YEAR 12 ATAR COURSE REVISED EDITION

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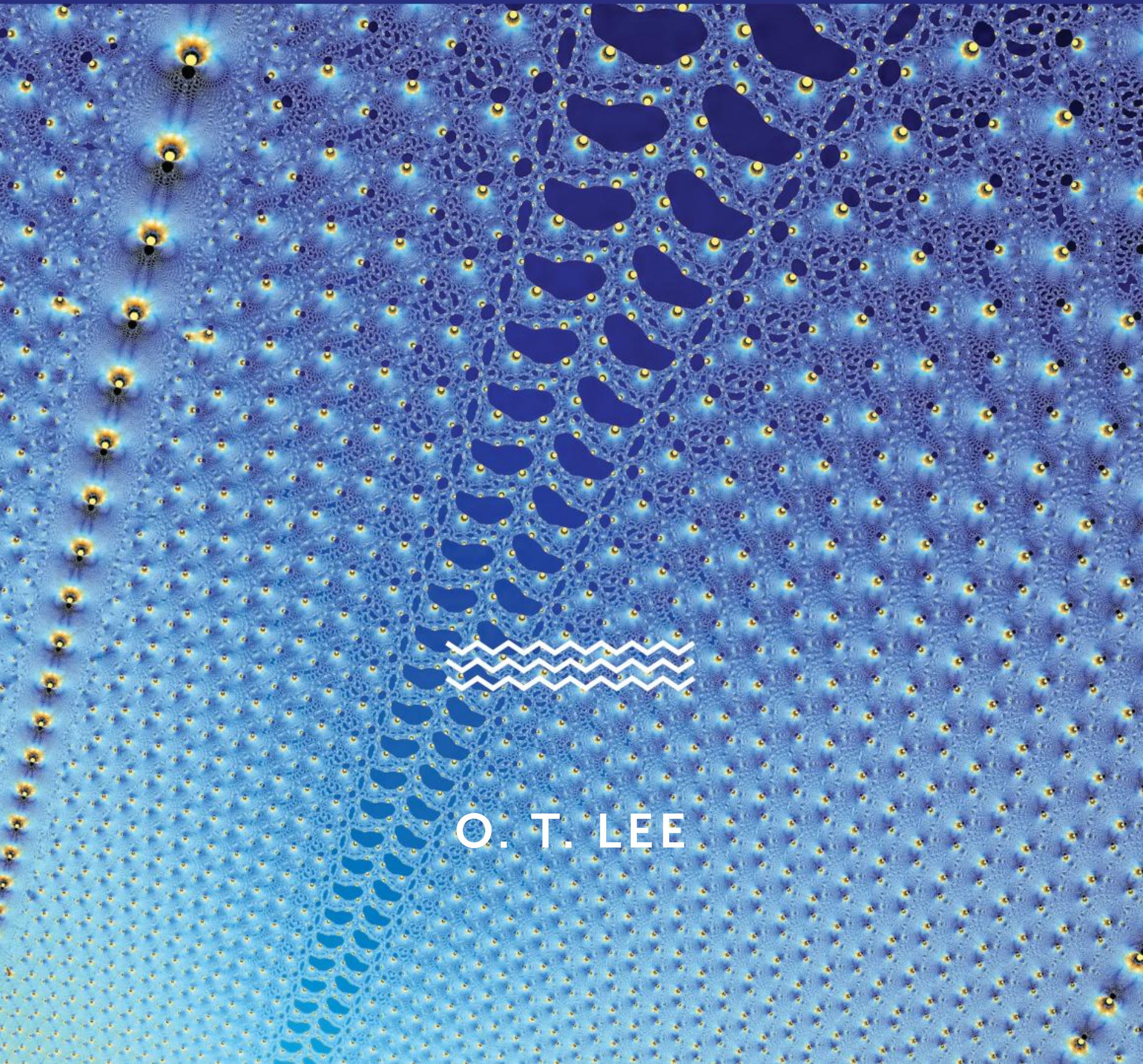
MATHEMATICS SPECIALIST

UNITS 3 & 4



O. T. LEE

ACADEMIC
TASK FORCE

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MATHEMATICS SPECIALIST

YEAR 12 ATAR COURSE
UNITS 3 & 4

SECOND EDITION

O. T. LEE

ACADEMIC GROUP
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Dr O. T. Lee is an author of many books which are used extensively in WA schools. Dr Lee is an exceptional, insightful teacher with wide-ranging experience as a WACE marker.

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Mathematics Specialist Revision Series Units 3 & 4

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Fully Worked Solutions

Mathematics Specialist Revision Series
Units 3 & 4

- The Mathematics Specialist Revision Series Units 3 & 4 provides a comprehensive set of revision/review questions for the new year 12 Mathematics Specialist Units 3 & 4 course.

- The review questions are written at test/examination level for both the Calculator Free and Calculator Assumed Sections and presented in a write-on format in topical order.

- This book exposes students to questions and problems at test/examination level.

- These questions are suitable for end-of-topic reviews and pre-test and pre-examination reviews.

- It is accompanied by a set of fully worked solutions with which students can measure their solutions. These solutions are often not the only solutions but they provide a model for students to work with. Students, interrogate your solutions to understand your errors and your successes. It may sometimes to possible to achieve a correct numerical answer with faulty reasoning!

- Do not memorise solutions. Understand the techniques and processes used in relation to the questions asked.

Notes

Complex numbers

- For $z = x + y\,i$, $\overline{z} = x - y\,i$.
  Note that, $z\overline{z} = x^2 + y^2$
- $\overline{u \pm v} = \overline{u} \pm \overline{v}$ and $\overline{u \times v} = \overline{u} \times \overline{v}$
- $\dfrac{1}{a+bi} = \dfrac{1}{a+bi} \times \dfrac{a-bi}{a-bi} = \dfrac{a-bi}{a^2+b^2}$
- $\dfrac{x+yi}{a+bi} = \dfrac{x+yi}{a+bi} \times \dfrac{a-bi}{a\text{-}bi} = \dfrac{(x+yi)(a-bi)}{a^2+b^2}$
- de Moivre's Theorem
  $z = cis\ \theta \Rightarrow z^n = cis\ n\theta$ for rational n.
- For $z = r\ cis\ \theta$:
    - $\overline{z} = r\ cis\ (-\theta)$
    - $\dfrac{1}{z} = \dfrac{1}{r}\ cis\ (-\theta)$
    - $iz = r\ cis\ (\theta + \dfrac{\pi}{2})$
- The nth roots of $z = r$ cis θ are given by
  $z_n = r^{1/n}\ cis\left(\dfrac{\theta+2k\pi}{n}\right) \quad k = 0, 1, 2, 3, \ldots$
- $\cos n\theta = \dfrac{1}{2}\left(z^n + \dfrac{1}{z^n}\right)$ • $\sin n\theta = \dfrac{1}{2i}\left(z^n - \dfrac{1}{z^n}\right)$
- If $z = a + bi$ is a root of a polynomial with all real coefficients, then $z = a - bi$ is also a root.
- $|z - a| = k$ represents a circle of radius k with centre at a.
- $|z - a| = |z - b|$ represents the perpendicular bisector of the line segment joining a and b.
- $\arg(z) = k$ represents a half-line inclined at an angle of k with the real axis.

Factor & Remainder Theorems

- If $f(k) = 0$, then $(x - k)$ is a linear factor of the polynomial $f(x)$. Conversely if $(x - k)$ is a factor of the polynomial $f(x)$ then $f(k) = 0$.
- When $f(x)$ is divided by $(ax - b)$,
  the remainder $R \equiv f\left(\dfrac{b}{a}\right)$.

Functions

- A function f is an *onto function* if its range is identical to its codomain.
- f is a one-to-one function
  if $f(a) = f(b)$ then $a = b$.
- Graphically, f is a one-to-one function if the graph of f passes the horizontal line test.
- f is a many-to-one function if $\exists$ a, b such that $a \neq b$ and $f(a) = f(b)$.
- The inverse function f^{-1} exists only if f is a *one-to-one and onto* function.
- If the codomain is not specified, then f^{-1} exists only if f is a *one-to-one* function.
    - If $fg(x) = gf(x) = x$, then $g = f^{-1}$.
    - The domain of f^{-1} = the range for f
      The range of f^{-1} = the natural (or restricted) domain of f.
- Graphically, f^{-1} exists only if the graph of f passes the horizontal line test.

Sketching Techniques

$y = f(x)$	$y = f^{-1}(x)$
Root at $x = a$	Vertical intercept at $y = a$
Vertical intercept at $y = a$	Root at $x = a$
Horizontal asymptote: $y = a$	Vertical Asymptote: $x = a$
Vertical Asymptote: $x = a$	Horizontal asymptote: $y = a$

$y = f(x)$	$y = \dfrac{1}{f(x)}$
Root at $x = a$	Vertical Asymptote at $x = a$
Vertical intercept $(0, a)$ where $a \neq 0$	Vertical intercept $(0, \dfrac{1}{a})$, $a \neq 0$
Vertical Asymptote: $x = b$	Root at $x = b$
Horizontal asymptote: $y = a$, $a \neq 0$	Horizontal asymptote: $y = \dfrac{1}{a}$
Local Minimum at (h, k) where $k \neq 0$	Local Maximum at $(h, \dfrac{1}{k})$ where $k \neq 0$
Local Maximum at (h, k) where $k \neq 0$	Local Minimum at $(h, \dfrac{1}{k})$ where $k \neq 0$

- If $y = |f(x)|$,
  $$y = \begin{cases} -f(x) & \text{over the interval for which } f(x) < 0 \\ f(x) & \text{over the interval for which } f(x) \geq 0 \end{cases}.$$
- The graph of $y = |f(x)|$ may be obtained from the graph of $y = f(x)$ by *reflecting* about the x-axis any part of $y = f(x)$ that is *below* the x-axis.
- If $y = f(|x|)$, $y = \begin{cases} f(-x) & x < 0 \\ f(x) & x \geq 0 \end{cases}$.

- The graph of $y = f(|x|)$ may be obtained from the graph of $y = f(x)$ as follows:
  - Parts of $y = f(x)$ to the right of the y-axis remain unchanged.
  - Parts of $y = f(x)$ to the left of the y-axis are completely removed and replaced by the reflection of the parts of $y = f(x)$ to the right of the y-axis.
- For the proper rational function $y = \dfrac{P(x)}{Q(x)}$:
  - the vertical asymptotes (poles) correspond to the roots of $Q(x) = 0$.
  - the horizontal asymptote corresponds to $\lim\limits_{x \to +\infty} \dfrac{P(x)}{Q(x)}$ and/or $\lim\limits_{x \to -\infty} \dfrac{P(x)}{Q(x)}$.
  - if $P(x)$ and $Q(x)$ share a common factor $(x + a)$, then the curve has a "hole" or a discontinuity (singularity) at $x = -a$.
- For $f(x) = \dfrac{P(x)}{Q(x)} + ax + b$, $\dfrac{P(x)}{Q(x)}$ is proper .
  - . The vertical asymptotes (poles) correspond to the roots of $Q(x) = 0$.
  - This function may or may not have any horizontal asymptote.
  - This function has an oblique asymptote with equation $y = ax + b$.

Vectors

Let $\boldsymbol{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and $\boldsymbol{v} = p\,\mathbf{i} + q\,\mathbf{j} + r\,\mathbf{k}$

- Magnitude of $\boldsymbol{u}$ $\quad |\boldsymbol{u}| = \sqrt{}(a^2 + b^2 + c^2)$
- Unit vector in the direction of $\boldsymbol{u}$ is $\hat{\boldsymbol{u}} = \dfrac{1}{|\boldsymbol{u}|}\,\boldsymbol{u}$
- $\boldsymbol{u} \,.\, \boldsymbol{v} = ap + bq + cr$
- $\boldsymbol{u} \times \boldsymbol{v} = < \begin{vmatrix} b & q \\ c & r \end{vmatrix}, -\begin{vmatrix} a & p \\ c & r \end{vmatrix}, \begin{vmatrix} a & p \\ b & q \end{vmatrix} >$

  $= < br - cq, -(ar - pc), aq - bp >$
  - $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$
  - $\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$.
- If θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$:
  - $\cos\theta = \dfrac{\boldsymbol{u} \,.\, \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$
  - $\sin\theta = \dfrac{|\boldsymbol{u} \times \boldsymbol{v}|}{|\boldsymbol{u}||\boldsymbol{v}|}$
- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel $\Rightarrow \boldsymbol{u} = \lambda\,\boldsymbol{v}$ or $|\boldsymbol{u} \times \boldsymbol{v}| = 0$ where λ is a constant
- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular $\Leftrightarrow \boldsymbol{u} \bullet \boldsymbol{v} = 0$
- If $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w}$, then $\boldsymbol{u} \bullet \boldsymbol{w} = 0$ and $\boldsymbol{v} \bullet \boldsymbol{w} = 0$.

Projections

- The scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$

  $= |\boldsymbol{u}|\cos\theta = \boldsymbol{u} \bullet \hat{\boldsymbol{v}}$.
- The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$,

  $\text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} = (|\boldsymbol{u}|\cos\theta)\,\hat{\boldsymbol{v}} = (\boldsymbol{u} \bullet \hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}$.

- Area of the parallelogram is $|\boldsymbol{u} \times \boldsymbol{v}|$.

  The area of a triangle is $\dfrac{1}{2}\,|\boldsymbol{u} \times \boldsymbol{v}|$.
- Volume of parallelepiped = $|(\boldsymbol{u} \times \boldsymbol{v}) \bullet \boldsymbol{w}|$.

Lines

- For a line passing through the point with position vector $\boldsymbol{a}$ and parallel to vector $\boldsymbol{d}$:
  - Vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda\,\boldsymbol{d}$
  - Cartesian equation $\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$

Planes

- Vector equation of a plane passing through $\boldsymbol{a}$ and perpendicular to $\boldsymbol{n}$ is $\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{a} \bullet \boldsymbol{n}$.
- Vector equation of a plane passing through the point with position vector $\boldsymbol{a}$ and containing non-parallel vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ is $\boldsymbol{r} = \boldsymbol{a} + \alpha\boldsymbol{b} + \beta\boldsymbol{c}$.

Sphere centre a and radius k:

- Vector equation $|\boldsymbol{r} - \boldsymbol{a}| = k$
- Cartesian equation

  $(x - a_1)^2 + (y - a_2)^2 + (z - a_3)^2 = k^2$

Angles

- The angle between line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$ and plane $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$ is $\sin^{-1}(\hat{\boldsymbol{d}} \bullet \hat{\boldsymbol{n}})$.
- The angle between the planes $\boldsymbol{r} \bullet \boldsymbol{m} = \rho_1$ and $\boldsymbol{r} \bullet \boldsymbol{n} = \rho_2$ is $\cos^{-1}(\hat{\boldsymbol{m}} \bullet \hat{\boldsymbol{n}})$.

Shortest Distance

- The shortest distance between the point $\boldsymbol{b}$ and line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$ is $|(\boldsymbol{b} - \boldsymbol{a}) \times \hat{\boldsymbol{d}}|$.
- The shortest distance between two non-intersecting lines $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}_1$ and $\boldsymbol{r} = \boldsymbol{b} + \lambda\,\boldsymbol{d}_2$ is $|(\boldsymbol{b} - \boldsymbol{a}) \bullet \hat{\boldsymbol{n}}|$ where $\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2$.
- The shortest distance between the plane $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$ and the origin is $\left| \dfrac{\rho}{|\hat{\boldsymbol{n}}|} \right|$.
- The shortest distance between $< p, q, r >$ and the plane $\boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{cz} = \boldsymbol{d}$ is $\left| \dfrac{ap + bq + r - d}{\sqrt{a^2 + b^2 + c^2}} \right|$.
- The shortest distance between the point $\boldsymbol{a}$ and the plane $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$ is $|(\boldsymbol{b} - \boldsymbol{a}) \bullet \hat{\boldsymbol{n}}|$.

  where $\boldsymbol{b}$ is any point on the plane.
- The shortest distance between two non-intersecting planes $\boldsymbol{r} \bullet \boldsymbol{m} = \rho_1$ and $\boldsymbol{r} \bullet \boldsymbol{n} = \rho_2$ is $|(\boldsymbol{b} - \boldsymbol{a}) \bullet \hat{\boldsymbol{n}}|$ where $\boldsymbol{a}$ is any point on plane 1 and $\boldsymbol{b}$ is any point on plane 2.

Vector Applications

- To travel from point $\boldsymbol{a}$ to $\boldsymbol{b}$, with wind $\boldsymbol{w}$, required velocity $\boldsymbol{v}$ is given by:
  $(\boldsymbol{b} - \boldsymbol{a})$ = journey time × $(\boldsymbol{v} + \boldsymbol{w})$.
- At closest approach ${}_{\mathrm{A}}\boldsymbol{r}_{\mathrm{B}}(t) \bullet {}_{\mathrm{A}}\boldsymbol{v}_{\mathrm{B}} = 0$.
- For collision ${}_{\mathrm{A}}\boldsymbol{r}_{\mathrm{B}}(0) = t\ {}_{\mathrm{B}}\boldsymbol{v}_{\mathrm{A}}$.

Systems of equations

- For the augmented matrix $\left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ j & k & m & n \end{array}\right)$

$$\begin{array}{r} \\ \textit{combine } R1 \,\&\, R2 \textit{ to get this zero} \rightarrow \\ \textit{combine } R1 \,\&\, R3 \textit{ to get this zero} \rightarrow \end{array} \left(\begin{array}{ccc|c} a & b & c & d \\ 0 & f & g & h \\ 0 & 0 & m & n \end{array}\right).$$

combine new R2 & new R3 to get this zero ↑

- For $\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & f & g & h \\ 0 & 0 & m & n \end{array}\right)$.
  - $m = 0$ and $n \neq 0 \Rightarrow$ system has no solutions
  - $m = n = 0 \Rightarrow$ system has infinite solutions
  - $m \neq 0 \Rightarrow$ system has unique solutions

Limits

- $\lim\limits_{x \to 0} \cos(x) = 1$, $\lim\limits_{x \to 0} \dfrac{1 - \cos(x)}{x} = 0$
- $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$, $\lim\limits_{x \to 0} \dfrac{\tan(x)}{x} = 1$

Differentiation

- $f'(x) = \dfrac{d}{dx} f(x) = \lim\limits_{h \to 0} \left[\dfrac{f(x+h) - f(x)}{h} \right]$
- $y = [f(x)]^n \Rightarrow y' = n \times [f(x)]^{n-1} \times f'(x)$
- $y = u\,v \Rightarrow y' = u'\,v + u\,v'$
- $y = \dfrac{u}{v} \Rightarrow y' = \dfrac{v\,u' - u\,v'}{v^2}$
- $y = a\,x^n \Rightarrow y' = n \times \mathrm{a}\,x^{n-1}$
- $y = e^{f(x)} \Rightarrow y' = f'(x) \times e^{f(x)}$
- $y = \mathit{ln}\, f(x) \Rightarrow y' = \dfrac{f'(x)}{f(x)}$
- $y = \sin x \Rightarrow y' = \cos x$
- $y = \cos x \Rightarrow y' = -\sin x$
- $y = \tan x \Rightarrow y' = \sec^2 x$
- $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ • $\dfrac{d}{dx} f(y) = \dfrac{d}{dy} f(y) \times \dfrac{dy}{dx}$.

Small Increments

- $\dfrac{\delta y}{\delta x} \approx \left. \dfrac{dy}{dx} \right|_{x = x_0, y = f(x_0)}$

Related Rates

- If $y = f(x)$ and $x = g(t)$, then $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$.

Integration

- $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1} + \mathrm{C} \quad [n \neq -1]$
- $\displaystyle\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1) \times a} + \mathrm{C} \quad [n \neq -1]$
- $\displaystyle\int f'(x) \times \left[f(x) \right]^n \, dx = \frac{\left[f(x) \right]^{n+1}}{n+1} + \mathrm{C}$
  where $n \neq -1$.
- $\displaystyle\int e^{mx} \, dx = \frac{e^{mx}}{m} + \mathrm{C}$
- $\displaystyle\int f'(x)\, e^{f(x)} \, dx = e^{f(x)} + \mathrm{C}$
- $\displaystyle\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + \mathrm{C}$
- $\displaystyle\int \cos\,(ax+b)\, dx = \frac{\sin\,(ax+b)}{a} + \mathrm{C}$
- $\displaystyle\int \sin\,(ax+b)\, dx = -\frac{\cos\,(ax+b)}{a} + \mathrm{C}$
- $\displaystyle\int \sec^2\,(ax+b)\, dx = \frac{\tan\,(ax+b)}{a} + \mathrm{C}$
- $\displaystyle\int f(x)\, dx = \int h(u) \cdot \frac{dx}{du}\, du$
- $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a) \qquad$ where $F'(x) = f(x)$

Area between curves

$$\mathrm{A} = \int_a^b \left| f(x) - g(x) \right| dx$$

Volume of revolution

- $\mathrm{V} = \pi \displaystyle\int_a^b [f(x)]^2 \, dx$ or $\mathrm{V} = \pi \displaystyle\int_a^b [f(y)]^2 \, dy$

Numerical Integration

- Box Method

Left-box method	$w \times \sum\limits_{i=0}^{n-1} f(x_i)$
Right-box method	$w \times \sum\limits_{i=0}^{n-1} f(x_{i+1})$
Middle-box/Mid-point Method	$w \times \sum\limits_{i=0}^{n-1} f\left(\dfrac{x_i + x_{i+1}}{2} \right)$

Width of strip, $w = \dfrac{b-a}{n}$, $x_i = a + iw$

- Trapezium Rule

  Area = $\frac{w}{2}\times\sum_{i=0}^{n-1}[\,f(x_i)+f(x_{i+1})\,]$

  $w = \frac{b-a}{n}$ and $x_i = a + iw$

Differential Equations

- $f\,(y)\frac{dy}{dx} = f\,(x)$, use separation of variables.
- $\frac{dy}{dt} = ky \quad \Leftrightarrow \quad y = y_0\, e^{kt}$,

Simple Harmonic Motion

- $\frac{d^2x}{dt^2} = -\omega^2 x$ is given by

  $x = A \sin(\omega t + \alpha)$ or $x = A \cos(\omega t + \alpha)$.

  Amplitude $|A|$ and period $T = \frac{2\pi}{\omega}$.
- $v^2 = \omega^2 (A^2 - x^2)$

Logistic Model

- For $\frac{dy}{dt} = ky\left(1 - \frac{y}{b}\right) \equiv \frac{k}{b}y(b-y)$

  $\Rightarrow \quad y = \frac{b}{1+Ae^{-kt}}$:
  - the initial value for y, $y(0) = \frac{b}{1+A}$
  - the constant k is the growth constant
  - the constant b is the limiting value for y.

Rectilinear Motion

- Displacement at time t, $x = \int v \; dt$

  Velocity $v = \frac{dx}{dt} = \int a \; dt$

  Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$
- Distance travelled for $a \le t \le b = \int_a^b |\,v\,| \; dt$.

Vector Calculus

- The velocity vector is $\boldsymbol{v}(t) = \boldsymbol{r}'(t)$.
- The speed of P at time t is given by $|\,\boldsymbol{v}(t)\,|$.
- The acceleration vector is $\boldsymbol{a}(t) = \boldsymbol{r}''(t)$.
- $\boldsymbol{r}(t) = \int \boldsymbol{v}(t) \; dt$ and $\boldsymbol{v}(t) = \int \boldsymbol{a}(t) \; dt$
- Direction of particle is given by $\tan\theta = \frac{v_y}{v_x}$.
- $\int_a^b \boldsymbol{v}(t) \; dt$ represents the *change in displacement*
- $\left|\int_a^b \boldsymbol{v}(t) \; dt\right|$ = the *magnitude* of change of displacement.
- $\int_a^b |\,\boldsymbol{v}(t)\,| \; dt$ = *distance travelled along the path.*

Uniform Circular Motion

- $\boldsymbol{a} = -\omega^2\boldsymbol{r}$, $\boldsymbol{r} = <\pm a \sin \omega t + p, \pm a \cos \omega t + q>$.
  - $\boldsymbol{a} \bullet \boldsymbol{v} = \boldsymbol{r} \bullet \boldsymbol{v} = 0 \;\; \forall\, t$.
  - Period of motion = $\frac{2\pi}{\omega}$.

Elliptical motion $a \neq b$

- $\boldsymbol{a} = -\omega^2\boldsymbol{r}$, $\boldsymbol{r} = <\pm a \sin \omega t + p, \pm b \cos \omega t + q>$
  - $\boldsymbol{a} \bullet \boldsymbol{v} = \boldsymbol{r} \bullet \boldsymbol{v} = 0$ for $t = \frac{n\pi}{2\omega}$ where $n \in \mathbb{Z}^+$.
  - Period of motion = $\frac{2\pi}{\omega}$.

Projectile Motion

- $\boldsymbol{v}(0) = <v \cos\theta, v \sin\theta>$.

  $x = v \cos\theta\; t \;\; y = -\frac{gt^2}{2} + v \sin\theta\; t$
- The Cartesian equation of particle projected with a speed of v at an angle θ to the x-axis is $y = = \left(-\frac{g \sec^2\theta}{2v^2}\right)x^2 + \tan\theta\, x$.

Sampling Distribution

- Sampling distribution has mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$.

  As $n \to \infty$, sampling distribution becomes increasingly normally distributed.

Confidence Intervals

- A 90% confidence interval for μ is:

  $$\bar{x} - 1.645 \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.645 \times \frac{\sigma}{\sqrt{n}}$$
- A 95% confidence interval for μ is:

  $$\bar{x} - 1.96 \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \times \frac{\sigma}{\sqrt{n}}$$
- A 99% confidence interval for μ is:

  $$\bar{x} - 2.576 \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.576 \times \frac{\sigma}{\sqrt{n}}$$
- A 100c % confidence interval for μ is:

  $$\bar{x} - z_c \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_c \times \frac{\sigma}{\sqrt{n}}$$

  where $\mathrm{P}(-z_c < \mathrm{Z} < z_c) = c$.

01 Complex Numbers I

Calculator Free

1. [5 marks: 1, 2, 2] [TISC]

Given the complex numbers $z_1 = 2 - i$, $z_2 = i$ and $z_3 = 2a\,i$, find:

(i) $z_1\,\overline{z_2}$

(ii) $|z_1 + z_3|$

(iii) $arg\,(\frac{z_3}{2a\,z_2})$.

---

2. [5 marks: 1, 2, 2] [TISC]

Let the complex numbers $z_1 = a + 2i$, $z_2 = 3$ and $z_3 = \sqrt{3} - i$.

(a) Express the following in the form $x + yi$.

(i) ${z_1}^2$

(ii) $\frac{z_1}{z_3}$

(b) Determine in **exact form** $arg\,(i\,z_2) + arg\,(z_3)$.

Calculator Free

3. [7 marks: 2, 2, 3] [TISC]

Let the complex numbers $z_1 = a - 2i$, $z_2 = 1 + i$ and $z_3 = -4i$, where a is a *real number*. Determine all possible values of a if:

(a) $z_1 \times z_2 = z_3$

(b) $\dfrac{z_1}{z_3} = \dfrac{1}{2} z_2$

(c) $arg\,(z_1) + arg\,(z_2) = -\dfrac{\pi}{4}$.

---

4. [9 marks: 2, 2, 3, 2] [TISC]

Let the complex numbers $z_1 = 2 + a\,i$ and $z_2 = 1 - 2\,i$, where a is a *real number*.

Determine all possible values of a if:

(a) $z_1 \times \overline{z_1} = 2\,a^2$

(b) $z_1 = i\,\overline{z_1}$

Calculator Free

4. (c) $\boldsymbol{Re}\,(z_1^{\,2}) = \boldsymbol{Re}\,(z_2^{\,2})$

(d) $\boldsymbol{arg}\,(z_1) = \boldsymbol{arg}\,(\overline{z_1})$

---

5. [11 marks: 2, 3, 3, 3] [TISC]

Let the complex numbers $z_1 = a + 5\,i$, $z_2 = 3 - 4\,i$ and $z_3 = 1 + i\,\sqrt{3}$ where *a* is a *real number*.

(a) Find *a* if $|z_1| = |z_2|$

(b) Find the exact value of *a* if $\tan[\boldsymbol{arg}\,(\overline{z_1})] = \tan[\boldsymbol{arg}\,(z_3)]$.

(c) Explain clearly why there is no solution for *a* if $|\, z_1 - z_2 \,| = |z_3|$.

(d) Find the value of *a* if $\mathbf{Im}\left(\dfrac{z_1}{z_2}\right) = 0$

Calculator Free

6. [9 marks: 3, 3, 3]

The complex number z has a modulus of 2 and an argument of $\frac{2\pi}{3}$

(a) State the complex number z^4 in Cartesian form.

(b) State the complex number $\frac{z}{i}$ in *cis* form.

(c) Given that $w \times z = 2i$, determine the complex number w.
Give your answer in polar form.

7. [7 marks: 2, 2, 3]

Let $u = a \text{ } cis \text{ } \alpha$ and $v = a \text{ } cis \text{ } \beta$ where $a > 0$ and α and β are acute.
Express each of the following in *cis* form.

(a) $\frac{1}{u}$

Calculator Free

7. (b) $(\bar{v}\,u\,v)^8$

(c) $u + \bar{u}$

---

8. [11 marks: 3, 4, 4] [TISC]

(a) Given that $4a - 4a\,i = r\,cis\,\theta$, find r and θ in terms of $a > 0$ where appropriate.

(b) Given that $r\,cis\left(\frac{-5\pi}{6}\right) = x + 5a\,i$, find r and x in terms of $a < 0$.

(c) Simplify $\left[\sqrt{3}\,cis\left(\frac{5\pi}{6}\right)\right]^3 \times \sqrt{3\,cis\left(\frac{\pi}{4}\right)}$

Calculator Free

9. [7 marks: 3, 4]

Let $x = cis\ \frac{\pi}{3}$, $y = \left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$ and $z = -1 - 3i$.

(a) Find $\frac{xy}{z-2}$ giving your answer in polar form.

(b) Find $\sqrt{3}\,x + \sqrt{2}\,y^2 + z$ giving your answer in *cis* form.

---

10. [7 marks: 2, 5]

(a) Express $2\sqrt{3}\ cis\left(\frac{-\pi}{3}\right)$ in Cartesian (rectangular) form.

Calculator Free

10. (b) Given $u + v = \sqrt{3} + i$ and $u - v = 2\sqrt{3}\, cis\left(\frac{-\pi}{3}\right)$.

Determine the complex numbers *u* and *v* giving your answer in polar form.

---

11. [6 marks: 1, 2, 3]

Let $a = -1 + \sqrt{3}\, i$ and $b = -1 - i$.

(b) Find *ab* in exact Cartesian form.

(c) Find *ab* in exact *cis* form.

(c) Use your answers in (a) and (b) to find $\sin\left(\frac{\pi}{12}\right)$ in exact form.

02 Complex Numbers II

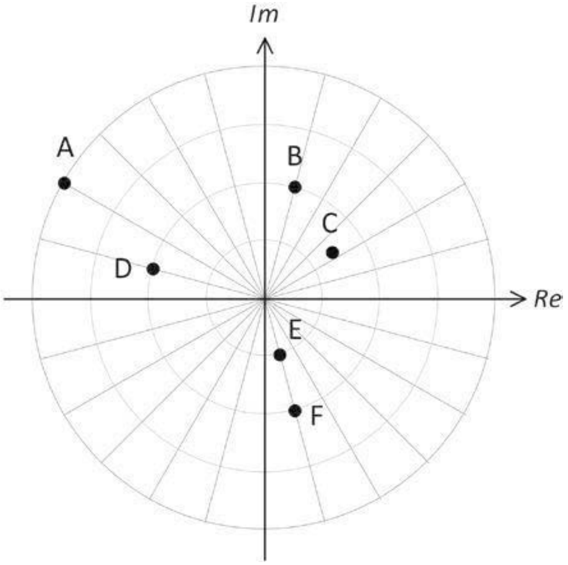
Calculator Free

1. [6 marks] [TISC]

The diagram below shows an Argand diagram marked with the points A, B, C, D, E and F. These points correspond to the complex numbers (not in matching order): z $-iz$ iz $2z$ $\bar{z}$ z^2 $\dfrac{1}{z}$ $\dfrac{2}{z}$ $\sqrt{z}$

Complete the table below matching these points with one of the complex numbers listed.

Point	Complex Number
A	
B	
C	
D	
E	
F	



2. [4 marks: 2, 2]

Let $z = r\ cis\ \theta$. Describe what happens to the magnitude and direction of the vector representation of z:

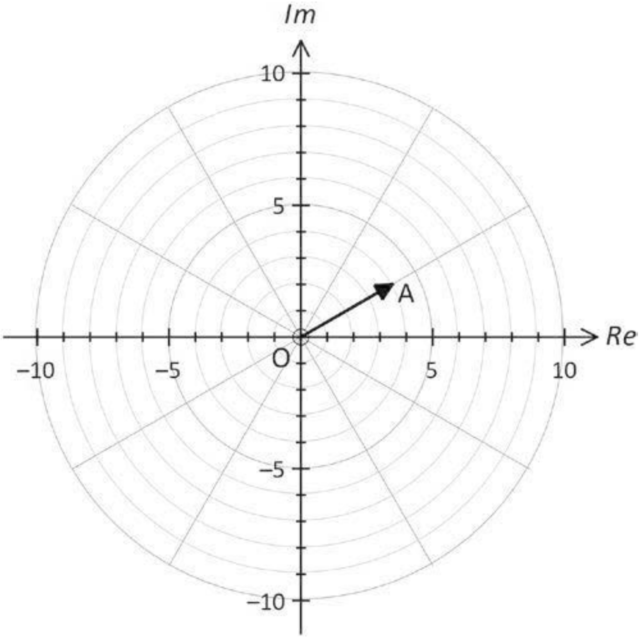
(a) when z is multiplied with i.

(b) when z is multiplied with $2\,cis\left(\dfrac{\pi}{7}\right)$

Calculator Free

3. [7 marks: 1, 2, 2, 2]

The vector **OA** in the Argand diagram below represents the complex number a.
Let the complex number $b = 2\,cis\left(\dfrac{2\pi}{3}\right)$.



(a) State the modulus and argument of a.

(b) On the Argand diagram above, plot the point P which represents the product of the complex numbers a and b.

(c) Use linear transformations to describe what happens to **OA** when a is multiplied by b.

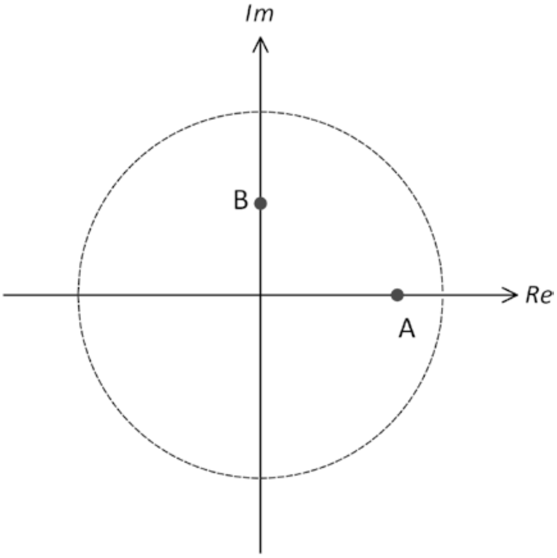
(d) On the Argand diagram above, plot the point Q which represents the sum of the complex numbers a and b.

Calculator Free

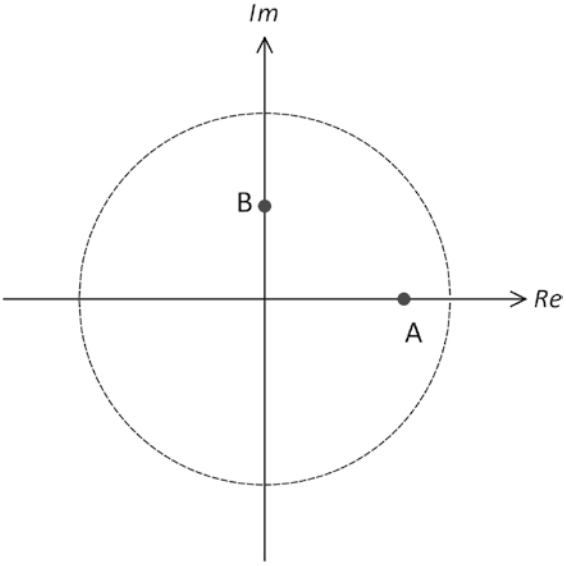
4. [8 marks: 3, 5]

In each of the Argand diagrams below, the point A represents the complex number a and the point B represents the complex number b.

(a) Mark and label in the diagram below, the points representing the complex numbers $a + b$ and $2b - a$.



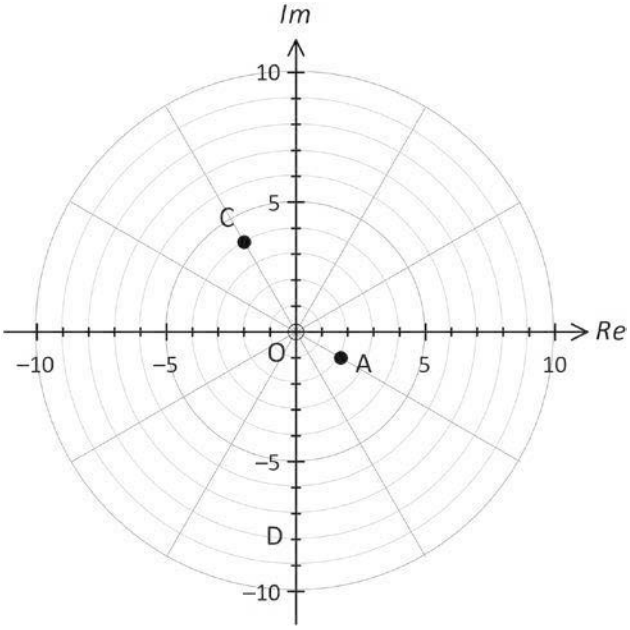
(b) The centre of the circle drawn is at the origin and the radius of the circle is 1 unit. Mark and label in the diagram below, the points representing the complex numbers $a \times b$ and $\dfrac{a}{b}$.



Calculator Free

5. [8 marks: 2, 2, 2, 2]

The points A and C on the diagram below represents the complex numbers a and c. The complex number b is such that $ab = c$.



(a) On the diagram above, plot the point representing $\overline{a-c}$.

(b) On the diagram above, plot the point B that represents the complex number b.

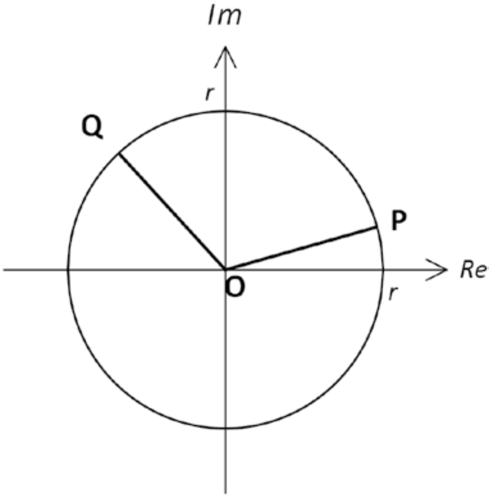
(c) On the diagram above, plot the point D which represents the complex number ab^2.

(d) Use the language of linear transformations to describe what happens to the complex number a when it is multiplied by b^4.

Calculator Free

6. [7 marks: 1, 3, 3]

The points P, Q and R represent the complex numbers w, z and $w + z$ respectively. The points P, Q and R lie on a circle with centre O and of radius r. The Argand diagram below shows the location of the points P and Q.



(a) On the diagram above, mark the point R representing the complex number $w + z$.

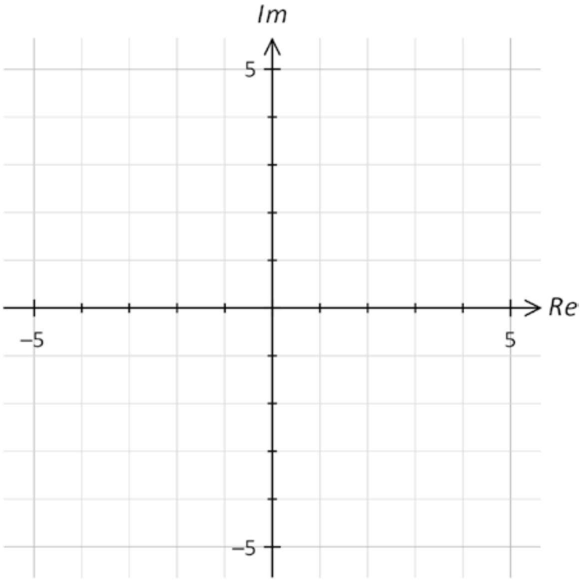
(b) Prove that $\angle POQ = \dfrac{2\pi}{3}$.

(c) If $w = r\ cis\ \theta$, find z in terms of r and θ and hence prove that $z^3 = w^3$.

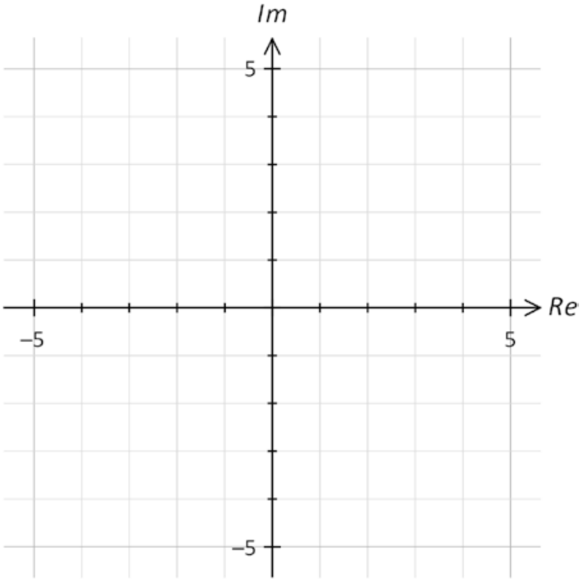
Calculator Free

7. [6 marks: 3, 3]

(a) Sketch the locus of the complex number z satisfying the inequality $|z - 3 + i| \geq |z + 5 - i|$



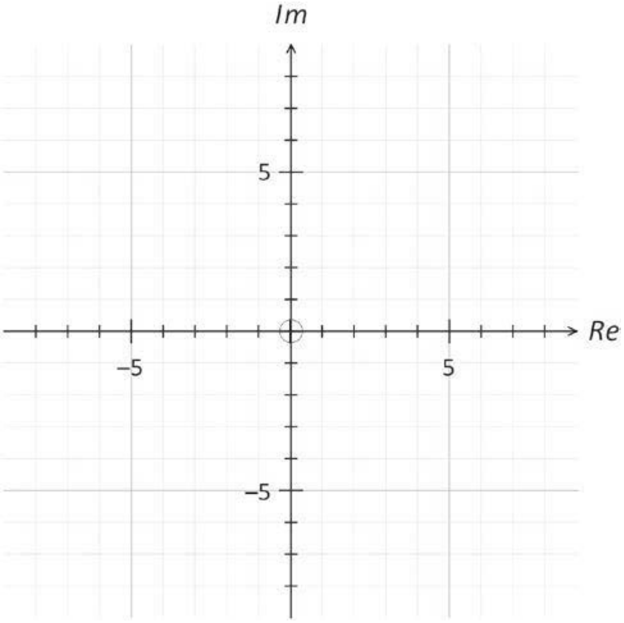
(b) Sketch the locus of the complex number z satisfying the inequality $\frac{-\pi}{4} \leq \arg(z) \leq \tan^{-1}(3)$



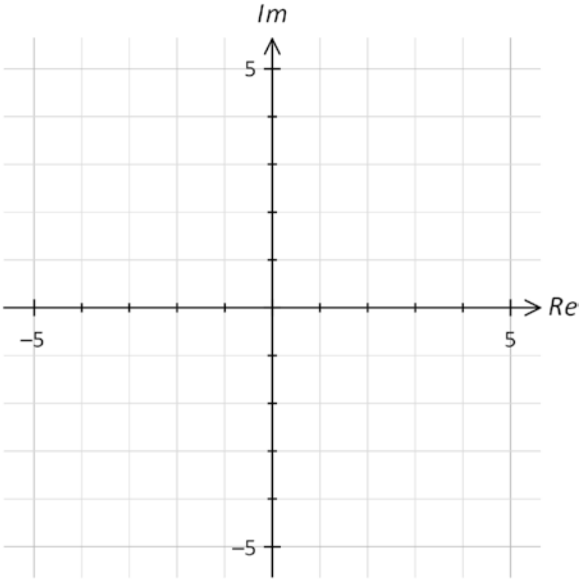
Calculator Free

8. [4 marks: 2, 2] [TISC]

(a) Sketch the region in the Argand Plane defined by
$\{ z : | z | = arg\,(z)$ where $0 \le arg\,(z) \le 2\pi\}$. 0



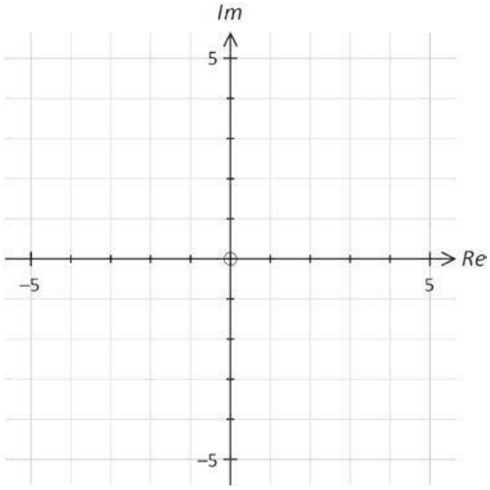
(b) Sketch the region in the Argand Plane defined by
$\{z : z = \cos\theta + i \sin\theta$ where $-\pi < \theta \le \pi\}$.



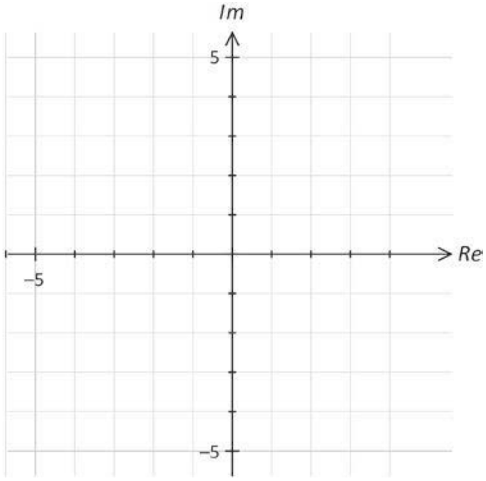
Calculator Free

9. [9 marks: 3, 3, 3]

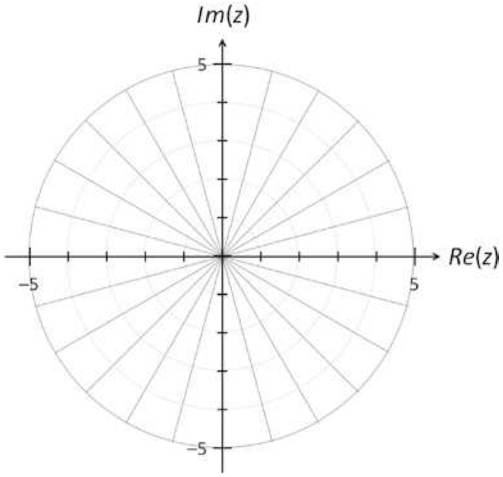
(a) On the Argand diagram provided sketch the locus of the complex number z
where $\tan^{-1}(-3) \le arg(z) \le \tan^{-1} 2$.



(b) On the Argand diagram provided sketch the locus of the complex number z
where $|z + 2 + i| \ge |z - 2 - 3i|$ 0
and $1 \le |z - 1| \le 2$.



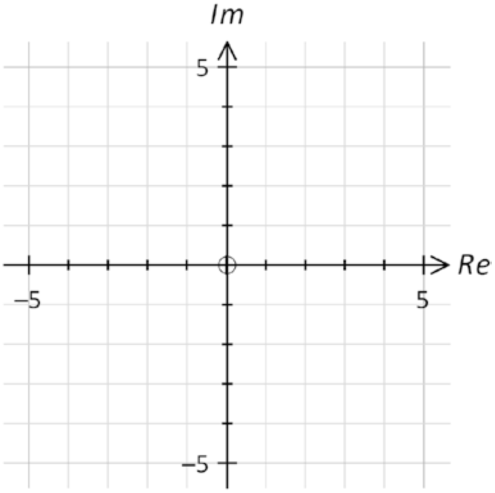
(c) Let $p = 5\ cis\ \dfrac{\pi}{12}$ and $q = 4\ cis\ \dfrac{-3\pi}{4}$.
On the axes provided, plot the points p and q and sketch the locus of the complex number z satisfying $|z - p| = |z - q|$.



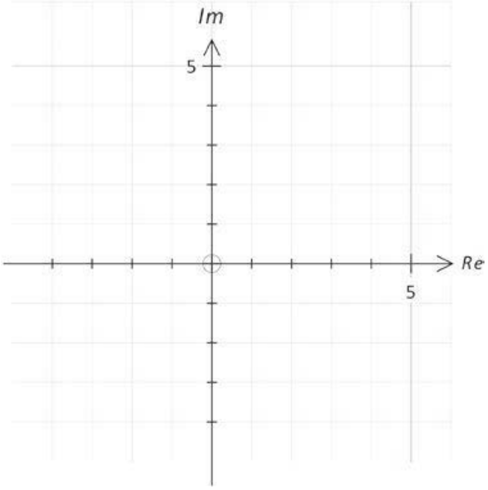
Calculator Free

10. [9 marks: 2, 3, 4] [TISC]

(a) In the Argand diagram provided, sketch the region defined by $\{ z : arg(\bar{z}) = \frac{-\pi}{4}\}$.



(b) Sketch on the diagram below the locus of the point z defined by: $\{z : |z + 2 + 2i| \leq 2 \cup 0 \leq arg\ (z) \leq \frac{\pi}{2}\}$.

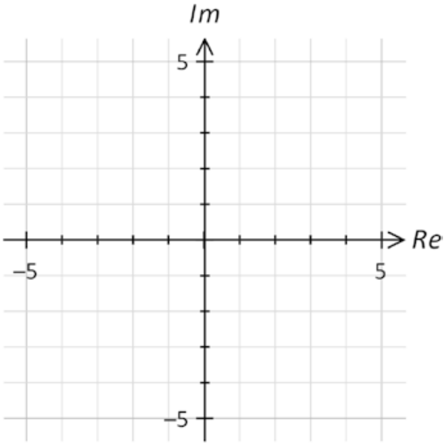


(c) Find, in its simplest form the Cartesian equation of the locus of the point z defined by $|z - 1 - i|$ = **Re** $(z + 3 + 4i)$.

Calculator Free

11. [12 marks: 2, 3, 7] [TISC]

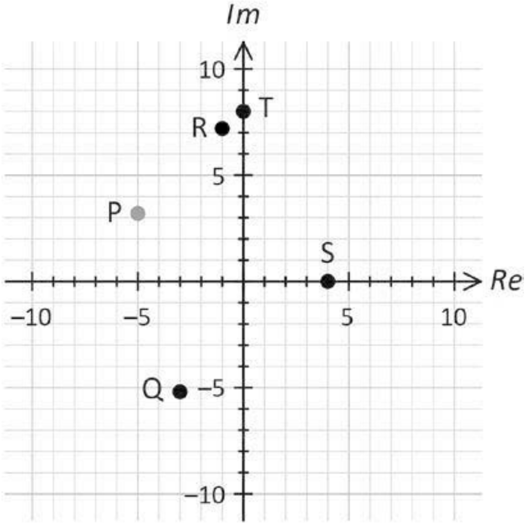
(a) Sketch on the diagram provided the . locus of the point z defined by: $\{z : |z - 2i| \geq 1\}$.



✓

(b) Find, in simplest form the Cartesian equation of the locus of the point z defined by $|z - 1| = |z - 1 + 2i|$.

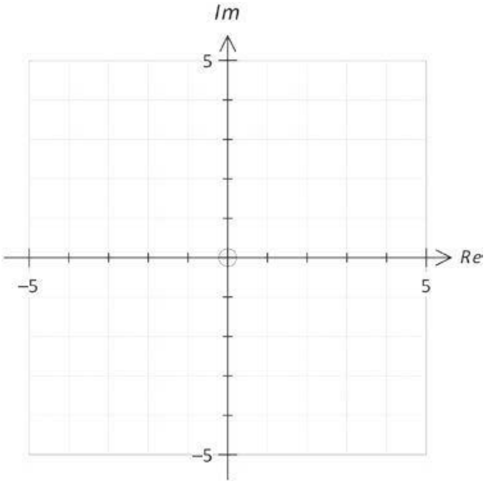
(c) Consider the complex numbers $u = 2 + 2i$ and $v = -3 + 3\sqrt{3}\,i$. The accompanying Argand diagram shows the points P, Q, R, S and T. Describe the complex numbers represented by each of the points Q, R, S and T using the complex numbers u and v and/or their conjugates. For example, the point P represents $v - u$.



Calculator Free

12. [8 marks: 2, 3, 3] [TISC]

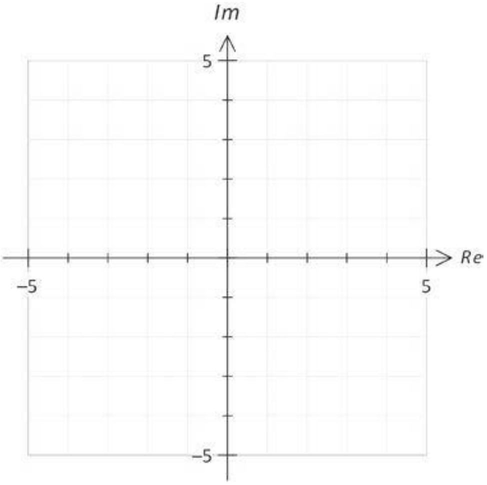
(a) Sketch the region in the Argand Plane defined by $\{z : \tan[arg(z)] = 1\}$.



(b) Consider the region in the Argand Plane defined by $\{z : z^2 = 2i\}$.
Let $z = x + iy$ where x and y are real numbers.

(i) Show that the Cartesian equation of this region is given by $x^4 = 1$.

(ii) Hence, show that this region consists of exactly *two* points. Mark these two points clearly on the axes provided.



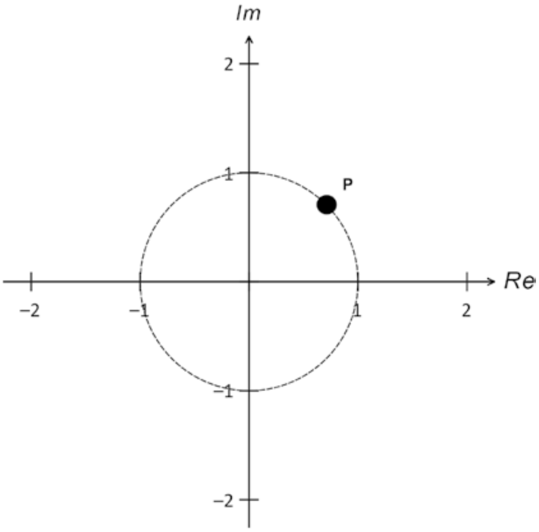
Calculator Free

13. [13 marks: 8, 5] [TISC]

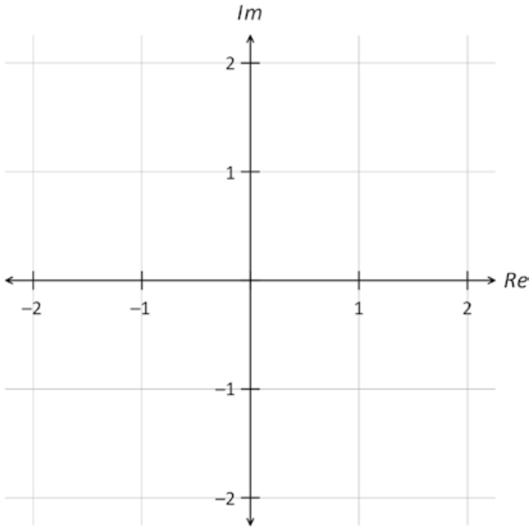
(a) The complex number z where $|z| = 1$, is represented by the point P as marked in the accompanying Argand diagram. Mark and label clearly on the diagram given the points representing the complex numbers:

(i) $-z$ (ii) $-i\,z$

(iii) $z + \overline{z}$ (iv) $z \times \overline{z}$



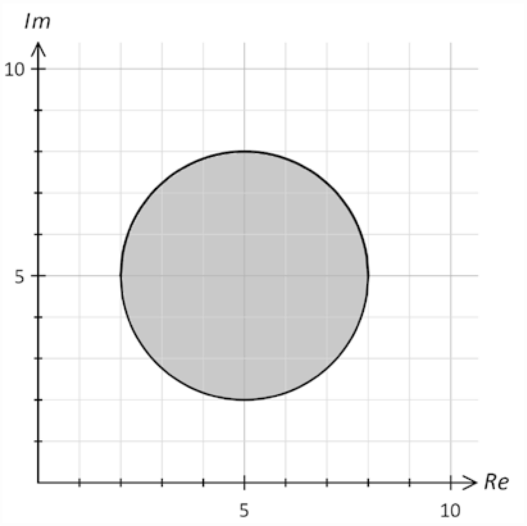
(b) The locus of the complex number z satisfies the equation $|z - 1| = |\overline{z}|$. Find the Cartesian equation of the locus and hence sketch the locus of z in the Argand diagram provided.



Calculator Free

14. [7 marks: 2, 2, 3]

The accompanying diagram shows the locus of the complex number z.



(a) Use complex number concepts to state the equation of the locus of z.

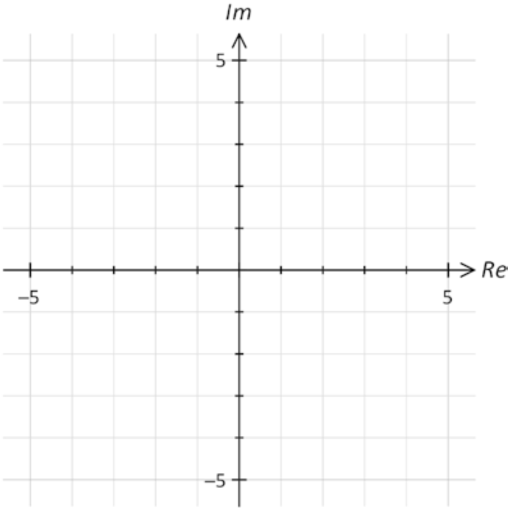
(b) Given that $a \leq |z| \leq b$, state the values for a and b.

(c) Given that $\arg(z) \geq \alpha$, determine the value of α.

Calculator Free

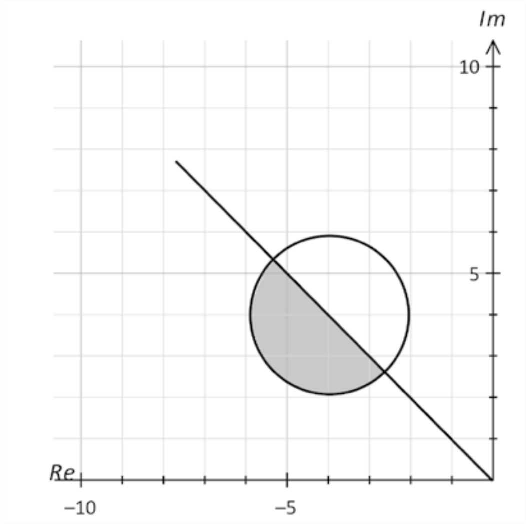
15. [8 marks: 3, 2, 3]

(a) On the Argand diagram given sketch the locus of the complex number z where $|z + i| \geq |z - 2 - i|$ and $0 \leq arg(z) \leq \tan^{-1} 2$.



(b) The given diagram shows the locus of the complex number z where $arg(z) \geq \frac{3\pi}{4}$ and $|z + 4 - 4i| \leq 2$.

(i) Determine the minimum value of $|z|$.



(ii) Determine the maximum value of $arg(z)$.

Calculator Free

16. [7 marks: 3, 2, 2] [TISC]

The complex number z is defined by $z = \dfrac{a+4i}{i} + \dfrac{4}{1+i}$ where a is a real constant.

(a) Rewrite z in the form $x + y\,i$ where x and y are real.

(b) Find the value of a if z lies on the line $Im(z) = -\,Re(z)$.

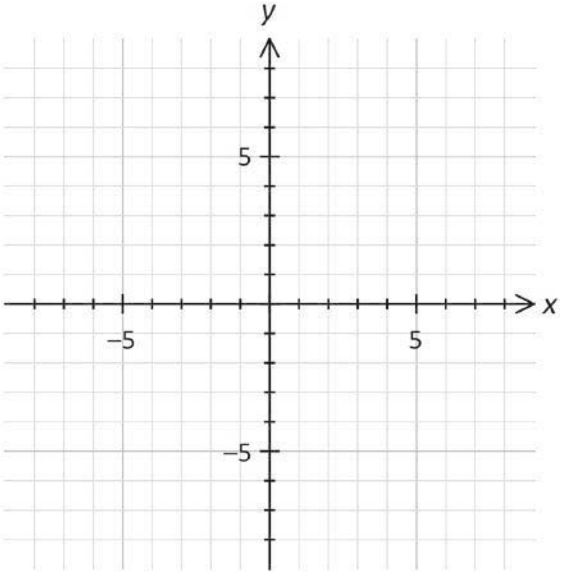
(c) Show that z cannot lie on the curve $arg(z) = \dfrac{3\pi}{4}$.

Calculator Assumed

17. [6 marks: 4, 2]

The complex number $z = x + iy$ has locus described by
$|z + 5 - 4i| + |z - 3 - 4i| = 8$ and $|z - 3 - 4i| \geq 2$.

(a) In the given diagram below draw the locus of z.



(b) Determine in its simplest form the Cartesian equation of this locus.

---

18. [7 marks: 3, 4]

(a) The locus of a complex number $z = x + iy$ is given by $|z| = |2z - 3i|$. Determine the Cartesian equation of this locus, giving your answer in its simplest form.

Calculator Assumed

18. (b) The complex number z has locus described by $|z + 2 - i| = \frac{1}{2}\,|z - 1 + 2i|$.

Determine in its simplest form the Cartesian equation of this locus.

---

19. [8 marks: 3, 5] [TISC]

Let $w = x + y\,i$.

(a) If $\left|\frac{w}{1-w}\right| = 1$, show that w lies on the line with equation $x = \frac{1}{2}$.

(b) If $\left|\frac{w}{1-w}\right| = 3$, show that w lies on a circle. Find the equation of this circle.

03 Complex Numbers III

Calculator Free

1. [7 marks: 1, 3, 3] [TISC]

Given $u = 4\ cis\left(\frac{\pi}{3}\right)$ and $v = 2\ cis\left(\frac{k\pi}{3}\right)$ where k is a real number.

(a) If $2 < k < 4$, find $\frac{u}{v}$ in $r\ cis\ \theta$ form where $-\pi < \theta \le \pi$.

(b) Find $u \times v$ in $r\ cis\ \theta$ form where $2 < k < 4$ and $-\pi < \theta \le \pi$.

(c) Find k given that v is one of the square roots of u.

Calculator Free

2. [9 marks: 3, 3, 3] [TISC]

(a) Simplify $\dfrac{a^2\left[\cos\left(\dfrac{5\pi}{6}\right) - i\sin\left(\dfrac{5\pi}{6}\right)\right]}{4a\left[\cos\left(\dfrac{11\pi}{12}\right) + i\sin\left(\dfrac{11\pi}{12}\right)\right]}$, giving your answer in exact *cis* form.

(b) Simplify $\left[cis\left(\dfrac{\frac{\pi}{3}+2k\pi}{5}\right)\right]^5$, where $k = 0, 1, 2, 3, 4, 5, \ldots$.

Give your answer in exact *cis* form.

(c) Solve exactly for θ where $-\pi < \theta \le \pi$ in

$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = 1$.

Calculator Free

3. [6 marks: 3, 3] [TISC]

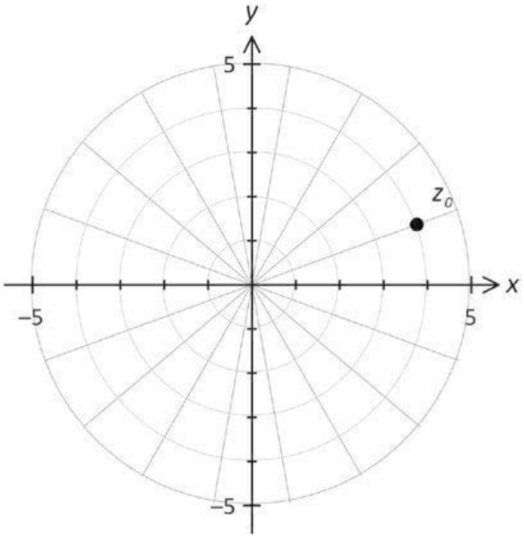
(a) Find n given that $\dfrac{1}{\cos 3\theta + i \sin 3\theta} = [cis\ \theta\ ]^n$

(b) Given that $\left|\dfrac{z-2}{z+2}\right| = 1$, where $z \neq 0$, show that z is completely imaginary.

---

4. [7 marks: 2, 3, 2]

Consider the equation $z^6 = k$ where $k \in \mathbb{C}$. One of the roots of this equation is z_0. The accompanying Argand diagram shows the position of z_0.



(a) On the Argand diagram given above, plot the remaining roots of this equation.

Calculator Free

4. (b) A polygon is drawn with plots of the roots as vertices.
Determine with reasons:
(i) the exact perimeter of this polygon.

(ii) the exact area of this polygon.

---

5. [9 marks: 4, 5] [TISC]

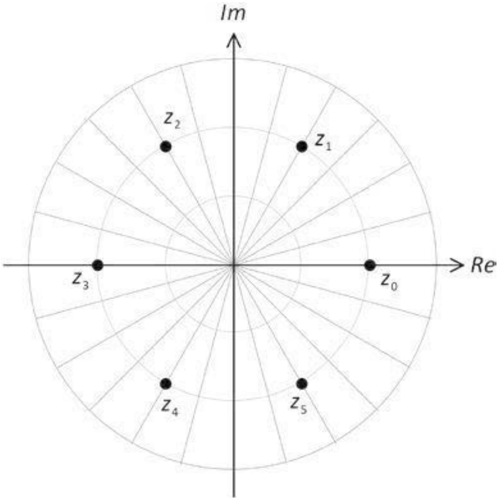
(a) Solve $z^4 = 1 + i$. Leave your answers in exact polar form.

(b) Consider the equation $z^n = a + bi$. When plotted on an Argand diagram, two immediate adjacent roots are $cis\left(\frac{\pi}{12}\right)$ and $cis\left(\frac{7\pi}{12}\right)$. Find the value(s) of n, and corresponding exact values of a and b. Justify your answer.

Calculator Free

6. [10 marks: 4, 3, 3]

The accompanying diagram shows the complex numbers z_0, z_1, z_2, z_3, z_4 and z_5 plotted as vertices of a regular hexagon centred at the pole (origin).



(a) Given that $z_0 = r\,cis\,0$, show that

$$z_1 = z_0\,cis\left(\frac{\pi}{3}\right)$$

(b) Hence, show the use of the relationship in (a) to find z_5 in terms of z_0.

(c) Determine the value of $z_0 + z_1 + z_2 + z_3 + z_4 + z_5$.

Calculator Free

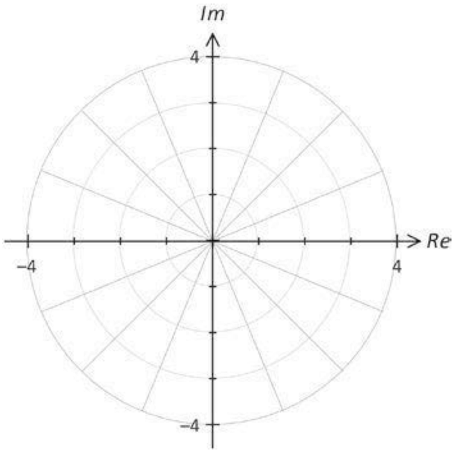
7. [12 marks: 4, 3, 2, 3]

Consider the equation $z^4 - 16i = 0$ for $z \in \mathbb{C}$.

(a) Solve the given equation, giving your answers in *cis* form.

(b) Show that the Cartesian roots of this equation can be written in the form $\pm u$ and $\pm v$.

(c) On the Argand diagram provided, plot the points corresponding to the roots of this equation.



(d) A polygon is formed by using the roots of the given equation as its vertices. Prove that this polygon is a square.

Calculator Free

8. [12 marks: 4, 3, 3, 2]

Consider the equation $iz^6 - 1 = 0$ for $z \in \mathbb{C}$.

(a) Solve the given equation, giving your answers in *cis* form.

(b) Calculate the product of all the roots of the equation.

(c) A polygon is formed by using the roots of the given equation as its vertices. Determine the perimeter of this polygon. Justify your answer

(d) Determine with reasons the perimeter of the polygon with vertices formed by the roots of the equation $i\,z^n - 1 = 0$ as $n \to \infty$.

Calculator Free

9. [10 marks: 3, 3, 4]

(a) Determine the imaginary part of the expansion of $(\cos x - i \sin x)^5$.
Express your answer in terms of the powers of $\sin x$.

(b) Show how your answer in (a) can be used to prove that
$\sin 5x = 16 \sin^5 x - 20 \sin^3 x + 5 \sin x$.

(c) Hence, solve the equation $32y^5 - 40y^3 + 10y - \sqrt{2} = 0$,
giving your answers in trigonometric form.

Calculator Free

10. [9 marks: 4, 2, 3]

(a) If $z = \cos\theta + i\sin\theta$, show that $\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$ and $\sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$.

(b) Hence, show that $\tan\theta = i\left(\frac{1-z^2}{1+z^2}\right)$.

(c) Use the result in (a) to prove that $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

Calculator Free

11. [7 marks: 3, 2, 2] [TISC]

Let $w = z + \frac{1}{z}$.

It can be shown that $w^3 + w^2 - 2w - 2 = \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right)$.

Given that $z = cis\,\theta$, a commonly used result is $z^n + \frac{1}{z^n} = 2\cos n\theta$.

(a) Show that solving $w^3 + w^2 - 2w - 2 = 0$ is equivalent to solving $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

(b) The solutions to $w^3 + w^2 - 2w - 2 = 0$ are $-\sqrt{2}$, -1 and $\sqrt{2}$.

Explain clearly why the solution $w = -1$ implies that $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

(c) Hence, find one solution to $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

Calculator Assumed

12. [9 marks: 2, 2, 3, 2]

(a) Let $z_0 = 2\,cis\left(\frac{\pi}{5}\right)$.

(i) Show that $z_0^{\,5} = -32$.

(ii) Hence, find four other complex numbers in polar form where $-\pi < \theta \leq \pi$ such that $z^5 = -32$.

(b) Determine $cis\left(\frac{\theta}{4}\right) + cis\left(-\frac{\theta}{4}\right)$ in the form $a + bi$.

(c) Use your answer in (b) to prove that $2\,cis\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right) = 1 + cis\left(\frac{\theta}{2}\right)$.

Calculator Assumed

13. [13 marks: 4, 2, 2, 5] [TISC]

Let $z = cis\ \theta$.

(a) Prove that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

(b) If $w = z + \dfrac{1}{z}$, prove that

$$w^3 + w^2 - 2w - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right).$$

Calculator Assumed

13. (c) Use parts (a) and (b) to show that the equation $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ can be rewritten as $w^3 + w^2 - 2w - 2 = 0$.

.

(d) Given that $-\pi < \theta \leq \pi$, use part (c) to solve for θ where

$$\cos\theta + \cos 2\theta + \cos 3\theta = 0.$$

Calculator Assumed

14. [8 marks: 4, 4]

(a) Solve for z in $z+z^2+z^3+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}$ = 0, given that $z \neq 0$ and $(1 + z + z^2) \neq 0$.

Give your answers in exact *cis* form.

(b) For $z = cis\ \theta$ where $-\pi < \theta \leq \pi$, it may be proven that $z^n + \frac{1}{z^n} = 2\cos(n\theta)$

where n is a positive integer. Use this result to solve for θ in

$$\cos \theta + \cos 2\theta + \cos 3\theta = 0 \text{ for } \theta \neq \frac{2\pi}{3} \text{ and } -\pi < \theta \leq \pi$$

Calculator Assumed

15. [11 marks: 4, 3, 2, 2]

Consider the equation $z^6 + 1 = 0$ for $z \in \mathbb{C}$.

(a) Use de' Moivre's Theroem to solve this equation. Give answers in *cis* form.

(b) Show that the solutions can be written in the form
$w,\ w^3, w^5, w^7, w^9$ and w^{11}.

(c) Prove that $w + w^3 + w^5 + w^7 + w^9 + w^{11} = 0$.

(d) Hence, deduce that $w^2 + w^4 + w^6 + w^8 + w^{10} = -1$.

04 The Factor & Remainder Theorems

Calculator Free

1. [5 marks]

Given that $x^2 + x + 1$ is a factor of the $f(x) = 2x^5 + x^4 - 4x^3 - 8x^2 - 7x - 2$, where $x \in \mathbb{R}$, determine the quotient when $f(x)$ is divided by $x^2 + x + 1$.

---

2. [5 marks]

Determine the quotient and remainder when $x^5 + 2x^3 - x^2 + 2x + 1$ is divided by $x^2 + 1$ for $x \in \mathbb{R}$.

Calculator Free

3. [7 marks]

$(x^2 + 4)$ is a factor of the polynomial $f(x) = 2x^5 + ax^4 + bx^3 + cx^2 - 8x + 12$ for $x \in \mathbb{C}$. When $f(x)$ is divided by $(x - 2)$ the remainder is 24. Determine the values of *a*, *b* and *c*.

4. [7 marks]

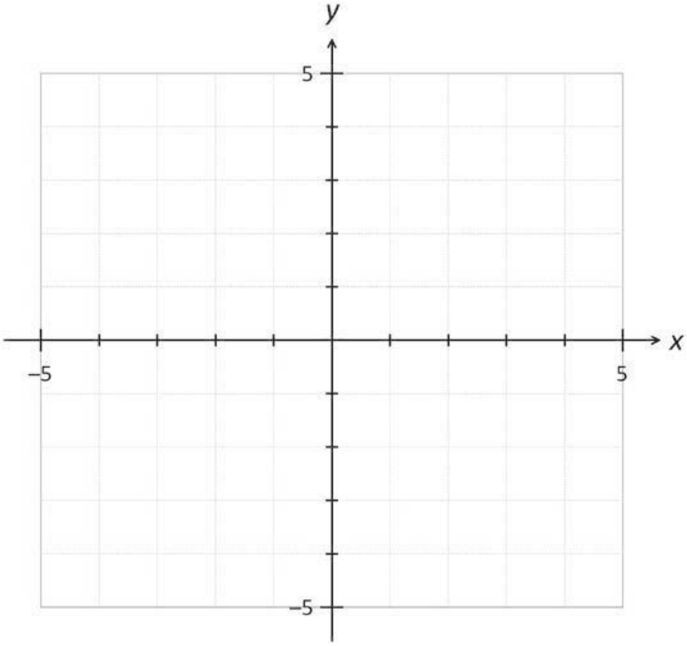
The polynomial $f(x) = x^5 + ax^4 + bx^3 + cx^2 + 6x + 4$ for $x \in \mathbb{R}$ has a of factor $x + 2$ and leaves a remainder of $2x + 1$ when divided by $x^2 - 1$. Determine the values of *a*, *b* and *c*.

Calculator Free

5. [10 marks: 6, 4]

(a) Factorise $x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$ for $x \in \mathbb{R}$

(b) On the axes provided below, sketch the curve with equation $y = x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$. Indicate all intercepts.
(The curve has several stationary points including (–1.7, 1.1) and (0.1, 2.1).)



Calculator Free

6. [11 marks: 4, 7]

(a) Solve for $3x^4 + 2x^3 - 13x^2 - 8x + 4 = 0$ for $x \in \mathbb{R}$.

(b) Hence, or otherwise solve $4\cos^4\theta - 8\cos^3\theta - 13\cos^2\theta + 2\cos\theta + 3 = 0$ for $-\pi < \theta \le \pi$. Explain clearly how you obtained your answer.

Calculator Free

7. [5 marks]

Given that $z = k + i$ is a root of the equation $2z^3 - 3z^2 + 2z + a = 0$, determine the values of the real constants a and k.

---

8. [6 marks: 1, 1, 4]

Consider $f(z) = z^5 + z^4 - z^3 - z^2 - 2z$.

(a) Determine the remainder when $f(z)$ is divided by $(z + 1)$.

(b) Determine $f(i)$.

(c) Solve the equation $f(z) = 2$.

Calculator Free

9. [8 marks]

Solve $x^6 - x^4 + x^2 - 1 = 0$ for $x \in \mathbb{C}$.

---

10. [6 marks]

Solve $x^3 + (1 + i)x^2 + (2 + i)x + 2 = 0$ for $x \in \mathbb{C}$

Calculator Free

11. [13 marks: 2, 2, 3, 6]

(a) The roots of the equation $ax^2 + bx + c = 0$ where a, b and c are real numbers are α and β.

(i) Use the quadratic formula to show the sum of the roots $\alpha + \beta = -\dfrac{b}{a}$.

(ii) Show that the product of the roots $\alpha \times \beta = \dfrac{c}{a}$.

(b) A quadratic equation with all real coefficients has a solution $x = 2 + 3i$. Determine this equation.

(c) $x = i$ and $x = 1 - i$ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where the coefficients a, b, c, d and e are real constants. Determine the values of a, b, c, d and e.

Calculator Free

12. [11 marks: 3, 3, 5]

(a) Prove that if $(x-a)^2$ is a factor of the real polynomial $f(x)$, then $(x-a)$ is a factor of $f'(x)$ where $f'(x)$ is the derivative of $f(x)$ with respect to x.

(b) $(2x-1)^2$ is a factor of $4x^4 - kx^3 - 3x^2 + kx - 1$. Determine the value of k.

(c) $(x+2)^2$ is a factor of $2x^4 + ax^3 + bx^2 - 4$. Determine the values of a and b.

05 Functions I

Calculator Free

1. [4 marks: 2, 2]

Determine analytically if each of the following functions are one-to-one or many-to-one functions.

(a) $f(x) = \dfrac{1}{x^2}$

(b) $f(x) = ln\,(1 + x)$

2. [6 marks: 3, 3]

Find the largest possible domain for each of the following functions to be one-to-one functions. In each case, state the corresponding range.

(a) $f(x) = x(x - 1)$

(b) $f(x) = -1 + \dfrac{1}{2}\sqrt{36 - 9(x - 1)^2}$

Calculator Free

3. [5 marks: 2, 1, 2] [TISC]

Let $f(x) = x + 1$ for $x \in \mathbb{R}$,

(a) Show that $g(x) = f(-|x|)$ is not a one-to-one function.

(b) Let $h(x) = f\left(\frac{1}{x}\right)$ for $x \in \{x : x \in \mathbb{R}, x \neq 0\}$.

(i) Determine the range for $h(x)$.

(ii) Determine algebraically if $h(x)$ is a one-to-one or many to one function.

---

4. [5 marks: 3, 2]

Given that $f(x) = x^2 - 2$ and $g(x) = \sqrt{x+2}$.

(a) Find the rule for $gf(x)$.

(b) State the natural domain and range for $gf(x)$.

Calculator Free

5. [5 marks: 1, 2, 2]

Given that $f(x) = \frac{1}{x+1}$ and $g(x) = x - 4$.

(a) State the natural domain for g.

(b) Explain clearly why the domain for g has to be restricted if the $f\,g$ is to be a function.

(c) State the largest possible domain for fg and the corresponding range.

---

6. [6 marks: 3, 3]

Let $f(x) = \frac{1}{\sqrt{x-3}}$. Let $g(x) = x^2 - 1$ where $x > 0$.

(a) Determine the largest possible domain for $g(x)$ so that $f(g(x))$ is a function.

(b) Determine the rule for $g(f(x))$, state its domain and range.

Calculator Free

7. [11 marks: 5, 3, 3]

(a) Let $f(x) = \sqrt{9 - x}$ and $g(x) = 3^{x+1}$. Find the domain and range for $f(g(x))$.

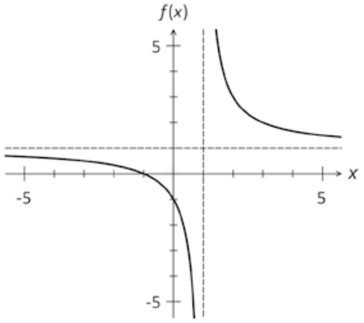
(b) Given that $f(g(x)) = x^2$ and $f(x) = e^x - 1$, determine $g(x)$.

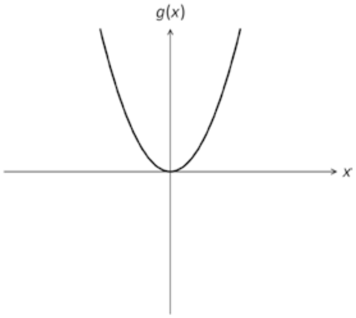
(c) Given that $f(g(x)) = \dfrac{x}{x+1}$ and $g(x) = 4 - x$, determine $f(x)$.

Calculator Free

8. [4 marks: 2, 2] [TISC]

The graphs of functions $f\,(x)$ and $g\,(x)$ are drawn below.





(a) Find the asymptote(s) of $g\,f\,(x)$.

(b) Find the range for $g\,f\,(x)$.

---

9. [7 marks: 2, 2, 3] [TISC]

(a) Given that $f \circ f\,(x) = x + 4$, find $f\,(x)$.

(b) Given that $g \circ g\,(x) = x^4$, find $g\,(x)$.

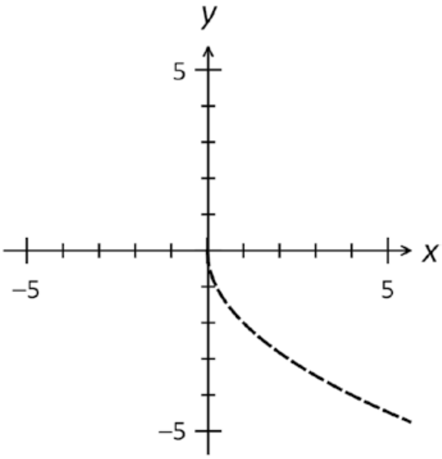
(c) Explain clearly why it is not possible to find a real valued function $h\,(x)$ such that $h \circ h\,(x) = -\,x$. [Hint: Let $h\,(x) = ax + b$ where $a, b \in \mathbb{R}$.]

Calculator Free

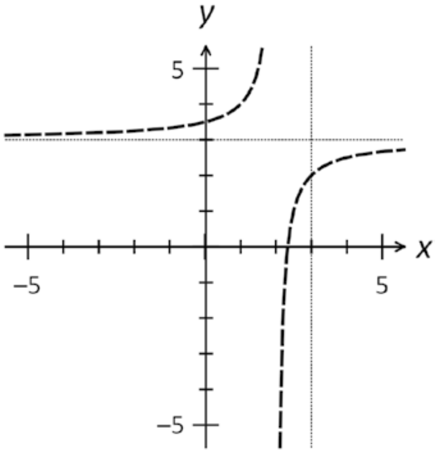
10. [6 marks: 2, 4]

The graph of $y = f(x)$ is shown in the accompanying diagrams. In each case, sketch the graph for $f^{-1}(x)$.

(a)



(b)



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11. [8 marks: 2, 6]

Consider the function $f(x) = \dfrac{1 - x}{2 + x}$.

(a) State the natural domain and range for $f(x)$.

(b) Find the rule for $f^{-1}(x)$. State the domain and range for $f^{-1}(x)$.

Calculator Free

12. [8 marks: 4, 4]

Let $f(x) = \ln(1 - x)$ and $g(x) = \dfrac{1}{x}$.

(a) Find the largest possible domain for f so that $g \circ f$ is a function.
State the accompanying range.

(b) Find x such that $g \circ f(x) = g^{-1} \circ f(x)$. Justify your answer.

---

13. [7 marks: 3, 1, 3]

Given that $f(x) = (x - 1)^2$ where x is a real number.

(a) Find $f(0)$ and $f(2)$. Hence, show that $f(x)$ does not have an inverse function.

(c) Find the largest possible positive domain for $f(x)$,
so that $f(x)$ has an inverse function.

(d) For the domain in (c), find the rule for the inverse for $f(x)$.

Calculator Free

14. [6 marks: 2, 1, 3]

Consider $f(x) = e^{|x-2|}$ where $x \in \mathbb{R}$.

(a) Show that the inverse of f is not a function.

(b) One largest possible domain for $f(x)$ to have an inverse which is a function is $\{x \mid x \in \mathbb{R}, -\infty < x \le a\}$. Determine the value of a.

(c) For the domain specified in part (b), find the rule for $f^{-1}(x)$.

---

15. [6 marks: 3, 3]

(a) $f(x) = \sin 2x$ is a one-to-one function within the domain $-a \le x \le a$. Determine the largest possible value for $|a|$. Hence, determine the rule for $f^{-1}(x)$ and state the corresponding range.

Calculator Free

15. (b) $g\,(x) = \cos\dfrac{x}{2}$ is a one-to-one function within the domain $0 \le x \le b$. Determine the largest possible value for b. Hence, determine the rule for $g^{-1}(x)$ and state the corresponding range.

16. [8 marks: 4, 4]

Let $f\,(x) = \dfrac{1+x}{2-x}$ where $x \in \{x : x \in \mathbb{R}, 2 < x < 5\}$ and $g\,(x) = \dfrac{2x-1}{x+1}$ where $x \neq -1$.

(a) Determine $f\,(g\,(x))$.

(b) Determine the domain of $g\,(x)$ so that f is the inverse function for g.

Calculator Free

17. [10 marks: 2, 3, 5]

Let $f(x) = -1+\sqrt{\left(\frac{x}{4}\right)}$ and $g(x) = (2x + 2)^2$.

(a) Determine $g(f(x))$.

(b) Determine $f(g(x))$.

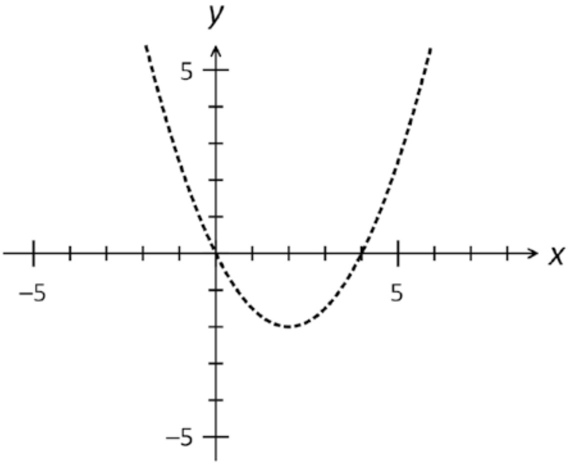
(c) Determine with reasons, the domain of f and the domain of g so that these functions are inverses of each other.

06 Functions II

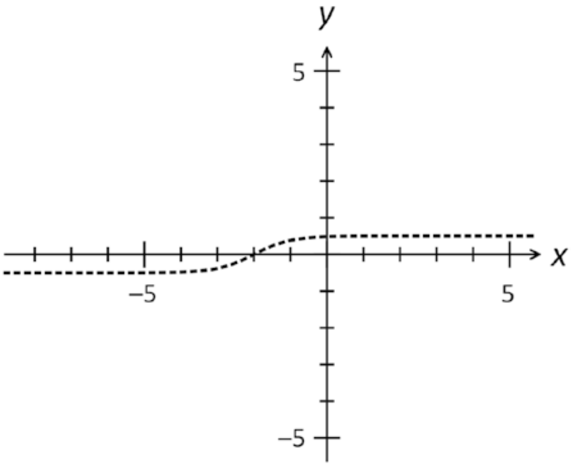
Calculator Free

1. [12 marks: 4, 4, 4]

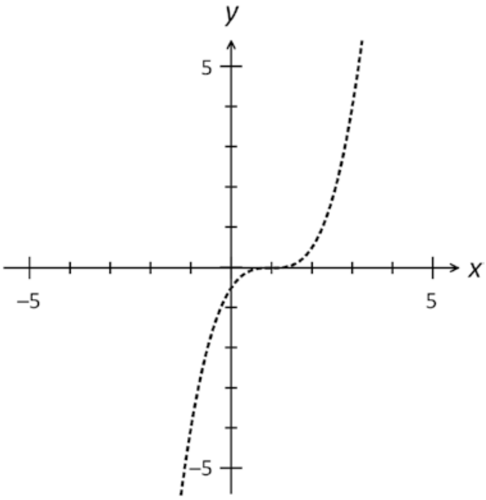
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = \dfrac{1}{f(x)}$.



(b) The sketch of $y = \dfrac{1}{f(x)}$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(x)$.



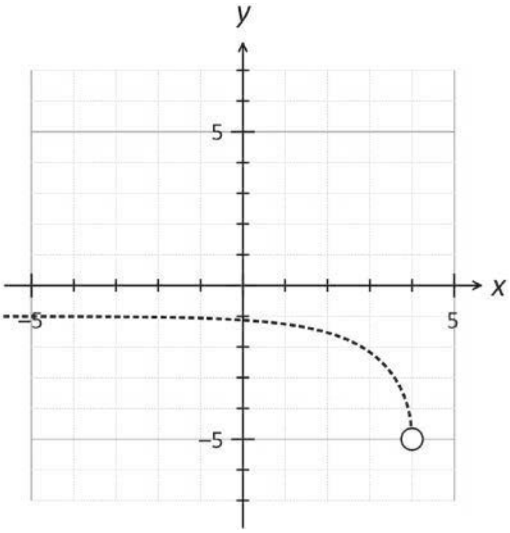
(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y^2 = f(x)$.



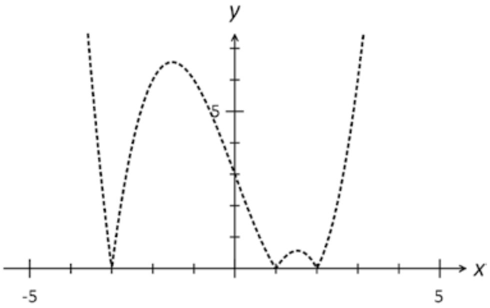
Calculator Free

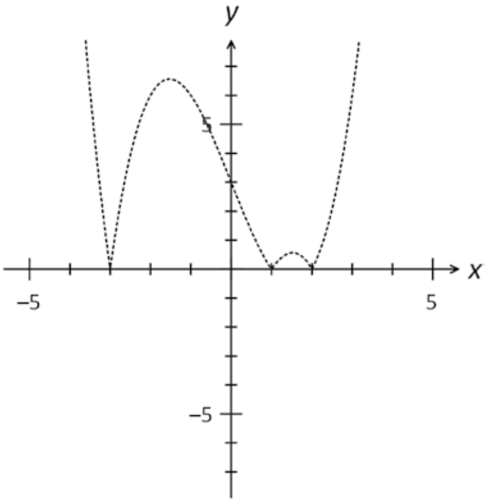
2. [8 marks: 4, 4]

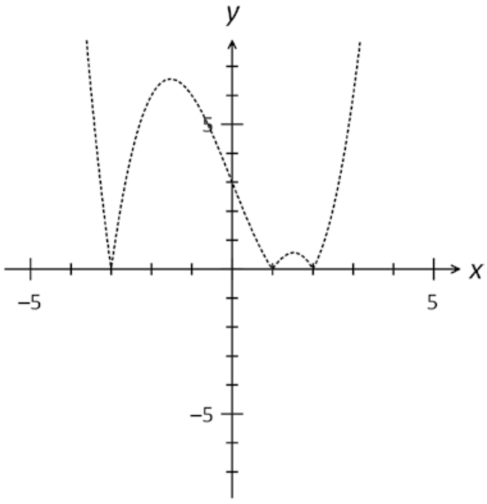
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = |f(x)|$.



(b) The sketch of $y = |f(x)|$ is given in the accompanying diagram. Sketch on the axes provided below the two possible graphs of $y = f(x)$.



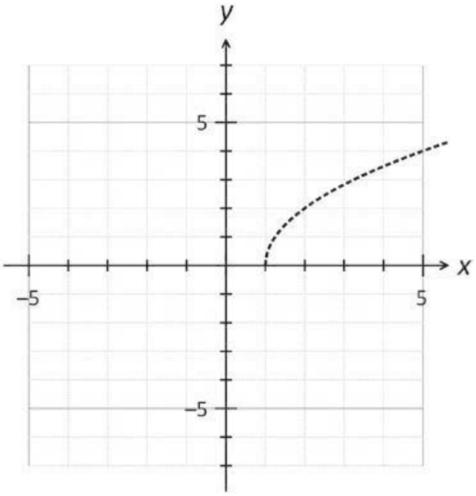




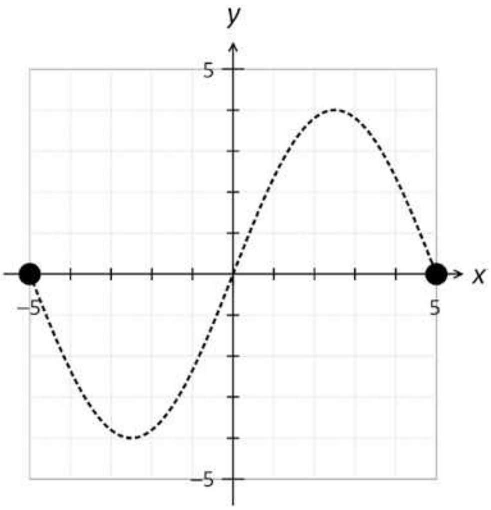
Calculator Free

3. [9 marks: 3, 3, 3]

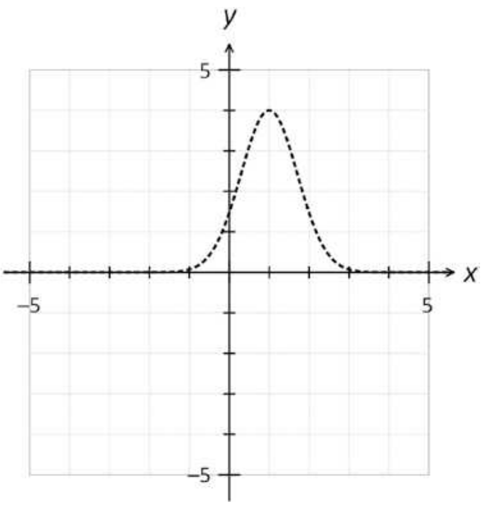
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(|x|)$.



(b) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $|y| = f(x)$.



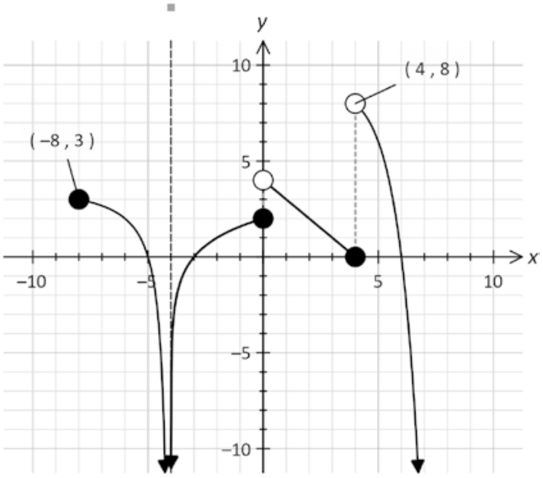
(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $|y| = f(|x|)$.



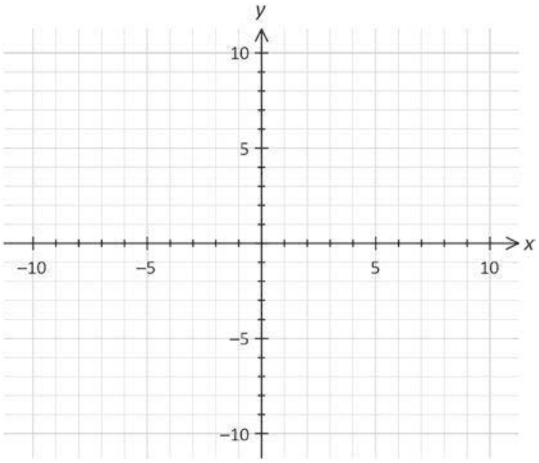
Calculator Free

4. [6 marks: 3, 3] [TISC]

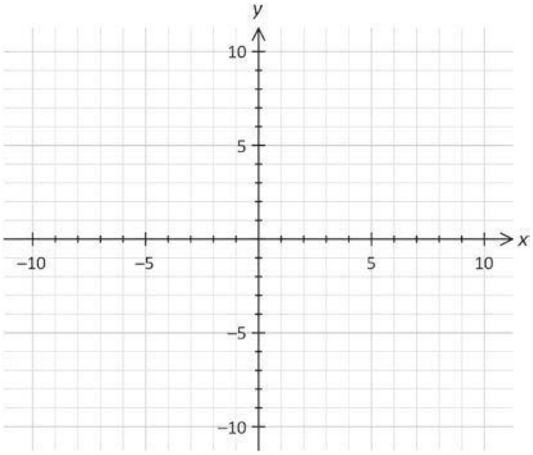
The accompanying diagram shows the graph of the function $y = f(x)$.



(b) On the accompanying diagram, for $x > 0$, sketch the graph of $y = \dfrac{16}{f(x)}$.



(c) On the accompanying diagram, for $x < 0$, sketch the graph of $y = f(|x|)$.

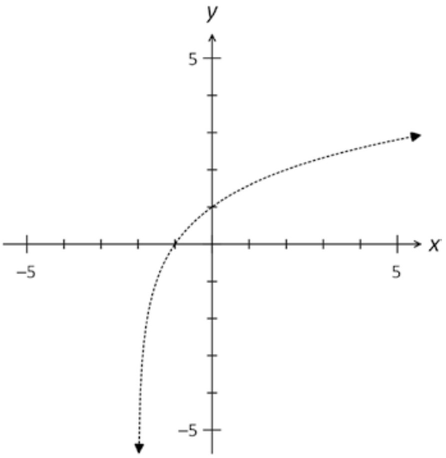


Calculator Free

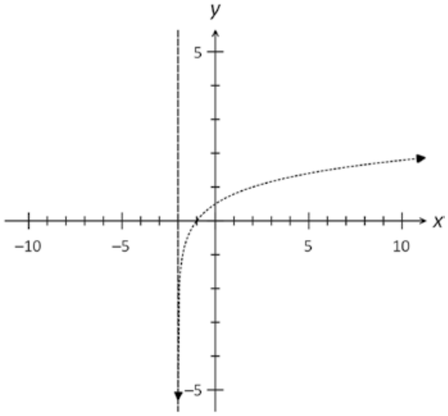
5. [7 marks: 3, 4]

The accompanying diagrams each show the graph of $y = f(x)$.

(a) Sketch on the same axes, the graph of $y = f^{-1}(x)$.



(b) Sketch on the same axes, the graph of $y = \dfrac{1}{f(|x|)}$.



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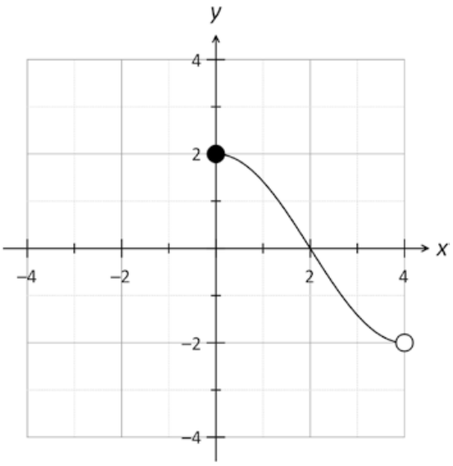
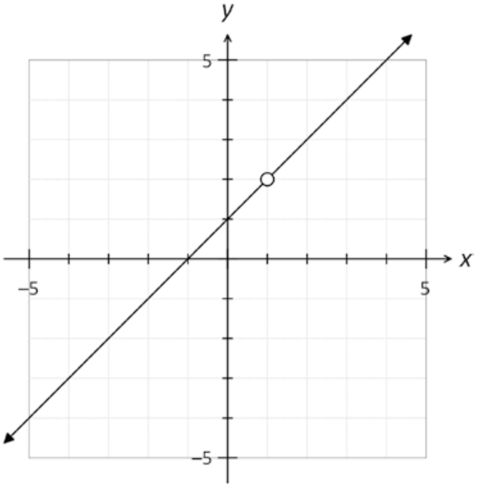
6. [7 marks: 3, 4]

(a) The accompanying diagram shows the graph of $y = f^{-1}(x)$. Sketch on the same axes, the graph of $y = f(x)$.



Calculator Free

6. (b) The accompanying diagram shows the graph of $y = \dfrac{1}{f(x)}$. Sketch on the same axes, the graph of $y = f(x)$.



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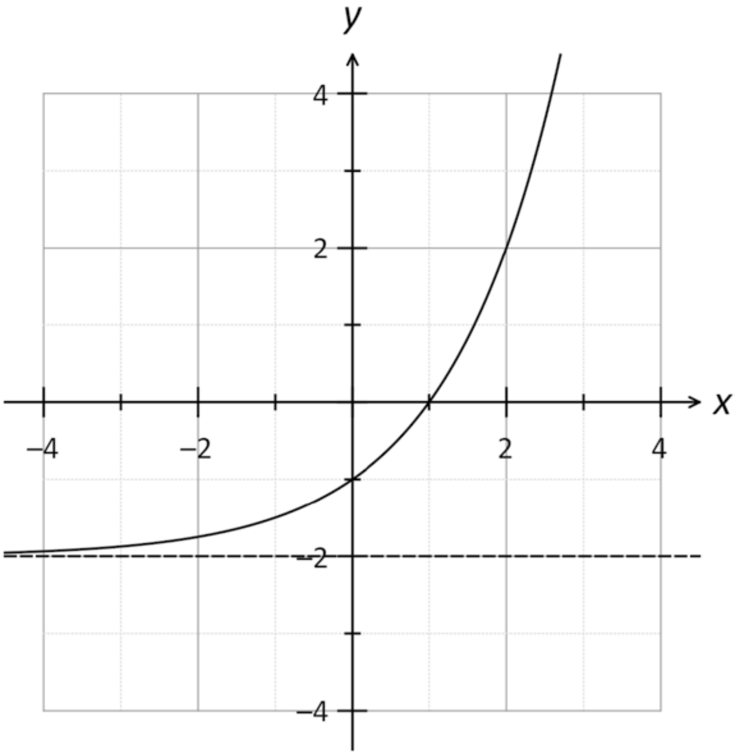
7. [6 marks: 3, 3]

The diagram below shows the graph of $y = f(x)$. Let $g(x) = \dfrac{1}{f(x)}$.



(a) On the axes above sketch and label the graph of $y = g(x)$.

(b) On the diagram above, sketch and label the graph of $y = g^{-1}(x)$.

Calculator Free

8. [16 marks: 4, 4, 4, 4]

The graph of $y = f(x)$ has intercepts at (2, 0) and (0, –2) and asymptotes with equations $x = 1$ and $y = -1$.

(a) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = \dfrac{1}{f(x)}$.

(b) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = |f(x)|$.

(c) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y^2 = f(x)$.

(d) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = f(|x|)$.

Calculator Free

9. [7 marks: 3, 4]

Let $f(x) = |x - 1| + |x + 2| - |x|$.

