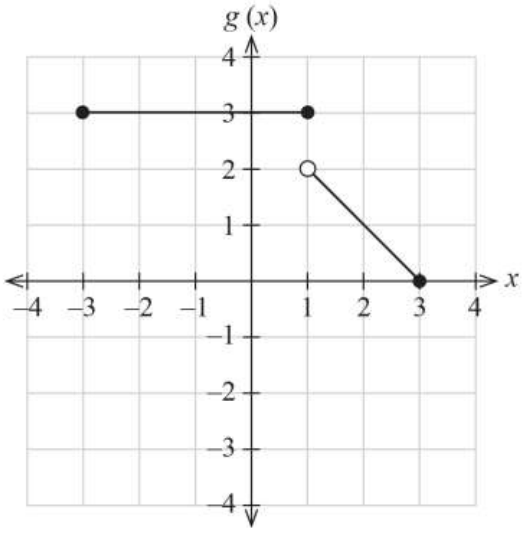
Solution
The x intercepts are $x=-1$, $x=4$ $\therefore$ $a=1$, $b=4$ Vertical asymptote is $x=3$ $\therefore$ $c=3$ Horizontal asymptote is $y=2$ $\therefore$ $k=2$
Specific behaviours
✓ states the value for a and b correctly ✓ states the value for c correctly ✓ states the value for k correctly ✓ provides justification for at least one of the values for a, b, c, k

2023 Section 1 Question 4

Functions and sketching graphs

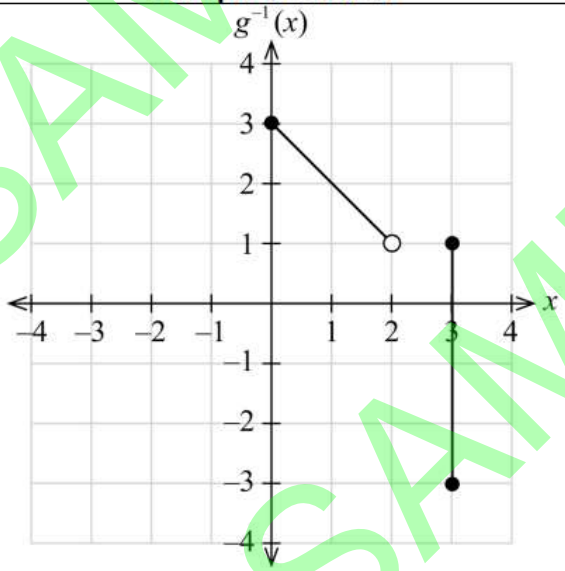
The graphs of functions $f(x)$ and $g(x)$ are shown.





(a) Sketch the graph of $y = g^{-1}(x)$ on the axes below. (2 marks)

Solution
Inverse of $g(x)$ does not exist as $g(x)$ is not a one-to-one function. (Graph cannot be sketched).
Specific behaviours
✓✓ identifies inverse of $g(x)$ does not exist or identifies graph cannot be sketched

Accepted solution

Specific behaviours
✓ indicates $x = 3$ for $-3 \le y \le 1$
✓ indicates $y = 3 - x$ for $0 \le x < 2$

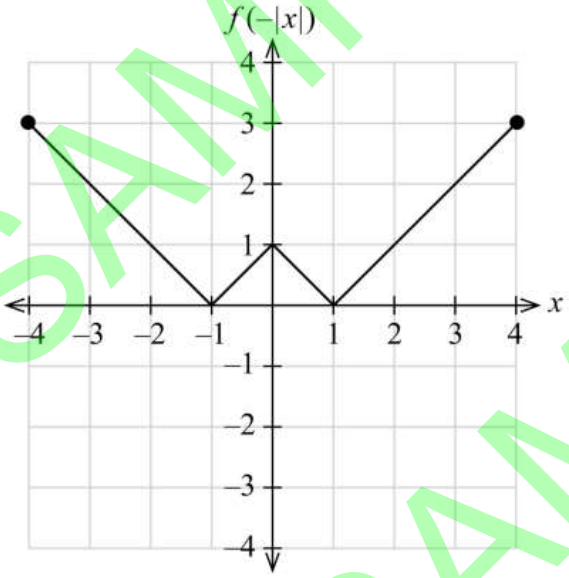
(b) State the value for $g\left(f^{-1}(0)\right)$. (2 marks)

Solution
Let $f^{-1}(0)=x \quad \therefore \ f(x)=0$ From the graph of $y=f(x)$ hence $x=-1$. $g\left(f^{-1}(0)\right) = g(-1) = 3$
Specific behaviours
✓ states that $f^{-1}(0)=-1$ ✓ evaluates $g\left(f^{-1}(0)\right)$ correctly

(c) Determine the set of values of x such that $f(g(x))$ is defined. (2 marks)

Solution
For $f(g(x))$ to be defined then $R_g \subseteq D_f$ i.e. the range of g must be part of the domain of f. This will occur when $g(x) \leq 2$ i.e. $1 < x \leq 3$.
Specific behaviours
✓ states that $R_g \subseteq D_f$ ✓ states the correct set of values for x

(d) Sketch the graph of $y=f(-|x|)$ on the axes below. (2 marks)

