PEARSON

MATHEMATICAL METHODS

12

QUEENSLAND

UNITS 3 & 4

STUDENT BOOK

QCE 2019 SYLLABUS

PEARSON

MATHEMATICAL METHODS

QUEENSLAND

STUDENT BOOK

12

UNITS 3 & 4

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Supporting the integrating of technology

Students are supported with the integration of technology in a number of ways. The eBook includes 'How to' user guides covering all basic functionality for the following three graphing calculators:

- TI-84 Plus CE
- TI-Nspire CX (non CAS)
- CASIO fx-CG50AU

Throughout the student book are Technology worked examples strategically placed within the theory for both the TI-Nspire CX (non CAS) and CASIO fx-CG50AU. The examples clearly demonstrate how the technology can be used effectively and efficiently for the content being covered in that chapter.

Graphing calculators are not the only technology integrated throughout the Pearson Queensland senior mathematics series. Spreadsheets, Desmos and interactive widgets have been included to provide students with the opportunity to visualise concepts, consolidate their understanding and make mathematical connections.

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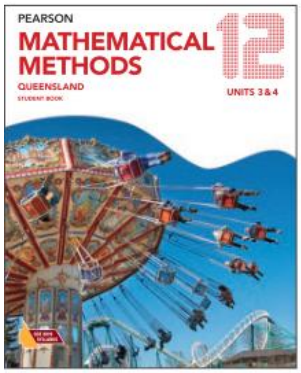


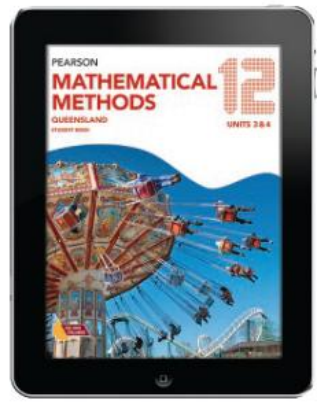
Mathematical Methods 12
Student book

Student book

The student book has been written by local authors, ensuring quality content and complete curriculum coverage for Queensland, enabling students to prepare with ease and confidence. We have covered the breadth of the content within our exercise questions, from simpler skills-focused questions to those using unfamiliar contexts and application of the theory learnt. The theory, worked examples and question sets are written in line with the assessment objectives, with the aim of familiarising students with QCE cognitive verbs in the process of dependent and guided instruction. Additional interactives that help explain the theory and consolidate concepts have been included throughout all chapters.

Pearson Reader+

Pearson Reader+ is our next generation eBook. This is an electronic textbook that students can access on any device, online or offline, and is linked to features, interactives and visual media that will help consolidate their understanding of concepts and ideas, as well as other useful content developed specifically for senior mathematics. It supports students with appropriate online resources and tools for every section of the student book, providing access to exemplar worked solutions that demonstrate high levels of mathematical and everyday communication. Students will have the opportunity to learn independently through the Explore further tasks and Making connections interactive widgets, which have been designed to engage and support conceptual understanding. Additionally, teachers have access to syllabus maps, a teaching program, sample exams, problem-solving and modelling tasks, and additional banks of questions for extra revision.



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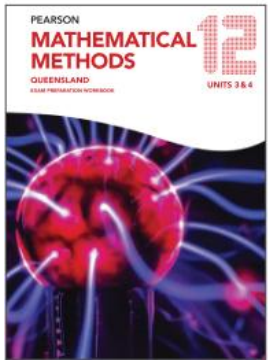


Mathematical Methods 12
Exam preparation workbook

Exam preparation workbook

Additional component for Year 12 only

The exam preparation workbook provides additional support in preparing students for the external exam. It has been constructed to guide the students through a sequence of preparatory steps and build confidence leading up to the external exam.

How to use this book

Pearson Mathematical Methods 12 Queensland Units 3 & 4

This Queensland senior mathematics series has been written by a team of experienced Queensland teachers for the QCE 2019 syllabus. It offers complete curriculum coverage, rich content and comprehensive teachers support.

Additional information

These interactives appear in the eBook in two forms, as videos explaining specific concepts or as interactive questions to check students' understanding.

Key information

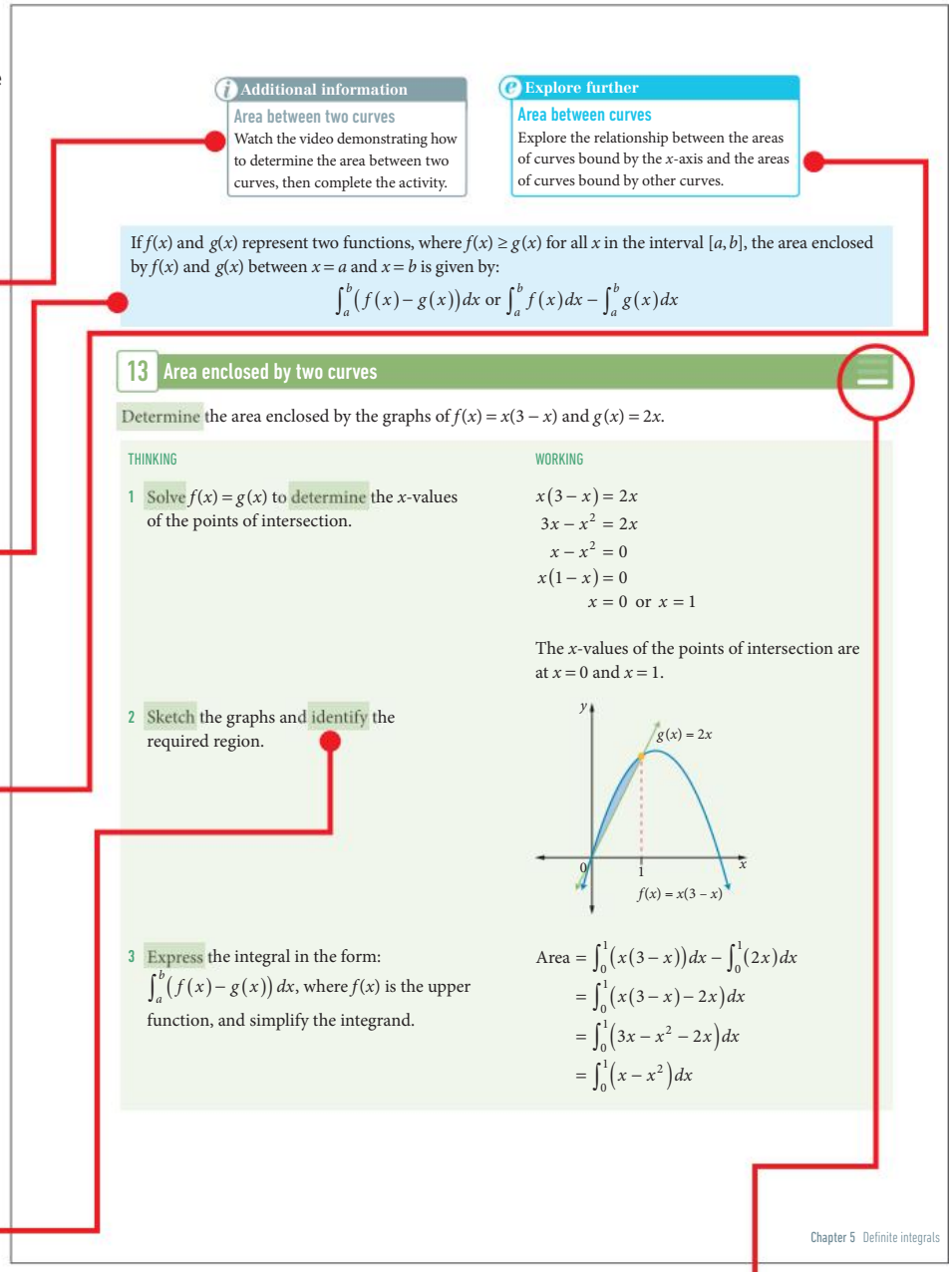
Key information and rules are highlighted throughout the chapter.

Explore further

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skills base. These activities should take approximately 5–15 minutes to complete.

Meeting the needs of the QCE Syllabus

The authors have integrated both the cognitive verbs and the language of the syllabus objectives throughout the worked examples and questions.



Additional information
Area between two curves
Watch the video demonstrating how to determine the area between two curves, then complete the activity.

Explore further
Area between curves
Explore the relationship between the areas of curves bound by the x-axis and the areas of curves bound by other curves.

If $f(x)$ and $g(x)$ represent two functions, where $f(x) \geq g(x)$ for all x in the interval $[a, b]$, the area enclosed by $f(x)$ and $g(x)$ between $x = a$ and $x = b$ is given by:

$$\int_a^b (f(x) - g(x))\,dx \text{ or } \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

13 Area enclosed by two curves

Determine the area enclosed by the graphs of $f(x) = x(3 - x)$ and $g(x) = 2x$.

THINKING

1 Solve $f(x) = g(x)$ to determine the x-values of the points of intersection.

WORKING

$x(3 - x) = 2x$
$3x - x^2 = 2x$
$x - x^2 = 0$
$x(1 - x) = 0$
$x = 0$ or $x = 1$

The x-values of the points of intersection are at $x = 0$ and $x = 1$.

2 Sketch the graphs and identify the required region.

3 Express the integral in the form: $\int_a^b (f(x) - g(x))\,dx$, where $f(x)$ is the upper function, and simplify the integrand.

$$\text{Area} = \int_0^1 (x(3 - x))\,dx - \int_0^1 (2x)\,dx$$
$$= \int_0^1 (x(3 - x) - 2x)\,dx$$
$$= \int_0^1 (3x - x^2 - 2x)\,dx$$
$$= \int_0^1 (x - x^2)\,dx$$

Chapter 5 Definite integrals

Every worked example and question is graded

Every example and question is graded using the three levels of difficulty, as specified in the QCE syllabus:

- simple familiar (1 bar)
- complex familiar (2 bars)
- complex unfamiliar (3 bars)

The visibility of this grading helps ensure all levels of difficulty are well covered.

Making connections

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their understanding.

Making connections

The exponential function and its inverse

Explore the properties of the exponential graph and its inverse, the logarithmic graph.

Technology worked examples

These worked examples offer support in using technology such as spreadsheets, graphing calculators and graphing software, and include technology-focused worked examples and activities.

Technology worked example

Solving exponential equations

Tech-free questions

These questions are designed to provide students with the opportunity to practise algebraic manipulations to prepare them for technology-free examination papers.

Highlighting common errors

Throughout the exercises, authors have integrated questions designed to highlight common errors frequently made by students. Explanations are given in the worked solutions.

EXERCISE 3.1

Logarithmic functions and their properties

1 Simplify each of the following expressions.

(a) $\log_4(1)$ (b) $2\log_4(1)$ (c) $\log_{10}(10)$ (d) $3\log_4(4)$

2 Which expression is equal to $\log_2(x) - \log_2(3x+2)$?

A $\log_2(4x+2)$ B $\log_2\left(3+\frac{3}{x}\right)$ C $\log_2(2x+2)$ D $\log_2\left(\frac{x}{3x+2}\right)$

3 Simplify each of the following expressions.

(a) $\log_5(3)+\log_5(5)$ (b) $\log_8(4)+\log_8(2)$ (c) $\log_3(5)-\log_3(4)$

(d) $\log_2(8)-\log_2(4)$ (e) $\log_{10}(6)-\log_{10}\left(\frac{3}{5}\right)$

(f) $\log_{10}(6)+\log_{10}(5)-\log_{10}(3)$ (g) $\log_2(12)-\log_2(2)-\log_2(3)$

4 Consider the expression $3\log_e(x)+2\log_e(x)+\log_e(4x)$.

(a) Which expression is equivalent to the original?

A $\log_e(2x^2+4x)$ B $\log_e(x^3+x^2+4x)$ C $\log_e(4x^6)$ D $6\log_e(4x)$

(b) Explain the common error made by a student who obtained an expression with a coefficient of 6 for part (a).