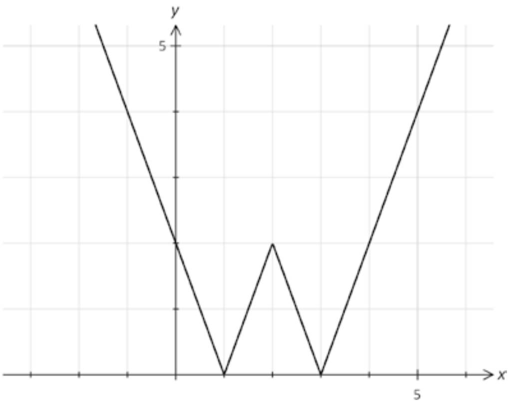
(a) Rewrite $f(x)$ in piecewise defined form.

(b) Hence or otherwise, solve $|x - 1| + |x + 2| - |x| \geq 5$.

---

10. [4 marks]

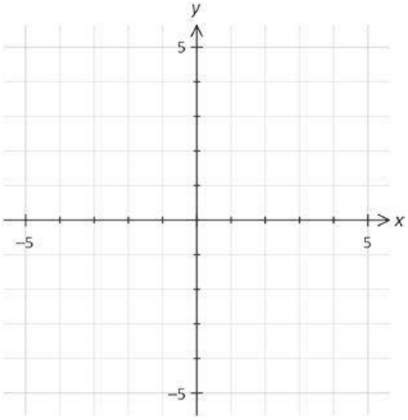
The given diagram shows the graph of $y = |a + b|kx + c||$. Determine the values of constants a, b, c and k where $k < 0$.



Calculator Free

11. [7 marks: 3, 4]

(a) On the given axes, sketch the graph of $y = \left| |x| + 2x \right|$.



(b) Solve algebraically $\left| |x| + 2x \right| = 4 - |x|$.

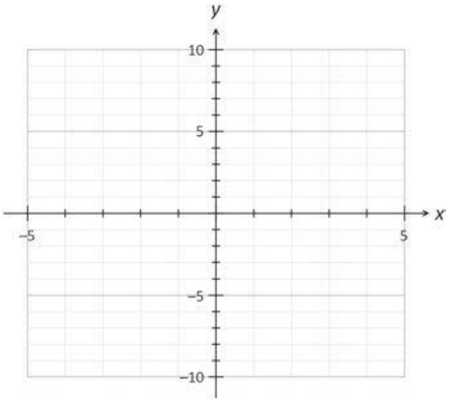
---

12. [7 marks: 3, 3, 1] [TISC]

Let $f(x) = x^2 - 3|x| - 4$.

(a) Rewrite $f(x)$ in piecewise defined form.

(b) In the axes provided, sketch the graph of $y = x^2 - 3|x| - 4$.



(c) Use your sketch to explain why $f(x)$ does not have an inverse function.

07 Rational Functions

Calculator Free

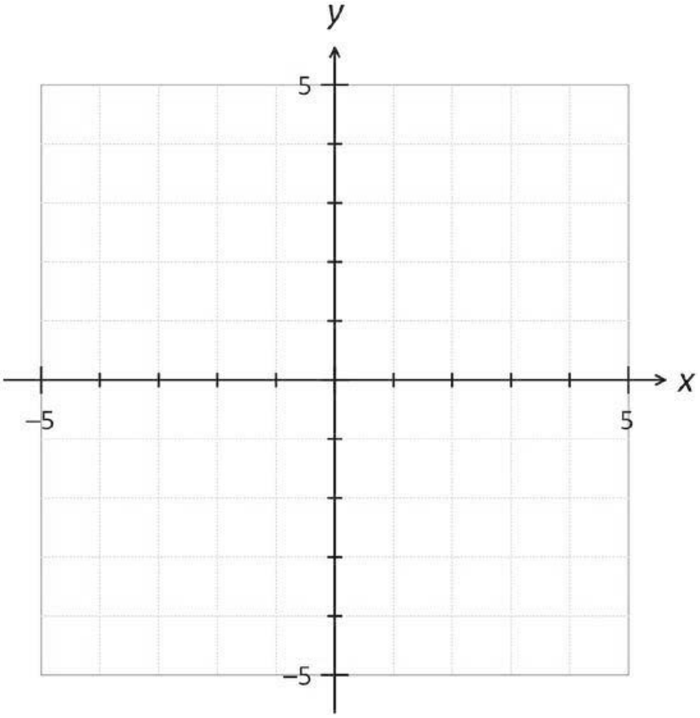
1. [9 marks: 3, 3, 3]

Consider the curve with equation $y = \dfrac{(x+1)(x-2)^2}{x^2 - x - 2}$.

(a) Determine the coordinates of the point(s) of discontinuity in this curve.

(b) Explain clearly why this curve does not have any horizontal asymptotes.

(c) Draw a sketch of this curve showing all the relevant features.



Calculator Free

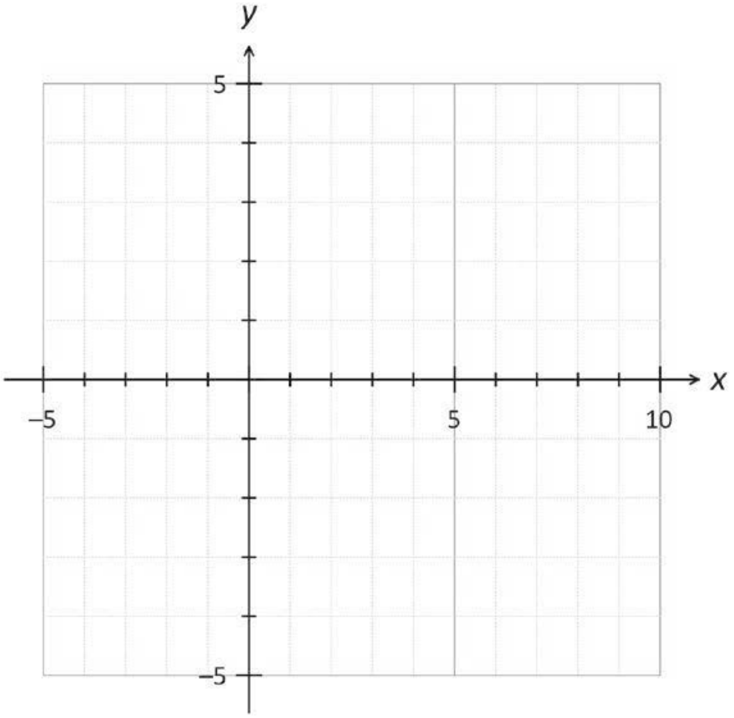
2. [9 marks: 2, 3, 4]

Consider the curve with equation $y = \dfrac{x^2 + x - 2}{x^2 - 2x - 8}$.

(a) State the equation of all asymptotes.

(b) Identify the point of discontinuity on this curve.

(c) Sketch this curve on the axes provided below.

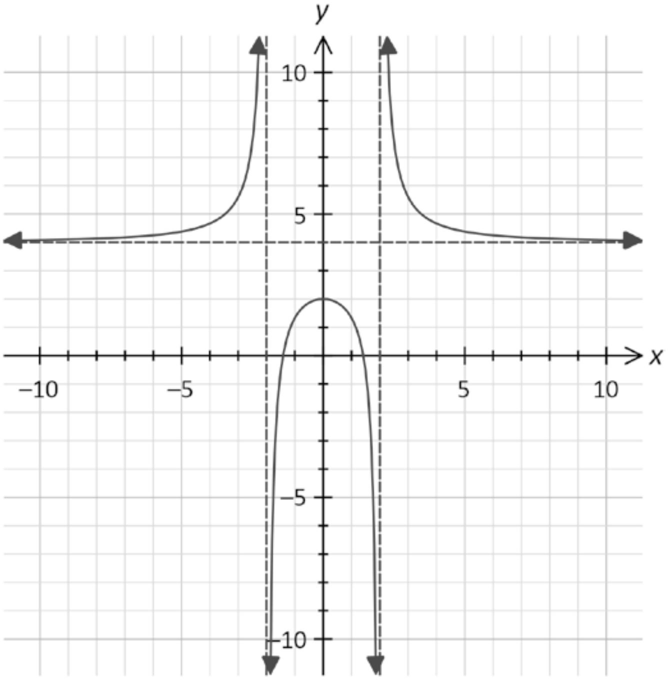


Calculator Free

3. [7 marks: 3, 4]

The diagram below shows the graph of $y = f(x)$ where $f(x) = \dfrac{ax^2 + b}{x^2 + c}$,

where *a*, *b* and *c* are real constants. The graph has a turning point at (0, 2).
The graph has asymptotes with equation $x = -2$, $x = 2$ and $y = 4$.



(a) Determine the values of the constants *a*, *b* and *c*.

(b) State the largest possible domain for $f(x)$ so that $f^{-1}(x)$ exists.

For this domain, on the axis above, sketch the graph of $y = f^{-1}(x)$.

Calculator Free

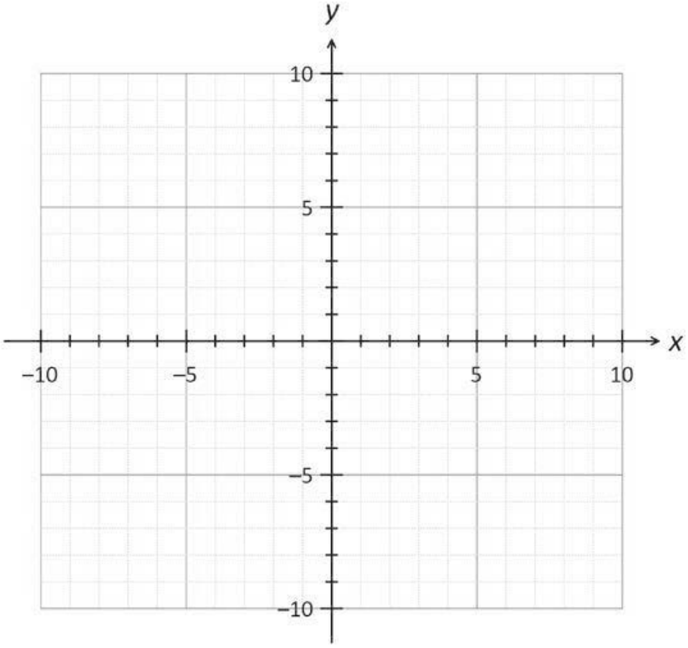
4. [11 marks: 4, 2, 5]

Consider the curve with equation $y = \dfrac{x^2 + x - 6}{x - 1}$.

(a) Rewrite the equation of the curve in the form $y \equiv \dfrac{P(x)}{Q(x)} + ax + b$ where $\dfrac{P(x)}{Q(x)}$ is a rational proper fraction and a and b are real constants.

(b) State the equations of all asymptotes of this curve.

(c) On the axes provided below sketch the graph of $y = \dfrac{x^2 + x - 6}{x - 1}$. Indicate all intercepts and asymptotes.



Calculator Free

5. [6 marks]

Consider the curve with equation $y = \dfrac{x^3 + 2x^2 + x - 4}{2(x^2 - 1)}$. Identify the equations of all asymptotes and points of discontinuities of this curve.

---

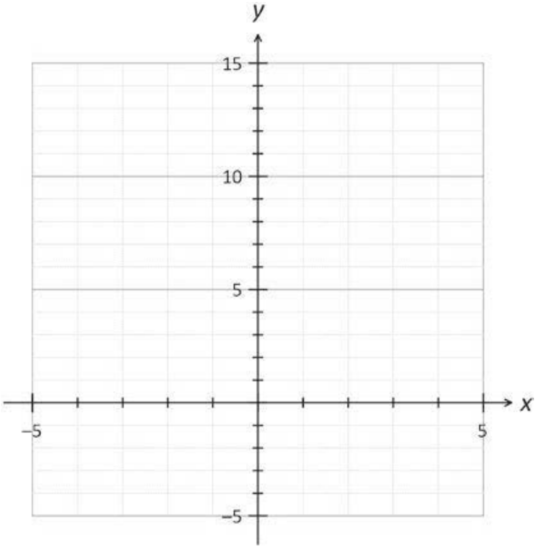
6. [5 marks]

Sketch the graph of the rational function $f(x)$ with the following properties:

- $f(0) = f\left(\dfrac{3}{2}\right) = 0,$
- $f'(1) = f'(3) = 0$
- $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to +\infty} f(x) = 0$
- $\lim\limits_{x \to 2^-} f(x) \to -\infty,$

  $\lim\limits_{x \to 2^+} f(x) \to \infty$
- $\lim\limits_{x \to \infty} f(x) = 2x + 1,$

  $\lim\limits_{x \to -\infty} f(x) = 2x + 1$

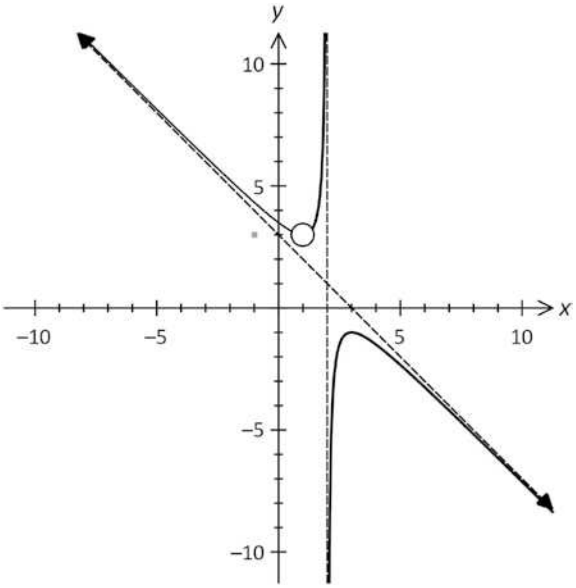


Calculator Free

7. [9 marks: 3, 3, 3]

The diagram below shows the graph of $y = f(x)$ where $f(x) = \dfrac{p(x)}{q(x)}$.

$p(x)$ is a polynomial of degree 3. $q(x)$ is a polynomial of degree 2 with a leading coefficient of 1. The polynomials $p(x)$ and $q(x)$ are non-reduced, that is, they may have factors in common. The graph has a point of discontinuity at (1, 3).
The graph as asymptotes with equation $x = 2$ and $y = -x + 3$



(a) Determine with reasons the polynomial $q(x)$.

(b) Determine the polynomial $p(x)$.

(c) On the axes above, sketch the graph of $y = f(|x|)$.

Calculator Free

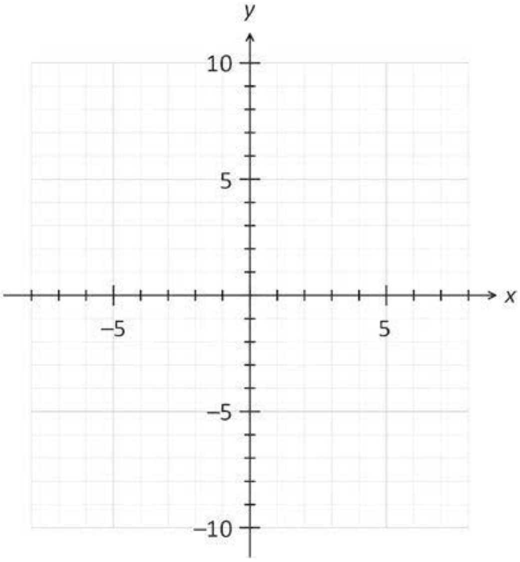
8. [13 marks: 3, 2, 4, 4] [TISC]

(a) The function $f(x) = \dfrac{x^3+1}{x^2-1}$ can be rewritten as $px + \dfrac{qx+r}{x^2-1}$. Find p, q and r.

(b) State the equations of all the asymptotes of the curve $y = f(x)$.

(c) The curve with equation $y = f(x)$ has a maximum point at (0, –1).
Use Calculus to show that the curve has a local minimum point at (2, 3).

(d) Sketch the graph of $y = \dfrac{x^3+1}{x^2-1}$.



Calculator Free

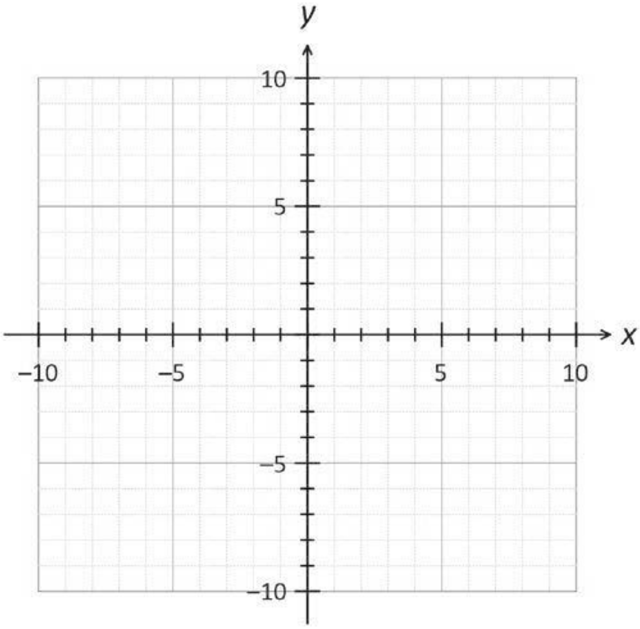
9. [12 marks: 4, 4, 4] [TISC]

Consider the curve with equation $y = \dfrac{x^3}{x^2 - 4}$.

(a) Find the equation of the asymptote(s).

(b) Use differentiation to find the number of stationary points on this curve.

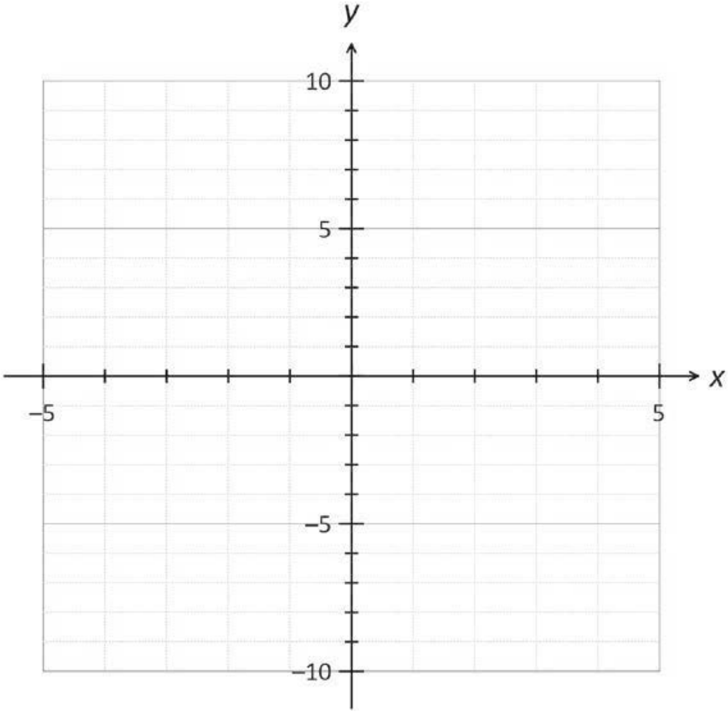
(c) Sketch this curve on the axes below.



Calculator Free

10. [7 marks]

On the axes provided below, sketch the curve with equation
$y = x + \frac{(x-1)}{(x-1)(x-2)}$. Indicate clearly all essential features of this curve.



Calculator Assumed

11. [15 marks: 2, 2, 4, 3, 4]

Consider the curve with equation $y = \left| \dfrac{x-1}{x^2+1} \right|$.

(a) Explain why the equation of this curve can also be written as $y = \dfrac{|x-1|}{x^2+1}$.

(b) The equation of this curve can be rewritten as $y = \begin{cases} \dfrac{ax+b}{x^2+1} & x<1 \\ \dfrac{x-1}{x^2+1} & x \geq 1 \end{cases}$

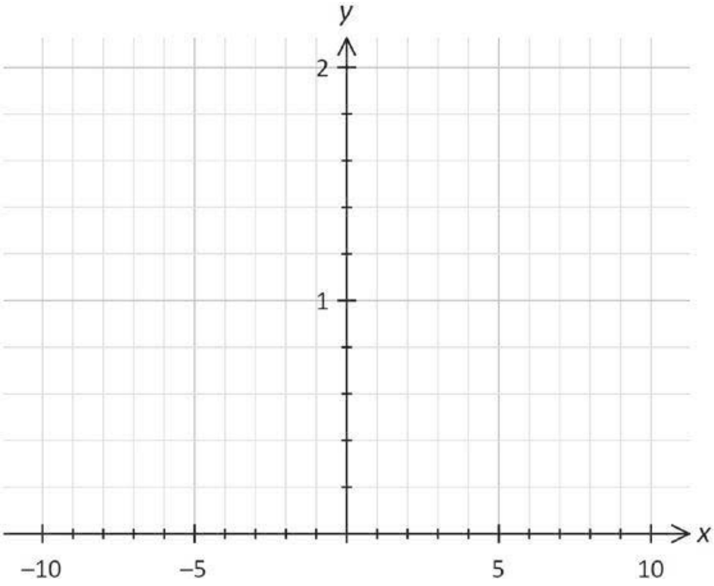
where a and b are constants. Find a and b.

(c) Use differentiation to verify that this curve has a maximum point at $x = 1 + \sqrt{2}$.

Calculator Free

11. (d) Find the minimum point of this curve. Justify your answer.

(e) On the axes provided below, sketch this curve.



08 Vectors I

Calculator Free

1. [8 marks: 2, 1, 1, 4]

Given that $\boldsymbol{a} = 3\,\mathbf{i} + 10\,\mathbf{j} + 8\,\mathbf{k}$ and $\boldsymbol{b} = 2\,\mathbf{j} + 3\,\mathbf{k}$, find:
(a) $|\boldsymbol{a} - 2\boldsymbol{b}|$

(b) the unit vector parallel to $\boldsymbol{a} - 2\boldsymbol{b}$

(c) a vector that is parallel to $\boldsymbol{a} - 2\boldsymbol{b}$ but with a magnitude of 10

(d) $\boldsymbol{a}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ where $\boldsymbol{p} = -\mathbf{i} + 4\,\mathbf{k}$ and $\boldsymbol{q} = 2\mathbf{i} + 5\,\mathbf{j} + 2\mathbf{k}$.

---

2. [6 marks]

$\mathbf{OA} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{OB} = 2\mathbf{i} + b\,\mathbf{j} - \mathbf{k}$ and $\mathbf{OC} = 6\mathbf{i} + 3\mathbf{j} + c\,\mathbf{k}$.
Find b and c if A, B and C are collinear.

Calculator Free

3. [6 marks]

Vector $\alpha\mathbf{i} + \beta\mathbf{j} + \sqrt{2\alpha\beta}\,\mathbf{k}$ has a magnitude of 10 and is parallel to vector $3\mathbf{i} + 4\mathbf{j} + 2\sqrt{6}\mathbf{k}$. Find all possible values of α and β.

---

4. [3 marks]

The points P and Q have position vectors $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ respectively. The point K is such that $\mathbf{PQ} = -2\mathbf{QK}$. Find the position vector of K.

---

5. [6 marks: 2, 2, 2]

Given that $\boldsymbol{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, find:

(a) $(\boldsymbol{a} - \boldsymbol{b}) \bullet \boldsymbol{c}$

(b) $(\boldsymbol{a} - \boldsymbol{b}) \bullet (\boldsymbol{a} - \boldsymbol{c})$

(c) m if $\boldsymbol{b} \bullet (2\mathbf{i} - m\mathbf{j} + 3\mathbf{k}) = -1$

Calculator Free

6. [5 marks: 3, 1, 1]

(a) Find the acute angle between the vectors $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

Hence, or otherwise, find the acute angle between:
(b) $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} - \mathbf{j}$

(c) $-4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ and $-5\mathbf{i} - 5\mathbf{j}$.

---

7. [4 marks]

Find the value(s) of α if the angle between the vectors $\mathbf{i} + \mathbf{k}$ and $2\,\mathbf{i} + \alpha\,\mathbf{j}$ is 60°.

---

8. [6 marks]

$\boldsymbol{u} = a\,\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ has the same magnitude as $\boldsymbol{v} = (7 - b)\,\mathbf{i} + (a + c)\,\mathbf{j} + (4b - c)\,\mathbf{k}$.
Find the values of a, b and c if $\boldsymbol{u}$ and $\boldsymbol{v}$ act in opposite directions.

Calculator Free

9. [6 marks: 3, 3]

(a) Find a unit vector that is parallel to –**i** + **j** – **k** and perpendicular to –**i** + 2**j** + 3**k** .

(b) Find a unit vector that is perpendicular to both –**i** + **j** – **k** and –**i** + 2**j** + 3**k** .

---

10. [8 marks: 3, 1, 4]

Let $\boldsymbol{u}$ = < 2, 1, 2 >, $\boldsymbol{v}$ = < 1, 0, 1> and $\boldsymbol{w}$ = < –1, 1, 0 >.

(a) Use the result $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta$, where θ is the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ to calculate the sine of the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(b) Determine the area of the parallelogram with sides parallel to v and w.

(c) The scalar projection of vector $\boldsymbol{a}$ onto vector $\boldsymbol{b}$ is given by $a \bullet \hat{b}$.
Find the vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v} \times \boldsymbol{w}$.

Calculator Assumed

11. [4 marks]

Prove that the line segments congruent to the vectors $\begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix}$, $\begin{pmatrix} -6 \\ -14 \\ -9 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -13 \\ 2 \end{pmatrix}$ form a right angled triangle.

---

12. [6 marks]

The sides of an equilateral triangle are congruent with the vectors $\boldsymbol{u} = \begin{pmatrix} \frac{-1}{2} \\ 0 \\ \frac{-\sqrt{3}}{2} \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{-\sqrt{3}}{2} \end{pmatrix}$ and $\boldsymbol{w}$. Find $\boldsymbol{w}$.

09 Vectors II

Calculator Free

1. [9 marks: 3, 3, 3]

The points A and B have position vectors $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively.

(a) Find the position vector of the point K which is 6 units from A in the direction $-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(b) Find the position vector of the point L which is 21 units from A in the direction AB.

(c) Find the area of ΔKAL. Show clearly how you obtained your answer.

Calculator Free

2. [10 marks: 3, 3, 4]

Let $\boldsymbol{u} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\boldsymbol{v} = -\mathbf{i} + \mathbf{k}$ and $\boldsymbol{w} = < 2, -2, 2 >$.

(a) Find the acute angle between ***u*** and ***v***.

(b) Find the area of a triangle with sides parallel and congruent to ***u*** and ***v***.

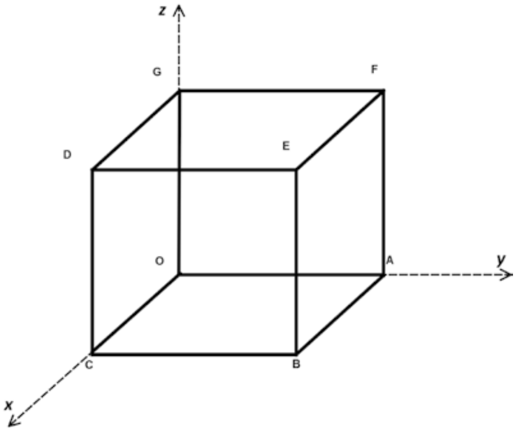
(c) A triangular pyramid has sides parallel and congruent to ***u, v*** and ***w***. Determine the volume of this triangular pyramid. Show clearly how you obtained your answer.

Calculator Assumed

3. [11 marks: 2, 2, 4, 3]

A rectangular box OABCDEFG rests on the *x*-*y* plane as shown. The vertex O has position vector < 0, 0, 0 > cm.
CB = 10 cm, AB = BE = 8 cm.

(a) State the position vectors of the vertices C and E of this box.



(b) Use vector methods to find the length of the diagonal BG.

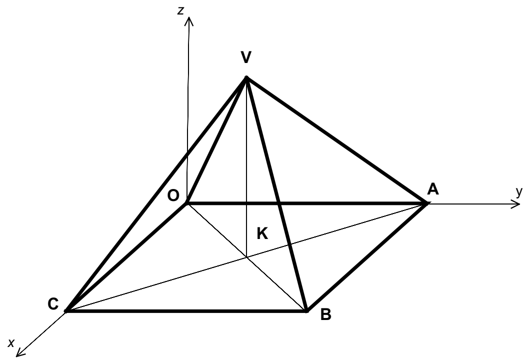
(c) Use a cross-product method to calculate exactly sin(∠GBF).

(d) Calculate the exact area of ΔBGF.

Calculator Assumed

4. [10 marks: 2, 8]

A regular pyramid VOABC rests on the *x*-*y* plane as shown. The vertices O, A, and C have position vectors < 0, 0, 0 >, < 0, 4, 0 > and < 6, 0, 0 > cm respectively. The vertex V is 5 cm vertically above the rectangular base OABC.



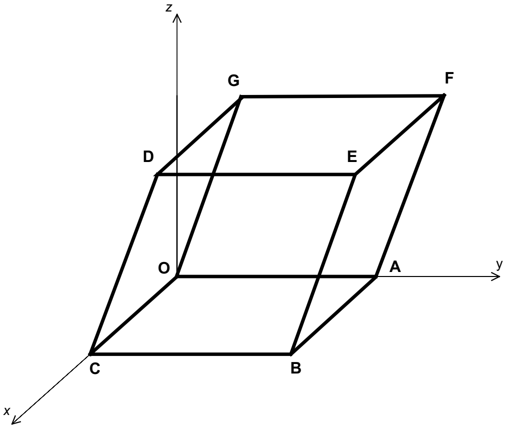
(a) State the position vector of V.

(b) Use a vector method to calculate the exact total surface area of this pyramid.

Calculator Assumed

5. [11 marks: 4, 3, 2, 2]

A parallelepiped OABCDEFG rests on the *x*-*y* plane as shown. [All six faces are parallelograms with opposite faces congruent.] The vertices O, A, C and G have position vectors < 0, 0, 0 >, < 0, 3, 0 >, < 2, 0, 0 > and < 0, 1, 2 > cm respectively.



(a) State the position vectors of the vertices D and E of this box.

(b) Determine the angle between **GE** and **AB**.

(c) The volume of the parallelepiped = Area of base × Perpendicular height.

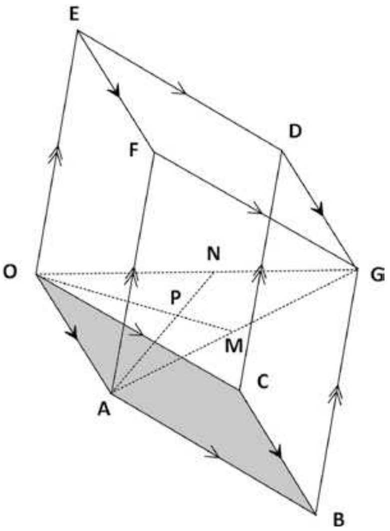
(i) Find the area of the base OABC.

(ii) Find the volume of the parallelepiped.

Calculator Assumed

6. [13 marks: 2, 4, 2, 5]

The given diagram shows a parallelepiped OABCDEFG. The opposite faces of the solid are congruent parallelograms which are parallel to each other. **OA** = ***a***, **OC** = ***c*** and **OE** = ***e***. The point M is the midpoint of AG and the point N is the midpoint of OG. OM and AN intersect at P. Let **OP** = α**OM** and **AP** = β**AN**.



(a) Express **AG** and **OG** in terms of ***a***, ***c*** and/or ***e***.

(b) Express **OM** and **AN** in terms of ***a***, ***c*** and/or ***e***.

(c) Express **OP** and **AP** in terms of ***a***, ***c*** and/or ***e***.

(d) Determine the values of α and β.

10 Vectors III (Lines)

Calculator Assumed

1. [3 marks: 1, 2]

Find the vector equation of a straight line passing through (4, –1, 2) and:

(a) parallel to the vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

(b) the point with position vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

---

2. [6 marks: 3, 3]

The vector equation of a line is given by $\boldsymbol{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(a) Determine with reasons if the point (–1, 4, –6) lies on the line.

(b) Given that the point with position vector $< -2k, k, k + 5 >$ lies on this line, find the value(s) of k.

Calculator Assumed

3. [7 marks: 2, 2, 3]

The parametric equation of a line is given by $x = \dfrac{1 + \lambda}{2}$, $y = \dfrac{-1 + 2\lambda}{2}$, $z = \dfrac{4 - 3\lambda}{5}$

(a) Determine the Cartesian equation of this line.

(b) Find the vector equation of this line.

(c) The points A, B and C on this line are such that $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$ respectively. Find the ratio with which the point B divides the line AC. Show clearly how you obtained your answer.

---

4. [8 marks: 4, 4]

The point A $(a, -a, 0)$ lies on the line L with equation $\boldsymbol{r} = \begin{pmatrix} b \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ b \\ -1 \end{pmatrix}$.

(a) Find the value(s) of a and b.

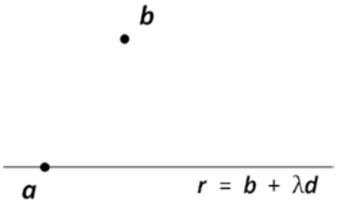
Calculator Assumed

4. (b) Use a vector method to find the shortest distance between the point P (1, 0, 3) and the line L.

---

5. [8 marks: 3, 5]

(a) Use the definition $|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}|\,|\boldsymbol{v}| \sin\theta$ where θ is the acute angle between ***u*** and ***v*** to prove that the shortest distance between the point with position vector ***b*** to the line with equation $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$ is given by $|(\boldsymbol{b} - \boldsymbol{a}) \times \hat{\boldsymbol{d}}|$.



Calculator Assumed

5. (b) ABCD is a trapezium. The points A, B, and D have position vectors $< 1, 0\ 1 >$, $< 0, 2, 0 >$ and $< -1, 1\ 1 >$ respectively. The point C is such that **BC** is parallel to **AD** and has magnitude 3 times that of **AD**.
Calculate the area of the trapezium ABCD.

---

6. [9 marks: 2, 4, 3]

The lines L1 and L2 have equations $\boldsymbol{r} = < 1, 1, 2 > + \lambda < -2, 1, -3 >$ and $\boldsymbol{r} = < 2, 1, 4 > + \mu < 1, -1, 1 >$ respectively. The lines L1 and L2 intersect at K.

(a) Find the acute angle between L1 and L2.

(b) Use a vector method to find the position vector of K.

(c) Find the vector equation of the line passing through the point K and perpendicular to L1 and L2.

Calculator Assumed

7. [8 marks: 2, 2, 2, 2]

Lines L1 and L2 have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ respectively.

(a) Find the parametric equations of the lines L1 and L2.

(b) Show that L1 and L2 are skew lines (non-intersecting and non-parallel)..

(c) Find the acute angle between L1 and L2.

(d) Find the vector equation of a line parallel to L1 but which intersects L2.

Calculator Assumed

8. [11 marks: 3, 1, 1, 4, 2]

The lines L1 and L2 have equations $\boldsymbol{r}$ = < 0, 1, –1 > + λ < 1, 1, 1 > and $\boldsymbol{r}$ = < 1, 0, –2 > + μ < 1, 2, α > respectively.

(a) Find the value(s) of α if the lines L1 and L2 are non-intersecting.

(b) Let α = 1. The points P and Q are points respectively on lines L1 and L2.

(i) Determine the position vectors of P and Q in terms of λ and/or μ.

(ii) Find **PQ** in terms of λ and/or μ.

(iii) Find λ and μ if **PQ** is perpendicular to both L1 and L2.

(iv) Hence, find the shortest distance between L1 and L2.

Calculator Assumed

9. [8 marks: 4, 2, 2, 2]

The line L1 has equation $\boldsymbol{r} = < -4 + \lambda, -2\lambda, \lambda - 2 >$. The point B (–4, 0, –2) lies on the line L1. The point A has position vector < 2, –2, 0 >.

(a) Find the position vector of the point K on the line L1 such that AK is perpendicular to L1.

(b) Hence, find the shortest distance between A and the line L1.

(c) Find the size of $\angle$BAK.

(d) Find the area of ΔAKB.

Calculator Assumed

10. [11 marks: 4, 1, 4, 2]

The point A with position vector $< 1 + m, -m, -1 >$ lies on line L1 with equation $\boldsymbol{r} = < 1, 0, -1 > + \lambda < 1, -1, 0 >$. The point B with position vector $< 2 - 2n, 1 + n, n >$ lies on line L2 with equation $\boldsymbol{r} = < 2, 1, 0 > + \mu < -2, 1, 1 >$.

(a) Show that L1 and L2 are skew lines (non-intersecting and non-parallel).

(b) Find **BA** in terms of m and n.

(c) Find the value(s) of m and n if **BA** is perpendicular to both L1 and L2.

(d) Hence, find the shortest distance between the lines L1 and L2.

Calculator Assumed

11. [10 marks: 4, 4, 1]

The point A (1, 0, –2) lies on the line L1 which is parallel to $\boldsymbol{d}_1$ = < 4, –2, –4 >.
The point B (6, –1, –10) lies on the line L2 which is parallel to $\boldsymbol{d}_2$ = < 2, 1, 2 >.

(a) Prove that the lines L1 and L2 are skew lines
(neither parallel nor intersecting).

(b) Determine the scalar projection of **BA** onto $\boldsymbol{d}_1 \times \boldsymbol{d}_2$.

(c) Hence, find the shortest distance between the lines L1 and L2.

11 Vectors IV (Planes)

Calculator Free

1. [4 marks: 1, 3]

Find in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \rho$, the vector equation of a plane:

(a) passing through (–1, 3, 4) and perpendicular to the vector $-2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

(b) containing the line $\boldsymbol{r} = < 5 - 2\lambda, 2 + 6\lambda, 4\lambda >$ and perpendicular to the vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

---

2. [7 marks: 2, 5] [TISC]

The line L with equation $\boldsymbol{r} = 2\boldsymbol{i} + 2\lambda\boldsymbol{j} + (1 - \lambda)\boldsymbol{k}$ is perpendicular to the plane P.

(a) The point A with position vector $-\boldsymbol{i} + \boldsymbol{k}$ lies on the plane P.
Determine the vector equation of the plane P.

(b) B is a point on line L. The distance between A and B is $\sqrt{14}$ units.
Determine the position vector of the point B.

Calculator Free

3. [4 marks]

Find in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \rho$, the vector equation of a plane containing the lines $\boldsymbol{r} = \mathbf{i} - \mathbf{k} + \lambda\ (\mathbf{i} - \mathbf{k}\ )$ and $\boldsymbol{r} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda\ (-\mathbf{i} + \mathbf{j} + \mathbf{k}\ )$.

---

4. [4 marks]

Find the vector equation of the plane passing through the line with equation $\boldsymbol{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(4\mathbf{i} - \mathbf{k})$. The points with position vectors $< -1, 0, 0 >$ and $< 2, 1, -1 >$ lie on this plane. Give your answer in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \rho$.

---

5. [7 marks: 3, 4]

The points with position vectors $< 1, 0, 2 >$, $< -1, 1, 0 >$ and $< 0, 1, -1 >$ lie on a plane.

(a) Determine the vector equation of the plane in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$.

(b) Determine the Cartesian equation of the plane.

Calculator Free

6. [6 marks: 3, 2, 1]

Plane P has equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Point A has position vector $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

The line L passes through the point A and is perpendicular to P.

(a) Determine the vector equation of line L.

(b) Determine the Cartesian equation of the plane P.

(c) State the position vector of the point of intersection between L and P.

---

7. [8 marks: 3, 5]

The plane P has equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

(a) Determine with reasons if the point A with position vector $2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$ lies on the plane P

Calculator Free

7. (b) Plane Q has equation $2x - y + z = 1$.
Calculate the acute angle between planes P and Q.

8. [8 marks: 5, 3]

The points A, B and C have position vectors < 1, 0, 2 >, < 4, 1, –2 > and < 0, 2, 1 > respectively.

(a) Determine the vector equation of the plane passing through points A, B and C. Give your answer in the form $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$

(b) The area of a triangle spanned by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by $\frac{1}{2}|\,\boldsymbol{u} \times \boldsymbol{v}\,|$.

Use this result or otherwise to calculate the area of ΔABC.

Calculator Assumed

9. [5 marks: 3, 2]

Consider the line L with equation $\boldsymbol{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} - \mathbf{k})$ and the plane P with equation $\boldsymbol{r} \bullet < 2, 3, -4 > = 0$.

(a) Find the position vector of the point of intersection between L and P

(b) Find the acute angle between the L and P.

---

10. [9 marks: 2, 2, 3, 2]

Consider the plane Π_1 with equation $\boldsymbol{r} \cdot < 1, 0, 1 > = 2$ and the line L with equation $\boldsymbol{r} = < 3, 1, -1 > + \lambda < -2, 1, 2 >$. Plane Π_2 has equation $\boldsymbol{r} \cdot < 2, 2, 1 > = \rho$.

(a) Show that the line L lies on the plane Π_1.

(b) Find ρ if the line L also lies on the plane Π_2.

(c) For the value of ρ in part (b), find the vector equation of the line of intersection between the two planes.

Calculator Assumed

10\. (d) For the value of ρ in part (b), find the acute angle between the planes Π_1 and Π_2.

11\. [9 marks: 1, 1, 2, 2, 3]

The points A and B have position vectors $< 1, -2, 1 >$ and $< 0, 0, 1 >$ respectively. The plane Π has equation $\boldsymbol{r} \bullet < 2, -1, 1 > = 1$.

(a) Find a unit vector perpendicular to the plane Π.

(b) Show that the point B lies on the plane Π.

(c) Find $|\mathbf{BA}|$.

(d) Find the acute angle between **BA** and the normal to the plane Π.

(e) Hence, find the minimum distance between the point A and the plane Π.

Calculator Assumed

12. [4 marks]

Find the value(s) of m if the line $\boldsymbol{r} = < 2 + m\lambda, -3, 1 + \lambda >$ is inclined to the plane $\boldsymbol{r} \bullet < 0, -1, 1 > = 10$ at an angle of 30°.

---

13. [6 marks: 1, 1, 1, 1, 2]

The planes Π_1 and Π_2 have equations $\boldsymbol{r} \bullet < 1, -1, 2 > = 5$ and $\boldsymbol{r} \bullet < 1, -1, 2 > = 8$ respectively.

(a) Show that the two given planes are parallel.

(b) Find a unit vector normal to these two given planes.

(c) The points A and B with position vectors $< 0, -5, 0 >$ and $< 0, -8, 0 >$ lie on Π_1 and Π_2 respectively. Find $|\mathbf{AB}|$.

(d) Find the acute angle between **AB** and the normal to the two given planes.

(e) Hence, find the distance between the two given planes.

Calculator Assumed

14. [11 marks: 2, 3, 3, 3]

The plane P has equation $\boldsymbol{r} \bullet (2\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}) = -2$.
The line L has equation $\dfrac{x}{-1} = \dfrac{y-2}{1} = \dfrac{z+1}{2}$.

(a) Find the vector equation for line L

(b) Show that the line L does not intersect the plane P

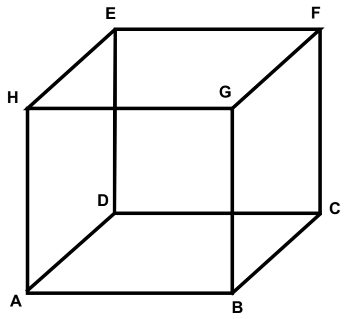
(c) Find the Cartesian equation of the plane Q containing line L and parallel to P.

(d) Determine with reasons if planes P and Q are on the same side or on opposite sides of the origin (0, 0, 0).

Calculator Assumed

15. [13 marks: 4, 2, 2, 3, 2]

ABCDEFGH is a rectangular prism. The vertices A, B, C and G have position vectors < 2, 2, 2 >, < 1, 1, 1 >, < 2, 1, 0 > and < 0, 3, 0 > respectively.



(a) Find the position vectors of the vertices D and E.

(b) Find the vector equation of the edge AB.

(c) Find the vector equation of the plane ABCD in the form $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$.

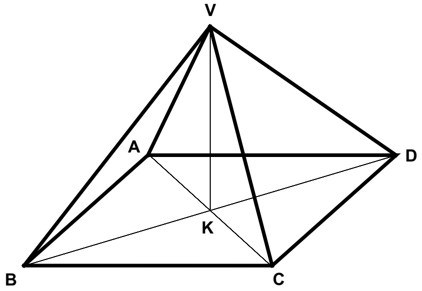
(d) Use vectors to find the acute angle between the planes ADFG and ABCD.

(e) Find the volume of the rectangular prism.

Calculator Assumed

16. [11 marks: 2, 2, 2, 3, 2] [TISC]

The accompanying diagram shows a rectangular pyramid VABCD. The vertices B, C and D have position vectors **i** + **j** + **k**, 3**j** + 3**k** and 2**i** + 2**j** + 5**k** respectively. The vertex V with position vector $-\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 4\mathbf{k}$ is vertically above K, the centre of the rectangular base ABCD.



(a) Use vectors to find the position vector of the vertex A.

(b) Find **KV**.

(c) Find the vector equation of the line passing through C and V.

Calculator Assumed

16. (d) Find the vector equation of the plane ABCD in the form $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$.

(e) Find the acute angle between the line VC and the plane ABCD.

---

17. [10 marks: 4, 4, 2]

A tetrahedron has vertices V, A, B and C with position vectors < –1, –1, 9 >, < –2, 2, 4 >, < 0, –4, 4 > and < 0, –1, 2 > respectively.

(a) Find the area of the base ABC.

(b) Find the shortest distance between the vertex V and the base ABC.

(c) Hence, determine the volume of the tetrahedron VABC.

Calculator Assumed

18. [11 marks: 5, 6]

An object A is acted on by three forces $12\boldsymbol{i} + 10\boldsymbol{j} + 5\boldsymbol{k}$ N, $-5\boldsymbol{i} + 8\boldsymbol{j} - 10\boldsymbol{k}$ N and $6\boldsymbol{i} - 8\boldsymbol{j} + 4\boldsymbol{k}$ N.

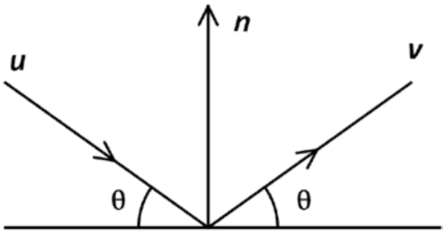
(a) Determine the magnitude of the resultant force and the acute angle it makes with the x-y plane.

(b) A fourth force $\boldsymbol{F}$ of magnitude $\sqrt{306}$ N acts on A so that the resultant now acts in the same direction as the vector $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. Determine $\boldsymbol{F}$.

Calculator Assumed

19. [11 marks: 2, 2, 2, 2, 3]

A ball travelling with velocity ***u*** = < 2, 1, –2 > ms^{-1}, hits the surface of a plane P with equation ***r*** • < 2, –2, –1 > = 10 at an angle of θ° with the plane. It rebounds with velocity ***v*** at an angle of θ° with the plane. ***u***, ***v*** and ***n*** the normal to the plane P all lie on the same plane.



(a) Determine the unit vector normal to the plane.

(b) Determine the exact value for cos(90° – θ°).

(c) Determine ***a***, the component of ***u*** is perpendicular to the plane P.

(d) Determine ***b***, the component of ***u*** that is parallel to the plane P.

(e) Determine ***v*** if $|\boldsymbol{v}| = 0.9|\boldsymbol{u}|$.

12 Vectors V (Spheres)

Calculator Free

1. [10 marks: 2, 2, 2, 4]

The sphere S has centre (1, –1, 2) and radius 5. The point P with coordinates (5, 2, 2) lies on this sphere.

(a) Determine the vector and Cartesian equations of this sphere.

(b) PQ is a diameter of this sphere, determine the coordinates of Q.

(c) The sphere intersects the y-z plane in the form of a circle. Determine the Cartesian equation of this circle.

(d) Determine the value(s) of k if the point with position vector $< 1, 3, k >$ is outside this sphere.

Calculator Assumed

2. [8 marks: 4, 4]

The sphere S has Cartesian equation $9x^2 + 9y^2 + 9z^2 - 6x + 54y + 18z + 10 = 0$.

(a) Determine the position vector of the centre of this sphere and its radius.

(b) Determine with reasons if the plane with equation $\boldsymbol{r} \bullet < 2, 1, 2 > = 2$ will intersect this sphere.

---

3. [10 marks: 4, 6]

The line with equation $x - 1 = \dfrac{y-2}{2} = 1 - z$ intersects the sphere with equation $x^2 + y^2 + z^2 + 2x + 4y - 2z - 30 = 0$ at the points A and B.

(a) Determine the vector equations of the line and the sphere.

Calculator Assumed

3. (b) Hence, or otherwise, determine the exact length of the chord AB.

---

4. [9 marks: 5, 4]

(a) The point A has position vector < 2, 1, –2 >.
The plane P has equation $\boldsymbol{r} \bullet < 1, 2, 2 > = 1$.
The sphere S has radius 6. The plane P is tangential to the sphere S at the point A. Determine the vector equation of the sphere S.

Calculator Assumed

4. (b) The line L has equation $\frac{x-1}{2} = \frac{y+1}{-1} = z-2 = \lambda$.

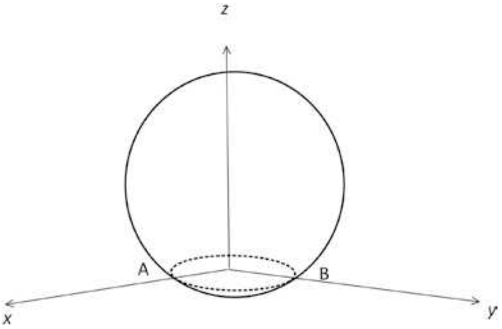
A sphere S has equation $(x-1)^2 + (y+1)^2 + (z-2)^2 = 6$

Determine the point(s) of intersection between the line L and the sphere S.

---

5. [6 marks: 1, 2, 3]

The sphere shown in the given diagram intersects the x-y plane. The circle of intersection cuts the x-axis and y-axis at the points A and B respectively and has vector equation $|\boldsymbol{r}| = 3$.



(a) Express the equation of the circle of intersection in Cartesian form.

(b) Explain clearly why C, the centre of the sphere, must lie on the z-axis.

(c) The sphere has a radius of 5 units. Determine a possible vector equation of the sphere.

Calculator Assumed

6. [8 marks: 4, 4]

Consider the sphere S with equation $\left| \boldsymbol{r} - <1, 1, 1> \right| = 3$.

(a) Prove that the line L with equation $\boldsymbol{r} = <3, -1, 2> + \lambda <1, 1, 0>$ is a tangent to the sphere S.

(b) Prove that the plane with equation $\boldsymbol{r} \bullet <2, 2, 1> = 14$ is tangential to S.

Calculator Assumed

7. [7 marks: 4, 3]

Consider the sphere S with equation $\left| \boldsymbol{r} - <1, 0, 1> \right| = 2$

(a) Determine the equation of the curve of intersection of S with the *x*-*y* plane. Describe this curve.

(b) Determine the equation of the curve of intersection of S with the sphere T with equation $\left| \boldsymbol{r} - <1, 0, 2> \right| = \sqrt{3}$. Describe this curve.

13 Vectors VI (Vector Functions)

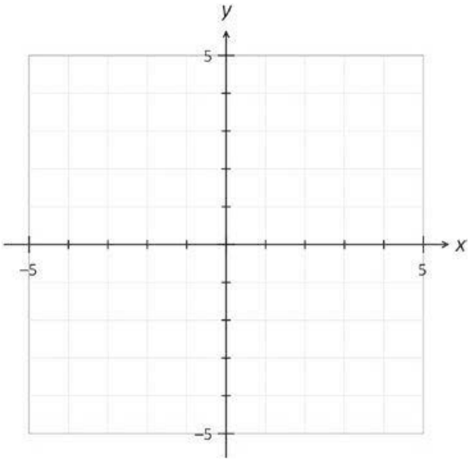
Calculator Free

1. [9 marks: 3, 4, 2]

The position vector of a particle P at time t seconds is given by $\boldsymbol{r}(t) = < 1 + 2\sin(\pi t), 2 - 3\cos(\pi t) >$ cm where $0 \le t \le 2$.

(a) Determine the Cartesian equation of the path of P.

(b) On the axes provided, sketch the graph of the path of P. Show the direction of motion of P.



(c) Determine the shortest distance the particle gets to point with position vector $< 1, 2 >$.

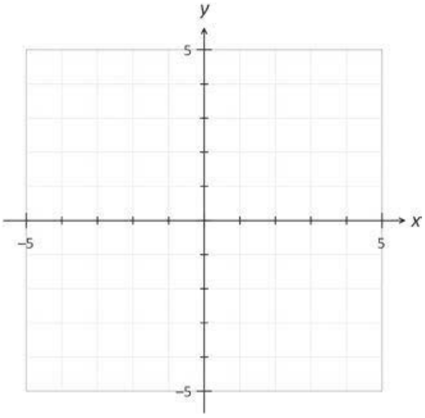
Calculator Free

2. [6 marks: 4, 2]

The position vector of a particle P at time t seconds is given by
$\boldsymbol{r}(t) = < 1 + \tan^2 t, 2 - \sec^2 t >$ cm where $t \geq 0$.

(a) Determine the Cartesian equation of the path of P.

(b) On the axes provided, sketch the graph of the path of P.



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3. [5 marks: 2, 3] [TISC]

Object P leaves point A with position vector <–4, 8 > km at 1200 hours. P travels with a constant velocity $\boldsymbol{v}$ and its position vector at 1400 hours is < 10, – 8 > km.

(a) Show that $\boldsymbol{v} = 7\,\mathbf{i} - 4\,\mathbf{j} - 4\,\mathbf{k}$.

(b) What time will P cross the line with equation $\boldsymbol{r} = \mathbf{i} + 6\,\mathbf{j} + 4\,\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 4\,\mathbf{k})$?

Calculator Assumed

4. [11 marks: 2, 3, 3, 3]

An eagle flies with a constant velocity of < 2, 4, 1 > metres per minute.
At 7 am, the eagle flies past the point A with position vector < 10, –10, 2 > . Find:

(a) the position vector of the eagle at 7.15 am (same day)

(b) when the eagle is 50 m from A

(c) when and where the eagle crosses the x-z plane.

(d) using a vector method, the closest distance between the eagle and the point B with position vector < 20, 5, 8 > m.

Calculator Assumed

5. [5 marks: 1, 1, 3] [TISC]

Particle P leaves the point with position vector 2 **i** – 3 **j** +5 **k** km at 0700 hours. P travels with a constant velocity of 10 **i** – 10 **j** – 5 **k** km per hour.

(a) Find the speed of P.

(b) Find the position vector of the location of P at 0900 hours.

(c) Another particle Q leaves the point with position vector –20 **i** +10 **j** – 30 **k** km at 0900 hours. Q travels with a constant velocity of 30 **i** – 30 **j** – 15 **k**. Determine with reasons if P and Q will collide.

---

6. [5 marks]

The position vectors of particles A and B at time t seconds are respectively

$\boldsymbol{r}_\text{A} = \begin{pmatrix} 2+t \\ 3-t \\ -2+4t \end{pmatrix}$ and $\boldsymbol{r}_\text{B} = \begin{pmatrix} -4+3t \\ 4-2t \\ -1+7t \end{pmatrix}$. Find when and where A crosses the path of B.

Calculator Assumed

7. [11 marks: 2, 3, 3, 3]

At 0900 hours, the position and velocity vectors of particles P, Q and R are respectively $< 0, -1, -2 >$ m , $< 1, 2, 2 >$ ms^{-1}; $< 4, 1, 6 >$ m , $< -1, \alpha, -2 >$ ms^{-1} and $< -4, 1, -2 >$ m and $< 3, 1, \beta >$ ms^{-1}.

(a) Find the position vectors of P, Q and R, *t* seconds after 0900 hours.

(b) If these velocities were maintained, all three particles will collide simultaneously. Use vector methods to find the value(s) of α and β.

(c) If these velocities were maintained, only P and Q will collide. Use vector methods to find the value(s) of α and β.

(d) If these velocities were maintained, only two of the three particles will collide. Use vector methods to find the value(s) of α and β.

Calculator Assumed

8. [13 marks: 3, 4, 2, 4]

The position vectors of objects P and Q at time t seconds is given by
$\mathbf{OP}(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ metres and $\mathbf{OQ}(t) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ metres respectively.

(a) Show that P and Q do not collide.

(b) Calculate the closest distance between P and Q.
Show how you obtained your answer.

(c) State the vector equations of the paths of P and Q.

Calculator Assumed

8. (d) Calculate the closest distance between the paths of P and Q.

9. [10 marks: 5, 5]

Particle A leaves the point < 1, a, 2 > m with constant velocity < –1, 2, 5 > ms^{-1}. Simultaneously, particle B leaves the point < –3, 1, 10 > m with constant velocity < 1, 1, u > ms^{-1}. a and u are real constants.

(a) For a = –1, determine the value(s) of u for which the particles do not collide.

(b) For a = –4, determine the value(s) of u for which the paths of A and B will intersect.

Calculator Assumed

10. [11 marks, 3, 4, 4]

Particles A and B start moving at 0800 hours. The position vectors of A and B t hours after 0800 hours are respectively $\boldsymbol{r}_{\text{A}} = \begin{pmatrix} 10+4t \\ -20+2t \\ 4-t \end{pmatrix}$ and $\boldsymbol{r}_{\text{B}} = \begin{pmatrix} 20+t \\ 20-3t \\ -11+t \end{pmatrix}$.

(a) Prove that A and B will not collide.

(b) The paths traced by A and B respectively meet at the point P.
Determine the position vector of the point P.

(c) Determine with reasons, which of the two particles A or B is the first to reach P. State the exact distance between the two particles at the time the first particle reaches P.

Calculator Assumed

11. [8 marks: 5, 3]

A drone located at P is to be flown to Q. The position vector of Q relative to P is $< 100, 80, 40 >$ m. A wind is blowing with constant velocity $< 0.8, 1.3, 0.4 >$ ms^{-1}.

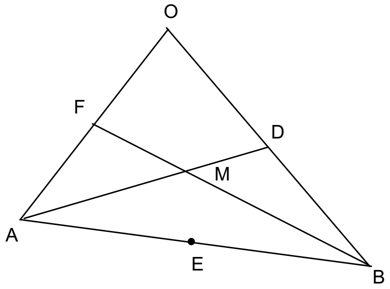
(a) Determine $\boldsymbol{v}$ the velocity of the drone if it is flown from P to Q at its maximum speed of 1.3 ms^{-1}.

(b) Comment briefly on the significance of your answer in (a).

14 Geometric Proofs using Vectors

Calculator Assumed

1. [13 marks: 2, 2, 5, 3, 1]



OAB is a triangle with **OA** = ***a*** and **OB** = ***b***. D, E and F are the midpoints of OB, AB, and OA respectively. **AM** = α**AD** and **MF** = β**BF**.

(a) Find **AD** and **BF** in terms of ***a*** and ***b***.

(b) Find **AM** and **MF** in terms of ***a*** and ***b***.

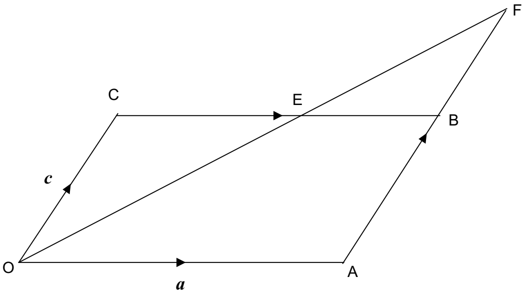
(c) Use your answers in (b) to find α and β.

(d) Show that **OM** = μ**OE** giving the value of μ .

(e) Comment on the significance of the location of M in terms of the lines OE, AD and BF.

Calculator Assumed

2. [8 marks]



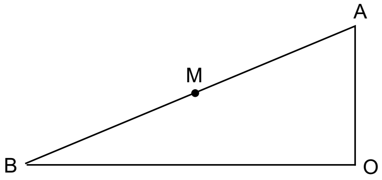
OABC is a parallelogram. The point E divides CB in the ratio α : β. That is, the point E is such that **EB** = $\frac{\beta}{\alpha+\beta}$**CB**. OE extended meets the AB extended at F. Use vector ethods to prove that:

Area of ΔFEB = $\left(\frac{\beta}{\alpha+\beta}\right)^2$ × Area of ΔFOA.

[Hint: Let **EF** = λ**OF** and **BF** = μ**AF**.]

Calculator Assumed

3. [8 marks: 1, 3, 4]



OAB is a right angled triangle with ∠ AOB = 90°.
OA = $\boldsymbol{a}$ and **OB** = $\boldsymbol{b}$. M is the midpoint of AB.

(a) Explain why $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

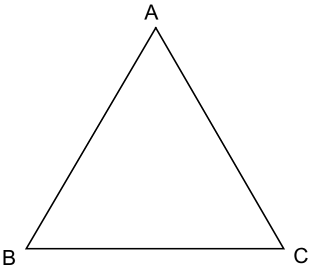
(b) Find $|\mathbf{BM}|^2$ in terms of a and b, where $|\boldsymbol{a}| = a$ and $|\boldsymbol{b}| = b$.

(c) Hence, prove that M is the centre of a circle passing through A, B and O.

Calculator Assumed

4. [12 marks: 2, 3, 4, 3]

ABC is an isosceles triangle with AB = AC.
Also **BA** = $\boldsymbol{a}$ and **CB** = $\boldsymbol{b}$.



(a) Show that $|\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a}|$

(b) Show that $|\boldsymbol{b}|^2 = -2\boldsymbol{a} \cdot \boldsymbol{b}$.

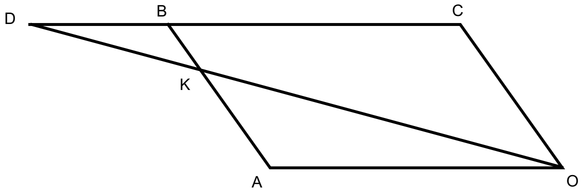
(c) Show that $\cos C = \dfrac{-\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$.

(d) Hence, using a vector method, prove that the base angles of an isosceles triangle are equal.

Calculator Assumed

5. [5 marks: 2, 3] [TISC]

OABC is a parallelogram with **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. The point K divides AB in the ratio 2 : 1. OK extended meets the line CB extended at D. **OK** = α**OD** and **CD** = β**CB**.



(a) Find **AK** and **OK** in terms of **a** and **c**.

(b) Use your answer in (a) to prove that B divides the line CD in the ratio 2 : 1.

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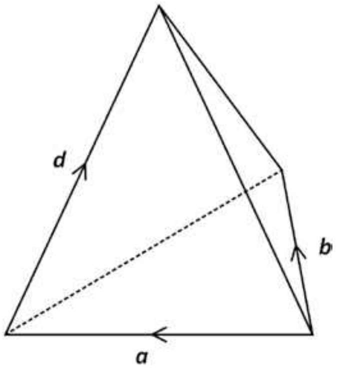
6. [7 marks: 2, 5]

The accompanying diagram shows a tetrahedron. Two of the adjacent edges of the tetrahedron are represented by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. A third edge is represented by the vector $\boldsymbol{d}$.

(a) Prove that the area of the face bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $A = \frac{1}{2}\,|\boldsymbol{a} \times \boldsymbol{b}|$.



Calculator Assumed

6. (b) Prove that the volume of the tetrahedron is given by V = $\frac{1}{6}\,|\boldsymbol{d} \bullet (\boldsymbol{a} \times \boldsymbol{b})|$.

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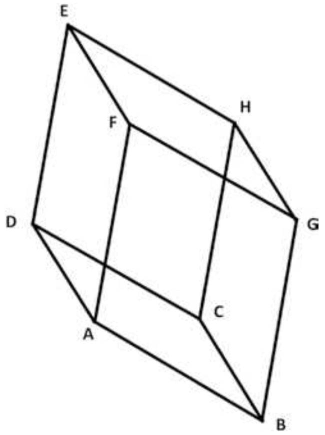
7. [5 marks]

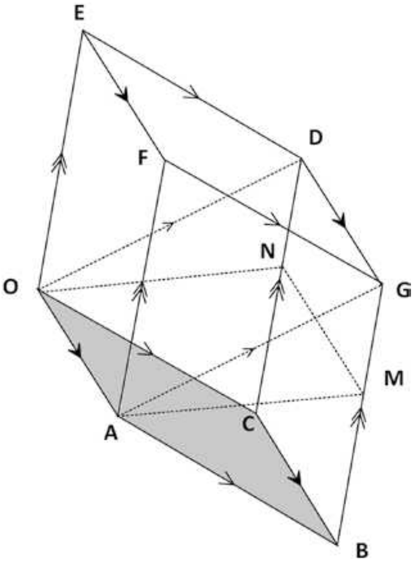
The accompanying diagram shows a parallelepiped. The equations of the faces ABCD and EFGH are respectively $x + 7y - 5z = 23$ and $x + 7y - 5z = 38$. Prove that the perpendicular distance between the faces ABCD and EFGH is $\sqrt{3}$.



Calculator Assumed

8. [7 marks: 3, 4]

The accompanying diagram shows a parallelepiped OABCDEFG. The opposite faces of the solid are congruent parallelograms which are parallel to each other. **OA** = $\boldsymbol{a}$, **OC** = $\boldsymbol{c}$ and **OE** = $\boldsymbol{e}$. The point M is such that **BM** = α**BG**. The point N is such that **CN** = α**CD**.



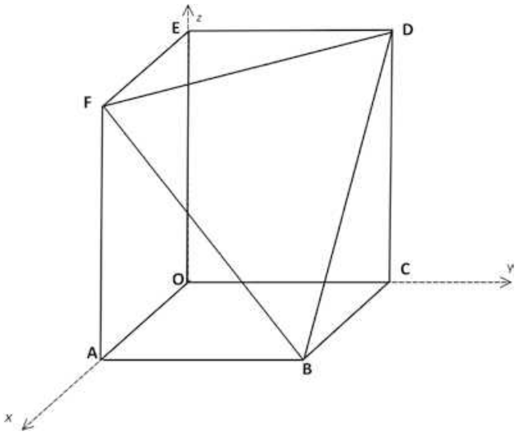
(a) Prove that **NM** = $\boldsymbol{a}$.

(b) The volume of parallelepiped OABCDEFG is given by $|(\boldsymbol{a} \times \boldsymbol{c}) \bullet \boldsymbol{e}|$. Use this result to prove that $\alpha = \dfrac{3}{4}$ given that the ratio of the volume of ODGABC to the volume of ONMABC is 4 : 3.

Calculator Assumed

9. [6 marks]

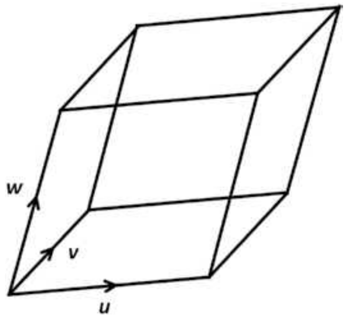
The accompanying diagram shows a rectangular prism with a section removed. O is the origin of the *x*-*y*-*z* axes. The position vectors of vertices A, C and E are ***a***, ***c*** and ***e*** respectively.



Prove that the volume of this solid is given by V = $\frac{5}{6}\,|\boldsymbol{e} \bullet (\boldsymbol{a} \times \boldsymbol{c})|$.

Calculator Assumed

10. [6 marks]



The edges of a parallelepiped are parallel and congruent to the vectors ***u***, ***v*** and ***w*** as shown in the accompanying diagram.
Determine the volume V of the parallelepiped.
Hence, deduce that:

$$|\boldsymbol{w} \bullet (\boldsymbol{u} \times \boldsymbol{v})| = |\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})| = |\boldsymbol{v} \bullet (\boldsymbol{w} \times \boldsymbol{u})|$$

15 Systems of Linear Equations

Calculator Free

1. [9 marks: 1, 2, 2, 4]

   A system of 3 linear equations in x, y and z is reduced to
   $x - y - 2z = 0$ and $pz = (p - 1)(p - 2)$ where p is a constant.

   (a) Determine the nature of the solutions to this system when $p = 3$.

   (b) Find the value(s) of p for which the system has no solutions.

   (c) Find the value(s) of p for which the system has an infinite number of solutions.

   (d) For the value(s) of p for which the system has an infinite number of solutions, determine these solutions.

Calculator Free

2. [5 marks: 4, 1]

Consider the set of equations:
$$\begin{aligned} x + 2y - 3z &= -12 \\ x - y + 2z &= 9 \\ 3x + 3y + kz &= -2 \end{aligned}$$

(a) Show that this system cannot have an infinite number of solutions.

(b) Hence or otherwise, find the value(s) of k for which the system has no solution.

3. [7 marks: 4, 3] [TISC]

Consider the set of equations:
$$\begin{aligned} x + 2y + 2z &= 4 \\ x + 8y &= 4 \\ x - 4y + kz &= k \end{aligned}$$

(a) Show that this set will always have at least one solution.

Calculator Free

3. (b) Find the unique solution to this set of equations.

---

4. [9 marks: 6, 3]

Consider the set of equations:

$$\begin{aligned} x + 2y + 3z &= 5 \\ 3x + 6y + az &= 14 \\ -2x - 4y + bz &= -9 \end{aligned}$$

where a and b are real constants.

(a) Determine if possible, the value(s) of a and b for this set of equations to have at least one solution.

(b) Determine the number of solutions for this set of equations if $a = 8$ and $b = -5$. Justify your answer.

Calculator Free

5. [11 marks: 5, 3, 3]

Consider the set of equations:

$$\begin{aligned} x + 2y + z &= p \\ p^2x + 8y + 4z &= 2 + 3p \\ x + 2y &= 0 \end{aligned}$$

where p is a real constant.

(a) Determine where possible, the value(s) of p and q for this set of equations to have: (i) no solution (ii) more than one solution (iii) a unique solution.

(b) In the case where the system has a unique solution, find x, y and z in terms of p.

(c) In the case where the system has an infinite number of solutions, find x, y and z.

Calculator Free

6. [8 marks: 3, 1, 1, 1, 2]

Consider the equations: $x + y + z = 2$ $2x - y - z = 4$ $-x + py + qz = r$ where p, q and r are real constants.

(a) Show that $z = \dfrac{r+2}{q-p}$.

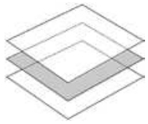
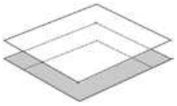
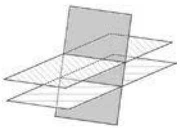
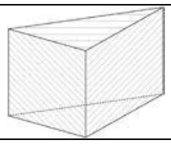
(b) Determine the values of p, q and r so that this system has:

(i) a unique set of solutions.

(ii) has no solutions.

(iii) inifinite number of solutions.

(c) Use your answers in part (b) to determine which of the following sketches describes the geometrical situation when the system has no solutions.

Sketch A	Sketch B	Sketch C	Sketch D
			

Calculator Free

7. [8 marks: 4, 2, 2]

Planes A, B and C respectively have equations:

$x + y - z = -2$ $x - y + z = 6$ $2x + y + pz = q$

where p and q are real constants.

(a) Determine the values of p and q if these three planes meet at more than one point.

(b) For the case where these planes meet more than once, provide two sets of *positive integer* solutions.

(c) For the case where these planes meet more than once, draw a sketch of the relative positions of these planes.

Calculator Free

8. [9 marks: 5, 4]

The planes P, Q and R respectively have equations:

$$x + y - z = 1$$
$$x - y - z = 1$$
$$2x - y + pz = q$$

where p and are real constants.

(a) Determine the value(s) of p and q if these planes form the lateral faces of a triangular prism.

(b) Determine the value(s) of p and q if these planes meet along a common line. Determine the vector equation of this line.

16 Differentiation

Calculator Free

1. [12 marks: 3, 3, 3, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = 5^x$

(b) $y = \sin^5(1 - \sqrt{x})$

(c) $y = \int_0^{e^{2x}} ln(1-x^2)\ dx$

(d) $y = \dfrac{1-x}{\tan 3x}$

Calculator Free

2. [10 marks: 1, 3, 3, 3]

Find $\frac{dy}{dt}$ for each of the following. You do not need to simplify your answer.

(a) $y = \pi^3$

(b) $y = \tan^3(\pi t^4)$

(c) $y = \int_1^{t^3} \sin^5 2x \, \mathrm{d}x + t \int_0^1 5 \, dt$

(d) $y = \dfrac{\cos(2 - e^{2t})}{t}$

Calculator Free

3. [7 marks: 1, 2, 2, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \tan(60^{\circ})$

(b) $y = \tan(1 - \sqrt{x})$

(c) $y = \int_{0}^{\pi x} 1 + \cos^{4}(t)\, dt$

(d) $y = x^{2}\, ln\,(\sin 2x)$

Calculator Free

4. [9 marks: 1, 3, 2, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = ln\ e^{2x}$

(b) $y = \cos^3 (3 + \frac{1}{x})$

(c) $y = \int_0^{\tan x} e^{1+t^2}\ dt$

(d) $y = e^{\sin x} \cos x$

Calculator Free

5. [9 marks: 1, 3, 2, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = ln\ 2^x$

(b) $y = sin^5\ (1 + ln\ x\ )$

(c) $y = \int_0^{x^2} \tan(1+2t)\ dt$

(d) $y = (1 + x^2)\ ln\ \sqrt{x+1}$

Calculator Free

6. [12 marks: 2, 3, 4, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \sqrt{3x}$

(b) $y = e^{\tan(1-2x)}$

(c) $y = \sin^2(2x)\cos^3(1-x)$

(d) $y = \frac{\sin(2x)}{ln\cos(3x)}$

Calculator Free

7. [10 marks: 1, 3, 3, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \frac{1}{(1+e)^2}$

(b) $y = \frac{ln(1+\sin x)}{x}$

(c) $y = e^{\tan(\frac{\pi x}{4})}\cos(\frac{\pi x}{4})$

(d) $y = \frac{\sin^2(\pi x)}{\cos(1+x)}$

Calculator Assumed

8. [10 marks: 3, 4, 3]

Find $\frac{dy}{dx}$ in terms of x, for each of the following.

(a) $x = t^2$ and $y = e^{t^3}$

(b) $x = \cos 2\theta$ and $y = \sin 2\theta$

(c) $x = 1 + t$ and $y = \frac{1-t}{1+t}$

Calculator Assumed

9. [11 marks: 3, 4, 4]

Find $\frac{dy}{dx}$ in terms of x, for each of the following.

(a) $x = t^2$ and $y = \ln(1 - t)$

(b) $x = 1 + \cos\theta$ and $y = 2 - \sin\theta$

(c) $x = \frac{1-t^2}{1+t^2}$ and $y = 1 + t$

Calculator Assumed

10. [11 marks: 3, 4, 4]

Find $\frac{dy}{dx}$ in terms of x, for each of the following.

(a) $x = e^{2t}$ and $y = ln\,(1 + t)$

(b) $x = 1 - 3\sin\theta$ and $y = 3 + 4\cos\theta$

(c) $x = \frac{1-2t}{1+2t}$ and $y = \frac{t^2}{1+2t}$.

17 Implicit Differentiation

Calculator Free

1. [10 marks: 3, 3, 4] [TISC]

A curve has equation given by $x^2 + xy = y^2 - 5$.

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(b) The tangent to the curve at the point P (p, q) is parallel to the line $y = x + 5$. Show that $q = 3p$.

(c) Hence, or otherwise, find the coordinates of the point(s) on the curve with gradient 1.

---

2. [11 marks: 5, 6]

(a) Given $\left(1+xy^2\right)^2 + \cos(x + y) = 0$, find $\frac{dy}{dx}$.

Calculator Free

2. (b) Consider the curve with equation $2y^3 - 3y^2 - 3x^2 - 12x = 9$. Find the equation of the tangent to the curve that is parallel to the y-axis.

---

3. [7 marks: 4, 3] [TISC]

(a) Given that $y = \sqrt{\dfrac{1+2x}{1+x^2}}$, use *logarithmic differentiation* to find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

(b) Find the equation of the line which is *perpendicular to the tangent* to this curve at the point (2, 1).

Calculator Free

4. [7 marks: 4, 3]

(a) Given that $y = \sqrt{\dfrac{1+\cos(x)}{1-\sin(x)}}$, use *logarithmic differentiation* to find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

(b) Find the equation of the tangent to this curve at the point where $x = 0$.

---

5. [8 marks: 4, 4]

A curve has equation $3x^2 + 3y^2 + 2xy - 6x - 2y = 1$.

(a) Determine the equation of the tangent to this curve at the point (0, 1).

Calculator Free

5. (b) Determine the x-coordinate of the point(s) on the curve with tangents that are parallel to the line with equation $y = -x$.

---

6. [8 marks: 4, 4] [TISC]

A curve has equation $\sin(xy) = -\cos(x)$ for $0 \le x \le \dfrac{\pi}{2}$ and $-\dfrac{\pi}{2} \le y \le 0$.

(a) Find an expression for $\dfrac{dy}{dx}$.

(b) Show that the tangent to this curve at the point where $y = 0$ has equation $y = \dfrac{2}{\pi}x - 1$.

Calculator Assumed

7. [8 marks: 3, 2, 3] [TISC]

A curve has equation $\sqrt{x+y} = x$.

(a) Find an expression for $\frac{dy}{dx}$.

(b) Show that the tangent to this curve at the point (2, 2) is $3x - y = 4$.

(c) Find the point(s) (a, b) on the curve, where a and b are <u>*integers*</u>, such that the gradient of the curve is 1. Justify your answer.

---

8. [7 marks: 4, 3] [TISC]

(a) Consider the curve with equation $\ln(y + 1) = x\,y$, determine $\frac{dy}{dx}$.

Calculator Assumed

8. (b) Find the equation of the line passing through the point (1, 1) and parallel to the tangent to this curve at the point ($ln\, 2$, 1).

---

9. [7 marks]

Consider the curve with equation $x^2 + xy + y^2 = \frac{75}{4}$. Determine the point of intersection of the tangents to this curve at the points where $x = \frac{5}{2}$.

Calculator Assumed

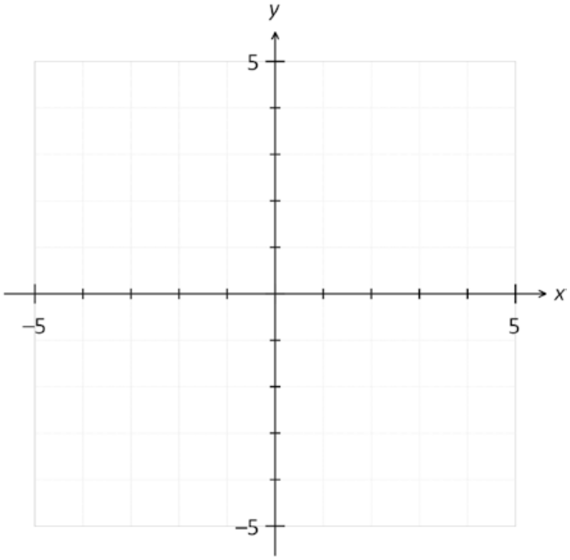
10. [11 marks: 4, 3, 4]

Consider the curve with equation $y^2 = \dfrac{x^4}{x^2 - 1}$.

(a) Determine $\lim\limits_{x \to \pm\infty} y$. Hence, find the equation of the oblique asymptotes.

(b) Use differentiation to verify that $(\sqrt{2}\,, 2)$ is a minimum point on this curve.

(c) Sketch the curve $y^2 = \dfrac{x^4}{x^2 - 1}$.

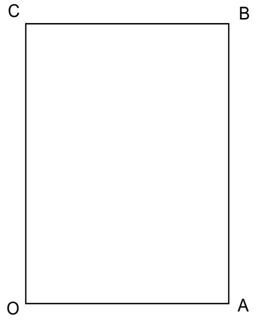


18 Related Rates

Calculator Assumed

1. [5 marks:]

Rectangle OABC has a constant area 100 mm^2. The lengths of the two parallel sides OA and CB increases at a constant rate of 1 mm per second. The lengths of the other two parallel sides OC and AB reduces at a constant rate. These changes occur in such a way that the shape of OABC remains a rectangle at all times. Find the rate of change of the perimeter of OABC when the length of the OA is 20 mm.



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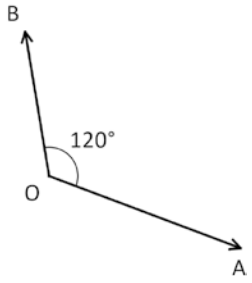
2. [5 marks]

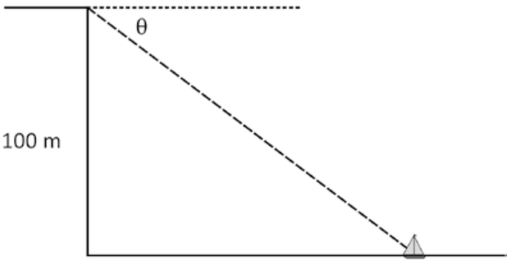
Two straight roads OA and OB, are inclined at an angle of 120^o to each other. At time $t = 0$, cyclist P is at a point on OA 50 metres from O while cyclist Q is at O. P moves away from O, along the road OA at a speed of 10 ms^{-1}. Q moves away from O, along the road OB at a speed of 12 ms^{-1}. Calculate the rate at which the distance between P and Q is changing after 5 seconds.



Calculator Assumed

3. [6 marks] [TISC]

From the top of a 100 m high vertical cliff, the angle of depression of a yacht is θ radians. The yacht is moving away from the cliff at a speed of 10 metres per minute. Find exactly the rate of change of the angle of depression when the yacht is 200 m away from the base of the cliff.



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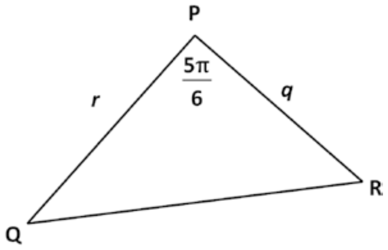
4. [8 marks: 4, 4]

Triangle PQR is such that PR = q and PQ = r. Angle QPR is a fixed angle and is of size $\frac{5\pi}{6}$ radians. The sides PQ and PR vary with time t. Let A be the area of triangle PQR.



(a) Find an expression for $\frac{dA}{dt}$.

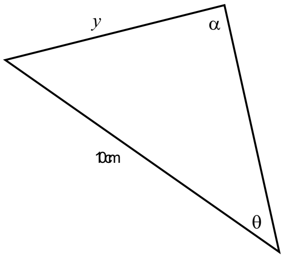
Calculator Assumed

4. (b) Given that when PQ = 5 cm , PQ is decreasing at a rate of 0.1 cm s^{-1}, and when PR = 8 cm, PR is increasing at a rate of 0.2 cm s^{-1}, find the rate at which the area of triangle PQR is changing.

---

5. [6 marks] [TISC]

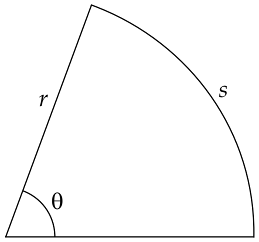
In the triangle shown, the angles α and θ vary with time t minutes. The angle α increases at the constant rate of 0.1 radians per minute. The angle θ decreases at a constant rate of 0.1 radians per minute. Find the rate of change of y with respect to time when $\alpha = \theta = \frac{\pi}{4}$.



Calculator Assumed

6. [6 marks] [TISC]

The given diagram shows a circular sector of radius r cm and arc length s cm with a central angle of θ radians. Find the rate of change in the area at the instant in time when the radius is 5 cm and decreasing at a rate of 1 cm per second and the central angle is 1.2 radians and increasing at a rate of 0.01 radians per second.



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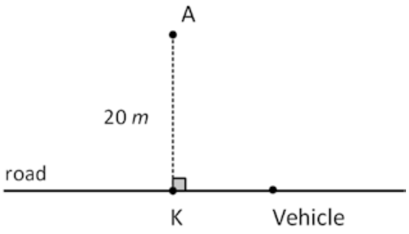
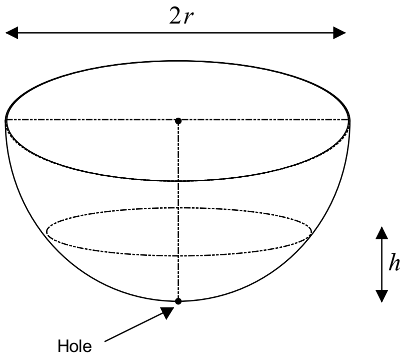
7. [6 marks]

A surveillance camera is located at A, 20 *m* from a road. The point K on the road is directly opposite to A. A vehicle travels along the road at a speed of 10 ms^{-1}. As the car travels along the road, the camera turns and follows the car. How fast is the camera turning as the car is moving away from A and the car is 40 *m* from the camera?



Calculator Assumed

8. [10 marks: 4, 6] [TISC]



The given diagram shows a hemispherical bowl with diameter $2r$ cm. Water leaks out of the bowl through a hole at the bottom of the bowl. The depth of the water at any time t seconds is h cm and the volume of water in the bowl at any time t seconds is given by $V = \pi\left(rh^2 - \frac{h^3}{3}\right)$ cm^3. The rate at which water leaks out from the hole is given by $\frac{dV}{dt} = k\sqrt{h}$ where k is a constant.

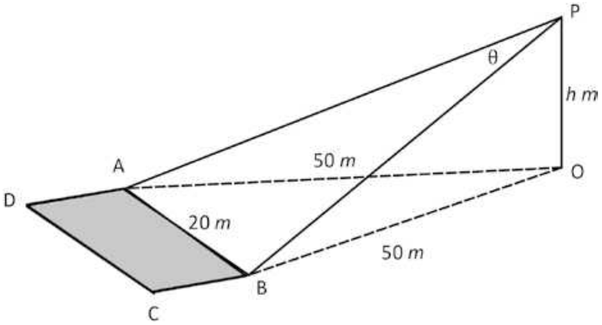
(a) Show that $\frac{dh}{dt} = \frac{k}{\pi(2rh^{\frac{1}{2}} - h^{\frac{3}{2}})}$

(b) Use calculus methods to find h in terms of r when the rate at which the water level falls is a minimum.

Calculator Assumed

9. [7 marks: 3, 4] [TISC]

ABCD is a rectangular garden patch on level ground, with AB = 20 m. A cameraman is located at P which is h metres above the ground. The point O is vertically below P. The cameraman is moved vertically upwards at a rate of 0.5 metres per second. The edge AB subtends an angle of θ radians at P.



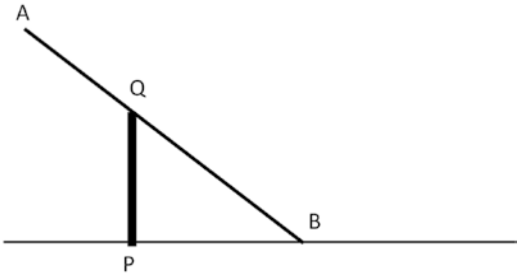
(a) Use the cosine rule to show that $(2h^2 + 5000)(1 - \cos\theta) = 400$.

(b) Find the rate at which θ changes when h = 20 m.

Calculator Assumed

10. [6 marks]

The diagram below shows a 10 m long pole AB resting between the top of a 2 m high wall PQ and the ground. The bottom of the pole B is being pushed towards P the foot of the wall at a constant rate of 0.1 ms^{-1}. The pole is always in contact with Q the top of the wall. Calculate how fast A, the free end of the pole, is moving vertically when B is 2 m away from P.



Calculator Assumed

11. [8 marks: 5, 3]

Figure 1 below shows a container in the shape of a frustum of a cone.
This shape is formed when an inverted cone of height 20 cm and base diameter 20 cm is truncated parallel to the base of the cone at a height 10 cm above the vertex of the cone (see Figure 2). The frustum is of height 10 cm as shown in Figure 1. Water is poured into the container (Figure 1) at a rate of 10 cm^3 per minute.

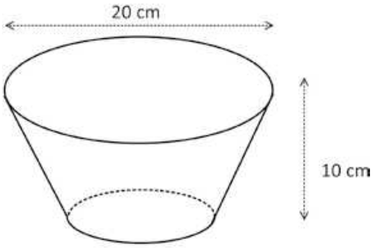


Figure 1

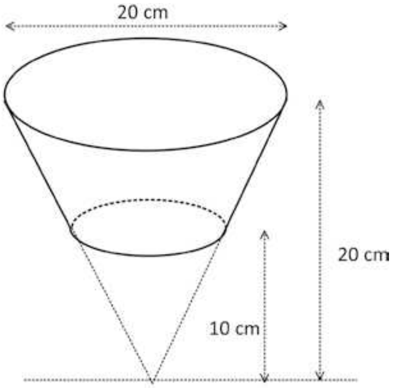


Figure 2

(a) Show that when the water level is h cm from the base of the container (Figure 1), the volume of the water in the container is given by

$$V = \frac{\pi(10+h)^3}{12} - \frac{250\pi}{3}$$

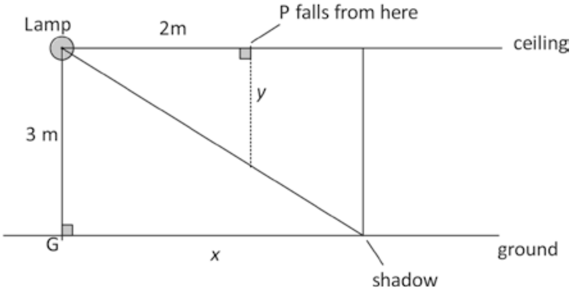
Calculator Assumed

11. (b) Calculate the *exact* rate of increase of the height of the water level in the container (Figure 1) when the water level is 2 cm above its base.

---

12. [5 marks] [TISC]

A light lamp is fixed to a ceiling 3 metres vertically above the point G on the ground. An object P is attached 2 metres away from the lamp to the same ceiling. P becomes loose and falls to the ground with constant acceleration. When P has fallen y metres, the shadow of P cast on the ground by the light from the lamp is x metres from G. When P has fallen a distance of 1.225 m, its speed is 4.9 ms^{-1}. Determine the speed and direction with which the shadow is moving at this instant.



19 Integration I

Calculator Free

1. [10 marks: 2, 3, 5]

Find:

(a) $\int \sin 2x \cos 2x \ dx$

(b) $\int \frac{4\sin(x)}{\cos^3(x)\tan(x)} \ dx$

(c) $\int_0^2 \frac{1}{\sqrt{16-x^2}} \ dx$

Calculator Free

2. [15 marks: 2, 3, 4, 6]

Find:

(a) $\int_1^{\pi} \frac{\pi}{4}\, d\theta$

(b) $\int \cos^2(\pi x)\ dx$

(c) $\int \frac{\cos\theta - \cos^3\theta}{\sin^3\theta}\ d\theta$

(d) $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\ dx$

Calculator Free

3. [12 marks: 2, 3, 3, 4]

Find:

(a) $\int \sin\left(\frac{\pi t}{3} - \frac{\pi}{6}\right)\ dt$

(b) $\int \frac{\sin 2x \cos 2x}{\cos^2 2x - \sin^2 2x}\ dx$

(c) $\int \tan 2x + \tan^3 2x\ dx$

(d) Find $\int \frac{-3}{\sqrt{9 - x^2}}\ dx$

Calculator Free

4. [13 marks: 2, 2, 5, 4]

Find:

(a) $\displaystyle\int_0^1 \sin\frac{\pi}{6}\,dx$

(b) $\displaystyle\int \frac{1}{\cos^2(5x)}\,dx$

(c) $\displaystyle\int_{\frac{1}{3}}^{\frac{1}{2}} \sin(\pi x)\cos^2(\pi x)\;dx$

(d) $\displaystyle\int \frac{1-2\sin^2 x}{(\sin x-\cos x)^2}\,dx$

Calculator Free

5. [13 marks: 3, 4, 6]

Determine each of the following integrals.

(a) $\int \sin(\pi x) + \tan^2(x)\ dx$.

(b) $\int x\sqrt{1-x^2}\ dx$ using the <u>substitution</u> $x = \cos t$.

(c) $\int \sqrt{1-4x^2}\ dx$.

Calculator Free

6. [11 marks: 3, 3, 5]

Evaluate the following integrals.

(a) $\int\left(\cos\frac{\pi x}{2}-\sin\frac{\pi x}{2}\right)\left(\cos\frac{\pi x}{2}+\sin\frac{\pi x}{2}\right)\ dx$.

(b) $\int\frac{1+\tan^2(2x)}{2+\tan(2x)}\ dx$.

(c) $\int\ \frac{ln\,x}{x-x\,ln\,x}\ dx$

Calculator Free

7. [12 marks: 2, 4, 6]

(a) Evaluate $\int \frac{\sin x + \cos x}{\cos x}\ dx$.

(b) Evaluate $\int (1 + 2\sin x)^2\ dx$.

(c) Use the substitution $x = \sin \theta$ to determine $\int_0^1 x^2 \sqrt{1 - x^2}\ dx$

Calculator Free

8. [10 marks: 3, 3, 4]

(a) Evaluate $\int_{0}^{\frac{\pi}{8}} \tan^2 2x \ dx$.

(b) Evaluate $\int_{0}^{\frac{\pi}{12}} \sin^2 3x \ dx$.

(c) Use an appropriate substitution to evaluate $\int \frac{2x}{(1+x)^2} \ dx$.

Calculator Free

9. [12 marks: 5, 7]

(a) Determine $\int \sqrt{1+\sqrt{x}}\ dx$.

(b) Determine $\frac{d}{dx}\left[e^{\cos^2(x)}\right]$. Hence, find $\int \sin 2x\left[\cos^2 x + e^{\cos^2(x)}\right]dx$.

Calculator Free

10. [11 marks: 4, 7]

(a) Determine $\int \frac{x}{\sqrt{x-1}}\, dx$

(b) Determine $\frac{d}{dx}\left[e^x \sin(x)\right]$ and $\frac{d}{dx}\left[e^x \cos(x)\right]$. Hence, find $\int e^x \cos(x)\, dx$.

Calculator Free

11. [6 marks: 3, 3]

(a) Given that $y = a^x$, use implicit differentiation to prove that $\frac{dy}{dx} = a^x \ln a$.

(b) Hence, show how the result in (a) may be used to determine $\int 10^x \, dx$

---

12. [10 marks: 5, 5]

The Wallis formula for definite integrals of powers of cos x between $x = 0$ and $x = \frac{\pi}{2}$ is $\int_0^{\frac{\pi}{2}} \cos^n x \; dx = \begin{cases} \dfrac{(2)(4)(6)\ldots(n-3)(n-1)}{(1)(3)(5)\ldots(n-2)(n)} & \text{for } n=2k+1 \\ \dfrac{(1)(3)(5)\ldots(n-3)(n-1)}{(2)(4)(6)\ldots(n-2)(n)} \times \dfrac{\pi}{2} & \text{for } n=2k \end{cases}$.

(a) Use integration to show that Wallis' formula is true for $n = 5$.

Calculator Free

12. (b) Use integration to show that Wallis’ formula is true for $n = 4$.

20 Integration II

Calculator Free

1. [10 marks: 3, 2, 2, 3]

(a) Decompose $\dfrac{1-2x-2x^2}{(x+2)(x^2-1)}$ into its partial fractions.

(b) Determine $\dfrac{d}{dx}\left(\dfrac{1-2x-2x^2}{(x+2)(x^2-1)}\right)$.

(c) Determine $\displaystyle\int \frac{1-2x-2x^2}{(x+2)(x^2-1)}\,dx$.

(d) Evaluate in its simplest form the exact value of $\displaystyle\int_2^3 \frac{1-2x-2x^2}{(x+2)(x^2-1)}\,dx$.

Calculator Free

2. [10 marks: 3, 2, 5]

(a) Find $\int \frac{4x}{4-x}\,dx$.

(b) Find $\int \frac{4x}{4-x^2}\,dx$

(c) Find $\int \frac{x^3+1}{6-x-x^2}\,dx$.

Calculator Free

3. [9 marks: 6, 3]

(a) Determine $\int \frac{x^2+2x-1}{x^3+x^2-x-1}\,dx$

(b) Evaluate $\int_2^3 \frac{3x^2+2x-1}{x^3+x^2-x-1}\,dx$. Simplify your answer.

Calculator Free

4. [10 marks: 6, 4]

(a) Determine $\int \frac{2x^2+x+1}{(x^2+1)(x+1)}\,dx$

(b) Use your answer in (a) to find $\int \frac{4x^2+3x+1}{(x^2+1)(x+1)}\,dx$.

Calculator Free

5. [10 marks: 3, 7]

(a) Use the substitution $x = 2\tan\theta$ to evaluate $\int \frac{1}{x^2+4}\,dx$.

(b) Determine $\int \frac{2x^2+3}{(x^2+4)(x-1)}\,dx$.

Calculator Free

6. [10 marks: 3, 7]

(a) Use the substitution tan θ = x + 1 to evaluate $\int \frac{1}{x^2+2x+2}\,dx$.

(b) Determine $\int \frac{2}{x\,(x^2+2x+2)}\,dx$.

Calculator Free

7. [8 marks: 2, 6]

(a) Given that $t = \tan\theta$, find an expression for $\frac{d\theta}{dt}$ giving your answer in terms of t.

(b) Hence or otherwise determine $\int \frac{2x^2 - x + 9}{(x-1)(x^2+4)}\, dx$.

Calculator Free

8. [9 marks: 3, 2, 2, 2]

Consider $f(x) = \dfrac{2+x}{(x+1)^2}$.

(a) Determine $\int f(x)\,dx$.

Evaluate where possible each of the following integrals. If the integral cannot be evaluated, explain why it cannot be evaluated.

(b) $\displaystyle\int_0^2 f(x)\,dx$

(c) $\displaystyle\int_{-2}^0 f(x)\,dx$

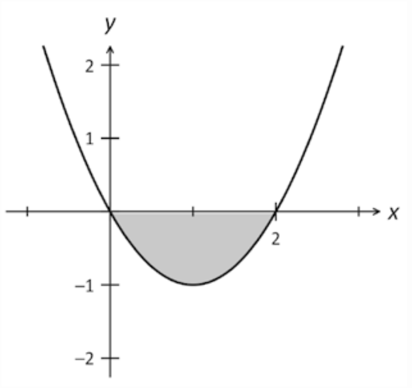
(d) $\displaystyle\int_{-2}^0 f(-x)\,dx$

21 Numerical Integration

Calculator Assumed

1. [9 marks: 4, 2, 3]

The shaded region in the accompanying diagram is trapped between the curve $y = f(x)$ where $f(x) = x(x - 2)$ and the x-axis. The area of this region is to be estimated using 50 trapeziums of uniform width.



(a) Calculate the area of the second trapezium, showing all working.

(b) Write an expression for the area of the nth trapezium (for $1 \le n \le 50$).

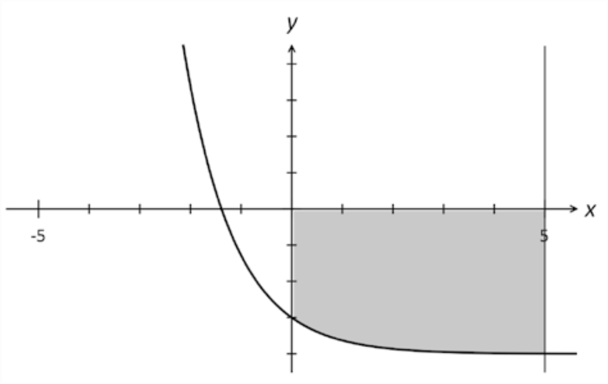
(c) Estimate the area of the shaded region. Show clearly how you obtained your answer.

Calculator Assumed

2. [12 marks: 5, 3, 4]

The shaded region R in the accompanying diagram is trapped between the curve with equation $y = -4 + e^{-x}$, the x-axis, the y-axis and the line $x = 5$.

The area of R is to be estimated by using 100 trapezoidal strips of uniform width.



(a) State the area of strip n for $1 \le n \le 100$.

(b) Estimate the area of R correct to 4 decimal places.

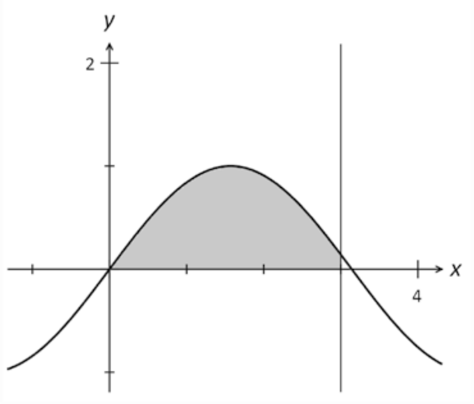
(c) Estimate the percentage difference in the area of R if 50 trapezoidal strips of uniform width was used instead of 100 strips.

Calculator Assumed

3. [8 marks: 4, 4]

The shaded region R in the accompanying diagram is trapped between the curve with equation $y = \sin(x)$, the x-axis and the line $x = 3$.

Simpson's Rule for estimating the area trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ using $2n$ strips of uniform width is given by the formula:



$$\text{Area} = \left(\frac{w}{3}\right) \times \left[f(a) + f(b) + 4 \sum_{i=0}^{i=n-1} f\left(a + (2i+1)w\right) + 2 \sum_{i=1}^{i=n-1} f\left(a + 2iw\right) \right]$$

where $w = \dfrac{b-a}{2n}$ is the uniform width of a strip.

(a) Apply Simpson's Rule to estimate the area of R using 2 strips.
Give your answer correct to 2 decimal places.

(b) Apply Simpson's Rule to estimate the area of R using 100 strips.
Give your answer correct to 2 decimal places.

Calculator Assumed

4. [11 marks: 5, 5, 1]

Let $f(x) = e^{0.05x^2}$. Consider $I = \int_0^8 e^{0.05x^2}\ dx$.

(a) Apply the middle-box (mid-point) method using 100 rectangular strips of uniform width to estimate the value of I correct to 4 decimal places.

[Note: For the strip with $a \le x \le b$, the height of the strip is $f\left(\frac{a+b}{2}\right)$.]

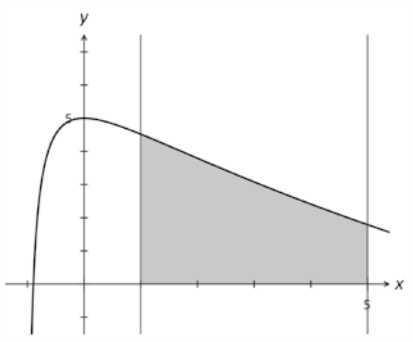
(b) Apply the trapezium rule using 100 trapezoidal strips of uniform width to estimate the value of I correct to 4 decimal places.

(c) Calculate the percentage difference in the value of I if the trapezium rule was used instead of the centre-box method (for 100 strips each).

Calculator Assumed

5. [13 marks: 3, 3, 1, 4, 2]

The shaded region R in the diagram below shows the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = 1$ and $x = 5$.



x	$f(x)$
1	4.52
1.2	4.38
1.4	4.23
1.6	4.09
1.8	3.94
2.0	3.79
2.2	3.65
2.4	3.5
2.6	3.36
2.8	3.22
3.0	3.08
3.2	2.94
3.4	2.80≈
3.6	2.67
3.8	2.54
4.0	2.41
4.2	2.28
4.4	2.16
4.6	2.03
4.8	1.91
5.0	1.79

The accompanying table shows the value of the function $f(x)$ for various values of x.

(a) Estimate the area of R using 20 inscribed uniform rectangular strips.

(b) Estimate the area of R using 20 circumscribed uniform rectangular strips.

Calculator Assumed

5. (c) Use your answers in (a) and (b) to provide a more accurate estimate for the area of R.