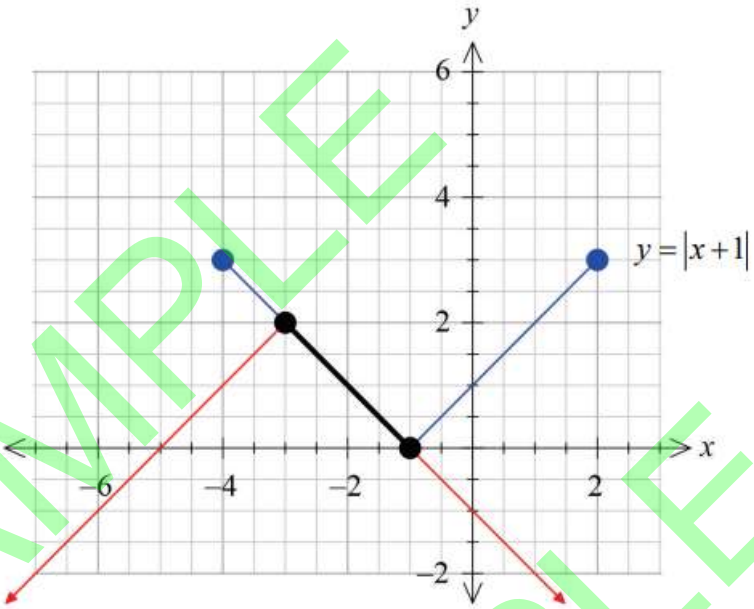


Solution
Shown above.
Specific behaviours
✓ indicates symmetry about $x=0$ ✓ indicates the correct set of points for $-4 \leq x \leq 4$

(e) The equation $|x+1| = k - |x+a|$ has an infinite number of solutions, with the solution set being $-3 \le x \le -1$. Determine the values of the constants a and k. (3 marks)

Solution

Consider the graph of $y = |x+1|$ and $y = k - |x+a|$ so that they intersect only when $-3 \le x \le -1$.



This intersection will occur when we consider $y = 2 - |x+3|$.

Hence $k = 2$ and $a = 3$.

Specific behaviours

- ✓ states the correct value for k
- ✓ states the correct value for a
- ✓ provides appropriate justification (considers the graphs of absolute value functions that yields an intersection only for $-3 \le x \le -1$)

Accept other relevant answers.

2022 Section 1 Question 1

Functions and sketching graphs

Consider functions $f(x) = \sqrt{4-x}$ and $g(x) = \frac{1}{x^2}$.

(a) Determine the exact value of $g(f(-5))$. (2 marks)

Solution

$$g(f(-5)) = g\left(\sqrt{4-(-5)}\right) = g(3) = \frac{1}{9}$$

Specific behaviours

- ✓ determines $f(-5)$ correctly
- ✓ obtains the correct value for $g(f(-5))$

(b) Determine the domain for $f(g(x))$. (3 marks)

Solution
$f(g(x)) = \sqrt{4-\frac{1}{x^2}}$ This will be defined when $4-\frac{1}{x^2} \geq 0$, $x \neq 0$ since $g(x)$ must exist. i.e. $\frac{1}{x^2} \leq 4$ i.e. $x^2 \geq \frac{1}{4}$ $\therefore D_{fog} = \{x \mid x \geq \frac{1}{2} \cup x \leq -\frac{1}{2}\}$
Specific behaviours
✓ identifies that $f(g(x))$ is defined when $4-\frac{1}{x^2} \geq 0$ ✓ states $x \geq \frac{1}{2}$ ✓ states $x \leq -\frac{1}{2}$

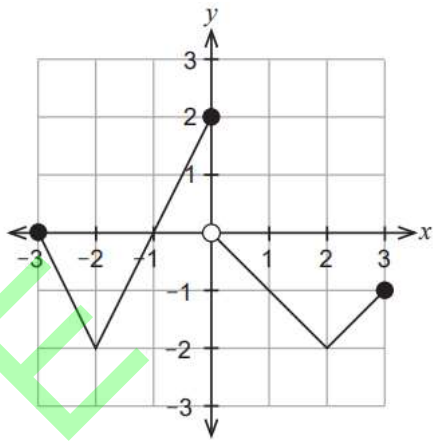
(c) Explain why function g is not a one-to-one function. (1 mark)

Solution
$g(-2) = g(2) = \frac{1}{4}$ This shows that g maps two values of x to a single value. Hence g is NOT a one-to-one function BUT is a MANY-to-one function.
Specific behaviours
✓ justifies why g is not a one-to-one function

2022 Section 1 Question 2

Functions and sketching graphs

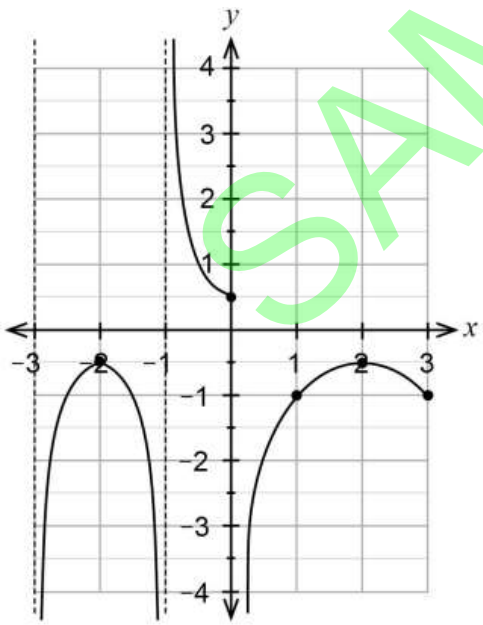
The graph of $y = f(x)$ is shown below.



(a) Solve the equation $|f(x)| = x$. (2 marks)

Solution
Equation requires the intersection between $y = \|f(x)\|$ and $y = x$.
This occurs when $0 < x \leq 2$.
Specific behaviours
✓ excludes $x = 0$ and includes $x = 2$ in the solution
✓ states the correct interval of real values for x

(b) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below. (5 marks)