5 Simplify each of the following expressions.

(a) $\log_2(2^3)$ (b) $\log_3(27)$ (c) $\log_2\left(\frac{1}{2}\right)$

(d) $\log_{10}(0.01)$ (e) $3\log_8(2)$

(f) $2\log_e(\sqrt{e})$ (g) $-2\log_2\left(\frac{1}{4}\right)$

6 Convert each of the following to logarithmic form.

(a) $3^4 = 81$ (b) $10^3 = 1000$ (c) $2^{-2} = \frac{1}{4}$ (d) $10^{-3} = 0.001$

7 Convert each of the following to exponential form.

(a) $\log_2(16) = 4$ (b) $\log_3(27) = 3$ (c) $\log_2\left(\frac{1}{8}\right) = -3$ (d) $\log_{10}(0.1) = -1$

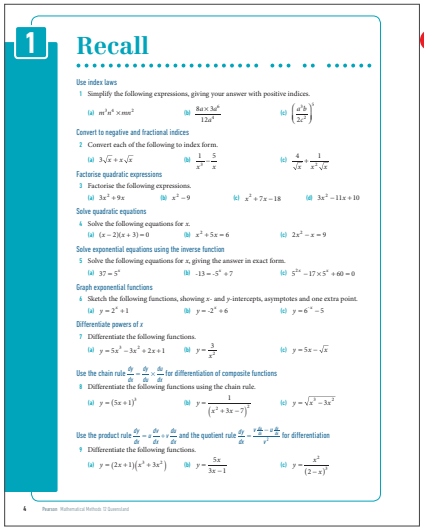
Warning boxes

Warning boxes are located throughout the chapter to alert students to common errors and misconceptions.

WARNING

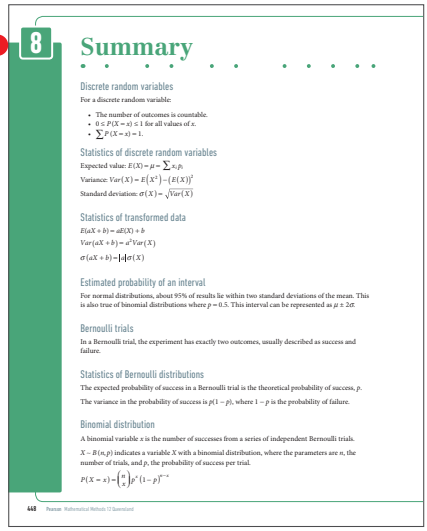
It is a common misconception that these expressions are equivalent.

$$\frac{\log_a(m)}{\log_a(n)} \neq \log_a(m) - \log_a(n) \qquad \log_a(m+n) \neq \log_a(m) + \log_a(n)$$



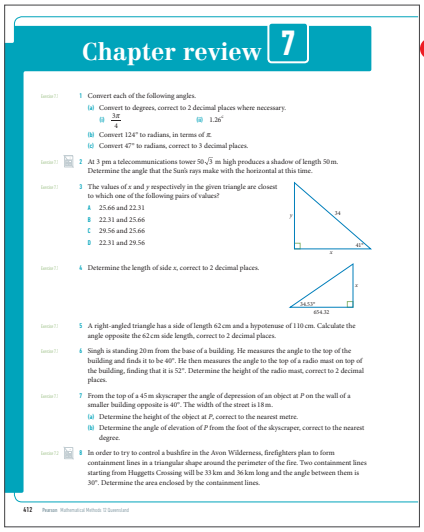
Recall

Each chapter begins with a review of assumed knowledge for the chapter.



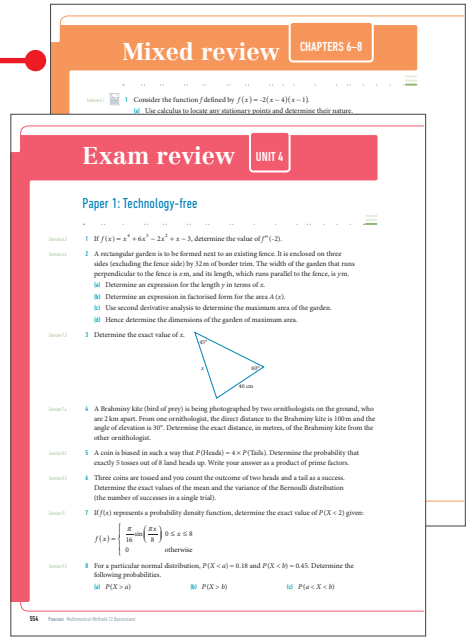
Summary

At the end of each chapter, there is a summary of the key facts and rules discussed in the chapter.



Chapter review

Every chapter review follows the QCAA examination proportions for level of difficulty, which is 60% simple familiar, 20% complex familiar and 20% complex unfamiliar.



Mixed and exam review

Exam reviews provide cumulative practice of content already covered, to prepare students for the end-of-year exam. They have been placed at the end of each unit. Mixed reviews provide cumulative reviews placed midway through each unit.

1

Derivatives of exponential functions

Recall 4

1.1 Euler's number and index laws 5

1.2 Graphing exponential functions 14

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1 Recall

Use index laws

1 Simplify the following expressions, giving your answer with positive indices.

(a) $m^3n^4 \times mn^2$ (b) $\dfrac{8a \times 3a^6}{12a^4}$ (c) $\left(\dfrac{a^3b}{2c^2}\right)^5$

Convert to negative and fractional indices

2 Convert each of the following to index form.

(a) $3\sqrt{x} + x\sqrt{x}$ (b) $\dfrac{1}{x^3} - \dfrac{5}{x}$ (c) $\dfrac{4}{\sqrt{x}} + \dfrac{1}{x^2\sqrt{x}}$

Factorise quadratic expressions

3 Factorise the following expressions.

(a) $3x^2 + 9x$ (b) $x^2 - 9$ (c) $x^2 + 7x - 18$ (d) $3x^2 - 11x + 10$

Solve quadratic equations

4 Solve the following equations for x.

(a) $(x-2)(x+3) = 0$ (b) $x^2 + 5x = 6$ (c) $2x^2 - x = 9$

Solve exponential equations using the inverse function

5 Solve the following equations for x, giving the answer in exact form.

(a) $37 = 5^x$ (b) $-13 = -5^x + 7$ (c) $5^{2x} - 17 \times 5^x + 60 = 0$

Graph exponential functions

6 Sketch the following functions, showing x- and y-intercepts, asymptotes and one extra point.

(a) $y = 2^x + 1$ (b) $y = -2^x + 6$ (c) $y = 6^{-x} - 5$

Differentiate powers of x

7 Differentiate the following functions.

(a) $y = 5x^3 - 3x^2 + 2x + 1$ (b) $y = \dfrac{3}{x^2}$ (c) $y = 5x - \sqrt{x}$

Use the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ for differentiation of composite functions

8 Differentiate the following functions using the chain rule.

(a) $y = (5x+1)^3$ (b) $y = \dfrac{1}{(x^2 + 3x - 7)^2}$ (c) $y = \sqrt{x^3 - 3x^2}$

Use the product rule $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ and the quotient rule $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ for differentiation

9 Differentiate the following functions.

(a) $y = (2x+1)(x^3 + 3x^2)$ (b) $y = \dfrac{5x}{3x-1}$ (c) $y = \dfrac{x^2}{(2-x)^3}$

Euler's number and index laws

1.1

The base, *e*

Many natural phenomena, such as the growth of algal blooms or mouse plagues, and natural decay such as the drying of lakes in a drought, can be modelled using exponential functions.

Scientists often choose the form $A = A_0 e^{kt}$, where e is Euler's number, because it is easy to use with rates of change. Like π, Euler's number is an irrational number. Differentiation of exponential functions is covered later in this chapter.

The number e is defined as $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

$n = 1$: $\left(1 + \frac{1}{1}\right)^1 = 2$

$n = 2$: $\left(1 + \frac{1}{2}\right)^2 = 2.25$

$n = 3$: $\left(1 + \frac{1}{3}\right)^3 = 2.37\overline{037}$

$n = 4$: $\left(1 + \frac{1}{4}\right)^4 = 2.441\,406\,25$

$n = 10$: $\left(1 + \frac{1}{10}\right)^{10} \approx 2.593\,74$

$n = 1000$: $\left(1 + \frac{1}{1000}\right)^{1000} \approx 2.716\,92$

$n = 1\,000\,000$: $\left(1 + \frac{1}{1\,000\,000}\right)^{1\,000\,000} \approx 2.718\,28$

As $n \to \infty$, e becomes 2.718 281 828 459…

The value of e can only be stated approximately using decimals. To 4 decimal places, its approximation is $e = 2.7183$.

Additional information

Euler's number

Try this activity to see an alternative method for evaluating Euler's number, *e*.

Index laws

With numbers where e is the base, the index laws still apply.

Multiplying numbers with indices	$e^m \times e^n = e^{(m+n)}$
Dividing numbers with indices	$\frac{e^m}{e^n} = e^{(m-n)}$
Power of zero	$e^0 = 1$
Negative powers	$\frac{1}{e^m} = e^{-m}$
Power of a power	$\left(e^m\right)^n = e^{(m \times n)}$
Rational indices	$\sqrt[n]{e} = e^{\frac{1}{n}}$

Evaluating a decimal approximation

Most calculators have e^x above the ln (natural logarithm) button.

Remember that $e = e^1$, so expressions involving e, written without a power, will require you to input e^1 when using e^x on the calculator.

1 Approximate and calculate the value of expressions with *e*

Calculate each of the following expressions. Give your answer correct to 2 decimal places.

(a) e^3

THINKING	WORKING
1 Estimate the value of the expression.	$2 < e < 3$ Therefore: $2^3 < e^3 < 3^3$ $8 < e^3 < 27$
2 Calculate the value of the expression using technology.	$e^3 = 20.085\ldots$ $= 20.09\ (2\text{ d.p.})$

(b) $\sqrt{e}$

1 Estimate the value of the expression.	$2 < e < 3$ Therefore: $\sqrt{2} < \sqrt{e} < \sqrt{3}$ $1.41\ldots < \sqrt{e} < 1.73\ldots$
2 Calculate the value of the expression using technology, where $e^1 = e$.	$\sqrt{e} = 1.648\ldots$ $= 1.65\ (2\text{ d.p.})$

(c) $e^3 + 2e$

1 Estimate the value of the expression.	$2 < e < 3$ Therefore: $2^3 + 2 \times 2 < e^3 + 2e < 3^3 + 2 \times 3$ $8 + 4 < e^3 + 2e < 27 + 6$ $12 < e^3 + 2e < 33$
2 Calculate the value of the expression using technology, where $e^1 = e$.	$e^3 + 2e = 25.522\ldots$ $= 25.52\ (2\text{ d.p.})$

Solving equations containing exponential expressions

When two powers are equal, if the bases are the same, then the indices must be equal.
If $e^x = e^5$ then $x = 5$.

Additional information

Solving exponential equations

Watch this video and try the activities to reinforce techniques used to solve basic exponential equations.

2 Solve exponential equations by equating indices

Solve the following equations for x.

(a) $e^x = e$

THINKING	WORKING
1 Recall the equation by writing e as e^1.	$e^x = e^1$
2 Since the bases are the same, equate the indices.	$x = 1$

(b) $e^{x-1} = e^3$

Since the bases are the same, equate the indices and solve for x.	$e^{x-1} = e^3$ $\therefore x - 1 = 3$ $x = 4$

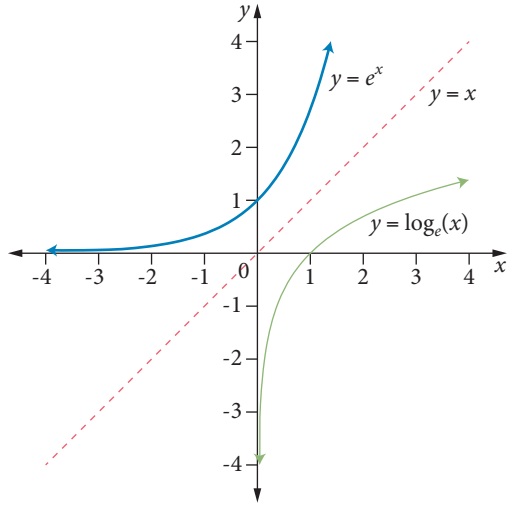
(c) $\dfrac{1}{\sqrt{e}} = \dfrac{e^{2x}}{e}$

1 Use index laws to write the equation in the form $e^a = e^b$.	$\dfrac{1}{\sqrt{e}} = \dfrac{e^{2x}}{e}$ $\dfrac{1}{e^{\frac{1}{2}}} = e^{2x} \times e^{-1}$ $e^{-\frac{1}{2}} = e^{2x-1}$
2 Since the bases are the same, equate the indices and solve for x.	$\therefore 2x - 1 = -\dfrac{1}{2}$ $2x = \dfrac{1}{2}$ $x = \dfrac{1}{4}$

Solving exponential equations using the inverse function

Two functions are said to be inverse functions if their graphs are the reflection of each other across the line $y = x$.