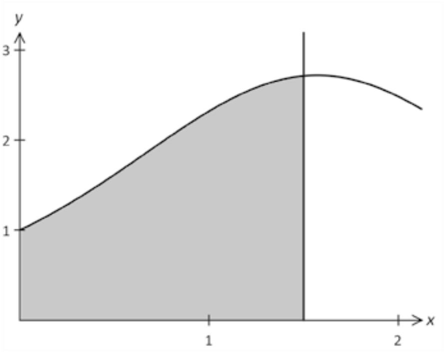
(d) Estimate the area of R using 10 uniform trapezoidal strips.

(e) If greater accuracy is used (in terms of the number of decimal places used), determine with reasons which of the procedures in (c) or (d) will provide a more accurate estimate for the area of R?

Calculator Assumed

6. [9 marks: 3, 4, 2]

The shaded region R in the accompanying diagram is trapped between the curve $y = f(x)$ where $f(x) = e^{\sin x}$, the x-axis, the y-axis and the line $x = 1.5$. The area of region R is to be estimated using Simpson's rule with six strips. Simpson's Rule for estimating the area A of a region trapped between the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ using n uniform strips (where n is even) is given by:



$$A = \left(\frac{w}{3}\right) \times \left[f(x_0) + 4\left[f(x_1) + f(x_3) + f(x_{n-1})\right] + 2\left\{f(x_2) + 2f(x_4) + \ldots + 2f(x_{n-2})\right\} + f(x_n) \right]$$

where w is the width of a strip, $x_0 = a$ and $x_n = b$. [Note: $f(x) \geq 0$ for $a \leq x \leq b$]

(a) Complete the table below where x_n are the x-coordinates of the boundaries of the strips. Give answers correct to 6 decimal places.

n	0	1	2	3	4	5	6
x_n							
$f(x_n)$							

(b) Show use of Simpson's rule with six strips to estimate the area of region R, correct to 5 decimal places.

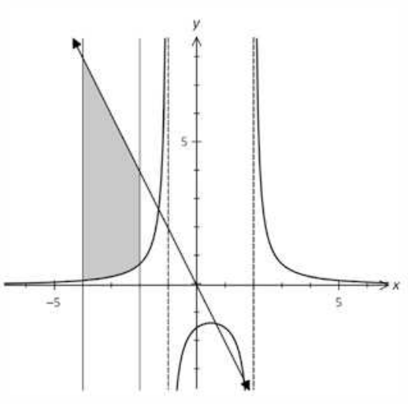
(b) Suggest how a more accurate estimate may be obtained by using Simpson's rule.

22 Applications of Integration

Calculator Free

1. [6 marks]

In the given diagram, the shaded region is trapped between the line $y = -2x$, the curve $y = \dfrac{3}{(x+1)(x-2)}$, and the lines $x = -4$ and $x = -2$. Determine the area of this region.



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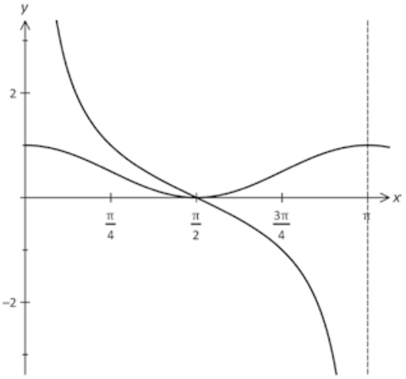
2. [6 marks]

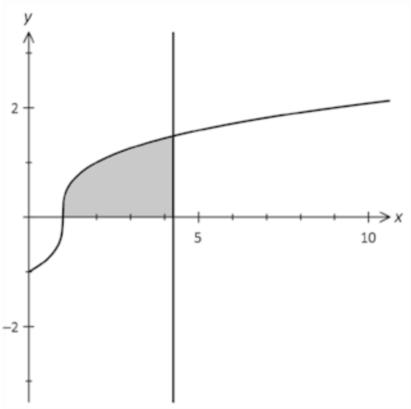
The given diagram, shows the graphs of $y = \cos^2(x)$ and $y = \cot(x)$. Calculate the area of the region trapped between the curves $y = \cos^2(x)$ and $y = \cot(x)$ and the line $x = \dfrac{3\pi}{4}$



Calculator Free

3. [6 marks: 3, 3] [TISC]

The shaded region shown in the diagram given is trapped between the curve $y = \sqrt[3]{x-1}$, the x-axis and the line $x = m$. The shaded region is rotated about the x-axis to form a solid.



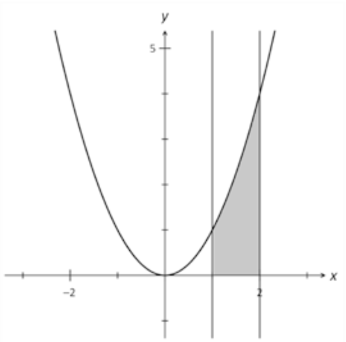
(a) Determine in terms of m, the volume of the solid formed.

(b) The line $x = m$ starts moving to the right starting from $x = 1$ at a speed of 2 units per minute. Calculate the rate of increase of the volume of the solid when $t = 4$ minutes.

---

4. [5 marks]

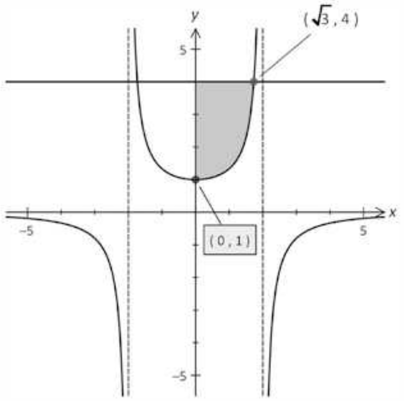
The shaded region shown in the accompanying diagram is trapped between the curve $y = x^2$, the lines $x = 1$, $x = 2$ and the x-axis. This region is rotated about the y-axis to form a solid of revolution. Determine the volume of the solid formed.



Calculator Free

5. [10 marks: 6, 4]

The shaded region R in the given diagram is trapped between the curve with equation $y = \dfrac{4}{4-x^2}$, the y-axis and the line $y = 4$.



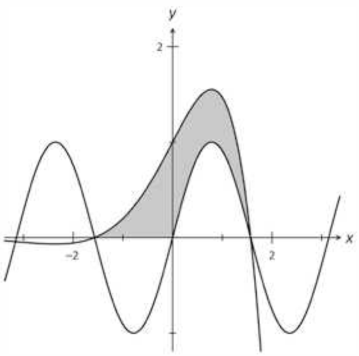
(a) Calculate the area of region R.

(b) Region R is rotated about the y-axis to form a solid of revolution. Determine the volume of this solid.

Calculator Assumed

6. [5 marks]

The shaded region shown in the given diagram is trapped between the x-axis and the curves $y = \sin 2x$, $y = e^x \cos x$. This region is rotated about the x-axis to form a solid. Use integrals to express the volume of the solid formed. Hence, evaluate the volume of this solid correct to 2 decimal places.



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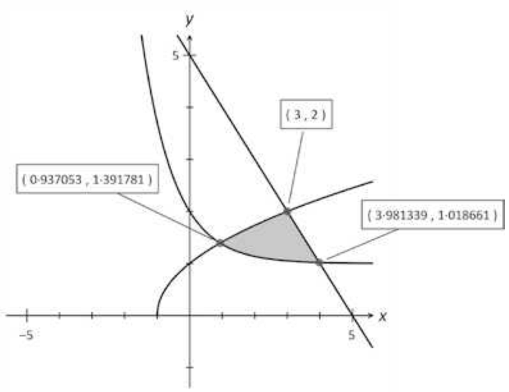
7. [6 marks]

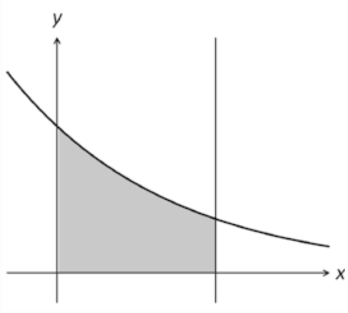
The region trapped between the curves with equations $y = \sqrt{x+1}$, $y = 1 + e^{-x}$ and $x + y = 5$ is rotated about the x-axis to form a solid of revolution.
Use integrals to express the volume of the solid formed. Hence, evaluate the volume of this solid.



Calculator Assumed

8. [9 marks: 3, 2, 2, 2] [TISC]

The shaded region R in accompanying diagram is the region trapped by the curve $y = 10\,e^{kx}$, the x-axis and the lines $x = 0$ and $x = 5$.



(a) Show that the area of this region is given by

$$\text{Area} = \frac{10}{k}(e^{5k} - 1).$$

(b) Find the value of k if the area of R is given by Area $= 50(1 - \frac{1}{e})$.

(c) The area of the region trapped by the curve $y = 10\,e^{kx}$, the x-axis and the lines $x = 0$ and $x = b$ (where $b > 5$) is given by Area $= \frac{10}{k}\,(e^{10k} - 1)$.

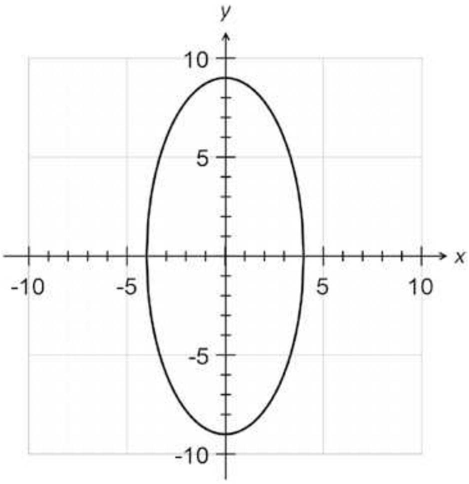
(i) Find in terms of b and k, the area of the region trapped by the curve $y = 10\,e^{kx}$, the x-axis and the lines $x = 5$ and $x = b$ (where $b > 5$) .

(ii) Find b.

Calculator Assumed

9. [10 marks: 7, 3]

The accompanying diagram shows an ellipse with equation $\frac{x^2}{16}+\frac{y^2}{81}=1$.



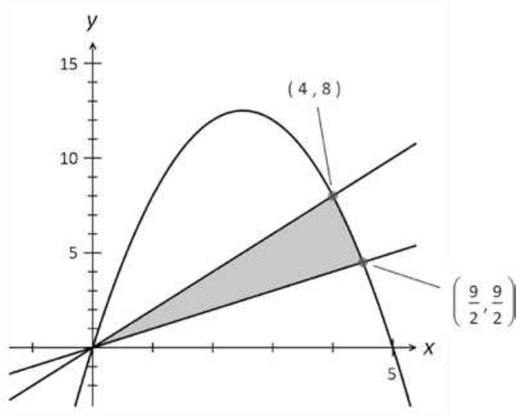
(a) Use integration to show that the area of the ellipse is 36π units^2. Show clearly the integrals being evaluated.

(b) Calculate the volume of the solid formed when the ellipse is rotated about the *x*-axis. State clearly any mathematical expression(s) used.

Calculator Assumed

10. [8 marks: 4, 4] [TISC]

The accompanying diagram shows the sketch of the curve $y = 2x(5 - x)$ and the lines $y = x$ and $y = 2x$.



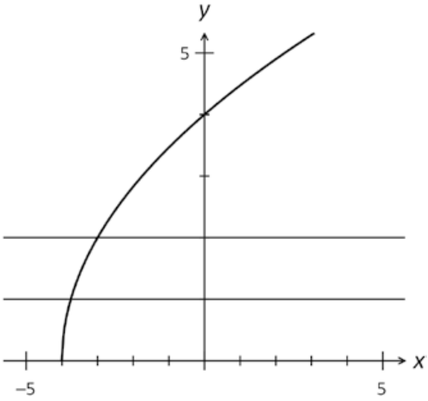
(a) Find the area of the shaded region. Show clearly how you obtained your answer. Show clearly all the integral expressions used.

(b) The region trapped by $y = x$, $y = 2x(5 - x)$ and the x-axis is rotated 2π radians about the x-axis. Find the volume of the solid formed. Show clearly how you obtained your answer.

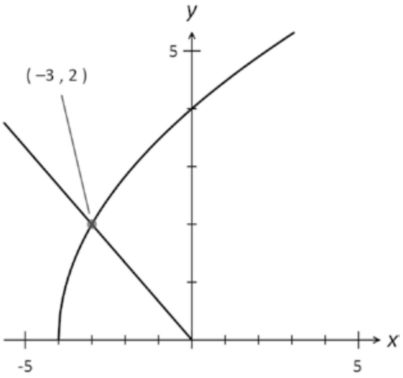
Calculator Assumed

11. [8 marks: 4, 4] [TISC]

(a) The accompanying diagram shows the sketch of the curve $y = 2\sqrt{x+4}$ and the lines $y = 1$ and $y = 2$. Use calculus techniques to find <u>exactly</u> the area of the region trapped between the curve $y = 2\sqrt{x+4}$, the y-axis and the lines $y = 1$ and $y = 2$.



(b) The accompanying diagram shows the sketch of the curve $y = 2\sqrt{x+4}$ and the line $y = \dfrac{-2x}{3}$. The region trapped by $y = 2\sqrt{x+4}$, the y-axis and the line $y = \dfrac{-2x}{3}$ is rotated about the y-axis. Find the volume of the solid formed. State clearly any integrals used.

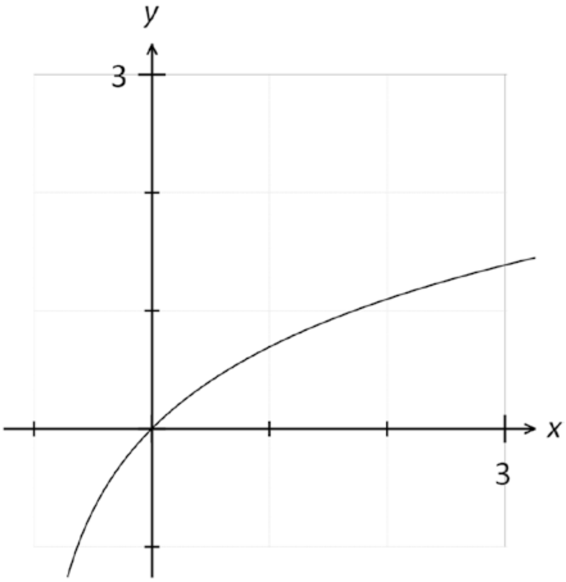


Calculator Assumed

12. [7 marks: 2, 1, 1, 3] [TISC]

(a) Use calculus to verify that $\frac{d}{dx}(x+1)\left[ln(x+1)-1\right]=ln(x+1)$

(b) Let $f(x) = ln(x + 1)$. The diagram below shows the graph of $y = f(x)$.



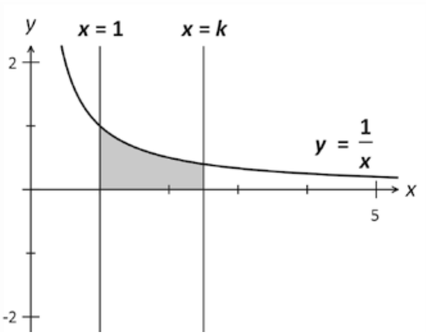
(i) Sketch on the same set of axes, the graph of the $y = f^{-1}(x)$.

(ii) On the same set of axes above, shade the region trapped by the curves $y = f(x)$, $y = f^{-1}(x)$, $x = 2$ and $y = 2$.

(c) Use integration, to find the area of the region described in (b). Show clearly the functions being integrated and their results. Give your answers to 3 significant figures.

Calculator Assumed

13. [7 marks: 3, 3, 1] [TISC]



The region trapped between the curve $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = k$ where $k > 1$, is rotated 2π radians about the x-axis to form a solid.

(a) Find the exact volume of the solid formed as $k \to \infty$.

(b) The <u>*surface area*</u> of the solid formed is given by $S = \int_1^k \frac{2\pi}{x}\sqrt{1+\left(\frac{1}{x}\right)^4}\ dx$.

Use the relationship $\int_1^k \frac{2\pi}{x}\sqrt{1+\left(\frac{1}{x}\right)^4}\ dx > \int_1^k \frac{2\pi}{x}\,dx$ to determine the surface area of the solid formed as $k \to \infty$.

(c) Comment on your answers in (b) and (c).

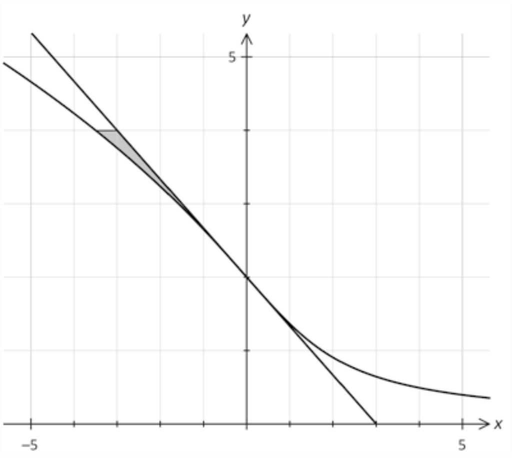
Calculator Assumed

14. [7 marks: 3, 4] [TISC]

Consider the curve with equation $y^3 + 4xy = 8$ for $y > 0$.

(a) Show that the tangent to the curve at the point (0, 2) has equation $y = \frac{-2x}{3} + 2$.

The given diagram shows the graph of $y^3 + 4xy = 8$ for $y > 0$. The diagram also shows the tangent to the curve at the point (0, 2).



(b) The shaded region is trapped between the curve $y^3 + 4xy = 8$, the tangent to the curve at (0, 2) and the lines $y = 2$ and $y = 4$. This region is rotated about the y-axis to form a solid of revolution. Show the exact volume of this solid.

23 Differential Equations I

Calculator Assumed

1. [7 marks]

A tablet contains 100 mg of drug X. The amount of drug (A mg) left in a patient's body changes according to the formula $\dfrac{dA}{dt} = -0.08664\,A$, where t is time in hours. Jane takes the first tablet at 8.00 am and another tablet at 4.00 pm on the same day. How much of drug X (to the nearest mg) is in Jane's body at 8.00 pm on the same day?

---

2. [8 marks: 5, 1, 2] [TISC]

A scientist suspects that a colony of bacteria grows in such a way that its population growth $\dfrac{dP}{dt}$ is proportional to its population P at time t hours. At $t = 5$ hours, there were 200 bacteria and at $t = 8$ hours, there were 300 bacteria.

(a) Use integration to find an expression for P in terms of t.

Calculator Assumed

2. (b) Give a reason why this model of the population growth of the bacteria cannot be correct.

Another scientist uses another equation, $P = \dfrac{5\,000}{1+e^{-t}}$ to describe P in terms of t.

(c) Explain why this is a better model than the first one.

---

3. [9 marks: 6, 3] [TISC]

A room is being heated up by a new type of heater. The rate of change of the temperature of the room (θ) is given by $\dfrac{d\theta}{dt} = 0.25(\theta - 5)$ degrees Celsius per minute, where k is a constant and t is time in minutes. The manufacturer of the heater claims that the heater can raise the temperature of the room from 10° to 22° within 5 minutes.

(a) Use calculus methods to test if the manufacturer's claim is true.
Show clearly the method you used.

(b) How long will this heater take to heat a room from 15° Celsius to 22° Celsius? Justify your answer.

Calculator Assumed

4. [6 marks: 3, 3] [TISC]

The rate of change of the quantity of fluid in a container is given by $\frac{dQ}{dt} = k\,(Q + 50)$ litres per minute, where k is a constant and t is time in minutes.

(a) Use integration to show that $Q = A\,e^{k\,t} - 50$ where A is a constant.

(b) Find the rate of change of the quantity of fluid if $Q = 100$ litres when $t = 10$ minutes and $A = 40$.

---

5. [10 marks: 3, 5, 2]

A tank contains 100 L of a salt solution with a concentration of 4 g/L. Fresh salt solution with a concentration of 10 g/L flows into the tank at a rate of 5 L per minute. The concentration of salt in the tank is kept uniform by constant stirring. The mixture flows out of the container at a rate of 5 L per minute. The amount of salt at time t minutes is S g.

(a) Show that $\frac{dS}{dt} = a - \frac{S}{b}$, giving the values of the constants a and b.

Calculator Assumed

5. (b) Hence, use integration to show that $S = m - ne^{-kt}$, giving the values of the constants m, n and k.

(c) Find the long term concentration of salt in the tank. Justify your answer.

---

6. [7 marks]

A tank contains salt solution with 250g of salt dissolved in 100 L of water. Another solution with a salt concentration of 2 g/L flows into the tank at a rate of 1 L/minute. The salt mixture in the tank is kept uniform by constant stirring. 1 L of the mixture flows out through a valve every minute keeping the amount of solution in the tank at a constant 100 L. The amount of salt in the tank at time t minutes is S g. Use a Calculus method to find an expression for S in terms of t.

Calculator Assumed

7. [9 marks: 4, 5]

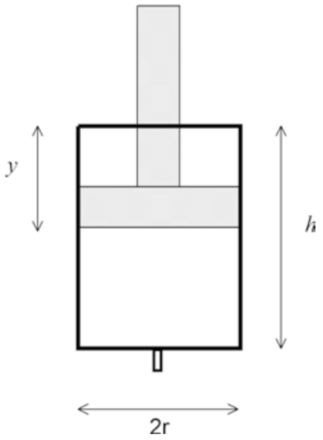
(a) Given the equation $\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x}$, solve for y given that when $x = e$, $y = 1$.

(b) Given that $\frac{dy}{dx} = \frac{1+y^2}{2xy}$, find y in terms of x given that when $x = 1$, $y = -1$.

Calculator Assumed

8. [10 marks: 5, 5]

The accompanying diagram is a schematic diagram of a piston within a cylindrical chamber of length h cm and fixed diameter $2r$ cm. The head of the piston is cylindrical in shape and fits exactly into the chamber. At time $t = 0$ seconds, the chamber is completely filled with fluid. Assume that the fluid is incompressible. As the piston head is forced into the chamber, fluid is forced out through a nozzle at the base of the chamber at a rate of $(h - y)$ cm^{3}s^{-1} where y cm is the distance the piston head has moved into the chamber, $0 \leq y < h$.



(a) Show that the rate with which the piston is being forced into the chamber when it is 25% full is given by $\dfrac{dy}{dt} = \dfrac{h}{4\pi r^2}$.

(b) Find y in terms of t.

Calculator Assumed

9. [9 marks: 2, 3, 4] [TISC]

Consider the differential equation $\frac{d^2y}{dx^2} = k\frac{dy}{dx}$ where k is a constant.

(a) Given $u = \frac{dy}{dx}$, show that $\frac{du}{dx} = ku$.

(b) Show that $u = \mathrm{A}e^{kx}$ where A is a constant.

(c) Find y in terms of x if when $x = 0$, $y = 2000$, $\frac{dy}{dx} = 1000$ and $\frac{d^2y}{dx^2} = 1000$.

Calculator Assumed

10. [9 marks: 2, 6, 1]

(a) Decompose $\dfrac{1}{P(500-P)}$ into its partial fractions.

(b) The population P of a colony of native rodents at time t years is modelled by the differential equation $\dfrac{dP}{dt}=0.01P\left(1-\dfrac{P}{500}\right)$ where $P(0) = 250$.

Use integration to show that $P=\dfrac{A}{1+Be^{-kt}}$, stating the values of A, B and k.

(c) Use your answer in (b) to discuss the long-term growth trend of this colony.

Calculator Assumed

11. [10 marks: 6, 2, 2]

The concentration C of a chemical in a solution is modelled by the equation $\frac{dC}{dt} = 0.000\,25C\,(200 - C)$ where t is time in minutes. The initial concentration of the chemical is 10 g/L.

(a) Use a calculus method to determine an expression for C in terms of t.

(b) Determine the time (to the nearest minute) it takes for the concentration to reach half its maximum concentration.

(c) Determine the time (to the nearest minute) it takes for the concentration to effectively reach its maximum concentration.

Calculator Assumed

12. [8 marks: 6, 2]

The number of chickens on a farm infected by a fatal strain of bird virus is modelled by $\frac{dP}{dt} = 0.1P - 0.000\,1P^2$ where t is time in days after the detection of the virus strain among 10 chickens.

(a) Use a calculus method to determine P in terms of t.
Give your answer in the form with a constant in the numerator.

(b) All the surviving chickens on the farm were culled when the virus had spread to more than 100 chickens on the farm. After how many days did the culling occur?

Calculator Assumed

13. [7 marks: 6, 1]

The variable Q changes with time t hours, according to the differential equation $\frac{dQ}{dt} = 0.000\,01Q(2000 - Q)$. The initial value of Q, $Q_0 = 10$.

(a) Use integration to show that $Q = \dfrac{kQ_0}{Q_0 + (k - Q_0)e^{-rkt}}$,

stating the values of the constants k and r.

(b) Determine the limiting value for Q.

Calculator Assumed

14. [8 marks: 2, 1, 5]

$P = \dfrac{5000\,e^{0.02t}}{e^{0.02t} + 49}$ is the solution to logistic differential equation $\dfrac{dP}{dt} = aP\,(1 - bP)$ where a and b are real constants.

(a) Determine with reasons the limiting value of P.

(b) State the initial value of P.

(c) Implicitly differentiate P to determine the values of a and b.

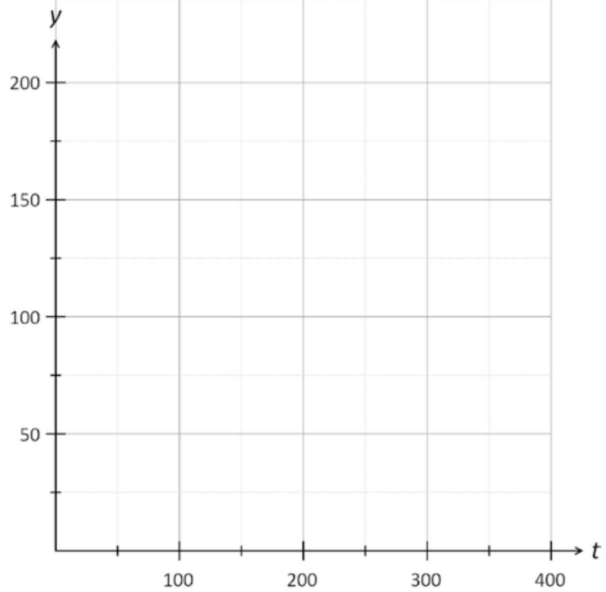
Calculator Assumed

15. [9 marks: 2, 3, 4]

In a certain fare promotion for an airline route, the number of tickets sold may be modelled by the logistic function $y = f(t)$ with a growth rate of 0.025 tickets per minute and an initial value of 10 tickets with a limiting value of 200 tickets.

(a) State the logistic function and the associated logistic differential equation.

(b) In the axes provided below, sketch $y = f(t)$.

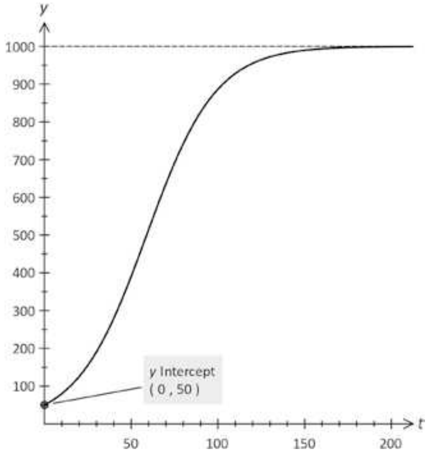


(c) Use calculus to determine when the tickets are being sold at the fastest rate and state this rate.

Calculator Assumed

16. [7 marks: 4, 3] [TISC]

The diagram given shows the graph of a logistic function $y = \dfrac{m}{1+ne^{-0.05t}}$.



(a) Determine with reasons the values of m and n.

(b) The graph given above is a particular solution to a first order differential equation of the form $\dfrac{dy}{dt} = f(y)$. State the differential equation and its initial condition.

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17. [10 marks: 1, 1, 2, 2, 2, 2] [TISC]

The variable y changes with respect to time t according to the differential equation $\dfrac{dy}{dt} = 0.05y$ with $y(0) = 1000$.

(a) Complete the table below.

y	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
$\dfrac{dy}{dt}$										

(b) Hence, describe the way the $\dfrac{dy}{dt}$ changes with respect to y.

Calculator Assumed

17. (c) Use your answers in parts (a) and (b) to explain clearly why as $t \to \infty$, $y \to \infty$.

To limit the growth of the variable y, a "correction factor" $C = -\,0.000\,005\,y^2$ is added to the differential equation so that it now becomes

$$\frac{dy}{dt} = 0.05y - 0.000\,005\,y^2 \text{ with } y(0) = 1000.$$

(d) Complete the table below for the new differential equation.

y	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
$\frac{dy}{dt}$										

(e) Hence, describe the way the $\frac{dy}{dt}$ changes with respect to y.

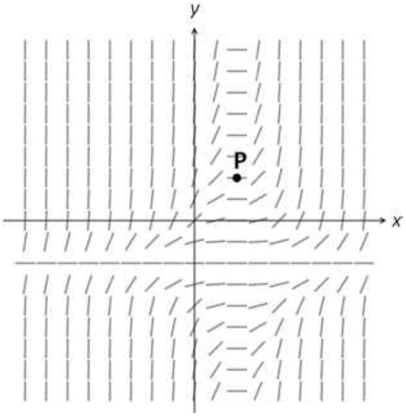
(f) Use your answers in parts (d) and (e) to explain how this "correction factor" limits the growth of y.

24 Differential Equations II

Calculator Assumed

1. [3 marks] [TISC]

The diagram given shows the slope field for a differential equation. On the diagram given draw the solution curve to the differential equation which passes through the point **P**.



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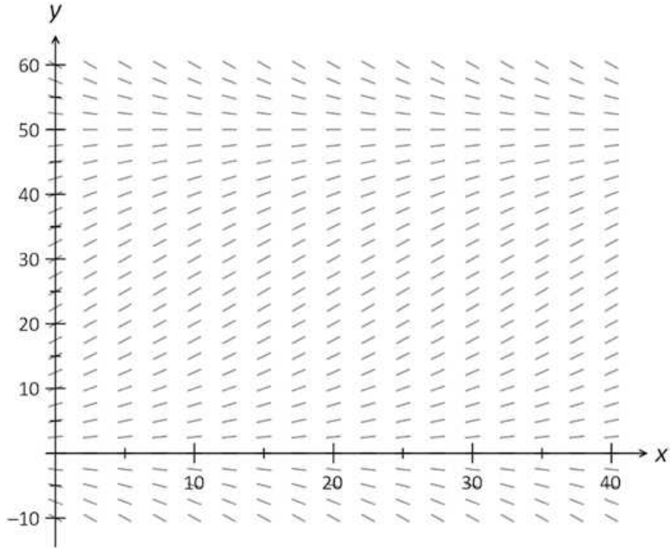
2. [9 marks: 3, 2, 2, 2]

The given diagram shows the slope field of a differential equation $\dfrac{dy}{dx} = f(x, y)$.



(a) On the slope field given draw in the curve for the particular solution with initial condition (15, 40).

(b) Mark and state the coordinates of the point on the particular curve where the gradient is steepest.

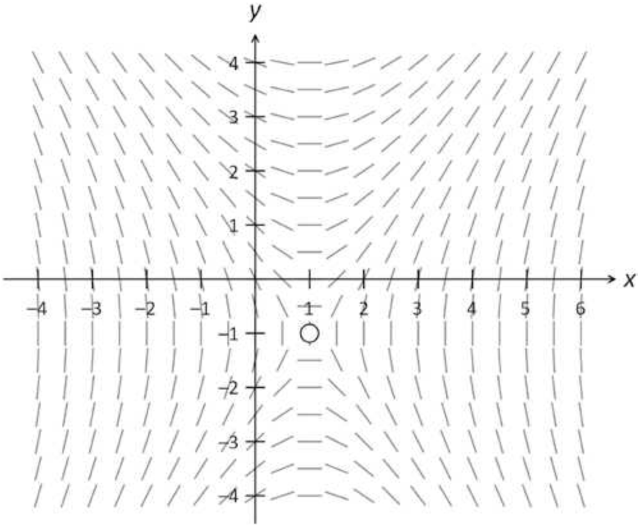
(c) Explain why $f(x, y)$ corresponding to this slope field is independent of x.

(d) Determine with reasons if there is an isocline with a gradient of 3.

Calculator Assumed

3. [9 marks: 3, 2, 1, 3]

The given diagram shows the slope field of a differential equation $\frac{dy}{dx} = f(x, y)$.



(a) On the slope field given draw in the curve representing the particular solution with initial condition (2, 1).

(b) For the particular curve in (a), estimate the minimum value for y and the corresponding value of x.

(c) State the equation of the isocline with infinite gradient

(d) Determine with reasons a possible expression for $f(x, y)$.

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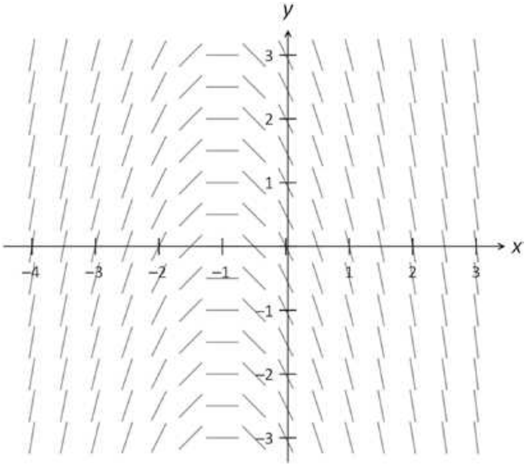
4. [7 marks: 2, 2, 3]

The given diagram shows the slope field of a differential equation $\frac{dy}{dx} = f(x, y)$.



(a) On the slope field given draw in the curve representing the particular solution with initial condition (1, –1). Determine the coordinates of the stationary point(s) on this curve.

Calculator Assumed

4. (b) Determine with reasons, which of the following equations best describe the differential equation.

A. $y' = -x - 1$ B. $y' = -2y - 2$ C. $y' = -2 - 2x$ D. $y' = 2x + 2$.

(c) For the equation you have chosen, state the equation of the isocline with gradient 4. Draw this isocline on the diagram given.

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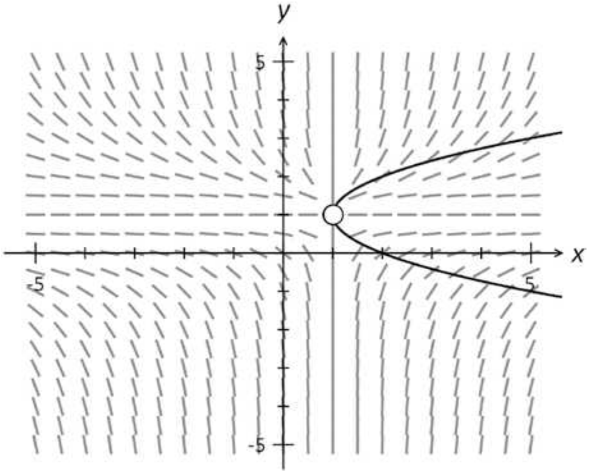
5. [8 marks: 3, 5]

The given diagram shows the slope field of a differential equation $\dfrac{dy}{dx} = f(x, y)$.



(a) Determine with reasons, which of the following equations best describe the differential equation.

A. $y' = \dfrac{(x-1)^2}{y-1}$ B. $y' = \dfrac{x-1}{y-1}$

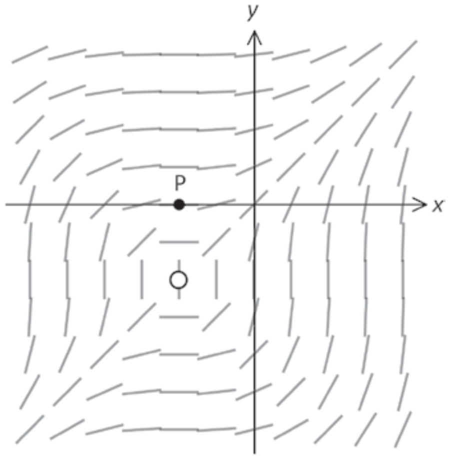
C. $y' = \dfrac{(x+1)^2}{y-1}$ D. $y' = \dfrac{(y-1)^2}{x-1}$

(b) For the equation you have chosen, state the equation of the isocline with gradient 1. Draw this isocline on the diagram given.

Calculator Assumed

6. [7 marks: 3, 4]

The given diagram shows the slope field for $\frac{dy}{dx} = f(x, y)$.



(a) Draw the integral curve passing through the point P.

(b) Suggest with reasons a possible expression for $f(x, y)$.

---

7. [6 marks: 1, 3, 1, 1]

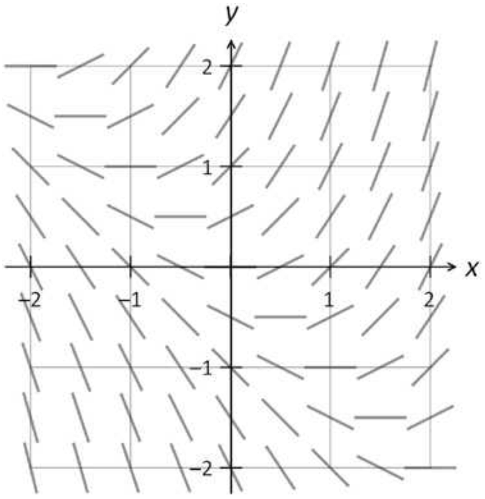
The table below gives the gradients for the differential equation $\frac{dy}{dx} = f(x, y)$ for various values of x and y.

y \ x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
−2	-4	−3.5	−3	−2.5	−2	−1.5	−1	−0.5	0
−1.5	−3.5	−3	−2.5	−2	−1.5	−1	−0.5	0	0.5
−1	−3	−2.5	−2	−1.5	−1	−0.5	0	0.5	1
−0.5	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5
0	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
0.5	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5
1	−1	−0.5	0	0.5	1	1.5	2	2.5	3
1.5	−0.5	0	0.5	1	1.5	2	2.5	3	3.5
2	0	0.5	1	1.5	2	2.5	3	3.5	4

(a) State the equation of the isocline with gradient –1.

Calculator Assumed

7. (b) Using the scale on the axes provided, sketch the slope field.



(c) In the diagram in (b), draw the isocline with gradient 0.

(d) Determine $f(x, y)$.

8. [3 marks]

Let $\frac{dy}{dx} = \frac{e^{x^2}}{y^2+1}$. Given that when $x = 1$, $y = 1$, use the method of small changes to estimate the value of y when $x = 1.01$. Give your answer to 5 decimal places.

9. [4 marks] [TISC]

Consider the differential equation $\frac{dy}{dx} = \frac{y^2+1}{x}$. Given that when $x = 1$, $y = 2$, use the incremental formula with $\delta x = 0.01$ to estimate the value of y when $x = 1.02$. Give your answer to 4 decimal places.

Calculator Assumed

10. [8 marks: 5, 3] [TISC]

A curve has equation curve $x^2y + \sqrt{3+y^2} = 3$.

(a) Find the equation of the tangent to this curve at the point (–1, 1).

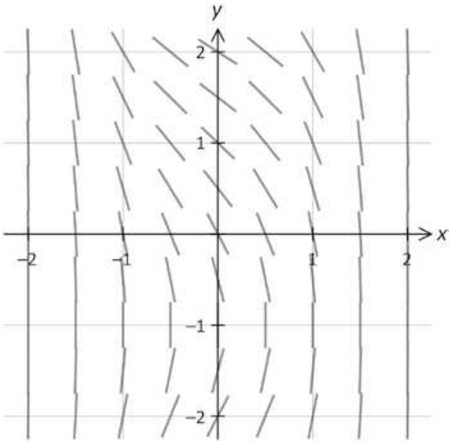
(b) Use the method of incremental change to find the change in y when x changes from –1.00 to –1.01.

---

11. [6 marks: 2, 4]

Consider the differential equation $\frac{dy}{dx} = \frac{e^{x^2}}{y}$

(a) Explain why the slope field shown in the given diagram cannot be the slope field for the differential equation given above.



Calculator Assumed

11. (b) A particular curve to this differential equation passes through the point (0, 1). Use the method of small increments to show that this curve also passes through the point with coordinates $\left(0.2,\ 1.1+\dfrac{e^{0.1^2}}{11}\right)$

---

12. [7 marks: 1, 2, 4]

Consider the differential equation $\dfrac{dy}{dx}=\dfrac{x+1}{y-1}$.

(a) Identify the coordinates of the point(s) in the slope field of this differential equation with indeterminate gradient.

(b) A nullcline is a curve with passes through all slope fields with zero gradients. State the equation of the nullcline for this differential equation.

(c) A particular solution to this differential equation passes through the point (1, 2). Use the method of small increments with increment2 of 0.1 to estimate the value of y when x = 1.2.

25 Rectilinear Motion

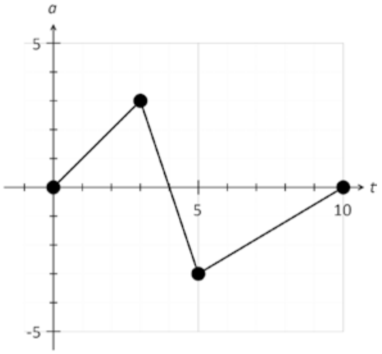
Calculator Assumed

1. [5 marks]

Object P travels along a straight line. At time $t = 0$ seconds, it passes a fixed point O with velocity 10 ms^{-1} and undergoes a constant acceleration of 2 ms^{-2}. Use a calculus method to find the velocity of P when it is 24 m from O.

2. [7 marks]

Object P starts from rest and travels uninterrupted along a straight line. The graph of its acceleration a ms^{-2} plotted against time t seconds for $0 \le t \le 10$ is shown in the given diagram. Find with reasons maximum and minimum velocity of the object for $0 \le t \le 10$.



Calculator Assumed

3. [9 marks: 3, 3, 3]

The acceleration of a particle P undergoing rectilinear motion is given by $a = \sqrt{1+2t}\ \text{ms}^{-2}$. The initial velocity of P is $1/3\ \text{ms}^{-1}$.

(a) Calculate the exact velocity of P when t = 12 seconds

(b) Find the exact average speed of P during the first twelve seconds.

(c) Find the average acceleration of P during the first twelve seconds.

Calculator Assumed

4. [10 marks: 5, 5]

A vehicle travels along a straight stretch of a highway. The driver notices a car stalled on the highway k metres ahead and applies the brakes of the vehicle. The acceleration of the vehicle t seconds after the brakes are applied is given by

$$a = -10\ e^{-0.1t}.$$

(a) Determine an expression for the displacement of the vehicle t seconds after the brakes are applied.

(b) The vehicle comes to a complete stop after 3 seconds just behind the stalled car. Find k and the initial speed of the vehicle.

Calculator Free

5. [4 marks]

A particle P moves along a straight line. Its acceleration (cms^{-2}) t seconds after it passes a fixed point O is given by $a(t) = x$ where $x(t)$ is the displacement of P, t seconds after passing O. P starts from O with a velocity of $v = -2$ cms^{-1}. Determine an expression for the velocity v in terms of x.

6. [6 marks]

The acceleration a of a particle at time t is given by $a = 10(1 - 4v^2)$ where v is the velocity of the particle. Determine v in terms of x given that $t = 0$, $x = 0$.

Calculator Free

7 [6 marks]

A particle moves in a straight line. The particle starts from a fixed point O in the positive direction. Its displacement after t seconds is given by x cm. Its velocity when it is x cm from the point O is given by $v^2 = 36 - 4x^2$. Use integration with an appropriate substitution to prove that $x = 3 \sin 2t$.

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8. [6 marks]

A particle P moves along a straight line. Its acceleration (cms^{-2}) t seconds after it passes a fixed point O is given by $a = -4\,e^{-2x}$ where $x(t)$ is the displacement of P, t seconds after passing O. P starts from O with a velocity of 2 cms^{-1}.
Given that $v \geq 0 \ \forall \ t$, determine $x(t)$.

Calculator Assumed

9. [9 marks: 4, 5]

An object P of constant mass m kg moves in a straight line with velocity v ms^{-1}.
The acceleration of P at time t seconds is given by $a = e^{-0.1\,t}$ ms^{-2} .
The kinetic energy possessed by P is given by $E = \dfrac{1}{2}mv^2$.

(a) P has a mass of 4 000 kg and at $t = 5$ seconds, P is travelling at 10 ms^{-1}.
Find the rate of kinetic energy change when $t = 5$ seconds.

(b) Calculate the distance travelled by the object in (a) in the first 10 seconds.

26 Simple Harmonic Motion

Calculator Assumed

1. [8 marks: 3, 5] [TISC]

A particle P travels along the x-axis between $x = -4$ and $x = 4$.
Its velocity v is given by $v^2 = 9(A^2 - x^2)$.

(a) Prove that P undergoes simple harmonic motion.

(b) Complete the table below to describe the motion of P.

Coordinates of the mean position	
Amplitude of Motion	
Period of Motion	
Maximum Speed	
Magnitude of maximum acceleration	

2. [11 marks: 3, 2, 3, 3] [TISC]

A particle P moves along the x-axis and its velocity at any time t seconds is given by $v_P = \frac{2\pi}{3} \cos\left(\frac{\pi t}{3}\right)$ ms^{-1}. The particle starts from the origin and its displacement from the origin O is represented by r_P

(a) Show that $-a \le r_P \le a$ and state the value of a.

Calculator Assumed

2. (b) Show that the particle undergoes simple harmonic motion.

(c) An observer is located at M with coordinates (6, 8). The distance between the particle and M at any time t is represented by *s*. Find the maximum and minimum value for *s*.

(d) A second particle Q moves along the *x*-axis such that its displacement from the origin is given by $r_Q = 8 - \sin\left(\frac{\pi t}{3}\right)$ metres. The distance between the particles P and Q at any time *t* seconds is given by *h* metres. Find the minimum and maximum value of *h*. Justify your answer.

Calculator Assumed

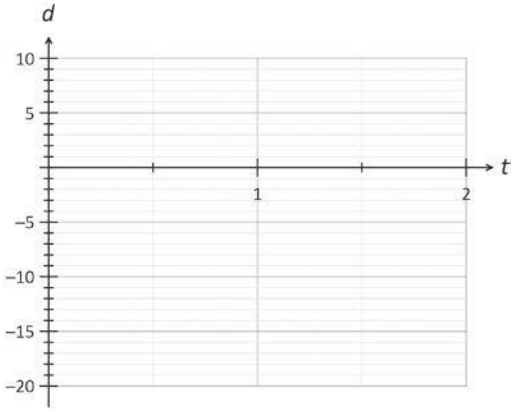
3. [13 marks: 5, 1, 3, 2, 2] [TISC]

Points A and B have coordinates (–4, 0) and (5, 0) respectively. Particles P and Q start moving along the *x*-axis at the same time. The displacement of P, *t* seconds later, is given by $r_P = -4 + 6\cos(\pi t)$. The displacement of Q, *t* seconds later, is given by $r_Q = 5 + 5\sin(2\pi t)$. It is known that $0 \le t \le 2$ seconds.

(a) Show that P undergoes simple harmonic motion about its mean position. State clearly the location of the mean position, its amplitude and period.

(b) Show that the separation, at time *t* seconds, between P and Q is given by $d = 6\cos(\pi t) - 5\sin(2\pi t) - 9$.

(c) Draw a sketch of *d* against *t* for $0 \le t \le 2$ on the axes provided below and use this sketch to explain why *d* varies with *t* in a manner that is **NOT** simple harmonic.



(d) Determine when P and Q collide for the first time and the location of the point of collision.

(e) Determine the maximum separation between P and Q stating when it occurs for $0 \le t \le 2$ seconds.

Calculator Assumed

4. [10 marks: 3, 3, 1, 3]

A particle P travels in a straight line. Let x be the displacement of P from a fixed point O at time t seconds. The velocity of P at time t seconds v is given by $v^2 = 4\pi^2(4 - x^2)$ cm s^{-1}.

(a) Show that P undergoes simple harmonic motion.

It is also known that P starts with $a = 4\pi^2$ cms^{-2}.

(b) Find x in the form $x = A\sin(kt + \alpha)$ where $\alpha \geq 0$.

(c) Find t when $x = -2$ cm.

(d) Find β given that $x \leq \beta$ for 20% of its period.

Calculator Assumed

5. [13 marks: 1, 1, 4, 4, 3] [TISC]

The displacement of a particle P at time t seconds, from a fixed point O, is given by $x = A \cos(\omega t)$ metres.

(a) Find v, the velocity of P at time t seconds.

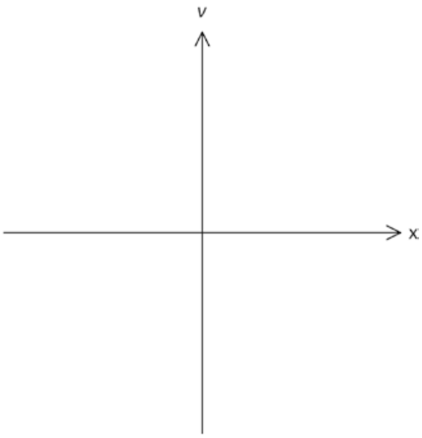
(b) Find x in terms of a, the acceleration of P at time t seconds.

(c) Differentiate v implicitly with respect to x to show that $v \dfrac{dv}{dx} = a$.

(d) Use an appropriate technique to integrate $v \dfrac{dv}{dx} = a$ with respect to the appropriate variables to show that $v^2 = \omega^2(A^2 - x^2)$.

Calculator Assumed

5. (e) On the axes given, sketch the graph of v against x. Show clearly the intercepts of the curve.



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6. [10 marks: 4, 6] [TISC]

The equation of motion of a particle P is given by $\frac{d^2x}{dt^2} = -16\pi^2 x$.

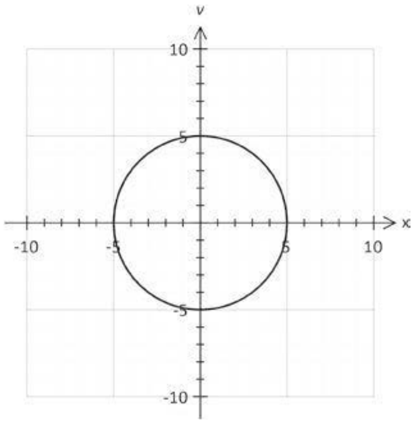
(a) Find x in terms of t given that when $t = 0$, $x = 8$, and $\frac{dx}{dt} = 0$.

(b) Find when within the first cycle P travels at half its maximum speed. Give your answer in exact form. Justify your answer.

Calculator Assumed

7. [10 marks: 6, 4]

The diagram below shows the velocity- displacement graph of a particle undergoing rectilinear motion. Displacement is measured in cm and velocity in cms^{-1} .



(a) Given that the shape of the graph is a circle of radius 5, use calculus to show that the motion of the particle is simple harmonic. State the period and amplitude of the motion.

(b) Given that the particle starts from the mean position of its motion, find the percentage of time within a cycle when the speed of the particle is no more than 4 ms^{-1}.

Calculator Assumed

8. [7 marks: 3, 4]

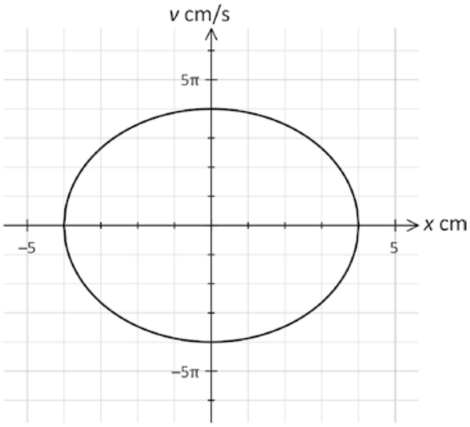
(a) The equation of motion of a body is given by $\frac{d^2x}{dt^2} + 25x = 0$, where x is the distance of the body (in m) to a fixed point C and t is time in minutes. It is known that the body starts moving from C with a velocity of –2 m per minute. Show clearly that $x = \text{A} \sin(\omega t + \alpha)$ giving the values of A, ω (where $\omega > 0$) and α.

(b) The equation of motion of another body is given by $x = 4\cos(2t + \frac{\pi}{6})$ metres where t is time in minutes. Find k, if $x \geq k$ for 90% of a cycle.

Calculator Assumed

9. [11 marks: 3, 3, 3, 2] [TISC]

(a) A particle P undergoes simple harmonic motion. It starts from the mean position with a velocity of 4π cms^{-1}. The given diagram shows the velocity-displacement graph of P.



(i) Determine the period and amplitude of the motion.

(ii) Determine the value(s) of t, where $0 \le t \le 2$ seconds, when it is travelling with a speed of 2π cms^{-1}.

(b) A second particle Q travels in a straight line and has an initial velocity of 4π cms^{-1}. Let x be the displacement of Q along this line from the point when it starts moving. The acceleration of Q is given by $a = \pi^2 x$ cms^{-2}. Use calculus to show that velocity of Q is given by $v^2 = \pi^2\left(16 + x^2\right)$.

(c) Compare the motions of P and Q.
State an important difference in the speeds achieved by P and Q.

Calculator Assumed

10. [13 marks: 2, 4, 5, 2] [TISC]

The amount of rainfall received by a town is modelled by

$r = 5 - \frac{5}{2}\sin\left[\frac{2\pi t}{365}\right]$ millimetres, for $0 \le t \le 365$ days.

(a) Determine the approximate number of days the town receives more than 7 millimetres of rainfall.

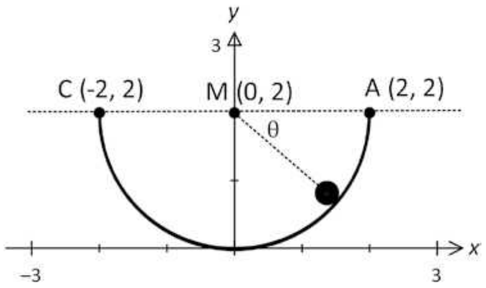
(b) Find on which day(s) the amount of rainfall received is decreasing at a rate of –0.025 millimetres per day

(c) Show that the *variation* of rainfall received is simple harmonic. State its period and amplitude.

(d) Use an appropriate method to find the total amount of rainfall received for $0 \le t \le 365$ days.

Calculator Assumed

11. [11 marks: 5, 3, 3]



A steel bearing B of negligible radius rolls to and fro in the vertical plane between the points A (2, 2) and C (–2, 2) along the path with equation $y = 2 - \sqrt{4-x^2}$ located on the frictionless inner surface of a hemispherical dish of radius 2 cm. The steel bearing is released from rest at point A. M is a point with coordinates (0, 2). Let θ be the angle between BM and the horizontal line through C, M and A. θ changes at a constant rate of $\frac{\pi}{2}$ radians per second.

(a) Show that the horizontal displacement of B from M displays variation that is simple harmonic.

(c) Determine the rate of change of x when x is 1 cm.

(d) The magnitude of the horizontal displacement x is within k cm from M for 60% of its period. Determine the value of k.

27 Vector Calculus I

Calculator Assumed

1. [16 marks: 2, 2, 3, 3, 3, 3] [TISC]

If air resistance is ignored, the velocity vector, $v(t)$, of a particle P at time t seconds, is given by $v(t) = 40\sqrt{3}\,\mathbf{i} + (40 - 9.8t)\,\mathbf{j}$, where the components are measured in ms^{-1}. It is known that when $t = 0$, its displacement $\boldsymbol{r}(0) = 2\,\mathbf{i} + 4\,\mathbf{j}$ metres. Ground level is modelled by $\mathbf{j} = 0$.

(a) Find the time it takes the particle to reach its highest point.

(b) Verify that the position vector of the particle, t seconds after it is thrown, is given by. $\boldsymbol{r}(t) = (2 + 40\sqrt{3}\ t)\,\mathbf{i} + (4 + 40t - 4.9t^2)\,\mathbf{j}$.

(c) Find the position vector of the point where it hits the ground.

(d) Find when P moves in a direction that is parallel to $2\mathbf{i} - \mathbf{j}$.

(e) Find when the speed of P is 70 ms^{-1} .

Calculator Assumed

1. (f) Use the information in part (b) to find the parametric equation of the path of P. Hence, show that the Cartesian equation of the path of P is in the form of a quadratic equation. You are NOT required to simplify the Cartesian equation.

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2. [13 marks: 3, 2, 3, 3, 2] [TISC]

The velocity vector of particle P at time t seconds (for $0 \le t \le 5$) is given by $\boldsymbol{v}_P\ (t) = 2t\ \mathbf{i} + (4t^3 - 10t\ )\ \mathbf{j}$ centimetres per second. It is known that when $t = 0$, the particle is at a point with position vector $0\ \mathbf{i} + 4\ \mathbf{j}$ centimetres.

(a) Find the exact value(s) of t when P is moving at an angle of 45° to the horizontal.

(b) Find the speed of P when it is moving at an angle of 45° above the horizontal.

(c) Find the position vector of P when it is moving parallel to the horizontal for the first time.

Calculator Assumed

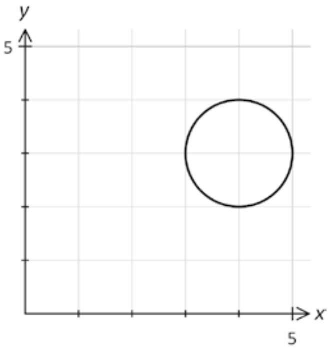
2. (d) Find the magnitude of the minimum acceleration of P.

(e) Find the distance travelled by P along its path of motion in the first 5 seconds.

---

3. [9 marks: 3, 4, 2]

The position vector of a particle P at time t minutes is given by $\boldsymbol{r}(t) = < 4 + \cos t, 3 + \sin t >$ metres. The path traced by P is shown in the given diagram.



(a) Show that P is *first* furthest from the origin at

$t = \tan^{-1}\left(\frac{3}{4}\right)$ minutes.

(b) Determine the speed and direction of motion of P when it is *first* furthest away from the origin.

(c) Show that the acceleration of P is always perpendicular to its velocity.

Calculator Assumed

4. [10 marks: 4, 1, 5] [TISC]

A particle P travels in the x-y plane. The position vector of P at time t seconds given by $\boldsymbol{r}(t) = a \sin 2t\, \boldsymbol{i} + b \cos 2t\, \boldsymbol{j}$ ms^{-1} where a and b are positive real constants. P starts from the point with position vector $2\,\boldsymbol{j}$.

(a) Determine the Cartesian equation (in terms of a) of the path of P.

(b) State the value(s) of a if the path of P is not a circle.

(c) If the path of P is not a circle, find t when its acceleration is perpendicular to its velocity.

---

5. [13 marks: 2, 2, 2, 2, 5]

The velocity vector of particle P at time t hours is given by $\boldsymbol{v_P}(t) = 3 \cos t\, \mathbf{i} - 4 \sin t\, \mathbf{j}$ kilometres per hour. It is known that when $t = 0$, the particle is at a point with position vector $0\,\mathbf{i} + 8\,\mathbf{j}$.

(a) Find $\boldsymbol{r_P}(t)$, the position vector of P at time t hours.

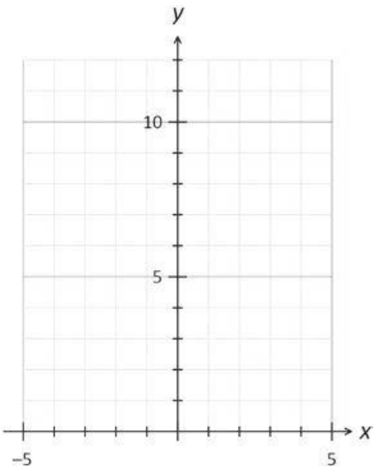
Calculator Assumed

5. (b) Show that the speed of P at time t hours is given by $\sqrt{9+7\sin^2(t)}$

(c) Find the position vector of P when its speed is maximised for the first time.

(d) A second particle Q has position vector at time t hours given by $\boldsymbol{r}_Q(t) = (2\cos t)\,\mathbf{i} + (2\sin t + 8)\mathbf{j}$. P and Q start moving at the same time.

(i) Sketch on the axes provided below the paths of particles P and Q.

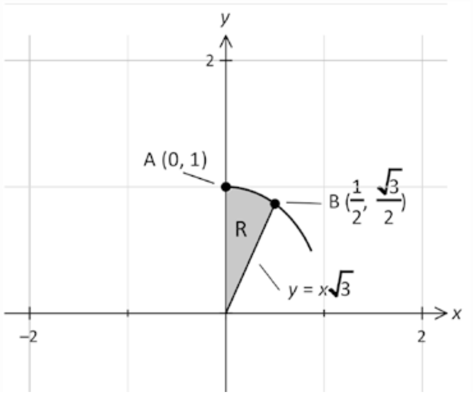


(ii) Show that while the paths of P and Q cross twice, the two particles do not collide.

Calculator Assumed

6. [11 marks: 2, 3, 6] [TISC]

The position vector of a particle P at time t seconds is given by $\boldsymbol{r} = < \sin t, \cos t >$ cm. The given diagram shows part of the path traced by P. The points A (0, 1) and B $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ correspond to the positions of P at times $t = 0$ and $t = \frac{\pi}{6}$ seconds respectively. The shaded region R shows the area swept by the vector $\boldsymbol{r}$ between A and B.



(a) Show that the Cartesian equation of the path of P is given by $y^2 = 1 - x^2$.

(b) Use integrals to determine the length of the curve between *A* and *B*.

(c) Use the method of integral substitution to determine the *exact* area of R.

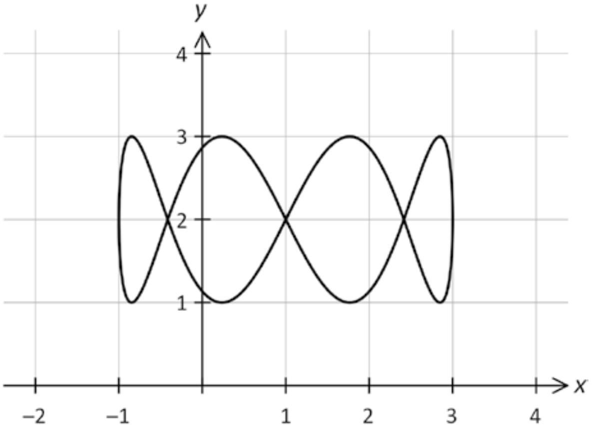
Calculator Assumed

7. [8 marks: 3, 2, 3]

The particle P moves in the *x*-*y* plane. It starts from the point with position vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ cm with velocity $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ cms^{-1}. Its acceleration at time *t* seconds is given by $\boldsymbol{a}(t) = \begin{pmatrix} -2\sin(t) \\ -16\sin(4t) \end{pmatrix}$ cms^{-2}.

(a) Find an expression for $\boldsymbol{r}(t)$, the position vector of P at time *t* seconds.

(b) The given diagram shows the path traced by P. On the diagram provided, locate the position of P and use an arrow to indicate the direction of motion of P at $t = \dfrac{\pi}{2}$ seconds.



(c) Determine one instance (one value of *t*) when P is travelling parallel to the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$.

Calculator Assumed

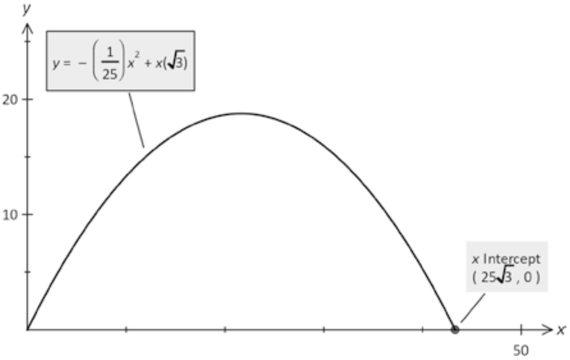
8. [8 marks]

The particle P moves in the *x*-*y* plane. It starts from the point with position vector $< 0, -1 >$ cm with velocity $< 0, 2 >$ cms^{-1}. Its acceleration at time *t* seconds is given by $\boldsymbol{a}(t) = < \cos(t), -2\sin(t) >$ cms^{-2}. Determine the perimeter of the path traced by P.

---

9. [11 marks: 1, 4, 3, 3]

A particle P moves in the *x*-*y* plane. P starts at the origin and its position vector at time *t* seconds is given by $\boldsymbol{r}(t) = (20\cos\theta)t\,\boldsymbol{i} + [(20\sin\theta)\,t - 4t^2]\,\boldsymbol{j}$ metres. The path taken by P has equation $y = \dfrac{-x^2}{25} + x\sqrt{3}$ and is drawn in the given diagram



(a) Find an expression $\boldsymbol{v}(t)$, the velocity of P at time *t* seconds.

Calculator Assumed

9. (b) Find the gradient of the path at the point of projection.

Hence or otherwise prove that $\theta = \frac{\pi}{3}$.

(c) Find when and where P is moving in a direction that is perpendicular to its angle of projection.

(d) Calculate the total distance travelled by P from the point of projection until the point where it meets the *x*-axis the second time.

Calculator Assumed

10. [11 marks: 5, 3, 3]

Particle P is projected from the origin and simultaneously particle Q is projected from the point with position vector <100, *h* > *m*. Both P and Q travel in the *x*-*y* plane. The position vector of P at time *t* seconds is given by $\boldsymbol{r}(t) = <40t\cos 40^{\circ}, 40t\sin 40^{\circ} - 4.9t^2> m$. The velocity vector of Q at time *t* seconds is given by $v(t) = <-10\cos 20^{\circ}, 10\sin 20^{\circ} - 9.8t> ms^{-1}$.

(a) Determine the value of *h* for P and Q to collide.

(b) For the value of *h* obtained in part (a), determine the angle of impact between P and Q.

(c) For the value of *h* obtained in part (a), determine the difference in the distance from the origin to the point of collision and the distance travelled by P to the point of collision.

28 Vector Calculus II

Calculator Assumed

1. [9 marks: 5, 2, 2]

A particle P is projected from the top of a cliff of height 100 m. Define x and y metres as the horizontal and vertical displacements of the particle from its point of projection after t seconds. Its equation of motion at any time t seconds is given by $\frac{d\boldsymbol{r}}{dt} = < 20\sqrt{2}\ ,\ 20\sqrt{2} - 10t >$ where $\boldsymbol{r} = < x, y >$.

(a) The equation of the path of P is given by $y = ax^2 + bx + c$. Find a, b and c.

(b) Find the highest point above ground level reached by P.

(c) The particle hits the ground at K. Find the distance between K and the foot of the cliff.

Calculator Assumed

2. [8 marks: 4, 4]

A missile is fired at a target from the origin O, with equations of motion, t seconds after it was fired, given by $\frac{dx}{dt} = u\cos\theta$ and $\frac{dy}{dt} = u\sin\theta - 10\,t$, where the constants u and θ are respectively the speed of projection and the angle of projection (with the horizontal axis).

(a) Prove that the equation of the path is $y = x\tan\theta - \frac{5x^2}{u^2}(1 + \tan^2\theta)$.

(b) Prove that for the missile to hit a target located at the point with position vector < a, b >, $a^2u^4 - 20\,ba^2u^2 - 100\,a^4 \geq 0$.

Calculator Assumed

3. [9 marks: 7, 2]

A particle P moving in the *x*-*y* plane starts from the point with position vector $< 0, -\frac{1}{\pi} >$ m with equation of motion given by $\frac{d\boldsymbol{r}}{dt} = < \cos(\pi t), \sin(\pi t) >$ where $\boldsymbol{r} = < x, y >$ m and t is time in minutes.

(a) Show that the path of P is a circle.
State the centre and radius of this circular path.

(b) Find the period of the motion for P. Justify your answer.

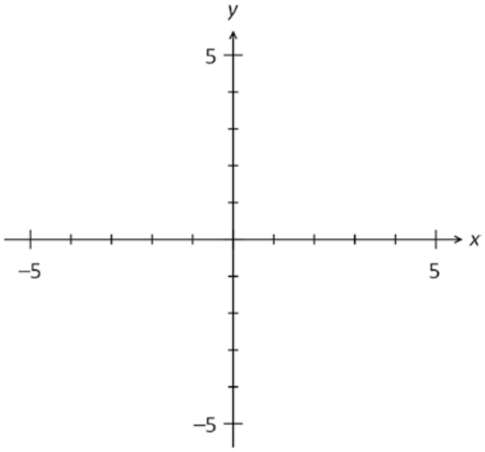
Calculator Assumed

4. [10 marks: 4, 3, 3]

A particle P moves in the x-y plane and starts from the point (4, 0). The velocity of P at time t is given by $\frac{dx}{dt} = 4y$ and $\frac{dy}{dt} = -x$.

(a) Show that the path of P is an ellipse.

(b) Sketch the path of P and indicate its direction of motion.



(c) Find the period of the motion. Justify your answer.

Calculator Assumed

5. [9 marks: 2, 4, 3]

A particle moves in the x-y plane. Its position at time t seconds is given by

$$x = 2\cos t + \cos 2t \quad \text{and} \quad y = 2\sin t + \sin 2t$$

where x and y are both measured in metres.

(a) Find an expression for $\dfrac{dy}{dx}$ in terms of t.

(b) Find t for $0 \le t \le 2\pi$, when $\dfrac{dy}{dx} = 0$.

(c) Use your answer in (b) to find the coordinates of point with the highest y-coordinate.

Calculator Assumed

6. [10 marks: 4, 3, 3]

A particle moves in the x-y plane.
Its position (x, y) at time t seconds satisfies the pair of differential equations:
$\frac{dx}{dt} = 10\,t$ and $\frac{dy}{dt} = 10(\sqrt{3} - t)$ where x and y are both measured in metres.
The particle starts moving from the point with coordinates (0, 4).

(a) Determine the position (x, y) of the particle at time $t = 2$ seconds.

(b) Determine the Cartesian equation of the path travelled by the particle.

(c) The distance travelled by the particle in the interval $a \le t \le b$ is given by $S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. Calculate the distance travelled in the first two seconds.

Calculator Assumed

7. [6 marks: 2, 1, 3]

A particle P starts moving from the origin in the x-y plane. Its position vector and velocity at time t seconds is given by $\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ cm and $\boldsymbol{v} = \begin{pmatrix} 1+y \\ 2-x \end{pmatrix}$ cms^{-1} respectively.

(a) Show that $\dfrac{dx}{dt} = 1 + y$ and $\dfrac{dy}{dt} = 2 - x$.

(b) Hence, determine an expression for $\dfrac{dy}{dx}$ in terms of x and y.

(c) Use your answer in (b) to prove that the path of P is a circle with equation $(x-2)^2 + (y+1)^2 = 5$

29 Sampling Distributions

Calculator Assumed

1. [4 marks: 1, 2, 1]

   Bryce sells rice in bags with a mean of 9.95 kg and standard deviation of 0.05 kg. Samples of size *n* are selected and the mean mass of each sample calculated.
   (a) Describe the probability distribution for the mean mass of the samples.

   (b) Find *n* if the sampling distribution has a standard deviation of 0.008.

   (c) Comment on the value of the standard deviation of the sampling distribution as the sample size *n* increases.

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2. [8 marks: 3, 3, 2]

   SugarWest sells sugar in packs with mean 995 g with standard deviation 15 g. Samples of size *n* are selected and the mean mass of each sample calculated.
   (a) Find the probability that the mean mass:
   (i) of a sample of 40 packs exceeds 1000 g.

   (ii) of a sample of 100 packs exceeds 1000 g.

   (b) Comment on your answers in (a) and (b).

Calculator Assumed

3. [8 marks: 2, 4, 2]

(a) The waiting time at a set of traffic lights may be modelled by a uniform distribution with mean 30 seconds and variance 300. A random sample of 100 waiting times was taken. Describe the distribution of sample mean waiting time $\overline{X}$.

(b) $\overline{W}$ the sample mean waiting times of samples of size *n* has an approximate normal distribution. One sample of size *n* had a sample mean waiting time of 28 seconds and sample standard deviation 3.8 seconds. Estimate the value of *n* if in 1 000 samples of size *n*, 20 samples had mean waiting times in excess of 29 seconds.

(c) 1 000 samples of 50 waiting times each were taken and the mean of each sample calculated. Let $\overline{Y}$ represent the sample mean waiting time. The sample means had a mean of 29.2 seconds and a standard deviation of 4.2 seconds. State with reasons the distribution of $\dfrac{\overline{Y}-29.2}{4.2}$ and its parameters.

Calculator Assumed

4. [9 marks: 3, 3, 3]

The time taken to serve customers at a drive through counter of a fast food restaurant has mean 3 minutes with variance 3 minutes.

(a) Estimate the probability that:

(i) the time taken to serve 20 successive customers exceeds 61 minutes.

(ii) no more than 100 minutes is required to serve 30 successive customers.

(b) The probability that the time taken to record 50 successive transactions does not exceed *k* minutes is approximately 0.1. Find *k*.

---

5. [6 marks: 2, 2, 2] [TISC]

The mean mass of a variety of mangoes is 879 g with standard deviation 5 g.

(a) Estimate the probability that a sample of 50 such mangoes have a mean mass not exceeding 880 g.

Calculator Assumed

5. (b) 90% of samples of 50 mangoes each have mean sample masses above *k* g. Find *k*.

(c) Estimate the probability that the total mass of a sample of 50 such mangoes exceed 43.94 kg.

---

6. [7 marks: 1, 4, 2] [TISC]

The time taken for a child to complete a particular puzzle is normally distributed with mean 3 minutes with standard deviation 20 seconds.

(a) A sample of fifty children of the same age took 2 hours and 35 minutes to complete the puzzle. Find the mean time, in seconds, for this sample.

(b) Estimate the probability that a second sample of 50 children of the same age will take a total of more than 2 hours and 35 minutes to complete the puzzle.

(c) Children who complete the puzzle under *k* seconds are classified "highly gifted". If 0.01% of all children are classified highly gifted, find *k*.

Calculator Assumed

7. [6 marks: 3, 3] [TISC]

The mean time taken by an airline to fly between two cities is 240 minutes with a standard deviation of 15 minutes.

(a) Calculate the probability that in 80 flights between these two cities, the mean flight time is less than 241 minutes. State any assumptions you may need to make.

(b) The airline flies n times between these two cities. The probability that the mean time for these flights is less than 241 minutes is 0.7929. Calculate the value of n. Show clearly how you obtained your answer.

30 Point & Interval Estimates for μ

Calculator Free

1. [9 marks: 2, 3, 4] [TISC]

The time taken to complete a task T has mean μ minutes and standard deviation 10 minutes. For Z as the standard normal variable with mean 0 and standard deviation 1, $P(-2.5 < Z < 2.5) \approx 0.988$.

(a) A sample of 100 students completed task T with a mean time of 102 minutes. Find a 98.8% confidence interval for μ.

(b) Another sample of n students (where $n \geq 30$) is chosen. Find n if we are to be 98.8% confident that the sample mean is to differ from μ by no more than 1.25 minutes.

(c) Given that $\mu = 100$ minutes, estimate the probability that a sample of 100 students will complete the task with a mean time exceeding 102.5 minutes.

Calculator Assumed

2. [8 marks: 2, 3, 3]

Let μ and σ respectively be the mean wing-span of the Australian wedge tail eagle and its associated standard deviation.

(a) The wing-spans of a random sample of 100 eagles are measured. The mean wing-span for this sample is 2.2 m with sample standard deviation 0.12 m.

(i) Find a point estimate for μ and σ. Justify your answer.

(ii) Find a 98% confidence interval for μ.

(b) For σ = 0.12, find the sample size *n* such that the 90% confidence interval for μ differs from the sample mean by no more than 0.05 m.

---

3. [9 marks: 2, 4, 3]

The time required to refuel a bus has mean μ minutes and standard deviation 1 minute. A sample of 50 buses took 248 minutes to refuel.

(a) Use the data from the sample provided to describe the sampling distribution of sample means of size 50.

Calculator Assumed

3. (b) Use the data from the sample provided to calculate a 95% confidence interval for μ. Explain what this 95% confidence interval means.

(c) A second sample of 100 buses provided a confidence interval for μ as $4.58 \le \mu \le 5.02$. Determine the confidence level associated with this interval.

4. [9 marks: 3, 3, 3] [TISC]

The amount of spring water in each bottle sold by a manufacturer has a mean of μ mL with a standard deviation of 1 mL.

(a) A randomly chosen sample of 100 bottles had a mean volume of 999.8 mL with a standard deviation of 1 mL. Use this sample to determine a 95% confidence interval for μ.

(b) Another random sample of 200 bottles gave a confidence interval with a margin of error of 0.15. Determine the level of confidence of this interval.

(c) Calculate the minimum sample size for a 90% confidence interval for μ with a margin of error of less than 0.1.

Calculator Assumed

5. [13 marks: 2, 4, 1, 3, 2, 1]

Let μ and σ respectively be the mean mass of locally farmed yabbies and its associated standard deviation.

(a) The masses of a random sample of 200 yabbies are measured. The mean mass for this sample is 95.6 g with a sample standard deviation of 9.2 g.

(i) Find a point estimate for μ and σ. Justify your answer.

(ii) Find a 95% and a 99% confidence interval for μ.

(iii) Comment on the statistical implications of the differing widths of the two intervals in (ii).

(b) For σ = 9.2, find the sample size *n* such that the 90% confidence interval for μ:

(i) has width 3 g.

(i) has width 1 g.

(iii) Comment on the statistical implications of the differing widths of the two intervals in (i) & (ii) for a given level of confidence.

Calculator Assumed

6. [10 marks: 1, 2, 3, 2, 2]

The heights of year 8 students in a state is known to be normally distributed with mean μ cm and population standard deviation 11.5 cm. A random sample of size 20 is selected. The mean height of the students in this sample is 152 cm.

(a) Find a point estimate for μ.

(b) State the probability distribution for the sample mean heights.

(c) Show clearly how a 95% confidence interval for μ is derived using the probability distribution for the sample means.

(d) Explain what a 99% confidence interval for μ is.

(e) The mean height of another sample was measured to be 159 cm. Use your answer in (c) to determine with reasons if the students in the second sample are unusually tall for their age.

Calculator Assumed

7. [8 marks: 1, 1, 4, 2]

The age of mathematics teachers in a certain geographical region is known to be normally distributed with a mean of μ and a population standard deviation of 5 years 3 months.

(a) A sample of 25 teachers is taken. The mean age of teachers in this sample is 54 years 2 months.

(i) Find a point estimate for μ.

(ii) Given that all 25 teachers from this sample were selected from just one suburb, determine mathematically if your estimate in (i) is reliable.

(b) A stratified sample of 25 teachers is selected from this region and the mean age of the teachers in this sample is 42 years 9 months.

(i) Find a 90% and a 99% confidence interval for μ.

(ii) The mean age for a second sample is exactly 40 years and 4 months old. Determine with reasons if the teachers in this sample are younger than those in the first sample.

Calculator Assumed

8. [10 marks: 2, 4, 2, 2]

Let μ and σ respectively be the mean telephone waiting time for calls to a certain bank and its associated standard deviation.

(a) The waiting times for a random sample of 150 customers were recorded. The mean waiting time was 22.6 minutes with a sample standard deviation of 2.7 minutes. Find a 99% confidence interval for μ.

(b) The waiting times for a second random sample of 150 customers were recorded. The mean waiting time was $\bar{x}$ minutes with a sample standard deviation of s minutes. A 99% confidence interval for μ is $17.78 < \mu < 19.22$ minutes. Find $\bar{x}$ and s.

(c) Use your answers in (a) and (b) to explain clearly why for a given population, there exists more than one 99% confidence interval estimates of μ.

(d) For $\sigma = 2.7$, find the sample size n such that the 95% confidence interval for μ differs from the sample mean by no more than 0.5 minutes.

Calculator Assumed

9. [7 marks: 2, 3, 2] [TISC]

Manufacturers must label food “genetically modified” if more than 1% of ingredients have been genetically modified. The manufacturer tests a sample of 100 one kg packs of this food product and calculates the mean mass of genetically modified ingredients included in the packs is 9.8 g. Assume that the population standard deviation of the amount of genetically modified ingredients is 0.88 g per kg.

(a) Calculate a 99% confidence interval for the amount of genetically modified ingredients in 1 kg of the manufactured food.

(b) Find the minimum size of a second sample if the 99% confidence limits is not to exceed 10 g.

(c) The manufacturer decides to label his food product with the warning "May contain genetically modified ingredients". What percentage of samples would have a mean exceeding 10 g per kilogram?

Calculator Assumed

10. [10 marks: 4, 4, 2] [TISC]

The mean amount of blood drawn out from an individual donor at a blood bank is μ mL with standard deviation 20 mL.

(a) On a particular day, 50 donors at the blood bank donated a total of 49.9 L of blood. Determine a 95% confidence interval for μ.

(b) How many donors should be in a sample if a 90% confidence interval for μ is $995 \leq \mu \leq 1000$ mL?

(c) A 99% confidence interval for μ is $998 \leq \mu \leq 1000$ mL.
A student concluded that the probability that the mean amount of blood donated by an individual donor is between 998 mL and 1000 mL is 0.99.
Is the student correct? Justify your answer.

Calculator Assumed

11. [9 marks 2, 3, 2, 2] [TISC]

The mass of sugar in a 200 g chocolate bar is normally distributed with mean μ g and standard deviation σ g.

(a) A randomly chosen sample of 400 chocolate bars (200 g each) had a mean sugar mass of 100.5 g with a standard deviation of 0.8 g. Use this sample to determine a 90% confidence interval for μ.

(b) Another random sample of 400 chocolate bars (200 g each) gave a 90% confidence for μ as $99.9088 < \mu < 100.0912$.

(i) Determine the mean mass of sugar for this sample and the accompanying standard deviation.

(ii) Determine with reasons if the sample in (a) and sample in (b) are more likely to be from the same population or more likely to be from different populations.

(c) Given that $\mu = 100$ g and $\sigma = 0.8$ g, calculate the minimum sample size for a 90% confidence interval for μ with an interval width of less than 0.1.

Calculator Assumed

12. [10 marks: 2, 3, 3, 1, 1]

The time it takes Tracy to get to work each day is a continuous random variable T with mean μ minutes and standard deviation σ minutes. Tracy took a total of 21 hours to travel to work for her last 100 trips with a sample standard deviation of 1.5 minutes.

(a) State a single value estimate for μ and σ.

(b) Calculate a 95% confidence interval for μ.

(c) For another 100 trips, the sample standard deviation was 1.8 minutes and Tracy obtained $12.0965 \le \mu \le 12.9035$ as a confidence interval for μ.

(i) Determine the confidence level for this interval.

(ii) Determine with reasons if the second sample is statistically different from the first sample.

(d) Tracy calculated a total of thirty 90% confidence intervals for μ. How many of these intervals are expected to contain μ?

Fully Worked Solutions

01 Complex Numbers I

Calculator Free

1. [5 marks: 1, 2, 2] [TISC]

Given the complex numbers $z_1 = 2 - i,\ z_2 = i$ and $z_3 = 2a\,i$, find:

(i) $z_1\overline{z_2}$

$$\boxed{(2-i)(-i) = -1-2i \quad \checkmark}$$

(ii) $|z_1 + z_3|$

$$\boxed{\begin{aligned} |(2-i)+2ai| &= |2+(2a-1)\,i| \quad \checkmark \\ &= \sqrt{[4+(2a-1)^2]} \quad \checkmark \end{aligned}}$$

(iii) $arg\left(\dfrac{z_3}{2a\,z_2}\right)$.

$$\boxed{\begin{aligned} arg\left(\frac{2ai}{2a\,i}\right) &= \arg(1) \quad \checkmark \\ &= 0. \quad \checkmark \end{aligned}}$$

---

2. [5 marks: 1, 2, 2] [TISC]

Let the complex numbers $z_1 = a + 2i,\ z_2 = 3$ and $z_3 = \sqrt{3} - i$.

(a) Express the following in the form $x + yi$.

(i) $z_1^{\,2}$

$$\boxed{(a+2i)\,(a+2i) = (a^2-4) + 4ai \quad \checkmark}$$

(ii) $\dfrac{z_1}{z_3}$

$$\boxed{\frac{a+2i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i} = \frac{a\sqrt{3}\ -2}{4} + \frac{a+2\sqrt{3}}{4}\,i \quad \checkmark\checkmark}$$

(b) Determine in **exact form** $arg\,(i\,z_2) + arg\,(z_3)$.

$$\boxed{\begin{aligned} &arg\,(3\,i) + arg\,(\sqrt{3}-i) \\ &= \frac{\pi}{2} + \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}. \quad \checkmark\checkmark \end{aligned}}$$

Calculator Free

3. [7 marks: 2, 2, 3] [TISC]

Let the complex numbers $z_1 = a - 2i,\ z_2 = 1 + i$ and $z_3 = -4i$, where a is a *real number*. Determine all possible values of a if:

(a) $z_1 \times z_2 = z_3$

$$\boxed{\begin{aligned} &(a-2i)(1+i) = -4i \quad \checkmark \\ &(a+2) + (a-2)i = -4i \\ &\Rightarrow\ a = -2 \quad \checkmark \end{aligned}}$$

(b) $\dfrac{z_1}{z_3} = \dfrac{1}{2}z_2$

$$\boxed{\begin{aligned} &\frac{a-2i}{-4i} = \frac{1}{2}(1+i) \\ &a - 2i = 2 - 2i \quad \checkmark \\ &\Rightarrow\ a = 2 \quad \checkmark \end{aligned}}$$

(c) $arg\,(z_1) + arg\,(z_2) = -\dfrac{\pi}{4}$.

$$\boxed{\begin{aligned} &arg(z_2) = \frac{\pi}{4}. \quad \checkmark \\ &\Rightarrow\ arg(z_1) + \frac{\pi}{4} = -\frac{\pi}{4} \quad \checkmark \\ &\qquad arg(z_1) = -\frac{\pi}{2} \\ &\text{Hence, } a = 0. \quad \checkmark \end{aligned}}$$

---

4. [9 marks: 2, 2, 3, 2] [TISC]

Let the complex numbers $z_1 = 2 + a\,i$ and $z_2 = 1 - 2\,i$, where a is a *real number*. Determine all possible values of a if:

(a) $z_1 \times \overline{z_1} = 2\,a^2$

$$\boxed{\begin{aligned} (2+ai)(2-ai) &= 2a^2 \\ 4 + a^2 &= 2a^2 \quad \checkmark \\ a &= \pm 2 \quad \checkmark \end{aligned}}$$

(b) $z_1 = i\,\overline{z_1}$

$$\boxed{\begin{aligned} (2+ai) &= i(2-ai) \\ 2 + ai &= 2i + a \quad \checkmark \\ a &= 2 \quad \checkmark \end{aligned}}$$

Calculator Free

4. (c) $\boldsymbol{Re}\,(z_1^{\,2}) = \boldsymbol{Re}\,(z_2^{\,2})$

$$\boxed{\begin{array}{ll} z_1^2 = 4 - a^2 + 4ai & \\ z_2^2 = 1 - 4 - 4i & \checkmark \\ 4 - a^2 = -3 & \checkmark \\ \quad\Rightarrow \quad a = \pm\sqrt{7} & \checkmark \end{array}}$$

(d) $\boldsymbol{arg}\,(z_1) = \boldsymbol{arg}\,(\overline{z_1})$

$$\boxed{\begin{array}{ll} \tan^{-1}\left(\dfrac{a}{2}\right) = \tan^{-1}\left(-\dfrac{a}{2}\right) & \checkmark \\ \quad\Rightarrow \quad a = 0 & \checkmark \end{array}}$$

---

5. [11 marks: 2, 3, 3, 3] [TISC]

Let the complex numbers $z_1 = a + 5\,i$, $z_2 = 3 - 4\,i$ and $z_3 = 1 + i\,\sqrt{3}$ where a is a *real number*.

(a) Find a if $|z_1| = |z_2|$

$$\boxed{\begin{array}{ll} a^2 + 25 = 25 & \checkmark \\ \quad\Rightarrow \quad a = 0 & \checkmark \end{array}}$$

(b) Find the exact value of a if $\tan[\boldsymbol{arg}\,(\overline{z_1})] = \tan[\boldsymbol{arg}\,(z_3)]$.

$$\boxed{\begin{array}{ll} \tan[arg(\overline{z})] = \tan[arg(z_3)] & \checkmark \\ -\dfrac{5}{a} = \dfrac{\sqrt{3}}{1} \;\Rightarrow\; a = -\dfrac{5\sqrt{3}}{3} & \checkmark\checkmark \end{array}}$$

(c) Explain clearly why there is no solution for a if $|\,z_1 - z_2\,| = |z_3|$.

$$\boxed{\begin{array}{ll} (a-3)^2 + (5+4)^2 = 4 & \checkmark \\ \Rightarrow\; (a-3)^2 < 0 & \checkmark \\ \text{But } a \text{ is Real. Hence, there is no real solution for } a. & \checkmark \end{array}}$$

(d) Find the value of a if $\mathbf{Im}\left(\dfrac{z_1}{z_2}\right) = 0$

$$\boxed{\begin{array}{ll} \left(\dfrac{z_1}{z_2}\right) = \dfrac{a+5i}{3-4i} = \dfrac{3a - 20 + (4a+15)i}{25} & \checkmark \\ \Rightarrow\; 4a + 15 = 0 \;\Rightarrow\; a = -\dfrac{15}{4} & \checkmark\checkmark \end{array}}$$

Calculator Free

6. [9 marks: 3, 3, 3]

The complex number z has a modulus of 2 and an argument of $\dfrac{2\pi}{3}$

(a) State the complex number z^4 in Cartesian form.

$$\boxed{\begin{array}{ll} z^4 = \left[2\,cis\left(\dfrac{2\pi}{3}\right)\right]^4 & \checkmark \\ \quad = 16\,cis\left(\dfrac{8\pi}{3}\right) = 16\,cis\left(\dfrac{2\pi}{3}\right) & \checkmark \\ \quad = -8 + 8\sqrt{3}\;i & \checkmark \end{array}}$$

(b) State the complex number $\dfrac{z}{i}$ in *cis* form.

$$\boxed{\begin{array}{ll} \dfrac{z}{i} = -i\,z & \checkmark \\ \quad = cis\left(\dfrac{-\pi}{2}\right) \times 2\,cis\left(\dfrac{2\pi}{3}\right) & \checkmark \\ \quad = 2\,cis\left(\dfrac{\pi}{6}\right) & \checkmark \end{array}}$$

(c) Given that $w \times z = 2i$, determine the complex number w.
Give your answer in polar form.

$$\boxed{\begin{array}{ll} w \times z = 2\,cis\,\pi & \checkmark \\ \quad w = \dfrac{2\,cis\,\pi}{2\,cis\left(\dfrac{2\pi}{3}\right)} & \checkmark \\ \quad\;\; = cis\left(\dfrac{\pi}{3}\right) & \checkmark \end{array}}$$

---

7. [7 marks: 2, 2, 3]

Let $u = a\;cis\;\alpha$ and $v = a\;cis\;\beta$ where $a > 0$ and α and β are acute.
Express each of the following in *cis* form.

(a) $\dfrac{1}{u}$

$$\boxed{\begin{array}{ll} \dfrac{1}{u} = \dfrac{cis\,0}{a\;cis\,(-\alpha)} & \checkmark \\ \quad \dfrac{1}{a}\,cis\,(\alpha) & \checkmark \end{array}}$$

Calculator Free

7. (b) $(\bar{v}uv)^8$

$$\begin{aligned}(\bar{v}uv)^8 &= (\bar{v}vu)^8 \\ &= a^2u^8 \quad ✓ \\ &= a^{10}\ cis\,(6\alpha) \quad ✓\end{aligned}$$

(c) $u + \bar{u}$

$$\begin{aligned}u + \bar{u} &= a\,cis\,(\alpha) + a\,cis\,(-\alpha) \quad ✓ \\ &= a\,[\cos\alpha + i\sin\alpha + (\cos(-\alpha) + i\sin(-\alpha))] \quad ✓ \\ &= (2\,a\cos\alpha) \\ &= (2\,a\cos\alpha)\,cis\,0 \quad ✓\end{aligned}$$

---

8. [11 marks: 3, 4, 4] [TISC]

(a) Given that $4a - 4a\,i = r\,cis\,\theta$, find r and θ in terms of $a > 0$ where appropriate.

$$\begin{aligned}r &= \sqrt{(4a)^2+(-4a)^2} \quad \surd[(4a)^2+(-4a)^2] \\ &= 4a\sqrt{2} \quad ✓ \\ \tan\theta &= -1 \text{ (in Quadrant 4 as } a > 0) \quad ✓ \\ \Rightarrow\ \theta &= -\frac{\pi}{4}. \quad ✓\end{aligned}$$

(b) Given that $r\,cis\left(\frac{-5\pi}{6}\right) = x + 5a\,i$, find r and x in terms of $a < 0$.

$$\begin{aligned}\tan\left(\frac{-5\pi}{6}\right) &= \frac{5a}{x} \quad ✓ \\ \frac{1}{\sqrt{3}} &= \frac{5a}{x} \quad ✓ \\ \Rightarrow\ x &= 5a\sqrt{3} \quad ✓ \\ r &= \sqrt{(5a\sqrt{3})^2 + (5a)^2} = 10\,|a| \quad ✓\end{aligned}$$

(c) Simplify $\left[\sqrt{3}\,cis\left(\frac{5\pi}{6}\right)\right]^3 \times \sqrt{3\,cis\left(\frac{\pi}{4}\right)}$

$$\begin{aligned}&3\sqrt{3}\ cis\left(\frac{15\pi}{6}\right) \times \sqrt{3}\ cis\left(\frac{\pi}{8}\right) \quad ✓✓ \\ &= 9\,cis\left(\frac{21\pi}{8}\right) \quad ✓ \\ &= 9\,cis\left(\frac{5\pi}{8}\right) \quad ✓\end{aligned}$$

Calculator Free

9. [7 marks: 3, 4]

Let $x = cis\,\frac{\pi}{3}$, $y = \left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$ and $z = -1 - 3i$.

(a) Find $\frac{xy}{z-2}$ giving your answer in polar form.

$$\begin{aligned}\frac{xy}{z-2} &= \frac{\left(cis\frac{\pi}{3}\right)\left(cis\frac{\pi}{8}\right)}{3cis\left(\frac{-3\pi}{4}\right)} \quad ✓ \\ &= \frac{1}{3}cis\left(\frac{\pi}{3}+\frac{\pi}{8}+\frac{3\pi}{4}\right) \quad ✓ \\ &= \frac{1}{3}cis\left(\frac{29\pi}{24}\right) = \frac{1}{3}cis\left(\frac{-19\pi}{24}\right) \quad ✓\end{aligned}$$

(b) Find $\sqrt{3}\,x + \sqrt{2}\,y^2 + z$ giving your answer in *cis* form.

$$\begin{aligned}\text{Expression} &= \sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) + \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) + -1 - 3i \quad ✓ \\ &= \sqrt{3}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) - 1 - 3i \quad ✓ \\ &= \frac{\sqrt{3}}{2} - i\frac{1}{2} \quad ✓ \\ &= cis\left(\frac{-\pi}{6}\right) \quad ✓\end{aligned}$$

---

10. [7 marks: 2, 5]

(a) Express $2\sqrt{3}\,cis\left(\frac{-\pi}{3}\right)$ in Cartesian (rectangular) form.

$$\begin{aligned}2\sqrt{3}\,cis\left(\frac{-\pi}{3}\right) &= 2\sqrt{3}\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right) \quad ✓ \\ &= 2\sqrt{3}\left(\frac{1}{2} + \frac{-i\sqrt{3}}{2}\right) \\ &= \sqrt{3} - 3i \quad ✓\end{aligned}$$

Calculator Free

10. (b) Given $u + v = \sqrt{3} + i$ and $u - v = 2\sqrt{3}\,cis\left(\frac{-\pi}{3}\right)$.

Determine the complex numbers u and v giving your answer in polar form.

$$\begin{aligned} u + v &= \sqrt{3} + i && (1) \quad ✓ \\ u - v &= \sqrt{3} - 3i && (2) \quad ✓ \\ (1) + (2) \quad 2u &= 2\sqrt{3} - 2i && \\ u &= \sqrt{3} - i && ✓ \\ \Rightarrow\ u &= 2\,cis\left(\frac{-\pi}{6}\right) && ✓ \\ (1) - (2) \quad 2v &= 4i \ \Rightarrow\ v = 2i && \\ \Rightarrow\ v &= 2\,cis\left(\frac{\pi}{2}\right) && ✓ \end{aligned}$$

---

11. [6 marks: 1, 2, 3]

Let $a = -1 + \sqrt{3}\,i$ and $b = -1 - i$.

(b) Find ab in exact Cartesian form.

$$ab = (1 + \sqrt{3}) + (1 - \sqrt{3})\,i \quad ✓$$

(c) Find ab in exact *cis* form.

$$\begin{aligned} ab &= 2cis\left(\frac{2\pi}{3}\right) \times \sqrt{2}\,cis\left(\frac{-3\pi}{4}\right) && ✓ \\ &= 2\sqrt{2}\,cis\left(\frac{-\pi}{12}\right) && ✓ \end{aligned}$$

(c) Use your answers in (a) and (b) to find $\sin\left(\frac{\pi}{12}\right)$ in exact form.

From (b): $ab = 2\sqrt{2}\cos\left(\frac{-\pi}{12}\right) + \left(2\sqrt{2}\sin\left(\frac{-\pi}{12}\right)\right)i$

From (a): $ab = (1 + \sqrt{3}) + (1 - \sqrt{3})\,i$

Compare Im parts: $2\sqrt{2}\sin\left(\frac{-\pi}{12}\right) = (1 - \sqrt{3})$ ✓

$$\sin\left(\frac{-\pi}{12}\right) = \frac{\sqrt{2} - \sqrt{6}}{4} \quad ✓$$

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4} \quad ✓$$

---

02 Complex Numbers II

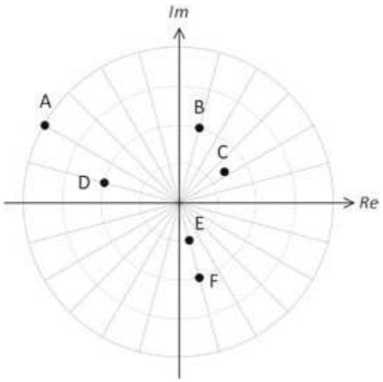
Calculator Free

1. [6 marks] [TISC]

The diagram below shows an Argand diagram marked with the points A, B, C, D, E and F. These points correspond to the complex numbers (not in matching order): $z \quad -iz \quad iz \quad 2z \quad \bar{z} \quad z^2 \quad \frac{1}{z} \quad \frac{2}{z} \quad \sqrt{z}$

Complete the table below matching these points with one of the complex numbers listed.

Point	Complex Number
A	z^2 ✓
B	z ✓
C	$\sqrt{z}$ ✓
D	iz ✓
E	$\frac{2}{z}$ ✓
F	$\bar{z}$ ✓



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2. [4 marks: 2, 2]

Let $z = r\,cis\,\theta$. Describe what happens to the magnitude and direction of the vector representation of z:

(a) when z is multiplied with i.

Magnitude of vector is unchanged. ✓

Vector is rotated $\frac{\pi}{2}$ radians anti-clockwise about the origin. ✓

(b) when z is multiplied with $2\,cis\left(\frac{\pi}{7}\right)$

Magnitude of vector is doubled. ✓

Vector is rotated $\frac{\pi}{7}$ radians anti-clockwise about the origin. ✓

Calculator Free

3. [7 marks: 1, 2, 2, 2]

The vector **OA** in the Argand diagram below represents the complex number a.

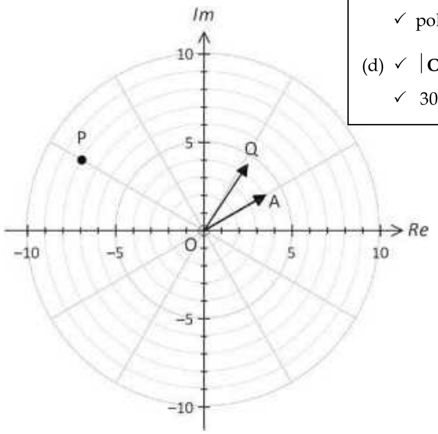
Let the complex number $b = 2\,cis\left(\frac{2\pi}{3}\right)$.

(b) ✓ polar distance = 8

✓ polar angle 120^{o}.

(d) ✓ $|\mathbf{OQ}| > |\mathbf{OA}|$

✓ $30^{o} < arg(a + b) < 60^{o}$.



(a) State the modulus and argument of a.

$|a| = 4$ and $arg(a) = 30^{o}$ or $\frac{\pi}{6}$ ✓

(b) On the Argand diagram above, plot the point P which represents the product of the complex numbers a and b.

(c) Use linear transformations to describe what happens to **OA** when a is multiplied by b.

Magnitude of **OA** is dilated by a factor of 2. ✓
OA is rotated 120^{o} anti-clockwise about the origin O. ✓

(d) On the Argand diagram above, plot the point Q which represents the sum of the complex numbers a and b.

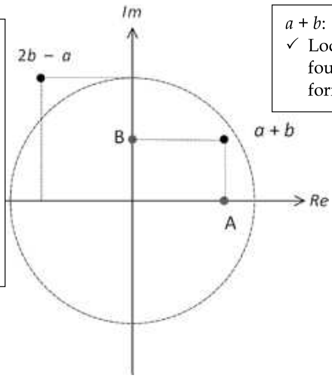
Calculator Free

4. [8 marks: 3, 5]

In each of the Argand diagrams below, the point A represents the complex number a and the point B represents the complex number b.

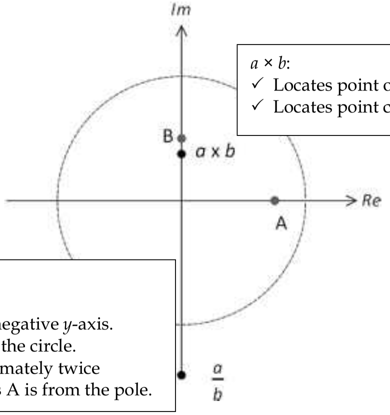
(a) Mark and label in the diagram below, the points representing the complex numbers $a + b$ and $2b - a$.

$b - 2a$:
✓ Locates point in quadrant 2 outside the circle.
✓ Locates point as fourth vertex of rectangle formed by the pole, the reflection of the point A about the y-axis and the point where circle intersects the positive y-axis.



$a + b$:
✓ Locates point as the fourth vertex of rectangle formed by the pole and A

(b) The centre of the circle drawn is at the origin and the radius of the circle is 1 unit. Mark and label in the diagram below, the points representing the complex numbers $a \times b$ and $\frac{a}{b}$.



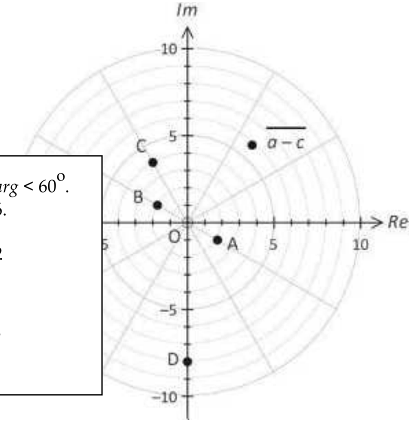
$a \times b$:
✓ Locates point on the positive y-axis.
✓ Locates point closer to B than the pole.

$\frac{a}{b}$:
✓ Locates point on the negative y-axis.
✓ Locates point outside the circle.
✓ Locates point approximately twice as far from the pole as A is from the pole.

Calculator Free

5. [8 marks: 2, 2, 2, 2]

The points A and C on the diagram below represents the complex numbers a and c. The complex number b is such that $ab = c$.



(a) ✓ Point with $30^o < arg < 60^o$.
✓ Point with $|z| \approx 6$.