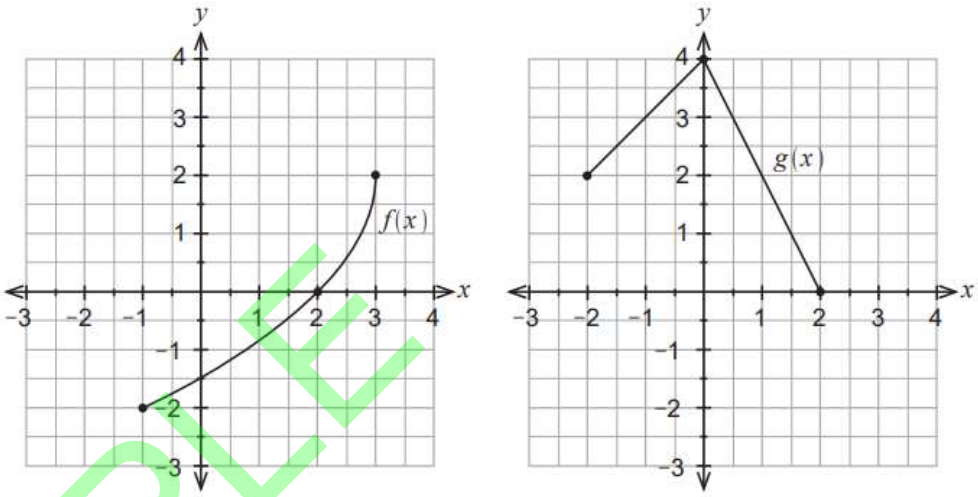


Solution
See graph axes.
Specific behaviours
✓ indicates vertical asymptotes at $x = -3, -1, 0$
✓ indicates correct function behaviour as $x \to -3$ and $x \to 0^+$
✓ indicates correct function behaviour as $x \to -1$
✓ indicates the correct curvature
✓ indicates at least one of the 5 highlighted points

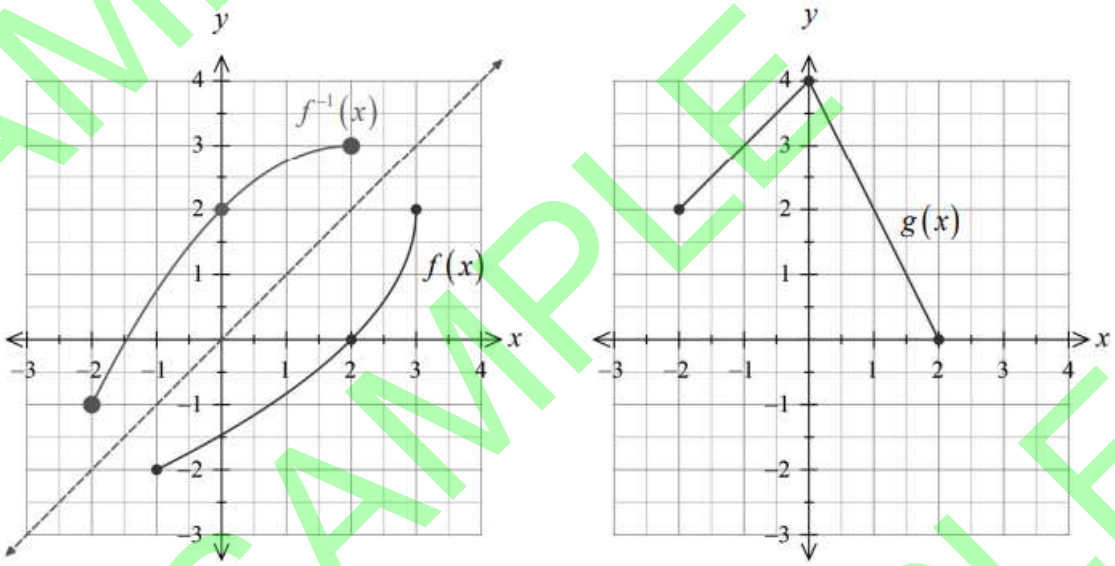
2021 Section 1 Question 2

Functions and sketching graphs

The graphs of functions f and g are shown below.



(a) Sketch the graph of function f^{-1} on the same axes used for function f. (2 marks)



Solution
See above graph axes.
Specific behaviours
✓ indicates a concave down curve that is a reflection of $y = f(x)$ about $y = x$
✓ indicates all the points $(-2,-1)$, $(0,2)$ and $(2,3)$

(b) Explain why the inverse of g is not a function. (1 mark)

Solution
Function g is not a one-to-one function over its domain OR does not pass the horizontal line test.
Specific behaviours
✓ refers to function g not being a one-to-one function

The defining rule for function f is $f(x) = 2 - 2\sqrt{3-x}$ where $-1 \le x \le 3$.

(c) Determine the rule for $y = f^{-1}(x)$. (3 marks)

Solution
$f: \ y = 2 - 2\sqrt{3-x}$ Hence $f^{-1}: \ x = 2 - 2\sqrt{3-y}$ $\sqrt{3-y} = \dfrac{2-x}{2}$ $\therefore \ 3-y = \left(\dfrac{2-x}{2}\right)^2$ $\therefore \ f^{-1}(x) = 3 - \left(\dfrac{2-x}{2}\right)^2$
Specific behaviours
✓ interchanges x, y to obtain the rule for the inverse ✓ obtains the correct expression for $\sqrt{3-y}$ ✓ obtains the correct defining rule for $y = f^{-1}(x)$

(d) Determine the exact value for $g(f(0))$. (2 marks)

Solution
$g(f(0)) = g(2-2\sqrt{3})$ $= (2-2\sqrt{3}) + 4$ since $-2 \le 2-2\sqrt{3} \le 0$ $= 6-2\sqrt{3}$
Specific behaviours
✓ evaluates $f(0)$ correctly ✓ determines the exact value $6-2\sqrt{3}$ correctly

(e) Determine the domain for the function $y = f(g(x))$. Justify your answer. (3 marks)