The inverse of an exponential function is the logarithmic function to the same base. The graph on the right shows that $\log_e(x)$ is the reflection of e^x across the line $y = x$. This also shows that logarithms of negative numbers do not exist.



C Making connections

The exponential function and its inverse

Explore the properties of the exponential graph and its inverse, the logarithmic graph.

Hence, when working with equations involving numbers expressed as a power of e, the exact answer can be found using an inverse operation that will involve $\log_e$, which is sometimes expressed as 'ln' (natural log). For example, if $a^x = y$, then $x = \log_a(y)$. So if $e^x = y$, then $x = \log_e(y)$, or $x = \ln(y)$.

3 Solve exponential equations using the natural logarithm

Solve for x in exact form, and then approximate the value, correct to 2 decimal places.

(a) $e^x = 2$

THINKING	WORKING
1 Convert to a logarithm to solve for x.	$e^x = 2$ $\log_e(e)^x = \log_e(2)$ $x \log_e(e) = \log_e(2)$ $x = \log_e(2)$
2 Use the ln key to approximate the value.	$\ln(2) = 0.693\ldots$ $= 0.69\ (2\text{ d.p.})$
3 Evaluate the reasonableness of your solution by substituting the answer into the original equation.	$e^{0.69} = 1.99\ldots$ ≈ 2

(b) $3e^{5x} - 10 = 0$

1 Solve for the power.	$3e^{5x} - 10 = 0$ $3e^{5x} = 10$ $e^{5x} = \frac{10}{3}$

2 Convert to a logarithm, then solve for x in exact form.	$\log_e(e)^{5x} = \log_e\left(\frac{10}{3}\right)$ $5x = \log_e\left(\frac{10}{3}\right)$ $x = \frac{1}{5}\log_e\left(\frac{10}{3}\right)$
3 Use the ln key to approximate the value.	$\frac{1}{5}\ln\left(\frac{10}{3}\right) = 0.240\ldots$ $= 0.24\,(2\text{ d.p.})$
4 Evaluate the reasonableness of your solution by substituting the answer into the original equation.	$3e^{5\times0.24} = 3e^{1.2}$ $= 9.96\ldots$ ≈ 10

More complex exponential equations can be solved using quadratic factorising as a first step.

4 Solve more complex exponential equations

Solve for x, in exact form.

(a) $e^{2x} - 5e^x + 6 = 0$

THINKING	WORKING
	$e^{2x} - 5e^x + 6 = 0$
1 Let $u = e^x$ and express in the form $au^2 + bu + c = 0$.	$u^2 - 5u + 6 = 0$
2 Factorise the quadratic expression.	$(u-2)(u-3) = 0$
3 Replace u with e^x.	$\left(e^x - 2\right)\left(e^x - 3\right) = 0$
4 Use the null factor law to solve for e^x.	$e^x = 2$ or $e^x = 3$
5 Solve for x.	$x = \log_e(2)$ or $x = \log_e(3)$ The solution can also be written as $x = \ln(2)$ or $x = \ln(3)$.

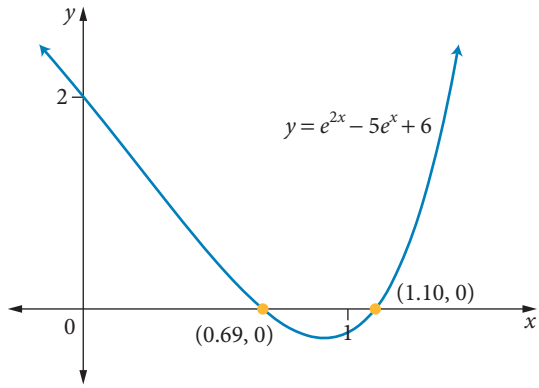
(b) $\frac{8}{e^x} - e^x = 2$

1 Multiply all terms by e^x and write the equation in the form $a\left(e^x\right)^2 + b\left(e^x\right) + c = 0$.	$\frac{8}{e^x} - e^x = 2$ $8 - \left(e^x\right)^2 = 2e^x$ $\left(e^x\right)^2 + 2e^x - 8 = 0$
2 Factorise the left-hand side.	$\left(e^x + 4\right)\left(e^x - 2\right) = 0$
3 Use the null factor law to solve for e^x.	$e^x = -4$ or $e^x = 2$

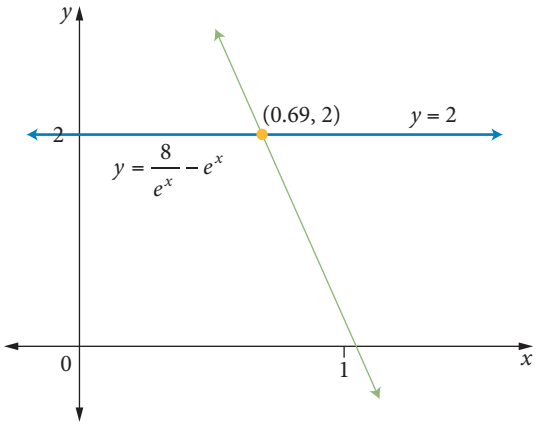
4 If there is an answer that needs to be excluded, always give a reason.	$e^x = -4$ is rejected because e^x is always positive.
5 Solve for x.	$e^x = 2$ $x = \log_e(2)$

Using technology to solve exponential equations

For complex equations for which algebraic techniques cannot be found, you can use the 'solve' function on a graphics calculator to find approximate solutions. Drawing one or more graphs is an excellent method to ensure all solutions are found. To solve $e^{2x} - 5e^x + 6 = 0$, graph $y = e^{2x} - 5e^x + 6$ and determine the x-intercepts, where $y = 0$.



To solve $\frac{8}{e^x} - e^x = 2$, graph $y = \frac{8}{e^x} - e^x$ and $y = 2$, then determine the x-value of the point of intersection.



t Technology worked example
Solving exponential equations

Exponential functions in real contexts

In some situations, decimal approximations are appropriate as solutions to exponential equations.

5 Model exponential growth

The value of a particular investment sitting in the bank can be given by $v(t) = 10\,000 \times e^{0.095t}$, where v is the value at any time t years after the money is deposited.

(a) Determine the initial amount deposited.

THINKING	WORKING
1 Substitute $t = 0$ and calculate $v(0)$. Recall that $e^0 = 1$.	$v(t) = 10\,000 \times e^{0.095t}$ $v(0) = 10\,000 \times e^{0.095 \times 0}$ $= 10\,000 \times 1$ $= 10\,000$

2 Interpret the answer.	The initial deposit was \$10 000.

(b) Determine the value of the account after 5 years. Give your answer to the nearest dollar.

1 Substitute $t = 5$ and calculate $v(5)$.	$v(5) = 10000 \times e^{0.095 \times 5}$ $= 16080$
2 Interpret the answer.	The value of the account after 5 years is \$16 080.

(c) How long, to the nearest year, would the investment need to be maintained to be able to purchase a \$30 000 car outright using only this investment?

1 Substitute the known value for v, then solve for t.	Let $v = 30000$: $10000 \times e^{0.095t} = v(t)$ $10000 \times e^{0.095t} = 30000$ $e^{0.095t} = 3$ $0.095t = \log_e(3)$ $t = \dfrac{\log_e(3)}{0.095}$ $t = 11.5\ldots$
2 Interpret the answer, rounding up to the nearest year.	It will take 12 years to be able to purchase the car outright.
3 Discuss the limitations of the model.	A car model depreciates in value with age. An item currently worth \$30 000 increases in value with inflation at an approximate rate of 2.3% per annum, giving a required investment total after 12 years of $30000 \times (1 + 0.023)^{12} \approx \$39412$. An investment total of \$30 000 in 12 years has a current value of approximately $30000 \times (1 - 0.023)^{12} \approx \$22691$. The assumption for the purchase price of \$30 000 is that it must be the 'drive-away' price; that is, there are no additional costs, which may include registration, transfer of title or stamp duty.

EXERCISE 1.1 Euler's number and index laws

Worked Example

1 Calculate the value of each expression. Give your answers correct to 2 decimal places.

(a) e^4 (b) $\sqrt{e+1}$ (c) $e^{\frac{2}{3}}$ (d) $e-1$

(e) e^2+3 (f) $6e-e^2$ (g) $\sqrt[5]{2e}$ (h) $e^{\frac{1}{2}}-e$

2 Solve the following equations for x.

(a) $e^x = e^3$ (b) $e^{2x} = e^6$ (c) $e^{x+1} = e^{-3}$ (d) $\frac{1}{e} = e^{x-1}$

(e) $\frac{1}{e^2} = e^{2x+1}$ (f) $\sqrt{e} = e^{x+1}$ (g) $e^{x-2} = e^{1-x}$ (h) $\sqrt[x]{e} = e^{4x}$

3 Solve the following equations for x, giving your answers in exact form.

(a) $e^x = 5$ (b) $e^{2x} = 13$ (c) $e^x - 2 = 10$

(d) $2e^{3x+1} = 6$ (e) $e^{3x} = -5$ (f) $e^{\frac{x}{3}} = 7$

4 Solve the following equations for x, correct to 2 decimal places.

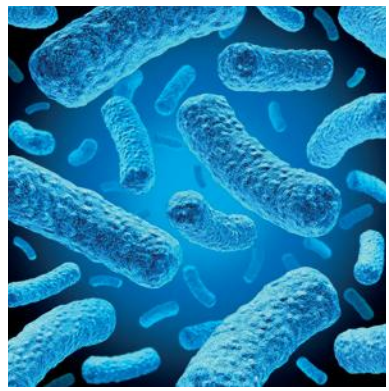
(a) $e^x = 3$ (b) $e^{2x} = 8$ (c) $e^{2x-1} = 6$ (d) $5e^x = 3$

5 The number of bacteria in a culture grows with time according to the rule $N = 50e^{0.3t}$, where t is the time in hours since the bacteria was introduced to the culture.

(a) Determine the initial number of bacteria present.

(b) Determine the number of bacteria present after 4 hours.

(c) Determine the time taken, in hours and minutes, for the bacterial colony to double in size.



6 Scientists are monitoring the population of a sub-species of tiger that is critically endangered. The population can be modelled by $N = N_0 \times e^{-kt}$, where N is the number of tigers and t is the number of years since monitoring began. When $t = 1$, $N = 1500$ and when $t = 5$, $N = 800$.

(a) From the information given, construct two simultaneous equations in terms of N_0 and k.

(b) Solve for N_0 and k, correct to 4 significant figures.

(c) Once the population reaches 14, it will be impossible to recreate a viable population in the wild due to the lack of genetic variation. How many years do scientists have remaining to intervene before it is too late to save the sub-species?

7 A person takes out a personal loan for \$15 000. The amount owing on the loan can be modelled by $A(t) = 15000e^{0.0038t}$, where $A(t)$ is the amount owed to the bank without any payments taken into account, and t is the number of months since the loan was issued. The repayments expected are \$310 per month.

(a) Determine an expression for $R(t)$, the amount of money repaid after t months.

(b) Determine an expression for $B(t)$, the balance owing after t months.

(c) Determine the time taken for the loan to be completely paid off.

8 Solve the following equations for x, giving your answers in exact form.

(a) $\left(e^x - 1\right)\left(e^x - 2\right) = 0$ **(b)** $\left(e^x - 4\right)\left(2e^x - 3\right) = 0$ **(c)** $\left(e^x\right)^2 - 9 = 0$

9 Consider the equation $2e^x + 9 = \dfrac{5}{e^x}$.

(a) Select the correct solution.