(b) ✓ polar distance = 2
✓ polar angle 150^o.

(c) ✓ polar distance = 8
✓ polar angle 150^o.

(a) On the diagram above, plot the point representing $\overline{a-c}$.

(b) On the diagram above, plot the point B that represents the complex number b.

(c) On the diagram above, plot the point D which represents the complex number ab^2.

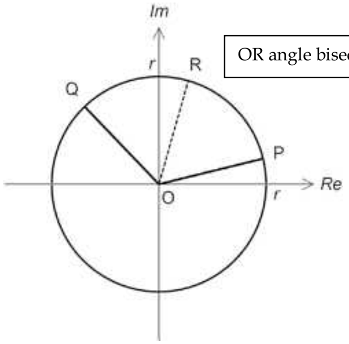
(d) Use the language of linear transformations to describe what happens to the complex number a when it is multiplied by b^4.

Modulus of a is dilated by a factor of $2^4 = 16$ ✓
and argument of a is rotated by 600^o anticlockwise ✓
(or $\frac{10\pi}{3}$ radians) about O.

Calculator Free

6. [7 marks: 1, 3, 3]

The points P, Q and R represent the complex numbers w, z and $w + z$ respectively. The points P, Q and R lie on a circle with centre O and of radius r. The Argand diagram below shows the location of the points P and Q.



OR angle bisector of ∠POQ. ✓

(a) On the diagram above, mark the point R representing the complex number $w + z$.

(b) Prove that $\angle POQ = \frac{2\pi}{3}$.

$|OP| = |OQ| = |OR| = r$
Also, $|OQ| = |PR| = r$. ✓

Hence, ΔPOR and ΔROQ are equilateral. ✓
$\Rightarrow \angle POR = \angle ROQ = \frac{\pi}{3}$ ✓
$\Rightarrow \angle POQ = \frac{2\pi}{3}$.

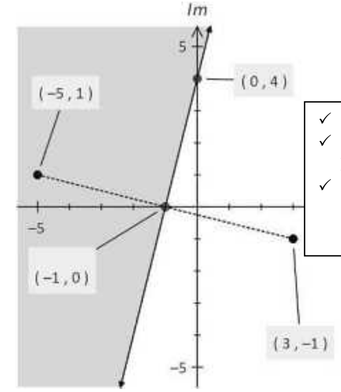
(c) If $w = r\ cis\ \theta$, find z in terms of r and θ and hence prove that $z^3 = w^3$.

$z = r\ cis\ (\theta + \frac{2\pi}{3}) = r\ cis\ \theta \times cis\ \frac{2\pi}{3}$ ✓
$= w \times cis\ \frac{2\pi}{3}$ ✓
Hence, $z^3 = \left(w \times cis \frac{2\pi}{3}\right)^3$
$= w^3 \times cis\ 2\pi$ ✓
$= w^3$.

Calculator Free

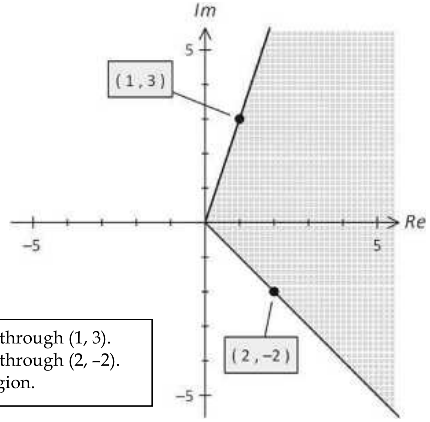
7. [6 marks: 3, 3]

(a) Sketch the locus of the complex number z satisfying the inequality
$|z - 3 + i| \geq |z + 5 - i|$



✓ Shades a half-plane.
✓ Boundary is perpendicular to line joining (–5, 1) and (3, –1).
✓ Boundary is perpendicular bisector of the line joining (–5, 1) and (3, –1).

(b) Sketch the locus of the complex number z satisfying the inequality
$\frac{-\pi}{4} \leq \arg(z) \leq \tan^{-1}(3)$

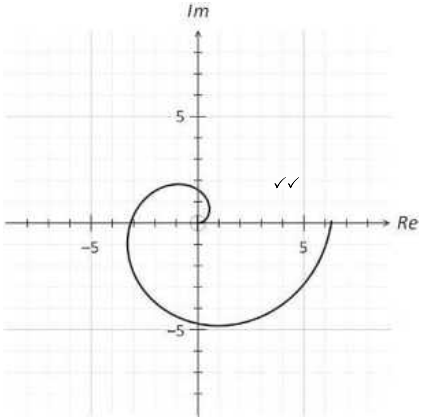


✓ Boundary passes through (1, 3).
✓ Boundary passes through (2, –2).
✓ Shades correct region.

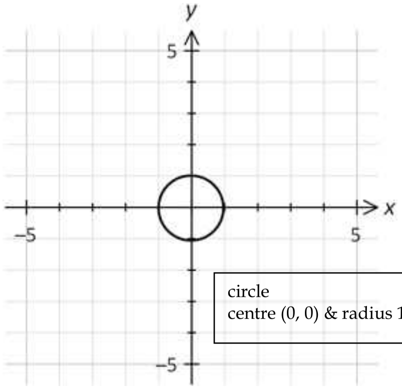
Calculator Free

8. [4 marks: 2, 2] [TISC]

(a) Sketch the region in the Argand Plane defined by
$\{ z : | z | = arg\ (z)$ where $0 \leq arg\ (z) \leq 2\pi \}$. 0



(b) Sketch the region in the Argand Plane defined by
$\{z : z = \cos\theta + i \sin\theta$ where $-\pi < \theta \leq \pi\}$.

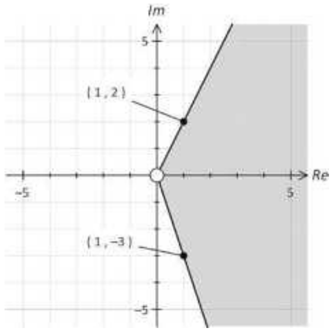


circle ✓
centre (0, 0) & radius 1 ✓

Calculator Free

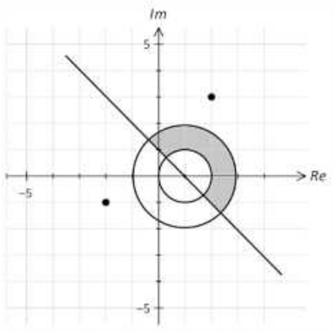
9. [9 marks: 3, 3, 3]

(a) On the Argand diagram provided sketch the locus of the complex number z where $\tan^{-1}(-3) \le arg(z) \le \tan^{-1} 2$.



- ✓ Shades plane between two half lines.
- ✓ First half-line has correct gradient.
- ✓ Second half-line has correct gradient.

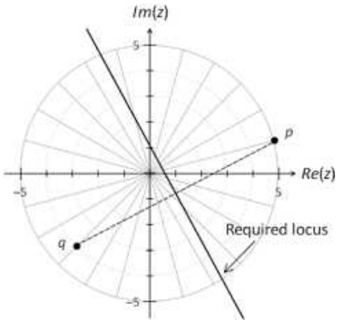
(b) On the Argand diagram provided sketch the locus of the complex number z where $|z+2+i| \ge |z-2-3i|$ 0 and $1 \le |z-1| \le 2$.



- ✓ Line $y = -x + 1$
- ✓ Concentric circles centre (1, 0) and radii 1 and 2.
- ✓ Half annulus shaded correctly.

(c) Let $p = 5\ cis\ \dfrac{\pi}{12}$ and $q = 4\ cis\ \dfrac{-3\pi}{4}$.

On the axes provided, plot the points p and q and sketch the locus of the complex number z satisfying $|z-p| = |z-q|$.



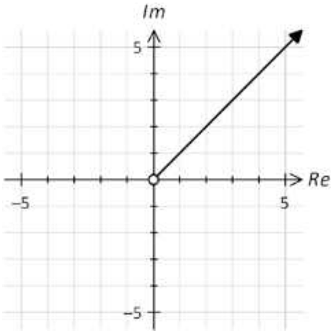
- ✓ Points p and q plotted correctly.
- ✓ Locus is a line with negative gradient and is perpendicular to the line joiningp with q.
- ✓ Line has passes through midpoint of line joining p with q.

Calculator Free

10. [9 marks: 2, 3, 4] [TISC]

(a) In the Argand diagram provided, sketch the region defined by

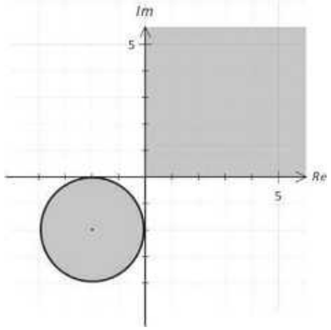
$\{ z : arg(\bar{z}) = \dfrac{-\pi}{4} \}$.



- ✓ Half-line through origin.
- ✓ Gradient = 1 and $x > 0$

(b) Sketch on the diagram below the locus of the point z defined by:

$\{z : |z+2+2i| \le 2 \cup 0 \le arg\ (z) \le \dfrac{\pi}{2}\}$.



- ✓ Quadrant 1 is shaded.
- ✓ Disc is shaded.
- ✓ Disc has centre (–2, –2) & radius 2

(c) Find, in its simplest form the Cartesian equation of the locus of the point z defined by $|z-1-i| =$ **Re** $(z+3+4i)$.

$|(x-1)+(y-1)i| = \text{Re}(x+3+4i)$ ✓

$\sqrt{(x-1)^2+(y-1)^2} = x+3$ ✓

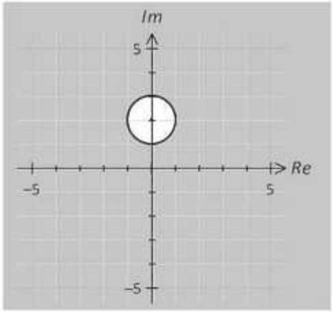
$(x-1)^2+(y-1)^2 = (x+3)^2$ ✓

$y^2 - 2y - 8x - 7 = 0$ ✓

Calculator Free

11. [12 marks: 2, 3, 7] [TISC]

(a) Sketch on the diagram provided the . locus of the point z defined by: $\{z : |z - 2i| \geq 1\}$.

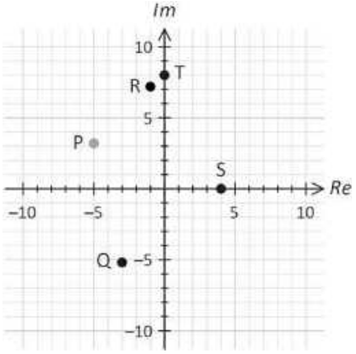


✓ Circle centre (0, 2) and radius 1.
✓ Shades region outside circle.

(b) Find, in simplest form the Cartesian equation of the locus of the point z defined by $|z - 1| = |z - 1 + 2i|$.

Let $z = x + yi$.
$|(x - 1) + yi| = |(x - 1) + (y + 2)i|$ ✓
$(x - 1)^2 + y^2 = (x - 1)^2 + (y + 2)^2$ ✓
$y^2 = y^2 + 4y + 4$
$y = -1$ ✓

(c) Consider the complex numbers $u = 2 + 2i$ and $v = -3 + 3\sqrt{3}\,i$. The accompanying Argand diagram shows the points P, Q, R, S and T. Describe the complex numbers represented by each of the points Q, R, S and T using the complex numbers u and v and/or their conjugates. For example, the point P represents $v - u$.



Q: $\overline{v}$ ✓

R: $u + v$ ✓✓

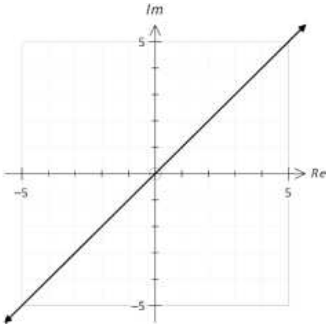
S: $u + \overline{u}$ ✓✓

T: u^2 or $2(u - \overline{u})$ ✓✓

Calculator Free

12. [8 marks: 2, 3, 3] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{z : \tan[arg(z)] = 1\}$.



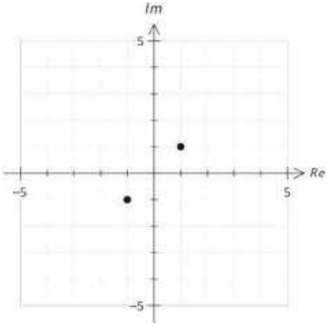
Any positive line through (0, 0). ✓
Line $y = x$ ✓

(b) Consider the region in the Argand Plane defined by $\{z : z^2 = 2i\}$. Let $z = x + i\,y$ where x and y are real numbers.

(i) Show that the Cartesian equation of this region is given by $x^4 = 1$.

$(x + iy)^2 = 2i$
$(x^2 - y^2) + 2xyi = 2i$
$\Rightarrow x^2 - y^2 = 0$ and $xy = 1$ ✓✓
$x^2 - \left(\frac{1}{x}\right)^2 = 0$
$\Rightarrow x^4 = 1$ ✓

(ii) Hence, show that this region consists of exactly *two* points. Mark these two points clearly on the axes provided.



Since, x is real, $x^4 = 1 \Rightarrow x = \pm 1$. ✓
Hence, points are (1, 1) and (–1, –1). ✓✓

Calculator Free

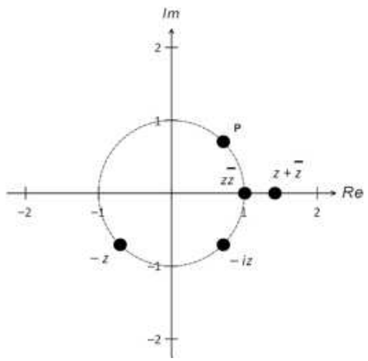
13. [13 marks: 8, 5] [TISC]

(a) The complex number z where $|z| = 1$, is represented by the point P as marked in the accompanying Argand diagram. Mark and label clearly on the diagram given the points representing the complex numbers:

(i) $-z$ (ii) $-i\,z$

(iii) $z + \bar{z}$ (iv) $z \times \bar{z}$

2 marks each.



(b) The locus of the complex number z satisfies the equation $|z-1| = |\bar{z}|$. Find the Cartesian equation of the locus and hence sketch the locus of z in the Argand diagram provided.

Let $z = x + yi$.

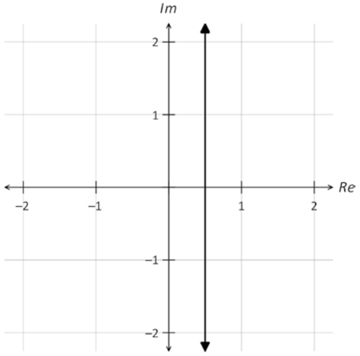
$|(x-1) + yi| = |x - yi|$ ✓

$(x-1)^2 + y^2 = x^2 + y^2$ ✓

$x^2 - 2x + 1 = x^2$ ✓

$x = \frac{1}{2}$ ✓

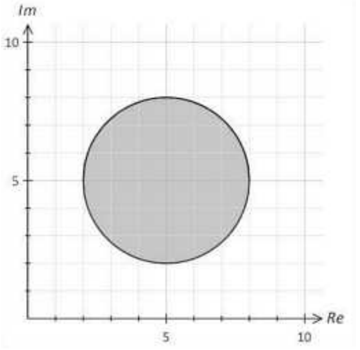
✓ Sketch.



Calculator Free

14. [7 marks: 2, 2, 3]

The accompanying diagram shows the locus of the complex number z.



(a) Use complex number concepts to state the equation of the locus of z.

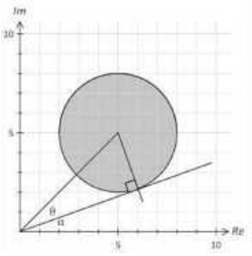
$|z - (5 + 5i)| \le 3$ ✓✓

(b) Given that $a \le |z| \le b$, state the values for a and b.

$a = 5\sqrt{2} - 3$ ✓

$b = 5\sqrt{2} + 3$ ✓

(c) Given that $\arg(z) \ge \alpha$, determine the value of α.



✓ Diagram.

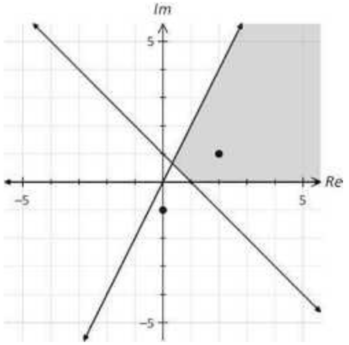
$\theta = \sin^{-1}\left(\frac{3}{5\sqrt{2}}\right)$ ✓

Hence $\alpha = \frac{\pi}{4} - \sin^{-1}\left(\frac{3}{5\sqrt{2}}\right)$ ✓

Calculator Free

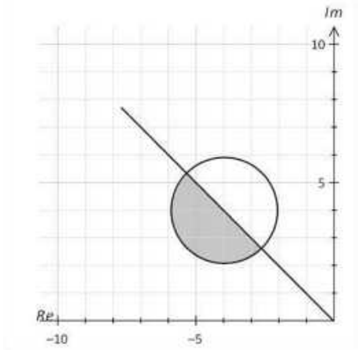
15. [8 marks: 3, 2, 3]

(a) On the Argand diagram given sketch the locus of the complex number z where $|z+i| \geq |z-2-i|$ and $0 \leq arg(z) \leq \tan^{-1} 2$.



✓ $x + y \geq 1$
✓ $y \leq 2x$
✓ $y \geq 0$

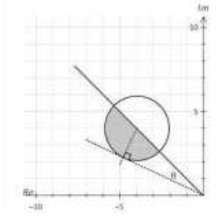
(b) The given diagram shows the locus of the complex number z where $arg(z) \geq \dfrac{3\pi}{4}$ and $|z + 4 - 4i| \leq 2$.



(i) Determine the minimum value of $|z|$.

Distance from centre of circle to origin = $4\sqrt{2}$ ✓
Radius of circle = 2
Hence, min $|z| = 4\sqrt{2} - 2$ ✓

(ii) Determine the maximum value of $arg(z)$.



✓ Diagram.

$\theta = \sin^{-1}\left(\dfrac{2}{4\sqrt{2}}\right)$ ✓

Hence $arg(z) \leq \dfrac{3\pi}{4} + \sin^{-1}\left(\dfrac{\sqrt{2}}{4}\right)$ ✓

Calculator Free

16. [7 marks: 3, 2, 2] [TISC]

The complex number z is defined by $z = \dfrac{a+4i}{i} + \dfrac{4}{1+i}$ where a is a real constant.

(a) Rewrite z in the form $x + y\,i$ where x and y are real.

$$\begin{aligned} z &= \frac{a+4i}{i} + \frac{4}{1+i} \\ &= \frac{(a+4i)i}{i \times i} + \frac{4}{1+i} \times \frac{1-i}{1-i} && ✓ \\ &= \frac{-4+ai}{-1} + \frac{4-4i}{2} && ✓ \\ &= 6-(a+2)i && ✓ \end{aligned}$$

(b) Find the value of a if z lies on the line $Im(z) = -Re(z)$.

$-(a+2) = -6$ ✓
$a = 4$ ✓

(c) Show that z cannot lie on the curve $arg(z) = \dfrac{3\pi}{4}$.

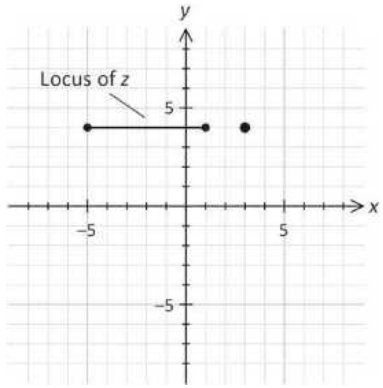
If $\arg(z) = \dfrac{3\pi}{4}$, then $\text{Re}(z) \leq 0$. ✓
but $\text{Re}(z) = 6 > 0$.
Hence, z cannot lie on $\arg(z) = \dfrac{3\pi}{4}$. ✓

Calculator Assumed

17. [6 marks: 4, 2]

The complex number $z = x + iy$ has locus described by $|z + 5 - 4i| + |z - 3 - 4i| = 8$ and $|z - 3 - 4i| \geq 2$.

(a) In the given diagram below draw the locus of z.



- ✓ Line segment.
- ✓ Horizontal line segment with:
- ✓ End point (–5, 4).
- ✓ End point (1, 4)

(b) Determine in its simplest form the Cartesian equation of this locus.

$y = 4, -5 \leq x \leq 1$ ✓✓

---

18. [7 marks: 3, 4]

(a) The locus of a complex number $z = x + iy$ is given by $|z| = |2z - 3i|$. Determine the Cartesian equation of this locus, giving your answer in its simplest form.

$|x + iy| = |2x + 2yi - 3y|$ ✓

$x^2 + y^2 = (2x)^2 + (2y-3)^2$

$3x^2 + 3y^2 - 12y + 9 = 0$ ✓

$x^2 + (y-2)^2 = 1$ ✓

Calculator Assumed

18. (b) The complex number z has locus described by $|z + 2 - i| = \frac{1}{2}|z - 1 + 2i|$.

Determine in its simplest form the Cartesian equation of this locus.

$|z + 2 - i| = \frac{1}{2}|z - 1 + 2i|$

$\Rightarrow 2|z + 2 - i| = |z - 1 + 2i|$

$2\,|(x + 2) + (y - 1)i| = |x - 1 + (y + 2)i|$ ✓

$4\left[(x+2)^2 + (y-1)^2\right] = (x-1)^2 + (y+2)^2$ ✓

$3(x^2 + y^2 + 6x - 4y + 5) = 0$

$x^2 + y^2 + 6x - 4y + 5 = 0$ ✓

$(x + 3)^2 + (y - 2)^2 = 8$ ✓

---

19. [8 marks: 3, 5] [TISC]

Let $w = x + y\,i$.

(a) If $\left|\frac{w}{1-w}\right| = 1$, show that w lies on the line with equation $x = \frac{1}{2}$.

$\left|\frac{w}{1-w}\right| = 1 \Rightarrow |w| = |1 - w|$ ✓

$x^2 + y^2 = (1 - x)^2 + y^2$ ✓

$x^2 + y^2 = 1 - 2x + x^2 + y^2$

$\Rightarrow \quad x = \frac{1}{2}$ ✓

(b) If $\left|\frac{w}{1-w}\right| = 3$, show that w lies on a circle. Find the equation of this circle.

$\left|\frac{w}{1-w}\right| = 3 \Rightarrow |w| = 3|1 - w|$ ✓

$x^2 + y^2 = 9[(1 - x)^2 + y^2]$ ✓

$8x^2 - 18x + 8y^2 = -9$

$(x - \frac{9}{8})^2 + y^2 = \frac{9}{64}$ ✓

This is the equation of a circle

centre $(\frac{9}{8}, 0)$ and radius $\frac{3}{8}$ ✓✓

03 Complex Numbers III

Calculator Free

1. [7 marks: 1, 3, 3] [TISC]

Given $u = 4\,cis\left(\frac{\pi}{3}\right)$ and $v = 2\,cis\left(\frac{k\pi}{3}\right)$ where k is a real number.

(a) If $2 < k < 4$, find $\frac{u}{v}$ in $r\,cis\,\theta$ form where $-\pi < \theta \le \pi$.

$$\frac{u}{v} = 2\,cis\left(\frac{\pi}{3} - \frac{k\pi}{3}\right) \quad ✓$$

(b) Find $u \times v$ in $r\,cis\,\theta$ form where $2 < k < 4$ and $-\pi < \theta \le \pi$.

$$\begin{aligned} uv &= 8\,cis\left(\frac{\pi}{3} + \frac{k\pi}{3}\right) && ✓\\ &= 8\,cis\left(\frac{\pi(k+1)}{3}\right) && ✓\\ &= 8\,cis\left(\frac{\pi(k+1)}{3} - 2\pi\right) && ✓ \end{aligned}$$

as $\left(\frac{\pi(k+1)}{3}\right)$ is outside the principal domain for $2 \le k \le 4$.

(c) Find k given that v is one of the square roots of u.

$u = 4\,cis\left(\frac{\pi}{3}\right)$.

$$\begin{aligned} \sqrt{u} &= 2\,cis\left(\frac{\frac{\pi}{3} + 2n\pi}{2}\right) && ✓\\ &= 2\,cis\left(\frac{\pi}{6}\right) \text{ or } 2\,cis\left(\frac{7\pi}{6}\right) \\ &= 2\,cis\left(\frac{\pi}{6}\right) \text{ or } 2\,cis\left(\frac{-5\pi}{6}\right) \end{aligned}$$

Hence, $k = \frac{1}{2}$ or $\frac{-5}{2}$. ✓✓

Calculator Free

2. [9 marks: 3, 3, 3] [TISC]

(a) Simplify $\dfrac{a^2\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)\right]}{4a\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]}$, giving your answer in exact *cis* form.

$$\begin{aligned} \frac{a^2\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)\right]}{4a\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]} &= \frac{a^2 cis\left(-\frac{5\pi}{6}\right)}{4a\ cis\left(\frac{11\pi}{12}\right)} && ✓\\ &= \frac{a}{4} cis\left(-\frac{7\pi}{4}\right) && ✓\\ &= \frac{a}{4} cis\left(\frac{\pi}{4}\right) \text{ or } -\frac{a}{4} cis\left(-\frac{3\pi}{4}\right) && ✓ \end{aligned}$$

(b) Simplify $\left[cis\left(\dfrac{\frac{\pi}{3} + 2k\pi}{5}\right)\right]^5$, where $k = 0, 1, 2, 3, 4, 5, \ldots$.

Give your answer in exact *cis* form.

$$\begin{aligned} \left[cis\left(\frac{\frac{\pi}{3} + 2k\pi}{5}\right)\right]^5 &= \left[cis\left(5 \times \frac{\frac{\pi}{3} + 2k\pi}{5}\right)\right] && ✓\\ &= cis\left(\frac{\pi}{3} + 2k\pi\right) && ✓\\ &= cis\left(\frac{\pi}{3}\right) \times cis\,(2k\pi) \\ &= cis\left(\frac{\pi}{3}\right) && ✓ \end{aligned}$$

(c) Solve exactly for θ where $-\pi < \theta \le \pi$ in
$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = 1$.

$$\begin{aligned} &(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = 1 \\ \Rightarrow\ & cis\,\theta \times cis\,\theta = 1 \\ & cis\,2\theta = 1 && ✓\\ & 2\theta = 0, 2\pi && ✓\\ \Rightarrow\ & \theta = 0, \pi && ✓ \end{aligned}$$

Calculator Free

3. [6 marks: 3, 3] [TISC]

(a) Find n given that $\dfrac{1}{\cos 3\theta + i\,\sin 3\theta} = [cis\,\theta\,]^n$

$$\frac{1}{\cos 3\theta + i\,\sin 3\theta} = cis\,0 - cis\,3\theta \quad ✓$$
$$= cis\,(-3\theta)$$
$$= [cis\,\theta]^{-3} \quad ✓$$

Hence, $n = -3$. ✓

(b) Given that $\left|\dfrac{z-2}{z+2}\right| = 1$, where $z \neq 0$, show that z is completely imaginary.

Let $z = x + iy$.

$\left|\dfrac{z-2}{z+2}\right| = 1 \Rightarrow |z-2| = |z+2|$ ✓

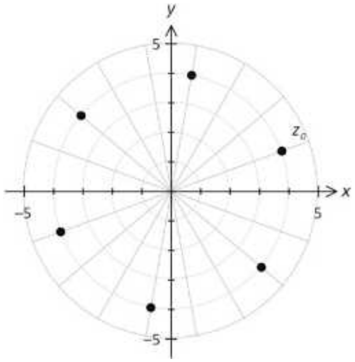
$$(x-2)^2 + y^2 = (x+2)^2 + y^2 \quad ✓$$
$$x^2 - 4x + 4 = x^2 + 4x + 4$$
$$x = 0 \quad ✓$$

Hence, $z = iy$ which is completely imaginary.

---

4. [7 marks: 2, 3, 2]

Consider the equation $z^6 = k$ where $k \in \mathbb{C}$. One of the roots of this equation is z_0. The accompanying Argand diagram shows the position of z_0.



(a) On the Argand diagram given above, plot the remaining roots of this equation.

✓ Total of six points plotted on circle of radius 4.

✓ Angular separation between points is 60^o or $\dfrac{\pi}{3}$.

Calculator Free

4. (b) A polygon is drawn with plots of the roots as vertices. Determine with reasons:

(i) the exact perimeter of this polygon.

Polygon is a regular hexagon and comprises six congruent equilateral triangles. ✓
Sides of triangles are of length 4. ✓
Hence, perimeter of polygon = 6 × 4 = 24 ✓

(ii) the exact area of this polygon.

$$\text{Area of polygon} = 6 \times \frac{1}{2} \times 4 \times 4 \times \sin\left(\frac{\pi}{3}\right) \quad ✓$$
$$= 24\sqrt{3} \quad ✓$$

---

5. [9 marks: 4, 5] [TISC]

(a) Solve $z^4 = 1 + i$. Leave your answers in exact polar form.

$$z^4 = \sqrt{2}\,cis\,\frac{\pi}{4}$$
$$z = \left(\sqrt{2}\,cis\,\frac{\pi}{4}\right)^{1/4} = 2^{1/8}\,cis\left(\frac{\frac{\pi}{4} + 2n\pi}{4}\right) \quad ✓✓$$
$$\Rightarrow \quad z = 2^{1/8}\,cis\left(\frac{\pi}{16}\right),\ 2^{1/8}\,cis\left(\frac{9\pi}{16}\right), \quad ✓$$
$$2^{1/8}\,cis\left(-\frac{15\pi}{16}\right),\ 2^{1/8}\,cis\left(-\frac{7\pi}{16}\right) \quad ✓$$

(b) Consider the equation $z^n = a + bi$. When plotted on an Argand diagram, two immediate adjacent roots are $cis\left(\dfrac{\pi}{12}\right)$ and $cis\left(\dfrac{7\pi}{12}\right)$. Find the value(s) of n, and corresponding exact values of a and b. Justify your answer.

Angular difference between roots = $\frac{\pi}{2}$. ✓

Hence, number of roots = $\dfrac{2\pi}{\left(\frac{\pi}{2}\right)} = 4k$ where k is a positive integer. ✓

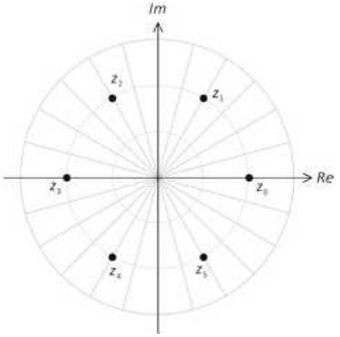
As the roots are adjacent and immediate, $\Rightarrow k = 1$

Hence: $n = 4$, $a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$ ✓✓✓

Calculator Free

6. [10 marks: 4, 3, 3]

The accompanying diagram shows the complex numbers z_0, z_1, z_2, z_3, z_4 and z_5 plotted as vertices of a regular hexagon centred at the pole (origin).



(a) Given that $z_0 = r\,cis\,0$, show that

$$z_1 = z_0\,cis\left(\frac{\pi}{3}\right)$$

$$\begin{aligned} z_0 = r\,cis\,0 \;\Rightarrow\; z_1 &= r\,cis\,\frac{\pi}{3} && \checkmark \\ &= r\,cis\left(0+\frac{\pi}{3}\right) && \checkmark \\ &= r\,cis\,0 \times cis\,\frac{\pi}{3} && \checkmark \\ &= z_0\;cis\,\frac{\pi}{3} && \checkmark \end{aligned}$$

(b) Hence, show the use of the relationship in (a) to find z_5 in terms of z_0.

$z_5 = z_4\;cis\,\frac{\pi}{3}$ $= z_3\;cis\,\frac{\pi}{3}\;cis\,\frac{\pi}{3} = z_3\;cis\,\frac{2\pi}{3}$ $= z_2\;cis\,\frac{\pi}{3}\;cis\,\frac{2\pi}{3} = z_2\;cis\,\frac{3\pi}{3}$ $= z_1\;cis\,\frac{\pi}{3}\;cis\,\frac{3\pi}{3} = z_1\;cis\,\frac{-2\pi}{3}$ $= z_0\;cis\,\frac{\pi}{3}\;cis\,\frac{-2\pi}{3} = z_0\;cis\,\frac{-\pi}{3}$	✓ Uses the iterative relationship $z_{n+1} = z_n\;cis\,\frac{\pi}{3}$. ✓ Correct number of iterations ✓ Correct answer

(c) Determine the value of $z_0 + z_1 + z_2 + z_3 + z_4 + z_5$.

Given $z_0 = r\,cis\,0$.
Hence:

$$\begin{aligned} & z_0 + z_1 + z_2 + z_3 + z_4 + z_5 \\ &= r\,cis\,0 + r\,cis\,\frac{\pi}{3} + r\,cis\,\frac{2\pi}{3} + r\,cis\,\pi + r\,cis\,\frac{-2\pi}{3} + r\,cis\,\frac{-\pi}{3} && \checkmark \\ &= (r\,cis\,0 + r\,cis\,\pi) + (r\,cis\,\frac{\pi}{3} + r\,cis\,\frac{2\pi}{3}) + (r\,cis\,\frac{-2\pi}{3} + r\,cis\,\frac{-\pi}{3}) && \checkmark \\ &= 0 && \checkmark \end{aligned}$$

Calculator Free

7. [12 marks: 4, 3, 2, 3]

Consider the equation $z^4 - 16i = 0$ for $z \in \mathbb{C}$.

(a) Solve the given equation, giving your answers in *cis* form.

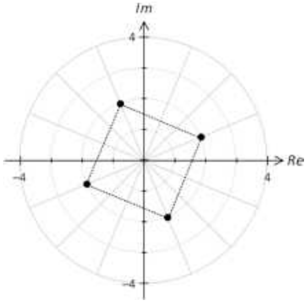
$$\begin{aligned} & z^4 = 16i = 16\;cis\left(\frac{\pi}{2}\right) && \checkmark \\ & z_0 = 2\,cis\left(\frac{\pi}{8}\right) && \checkmark \\ & z_1 = 2\,cis\left(\frac{\pi}{8}+\frac{4\pi}{8}\right) = 2\,cis\left(\frac{5\pi}{8}\right) && \checkmark \\ & z_2 = 2\,cis\left(\frac{-3\pi}{8}\right),\; z_3 = 2\,cis\left(\frac{-7\pi}{8}\right) && \checkmark \end{aligned}$$

(b) Show that the Cartesian roots of this equation can be written in the form $\pm\,u$ and $\pm\,v$.

$$\begin{aligned} & z_0 = 2\,cis\left(\frac{\pi}{8}\right) = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) && \checkmark \\ & z_3 = 2\;cis\left(\frac{-7\pi}{8}\right) = 2\left(-\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right) = -\,z_0 && \checkmark \\ & z_1 = 2\;cis\left(\frac{5\pi}{8}\right) = 2\left(-\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right) \\ & z_2 = 2\;cis\left(\frac{-3\pi}{8}\right) = 2\left(\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right) = -\,z_1 && \checkmark \end{aligned}$$

Hence roots are $\pm\,z_0$ and $\pm\,z_1$.

(c) On the Argand diagram provided, plot the points corresponding to the roots of this equation.



✓ Plots 4 points on circle of radius 2 with angular separation of 90°.
✓ Location of 1 point correct.

(d) A polygon is formed by using the roots of the given equation as its vertices. Prove that this polygon is a square.

As angular separation between roots = 90°; diagonals are perpendicular ✓
and diagonals are of equal length = 4. ✓
Hence, polygon is a square.
Area = $4 \times \frac{1}{2} \times 2 \times 2 = 8$ ✓

Calculator Free

8. [12 marks: 4, 3, 3, 2]

Consider the equation $iz^6 - 1 = 0$ for $z \in \mathbb{C}$.

(a) Solve the given equation, giving your answers in *cis* form.

$$z^6 = -i = cis\left(\frac{-\pi}{2}\right) \quad \checkmark$$
$$z_0 = cis\left(\frac{-\pi}{12}\right) \quad \checkmark$$
$$z_1 = cis\left(\frac{-\pi}{12}+\frac{4\pi}{12}\right) = cis\left(\frac{3\pi}{12}\right) \quad \checkmark$$
$$z_2 = cis\left(\frac{7\pi}{12}\right),\ z_3 = cis\left(\frac{11\pi}{12}\right)$$
$$z_4 = cis\left(\frac{-5\pi}{12}\right),\ z_5 = cis\left(\frac{-9\pi}{12}\right) \quad \checkmark$$

(b) Calculate the product of all the roots of the equation.

$$\text{Product} = \prod_{i=0}^{6} z_i$$
$$= cis\left(\frac{-\pi}{12}+\frac{3\pi}{12}+\frac{7\pi}{12}+\frac{11\pi}{12}+\frac{-5\pi}{12}+\frac{-9\pi}{12}\right) \quad \checkmark$$
$$= cis\left(\frac{\pi}{2}\right) = i \quad \checkmark\checkmark$$

(c) A polygon is formed by using the roots of the given equation as its vertices. Determine the perimeter of this polygon. Justify your answer

Polygon consists of six congruent isosceles triangles with side length 1 and vertex angle $\frac{2\pi}{6} = 60^{o}$. ✓
Which makes these triangles equilateral. ✓
Hence, perimeter of polygon = 6 ✓

(d) Determine with reasons the perimeter of the polygon with vertices formed by the roots of the equation $i\,z^n - 1 = 0$ as $n \to \infty$.

As $n \to \infty$, the polygon tends to a circle of radius 1. ✓
Hence, perimeter = 2π. ✓

Calculator Free

9. [10 marks: 3, 3, 4]

(a) Determine the imaginary part of the expansion of $(\cos x - i\sin x)^5$. Express your answer in terms of the powers of sin *x*.

$$\text{Im}((\cos x - i\sin x)^5)$$
$$= 5\sin^5 x + 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x \quad \checkmark$$
$$= 5\sin^5 x + 5(1-\sin^2 x)^2 \sin x - 10(1-\sin^2 x)\sin^3 x \quad \checkmark$$
$$= -(16\sin^5 x - 20\sin^3 x + 5\sin x) \quad \checkmark$$

(b) Show how your answer in (a) can be used to prove that $\sin 5x = 16\sin^5 x - 20\sin^3 x + 5\sin x$.

$$(\cos x - i\sin x)^5 = (\cos(-x) + i\sin(-x))^5$$
$$\text{Im}((\cos x - i\sin x)^5) = \text{Im}((\cos(-x) + i\sin(-x))^5)$$
$$= \text{Im}(\cos(-5x) + i\sin(-5x)) \quad \checkmark$$
$$= -\sin(5x) \quad \checkmark$$

But from (a):

$$\text{Im}((\cos x - i\sin x)^5) = -(16\sin^5 x - 20\sin^3 x + 5\sin x) \quad \checkmark$$

Hence: $\sin(5x) = 16\sin^5 x - 20\sin^3 x + 5\sin x$

(c) Hence, solve the equation $32y^5 - 40y^3 + 10y - \sqrt{2} = 0$, giving your answers in trigonometric form.

Let $y = \sin x$.
Hence, equation is
$32\sin^5 x - 40\sin^3 x + 10\sin x - \sqrt{2} = 0$
Rewrite as:

$$16\sin^5 x - 20\sin^3 x + 6\sin x = \frac{\sqrt{2}}{2} \quad \checkmark$$

Hence: $\sin 5x = \frac{\sqrt{2}}{2}$ ✓

$$x = \frac{\pi}{20}, \frac{3\pi}{20}, \frac{9\pi}{20}, \frac{25\pi}{20}, \frac{27\pi}{20} \quad \checkmark$$
$$y = \sin\frac{\pi}{20}, \sin\frac{3\pi}{20}, \sin\frac{9\pi}{20}, \sin\frac{25\pi}{20}, \sin\frac{27\pi}{20} \quad \checkmark$$

Calculator Free

10. [9 marks: 4, 2, 3]

(a) If $z = \cos\theta + i\sin\theta$, show that $\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$ and $\sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$.

$$\begin{aligned}
z^n &= (\cos\theta + i\sin\theta)^n \\
&= \cos n\theta + i\sin n\theta \qquad \text{I} \quad ✓\\
z^{-n} &= (\cos\theta + i\sin\theta)^{-n} \\
&= \cos(-n\theta) + i\sin(-n\theta) \\
&= \cos n\theta - i\sin n\theta \qquad \text{II} \quad ✓
\end{aligned}$$

$$\begin{aligned}
\text{I + II} \quad & \left(z^n + \frac{1}{z^n}\right) = 2\cos n\theta \quad ✓\\
& \Rightarrow \cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)\\
\text{I − II} \quad & \left(z^n - \frac{1}{z^n}\right) = 2i\sin n\theta \quad ✓\\
& \Rightarrow \sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)
\end{aligned}$$

(b) Hence, show that $\tan\theta = i\left(\frac{1-z^2}{1+z^2}\right)$.

$$\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta}\\
&= \frac{\frac{1}{2i}\left[z - \frac{1}{z}\right]}{\frac{1}{2}\left[z + \frac{1}{z}\right]} = \frac{\left[z - \frac{1}{z}\right]}{\left[i\,(z + \frac{1}{z})\right]} \quad ✓\\
&= -i\left(\frac{z^2-1}{z^2+1}\right) = i\left(\frac{1-z^2}{1+z^2}\right) \quad ✓
\end{aligned}$$

(c) Use the result in (a) to prove that $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

$$\begin{aligned}
\text{LHS} &= \cos^2\theta - \sin^2\theta\\
&= \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^2 - \left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2 \quad ✓✓\\
&= \left[\frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right)\right] - \left[-\frac{1}{4}\left(z^2 - 2 + \frac{1}{z^2}\right)\right]\\
&= \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) \quad ✓\\
&= \cos 2\theta = \text{RHS}
\end{aligned}$$

Calculator Free

11. [7 marks: 3, 2, 2] [TISC]

Let $w = z + \frac{1}{z}$.

It can be shown that $w^3 + w^2 - 2w - 2 = \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right)$.

Given that $z = cis\,\theta$, a commonly used result is $z^n + \frac{1}{z^n} = 2\cos n\theta$.

(a) Show that solving $w^3 + w^2 - 2w - 2 = 0$ is equivalent to solving $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

$$\begin{aligned}
w^3 + w^2 - 2w - 2 &= \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) \quad ✓\\
&= 2\cos 3\theta + 2\cos 2\theta + 2\cos\theta. \quad ✓✓
\end{aligned}$$

Hence, $w^3 + w^2 - 2w - 2 = 0$ is equivalent to $2\cos 3\theta + 2\cos 2\theta + 2\cos\theta = 0$.

(b) The solutions to $w^3 + w^2 - 2w - 2 = 0$ are $-\sqrt{2}$, -1 and $\sqrt{2}$.

Explain clearly why the solution $w = -1$ implies that $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

$$\begin{aligned}
w = -1 \Rightarrow \left(z + \frac{1}{z}\right) &= -1 \quad ✓\\
z^2 + z + 1 &= 0 \quad ✓\\
z &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i
\end{aligned}$$

(c) Hence, find one solution to $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

One solution to $w^3 + w^2 - 2w - 2 = 0$ is $w = -1$.

$w = -1$ is equivalent to $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Since $z = cis\,\theta$: $cis\,\theta = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ ✓

$\cos\theta + i\sin\theta = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

$\Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$

and $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$

Hence, $\theta = \frac{2\pi}{3}$. ✓

Calculator Assumed

12. [9 marks: 2, 2, 3, 2]

(a) Let $z_0 = 2\,cis\left(\frac{\pi}{5}\right)$.

(i) Show that $z_0{}^5 = -32$.

$$z_0{}^5 = \left[2\,cis\left(\frac{\pi}{5}\right)\right]^5 = 2^5\,cis\left(\frac{\pi}{5}\times 5\right) \quad ✓$$
$$= 32\,cis\,(\pi) = -32 \quad ✓$$

(ii) Hence, find four other complex numbers in polar form where $-\pi < \theta \le \pi$ such that $z^5 = -32$.

$$z_1 = 2\,cis\left(\frac{\pi}{5}+\frac{2\pi}{5}\right) = 2\,cis\left(\frac{3\pi}{5}\right)$$
$$z_2 = 2\,cis\left(\frac{\pi}{5}+\frac{4\pi}{5}\right) = 2\,cis(\pi)$$
$$z_3 = 2\,cis\left(\frac{\pi}{5}+\frac{6\pi}{5}\right) = 2\,cis\left(-\frac{3\pi}{5}\right)$$
$$z_4 = 2\,cis\left(\frac{\pi}{5}+\frac{8\pi}{5}\right) = 2\,cis\left(-\frac{\pi}{5}\right) \quad ✓✓$$

(b) Determine $cis\left(\frac{\theta}{4}\right) + cis\left(-\frac{\theta}{4}\right)$ in the form $a + bi$.

$$cis\left(\frac{\theta}{4}\right) + cis\left(-\frac{\theta}{4}\right) = [\cos\left(\frac{\theta}{4}\right) + i\sin\left(\frac{\theta}{4}\right)] + [\cos\left(-\frac{\theta}{4}\right) + i\sin\left(-\frac{\theta}{4}\right)] \quad ✓$$
$$= [\cos\left(\frac{\theta}{4}\right) + i\sin\left(\frac{\theta}{4}\right)] + [\cos\left(\frac{\theta}{4}\right) - i\sin\left(\frac{\theta}{4}\right)] \quad ✓$$
$$= 2\cos\left(\frac{\theta}{4}\right) + 0\,i \quad ✓$$

(c) Use your answer in (b) to prove that $2\,cis\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right) = 1 + cis\left(\frac{\theta}{2}\right)$.

$$\text{LHS} = 2\,cis\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right)$$
$$= cis\left(\frac{\theta}{4}\right) \times [cis\left(\frac{\theta}{4}\right) + cis\left(-\frac{\theta}{4}\right)] \quad ✓$$
$$= cis\left(\frac{\theta}{2}\right) + cis\,0 \quad ✓$$
$$= cis\left(\frac{\theta}{2}\right) + 1 = \text{RHS}$$

Calculator Assumed

13. [13 marks: 4, 2, 2, 5] [TISC]

Let $z = cis\,\theta$.

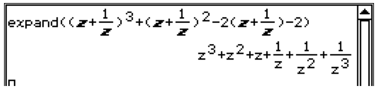
(a) Prove that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

$$z = cis\,\theta \;\Rightarrow\; z^n = cis\,n\theta \text{ and } \frac{1}{z^n} = cis\,(-n\theta) \quad ✓$$
$$\text{LHS} = z^n + \frac{1}{z^n}$$
$$= cis\,n\theta + cis\,(-n\theta) \quad ✓$$
$$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta + i\sin n\theta + \underset{✓}{\cos(n\theta)} - \underset{✓}{i\sin(n\theta)}$$
$$= 2\cos n\theta = \text{RHS}$$

(b) If $w = z + \frac{1}{z}$, prove that

$$w^3 + w^2 - 2w - 2 = \left(z+\frac{1}{z}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z^3+\frac{1}{z^3}\right).$$

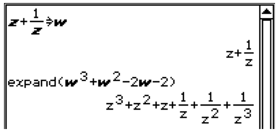
$$\text{LHS} = w^3 + w^2 - 2w - 2$$
$$= \left(z+\frac{1}{z}\right)^3 + \left(z+\frac{1}{z}\right)^2 - 2\left(z+\frac{1}{z}\right) - 2 \quad ✓$$
$$= z^3 + z^2 + z + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}$$
$$= \left(z+\frac{1}{z}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z^3+\frac{1}{z^3}\right) \quad ✓$$
$$= \text{RHS}$$



expand($(z+\frac{1}{z})^3+(z+\frac{1}{z})^2-2(z+\frac{1}{z})-2$)

$z^3+z^2+z+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}$

OR



$z+\frac{1}{z}$ ⇒ w

$z+\frac{1}{z}$

expand(w^3+w^2-2w-2)

$z^3+z^2+z+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}$

Calculator Assumed

13. (c) Use parts (a) and (b) to show that the equation $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ can be rewritten as $w^3 + w^2 - 2w - 2 = 0$.

.

From (a): $2\cos\theta = \left(z+\frac{1}{z}\right)$, $2\cos 2\theta = \left(z^2+\frac{1}{z^2}\right)$ and $2\cos 3\theta = \left(z^3+\frac{1}{z^3}\right)$

Hence: $\cos\theta + \cos 2\theta + \cos 3\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)+\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)+\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)$

Therefore: $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

$\Rightarrow \frac{1}{2}\left(z+\frac{1}{z}\right)+\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)+\frac{1}{2}\left(z^3+\frac{1}{z^3}\right) = 0$ ✓

$\left(z+\frac{1}{z}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z^3+\frac{1}{z^3}\right) = 0$

But from (b): $\left(z+\frac{1}{z}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z^3+\frac{1}{z^3}\right) = w^3 + w^2 - 2w - 2$ ✓

Hence: $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

is equivalent to $w^3 + w^2 - 2w - 2 = 0$

(d) Given that $-\pi < \theta \le \pi$, use part (c) to solve for θ where
$\cos\theta + \cos 2\theta + \cos 3\theta = 0$.

From (c): $\cos\theta + \cos 2\theta + \cos 3\theta = 0$
is equivalent to $w^3 + w^2 - 2w - 2 = 0$
Solving for w: $w = -1, \pm\sqrt{2}$ ✓
But $w = z + \frac{1}{z}$ $\Rightarrow$ $z + \frac{1}{z} = -1, \pm\sqrt{2}$ ✓
But $z + \frac{1}{z} = 2\cos\theta$ $\Rightarrow$ $2\cos\theta = -1, \pm\sqrt{2}$
$\cos\theta = -\frac{1}{2}, \pm\frac{\sqrt{2}}{2}$ ✓
Hence solution for (I) is: $\theta = \pm\frac{2\pi}{3}, \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ ✓✓

Calculator Assumed

14. [8 marks: 4, 4]

(a) Solve for z in $z+z^2+z^3+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3} = 0$, given that $z \ne 0$ and $(1 + z + z^2) \ne 0$.
Give your answers in exact *cis* form.

$z+z^2+z^3+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3} = 0$

$\frac{z^6+z^5+z^4+z^2+z+1}{z^3} = 0$

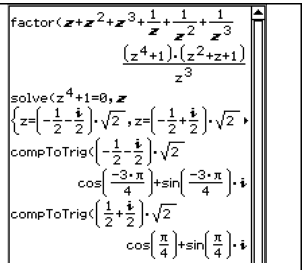
But $z \ne 0$. $\Rightarrow$ $z^6+z^5+z^4+z^2+z+1 = 0$ ✓

$(z^4+1)(z^2+z+1) = 0$ ✓

Hence: $z^4 = -1$ since $z^2+z+1 \ne 0$ ✓

For $z^4 = -1$

$z = cis\left(\frac{-\pi}{4}\right), cis\left(\frac{\pi}{4}\right), cis\left(\frac{3\pi}{4}\right), cis\left(\frac{-3\pi}{4}\right)$ ✓



$\text{factor}(z+z^2+z^3+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}$

$\frac{(z^4+1)\cdot(z^2+z+1)}{z^3}$

$\text{solve}(z^4+1=0,z$

$\left\{z=\left(-\frac{1}{2}-\frac{i}{2}\right)\cdot\sqrt{2}, z=\left(-\frac{1}{2}+\frac{i}{2}\right)\cdot\sqrt{2}\right.$ ▸

$\text{compToTrig}(\left(-\frac{1}{2}-\frac{i}{2}\right)\cdot\sqrt{2}$

$\cos\left(\frac{-3\cdot\pi}{4}\right)+\sin\left(\frac{-3\cdot\pi}{4}\right)\cdot i$

$\text{compToTrig}(\left(\frac{1}{2}+\frac{i}{2}\right)\cdot\sqrt{2}$

$\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)\cdot i$

(b) For $z = cis\,\theta$ where $-\pi < \theta \le \pi$, it may be proven that $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ where n is a positive integer. Use this result to solve for θ in
$\cos\theta + \cos 2\theta + \cos 3\theta = 0$ for $\theta \ne \frac{2\pi}{3}$ and $-\pi < \theta \le \pi$

$\cos\theta + \cos 2\theta + \cos 3\theta = 0$

$\Rightarrow \left(z+\frac{1}{z}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z^3+\frac{1}{z^3}\right) = 0$ ✓

$\frac{z^6+z^5+z^4+z^2+z+1}{z^3} = 0$

From (a): $(z^4+1)(z^2+z+1) = 0$

$\theta \ne \frac{2\pi}{3}$ $\Rightarrow$ $z^2+z+1 \ne 0$.

$\Rightarrow z^4 = -1$ ✓

$z = cis\left(\frac{\pm\pi}{4}\right), cis\left(\frac{\pm 3\pi}{4}\right)$. ✓

Hence: $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ ✓

Calculator Assumed

15. [11 marks: 4, 3, 2, 2]

Consider the equation $z^6 + 1 = 0$ for $z \in \mathbb{C}$.

(a) Use de' Moivre's Theroem to solve this equation. Give answers in *cis* form.

$$z^6 = -1 = cis(\pi + 2k\pi) \quad \checkmark$$
$$z = cis\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right) \quad \text{for } k = 0, 1, 2, 3, 4, 5 \quad \checkmark$$
$$z = cis\left(\frac{\pi}{6}\right),\ cis\left(\frac{3\pi}{6}\right),\ cis\left(\frac{5\pi}{6}\right), \quad \checkmark$$
$$cis\left(\frac{-\pi}{6}\right), cis\left(\frac{-3\pi}{6}\right),\ cis\left(\frac{-5\pi}{6}\right) \quad \checkmark$$

(b) Show that the solutions can be written in the form $w,\ w^3, w^5, w^7, w^9$ and w^{11}.

Let $w = cis\left(\frac{\pi}{6}\right)$. ✓

For the root: $z = cis\left(\frac{3\pi}{6}\right) = \left[cis\left(\frac{\pi}{6}\right)\right]^3 = w^3$ ✓

$$z = cis\left(\frac{5\pi}{6}\right) = \left[cis\left(\frac{\pi}{6}\right)\right]^5 = w^5$$
$$z = cis\left(\frac{-5\pi}{6}\right) = cis\left(\frac{7\pi}{6}\right) = \left[cis\left(\frac{\pi}{6}\right)\right]^7 = w^7$$
$$z = cis\left(\frac{-3\pi}{6}\right) = cis\left(\frac{9\pi}{6}\right) = \left[cis\left(\frac{\pi}{6}\right)\right]^9 = w^9$$
$$z = cis\left(\frac{-\pi}{6}\right) = cis\left(\frac{11\pi}{6}\right) = \left[cis\left(\frac{\pi}{6}\right)\right]^{11} = w^{11} \quad \checkmark$$

(c) Prove that $w + w^3 + w^5 + w^7 + w^9 + w^{11} = 0$.

Sum of roots $z = cis\left(\frac{\pi}{6}\right) + cis\left(\frac{-\pi}{6}\right) + cis\left(\frac{3\pi}{6}\right) + cis\left(\frac{-3\pi}{6}\right) + cis\left(\frac{5\pi}{6}\right) + cis\left(\frac{-5\pi}{6}\right)$ ✓

$= \sqrt{3} + 0 - \sqrt{3} = 0$ ✓

(d) Hence, deduce that $w^2 + w^4 + w^6 + w^8 + w^{10} = -1$.

From (c): $w + w^3 + w^5 + w^7 + w^9 + w^{11} = 0$

$w(1 + w^2 + w^4 + w^6 + w^8 + w^{10}) = 0$ ✓

But $w \neq 0 \;\Rightarrow\; w^2 + w^4 + w^6 + w^8 + w^{10} = -1.$ ✓

04 The Factor & Remainder Theorems

Calculator Free

1. [5 marks]

Given that $x^2 + x + 1$ is a factor of the $f(x) = 2x^5 + x^4 - 4x^3 - 8x^2 - 7x - 2$, where $x \in \mathbb{R}$, determine the quotient when $f(x)$ is divided by $x^2 + x + 1$.

$$\begin{array}{r l}
 & 2x^3 - x^2 - 5x - 2 \\
x^2+x+1 \,\big) & 2x^5 + x^4 - 4x^3 - 8x^2 - 7x - 2 \\
 & 2x^5 + 2x^4 + 2x^3 \\
 & \quad -x^4 - 6x^3 - 8x^2 - 7x - 2 \\
 & \quad -x^4 - x^3 - x^2 \\
 & \qquad -5x^3 - 7x^2 - 7x - 2 \\
 & \qquad -5x^3 - 5x^2 - 5x \\
 & \qquad\quad -2x^2 - 2x - 2 \\
 & \qquad\quad -2x^2 - 2x - 2 \\
 & \qquad\qquad 0 \\
 & \qquad\qquad \checkmark\checkmark\checkmark\checkmark
\end{array}$$

Hence, quotient is $2x^3 - x^2 - 5x - 2$ ✓

OR

By inspection:

$2x^5 + x^4 - 4x^3 - 8x^2 - 7x - 2$

$\equiv (x^2 + x + 1)(2x^3 + ax^2 + bx - 2)$ ✓✓

By further inspection:

$a = -1,\ b = -5$ ✓✓

Hence, quotient is $2x^3 - x^2 - 5x - 2$ ✓

---

2. [5 marks]

Determine the quotient and remainder when $x^5 + 2x^3 - x^2 + 2x + 1$ is divided by $x^2 + 1$ for $x \in \mathbb{R}$.

$$\begin{array}{r l}
 & x^3 + x - 1 \\
x^2+1 \,\big) & x^5 + 0x^4 + 2x^3 - x^2 + 2x + 1 \\
 & x^5 + 0x^4 + x^3 \\
 & \qquad x^3 - x^2 + 2x + 1 \\
 & \qquad x^3 + 0x^2 + x \\
 & \qquad\quad -x^2 + x + 1 \\
 & \qquad\quad -x^2 - 0x - 1 \\
 & \qquad\qquad x + 2 \\
 & \qquad\qquad \checkmark\checkmark\checkmark
\end{array}$$

Hence, quotient is $x^3 + x - 1$ ✓
remainder is $x + 2$. ✓

OR

By inspection:

$x^5 + 2x^3 - x^2 + 2x + 1$

$\equiv (x^2 + 1)(x^3 + ax^2 + bx + c) + (dx + e)$ ✓

By further inspection:

$a = 0, b = 1, c = -1, d = 1, e = 2$ ✓✓

Hence, quotient is $x^3 + x - 1$ ✓
remainder is $x + 2$. ✓

Calculator Free

3. [7 marks]

$(x^2 + 4)$ is a factor of the polynomial $f(x) = 2x^5 + ax^4 + bx^3 + cx^2 - 8x + 12$ for $x \in \mathbb{C}$. When $f(x)$ is divided by $(x - 2)$ the remainder is 24. Determine the values of a, b and c.

$f(2i) = 0 \Rightarrow 64i + 16a - 8bi - 4c - 16i + 12 = 0$ ✓

$16a - 4c + 12 + (64 - 8b - 16)i = 0$
$4a - c = -3$ I ✓
$\Rightarrow b = 6$ ✓

$f(2) = 24 \Rightarrow 64 + 16a + 48 + 4c - 16 + 12 = 24$ ✓
$4a + c = -21$ II ✓

I + II $8a = -24$
$a = -3$ ✓
$c = -9$ ✓

4. [7 marks]

The polynomial $f(x) = x^5 + ax^4 + bx^3 + cx^2 + 6x + 4$ for $x \in \mathbb{R}$ has a of factor $x + 2$ and leaves a remainder of $2x + 1$ when divided by $x^2 - 1$. Determine the values of a, b and c.

$x^5 + ax^4 + bx^3 + cx^2 + 6x + 4 \equiv (x^2 - 1)Q(x) + 2x + 1$ ✓
When $x = 1$: $1 + a + b + c + 10 = 3$
$a + b + c = -8$ I ✓

When $x = -1$: $-1 + a - b + c - 2 = -1$
$a - b + c = 2$ II ✓

I – II $b = -5$ ✓

$f(-2) = 0 \Rightarrow -32 + 16a + 40 + 4c - 12 + 4 = 0$ ✓
$4a + c = 0$ III

Subst. $c = -4a$ into I $a = 1$ ✓
$c = -4$ ✓

Calculator Free

5. [10 marks: 6, 4]

(a) Factorise $x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$ for $x \in \mathbb{R}$

Let $f(x) = x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$
$f(1) = 1 + 2 - 2 - 4 + 1 + 2 = 0$
$f(-1) = -1 + 2 + 2 - 4 - 1 + 2 = 0$
$f(2) = 32 + 32 - 16 - 16 + 2 + 2 \neq 0$
$f(-2) = -32 + 32 + 16 - 16 - 2 + 2 = 0$

Hence, by inspection:
$f(x) = (x - 1)(x + 1)(x + 2)(x^2 + ax - 1)$
$= (x^2 - 1)(x + 2)(x^2 + ax - 1)$
$= (x^3 + 2x^2 - x - 2)(x^2 + ax - 1)$

By further inspection: $a = 0$

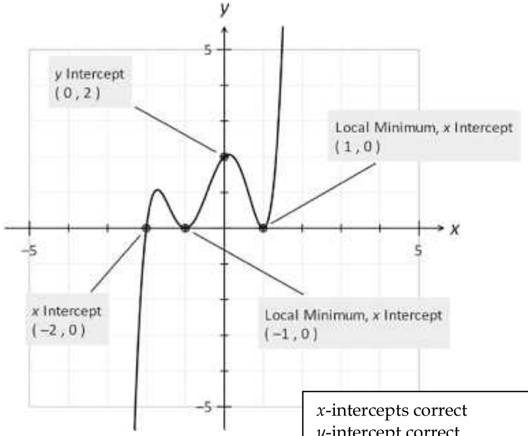
Hence, $f(x) = (x - 1)(x + 1)(x + 2)(x^2 - 1)$
$= (x - 1)^2(x + 1)^2(x + 2)$

Use of Factor Theorem to obtain first 2 factors. ✓✓

Next 3 factors obtained by polynomial division or inspection. ✓✓

All factors correct ✓✓

(b) On the axes provided below, sketch the curve with equation $y = x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$. Indicate all intercepts.
(The curve has several stationary points including (–1.7, 1.1) and (0.1, 2.1).)



x-intercepts correct ✓
y-intercept correct ✓
Min points at (–1, 0) & (1, 0) ✓
All correct ✓

Calculator Free

6. [11 marks: 4, 7]

(a) Solve for $3x^4 + 2x^3 - 13x^2 - 8x + 4 = 0$ for $x \in \mathbb{R}$.

Let $f(x) = 3x^4 + 2x^3 - 13x^2 - 8x + 4$
$f(-1) = 3 - 2 - 13 + 8 + 4 = 0$
$f(-2) = 48 - 16 - 52 + 16 + 4 = 0$

Hence, by inspection:

$f(x) = (x+1)(x+2)(3x^2 + ax + 2)$ ✓
$= (x^2 + 3x + 2)(3x^2 + ax + 2)$
$= (x^2 + 3x + 2)(3x^2 - 7x + 2)$
$= (x+1)(x+2)(3x-1)(x-2)$ ✓

Hence, $f(x) = 0 \Rightarrow x = -2, -1, \frac{1}{3}, 2$ ✓✓

(b) Hence, or otherwise solve $4\cos^4\theta - 8\cos^3\theta - 13\cos^2\theta + 2\cos\theta + 3 = 0$ for $-\pi < \theta \le \pi$. Explain clearly how you obtained your answer.

Let $x = \frac{1}{\cos\theta}$ in $f(x) = 3x^4 + 2x^3 - 13x^2 - 8x + 4 = 0$. ✓

Hence:

$$3\left(\frac{1}{\cos\theta}\right)^4 + 2\left(\frac{1}{\cos\theta}\right)^3 - 13\left(\frac{1}{\cos\theta}\right)^2 - 8\left(\frac{1}{\cos\theta}\right) + 4 = 0$$ ✓

$$3 + 2\cos\theta - 13\cos^2\theta - 8\cos^3\theta + 4\cos^4\theta = 0 \qquad \text{I}$$

Hence, solutions to I are given by:

$$\cos\theta = \frac{1}{x}$$ ✓

But solutions to $f(x) = 0$ are $x = -2, -1, \frac{1}{3}, 2$

Hence, solutions to I:

$$\cos\theta = -\frac{1}{2}, -1, 3, \frac{1}{2}$$ ✓

$$\theta = \pm\frac{2\pi}{3}, \pm\frac{\pi}{3}, \pi$$ ✓✓✓

Calculator Free

7. [5 marks]

Given that $z = k + i$ is a root of the equation $2z^3 - 3z^2 + 2z + a = 0$, determine the values of the real constants a and k.

Since $k + i$ is a root:

$2(k+i)^3 - 3(k+i)^2 + 2(k+i) + a = 0$ ✓

$2k^3 - 3k^2 - 4k + a + 3 + (6k^2 - 6k)i = 0$

Hence:

Im parts: $6k^2 - 6k = 0 \Rightarrow k = 0, 1$ ✓✓

For real parts: For $k = 0$, $a = -3$ ✓
$k = 1$, $a = 2$ ✓

---

8. [6 marks: 1, 1, 4]

Consider $f(z) = z^5 + z^4 - z^3 - z^2 - 2z$.

(a) Determine the remainder when $f(z)$ is divided by $(z + 1)$.

Remainder $= f(-1) = 2$ ✓

(b) Determine $f(i)$.

$f(i) = 2$ ✓

(c) Solve the equation $f(z) = 2$.

$f(z) = 2 \Rightarrow z^5 + z^4 - z^3 - z^2 - 2z - 2 = 0$
From (b), roots are -1 and $\pm i$. ✓
Hence factors are $(z^2 + 1)(z + 1)$. ✓
$z^5 + z^4 - z^3 - z^2 - 2z - 2 = (z^3 + z^2 + z + 1)(z^2 - 2)$ ✓
Hence: $z = -i, \pm i, \pm\sqrt{2}$. ✓

Calculator Free

9. [8 marks]

Solve $x^6 - x^4 + x^2 - 1 = 0$ for $x \in \mathbb{C}$.

Let $f(x) = x^6 - x^4 + x^2 - 1$
$f(-1) = 1 - 1 + 1 - 1 = 0$
$f(1) = 1 - 1 + 1 - 1 = 0$

Hence, by inspection:
$f(x) = (x+1)(x-1)Q(x)$ ✓✓
$= (x^2-1)(x^4+1)$ ✓

Hence, $f(x) = 0 \Rightarrow x = \pm 1$ ✓
or $x^4 = -1$

For $x^4 = -1 = cis\,\pi$ ✓

$$x = cis\left(\frac{\pi}{4}\right), cis\left(\frac{\pi}{4}+\frac{2\pi}{4}\right), cis\left(\frac{\pi}{4}+\frac{4\pi}{4}\right), cis\left(\frac{\pi}{4}+\frac{6\pi}{4}\right) \quad ✓$$
$$= cis\left(\frac{\pi}{4}\right), cis\left(\frac{3\pi}{4}\right), cis\left(-\frac{3\pi}{4}\right), cis\left(-\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2} \quad ✓✓$$

Hence, $x = \pm 1, \pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$

---

10. [6 marks]

Solve $x^3 + (1+i)x^2 + (2+i)x + 2 = 0$ for $x \in \mathbb{C}$

Let $f(x) = x^3 + (1+i)x^2 + (2+i)x + 2$
$f(-1) = -1 + (1+i) - (2+i) + 2 = 0$

Hence:
$x^3 + (1+i)x^2 + (2+i)x + 2 \equiv (x+1)(x^2+ax+2)$ ✓

Compare x^2 term: $1 + i = a + 1$
$a = i$

Equation is $(x+1)(x^2+ix+2) = 0$ ✓
$x = -1, \dfrac{-i \pm \sqrt{-1-8}}{2}$ ✓
$= -1, i, -2i$ ✓✓✓

Calculator Free

11. [13 marks: 2, 2, 3, 6]

(a) The roots of the equation $ax^2 + bx + c = 0$ where a, b and c are real numbers are α and β.

(i) Use the quadratic formula to show the sum of the roots $\alpha + \beta = -\frac{b}{a}$.

$$\alpha + \beta = \left(-\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right) + \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right) \quad ✓✓$$
$$= -\frac{b}{a}$$

(ii) Show that the product of the roots $\alpha \times \beta = \frac{c}{a}$.

$$\alpha \times \beta = \left(-\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right) \times \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right) \quad ✓$$
$$= \frac{b^2}{4a^2} - \frac{b^2-4ac}{4a^2} \quad ✓$$
$$= \frac{c}{a}$$

(b) A quadratic equation with all real coefficients has a solution $x = 2 + 3i$. Determine this equation.

Since coefficients of given equation are all real, the roots must appear as conjugate pairs.
Hence, roots are $x = 2 + 3i, 2 - 3i$. ✓
Sum of roots = 4
Product of roots = 13
Hence, equation is $x^2 - 4x + 13 = 0$ ✓✓

(c) $x = i$ and $x = 1 - i$ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where the coefficients a, b, c, d and e are real constants. Determine the values of a, b, c, d and e.

Hence, roots are $x = i, -i$ and $x = 1 - i, 1 + i$.
Therefore, equation is:
$(x-i)(x+i)(x-(1-i))(x-(1+i)) = 0$ ✓✓

$(x^2+1)(x^2-2x+2) = 0$ ✓
$x^4 - 2x^3 + 3x^2 - 2x + 2 = 0$

$\Rightarrow a = 1, b = -2, c = 3, d = -2, e = 2$ ✓✓✓

Calculator Free

12. [11 marks: 3, 3, 5]

(a) Prove that if $(x-a)^2$ is a factor of the real polynomial $f(x)$, then $(x-a)$ is a factor of $f'(x)$ where $f'(x)$ is the derivative of $f(x)$ with respect to x.

If $(x-a)^2$ is a factor of $f(x)$, then

$f(x) = (x-a)^2 \times Q(x)$. ✓

$f'(x) = 2(x-a) \times Q(x) + (x-a)^2 \times Q'(x)$ ✓

$f'(a) = 2(a-a) \times Q(a) + (a-a)^2 \times Q'(a)$

$= 0$ ✓

Hence, $(x-a)$ is a factor of $f'(x)$

(b) $(2x-1)^2$ is a factor of $4x^4 - kx^3 - 3x^2 + kx - 1$. Determine the value of k.

Let $f(x) = 4x^4 - kx^3 - 3x^2 + kx - 1$

$f\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{1}{4} - \frac{k}{8} - \frac{3}{4} + \frac{k}{2} - 1 = 0$ ✓✓

$\frac{3k}{8} = \frac{3}{2}$

$k = 4$ ✓

(c) $(x+2)^2$ is a factor of $2x^4 + ax^3 + bx^2 - 4$. Determine the values of a and b.

Let $f(x) = 2x^4 + ax^3 + bx^2 - 4$

$f(-2) = 0 \Rightarrow 32 - 8a + 4b - 4 = 0$

$2a - b = 7$ I ✓

$f'(x) = 8x^3 + 3ax^2 + 2bx$ ✓

$f'(-2) = -64 + 12a - 4b = 0$

$3a - b = 16$ II ✓

II – I $a = 9$ ✓

$b = 11$ ✓

05 Functions I

Calculator Free

1. [4 marks: 2, 2]

Determine analytically if each of the following functions are one-to-one or many-to-one functions.

(a) $f(x) = \frac{1}{x^2}$

$f(-1) = f(1) = 1$. ✓

Hence, $f(x)$ is a many-to-one function. ✓

(b) $f(x) = ln(1 + x)$

For $f(a) = f(b)$, $ln(1+a) = ln(1+b)$

$\Rightarrow a = b$ ✓

Hence, $f(x)$ is a one-to-one function. ✓

---

2. [6 marks: 3, 3]

Find the largest possible domain for each of the following functions to be one-to-one functions. In each case, state the corresponding range.

(a) $f(x) = x(x-1)$

$f(x)$ is symmetrical about $x = \frac{1}{2}$. ✓

Hence, largest possible domain is $[\frac{1}{2}, \infty)$ with corresponding range $[-\frac{1}{4}, \infty)$. ✓✓

OR $(-\infty, \frac{1}{2}]$ with range $[-\frac{1}{4}, \infty)$.

(b) $f(x) = -1 + \frac{1}{2}\sqrt{36 - 9(x-1)^2}$

$f(x)$ is symmetrical about $x = 1$. ✓

Hence, largest possible domain is $[1, 3]$ with corresponding range $[-1, 2]$. ✓✓

OR $[-2, 1]$ with range $[-1, 2]$.

Calculator Free

3. [5 marks: 2, 1, 2] [TISC]

Let $f(x) = x + 1$ for $x \in \mathbb{R}$,

(a) Show that $g(x) = f(-|x|)$ is not a one-to-one function.

$g(1) = f(-|1|) = f(-1) = 0$
$g(-1) = f(-|-1|) = f(-1) = 0$
Hence $g(1) = g(-1)$ but $1 \neq -1$. ✓✓
Hence $g(x)$ is not a one-to-one function.

(b) Let $h(x) = f\left(\frac{1}{x}\right)$ for $x \in \{x : x \in \mathbb{R}, x \neq 0\}$.

(i) Determine the range for $h(x)$.

$h(x) = f\left(\frac{1}{x}\right) = \frac{1}{x} + 1$
Range: $\mathbb{R}\backslash\{1\}$ ✓

(ii) Determine algebraically if $h(x)$ is a one-to-one or many to one function.

$h(a) = \frac{1}{a} + 1$ $h(b) = \frac{1}{b} + 1$ ✓
If $h(a) = h(b) \Rightarrow a = b$. ✓
Hence, $h(x)$ is one-to-one.

---

4. [5 marks: 3, 2]

Given that $f(x) = x^2 - 2$ and $g(x) = \sqrt{x+2}$.

(a) Find the rule for $g\,f(x)$.

$g\,f(x) = g(x^2 - 2)$
$= \sqrt{x^2 - 2 + 2}$ ✓
$= \sqrt{x^2}$ ✓
$= |x|$ ✓

(b) State the natural domain and range for $gf(x)$.

Domain $\mathbb{R}$. ✓
Range $[0, \infty)$ ✓

Calculator Free

5. [5 marks: 1, 2, 2]

Given that $f(x) = \frac{1}{x+1}$ and $g(x) = x - 4$.

(a) State the natural domain for g.

Domain $\mathbb{R}$ ✓

(b) Explain clearly why the domain for g has to be restricted if the $f\,g$ is to be a function.

$g(3) = -1$
$f\,g(3)$ is undefined. ✓
Hence, domain for $f\,g$ has to be restricted to $\{x : x \neq 3,\ x \in \mathbb{R}\}$ ✓

(c) State the largest possible domain for fg and the corresponding range.

Domain $\{x : x \neq 3,\ x \in \mathbb{R}\}$ ✓
Range $\{y : y \neq 0,\ y \in \mathbb{R}\}$ ✓

---

6. [6 marks: 3, 3]

Let $f(x) = \frac{1}{\sqrt{x-3}}$. Let $g(x) = x^2 - 1$ where $x > 0$.

(a) Determine the largest possible domain for $g(x)$ so that $f(g(x))$ is a function.

Range for g: $[-1, \infty)$. Domain for f: $(3, \infty)$. ✓
For $f(g(x))$ to be a function,
Range of $g \subset$ Domain of f.
Hence: $x^2 - 1 > 3$ ✓
$\Rightarrow x > 2$
$\Rightarrow$ Restricted domain for g is $(2, \infty)$. ✓

$f(g(x)) = \frac{1}{\sqrt{x^2 - 4}}$ ✓
Domain for $f(g(x))$ is $(-\infty, -2) \cup (2, \infty)$.
Domain for g is $(0, \infty)$. ✓
Restricted domain for g is $(2, \infty)$. ✓

(b) Determine the rule for $g(f(x))$, state its domain and range.

$g(f(x)) = g\left(\frac{1}{\sqrt{x-3}}\right))$
$= \frac{1}{x-3} - 1$ ✓
Domain: $(3, \infty)$ ✓
Range: $(-1, \infty)$ ✓

Calculator Free

7. [11 marks: 5, 3, 3]

(a) Let $f(x) = \sqrt{9-x}$ and $g(x) = 3^{x+1}$. Find the domain and range for $f(g(x))$.

$f(g(x)) = f(3^{x+1})$
$= \sqrt{9-3^{x+1}}$ ✓✓
Domain: $9 - 3^{x+1} \geq 0$
$3^2 \geq 3^{x+1}$
$2 \geq x+1$
Hence, domain is $\{x : x \leq 1,\ x \in \mathbb{R}\}$ ✓
Range: $\{y: 0 \leq y < 3,\ x \in \mathbb{R}\}$ ✓✓

(b) Given that $f(g(x)) = x^2$ and $f(x) = e^x - 1$, determine $g(x)$.

$f(g(x)) = e^{g(x)} - 1$ ✓
But, $f(g(x)) = x^2$
Hence: $e^{g(x)} - 1 = x^2$
$e^{g(x)} = x^2 + 1$ ✓
$g(x) = ln(x^2 + 1)$ ✓

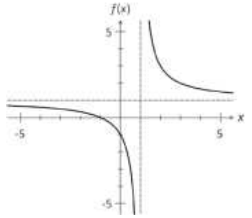
(c) Given that $f(g(x)) = \dfrac{x}{x+1}$ and $g(x) = 4 - x$, determine $f(x)$.

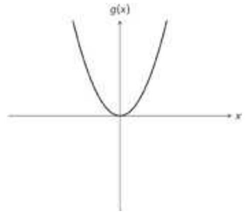
$f(g(x)) = \dfrac{x}{x+1}$
$f(4-x) = \dfrac{x}{x+1}$ ✓
Let $u = 4 - x \Rightarrow x = 4 - u$
Hence: $f(u) = \dfrac{x}{x+1}$ ✓
$= \dfrac{4-u}{4-u+1}$
$f(x) = \dfrac{4-x}{3-x}$ ✓

Calculator Free

8. [4 marks: 2, 2] [TISC]

The graphs of functions $f(x)$ and $g(x)$ are drawn below.





(a) Find the asymptote(s) of $g\,f(x)$.

$x = 1$ and $y = 1$ ✓✓

(b) Find the range for $g\,f(x)$.

$\{y:\ y \geq 0, y \neq 1,\ y \in \mathbb{R}\}$ ✓✓

---

9. [7 marks: 2, 2, 3] [TISC]

(a) Given that $f \circ f(x) = x + 4$, find $f(x)$.

By inspection: $f(x) = x + 2$ ✓✓

(b) Given that $g \circ g(x) = x^4$, find $g(x)$.

By inspection: $g(x) = x^2$ ✓✓

(c) Explain clearly why it is not possible to find a real valued function $h(x)$ such that $h \circ h(x) = -x$. [Hint: Let $h(x) = ax + b$ where $a, b \in \mathbb{R}$.]

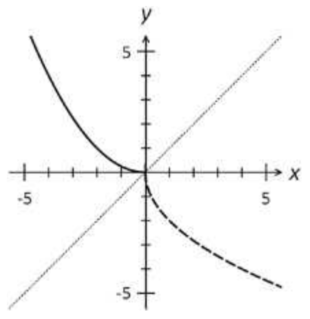
Let $h(x) = ax + b$ where $a, b \in \mathbb{R}$.
$\Rightarrow h(h(x)) = h(ax+b)$
$= a(ax+b) + b$
$= a^2x + (ab+b)$ ✓
But $h(h(x)) = -x$.
Compare x terms: $a^2 = -1$ ✓
$\Rightarrow a$ is not real. ✓
Hence, it is not possible.

Calculator Free

10. [6 marks: 2, 4]

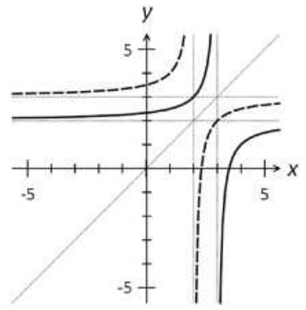
The graph of $y = f(x)$ is shown in the accompanying diagrams. In each case, sketch the graph for $f^{-1}(x)$.

(a)



Symmetrical about $y = x$ ✓
All correct ✓

(b)



Symmetrical about $y = x$ ✓
Asymptotes $y = 2$ & $x = 3$ ✓✓
All correct ✓

---

11. [8 marks: 2, 6]

Consider the function $f(x) = \dfrac{1 - x}{2 + x}$.

(a) State the natural domain and range for $f(x)$.

Domain $\{x:\ x \neq -2,\ x \in \mathbb{R}\}$ ✓
Range $\{y:\ y \neq -1,\ y \in \mathbb{R}\}$ ✓

(b) Find the rule for $f^{-1}(x)$. State the domain and range for $f^{-1}(x)$.

Let $y = \dfrac{1 - x}{2 + x}$.

$xy + x = 1 - 2y$ ✓

$x(1 + y) = 1 - 2y$ ✓

$\Rightarrow \quad x = \dfrac{1 - 2y}{1 + y}$. ✓

Hence, $f^{-1}(x) = \dfrac{1 - 2x}{1 + x}$ ✓

Domain $\{x: x \neq -1,\ x \in \mathbb{R}\}$ ✓
Range $\{y:\ y \neq -2,\ y \in \mathbb{R}\}$ ✓

Calculator Free

12. [8 marks: 4, 4]

Let $f(x) = ln\,(1 - x)$ and $g(x) = \dfrac{1}{x}$.

(a) Find the largest possible domain for f so that $g \circ f$ is a function. State the accompanying range.

$g\,(f(x)) = \dfrac{1}{f(x)} = \dfrac{1}{ln\,(1 - x)}$ ✓

Hence, domain is $\{x: x < 1, x \neq 0,\ x \in \mathbb{R}\}$ ✓✓
Range is $\{y:\ y \neq 0,\ y \in \mathbb{R}\}$ ✓

(b) Find x such that $g \circ f(x) = g^{-1} \circ f(x)$. Justify your answer.

$g(x) = \dfrac{1}{x} \Rightarrow g^{-1}(x) = \dfrac{1}{x}$ ✓

$g^{-1}(x) = g(x)$ ✓

Hence, $g(f(x)) \equiv g^{-1}(f(x))$. ✓

Therefore, $g(f(x)) = g^{-1}(f(x))$
$\forall\ x < 1$ with $x \neq 0$. ✓

---

13. [7 marks: 3, 1, 3]

Given that $f(x) = (x - 1)^2$ where x is a real number.

(a) Find $f(0)$ and $f(2)$. Hence, show that $f(x)$ does not have an inverse function.

$f(0) = 1 \qquad f(2) = 1$ ✓
Since $f(0) = f(2) = 1$, $f(x)$ is a many to one function. ✓✓
Hence, $f(x)$ does not have an inverse function.

(c) Find the largest possible positive domain for $f(x)$, so that $f(x)$ has an inverse function.

Domain $\{x: x \geq 1,\ x \in \mathbb{R}\}$ ✓

(d) For the domain in (c), find the rule for the inverse for $f(x)$.

Let $y = (x - 1)^2$.
$x = 1 \pm \sqrt{y}$ ✓
Since $x \geq 1$, $x = 1 + \sqrt{y}$. ✓
Hence, $f^{-1}(x) = 1 + \sqrt{x}$ ✓

Calculator Free

14. [6 marks: 2, 1, 3]

Consider $f(x) = e^{|x-2|}$ where $x \in \mathbb{R}$.

(a) Show that the inverse of f is not a function.

$f(1) = f(3) = e$ but $1 \neq 3$.
Hence, f is not a one-to-one function. ✓
Therefore inverse of f is not a function. ✓

(b) One largest possible domain for $f(x)$ to have an inverse which is a function is $\{x \mid x \in \mathbb{R}, -\infty < x \leq a\}$. Determine the value of a.

$a = 2$ ✓

(c) For the domain specified in part (b), find the rule for $f^{-1}(x)$.

For $\{x|x\in\mathbb{R}, -\infty < x \leq 2\}$: $|x-2| = -(x-2)$ ✓
$\Rightarrow \ y = e^{-(x-2)}$
$\ln y = 2 - x$ ✓
$x = 2 - \ln y$
Hence: $f^{-1}(x) = 2 - \ln x$ ✓

---

15. [6 marks: 3, 3]

(a) $f(x) = \sin 2x$ is a one-to-one function within the domain $-a \leq x \leq a$. Determine the largest possible value for $|a|$. Hence, determine the rule for $f^{-1}(x)$ and state the corresponding range.

Max value for $|a| = \frac{\pi}{4}$ ✓
$y = \sin 2x \ \Rightarrow \ x = \frac{1}{2}\sin^{-1} y$
Hence, $f^{-1}(x) = \frac{1}{2}\sin^{-1} x$. ✓
Range: $[-\frac{\pi}{4}, \frac{\pi}{4}]$ ✓

Calculator Free

15. (b) $g(x) = \cos\frac{x}{2}$ is a one-to-one function within the domain $0 \leq x \leq b$. Determine the largest possible value for b. Hence, determine the rule for $g^{-1}(x)$ and state the corresponding range.

Max value for $b = 2\pi$ ✓
$y = \cos\frac{x}{2} \ \Rightarrow \ x = 2\cos^{-1} y$
Hence, $g^{-1}(x) = 2\cos^{-1} x$. ✓
Range: $[0, 2\pi]$ ✓

---

16. [8 marks: 4, 4]

Let $f(x) = \frac{1+x}{2-x}$ where $x \in \{x : x \in \mathbb{R}, 2 < x < 5\}$ and $g(x) = \frac{2x-1}{x+1}$ where $x \neq -1$.

(a) Determine $f(g(x))$.

$$f(g(x)) = f\left(\frac{2x-1}{x+1}\right)) \quad ✓$$
$$= \frac{1+\left(\frac{2x-1}{x+1}\right)}{2-\left(\frac{2x-1}{x+1}\right)} \quad ✓$$
$$= \frac{\left(\frac{3x}{x+1}\right)}{\left(\frac{3}{x+1}\right)} \quad ✓$$
$$= x \quad ✓$$

(b) Determine the domain of $g(x)$ so that f is the inverse function for g.

From (a) $f(g(x)) = x$
$\Rightarrow$ f and g are inverse pairs. ✓
For f to be the inverse of g,
Range of g = Domain of f. ✓
Hence: Range of $g = \{x : x\in\mathbb{R}, \ 2 < x < 5\}$
$\Rightarrow \ 2 < \frac{2x-1}{x+1} < 5$ ✓
$x < -2$
Hence, domain for g: $\{x : x\in\mathbb{R}, \ x < -2\}$ ✓

Calculator Free

17. [10 marks: 2, 3, 5]

Let $f(x) = -1 + \sqrt{\left(\frac{x}{4}\right)}$ and $g(x) = (2x+2)^2$.

(a) Determine $g(f(x))$.

$g(f(x)) = g(-1 + \sqrt{\left(\frac{x}{4}\right)})$

$= \left[2\left(-1 + \sqrt{\left(\frac{x}{4}\right)}\right) + 2\right]^2$ ✓

$= \left[2\sqrt{\left(\frac{x}{4}\right)}\right]^2$

$= x$ ✓

(b) Determine $f(g(x))$.

$f(g(x)) = f((2x+2)^2)$

$= -1 + \sqrt{\frac{(2x+2)^2}{4}}$ ✓

$= -1 + \sqrt{(x+1)^2}$ ✓

$= -1 + |x+1|$ ✓

(c) Determine with reasons, the domain of f and the domain of g so that these functions are inverses of each other.

From (a):

$g(f(x)) = x$ if $x \geq 0$ ✓

Hence, g is the inverse of f if $x \geq 0$. ✓

From (a):

$f(g(x)) = -1 + |x+1|$

$= -1 + (x+1) = x$ if $x \geq -1$ ✓

Hence, domain of f is $[0, \infty)$, ✓

domain of g restricted to $[-1, \infty)$. ✓

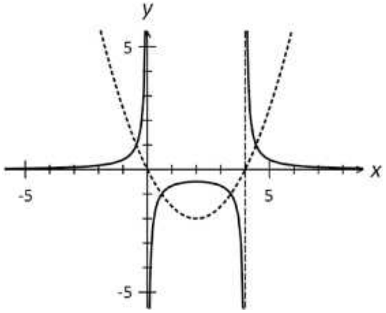
---

06 Functions II

Calculator Free

1. [12 marks: 4, 4, 4]

(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = \frac{1}{f(x)}$.



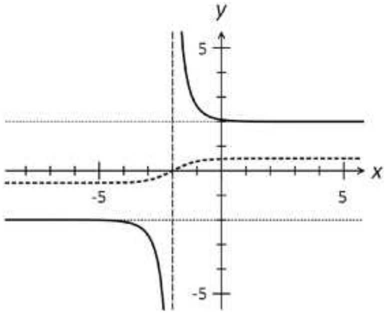
Asymptotes:

$y = 0, x = 0, x = 4$ ✓✓

Max point $(2, \frac{1}{2})$ ✓

All correct ✓

(b) The sketch of $y = \frac{1}{f(x)}$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(x)$.

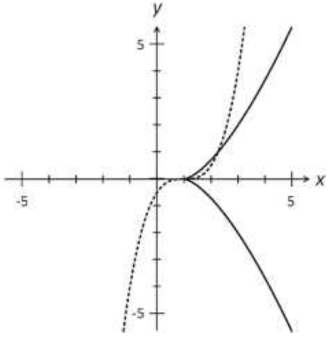


Asymptotes:

$y = -2, y = 2, x = -2$ ✓✓✓

All correct ✓

(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y^2 = f(x)$.



Domain: $[1, \infty)$ ✓

Symmetrical about the x-axis. ✓

Both graphs intersect when $y = 1$ ✓

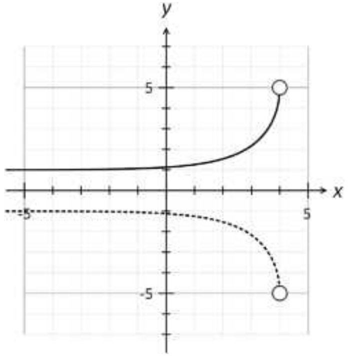
All correct. ✓

Calculator Free

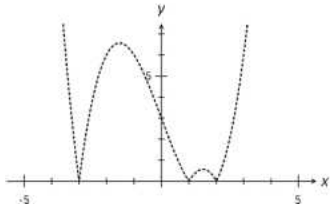
2. [8 marks: 4, 4]

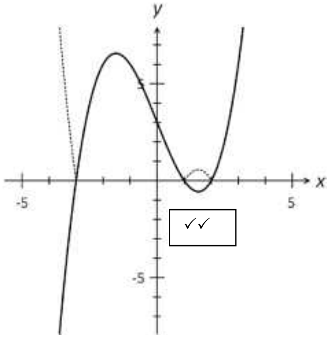
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = |f(x)|$.

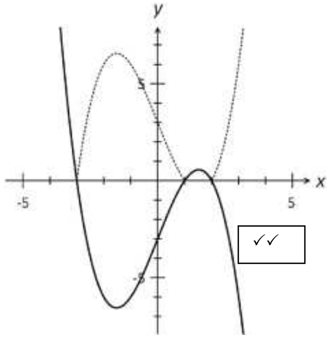
Reflected about x-axis ✓
End point: (5, 5) ✓
Asymptote: $y = 1$ ✓
All correct. ✓



(b) The sketch of $y = |f(x)|$ is given in the accompanying diagram. Sketch on the axes provided below the two possible graphs of $y = f(x)$.





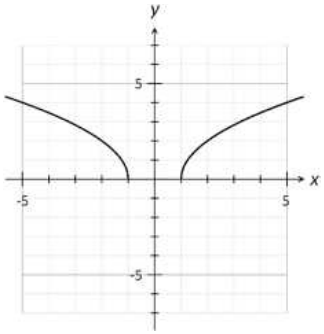


Calculator Free

3. [9 marks: 3, 3, 3]

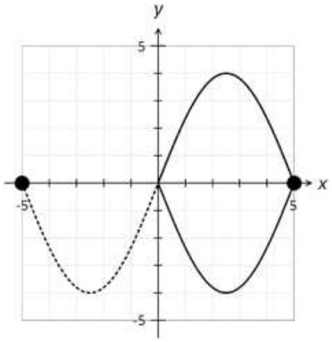
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(|x|)$.

Symmetrical about y-axis ✓
Domain: $(-\infty, -1] \cup [1, \infty)$ ✓
All correct ✓



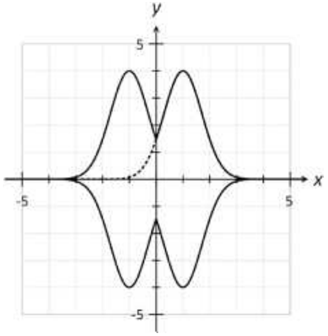
(b) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $|y| = f(x)$.

Domain: [0, 5] ✓
Range: [–4, 4] ✓
Symmetrical about x-axis ✓



(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $|y| = f(|x|)$.

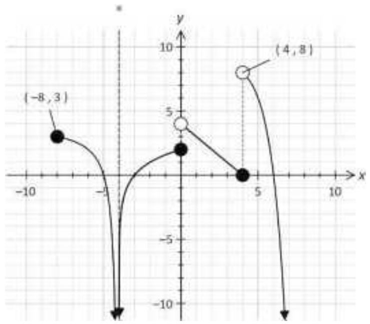
Symmetrical about x-axis ✓
Symmetrical about y-axis ✓
All correct ✓



Calculator Free

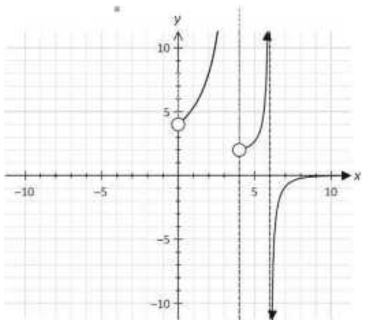
4. [6 marks: 3, 3] [TISC]

The accompanying diagram shows the graph of the function $y = f(x)$.



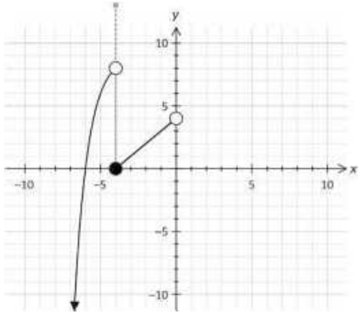
(b) On the accompanying diagram, for $x > 0$, sketch the graph of $y = \dfrac{16}{f(x)}$.

- ✓ Vertical asymptote at $x = 6$.
- ✓ Horizontal asymptote at $y = 0$.
- ✓ "Holes" at (0, 4) and (4, 2).



(c) On the accompanying diagram, for $x < 0$, sketch the graph of $y = f(|x|)$.

- ✓ Reflects portions of graph for $x > 0$ about the y-axis
- ✓ "Holes" at (0, 4) and (−4, 8).
- ✓ Roots at (−4, 0) and (−6, 0).



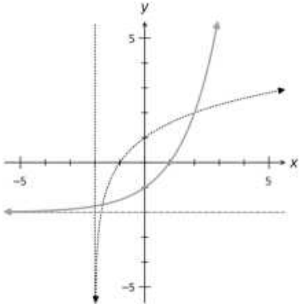
Calculator Free

5. [7 marks: 3, 4]

The accompanying diagrams each show the graph of $y = f(x)$.

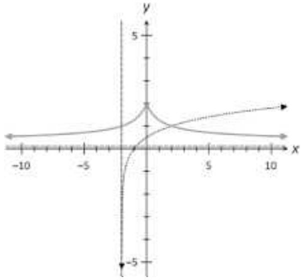
(a) Sketch on the same axes, the graph of $y = f^{-1}(x)$.

- ✓ Intercepts at (0, −1) & (1, 0).
- ✓ Horizontal asymptote $y = -2$.
- ✓ Curves intersect on line $y = x$.



(b) Sketch on the same axes, the graph of $y = \dfrac{1}{f(|x|)}$.

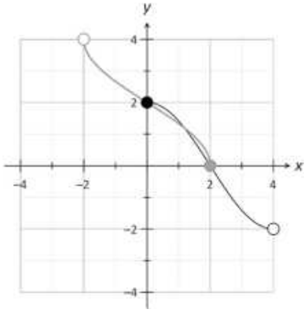
- ✓ Symmetrical about y-axis
- ✓ Horizontal asymptote: $y = 0$
- ✓ Global Max at (0, 2)
- ✓ Curves intersect along $y = 1$



6. [7 marks: 3, 4]

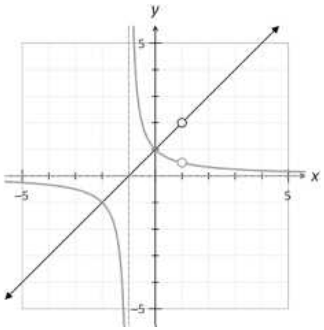
(a) The accompanying diagram shows the graph of $y = f^{-1}(x)$. Sketch on the same axes, the graph of $y = f(x)$.

- ✓ Open point at (−2, 4) and closed point at (2, 0).
- ✓ The two curves intersect on $y = x$.
- ✓ Symmetry about $y = x$.



Calculator Free

6. (b) The accompanying diagram shows the graph of $y = \dfrac{1}{f(x)}$. Sketch on the same axes, the graph of $y = f(x)$.

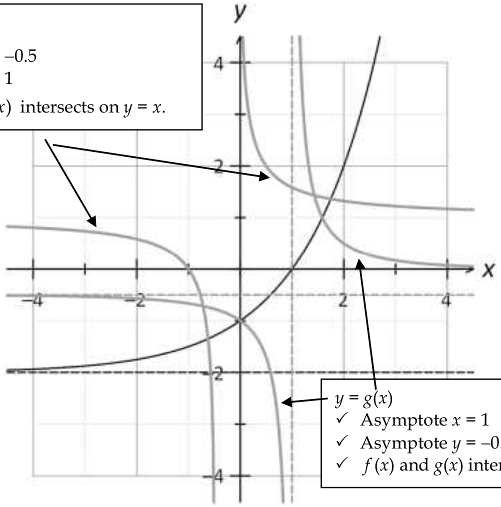


- ✓ Vertical asymptote $x = -1$.
- ✓ Horizontal asymptote: $y = 0$
- ✓ Discontinuity at (1, 0.5)
- ✓ Curves intersect along $y = \pm 1$

---

7. [6 marks: 3, 3]

The diagram below shows the graph of $y = f(x)$. Let $g(x) = \dfrac{1}{f(x)}$.



$y = g^{-1}(x)$
- ✓ Asymptote $x = -0.5$
- ✓ Asymptote $y = 1$
- ✓ $f(x)$ and $g^{-1}(x)$ intersects on $y = x$.

$y = g(x)$
- ✓ Asymptote $x = 1$
- ✓ Asymptote $y = -0.5$
- ✓ $f(x)$ and $g(x)$ intersects on $y = \pm 1$.

(a) On the axes above sketch and label the graph of $y = g(x)$.

(b) On the diagram above, sketch and label the graph of $y = g^{-1}(x)$.

Calculator Free

8. [16 marks: 4, 4, 4, 4]

The graph of $y = f(x)$ has intercepts at (2, 0) and (0, –2) and asymptotes with equations $x = 1$ and $y = -1$.

(a) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = \dfrac{1}{f(x)}$.

Intercepts: (1, 0) and $(0, -\frac{1}{2})$. ✓✓

Asymptotes: $x = 2, y = -1$ ✓✓

(b) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = |f(x)|$.

Intercepts: (2, 0), (0, 2) ✓✓

Asymptotes: $x = 1, y = 1$ ✓✓

(c) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y^2 = f(x)$.

Intercepts: (2, 0) ✓✓

Asymptotes: $x = 1$ ✓✓

(d) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = f(|x|)$.

Intercepts: (2, 0), (–2, 0), (0, –2) ✓✓

Asymptotes: $y = -1, x = -1, x = 1$ ✓✓

Calculator Free

9. [7 marks: 3, 4]

Let $f(x) = |x - 1| + |x + 2| - |x|$.

(a) Rewrite $f(x)$ in piecewise defined form.

Critical points: $x = -2, 0, 1$ ✓

$$f(x) = \begin{cases} -x - 1 & x < -2 \\ x + 3 & -2 \le x < 0 \\ -x + 3 & 0 \le x < 1 \\ x + 1 & x \ge 1 \end{cases}$$ ✓✓

(b) Hence or otherwise, solve $|x - 1| + |x + 2| - |x| \ge 5$.

$$\begin{cases} -x - 1 & x < -2 \\ x + 3 & -2 \le x < 0 \\ -x + 3 & 0 \le x < 1 \\ x + 1 & x \ge 1 \end{cases} \ge 5$$

For $x < -2$: $-x - 1 \ge 5 \Rightarrow x \le -6$ ✓

For $-2 \le x < 0$: $x + 3 \ge 5 \Rightarrow x \ge 2$
Hence: No solution

For $0 \le x < 1$: $-x + 3 \ge 5 \Rightarrow x \le -2$
Hence: No solution. ✓

For $x \ge 1$: $x + 1 \ge 5 \Rightarrow x \ge 4$ ✓

Therefore: $x \le -6$ or $x \ge 4$ ✓

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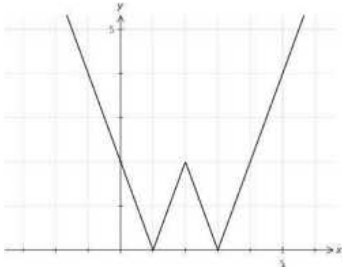
10. [4 marks]

The given diagram shows the graph of $y = |a + b|kx + c||$. Determine the values of constants a, b, c and k where $k < 0$.

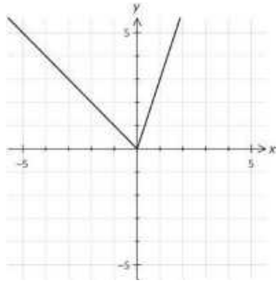


Primary Vertex at (2, 2). $\Rightarrow a = -2$ ✓
Gradient of pieces = ± 2 $\Rightarrow k = \pm 2$
But $k < 0$. $\Rightarrow k = -2$. ✓
Hence: $c = 4$ ✓
Hence: $y = |-2 + b|-2x + 4||$
$x = 1, y = 0$ $0 = |-2 + b|2|| \Rightarrow b = 1$ ✓

Calculator Free

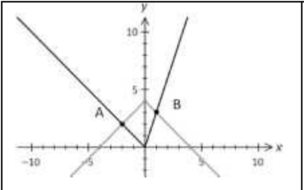
11. [7 marks: 3, 4]

(a) On the given axes, sketch the graph of $y = ||x| + 2x|$.



✓ V shaped with vertex at (0, 0).
✓ $y = x + 3$ for $x \ge 0$
✓ $y = -x$ for $x < 0$

(b) Solve algebraically $||x| + 2x| = 4 - |x|$.



At A: $-x = x + 4$ ✓
$x = -2$ ✓
At B: $3x = -x + 4$ ✓
$x = 1$ ✓

---

12. [7 marks: 3, 3, 1] [TISC]

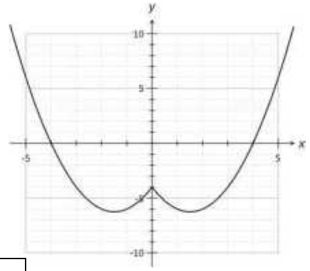
Let $f(x) = x^2 - 3|x| - 4$.

(a) Rewrite $f(x)$ in piecewise defined form.

Critical point: $x = 0$ ✓

$$f(x) = \begin{cases} x^2 + 3x - 4 & x < 0 \\ x^2 - 3x - 4 & x \ge 0 \end{cases}$$ ✓✓

(b) In the axes provided, sketch the graph of $y = x^2 - 3|x| - 4$.