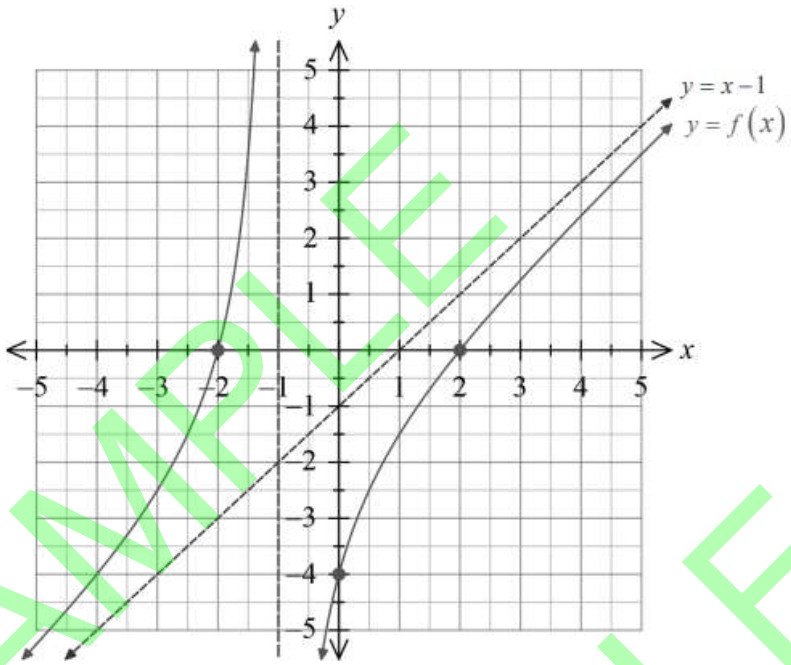
Solution
The range of g must be a SUBSET of the domain of f. $\therefore \ D_{fog} = \{x \mid -2 \le x \le -1, \ 0.5 \le x \le 2\}$ Note that for $-1 < x < 0.5$ $g(x) > 3$ which is not in the domain for function f.
Specific behaviours
✓ states that $-2 \le x \le -1$ ✓ states that $0.5 \le x \le 2$ ✓ justifies the chosen domain correctly

2021 Section 1 Question 4

Functions and sketching graphs

Consider the function $f(x) = \dfrac{x^2-4}{x+1} = x-1-\dfrac{3}{x+1}$.

Sketch the graph of the function $y = f(x)$ on the axes below. Indicate clearly the x and y intercepts and any asymptotes.



Solution

$f(x) = \dfrac{(x+2)(x-2)}{(x+1)} = x-1-\dfrac{3}{x+1}$

x intercepts occur when $x^2-4=0$ *i.e.* at $x=\pm 2$ $\quad$ y intercept $f(0)=-4$

Vertical asymptote is $x=-1$.

As $|x|\to\infty$, $f(x)\to x-1$ (inclined asymptote)

Sketch shown above.

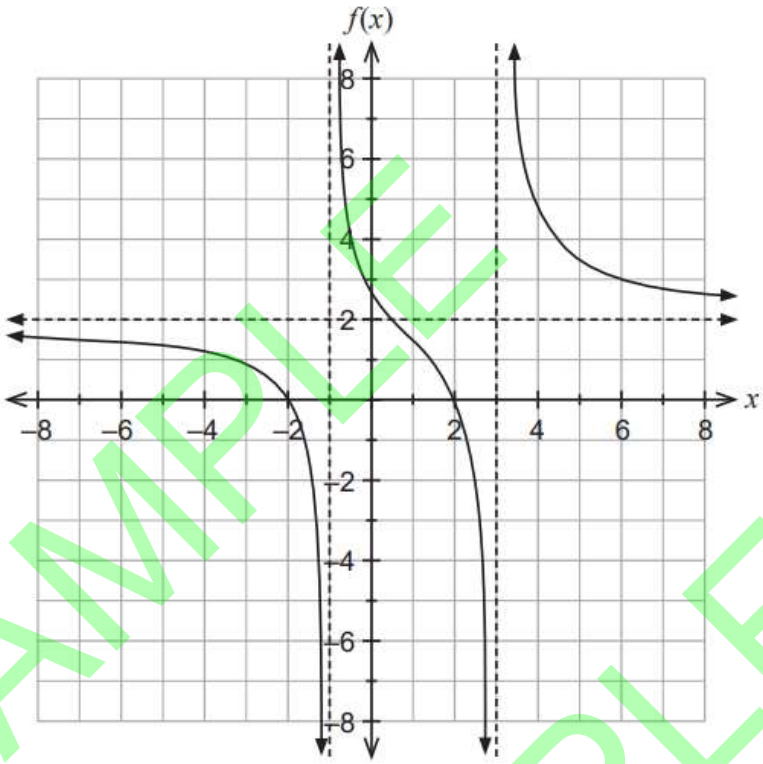
Specific behaviours

- ✓ indicates x intercepts at $x=\pm 2$
- ✓ indicates a vertical asymptote at $x=-1$
- ✓ indicates $f(0)=-4$
- ✓ indicates inclined asymptote $y=x-1$ i.e. $f(x)\to x-1$ for $|x|\to\infty$
- ✓ indicates correct curvature in the graph

2020 Section 1 Question 3

Functions and sketching graphs

The graph of $y = f(x)$ is shown on the axes below. The defining rule is given by $f(x) = \dfrac{a(x^2 - b)}{(x + c)(x - d)}$ where a, b, c and d are positive constants.



Determine the value of the constants a, b, c and d. Justify your answers. (6 marks)

a	b	c	d
2	4	1	3

Solution

Horizontal intercepts are $x = -2,\ x = 2 \ \ \therefore\ x^2 - b = (x+2)(x-2)$ i.e. $b = 4$