A $x = -\log_e(2)$ **B** $x = -\log_e(5)$

C $x = \dfrac{1}{2}$ **D** $x = \log_e\left(\dfrac{1}{2}\right)$ or $x = -\log_e(5)$

(b) Explain the common error made by a student who gave two solutions.

10 When u is substituted for e^{2x} in the equation $e^{4x} + e^{2x} = 2$, which of the following equations does it become?

A $u^4 + u^2 = 0$ **B** $u^2 + u = 2$ **C** $e^x = 0$ **D** $6u = \log_e(2)$

11 Solve the following equations for x, giving your answers in exact form.

(a) $\left(e^x\right)^2 - 5e^x + 4 = 0$ **(b)** $e^{2x} - 6e^x + 8 = 0$ **(c)** $e^{2x} + e^x = 6$

(d) $\dfrac{6}{e^x} - 5 = e^x$ **(e)** $3e^x + 12e^{-x} - 13 = 0$

Worked Example 4

12 Solve the following equations for x, correct to 2 decimal places.

(a) $e^{2x} - 7e^x + 12 = 0$ **(b)** $e^{2x} - e^x - 30 = 0$ **(c)** $2e^{2x} + 30 = 17e^x$ **(d)** $e^{2x} + 7e^x + 6 = 0$

13 Consider the expression $e^{-x} + e^x$.

(a) Convert this expression to a single fraction.

(b) Solve the equation $e^{-x} + e^x = 2$.

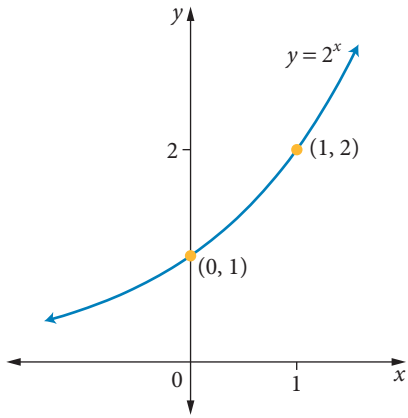
14 Consider the equation $6e^{-x} + e^x = k$. Determine values of k for which the equation has zero, one and two solutions, respectively. Give an example of a value of k to illustrate each situation.

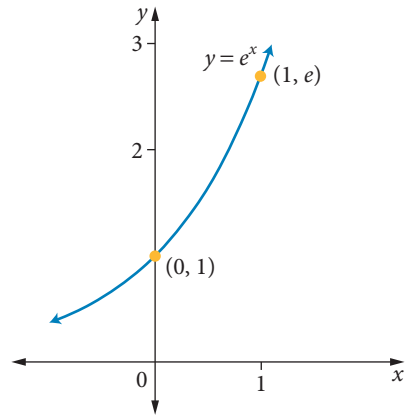
15 Solve the equation for x: $\dfrac{2}{e^x} = \dfrac{20}{e^{2x}} - 1$

1.2 Graphing exponential functions

The exponential function

The function $f(x) = e^x$ is called the exponential function. The rule of any other exponential function can be expressed in the form $f(x) = a^x$, where $a > 0$. The graphs of these functions behave in the same way as the graphs of other exponential functions studied previously. So the graph of $f(x) = e^x$ behaves the same way as $f(x) = 2^x$.

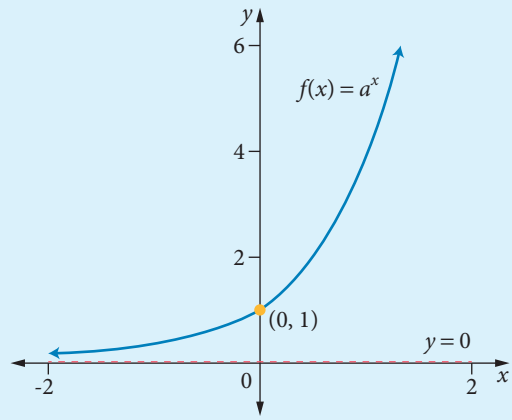




As can be seen from the diagram, $f(x) = e^x$ is a strictly increasing function.

Common properties of $f(x) = a^x$, $a \in (0, \infty) \setminus \{1\}$:

- The coordinates of the y-intercept are $(0, 1)$.
- The equation of the horizontal asymptote is $y = 0$, so there is no x-intercept.
- The domain is $\mathbb{R}$, so $x \in \mathbb{R}$.
- The range is $\mathbb{R}^+$, so $y \in \mathbb{R}^+$.



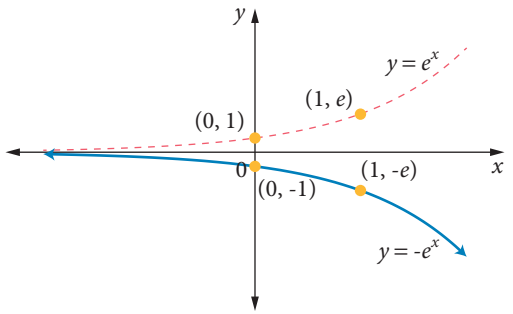
Making connections

Transformations of exponential functions

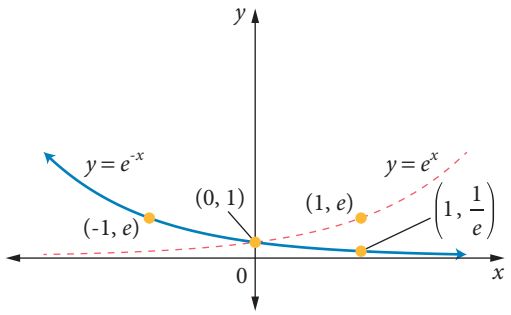
Move the sliders to observe how each parameter transforms the graph of the exponential function in the form $y = Ae^{b(x-c)} + d$.

Reflection in the axes

For any function $f(x)$, reflection in the x-axis occurs when graphing $y = -f(x)$. Points that appeared above the x-axis in the original are now graphed below the x-axis. To reflect in the y-axis, that is, to *flip* from the left side to the right side of the y-axis, graph $y = f(-x)$. The exponential function, $f(x) = -e^x$, will appear reflected in the x-axis, and $f(x) = e^{-x}$ will appear reflected in the y-axis.



Reflection in the x-axis occurs when graphing $y = -e^x$



Reflection in the y-axis occurs when graphing $y = e^{-x}$

Remember: when a number is raised to a negative power, it is the same as the reciprocal of the number raised to a positive power.

$$e^{-x} = \frac{1}{e^x}$$

Dilation parallel to the y-axis

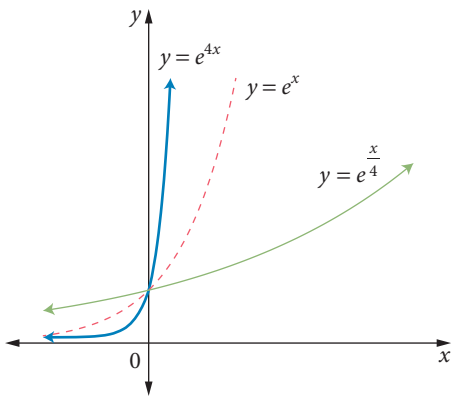
When graphing $y = A \times e^x$, the graph is dilated by a factor of $|A|$ parallel to the y-axis (or away from the x-axis). This affects both the steepness of the graph and the location of the y-intercept. As $|A|$ becomes larger, the graph becomes steeper and moves further from the x-axis. The sign of A determines whether the graph is positive or negative (reflected in the x-axis).

The domain and asymptote remain unchanged. If $A > 0$, the range remains unchanged.

Dilation parallel to the x-axis

When graphing $y = e^{bx}$, the graph is dilated by a factor of $\frac{1}{|b|}$ parallel to the x-axis. This affects the steepness of the graph, but not the location of the y-intercept, domain, range or asymptote.

So, $y = e^{4x}$ experiences a dilation by a factor of $\frac{1}{4}$ parallel to the x-axis, whereas $y = e^{\frac{x}{4}}$ experiences a dilation by a factor of 4.



Vertical translation

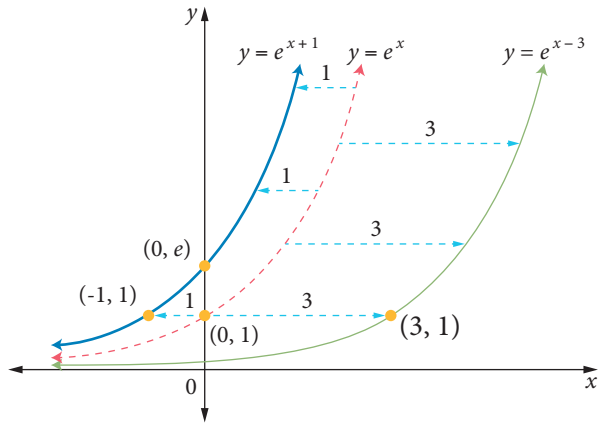
When graphing $y = e^x + d$, the graph is translated vertically d units up if $d > 0$, or vertically $|d|$ units down if $d < 0$. This means that the asymptote, the y-intercept and the range all change. The vertical translation parallel to the y-axis is given by the constant term d.

The asymptote is given by $y = d$, and the y-intercept is found by letting $x = 0$.

Horizontal translation

When graphing $y = e^{x-c}$, the graph is translated horizontally c units right if $c > 0$, or c units left if $c < 0$. The consequence of this is that the y-intercept changes. The asymptote, domain and range remain unchanged.

Notice that the left or right motion can seem counter-intuitive. A power of $x + 1$ moves the graph to the left 1 unit, and conversely, a power of $x - 3$, moves the graph to the right 3 units. The translation parallel to the x-axis is given by the value that would make the exponential expression zero. So, to determine the horizontal translation, let the power equal zero and solve for x.



6 Transformations of exponential functions

Determine the transformations of $y = e^x$ required to sketch the graphs of each of the following.

(a) $y = 3e^x - 2$

THINKING	WORKING
1 Express the function in the form $y = Ae^{b(x-c)} + d$ and identify the value of each parameter.	$y = 3e^{1(x-0)} - 2$ $A = 3$, $b = 1$, $c = 0$ and $d = -2$
2 Determine the dilations and/or reflections in the form $y = Ae^{bx}$.	$A = 3$, which gives a dilation by a factor of 3 parallel to the y-axis. $b = 1$, therefore there is no dilation parallel to the x-axis. Given that $A > 0$ and $b > 0$, there is no reflection in either axis.
3 Determine the horizontal translation c and vertical translation d in the form $y = Ae^{b(x-c)} + d$.	$c = 0$, therefore no horizontal translation has occurred from the graph of $y = e^x$. $d = -2$, which indicates a translation of 2 units in the negative direction of the y-axis (down).

(b) $y = 1 - e^{x+2}$

1 Express the function in the form $y = Ae^{b(x-c)} + d$ and identify the value of each parameter.	$y = -e^{1(x+2)} + 1$ $A = -1$, $b = 1$, $c = -2$ and $d = 1$

2 Determine the dilations and/or reflections in the form $y = Ae^{bx}$.	$A = -1$, which indicates a reflection in the x-axis. $b = 1$, therefore there is no dilation parallel to the x-axis.
3 Determine the horizontal translation c and vertical translation d in the form $y = Ae^{b(x-c)} + d$.	$c = -2$, which indicates a translation of 2 units left. $d = 1$, which indicates a translation of 1 unit up.

(c) $y = 5e^{3-2x}$

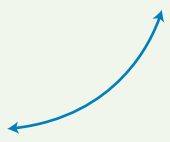
1 Express the function in the form $y = Ae^{b(x-c)} + d$ and identify the value of each parameter.	$y = 5e^{-2\left(x-\frac{3}{2}\right)} + 0$ $A = 5$, $b = -2$, $c = \frac{3}{2}$ and $d = 0$
2 Determine the dilations and/or reflections in the form $y = Ae^{bx}$.	$A = 5$, which indicates a dilation by a factor of 5 parallel to the y-axis. $b = -2$, given that $b < 0$ the graph is reflected in the y-axis, $\lvert b \rvert = 2$ gives a dilation by a factor of $\frac{1}{2}$ parallel to the x-axis.
3 Determine the horizontal translation c and vertical translation d in the form $y = Ae^{b(x-c)} + d$.	$c = \frac{3}{2}$, there is a horizontal translation of $\frac{3}{2}$ units to the right. $d = 0$ indicates that there is no vertical translation.