✓ (0, –4)
✓ Min points
✓ Symmetrical about y-axis

(c) Use your sketch to explain why $f(x)$ does not have an inverse function.

Graph of $f(x)$ fails the horizontal line test. ✓

07 Rational Functions

Calculator Free

1. [9 marks: 3, 3, 3]

Consider the curve with equation $y = \dfrac{(x+1)(x-2)^2}{x^2-x-2}$.

(a) Determine the coordinates of the point(s) of discontinuity in this curve.

$y = \dfrac{(x+1)(x-2)^2}{(x+1)(x-2)}$ ✓

$\Rightarrow$ y is undefined for $x = -1$ and $x = 2$. ✓

Hence: (–1, –3) and (2, 0). ✓

(b) Explain clearly why this curve does not have any horizontal asymptotes.

For $x \neq -1$ and $x \neq 2$: $y = x - 2$ ✓

$\lim\limits_{x \to \infty} y = x$ ✓

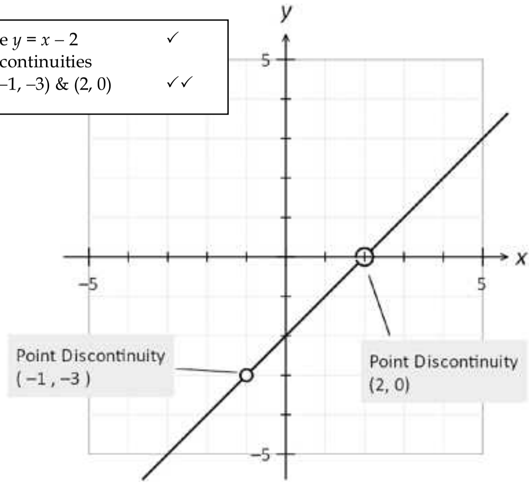
Hence, curve does not have a horizontal asymptote. ✓

(c) Draw a sketch of this curve showing all the relevant features.

Line $y = x - 2$ ✓

Discontinuities

at (–1, –3) & (2, 0) ✓✓



Calculator Free

2. [9 marks: 2, 3, 4]

Consider the curve with equation $y = \dfrac{x^2+x-2}{x^2-2x-8}$.

(a) State the equation of all asymptotes.

$y = \dfrac{(x-1)(x+2)}{(x-4)(x+2)}$

Asymptotes are: $x = 4$ and $y = 1$ ✓✓

(b) Identify the point of discontinuity on this curve.

$y = \dfrac{(x-1)(x+2)}{(x-4)(x+2)}$

$= \dfrac{(x-1)}{(x-4)}$ for $x \neq -2$ ✓✓

For $x \to -2 \Rightarrow y \to -\dfrac{1}{2}$

Hence, point of discontinuity is $(-2, -\dfrac{1}{2})$. ✓

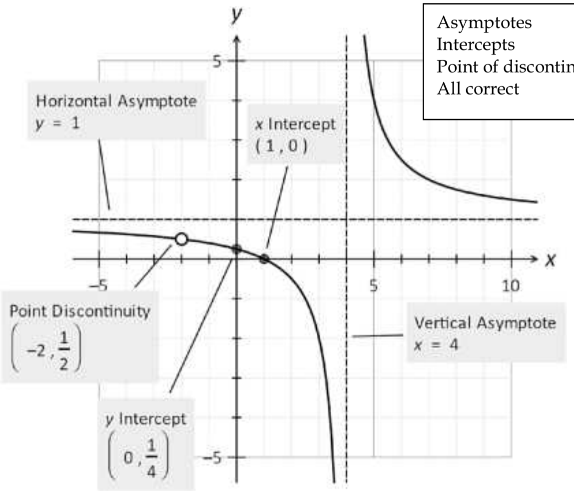
(c) Sketch this curve on the axes provided below.

Asymptotes ✓

Intercepts ✓

Point of discontinuity ✓

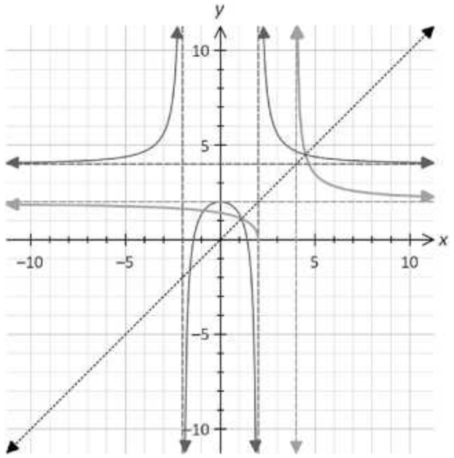
All correct ✓



Calculator Free

3. [7 marks: 3, 4]

The diagram below shows the graph of $y = f(x)$ where $f(x) = \dfrac{ax^2 + b}{x^2 + c}$,
where a, b and c are real constants. The graph has a turning point at (0, 2).
The graph has asymptotes with equation $x = –2$, $x = 2$ and $y = 4$.



(a) Determine the values of the constants a, b and c.

Vertical asymptotes $x = \pm 2 \Rightarrow c = –4$ ✓
Horizontal asymptote $y = 4 \Rightarrow a = 4$ ✓
Vertical intercept (0, 2) $\Rightarrow b = –8$ ✓

(b) State the largest possible domain for $f(x)$ so that $f^{-1}(x)$ exists.

For this domain, on the axis above, sketch the graph of $y = f^{-1}(x)$.

Domain $\mathbb{R}_0^+\backslash\{2\}$ or $\mathbb{R}_0^-\backslash\{-2\}$ ✓

Graph of $y = f^{-1}(x)$
- ✓ Root at (2, 0)
- ✓ Asymptotes $x = 4$ and $y = 2$.
- ✓ Graph is symmetrical about $y = x$ and intersects $y = f(x)$ on $y = x$.

Calculator Free

4. [11 marks: 4, 2, 5]

Consider the curve with equation $y = \dfrac{x^2 + x - 6}{x - 1}$.

(a) Rewrite the equation of the curve in the form $y \equiv \dfrac{P(x)}{Q(x)} + ax + b$ where $\dfrac{P(x)}{Q(x)}$ is a rational proper fraction and a and b are real constants.

$$\frac{x^2 + x - 6}{x - 1} = \frac{x(x-1) + 2x - 6}{x - 1} \quad ✓$$
$$= \frac{x(x-1) + 2(x-1) - 4}{x - 1} \quad ✓$$
$$= \frac{-4}{x-1} + x + 2 \quad ✓✓$$

OR

$$\begin{array}{r|l} & x + 2 \\ \hline x - 1 & x^2 + x - 6 \quad ✓✓ \\ & x^2 - x \\ \hline & 2x - 6 \\ & 2x - 2 \\ \hline & -4 \end{array}$$

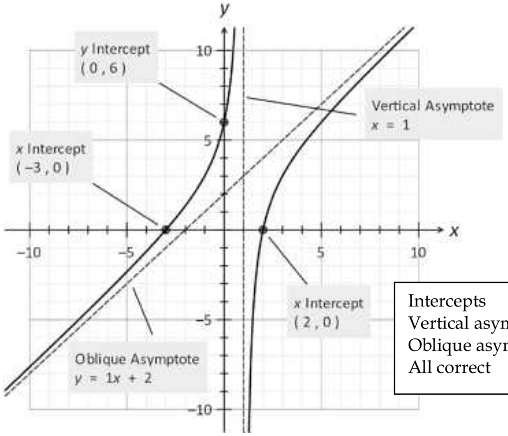
$$y = \frac{-4}{x-1} + x + 2 \quad ✓✓$$

(b) State the equations of all asymptotes of this curve.

Asymptotes: $x = 1$, ✓
$y = x + 2$ ✓

(c) On the axes provided below sketch the graph of $y = \dfrac{x^2 + x - 6}{x - 1}$.

Indicate all intercepts and asymptotes.



Intercepts	✓✓
Vertical asymptote	✓
Oblique asymptote	✓
All correct	✓

Calculator Free

5. [6 marks]

Consider the curve with equation $y = \dfrac{x^3+2x^2+x-4}{2(x^2-1)}$. Identify the equations of all asymptotes and points of discontinuities of this curve.

Rewrite $y = \dfrac{1}{2}\left(\dfrac{x^3+2x^2+x-4}{(x^2-1)}\right)$

$$x^2-1\,\overset{x+2}{\overline{)\,x^3+2x^2+x-4}} \quad ✓$$

$$\underline{x^3-x}$$
$$2x^2+2x-4$$
$$\underline{2x^2-2}$$
$$2x-2$$

Hence: $y = \dfrac{1}{2}\left(x+2+\dfrac{2x-2}{x^2-1}\right)$ ✓✓

$= \dfrac{x}{2}+1+\dfrac{(x-1)}{(x-1)(x+1)}$

Vertical asymptote is $x = -1$. ✓
Point of discontinuity (1, 2) ✓

Oblique asymptote is $y = \dfrac{x}{2} + 1$. ✓

---

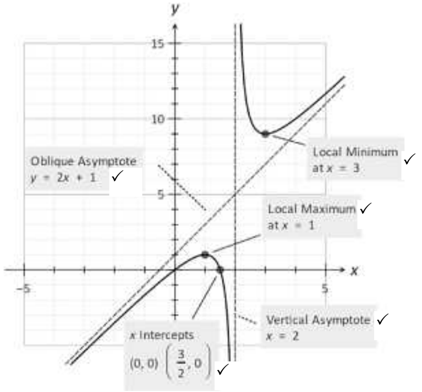
6. [5 marks]

Sketch the graph of the rational function $f(x)$ with the following properties:

- $f(0) = f\left(\dfrac{3}{2}\right) = 0$,
- $f'(1) = f'(3) = 0$
- $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = 0$
- $\lim\limits_{x\to2^-} f(x) \to -\infty$,

  $\lim\limits_{x\to2^+} f(x) \to \infty$
- $\lim\limits_{x\to\infty} f(x) = 2x+1$,

  $\lim\limits_{x\to-\infty} f(x) = 2x+1$

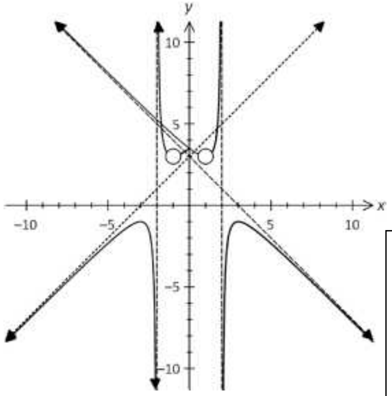


Calculator Free

7. [9 marks: 3, 3, 3]

The diagram below shows the graph of $y = f(x)$ where $f(x) = \dfrac{p(x)}{q(x)}$.

$p(x)$ is a polynomial of degree 3. $q(x)$ is a polynomial of degree 2 with a leading coefficient of 1. The polynomials $p(x)$ and $q(x)$ are non-reduced, that is, they may have factors in common. The graph has a point of discontinuity at (1, 3). The graph as asymptotes with equation $x = 2$ and $y = -x + 3$



✓ Reflects "hole" about the y-axis.
✓ Reflects vertical asymptote about the y-axis.
✓ Reflects entire graph for $x \geq 0$ about the y-axis.

(a) Determine with reasons the polynomial $q(x)$.

Since $x = 2$ is an asymptote, $(x - 2)$ must be a factor of q. ✓
Since there is a discontinuity at $x = 1$, $(x - 1)$ must be a factor of q. ✓
Since, q is of degree 2 with lead coefficient of 1,
$q(x) = (x - 1)(x - 2)$ ✓

(b) Determine the polynomial $p(x)$.

$y = -x + 3 + \dfrac{x-1}{(x-1)(x-2)}$ ✓
$p(x) = (-x + 3)(x - 1)(x - 2) - (x - 1)$ ✓
$= -x^3 + 6x^2 - 12x + 7$ ✓

(c) On the axes above, sketch the graph of $y = f(|x|)$.

Calculator Free

8. [13 marks: 3, 2, 4, 4] [TISC]

(a) The function $f(x) = \dfrac{x^3+1}{x^2-1}$ can be rewritten as $px + \dfrac{qx+r}{x^2-1}$. Find p, q and r.

> $x^3 + 1 = px(x^2 - 1) + qx + r$
> Compare x^3 coefficient: $p = 1$ ✓
> Subst. $x = 0$: $r = 1$ ✓
> Subst. $x = 1$: $q = 1$ ✓

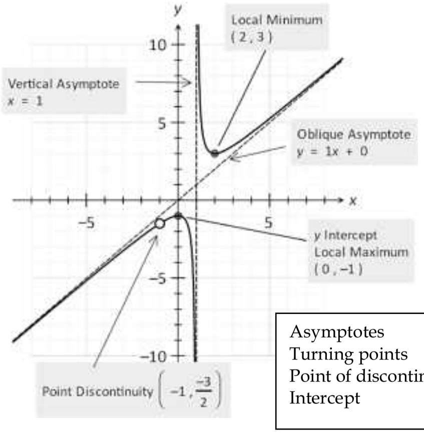
(b) State the equations of all the asymptotes of the curve $y = f(x)$.

> Vertical asymptote: $x = 1$ ✓
> Oblique asymptote: $y = x$ ✓

(c) The curve with equation $y = f(x)$ has a maximum point at (0, –1).
Use Calculus to show that the curve has a local minimum point at (2, 3).

> $\dfrac{dy}{dx} = \dfrac{3x^2(x^2-1)-(x^3+1)(2x)}{(x^2-1)^2}$ ✓
>
> When $x = 2$, $\dfrac{dy}{dx} = 0.$ ✓
>
> $\left.\dfrac{dy}{dx}\right|_{x=2^-} < 0$ and $\left.\dfrac{dy}{dx}\right|_{x=2^+} > 0$ ✓
>
> Hence, (2, 3) is a minimum point. ✓

(d) Sketch the graph of $y = \dfrac{x^3+1}{x^2-1}$.



> Asymptotes ✓
> Turning points ✓
> Point of discontinuity ✓
> Intercept ✓

Calculator Free

9. [12 marks: 4, 4, 4] [TISC]

Consider the curve with equation $y = \dfrac{x^3}{x^2-4}$.

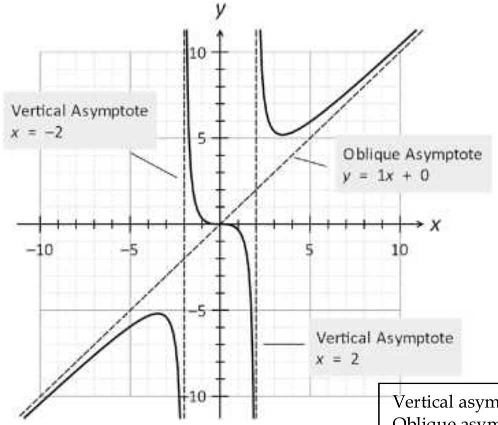
(a) Find the equation of the asymptote(s).

> Vertical Asymptotes: $x = -2, x = 2$ ✓✓
> By polynomial division: $y = x + \dfrac{4x}{x^2-4}$. ✓
> Hence, oblique asymptote: $y = x$ ✓

(b) Use differentiation to find the number of stationary points on this curve.

> $\dfrac{dy}{dx} = \dfrac{(x^2-4)(3x^2)-x^3(2x)}{(x^2-4)^3}$ ✓
>
> $\dfrac{dy}{dx} = 0 \Rightarrow 3x^4 - 12x^2 - 2x^4 = 0$ ✓
>
> $x^2(x^2-12) = 0$
>
> $x = 0, \pm 2\sqrt{3}$ ✓
>
> $y = \dfrac{x^3}{x^2-4}$ is defined for $x = 0$ and $\pm 2\sqrt{3}$.
>
> Hence, there are three stationary points. ✓

(c) Sketch this curve on the axes below.

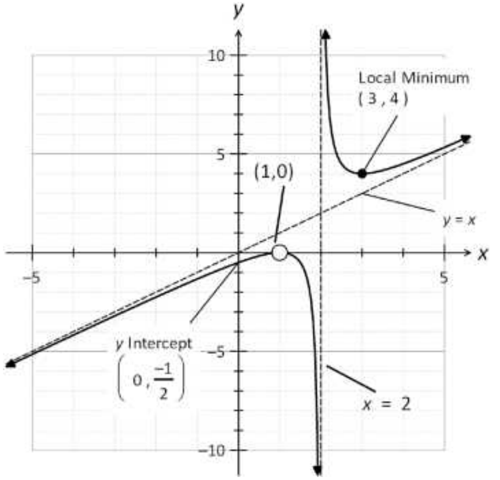


> Vertical asymptotes ✓
> Oblique asymptotes ✓
> Stationary points ✓
> All correct ✓

Calculator Free

10. [7 marks]

On the axes provided below, sketch the curve with equation $y = x + \frac{(x-1)}{(x-1)(x-2)}$. Indicate clearly all essential features of this curve.



Consider $y = x + \frac{(x-1)}{(x-1)(x-2)}$ $= x + \frac{1}{x-2}$ for $x \neq 1$	
Vertical asymptote: $x = 2$ Oblique asymptote: $y = x$ Discontinuity (1, 0)	✓ Draws correct vertical asymptote. ✓ Draws correct oblique asymptote. ✓ Indicates discontinuity at (1, 0).
Roots: $x + \frac{1}{x-2} = 0$ $x^2 - 2x + 1 = 0$ $x = 1$ But $x \neq 1$, hence no roots.	✓ Shows attempt to determine roots.
$x = 0, y = \frac{-1}{2}$	✓ Indicates y-intercept.
$\frac{dy}{dx} = 1 - \frac{1}{(x-2)^2} = 0$ $\Rightarrow x = 1, 3$ Stationary points (1, 0) & (3, 4).	✓ Shows attempt to determine stationary points. ✓ Obtains and indicates correct stationary points.

Calculator Assumed

11. [15 marks: 2, 2, 4, 3, 4]

Consider the curve with equation $y = \left|\frac{x-1}{x^2+1}\right|$.

(a) Explain why the equation of this curve can also be written as $y = \frac{|x-1|}{x^2+1}$.

$y = \left|\frac{x-1}{x^2+1}\right| = \frac{|x-1|}{|x^2+1|}$. ✓

But $x^2 + 1 > 0 \; \forall \; x$. ✓

Hence, $y = \frac{|x-1|}{x^2+1}$.

(b) The equation of this curve can be rewritten as $y = \begin{cases} \frac{ax+b}{x^2+1} & x < 1 \\ \frac{x-1}{x^2+1} & x \geq 1 \end{cases}$

where a and b are constants. Find a and b.

$a = -1, \; b = 1$ ✓✓

(c) Use differentiation to verify that this curve has a maximum point at $x = 1 + \sqrt{2}$.

For $x \geq 1$, $y = \frac{x-1}{x^2+1}$.

$\frac{dy}{dx} = \frac{(x^2+1) - (x-1)(2x)}{(x^2+1)^2}$ ✓

$= \frac{-x^2 + 2x + 1}{(x^2+1)^2}$ ✓

When $\frac{dy}{dx} = 0, x = 1 + \sqrt{2}$. ✓

$\left.\frac{dy}{dx}\right|_{x=(1+\sqrt{2})^-} > 0$ and $\left.\frac{dy}{dx}\right|_{x=(1+\sqrt{2})^+} < 0$ ✓

Hence, $x = 1 + \sqrt{2}$ gives a maximum point.

Calculator Free

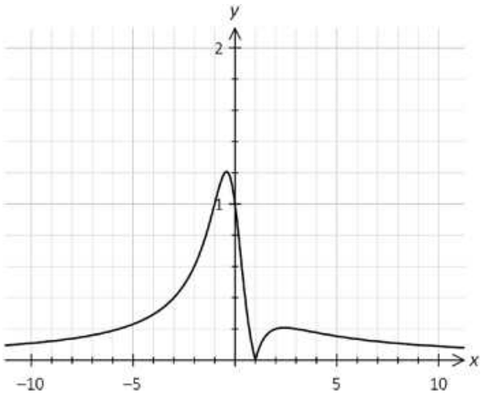
11. (d) Find the minimum point of this curve. Justify your answer.

$\left|\dfrac{x-1}{x^2+1}\right| \geq 0.$ ✓

$\dfrac{x-1}{x^2+1}$ = 0 when $x = 1$. ✓

Hence, minimum point is (1, 0). ✓

(e) On the axes provided below, sketch this curve.



✓ Two maximum points.
✓ Minimum point.
✓ Asymptote $y = 0$.
✓ General shape correct.

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08 Vectors I

Calculator Free

1. [8 marks: 2, 1, 1, 4]

Given that $\boldsymbol{a} = 3\,\mathbf{i} + 10\,\mathbf{j} + 8\,\mathbf{k}$ and $\boldsymbol{b} = 2\,\mathbf{j} + 3\,\mathbf{k}$, find:
(a) $|\boldsymbol{a} - 2\boldsymbol{b}|$

$\boldsymbol{a} - 2\boldsymbol{b} = < 3, 6, 2 >$ ✓
$|\boldsymbol{a} - 2\boldsymbol{b}| = \sqrt{(9 + 36 + 4)} = 7$ ✓

(b) the unit vector parallel to $\boldsymbol{a} - 2\boldsymbol{b}$

Required unit vector $= \frac{1}{7} < 3, 6, 2 >$ ✓

(c) a vector that is parallel to $\boldsymbol{a} - 2\boldsymbol{b}$ but with a magnitude of 10

Required vector $= \frac{10}{7} < 3, 6, 2 >$ ✓

(d) $\boldsymbol{a}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ where $\boldsymbol{p} = -\mathbf{i} + 4\,\mathbf{k}$ and $\boldsymbol{q} = 2\mathbf{i} + 5\,\mathbf{j} + 2\mathbf{k}$.

Let $< 3, 10, 8 > = \alpha < -1, 0\ 4 > + \beta < 2, 5, 2 >$ ✓
Hence, $-\alpha + 2\beta = 3$
$5\beta = 10$
$4\alpha + 2\beta = 8$ ✓
$\Rightarrow \beta = 2, \alpha = 1$
Hence, $\boldsymbol{a} = \boldsymbol{p} + 2\,\boldsymbol{q}$. ✓✓

2. [6 marks]

$\mathbf{OA} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{OB} = 2\mathbf{i} + b\,\mathbf{j} -$ $\mathbf{OC} = 6\mathbf{i} + 3\mathbf{j} + c\,\mathbf{k}$.
Find b and c if A, B and C are collinear.

$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = < 4, b-1, -2 >$ ✓

$\mathbf{AC} = \mathbf{OC} - \mathbf{OA} = < 8, 2, c-1 >$ ✓

But $\mathbf{AC} = \lambda\mathbf{AB}$
$\Rightarrow < 8, 2, c-1 > = \lambda < 4, b-1, -2 >$ ✓
Hence, $\lambda = 2$. ✓
$\Rightarrow b - 1 = 1 \quad \Rightarrow b = 2$ ✓
$c - 1 = -4 \quad \Rightarrow c = -3$ ✓

Calculator Free

3. [6 marks]

Vector $\alpha\mathbf{i} + \beta\mathbf{j} + \sqrt{2\alpha\beta}\,\mathbf{k}$ has a magnitude of 10 and is parallel to vector $3\mathbf{i} + 4\mathbf{j} + 2\sqrt{6}\mathbf{k}$. Find all possible values of α and β.

$< \alpha, \beta, \sqrt{(2\alpha\beta)} > = \lambda < 3, 4, 2\sqrt{6} >$ ✓

$\Rightarrow \quad \alpha = 3\lambda$ and $\beta = 4\lambda$ ✓

$\sqrt{\alpha^2+\beta^2+2\alpha\beta} = 10 \quad \Rightarrow \quad (\alpha+\beta)^2 = 100$ ✓

Hence, $(3\lambda + 4\lambda)^2 = 100 \Rightarrow \lambda = \pm\frac{10}{7}$ ✓

Hence, $\alpha = \pm\frac{30}{7}$ and $\beta = \pm\frac{40}{7}$. ✓

But $\sqrt{2\alpha\beta} \geq 0$, and $\alpha\mathbf{i} + \beta\mathbf{j} + \sqrt{2\alpha\beta}$ k are parallel to $3\mathbf{i} + 4\mathbf{j} + 2\sqrt{6}\mathbf{k}$;

hence $\alpha = \frac{30}{7}$ and $\beta = \frac{40}{7}$ ✓

---

4. [3 marks]

The points P and Q have position vectors $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ respectively. The point K is such that **PQ** = −2**QK** . Find the position vector of K.

PQ = −2**QK**

OQ − **OP** = −2[**OK** − **OQ**] ✓

2**OK** = **OQ** + **OP**

Hence, **OK** = $\frac{1}{2} < 2, 1, 6 >$ ✓✓

---

5. [6 marks: 2, 2, 2]

Given that $\boldsymbol{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, find:

(a) $(\boldsymbol{a} - \boldsymbol{b}) \bullet \boldsymbol{c}$

$< 2, -3, 1 > \bullet <-2, 1, -1 > = -8$ ✓✓

(b) $(\boldsymbol{a} - \boldsymbol{b}) \bullet (\boldsymbol{a} - \boldsymbol{c})$

$< 2, -3, 1 > \bullet < 3, -2, 3 > = 15$ ✓✓

(c) m if $\boldsymbol{b} \bullet (2\mathbf{i} - m\mathbf{j} + 3\mathbf{k}) = -1$

$< -1, 2, 1 > \bullet < 2, -m, 3 > = -1$

$\Rightarrow \quad -2 - 2m + 3 = -1$ ✓

$m = 1$ ✓

Calculator Free

6. [5 marks: 3, 1, 1]

(a) Find the acute angle between the vectors $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

$\cos\theta = \frac{<2,1,2>\bullet<1,1,0>}{(\sqrt{9})(\sqrt{2})}$ ✓

$= \frac{1}{\sqrt{2}}$ ✓

Hence, $\theta = 45^{\circ}$. ✓

Hence, or otherwise, find the acute angle between:

(b) $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} - \mathbf{j}$

$\theta = 45^{\circ}$ ✓

(c) $-4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ and $-5\mathbf{i} - 5\mathbf{j}$.

$\theta = 45^{\circ}$ ✓

---

7. [4 marks]

Find the value(s) of α if the angle between the vectors $\mathbf{i} + \mathbf{k}$ and $2\,\mathbf{i} + \alpha\,\mathbf{j}$ is 60°.

$\cos\theta = \frac{<1,0,1>\bullet<2,\alpha,0>}{(\sqrt{2})\sqrt{(4+\alpha^2)}} = \frac{1}{2}$ ✓✓

$\frac{2}{(\sqrt{2})\sqrt{(4+\alpha^2)}} = \frac{1}{2}$

$2\alpha^2 + 8 = 16$ ✓

$\alpha = \pm 2$ ✓

---

8. [6 marks]

$\boldsymbol{u} = a\,\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ has the same magnitude as $\boldsymbol{v} = (7 - b)\,\mathbf{i} + (a + c)\,\mathbf{j} + (4b - c)\,\mathbf{k}$. Find the values of a, b and c if $\boldsymbol{u}$ and $\boldsymbol{v}$ act in opposite directions.

$< 7 - b, a + c, 4b - c > = - < a, -1, 3 >$ ✓

$\Rightarrow \quad 7 - b = -a \quad \Rightarrow -a + b = 7$ I

$a + c = 1$ II

$4b - c = -3$ III ✓

II + III $\Rightarrow$ $a + 4b = -2$ IV ✓

I + IV $\Rightarrow$ $b = 1$ ✓

$a = -6, c = 7$ ✓✓

Calculator Free

9. [6 marks: 3, 3]

(a) Find a unit vector that is parallel to –**i** + **j** – **k** and perpendicular to –**i** + 2**j** + 3**k** .

$\lambda < -1, 1, -1 > \bullet < -1, 2, 3 > = 0$ for all $\lambda \neq 0$ ✓✓

Hence, required vector is $\frac{\sqrt{3}}{3} < -1, 1, -1 >$. ✓

(b) Find a unit vector that is perpendicular to both –**i** + **j** – **k** and –**i** + 2**j** + 3**k** .

Normal vector = $< -1, 1, -1 > \times < -1, 2, 3 >$ ✓

$= < 5, 4, -1 >$ ✓

Hence, unit vector of the form $\pm \frac{\sqrt{42}}{42} < 5, 4, -1 >$. ✓

---

10. [8 marks: 3, 1, 4]

Let $\boldsymbol{u} = < 2, 1, 2 >$, $\boldsymbol{v} = < 1, 0, 1>$ and $\boldsymbol{w} = < -1, 1, 0 >$.

(a) Use the result $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$, where θ is the acute angle between ***a*** and ***b*** to calculate the sine of the angle between *v* and *w*.

$v \times w = < -1, -1, 1 >$ ✓

$|v \times w| \quad \sqrt{3}, |v| = \sqrt{2}, |w| = \sqrt{2}$ ✓

$\sin\theta = \frac{\sqrt{3}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{3}}{2}$ ✓

(b) Determine the area of the parallelogram with sides parallel to *v* and *w*.

Area of parallelogram = $|v \times w| \quad \sqrt{3}$ ✓

(c) The scalar projection of vector ***a*** onto vector ***b*** is given by $a \bullet \hat{b}$.
Find the vector projection of ***u*** onto ***v*** × ***w***.

Let $c = v \times w$. $\Rightarrow$ $\hat{c} = \frac{\sqrt{3}}{3} < -1, -1, 1 >$ ✓

Scalar projection = $< 2, 1, 2 > \bullet \frac{\sqrt{3}}{3} < -1, -1, 1 > = -\frac{\sqrt{3}}{3}$ ✓

Vector projection = $-\frac{\sqrt{3}}{3}(\frac{\sqrt{3}}{3} < -1, -1, 1 >)$ ✓

$= \frac{1}{3} < 1, 1, -1 >$ ✓

Calculator Assumed

11. [4 marks]

Prove that the line segments congruent to the vectors $\begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix}, \begin{pmatrix} -6 \\ -14 \\ -9 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -13 \\ 2 \end{pmatrix}$

form a right angled triangle.

$< 3, 1, 11 > \cdot < -3, -13, 2 > = 0$

Hence, 2 sides are perpendicular. ✓

No two vectors are parallel. ✓

Also, $< 3, 1, 11 > + < -6, -14, -9 > = < -3, -13, 2 >$ ✓

Hence, these three vectors form a right angled triangle. ✓

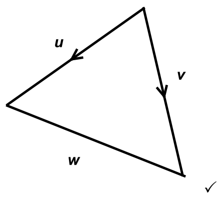
---

12. [6 marks]

The sides of an equilateral triangle are congruent with the vectors

$\boldsymbol{u} = \begin{pmatrix} \frac{-1}{2} \\ 0 \\ \frac{-\sqrt{3}}{2} \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{-\sqrt{3}}{2} \end{pmatrix}$ and ***w***. Find ***w***.

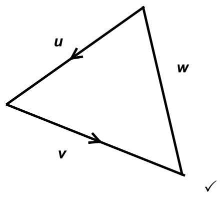
Possibility 1



$\Rightarrow$ $\boldsymbol{w} = \pm(\boldsymbol{u} - \boldsymbol{v})$

$= \pm < -1, 0, 0 >$ ✓

Possibility 2



$\Rightarrow$ $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$

$= < 0, 0, -\sqrt{3} >$. ✓

This gives $|\boldsymbol{w}| = \sqrt{3}$

But $|\boldsymbol{u}| = |\boldsymbol{v}| = 1$.

Hence, this does not give an equilateral triangle. ✓

Hence, $\boldsymbol{w} = \pm < 1, 0, 0 >$ ✓

09 Vectors II

Calculator Free

1. [9 marks: 3, 3, 3]

The points A and B have position vectors 3**i** + 4**j** – **k** and –3**i** + 2**j** + 2**k** respectively.

(a) Find the position vector of the point K which is 6 units from A in the direction –2**i** + 2**j** + **k**.

$\mathbf{AK} = 6 \times \frac{1}{3} < -2, 2, 1 >$ ✓✓
$\mathbf{OK} - \mathbf{OA} = 2 < -2, 2, 1 >$
Hence, $\mathbf{OK} = < -1, 8, 1 >$ ✓

(b) Find the position vector of the point L which is 21 units from A in the direction AB.

$\mathbf{AB} = < -6, -2, 3 >$ ✓
$\mathbf{AL} = 21 \times \frac{1}{7} < -6, -2, 3 >$ ✓
$\mathbf{OL} - \mathbf{OA} = 3 < -6, -2, 3 >$
Hence, $\mathbf{OL} = < -15, -2, 8 >$ ✓

(c) Find the area of ΔKAL. Show clearly how you obtained your answer.

$\text{Area} = \frac{1}{2} |\mathbf{AK} \times \mathbf{AL}|$
$= \frac{1}{2} |\, 2 < -2, 2, 1 > \times 3 < -6, -2, 3 > |$ ✓
$= 3 \, |< 8, 0, 16 >|$ ✓
$= 3 \times 8\sqrt{5}$
$= 24\sqrt{5} \text{ units}^2$ ✓

Calculator Free

2. [10 marks: 3, 3, 4]

Let $\boldsymbol{u} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\boldsymbol{v} = -\mathbf{i} + \mathbf{k}$ and $\boldsymbol{w} = < 2, -2, 2 >$.

(a) Find the acute angle between ***u*** and ***v***.

$\cos\theta = \dfrac{< 2, 1, -1 > \bullet < -1, 0, 1 >}{\sqrt{6}\sqrt{2}}$ ✓
$= -\dfrac{\sqrt{3}}{2}$ ✓
$\text{Acute angle} = \dfrac{\pi}{6}$ ✓

OR

$\sin\theta = \dfrac{|< 2, 1, -1 > \times < -1, 0, 1 >|}{\sqrt{6}\sqrt{2}}$ ✓
$= \dfrac{|< 1, -1, 1 >|}{\sqrt{6}\sqrt{2}} = \dfrac{\sqrt{3}}{2}$ ✓
$\text{Acute angle} = \dfrac{\pi}{6}$ ✓

(b) Find the area of a triangle with sides parallel and congruent to ***u*** and ***v***.

$\text{Area of } \Delta = \frac{1}{2} |\boldsymbol{u} \times \boldsymbol{v}|$ ✓
$= \frac{1}{2} |< 1, -1, 1 >|$ ✓
$= \frac{\sqrt{3}}{2} \text{ unit}^2$ ✓

OR

$\text{Area of } \Delta = \frac{1}{2} |\boldsymbol{u}|\,|\boldsymbol{v}| \sin\frac{\pi}{6}$ ✓
$= \frac{1}{2} \times \sqrt{6} \times \sqrt{2} \times \frac{1}{2}$ ✓
$= \frac{\sqrt{3}}{2} \text{ unit}^2$ ✓

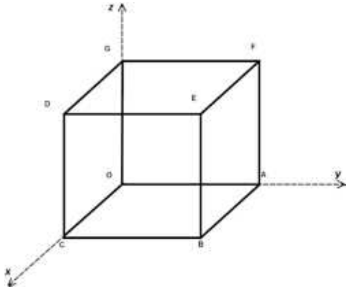
(c) A triangular pyramid has sides parallel and congruent to ***u***, ***v*** and ***w***. Determine the volume of this triangular pyramid. Show clearly how you obtained your answer.

$\boldsymbol{u} \times \boldsymbol{v} = < 1, -1, 1 >$
$\boldsymbol{w} = < 2, -2, 2 >$
Hence, ***w*** is parallel to $\boldsymbol{u} \times \boldsymbol{v}$. ✓
$\Rightarrow$ ***w*** is perpendicular to the Δ formed by ***u*** and ***v***. ✓
Hence, volume of pyramid $= \frac{1}{3} \times$ base area × height of pyramid
$= \frac{1}{3} \times \frac{\sqrt{3}}{2} \times |< 2, -2, 2 >|$ ✓
$= \frac{1}{3} \times \frac{\sqrt{3}}{2} \times 2\sqrt{3}$
$= 1 \text{ unit}^3.$ ✓

Calculator Assumed

3. [11 marks: 2, 2, 4, 3]

A rectangular box OABCDEFG rests on the x-y plane as shown. The vertex O has position vector < 0, 0, 0 > cm.
CB = 10 cm, AB = BE = 8 cm.



(a) State the position vectors of the vertices C and E of this box.

OC = < 8, 0, 0 > ✓

OE = < 8, 10, 8 > ✓

(b) Use vector methods to find the length of the diagonal BG.

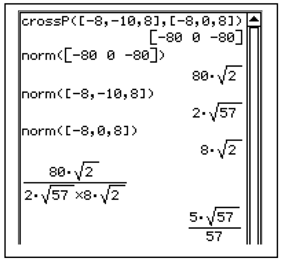
BG = **OG** – **OB**
= < 0, 0 , 8 > – < 8, 10, 0 > = < –8, –10, 8 > ✓
Hence, $|\mathbf{BG}| = 2\sqrt{57}$ cm ✓

(c) Use a cross-product method to calculate exactly sin(∠GBF).

BG = < –8, –10, 8 >
BF = **OF** – **OB**
= < 0, 10 , 8 > – < 8, 10, 0 >
= < –8, 0, 8 > ✓

$$\sin(\angle GBF) = \frac{|<-8,-10,8>\times<-8,0,8>|}{|<-8,-10,8>|\,|<-8,0,8>|} \quad ✓✓$$

$$= \frac{5\sqrt{57}}{57} \quad ✓$$



```
crossP([-8,-10,8],[-8,0,8])
                  [-80 0 -80]
norm([-80 0 -80])
                       80·√2
norm([-8,-10,8])
                       2·√57
norm([-8,0,8])
                        8·√2
   80·√2
 ─────────────
 2·√57 ×8·√2
                      5·√57
                      ─────
                        57
```

(d) Calculate the exact area of ΔBGF.

$$\text{Area of } \Delta BGF = \frac{1}{2}|<-8,-10,8>\times<-8,0,8>| \quad ✓$$

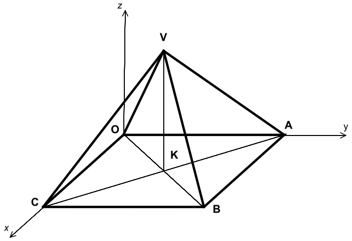
$$= \frac{1}{2}|80<-1,0,-1>| \quad ✓$$

$$= 40\sqrt{2}\ \text{unit}^2 \quad ✓$$

Calculator Assumed

4. [10 marks: 2, 8]

A regular pyramid VOABC rests on the x-y plane as shown. The vertices O, A, and C have position vectors < 0, 0, 0 >, < 0, 4, 0 > and < 6, 0, 0 > cm respectively. The vertex V is 5 cm vertically above the rectangular base OABC.



(a) State the position vector of V.

$$\mathbf{OK} = \tfrac{1}{2}\mathbf{OB} = \tfrac{1}{2}(\mathbf{OA} + \mathbf{AB})$$

$$= \tfrac{1}{2}(\mathbf{OA} + \mathbf{OC}) = <3, 2, 0> \quad ✓$$

OV = **OK** + **KV**
= < 3, 2, 0 > + < 0, 0, 5 > = < 3, 2, 5 > ✓

(b) Use a vector method to calculate the exact total surface area of this pyramid.

VC = **OC** – **OV** = < 3, –2, –5 > ✓

$$\text{Area of } \Delta VOC = \frac{1}{2}|<3,2,5>\times<3,-2,-5>| \quad ✓$$

$$= \frac{1}{2}|<0,30,-12>|$$

$$= 3\sqrt{29} \quad ✓$$

VB = **OB** – **OV** = < 3, 2, –5 > ✓

$$\text{Area of } \Delta VCB = \frac{1}{2}|<3,-2,-5>\times<3,2,-5>| \quad ✓$$

$$= \frac{1}{2}|<20,0,12>|$$

$$= 2\sqrt{34} \quad ✓$$

Area of base OABC = 4 × 6 = 24 ✓

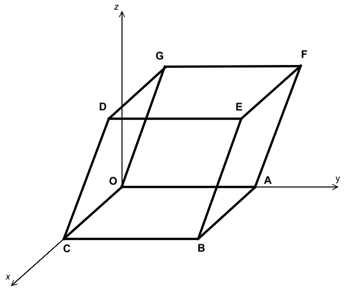
$$\text{Hence, total surface area} = 24 + (2\times3\sqrt{29}) + (2\times2\sqrt{34})$$

$$= 24 + 6\sqrt{29} + 4\sqrt{34}\ \ \text{unit}^2 \quad ✓$$

Calculator Assumed

5. [11 marks: 4, 3, 2, 2]

A parallelepiped OABCDEFG rests on the *x-y* plane as shown. [All six faces are parallelograms with opposite faces congruent.] The vertices O, A, C and G have position vectors < 0, 0, 0 >, < 0, 3, 0 >, < 2, 0, 0 > and < 0, 1, 2 > cm respectively.



(a) State the position vectors of the vertices D and E of this box.

OD = **OG** + **GD** = **OG** + **OC** ✓
= < 0, 1, 2 > + < 2, 0, 0 >
= < 2, 1, 2 > ✓

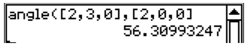
OE = **OD** + **DE** = **OD** + **OA** ✓
= < 2, 1, 2 > + < 0, 3, 0 >
= < 2, 4, 2 > ✓

(b) Determine the angle between **GE** and **AB**.

GE = **OE** – **OG** = < 2, 3, 0 > ✓

AB = **OC** = < 2, 0, 0 > ✓

Required Angle = 56.3^{o} ✓



```
angle([2,3,0],[2,0,0]
                56.30993247
```

(c) The volume of the parallelepiped = Area of base × Perpendicular height.
(i) Find the area of the base OABC.

OABC is a rectangle. ✓
⇒ Area = 2 × 3 = 6 cm^2. ✓

(ii) Find the volume of the parallelepiped.

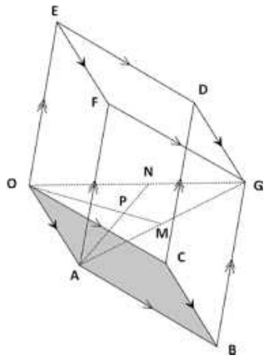
Perpendicular height = 2 ✓

Hence, volume = 6 × 2 = 12 cm^3. ✓

Calculator Assumed

6. [13 marks: 2, 4, 2, 5]

The given diagram shows a parallelepiped OABCDEFG. The opposite faces of the solid are congruent parallelograms which are parallel to each other. **OA** = ***a***, **OC** = ***c*** and **OE** = ***e***. The point M is the midpoint of AG and the point N is the midpoint of OG. OM and AN intersect at P. Let **OP** = α**OM** and **AP** = β**AN**.



(a) Express **AG** and **OG** in terms of ***a***, ***c*** and/or ***e***.

$\mathbf{AG} = \boldsymbol{c} + \boldsymbol{e}$ ✓
$\mathbf{OG} = \boldsymbol{a} + \boldsymbol{c} + \boldsymbol{e}$ ✓

(b) Express **OM** and **AN** in terms of ***a***, ***c*** and/or ***e***.

OM = **OA** + **AM**
$= \boldsymbol{a} + \frac{1}{2}\mathbf{AG} = \boldsymbol{a} + \frac{1}{2}(\boldsymbol{c}+\boldsymbol{e})$ ✓✓
AN = –**OA** + **ON**
$= -\boldsymbol{a} + \frac{1}{2}\mathbf{OG}$ ✓
$= -\boldsymbol{a} + \frac{1}{2}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{e}) = \frac{1}{2}(-\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{e})$ ✓

(c) Express **OP** and **AP** in terms of ***a***, ***c*** and/or ***e***.

$\mathbf{OP} = \alpha\mathbf{OM} = \alpha\boldsymbol{a} + \frac{\alpha}{2}(\boldsymbol{c}+\boldsymbol{e})$ ✓

$\mathbf{AP} = \beta\mathbf{AN} = \frac{\beta}{2}(-\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{e})$ ✓

(d) Determine the values of α and β.

OP = **OA** + **AP**
$\alpha\boldsymbol{a} + \frac{\alpha}{2}(\boldsymbol{c}+\boldsymbol{e}) = \boldsymbol{a} + \frac{\beta}{2}(-\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{e})$ ✓

Comparing coefficients of ***a***: $\alpha = 1 - \frac{\beta}{2}$ ✓

Comparing coefficients of ***c***: $\frac{\alpha}{2} = \frac{\beta}{2}$ ✓
$\Rightarrow \alpha = \beta$ ✓

Hence: $\alpha = 1 - \frac{\alpha}{2} \Rightarrow \alpha = \frac{2}{3} = \beta$ ✓

10 Vectors III (Lines)

Calculator Assumed

1. [3 marks: 1, 2]

Find the vector equation of a straight line passing through (4, –1, 2) and:
(a) parallel to the vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$\boldsymbol{r} = <4, -1, 2> + \lambda <1, 2, -1>$ ✓

(b) the point with position vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

$\boldsymbol{r} = <4, -1, 2> + \lambda\,[<4, -1, 2> - <1, -1, 2>]$ ✓
$= <4, -1, 2> + \lambda <3, 0, 0>$ ✓
(or equivalent)

---

2. [6 marks: 3, 3]

The vector equation of a line is given by $\boldsymbol{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k})$.
(a) Determine with reasons if the point (–1, 4, –6) lies on the line.

$<-1, 4, -6> = <1 - \lambda, 2 + \lambda, -3 - \lambda>$

Comparing x-components, $\lambda = 2$
Comparing y-components, $\lambda = 2$
Comparing z-components, $\lambda = 3$ ✓
$\Rightarrow$ The λ values are not consistent. ✓
Hence, (–1, 4, –6) does not line on the line. ✓

(b) Given that the point with position vector $<-2k, k, k + 5>$ lies on this line, find the value(s) of k.

$<-2k, k, k + 5> = <1 - \lambda, 2 + \lambda, -3 - \lambda>$
Comparing x-components, $\lambda = 1 + 2k$
Comparing y-components, $\lambda = k - 2 \;\Rightarrow\; 1 + 2k = k - 2 \;\Rightarrow\; k = -3$ ✓✓
Comparing z-components, $\lambda = -k - 8 \;\Rightarrow\; \lambda = -5$
For x and y components $\lambda = -5$. Consistency check passed! ✓

Hence, $k = -3$.

Calculator Assumed

3. [7 marks: 2, 2, 3]

The parametric equation of a line is given by $x = \frac{1 + \lambda}{2}$, $y = \frac{-1 + 2\lambda}{2}$, $z = \frac{4 - 3\lambda}{5}$

(a) Determine the Cartesian equation of this line.

$2x - 1 = \frac{2y + 1}{2} = \frac{-5z + 4}{3}$ ✓✓

(b) Find the vector equation of this line.

$\boldsymbol{r} = <\frac{1}{2}, -\frac{1}{2}, \frac{4}{5}> + \lambda <\frac{1}{2}, 1, -\frac{3}{5}>$ ✓✓

(c) The points A, B and C on this line are such that $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$ respectively. Find the ratio with which the point B divides the line AC. Show clearly how you obtained your answer.

$\boldsymbol{r} = <\frac{1}{2}, -\frac{1}{2}, \frac{4}{5}> + \lambda <\frac{1}{2}, 1, -\frac{3}{5}>$
For $\lambda = 0$, $\mathbf{OB} = <\frac{1}{2}, -\frac{1}{2}, \frac{4}{5}>$.
Let $\mathbf{d} = <\frac{1}{2}, 1, -\frac{3}{5}>$
For $\lambda = -1$, $\mathbf{OA} = \mathbf{OB} - \mathbf{d}$.
For $\lambda = 1$, $\mathbf{OC} = \mathbf{OB} + \mathbf{d}$. ✓
Clearly B is the midpoint of the line AC. ✓
Hence, B divides the line AC in the ratio 1 : 1. ✓

---

4. [8 marks: 4, 4]

The point A (a, $-a$, 0) lies on the line L with equation $\boldsymbol{r} = \begin{pmatrix} b \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ b \\ -1 \end{pmatrix}$.

(a) Find the value(s) of a and b.

$<a, -a, 0> = <b + 2\lambda, -2 + \lambda b, 3 - \lambda>$

Comparing z-components: $\lambda = 3$ ✓
Comparing x-components: $a = b + 6$
Comparing y-components: $-a = -2 + 3b$ ✓
Hence, $b = -1$, $a = 5$. ✓✓

Calculator Assumed

4. (b) Use a vector method to find the shortest distance between the point P (1, 0, 3) and the line L.

Point Q with position vector < –1, –2, 3 > lies on L.
QP = < 1, 0, 3 > – <–1, –2, 3 >
 = < 2, 2, 0 > ✓

Let θ be the angle between PQ and L.
Let direction vector of line L be $\boldsymbol{d}$.
Clearly $\boldsymbol{d}$ = < 2, –1, –1 >.
Shortest distance between P and L is:

$s = |\mathbf{PQ}| \sin\theta$ ✓

$= |\mathbf{PQ}| \dfrac{|\mathbf{PQ}\times \boldsymbol{d}|}{|\mathbf{PQ}||\boldsymbol{d}|}$

$= |\mathbf{PQ} \times \hat{\boldsymbol{d}}|$

$= |< 2, 2, 0 > \times \dfrac{\sqrt{6}}{6} < 2, -1, -1 >|$ ✓

$= \dfrac{\sqrt{6}}{6} | < -2, 2, -6 > |$

$= \dfrac{\sqrt{66}}{3}$ units. ✓

OP = < 1, 0, 3 >.
Let K be a point on the line.
Hence, **OK** = < –1 + 2λ, –2 – λ, 3 – λ >.
PK = **OK** – **OP**
 = < –2 + 2λ, –2 – λ, –λ >. ✓

Distance is a minimum when **KP** is perpendicular to the line.
That is, **PK** • < 2, –1, –1 > = 0.
< –2 + 2λ, –2 – λ, –λ > • < 2, –1, –1 > = 0 ✓

$\lambda = \dfrac{1}{3}$ ✓

$\Rightarrow$ **PK** $= < -\dfrac{4}{3}, -\dfrac{7}{3}, -\dfrac{1}{3} >$.

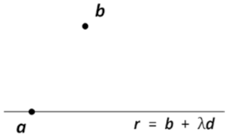
Hence, shortest distance = |**PK**|

$= \dfrac{\sqrt{66}}{3}$ units. ✓

---

5. [8 marks: 3, 5]

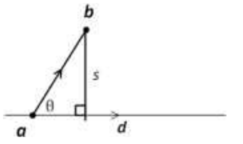
(a) Use the definition $|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| \sin\theta$ where θ is the acute angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ to prove that the shortest distance between the point with position vector $\boldsymbol{b}$ to the line with equation $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$ is given by $|(\boldsymbol{b} - \boldsymbol{a}) \times \hat{\boldsymbol{d}}|$.



Let angle between vector $\boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{d}$ be θ.
Shortest distance $s = |\boldsymbol{b} - \boldsymbol{a}| \sin\theta$ ✓

$= |\boldsymbol{b} - \boldsymbol{a}| \dfrac{(\boldsymbol{b} - \boldsymbol{a}) \times \boldsymbol{d}}{\boldsymbol{b} - \boldsymbol{a} \quad \boldsymbol{d}}$ ✓

$= |(\boldsymbol{b} - \boldsymbol{a}) \times \hat{\boldsymbol{d}}|$ ✓



Calculator Assumed

5. (b) ABCD is a trapezium. The points A, B, and D have position vectors < 1, 0 1 >, < 0, 2, 0 > and < –1, 1 1 > respectively. The point C is such that **BC** is parallel to **AD** and has magnitude 3 times that of **AD**. Calculate the area of the trapezium ABCD.

Clearly AD is parallel to BC ✓

$\mathbf{AD} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Height of trapezium is the perpendicular from B to side AD.

$\text{Height} = \dfrac{1}{\sqrt{5}} \left| \left[\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right] \times \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right|$ ✓

$= \dfrac{1}{\sqrt{5}} \left| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right| = \dfrac{\sqrt{14}}{\sqrt{5}}$ ✓

$\text{Area} = \dfrac{1}{2}\left(\sqrt{5} + 3\sqrt{5}\right) \times \dfrac{\sqrt{14}}{\sqrt{5}} = 2\sqrt{14}$ ✓✓

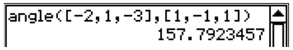
---

6. [9 marks: 2, 4, 3]

The lines L1 and L2 have equations $\boldsymbol{r}$ = < 1, 1, 2 > + λ < –2, 1, –3 > and $\boldsymbol{r}$ = < 2, 1, 4 > + μ < 1, –1, 1 > respectively. The lines L1 and L2 intersect at K.

(a) Find the acute angle between L1 and L2.

Angle between L1 and L2 = 157.7923° ✓
Hence, acute angle = 22.2°. ✓



```
angle([-2,1,-3],[1,-1,1])
                157.7923457
```

(b) Use a vector method to find the position vector of K.

At point of intersection:
$\boldsymbol{r}$ = < 1 –2λ, 1 + λ, 2 – 3λ > = < 2 + μ, 1 – μ, 4 + μ > ✓
Comparing x-components: 2λ + μ = –1
Comparing y-components: λ + μ = 0 ⇒ λ = –μ
μ = 1, λ = –1 ✓✓
Comparing z-components: 3λ + μ = –2: λ = –1, μ = 1 satisfies this eqn.
Hence, **OK** = < 3, 0, 5 >. ✓

(c) Find the vector equation of the line passing through the point K and perpendicular to L1 and L2.

Normal to L1 & L2 = < –2, 1, –3 > × < 1, –1, 1 > ✓
 = < –2, –1, 1 > ✓
Hence, required line has equation: $\boldsymbol{r}$ = < 3, 0, 5 > + α < –2, –1, 1 > . ✓

Calculator Assumed

7. [8 marks: 2, 2, 2, 2]

Lines L1 and L2 have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ respectively.

(a) Find the parametric equations of the lines L1 and L2.

For L1: $x = 1 - \lambda$, $y = 4 + 2\lambda$, $z = -1 + 3\lambda$ ✓

For L2: $x = -\mu$, $y = -2 + \mu$, $z = 2 - 2\mu$ ✓

(b) Show that L1 and L2 are skew lines (non-intersecting and non-parallel)..

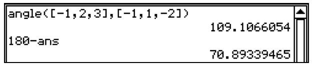
The direction vectors of L1 and L2, < –1, 2, 3 > and < –1, 1, –2 > are clearly non-parallel. ✓

When L1 and L2 intersect:
Comparing x-components: $\lambda - \mu = 1$
Comparing y-components: $2\lambda - \mu = -6 \Rightarrow \lambda = -7, \mu = -8$
Checking z-components: $-1 + 3(-7) \neq 2 - 2(-8) \Rightarrow$ L1 & L2 do not intersect. ✓

Hence, L1 and L2 are non-parallel and non-intersecting.

(c) Find the acute angle between L1 and L2.

Angle between L1 and L2
= Angle between < –1, 2, 3 > and < –1, 1, –2 > ✓
= 109.10660
$\Rightarrow$ Acute angle = 70.9^o. ✓



```
angle([-1,2,3],[-1,1,-2])
                        109.1066054
180-ans
                        70.89339465
```

(d) Find the vector equation of a line parallel to L1 but which intersects L2.

L2 passes through the point < 0, –2, 2 >.
L1 is parallel to < –1, 2, 3 >

Hence, required line is r = < 0, –2, 2 > + λ < –1, 2, 3 > . ✓✓

Calculator Assumed

8. [11 marks: 3, 1, 1, 4, 2]

The lines L1 and L2 have equations $\boldsymbol{r}$ = < 0, 1, –1 > + λ < 1, 1, 1 > and $\boldsymbol{r}$ = < 1, 0, –2 > + μ < 1, 2, α > respectively.

(a) Find the value(s) of α if the lines L1 and L2 are non-intersecting.

When L1 and L2 intersect:
Comparing x-components: $\lambda - \mu = 1$
Comparing y-components: $\lambda - 2\mu = -1 \Rightarrow \lambda = 3, \mu = 2$ ✓✓
Checking z-components: $-1 + 3 = -2 + 2\alpha \Rightarrow \alpha = 2$

Hence, if L1 and L2 are non-intersecting, $\alpha \neq 2$. ✓

(b) Let α = 1. The points P and Q are points respectively on lines L1 and L2.
(i) Determine the position vectors of P and Q in terms of λ and/or μ.

OP = < λ, 1 + λ, –1 + λ >
OQ = < 1 + μ, 2μ, –2 + μ > ✓

(ii) Find **PQ** in terms of λ and/or μ.

PQ = **OQ** – **OP**
= < 1 + μ – λ, –1 + 2μ – λ, –1 + μ – λ > ✓

(iii) Find λ and μ if **PQ** is perpendicular to both L1 and L2.

PQ perpendicular to L1:
< 1 + μ – λ, –1 + 2μ – λ, –1 + μ – λ > • < 1, 1, 1 > = 0
$\Rightarrow$ $3\lambda - 4\mu = -1$ ✓

PQ perpendicular to L2:
< 1 + μ – λ, –1 + 2μ – λ, –1 + μ – λ > • < 1, 2, 1 > = 0
$\Rightarrow$ $4\lambda - 6\mu = -2$ ✓

$\Rightarrow \lambda = 1, \mu = 1$ ✓✓

(iv) Hence, find the shortest distance between L1 and L2.

PQ = < 1 + μ – λ, –1 + 2μ – λ, –1 + μ – λ >
= < 1, 0, –1 > ✓

Hence, shortest distance = $|\mathbf{PQ}| = \sqrt{2}$. ✓

Calculator Assumed

9. [8 marks: 4, 2, 2, 2]

The line L1 has equation $\boldsymbol{r} = < -4 + \lambda, -2\lambda, \lambda - 2 >$. The point B $(-4, 0, -2)$ lies on the line L1. The point A has position vector $< 2, -2, 0 >$.

(a) Find the position vector of the point K on the line L1 such that AK is perpendicular to L1.

$\mathbf{OA} = < 2, -2, 0 >$
$\mathbf{OK} = < -4 + \lambda, -2\lambda, \lambda - 2 >$
$\Rightarrow \mathbf{AK} = \mathbf{OK} - \mathbf{OA} = < -6 + \lambda, 2 - 2\lambda, -2 + \lambda >$ ✓

AK perpendicular to L1:
$\Rightarrow < -6 + \lambda, 2 - 2\lambda, -2 + \lambda > \bullet < 1, -2, 1 > = 0$ ✓
$\Rightarrow \lambda = 2$ ✓

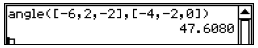
Hence, $\mathbf{OK} = < -2, -4, 0 >$. ✓

(b) Hence, find the shortest distance between A and the line L1.

Shortest distance $= |\mathbf{AK}|$ ✓
$= |< -4, -2, 0 >|$
$= 2\sqrt{5}$ units. ✓

(c) Find the size of ∠BAK.

$\mathbf{OB} = < -4, 0, -2 > \Rightarrow \mathbf{AB} = < -6, 2, -2 >$ ✓
Angle between **AB** and **AK**
$= 47.6080 = 47.6^{o}$. ✓



```
angle([-6,2,-2],[-4,-2,0])
                    47.6080
```

(d) Find the area of ΔAKB.

Area $= \frac{1}{2} \times |\mathbf{AB}| \times |\mathbf{AK}| \times \sin 47.6080$
$= \frac{1}{2} \times \sqrt{44} \times 2\sqrt{5} \times \sin 47.6080$ ✓
$= 10.95 \text{ unit}^2$. ✓

OR

Area $= \frac{1}{2} \times |\mathbf{BK}| \times |\mathbf{AK}|$
$= \frac{1}{2} \times \sqrt{24} \times 2\sqrt{5}$ ✓
$= 10.95 \text{ unit}^2$. ✓

Calculator Assumed

10. [11 marks: 4, 1, 4, 2]

The point A with position vector $< 1 + m, -m, -1 >$ lies on line L1 with equation $\boldsymbol{r} = < 1, 0, -1 > + \lambda < 1, -1, 0 >$. The point B with position vector $< 2 - 2n, 1 + n, n >$ lies on line L2 with equation $\boldsymbol{r} = < 2, 1, 0 > + \mu < -2, 1, 1 >$.

(a) Show that L1 and L2 are skew lines (non-intersecting and non-parallel).

Direction vectors of L1 and L2
are $< 1, -1, 0 >$ and $< -2, 1, 1 >$ respectively.
Clearly these vectors are not parallel. ✓
$\Rightarrow$ L1 and L2 are *not* parallel.

At point of intersection:
$< 1, 0, -1 > + \lambda < 1, -1, 0 > = < 2, 1, 0 > + \mu < -2, 1, 1 >$.
Comparing z-components: $\mu = -1$ ✓
Comparing y-components: $\lambda = -1 - \mu = 0$ ✓
Checking x-components: $1 + (-1) = 2 - 2(0)$ False! ✓
$\Rightarrow$ L1 and L2 are non-intersecting.

Hence, L1 and L2 are skew lines.

(b) Find **BA** in terms of m and n.

$\mathbf{OA} = < 1 + m, -m, -1 >$
$\mathbf{OB} = < 2 - 2n, 1 + n, n >$
$\Rightarrow \mathbf{BA} = \mathbf{OA} - \mathbf{OB} = < -1 + m + 2n, -1 - m - n, -1 - n >$ ✓

(c) Find the value(s) of m and n if **BA** is perpendicular to both L1 and L2.

BA perpendicular to L1:
$\Rightarrow < -1 + m + 2n, -1 - m - n, -1 - n > \bullet < 1, -1, 0 > = 0$
$\Rightarrow 2m + 3n = 0$ ✓

BA perpendicular to L2:
$\Rightarrow < -1 + m + 2n, -1 - m - n, -1 - n > \bullet < -2, 1, 1 > = 0$
$\Rightarrow -3m - 6n = 0$ ✓

Hence, $m = 0, n = 0$. ✓✓

(d) Hence, find the shortest distance between the lines L1 and L2.

When $m = n = 0$, **BA** perpendicular to L1 and L2.
$\Rightarrow$ Shortest distance between L1 and L2
$= |\mathbf{BA}(m = n = 0)|$ ✓
$= |< -1, -1, -1 >|$
$= \sqrt{3}$ units. ✓

Calculator Assumed

11. [10 marks: 4, 4, 1]

The point A (1, 0, –2) lies on the line L1 which is parallel to $\boldsymbol{d_1}$ = < 4, –2, –4 >.
The point B (6, –1, –10) lies on the line L2 which is parallel to $\boldsymbol{d_2}$ = < 2, 1, 2 >.

(a) Prove that the lines L1 and L2 are skew lines (neither parallel nor intersecting).

Direction vectors of L1 and L2
are < 4, –2, –4 > and < 2, 1, 2 > respectively.
Clearly these vectors are not parallel. ✓
⇒ L1 and L2 are *not* parallel.

At point of intersection:
< 1, 0, –2 > + λ < 4, –2, –4 > = < 6, –1, –10 > + μ < 2, 1, 2 >. ✓
Comparing *x*-components: 1 + 4λ = 6 + 2μ I
Comparing *y*-components: –2λ = –1 + μ II
Comparing *z*-components: –2 – 4λ = –10 + 2μ III
II & III represent "parallel lines" which are not coincident
and hence will yield no solutions. ✓
⇒ L1 and L2 are non-intersecting. ✓

(b) Determine the scalar projection of **BA** onto $\boldsymbol{d_1} \times \boldsymbol{d_2}$.

BA = < 1, 0, –2 > – < 6, –1, –10 >
= < –5, 1, 8 > ✓
$\boldsymbol{d_1} \times \boldsymbol{d_2}$ = < 0, –16, 8 > ✓

scalar projection = < –5, 1, 8 > • $\frac{\sqrt{5}}{5}$ < 0, –2, 1 > ✓

= $\frac{6\sqrt{5}}{5}$ ✓

(c) Hence, find the shortest distance between the lines L1 and L2.

Shortest distance = $\frac{6\sqrt{5}}{5}$ units ✓

11 Vectors IV (Planes)

Calculator Free

1. [4 marks: 1, 3]

Find in the form ***r*** **.** ***n*** = ρ, the vector equation of a plane:

(a) passing through (–1, 3, 4) and perpendicular to the vector –2**i** + **j** – 3**k**.

r **.** < –2, 1, –3 > = < –1, 3, 4 > **.** <–2, 1, –3 >
= –7. ✓

(b) containing the line ***r*** = < 5 – 2λ, 2 + 6λ, 4λ > and perpendicular to the vector **i** – **j** + 2**k**.

The point with position vector < 5, 2, 0 > lies on the given line. ✓
Hence, equation of plane
r **.** < 1, –1, 2 > = < 5, 2, 0 > **.** < 1, –1, 2 > ✓
= 3. ✓

---

2. [7 marks: 2, 5] [TISC]

The line L with equation ***r*** = 2***i*** + 2λ***j*** + (1 – λ)***k*** is perpendicular to the plane P.

(a) The point A with position vector –***i*** + ***k*** lies on the plane P. Determine the vector equation of the plane P.

Normal to plane is < 0, 2, –1 >. ✓
Equation of plane P: ***r*** • < 0, 2, –1 > = < –1, 0, 1 > • < 0, 2, –1 > = –1 ✓

(b) B is a point on line L. The distance between A and B is $\sqrt{14}$ units. Determine the position vector of the point B.

OB = < 2, 2λ, 1 – λ >
AB = **OB** – **OA**
= < 2, 2λ, 1 – λ > – < –1, 0, 1 > = < 3, 2λ, –λ > ✓
$|\mathbf{AB}| = \sqrt{9+4\lambda^2+\lambda^2} = \sqrt{9+5\lambda^2}$ ✓
Hence: $\sqrt{9+5\lambda^2} = \sqrt{14}$ ✓
λ = ±1 ✓
OB = < 2, 2, 0 > or < 2, –2, 2 > ✓

Calculator Free

3. [4 marks]

Find in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \rho$, the vector equation of a plane containing the lines $\boldsymbol{r} = \mathbf{i} - \mathbf{k} + \lambda\,(\mathbf{i} - \mathbf{k})$ and $\boldsymbol{r} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda\,(-\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Normal to plane = < 1, 0, –1 > × < –1, 1, 1 > ✓
= < 1, 0, 1 > ✓

< 1, 0, –1 > is a point on one of the lines and hence on the plane.
⇒ Vector equation of plane is $\boldsymbol{r}$ • < 1, 0, 1 > = < 1, 0, 1 > • < 1, 0, –1 > ✓
= 0. ✓

---

4. [4 marks]

Find the vector equation of the plane passing through the line with equation $\boldsymbol{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(4\mathbf{i} - \mathbf{k})$. The points with position vectors < –1, 0, 0 > and < 2, 1, –1 > lie on this plane. Give your answer in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \rho$.

< 2, 1, –1 > – < –1, 0, 0 > = < 3, 1, –1 > is a vector on the plane. ✓
Normal to plane = < 4, 0, –1 > × < 3, 1, –1 > ✓
= < 1, 1, 4 > ✓
< –1, 0, 0 > is a point on the plane.
Vector equation of plane is $\boldsymbol{r}$ • < 1, 1, 4 > = < –1, 0, 0 > • < 1, 1, 4 > = –1. ✓

---

5. [7 marks: 3, 4]

The points with position vectors < 1, 0, 2 >, < –1, 1, 0 > and < 0, 1, –1 > lie on a plane.

(a) Determine the vector equation of the plane in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$.

Two non-parallel vectors on the plane are:
< 1, 0, 2 > – < –1, 1, 0 > = < 2, –1, 2 > ✓
< –1, 1, 0 > – < 0, 1, –1 > = < –1, 0, 1 >. ✓
Hence, equation of plane is $\boldsymbol{r}$ = < 1, 0, 2 > + λ< 2, –1, 2 > + μ< –1, 0, 1 >. ✓

(b) Determine the Cartesian equation of the plane.

Normal to plane = < 2, –1, 2 > × < –1, 0, 1 > ✓
= < –1, –4, –1 > ✓
< 1, 0, 2 > is a point on the plane.
Vector equation of plane is $\boldsymbol{r}$ • < 1, 4, 1 > = < 1, 0, 2 > • < 1, 4, 1 > = 3. ✓
Hence, Cartesian equation is $x + 4y + z = 3$ ✓

Calculator Free

6. [6 marks: 3, 2, 1]

Plane P has equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Point A has position vector $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

The line L passes through the point A and is perpendicular to P.

(a) Determine the vector equation of line L.

Normal to plane $\boldsymbol{n} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ✓✓

Vector equation of line: $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ✓

(b) Determine the Cartesian equation of the plane P.

Vector equation of plane P: $\boldsymbol{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 3$ ✓

Hence, Cartesian equation of plane P: $x + y + z = 3$ ✓

(c) State the position vector of the point of intersection between L and P.

< 0, 2, 1 > ✓

---

7. [8 marks: 3, 5]

The plane P has equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

(a) Determine with reasons if the point A with position vector $2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$ lies on the plane P

$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ ✓

x-component: $\beta = 1$
z-component: $\alpha = -2$ ✓
For y-component: $-5 = 2\alpha - \beta$
For $\alpha = -2$, $\beta = 1$ $-5 = -5$. ✓
Hence, A lies on plane P.

Calculator Free

7. (b) Plane Q has equation $2x - y + z = 1$.
Calculate the acute angle between planes P and Q.

Normal to plane P is $\boldsymbol{n}_P = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$ ✓✓

Normal to plane Q is $\boldsymbol{n}_Q = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. ✓

Angle between planes is angle between $\boldsymbol{n}_P$ and $\boldsymbol{n}_Q$.

$$\cos\theta = \frac{\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right|} = \frac{-3}{6} \quad ✓$$

Hence, acute angle is 60^o. ✓

---

8. [8 marks: 5, 3]

The points A, B and C have position vectors < 1, 0, 2 >, < 4, 1, –2 > and < 0, 2, 1 > respectively.

(a) Determine the vector equation of the plane passing through points A, B and C. Give your answer in the form $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$

AB = < 3, 1, –4 > ✓
AC = < –1, 2, –1 > ✓
AB × **AC** = < 7, 7, 7 > ✓✓
Hence: $\boldsymbol{r} \bullet$ < 1, 1, 1 > = <1, 0, 2 > • < 1, 1, 1 >
= 3 ✓

(b) The area of a triangle spanned by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by $\frac{1}{2}|\boldsymbol{u} \times \boldsymbol{v}|$.

Use this result or otherwise to calculate the area of ΔABC.

Two spanning vectors are
AB = < 3, 1, –4 > and **AC** = < –1, 2, –1 >. ✓

AB × **AC** = < 7, 7, 7 > ✓
Hence:
$\text{Area} = \frac{1}{2}|<7, 7, 7>| = \frac{7\sqrt{3}}{2}$ ✓

Calculator Assumed

9. [5 marks: 3, 2]

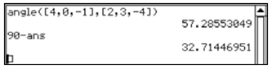
Consider the line L with equation $\boldsymbol{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} - \mathbf{k})$ and the plane P with equation $\boldsymbol{r} \bullet$ < 2, 3, –4 > = 0.

(a) Find the position vector of the point of intersection between L and P

< 2 + 4λ, 1, 1 – λ > . < 2, 3, –4 > = 0 ✓
$\Rightarrow \lambda = -\frac{1}{4}$ ✓
Hence, $< 1, 1, \frac{5}{4} >$. ✓

(b) Find the acute angle between the L and P.

Angle between L and the normal to P
= Angle between < 4, 0, –1 > & < 2, 3, –4 > = 57.2855 ✓
Hence, angle between L and P = 90 – 57.2855 = 32.7°. ✓



```
angle([4,0,-1],[2,3,-4])
                    57.28553049
90-ans
                    32.71446951
```

---

10. [9 marks: 2, 2, 3, 2]

Consider the plane Π_1 with equation $\boldsymbol{r}$. < 1, 0, 1 > = 2 and the line L with equation $\boldsymbol{r}$ = < 3, 1, –1 > + λ < –2, 1, 2 >. Plane Π_2 has equation $\boldsymbol{r}$. < 2, 2, 1 > = ρ.

(a) Show that the line L lies on the plane Π_1.

< 3 – 2λ, 1 + λ, –1 + 2λ > . < 1, 0, 1 > = 3 – 2λ + 0 – 1 + 2λ ✓
= 2. ✓
Hence L lies on the plane.

(b) Find ρ if the line L also lies on the plane Π_2.

< 3 – 2λ, 1 + λ, –1 + 2λ > . < 2, 2, 1 > = ρ ✓
ρ = 7. ✓

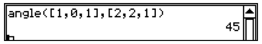
(c) For the value of ρ in part (b), find the vector equation of the line of intersection between the two planes.

Planes are not parallel. ✓
Therefore, intersection is a line common to both planes. ✓
Hence, $\boldsymbol{r}$ = < 3, 1, –1 > + λ < –2, 1, 2 >. ✓

Calculator Assumed

10. (d) For the value of ρ in part (b), find the acute angle between the planes Π_1 and Π_2.

Angle between the 2 planes
= Angle between the two normal vectors
= Angle between < 1, 0, 1 > and < 2, 2, 1 > ✓
= 45^o. ✓



```
angle([1,0,1],[2,2,1])
                    45
```

---

11. [9 marks: 1, 1, 2, 2, 3]

The points A and B have position vectors < 1, –2, 1 > and < 0, 0, 1 > respectively. The plane Π has equation $\boldsymbol{r} \bullet$ < 2, –1, 1 > = 1.

(a) Find a unit vector perpendicular to the plane Π.

Unit vector $\hat{\boldsymbol{n}} = \frac{1}{\sqrt{6}}$ <2, –1, 1 >. ✓

(b) Show that the point B lies on the plane Π.

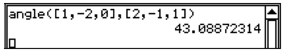
< 0, 0, 1 > • <2, –1, 1 > = 0 + 0 + 1 = 1
Hence, B lies on the plane. ✓

(c) Find |**BA**|.

BA = < 1, –2, 0 >. ✓
⇒ |**BA**| = √5 ✓

(d) Find the acute angle between **BA** and the normal to the plane Π.

Required angle = Angle between < 1, –2, 0 > and < 2, –1, 1 > ✓
= 43.0887 = 43.0^o. ✓



```
angle([1,-2,0],[2,-1,1])
             43.08872314
```

(e) Hence, find the minimum distance between the point A and the plane Π.

Minimum distance = $\hat{\boldsymbol{n}} \bullet$ **BA**
$= \frac{1}{\sqrt{6}}$ <2, –1, 1 > • < 1, –2, 0 > ✓✓
$= \frac{2\sqrt{6}}{3}$ units. ✓

Minimum distance
= |**BA**| cos 43.0887 ✓✓
= 1.6329 = 1.63 units ✓

Calculator Assumed

12. [4 marks]

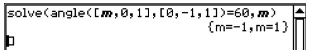
Find the value(s) of *m* if the line $\boldsymbol{r}$ = < 2 + *m*λ, –3, 1 + λ > is inclined to the plane $\boldsymbol{r} \bullet$ < 0, –1, 1 > = 10 at an angle of 30^o.

Angle between line and the normal to the plane = 90 – 30 = 60^o.
⇒ Angle between < *m*, 0, 1 > and < 0, –1, 1 > = 60^o. ✓

$$\Rightarrow \quad \cos 60 = \frac{<m, 0, 1> \bullet <0, -1, 1>}{\sqrt{2}\sqrt{m^2+1}}$$

$$\frac{1}{2} = \frac{1}{\sqrt{2}\sqrt{m^2+1}} \quad ✓$$

$$\Rightarrow \quad m^2 + 1 = 2 \;\Rightarrow\; m = \pm 1 \quad ✓✓$$



```
solve(angle([m,0,1],[0,-1,1])=60,m)
                         {m=-1,m=1}
```

---

13. [6 marks: 1, 1, 1, 1, 2]

The planes Π_1 and Π_2 have equations $\boldsymbol{r} \bullet$ < 1, –1, 2 > = 5 and $\boldsymbol{r} \bullet$ < 1, –1, 2 > = 8 respectively.

(a) Show that the two given planes are parallel.

Π_1 and Π_2 share the same normal vector < 1, –1, 2 >.
Hence, the two planes are parallel. ✓

(b) Find a unit vector normal to these two given planes.

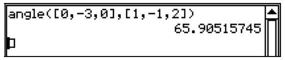
Unit vector $= \frac{1}{\sqrt{6}}$ < 1, –1, 2 >. ✓

(c) The points A and B with position vectors < 0, –5, 0 > and < 0, –8, 0 > lie on Π_1 and Π_2 respectively. Find |**AB**|.

|**AB**| = | < 0, –3, 0 > | = 3 units ✓

(d) Find the acute angle between **AB** and the normal to the two given planes.

Required angle
= Angle between < 0, –3, 0 > and < 1, –1, 2 >
= 65.9052 = 65.9^o. ✓



```
angle([0,-3,0],[1,-1,2])
             65.90515745
```

(e) Hence, find the distance between the two given planes.

Distance between the two planes = |**AB**| cos 65.9052 ✓
= 1.2247 = 1.2 units. ✓

Calculator Assumed

14. [11 marks: 2, 3, 3, 3]

The plane P has equation $\boldsymbol{r} \bullet (2\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}) = -2$.

The line L has equation $\dfrac{x}{-1} = \dfrac{y-2}{1} = \dfrac{z+1}{2}$.

(a) Find the vector equation for line L

$\boldsymbol{r} = 2\boldsymbol{j} - \boldsymbol{k} + \lambda\,(-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$ ✓✓

(b) Show that the line L does not intersect the plane P

Normal to plane P is $\boldsymbol{n} = < 2, 4, -1 >$

Direction vector of line $\boldsymbol{d} = < -1, 1, 2 >$. ✓

$\boldsymbol{n} \bullet \boldsymbol{d} = < 2, 4, -1 > \bullet < -1, 1, 2 > = 0$ ✓

Hence line is perpendicular to the normal vector of the plane.

Therefore, L is parallel to P and does not intersect P. ✓

(c) Find the Cartesian equation of the plane Q containing line L and parallel to P.

Equation of plane:

$$\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 9 \quad ✓✓$$

$\Rightarrow \; 2x + 4y - z = 9$ ✓

(d) Determine with reasons if planes P and Q are on the same side or on opposite sides of the origin (0, 0, 0).

Plane P: $2x + 4y - z = -2$

Point (0, 0, 2) is on P. ✓

Plane Q: $2x + 4y - z = 7$

Point (0, 0, –7) is on Q. ✓

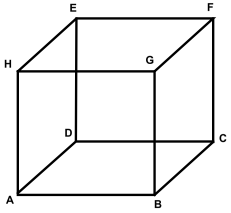
Points are on opposite sides of the origin.

⇒ P and Q are on opposite sides of the origin. ✓

Calculator Assumed

15. [13 marks: 4, 2, 2, 3, 2]

ABCDEFGH is a rectangular prism. The vertices A, B, C and G have position vectors < 2, 2, 2 >, < 1, 1, 1 >, < 2, 1, 0 > and < 0, 3, 0 > respectively.



(a) Find the position vectors of the vertices D and E.

AD = **BC** = **OC** – **OB** = < 1, 0, –1 >. ✓

OD = **OA** + **AD** = < 3, 2, 1 > ✓

DE = **BG** = **OG** – **OB** = < –1, 2, –1 >. ✓

OE = **OD** + **DE** = < 2, 4, 0 > ✓

(b) Find the vector equation of the edge AB.

$\boldsymbol{d}$ = **AB** = < –1, –1, –1 >

$\boldsymbol{r} = < 2, 2, 2 > + \lambda < 1, 1, 1 >$ ✓✓

or equivalent

(c) Find the vector equation of the plane ABCD in the form $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$.

$\boldsymbol{n}$ = **DE** = < –1, 2, –1 > ✓

$\boldsymbol{r} \bullet < -1, 2, -1 > = < 1, 1, 1 > \bullet < -1, 2, -1 >$

$= 0$ ✓

(d) Use vectors to find the acute angle between the planes ADFG and ABCD.

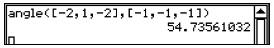
Required angle = ∠GAB

= Angle between **AG** and **AB** ✓

AG = **OG** – **OA** = < –2, 1, –2 > ✓

AB = < –1, –1, –1 >

Hence, ∠GAB = 54.7356 = 54.7°. ✓



```
angle([-2,1,-2],[-1,-1,-1])
                  54.73561032
```

(e) Find the volume of the rectangular prism.

Volume = $|\mathbf{AB}| \times |\mathbf{BC}| \times |\mathbf{BG}|$ ✓

$= \sqrt{3} \times \sqrt{2} \times \sqrt{6}$

= 6 unit^3. ✓

OR

Volume = $|\mathbf{BG} \bullet (\mathbf{AD} \times \mathbf{AB})|$ ✓

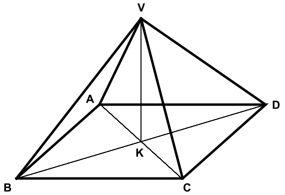
$= |< -1, 2, -1 > \bullet < 1, -2, 1 >|$

= 6 unit^3. ✓

Calculator Assumed

16. [11 marks: 2, 2, 2, 3, 2] [TISC]

The accompanying diagram shows a rectangular pyramid VABCD. The vertices B, C and D have position vectors **i** + **j** + **k**, 3**j** + 3**k** and 2**i** + 2**j** + 5**k** respectively. The vertex V with position vector $-\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 4\mathbf{k}$ is vertically above K, the centre of the rectangular base ABCD.



(a) Use vectors to find the position vector of the vertex A.

CD = **OD** – **OC**
= (2**i** + 2**j** + 5**k**) – (3**j** + 3**k**)
= 2**i** – **j** + 2**k** ✓

OA = **OB** + **BA** = **OB** + **CD**
= (**i** + **j** + **k**) + (2**i** – **j** + 2**k**)
= 3**i** + 3**k**. ✓

(b) Find **KV**.

$\mathbf{OK} = \frac{1}{2}(\mathbf{OB} + \mathbf{OD})$
$= \frac{1}{2}\,[(\mathbf{i} + \mathbf{j} + \mathbf{k}) + (2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})]$
$= \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + 3\mathbf{k}$ ✓

KV = **OV** – **OK**
$= (-\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 4\mathbf{k}) - (\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + 3\mathbf{k})$
= –2**i** – 2**j** + **k**. ✓

(c) Find the vector equation of the line passing through C and V.

CV = **OV** – **OC**
$= (-\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 4\mathbf{k}) - (3\mathbf{j} + 3\mathbf{k})$
$= -\frac{1}{2}\mathbf{i} - \frac{7}{2}\mathbf{j} + \mathbf{k}$. ✓
Hence, line has equation:
$\boldsymbol{r} = (3\mathbf{j} + 3\mathbf{k}) + \lambda(-\frac{1}{2}\mathbf{i} - \frac{7}{2}\mathbf{j} + \mathbf{k})$. ✓
(or equivalent)

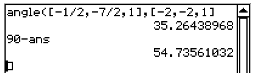
Calculator Assumed

16. (d) Find the vector equation of the plane ABCD in the form $\boldsymbol{r} \bullet \boldsymbol{n} = \rho$.

Vector normal to plane = **KV** = –2**i** – 2**j** + **k** ✓
Vector equation of plane:
$\boldsymbol{r} \bullet (-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = (3\mathbf{j} + 3\mathbf{k}) \bullet (-2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ ✓
= –3 (or equivalent) ✓

(e) Find the acute angle between the line VC and the plane ABCD.

Angle between **CV** and **KV** = 35.2644° ✓
Hence, angle between **CV** and the plane
= 90 – 35.2644 = 54.7° ✓



```
angle([-1/2,-7/2,1],[-2,-2,1]
                    35.26438968
90-ans
                    54.73561032
```

CA = < 3, 0, 3 > – < 0, 3, 3 > = < 3, –3, 0 >
$\mathbf{CV} = < -\frac{1}{2}, -\frac{7}{2}, 1 >$
$\cos\alpha = \frac{\mathbf{CV}\bullet\mathbf{CA}}{|\mathbf{CV}||\mathbf{CA}|} = 0.5774$ ✓
⇒ α = 54.7° . ✓

---

17. [10 marks: 4, 4, 2]

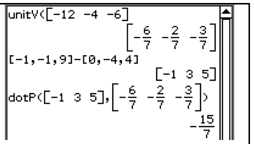
A tetrahedron has vertices V, A, B and C with position vectors < –1, –1, 9 >, < –2, 2, 4 >, < 0, –4, 4 > and < 0, –1, 2 > respectively.

(a) Find the area of the base ABC.

BA = < –2, 2, 4 > – < 0, –4, 4 > = < –2, 6, 0 > ✓
BC = < 0, –1, 2 > – < 0, –4, 4 > = < 0, 3, –2 > ✓
$\text{Area} = \frac{1}{2}\,|< -2, 6, 0 > \times < 0, 3, -2 >|$ ✓
$= \frac{1}{2}\,|< -12, -4, -6 >| = 7$ ✓

(b) Find the shortest distance between the vertex V and the base ABC.

Vector normal to plane is $\boldsymbol{n}$ = < –12, –4, –6 >. ✓
Displacement between V and B
BV = < –1, –1, 9 > – < 0, –4, 4 > = < –1, 3, 5 >. ✓
Hence, shortest distance
= scalar projection of **BV** onto $\boldsymbol{n}$. ✓
$= |< -1, 3, 5 > \bullet \frac{1}{14} < -12, -4, -6 >|$
$= \frac{15}{7}$ ✓



```
unitV([-12 -4 -6]
                 [-6/7 -2/7 -3/7]
[-1,-1,9]-[0,-4,4]
                       [-1 3 5]
dotP([-1 3 5],[-6/7 -2/7 -3/7])
                          -15/7
```

(c) Hence, determine the volume of the tetrahedron VABC.

$\text{Volume} = \frac{1}{3} \times 7 \times \frac{15}{7}$ ✓
$= 5\ \text{unit}^3$. ✓

Calculator Assumed

18. [11 marks: 5, 6]

An object A is acted on by three forces 12***i*** + 10***j*** + 5***k*** N, –5***i*** + 8***j*** – 10***k*** N and 6***i*** – 8***j*** + 4***k*** N.

(a) Determine the magnitude of the resultant force and the acute angle it makes with the *x-y* plane.

$$\text{Resultant Force} = \begin{pmatrix}12\\10\\5\end{pmatrix} + \begin{pmatrix}-5\\8\\-10\end{pmatrix} + \begin{pmatrix}6\\-8\\4\end{pmatrix} = \begin{pmatrix}13\\10\\-1\end{pmatrix} \quad ✓$$

Hence, magnitude = $3\sqrt{30}$ N. ✓

Normal to the *x-y* plane is $\begin{pmatrix}0\\0\\1\end{pmatrix}$. ✓

Angle between resultant and normal:

$$\theta = \text{angle}\left(\begin{pmatrix}13\\10\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 93.5 \quad ✓$$

Hence, acute angle = 3.5° ✓

(b) A fourth force ***F*** of magnitude $\sqrt{306}$ N acts on A so that the resultant now acts in the same direction as the vector 2***i*** – ***j*** + ***k***. Determine ***F***.

$$\boldsymbol{F} + \begin{pmatrix}13\\10\\-1\end{pmatrix} = \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix} \quad ✓$$

$$\boldsymbol{F} = \begin{pmatrix}2\lambda-13\\-\lambda-10\\\lambda+1\end{pmatrix} \Rightarrow |\boldsymbol{F}| = \sqrt{6(\lambda^2-5\lambda+45)} \quad ✓$$

Hence: $\sqrt{6(\lambda^2-5\lambda+45)} = \sqrt{306}$ ✓

$\lambda = 6$ or -1. ✓

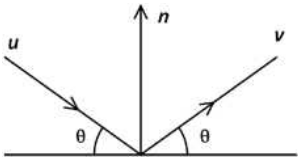
Reject λ = –1, as this means resultant would be in a direction opposing < 2, –1, 1 >. ✓

Hence, $F = \begin{pmatrix}-1\\-16\\7\end{pmatrix}$ ✓

Calculator Assumed

19. [11 marks: 2, 2, 2, 2, 3]

A ball travelling with velocity ***u*** = < 2, 1, –2 > ms^{-1}, hits the surface of a plane P with equation ***r*** • < 2, –2, –1 > = 10 at an angle of θ° with the plane. It rebounds with velocity ***v*** at an angle of θ° with the plane. ***u***, ***v*** and ***n*** the normal to the plane P all lie on the same plane.



(a) Determine the unit vector normal to the plane.

$$\boldsymbol{n} = \begin{pmatrix}2\\-2\\-1\end{pmatrix} \quad ✓ \qquad \hat{\boldsymbol{n}} = \frac{1}{3}\begin{pmatrix}2\\-2\\-1\end{pmatrix} \quad ✓$$

(b) Determine the exact value for cos(90° – θ°).

$$\cos(90-\theta) = \frac{\begin{pmatrix}2\\1\\-2\end{pmatrix} \bullet \begin{pmatrix}2\\-2\\-1\end{pmatrix}}{(3)(3)} = \frac{4}{9} \quad ✓✓$$

(c) Determine ***a***, the component of ***u*** is perpendicular to the plane P.

$$| \quad = |v| \cos(90-\theta) = 3 \times \frac{4}{9} = \frac{4}{3} \quad ✓$$

$$\boldsymbol{a} = \frac{4}{3}\hat{\boldsymbol{n}} = \frac{4}{9}\begin{pmatrix}2\\-2\\-1\end{pmatrix} \quad ✓$$

(d) Determine ***b***, the component of ***u*** that is parallel to the plane P.

$$\boldsymbol{b} = \boldsymbol{v} - \boldsymbol{a} \quad ✓$$

$$= \begin{pmatrix}2\\1\\-2\end{pmatrix} - \frac{4}{9}\begin{pmatrix}2\\-2\\-1\end{pmatrix} = \frac{1}{9}\begin{pmatrix}10\\17\\-14\end{pmatrix} \quad ✓$$

(e) Determine ***v*** if $|\boldsymbol{v}| = 0.9|\boldsymbol{u}|$.

$$\boldsymbol{v} = 0.9(\boldsymbol{b} - \boldsymbol{a}) \quad ✓$$

$$= 0.9\left[\frac{1}{9}\begin{pmatrix}10\\17\\-14\end{pmatrix} - \frac{4}{9}\begin{pmatrix}2\\-2\\-1\end{pmatrix}\right] \quad ✓$$

$$= \frac{1}{10}\begin{pmatrix}2\\25\\-10\end{pmatrix} \quad ✓$$

12 Vectors V (Spheres)

Calculator Free

1. [10 marks: 2, 2, 2, 4]

The sphere S has centre (1, –1, 2) and radius 5. The point P with coordinates (5, 2, 2) lies on this sphere.

(a) Determine the vector and Cartesian equations of this sphere.

Vector Equation: $| \boldsymbol{r} - < 1, -1, 2 > | = 5$ ✓
Cartesian Equation: $(x-1)^2 + (y+1)^2 + (z-2)^2 = 25$ ✓

(b) PQ is a diameter of this sphere, determine the coordinates of Q.

The centre of the sphere (1, –1, 2) is the midpoint of PQ.
Hence, coordinates of Q is (–3, –4, 2). ✓✓

(c) The sphere intersects the *y*-*z* plane in the form of a circle.
Determine the Cartesian equation of this circle.

When it intersects the *y*-*z* plane, $x = 0$. ✓
Hence: $(0-1)^2 + (y+1)^2 + (z-2)^2 = 25$
$(y+1)^2 + (z-2)^2 = 24$ ✓

(d) Determine the value(s) of *k* if the point with position vector < 1, 3, *k* > is outside this sphere.

Distance between K and centre of sphere
$= \sqrt{(1-1)^2 + (3+1)^2 + (k-2)^2}$
$= \sqrt{16 + (k-2)^2}$ ✓
Since point is outside the sphere
$\sqrt{16 + (k-2)^2} > 5$ ✓
Hence, $k > 5$ or $k < -1$ ✓✓

Calculator Assumed

2. [8 marks: 4, 4]

The sphere S has Cartesian equation $9x^2 + 9y^2 + 9z^2 - 6x + 54y + 18z + 10 = 0$.

(a) Determine the position vector of the centre of this sphere and its radius.

$9x^2 + 9y^2 + 9z^2 - 6x + 54y + 18z + 10 = 0$
$\Rightarrow x^2 + y^2 + z^2 - \frac{2x}{3} + 6y + 2z = -\frac{10}{9}$
$(x - \frac{1}{3})^2 + (y+3)^2 + (z+1)^2 = 9$ ✓✓
Hence, centre has position vector $< \frac{1}{3}, -3, -1 >$ ✓
and radius of sphere = 3 ✓

(b) Determine with reasons if the plane with equation $\boldsymbol{r} \bullet < 2, 1, 2 > = 2$ will intersect this sphere.

Vector normal to plane $\boldsymbol{n} = < 2, 1, 2 >$.
The point A < 0, 2, 0 > lies on the plane.
Separation vector between the centre C and A
$\mathbf{AC} = < \frac{1}{3}, -3, -1 > - < 0, 2, 0 > = < \frac{1}{3}, -5, -1 >$
Hence, shortest distance between the centre of the sphere and the plane
= scalar projection of **AC** onto ***n*** ✓
$= \left| < \frac{1}{3}, -5, -1 > \bullet \frac{1}{3} < 2, 1, 2 > \right|$ ✓
$= \frac{19}{9}$ which is less than 3. ✓
Hence, plane will intersect the sphere. ✓

---

3. [10 marks: 4, 6]

The line with equation $x - 1 = \dfrac{y-2}{2} = 1 - z$ intersects the sphere with equation $x^2 + y^2 + z^2 + 2x + 4y - 2z - 30 = 0$ at the points A and B.

(a) Determine the vector equations of the line and the sphere.

Vector equation of line: $\boldsymbol{r} = < 1, 2, 1 > + \lambda < 1, 2, -1 >$ ✓
Vector equation of sphere:
$(x+1)^2 + (y+2)^2 + (z-1)^2 = 36$ ✓✓
$| \boldsymbol{r} - < -1, -2, 1 > | = 6$ ✓

Calculator Assumed

3. (b) Hence, or otherwise, determine the exact length of the chord AB.

Points of intersection:

$|<1, 2, 1> + \lambda <1, 2, -1> - <-1, -2, 1>| = 6$ ✓

$\sqrt{2(3\lambda^2 + 10\lambda + 10)} = 6$

$\lambda = -4, \frac{2}{3}$ ✓

Hence, points of intersection have position vectors:

$<1, 2, 1> + (-4) <1, 2, -1> = <-3, -6, 5>$ ✓

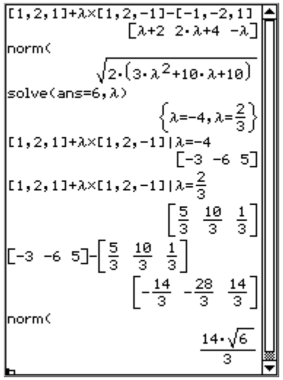
$<1, 2, 1> + \frac{2}{3} <1, 2, -1> = <\frac{5}{3}, \frac{10}{3}, \frac{1}{3}>$. ✓

Hence length of chord AB

$= |<-3, -6, 5> - <\frac{5}{3}, \frac{10}{3}, \frac{1}{3}>|$ ✓

$= |<-\frac{14}{3}, -\frac{28}{3}, \frac{14}{3}>|$

$= \frac{14\sqrt{6}}{3}$ ✓



```
[1,2,1]+λ×[1,2,-1]-[-1,-2,1]
                    [λ+2  2·λ+4  -λ]
norm(
               √2·(3·λ²+10·λ+10)
solve(ans=6,λ)
                       {λ=-4,λ=2/3}
[1,2,1]+λ×[1,2,-1]|λ=-4
                          [-3  -6  5]
[1,2,1]+λ×[1,2,-1]|λ=2/3
                     [5/3  10/3  1/3]
[-3 -6 5]-[5/3 10/3 1/3]
                 [-14/3  -28/3  14/3]
norm(
                             14·√6/3
```

---

4. [9 marks: 5, 4]

(a) The point A has position vector < 2, 1, –2 >.
The plane P has equation $\boldsymbol{r} \bullet <1, 2, 2> = 1$.
The sphere S has radius 6. The plane P is tangential to the sphere S at the point A. Determine the vector equation of the sphere S.

Normal to plane P is $\boldsymbol{n} = <1, 2, 2>$.

$|\boldsymbol{n}| = 3$ ✓

Let C be the centre of the sphere.

Hence $\mathbf{AC} = \lambda\boldsymbol{n}$. ✓

Since $|\boldsymbol{n}| = 3$ and $|\mathbf{AC}| = 6$, $\Rightarrow \lambda = 2$. ✓

$\Rightarrow \mathbf{AC} = <2, 4, 4>$.

Hence OC = < 2, 4, 4 > + < 2, 1, –2 >

= < 4, 5, 2 >. ✓

Hence, vector equation of sphere is:

$|\boldsymbol{r} - <4, 5, 2>| = 6$ ✓

Calculator Assumed

4. (b) The line L has equation $\frac{x-1}{2} = \frac{y+1}{-1} = z - 2 = \lambda$.

A sphere S has equation $(x-1)^2 + (y+1)^2 + (z-2)^2 = 6$

Determine the point(s) of intersection between the line L and the sphere S.

$x - 1 = 2\lambda$ $(x-1)^2 = 4\lambda^2$

$y + 1 = -\lambda$ $(y+1)^2 = \lambda^2$

$z - 2 = \lambda$ $(z-2)^2 = \lambda^2$ ✓

Subst. into equation of sphere: ✓

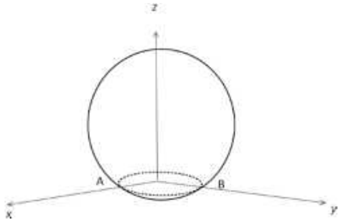
$6\lambda^2 = 6 \Rightarrow \lambda = \pm 1$ ✓

Hence, points of intersections have position vectors: < 3, –2, 3 > and < –1, 0, 1 >. ✓

---

5. [6 marks: 1, 2, 3]

The sphere shown in the given diagram intersects the *x*-*y* plane. The circle of intersection cuts the *x*-axis and *y*-axis at the points A and B respectively and has vector equation $|\boldsymbol{r}| = 3$.



(a) Express the equation of the circle of intersection in Cartesian form.

$x^2 + y^2 = 9$ ✓

(b) Explain clearly why C, the centre of the sphere, must lie on the *z*-axis.

Let centre of circle of intersection be O. Position vector of O = < 0, 0, 0 >.
The distance from C to any point of the circle of intersection = radius of sphere.
Hence, CO must be perpendicular to the plane of the circle of intersection. ✓
But O is the origin. Hence CO lies on the *z*-axis. ✓
Therefore, C must lie on the *z*-axis.

(c) The sphere has a radius of 5 units. Determine a possible vector equation of the sphere.

Let position vector of C, centre of sphere = < 0, 0, k >.

Position vector of A = < 3, 0, 0 >.

$\mathbf{CA} = <3, 0, -k>$ ✓

$|\mathbf{CA}| = 5 \Rightarrow 3^2 + k^2 = 25$ ✓

$k = \pm 4$

Hence, one possible equation is $|\boldsymbol{r} - <0, 0, 4>| = 5$ ✓

Calculator Assumed

6. [8 marks: 4, 4]

Consider the sphere S with equation $|\boldsymbol{r} - <1, 1, 1>| = 3$.

(a) Prove that the line L with equation $\boldsymbol{r} = <3, -1, 2> + \lambda <1, 1, 0>$ is a tangent to the sphere S.

When line and sphere intersects: $\lvert <3, -1, 2> + \lambda <1, 1, 0> - <1, 1, 1> \rvert = 3$ ✓ $\sqrt{2\lambda^2 + 9} = 3$ ✓ $\lambda = 0$ ✓ $\Rightarrow$ There is only one point of intersection. ✓ Therefore, the line is tangential to the sphere.	The point A < 3, –1, 2 > is on the line. Centre of sphere C is < 1, 1, 1 >. **AC** = < 1, 1, 1 > – < 3, –1, 2 > = < –2, 2, –1 >. Let θ be the angle between AC and line L Shortest distance between centre of sphere and line = $\lvert \mathbf{AC} \rvert \sin\theta = \lvert \mathbf{AC} \rvert \dfrac{\lvert \mathbf{AC} \times \boldsymbol{d} \rvert}{\lvert \mathbf{AC} \rvert \lvert \boldsymbol{d} \rvert}$ ✓ $= \left\lvert <-2, 2, -1> \times \dfrac{1}{\sqrt{2}} <1, 1, 0> \right\rvert$ ✓ = 3 = radius of sphere. ✓✓ Hence, line is tangential to sphere.

(b) Prove that the plane with equation $\boldsymbol{r} \bullet <2, 2, 1> = 14$ is tangential to S.

Normal to plane is $\boldsymbol{n} = <2, 2, 1>$.
The point A with position vector < 0, 0, 14 > is on the plane.
Centre of sphere C has position vector < 1, 1, 1 >.
AC = < 1, 1, 1 > – < 0, 0, 14 > = < 1, 1,–13 > ✓
Shortest distance between centre of sphere and plane
= scalar projection of **AC** onto $\boldsymbol{n}$ ✓
$= \left\lvert <1, 1, -13> \bullet \dfrac{1}{3} <2, 2, 1> \right\rvert$ ✓
= 3 = radius of sphere. ✓
Hence, plane is tangential to sphere.

Calculator Assumed

7. [7 marks: 4, 3]

Consider the sphere S with equation $|\boldsymbol{r} - <1, 0, 1>| = 2$

(a) Determine the equation of the curve of intersection of S with the *x-y* plane. Describe this curve.

Cartesian equation of S:
$(x-1)^2 + y^2 + (z-1)^2 = 4$ ✓
Equation of *x-y* plane is $z = 0$.
At intersection:
$(x-1)^2 + y^2 + (0-1)^2 = 4$ ✓
$(x-1)^2 + y^2 = 3$ ✓
Intersection is a circle with centre (1, 0) ✓
and radius $\sqrt{3}$.

(b) Determine the equation of the curve of intersection of S with the sphere T with equation $|\boldsymbol{r} - <1, 0, 2>| = \sqrt{3}$. Describe this curve.

Cartesian equation of S:
$(x-1)^2 + y^2 + (z-1)^2 = 4$ I
Cartesian equation of T:
$(x-1)^2 + y^2 + (z-2)^2 = 3$ II

II – I $z = 2$ ✓

Subst. into II $(x-1)^2 + y^2 = 3$ ✓
Intersection is a curve with centre (1, 0) ✓
and radius $\sqrt{3}$.

13 Vectors VI (Vector Functions)

Calculator Free

1. [9 marks: 3, 4, 2]

The position vector of a particle P at time t seconds is given by $\boldsymbol{r}(t) = < 1 + 2\sin(\pi t), 2 - 3\cos(\pi t) >$ cm where $0 \le t \le 2$.

(a) Determine the Cartesian equation of the path of P.

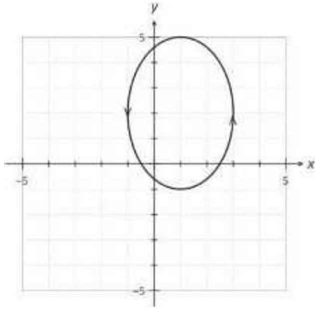
$x = 1 + 2\sin(\pi t) \Rightarrow \sin(\pi t) = \dfrac{x-1}{2}$ ✓

$y = 2 - 3\cos(\pi t) \Rightarrow \cos(\pi t) = \dfrac{y-2}{-3}$ ✓

$\sin^2(\pi t) + \cos^2(\pi t) = \left(\dfrac{x-1}{2}\right)^2 + \left(\dfrac{y-2}{-3}\right)^2$

$\Rightarrow \left(\dfrac{x-1}{2}\right)^2 + \left(\dfrac{y-2}{-3}\right)^2 = 1$ ✓

(b) On the axes provided, sketch the graph of the path of P. Show the direction of motion of P.