Vertical asymptotes are $x = -1,\ x = 3 \ \ \therefore\ c = 1,\ d = 3$

Horizontal asymptote is $y = 2 \ \ \therefore\ a = 2$

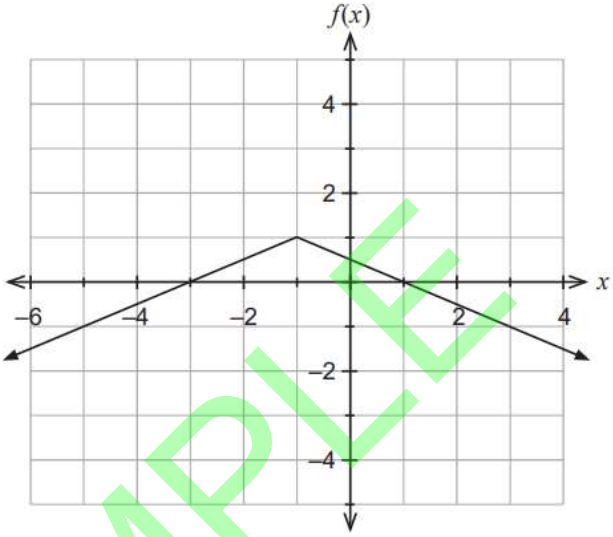
Specific behaviours

- ✓ states that $a = 2$
- ✓ justifies why $a = 2$ (refers to the horizontal asymptote $y = 2$)
- ✓ states that $b = 4$
- ✓ justifies why $b = 4$ (refers to two horizontal intercepts)
- ✓ states that $c = 1,\ d = 3$
- ✓ justifies why $c = 1,\ d = 3$ (refers to the two vertical asymptotes)

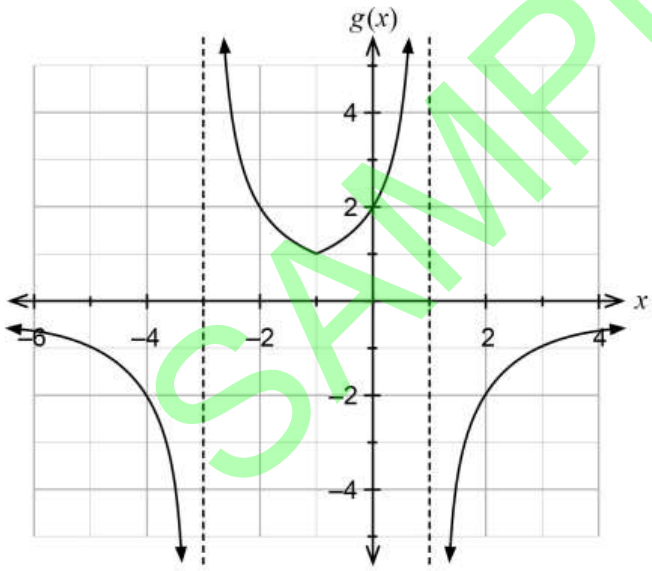
2020 Section 1 Question 5

Functions and sketching graphs

The graph of $f(x) = 1 - \dfrac{|x+1|}{2}$ is shown below.



(a) Sketch the graph of $g(x) = \dfrac{1}{f(x)}$ on the axes below. (4 marks)



Solution
Shown above.
Specific behaviours
✓ indicates vertical asymptotes at $x = -3$ and $x = 1$
✓ indicates $g(x) < 0$ as $\lvert x \rvert \to \infty$
✓ indicates $g(x) \geq 1$ for $-3 < x < 1$
✓ indicates correct graph curvature (cusp at $x = -1$ is not required)

(b) Hence give the domain and range for $h(x) = \dfrac{4}{2-|x+1|}$. (3 marks)

Solution
$h(x) = \dfrac{4}{2-\lvert x+1\rvert} = \dfrac{4}{2\left(1-\dfrac{\lvert x+1\rvert}{2}\right)}$ $= \dfrac{2}{f(x)} = 2g(x)$ Domain $D_h = D_g = \{x \mid x \neq -3,\ x \neq 1\}$ Range $R_h = \{y \mid y < 0,\ y \geq 2\}$
Specific behaviours
✓ states the correct domain (from function g) ✓ states the correct range component $y < 0$ ✓ states the correct range component $y \geq 2$

2020 Section 1 Question 6

Functions and sketching graphs

Consider $f(x) = 2\tan(x)$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

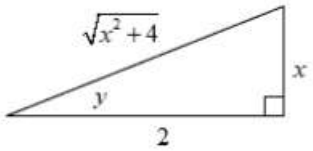
Let $g(x) = f^{-1}(x)$ be the inverse of function f.

(a) Determine the defining rule for $y = g(x)$. (2 marks)

Solution
Function f: $y = 2\tan(x)$ Hence f^{-1}: $x = 2\tan(y)$ i.e. $\frac{x}{2} = \tan(y)$ $\therefore\ y = \tan^{-1}\left(\frac{x}{2}\right)$ i.e. $g(x) = f^{-1}(x) = \tan^{-1}\left(\frac{x}{2}\right)$ *or* $\arctan\left(\frac{x}{2}\right)$
Specific behaviours
✓ interchanges x, y to form the rule for the inverse function ✓ expresses $f^{-1}(x)$ correctly in terms of x

(b) By using implicit differentiation show that $g'(x)$ can be written in the form $\frac{a}{x^2+b}$.

(4 marks)

Solution
$\frac{d}{dx}(x) = \frac{d}{dx}(2\tan(y))$ $1 = 2(\sec^2 y)\left(\frac{dy}{dx}\right)$ ... (1) $\therefore\ \frac{dy}{dx} = \frac{\cos^2 y}{2} = \frac{1}{2}\left(\frac{2}{\sqrt{x^2+4}}\right)^2 = \frac{2}{x^2+4}$ Given $\tan(y) = \frac{x}{2}$ $\cos(y) = \frac{2}{\sqrt{x^2+4}}$    OR $\frac{dy}{dx} = \frac{1}{2\sec^2 y} = \frac{1}{2(1+\tan^2 y)} = \frac{1}{2+2\left(\frac{x}{2}\right)^2} = \frac{2}{x^2+4}$
Specific behaviours
✓✓ differentiates implicitly correctly to obtain statement (1) ✓ obtains an expression for $\cos(y)$ or $\tan(y)$ correctly in terms of x ✓ obtains a correct simplified expression for $\frac{dy}{dx}$ correctly in the form $\frac{a}{x^2+b}$