7 Sketch exponential graphs

For the following functions, identify the asymptote and the coordinates of the x- and y-intercepts, sketch the graph, and identify the domain and range.

(a) $y = e^{2x+1} - 20$

THINKING	WORKING
1 Express the function in the form $y = Ae^{b(x-c)} + d$ and identify the value of each parameter.	$y = e^{2x+1} - 20$ $y = 1e^{2\left(x+\frac{1}{2}\right)} - 20$ $A = 1$, $b = 2$, $c = -\frac{1}{2}$ and $d = -20$
2 Determine the basic shape of the graph.	Given that $A > 0$ and $b > 0$, the basic function has not been reflected in either axis.



3 Determine the equation of the asymptote.

$d = -20$ indicates that there is a translation of 20 units in the negative direction of the y-axis (down).

The equation of the asymptote is $y = -20$.

4 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the exact value of y.

$$\begin{aligned} y &= e^{2\times 0+1} - 20 \\ &= e^1 - 20 \\ &\approx -17.28 \end{aligned}$$

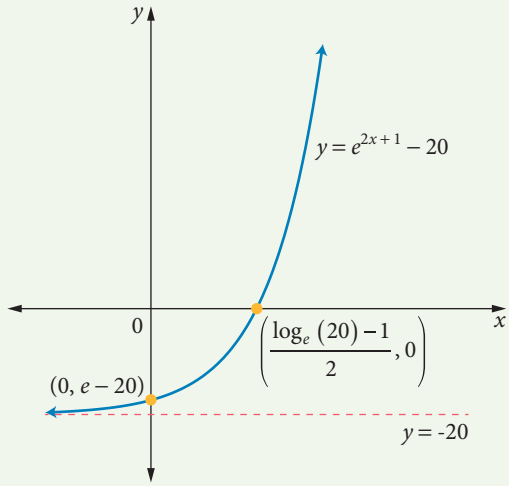
The coordinates of the y-intercept are $(0, e - 20)$.

5 Determine the coordinates of the x-intercept by letting $y = 0$ and solving for x.

$$\begin{aligned} 0 &= e^{2x+1} - 20 \\ 20 &= e^{2x+1} \\ \log_e(20) &= 2x + 1 \\ \log_e(20) - 1 &= 2x \\ \frac{\log_e(20) - 1}{2} &= x \\ x &\approx 1.00 \end{aligned}$$

The coordinates of the x-intercept are $\left(\frac{\log_e(20) - 1}{2}, 0\right)$.

6 Sketch the graph, showing the asymptote, the y-intercept and the x-intercept.



7 Identify the domain and range from the graph.

Domain: $x \in \mathbb{R}$

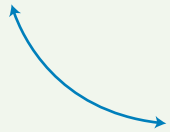
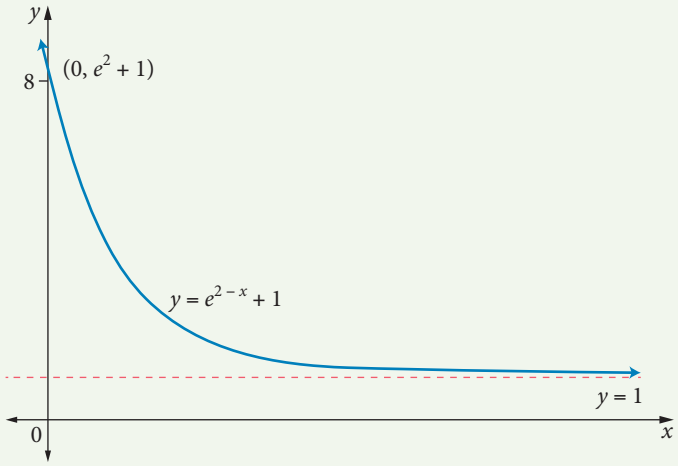
Range: $y \in (-20, \infty)$

(b) $y = e^{2-x} + 1$

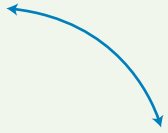
1 Express the function in the form $y = Ae^{b(x-c)} + d$ and identify the value of each parameter.

$$\begin{aligned} y &= e^{2-x} + 1 \\ y &= 1e^{-(x-2)} + 1 \end{aligned}$$

$A = 1$, $b = -1$, $c = 2$ and $d = 1$

2 Determine the basic shape of the graph.	For $b < 0$, there is a reflection in the y-axis. 
3 Determine the equation of the asymptote.	For $d = 1$, there is a translation of 1 unit up. The equation of the asymptote is $y = 1$.
4 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the exact value of y.	$y = e^{2-0} + 1$ $= e^2 + 1$ ≈ 8.39 The coordinates of the y-intercept are $\left(0, e^2 + 1\right)$.
5 Determine the coordinates of the x-intercept by letting $y = 0$ and solving for x.	$0 = e^{2-x} + 1$ $-1 = e^{2-x}$ e^{2-x} cannot be negative, so there is no x-intercept.
6 Sketch the graph, showing the asymptote and the y-intercept.	
7 Identify the domain and range from the graph.	Domain: $x \in \mathbb{R}$ Range: $y \in (1, \infty)$

(c) $y = 3 - 5 \times e^{3x}$

1 Express the function in the form $y = Ae^{b(x-c)} + d$ and identify the value of each parameter.	$y = 3 - 5 \times e^{3x}$ $= -5 \times e^{3(x-0)} + 3$ $A = -5$, $b = 3$, $c = 0$ and $d = 3$
2 Determine the basic shape of the graph.	Given that $A < 0$, there is a reflection in the x-axis. 
3 Determine the equation of the asymptote.	For $d = 3$, there is a translation of 3 units up. The equation of the asymptote is $y = 3$.

4 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the exact value of y.

$$y = -5 \times e^{3 \times 0} + 3$$
$$= -2$$

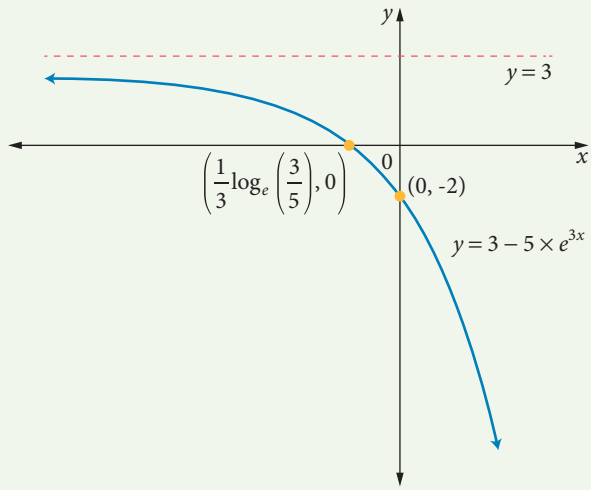
The coordinates of the y-intercept are $(0, -2)$.

5 Determine the coordinates of the x-intercept by letting $y = 0$ and solving for x.

$$0 = -5 \times e^{3x} + 3$$
$$-3 = -5 \times e^{3x}$$
$$\frac{3}{5} = e^{3x}$$
$$\log_e\left(\frac{3}{5}\right) = 3x$$
$$x = \frac{1}{3}\log_e\left(\frac{3}{5}\right)$$

The coordinates of the x-intercept are $\left(\frac{1}{3}\log_e\left(\frac{3}{5}\right), 0\right)$ or approximately $(-1.61, 0)$.

6 Sketch the graph, showing the asymptote, the y-intercept and the x-intercept.



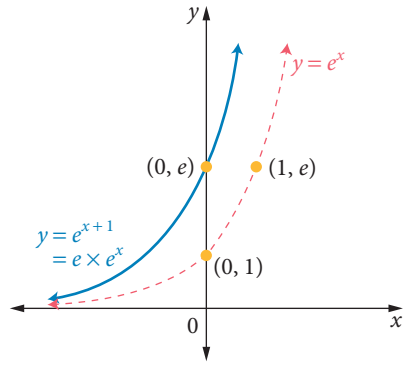
7 Identify the domain and range from the graph.

Domain: $x \in \mathbb{R}$

Range: $y \in (-\infty, 3)$

Identifying the rule for a graph

The rule for an exponential graph can generally be written in more than one simplified form. A horizontal translation of $y = e^x$ and a dilation of $y = e^x$ parallel to the y-axis can result in the same graph. The index laws state that if the indices have been added, then the original indicial expressions have been multiplied, so $e^{x+c} = e^x \times e^c$. Hence, $e^{x+c} = Ae^x$, where $A = e^c$.



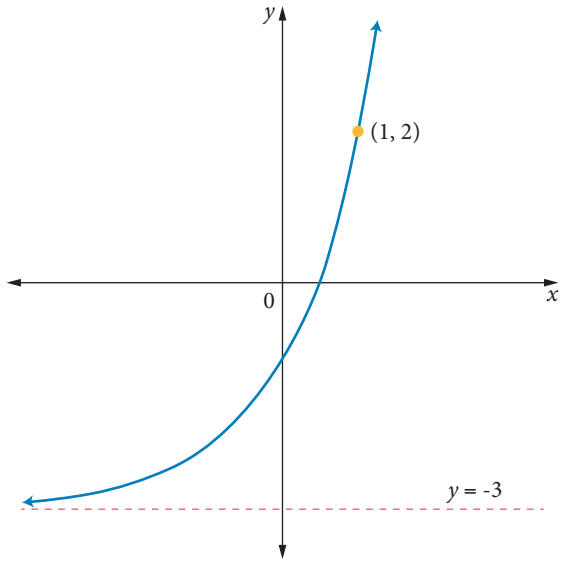
Explore further

Determine the rule of an exponential function

Explore the shapes and rules of exponential functions.

8 Determine an equation from a graph in the form $f(x) = Ae^x + d$

Express the rule of the graph below in the form $f(x) = Ae^x + d$, and then simplify using index laws.



THINKING	WORKING
1 Use the asymptote to identify the vertical translation, d.	The equation of the asymptote is $y = -3$. $\therefore d = -3$
2 Recall the equation and substitute the value of d.	$f(x) = Ae^x - 3$
3 Substitute the coordinates of a known point into the rule and solve for A.	The coordinates given are $(1, 2)$, let $x = 1$ and $y = 2$: $2 = Ae^1 - 3$ $5 = Ae$ $\frac{5}{e} = A$ $\therefore A = 5e^{-1}$
4 Substitute the value for A into the rule. Simplify using index laws.	$f(x) = 5e^{-1} \times e^x - 3$ $= 5e^{x-1} - 3$

Summary of transformations for $f(x) = e^x$:

$y = -f(x)$ reflects $y = f(x)$ in the x-axis.

$y = f(-x)$ reflects $y = f(x)$ in the y-axis.

$y = Af(x)$ dilates $y = f(x)$ by a factor of $|A|$ parallel to the y-axis.

$y = f(bx)$ dilates $y = f(x)$ by a factor of $\frac{1}{|b|}$ parallel to the x-axis.

$y = f(x - c)$ translates $y = f(x)$ horizontally c units.

$y = f(x) + d$ translates $y = f(x)$ vertically d units.

EXERCISE 1.2 Graphing exponential functions

Worked Example 6

1 Determine the transformations of $y = e^x$ that have been applied to each of the following rules.

(a) $y = -e^x$ **(b)** $y = 2e^x + 1$ **(c)** $y = e^{-x} - 3$ **(d)** $y = -10e^{2x}$

(e) $y = 6 - e^x$ **(f)** $y = 4 \times e^{x-1}$ **(g)** $y = e^{-x} - 5$ **(h)** $y = 2e^{5-2x} + 3$

(i) $y = \frac{1}{4}e^{x+2}$ **(j)** $y = 2e^{\frac{x-1}{2}} + 1$

2 Determine the rule for the image of the function $y = e^x$ after each of the following transformations have been applied.

(a) The function is reflected in the x-axis and dilated by a factor of 2 parallel to the x-axis.

(b) The function is translated 2 units to the right and 1 unit down.

(c) The function is reflected in the x-axis, then dilated by a factor of $\frac{1}{2}$ parallel to the x-axis, then translated 3 units up.