- ✓ Draws an ellipse with centre at (1, 2).
- ✓ Semi-major axis = 3
- ✓ Semi-minor axis = 2
- ✓ Anti-clockwise direction.

(c) Determine the shortest distance the particle gets to point with position vector < 1, 2 >.

< 1, 2 > is the centre of the ellipse. ✓
Hence, closest distance = length of semi-minor axis
= 2 ✓

Calculator Free

2. [6 marks: 4, 2]

The position vector of a particle P at time t seconds is given by $\boldsymbol{r}(t) = < 1 + \tan^2 t, 2 - \sec^2 t >$ cm where $t \ge 0$.

(a) Determine the Cartesian equation of the path of P.

$x = 1 + \tan^2 t \Rightarrow \tan^2 t = x - 1$

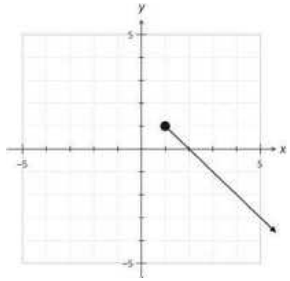
$y = 2 - \sec^2 t \Rightarrow \sec^2 t = 2 - y$ ✓

$\sec^2 t = 1 + \tan^2 t \Rightarrow 2 - y = 1 + x - 1$ ✓

$y = -x + 2$ where $x \ge 1$ ✓✓

(b) On the axes provided, sketch the graph of the path of P.

- ✓ Draws a line with gradient –1.
- ✓ Endpoint (1, 1) and $x \ge 1$.



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3. [5 marks: 2, 3] [TISC]

Object P leaves point A with position vector <–4, 8 > km at 1200 hours. P travels with a constant velocity $\boldsymbol{v}$ and its position vector at 1400 hours is < 10, – 8 > km.

(a) Show that $\boldsymbol{v} = 7\,\mathbf{i} - 4\,\mathbf{j} - 4\,\mathbf{k}$.

$-4\,\mathbf{i} + 8\,\mathbf{j} + 2\boldsymbol{v} = 10\,\mathbf{i} - 8\,\mathbf{k}$ ✓

$\boldsymbol{v} = 7\,\mathbf{i} - 4\,\mathbf{j} - 4\,\mathbf{k}$. ✓

(b) What time will P cross the line with equation $\boldsymbol{r} = \mathbf{i} + 6\,\mathbf{j} + 4\,\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 4\,\mathbf{k})$?

$\mathbf{OP} = -4\,\mathbf{i} + 8\,\mathbf{j} + t\,(7\,\mathbf{i} - 4\,\mathbf{j} - 4\,\mathbf{k})$
$\boldsymbol{r} = \mathbf{i} + 6\,\mathbf{j} + 4\,\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 4\,\mathbf{k})$

Solve simultaneously: ✓
Compare x-components $-4 + 7t = 1 + \lambda$
Compare y-components $8 - 4t = 6 - \lambda$ $\Rightarrow$ $t = 1, \lambda = 2$ ✓
Compare z-components $-4(1) = 4 - 2(4)$ True

Hence, P will cross the line when $t = 1$, ie 1300 hours. ✓

Calculator Assumed

4. [11 marks: 2, 3, 3, 3]

An eagle flies with a constant velocity of < 2, 4, 1 > metres per minute.
At 7 am, the eagle flies past the point A with position vector < 10, –10, 2 > . Find:

(a) the position vector of the eagle at 7.15 am (same day)

OE = **OA** + **AE**
= < 10, –10, 2 > + 15 < 2, 4, 1 > = < 40, 50, 17 > ✓✓

(b) when the eagle is 50 m from A

$|t < 2, 4, 1 >| = 50$ ✓
$21t^2 = 2500$
$t = 10.9$ minutes ✓
Hence, 10.9 minutes after 7 am. ✓

(c) when and where the eagle crosses the *x-z* plane.

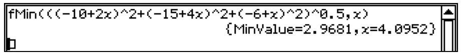
OE = < 10, –10, 2 > + *t* < 2, 4, 1 >
= < 10 + 2*t*, –10 + 4*t*, 2 + *t* >. ✓
When it crosses the *x-z* plane, *y*-component = 0.
⇒ $-10 + 4t = 0$ ✓
$t = 2.5$ minutes
Hence, 2.5 minutes after 7 am ✓

(d) using a vector method, the closest distance between the eagle and the point B with position vector < 20, 5, 8 > m.

OE = < 10 + 2*t*, –10 + 4*t*, 2 + *t* >.
OB = < 20, 5, 8 >
BE =< –10 + 2*t*, –15 + 4*t*, –6 + *t* >. ✓

$|\mathbf{BE}| = \sqrt{[(-10 + 2t)^2 + (-15 + 4t)^2 + (-6 + t)^2]}$ ✓

Using fMin, Min value for |**BE**| = 2.9681
≈ 3.0 m. ✓



```
fMin(((-10+2x)^2+(-15+4x)^2+(-6+x)^2)^0.5,x)
                         {MinValue=2.9681,x=4.0952}
```

Calculator Assumed

5. [5 marks: 1, 1, 3] [TISC]

Particle P leaves the point with position vector 2 **i** – 3 **j** +5 **k** km at 0700 hours. P travels with a constant velocity of 10 **i** – 10 **j** – 5 **k** km per hour.

(a) Find the speed of P.

Speed of P = |10 **i** – 10 **j** – 5 **k**|
$= \sqrt{10^2+10^2+5^2} = \sqrt{225} = 15$ ✓

(b) Find the position vector of the location of P at 0900 hours.

OP = 2 **i** – 3 **j** +5 **k** + 2(10 **i** – 10 **j** – 5 **k**)
= 22 **i** – 23 **j** – 5 **k** km ✓

(c) Another particle Q leaves the point with position vector –20 **i** +10 **j** – 30 **k** km at 0900 hours. Q travels with a constant velocity of 30 **i** – 30 **j** – 15 **k**. Determine with reasons if P and Q will collide.

At 0900, **OP** = 22 **i** – 23 **j** – 5 **k** km
OQ = –20 **i** +10 **j** – 30 **k** km.

Velocity of P = 10 **i** – 10 **j** – 5 **k**
Velocity of Q = 30 **i** – 30 **j** – 15 **k**.

Velocity of Q = 3 × Velocity of P. ✓
P and Q will be travelling on parallel paths from different start points. ✓
Hence, P and Q will never collide. ✓

When **OP** = **OQ**:
Compare *x*-components
$22 + 10t = -20 + 30t$ ⇒ $t = 2.1$ ✓
Compare *y*-components
$-23 - 10t = 10 - 30t$ ⇒ $t = 1.65$ ✓

As the *t* values are all different, P and Q do not collide. ✓

---

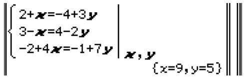
6. [5 marks]

The position vectors of particles A and B at time *t* seconds are respectively

$\boldsymbol{r}_{\rm A} = \begin{pmatrix} 2+t \\ 3-t \\ -2+4t \end{pmatrix}$ and $\boldsymbol{r}_{\rm B} = \begin{pmatrix} -4+3t \\ 4-2t \\ -1+7t \end{pmatrix}$. Find when and where A crosses the path of B.

$$\begin{pmatrix} 2+t \\ 3-t \\ -2+4t \end{pmatrix} = \begin{pmatrix} -4+3t \\ 4-2t \\ -1+7t \end{pmatrix}$$ ✓

⇒ $t = 9, \lambda = 5$ ✓✓
Hence, A crosses path of B at $t = 9$ sec ✓
at $\begin{pmatrix} 11 \\ -6 \\ 34 \end{pmatrix}$ ✓



```
⎧2+x=-4+3y
⎨3-x=4-2y
⎩-2+4x=-1+7y |x,y
                {x=9,y=5}
```

Calculator Assumed

7. [11 marks: 2, 3, 3, 3]

At 0900 hours, the position and velocity vectors of particles P, Q and R are respectively $< 0, -1, -2 >$ m , $< 1, 2, 2 >$ ms^{-1}; $< 4, 1, 6 >$ m , $< -1, \alpha, -2 >$ ms^{-1} and $< -4, 1, -2 >$ m and $< 3, 1, \beta >$ ms^{-1}.

(a) Find the position vectors of P, Q and R, *t* seconds after 0900 hours.

OP = $< 0, -1, -2 > + t < 1, 2, 2 >$
OQ = $< 4, 1, 6 > + t < -1, \alpha, -2 >$
OR = $< -4, 1, -2 > + t < 3, 1, \beta >$ ✓✓ [–1 per error]

(b) If these velocities were maintained, all three particles will collide simultaneously. Use vector methods to find the value(s) of α and β.

OP = OQ = OR
$< t, 2t - 1, 2t - 2 > = < -t + 4, \alpha t + 1, -2t + 6 > = < 3t - 4, t + 1, \beta t - 2 >$
Comparing *x*-components: $t = -t + 4 = 3t - 4 \Rightarrow t = 2$ ✓
Comparing *y*-components: $2t - 1 = \alpha t + 1 = t + 1$
when $t = 2$: $2\alpha + 1 = 3 \Rightarrow \alpha = 1$ ✓
Comparing *z*-components: $2t - 2 = -2t + 6 = \beta t - 2$
when $t = 2$: $2\beta - 2 = 2 \Rightarrow \beta = 2$ ✓

(c) If these velocities were maintained, only P and Q will collide. Use vector methods to find the value(s) of α and β.

P, Q and R collide at $t = 2$ seconds for $\alpha = 1$ and $\beta = 2$.
Hence, for P and Q to collide: $t = 2$ and $\alpha = 1$. ✓
For P and R not to collide at $t = 2$ seconds $\beta \neq 2$.
For Q and R not to collide at $t = 2$ seconds $\alpha \neq 1$ and/or $\beta \neq 2$. ✓

Hence, only P and Q will collide when $\alpha = 1$ and $\beta \neq 2$. ✓

(d) If these velocities were maintained, only two of the three particles will collide. Use vector methods to find the value(s) of α and β.

Only P and Q will collide when $\alpha = 1$ and $\beta \neq 2$.
Only P and R will collide when $\alpha \neq 1$ and $\beta = 2$. ✓
It is not possible for only Q and R to collide,
as Q and R collide when $t = 2$. ✓

Hence, only 2 of the 3 particles will collide when $\alpha \neq 1$ <u>or</u> $\beta \neq 2$. ✓

Calculator Assumed

8. [13 marks: 3, 4, 2, 4]

The position vectors of objects P and Q at time *t* seconds is given by

$\mathbf{OP}(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ metres and $\mathbf{OQ}(t) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ metres respectively.

(a) Show that P and Q do not collide.

For collision: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ ✓

x-component: $-1 + t = 2 - t \Rightarrow t = \frac{3}{2}$

y-component: $2t = 1 - t \Rightarrow t = \frac{1}{3}$ ✓

Hence, *t* for *x* and *y* components do not agree. ✓
Therefore P and Q do not collide

(b) Calculate the closest distance between P and Q.
Show how you obtained your answer.

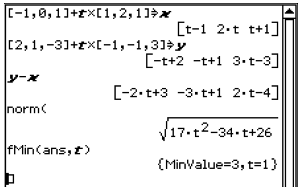
$\mathbf{PQ}(t) = \begin{pmatrix} 2-t \\ 1-t \\ -3+3t \end{pmatrix} - \begin{pmatrix} -1+t \\ 2t \\ 1+t \end{pmatrix} = \begin{pmatrix} 3-2t \\ 1-3t \\ -4+2t \end{pmatrix}$ ✓

$|\mathbf{PQ}(t)| = \sqrt{(3-2t)^2 + (1-3t)^2 + (-4+2t)^2}$ ✓

$= \sqrt{17t^2 - 34t + 26}$

Use $fMin(\sqrt{17t^2 - 34t + 26}, t)$ ✓

Hence, closest distance is 3 metres. ✓



```
[-1,0,1]+t×[1,2,1]⇒x
                    [t-1  2·t  t+1]
[2,1,-3]+t×[-1,-1,3]⇒y
                  [-t+2  -t+1  3·t-3]
y-x
            [-2·t+3  -3·t+1  2·t-4]
norm(
                  √(17·t²-34·t+26)
fMin(ans,t)
                 {MinValue=3,t=1}
```

(c) State the vector equations of the paths of P and Q.

$r_P = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ✓

$r_Q = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ ✓

Calculator Assumed

8. (d) Calculate the closest distance between the paths of P and Q.

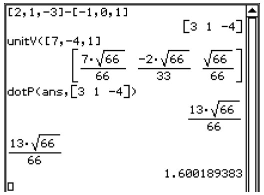
Vector normal to both paths $\boldsymbol{n} = \begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}-1\\-1\\3\end{pmatrix} = \begin{pmatrix}7\\-4\\1\end{pmatrix}$ ✓

A $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ is on the path of P. B $\begin{pmatrix}2\\1\\-3\end{pmatrix}$ is on the path of Q.

$\mathbf{AB} = \begin{pmatrix}2\\1\\-3\end{pmatrix} - \begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\-4\end{pmatrix}$. ✓

Shortest distance = scalar projection of **AB** onto $\boldsymbol{n}$

$= \begin{pmatrix}3\\1\\-4\end{pmatrix} \bullet \frac{1}{\sqrt{66}}\begin{pmatrix}7\\-4\\1\end{pmatrix} \approx 1.6$ metres ✓✓



```
[2,1,-3]-[-1,0,1]
                              [3 1 -4]
unitV([7,-4,1])
        [7·√66   -2·√66   √66]
        [ 66       33      66]
dotP(ans,[3 1 -4])
                              13·√66
                                66
13·√66
  66
                         1.600189383
```

---

9. [10 marks: 5, 5]

Particle A leaves the point < 1, a, 2 > m with constant velocity < –1, 2, 5 > ms^{-1}. Simultaneously, particle B leaves the point < –3, 1, 10 > m with constant velocity < 1, 1, u > ms^{-1}. a and u are real constants.

(a) For $a = -1$, determine the value(s) of u for which the particles do not collide.

$\begin{pmatrix}1-t\\-1+2t\\2+5t\end{pmatrix} = \begin{pmatrix}-3+t\\1+t\\10+ut\end{pmatrix}$ ✓✓

$t_x = 2,\ t_y = 2,\ t_z = \frac{8}{5-u}$ ✓

Since $t_x = t_y = 2$, for A and B not to collide:

$t_z \neq 2 \Rightarrow u \neq 1$ ✓✓

(b) For $a = -4$, determine the value(s) of u for which the paths of A and B will intersect.

$\begin{pmatrix}1-\mu\\-1+2\mu\\2+5\mu\end{pmatrix} = \begin{pmatrix}-3+\lambda\\1+\lambda\\10+u\lambda\end{pmatrix}$ ✓✓

Solving x and y components:

$1 - \mu = -3 + \lambda$

$-4 + 2\mu = 1 + \lambda$

$\mu = 3, \lambda = 1$ ✓

For the paths to intersect, comparing z components:

$2 + 15 = 10 + u \Rightarrow u = 7$ ✓✓

Calculator Assumed

10. [11 marks, 3, 4, 4]

Particles A and B start moving at 0800 hours. The position vectors of A and B t hours after 0800 hours are respectively $\boldsymbol{r}_A = \begin{pmatrix}10+4t\\-20+2t\\4-t\end{pmatrix}$ and $\boldsymbol{r}_B = \begin{pmatrix}20+t\\20-3t\\-11+t\end{pmatrix}$.

(a) Prove that A and B will not collide.

$\begin{pmatrix}10+4t\\-20+2t\\4-t\end{pmatrix} = \begin{pmatrix}20+t\\20-3t\\-11+t\end{pmatrix}$ ✓

$\Rightarrow t_x = \frac{10}{3}$ and $t_y = 8$ ✓

Since $t_x \neq t_y$, A and B do not collide. ✓

(b) The paths traced by A and B respectively meet at the point P. Determine the position vector of the point P.

At the point P: $\begin{pmatrix}10+4\lambda\\-20+2\lambda\\4-\lambda\end{pmatrix} = \begin{pmatrix}20+\mu\\20-3\mu\\-11+\mu\mu\end{pmatrix}$ ✓

$\Rightarrow \lambda = 5$ and $\mu = 10$ ✓✓

Hence: $\mathbf{OP} = \begin{pmatrix}30\\-10\\-1\end{pmatrix}$ ✓

(c) Determine with reasons, which of the two particles A or B is the first to reach P. State the exact distance between the two particles at the time the first particle reaches P.

From (b), A reaches P after 5 hours while B reaches P after 10 hours.
Hence, A is the first to reach P. ✓

$\boldsymbol{r}_A(5) = \text{OP} = \begin{pmatrix}30\\-10\\-1\end{pmatrix}$

$\boldsymbol{r}_B(5) = \begin{pmatrix}20+5\\20-3(5)\\-11+5\end{pmatrix} = \begin{pmatrix}25\\5\\-6\end{pmatrix}$ ✓

$\mathbf{BA}(5) = \begin{pmatrix}30\\-10\\-1\end{pmatrix} - \begin{pmatrix}25\\5\\-6\end{pmatrix} = \begin{pmatrix}5\\-15\\5\end{pmatrix}$ ✓

Distance from B to P = $|\mathbf{BA}(5)| = 5\sqrt{11}$ ✓

Calculator Assumed

11. [8 marks: 5, 3]

A drone located at P is to be flown to Q. The position vector of Q relative to P is < 100, 80, 40 > *m*. A wind is blowing with constant velocity < 0.8, 1.3, 0.4 > ms^{-1}.

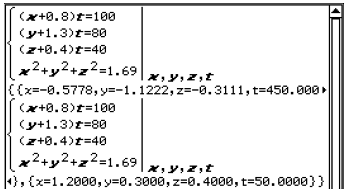
(a) Determine ***v*** the velocity of the drone if it is flown from P to Q at its maximum speed of 1.3 ms^{-1}.

Let velocity of drone $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Resultant velocity = $\begin{pmatrix} x+0.8 \\ y+1.3 \\ z+0.4 \end{pmatrix}$. ✓

Hence: $\begin{pmatrix} x+0.8 \\ y+1.3 \\ z+0.4 \end{pmatrix} \times t = \begin{pmatrix} 100 \\ 80 \\ 40 \end{pmatrix}$ ✓

$\Rightarrow$ $(x + 0.8)\,t = 100$
$(y + 1.3)\,t = 800$
$(z + 0.4)\,t = 40$
$x^2 + y^2 + z^2 = 1.69$ ✓



```
(x+0.8)t=100
(y+1.3)t=80
(z+0.4)t=40
x^2+y^2+z^2=1.69 | x,y,z,t
{{x=-0.5778,y=-1.1222,z=-0.3111,t=450.000▸
(x+0.8)t=100
(y+1.3)t=80
(z+0.4)t=40
x^2+y^2+z^2=1.69 | x,y,z,t
◂},{x=1.2000,y=0.3000,z=0.4000,t=50.0000}}
```

$v = \begin{pmatrix} 1.2 \\ 0.3 \\ 0.4 \end{pmatrix}$ with time of flight 50 sec. ✓

or $\begin{pmatrix} -0.5\overline{7} \\ -1.1\overline{2} \\ -0.3\overline{1} \end{pmatrix}$ with time of flight 450 sec. ✓

(b) Comment briefly on the significance of your answer in (a).

There are two possible answers for ***v***.

$v = \begin{pmatrix} 1.2 \\ 0.3 \\ 0.4 \end{pmatrix}$ is headed in the general direction of Q. ✓

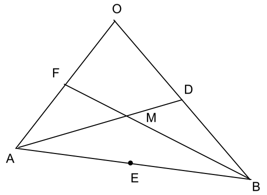
$v = \begin{pmatrix} -0.5\overline{7} \\ -1.1\overline{2} \\ -0.3\overline{1} \end{pmatrix}$ is headed in general direction away from Q. ✓

The second case is possible because speed of the wind (1.58 ms^{-1}) > speed of the drone (1.3 ms^{-1}). ✓

14 Geometric Proofs using Vectors

Calculator Assumed

1. [13 marks: 2, 2, 5, 3, 1]



OAB is a triangle with **OA** = ***a*** and **OB** = ***b***. D, E and F are the midpoints of OB, AB, and OA respectively. **AM** = α**AD** and **MF** = β**BF**.

(a) Find **AD** and **BF** in terms of ***a*** and ***b***.

AD = –***a*** + ½ ***b*** ✓
BF = ½ ***a*** – ***b*** ✓

(b) Find **AM** and **MF** in terms of ***a*** and ***b***.

AM = $\alpha(-a + \tfrac{1}{2}b) = -\alpha a + \dfrac{\alpha}{2}b$ ✓

MF = $\beta(\tfrac{1}{2}a - b) = \dfrac{\beta}{2}a - \beta b$ ✓

(c) Use your answers in (b) to find α and β.

AF = AM + MF ✓

$-\tfrac{1}{2}a = (-\alpha a + \dfrac{\alpha}{2}b) + (\dfrac{\beta}{2}a - \beta b) = (-\alpha + \dfrac{\beta}{2})\,a + (\dfrac{\alpha}{2} - \beta)\,b$ ✓

Compare coefficients for ***b*** and ***a*** vectors:

$\dfrac{\alpha}{2} - \beta = 0 \;\Rightarrow\; \beta = \dfrac{\alpha}{2}$ ✓

$-\alpha + \dfrac{\beta}{2} = -\dfrac{1}{2} \;\Rightarrow\; -\alpha + \dfrac{1}{4}\alpha = -\dfrac{1}{2} \;\Rightarrow\; \alpha = \dfrac{2}{3},\ \beta = \dfrac{1}{3}$ ✓✓

(d) Show that **OM** = μ**OE** giving the value of μ .

OM = OA + AM
$= a + (-\dfrac{2}{3}a + \dfrac{1}{3}b) = \dfrac{1}{3}(a + b)$ ✓
OE = OA + AE
$= a + \tfrac{1}{2}(-a + b) = \tfrac{1}{2}(a + b)$ ✓
Hence, **OM** = $\dfrac{2}{3}$ **OE** ✓

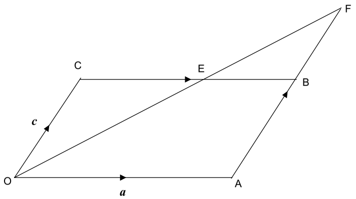
(e) Comment on the significance of the location of M in terms of the lines OE, AD and BF.

M is the intersection between OE, AD and BF.
Hence, the medians of a triangle meet at a point that is two-thirds down from the vertex. ✓

Calculator Assumed

2. [8 marks]

OABC is a parallelogram. The point E divides CB in the ratio α : β. That is, the point E is such that $\mathbf{EB} = \dfrac{\beta}{\alpha+\beta}\mathbf{CB}$. OE extended meets the AB extended at F. Use vector methods to prove that:



Area of ΔFEB = $\left(\dfrac{\beta}{\alpha+\beta}\right)^2$ × Area of ΔFOA.

[Hint: Let **EF** = λ**OF** and **BF** = μ**AF**.]

Let **EF** = λ**OF** and **BF** = μ**AF**.

$\mathbf{EF} = \mathbf{EB} + \mathbf{BF}$ ✓

$\lambda\mathbf{OF} = \dfrac{\beta}{\alpha+\beta}\mathbf{CB} + \mu\mathbf{AF}$

$= \dfrac{\beta}{\alpha+\beta}\mathbf{OA} + \mu(\mathbf{AO} + \mathbf{OF})$

$\lambda\mathbf{OF} = \dfrac{\beta}{\alpha+\beta}\mathbf{OA} + \mu(-\mathbf{OA} + \mathbf{OF})$

$(\lambda - \mu)\mathbf{OF} = (\dfrac{\beta}{\alpha+\beta} - \mu)\mathbf{OA}$ ✓✓

Since OF and OA are non-parallel:
$\lambda - \mu = 0 \Rightarrow \lambda = \mu$
and $\dfrac{\beta}{\alpha+\beta} - \mu = 0 \Rightarrow \mu = \dfrac{\beta}{\alpha+\beta}$ and $\lambda = \dfrac{\beta}{\alpha+\beta}$. ✓✓

Hence, $\mathbf{EF} = \dfrac{\beta}{\alpha+\beta}\mathbf{OF}$ and $\mathbf{BF} = \dfrac{\beta}{\alpha+\beta}\mathbf{AF}$

Area of ΔFOA = $\dfrac{1}{2} \times |\mathbf{OF}| \times |\mathbf{AF}| \times \sin E\hat{F}B$ ✓

Area of ΔFEB = $\dfrac{1}{2} \times |\mathbf{FE}| \times |\mathbf{FB}| \times \sin E\hat{F}B$

$= \dfrac{1}{2} \times \dfrac{\beta}{\alpha+\beta}|\mathbf{OF}| \times \dfrac{\beta}{\alpha+\beta}|\mathbf{AF}| \times \sin E\hat{F}B$ ✓

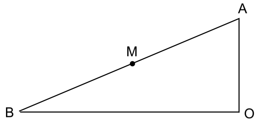
$= \left(\dfrac{\beta}{\alpha+\beta}\right)^2 \times \dfrac{1}{2} \times |\mathbf{OF}| \times |\mathbf{AF}| \times \sin E\hat{F}B$ ✓

$= \left(\dfrac{\beta}{\alpha+\beta}\right)^2$ × Area of ΔFOA.

Calculator Assumed

3. [8 marks: 1, 3, 4]

OAB is a right angled triangle with ∠ AOB = 90^o.
$\mathbf{OA} = \boldsymbol{a}$ and $\mathbf{OB} = \boldsymbol{b}$. M is the midpoint of AB.



(a) Explain why $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

OA is perpendicular to OB.
Hence, $\mathbf{OA} \cdot \mathbf{OB} = 0$.
Therefore, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. ✓

(b) Find $|\mathbf{BM}|^2$ in terms of a and b, where $|\boldsymbol{a}| = a$ and $|\boldsymbol{b}| = b$.

$\mathbf{BM} = \frac{1}{2}\mathbf{BA} = \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$ ✓
$|\mathbf{BM}|^2 = \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b}) \cdot \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$ ✓
$= \frac{1}{4}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\,\boldsymbol{a} \cdot \boldsymbol{b})$
$= \frac{1}{4}(a^2 + b^2)$ since $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ ✓

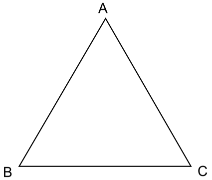
(c) Hence, prove that M is the centre of a circle passing through A, B and O.

$\mathbf{OM} = \boldsymbol{b} + \mathbf{BM} = \boldsymbol{b} + \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$
$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ ✓
$|\mathbf{OM}|^2 = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) \cdot \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$
$= \frac{1}{4}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\,\boldsymbol{a} \cdot \boldsymbol{b})$
$= \frac{1}{4}(a^2 + b^2)$ since $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ ✓
Hence, $|\mathbf{OM}| = |\mathbf{BM}| = |\mathbf{MA}|$. ✓
Therefore, M is equidistant from O, A and B.
M is then the centre of a circle passing through A, B and O. ✓

Calculator Assumed

4. [12 marks: 2, 3, 4, 3]

ABC is an isosceles triangle with AB = AC.
Also **BA** = $\boldsymbol{a}$ and **CB** = $\boldsymbol{b}$.



(a) Show that $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}|$

$\mathbf{CA} = \mathbf{CB} + \mathbf{BA}$
$= \boldsymbol{b} + \boldsymbol{a}$ ✓
But, $|\mathbf{CA}| = |\mathbf{BA}|$ ✓
$\Rightarrow |\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}|$

(b) Show that $|\boldsymbol{b}|^2 = -2\boldsymbol{a} \,.\, \boldsymbol{b}$.

$|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}|$
$\Rightarrow (\boldsymbol{a}+\boldsymbol{b}) \,.\, (\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a} \,.\, \boldsymbol{a}$ ✓
$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\,\boldsymbol{a} \,.\, \boldsymbol{b} = |\boldsymbol{a}|^2$ ✓
$\Rightarrow |\boldsymbol{b}|^2 = -2\,\boldsymbol{a} \,.\, \boldsymbol{b}$ ✓

(c) Show that $\cos C = \dfrac{-\boldsymbol{a} \,.\, \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$.

$\cos C = \dfrac{\mathbf{CA} \,.\, \mathbf{CB}}{|\mathbf{CA}||\mathbf{CB}|}$
$= \dfrac{(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{b}}{|\boldsymbol{a}+\boldsymbol{b}||\boldsymbol{b}|}$ ✓
$= \dfrac{(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ since $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}|$ ✓
$= \dfrac{\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$
$= \dfrac{\boldsymbol{a}.\boldsymbol{b}-2\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ since $|\boldsymbol{b}|^2 = -2\,\boldsymbol{a} \,.\, \boldsymbol{b}$ ✓
$= \dfrac{-\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ ✓

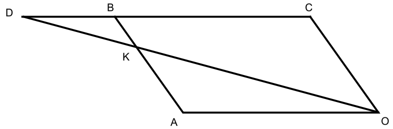
(d) Hence, using a vector method, prove that the base angles of an isosceles triangle are equal.

$\cos B = \dfrac{\mathbf{BA} \,.\, \mathbf{BC}}{|\mathbf{BA}||\mathbf{BC}|} = \dfrac{-\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ ✓
Since cos C = cos B, ∠ACB = ∠ABC. ✓
Hence, the base angles of an isosceles triangle are equal. ✓

Calculator Assumed

5. [5 marks: 2, 3] [TISC]

OABC is a parallelogram with **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. The point K divides AB in the ratio 2 : 1. OK extended meets the line CB extended at D. **OK** = α**OD** and **CD** = β**CB**.



(a) Find **AK** and **OK** in terms of **a** and **c**.

$\mathbf{AK} = \frac{2}{3}\mathbf{AB} = \frac{2}{3}\mathbf{OC} = \frac{2}{3}\boldsymbol{c}.$ ✓
$\mathbf{OK} = \mathbf{OA} + \mathbf{AK} = \boldsymbol{a} + \frac{2}{3}\boldsymbol{c}$ ✓

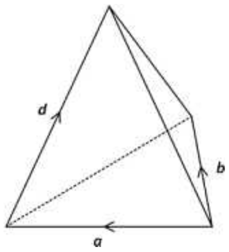
(b) Use your answer in (a) to prove that B divides the line CD in the ratio 2 : 1.

$\mathbf{OK} = \alpha\mathbf{OD}$
$\mathbf{OD} = \mathbf{OC} + \mathbf{CD} = \boldsymbol{c} + \beta\mathbf{CB} = \boldsymbol{c} + \beta(\boldsymbol{a})$
Hence, $\mathbf{OK} = \alpha[\,\boldsymbol{c} + \beta(\boldsymbol{a})]$
$= \alpha\boldsymbol{c} + \alpha\beta\boldsymbol{a}$. ✓
But from (a), $\mathbf{OK} = \boldsymbol{a} + \frac{2}{3}\boldsymbol{c}$
Hence, $\alpha\mathbf{c} + \alpha\beta\boldsymbol{a} = \boldsymbol{a} + \frac{2}{3}\boldsymbol{c}$ ✓
$\Rightarrow \quad = \frac{2}{3}$ and $\alpha\beta = 1$.
Therefore, $\beta = \frac{3}{2}$. ✓
Hence, B divides the line CD in the ratio 2 : 1.

---

6. [7 marks: 2, 5]

The accompanying diagram shows a tetrahedron. Two of the adjacent edges of the tetrahedron are represented by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. A third edge is represented by the vector $\boldsymbol{d}$.



(a) Prove that the area of the face bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $A = \dfrac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$.

Let the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ be θ.
Hence, Area $= \frac{1}{2} \times |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$ ✓
$= \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$ ✓

Calculator Assumed

6. (b) Prove that the volume of the tetrahedron is given by $V = \frac{1}{6}\left| \boldsymbol{d} \bullet (\boldsymbol{a} \times \boldsymbol{b}) \right|$.

Normal to face bounded by edges parallel and congruent to $\boldsymbol{a}$ and $\boldsymbol{b}$

$\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b}$. ✓

"Height" of tetrahedron = scalar projection of $\boldsymbol{d}$ onto $\boldsymbol{n}$.

$= \boldsymbol{d} \bullet \dfrac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|}$ ✓✓

Hence, volume of tetrahedron = $\frac{1}{3}$ × Base Area × Height

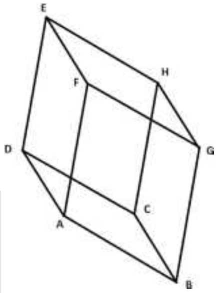
$= \frac{1}{3} \times \left| \frac{1}{2} |\boldsymbol{a} \times \boldsymbol{b}| \times \boldsymbol{d} \bullet \dfrac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|} \right|$ ✓✓

$= \frac{1}{6} \left| \boldsymbol{d} \bullet (\boldsymbol{a} \times \boldsymbol{b}) \right|$

---

7. [5 marks]

The accompanying diagram shows a parallelepiped. The equations of the faces ABCD and EFGH are respectively $x + 7y - 5z = 23$ and $x + 7y - 5z = 38$. Prove that the perpendicular distance between the faces ABCD and EFGH is $\sqrt{3}$.



Normal to both planes $\boldsymbol{n} = \begin{pmatrix} 1 \\ 7 \\ -5 \end{pmatrix}$ ✓

Point P on ABCD has position vector $\mathbf{OP} = \begin{pmatrix} 23 \\ 0 \\ 0 \end{pmatrix}$. ✓

Point Q on EFGH has position vector $\mathbf{OQ} = \begin{pmatrix} 38 \\ 0 \\ 0 \end{pmatrix}$. ✓

Shortest distance = Scalar projection of PQ onto n ✓

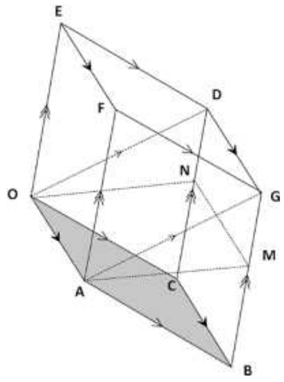
$$= \left| \left[\begin{pmatrix} 38 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 23 \\ 0 \\ 0 \end{pmatrix} \right] \bullet \frac{1}{\sqrt{75}} \begin{pmatrix} 1 \\ 7 \\ -5 \end{pmatrix} \right| \quad ✓$$

$= \sqrt{3}$

Calculator Assumed

8. [7 marks: 3, 4]

The accompanying diagram shows a parallelepiped OABCDEFG. The opposite faces of the solid are congruent parallelograms which are parallel to each other. **OA** = $\boldsymbol{a}$, **OC** = $\boldsymbol{c}$ and **OE** = $\boldsymbol{e}$. The point M is such that **BM** = α**BG**. The point N is such that **CN** = α**CD**.



(a) Prove that **NM** = $\boldsymbol{a}$.

NM = **NC** + **CB** + **BM** ✓

But **NC** = −α**CD** = −α**OE** = −α$\boldsymbol{e}$

CB = **OA** = a

BM = α**BG** = α**OE** = α$\boldsymbol{a}$ ✓

Hence:

NM = −α$\boldsymbol{e}$ + $\boldsymbol{a}$ + α$\boldsymbol{e}$

= $\boldsymbol{a}$ ✓

(b) The volume of parallelepiped OABCDEFG is given by $|(\boldsymbol{a} \times \boldsymbol{c}) \bullet \boldsymbol{e}|$.

Use this result to prove that $\alpha = \frac{3}{4}$ given that the ratio of the volume of ODGABC to the volume of ONMABC is 4 : 3.

ODGABC is half a parallelepiped.

⇒ Volume of ODGABC = $\frac{1}{2}|(\boldsymbol{a} \times \boldsymbol{c}) \bullet \boldsymbol{e}|$. ✓

Since **NM** = **OA** = **CB**,

ONMABC is half a parallelepiped. ✓

Volume of parallelepiped spanned by **OA**, **OC** and **BM** = $|(\boldsymbol{a} \times \boldsymbol{c}) \bullet \alpha\boldsymbol{e}|$

Hence, volume of ONMABC = $\frac{1}{2}|(\boldsymbol{a} \times \boldsymbol{c}) \bullet \alpha\boldsymbol{e}|$

$= \frac{\alpha}{2}|(\boldsymbol{a} \times \boldsymbol{c}) \bullet \boldsymbol{e}|$ ✓

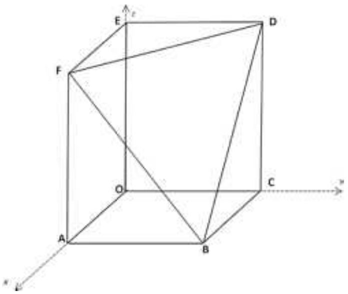
Hence: $\frac{1}{2}|(\boldsymbol{a} \times \boldsymbol{c}) \bullet \boldsymbol{e}| : \frac{\alpha}{2}|(\boldsymbol{a} \times \boldsymbol{c}) \bullet \boldsymbol{e}| = 4 : 3$ ✓

$\Rightarrow \alpha = \frac{3}{4}$

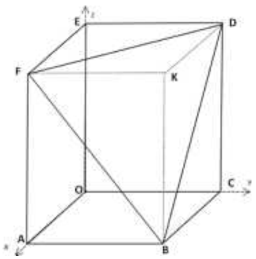
Calculator Assumed

9. [6 marks]

The accompanying diagram shows a rectangular prism with a section removed. O is the origin of the *x*-*y*-*z* axes. The position vectors of vertices A, C and E are $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{e}$ respectively.



Prove that the volume of this solid is given by $V = \frac{5}{6}\left|\boldsymbol{e} \bullet (\boldsymbol{a} \times \boldsymbol{c})\right|$.



Construct DK parallel and congruent to BC.
Construct FK parallel and congruent to OC.
Construct BK parallel and congruent to OE.

Hence, OABCDEFK is a rectangular prism.

DFKB is a triangular prism.

Volume of DFKB = $\frac{1}{3}$ × Base Area × Height

Base Area = Area of ΔDFK

$$= \frac{1}{2}\left|\mathbf{KD} \times \mathbf{KF}\right|$$

$$= \frac{1}{2}\left|\boldsymbol{a} \times \boldsymbol{c}\right| \quad ✓$$

Height *h* of DFKB is scalar projection of **BK** onto $\boldsymbol{a} \times \boldsymbol{c}$.

$$\Rightarrow \; h = \boldsymbol{e} \bullet \frac{\boldsymbol{a} \times \boldsymbol{c}}{\left|\boldsymbol{a} \times \boldsymbol{c}\right|}$$

Volume of DFHB = $\frac{1}{3}$ × Base Area × Height

$$= \frac{1}{3} \times \frac{1}{2}\left|\boldsymbol{a} \times \boldsymbol{c}\right| \times \boldsymbol{e} \bullet \frac{\boldsymbol{a} \times \boldsymbol{c}}{\left|\boldsymbol{a} \times \boldsymbol{c}\right|} \quad ✓✓$$

$$= \frac{1}{6}\left|\boldsymbol{e} \bullet (\boldsymbol{a} \times \boldsymbol{c})\right| \quad ✓$$

Volume of OABCDEFK = Base Area × Height

$$= \left|\boldsymbol{a} \times \boldsymbol{c}\right| \times \boldsymbol{e} \bullet \frac{\boldsymbol{a} \times \boldsymbol{c}}{\left|\boldsymbol{a} \times \boldsymbol{c}\right|}$$

$$= \left|\boldsymbol{e} \bullet (\boldsymbol{a} \times \boldsymbol{c})\right| \quad ✓$$

Hence, volume of solid = $\left|\boldsymbol{e} \bullet (\boldsymbol{a} \times \boldsymbol{c})\right| - \frac{1}{6}\left|\boldsymbol{e} \bullet (\boldsymbol{a} \times \boldsymbol{c})\right|$ ✓

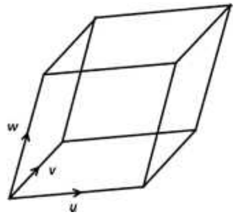
$$= \frac{5}{6}\left|\boldsymbol{e} \bullet (\boldsymbol{a} \times \boldsymbol{c})\right|$$

Calculator Assumed

10. [6 marks]

The edges of a parallelepiped are parallel and congruent to the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ as shown in the accompanying diagram.
Determine the volume V of the parallelepiped.
Hence, deduce that:

$$\left|\boldsymbol{w} \bullet (\boldsymbol{u} \times \boldsymbol{v})\right| = \left|\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})\right| = \left|\boldsymbol{v} \bullet (\boldsymbol{w} \times \boldsymbol{u})\right|$$



Let the base be formed by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.
Area of base = $\left|\boldsymbol{u} \times \boldsymbol{v}\right|$
Height $\boldsymbol{h}$ = scalar projection of $\boldsymbol{w}$ onto $\boldsymbol{u} \times \boldsymbol{v}$

$$= \boldsymbol{w} \bullet \frac{\boldsymbol{u} \times \boldsymbol{v}}{\left|\boldsymbol{u} \times \boldsymbol{v}\right|} \quad ✓$$

Hence, V = Base Area × Height

$$= \left|\boldsymbol{u} \times \boldsymbol{v}\right| \times \boldsymbol{w} \bullet \frac{\boldsymbol{u} \times \boldsymbol{v}}{\left|\boldsymbol{u} \times \boldsymbol{v}\right|} \quad ✓$$

$$= \left|\boldsymbol{w} \bullet (\boldsymbol{u} \times \boldsymbol{v})\right| \quad ✓$$

Let the base be formed by the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.
Then, V = $\left|\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})\right|$ ✓

Similarly if the base is formed by the vectors $\boldsymbol{w}$ and $\boldsymbol{u}$,
V = $\left|\boldsymbol{v} \bullet (\boldsymbol{w} \times \boldsymbol{u})\right|$ ✓

Therefore:

$$\left|\boldsymbol{w} \bullet (\boldsymbol{u} \times \boldsymbol{v})\right| = \left|\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})\right| = \left|\boldsymbol{v} \bullet (\boldsymbol{w} \times \boldsymbol{u})\right| \quad ✓$$

15 Systems of Linear Equations

Calculator Free

1. [9 marks: 1, 2, 2, 4]

A system of 3 linear equations in x, y and z is reduced to $x - y - 2z = 0$ and $pz = (p - 1)(p - 2)$ where p is a constant.

(a) Determine the nature of the solutions to this system when $p = 3$.

Infinite number of solutions. ✓

(b) Find the value(s) of p for which the system has no solutions.

$p = 0$ ✓✓

(c) Find the value(s) of p for which the system has an infinite number of solutions.

$p \neq 0$ ✓✓

(d) For the value(s) of p for which the system has an infinite number of solutions, determine these solutions.

For $p \neq 0$:

$$z = \frac{(p-1)(p-2)}{p}$$

$$x - y - 2z = 0$$

$$x - y = \frac{2(p-1)(p-2)}{p} \quad ✓$$

$$x = y + \frac{2(p-1)(p-2)}{p}$$

Hence:

$$\underset{✓}{x = t + \frac{2(p-1)(p-2)}{p}},\ \underset{✓}{y = t},\ \underset{✓}{z = \frac{(p-1)(p-2)}{p}} \quad \text{where } p, t \in \mathbb{R},\ p \neq 0.$$

Calculator Free

2. [5 marks: 4, 1]

Consider the set of equations:

$$\begin{aligned} x + 2y - 3z &= -12 \\ x - y + 2z &= 9 \\ 3x + 3y + kz &= -2 \end{aligned}$$

(a) Show that this system cannot have an infinite number of solutions.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -12 \\ 0 & 3 & -5 & -21 \\ 0 & 3 & -9-k & -34 \end{array}\right) \quad ✓$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -12 \\ 0 & 3 & -5 & -21 \\ 0 & 0 & k+4 & 13 \end{array}\right) \quad ✓$$

Clearly, the last row cannot be reduced to $(0\ \ 0\ \ 0\ |\ 0)$. ✓

Hence, the system cannot have an infinite number of solutions. ✓

(b) Hence or otherwise, find the value(s) of k for which the system has no solution.

$k = -4$ ✓

---

3. [7 marks: 4, 3] [TISC]

Consider the set of equations:

$$\begin{aligned} x + 2y + 2z &= 4 \\ x + 8y &= 4 \\ x - 4y + kz &= k \end{aligned}$$

(a) Show that this set will always have at least one solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 1 & 8 & 0 & 4 \\ 1 & -4 & k & k \end{array}\right)$$

$$\begin{array}{r} \\ R1-R2 \\ R1-R3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & -6 & 2 & 0 \\ 0 & 6 & 2-k & 4-k \end{array}\right) \quad ✓$$

$$\begin{array}{r} \\ \\ R2+R3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & -6 & 2 & 0 \\ 0 & 0 & 4-k & 4-k \end{array}\right) \quad ✓$$

Hence, when $k = 4$, this system has an infinite number of solutions. ✓

When $k \neq 4$, this system has a unique set of solutions. ✓

Hence, this system will have at least one solution.

Calculator Free

3. (b) Find the unique solution to this set of equations.

When $k \neq 4$, $z = \frac{4-k}{4-k} = 1$ ✓

Hence, $y = \frac{1}{3}$ ✓

$x = \frac{4}{3}$ ✓

---

4. [9 marks: 6, 3]

Consider the set of equations:

$$\begin{aligned} x + 2y + 3z &= 5 \\ 3x + 6y + az &= 14 \\ -2x - 4y + bz &= -9 \end{aligned}$$

where a and b are real constants.

(a) Determine if possible, the value(s) of a and b for this set of equations to have at least one solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 3 & 6 & a & 14 \\ -2 & -4 & b & -9 \end{array}\right)$$

$$\begin{array}{r} \\ 3R1-R2 \\ 2R1+R3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 9-a & 1 \\ 0 & 0 & b+6 & 1 \end{array}\right) \quad ✓$$

$$\Rightarrow \; z = \frac{1}{9-a} = \frac{1}{b+6} \quad ✓✓$$

Hence, for solution to exist, $a \neq 9$ and $b \neq 6$.

Also, $9 - a = b + 6$

$a + b = 3$.

Therefore, for solution to exist, $a \neq 9$ and $b \neq 6$ and $a + b = 3$. ✓✓✓

(b) Determine the number of solutions for this set of equations if $a = 8$ and $b = -5$. Justify your answer.

$$a = 8, b = -5 \;\Rightarrow\; \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right). \quad ✓$$

Since $R2 = R3$, system has an infinite number of solutions. ✓✓

Calculator Free

5. [11 marks: 5, 3, 3]

Consider the set of equations:

$$\begin{aligned} x + 2y + z &= p \\ p^2x + 8y + 4z &= 2 + 3p \\ x + 2y &= 0 \end{aligned}$$

where p is a real constant.

(a) Determine where possible, the value(s) of p and q for this set of equations to have: (i) no solution (ii) more than one solution (iii) a unique solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & p \\ p^2 & 8 & 4 & 2+3p \\ 1 & 2 & 0 & 0 \end{array}\right)$$

$$\begin{array}{r} \\ R2-4R1 \\ R1-R3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 1 & p \\ p^2-4 & 0 & 0 & 2-p \\ 0 & 0 & 1 & p \end{array}\right) \quad ✓✓$$

(i) $p = -2$ ✓

(ii) $p = 2$ ✓

(iii) $p \neq \pm 2$ ✓

(b) In the case where the system has a unique solution, find x, y and z in terms of p.

For $p \neq \pm 2$:

$z = p$ ✓

$(p^2 - 4)\,x = 2 - p \;\Rightarrow\; x = \frac{-1}{p+2}$ ✓

$\frac{-1}{p+2} + 2y + p = p \;\Rightarrow\; y = \frac{1}{2(p+2)}$ ✓

(c) In the case where the system has an infinite number of solutions, find x, y and z.

For $p = 2$:

$z = 2$ ✓

$x + 2y + z = 2$

$\Rightarrow\; x + 2y = 0$ ✓

Hence: $x = -2t,\ y = t,\ z = 2$ where $t \in \mathbb{R}$ ✓

Calculator Free

6. [8 marks: 3, 1, 1, 1, 2]

Consider the equations: $x + y + z = 2$ $2x - y - z = 4$ $-x + py + qz = r$
where p, q and r are real constants.

(a) Show that $z = \dfrac{r+2}{q-p}$.

Eqn 1 + Eqn 2: $3x = 6 \Rightarrow x = 2$ ✓

Eqn 1 becomes: $y + z = 0 \Rightarrow y = -z$

Eqn 3 becomes: $py + qz = r + 2$ ✓
$(q-p)z = r + 2$ ✓

$$z = \frac{r+2}{q-p}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & -1 & 4 \\ -1 & p & q & r \end{array}\right)$$

$$\begin{array}{r} \\ (2R1-R2)\div 3 \\ R1+R3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1+p & 1+q & r+2 \end{array}\right) \quad ✓✓$$

$$\begin{array}{r} \\ \\ (1+p)R2-R3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & p-q & -(r+2) \end{array}\right) \quad ✓$$

Hence: $z = \dfrac{r+2}{q-p}$

(b) Determine the values of p, q and r so that this system has:
(i) a unique set of solutions.

For unique set: $p \neq q$ and $r \in \mathbb{R}$ ✓

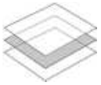
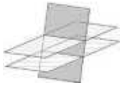
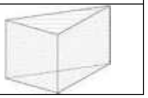
(ii) has no solutions.

For no solutions: $p = q$ and $r \neq -2$ ✓

(iii) inifinite number of solutions.

For infinite solutions: $p = q$ and $r = -2$ ✓

(c) Use your answers in part (b) to determine which of the following sketches describes the geometrical situation when the system has no solutions.

Sketch A	Sketch B	Sketch C	Sketch D
			

Sketch C (for $p = q = -1$ and $r \neq -2$) ✓
Sketch D (for $p = q \neq -1$ and $r \neq -2$) ✓

Calculator Free

7. [8 marks: 4, 2, 2]

Planes A, B and C respectively have equations:
$x + y - z = -2$ $x - y + z = 6$ $2x + y + pz = q$
where p and q are real constants.

(a) Determine the values of p and q if these three planes meet at more than one point.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 1 & -1 & 1 & 6 \\ 2 & 1 & p & q \end{array}\right)$$

$$\begin{array}{r} \\ R1-R2 \\ 2R1-R3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 2 & -2 & -8 \\ 0 & 1 & -2-p & -4-q \end{array}\right) \quad ✓✓$$

$$\begin{array}{r} \\ \\ R2-2R3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 2 & -2 & -8 \\ 0 & 0 & 2+2p & 2q \end{array}\right) \quad ✓$$

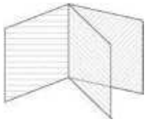
For more than one solution $p = -1$ and $q = 0$. ✓

(b) For the case where these planes meet more than once, provide two sets of <u>*positive integer*</u> solutions.

If $y = k$, $z = k + 4$ and $x = 2$
Two sets of positive integer solutions are:
$x = 2, y = 1, z = 5$ and $x = 2, y = 2, z = 6$ ✓✓

(c) For the case where these planes meet more than once, draw a sketch of the relative positions of these planes.

No two planes are parallel as the normal vectors are all distinct.
Hence:



✓✓

Calculator Free

8. [9 marks: 5, 4]

The planes P, Q and R respectively have equations:

$$x + y - z = 1$$
$$x - y - z = 1$$
$$2x - y + pz = q$$

where p and are real constants.

(a) Determine the value(s) of p and q if these planes form the lateral faces of a triangular prism.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 2 & -1 & p & q \end{array}\right)$$

$$\begin{array}{l} \\ R1-R2 \\ 2R1-R3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & -2-p & 2-q \end{array}\right) \qquad \checkmark\checkmark$$

$$R2\div 2\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & -2-p & 2-q \end{array}\right) \qquad \checkmark$$

$$\begin{array}{l} \\ \\ 3R2-R3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2+p & -2+q \end{array}\right) \qquad \checkmark$$

To form the lateral faces of a triangular prism:

$p = -2$ and $q \neq 2$ ✓

(b) Determine the value(s) of p and q if these planes meet along a common line. Determine the vector equation of this line.

Planes meet along a common line:

$p = -2$ and $q = 2$ ✓

Parametric equation of line:

$y = 0$

$x - z = -2$ ✓

Let $x = t \quad \Rightarrow \quad z = t + 2$

Hence: $x = t$

$y = 0$

$z = t + 2$ ✓

$$\text{Hence: } \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \checkmark$$

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16 Differentiation

Calculator Free

1. [12 marks: 3, 3, 3, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = 5^x$

$$y = 5^x = e^{ln\,5^x} = e^{x\,ln\,5} \qquad \checkmark$$
$$\Rightarrow \frac{dy}{dx} = ln\,5 \times e^{x\,ln\,5} \qquad \checkmark$$
$$= 5^x\,ln\,5 \qquad \checkmark$$

(b) $y = \sin^5(1 - \sqrt{x})$

$$\frac{dy}{dx} = \underset{\checkmark}{5 \times [\sin^4(1 - \sqrt{x})]} \times \underset{\checkmark}{[\cos(1 - \sqrt{x})]} \times \underset{\checkmark}{\left(-\frac{1}{2\sqrt{x}}\right)}$$

(c) $y = \int_0^{e^{2x}} ln\,(1 - x^2)\,dx$

$$\frac{dy}{dx} = \underset{\checkmark}{2}\,\underset{\checkmark}{e^{2x}}\,ln\,[1 - \underset{\checkmark}{\left(e^{2x}\right)^2}]$$

(d) $y = \dfrac{1-x}{\tan 3x}$

$$\frac{dy}{dx} = \frac{\overset{\checkmark}{(-1)\times\tan 3x} - \overset{\checkmark}{(1-x)(\sec^2 3x)\times 3}}{(\tan 3x)^2} \qquad \checkmark$$

Calculator Free

2. [10 marks: 1, 3, 3, 3]

Find $\frac{dy}{dt}$ for each of the following. You do not need to simplify your answer.

(a) $y = \pi^3$

$$\boxed{\frac{dy}{dt} = 0 \qquad \checkmark}$$

(b) $y = \tan^3(\pi t^4)$

$$\boxed{\frac{dy}{dt} = 3 \times \underset{\checkmark}{[\tan^2(\pi t^4)]} \times \underset{\checkmark}{[\sec^2(\pi t^4)]} \times \underset{\checkmark}{4\pi t^3}}$$

(c) $y = \int_1^{t^3} \sin^5 2x \, dx + t \int_0^1 5 \, dt$

$$\boxed{\frac{dy}{dt} = \underset{\checkmark}{[\sin^5(2t^3)]} \times \underset{\checkmark}{3t^2} + \underset{\checkmark}{5}}$$

(d) $y = \frac{\cos(2 - e^{2t})}{t}$

$$\boxed{\frac{dy}{dt} = \frac{\overset{\checkmark}{(t)\times[-\sin(2-e^{2t})]\times(-2e^{2t})} - \overset{\checkmark}{[\cos(2-e^{2t})]\times 1}}{(t)^2} \quad \checkmark}$$

Calculator Free

3. [7 marks: 1, 2, 2, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \tan(60^o)$

$$\boxed{\frac{dy}{dx} = 0 \qquad \checkmark}$$

(b) $y = \tan(1 - \sqrt{x})$

$$\boxed{\frac{dy}{dx} = \underset{\checkmark}{[\sec^2(1 - \sqrt{x})]} \times \underset{\checkmark}{\left(-\frac{1}{2\sqrt{x}}\right)}}$$

(c) $y = \int_0^{\pi x} 1 + \cos^4(t) \, dt$

$$\boxed{\frac{dy}{dx} = \underset{\checkmark}{[1 + \cos^4(\pi x)]} \times \underset{\checkmark}{\pi}}$$

(d) $y = x^2 \, ln\,(\sin 2x)$

$$\boxed{\frac{dy}{dx} = \underset{\checkmark}{2x\, ln\,(\sin 2x)} + x^2 \times \underset{\checkmark}{\frac{2\cos 2x}{\sin 2x}} \quad \checkmark}$$

Calculator Free

4. [9 marks: 1, 3, 2, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = ln\ e^{2x}$

$$\boxed{\begin{array}{l} y = 2x \\ \frac{dy}{dx} = 2 \qquad \checkmark \end{array}}$$

(b) $y = \cos^3(3 + \frac{1}{x})$

$$\boxed{\frac{dy}{dx} = 3\,[\underset{\checkmark}{\cos^2(3 + \tfrac{1}{x})}] \times [\underset{\checkmark}{-\sin(3 + \tfrac{1}{x})}] \times \underset{\checkmark}{-\frac{1}{x^2}}}$$

(c) $y = \int_0^{\tan x} e^{1+t^2}\, dt$

$$\boxed{\frac{dy}{dx} = \underset{\checkmark}{e^{1+\tan^2 x}} \times \underset{\checkmark}{\sec^2 x}}$$

(d) $y = e^{\sin x} \cos x$

$$\boxed{\frac{dy}{dx} = e^{\sin x} \times \underset{\checkmark}{[-\sin x]} + \underset{\checkmark}{[\cos x \times e^{\sin x}]} \times \cos x \qquad \checkmark}$$

Calculator Free

5. [9 marks: 1, 3, 2, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = ln\ 2^x$

$$\boxed{\begin{array}{l} y = x\ ln\ 2 \\ \frac{dy}{dx} = ln\ 2 \qquad \checkmark \end{array}}$$

(b) $y = sin^5(1 + ln\ x)$

$$\boxed{\frac{dy}{dx} = 5\,[\underset{\checkmark}{\sin^4(1 + ln\ x)}] \times [\underset{\checkmark}{\cos(1 + ln\ x)}] \times \underset{\checkmark}{\frac{1}{x}}}$$

(c) $y = \int_0^{x^2} \tan(1+2t)\, dt$

$$\boxed{\frac{dy}{dx} = \underset{\checkmark}{\tan(1 + 2x^2)} \times \underset{\checkmark}{2x}}$$

(d) $y = (1 + x^2)\, ln\ \sqrt{x+1}$

$$\boxed{\frac{dy}{dx} = \underset{\checkmark}{2x\ ln\ \sqrt{x+1}} + (1 + x^2) \times \underset{\checkmark}{\frac{1}{2(x+1)}} \qquad \checkmark}$$

Calculator Free

6. [12 marks: 2, 3, 4, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \sqrt{3x}$

$$\boxed{\begin{aligned} y &= (\sqrt{3})\sqrt{x} \\ \frac{dy}{dx} &= \frac{\sqrt{3}}{2\sqrt{x}} \quad \checkmark\checkmark \end{aligned}}$$

(b) $y = e^{\tan(1-2x)}$

$$\boxed{\frac{dy}{dx} = \underset{\checkmark}{\sec^2(1-2x)} \times \underset{\checkmark}{(-2)} \times \underset{\checkmark}{e^{\tan(1-2x)}}}$$

(c) $y = \sin^2(2x)\cos^3(1-x)$

$$\boxed{\frac{dy}{dx} = \underset{\checkmark}{[2\sin(2x)} \times \underset{\checkmark}{\cos(2x) \times 2]} \times \cos^3(1-x) + \sin^2(2x) \times \underset{\checkmark}{[3\cos^2(1-x)} \times \underset{\checkmark}{-\sin(1-x)] \times -1]}}$$

(d) $y = \frac{\sin(2x)}{ln\cos(3x)}$

$$\boxed{\frac{dy}{dx} = \frac{\overset{\checkmark}{[2\cos(2x)]} \times ln\cos(3x) - \sin(2x) \times \overset{\checkmark}{\frac{-3\sin(3x)}{\cos(3x)}}}{[ln\cos(3x)]^2} \quad \checkmark}$$

Calculator Free

7. [10 marks: 1, 3, 3, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \frac{1}{(1+e)^2}$

$$\boxed{\frac{dy}{dx} = 0 \qquad \checkmark}$$

(b) $y = \frac{ln(1+\sin x)}{x}$

$$\boxed{\frac{dy}{dx} = \frac{\overset{\checkmark}{x \times \frac{\cos x}{1+\sin x}} \overset{\checkmark}{- ln(1+\sin x)}}{\underset{\checkmark}{x^2}}}$$

(c) $y = e^{\tan(\frac{\pi x}{4})}\cos(\frac{\pi x}{4})$

$$\boxed{\frac{dy}{dx} = e^{\tan(\frac{\pi x}{4})} \overset{\checkmark}{\times \frac{\pi}{4} \times -\sin(\frac{\pi x}{4})} + \overset{\checkmark}{\frac{\pi}{4} \times \sec^2(\frac{\pi x}{4})} \, e^{\tan(\frac{\pi x}{4})} \cos(\frac{\pi x}{4}) \quad \checkmark}$$

(d) $y = \frac{\sin^2(\pi x)}{\cos(1+x)}$

$$\boxed{\frac{dy}{dx} = \frac{\overset{\checkmark}{\cos(1+x)\ 2\pi\sin(\pi x)\cos(\pi x)} + \overset{\checkmark}{\sin(1+x)\ \sin^2(\pi x)}}{\cos^2(1+x)} \quad \checkmark}$$

Calculator Assumed

8. [10 marks: 3, 4, 3]

Find $\frac{dy}{dx}$ in terms of x, for each of the following.

(a) $x = t^2$ and $y = e^{t^3}$

$$\frac{dx}{dt} = 2t$$
$$\frac{dy}{dt} = 3t^2\, e^{t^3} \quad ✓$$
$$\frac{dy}{dx} = \frac{3t^2\, e^{t^3}}{2t} \quad ✓$$
$$= \frac{3t\, e^{t^3}}{2} = \frac{\pm 3\sqrt{x}\, e^{\pm x^{3/2}}}{2} \quad ✓$$

(b) $x = \cos 2\theta$ and $y = \sin 2\theta$

$$\frac{dx}{d\theta} = -2\sin 2\theta \quad ✓$$
$$\frac{dy}{d\theta} = 2\cos 2\theta \quad ✓$$
$$\frac{dy}{dx} = -\frac{x}{y} \quad ✓$$
$$= \pm\frac{x}{\sqrt{1-x^2}} \quad ✓$$

(c) $x = 1 + t$ and $y = \frac{1-t}{1+t}$

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = \frac{-(1+t)-(1-t)}{(1+t)^2} = \frac{-2}{(1+t)^2} \quad ✓$$
$$\frac{dy}{dx} = \frac{-2}{(1+t)^2} \quad ✓$$
$$= -\frac{2}{x^2} \quad ✓$$

Calculator Assumed

9. [11 marks: 3, 4, 4]

Find $\frac{dy}{dx}$ in terms of x, for each of the following.

(a) $x = t^2$ and $y = \ln(1-t)$

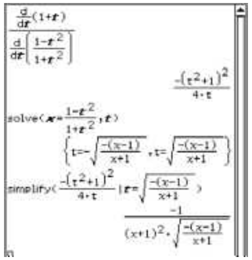
$$\frac{dx}{dt} = 2t$$
$$\frac{dy}{dt} = \frac{-1}{(1-t)} \quad ✓$$
$$\frac{dy}{dx} = \frac{-1}{2t(1-t)} \quad ✓$$
$$= \frac{\mp 1}{2\sqrt{x}\,(1\pm\sqrt{x})} \quad ✓$$

(b) $x = 1 + \cos\theta$ and $y = 2 - \sin\theta$

$$\frac{dx}{d\theta} = -\sin\theta \quad ✓$$
$$\frac{dy}{d\theta} = -\cos\theta \quad ✓$$
$$\frac{dy}{dx} = \frac{x-1}{2-y} \quad ✓$$
$$= \frac{x-1}{\pm\sqrt{1-(x-1)^2}} \quad ✓$$

(c) $x = \frac{1-t^2}{1+t^2}$ and $y = 1 + t$

$$\frac{dx}{dt} = \frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2} \quad ✓$$
$$\frac{dy}{dt} = 1 \quad ✓$$
$$\frac{dy}{dx} = \frac{-(1+t^2)^2}{4t} \quad ✓$$
$$= \pm\frac{\sqrt{1+x}}{(1+x)^2\sqrt{1-x}} = \pm\frac{1}{(1+x)^{3/2}\sqrt{1-x}} \quad ✓$$



$$\frac{\frac{d}{dt}(1+t)}{\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)}$$
$$\frac{-(t^2+1)^2}{4\cdot t}$$
$$\text{solve}\left(x=\frac{1-t^2}{1+t^2},t\right)$$
$$\left\{t=-\sqrt{\frac{-(x-1)}{x+1}},t=\sqrt{\frac{-(x-1)}{x+1}}\right\}$$
$$\text{simplify}\left(\frac{-(t^2+1)^2}{4\cdot t}\,\middle|\,t=\sqrt{\frac{-(x-1)}{x+1}}\right)$$
$$\frac{-1}{(x+1)^2\cdot\sqrt{\frac{-(x-1)}{x+1}}}$$

Calculator Assumed

10. [11 marks: 3, 4, 4]

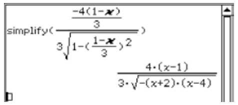
Find $\frac{dy}{dx}$ in terms of x, for each of the following.

(a) $x = e^{2t}$ and $y = \ln(1 + t)$

$$\frac{dx}{dt} = 2e^{2t}$$
$$\frac{dy}{dt} = \frac{1}{(1+t)} \quad \checkmark$$
$$\frac{dy}{dx} = \frac{1}{2e^{2t}(1+t)} \quad \checkmark$$
$$= \frac{1}{x(2 + \ln x)} \quad \checkmark$$

(b) $x = 1 - 3\sin\theta$ and $y = 3 + 4\cos\theta$

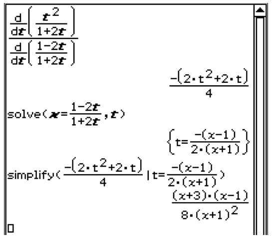
$$\frac{dx}{d\theta} = -3\cos\theta \quad \checkmark$$
$$\frac{dy}{d\theta} = -4\sin\theta \quad \checkmark$$
$$\frac{dy}{dx} = \frac{-4\left(\frac{1-x}{3}\right)}{\pm 3\sqrt{1-\left(\frac{1-x}{3}\right)^2}} \quad \checkmark$$
$$= \pm\frac{4(1-x)}{3\sqrt{(x-2)(4-x)}} \quad \checkmark$$



$$\text{simplify}\left(\frac{\frac{-4(1-x)}{3}}{3\sqrt{1-\left(\frac{1-x}{3}\right)^2}}\right)$$
$$\frac{4\cdot(x-1)}{3\cdot\sqrt{-(x+2)\cdot(x-4)}}$$

(c) $x = \frac{1-2t}{1+2t}$ and $y = \frac{t^2}{1+2t}$.

$$\frac{dx}{dt} = \frac{-4}{(1+2t)^2} \quad \checkmark$$
$$\frac{dy}{dt} = \frac{2t^2+2t}{(1+2t)^2} \quad \checkmark$$
$$\frac{dy}{dx} = \frac{-(2t^2+2t)}{4} \quad \checkmark$$
$$= \frac{(x+3)(x-1)}{8(1+x)^2} \quad \checkmark$$



$$\frac{\frac{d}{dt}\left(\frac{t^2}{1+2t}\right)}{\frac{d}{dt}\left(\frac{1-2t}{1+2t}\right)}$$
$$\frac{-(2\cdot t^2+2\cdot t)}{4}$$
$$\text{solve}\left(x=\frac{1-2t}{1+2t}, t\right)$$
$$\left\{t=\frac{-(x-1)}{2\cdot(x+1)}\right\}$$
$$\text{simplify}\left(\frac{-(2\cdot t^2+2\cdot t)}{4} \,\Big|\, t=\frac{-(x-1)}{2\cdot(x+1)}\right)$$
$$\frac{(x+3)\cdot(x-1)}{8\cdot(x+1)^2}$$

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17 Implicit Differentiation

Calculator Free

1. [10 marks: 3, 3, 4] [TISC]

A curve has equation given by $x^2 + xy = y^2 - 5$.

(a) Find $\frac{dy}{dx}$ in terms of x and y.

$$2x + \left(y + x\frac{dy}{dx}\right) = 2y\frac{dy}{dx} \quad \checkmark\checkmark$$
$$\frac{dy}{dx} = \frac{y+2x}{2y-x} \quad \checkmark$$

(b) The tangent to the curve at the point P (p, q) is parallel to the line $y = x + 5$. Show that $q = 3p$.

Since tangent is parallel to $y = x + 5$, $\frac{dy}{dx} = 1$. ✓

Hence, $2x + y = 2y - x \Rightarrow y = 3x$. ✓

Therefore, when $x = p$, $y = 3p \Rightarrow q = 3p$. ✓

(c) Hence, or otherwise, find the coordinates of the point(s) on the curve with gradient 1.

Since $y = 3x$: $x^2 + 3x^2 = 9x^2 - 5$ ✓

$x = \pm 1$. ✓

Hence, points are (1, 3) and (–1, –3). ✓✓

---

2. [11 marks: 5, 6]

(a) Given $\left(1+xy^2\right)^2 + \cos(x + y) = 0$, find $\frac{dy}{dx}$.

$$\overset{\checkmark}{2(1+xy^2)}\times\overset{\checkmark}{\left(y^2+2xy\frac{dy}{dx}\right)} - \overset{\checkmark}{\left(1+\frac{dy}{dx}\right)\sin(x+y)} = 0$$
$$2y^2(1+xy^2) + 4xy(1+xy^2)\frac{dy}{dx} - \sin(x+y) - \frac{dy}{dx}\sin(x+y) = 0$$
$$\frac{dy}{dx} = \frac{\sin(x+y) - 2y^2(1+xy^2)}{4xy(1+xy^2) - \sin(x+y)} \quad \checkmark\checkmark$$

Calculator Free

2. (b) Consider the curve with equation $2y^3 - 3y^2 - 3x^2 - 12x = 9$. Find the equation of the tangent to the curve that is parallel to the y-axis.

$6y^2\frac{dy}{dx} - 6y\frac{dy}{dx} - 6x - 12 = 0$ ✓

$\frac{dy}{dx} = \frac{6x+12}{6y^2 - 6y}$ ✓

When tangent is parallel to the y-axis, $\frac{dy}{dx} \to \infty$.

Hence, $6y^2 - 6y = 0 \Rightarrow y = 0$ or 1 ✓✓

When $y = 0$, $-3x^2 - 12x = 9 \Rightarrow x = -1, -3$. ✓

$y = 1$, $-3x^2 - 12x = 10 \Rightarrow x = -2 \pm \frac{\sqrt{6}}{3}$. ✓

Hence, tangents have equations: $x = -1$, $x = -3$, $x = -2 \pm \frac{\sqrt{6}}{3}$

---

3. [7 marks: 4, 3] [TISC]

(a) Given that $y = \sqrt{\frac{1+2x}{1+x^2}}$, use *logarithmic differentiation* to find an expression for $\frac{dy}{dx}$ in terms of x and y.

$\ln y = \frac{1}{2}[\ln(1 + 2x) - \ln(1 + x^2)]$ ✓

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{2}{1+2x} - \frac{2x}{1+x^2}\right]$ ✓✓

$\frac{dy}{dx} = y\left[\frac{1}{1+2x} - \frac{x}{1+x^2}\right]$ ✓

(b) Find the equation of the line which is *perpendicular to the tangent* to this curve at the point (2, 1).

At (2, 1), gradient of tangent $\frac{dy}{dx} = -\frac{1}{5}$. ✓

Hence, perpendicular line has gradient $m = 5$. ✓

$\Rightarrow$ Equation of perpendicular line is $y = 5x - 9$. ✓

Calculator Free

4. [7 marks: 4, 3]

(a) Given that $y = \sqrt{\frac{1+\cos(x)}{1-\sin(x)}}$, use *logarithmic differentiation* to find an expression for $\frac{dy}{dx}$ in terms of x and y.

$\ln y = \frac{1}{2}[\ \ln(1 + \cos x) - \ln(1 - \sin x)]$ ✓

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{-\sin x}{1+\cos x} + \frac{\cos x}{1-\sin x}\right]$ ✓✓

$\frac{dy}{dx} = \frac{y}{2}\left[\frac{-\sin x}{1+\cos x} + \frac{\cos x}{1-\sin x}\right]$ ✓

(b) Find the equation of the tangent to this curve at the point where $x = 0$.

When $x = 0$, $y = \sqrt{2}$ ✓

Gradient of tangent $\frac{dy}{dx} = \frac{\sqrt{2}}{2}$. ✓

$\Rightarrow$ Equation of tangent is $y = \frac{\sqrt{2}}{2}x + \sqrt{2}$. ✓

---

5. [8 marks: 4, 4]

A curve has equation $3x^2 + 3y^2 + 2xy - 6x - 2y = 1$.

(a) Determine the equation of the tangent to this curve at the point (0, 1).

$6x + 6y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} - 6 - 2\frac{dy}{dx} = 0$ ✓✓

At (0, 1): $\frac{dy}{dx} = 1$ ✓

Hence, equation of tangent is: $y = x + 1$ ✓

Calculator Free

5. (b) Determine the x-coordinate of the point(s) on the curve with tangents that are parallel to the line with equation $y = -x$.

From (a): $\frac{dy}{dx} = \frac{3-3x-y}{3y+x-1}$

For tangent parallel to the $y = x$: $\frac{dy}{dx} = -1$

Hence: $\frac{3-3x-y}{3y+x-1} = -1$ ✓

$\Rightarrow \quad y = x - 1$ ✓

$3x^2 + 3(x-1)^2 + 2x(x-1) - 6x - 2(x-1) = 1$ ✓

$2(x-1)^2 = 0 \Rightarrow x = 1$ ✓

---

6. [8 marks: 4, 4] [TISC]

A curve has equation $\sin(xy) = -\cos(x)$ for $0 \le x \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le y \le 0$.

(a) Find an expression for $\frac{dy}{dx}$.

$(x\frac{dy}{dx} + y)\cos(xy) = \sin(x)$

✓ ✓ ✓

$\frac{dy}{dx} = \frac{1}{x}\left(\frac{\sin(x)}{\cos(xy)} - y\right)$ ✓

(b) Show that the tangent to this curve at the point where $y = 0$ has equation $y = \frac{2}{\pi}x - 1$.

$y = 0 \Rightarrow \sin 0 = -\cos x$

$\cos x = 0$

$x = \frac{\pi}{2}$ ✓✓

At $(\frac{\pi}{2}, 0)$: $\frac{dy}{dx} = \frac{2}{\pi}$. ✓

Hence, equation of tangent is $y = \frac{2}{\pi}x - 1$. ✓

Calculator Assumed

7. [8 marks: 3, 2, 3] [TISC]

A curve has equation $\sqrt{x+y} = x$.

(a) Find an expression for $\frac{dy}{dx}$.

$\frac{1}{2}(x+y)^{-\frac{1}{2}}(1 + \frac{dy}{dx}) = 1$ ✓✓

$(1 + \frac{dy}{dx}) = 2(x+y)^{\frac{1}{2}}$

$\frac{dy}{dx} = 2(x+y)^{\frac{1}{2}} - 1$ ✓

(b) Show that the tangent to this curve at the point (2, 2) is $3x - y = 4$.

When $x = 2$, $y = 2$: $\frac{dy}{dx} = 3$. ✓

$\Rightarrow$ Equation of tangent is $y - 2 = 3(x - 2) \Rightarrow y = 3x - 4$ ✓

(c) Find the point(s) (a, b) on the curve, where a and b are *integers*, such that the gradient of the curve is 1. Justify your answer.

$\frac{dy}{dx} = 2(x+y)^{\frac{1}{2}} - 1 = 1$

$\Rightarrow \quad (x+y)^{\frac{1}{2}} = 1$ ✓

Since, x and y must be integers, possible answers are (0, 1) or (1, 0).

But (0, 1) is not on the curve. ✓

(1, 0) is on the curve. $\Rightarrow a = 1, b = 0$ ✓

---

8. [7 marks: 4, 3] [TISC]

(a) Consider the curve with equation $\ln(y+1) = x\,y$, determine $\frac{dy}{dx}$.

$\frac{1}{y+1}\frac{dy}{dx} = x\frac{dy}{dx} + y$ ✓✓

$\frac{dy}{dx}\left[\frac{1}{y+1} - x\right] = y$ ✓

$\frac{dy}{dx} = \frac{y(y+1)}{1 - x(y+1)}$ ✓

Calculator Assumed

8. (b) Find the equation of the line passing through the point (1, 1) and parallel to the tangent to this curve at the point ($ln\,2$, 1).

Gradient $m = \frac{2}{1-2\,ln\,2}$. ✓

Hence, equation of tangent is

$$y - 1 = \frac{2}{1-2\,ln\,2}\,(x-1) \quad ✓$$

$$y = \frac{2x}{1-2\,ln\,2} - \left(\frac{1+2\,ln\,2}{1-2\,ln\,2}\right) \quad ✓$$

---

9. [7 marks]

Consider the curve with equation $x^2 + xy + y^2 = \frac{75}{4}$. Determine the point of intersection of the tangents to this curve at the points where $x = \frac{5}{2}$.

When $x = \frac{5}{2}$:

$$y^2 + \frac{5y}{2} - \frac{25}{2} = 0$$

$$\Rightarrow\ y = \frac{5}{2} \text{ and } -5$$

Hence points are: $(\frac{5}{2}, \frac{5}{2})$ and $(\frac{5}{2}, -5)$. ✓✓

$$x^2 + xy + y^2 = \frac{75}{4}$$

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \quad ✓$$

$$\frac{dy}{dx} = \frac{-(2x+y)}{(x+2y)} \quad ✓$$

For $(\frac{5}{2}, \frac{5}{2})$: $\frac{dy}{dx} = -1$

Hence, equation of tangent is $x + y = 5$. ✓

For $(\frac{5}{2}, -5)$: $\frac{dy}{dx} = 0$

Hence, equation of tangent is $y = -5$. ✓

Hence, tangents meet at (10, –5). ✓

Calculator Assumed

10. [11 marks: 4, 3, 4]

Consider the curve with equation $y^2 = \frac{x^4}{x^2-1}$.

(a) Determine $\lim\limits_{x \to \pm\infty} y$. Hence, find the equation of the oblique asymptotes.

Using polynomial division: $y^2 = x^2 + 1 + \frac{1}{x^2-1}$ ✓

$$\Rightarrow\ y = \pm\sqrt{\left(x^2+1+\frac{1}{x^2-1}\right)} \quad ✓$$

Hence: $\lim\limits_{x\to\infty} y = \pm x$ and $\lim\limits_{x\to-\infty} y = \pm x$ ✓

$\Rightarrow$ Oblique asymptotes are $y = \pm x$ ✓

(b) Use differentiation to verify that $(\sqrt{2}, 2)$ is a minimum point on this curve.

$$2y\frac{dy}{dx} = \frac{4x^3(x^2-1) - x^4(2x)}{(x^2-1)^2} \quad ✓$$

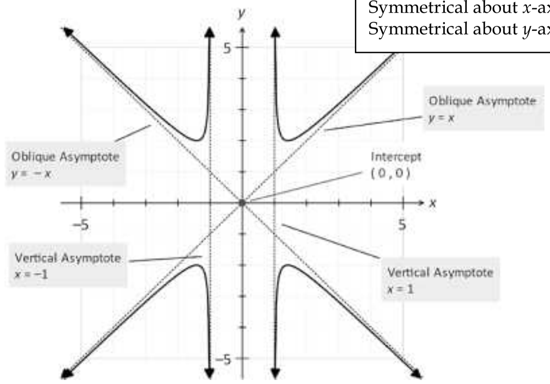
When $x = \sqrt{2}$, $y = 2$: $\frac{dy}{dx} = 0$. ✓

$\left.\frac{dy}{dx}\right|_{x=\sqrt{2}^-} < 0$ and $\left.\frac{dy}{dx}\right|_{x=\sqrt{2}^+} > 0$ ✓

Hence, $(\sqrt{2}, 2)$ is a minimum point.

(c) Sketch the curve $y^2 = \frac{x^4}{x^2-1}$.

Point (0, 0) ✓
Asymptotes ✓
Symmetrical about x-axis ✓
Symmetrical about y-axis ✓

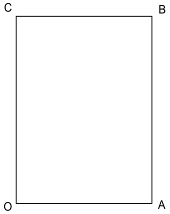


18 Related Rates

Calculator Assumed

1. [5 marks:]

Rectangle OABC has a constant area 100 mm^2. The lengths of the two parallel sides OA and CB increases at a constant rate of 1 mm per second. The lengths of the other two parallel sides OC and AB reduces at a constant rate. These changes occur in such a way that the shape of OABC remains a rectangle at all times. Find the rate of change of the perimeter of OABC when the length of the OA is 20 mm.



Let OA = x and AB = y.

Hence, Area $xy = 100 \Rightarrow y = \frac{100}{x}$. ✓

Perimeter P = $2x + 2y = 2x + \frac{200}{x}$ ✓

$\frac{dP}{dt} = (2 - \frac{200}{x^2}) \times \frac{dx}{dt}$. ✓

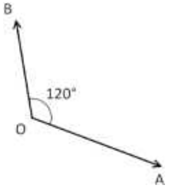
When $x = 20$, $\frac{dx}{dt} = 1$: $\Rightarrow$ $\frac{dP}{dt} = \frac{3}{2}$ mm s^{-1}. ✓

Perimeter increases at a rate of $\frac{3}{2}$ mm s^{-1}. ✓

---

2. [5 marks]

Two straight roads OA and OB, are inclined at an angle of 120^o to each other. At time $t = 0$, cyclist P is at a point on OA 50 metres from O while cyclist Q is at O. P moves away from O, along the road OA at a speed of 10 ms^{-1}. Q moves away from O, along the road OB at a speed of 12 ms^{-1}. Calculate the rate at which the distance between P and Q is changing after 5 seconds.



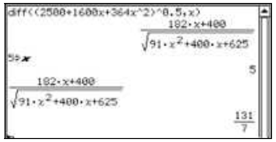
$s^2 = (50 + 10t)^2 + (12t)^2 - 2(50 + 10t)(12t) \cos 120^o$ ✓

$s = \sqrt{2500 + 1600t + 364t^2}$ ✓

$\frac{ds}{dt} = \frac{1}{2}(2500 + 1600t + 364t^2)^{-1/2}(1600 + 728t)$. ✓

When $t = 5$, $\frac{ds}{dt} = \frac{131}{7}$ ms^{-1}. ✓

P and Q are moving apart at $\frac{131}{7}$ ms^{-1}. ✓



diff((2500+1600x+364x^2)^0.5,x)

$\frac{182\cdot x+400}{\sqrt{91\cdot x^2+400\cdot x+625}}$

5⇒x

5

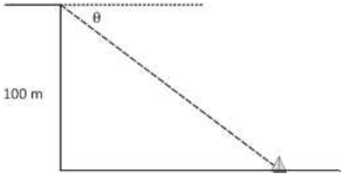
$\frac{182\cdot x+400}{\sqrt{91\cdot x^2+400\cdot x+625}}$

$\frac{131}{7}$

Calculator Assumed

3. [6 marks] [TISC]

From the top of a 100 m high vertical cliff, the angle of depression of a yacht is θ radians. The yacht is moving away from the cliff at a speed of 10 metres per minute. Find exactly the rate of change of the angle of depression when the yacht is 200 m away from the base of the cliff.



Let x: distance from yacht to base of cliff

$\tan \theta = \frac{100}{x}$ ✓

$\sec^2\theta \frac{d\theta}{dt} = -\frac{100}{x^2}\frac{dx}{dt}$ ✓✓

When $x = 200$, $\frac{dx}{dt} = 10$, $\sec \theta = \frac{\sqrt{5}}{2}$. ✓

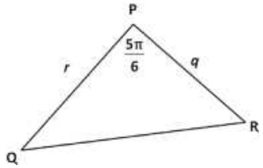
$\Rightarrow \frac{5}{4}\frac{d\theta}{dt} = -\frac{100}{200^2}(10)$ ✓

$\frac{d\theta}{dt} = -\frac{1}{50}$ rad/min. ✓

---

4. [8 marks: 4, 4]

Triangle PQR is such that PR = q and PQ = r. Angle QPR is a fixed angle and is of size $\frac{5\pi}{6}$ radians. The sides PQ and PR vary with time t. Let A be the area of triangle PQR.



(a) Find an expression for $\frac{dA}{dt}$.

$A = \frac{1}{2} \times r \times q \times \sin \frac{5\pi}{6}$ ✓

$= \frac{1}{4}qr$ ✓

$\frac{dA}{dt} = \frac{1}{4}\left[q\frac{dr}{dt} + r\frac{dq}{dt}\right]$ ✓✓

Calculator Assumed

4. (b) Given that when PQ = 5 cm , PQ is decreasing at a rate of 0.1 cm s^{-1}, and when PR = 8 cm, PR is increasing at a rate of 0.2 cm s^{-1}, find the rate at which the area of triangle PQR is changing.

$r = 5,\ \frac{dr}{dt} = -0.1,\ q = 8,\ \frac{dq}{dt} = 0.2$ ✓

$\frac{dA}{dt} = \frac{1}{4}\left[q\frac{dr}{dt} + r\frac{dq}{dt}\right]$

$= \frac{1}{4}\left[8(-0.1) + 5(0.2)\right]$ ✓

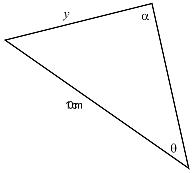
$= 0.05\ \text{cms}^{-1}$. ✓

A is increasing at a rate of 0.05 cms^{-1}. ✓

---

5. [6 marks] [TISC]

In the triangle shown, the angles α and θ vary with time t minutes. The angle α increases at the constant rate of 0.1 radians per minute. The angle θ decreases at a constant rate of 0.1 radians per minute. Find the rate of change of y with respect to time when $\alpha = \theta = \frac{\pi}{4}$.



$\frac{y}{\sin\theta} = \frac{10}{\sin\alpha} \Rightarrow y = \frac{10\sin\theta}{\sin\alpha}$ ✓

$\frac{dy}{dt} = 10\left[\frac{\sin\alpha\cos\theta\frac{d\theta}{dt} - \sin\theta\cos\alpha\frac{d\alpha}{dt}}{(\sin\alpha)^2}\right]$ ✓✓

When $\alpha = \frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$, $\frac{d\alpha}{dt} = 0.1\ \frac{d\theta}{dt} = -0.1$:

$\frac{dy}{dt} = 10\left[\frac{\sin\frac{\pi}{4}\cos\frac{\pi}{4}\times(-0.1) - \sin\frac{\pi}{4}\cos\frac{\pi}{4}\times 0.1}{\left(\sin\frac{\pi}{4}\right)^2}\right]$ ✓

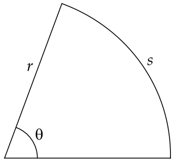
$= -2$ cm/minute ✓

That is, y is decreasing at a rate of 2 cm per minute. ✓

Calculator Assumed

6. [6 marks] [TISC]

The given diagram shows a circular sector of radius r cm and arc length s cm with a central angle of θ radians. Find the rate of change in the area at the instant in time when the radius is 5 cm and decreasing at a rate of 1 cm per second and the central angle is 1.2 radians and increasing at a rate of 0.01 radians per second.



$A = \frac{1}{2}r^2\theta$

$\frac{dA}{dt} = r\theta\frac{dr}{dt} + \frac{1}{2}r^2\frac{d\theta}{dt}$ ✓✓

$r = 5,\ \frac{dr}{dt} = -1,\ \theta = 1.2,\ \frac{d\theta}{dt} = 0.01$ ✓

Hence, $\frac{dA}{dt} = 5(1.2)(-1) + \frac{1}{2}(25)(0.01)$ ✓

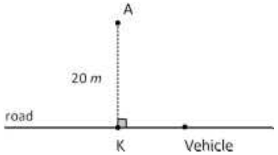
$= -5.875\ \text{cm}^2\ \text{s}^{-1}$. ✓

Area is decreasing at a rate of 5.875 $\text{cm}^2\ \text{s}^{-1}$. ✓

---

7. [6 marks]

A surveillance camera is located at A, 20 *m* from a road. The point K on the road is directly opposite to A. A vehicle travels along the road at a speed of 10 ms^{-1}. As the car travels along the road, the camera turns and follows the car. How fast is the camera turning as the car is moving away from A and the car is 40 *m* from the camera?



Let x: Horizontal distance from vehicle to K.

θ: Angle between AK and AV. ✓

$x = 20\tan\theta$. ✓

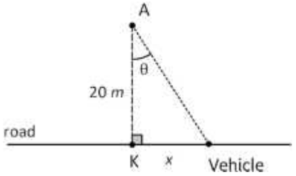
$\frac{dx}{dt} = \frac{20}{\cos^2\theta}\times\frac{d\theta}{dt}$ ✓

When AV = 40: $\cos\theta = \frac{1}{2}$, $\frac{dx}{dt} = 10$: ✓

$10 = 80\times\frac{d\theta}{dt}$

$\frac{d\theta}{dt} = 0.125$ radians per second ✓

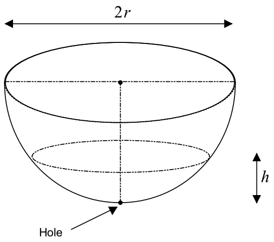
Camera is turning at 0.125 radians per second. ✓



Calculator Assumed

8. [10 marks: 4, 6] [TISC]

The given diagram shows a hemispherical bowl with diameter $2r$ cm. Water leaks out of the bowl through a hole at the bottom of the bowl. The depth of the water at any time t seconds is h cm and the volume of water in the bowl at any time t seconds is given by $V = \pi\left(rh^2 - \frac{h^3}{3}\right)$ cm^3. The rate at which water leaks out from the hole is given by $\frac{dV}{dt} = k\sqrt{h}$ where k is a constant.



(a) Show that $\frac{dh}{dt} = \frac{k}{\pi(2rh^{\frac{1}{2}} - h^{\frac{3}{2}})}$

$$V = \pi\left(r h^2 - \frac{h^3}{3}\right)$$

$$\frac{dV}{dt} = \pi\left(2r h - h^2\right)\frac{dh}{dt} \quad ✓✓$$

Hence: $k\sqrt{h} = \pi\left(2r h - h^2\right)\frac{dh}{dt}$ ✓

$$\Rightarrow \frac{dh}{dt} = \frac{k}{\pi(2rh^{\frac{1}{2}} - h^{\frac{3}{2}})} \quad ✓$$

(b) Use calculus methods to find h in terms of r when the rate at which the water level falls is a minimum.

Let $y = \frac{dh}{dt} = \frac{k}{\pi(2rh^{\frac{1}{2}} - h^{\frac{3}{2}})}$.

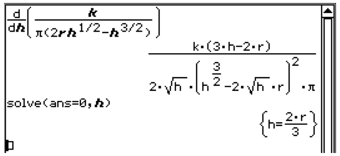
$$\frac{dy}{dh} = \frac{-k}{\pi(2rh^{\frac{1}{2}} - h^{\frac{3}{2}})^2} \times \left(\frac{r}{\sqrt{h}} - \frac{3\sqrt{h}}{2}\right) \quad ✓✓✓$$

$$\frac{dy}{dh} = 0 \Rightarrow \left(\frac{r}{\sqrt{h}} - \frac{3\sqrt{h}}{2}\right) = 0 \quad ✓$$

$$h = \frac{2r}{3} \quad ✓$$

$\left.\frac{dy}{dh}\right|_{h=\frac{2r}{3}^-} < 0$ and $\left.\frac{dy}{dh}\right|_{h=\frac{2r}{3}^+} > 0$.

$\Rightarrow$ $y = \frac{dh}{dt}$ is minimum when $h = \frac{2r}{3}$. ✓



$$\frac{d}{dh}\left(\frac{k}{\pi(2rh^{1/2}-h^{3/2})}\right)$$
$$\frac{k\cdot(3\cdot h-2\cdot r)}{2\cdot\sqrt{h}\cdot\left(h^{\frac{3}{2}}-2\cdot\sqrt{h}\cdot r\right)^2\cdot\pi}$$

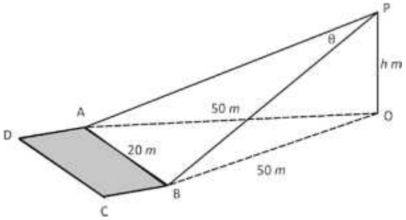
```
solve(ans=0,h)
```

$$\left\{h=\frac{2\cdot r}{3}\right\}$$

Calculator Assumed

9. [7 marks: 3, 4] [TISC]

ABCD is a rectangular garden patch on level ground, with AB = 20 m. A cameraman is located at P which is h metres above the ground. The point O is vertically below P. The cameraman is moved vertically upwards at a rate of 0.5 metres per second. The edge AB subtends an angle of θ radians at P.



(a) Use the cosine rule to show that $(2h^2 + 5000)(1 - \cos\theta) = 400$.

In ΔOBP: $BP = \sqrt{h^2+50^2}$

Similarly, $AP = \sqrt{h^2+50^2}$. ✓

In ΔABP: $AB^2 = BP^2 + AP^2 - 2 \times BP \times AP \times \cos\theta$

$20^2 = (h^2 + 2500) + (h^2 + 2500) - 2(h^2 + 2500)\cos\theta$ ✓

$400 = 2h^2 + 5000 - 2(h^2 + 2500)\cos\theta$ ✓

$(2h^2 + 5000)(1 - \cos\theta) = 400$

(b) Find the rate at which θ changes when h = 20 m.

$(2h^2 + 5000)(1 - \cos\theta) = 400$

Differentiate implicitly with respect to time t:

$4h\frac{dh}{dt}(1 - \cos\theta) + (2h^2 + 5000)\sin\theta\,\frac{d\theta}{dt} = 0$ ✓✓

When $h = 20$, $\frac{dh}{dt} = 0.5$, $\theta = 0.3736$ radians (21.40°) ✓

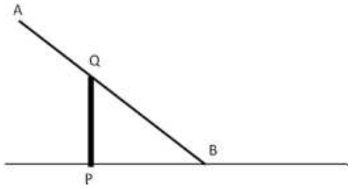
$2.7592 + 2116.6010 \times \frac{d\theta}{dt} = 0$

$\frac{d\theta}{dt} = -0.0013$ radians per second

θ is decreasing at a rate of 0.0013 radians per second ✓

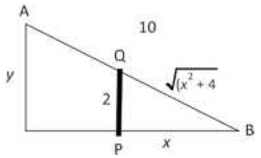
Calculator Assumed

10. [6 marks]



The diagram below shows a 10 m long pole AB resting between the top of a 2 m high wall PQ and the ground. The bottom of the pole B is being pushed towards P the foot of the wall at a constant rate of 0.1 ms^{-1}. The pole is always in contact with Q the top of the wall. Calculate how fast A, the free end of the pole, is moving vertically when B is 2 m away from P.

Let y: Vertical distance of A to the ground.
x: Horizontal distance BP. ✓
Using similar triangles:
$$\frac{y}{10} = \frac{2}{\sqrt{x^2+4}}$$ ✓
$$\frac{1}{10}\times\frac{dy}{dt} = \frac{-1}{(x^2+4)^{\frac{3}{2}}}\times 2x\frac{dx}{dt}$$ ✓



When $x = 2$, $\frac{dx}{dt} = -0.1$:
$$\frac{dy}{dt} = \frac{-20}{(4^2+4)^{\frac{3}{2}}}\times 2\times(-0.1)$$ ✓
$$\approx 0.1768$$ ✓
A is moving vertically at 0.18 ms^{-1}. ✓

Calculator Assumed

11. [8 marks: 5, 3]

Figure 1 below shows a container in the shape of a frustum of a cone.
This shape is formed when an inverted cone of height 20 cm and base diameter 20 cm is truncated parallel to the base of the cone at a height 10 cm above the vertex of the cone (see Figure 2). The frustum is of height 10 cm as shown in Figure 1. Water is poured into the container (Figure 1) at a rate of 10 cm^3 per minute.

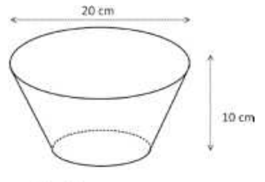


Figure 1

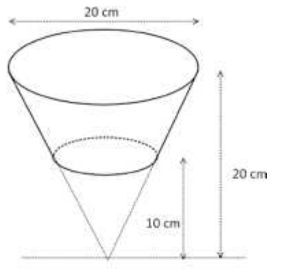


Figure 2

(a) Show that when the water level is h cm from the base of the container (Figure 1), the volume of the water in the container is given by
$$V = \frac{\pi(10+h)^3}{12} - \frac{250\pi}{3}$$

Base radius of truncated cone = 5 cm ✓
Volume of truncated cone $V_t = \frac{1}{3}\pi\times 5^2\times 10$
$= \frac{250\pi}{3}$ ✓
When height of water level is h cm,
height of "complete cone" is $(10 + h)$.
Base radius of "complete cone" is $\frac{10+h}{2}$ cm ✓
Volume of "complete cone" $V_c = \frac{1}{3}\pi\times\left(\frac{10+h}{2}\right)^2\times(10+h)$
$= \frac{\pi(10+h)^3}{12}$ ✓
Hence, volume of water $V = \frac{\pi(10+h)^3}{12} - \frac{250\pi}{3}$ ✓

Calculator Assumed

11. (b) Calculate the *exact* rate of increase of the height of the water level in the container (Figure 1) when the water level is 2 cm above its base.

$$V = \frac{\pi(10+h)^3}{12} - \frac{250\pi}{3}$$

$$\frac{dV}{dt} = \frac{3\pi(10+h)^2}{12} \times \frac{dh}{dt} \quad ✓$$

When water level is 2 cm above base, $h = 2$ cm.

$$10 = \frac{3\pi(10+2)^2}{12} \times \frac{dh}{dt} \quad ✓$$

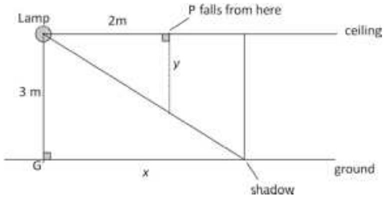
$$\frac{dh}{dt} = \frac{5}{18\pi}$$

Hence, rate of increase = $\frac{5}{18\pi}$ per minute. ✓

---

12. [5 marks] [TISC]

A light lamp is fixed to a ceiling 3 metres vertically above the point G on the ground. An object P is attached 2 metres away from the lamp to the same ceiling. P becomes loose and falls to the ground with constant acceleration. When P has fallen y metres, the shadow of P cast on the ground by the light from the lamp is x metres from G. When P has fallen a distance of 1.225 m, its speed is 4.9 ms^{-1}. Determine the speed and direction with which the shadow is moving at this instant.



Using similar triangles:

$$\frac{x}{3} = \frac{2}{y} \quad ✓$$

$$\frac{dx}{dt} = \frac{-6}{y^2} \times \frac{dy}{dt} \quad ✓$$

$$y = 1.225 \quad \frac{dy}{dt} = 4.9 \Rightarrow \frac{dx}{dt} = -19.59\ ms^{-1} \quad ✓$$

Hence, shadow is moving towards G ✓

at a speed of 19.59 ms^{-1} ✓

19 Integration I

Calculator Free

1. [10 marks: 2, 3, 5]

Find:

(a) $\int \sin 2x \cos 2x \ dx$

$$\text{I} = \frac{1}{2}\int 2\cos 2x \sin 2x \ dx \quad ✓$$

$$= \frac{1}{2}\left[\frac{(\sin 2x)^2}{2}\right] + \text{C} = \frac{(\sin 2x)^2}{4} + \text{C} \quad ✓$$

(b) $\int \frac{4\sin(x)}{\cos^3(x)\tan(x)} dx$

$$\text{I} = \int \frac{4\sin(x)}{\cos^3(x)\frac{\sin(x)}{\cos(x)}} dx \quad ✓$$

$$= \int 4\sec^2(x)\, dx \quad ✓$$

$$= 4\tan(x) + C \quad ✓$$

(c) $\int_0^2 \frac{1}{\sqrt{16-x^2}} dx$

Let $x = 4\sin\theta \Rightarrow dx = 4\cos\theta\, d\theta$. ✓

When $x = 0$, $\sin\theta = 0 \Rightarrow \theta = 0$

$x = 2$, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ ✓

$$\text{I} = \int_0^{\pi/6} \frac{1}{\sqrt{16-16\sin^2\theta}} 4\cos\theta\, d\theta$$

$$= \int_0^{\pi/6} 1\, d\theta \quad ✓$$

$$= [\,\theta\,]_0^{\pi/6} = \frac{\pi}{6} \quad ✓✓$$

Calculator Free

2. [15 marks: 2, 3, 4, 6]

Find:

(a) $\int_1^{\pi} \frac{\pi}{4}\, d\theta$

$$I = \left[\frac{\pi}{4}\theta\right]_1^{\pi} \quad ✓$$
$$= \frac{\pi^2}{4} - \frac{\pi}{4} \quad ✓$$

(b) $\int \cos^2(\pi x)\ dx$

$$I = \frac{1}{2}\int 1 + \cos 2\pi x\ \ dx \quad ✓$$
$$= \frac{1}{2}\left[x + \frac{\sin 2\pi x}{2\pi}\right] + C \quad ✓✓$$

(c) $\int \frac{\cos\theta - \cos^3\theta}{\sin^3\theta}\ d\theta$

$$I = \int \frac{\cos\theta\,(1-\cos^2\theta)}{\sin^3\theta}\ d\theta \quad ✓$$
$$= \int \frac{\cos\theta\,(\sin^2\theta)}{\sin^3\theta}\ d\theta \quad ✓$$
$$= \int \frac{\cos\theta}{\sin\theta}\ d\theta \quad ✓$$
$$= ln|\sin\theta| + C \quad ✓$$

(d) $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\ dx$

Let $x = \sin\theta \ \Rightarrow\ dx = \cos\theta\ d\theta$. ✓

When $x = 0$, $\sin\theta = 0 \ \Rightarrow\ \theta = 0$; $x = \frac{1}{2}$, $\sin\theta = \frac{1}{2} \ \Rightarrow\ \theta = \frac{\pi}{6}$ ✓

$$I = \int_0^{\pi/6} \sqrt{1-\sin^2\theta}\ \ \cos\theta\ d\theta = \int_0^{\pi/6} \cos^2\theta\ d\theta \quad ✓$$
$$= \frac{1}{2}\int_0^{\pi/6} 1 + \cos 2\theta\ \ d\theta \quad ✓$$
$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/6} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} \quad ✓✓$$

Calculator Free

3. [12 marks: 2, 3, 3, 4]

Find:

(a) $\int \sin\left(\frac{\pi t}{3} - \frac{\pi}{6}\right)\ dt$

$$I = \frac{-\cos\left(\frac{\pi t}{3} - \frac{\pi}{6}\right)}{\frac{\pi}{3}} = \frac{-3\cos\left(\frac{\pi t}{3} - \frac{\pi}{6}\right)}{\pi} + C \quad ✓✓$$

(b) $\int \frac{\sin 2x \cos 2x}{\cos^2 2x - \sin^2 2x}\, dx$

$$I = \frac{1}{2}\int \frac{\sin 4x}{\cos 4x}\ \ dx \quad ✓✓$$
$$= -\frac{1}{8}\ ln|\cos 4x| + C \quad ✓$$

(c) $\int \tan 2x + \tan^3 2x\, dx$

$$I = \int \tan 2x\,(1 + \tan^2 2x)\ dx \quad ✓$$
$$= \frac{\tan^2 2x}{2(2)} + C \quad ✓$$
$$= \frac{\tan^2 2x}{4} + C \quad ✓$$

(d) Find $\int \frac{-3}{\sqrt{9-x^2}}\ dx$

Let $x = 3\cos\theta \ \Rightarrow\ dx = -3\sin\theta\ d\theta$. ✓

$$I = \int \frac{-3}{\sqrt{9 - 9\cos^2\theta}} \times -3\sin\theta\ d\theta \quad ✓$$
$$= \int 3\ d\theta \quad ✓$$
$$= 3\ \cos^{-1}\left(\frac{x}{3}\right) + C \quad ✓$$

Calculator Free

4. [13 marks: 2, 2, 5, 4]

Find:

(a) $\displaystyle\int_0^1 \sin\frac{\pi}{6}\,dx$

$$I = \left[x\sin\frac{\pi}{6}\right]_0^1$$ ✓

$$= \sin\frac{\pi}{6} = \frac{1}{2}$$ ✓

(b) $\displaystyle\int \frac{1}{\cos^2(5x)}\,dx$

$$I = \int \sec^2(5x)\,dx$$

$$= \frac{1}{5}\tan(5x) + C$$ ✓✓

(c) $\displaystyle\int_{\frac{1}{3}}^{\frac{1}{2}} \sin(\pi x)\cos^2(\pi x)\,dx$

$$I = \frac{-1}{\pi}\int_{\frac{1}{3}}^{\frac{1}{2}} -\pi\sin(\pi x)\,[\cos(\pi x)]^2\,dx$$ ✓✓

$$= \frac{-1}{\pi}\left[\frac{\cos^3(\pi x)}{3}\right]_{\frac{1}{3}}^{\frac{1}{2}}$$ ✓

$$= \frac{-1}{\pi}\left[\frac{-1}{24}\right] = \frac{1}{24\pi}$$ ✓✓

(d) $\displaystyle\int \frac{1-2\sin^2 x}{(\sin x-\cos x)^2}\,dx$

$$I = \int \frac{1-2\sin^2 x}{\sin^2 x+\cos^2 x+2\sin x\cos x}\,dx$$ ✓

$$= \int \frac{\cos 2x}{1+\sin 2x}\,dx$$ ✓✓

$$= \frac{1}{2}\,ln\,(1+\sin 2x) + C$$ ✓

$$I = \int \frac{\cos^2 x-\sin^2 x}{(\sin x-\cos x)^2}\,dx$$ ✓

$$= \int \frac{\cos x+\sin x}{(\sin x-\cos x)}\,dx$$ ✓✓

$$= ln\,(\sin x - \cos x) + C$$ ✓

Calculator Free

5. [13 marks: 3, 4, 6]

Determine each of the following integrals.