(c) Show that $\frac{3x^2+2x+6}{(x^2+4)(x-3)}$ can be expressed as $\frac{q}{x^2+4}+\frac{r}{x-3}$ and hence determine the values for q and r. (3 marks)

Solution
It is required that $q(x-3)+r(x^2+4)=3x^2+2x+6$ Hence $rx^2+qx+(4r-3q)=3x^2+2x+6$ Equating co-efficients we obtain: $r=3 \quad \ldots(1)$ $q=2 \quad \ldots(2)$ $4r-3q=6 \quad \ldots(3)$ Testing $q=2,\ r=3$ in equation (3): $4(3)-3(2)=6$ is true. Solving gives $q=2,\ r=3$.
Specific behaviours
✓ forms the equivalence of numerators correctly ✓ solves for q, r correctly ✓ tests the consistency of q, r to obtain the constant 6

(d) Hence determine $\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx$. (4 marks)

Solution
$\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx = \int \frac{2}{x^2+4}+\frac{3}{x-3}\,dx$ $= \tan^{-1}\left(\frac{x}{2}\right)+3\ln\lvert x-3\rvert+c$
Specific behaviours
✓ re-writes the integrand in terms of the partial fractions correctly ✓ anti-differentiates correctly using the logarithm of an absolute value ✓ uses the result of part (b) to correctly anti-differentiate ✓ uses a constant of integration

2019 Section 1 Question 4

Functions and sketching graphs

Functions f, g and h are defined such that:

$$f(x)=\frac{1}{x-1},\ g(x)=x^2,\ h(x)=\sqrt{x}.$$

(a) Determine the defining rule for $f(h(x))$. (1 mark)

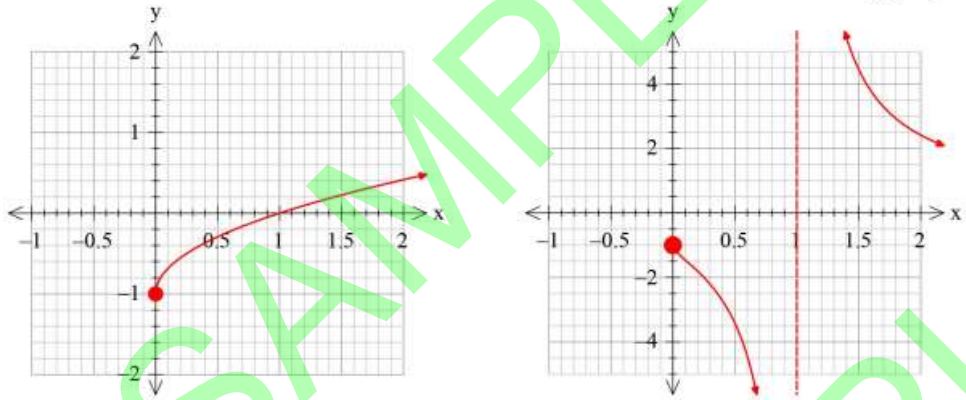
Solution
$f(h(x))=\frac{1}{\sqrt{x}-1}$
Specific behaviours
✓ states the correct defining rule

(b) Determine the domain for $f(h(x))$. (2 marks)

Solution
$D_{foh} = \{x \mid x \geq 0,\ x \neq 1\}$
Specific behaviours
✓ states $x \geq 0$ ✓ states $x \neq 1$

(c) Determine the range for $f(h(x))$. (2 marks)

Solution
When $x > 1$ $\quad f(h(x)) > 0$ When $0 \leq x < 1$ $\quad -1 \leq \sqrt{x} - 1 < 0$ $\quad \therefore\ -1 \geq \dfrac{1}{\sqrt{x}-1}$ Hence $R_{foh} = \{y \mid y > 0 \cup y \leq -1\}$
Specific behaviours
✓ states $y > 0$ ✓ states $y \leq -1$

Alternative Solution
Graph $y = \sqrt{x} - 1$ and then graph its reciprocal function $y = f(h(x)) = \dfrac{1}{\sqrt{x}-1}$    Hence $R_{foh} = \{y \mid y > 0 \cup y \leq -1\}$
Specific behaviours
✓ states $y > 0$ ✓ states $y \leq -1$

(d) Is it true that $f(h(g(x))) = \dfrac{1}{x-1} = f(x)$? Justify your answer. (2 marks)

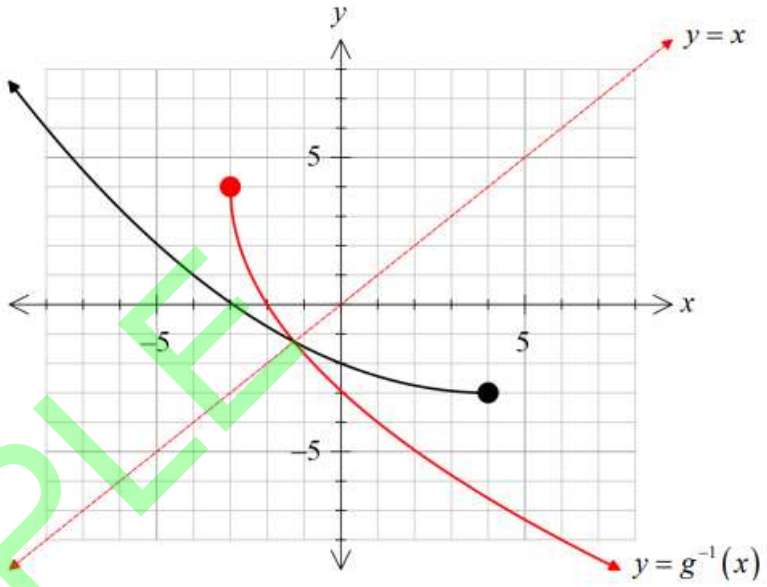
Solution
The statement is FALSE. $h(g(x)) = \sqrt{x^2} = \lvert x \rvert \geq 0$ Hence $f(h(g(x))) = \dfrac{1}{\sqrt{x^2}-1} = \dfrac{1}{\lvert x \rvert - 1}$ $\quad D_{fohog} = \{x \mid x \in \mathbb{R},\ x \neq \pm 1\}$ $\quad R_{fohog} = \{y \mid y > 0 \cup y \leq -1\}$ But $f(x) = \dfrac{1}{x-1}$ $\quad D_{fohog} = \{x \mid x \in \mathbb{R},\ x \neq 1\}$ $\quad R_{fohog} = \{y \mid y > 0\}$ $\therefore \quad f(h(g(x))) \neq f(x)$ as they have different DOMAIN and RANGE values.
Specific behaviours
✓ states that the statement is false ✓ justifies the statement is false