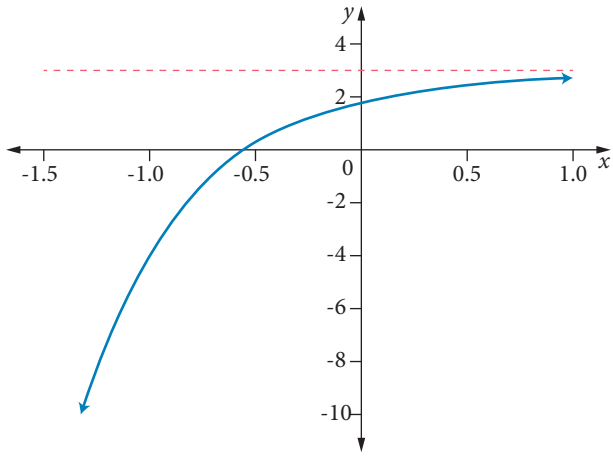
3 Determine the rule that best matches each graph.

(a) **A** $y = 3 + e^{-x}$

B $y = 3 - e^{-2x}$

C $y = 3 - e^{2x}$

D $y = e^{-2x} + 3$

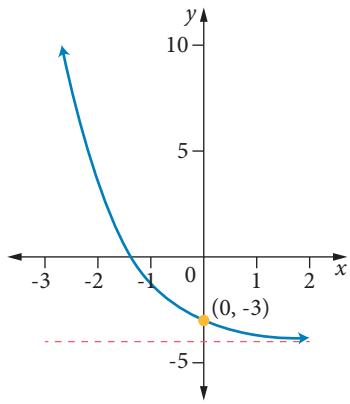


(b) **A** $y = -e^x - 4$

B $y = -e^{-x} - 4$

C $y = 2e^{-x} - 4$

D $y = e^{-x} - 4$



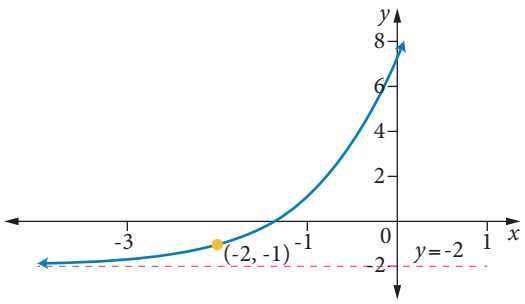
(c) A decreasing exponential graph with a horizontal asymptote at $y = 2$ and a y-intercept of $(0, 4)$.

A $f(x) = 2e^x + 2$ **B** $f(x) = e^{-x-2} + 2$ **C** $f(x) = e^{2-x} + 2$ **D** $f(x) = 2e^{-x} + 2$

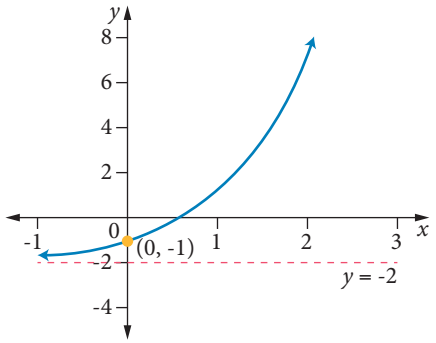
4 Consider the function $f(x) = e^{x-2} - 2$.

(a) Select the graph that represents the function.

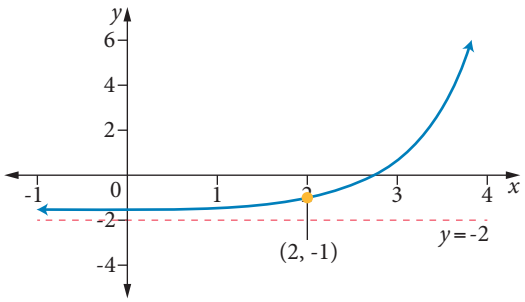
A



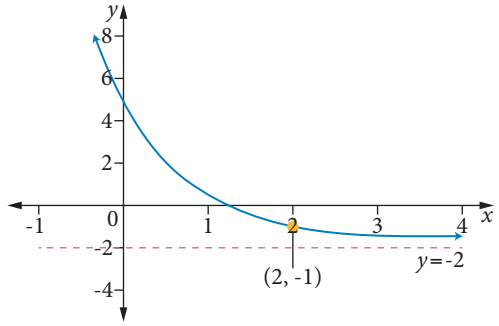
B



C



D



(b) Explain the common error made by a student who chose the first incorrect option.

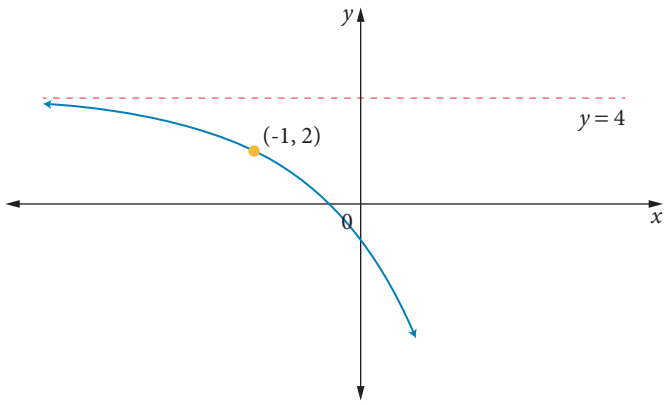
5 The graph of $y = e^x$ undergoes several transformations. It is reflected in the x-axis, then dilated by a factor of 2 parallel to the x-axis, then translated 1 unit to the right and 2 units up.

(a) Determine the rule for the new function, $f(x)$.

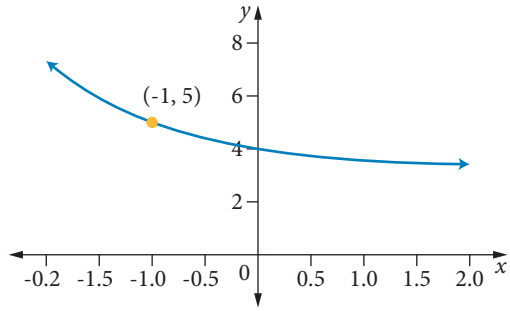
(b) Evaluate the reasonableness of your solution to part **(a)** by sketching a graph of the original function and the new function on the same set of axes.

Worked Example 8

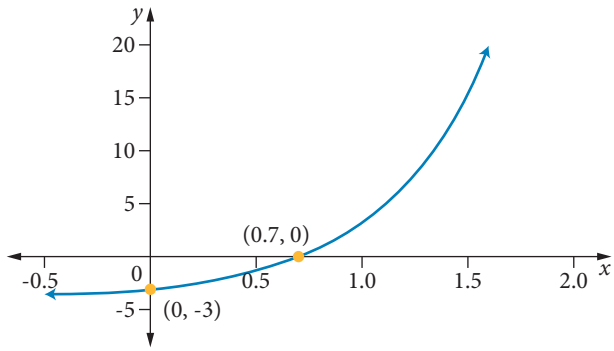
6 Determine a rule for the following graph in the form $y = Ae^x + d$. Simplify your answer using index laws.



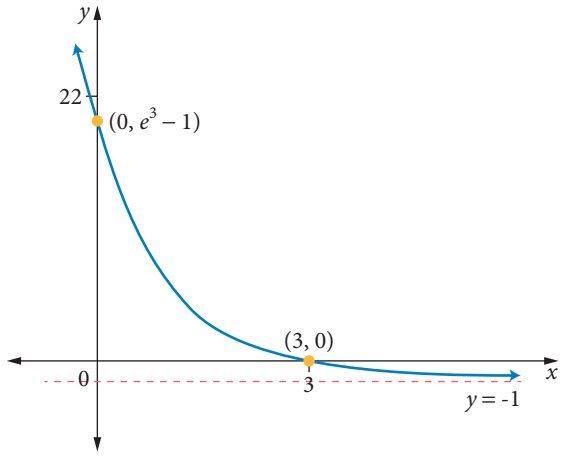
7 From the points indicated, determine a rule for the following graph in the form $y = Ae^{-x} + d$.



8 The graph on the right has a rule that can be expressed as $y = e^{bx} + d$. Using the points on the graph, establish the rule, correct to 2 decimal places.



9 Determine an equation for the graph in the form $y = e^{ax+b} + k$.



Worked Example

10 For each of the following equations, sketch the graph showing the equation of the asymptote and the coordinates of the axis intercepts. If there is only one intercept, add an additional point. Determine the domain and range.

(a) $y = -e^{x} + 1$ (b) $y = e^{-x} - 3$ (c) $y = 4e^{x}$ (d) $y = -2e^{x} + 6$

11 For each of the following equations, sketch the graph, identifying the asymptote and the axis intercepts. If there is only one intercept, add an additional point. Determine the domain and range.

(a) $y = 1 + e^{x-1}$ (b) $y = -e^{x+3}$ (c) $y = 6e^{x} - 5$

(d) $y = e^{3x-1} + 2$ (e) $y = -\frac{1}{3}e^{2-x} + 1$

12 Sketch the following functions, clearly labelling all key features of the graph.

(a) $f(x) = e^{x+1} - 3$ **(b)** $f(x) = 5e^{3-x} + 1$ **(c)** $f(x) = 3e^{x-2} - 5$

(d) $f(x) = -2e^{2x} + 4$ **(e)** $f(x) = 3 - e^{2-3x}$ **(f)** $f(x) = -2e^{1+x} + 5$

13 The graph of $y = e^x$ undergoes the described transformations to become the function $f(x)$. In each case, determine the rule for $f(x)$, and then sketch the graph, showing the asymptote and axis intercepts. If there is only one intercept, plot an additional point.

(a) The graph is dilated by a factor of 2 parallel to the y-axis and translated 3 units to the right.

(b) The graph is dilated by a factor of $\frac{1}{4}$ parallel to the y-axis and reflected in the y-axis.

(c) The graph is reflected in the x-axis and translated 5 units up.

(d) The graph is dilated by a factor of $\frac{1}{2}$ parallel to the x-axis and reflected in the x-axis.

(e) The graph is reflected in the x-axis and in the y-axis.

(f) The graph is dilated by a factor of 3 parallel to the x-axis and translated 7 units down.

(g) The graph is dilated by a factor of $\frac{1}{2}$ from the y-axis (parallel to the x-axis) and translated 3 units to the left.

(h) The graph is dilated by a factor of 3 from the x-axis (parallel to the y-axis), reflected in the y-axis and translated 1 unit to the right.

14 Let $f(x) = \left(\frac{1}{e}\right)^x$. Sketch the graph of $g(x) = -2f(3x - 2) + 4$, showing the equation of the asymptote and the axis intercepts. Determine the domain and range of g.

15 Let $f(x) = e^x$. The graph of $f(x)$ was reflected in the y-axis, dilated by a factor of 2 from the y-axis and translated horizontally 3 units to the left to form $g(x)$.

(a) Sketch the graph of $g(x)$ and label all key features on your graph.

(b) On the same set of axes, sketch the graph of $y = -2\ln(x) - 3$, determining its domain and range. Plot the point of intersection, correct to 2 decimal places, and the axis points as exact values.

1.3 Differentiating exponential functions

The gradient of $y = a^x$ at $x = 0$

To determine the gradient of a function $f(x)$ from first principles, you use $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ where h indicates the small change in x.

For an exponential function $y = a^x$, where $a > 0$, the gradient at $x = 0$ is given by:

$f(x) = a^x$

$f(0) = a^0$

$f(0+h) = a^{0+h}$

Therefore:

$$f'(0) = \lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$$

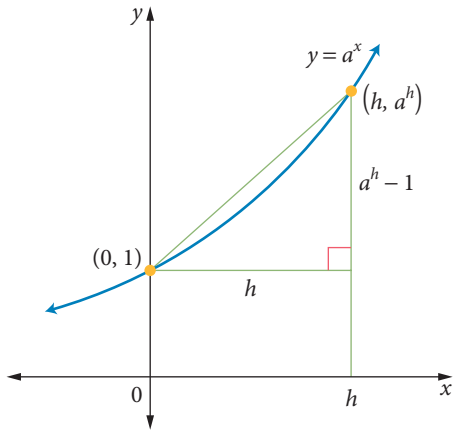
$$= \lim_{h \to 0} \frac{a^h - 1}{h}$$

Explore further

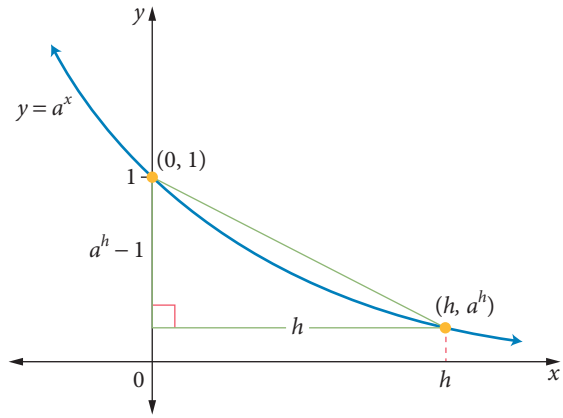
The gradient of exponential functions

Explore the gradient of exponential functions using tangents and secants.