(a) $\displaystyle\int \sin(\pi x)+\tan^2(x)\,dx$.

$$I = \int \sin(\pi x) - 1 + [1+\tan^2(x)]\,dx$$ ✓

$$= \frac{\cos(\pi x)}{\pi} - x + \tan(x) + C$$ ✓✓

(b) $\displaystyle\int x\sqrt{1-x^2}\,dx$ using the <u>substitution</u> $x = \cos t$.

Let $x = \cos t \Rightarrow dx = -\sin t\,dt$. ✓

$$I = \int \cos t\sqrt{1-\cos^2 t}\;\;-\sin t\,dt$$

$$= -\int \cos t\,\sin^2 t\,dt$$ ✓

$$= -\left[\frac{\sin^3 t}{3}\right] + C$$ ✓

$$= -\frac{(1-x^2)^{3/2}}{3} + C$$ ✓

(c) $\displaystyle\int \sqrt{1-4x^2}\,dx$.

Let $x = \frac{1}{2}\sin\theta \Rightarrow dx = \frac{1}{2}\cos\theta\,d\theta$. ✓

$$I = \int \sqrt{1-\sin^2\theta}\;\frac{1}{2}\cos\theta\,d\theta = \frac{1}{2}\int \cos^2\theta\,d\theta$$ ✓

$$= \frac{1}{4}\int 1+\cos 2\theta\,d\theta$$ ✓

$$= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$ ✓

$$= \frac{1}{4}\left[\sin^{-1}2x + \frac{2\sin\theta\cos\theta}{2}\right] + C$$ ✓

$$= \frac{1}{4}\left[\sin^{-1}2x + 2x\sqrt{1-4x^2}\right] + C$$ ✓

Calculator Free

6. [11 marks: 3, 3, 5]

Evaluate the following integrals.

(a) $\int \left(\cos\frac{\pi x}{2} - \sin\frac{\pi x}{2}\right)\left(\cos\frac{\pi x}{2} + \sin\frac{\pi x}{2}\right)\ dx$.

$$\begin{aligned} I &= \int \cos^2\left(\frac{\pi x}{2}\right) - \sin^2\left(\frac{\pi x}{2}\right)\ dx && \checkmark \\ &= \int \cos(\pi x)\ dx && \checkmark \\ &= \frac{\sin(\pi x)}{\pi} + C && \checkmark \end{aligned}$$

(b) $\int \frac{1+\tan^2(2x)}{2+\tan(2x)}\ dx$.

$$\begin{aligned} I &= \frac{1}{2}\int \frac{2\sec^2(2x)}{2+\tan(2x)}\ dx && \checkmark\checkmark \\ &= \frac{1}{2}\ ln\left|2+\tan(2x)\right| + C && \checkmark \end{aligned}$$

(c) $\int \frac{ln\,x}{x - x\,ln\,x}\ dx$

$$\begin{aligned} &\text{Let } u = x - x\,ln\,x && \checkmark \\ &\frac{du}{dx} = [1 - (ln\,x + 1)] = -\,ln\,x\,. \\ &\Rightarrow dx = \frac{1}{-\,ln\,x}\ du && \checkmark \\ &I = \int \frac{ln\,x}{u} \times \frac{-1}{ln\,x} du && \checkmark \\ &\quad = -\int \frac{1}{u}\ du && \checkmark \\ &\quad = -\,ln\ |u| + C \\ &\quad = -\,ln\ |x(1 - ln\,x)| + C && \checkmark \end{aligned}$$

Calculator Free

7. [12 marks: 2, 4, 6]

(a) Evaluate $\int \frac{\sin x + \cos x}{\cos x}\ dx$.

$$\begin{aligned} I &= \int 1 + \frac{\sin x}{\cos x}\ dx && \checkmark \\ &= x - ln\left|\cos x\right| + C && \checkmark \end{aligned}$$

(b) Evaluate $\int (1+2\sin x)^2\ dx$.

$$\begin{aligned} I &= \int 1 + 4\sin x + 4\sin^2 x\ dx && \checkmark \\ &= \int 1 + 4\sin x + \frac{4(1-\cos 2x)}{2}\ dx && \checkmark \\ &= \int 3 + 4\sin x - 2\cos 2x\ dx \\ &= 3x - 4\cos x - \sin 2x + C && \checkmark\checkmark \end{aligned}$$

(c) Use the substitution $x = \sin\theta$ to determine $\int_0^1 x^2\sqrt{1-x^2}\ dx$

$$\begin{aligned} &x = \sin\theta \ \Rightarrow\ dx = \cos\theta\, d\theta && \checkmark \\ &x = 0 \ \Rightarrow\ \theta = 0 \quad \text{and} \quad x = 1 \ \Rightarrow\ \theta = \frac{\pi}{2} && \checkmark \\ &I = \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta} \times \cos\theta\ d\theta \\ &\quad = \int_0^{\frac{\pi}{2}} \sin^2\theta\ \cos^2\theta\ d\theta && \checkmark \\ &\quad = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\ d\theta \\ &\quad = \frac{1}{8}\int_0^{\frac{\pi}{2}} 1 - \cos 2\theta\ d\theta && \checkmark \\ &\quad = \frac{1}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} && \checkmark \\ &\quad = \frac{\pi}{16} && \checkmark \end{aligned}$$

Calculator Free

8. [10 marks: 3, 3, 4]

(a) Evaluate $\int_{0}^{\frac{\pi}{8}} \tan^2 2x \; dx$.

$$I = \int_{0}^{\frac{\pi}{12}} \sec^2(2x) - 1 \; dx \qquad ✓$$

$$= \left[\frac{\tan(2x)}{2} - x\right]_0^{\frac{\pi}{8}} \qquad ✓$$

$$= \frac{1}{2} - \frac{\pi}{8} \qquad ✓$$

(b) Evaluate $\int_{0}^{\frac{\pi}{12}} \sin^2 3x \; dx$.

$$I = \int_{0}^{\frac{\pi}{12}} \frac{1 - \cos 6x}{2} \; dx \qquad ✓$$

$$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right]_0^{\frac{\pi}{12}} \qquad ✓$$

$$= \frac{\pi}{24} - \frac{1}{12} \qquad ✓$$

(c) Use an appropriate substitution to evaluate $\int \frac{2x}{(1+x)^2} \; dx$.

Let $u = (1 + x) \Rightarrow du = dx$ ✓

$$I = 2\int \frac{u-1}{u^2} \; dx$$

$$= 2\int \frac{1}{u} - \frac{1}{u^2} \; dx \qquad ✓$$

$$= 2\,ln\,u + \frac{2}{u} + C \qquad ✓$$

$$= 2\,ln\,|1 + x| + \frac{2}{1+x} + C \qquad ✓$$

Calculator Free

9. [12 marks: 5, 7]

(a) Determine $\int \sqrt{1+\sqrt{x}} \; dx$.

Let $u = 1 + \sqrt{x}$

$dx = 2\sqrt{x}\, du = 2(u - 1)\, du$ ✓✓

$$I = \int u^{\frac{1}{2}} \times 2(u-1) \; du$$

$$= \int 2u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \; du \qquad ✓$$

$$= \frac{4u^{\frac{5}{2}}}{5} - \frac{4u^{\frac{3}{2}}}{3} + C \qquad ✓$$

$$= \frac{4(1+\sqrt{x})^{\frac{5}{2}}}{5} - \frac{4(1+\sqrt{x})^{\frac{3}{2}}}{3} + C \qquad ✓$$

(b) Determine $\frac{d}{dx}\left[e^{\cos^2(x)}\right]$. Hence, find $\int \sin 2x\left[\cos^2 x + e^{\cos^2(x)}\right] dx$.

$$\frac{d}{dx}\left[e^{\cos^2(x)}\right] = -2\sin(x)\cos(x)\; e^{\cos^2(x)} \qquad ✓$$

Hence:

$$I = \int \sin 2x \cos^2 x + \sin 2x\, e^{\cos^2(x)} \; dx$$

$$= \int 2\sin x \cos^3 x + 2\sin x \cos x\, e^{\cos^2(x)} \; dx \qquad ✓✓$$

$$= -2\int -\sin x \cos^3 x \; dx + -\int -2\sin x \cos x\, e^{\cos^2(x)} \; dx \qquad ✓✓$$

$$= -2\left(\frac{\cos^4 x}{4}\right) - \left(e^{\cos^2(x)}\right) + C$$

$$= -\left(\frac{\cos^4 x}{2}\right) - \left(e^{\cos^2(x)}\right) + C \qquad ✓✓$$

Calculator Free

10. [11 marks: 4, 7]

(a) Determine $\int \frac{x}{\sqrt{x-1}}\,dx$

Let $u = x - 1 \Rightarrow dx = du$ ✓

$I = \int \frac{u+1}{u^{\frac{1}{2}}}\,du$

$= \int u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$ ✓

$= \frac{2u^{\frac{3}{2}}}{3} + 2u^{\frac{1}{2}} + C$ ✓

$= \frac{2(x-1)^{\frac{3}{2}}}{3} + 2(x-1)^{\frac{1}{2}} + C$ ✓

(b) Determine $\frac{d}{dx}\left[e^x \sin(x)\right]$ and $\frac{d}{dx}\left[e^x \cos(x)\right]$. Hence, find $\int e^x \cos(x)\,dx$.

$\frac{d}{dx}\left[e^x \sin(x)\right] = e^x \sin(x) + e^x \cos(x)$ (A) ✓

$\frac{d}{dx}\left[e^x \cos(x)\right] = e^x \cos(x) - e^x \sin(x)$ (B) ✓

From (A): $\int\left[e^x \sin(x) + e^x \cos(x)\right] dx = e^x \sin(x) + C$

$\Rightarrow \int e^x \cos(x)\,dx = e^x \sin(x) - \int e^x \sin(x)\,dx + C$ I ✓

From (B): $\int\left[e^x \cos(x) - e^x \sin(x)\right] dx = e^x \cos(x) + D$

$\Rightarrow \int e^x \cos(x)\,dx = e^x \cos(x) + \int e^x \sin(x)\,dx + D$ II ✓

I + II

$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) + F$ ✓✓

Hence: $\int e^x \cos(x)\,dx = \frac{1}{2}\left[e^x \sin(x) + e^x \cos(x)\right] + K$ ✓

Calculator Free

11. [6 marks: 3, 3]

(a) Given that $y = a^x$, use implicit differentiation to prove that $\frac{dy}{dx} = a^x \ln a$.

$\ln y = \ln a^x = x \ln a$ ✓

$\frac{1}{y}\frac{dy}{dx} = \ln a$ ✓

$\frac{dy}{dx} = y \ln a = a^x \ln a$ ✓

(b) Hence, show how the result in (a) may be used to determine $\int 10^x\,dx$

$\frac{d}{dx}10^x = 10^x \ln 10$ ✓

Hence: $\int 10^x \ln 10\,dx = 10^x + C$ ✓

$\int 10^x\,dx = \frac{10^x}{\ln 10} + D$ ✓

---

12. [10 marks: 5, 5]

The Wallis formula for definite integrals of powers of cos x between $x = 0$ and $x = \frac{\pi}{2}$ is

$$\int_0^{\frac{\pi}{2}} \cos^n x\;dx = \begin{cases} \frac{(2)(4)(6)\ldots(n-3)(n-1)}{(1)(3)(5)\ldots(n-2)(n)} & \text{for } n=2k+1 \\ \frac{(1)(3)(5)\ldots(n-3)(n-1)}{(2)(4)(6)\ldots(n-2)(n)}\times\frac{\pi}{2} & \text{for } n=2k \end{cases}.$$

(a) Use integration to show that Wallis' formula is true for $n = 5$.

$\text{RHS} = \frac{(2)(4)}{(1)(3)} = \frac{8}{15}$ ✓

$\text{LHS} = \int_0^{\frac{\pi}{2}} \cos^5 x\;dx = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)^2 \cos x\;dx$ ✓

$= \int_0^{\frac{\pi}{2}} \cos x - 2\cos x \sin^2 x + \sin^4 x \cos x\;dx$ ✓

$= \left[\sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}} = \frac{8}{15}$ ✓✓

Hence, true for $n = 5$.

Calculator Free

12. (b) Use integration to show that Wallis' formula is true for $n = 4$.

$$\text{RHS} = \frac{(1)(3)}{(2)(4)} \times \frac{\pi}{2} = \frac{3\pi}{16} \quad ✓$$

$$\text{LHS} = \int_0^{\frac{\pi}{2}} \cos^4 x \ dx = \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 \ dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right)^2 \ dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} 1 + 2\cos 2x + \cos^2 2x \ dx \quad ✓$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 1 + 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right) \ dx \quad ✓$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2} \ dx$$

$$= \frac{1}{4}\left[\frac{3x}{2} + \sin 2x + \frac{\sin 4x}{8}\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{16} \quad ✓✓$$

Hence, true for $n = 4$.

---

20 Integration II

Calculator Free

1. [10 marks: 3, 2, 2, 3]

(a) Decompose $\dfrac{1-2x-2x^2}{(x+2)(x^2-1)}$ into its partial fractions.

$$\frac{1-2x-2x^2}{(x+2)(x^2-1)} \equiv \frac{-1}{x+2} + \frac{-\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} \quad ✓✓✓$$

(b) Determine $\dfrac{d}{dx}\left(\dfrac{1-2x-2x^2}{(x+2)(x^2-1)}\right)$.

$$\frac{d}{dx}\left(\frac{1-2x-2x^2}{(x+2)(x^2-1)}\right) = \frac{1}{(x+2)^2} + \frac{1}{2(x-1)^2} + \frac{1}{2(x+1)^2} \quad ✓✓$$

(c) Determine $\displaystyle\int \frac{1-2x-2x^2}{(x+2)(x^2-1)}\,dx$.

$$\text{I} = -\int \frac{1}{x+2}\,dx - \frac{1}{2}\int \frac{1}{x-1} + \frac{1}{x+1}\,dx$$

$$= -ln|x+2| - \frac{1}{2}ln|x^2-1| + \text{C} \quad ✓✓$$

(d) Evaluate in its simplest form the exact value of $\displaystyle\int_2^3 \frac{1-2x-2x^2}{(x+2)(x^2-1)}\,dx$.

$$\text{I} = -\frac{1}{2}\left[2\,ln|x+2| + ln|x^2-1|\right]_2^3 \quad ✓$$

$$= -\frac{1}{2}\left[(2\,ln\,5 + ln\,8) - (2\,ln\,4 + ln\,3)\right] \quad ✓$$

$$= -\frac{1}{2}ln\left(\frac{25\times 8}{16\times 3}\right) = \frac{1}{2}ln\left(\frac{6}{25}\right) \quad ✓$$

Calculator Free

2. [10 marks: 3, 2, 5]

(a) Find $\int \frac{4x}{4-x}\,dx$.

$\frac{4x}{4-x} \equiv -4 + \frac{16}{4-x}$ ✓

$I = \int -4 + \frac{16}{4-x}\,dx$

$= -4x - 16\,ln\,|4-x| + C$ ✓✓

OR

Let $u = 4 - x \;\Rightarrow\; dx = -du$ ✓

$I = -\int \frac{4(4-u)}{u}\,du$

$= -\int \frac{16}{u} - 4\,du$ ✓

$= 4u - 16\,ln\,|u| + C$

$= -4x - 16\,ln\,|4-x| + D$ ✓

(b) Find $\int \frac{4x}{4-x^2}\,dx$

$I = -2\int \frac{-2x}{4-x^2}\,dx$ ✓

$= -2\,ln|4-x^2| + C$ ✓

(c) Find $\int \frac{x^3+1}{6-x-x^2}\,dx$.

$\frac{x^3+1}{6-x-x^2} \equiv -x+1+\frac{7x-5}{6-x-x^2}$ ✓

$\equiv -x+1+\frac{7x-5}{(x+3)(2-x)}$

$\equiv -x+1+\frac{\left(\frac{-26}{5}\right)}{(x+3)}+\frac{\left(\frac{9}{5}\right)}{(2-x)}$ ✓✓

$\int \frac{x^3+1}{6-x-x^2}\,dx = -\frac{x^2}{2}+x-\frac{26}{5}ln|x+3| - \frac{9}{5}ln|2-x| + C.$ ✓✓

Calculator Free

3. [9 marks: 6, 3]

(a) Determine $\int \frac{x^2+2x-1}{x^3+x^2-x-1}\,dx$

$x^3 + x^2 - x - 1 \equiv (x-1)(x^2+2x+1)$

$\equiv (x-1)(x+1)^2$ ✓

Hence: $\frac{2x^2+x-2}{x^3+x^2-x-1} \equiv \frac{2x^2+x-2}{(x-1)(x+1)^2}$

$\frac{x^2+2x-1}{(x-1)(x+1)^2} \equiv \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$ ✓

$x^2+2x-1 \equiv A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

For $x = 1$: $2 = 4A \;\Rightarrow\; A = \frac{1}{2}$ ✓

For $x = -1$: $-2 = -2C \;\Rightarrow\; C = 1$ ✓

For $x = 0$: $-1 = \frac{1}{2} - B - 1 \;\Rightarrow\; B = \frac{1}{2}$ ✓

Therefore:

$I = \int \frac{1}{2(x-1)}+\frac{1}{2(x+1)}+\frac{1}{(x+1)^2}\,dx$

$= \frac{1}{2}ln|x-1| + \frac{1}{2}ln|x+1| - \frac{1}{(x+1)} + C$ ✓

(b) Evaluate $\int_2^3 \frac{3x^2+2x-1}{x^3+x^2-x-1}\,dx$. Simplify your answer.

$\int_2^3 \frac{3x^2+2x-1}{x^3+x^2-x-1}\,dx = \left[ln\left|x^3+x^2-x-1\right|\right]_2^3$ ✓

$= ln\,32 - ln\,9$ ✓

$= ln\left(\frac{32}{9}\right)$ ✓

Calculator Free

4. [10 marks: 6, 4]

(a) Determine $\int \frac{2x^2+x+1}{(x^2+1)(x+1)}\,dx$

$$\frac{2x^2+x+1}{(x^2+1)(x+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{(x^2+1)} \quad \checkmark$$

$$2x^2+x+1 \equiv A(x^2+1)+(Bx+C)(x+1)$$

For $x = -1$: $2 = 2A \Rightarrow A = 1$ ✓

For $x = 0$: $1 = A + C \Rightarrow C = 0$ ✓

x^2 terms: $2 = A + B \Rightarrow B = 1$ ✓

Therefore:

$$\begin{aligned} I &= \int \frac{1}{(x+1)} + \frac{x}{(x^2+1)}\,dx \\ &= ln|x+1| + \frac{1}{2}ln|x^2+1| + C \quad \checkmark\checkmark \end{aligned}$$

(b) Use your answer in (a) to find $\int \frac{4x^2+3x+1}{(x^2+1)(x+1)}\,dx$.

$$\begin{aligned} \frac{4x^2+3x+1}{(x^2+1)(x+1)} &\equiv \frac{2x^2+x+1+2(x^2+x)}{(x^2+1)(x+1)} \\ &\equiv \frac{2x^2+x+1}{(x^2+1)(x+1)} + \frac{2(x^2+x)}{(x^2+1)(x+1)} \quad \checkmark \\ &\equiv \frac{2x^2+x+1}{(x^2+1)(x+1)} + \frac{2x(x+1)}{(x^2+1)(x+1)} \\ &\equiv \frac{2x^2+x+1}{(x^2+1)(x+1)} + \frac{2}{(x^2+1)} \quad \text{for } x \neq -1 \quad \checkmark \end{aligned}$$

Hence:

$$\begin{aligned} I &= \int \frac{2x^2+x+1}{(x^2+1)(x+1)} + \frac{3x}{(x^2+1)}\,dx \\ &= ln|x+1| + \frac{1}{2}ln|x^2+1| + ln|x^2+1| + C \quad \checkmark \\ &= ln|x+1| + \frac{3}{2}ln|x^2+1| + C \quad \checkmark \end{aligned}$$

Calculator Free

5. [10 marks: 3, 7]

(a) Use the substitution $x = 2\tan\theta$ to evaluate $\int \frac{1}{x^2+4}\,dx$.

$x = 2\tan\theta \Rightarrow dx = 2(1+\tan^2\theta)\,d\theta$ ✓

Hence:

$$\begin{aligned} I &= \int \frac{1}{4\tan^2\theta+4}\,2(1+\tan^2\theta)d\theta \\ &= \int \frac{1}{2}\,d\theta \quad \checkmark \\ &= \frac{\theta}{2} + C \\ &= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \quad \checkmark \end{aligned}$$

(b) Determine $\int \frac{2x^2+3}{(x^2+4)(x-1)}\,dx$.

$$\frac{2x^2+3}{(x^2+4)(x-1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{(x^2+4)} \quad \checkmark$$

$$2x^2+3 \equiv A(x^2+4)+(Bx+C)(x-1)$$

For $x = 1$: $5 = 5A \Rightarrow A = 1$ ✓

For $x = 0$: $3 = 4A - C \Rightarrow C = 1$ ✓

x^2 terms: $2 = A + B \Rightarrow B = 1$ ✓

Therefore:

$$\begin{aligned} I &= \int \frac{1}{(x-1)} + \frac{x+1}{(x^2+4)}\,dx \\ &= \int \frac{1}{(x-1)} + \frac{x}{(x^2+4)} + \frac{1}{(x^2+4)}\,dx \quad \checkmark \\ &= ln|x+1| + \frac{1}{2}ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \quad \checkmark\checkmark \end{aligned}$$

Calculator Free

6. [10 marks: 3, 7]

(a) Use the substitution tan θ = x + 1 to evaluate $\int \frac{1}{x^2+2x+2}\,dx$.

$x = \tan\theta - 1 \Rightarrow \quad dx = (1 + \tan^2\theta)\,d\theta$ ✓

Hence:

$$\begin{aligned} \text{I} &= \int \frac{1}{(\tan\theta-1)^2+2\tan\theta}\,(1+\tan^2\theta)d\theta \\ &= \int 1\ d\theta \quad ✓ \\ &= \theta + \text{C} \\ &= \tan^{-1}(x+1) + \text{C} \quad ✓ \end{aligned}$$

(b) Determine $\int \frac{2}{x(x^2+2x+2)}\,dx$.

$$\frac{2}{x(x^2+2x+2)} \equiv \frac{A}{x} + \frac{Bx+C}{(x^2+2x+2)} \quad ✓$$

$2 \equiv A(x^2 + 2x + 2) + x(Bx + C)$

For x = 0: $2 = 2A \quad \Rightarrow \quad A = 1$ ✓

x^2 terms: $0 = A + B \quad \Rightarrow \quad B = -1$ ✓

x terms: $0 = 2A + C \quad \Rightarrow \quad C = -2$ ✓

Therefore:

$$\begin{aligned} \text{I} &= \int \frac{1}{x} - \frac{x+2}{(x^2+2x+2)}\,dx \\ &= \int \frac{1}{x} - \frac{x+1}{(x^2+2x+2)} - \frac{1}{(x^2+2x+2)}\,dx \quad ✓ \\ &= ln|x| - \frac{1}{2}ln|x^2+2x+2| - \tan^{-1}(x+1) + \text{C} \quad ✓✓ \end{aligned}$$

Calculator Free

7. [8 marks: 2, 6]

(a) Given that t = tan θ, find an expression for $\frac{d\theta}{dt}$ giving your answer in terms of t.

Differentiate implicitly with respect to t:

$$\begin{aligned} 1 &= (1+\tan^2\theta)\,\frac{d\theta}{dt} \quad ✓ \\ \frac{d\theta}{dt} &= \frac{1}{1+\tan^2\theta} \\ &= \frac{1}{1+t^2} \quad ✓ \end{aligned}$$

(b) Hence or otherwise determine $\int \frac{2x^2-x+9}{(x-1)(x^2+4)}\,dx$.

$$\frac{2x^2-x+9}{(x-1)(x^2+4)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+4)} \quad ✓$$

$2x^2 - x + 9 = A(x^2 + 4) + (Bx + C)(x - 1)$

$x = 1$: $A = 2$ ✓

Compare x^2 term: $A + B = 2 \quad \Rightarrow \quad B = 0$

Compare x term: $-B + C = -1 \quad \Rightarrow \quad C = -1$ ✓

$$\begin{aligned} \text{Hence, I} &= \int \frac{2}{(x-1)} - \frac{1}{(x^2+4)}\,dx \\ &= 2\,ln|x-1| - \int \frac{1}{x^2+4}\,dx + C \quad ✓ \end{aligned}$$

For $\int \frac{1}{(x^2+4)}\,dx$:

Let $x = 2u \quad \Rightarrow \quad dx = 2du$

$$\begin{aligned} \Rightarrow \int \frac{1}{x^2+4}\,dx &= \int \frac{2}{4u^2+4}\,du \quad ✓ \\ &= \frac{1}{2}\int \frac{1}{u^2+1}\,du \\ &= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + K \quad ✓ \end{aligned}$$

Hence:

$$\text{I} = 2\,ln|x-1| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + D$$

Calculator Free

8. [9 marks: 3, 2, 2, 2]

Consider $f(x) = \dfrac{2+x}{(x+1)^2}$.

(a) Determine $\int f(x)\,dx$.

$$\frac{2+x}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \quad ✓$$
$$= \frac{A(x+1)+B}{(x+1)^2}$$

Compare x terms: $A = 1$

Compare constants: $A + B = 2 \Rightarrow B = 1$

$$\int f(x)\,dx = \int \frac{1}{(x+1)} + \frac{1}{(x+1)^2}\,dx \quad ✓$$
$$= ln|x+1| - \frac{1}{x+1} \quad ✓$$

Evaluate where possible each of the following integrals. If the integral cannot be evaluated, explain why it cannot be evaluated.

(b) $\int_0^2 f(x)\,dx$

$$\int_0^2 f(x)\,dx = \left[ln|x+1| - \frac{1}{x+1}\right]_0^2$$
$$= ln\,3 - \frac{1}{3} - (-1) = ln\,3 + \frac{2}{3} \quad ✓✓$$

(c) $\int_{-2}^0 f(x)\,dx$

Integral cannot be evaluated ✓
as $f(x)$ is discontinuous at $x = -1$. ✓

(d) $\int_{-2}^0 f(-x)\,dx$

Let $u = -x$. $\Rightarrow$ $du = -\,dx$

$$\int_{-2}^0 f(-x)\,dx = -\int_2^0 f(u)\,du \quad ✓$$
$$= ln\,3 + \frac{2}{3} \quad ✓$$

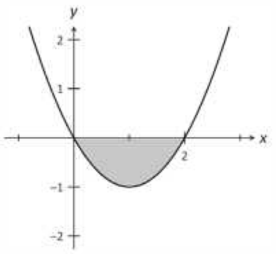
---

21 Numerical Integration

Calculator Assumed

1. [9 marks: 4, 2, 3]

The shaded region in the accompanying diagram is trapped between the curve $y = f(x)$ where $f(x) = x(x-2)$ and the x-axis. The area of this region is to be estimated using 50 trapeziums of uniform width.



(a) Calculate the area of the second trapezium, showing all working.

Width of trapezium = 0.04 ✓

Area of 2nd trapezium

$$= \left|\frac{1}{2}\left[f(0.04) + f(0.04\times 2)\right]\times 0.04\right| \quad ✓✓$$
$$= 0.00464 \quad ✓$$

(b) Write an expression for the area of the nth trapezium (for $1 \le n \le 50$).

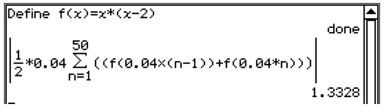
Area of nth trapezium

$$= \left|\frac{1}{2}\left[f(0.04\times(n-1)) + f(0.04\times n)\right]\times 0.04\right| \quad ✓✓$$

(c) Estimate the area of the shaded region. Show clearly how you obtained your answer.

Area of shaded region

$$= \left|\frac{1}{2}\times 0.04\times\sum_{n=1}^{50}\left[f(0.04\times(n-1)) + \ \ f(0.04\times n)\right]\right| \quad ✓✓$$
$$= 1.3328 \quad ✓$$



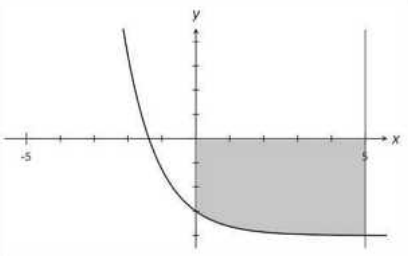
```
Define f(x)=x*(x-2)
                                        done
|1    50                                |
|-*0.04 Σ ((f(0.04×(n-1))+f(0.04*n)))  |
|2    n=1                               |
                                      1.3328
```

Calculator Assumed

2. [12 marks: 5, 3, 4]

The shaded region R in the accompanying diagram is trapped between the curve with equation $y = -4 + e^{-x}$, the x-axis, the y-axis and the line $x = 5$.

The area of R is to be estimated by using 100 trapezoidal strips of uniform width.



(a) State the area of strip n for $1 \le n \le 100$.

Width of strip = 0.05 ✓

Lengths of parallel sides of trapezium

$-4 + e^{-0.05(n-1)}$ and $-4 + e^{-0.05n}$. ✓✓

Hence, area of trapezium

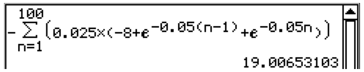
$= -\frac{1}{2} \times 0.05 \times \left[(-4 + e^{-0.05(n-1)}) + (-4 + e^{-0.05n})\right]$ ✓

$= -0.025\left[-8 + e^{-0.05(n-1)} + e^{-0.05n}\right]$ ✓

(b) Estimate the area of R correct to 4 decimal places.

$\text{Area} \approx -\sum_{n=1}^{n=100} 0.025\left[-8 + e^{-0.05(n-1)} + e^{-0.05n}\right]$ ✓✓

≈ 19.0065 ✓



```
 100
-Σ (0.025×(-8+e^-0.05(n-1)+e^-0.05n))
 n=1
                         19.00653103
```

(c) Estimate the percentage difference in the area of R if 50 trapezoidal strips of uniform width was used instead of 100 strips.

$\text{Area} \approx -\sum_{n=1}^{n=50} 0.05\left[-8 + e^{-0.1(n-1)} + e^{-0.1n}\right]$ ✓✓

≈ 19.0059 ✓

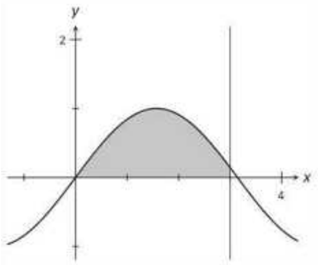
$\text{Difference} \approx \frac{19.0059 - 19.0065}{19.0059} \times 100 \approx -0.003\ \%$ ✓

Calculator Assumed

3. [8 marks: 4, 4]

The shaded region R in the accompanying diagram is trapped between the curve with equation $y = \sin(x)$, the x-axis and the line $x = 3$.

Simpson's Rule for estimating the area trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ <u>using $2n$ strips</u> of uniform width is given by the formula:



$$\text{Area} = \left(\frac{w}{3}\right) \times \left[f(a) + f(b) + 4\sum_{i=0}^{i=n-1} f\left(a + (2i+1)w\right) + 2\sum_{i=1}^{i=n-1} f\left(a + 2iw\right)\right]$$

where $w = \frac{b-a}{2n}$ is the uniform width of a strip.

(a) Apply Simpson's Rule to estimate the area of R using 2 strips. Give your answer correct to 2 decimal places.

For 2 strips: $n = 1$

$w = 1.5$ ✓

$\text{Area} \approx \left(\frac{1.5}{3}\right) \times \left[\sin(0) + \sin(3) + 4\sum_{i=0}^{i=0} \sin\left(0 + 1.5(2i+1)\right) + 2\sum_{i=1}^{i=0} \sin(0 + 1.5 \times 2i)\right]$

$\approx \left(\frac{1.5}{3}\right) \times \left[\sin(0) + \sin(3) + 4\sin(1.5) + 2 \times 0\right]$ ✓✓

$\approx 2.0655 \approx 2.07$ ✓

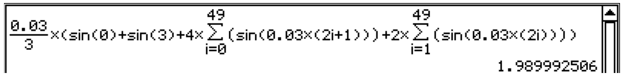
(b) Apply Simpson's Rule to estimate the area of R using 100 strips. Give your answer correct to 2 decimal places.

For 100 strips: $n = 50$

$w = 0.03$ ✓

$\text{Area} \approx \left(\frac{0.03}{3}\right) \times \left[\sin(0) + \sin(3) + 4\sum_{i=0}^{i=49} \sin\left(0 + 0.03(2i+1)\right) + 2\sum_{i=1}^{i=49} \sin(0 + 0.03 \times 2i)\right]$ ✓✓

$\approx 1.98999 \approx 1.99$ ✓



```
0.03                  49                          49
----×(sin(0)+sin(3)+4×Σ (sin(0.03×(2i+1)))+2×Σ (sin(0.03×(2i))))
 3                    i=0                         i=1
                                                      1.989992506
```

Calculator Assumed

4. [11 marks: 5, 5, 1]

Let $f(x) = e^{0.05x^2}$. Consider $I = \int_0^8 e^{0.05x^2}\,dx$.

(a) Apply the middle-box (mid-point) method using 100 rectangular strips of uniform width to estimate the value of *I* correct to 4 decimal places.

[Note: For the strip with $a \le x \le b$, the height of the strip is $f\left(\frac{a+b}{2}\right)$.]

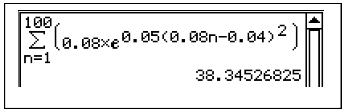
For 100 strips: width of strip = 0.08 ✓

Height of strip $n = f\left(0.04 + 0.08(n-1)\right)$

$= e^{0.05(0.08n-0.04)^2}$ ✓✓

Hence, $I \approx \sum_{n=1}^{n=100} 0.08 \times e^{0.05(0.08n-0.04)^2}$ ✓

$\approx 38.34526825 \approx 38.3453$ ✓



$\sum_{n=1}^{100}\left(0.08\times e^{0.05(0.08n-0.04)^2}\right)$

38.34526825

(b) Apply the trapezium rule using 100 trapezoidal strips of uniform width to estimate the value of *I* correct to 4 decimal places.

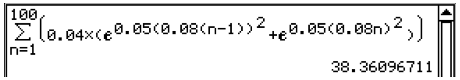
Width of strip = 0.08 ✓

Lengths of parallel sides of trapezium

$e^{0.05(0.08(n-1))^2}$ and $e^{0.05(0.08n)^2}$. ✓✓

Area $\approx \sum_{n=1}^{n=100} 0.04 \times \left[e^{0.05(0.08(n-1))^2} + e^{0.05(0.08n)^2}\right]$ ✓

$\approx 38.36096711 \approx 38.3610$ ✓



$\sum_{n=1}^{100}\left(0.04\times(e^{0.05(0.08(n-1))^2}+e^{0.05(0.08n)^2})\right)$

38.36096711

(c) Calculate the percentage difference in the value of I if the trapezium rule was used instead of the centre-box method (for 100 strips each).

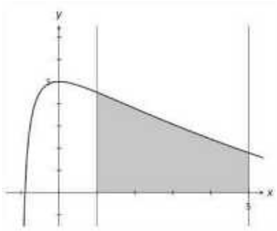
$\text{Difference} = \frac{38.3610 - 38.3453}{38.3453} \times 100$

$= 0.04\%$ ✓

Calculator Assumed

5. [13 marks: 3, 3, 1, 4, 2]

The shaded region R in the diagram below shows the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = 1$ and $x = 5$.



x	$f(x)$
1	4.52
1.2	4.38
1.4	4.23
1.6	4.09
1.8	3.94
2.0	3.79
2.2	3.65
2.4	3.5
2.6	3.36
2.8	3.22
3.0	3.08
3.2	2.94
3.4	2.80≈
3.6	2.67
3.8	2.54
4.0	2.41
4.2	2.28
4.4	2.16
4.6	2.03
4.8	1.91
5.0	1.79

The accompanying table shows the value of the function $f(x)$ for various values of x.

(a) Estimate the area of R using 20 inscribed uniform rectangular strips.

Width of strip = 0.2

Area $\approx \sum_{n=1}^{n=20} 0.2 \times \left[f(1+0.2n)\right]$

Area $\approx 0.2(f(1.2) + f(1.4) + f(1.6) + \ldots + f(5)])$ ✓✓

$\approx 0.2 \times (60.77)$

$\approx 12.154 \approx 12.2$ ✓

(b) Estimate the area of R using 20 circumscribed uniform rectangular strips.

Width of strip = 0.2

Area $\approx \sum_{n=1}^{n=20} 0.2 \times \left[f(1+0.2(n-1))\right]$

Area $\approx 0.2(f(1.0) + f(1.2) + f(1.4) + \ldots + f(4.8)])$ ✓✓

$\approx 0.2 \times (63.50)$

≈ 12.7 ✓

Calculator Assumed

5. (c) Use your answers in (a) and (b) to provide a more accurate estimate for the area of R.

$$\text{Area} \approx \frac{12.154 + 12.7}{2} \quad ✓$$
$$\approx 12.427 \approx 12.4$$

(d) Estimate the area of R using 10 uniform trapezoidal strips.

Width of strip = 0.4

$$\text{Area} \approx \sum_{n=1}^{n=10} \frac{0.4}{2} \times \left[f(1 + 0.4(n-1)) + f(1 + 0.4n) \right]$$

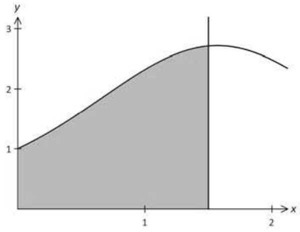
$$\text{Area} \approx \frac{0.4}{2} (f(1) + f(5) + 2\,[\,f(1.4) + f(1.8) + f(2.2) + f(2.6) + f(3) + f(3.4) + f(3.8) + f(4.2) + f(4.6)]) \quad ✓✓$$
$$\approx 0.2 \times (4.52 + 1.79 + 2(27.91)) \quad ✓$$
$$\approx 12.426 \approx 12.4 \quad ✓$$

(e) If greater accuracy is used (in terms of the number of decimal places used), determine with reasons which of the procedures in (c) or (d) will provide a more accurate estimate for the area of R?

The trapezium method and the average of the inscribed and circumscribed rectangles are mathematically equivalent.
As the procedure in (c) involves more strips, it will provide a more accurate estimate. ✓✓

Calculator Assumed

6. [9 marks: 3, 4, 2]



The shaded region R in the accompanying diagram is trapped between the curve $y = f(x)$ where $f(x) = e^{\sin x}$, the x-axis, the y-axis and the line $x = 1.5$. The area of region R is to be estimated using Simpson's rule with six strips. Simpson's Rule for estimating the area A of a region trapped between the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ using n uniform strips (where n is even) is given by:

$$A = \left(\frac{w}{3}\right) \times \left[f(x_0) + 4\left[f(x_1) + f(x_3) + f(x_{n-1}) \right] + 2\left\{ f(x_2) + 2\,f(x_4) + \ldots + 2\,f(x_{n-2}) \right\} + f(x_n) \right]$$

where w is the width of a strip, $x_0 = a$ and $x_n = b$. [Note: $f(x) \geq 0$ for $a \leq x \leq b$]

(a) Complete the table below where x_n are the x-coordinates of the boundaries of the strips. Give answers correct to 6 decimal places.

n	0	1	2	3	4	5	6
x_n	0	0.25	0.5	0.75	1	1.25	1.5
$f(x_n)$	1	1.280696	1.615146	1.977115	2.319777	2.583086	2.711481

✓ All x_n correct. ✓ At least 3 $f(x_n)$ correct. ✓ All $f(x_n)$ correct.

(b) Show use of Simpson's rule with six strips to estimate the area of region R, correct to 5 decimal places.

Width of strip = 0.25 ✓

$$A = \frac{0.25}{3} \times \left[f(x_0) + 4(f(x_1) + f(x_3) + f(x_5)) + 2(f(x_2) + f(x_4)) + f(x_6) \right] \quad ✓$$

$$A = \frac{0.25}{3} [1 + 4(1.280696 + 1.977115 + 2.583086) + 2(1.615146 + 2.319777) + 2.711481] \quad ✓$$

$$= \frac{0.25}{3} \times 34.944915 \approx 2.91208 \quad ✓$$

(b) Suggest how a more accurate estimate may be obtained by using Simpson's rule.

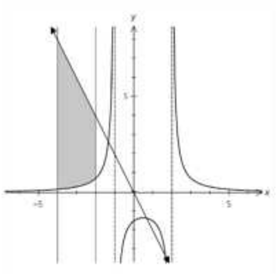
Reduce the width of the strips by increasing the number of strips ✓
with the total number of strips kept even. ✓

22 Applications of Integration

Calculator Free

1. [6 marks]

In the given diagram, the shaded region is trapped between the line $y = -2x$, the curve $y = \dfrac{3}{(x+1)(x-2)}$, and the lines $x = -4$ and $x = -2$. Determine the area of this region.



$$\text{Area} = \int_{-4}^{-2} -2x\ dx - \int_{-4}^{-2} \frac{3}{(x+1)(x-2)}\ dx \quad ✓$$

$$\int_{-4}^{-2} -2x\ dx = \left[-x^2\right]_{-4}^{-2} = 12 \quad ✓$$

$$\int_{-4}^{-2} \frac{3}{(x+1)(x-2)}\ dx = \int_{-4}^{-2} \frac{-1}{x+1} + \frac{1}{x-2}\ dx \quad ✓$$

$$= \left[\ ln|x-2| - ln|x+1|\ \right]_{-4}^{-2} \quad ✓$$

$$= (ln\,4 - ln\,1) - (ln\,6 - ln\,3) = ln\,2 \quad ✓$$

$$\text{Area} = 12 - ln\,2 \quad ✓$$

---

2. [6 marks]

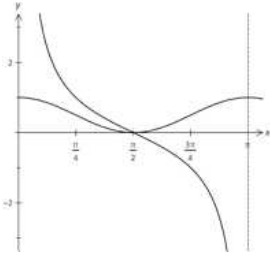
The given diagram, shows the graphs of $y = \cos^2(x)$ and $y = \cot(x)$. Calculate the area of the region trapped between the curves $y = \cos^2(x)$ and $y = \cot(x)$ and the line $x = \dfrac{3\pi}{4}$

$$\text{Area} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2 x - \cot x\ \ dx \quad ✓$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{1+\cos 2x}{2} - \frac{\cos x}{\sin x}\ dx \quad ✓✓$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4} - ln\left|\sin x\right|\ \right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \quad ✓✓$$

$$= \frac{\pi}{8} - \frac{1}{4} + \frac{1}{2}ln\,2 \quad ✓$$

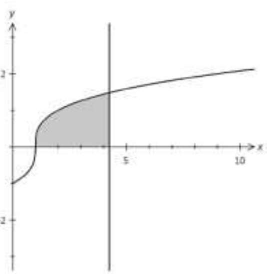


Calculator Free

3. [6 marks: 3, 3] [TISC]

The shaded region shown in the diagram given is trapped between the curve $y = \sqrt[3]{x-1}$, the x-axis and the line $x = m$. The shaded region is rotated about the x-axis to form a solid.

(a) Determine in terms of m, the volume of the solid formed.



$$\text{Volume} = \pi\int_1^m [(x-1)^{\frac{1}{3}}]^2\ \ dx \quad ✓✓$$

$$= \frac{3\pi\,(m-1)^{\frac{5}{3}}}{5} \quad ✓$$

(b) The line $x = m$ starts moving to the right starting from $x = 1$ at a speed of 2 units per minute. Calculate the rate of increase of the volume of the solid when $t = 4$ minutes.

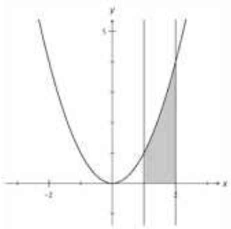
$$\frac{dV}{dt} = \pi\,(m-1)^{\frac{2}{3}} \times \frac{dm}{dt} \quad ✓$$

$$= \pi\,(9-1)^{\frac{2}{3}} \times 2 = 8\pi \quad ✓✓$$

---

4. [5 marks]

The shaded region shown in the accompanying diagram is trapped between the curve $y = x^2$, the lines $x = 1$, $x = 2$ and the x-axis. This region is rotated about the y-axis to form a solid of revolution. Determine the volume of the solid formed.

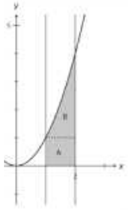


$$V_A = \pi\times 2^2 \times 1 - \pi \times 1^2 \times 1 = 3\pi \quad ✓$$

$$V_B = \pi \times 2^2 \times 3 - \pi\int_1^4 y\ dy \quad ✓$$

$$= 12\pi - \frac{\pi}{2}\left[y^2\right]_1^4 = 12\pi - \frac{15\pi}{2} \quad ✓$$

$$V = 3\pi + 12\pi - \frac{15\pi}{2} = \frac{15\pi}{2} \quad ✓✓$$



$$\text{Use } V = 2\pi\int_a^b xy\ dx \quad ✓$$

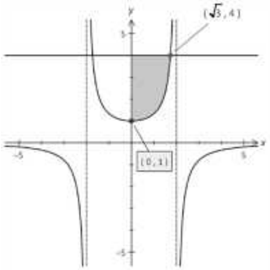
$$V = 2\pi\int_1^2 x{\times}x^2\ dx \quad ✓✓$$

$$= \frac{\pi}{2}\left[x^4\right]_1^2 = \frac{15\pi}{2} \quad ✓✓$$

Calculator Free

5. [10 marks: 6, 4]

The shaded region R in the given diagram is trapped between the curve with equation $y = \dfrac{4}{4-x^2}$, the y-axis and the line $y = 4$.



(a) Calculate the area of region R.

$$\text{Area} = 4 \times \sqrt{3} - \int_0^{\sqrt{3}} \frac{4}{4-x^2}\ dx \qquad ✓✓$$

$$\int_0^{\sqrt{3}} \frac{4}{4-x^2}\ dx = \int_0^{\sqrt{3}} \frac{1}{2+x} + \frac{1}{2-x}\ dx \qquad ✓$$

$$= \Big[ln\left|2+x\right| - ln\left|2-x\right|\Big]_0^{\sqrt{3}} \qquad ✓$$

$$= ln\left|2+\sqrt{3}\right| - ln\left|2-\sqrt{3}\right| \qquad ✓$$

$$\text{Hence, Area} = 4\sqrt{3} + ln\left|2+\sqrt{3}\right| - ln\left|2-\sqrt{3}\right| \qquad ✓$$

(b) Region R is rotated about the y-axis to form a solid of revolution. Determine the volume of this solid.

$$y = \frac{4}{4-x^2} \ \Rightarrow\ x^2 = 4 - \frac{4}{y} \qquad ✓$$

$$\text{Hence, volume } V = \pi \int_1^4 4 - \frac{4}{y}\ dy \qquad ✓$$

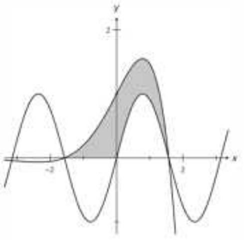
$$= \pi\Big[4y - 4\,ln\left|y\right|\Big]_1^4 \qquad ✓$$

$$= \pi\Big[12 - 4\,ln\,4\Big] \qquad ✓$$

Calculator Assumed

6. [5 marks]

The shaded region shown in the given diagram is trapped between the x-axis and the curves $y = \sin 2x$, $y = e^x \cos x$. This region is rotated about the x-axis to form a solid. Use integrals to express the volume of the solid formed. Hence, evaluate the volume of this solid correct to 2 decimal places.

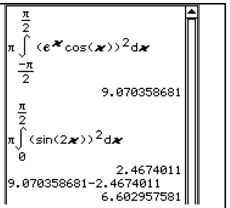


Point of intersection:

$$\sin 2x = e^x \cos x \ \Rightarrow\ x = \pm\frac{\pi}{2} \qquad ✓$$

$$V = \pi \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (e^x \cos x)^2\ dx - \pi \int_0^{\frac{\pi}{2}} (\sin 2x)^2\ dx \qquad ✓✓$$

$$= 9.07036 - 2.46740 \approx 6.60 \qquad ✓✓$$

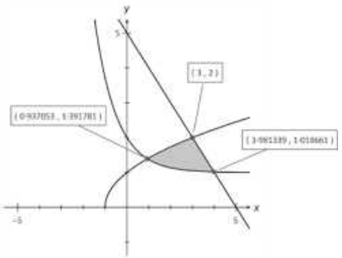


```
  π
  2
π∫ (e^x cos(x))^2dx
 -π
  2
                9.070358681
  π
  2
π∫(sin(2x))^2dx
 0
                  2.4674011
9.070358681-2.4674011
                6.602957581
```

---

7. [6 marks]

The region trapped between the curves with equations $y = \sqrt{x+1}$, $y = 1 + e^{-x}$ and $x + y = 5$ is rotated about the x-axis to form a solid of revolution.
Use integrals to express the volume of the solid formed. Hence, evaluate the volume of this solid.

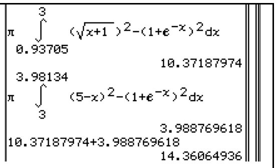


$$V = \pi \int_{0.93705}^{3} (\sqrt{x+1})^2 - (1+e^{-x})^2\ dx \qquad ✓✓$$

$$+\ \pi \int_{3}^{3.98134} (5-x)^2 - (1+e^{-x})^2\ dx \qquad ✓✓$$

$$= 10.3719 + 3.9888 \qquad ✓$$

$$= 14.3607 \qquad ✓$$

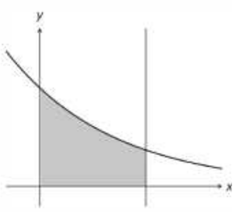


```
     3
π    ∫     (√x+1 )^2-(1+e^-x)^2dx
 0.93705
                       10.37187974
 3.98134
π    ∫     (5-x)^2-(1+e^-x)^2dx
     3
                       3.988769618
10.37187974+3.988769618
                       14.36064936
```

Calculator Assumed

8. [9 marks: 3, 2, 2, 2] [TISC]

The shaded region R in accompanying diagram is the region trapped by the curve $y = 10\,e^{kx}$, the x-axis and the lines $x = 0$ and $x = 5$.



(a) Show that the area of this region is given by Area = $\frac{10}{k}(e^{5k} - 1)$.

$$\text{Area} = \int_0^5 10\,e^{kx}\,dx \quad ✓$$
$$= \frac{10}{k}\Big[e^{kx}\Big]_0^5 \quad ✓$$
$$= \frac{10}{k}\Big[e^{5k} - 1\Big] \quad ✓$$

(b) Find the value of k if the area of R is given by Area = $50(1 - \frac{1}{e})$.

$$\frac{10}{k}\Big[e^{5k} - 1\Big] = 50(1 - \frac{1}{e})$$
$$= -50(\frac{1}{e} - 1) \quad ✓$$
$$\Rightarrow 5k = -1 \;\Rightarrow\; k = -\frac{1}{5} \quad ✓$$

(c) The area of the region trapped by the curve $y = 10\,e^{kx}$, the x-axis and the lines $x = 0$ and $x = b$ (where $b > 5$) is given by Area = $\frac{10}{k}(e^{10k} - 1)$.

(i) Find in terms of b and k, the area of the region trapped by the curve $y = 10\,e^{kx}$, the x-axis and the lines $x = 5$ and $x = b$ (where $b > 5$) .

$$\text{Area} = \frac{10}{k}\Big[e^{bk} - 1\Big] - \frac{10}{k}\Big[e^{5k} - 1\Big] \quad ✓$$
$$= \frac{10}{k}\Big[e^{bk} - e^{5k}\Big] \quad ✓$$

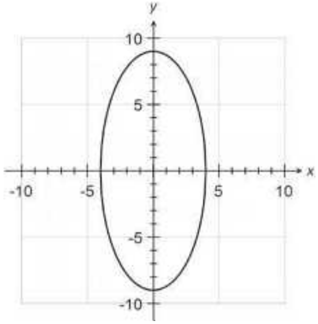
(ii) Find b.

$$\text{Area to } b = \frac{10}{k}\Big[e^{bk} - 1\Big]. \quad ✓$$
$$\text{But Area} = \frac{10}{k}\Big[e^{10k} - 1\Big].$$
$$\Rightarrow b = 10 \quad ✓$$

Calculator Assumed

9. [10 marks: 7, 3]

The accompanying diagram shows an ellipse with equation $\frac{x^2}{16} + \frac{y^2}{81} = 1$.



(a) Use integration to show that the area of the ellipse is 36π units^2. Show clearly the integrals being evaluated.

$$\frac{x^2}{16} + \frac{y^2}{81} = 1 \;\Rightarrow\; y = \pm\frac{9}{4}\sqrt{16 - x^2} \quad ✓$$
$$\text{Area} = 4 \times \frac{9}{4}\int_0^4 \sqrt{16 - x^2}\;dx = 9\int_0^4 \sqrt{16 - x^2}\;dx \quad ✓$$

Let $x = 4\sin\theta \Rightarrow dx = 4\cos\theta\; d\theta$ ✓

When $x = 0$, $\theta = 0$; when $x = 4$, $\theta = \frac{\pi}{2}$. ✓

$$\text{Area} = 9\int_0^{\pi/2} \sqrt{16 - 16\sin^2\theta} \times 4\cos\theta\, d\theta$$
$$= 144\int_0^{\pi/2} \cos^2\theta\;\; d\theta \quad ✓$$
$$= \frac{144}{2}\int_0^{\pi/2} 1 + \cos 2\theta\;\; d\theta \quad ✓$$
$$= 72\Big[\theta + \frac{\sin 2\theta}{2}\Big]_0^{\pi/2} = 36\pi. \quad ✓$$

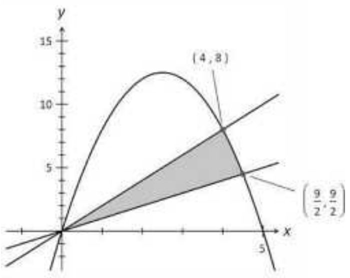
(b) Calculate the volume of the solid formed when the ellipse is rotated about the x-axis. State clearly any mathematical expression(s) used.

$$\text{Volume} = 2 \times \;\pi\int_0^4 \Big[\frac{9}{4}\sqrt{16 - x^2}\Big]^2 dx \quad ✓✓$$
$$= 2 \times 216\pi$$
$$= 432\pi \quad ✓$$

Calculator Assumed

10. [8 marks: 4, 4] [TISC]

The accompanying diagram shows the sketch of the curve $y = 2x(5 - x)$ and the lines $y = x$ and $y = 2x$.



(a) Find the area of the shaded region. Show clearly how you obtained your answer. Show clearly all the integral expressions used.

$$\text{Area} = \int_0^4 2x - x\,dx \;+\; \int_4^{9/2} 2x(5-x) - x\,dx \qquad \checkmark\checkmark\checkmark$$
$$= 8 + \frac{25}{24}$$
$$= \frac{217}{24} \qquad \checkmark$$

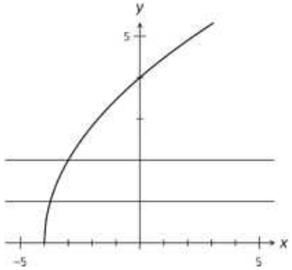
(b) The region trapped by $y = x$, $y = 2x(5 - x)$ and the x-axis is rotated 2π radians about the x-axis. Find the volume of the solid formed. Show clearly how you obtained your answer.

$$\text{Volume} = \pi\int_0^{4.5} x^2\,dx \;+\; \pi\int_{4.5}^{5} [2x(5-x)]^2\,dx \qquad \checkmark\checkmark\checkmark$$
$$= \frac{243\pi}{8} + \frac{107\pi}{30}$$
$$\approx 106.63 \qquad \checkmark$$

Calculator Assumed

11. [8 marks: 4, 4] [TISC]

(a) The accompanying diagram shows the sketch of the curve $y = 2\sqrt{x+4}$ and the lines $y = 1$ and $y = 2$. Use calculus techniques to find <u>exactly</u> the area of the region trapped between the curve $y = 2\sqrt{x+4}$, the y-axis and the lines $y = 1$ and $y = 2$.

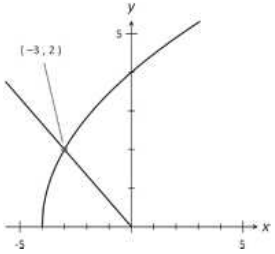


$$y = 2\sqrt{x+4} \;\Rightarrow\; x = \frac{y^2}{4} - 4 \qquad \checkmark$$

Hence:

$$\text{Area} = -\int_1^2 \frac{y^2}{4} - 4\,dy \qquad \checkmark$$
$$= -\left[\frac{y^3}{12} - 4y\right]_1^2 \qquad \checkmark$$
$$= \frac{41}{12} \qquad \checkmark$$

(b) The accompanying diagram shows the sketch of the curve $y = 2\sqrt{x+4}$ and the line $y = \frac{-2x}{3}$. The region trapped by $y = 2\sqrt{x+4}$, the y-axis and the line $y = \frac{-2x}{3}$ is rotated about the y-axis. Find the volume of the solid formed. State clearly any integrals used.



$$\text{Volume} = \frac{1}{3}\pi(3)^2 \times 2 \;+\; \pi\int_2^4 [\frac{y^2}{4} - 4]^2\,dy \qquad \checkmark\checkmark\checkmark$$
$$= 6\pi + \frac{106\pi}{15}$$
$$= \frac{196\pi}{15} \qquad \checkmark$$

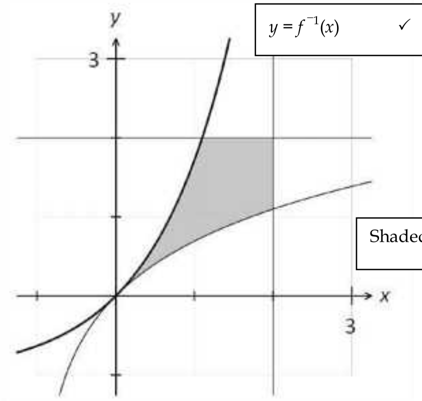
Calculator Assumed

12. [7 marks: 2, 1, 1, 3] [TISC]

(a) Use calculus to verify that $\frac{d}{dx}(x+1)\left[ln(x+1)-1\right]=ln(x+1)$

$$\frac{d}{dx}(x+1)\left[ln(x+1)-1\right]=\frac{d}{dx}\left[(x+1)\,ln(x+1)-(x+1)\right]$$
$$= ln\,(x+1) + (x+1)\frac{1}{x+1}-1 \quad ✓✓$$
$$= ln\,(x+1)$$

(b) Let $f(x) = ln(x + 1)$. The diagram below shows the graph of $y = f(x)$.



$y = f^{-1}(x)$ ✓

Shaded region. ✓

(i) Sketch on the same set of axes, the graph of the $y = f^{-1}(x)$.

(ii) On the same set of axes above, shade the region trapped by the curves $y = f(x)$, $y = f^{-1}(x)$, $x = 2$ and $y = 2$.

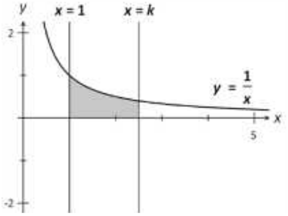
(c) Use integration, to find the area of the region described in (b). Show clearly the functions being integrated and their results. Give your answers to 3 significant figures.

$$\text{Area} = 2\times 2 - 2\times \int_0^2 ln(x+1)\ dx \quad ✓$$
$$= 4 - 2\left[(x+1)ln(x+1)-(x+1)\right]_0^2 \quad ✓$$
$$= 4 - 2\times 1.2958$$
$$= 1.4083$$
$$= 1.41 \text{ (3 SF)} \quad ✓$$

Calculator Assumed

13. [7 marks: 3, 3, 1] [TISC]

The region trapped between the curve $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = k$ where $k > 1$, is rotated 2π radians about the x-axis to form a solid.



(a) Find the exact volume of the solid formed as $k \to \infty$.

$$\text{Volume} = \pi\int_1^k\left[\frac{1}{x}\right]^2 dx \quad ✓$$
$$= \pi\left[-\frac{1}{x}\right]_1^k$$
$$= \pi\left[1-\frac{1}{k}\right] \quad ✓$$

Clearly as $k \to \infty$, Volume $\to \pi$. ✓

(b) The <u>*surface area*</u> of the solid formed is given by $S = \int_1^k \frac{2\pi}{x}\sqrt{1+\left(\frac{1}{x}\right)^4}\ dx$.

Use the relationship $\int_1^k \frac{2\pi}{x}\sqrt{1+\left(\frac{1}{x}\right)^4}\ dx > \int_1^k \frac{2\pi}{x}\,dx$ to determine the surface area of the solid formed as $k \to \infty$.

$$\int_1^k \frac{2\pi}{x}\,dx = 2\pi[ln\,x]_1^k$$
$$= 2\pi\,ln\,k \quad ✓$$

As $k \to \infty$, $\int_1^k \frac{2\pi}{x}\,dx \to \infty$. ✓

Hence, as $k \to \infty$, S $\to \infty$. ✓

(c) Comment on your answers in (b) and (c).

This solid has finite volume but infinite surface area! ✓

Calculator Assumed

14. [7 marks: 3, 4] [TISC]

Consider the curve with equation $y^3 + 4xy = 8$ for $y > 0$.

(a) Show that the tangent to the curve at the point (0, 2) has equation $y = \frac{-2x}{3} + 2$.

Differentiate implicitly:

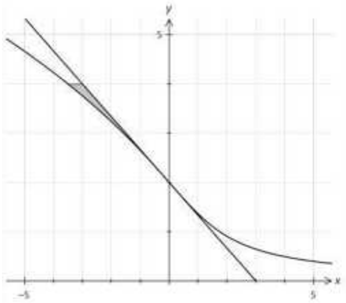
$3y^2 \frac{dy}{dx} + 4y + 4x\frac{dy}{dx} = 0$ ✓

$(0, 2) \Rightarrow \frac{dy}{dx} = \frac{-2}{3}$ ✓

Hence tangent has equation

$y = \frac{-2x}{3} + 2$. ✓

The given diagram shows the graph of $y^3 + 4xy = 8$ for $y > 0$. The diagram also shows the tangent to the curve at the point (0, 2).



(b) The shaded region is trapped between the curve $y^3 + 4xy = 8$, the tangent to the curve at (0, 2) and the lines $y = 2$ and $y = 4$. This region is rotated about the y-axis to form a solid of revolution. Show the exact volume of this solid.

$y^3 + 4xy = 8 \Rightarrow x = \frac{8 - y^3}{4y}$ ✓

$\text{Volume} = \pi \int_2^4 \left(\frac{8 - y^3}{4y}\right)^2 dy - \frac{1}{3} \times \pi \times 3^2 \times 2$ ✓✓

$= \frac{37\pi}{5} - 6\pi = \frac{7\pi}{5}$ ✓

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23 Differential Equations I

Calculator Assumed

1. [7 marks]

A tablet contains 100 mg of drug X. The amount of drug (A mg) left in a patient's body changes according to the formula $\frac{dA}{dt} = -0.08664\,A$, where t is time in hours. Jane takes the first tablet at 8.00 am and another tablet at 4.00 pm on the same day. How much of drug X (to the nearest mg) is in Jane's body at 8.00 pm on the same day?

$\int \frac{1}{A}\, dA = \int -0.08664\, dt$ ✓

$\ln A = -0.08664t + C$ ✓

$A = B\, e^{-0.08664\,t}$ ✓

When $t = 0, A = 100: \Rightarrow B = 100$ ✓

Hence, $A = 100\, e^{-0.08664\,t}$

Amount of drug at 8pm $= 100\, e^{-0.08664(12)} + 100\, e^{-0.08664(4)}$ ✓✓

$= 106.0684 \approx 106$ mg. ✓

---

2. [8 marks: 5, 1, 2] [TISC]

A scientist suspects that a colony of bacteria grows in such a way that its population growth $\frac{dP}{dt}$ is proportional to its population P at time t hours.
At $t = 5$ hours, there were 200 bacteria and at $t = 8$ hours, there were 300 bacteria.

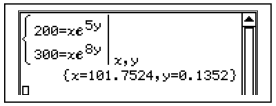
(a) Use integration to find an expression for P in terms of t.

$\frac{dP}{dt} = kP$ ✓

$\Rightarrow \int \frac{1}{P}\, dP = \int k\, dt$ ✓

$\ln P = kt + C$

$\Rightarrow P = A\, e^{kt}$ ✓



```
200=xe^5y
300=xe^8y |x,y
   {x=101.7524,y=0.1352}
```

$t = 5, P = 200 \Rightarrow 200 = A\, e^{5k}$ I

$t = 8, P = 300 \Rightarrow 300 = A\, e^{8k}$ II

II/I $e^{3k} = \frac{3}{2} \Rightarrow k = 0.1352 \quad A = 101.8 \approx 102.$ ✓

Hence, $P = 102\, e^{0.1352t}$ ✓

Calculator Assumed

2. (b) Give a reason why this model of the population growth of the bacteria cannot be correct.

This model allows for indefinite population growth which is not possible. ✓

Another scientist uses another equation, $P = \dfrac{5000}{1+e^{-t}}$ to describe P in terms of t.

(c) Explain why this is a better model than the first one.

As $t \to \infty$, $P \to 5\,000$. ✓
Hence, this model prescribes a limit to population size, which is more realistic. ✓

3. [9 marks: 6, 3] [TISC]

A room is being heated up by a new type of heater. The rate of change of the temperature of the room (θ) is given by $\dfrac{d\theta}{dt} = 0.25(\theta - 5)$ degrees Celsius per minute, where k is a constant and t is time in minutes. The manufacturer of the heater claims that the heater can raise the temperature of the room from 10^o to 22^o within 5 minutes.

(a) Use calculus methods to test if the manufacturer's claim is true. Show clearly the method you used.

$\displaystyle\int \frac{1}{\theta - 5}\,d\theta = \int 0.25\,dt$ ✓
$\ln(\theta - 5) = 0.25t + C$ ✓
$\theta = 5 + A\,e^{0.25t}$ ✓
$t = 0,\ \theta = 10 \ \Rightarrow\ A = 5$
$\theta = 5 + 5\,e^{0.25t}$ ✓
$t = 5,\ \theta = 22.45^o > 22^o$. ✓
Hence, claim is true. ✓

(b) How long will this heater take to heat a room from 15^o Celsius to 22^o Celsius? Justify your answer.

$\theta = 5 + 5\,e^{0.25t}$
$\theta = 15 \ \Rightarrow\ t = 2.7726$ min ✓
$\theta = 22 \ \Rightarrow\ t = 4.8951$ min ✓
Hence, time taken = 2.12 min. ✓

Calculator Assumed

4. [6 marks: 3, 3] [TISC]

The rate of change of the quantity of fluid in a container is given by $\dfrac{dQ}{dt} = k\,(Q + 50)$ litres per minute, where k is a constant and t is time in minutes.

(a) Use integration to show that $Q = A\,e^{k\,t} - 50$ where A is a constant.

$\displaystyle\int \frac{1}{Q+50}\,dQ = \int k\,dt$ ✓
$\ln(Q + 50) = kt + C$ ✓
$Q + 50 = e^{kt + C}$
$Q = Ae^{kt} - 50.$ ✓

(b) Find the rate of change of the quantity of fluid if $Q = 100$ litres when $t = 10$ minutes and $A = 40$.

$Q = Ae^{kt} - 50$
When $t = 10$, $A = 40$, $Q = 100$;
$100 = 40\,e^{10k} - 50$ ✓
$k = 0.1322$ ✓
Hence, $\dfrac{dQ}{dt} = k\,(Q + 50)$
$= 0.1322\,(100 + 50) = 19.83$ L/min. ✓

5. [10 marks: 3, 5, 2]

A tank contains 100 L of a salt solution with a concentration of 4 g/L. Fresh salt solution with a concentration of 10 g/L flows into the tank at a rate of 5 L per minute. The concentration of salt in the tank is kept uniform by constant stirring. The mixture flows out of the container at a rate of 5 L per minute. The amount of salt at time t minutes is S g.

(a) Show that $\dfrac{dS}{dt} = a - \dfrac{S}{b}$, giving the values of the constants a and b.

$\dfrac{dS}{dt}$ = Rate with which salt is "flowing in" – Rate with which salt is "flowing out".
$= 10 \times 5 - \dfrac{S}{100} \times 5$ ✓✓
$= 50 - \dfrac{S}{20}$. ✓

Calculator Assumed

5. (b) Hence, use integration to show that $S = m - ne^{-kt}$, giving the values of the constants *m*, *n* and *k*.

$$\frac{dS}{dt} = 50 - \frac{S}{20}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{1000 - S}{20}$$

$$\int \frac{1}{1000 - S}\, dS = \int \frac{1}{20}\, dt \qquad ✓✓$$

$$-ln\,(1000 - S) = \frac{1}{20}t + C \qquad ✓$$

$$1000 - S = A\,e^{-\frac{t}{20}} \qquad ✓$$

When $t = 0, S = 400, \Rightarrow A = 600$

$$\text{Hence,} \qquad S = 1000 - 600\,e^{-\frac{t}{20}} \qquad ✓$$

(c) Find the long term concentration of salt in the tank. Justify your answer.

As $t \to \infty,\ e^{-\frac{t}{20}} \to 0$. ✓

Hence, as $t \to \infty$, $S \to 1000$.

$\Rightarrow$ $t \to \infty$, the concentration $\to 1000/100 = 10$ g/L. ✓

---

6. [7 marks]

A tank contains salt solution with 250g of salt dissolved in 100 L of water. Another solution with a salt concentration of 2 g/L flows into the tank at a rate of 1 L/minute. The salt mixture in the tank is kept uniform by constant stirring. 1 L of the mixture flows out through a valve every minute keeping the amount of solution in the tank at a constant 100 L. The amount of salt in the tank at time *t* minutes is *S* g. Use a Calculus method to find an expression for *S* in terms of *t*.

$$\frac{dS}{dt} = \text{Inflow Rate} - \text{Outflow Rate}$$

$$= 2 - \frac{S}{100} \qquad ✓✓$$

$$\frac{dS}{dt} = 2 - \frac{S}{100} = \frac{200 - S}{100} \qquad ✓$$

$$\int \frac{1}{200 - S}\, dS = \int \frac{1}{100}\, dt \qquad ✓$$

$$-ln\,(200 - \mathrm{S}) = 0.01t + C \qquad ✓$$

$$S = 200 - A\,e^{-0.01t} \qquad ✓$$

$$S(0) = 250 \;\Rightarrow\; S = 200 + 50\,e^{-0.01t} \qquad ✓$$

Calculator Assumed

7. [9 marks: 4, 5]

(a) Given the equation $\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x}$, solve for *y* given that when $x = e, y = 1$.

$$\int \frac{1}{y^2}\, dy = \int \frac{1}{x}\, dx \qquad ✓$$

$$-\frac{1}{y} = ln\,x + \mathrm{C} \qquad ✓$$

$$x = e, y = 1 \qquad \Rightarrow \mathrm{C} = -2 \qquad ✓$$

$$-\frac{1}{y} = ln\,x - 2$$

$$\text{Hence, } y = \frac{1}{2 - ln\,x} \qquad ✓$$

(b) Given that $\frac{dy}{dx} = \frac{1+y^2}{2xy}$, find *y* in terms of *x* given that when $x = 1, y = -1$.

$$\frac{dy}{dx} = \frac{1+y^2}{2xy}$$

$$\int \frac{2y}{1+y^2}\, dy = \int \frac{1}{x}\, dx \qquad ✓$$

$$ln\,(1 + y^2) = ln\,x + \mathrm{C} = ln\,x + ln\,\mathrm{A} \qquad ✓$$

$$1 + y^2 = Ax$$

$x = 1, y = -1$:

$$\mathrm{A} = 2 \qquad ✓$$

$$\text{Hence,} \quad y^2 = 2x - 1 \qquad ✓$$

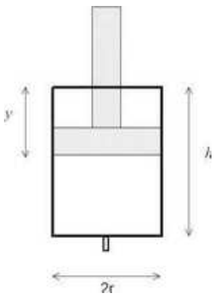
$$\Rightarrow \quad y = -\sqrt{2x-1} \qquad ✓$$

Reject $\sqrt{2x-1}$ as $x = 1, y = -1$.

Calculator Assumed

8. [10 marks: 5, 5]

The accompanying diagram is a schematic diagram of a piston within a cylindrical chamber of length h cm and fixed diameter $2r$ cm. The head of the piston is cylindrical in shape and fits exactly into the chamber. At time $t = 0$ seconds, the chamber is completely filled with fluid. Assume that the fluid is incompressible. As the piston head is forced into the chamber, fluid is forced out through a nozzle at the base of the chamber at a rate of $(h - y)$ cm^{3}s^{-1} where y cm is the distance the piston head has moved into the chamber, $0 \le y < h$.



(a) Show that the rate with which the piston is being forced into the chamber when it is 25% full is given by $\dfrac{dy}{dt} = \dfrac{h}{4\pi r^2}$.

Let V: Volume of fluid forced out of chamber at time t seconds.

$V = -\pi r^2 y \quad \Rightarrow \quad \dfrac{dV}{dt} = -\pi r^2 \dfrac{dy}{dt}$ ✓

$-(h-y) = -\pi r^2 \dfrac{dy}{dt}$ ✓

$\dfrac{dy}{dt} = \dfrac{h-y}{\pi r^2}$ ✓

When chamber is 25% full, $y = \dfrac{3h}{4}$. ✓

$\Rightarrow \quad \dfrac{dy}{dt} = \dfrac{h - \frac{3h}{4}}{\pi r^2} = \dfrac{h}{4\pi r^2}$ ✓

(b) Find y in terms of t.

$\dfrac{dy}{dt} = \dfrac{h-y}{\pi r^2}$

$\displaystyle\int \frac{1}{h-y}\,dy = \frac{1}{\pi r^2}\int 1\,dt$ ✓

$-\ln|h-y| = \dfrac{t}{\pi r^2} + C$ ✓

$h - y = A\,e^{\frac{-t}{\pi r^2}}$ ✓

When $t = 0, y = 0 \;\Rightarrow\; A = h$ ✓

Hence, $\quad y = h(1 - e^{\frac{-t}{\pi r^2}})$. ✓

Calculator Assumed

9. [9 marks: 2, 3, 4] [TISC]

Consider the differential equation $\dfrac{d^2y}{dx^2} = k\dfrac{dy}{dx}$ where k is a constant.

(a) Given $u = \dfrac{dy}{dx}$, show that $\dfrac{du}{dx} = ku$.

$u = \dfrac{dy}{dx} \;\Rightarrow\; \dfrac{du}{dx} = \dfrac{d^2y}{dx^2}$. ✓

But $\dfrac{d^2y}{dx^2} = k\dfrac{dy}{dx}$

$\Rightarrow \quad \dfrac{du}{dx} = k\dfrac{dy}{dx}$ ✓

$\dfrac{du}{dx} = ku$.

(b) Show that $u = \mathrm{A}e^{kx}$ where A is a constant.

$\dfrac{du}{dx} = ku \;\Rightarrow\; \displaystyle\int \frac{1}{u}\,du = \int k\,du$ ✓

$\ln u = kx + \mathrm{C}$ ✓

$\Rightarrow \quad u = \mathrm{A}\,e^{kx}$ ✓

(c) Find y in terms of x if when $x = 0$, $y = 2000$, $\dfrac{dy}{dx} = 1000$ and $\dfrac{d^2y}{dx^2} = 1000$.

$\dfrac{d^2y}{dx^2} = k\dfrac{dy}{dx}$.

But $\dfrac{dy}{dx} = 1000$ when $\dfrac{d^2y}{dx^2} = 1000. \;\Rightarrow\; k = 1.$ ✓

Hence, $u = \mathrm{A}\,e^{x}$.

$\Rightarrow \quad \dfrac{dy}{dx} = \mathrm{A}\,e^{x}$

But $x = 0$, $\dfrac{dy}{dx} = 1000 \;\Rightarrow\; \mathrm{A} = 1000$ ✓

Therefore, $\dfrac{dy}{dx} = 1000\,e^{x}$.

$\Rightarrow \quad y = 1000\,e^{x} + \mathrm{C}.$

$x = 0, y = 2000 \;\Rightarrow\; \mathrm{C} = 1000.$ ✓

Hence, $y = 1000(1 + e^{x})$. ✓

Calculator Assumed

10. [9 marks: 2, 6, 1]

(a) Decompose $\dfrac{1}{P(500-P)}$ into its partial fractions.

$$\frac{1}{P(500-P)} \equiv \frac{0.002}{P} + \frac{0.002}{500-P} \qquad ✓✓$$

(b) The population P of a colony of native rodents at time *t* years is modelled by the differential equation $\dfrac{dP}{dt} = 0.01P\left(1 - \dfrac{P}{500}\right)$ where $P(0) = 250$.

Use integration to show that $P = \dfrac{A}{1+Be^{-kt}}$, stating the values of *A*, *B* and *k*.

$$\frac{dP}{dt} = 0.01P\left(1-\frac{P}{500}\right) = 0.000\,02\,P\,(500-P) \qquad ✓$$

Separate the variables:

$$\int \frac{1}{P\,(500-P)}\,dP = 0.000\,02\int dt \qquad ✓$$

$$\int \frac{0.002}{P} + \frac{0.002}{500-P}\,dP = 0.000\,02\int dt$$

$$0.002\,ln\,P - 0.002\,ln\,(500-P) = 0.000\,02t + C \qquad ✓$$

$$\frac{P}{(500-P)} = A\,e^{0.01t} \qquad ✓$$

$P(0) = 250 \;\Rightarrow\; A = 1$ ✓

Hence, $\dfrac{P}{(500-P)} = e^{0.01t}$

$$P = \frac{500}{1+e^{-0.01t}} \qquad ✓$$

(c) Use your answer in (b) to discuss the long-term growth trend of this colony.

As $t \to \infty$, $P \to 500$.
Hence, maximum size of colony is 500. ✓

Calculator Assumed

11. [10 marks: 6, 2, 2]

The concentration *C* of a chemical in a solution is modelled by the equation $\dfrac{dC}{dt} = 0.000\,25C\,(200-C)$ where *t* is time in minutes. The initial concentration of the chemical is 10 g/L.

(a) Use a calculus method to determine an expression for *C* in terms of *t*.

Separate the variables:

$$\int \frac{1}{C\,(200-C)}\,dC = 0.000\,25\int 1\,dt \qquad ✓$$

$$\int \frac{0.005}{C} + \frac{0.005}{200-C}\,dC = 0.000\,25\int dt \qquad ✓$$

$$0.005\,ln\,C - 0.005\,ln\,(200-C) = 0.000\,25t + K \qquad ✓$$

$$\frac{C}{(200-C)} = A\,e^{0.05t} \qquad ✓$$

$C(0) = 10 \;\Rightarrow\; A = \dfrac{1}{19}$ ✓

Hence, $\dfrac{C}{(200-C)} = \dfrac{1}{19}e^{0.05t}$

$$\Rightarrow\; C = \frac{200e^{-0.05t}}{19+e^{0.05t}} \qquad ✓$$

$$\text{or } C = \frac{200}{1+19e^{-0.05t}}$$

(b) Determine the time (to the nearest minute) it takes for the concentration to reach half its maximum concentration.

Maximum concentration = 200 ✓

Hence, $100 = \dfrac{200}{1+19e^{-0.05t}}$

$t \approx 59$ minutes ✓

(c) Determine the time (to the nearest minute) it takes for the concentration to effectively reach its maximum concentration.

Maximum concentration = 200
Assume effective maximum concentration is 199.99.

Hence, $199.99 = \dfrac{200}{1+19e^{-0.05t}}$ ✓

$t \approx 257$ minutes ✓

[Open-ended question.]

Calculator Assumed

12. [8 marks: 6, 2]

The number of chickens on a farm infected by a fatal strain of bird virus is modelled by $\frac{dP}{dt} = 0.1P - 0.000\,1P^2$ where *t* is time in days after the detection of the virus strain among 10 chickens.

(a) Use a calculus method to determine *P* in terms of *t*.
Give your answer in the form with a constant in the numerator.

$$\frac{dP}{dt} = 0.0001P(1000 - P) \quad ✓$$

Separate the variables:

$$\int \frac{1}{P(1000-P)}\,dP = 0.0001\int dt$$

$$\int \frac{0.001}{P} + \frac{0.001}{1000-P}\,dP = 0.0001\int dt \quad ✓$$

$$0.001\ ln\,P - 0.001\ ln\,(1\,000 - P) = 0.000\,1t + C \quad ✓$$

$$\frac{P}{(1000-P)} = A\,e^{0.1t} \quad ✓$$

$$P(0) = 10 \ \Rightarrow\ A = \frac{1}{99} \quad ✓$$

Hence,

$$\frac{P}{(1000-P)} = \frac{1}{99}e^{0.1t}$$

$$P = \frac{1000e^{0.1t}}{99 + e^{0.1t}}$$

$$P = \frac{1000}{1 + 99e^{-0.1t}} \quad ✓$$

(b) All the surviving chickens on the farm were culled when the virus had spread to more than 100 chickens on the farm. After how many days did the culling occur?

Hence:

$$100 = \frac{1\,000}{1 + 99e^{-0.1t}} \quad ✓$$

$$t \approx 24 \text{ days} \quad ✓$$

Calculator Assumed

13. [7 marks: 6, 1]

The variable *Q* changes with time *t* hours, according to the differential equation $\frac{dQ}{dt} = 0.000\,01Q(2000 - Q)$. The initial value of Q, $Q_0 = 10$.

(a) Use integration to show that $Q = \frac{kQ_0}{Q_0 + (k - Q_0)e^{-rkt}}$,
stating the values of the constants *k* and *r*.

$$\frac{dQ}{dt} = 0.000\,01Q(2000 - Q)$$

$$\int \frac{1}{Q(2000-Q)}\,dQ = \int 0.000\,01\,dt \quad ✓$$

$$\frac{1}{2000}\int \frac{1}{Q} + \frac{1}{2000-Q}\,dQ = \int 0.000\,01\,dt \quad ✓$$

$$ln\,Q - ln\,(2000 - Q) = 0.02t + C \quad ✓$$

$$ln\left(\frac{2000-Q}{Q}\right) = -0.02t + D$$

$$\frac{2000-Q}{Q} = A\,e^{-0.02t} \quad ✓$$

$$Q = \frac{2000}{1 + Ae^{-0.02t}}$$

$$= \frac{2000}{1 + 199e^{-0.02t}} \quad ✓$$

$$= \frac{10 \times 2000}{10 + 1990e^{-0.000\,01 \times 2000t}}$$

$$= \frac{10 \times 2000}{10 + (2000 - 10)e^{-0.000\,01 \times 2000t}} \quad ✓$$

$$k = 2000,\ r = 0.000\,01$$

(b) Determine the limiting value for *Q*.

As $t \to \infty$, $Q \to 2\,000$. ✓

Calculator Assumed

14. [8 marks: 2, 1, 5]

$P = \dfrac{5000\,e^{0.02t}}{e^{0.02t}+49}$ is the solution to logistic differential equation $\dfrac{dP}{dt} = aP\,(1 - bP)$
where a and b are real constants.

(a) Determine with reasons the limiting value of P.

$$P = \frac{5000\,e^{0.02t}}{e^{0.02t}+49}$$

$$= \frac{5000}{1+49e^{-0.02t}} \quad ✓$$

As $t \to \infty$, $e^{-0.02t} \to 0$ and $P \to 5\,000$ ✓
Hence, limiting value of P is 5 000.

(b) State the initial value of P.

When $t = 0$, $P = \dfrac{5000}{1+49} = 100$ ✓

(c) Implicitly differentiate P to determine the values of a and b.

$$P = \frac{5000\,e^{0.02t}}{e^{0.02t}+49}$$

$$\Rightarrow\ P\,(e^{0.02t} + 49) = 5000\,e^{0.02t} \quad ✓$$

$$\frac{dP}{dt}(e^{0.02t} + 49) + P\,(0.02\,e^{0.02t}) = 5000 \times 0.02\,e^{0.02t} \quad ✓$$

$$\frac{dP}{dt}(e^{0.02t} + 49) = 5000 \times 0.02\,e^{0.02t} - P\,(0.02\,e^{0.02t})$$

$$= 0.02\,e^{0.02t}(5000 - P)$$

$$\frac{dP}{dt} = \frac{0.02\,e^{0.02t}}{e^{0.02t}+49}(5000 - P) \quad ✓$$

$$= 0.02 \times \frac{P}{5000}(5000 - P)$$

$$= 0.02\,P(1 - \frac{P}{5000}) \quad ✓✓$$

Hence: $a = 0.02,\ b = \dfrac{1}{5000}$

Calculator Assumed

15. [9 marks: 2, 3, 4]

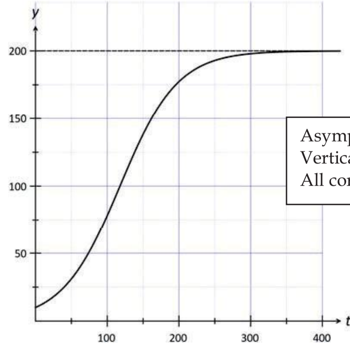
In a certain fare promotion for an airline route, the number of tickets sold may be modelled by the logistic function $y = f(t)$ with a growth rate of 0.025 tickets per minute and an initial value of 10 tickets with a limiting value of 200 tickets.

(a) State the logistic function and the associated logistic differential equation.

Logistic function: $y = \dfrac{200}{1+19e^{-0.025t}}$ ✓

Differential equation: $\dfrac{dy}{dt} = 0.025y(1 - \dfrac{y}{200})$ ✓

(b) In the axes provided below, sketch $y = f(t)$.



Asymptote ✓
Vertical intercept ✓
All correct ✓

(c) Use calculus to determine when the tickets are being sold at the fastest rate and state this rate.

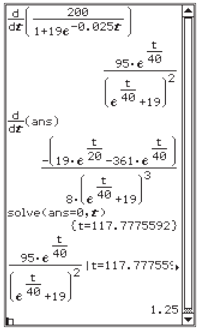
$$\frac{dy}{dt} = \frac{95\,e^{0.05t}}{(e^{0.025t}+19)^2}$$

$$\frac{d^2y}{dt^2} = \frac{-19\,e^{0.05t} + 361\,e^{0.025t}}{8(e^{0.025t}+19)^3} \quad ✓$$

$$\frac{d^2y}{dt^2} = 0 \ \Rightarrow\ t \approx 118 \text{ minutes} \quad ✓✓$$

$$\text{Rate} = \left.\frac{95\,e^{0.05t}}{(e^{0.025t}+19)^2}\right|_{t\,=\,117.7776}$$

$= 1.25$ tickets per minute. ✓



$$\frac{d}{dt}\left(\frac{200}{1+19e^{-0.025t}}\right)$$

$$\frac{95\cdot e^{\frac{t}{40}}}{\left(e^{\frac{t}{40}}+19\right)^2}$$

$$\frac{d}{dt}(\text{ans})$$

$$\frac{-\left(19\cdot e^{\frac{t}{20}}-361\cdot e^{\frac{t}{40}}\right)}{8\cdot\left(e^{\frac{t}{40}}+19\right)^3}$$

```
solve(ans=0,t)
                {t=117.7775592}
```

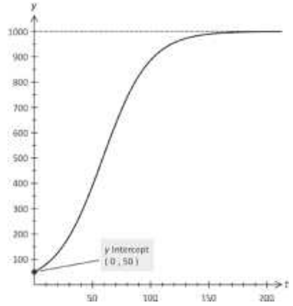
$$\frac{95\cdot e^{\frac{t}{40}}}{\left(e^{\frac{t}{40}}+19\right)^2}\,|t=117.77755$$

```
                          1.25
```

Calculator Assumed

16. [7 marks: 4, 3] [TISC]

The diagram given shows the graph of a logistic function $y = \dfrac{m}{1+ne^{-0.05t}}$.



(a) Determine with reasons the values of m and n.

From given function as $t \to \infty$, $y \to m$.
From given graph as $t \to \infty$, $y \to 1000$. ✓
Hence: $m = 1000$ ✓

From graph when $t = 0$, $y = 50$.
Hence: $50 = \dfrac{1000}{1+n}$ ✓
$\Rightarrow$ $n = 19$ ✓

(b) The graph given above is a particular solution to a first order differential equation of the form $\dfrac{dy}{dt} = f(y)$. State the differential equation and its initial condition.

$\dfrac{dy}{dt} = 0.05y\left(1 - \dfrac{1000}{y}\right)$ ✓✓ Initial condition: $t = 0, y = 50$ ✓	$\dfrac{dy}{dt} = 0.00005y(1000 - y)$ ✓✓ Initial condition: $t = 0, y = 50$ ✓

---

17. [10 marks: 1, 1, 2, 2, 2, 2] [TISC]

The variable y changes with respect to time t according to the differential equation $\dfrac{dy}{dt} = 0.05y$ with $y(0) = 1000$.

(a) Complete the table below.

y	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
$\dfrac{dy}{dt}$	50	100	150	200	250	300	350	400	450	500

✓ All correct.

(b) Hence, describe the way the $\dfrac{dy}{dt}$ changes with respect to y.

$\dfrac{dy}{dt}$ changes linearly with respect to y. ✓

Calculator Assumed

17. (c) Use your answers in parts (a) and (b) to explain clearly why as $t \to \infty$, $y \to \infty$.

The initial value of y is 1000 and the initial rate of change with respect to time, $\dfrac{dy}{dt} = 50 > 0$.
Hence, the value of y will be increasing with time. ✓
But as the rate of change $\dfrac{dy}{dt}$ increases without limit as y increases, hence, y will increase even faster without limit. ✓

To limit the growth of the variable y, a "correction factor" $C = -0.000\,005y^2$ is added to the differential equation so that it now becomes $\dfrac{dy}{dt} = 0.05y - 0.000\,005y^2$ with $y(0) = 1000$.

(d) Complete the table below for the new differential equation.

y	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
$\dfrac{dy}{dt}$	45	80	105	120	125	120	105	80	45	0

✓ ≥ 5 correct values
✓ All values correct

(e) Hence, describe the way the $\dfrac{dy}{dt}$ changes with respect to y.

$\dfrac{dy}{dt}$ increases to a maximum of 125 as y increases ✓
and then decreases until it becomes 0 when $y = 10000$. ✓

(f) Use your answers in parts (d) and (e) to explain how this "correction factor" limits the growth of y.

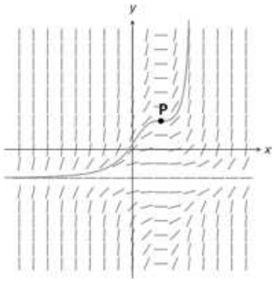
The initial value of y is 1000 and the initial rate of change with respect to time, $\dfrac{dy}{dt} = 45 > 0$.
Hence, the value of y will be increasing with time. ✓
As the rate of change $\dfrac{dy}{dt}$ increases and then finally decreases to 0, y will increase and then the increases get smaller until it stops increasing when $y = 10\,000$, the value of y now becomes constant. ✓

24 Differential Equations II

Calculator Assumed

1. [3 marks] [TISC]

The diagram given shows the slope field for a differential equation. On the diagram given draw the solution curve to the differential equation which passes through the point **P**.



- ✓ Integral curve has a horizontal inflection point at **P**.
- ✓ Integral curve has a horizontal asymptote corresponding to the isocline of gradient 0.
- ✓ Curve follows "curvature" of gradient field.

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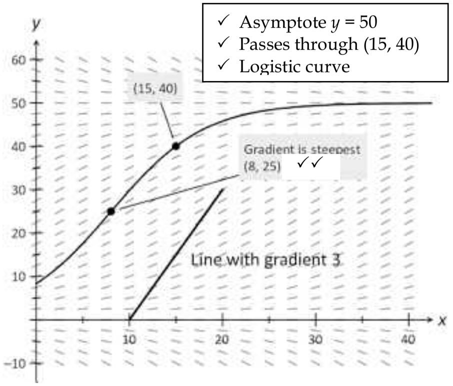
2. [9 marks: 3, 2, 2, 2]

The given diagram shows the slope field of a differential equation $\dfrac{dy}{dx} = f(x, y)$.



- ✓ Asymptote $y = 50$
- ✓ Passes through (15, 40)
- ✓ Logistic curve

(a) On the slope field given draw in the curve for the particular solution with initial condition (15, 40).

(b) Mark and state the coordinates of the point on the particular curve where the gradient is steepest.

(c) Explain why $f(x, y)$ corresponding to this slope field is independent of x.

For a given value of y, the slopes are constant. ✓
Hence, the gradients $\dfrac{dy}{dx}$ are independent of x. ✓

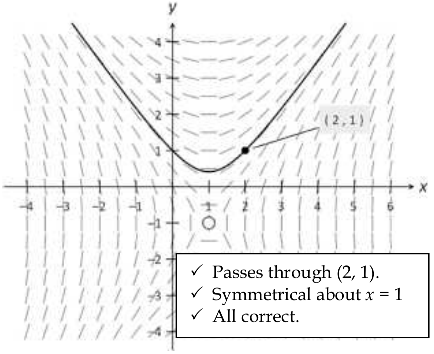
(d) Determine with reasons if there is an isocline with a gradient of 3.

Construct line with gradient 3.
From sketch drawn, there are no slope lines parallel to the line with gradient 3. ✓✓

Calculator Assumed

3. [9 marks: 3, 2, 1, 3]

The given diagram shows the slope field of a differential equation $\dfrac{dy}{dx} = f(x, y)$.



- ✓ Passes through (2, 1).
- ✓ Symmetrical about $x = 1$
- ✓ All correct.

(a) On the slope field given draw in the curve representing the particular solution with initial condition (2, 1).

(b) For the particular curve in (a), estimate the minimum value for y and the corresponding value of x.

Minimum value for $y \approx 0.4$ when $x = 1$ ✓✓

(c) State the equation of the isocline with infinite gradient

Isocline: $y = -1$ ✓

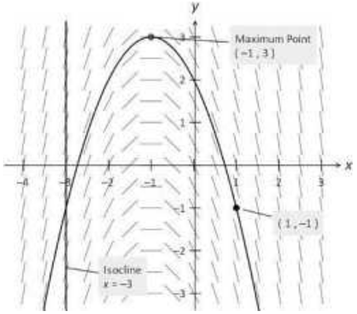
(d) Determine with reasons a possible expression for $f(x, y)$.

Gradient is 0 for $x = 1$ independent of y-values.
Gradient changes from negative to positive about $x = 1$. ✓
Gradient is infinite for $y = -1$ independent of x-values.
Gradient changes from positive to negative about $y = -1$. ✓
Hence, $\dfrac{dy}{dx} = \dfrac{k(x-1)^m}{(y+1)^n}$ where $k \in \mathbb{Z}^+$, m and n are odd positive integers. ✓

---

4. [7 marks: 2, 2, 3]

The given diagram shows the slope field of a differential equation $\dfrac{dy}{dx} = f(x, y)$.



(a) On the slope field given draw in the curve representing the particular solution with initial condition (1, −1). Determine the coordinates of the stationary point(s) on this curve.

✓ Draws a parabola through (1, −1).
Maximum point (−1, 3). ✓

Calculator Assumed

4. (b) Determine with reasons, which of the following equations best describe the differential equation.
A. $y' = -x - 1$ B. $y' = -2y - 2$ C. $y' = -2 - 2x$ D. $y' = 2x + 2$.

Slopes are the same for a given value of x.
Hence, differential equation is independent of y. ✓
Gradient is 0 at $x = -1$.
Hence differential equation is $y' = -2 - 2x$ ✓

(c) For the equation you have chosen, state the equation of the isocline with gradient 4. Draw this isocline on the diagram given.

$y' = -2 - 2x = 4 \Rightarrow x = -3$ ✓✓
✓ Draws isocline of gradient 4 ($x = -3$) ✓

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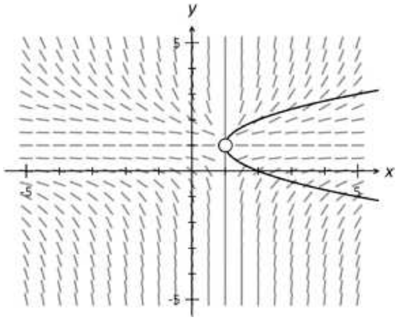
5. [8 marks: 3, 5]

The given diagram shows the slope field of a differential equation $\dfrac{dy}{dx} = f(x, y)$.



(a) Determine with reasons, which of the following equations best describe the differential equation.

A. $y' = \dfrac{(x-1)^2}{y-1}$ B. $y' = \dfrac{x-1}{y-1}$

C. $y' = \dfrac{(x+1)^2}{y-1}$ D. $y' = \dfrac{(y-1)^2}{x-1}$

Singularity at (1, 1). This eliminates C. ✓
Slopes are positive for $x > 1 \cap y < 1$.
$\Rightarrow$ This eliminates A and B. ✓
Hence, differential equation is $y' = \dfrac{(y-1)^2}{x-1}$. ✓

(b) For the equation you have chosen, state the equation of the isocline with gradient 1. Draw this isocline on the diagram given.

$y' = \dfrac{(y-1)^2}{x-1} = 1 \Rightarrow (y-1)^2 = x - 1$ ✓

$\Rightarrow y = 1 \pm \sqrt{x-1}$ where $x \neq 1 \cap y \neq 1$. ✓✓

✓✓ Isoclines drawn.

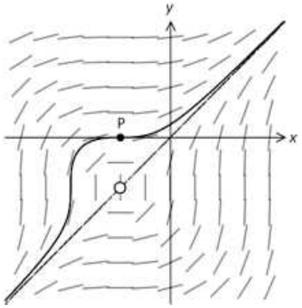
Calculator Assumed

6. [7 marks: 3, 4]

The given diagram shows the slope field for $\dfrac{dy}{dx} = f(x, y)$.

(a) Draw the integral curve passing through the point P.

(b) Suggest with reasons a possible expression for $f(x, y)$.



Let singularity be (a, b).

$f(x, y) = \dfrac{(x-a)^{2m}}{(y-b)^{2n}} + k$ ✓

- All powers are *even* as slopes are all *positive*. ✓
- Isocline of zero gradients is a vertical line, hence all x-terms in numerator. ✓
- Isocline of infinite gradients is a horizontal line, hence all y-terms in denominator. ✓

✓ Passes through P.
✓ Oblique asymptote through origin and singularity.
✓ Correct curve.

---

7. [6 marks: 1, 3, 1, 1]

The table below gives the gradients for the differential equation $\dfrac{dy}{dx} = f(x, y)$ for various values of x and y.

y x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
−2	-4	−3.5	−3	−2.5	−2	−1.5	−1	−0.5	0
−1.5	−3.5	−3	−2.5	−2	−1.5	−1	−0.5	0	0.5
−1	−3	−2.5	−2	−1.5	−1	−0.5	0	0.5	1
−0.5	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5
0	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
0.5	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5
1	−1	−0.5	0	0.5	1	1.5	2	2.5	3
1.5	−0.5	0	0.5	1	1.5	2	2.5	3	3.5
2	0	0.5	1	1.5	2	2.5	3	3.5	4

(a) State the equation of the isocline with gradient −1.

Equation is $y = -x - 1$ ✓

Calculator Assumed

7. (b) Using the scale on the axes provided, sketch the slope field.

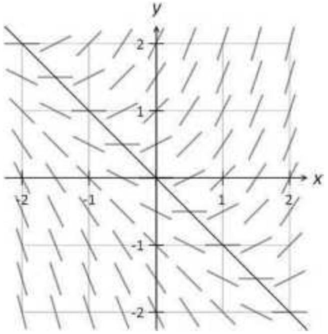
✓✓✓ Slopes correct

(c) In the diagram in (b), draw the isocline with gradient 0.

✓ Isocline $y = -x$

(d) Determine $f(x, y)$.

$f(x, y) = x + y$ ✓



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8. [3 marks]

Let $\frac{dy}{dx} = \frac{e^{x^2}}{y^2+1}$. Given that when $x = 1$, $y = 1$, use the method of small changes to estimate the value of y when $x = 1.01$. Give your answer to 5 decimal places.

For (1, 1) with $\delta x = 0.01$: $\delta y \approx \frac{e^1}{1^2+1} \times 0.01$ ✓✓

$= 0.01359$ ✓

Hence, when $x = 1.01$ $y \approx 1.01359$

---

9. [4 marks] [TISC]

Consider the differential equation $\frac{dy}{dx} = \frac{y^2+1}{x}$. Given that when $x = 1$, $y = 2$, use the incremental formula with $\delta x = 0.01$ to estimate the value of y when $x = 1.02$. Give your answer to 4 decimal places.

When $x = 1, y = 2$: $\frac{dy}{dx} = 5$

$\delta y \approx 5 \times 0.01 \approx 0.05$ ✓

$\Rightarrow$ When $x = 1.01$, $y \approx 2.05$. ✓

When $x = 1.01, y \approx 2.05$: $\frac{dy}{dx} \approx \frac{2.05^2+1}{1.01}$

$\approx 5.150\,990$

$\delta y \approx 5.150\,990 \times 0.01 \approx 0.051\,509\,90$ ✓

$\Rightarrow$ When $x = 1.02$, $y \approx 2.05 + 0.051\,509\,90$

≈ 2.1015 ✓

Calculator Assumed

10. [8 marks: 5, 3] [TISC]

A curve has equation curve $x^2y + \sqrt{3+y^2} = 3$.