Alternative Solution
The statement is FALSE. This would be true if $h(g(x)) = x$ i.e. true if $\sqrt{x^2} = x$. But actually $\sqrt{x^2} = \lvert x \rvert \neq x$.
Specific behaviours
✓ states that the statement is false ✓ justifies the statement is false

2019 Section 1 Question 5

Functions and sketching graphs

The graph of $y = g(x)$ is shown below.



(a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. (3 marks)

Solution
See above graph axes.
Specific behaviours
✓ indicates the points $(-3,4)$, $(-2,0)$ and $(2,-5)$
✓ indicates the range $y \le 4$ (arrow on graph not required)
✓ indicates symmetry of $y = g^{-1}(x)$ with $y = g(x)$ about the line $y = x$

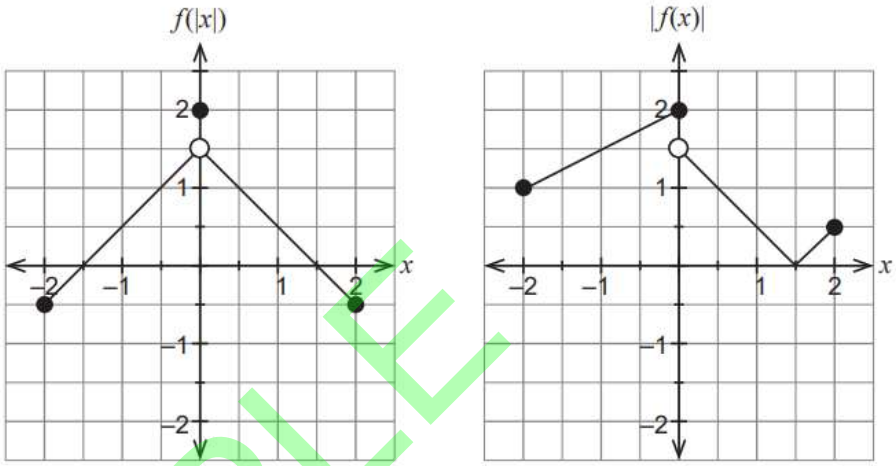
(b) Given that $g(x) = \frac{1}{16}(x-4)^2 - 3$ where $x \le 4$, determine the defining rule for $y = g^{-1}(x)$. (3 marks)

Solution
$g: \quad y = \frac{1}{16}(x-4)^2 - 3 \qquad g^{-1}: \quad x = \frac{1}{16}(y-4)^2 - 3$ $\therefore \quad 16(x+3) = (y-4)^2 \quad \ldots (1)$ $y - 4 = \pm 4\sqrt{x+3}$ Since $R_{g^{-1}} = D_g \ (x \le 4)$ then $g^{-1}(x) = 4 - 4\sqrt{x+3}$
Specific behaviours
✓ interchanges the x, y coordinates to obtain the inverse
✓ manipulates the equation correctly to obtain statement 1
✓ writes the correct defining rule

2019 Section 1 Question 7

Functions and sketching graphs

The graphs of $y = f(|x|)$ and $y = |f(x)|$ are shown below.

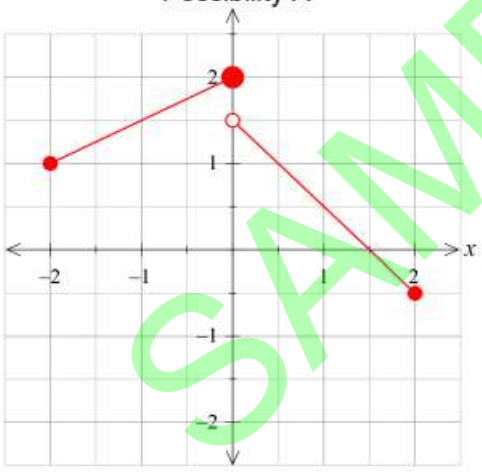


Given that $y = f^{-1}(x)$ is also a function, sketch a possible graph for $y = f(x)$ on the axes below. Justify your answer considering $y = f^{-1}(x)$.

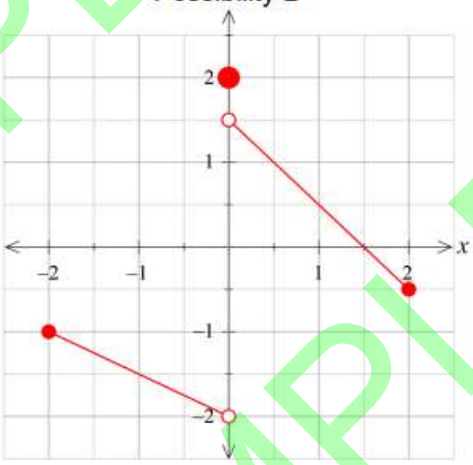
Solution

There are many possibilities for $y = f(x)$. Two of these are:

Possibility A



Possibility B



Since $y = f^{-1}(x)$ is a function then $y = f(x)$ over its domain $-2 \le x \le 2$ must be a ONE-TO-ONE function (which does not occur with possibility A or D). Hence $y = f(x)$ could be possibility B or C. Alternatively, function $f(x)$ must satisfy the 'horizontal' line test.