For $a > 1$:



For $0 < a < 1$:



9 Estimate $\lim_{h \to 0} \frac{a^h - 1}{h}$ for different values of $a > 0$

Estimate $\lim_{h \to 0} \frac{a^h - 1}{h}$ for the given values of a, correct to 4 decimal places. Interpret the answer.

(a) $a = 5$

THINKING	WORKING
1 Substitute very small values of h into $\frac{a^h - 1}{h}$ for the given value of a. Stop when the values are indistinguishable at the required accuracy.	$h = 0.1$: $\frac{5^{0.1} - 1}{0.1} = 1.7461\ldots$ $h = 0.01$: $\frac{5^{0.01} - 1}{0.01} = 1.6224\ldots$ $h = 0.001$: $\frac{5^{0.001} - 1}{0.001} = 1.6107\ldots$ $h = 0.0001$: $\frac{5^{0.0001} - 1}{0.0001} = 1.6095\ldots$

2 Interpret the value of the limit, rounding as required.	$\lim_{h\to 0}\frac{5^h-1}{h}=1.6095\,(4\text{ d.p.})$
3 Interpret the answer.	The gradient of $y=5^x$ at $(0,1)$ is 1.6095.

(b) $a=0.2$

1 Substitute very small values of h into $\frac{a^h-1}{h}$ for the given value of a. Stop when the values are indistinguishable at the required accuracy.	$h=0.1$: $\frac{0.2^{0.1}-1}{0.1}=-1.4866\ldots$ $h=0.01$: $\frac{0.2^{0.01}-1}{0.01}=-1.5965\ldots$ $h=0.001$: $\frac{0.2^{0.001}-1}{0.001}=-1.6081\ldots$ $h=0.0001$: $\frac{0.2^{0.0001}-1}{0.0001}=-1.6093\ldots$
2 Make a conclusion about the value of the limit, rounding as required. Notice that the gradient for $y=0.2^x$ at $x=0$ is the opposite of $y=5^x$ because 0.2 is the reciprocal of 5.	$\lim_{h\to 0}\frac{0.2^h-1}{h}=-1.6093\,(4\text{ d.p.})$
3 Interpret the answer.	The gradient of $y=0.2^x$ at $(0,1)$ is -1.6093.

An additional interpretation of the results from Worked Example 9 is that $5\approx e^{1.61}$ and $\frac{1}{5}\approx e^{-1.61}$, so that $5^x=e^{1.61x}$ and $\left(\frac{1}{5}\right)^x=e^{-1.61x}$.

The gradient of $y=e^x$ at $x=0$

To simplify the process of differentiating exponential functions generally, a value of a is needed such that $\lim_{h\to 0}\frac{a^h-1}{h}=1$.

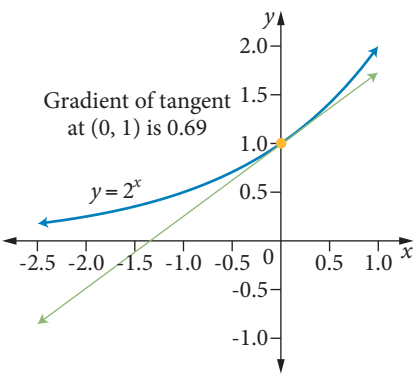
To show that $e=2.718\ldots$ is the unique value for which this occurs, it is possible to consider graphs of other exponential functions.

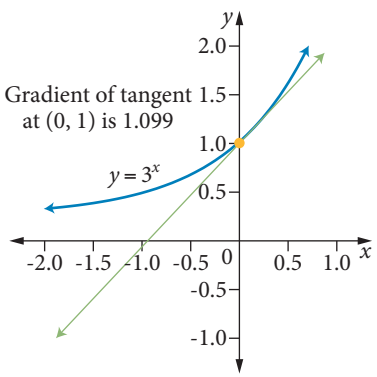
Looking at the graphs of $f(x)=2^x$, $f(x)=3^x$ and $f(x)=e^x$, and the tangents at the point where $x=0$, shows that $f(x)=e^x$ is the function that has a gradient of 1.

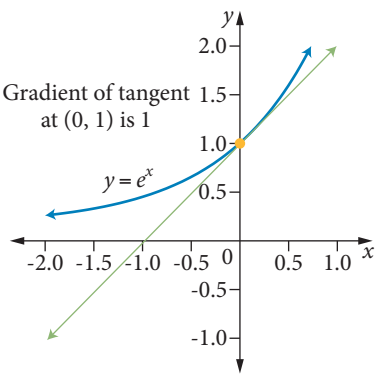
Additional information

Explore the gradient of the exponential function

Explore the gradient of $f(x)=e^x$ at key points $x=0$ and $x=1$ by looking at different intervals on either side of these points.







Check on a spreadsheet, or by keying in values using a calculator, that: $\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}=1$.

Note that on an Excel spreadsheet, e^h is written in the formula as EXP(cell containing h).

	A	B
1	h	(EXP(h) – 1)/h
2	1	1.718281828
3	0.1	1.051709181
4	0.01	1.005016708
5	0.001	1.000500167
6	0.0001	1.000050002
7	0.00001	1.000005
8	0.000001	1.0000005

$$\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}=1$$

Differentiating $y=e^x$ from first principles

Using $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to differentiate $f(x)=e^x$ using first principles would give:

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\&=\lim_{h\to 0}\frac{e^x e^h-e^x}{h}\\&=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}\\&=e^x\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}\end{aligned}$$

Because $\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}=1$:

$f'(x)=e^x\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}$ becomes $f'(x)=e^x\times 1$.

Hence if $f(x)=e^x$, then $f'(x)=e^x$.

The gradient at any point along the exponential function $f(x)=e^x$ has the same numerical value as the function values.

Additional information

Tangents to the natural exponential function

This activity enables you to compare the values of $f(x)$ and $f'(x)$ for different values of x, where $f(x)=e^x$.

If $f(x)=e^x$, then $f'(x)=e^x$, or if $y=e^x$, then $\frac{dy}{dx}=e^x$.

Derivatives using the chain rule

To determine the derivative of more complex exponential functions, such as $y=e^{kx}$, you will need to use the chain rule, as it is a composite function.

Let $y=e^u$, where $u=kx$.

$\frac{dy}{du}=e^u$ and $\frac{du}{dx}=k$

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\\&=e^u\times k\\&=ke^{kx}\end{aligned}$$

10 Differentiate exponential expressions

Differentiate the following functions with respect to x.

(a) $y = e^{3x}$

THINKING	WORKING
1 Recall the rule for the function.	$y = e^{3x}$
2 Express the function as $y = e^u$ and create an expression for u in terms of x.	Let $y = e^u$ and $u = 3x$.
3 Determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 3$
4 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ to differentiate the function.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= e^u \times 3$ $= 3e^u$ $= 3e^{3x}$

(b) $y = 4e^{-x}$

1 Recall the rule for the function.	$y = 4e^{-x}$
2 Express the function as $y = e^u$ and create an expression for u in terms of x.	Let $y = 4e^u$ and $u = -x$.
3 Determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = 4e^u$ and $\frac{du}{dx} = -1$
4 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ to differentiate the function.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= 4e^u \times -1$ $= -4e^u$ $= -4e^{-x}$

(c) $y = e^{5x-2}$

1 Recall the rule for the function.	$y = e^{5x-2}$
2 Express the function as $y = e^u$ and create an expression for u in terms of x.	Let $y = e^u$ and $u = 5x - 2$.
3 Determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 5$

4 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ to differentiate the function.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= e^u \times 5$ $= 5e^u$ $= 5e^{5x-2}$

Look over the results you obtained in Worked Example 10. Since u is assigned to the power, and the derivative of e^x is e^x, it is possible to begin to see that the derivative of the power multiplied by the original function will give the derivative of an exponential function. This is a useful shortcut.

If $f(x) = e^{mx}$, then $f'(x) = me^{mx}$.

If $f(x) = ae^{mx+c}$, then $f'(x) = ame^{mx+c}$.

If $f(x) = ae^{g(x)}$, then $f'(x) = a \times g'(x)e^{g(x)}$.

More complex derivatives

You sometimes have a choice of methods for differentiating complex expressions. If possible, separating a quotient into individual terms may simplify the process.

The derivative of the sum of the terms is given by the sum of the derivatives of these terms, that is if $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

11 Further differentiation

Determine the derivative of the following functions.

(a) $f(x) = \frac{e^{3x} + 3}{e^x}$

THINKING	WORKING
1 Transpose the expression into two terms. Simplify using the index laws.	$f(x) = \frac{e^{3x} + 3}{e^x}$ $= \frac{e^{3x}}{e^x} + \frac{3}{e^x}$ $= e^{2x} + 3e^{-x}$
2 Determine the derivative of each term.	$f'(x) = 2e^{2x} - 3e^{-x}$
3 Transpose the equation into the format of the question, i.e. as a single quotient.	$f'(x) = 2e^{2x} - \frac{3}{e^x}$ $= \frac{2e^{3x} - 3}{e^x}$

(b) $f(x) = \frac{1}{e^{x^2-3x}}$

1 Use index laws to write $f(x)$ as an expression without a denominator.	$f(x) = \frac{1}{e^{x^2-3x}}$ $= e^{-x^2+3x}$

2 Express the function as $y = e^u$ and create an expression for u in terms of x.	Let $y = e^u$ and $u = -x^2 + 3x$.
3 Determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = -2x + 3$
4 Apply the chain rule pattern to differentiate. If $f(x) = e^{g(x)}$, then $f'(x) = g'(x)e^{g(x)}$.	$f'(x) = \frac{dy}{du} \times \frac{du}{dx}$ $= (-2x + 3)e^u$ $= (-2x + 3)e^{-x^2+3x}$
5 Transpose the answer in the format of the question, i.e. as a single quotient.	$f'(x) = -\frac{2x - 3}{e^{x^2-3x}}$ or $f'(x) = \frac{3 - 2x}{e^{x^2-3x}}$

Tangent to a curve

A tangent is a straight line that touches a graph at one point only, having the same gradient as the graph at that point; that is, the gradient of the tangent to the curve $f(x)$ at the point (x_1, y_1) is the same as the instantaneous gradient of the curve $f(x)$ at the point (x_1, y_1). Differentiation enables the gradient of a curve to be found at any point (x_1, y_1) on the curve, so the equation to the tangent can be found.

12 Determine the equation of a tangent

Determine the equation to the tangent to the graph of $f(x) = e^{2x}$, at the point where $x = 1$.

THINKING	WORKING
1 Determine the coordinates of the point where $x = 1$.	$f(x) = e^{2x}$ $f(1) = e^2$ The coordinates of the point on the curve at $x = 1$ are $(1, e^2)$.
2 Determine the equation of the gradient function.	$f'(x) = 2e^{2x}$
3 Substitute the x-value into the equation of the gradient function and interpret the value.	$f'(1) = 2e^{2\times1}$ $= 2e^2$ The gradient of the tangent to the curve $f(x) = e^{2x}$ at the point $(1, e^2)$ is $2e^2$.
4 Substitute the values for the gradient and the point into the general equation for a straight line, $y - y_1 = m(x - x_1)$, and solve for y.	$y - e^2 = 2e^2(x - 1)$ $y = 2e^2x - 2e^2 + e^2$ $y = 2e^2x - e^2$
5 Interpret the answer.	The tangent to the curve $f(x) = e^{2x}$ at the point $(1, e^2)$ has the equation $y = 2e^2x - e^2$.