(a) Find the equation of the tangent to this curve at the point (–1, 1).

$2xy + x^2\frac{dy}{dx} + \frac{1}{2}(3+y^2)^{-\frac{1}{2}}\ (2y\frac{dy}{dx}) = 0$ ✓✓✓

$x = -1, y = 1 \Rightarrow -2 + \frac{dy}{dx} + \frac{1}{2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{4}{3}$ ✓

Equation of tangent is: $y = \frac{4x}{3} + \frac{7}{3}$ ✓

(b) Use the method of incremental change to find the change in y when x changes from –1.00 to –1.01.

$\delta y \approx \left.\frac{dy}{dx}\right|_{x=-1, y=1} \times \delta x$ ✓

$\approx \frac{4}{3} \times (-0.01) \approx -0.013$ ✓✓

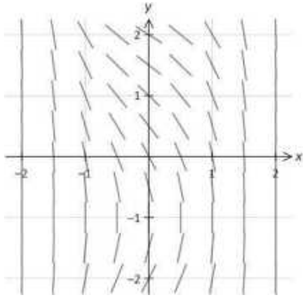
---

11. [6 marks: 2, 4]

Consider the differential equation $\frac{dy}{dx} = \frac{e^{x^2}}{y}$

(a) Explain why the slope field shown in the given diagram cannot be the slope field for the differential equation given above.

For given DE:
Infinite gradients for y =0.
Gradients are positive for $y > 0$. ✓
But sketch has non infinite gradients
at $y = 0$ and negative gradients for $y > 0$. ✓



Calculator Assumed

11. (b) A particular curve to this differential equation passes through the point (0, 1). Use the method of small increments to show that this curve also passes through the point with coordinates $\left(0.2,\ 1.1+\dfrac{e^{0.1^2}}{11}\right)$

At (0, 1); $\delta y \approx \dfrac{e^0}{1} \times 0.1 \approx 0.1$ ✓

Hence, when $x = 0.1$, $y \approx 1.1$ ✓

At (0.1, 1.1), $\delta y \approx \dfrac{e^{0.1^2}}{1.1} \times 0.1 = \dfrac{e^{0.1^2}}{11}$ ✓

Hence, when $x = 0.2$, $y \approx 1.1 + \dfrac{e^{0.1^2}}{11}$ ✓

---

12. [7 marks: 1, 2, 4]

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x+1}{y-1}$.

(a) Identify the coordinates of the point(s) in the slope field of this differential equation with indeterminate gradient.

(–1, 1) ✓

(b) A nullcline is a curve with passes through all slope fields with zero gradients. State the equation of the nullcline for this differential equation.

$\dfrac{dy}{dx} = 0 \Rightarrow x = -1$ with $y \neq 1$ ✓✓

(c) A particular solution to this differential equation passes through the point (1, 2). Use the method of small increments with increment2 of 0.1 to estimate the value of y when $x = 1.2$.

At (1, 2); $\delta y \approx \dfrac{1+1}{2-1}\ 2 \times 0.1 \approx 0.2$ ✓

Hence, when $x = 1.1$, $y \approx 2.2$ ✓

At (1.1, 2.2), $\delta y \approx \dfrac{2.1}{1.2} \times 0.1 = 0.175$ ✓

Hence, when $x = 1.2$, $y \approx 2.2 + 0.175 \approx 2.375$ ✓

---

25 Rectilinear Motion

Calculator Assumed

1. [5 marks]

Object P travels along a straight line. At time $t = 0$ seconds, it passes a fixed point O with velocity 10 ms^{-1} and undergoes a constant acceleration of 2 ms^{-2}. Use a calculus method to find the velocity of P when it is 24 m from O.

$\dfrac{dv}{dt} = 2 \Rightarrow v = \dfrac{dx}{dt} = 2t + C$ ✓

$t = 0, v = 10 \Rightarrow C = 10$

Hence, $\dfrac{dx}{dt} = 2t + 10$ ✓

$x = t^2 + 10t + D$

When $x = 0, t = 0 \Rightarrow D = 0$

Hence, $x = t^2 + 10t$. ✓

When $x = 24$, $t^2 + 10t = 24$

$t^2 + 10t - 24 = 0$

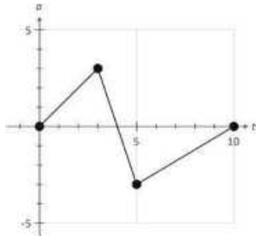
$t = 2$ (reject –12) ✓

Hence, when $t = 2$, $v = 2 \times 2 + 10 = 14\ \text{ms}^{-1}$. ✓

---

2. [7 marks]

Object P starts from rest and travels uninterrupted along a straight line. The graph of its acceleration a ms^{-2} plotted against time t seconds for $0 \le t \le 10$ is shown in the given diagram. Find with reasons maximum and minimum velocity of the object for $0 \le t \le 10$.



$a = \dfrac{dv}{dt} = 0$ when $t = 0$, 4 and 10.

For $t < 4$, $a > 0$ and for $t > 4$, $a < 0$. ✓

$\Rightarrow$ v has a local maximum at $t = 4$. ✓

$v(4)$ = Area under graph for $0 \le t \le 4$. ✓

$= \dfrac{1}{2} \times 4 \times 3 = 6\ \text{ms}^{-1}$. ✓

v has a minimum at $t = 10$. ✓

$v(10)$ = Area under graph for $0 \le t \le 10$. ✓

$= \dfrac{1}{2} \times 4 \times 3 - \dfrac{1}{2} \times 6 \times 3 = -3\ \text{ms}^{-1}$. ✓

Calculator Assumed

3. [9 marks: 3, 3, 3]

The acceleration of a particle P undergoing rectilinear motion is given by $a = \sqrt{1+2t}\ \text{ms}^{-2}$. The initial velocity of P is $1/3\ \text{ms}^{-1}$.

(a) Calculate the exact velocity of P when $t = 12$ seconds

$\frac{dv}{dt} = \sqrt{(1 + 2t)}$

$v = \frac{(1+2t)^{3/2}}{2\times\frac{3}{2}} + \text{C} = \frac{(1+2t)^{3/2}}{3} + \text{C}$ ✓

$v(0) = \frac{1}{3} \Rightarrow \text{C} = 0$ ✓

Hence, $v(12) = \frac{125}{3}\ \text{ms}^{-1}$. ✓

(b) Find the exact average speed of P during the first twelve seconds.

Since $v > 0$ for $t \geq 0$,
distance travelled in the first 12 seconds

$= \int_0^{12} \frac{(1+2t)^{3/2}}{3}\, dt$ ✓

$= \left[\frac{(1+2t)^{5/2}}{15}\right]_0^{12}$

$= \frac{3125}{15} - \frac{1}{15} = \frac{3124}{15}$ ✓

Hence, average speed $= \frac{\frac{3124}{15}}{12} = \frac{781}{45}\ \text{ms}^{-1}$. ✓

(c) Find the average acceleration of P during the first twelve seconds.

Average acceleration $= \frac{\frac{125}{3} - \frac{1}{3}}{12}$ ✓✓

$= \frac{31}{9}\ \text{ms}^{-2}$. ✓

Calculator Assumed

4. [10 marks: 5, 5]

A vehicle travels along a straight stretch of a highway. The driver notices a car stalled on the highway k metres ahead and applies the brakes of the vehicle. The acceleration of the vehicle t seconds after the brakes are applied is given by

$$a = -10\ e^{-0.1t}.$$

(a) Determine an expression for the displacement of the vehicle t seconds after the brakes are applied.

$a = -10\ e^{-0.1t}$

$v = \int -10\, e^{-0.1t}\ dt$ ✓

$= 100e^{-0.1t} + C$ ✓

$s = \int 100\,e^{-0.1t} + C\ \ dt$

$= -1000\ e^{-0.1t} + Ct + D$ ✓✓

$s(0) = 0 \Rightarrow D = 1000$

Hence, $s = -1000\ e^{-0.1t} + Ct + 1000$ ✓

(b) The vehicle comes to a complete stop after 3 seconds just behind the stalled car. Find k and the initial speed of the vehicle.

Vehicle comes to a stop after 3 seconds.

$v(3) = 0 \Rightarrow 100e^{-0.1(3)} + C = 0$ ✓

$C = -74.0818$ ✓

$s(t) = -1000\ e^{-0.1t} - 74.0818t + 1000$

$k = s(3) = -1000\ e^{-0.1(3)} - 74.0818(3) + 1000$ ✓

$= 36.9\ \text{m}$ ✓

Initial speed of vehicle $v(0) = 100 - 74.0818$

$\approx 25.9\ \text{ms}^{-1}$ ✓

Calculator Free

5. [4 marks]

A particle P moves along a straight line. Its acceleration (cms^{-2}) t seconds after it passes a fixed point O is given by $a(t) = x$ where $x(t)$ is the displacement of P, t seconds after passing O. P starts from O with a velocity of $v = -2$ cms^{-1}. Determine an expression for the velocity v in terms of x.

$$v\frac{dv}{dx} = x$$

$$\int v \ dv = \int x \ dx \quad ✓$$

$$\frac{v^2}{2} = \frac{x^2}{2} + C \quad ✓$$

$$v = \pm\sqrt{x^2 + D} \quad ✓$$

$$v(0) = 2 \ \Rightarrow \ v = -\sqrt{x^2+4} \quad ✓$$

---

6. [6 marks]

The acceleration a of a particle at time t is given by $a = 10(1-4v^2)$ where v is the velocity of the particle. Determine v in terms of x given that $t = 0$, $x = 0$.

$$a = \frac{dv}{dx} = 10(1-4v^2) \quad ✓$$

$$\int \frac{v}{(1-4v^2)}\, dv = \int 10\, dx \quad ✓$$

$$-\frac{1}{8}\, ln\,(1-4v^2) = 10x + \text{C} \quad ✓$$

$$1 - 4v^2 = A\ e^{-80x} \quad ✓$$

$$t = 0, x = 0 \ \Rightarrow \ A = 1$$

$$\text{Hence, } 1 - 4v^2 = e^{-80x} \quad ✓$$

$$v^2 = \frac{1}{4}(1 - e^{-80x})$$

$$v = \pm\frac{1}{2}\sqrt{1-e^{-80x}}\,. \quad ✓$$

Calculator Free

7 [6 marks]

A particle moves in a straight line. The particle starts from a fixed point O in the positive direction. Its displacement after t seconds is given by x cm. Its velocity when it is x cm from the point O is given by $v^2 = 36 - 4x^2$. Use integration with an appropriate substitution to prove that $x = 3 \sin 2t$.

$$v = \frac{dx}{dt} = \sqrt{36-4x^2} \quad ✓$$

$$\int \frac{1}{\sqrt{36-4x^2}}\, dx = \int 1\, dt \quad ✓$$

$$\text{Let } x = 3\sin\theta \ \Rightarrow \ dx = 3\cos\theta\, d\theta \quad ✓$$

$$\int \frac{3\cos\theta}{6\cos\theta}\, d\theta = \int 1\, dt \quad ✓$$

$$\theta = 2t + \alpha \quad ✓$$

$$x = 3\sin(2t+\alpha)$$

$$x(0) = 0 \ \Rightarrow \ x = 3\sin 2t. \quad ✓$$

---

8. [6 marks]

A particle P moves along a straight line. Its acceleration (cms^{-2}) t seconds after it passes a fixed point O is given by $a = -4\,e^{-2x}$ where $x(t)$ is the displacement of P, t seconds after passing O. P starts from O with a velocity of 2 cms^{-1}. Given that $v \geq 0 \ \forall \ t$, determine $x(t)$.

$$v\frac{dv}{dx} = -4\,e^{-2x}$$

$$\int v \ dv = \int -4e^{-2x}\ dx \quad ✓$$

$$\frac{v^2}{2} = 2\,e^{-2x} \quad ✓$$

$$\text{As } v(t) \geq 0\text{:} \quad v = 2\,e^{-2x}$$

$$\frac{dx}{dt} = 2\,e^{-x} \quad ✓$$

$$\int \frac{1}{e^{-x}}\ dx = \int 2\ dt \quad ✓$$

$$e^x = 2t + 1 \quad ✓$$

$$x = ln\,(2t+1) \quad ✓$$

Calculator Assumed

9. [9 marks: 4, 5]

An object P of constant mass m kg moves in a straight line with velocity v ms^{-1}.
The acceleration of P at time t seconds is given by $a = e^{-0.1\,t}$ ms^{-2} .
The kinetic energy possessed by P is given by $E = \frac{1}{2}mv^2$.

(a) P has a mass of 4 000 kg and at $t = 5$ seconds, P is travelling at 10 ms^{-1}.
Find the rate of kinetic energy change when $t = 5$ seconds.

$$E = 2000\,v^2 \quad ✓$$
$$\frac{dE}{dt} = 4000\,v\frac{dv}{dt} \quad ✓$$
$$\text{When } t = 5,\ a = \frac{dv}{dt} = e^{-0.5} \quad ✓$$
$$\frac{dE}{dt} = 4\,000 \times 10 \times e^{-0.5}$$
$$\approx 24\,261.2 \text{ J per second.} \quad ✓$$

(b) Calculate the distance travelled by the object in (a) in the first 10 seconds.

$$\frac{dv}{dt} = e^{-0.1\,t} \quad ✓$$
$$\Rightarrow\ v = -10\,e^{-0.1\,t} + C. \quad ✓$$
$$v(5) = 10\,e^{-0.5}$$
$$\Rightarrow\ v = -10\,e^{-0.1\,t} + 20\,e^{-0.5} \quad ✓$$

Distance travelled in first 10 seconds

$$= \int_0^{10} \left| -10e^{-0.1\,t} + 20e^{-0.5} \right| dt \quad ✓$$
$$= 58.0941 \approx 58.1 \text{ m} \quad ✓$$

26 Simple Harmonic Motion

Calculator Assumed

1. [8 marks: 3, 5] [TISC]

A particle P travels along the x-axis between $x = -4$ and $x = 4$.
Its velocity v is given by $v^2 = 9(A^2 - x^2)$.

(a) Prove that P undergoes simple harmonic motion.

$$2 \times v \times \frac{dv}{dt} = 9 \times -2\,x \times \frac{dx}{dt} \quad ✓$$
$$2 \times v \times \frac{dv}{dt} = -18\,x \times v \quad ✓$$
$$\frac{dv}{dt} = -\,9x \ \Rightarrow\ \frac{d^2x}{dt^2} = -9x \quad ✓$$

Hence, motion is simple harmonic.

(b) Complete the table below to describe the motion of P.

1 mark each.

Coordinates of the mean position	(0, 0)
Amplitude of Motion	4
Period of Motion	$\frac{2\pi}{3}$
Maximum Speed	12
Magnitude of maximum acceleration	36

---

2. [11 marks: 3, 2, 3, 3] [TISC]

A particle P moves along the x-axis and its velocity at any time t seconds is given by $v_P = \frac{2\pi}{3}\cos\left(\frac{\pi t}{3}\right)$ ms^{-1}. The particle starts from the origin and its displacement from the origin O is represented by r_P

(a) Show that $-a \le r_P \le a$ and state the value of a.

$$r_P = \frac{2\pi}{3} \times \frac{3}{\pi}\sin\left(\frac{\pi t}{3}\right) + C$$
$$= 2\sin\left(\frac{\pi t}{3}\right) + C. \quad ✓$$
$$t = 0,\ r_P = 0 \ \Rightarrow\ r_P = 2\sin\left(\frac{\pi t}{3}\right). \quad ✓$$
$$\Rightarrow\ -2 \le r_P \le 2. \quad ✓$$

Calculator Assumed

2. (b) Show that the particle undergoes simple harmonic motion.

$r_P = 2\sin\left(\frac{\pi t}{3}\right) \Rightarrow \dot{r}_P = \frac{2\pi}{3}\cos\left(\frac{\pi t}{3}\right)$ ✓

$\ddot{r}_P = -2\left(\frac{\pi}{3}\right)^2 \sin\left(\frac{\pi t}{3}\right)$

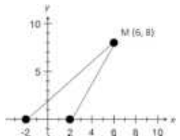
$\Rightarrow \ddot{r}_P = -\left(\frac{\pi}{3}\right)^2 r_P$ ✓

Hence, motion is simple harmonic.

(c) An observer is located at M with coordinates (6, 8). The distance between the particle and M at any time t is represented by *s*. Find the maximum and minimum value for *s*.

$s_{\max} = \sqrt{(8^2 + 8^2)} = 8\sqrt{2}$ ✓✓

$s_{\min} = \sqrt{(4^2 + 8^2)} = 4\sqrt{5}$ ✓



(d) A second particle Q moves along the *x*-axis such that its displacement from the origin is given by $r_Q = 8 - \sin\left(\frac{\pi t}{3}\right)$ metres. The distance between the particles P and Q at any time *t* seconds is given by *h* metres. Find the minimum and maximum value of *h*. Justify your answer.

Period of P and Q are the same.
At $t = 0$, both are in alignment.

Hence, $h = (8 - \sin\left(\frac{\pi t}{3}\right)) - 2\sin\left(\frac{\pi t}{3}\right)$

$= 8 - 3\sin\left(\frac{\pi t}{3}\right).$ ✓

$\Rightarrow 5 \le h \le 11$ ✓✓

Calculator Assumed

3. [13 marks: 5, 1, 3, 2, 2] [TISC]

Points A and B have coordinates (–4, 0) and (5, 0) respectively. Particles P and Q start moving along the *x*-axis at the same time. The displacement of P, *t* seconds later, is given by $r_P = -4 + 6\cos(\pi t)$. The displacement of Q, *t* seconds later, is given by $r_Q = 5 + 5\sin(2\pi t)$. It is known that $0 \le t \le 2$ seconds.

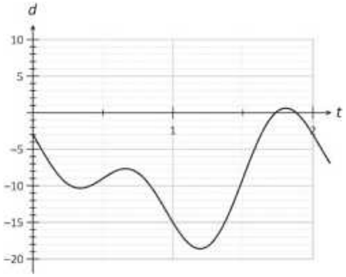
(a) Show that P undergoes simple harmonic motion about its mean position. State clearly the location of the mean position, its amplitude and period.

Mean Position A(–4, 0). ✓
Amplitude = 6 ✓
Period = 2 seconds ✓
$h = 6\cos \pi t$ ✓
$\dot{h} = -6\pi \sin \pi t$
$\ddot{h} = -6\pi^2 \cos \pi t \Rightarrow \ddot{h} = -\pi^2 h$
That is, motion is simple harmonic. ✓

(b) Show that the separation, at time *t* seconds, between P and Q is given by $d = 6\cos(\pi t) - 5\sin(2\pi t) - 9$.

$r_P - r_Q = (-4 + 6\cos \pi t) - (5 + 5\sin 2\pi t)$
$= 6\cos \pi t - 5\sin 2\pi t - 9$ ✓

(c) Draw a sketch of *d* against *t* for $0 \le t \le 2$ on the axes provided below and use this sketch to explain why *d* varies with *t* in a manner that is **NOT** simple harmonic.



✓✓ Sketch.
Graph is not sinusoidal. ✓
Hence, motion is not simple harmonic.

(d) Determine when P and Q collide for the first time and the location of the point of collision.

$d = 0 \Rightarrow t = 1.734$ seconds ✓
Collision at (0.025, 0). ✓

(e) Determine the maximum separation between P and Q stating when it occurs for $0 \le t \le 2$ seconds.

Maximum separation = 18.6 m ✓
at $t = 1.194$ seconds. ✓

Calculator Assumed

4. [10 marks: 3, 3, 1, 3]

A particle P travels in a straight line. Let x be the displacement of P from a fixed point O at time t seconds. The velocity of P at time t seconds v is given by $v^2 = 4\pi^2(4 - x^2)$ cm s^{-1}.

(a) Show that P undergoes simple harmonic motion.

$v^2 = 4\pi^2(4 - x^2)$ $2v\dfrac{dv}{dt} = -8\pi^2 x \times \dfrac{dx}{dt}$ ✓ $= -8\pi^2 x \times v$ ✓ $\dfrac{dv}{dt} = a = -4\pi^2 x$ ✓ Hence, motion is simple harmonic.	$v^2 = 4\pi^2(4 - x^2)$ $2v\dfrac{dv}{dx} = -8\pi^2 x$ ✓ $v\dfrac{dv}{dx} = a = = -4\pi^2 x$ ✓✓ Hence, motion is simple harmonic.

It is also known that P starts with $a = 4\pi^2$ cms^{-2}.

(b) Find x in the form $x = A\sin(kt + \alpha)$ where $\alpha \geq 0$.

$t = 0 \quad a = 4\pi^2 \Rightarrow x = -1$ ✓

$-1 = \pm A\sin(\alpha)$

From $v^2 = 4\pi^2(4 - x^2)$; $A = \pm 2$

But $\alpha \geq 0 \Rightarrow A = -2$ ✓

$\alpha = \dfrac{\pi}{6}$ ✓

Hence: $x = -2\sin(2\pi t + \dfrac{\pi}{6})$

(c) Find t when $x = -2$ cm.

$x = -2$ when $2\pi t + \dfrac{\pi}{6} = \dfrac{\pi}{2} \Rightarrow t = \dfrac{1}{6}$ ✓

(d) Find β given that $x \leq \beta$ for 20% of its period.

Period = 1 second.

20% of period = 0.2 seconds ✓

Hence: $x = \beta$ when $t = \dfrac{1}{6} + 0.1 = \dfrac{4}{15}$ ✓

$\beta = -2\sin(2\pi \times \dfrac{4}{15} + \dfrac{\pi}{6}) = -1.6180$ ✓

Calculator Assumed

5. [13 marks: 1, 1, 4, 4, 3] [TISC]

The displacement of a particle P at time t seconds, from a fixed point O, is given by $x = A\cos(\omega t)$ metres.

(a) Find v, the velocity of P at time t seconds.

$v = -A\omega\sin\omega t$ ✓

(b) Find x in terms of a, the acceleration of P at time t seconds.

$a = -A\omega^2\cos\omega t = -\omega^2 x$ ✓

(c) Differentiate v implicitly with respect to x to show that $v\dfrac{dv}{dx} = a$.

$\dfrac{dv}{dx} = -A\omega^2\cos\omega t\,\dfrac{dt}{dx}$ ✓

$= -A\omega^2\cos\omega t\,\dfrac{1}{\left(\dfrac{dx}{dt}\right)}$ ✓

$= -A\omega^2\cos\omega t\,\dfrac{1}{v}$ ✓

$\Rightarrow v\dfrac{dv}{dx} = -A\omega^2\cos\omega t = a$ ✓

(d) Use an appropriate technique to integrate $v\dfrac{dv}{dx} = a$ with respect to the appropriate variables to show that $v^2 = \omega^2(A^2 - x^2)$.

Since $a = -\omega^2 x$ and $v\dfrac{dv}{dx} = a$,

$v\dfrac{dv}{dx} = -\omega^2 x$ ✓

$\Rightarrow v\,dv = -\omega^2 x\,dx$

$\int v\,dv = \int -\omega^2 x\,dx$ ✓

$\dfrac{v^2}{2} = -\omega^2\dfrac{x^2}{2} + C$

$v^2 = -\omega^2 x^2 + K$ ✓

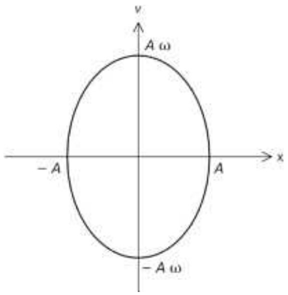
$x = A, v = 0 \Rightarrow K = \omega^2 A^2$ ✓

Hence, $v^2 = \omega^2(A^2 - x^2)$.

Calculator Assumed

5. (e) On the axes given, sketch the graph of v against x. Show clearly the intercepts of the curve.

✓ Ellipse
✓✓ Intercepts marked



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6. [10 marks: 4, 6] [TISC]

The equation of motion of a particle P is given by $\frac{d^2x}{dt^2} = -16\pi^2 x$.

(a) Find x in terms of t given that when $t = 0$, $x = 8$, and $\frac{dx}{dt} = 0$.

$x = \text{A} \cos(4\pi t + \alpha)$ ✓
$\dot{x} = -4\pi \text{A} \sin(4\pi t + \alpha)$ ✓
$\dot{x}(0) = 0 \Rightarrow -4\pi \text{A} \sin \alpha = 0$
$\alpha = 0$ ✓
$x(0) = 8 \Rightarrow \text{A} \cos 0 = 8$
$\text{A} = 8$ ✓
Hence, $x = 8 \cos(4\pi t)$

(b) Find when within the first cycle P travels at half its maximum speed. Give your answer in exact form. Justify your answer.

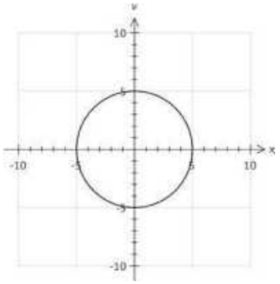
$\dot{x} = -32\pi \sin(4\pi t)$

Period of cycle = $\frac{2\pi}{4\pi} = \frac{1}{2}$.
Maximum speed = 32π. ✓
When speed = 16π,
$|-32\pi \sin(4\pi t)| = 16\pi$ ✓
$|\sin(4\pi t)| = \frac{1}{2}$
$\sin(4\pi t) = \pm \frac{1}{2}$ ✓
$4\pi t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ ✓
$t = \frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}$ ✓✓

Calculator Assumed

7. [10 marks: 6, 4]

The diagram below shows the velocity- displacement graph of a particle undergoing rectilinear motion. Displacement is measured in cm and velocity in cms^{-1}.



(a) Given that the shape of the graph is a circle of radius 5, use calculus to show that the motion of the particle is simple harmonic. State the period and amplitude of the motion.

Equation of curve: $v^2 + x^2 = 25$ ✓
$v^2 = 25 - x^2$
Differentiate with respect to time t:
$2v \frac{dv}{dt} = -2x \frac{dx}{dt}$ ✓✓
$\frac{dv}{dt} = -x$ ✓
That is: $\frac{d^2x}{dt^2} = -x$.
Hence, motion of particle is simple harmonic.

Period of motion = 2π seconds ✓
Amplitude of motion = 5 cm. ✓

(b) Given that the particle starts from the mean position of its motion, find the percentage of time within a cycle when the speed of the particle is no more than 4 ms^{-1}.

For $|v| \le 4$, from graph $-5 \le x \le -3$ and $3 \le x \le 5$ ✓

Let $x = 5 \sin t$
For $3 \le 5 \sin t \le 5$
$0.6435 \le t \le 2.4981$
$\Delta t = 1.8546$ seconds
By symmetry, total time = 1.8546×2
= 3.7092 seconds ✓✓

Hence, % of cycle = $\frac{3.7092}{2\pi} \times 100$
≈ 59% ✓

Calculator Assumed

8. [7 marks: 3, 4]

(a) The equation of motion of a body is given by $\frac{d^2x}{dt^2} + 25x = 0$, where x is the distance of the body (in m) to a fixed point C and t is time in minutes. It is known that the body starts moving from C with a velocity of –2 m per minute. Show clearly that $x = A \sin(\omega t + \alpha)$ giving the values of A, ω (where $\omega > 0$) and α.

$\frac{d^2x}{dt^2} = -25x = -(5)^2 x$.

Particle starts moving from the mean position. Hence, $\alpha = 0$.

$\Rightarrow \quad x = A \sin 5t$. ✓

Velocity $v = \frac{dx}{dt} = 5A \cos 5t$. ✓

When $t = 0, v = -2 \quad \Rightarrow 5A = -2 \quad \Rightarrow A = -\frac{2}{5}$ ✓

Hence, $x = -\frac{2}{5} \sin 5t$.

(b) The equation of motion of another body is given by $x = 4\cos(2t + \frac{\pi}{6})$ metres where t is time in minutes. Find k, if $x \geq k$ for 90% of a cycle.

Period of motion = $\frac{2\pi}{2} = \pi$. ✓

$x \geq k$ for 90% of a cycle

$\Rightarrow \quad x < k$ for 10% of cycle

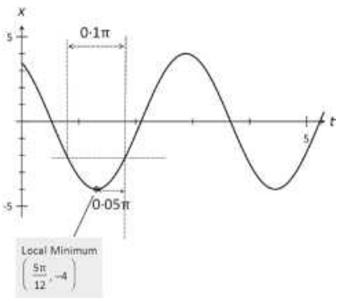
$x < k$ for 0.1π minutes

Minimum point at $t = \frac{5\pi}{12}$

Hence, $x = k$ for $t = \frac{5\pi}{12} + \frac{0.1\pi}{2} = \frac{7\pi}{15}$ ✓

$\Rightarrow \quad k = 4\cos[2(\frac{7\pi}{15}) + \frac{\pi}{6}]$ ✓

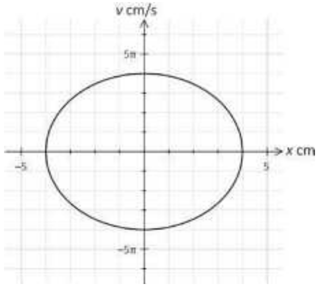
$= 3.8042 \approx 3.8$ ✓



Calculator Assumed

9. [11 marks: 3, 3, 3, 2] [TISC]

(a) A particle P undergoes simple harmonic motion. It starts from the mean position with a velocity of 4π cms^{-1}. The given diagram shows the velocity-displacement graph of P.



(i) Determine the period and amplitude of the motion.

Equation of graph is of the form:

$v^2 = \omega^2 (A^2 - x^2)$ ✓

$(4, 0) \Rightarrow A = \pm 4$

$\Rightarrow$ Amplitude is 4 cm ✓

$(0, 4\pi) \Rightarrow 16\pi^2 = 16\omega^2$

$\omega = \pi$

$\Rightarrow$ Period = 2 seconds ✓

(ii) Determine the value(s) of t, where $0 \leq t \leq 2$ seconds, when it is travelling with a speed of 2π cms^{-1}.

$x = 4\sin(\pi t) \quad \Rightarrow \quad v = 4\pi\cos(\pi t)$ ✓

$v = \pm 2\pi \quad \Rightarrow \quad \cos(\pi t) = \pm\frac{1}{2}$ ✓

$t = \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \;\&\; \frac{5}{3}$ ✓

(b) A second particle Q travels in a straight line and has an initial velocity of 4π cms^{-1}. Let x be the displacement of Q along this line from the point when it starts moving. The acceleration of Q is given by $a = \pi^2 x$ cms^{-2}. Use calculus to show that velocity of Q is given by $v^2 = \pi^2(16 + x^2)$.

$v\frac{dv}{dx} = \pi^2 x$ ✓

$\int v\, dv = \int \pi^2 x\, dx$ ✓

$\frac{v^2}{2} = \frac{\pi^2 x^2}{2} + C$

$x = 0, v = 4\pi \quad \Rightarrow \quad v^2 = \pi^2 x + 16\pi^2$ ✓

$= \pi^2(16 + x^2)$

(c) Compare the motions of P and Q.
State an important difference in the speeds achieved by P and Q.

For P: $0 \leq$ speed $\leq 4\pi$ cms^{-1} For Q: speed $\geq 4\pi$. ✓
Speed for P is bounded while speed for Q is unbounded. ✓

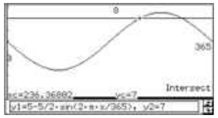
Calculator Assumed

10. [13 marks: 2, 4, 5, 2] [TISC]

The amount of rainfall received by a town is modelled by
$r = 5 - \frac{5}{2}\sin\left[\frac{2\pi t}{365}\right]$ millimetres, for $0 \le t \le 365$ days.

(a) Determine the approximate number of days the town receives more than 7 millimetres of rainfall.

No. of days ≈ 311 – 236
≈ 75 ✓✓



(b) Find on which day(s) the amount of rainfall received is decreasing at a rate of –0.025 millimetres per day

$$\frac{dr}{dt} = -\frac{5}{2} \times \frac{2\pi}{365}\cos\left[\frac{2\pi t}{365}\right] \quad ✓$$

$$\frac{dr}{dt} = -0.025 \Rightarrow \cos\left[\frac{2\pi t}{365}\right] = 0.5809 \quad ✓$$

$$t = 55.2,\ 309.8$$

That is on the 56th and 310th day. ✓✓

(c) Show that the *variation* of rainfall received is simple harmonic. State its period and amplitude.

$$\text{Let } h = \frac{5}{2}\sin\left[\frac{2\pi t}{365}\right] \Rightarrow \dot{v} = \frac{5}{2} \times \frac{2\pi}{365}\cos\left[\frac{2\pi t}{365}\right] \quad ✓$$

$$\ddot{h} = -\frac{5}{2} \times \left(\frac{2\pi}{365}\right)^2 \sin\left[\frac{2\pi t}{365}\right] \quad ✓$$

$$= -\left(\frac{2\pi}{365}\right)^2 h. \quad ✓$$

Hence, variation is simple harmonic. Period = 365 days, Amplitude = $\frac{5}{2}$.✓✓

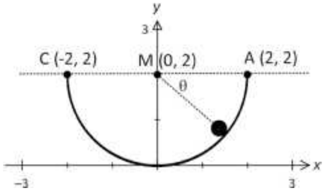
(d) Use an appropriate method to find the total amount of rainfall received for $0 \le t \le 365$ days.

$$T = \int_0^{365} 5 - \frac{5}{2}\sin\left[\frac{2\pi t}{365}\right] dt \quad ✓$$

$$= 1825 \text{ mm.} \quad ✓$$

Calculator Assumed

11. [11 marks: 5, 3, 3]



A steel bearing B of negligible radius rolls to and fro in the vertical plane between the points A (2, 2) and C (–2, 2) along the path with equation $y = 2 - \sqrt{4 - x^2}$ located on the frictionless inner surface of a hemispherical dish of radius 2 cm. The steel bearing is released from rest at point A. M is a point with coordinates (0, 2). Let θ be the angle between BM and the horizontal line through C, M and A. θ changes at a constant rate of $\frac{\pi}{2}$ radians per second.

(a) Show that the horizontal displacement of B from M displays variation that is simple harmonic.

Let x represent the horizontal displacement of B from M at time t seconds.

$$\theta = \frac{\pi}{2} \times t \quad \Rightarrow \quad x = 2\cos\theta = 2\cos\left(\frac{\pi t}{2}\right) \quad ✓✓$$

$$\frac{dx}{dt} = -\pi\sin\left(\frac{\pi t}{2}\right) \quad ✓$$

$$\frac{d^2x}{dt^2} = -\frac{\pi^2}{2}\cos\left(\frac{\pi t}{2}\right) = -\left(\frac{\pi}{2}\right)^2 x \quad ✓✓$$

Hence, x satisfies the equation of a motion of a body undergoing SHM.

(c) Determine the rate of change of x when x is 1 cm.

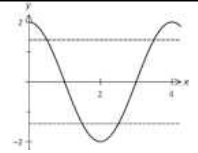
$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{\pi}{2}\right)^2 (4 - x^2) \quad ✓$$

$$\text{When } x = 1: \quad \left(\frac{dx}{dt}\right)^2 = 3\left(\frac{\pi}{2}\right)^2 \quad ✓$$

$$\frac{dx}{dt} = \pm\frac{\pi\sqrt{3}}{2} \text{ cm/second} \quad ✓$$

(d) The magnitude of the horizontal displacement x is within k cm from M for 60% of its period. Determine the value of k.

Period of motion = 4 seconds ✓
Hence, $|x|$ is outside k cm from M
for 0.4 × 4 = 1.6 seconds
By symmetry, $k \le |x| \le 2$ for 1.6 ÷ 4 = 0.4 sec. ✓
Hence, $k = 2\cos\left(\frac{\pi \times 0.4}{2}\right) = 1.62$ cm ✓



27 Vector Calculus I

Calculator Assumed

1. [16 marks: 2, 2, 3, 3, 3, 3] [TISC]

If air resistance is ignored, the velocity vector, $v(t)$, of a particle P at time t seconds, is given by $\boldsymbol{v}(t) = 40\sqrt{3}\,\mathbf{i} + (40 - 9.8t)\,\mathbf{j}$, where the components are measured in ms^{-1}. It is known that when $t = 0$, its displacement $\boldsymbol{r}(0) = 2\,\mathbf{i} + 4\,\mathbf{j}$ metres. Ground level is modelled by $\mathbf{j} = 0$.

(a) Find the time it takes the particle to reach its highest point.

$$\begin{aligned} 40 - 9.8t &= 0 && \checkmark \\ t &= 4.08 \text{ seconds} && \checkmark \end{aligned}$$

(b) Verify that the position vector of the particle, t seconds after it is thrown, is given by. $\boldsymbol{r}(t) = (2 + 40\sqrt{3}\ t)\,\mathbf{i} + (4 + 40t - 4.9t^2)\,\mathbf{j}$.

$$\begin{aligned} \boldsymbol{r}(t) &= \int <40\sqrt{3}, 40 - 9.8t>\,dt \\ &= (40\sqrt{3}\ t)\,\mathbf{i} + (40t - 4.9t^2)\,\mathbf{j} + \mathbf{C} && \checkmark \\ \boldsymbol{r}(0) &= <2, 4> \ \Rightarrow\ \mathbf{C} = <2, 4> \\ \text{Hence:}\quad \boldsymbol{r}(t) &= (2 + 40\sqrt{3}\ t)\,\mathbf{i} + (4 + 40t - 4.9t^2)\,\mathbf{j} && \checkmark \end{aligned}$$

(c) Find the position vector of the point where it hits the ground.

$$\begin{aligned} 4 + 40t - 4.9t^2 &= 0 && \checkmark \\ t &= 8.262 && \checkmark \\ \boldsymbol{r}(t) &= <572.41, 0> \text{ metres} && \checkmark \end{aligned}$$

(d) Find when P moves in a direction that is parallel to $2\mathbf{i} - \mathbf{j}$.

$$\begin{aligned} <40\sqrt{3},\ 40 - 9.8t> &= k<2, -1> && \checkmark \\ k &= 20\sqrt{3} && \checkmark \\ \Rightarrow\quad 40 - 9.8t &= -20\sqrt{3} \\ t &= 7.62 \text{ seconds} && \checkmark \end{aligned}$$

(e) Find when the speed of P is $70\ \text{ms}^{-1}$.

$$\begin{aligned} \sqrt{(40\sqrt{3})^2 + (40 - 9.8t)^2} &= 70 && \checkmark \\ t &= 3.06, 5.10 \text{ seconds.} && \checkmark\checkmark \end{aligned}$$

Calculator Assumed

1. (f) Use the information in part (b) to find the parametric equation of the path of P. Hence, show that the Cartesian equation of the path of P is in the form of a quadratic equation. You are NOT required to simplify the Cartesian equation.

$$\begin{aligned} &x = 2 + 40\sqrt{3}\ t \qquad y = 4 + 40t - 4.9t^2 && \checkmark \\ &t = \frac{x-2}{40\sqrt{3}} && \checkmark \\ &\text{Hence:}\quad y = 4 + 40\frac{x-2}{40\sqrt{3}} - 4.9\left(\frac{x-2}{40\sqrt{3}}\right)^2 \\ &\text{which is in the form of a quadratic equation.} && \checkmark \end{aligned}$$

---

2. [13 marks: 3, 2, 3, 3, 2] [TISC]

The velocity vector of particle P at time t seconds (for $0 \le t \le 5$) is given by $\boldsymbol{v}_P(t) = 2t\,\mathbf{i} + (4t^3 - 10t)\,\mathbf{j}$ centimetres per second. It is known that when $t = 0$, the particle is at a point with position vector $0\,\mathbf{i} + 4\,\mathbf{j}$ centimetres.

(a) Find the exact value(s) of t when P is moving at an angle of 45° to the horizontal.

$$\begin{aligned} &\text{Hence;}\quad \frac{4t^3 - 10t}{2t} = |\tan 45| = \pm 1 && \checkmark \\ &t = \sqrt{2},\ \sqrt{3} \qquad (\text{reject } -\sqrt{2}\ \&\ -\sqrt{3}) && \checkmark\checkmark \end{aligned}$$

(b) Find the speed of P when it is moving at an angle of 45° above the horizontal.

$$\begin{aligned} &\text{P moves at an angle of } 45^\circ \text{ above the horizontal at } t = \sqrt{3}. && \checkmark \\ &\text{Hence:}\quad \text{speed} = |v(\sqrt{3})| = |<2\sqrt{3}, 2\sqrt{3}>| = 2\sqrt{6}\ \text{ms}^{-1}. && \checkmark \end{aligned}$$

(c) Find the position vector of P when it is moving parallel to the horizontal for the first time.

$$\begin{aligned} &\text{Moving horizontally:}\quad \frac{4t^3 - 10t}{2t} = \tan 0 \\ &\qquad t = \sqrt{\frac{5}{2}} && \checkmark \\ &\boldsymbol{r}(t) = \int <2t, -4t^3 - 10t>\,dt \\ &\qquad = <t^2, t^4 - 5t^2 + 4> && \checkmark \\ &\text{Hence:}\quad \boldsymbol{r}\left(\sqrt{\frac{5}{2}}\right) = <\frac{5}{2}, -\frac{9}{4}> \text{cm} && \checkmark \end{aligned}$$

Calculator Assumed

2. (d) Find the magnitude of the minimum acceleration of P.

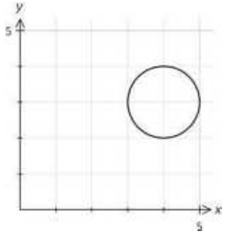
$\boldsymbol{a}(t) = < 2, 12t^2 - 10 >$ ✓
$|\boldsymbol{a}(t)| = \sqrt{4 + (12t^2 - 10)^2}$ ✓
$|\boldsymbol{a}(t)|$ has a minimum value of 2. ✓

(e) Find the distance travelled by P along its path of motion in the first 5 seconds.

Distance travelled along path $= \int_0^5 \left| <2t, 4t^3 - 10t> \right| dt$ ✓
$= 514.45$ cm ✓

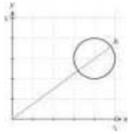
3. [9 marks: 3, 4, 2]

The position vector of a particle P at time t minutes is given by $\boldsymbol{r}(t) = < 4 + \cos t, 3 + \sin t >$ metres. The path traced by P is shown in the given diagram.



(a) Show that P is *first* furthest from the origin at $t = \tan^{-1}\left(\frac{3}{4}\right)$ minutes.

P is furthest from the origin when it is at point A.



$\Rightarrow \frac{3 + \sin t}{4 + \cos t} = \frac{3}{4}$ ✓✓

$\tan t = \frac{3}{4}$ ✓

$\Rightarrow t = \tan^{-1}\left(\frac{3}{4}\right)$

(b) Determine the speed and direction of motion of P when it is *first* furthest away from the origin.

$v(t) = < -\sin t, \cos t >$ ✓
$v(\tan^{-1}\left(\frac{3}{4}\right)) = < \frac{-3}{5}, \frac{4}{5} >$ ✓
speed = 1 metre/minute ✓
Direction: $\theta = \tan^{-1}\left(\frac{-4}{3}\right) \approx 126.9^\circ$ with the positive x-axis. ✓

(c) Show that the acceleration of P is always perpendicular to its velocity.

$\boldsymbol{a}(t) = < -\cos t, -\sin t >$ ✓
$\boldsymbol{a} \bullet \boldsymbol{v} = < -\cos t, -\sin t > \bullet < -\sin t, \cos t > = 0$ ✓
Since $\boldsymbol{a} \bullet \boldsymbol{v} = 0\ \forall\ t$, acceleration is always perpendicular to the velocity.

Calculator Assumed

4. [10 marks: 4, 1, 5] [TISC]

A particle P travels in the x-y plane. The position vector of P at time t seconds given by $\boldsymbol{r}(t) = a \sin 2t\, \boldsymbol{i} + b \cos 2t\, \boldsymbol{j}$ ms^{-1} where a and b are positive real constants. P starts from the point with position vector $2\boldsymbol{j}$.

(a) Determine the Cartesian equation (in terms of a) of the path of P.

$\boldsymbol{r}(0) = < 0, b > \Rightarrow b = 2$ ✓
$\boldsymbol{r}(t) = < a \sin 2t, 2 \cos 2t >$
Parametric equation of path:
$x = a \sin 2t \Rightarrow \frac{x}{a} = \sin 2t$ ✓
$y = 2 \cos 2t \Rightarrow \frac{y}{2} = \cos 2t$ ✓
$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ ✓

(b) State the value(s) of a if the path of P is not a circle.

If path is not a circle $a \neq 2$. ✓

(c) If the path of P is not a circle, find t when its acceleration is perpendicular to its velocity.

$v(t) = < 2a \cos 2t, -4 \sin 2t >$ ✓
$\boldsymbol{a}(t) = < -4a \sin 2t, -8 \cos 2t >$ ✓
$v(t) \bullet \boldsymbol{a}(t) = -8a^2 \sin 2t \cos 2t + 32 \sin 2t \cos 2t$
$= (16 - 4a^2) \sin 4t$ ✓
For $v(t)$ perpendicular to $\boldsymbol{a}(t)$: $(16 - 4a^2) \sin 4t = 0$ ✓
$t = \frac{n\pi}{4}$ for $n \in \mathbb{Z}_0^+$ ✓

5. [13 marks: 2, 2, 2, 2, 5]

The velocity vector of particle P at time t hours is given by $\boldsymbol{v}_P(t) = 3 \cos t\, \mathbf{i} - 4 \sin t\, \mathbf{j}$ kilometres per hour. It is known that when $t = 0$, the particle is at a point with position vector $0\,\mathbf{i} + 8\,\mathbf{j}$.

(a) Find $\boldsymbol{r}_P(t)$, the position vector of P at time t hours.

$\boldsymbol{r}(t) = \int <3\cos t, -4\sin t> dt$
$= < 3 \sin t, 4 \cos t + 4 >$ ✓✓

Calculator Assumed

5. (b) Show that the speed of P at time t hours is given by $\sqrt{9+7\sin^2(t)}$

$$\begin{aligned}\text{Speed} &= |<3\cos t, 4\sin t>| \\ &= \sqrt{9\cos^2 t + 16\sin^2 t} \quad ✓ \\ &= \sqrt{9(1-\sin^2 t)+16\sin^2 t} \quad ✓ \\ &= \sqrt{9+7\sin^2 t}\end{aligned}$$

(c) Find the position vector of P when its speed is maximised for the first time.

Speed = $\sqrt{9+7\sin^2 t}$

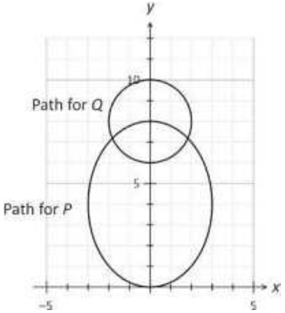
Speed is maximum when $\sin t = 1$. ✓

Hence: $r = <3, 4>$ ✓

(d) A second particle Q has position vector at time t hours given by $\boldsymbol{r}_Q\,(t) = (2\cos t)\,\mathbf{i} + (2\sin t + 8)\,\mathbf{j}$. P and Q start moving at the same time.

(i) Sketch on the axes provided below the paths of particles P and Q.

Path for P ✓ Path for Q ✓



(ii) Show that while the paths of P and Q cross twice, the two particles do not collide.

From sketch above, the paths of P and Q intersect twice. ✓

For collision:

$<3\sin t, 4\cos t + 4> = <2\cos t, 2\sin t + 8>$ ✓

x-comp: $3\sin t = 2\cos t$

$\Rightarrow t = 0.5880, 3.7296$ ✓

For $t = 0.5880$:

For P: y-comp. = 7.33

For Q: y-comp. = 9.11

Hence, P and Q do not collide. ✓

For $t = 3.7296$:

For P: y-comp. = 0.67

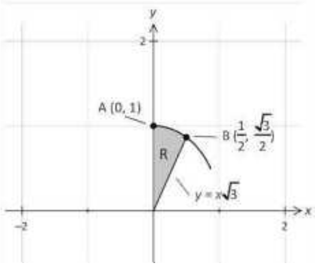
For Q: y-comp. = 6.89 ✓

Hence, P and Q do not collide.

Calculator Assumed

6. [11 marks: 2, 3, 6] [TISC]

The position vector of a particle P at time t seconds is given by $\boldsymbol{r} = <\sin t, \cos t>$ cm. The given diagram shows part of the path traced by P. The points A (0, 1) and B $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ correspond to the positions of P at times $t = 0$ and $t = \frac{\pi}{6}$ seconds respectively. The shaded region R shows the area swept by the vector $\boldsymbol{r}$ between A and B.



(a) Show that the Cartesian equation of the path of P is given by $y^2 = 1 - x^2$.

$x = \sin t \quad y = \cos t$ ✓

$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$ ✓

Hence: $y^2 = 1 - x^2$

(b) Use integrals to determine the length of the curve between A and B.

$v = <\cos t, -\sin t>$ ✓

$$\widehat{AB} = \int_0^{\frac{\pi}{6}} |<\cos t, -\sin t>|\,dt \quad ✓$$

$$= \frac{\pi}{6} \quad ✓$$

(c) Use the method of integral substitution to determine the *exact* area of R.

$$\text{Area} = \int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx - \left(\frac{1}{2}\times\frac{1}{2}\times\frac{\sqrt{3}}{2}\right) \quad ✓$$

Let $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$ ✓

$$\begin{aligned}\text{Area} &= \int_0^{\frac{\pi}{6}} \cos^2\theta\,d\theta - \frac{\sqrt{3}}{8} \quad ✓ \\ &= \frac{1}{2}\int_0^{\frac{\pi}{6}} 1+\cos 2\theta\,d\theta - \frac{\sqrt{3}}{8} \quad ✓ \\ &= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}} - \frac{\sqrt{3}}{8} \quad ✓ \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} = \frac{\pi}{12} \quad ✓\end{aligned}$$

Calculator Assumed

7. [8 marks: 3, 2, 3]

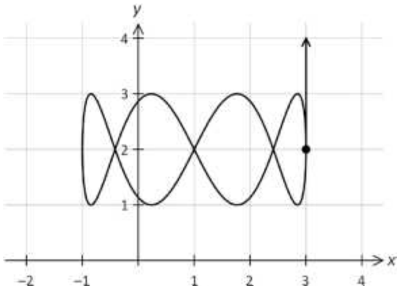
The particle P moves in the *x*-*y* plane. It starts from the point with position vector $\begin{pmatrix}1\\2\end{pmatrix}$ cm with velocity $\begin{pmatrix}2\\4\end{pmatrix}$ cms^{-1}. Its acceleration at time *t* seconds is given by $\boldsymbol{a}(t) = \begin{pmatrix}-2\sin(t)\\-16\sin(4t)\end{pmatrix}$ cms^{-2}.

(a) Find an expression for $\boldsymbol{r}(t)$, the position vector of P at time *t* seconds.

$\boldsymbol{v}(t) = \begin{pmatrix}2\cos(t)\\4\cos(4t)\end{pmatrix}$ ✓

$\boldsymbol{r}(t) = \begin{pmatrix}2\sin(t)+1\\\sin(4t)+2\end{pmatrix}$ ✓✓

(b) The given diagram shows the path traced by P. On the diagram provided, locate the position of P and use an arrow to indicate the direction of motion of P at $t = \dfrac{\pi}{2}$ seconds.



$\boldsymbol{r}(\frac{\pi}{2}) = <3, 2>$ $\quad v(\frac{\pi}{2}) = <0, 4>$

✓ Locates P at < 3, 2 >.

✓ From < 3, 2 > draw a vertical arrow in the direction of the positive *y*-axis.

(c) Determine one instance (one value of *t*) when P is travelling parallel to the vector $\begin{pmatrix}-2\\4\end{pmatrix}$.

$\begin{pmatrix}2\cos(t)\\4\cos(4t)\end{pmatrix} = \lambda\begin{pmatrix}-2\\4\end{pmatrix}$ ✓

For $\lambda = 1$: $\begin{pmatrix}2\cos(t)\\4\cos(4t)\end{pmatrix} = \begin{pmatrix}-2\\4\end{pmatrix}$ ✓

$\Rightarrow \cos(t) = -1$

and $\cos(4t) = 1$

Hence: $t = \pi$ ✓

Calculator Assumed

8. [8 marks]

The particle P moves in the *x*-*y* plane. It starts from the point with position vector < 0, –1 > cm with velocity < 0, 2 > cms^{-1}. Its acceleration at time *t* seconds is given by $\boldsymbol{a}(t)$ = < cos(*t*), – 2 sin(*t*) > cms^{-2}. Determine the perimeter of the path traced by P.

$v(t) = <\sin(t), 2\cos(t)>$ ✓

$\boldsymbol{r}(t) = <-\cos(t) + 1, 2\sin(t) - 1>$ ✓✓

Parametric equation of path:

$x = -\cos(t) + 1 \quad \Rightarrow \quad \cos(t) = -(x-1)$

$y = 2\sin(t) - 1 > \quad \Rightarrow \quad \sin(t) = \dfrac{y+1}{2}$ ✓

$\cos^2(t) + \sin^2(t) = 1 \quad \Rightarrow \quad (x-1)^2 + \dfrac{(y+1)^2}{4} = 1$ ✓

Path traced by P is an ellipse (closed path).
Required perimeter is distance travelled by P along its path in one cycle of period = 2π seconds. ✓

$\text{Perimeter} = \displaystyle\int_0^{2\pi} \left|<\sin(t), 2\cos(t)>\right| dt$ ✓

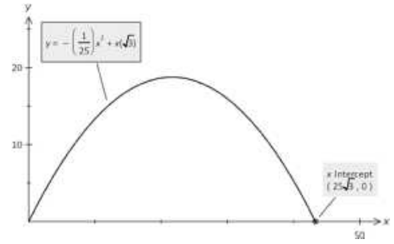
$= \displaystyle\int_0^{2\pi} \sqrt{\sin^2(t) + 4\cos^2(t)}\, dt$

$= 9.6884$ ✓

---

9. [11 marks: 1, 4, 3, 3]

A particle P moves in the *x*-*y* plane. P starts at the origin and its position vector at time *t* seconds is given by $\boldsymbol{r}(t) = (20\cos\theta)t\,\boldsymbol{i} + [(20\sin\theta)\,t - 4t^2]\,\boldsymbol{j}$ metres. The path taken by P has equation $y = \dfrac{-x^2}{25} + x\sqrt{3}$ and is drawn in the given diagram



(a) Find an expression $v(t)$, the velocity of P at time *t* seconds.

$v(t) = <20\cos\theta, 20\sin\theta - 8t>$ ✓

Calculator Assumed

9. (b) Find the gradient of the path at the point of projection.

Hence or otherwise prove that $\theta = \dfrac{\pi}{3}$.

$$\frac{dy}{dx} = \frac{-2x}{25} + \sqrt{3}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \sqrt{3} \quad ✓$$

But $\left.\dfrac{dy}{dx}\right|_{x=0}$ = direction of $v(0)$. ✓

Direction of = $v(0) = \tan^{-1}\left(\dfrac{20\sin\theta}{20\cos\theta}\right) = \theta$ ✓

Hence, $\tan\theta = \sqrt{3} \Rightarrow \theta = \dfrac{\pi}{3}$ ✓

(c) Find when and where P is moving in a direction that is perpendicular to its angle of projection.

$v(t) = < 10, 10\sqrt{3} - 8t >$

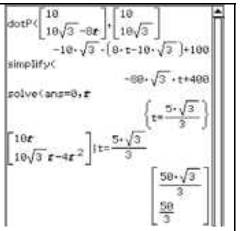
$v(0) = < 10, 10\sqrt{3} >$

$v(t) \bullet v(0) = 0$ ✓

$400 - 80\sqrt{3}\,t = 0 \Rightarrow t = \dfrac{5\sqrt{3}}{3}$ ✓

$r(\frac{5\sqrt{3}}{3}) = \left.<10t, 10\sqrt{3}\,t - 4t^2>\right|_{t=\frac{5\sqrt{3}}{3}}$

$= <\dfrac{50\sqrt{3}}{3}, \dfrac{50}{3}>$ ✓



$\text{dotP}\left(\begin{bmatrix}10\\10\sqrt{3}-8t\end{bmatrix}, \begin{bmatrix}10\\10\sqrt{3}\end{bmatrix}\right)$

$-10\cdot\sqrt{3}\cdot(8\cdot t - 10\cdot\sqrt{3}) + 100$

simplify(

$-80\cdot\sqrt{3}\cdot t + 400$

solve(ans=0,t

$\left\{t = \frac{5\cdot\sqrt{3}}{3}\right\}$

$\begin{bmatrix}10t\\10\sqrt{3}\,t-4t^2\end{bmatrix} | t = \frac{5\cdot\sqrt{3}}{3}$

$\begin{bmatrix}\frac{50\cdot\sqrt{3}}{3}\\ \frac{50}{3}\end{bmatrix}$

(d) Calculate the total distance travelled by P from the point of projection until the point where it meets the x-axis the second time.

Time of flight T = $\dfrac{25\sqrt{3}}{10} = \dfrac{5\sqrt{3}}{2}$ ✓

Distance travelled: D = $\displaystyle\int_0^{\frac{5\sqrt{3}}{2}} \left|<10, 10\sqrt{3} - 8t>\right|\, dt$ ✓

= 59.76 m ✓

Calculator Assumed

10. [11 marks: 5, 3, 3]

Particle P is projected from the origin and simultaneously particle Q is projected from the point with position vector <100, h > m. Both P and Q travel in the x-y plane. The position vector of P at time t seconds is given by $r(t) = < 40t\cos40^{\circ}, 40t\sin40^{\circ} - 4.9t^2 >$ m. The velocity vector of Q at time t seconds is given by $v(t) = < -10\cos20^{\circ}, 10\sin20^{\circ} - 9.8t >$ ms^{-1}.

(a) Determine the value of h for P and Q to collide.

$r_Q = \begin{pmatrix}100 - 10t\cos20^{\circ}\\ 10t\sin20^{\circ} - 4.9t^2 + h\end{pmatrix}$ ✓

At collision: $\begin{pmatrix}40t\cos40^{\circ}\\ 40t\sin40^{\circ} - 4.9t^2\end{pmatrix} = \begin{pmatrix}100 - 10t\cos20^{\circ}\\ 10t\sin20^{\circ} - 4.9t^2 + h\end{pmatrix}$

x-component:

$40t\cos40 = 100 - 10t\cos20$ ✓

$t = 2.4976$ ✓

y-component:

$\left.40t\sin40^{\circ} - 4.9t^2\right|_{t=2.4976} = \left.10t\sin20^{\circ} - 4.9t^2 + h\right|_{t=2.4976}$ ✓

$33.6508 = -22.0239 + h$

$h = 55.6747$ ✓

(b) For the value of h obtained in part (a), determine the angle of impact between P and Q.

$v_P = \begin{pmatrix}40\cos40^{\circ}\\ 40\sin40^{\circ} - 9.8t\end{pmatrix}$ ✓

$v_P(2.4976) = \begin{pmatrix}30.6418\\ 1.2350\end{pmatrix}$ $\quad v_Q(2.4976) = \begin{pmatrix}-93.9693\\ -21.0563\end{pmatrix}$ ✓

$angle\left(\begin{pmatrix}30.6418\\ 1.2350\end{pmatrix}, \begin{pmatrix}-93.9693\\ -21.0563\end{pmatrix}\right) = 169.7^{\circ}$ ✓

(c) For the value of h obtained in part (a), determine the difference in the distance from the origin to the point of collision and the distance travelled by P to the point of collision.

Point of collision $\left.\begin{pmatrix}40t\cos40^{\circ}\\ 40t\sin40^{\circ} - 4.9t^2\end{pmatrix}\right|_{t=2.4976} = \begin{pmatrix}76.5309\\ 33.6508\end{pmatrix}$

Distance to origin = 83.6024 ✓

Distance travelled = $\displaystyle\int_0^{2.4976} \left|\begin{pmatrix}40\cos40^{\circ}\\ 40\sin40^{\circ} - 9.8t\end{pmatrix}\right| dt = 85.1563$ ✓

Hence, path is longer by 1.55 m. ✓

28 Vector Calculus II

Calculator Assumed

1. [9 marks: 5, 2, 2]

A particle P is projected from the top of a cliff of height 100 m. Define x and y metres as the horizontal and vertical displacements of the particle from its point of projection after t seconds. Its equation of motion at any time t seconds is given by $\frac{d\mathbf{r}}{dt} = < 20\sqrt{2}\ ,\ 20\sqrt{2} - 10t >$ where $\mathbf{r} = < x, y >$.

(a) The equation of the path of P is given by $y = ax^2 + bx + c$. Find a, b and c.

$\frac{dy}{dt} = 20\sqrt{2} - 10t \ \Rightarrow\ y = 20\sqrt{2}\,t - 5t^2 + \text{D}$

$y(0) = 0 \qquad \Rightarrow\ y = 20\sqrt{2}\,t - 5t^2.$ ✓

$\frac{dx}{dt} = 20\sqrt{2} \qquad \Rightarrow\ x = 20\sqrt{2}\,t + \text{C}$

$x(0) = 0 \qquad \Rightarrow\ x = 20\sqrt{2}\,t \ \Rightarrow\ t = \frac{x\sqrt{2}}{40}.$ ✓

Hence, $y = 20\sqrt{2} \times \frac{x\sqrt{2}}{40} - 5\left(\frac{x\sqrt{2}}{40}\right)^2 = x - \frac{1}{160}x^2.$ ✓

$\Rightarrow a = 1, b = -\frac{1}{160}, c = 0.$ ✓✓

(b) Find the highest point above ground level reached by P.

$y = x - \frac{1}{160}x^2.$

Max point is (80, 40). ✓

Hence, highest point is 140 m above ground level. ✓

(c) The particle hits the ground at K. Find the distance between K and the foot of the cliff.

When it hits the ground $y = -100$.

$\Rightarrow\ x - \frac{1}{160}x^2 = -100$ ✓

$x = 229.6663$ or -69.6663 (reject)

Hence, distance is 229.7 m. ✓

Calculator Assumed

2. [8 marks: 4, 4]

A missile is fired at a target from the origin O, with equations of motion, t seconds after it was fired, given by $\frac{dx}{dt} = u\cos\theta$ and $\frac{dy}{dt} = u\sin\theta - 10t$, where the constants u and θ are respectively the speed of projection and the angle of projection (with the horizontal axis).

(a) Prove that the equation of the path is $y = x\tan\theta - \frac{5x^2}{u^2}(1 + \tan^2\theta)$.

$\frac{dy}{dt} = u\sin\theta - 10t \ \Rightarrow\ y = u\sin\theta\ t - 5t^2$ as $y(0) = 0$ ✓

$\frac{dx}{dt} = u\cos\theta \qquad \Rightarrow\ x = u\cos\theta\ t$ as $x(0) = 0$

$\Rightarrow\ t = \frac{x}{u\cos\theta}.$ ✓

Hence, $y = u\sin\theta \times \frac{x}{u\cos\theta} - 5\left(\frac{x}{u\cos\theta}\right)^2.$ ✓

$y = x\tan\theta - \frac{5x^2}{u^2\cos^2\theta}$ ✓

$y = x\tan\theta - \frac{5x^2}{u^2}(1 + \tan^2\theta).$

(b) Prove that for the missile to hit a target located at the point with position vector < a, b >, $a^2u^4 - 20\,ba^2u^2 - 100\,a^4 \geq 0$.

$y = x\tan\theta - \frac{5x^2}{u^2}(1 + \tan^2\theta).$

$x = a, y = b$: $\ b = a\tan\theta - \frac{5a^2}{u^2}(1 + \tan^2\theta)$ ✓

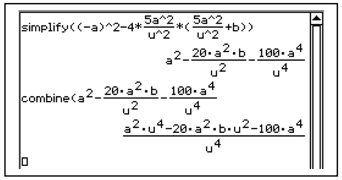
$\frac{5a^2}{u^2}\tan^2\theta - a\tan\theta + \frac{5a^2}{u^2} + b = 0.$ ✓

For equation to have real roots, $\Delta \geq 0$.

Hence, $(-a)^2 - 4 \times \frac{5a^2}{u^2} \times (\frac{5a^2}{u^2} + b) \geq 0$ ✓

$a^2 - \frac{20a^2b}{u^2} - \frac{100a^4}{u^4} \geq 0$ ✓

$a^2u^4 - 20\,ba^2u^2 - 100\,a^4 \geq 0$



simplify((-a)^2-4*$\frac{5a\text{^}2}{u\text{^}2}$*($\frac{5a\text{^}2}{u\text{^}2}$+b))

$a^2 - \frac{20\cdot a^2\cdot b}{u^2} - \frac{100\cdot a^4}{u^4}$

combine($a^2 - \frac{20\cdot a^2\cdot b}{u^2} - \frac{100\cdot a^4}{u^4}$

$\frac{a^2\cdot u^4 - 20\cdot a^2\cdot b\cdot u^2 - 100\cdot a^4}{u^4}$

Calculator Assumed

3. [9 marks: 7, 2]

A particle P moving in the *x*-*y* plane starts from the point with position vector $< 0, -\frac{1}{\pi} >$ m with equation of motion given by $\frac{d\boldsymbol{r}}{dt} = < \cos(\pi t), \sin(\pi t) >$ where $\boldsymbol{r} = < x, y >$ m and *t* is time in minutes.

(a) Show that the path of P is a circle.
State the centre and radius of this circular path.

$\frac{dx}{dt} = \cos(\pi t) \quad \Rightarrow \quad x = \frac{1}{\pi}\sin(\pi t) + C$ ✓

$t = 0, x = 0 \Rightarrow C = 0 \quad \Rightarrow \quad x = \frac{1}{\pi}\sin(\pi t)$

$\sin(\pi t) = \pi x$ I ✓

$\frac{dy}{dt} = \sin(\pi t) \quad \Rightarrow \quad y = -\frac{1}{\pi}\cos(\pi t) + D.$ ✓

$t = 0, y = -\frac{1}{\pi} \Rightarrow D = 0 \Rightarrow y = -\frac{1}{\pi}\cos(\pi t).$

$\cos(\pi t) = -\pi y$ II ✓

Combine I and II $\sin^2(\pi t) + \cos^2(\pi t) = 1$

$(\pi x)^2 + (-\pi y)^2 = 1$ ✓

$\Rightarrow \quad x^2 + y^2 = \frac{1}{\pi^2}$

Therefore the path of P is a circle with centre at (0, 0) and radius $\frac{1}{\pi}$ m.
✓ ✓

(b) Find the period of the motion for P. Justify your answer.

Period for *x* and $y = \frac{2\pi}{\pi} = 2$ minutes ✓
Hence, period for whole motion is 2 minutes. ✓

Calculator Assumed

4. [10 marks: 4, 3, 3]

A particle P moves in the *x*-*y* plane and starts from the point (4, 0). The velocity of P at time *t* is given by $\frac{dx}{dt} = 4y$ and $\frac{dy}{dt} = -x$.

(a) Show that the path of P is an ellipse.

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-x}{4y}$ ✓

$\int 4y\, dy = \int -x\, dx$ ✓

$\frac{4y^2}{2} = -\frac{x^2}{2} + C$

$x^2 + 4y^2 = D$ ✓

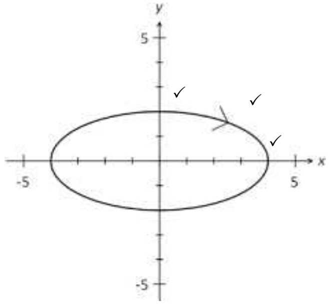
$x = 4, y = 0 \quad \Rightarrow D = 16$

Hence: $x^2 + 4y^2 = 16$ ✓

$\left(\frac{x}{2}\right)^2 + y^2 = 4$

Therefore, the path is an ellipse.

(b) Sketch the path of P and indicate its direction of motion.



(c) Find the period of the motion. Justify your answer.

$\frac{dx}{dt} = 4y: \Rightarrow \frac{d^2x}{dt^2} = 4\frac{dy}{dt} \Rightarrow \frac{d^2x}{dt^2} = -4x = -(2)^2x .$ ✓

Similarly, $\frac{dy}{dt} = -x: \Rightarrow \frac{d^2y}{dt^2} = -\frac{dx}{dt} \Rightarrow \frac{d^2y}{dt^2} = -4y = -(2)^2y .$ ✓

Hence, *x* and *y* undergo simple harmonic motion with ω = 2 and hence period = $\frac{2\pi}{2} = \pi$. ✓

Calculator Assumed

5. [9 marks: 2, 4, 3]

A particle moves in the *x*-*y* plane. Its position at time *t* seconds is given by
$x = 2\cos t + \cos 2t$ and $y = 2\sin t + \sin 2t$
where *x* and *y* are both measured in metres.

(a) Find an expression for $\frac{dy}{dx}$ in terms of *t*.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt}\\ &= \frac{2\cos t + 2\cos 2t}{-2\sin t - 2\sin 2t} \quad ✓✓\\ &= -\left(\frac{\cos t + \cos 2t}{\sin t + \sin 2t}\right)\end{aligned}$$

(b) Find *t* for $0 \le t \le 2\pi$, when $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 0 \Rightarrow -\left(\frac{\cos t + \cos 2t}{\sin t + \sin 2t}\right) = 0$

$\cos t + \cos 2t = 0$ ✓

$t = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ ✓

Reject $t = \pi$, as $\sin \pi + \sin 2\pi = 0$ ✓

Hence, $\frac{dy}{dx} = 0$ for $t = \frac{\pi}{3}, \frac{5\pi}{3}$ ✓

(c) Use your answer in (b) to find the coordinates of point with the highest *y*-coordinate.

$y = 2\sin t + \sin 2t$

When $t = \frac{\pi}{3}, y = \frac{3\sqrt{3}}{2}$ ✓

$t = \frac{5\pi}{3}, y = -\frac{3\sqrt{3}}{2}$ ✓

Hence, *y* is highest when $t = \frac{\pi}{3}$

Hence, the highest point is $\left(\frac{1}{2}, \frac{3\sqrt{3}}{2}\right)$. ✓

Calculator Assumed

6. [10 marks: 4, 3, 3]

A particle moves in the *x*-*y* plane.
Its position (*x*, *y*) at time *t* seconds satisfies the pair of differential equations:
$\frac{dx}{dt} = 10t$ and $\frac{dy}{dt} = 10(\sqrt{3} - t)$ where *x* and *y* are both measured in metres.
The particle starts moving from the point with coordinates (0, 4).

(a) Determine the position (*x*, *y*) of the particle at time *t* = 2 seconds.

$\frac{dx}{dt} = 10t \Rightarrow x = 5t^2$ ✓

$\frac{dy}{dt} = 10(\sqrt{3} - t) \Rightarrow y = 10\sqrt{3}\,t - 5t^2 + 4$ ✓

Hence, when *t* = 2:

$x = 20$ m, $y = 20\sqrt{3} - 16 = 18.64$ m ✓✓

(b) Determine the Cartesian equation of the path travelled by the particle.

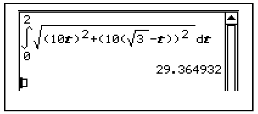
$x = 5t^2 \Rightarrow t = \sqrt{\frac{x}{5}}$ ✓

$$\begin{aligned}y &= 10\sqrt{3}\,t - 5t^2 + 4\\ &= 10\sqrt{3} \times \sqrt{\frac{x}{5}} - 5 \times \left(\frac{x}{5}\right) + 4 \quad ✓\\ &= 2\sqrt{15x} - x + 4 \quad ✓\end{aligned}$$

(c) The distance travelled by the particle in the interval $a \le t \le b$ is given by $S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$. Calculate the distance travelled in the first two seconds.

$S = \int_0^2 \sqrt{(10t)^2 + \left(10(\sqrt{3} - t)\right)^2}\ dt$ ✓✓

$= 29.36$ m ✓



$\int_0^2 \sqrt{(10t)^2+(10(\sqrt{3}-t))^2}\,dt$

29.364932

Calculator Assumed

7. [6 marks: 2, 1, 3]

A particle P starts moving from the origin in the x-y plane. Its position vector and velocity at time t seconds is given by $\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ cm and $\boldsymbol{v} = \begin{pmatrix} 1+y \\ 2-x \end{pmatrix}$ cms^{-1} respectively.

(a) Show that $\frac{dx}{dt} = 1 + y$ and $\frac{dy}{dt} = 2 - x$.

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} \quad ✓$$

$$\text{But } \boldsymbol{v} = \begin{pmatrix} 1+y \\ 2-x \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1+y \\ 2-x \end{pmatrix} \quad ✓$$

$$\text{Hence:} \quad \frac{dx}{dt} = 1 + y \text{ and } \frac{dy}{dt} = 2 - x$$

(b) Hence, determine an expression for $\frac{dy}{dx}$ in terms of x and y.

$$\frac{dx}{dt} = 1 + y \text{ and } \frac{dy}{dt} = 2 - x$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2-x}{1+y} \quad ✓$$

(c) Use your answer in (b) to prove that the path of P is a circle with equation $(x-2)^2 + (y+1)^2 = 5$

$$\frac{dy}{dx} = \frac{2-x}{1+y}$$

Separate the variables:

$$\int (1+y)\ dy = \int (2-x)\ dx \quad ✓$$

$$\frac{(1+y)^2}{2} = \frac{(2-x)^2}{-2} + C \quad ✓$$

$$x = 0, y = 0 \ \Rightarrow \ C = \frac{5}{2} \quad ✓$$

$$(x-2)^2 + (y+1)^2 = 5$$

29 Sampling Distributions

Calculator Assumed

1. [4 marks: 1, 2, 1]

Bryce sells rice in bags with a mean of 9.95 kg and standard deviation of 0.05 kg. Samples of size n are selected and the mean mass of each sample calculated.

(a) Describe the probability distribution for the mean mass of the samples.

Normally distributed with mean 9.95 kg and standard deviation = $\frac{0.05}{\sqrt{n}}$ ✓

(b) Find n if the sampling distribution has a standard deviation of 0.008.

$$\frac{0.05}{\sqrt{n}} = 0.008 \ \Rightarrow \ n = 39 \quad ✓✓$$

(c) Comment on the value of the standard deviation of the sampling distribution as the sample size n increases.

As n increases, standard deviation of the sampling distribution decreases. ✓

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2. [8 marks: 3, 3, 2]

SugarWest sells sugar in packs with mean 995 g with standard deviation 15 g. Samples of size n are selected and the mean mass of each sample calculated.

(a) Find the probability that the mean mass:

(i) of a sample of 40 packs exceeds 1000 g.

Let $\overline{W}$: mean mass of a sample of size 40.

$\overline{W} \sim \mathrm{N}(995, \frac{225}{40})$. ✓✓

$P(\overline{W} \geq 1000) = 0.01751$. ✓

(ii) of a sample of 100 packs exceeds 1000 g.

Let $\overline{W}$: mean mass of a sample of size 100.

$\overline{W} \sim \mathrm{N}(995, \frac{225}{100})$. ✓✓

$P(\overline{W} \geq 1000) = 0.0004291$ ✓

(b) Comment on your answers in (a) and (b).

As n increases from 40 to 100, the standard deviation of the sampling distribution diminishes and hence, probability values for intervals away from the mean decreases. ✓✓

Calculator Assumed

3. [8 marks: 2, 4, 2]

(a) The waiting time at a set of traffic lights may be modelled by a uniform distribution with mean 30 seconds and variance 300. A random sample of 100 waiting times was taken. Describe the distribution of sample mean waiting time $\overline{X}$.

Since $n = 100 > 30$:
$\overline{X}$ is approximately normal ✓
with mean 30 seconds
and standard error $\sqrt{\frac{300}{100}} \approx 1.73205$ sec. ✓

(b) $\overline{W}$ the sample mean waiting times of samples of size n has an approximate normal distribution. One sample of size n had a sample mean waiting time of 28 seconds and sample standard deviation 3.8 seconds. Estimate the value of n if in 1 000 samples of size n, 20 samples had mean waiting times in excess of 29 seconds.

$\overline{W} \sim N(28, \left(\frac{3.8}{\sqrt{n}}\right)^2)$ ✓

$P(\overline{W} > 29) = 0.02 \quad \Rightarrow \quad P(Z > \frac{29-28}{\left(\frac{3.8}{\sqrt{n}}\right)}) = 0.02$ ✓

But $P(Z > 2.05375) = 0.02 \quad \Rightarrow \quad \frac{29-28}{\left(\frac{3.8}{\sqrt{n}}\right)} = 2.05375$ ✓

$n = 60.9 \approx 61$ ✓

(c) 1 000 samples of 50 waiting times each were taken and the mean of each sample calculated. Let $\overline{Y}$ represent the sample mean waiting time. The sample means had a mean of 29.2 seconds and a standard deviation of 4.2 seconds. State with reasons the distribution of $\frac{\overline{Y}-29.2}{4.2}$ and its parameters.

As the number of samples is large, $\frac{\overline{Y}-29.2}{4.2}$ is approximately normal. ✓
Mean ≈ 0 and standard deviation ≈ 1 ✓

Calculator Assumed

4. [9 marks: 3, 3, 3]

The time taken to serve customers at a drive through counter of a fast food restaurant has mean 3 minutes with variance 3 minutes.

(a) Estimate the probability that:
   (i) the time taken to serve 20 successive customers exceeds 61 minutes.

$\overline{T} \sim \text{Normal}\ (\mu = 3, \sigma^2 = \frac{3}{20})$. ✓
$P(\overline{T} > \frac{61}{20}) = 0.4486$. ✓✓

   (ii) no more than 100 minutes is required to serve 30 successive customers.

$\overline{T} \sim \text{Normal}\ (\mu = 3, \sigma^2 = \frac{3}{30})$. ✓
$P(\overline{T} < \frac{100}{30}) = 0.8541$. ✓✓

(b) The probability that the time taken to record 50 successive transactions does not exceed k minutes is approximately 0.1. Find k.

$\overline{T} \sim \text{Normal}\ (\mu = 3, \sigma^2 = \frac{3}{50})$. ✓
$P(\overline{T} < \frac{k}{50}) = 0.1$
$\frac{k}{50} = 2.6861$ ✓
$k = 134.3$ minutes ✓

5. [6 marks: 2, 2, 2] [TISC]

The mean mass of a variety of mangoes is 879 g with standard deviation 5 g.

(a) Estimate the probability that a sample of 50 such mangoes have a mean mass not exceeding 880 g.

X: Mean mass of mango
As sample size is large, $\overline{X} \sim N(879, \frac{25}{50})$ ✓
Hence, $P(\overline{X} \le 880) = 0.9214$ ✓

Calculator Assumed

5. (b) 90% of samples of 50 mangoes each have mean sample masses above *k* g. Find *k*.

Hence, $P(\overline{X} > k) = 0.9$ ✓
$k = 878.09$ g ✓

(c) Estimate the probability that the total mass of a sample of 50 such mangoes exceed 43.94 kg.

Mean mass of sample = 43 940/50 = 878.8 g ✓
Hence, $P(\overline{X} > 878.8) = 0.6114$ ✓

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6. [7 marks: 1, 4, 2] [TISC]

The time taken for a child to complete a particular puzzle is normally distributed with mean 3 minutes with standard deviation 20 seconds.

(a) A sample of fifty children of the same age took 2 hours and 35 minutes to complete the puzzle. Find the mean time, in seconds, for this sample.

Sample mean = $\frac{155}{50}$ = 3.1 minutes = 186 seconds ✓

(b) Estimate the probability that a second sample of 50 children of the same age will take a total of more than 2 hours and 35 minutes to complete the puzzle.

$\overline{X}$: Mean time to complete puzzle.
As sample size is large, by the CTL, ✓
$\overline{X} \sim N(\mu = 180 \text{ seconds}, \sigma^2 = \frac{20^2}{50})$ ✓
Target mean time for second sample = $\frac{155 \times 60}{50}$ = 186 seconds ✓
Hence, $P(\overline{X} \geq 186 \text{ seconds}) = 0.01695$ ✓

(c) Children who complete the puzzle under *k* seconds are classified "highly gifted". If 0.01% of all children are classified highly gifted, find *k*.

X: time to complete puzzle.
$X \sim N(\mu = 180 \text{ seconds}, \sigma^2 = 20^2)$
Hence, $P(\overline{X} < k) = 0.0001$ ✓
$k = 105.6$ seconds ✓

Calculator Assumed

7. [6 marks: 3, 3] [TISC]

The mean time taken by an airline to fly between two cities is 240 minutes with a standard deviation of 15 minutes.

(a) Calculate the probability that in 80 flights between these two cities, the mean flight time is less than 241 minutes. State any assumptions you may need to make.

As sample size $n = 80$ is large,
sample means will have an approximate normal distribution ✓
with mean 240 and standard deviation $\frac{15}{\sqrt{80}}$. ✓
$P(\overline{X} < 241) = 0.7245$ ✓

(b) The airline flies *n* times between these two cities. The probability that the mean time for these flights is less than 241 minutes is 0.7929.
Calculate the value of *n*. Show clearly how you obtained your answer.

$\overline{X} \sim N(240, \left(\frac{15}{\sqrt{n}}\right)^2)$ ✓
$P(\overline{X} < 241) = 0.7929$
$P(Z < \frac{241 - 240}{\left(\frac{15}{\sqrt{n}}\right)}) = 0.7929$ ✓
$\Rightarrow \frac{241 - 240}{\left(\frac{15}{\sqrt{n}}\right)} = 0.81652$
$n = 150$ ✓

30 Point & Interval Estimates for μ

Calculator Free

1. [9 marks: 2, 3, 4] [TISC]

The time taken to complete a task T has mean μ minutes and standard deviation 10 minutes. For Z as the standard normal variable with mean 0 and standard deviation 1, $P(-2.5 < Z < 2.5) \approx 0.988$.

(a) A sample of 100 students completed task T with a mean time of 102 minutes. Find a 98.8% confidence interval for μ.

98.8% confidence interval for μ is:

$$102 \pm 2.5 \times \frac{10}{\sqrt{100}} \quad ✓$$

$$102 \pm 2.5 \quad ✓$$

That is $99.5 \le \mu \le 104.5$ minutes

(b) Another sample of n students (where $n \ge 30$) is chosen. Find n if we are to be 98.8% confident that the sample mean is to differ from μ by no more than 1.25 minutes.

$$\mu - \overline{X} = \pm 2.5 \times \left(\frac{10}{\sqrt{n}}\right)$$

Hence, $2.5 \times \left(\frac{10}{\sqrt{n}}\right) \le 1.25$ ✓

$$\sqrt{n} \ge \frac{2.5 \times 10}{1.25} \quad ✓$$

$$\ge 20$$

$$n \ge 400. \quad ✓$$

(c) Given that μ = 100 minutes, estimate the probability that a sample of 100 students will complete the task with a mean time exceeding 102.5 minutes.

As n is large, $\overline{X} \sim N(\mu = 100, \sigma = \frac{10}{\sqrt{100}} = 1)$. ✓

Hence, $P(\overline{X} > 102.5) = P(Z > \frac{102.5 - 100}{1})$ ✓

$$= P(Z > 2.5)$$

$$= P(Z > 0) - P(0 \le Z \le 2.5) \quad ✓$$

$$= 0.5 - \frac{0.988}{2}$$

$$= 0.5 - 0.494$$

$$= 0.006 \quad ✓$$

Calculator Assumed

2. [8 marks: 2, 3, 3]

Let μ and σ respectively be the mean wing-span of the Australian wedge tail eagle and its associated standard deviation.

(a) The wing-spans of a random sample of 100 eagles are measured. The mean wing-span for this sample is 2.2 m with sample standard deviation 0.12 m.

(i) Find a point estimate for μ and σ. Justify your answer.

Since, sample size is large, ✓
Point estimate for μ = sample mean = 2.2 m
Point estimate for σ = sample std. dev. = 0.12 m ✓

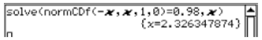
(ii) Find a 98% confidence interval for μ.

$P(-k < Z < k) = 0.98 \Rightarrow k = 2.32635$ ✓
Use sample std. dev. as an estimate for σ.
Hence, a 98% confidence interval is

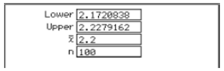
$$2.2 - 2.32635 \times \frac{0.12}{\sqrt{100}} < \mu < 2.2 + 2.32635 \times \frac{0.12}{\sqrt{100}} \quad ✓$$

$$\Rightarrow \quad 2.1720838 < \mu < 2.2279162$$

$$2.17\text{m} < \mu < 2.23\text{ m} \quad ✓$$



```
solve(normCDf(-x,x,1,0)=0.98,x)
                    {x=2.326347874}
```



Lower 2.1720838
Upper 2.2279162
$\overline{x}$ 2.2
n 100

(b) For σ = 0.12, find the sample size n such that the 90% confidence interval for μ differs from the sample mean by no more than 0.05 m.

$$1.645 \times \frac{0.12}{\sqrt{n}} \le 0.05 \quad ✓✓$$

$$n \ge 16 \quad ✓$$

---

3. [9 marks: 2, 4, 3]

The time required to refuel a bus has mean μ minutes and standard deviation 1 minute. A sample of 50 buses took 248 minutes to refuel.

(a) Use the data from the sample provided to describe the sampling distribution of sample means of size 50.

Since $n = 50 > 30$, sampling distribution is approximately normal ✓
with mean $= \frac{248}{50} = 4.96$ and standard deviation $\frac{1}{\sqrt{50}} \approx 0.14142$. ✓

Calculator Assumed

3. (b) Use the data from the sample provided to calculate a 95% confidence interval for μ. Explain what this 95% confidence interval means.

$4.96 \pm 1.96 \times \frac{1}{\sqrt{50}}$ ✓✓

$4.68 \le \mu \le 5.24$ ✓

We are 95% confident that μ lies within a 95% confidence interval. ✓

(c) A second sample of 100 buses provided a confidence interval for μ as $4.58 \le \mu \le 5.02$. Determine the confidence level associated with this interval.

Margin of error $= \frac{5.02 - 4.58}{2} = 0.22$ ✓

Hence, $z \times \frac{1}{\sqrt{100}} = 0.22 \Rightarrow z = 2.2$ ✓

$P(-2.2 \le Z \le 2.2) = 0.9722$

Hence, confidence level of 97.2%. ✓

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4. [9 marks: 3, 3, 3] [TISC]

The amount of spring water in each bottle sold by a manufacturer has a mean of μ mL with a standard deviation of 1 mL.

(a) A randomly chosen sample of 100 bottles had a mean volume of 999.8 mL with a standard deviation of 1 mL. Use this sample to determine a 95% confidence interval for μ.

$999.8 - 1.96 \times \frac{1}{\sqrt{100}} < \mu < 999.8 + 1.96 \times \frac{1}{\sqrt{100}}$ ✓✓

That is: $999.604 < \mu < 999.996 \text{ cm}^3$ ✓

(b) Another random sample of 200 bottles gave a confidence interval with a margin of error of 0.15. Determine the level of confidence of this interval.

Error = 0.15 $\Rightarrow z \times \frac{1}{\sqrt{200}} = 0.15$ ✓

$z = 2.1213$ ✓

$P(-2.1213 \le Z \le 2.1213) = 0.9661$

Hence, a 96.6% confidence interval. ✓

(c) Calculate the minimum sample size for a 90% confidence interval for μ with a margin of error of less than 0.1.

Error = 0.1 $\Rightarrow 1.645 \times \frac{1}{\sqrt{n}} < 0.1$ ✓✓

$n > 270.6 \Rightarrow n \ge 271$ ✓

Calculator Assumed

5. [13 marks: 2, 4, 1, 3, 2, 1]

Let μ and σ respectively be the mean mass of locally farmed yabbies and its associated standard deviation.

(a) The masses of a random sample of 200 yabbies are measured. The mean mass for this sample is 95.6 g with a sample standard deviation of 9.2 g.

(i) Find a point estimate for μ and σ. Justify your answer.

Since, sample size is large, ✓
Point estimate for μ = sample mean = 95.6 g
Point estimate for σ = sample std. dev. = 9.2 g ✓

(ii) Find a 95% and a 99% confidence interval for μ.

Use sample std. dev. as an estimate for σ.
A 95% confidence interval for μ is
$94.32 < \mu < 96.88$ ✓✓
A 99% confidence interval for μ is
$93.92 < \mu < 97.28$ ✓✓

(iii) Comment on the statistical implications of the differing widths of the two intervals in (ii).

A confidence interval of larger width has a higher confidence level. ✓

(b) For σ = 9.2, find the sample size *n* such that the 90% confidence interval for μ:

(i) has width 3 g.

Hence, $1.645 \times \frac{9.2}{\sqrt{n}} = 1.5$ ✓✓

$n = 102$ ✓

(i) has width 1 g.

$1.645 \times \frac{9.2}{\sqrt{n}} = 0.5$ ✓

$n = 916$ ✓

(iii) Comment on the statistical implications of the differing widths of the two intervals in (i) & (ii) for a given level of confidence.

A confidence interval of smaller width requires a larger sample size. ✓

Calculator Assumed

6. [10 marks: 1, 2, 3, 2, 2]

The heights of year 8 students in a state is known to be normally distributed with mean μ cm and population standard deviation 11.5 cm. A random sample of size 20 is selected. The mean height of the students in this sample is 152 cm.

(a) Find a point estimate for μ.

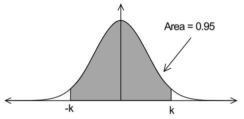
Point estimate for μ = sample mean
= 152 cm ✓

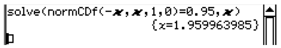
(b) State the probability distribution for the sample mean heights.

Normally distributed with mean 152 cm ✓
and standard deviation = $\frac{11.5}{\sqrt{20}}$ = 2.5715 ✓

(c) Show clearly how a 95% confidence interval for μ is derived using the probability distribution for the sample means.

P(–*k* < Z < *k*) = 0.95 ⇒ *k* = 1.96 ✓
Hence, a 95% confidence interval is
$152 - 1.96 \times \frac{11.5}{\sqrt{20}} < \mu < 152 + 1.96 \times \frac{11.5}{\sqrt{20}}$ ✓
⇒ 146.96 < μ < 157.04. ✓





```
solve(normCDf(-x,x,1,0)=0.95,x)
                {x=1.959963985}
```

(d) Explain what a 99% confidence interval for μ is.

If samples of size *n* are repeatedly taken and the sample means $\bar{x}$ calculated, 99% of intervals calculated using the formula ✓

$$\bar{x} - 2.576 \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.576 \times \frac{\sigma}{\sqrt{n}}$$ will contain μ. ✓

(e) The mean height of another sample was measured to be 159 cm. Use your answer in (c) to determine with reasons if the students in the second sample are unusually tall for their age.

Students' mean height is beyond the upper limit of the 95% confidence interval. ✓
Hence, students' are unusually tall for their age. ✓

Calculator Assumed

7. [8 marks: 1, 1, 4, 2]

The age of mathematics teachers in a certain geographical region is known to be normally distributed with a mean of μ and a population standard deviation of 5 years 3 months.

(a) A sample of 25 teachers is taken. The mean age of teachers in this sample is 54 years 2 months.

(i) Find a point estimate for μ.

Point estimate for μ = sample mean
= 54 years 2 months ✓

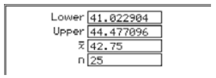
(ii) Given that all 25 teachers from this sample were selected from just one suburb, determine mathematically if your estimate in (i) is reliable.

Point estimate for μ is likely to be unreliable as there is statistical bias. ✓

(b) A stratified sample of 25 teachers is selected from this region and the mean age of the teachers in this sample is 42 years 9 months.

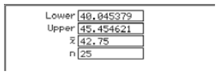
(i) Find a 90% and a 99% confidence interval for μ.

90% confidence interval for μ is:
41.022904 < μ < 44.477096 ✓
That is:
41 years 0 months < μ < 44 years 6 months. ✓



```
Lower 41.022904
Upper 44.477096
    x̄ 42.75
    n 25
```

99% confidence interval for μ is:
40.045379 < μ < 45.454621 ✓
That is:
40 years 1 month < μ < 45 years 5 months. ✓



```
Lower 40.045379
Upper 45.454621
    x̄ 42.75
    n 25
```

(ii) The mean age for a second sample is exactly 40 years and 4 months old. Determine with reasons if the teachers in this sample are younger than those in the first sample.

Mean age for second sample lies within the 99% confidence interval but outside the 90% confidence interval. ✓
Hence, yes if the 90% confidence interval is used but no if the 99% confidence interval is used. ✓

Calculator Assumed

8. [10 marks: 2, 4, 2, 2]

Let μ and σ respectively be the mean telephone waiting time for calls to a certain bank and its associated standard deviation.

(a) The waiting times for a random sample of 150 customers were recorded. The mean waiting time was 22.6 minutes with a sample standard deviation of 2.7 minutes. Find a 99% confidence interval for μ.

A 99% confidence interval for μ is
$22.03 < \mu < 23.17$ minutes ✓✓

(b) The waiting times for a second random sample of 150 customers were recorded. The mean waiting time was $\bar{x}$ minutes with a sample standard deviation of *s* minutes. A 99% confidence interval for μ is $17.78 < \mu < 19.22$ minutes. Find $\bar{x}$ and *s*.

Use *s* as an estimate for σ.

$$\bar{x} = \frac{17.78 + 19.22}{2} = 18.5$$ ✓

$$\text{Error} = 2.576 \times \frac{s}{\sqrt{150}} = 19.22 - 18.5$$ ✓✓

$$\Rightarrow \quad s = 3.4232$$ ✓

(c) Use your answers in (a) and (b) to explain clearly why for a given population, there exists more than one 99% confidence interval estimates of μ.

From a population of size N, there will be many random samples of size *n* (*n* < N). Hence, there will be many estimates for μ and σ, giving rise to many such 99% confidence interval estimates for μ. ✓ ✓

(d) For σ = 2.7, find the sample size *n* such that the 95% confidence interval for μ differs from the sample mean by no more than 0.5 minutes.

$$1.96 \times \frac{2.7}{\sqrt{n}} \le 0.5$$ ✓

$$n \ge 112$$ ✓

Calculator Assumed

9. [7 marks: 2, 3, 2] [TISC]

Manufacturers must label food "genetically modified" if more than 1% of ingredients have been genetically modified. The manufacturer tests a sample of 100 one kg packs of this food product and calculates the mean mass of genetically modified ingredients included in the packs is 9.8 g. Assume that the population standard deviation of the amount of genetically modified ingredients is 0.88 g per kg.

(a) Calculate a 99% confidence interval for the amount of genetically modified ingredients in 1 kg of the manufactured food.

$P(-k < Z < k) = 0.99 \Rightarrow k = 2.5758$
Hence, a 99% confidence interval is

$$9.8 - 2.5758 \times \frac{0.88}{\sqrt{100}} < \mu < 9.8 + 2.5758 \times \frac{0.88}{\sqrt{100}}$$ ✓

$$\Rightarrow \quad 9.57 < \mu < 10.03$$ ✓

(b) Find the minimum size of a second sample if the 99% confidence limits is not to exceed 10 g.

Let *n*: size of sample
Width of interval, $w < 10 - 9.8$
$\Rightarrow \quad w < 0.2$
Hence, a 99% confidence interval is

$$2.578 \times \frac{0.88}{\sqrt{n}} < 0.2$$ ✓✓

$$\Rightarrow \quad n > 128.7$$

$$n \ge 129$$ ✓

(c) The manufacturer decides to label his food product with the warning "May contain genetically modified ingredients". What percentage of samples would have a mean exceeding 10 g per kilogram?

$\bar{X}$: Mean weight of genetically modified ingredients
As sample size is large,

by the CTL, $\bar{X} \sim N(9.8, \frac{0.88^2}{100})$ ✓

Hence, $P(\bar{X} \ge 10) = 0.01152$ ✓

Hence, 1.2% of samples would have means exceeding 10 g of genetically modified ingredients.
(Significant at 1% level)

Calculator Assumed

10. [10 marks: 4, 4, 2] [TISC]

The mean amount of blood drawn out from an individual donor at a blood bank is μ mL with standard deviation 20 mL.

(a) On a particular day, 50 donors at the blood bank donated a total of 49.9 L of blood. Determine a 95% confidence interval for μ.

Sample mean = $\frac{49.9 \times 1000}{50}$ = 998 mL ✓
95% Confidence Interval:
$998 \pm 1.96 \times \frac{20}{\sqrt{50}}$ ✓✓
998 ± 5.5437
$992.5 \le \mu \le 1003.5$ mL ✓

(b) How many donors should be in a sample if a 90% confidence interval for μ is $995 \le \mu \le 1000$ mL?

Sample mean = $\frac{995 + 1000}{2}$ = 997.5 mL ✓
Error ≤ 1000 – 997.5 = 2.5 mL ✓
Hence: $1.645 \times \frac{20}{\sqrt{n}} \le 2.5$ ✓
$n \ge 173.2$
⇒ At least 174. ✓

(c) A 99% confidence interval for μ is $998 \le \mu \le 1000$ mL.
A student concluded that the probability that the mean amount of blood donated by an individual donor is between 998 mL and 1000 mL is 0.99.
Is the student correct? Justify your answer.

Student is incorrect. ✓
The probability that $998 \le \mu \le 1000$ is either 0 or 1. ✓

Calculator Assumed

11. [9 marks 2, 3, 2, 2] [TISC]

The mass of sugar in a 200 g chocolate bar is normally distributed with mean μ g and standard deviation σ g.

(a) A randomly chosen sample of 400 chocolate bars (200 g each) had a mean sugar mass of 100.5 g with a standard deviation of 0.8 g. Use this sample to determine a 90% confidence interval for μ.

$100.5 - 1.645 \times \frac{0.8}{\sqrt{400}} < \mu < 100.5 + 1.645 \times \frac{0.8}{\sqrt{400}}$ ✓
That is: $100.4342 < \mu < 100.5658$ g ✓

(b) Another random sample of 400 chocolate bars (200 g each) gave a 90% confidence for μ as $99.9088 < \mu < 100.0912$.

(i) Determine the mean mass of sugar for this sample and the accompanying standard deviation.

Mean sample mass = $\frac{99.9088 + 100.0912}{2}$ = 100 g ✓
Margin of error = 100.0912 – 100 = 0.0912
Hence: $1.645 \times \frac{s}{\sqrt{400}} = 0.0912$ ✓
$s = 1.1088$ g ✓

(ii) Determine with reasons if the sample in (a) and sample in (b) are more likely to be from the same population or more likely to be from different populations.

More likely to be from different populations. ✓
Reason:
Sample mean in (a) is 100.5 g which is
outside the 90% confidence interval for sample (b). ✓

(c) Given that μ = 100 g and σ = 0.8 g, calculate the minimum sample size for a 90% confidence interval for μ with an interval width of less than 0.1.

Margin of error = $\frac{0.1}{2} = 0.05$
Hence: $1.645 \times \frac{0.8}{\sqrt{n}} < 0.05$ ✓
$n > 692.7 \Rightarrow n \ge 693$ ✓

Calculator Assumed

12. [10 marks: 2, 3, 3, 1, 1]

The time it takes Tracy to get to work each day is a continuous random variable T with mean μ minutes and standard deviation σ minutes. Tracy took a total of 21 hours to travel to work for her last 100 trips with a sample standard deviation of 1.5 minutes.

(a) State a single value estimate for μ and σ.

Estimate for $\mu = \frac{21 \times 60}{100} = 12.6$ minutes. ✓

Estimate for $\sigma = 1.5$ minutes ✓

(b) Calculate a 95% confidence interval for μ.

$12.6 \pm 1.96 \times \frac{1.5}{\sqrt{100}}$ ✓✓

$12.306 \le \mu \le 12.894$ ✓

(c) For another 100 trips, the sample standard deviation was 1.8 minutes and Tracy obtained $12.0965 \le \mu \le 12.9035$ as a confidence interval for μ.

(i) Determine the confidence level for this interval.

Margin of error $= \frac{12.9035 - 12.0965}{2} = 0.4035$ ✓

Hence: $z \times \frac{1.8}{\sqrt{100}} = 0.4035$

$z = 2.24167$ ✓

$P(-2.24167 \le Z \le 2.24167) = 0.9750$ ✓

Hence, 97.5% confidence level.

(ii) Determine with reasons if the second sample is statistically different from the first sample.

Sample mean of first sample is inside the confidence interval for the second sample. Hence, there is no reason to suggest that the two samples are different. ✓

(d) Tracy calculated a total of thirty 90% confidence intervals for μ. How many of these intervals are expected to contain μ?

Expected number = 30 × 0.9 = 27 ✓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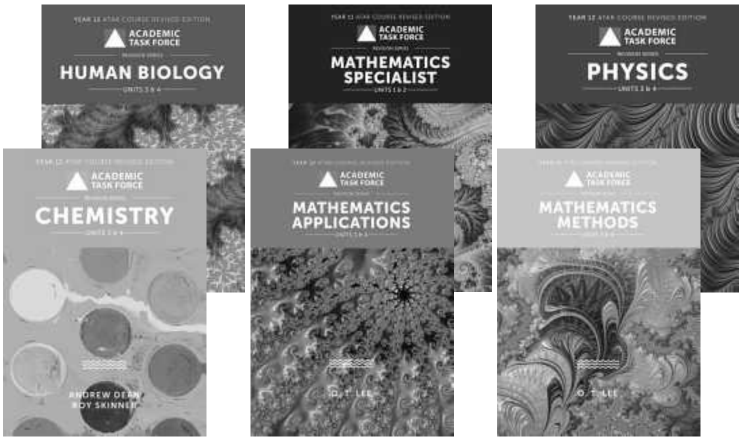
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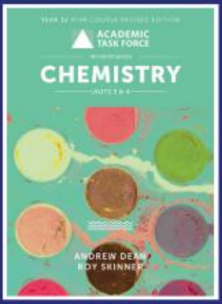

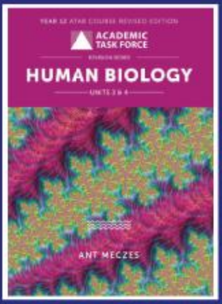

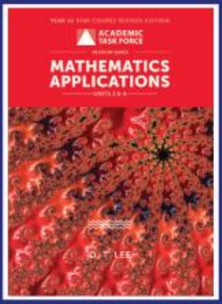

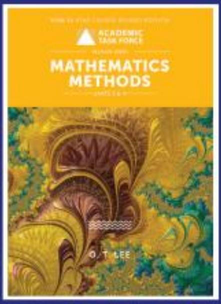

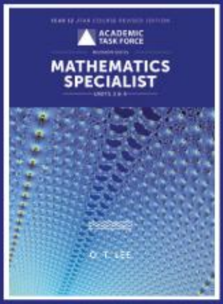

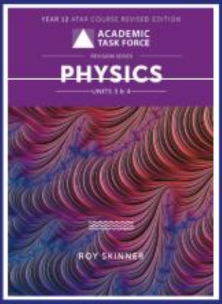

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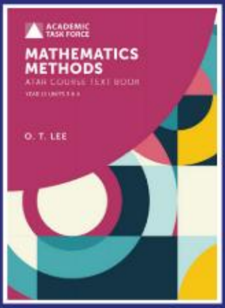

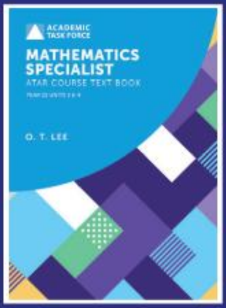

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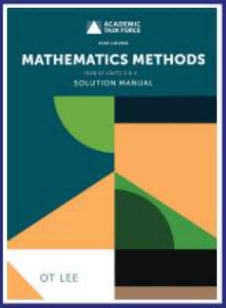

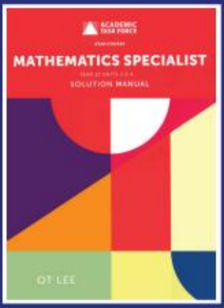

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