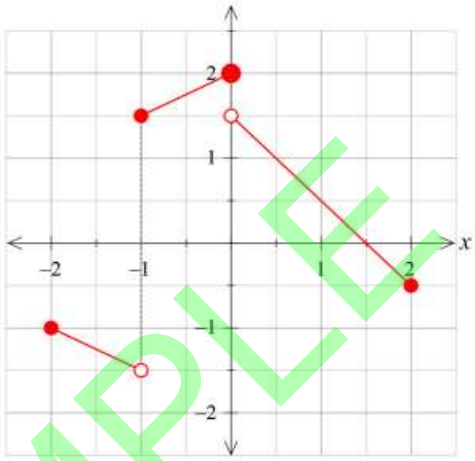
Specific behaviours

- ✓ indicates the points $(0,2)$, $(1.5,0)$, $(2,-0.5)$
- ✓ indicates $y = 1.5 - x$ for $0 < x \le 2$
- ✓ indicates $y = 0.5x + 2$ OR $y = -0.5x - 2$ for $-2 \le x < 0$ or equivalent to obtain $y = f(|x|)$ and $y = |f(x)|$ correctly
- ✓ justifies that $y = f(x)$ must be a one-to-one function so that $y = f^{-1}(x)$ is a function

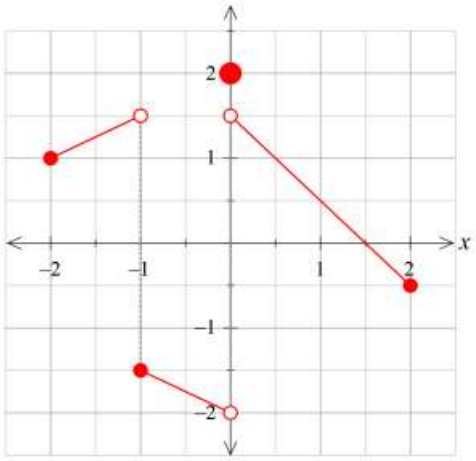
Alternative Solution

Other possibilities for $y = f(x)$:

Possibility C



Possibility D

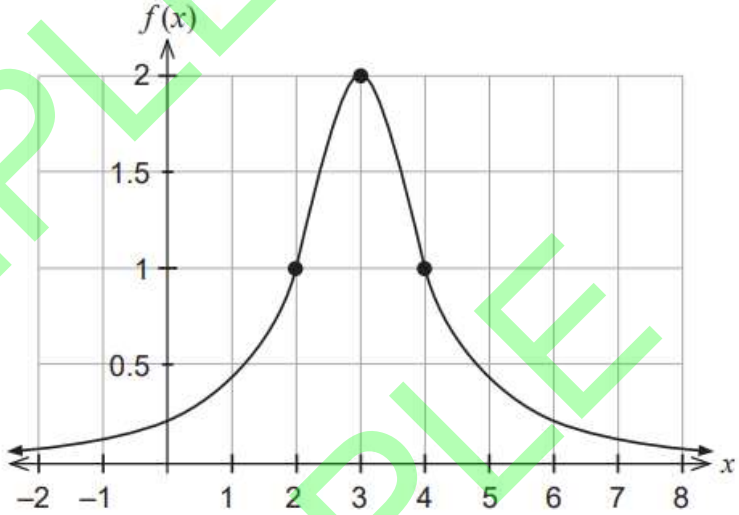


2022 Section 2 Question 15

Functions and sketching graphs

The graph of a rational function f is shown below. Function f has the form $f(x)=\dfrac{k}{q(x)}$ with the following properties:

- f has no x intercepts or vertical asymptotes
- $f(x)\to 0$ for $|x|\to\infty$
- f is symmetric about $x=3$
- function q is quadratic
- k is a constant.



Determine the defining rule for f. (4 marks)

Solution

Since q is quadratic with symmetry about $x=3$ $q(x)=a(x-3)^2+b$

As there are no vertical asymptotes then $q(x)\neq 0$ $\therefore$ $a,b>0$ using $k>0$.

The maximum of f will occur when q is a minimum.

Using $f(3)=2$ $\quad 2=\dfrac{k}{a(3-3)^2+b}$ $\quad$ *i.e.* $k=2b$ ... (1)

Using $f(2)=1$ $\quad 1=\dfrac{k}{a(2-3)^2+b}$ $\quad$ *i.e.* $a+b=k$ ... (2)

Solving (1),(2) simultaneously we obtain $a=b$, $k=2b$

i.e. $f(x)=\dfrac{2b}{b(x-3)^2+b}=\dfrac{2}{(x-3)^2+1}=\dfrac{2}{x^2-6x+10}$

Specific behaviours

- ✓ forms a quadratic rule for $q(x)$ that has symmetry about $x=3$
- ✓ forms a quadratic rule for $q(x)$ that does NOT have any x intercepts $(a,b>0)$
- ✓ forms correct relationships between k and other constants
- ✓ determines the correct defining rule for $f(x)$

Alternative Solution

Since q is quadratic then let $q(x)=ax^2+bx+c$

Using $f(3)=2$ $\quad 2 = \dfrac{k}{9a+3b+c}$ ... (1)

$f(2)=1$ $\quad 1 = \dfrac{k}{4a+2b+c}$ ... (2)

$f(4)=1$ $\quad 1 = \dfrac{k}{16a+4b+c}$ ... (3)

Solving $(1),(2),(3)$, simultaneously we obtain $a=a,\ b=-6a,\ c=10a,\ k=2a$

i.e. $f(x) = \dfrac{2a}{ax^2-6ax+10a} = \dfrac{2a}{a(x^2-6x+10)} = \dfrac{2}{x^2-6x+10}$

Specific behaviours

- ✓ uses the rule $q(x)=ax^2+bx+c$ and $f(3)=2, f(2)=f(4)=1$ to form three equations relating k,a,b,c
- ✓ solves simultaneously to obtain relationships between variables
- ✓ factors out the common factor between variables
- ✓ determines the correct defining rule for $f(x)$