EXERCISE 1.3 Differentiating exponential functions

Worked Example

9

1 Estimate $\lim_{h\to 0}\frac{a^h-1}{h}$ for the given values of a, correct to 2 decimal places. Interpret your answers.

(a) $a=3$ (b) $a=0.33$ (c) $a=2$

(d) $a=0.5$ (e) $a=7$ (f) $a=4$

10

2 Differentiate each of the following.

(a) $y=e^{2x}$ (b) $y=e^{\frac{1}{4}x}$ (c) $f(x)=e^{\frac{2x}{3}}$ (d) $y=3e^{-x}$

(e) $h(x)=5e^{3x}$ (f) $y=-3e^{-5x}$ (g) $y=4e^{0.25x}$ (h) $y=2e^{-3x}$

3 Differentiate the following functions with respect to x.

(a) $f(x)=x^3+e^{2x}-1$ (b) $y=2e^{-x}-5e^{2x}$ (c) $g(x)=x^3-5x^2+3e^{2x}$

(d) $y=2e^{-3x}+5e^{4x}$ (e) $y=3e^x+2e$

4 Write an expression for the derivative of each of the following functions.

(a) $f(x)=e^{5x+2}$ (b) $f(x)=e^{x^2-4}$ (c) $f(x)=2e^{8x-7}$

(d) $f(x)=5e^{\frac{x}{5}+3}$ (e) $f(x)=e^{x^2-7}$ (f) $f(x)=-e^{7-2x-x^2}$

(g) $f(x)=\frac{1}{3}e^{x^3+3x}$ (h) $f(x)=2e^{4-3x+x^2}$ (i) $f(x)=-2e^{(x-3)^2}$

5 Which of the following is the derivative $f'(x)$ for $f(x)=e^{5x-7}$?

A $(5x-7)e^{5x-7}$ B $5e^{5x-7}$ C $5e^{5x}$ D $5xe^{5x-7}$

6 Consider the function $f(x)=2e^{x^2+5x+6}$.

(a) Which is the derivative, $f'(x)$?

A $2(2x+5)e^{x^2+5x+6}$ B $2(2x+5)e^{2x+5}$

C $2\left(x^2+5x+6\right)e^{x^2+5x+6}$ D $2(2x+5)$

(b) Explain the common error made by a student in part (a) who gave an answer with an index of $2x+5$.

7 Determine the expression for $\frac{dy}{dx}$ for each of the following functions.

(a) $y=e^{3x}-\frac{2}{e^x}+3$ (b) $y=5e^{2x}+\frac{1}{e^{3x}}+3x^2$ (c) $y=\frac{e^{4x}}{3e^x}$

8 Consider the function $f(x)=3e^{2x+5}$.

(a) Determine an expression for $f'(x)$.

(b) Calculate $f(1)$.

(c) Calculate $f'(1)$.

(d) Determine the equation of the tangent to the graph of $f(x)$ at the point where $x=1$.

Worked Example 11

9 Determine the expression for $\frac{dy}{dx}$ for each of the following functions.

(a) $y = \frac{e^{3x} - e^{-x}}{e^{x}}$ **(b)** $y = \frac{3}{e^{2x-5}}$ **(c)** $y = \frac{1}{e^{x^2-x-1}}$

10 Determine an expression for the derivative of each of the following functions.

(a) $f(x) = e^{(x-3)^2}$ **(b)** $f(x) = e^{\sqrt{3x-1}}$ **(c)** $f(x) = \left(e^{x} + 4\right)^3$

(d) $f(x) = 2e^{(2x+5)^4}$ **(e)** $f(x) = \frac{2}{e^{(x-3)^2}}$

11 Determine the equation of the tangent to each of the following functions at the given x-value. 12

(a) $f(x) = -e^{x}$, where $x = -2$

(b) $f(x) = e^{x^2+1}$, where $x = 0$

12 For $f(x) = e^{x}$, demonstrate that a tangent at $x = 1$ passes through the origin.

13 Consider the function $f(x) = e^{2x^2+3x}$.

(a) Determine the equation of the tangent where $x = 0$.

(b) Sketch the tangent and $f(x)$ on the same set of axes.

14 Consider the function $f(x) = e^{-x} + e^{2x}$.

(a) Sketch $y = e^{-x}$, $y = e^{2x}$ and $f(x) = e^{-x} + e^{2x}$ on the same set of axes.

(b) Determine an expression for $f'(x)$.

(c) Sketch $y = -e^{-x}$, $y = 2e^{2x}$ and $f'(x)$ on the same set of axes.

(d) Calculate the value of x for which $f'(x) = 0$, correct to 2 decimal places.

(e) Determine the point on the function $f(x)$ with the x-value found in the part **(d)**. Interpret your answer.

15 Determine the equation of the tangent to the graph of $y = e^{2-2x} - 2e^{-x}$ at $x = 1$. Give your answer correct to 2 decimal places.

16 Given that $f(x) = e^{ax^2-3x+1}$ has a stationary point at $x = \frac{3}{4}$, determine the value of a.

17 Estimate the equations to the tangent of $y = 6^{x}$ for the given values of x, correct to 2 decimal places.

(a) $x = 0$ **(b)** $x = 3$

18 The second derivative $f''(x)$ is the derivative of the (first) derivative $f'(x)$ of a function. Determine the first and second derivatives of $f(x) = 2e^{x^3} - 6e^{\sqrt{x}}$.

1.4 Applications of exponential functions

Exponential growth and decay

Exponential growth applies to many natural phenomena, such as the growth of an algal bloom or median house prices during a period of steady percentage change; in fact, it can refer to any quantity that increases exponentially over time.

Exponential growth can be modelled by $N(t) = N_0 e^{kt}$, where N_0 represents the initial or starting number (or value) and t represents time.

Making connections

Exponential growth and decay

Explore the behaviour of exponential growth $k > 0$ and decay $k < 0$.

Rate of change

If $N(t) = N_0 e^{kt}$ then the rate of change can be found by differentiating.

$$\begin{aligned} N'(t) &= N_0 \times ke^{kt} \\ &= k \times N_0 e^{kt} \\ &= k \times N(t) \end{aligned}$$

The rate of change $N'(t)$ is proportional to the function value $N(t)$, which is proportional to the original number N_0, and k is the rate constant.

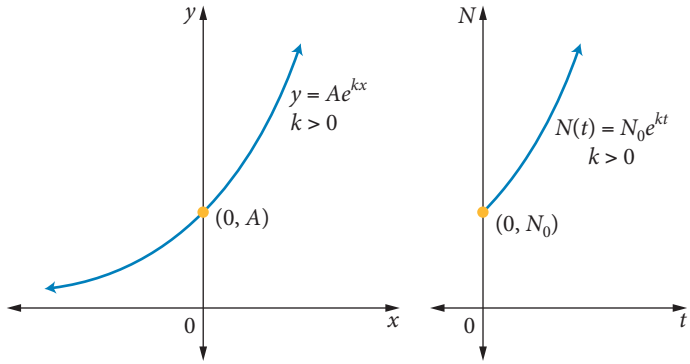
If $k > 0$, then $N'(t) > 0$, and the equation represents growth.

Conversely if $k < 0$, then $N'(t) < 0$ and it represents decline.

If $k = 0$, then the original function becomes $N(t) = N_0$, or a constant, since $e^0 = 1$, and the derivative will be $N'(t) = 0$.

Exponential growth

Since $t \geq 0$ in most situations, just the first quadrant part of the exponential function is relevant.



13 Determine the rule of an exponential model

Under ideal conditions, flies reproduce rapidly. After introducing four flies to a contained laboratory environment, the following data was collected.

Weeks (t)	0	1	2
Number of flies ($N(t)$)	4	133	4400

Use the data from the table to determine an exponential growth function representing the population of the flies. Evaluate the reasonableness of the model, stating the strengths and limitations.

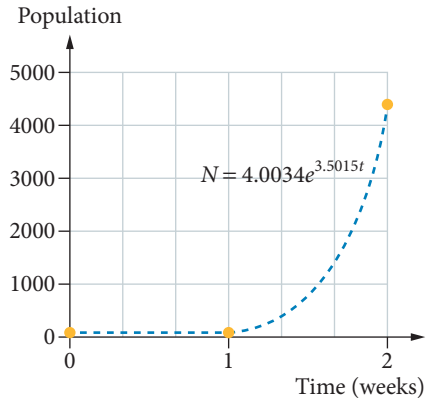
THINKING	WORKING
1 Substitute the initial value into an exponential growth formula.	Let $N(t) = N_0 e^{kt}$, with $N_0 = 4$. $N(t) = 4e^{kt}$
2 Substitute a pair of values and solve for k.	For $(1, 133)$: $133 = 4e^{k\times 1}$ $e^k = \frac{133}{4}$ $k = \log_e\left(\frac{133}{4}\right)$ $k = 3.504\ldots$
3 Verify the value of k using the other pair of values.	For $(2, 4400)$: $4400 = 4e^{k\times 2}$ $e^{2k} = \frac{4400}{4}$ $e^{2k} = 1100$ $2k = \log_e(1100)$ $k = \frac{\log_e(1100)}{2}$ $k = 3.501\ldots$
4 Determine the equation of the final function. Comment on its usefulness.	$N(t) = 4e^{3.50t}$ The model is suitable for the degree of accuracy (2 decimal places for k) given. The model could not be used outside of the given timeframe unless it was known that sufficient nutrients and space were available.

The data for Worked Example 13 can be entered into a spreadsheet and an exponential curve of best fit calculated.

You can specify the axis value, and so can choose the value of N_0.

If this was done in the example, N_0 would be 4 rather than 4.0034 and the other parameters would be adjusted accordingly.

t	N
0	4
1	133
2	4400



Technology worked example

Determine an exponential equation from data points

Additional information

Exponential growth

The activity gives you the opportunity to explore two examples of exponential growth. Use technology to evaluate the missing values in these exponential growth problems.

Population increasing by any fixed rate

If an entomologist introduces a beetle colony of 500 beetles to a new and fertile environment, and finds that the population doubles after 5 weeks, to 1000 beetles, she might suspect there is exponential growth. If continued monitoring showed that after a further 5 weeks it has doubled again to 2000 beetles, and this trend continues, this would verify the trend.

An exponential equation in the form $N(t) = N_0 e^{kt}$, for the size of the population N at time t (weeks), gives $N(t) = 500e^{kt}$. Substituting $(5, 1000)$:

$$1000 = 500e^{5k}$$
$$e^{5k} = 2$$
$$5k = \log_e(2)$$
$$k = \frac{1}{5}\log_e(2)$$

Substituting and simplifying using the inverse property $e^{\log_e(k)} = k$ gives:

$$N(t) = 500e^{\frac{\log_e(2)}{5}t}$$
$$= 500\left(e^{\log_e(2)}\right)^{\frac{t}{5}}$$
$$= 500 \times 2^{\frac{t}{5}}$$

When a population doubles its size in d units of time (hours, days, weeks, years etc.), then the standard exponential growth equation, $N(t) = N_0 e^{kt}$, can be expressed as an exponential equation in the form:

$$N(t) = N_0 \times 2^{\frac{t}{d}}$$

Additional information

Population doubling

View the short video to see how the exponential growth equation works for any case where the population doubles over a fixed period.

Additional information

Determining the power equation

Try the interactive activity to further explore this exponential growth relationship.
Identify the power equation in each case.

Consider a case where the population of a growing community of rabbits triples every 10 years. Using the same reasoning as above, you can show that for this case, $N(t) = N_0 e^{kt}$ can be expressed as $N(t) = N_0 \times 3^{\frac{t}{10}}$, where t is measured in years.

The exponential equation to show the population growth when the population increases by a fixed factor, m, over a particular time, d, where t has the same units as d is:

$$N(t) = N_0 \times m^{\frac{t}{d}}$$

14 Exponential growth increasing by a fixed factor over time

An initial population of 2000 increases by 7% every 9 days.

(a) Determine a suitable exponential equation to model the population growth.

THINKING	WORKING
1 Recall the general form of the equation.	$N(t) = N_0 \times m^{\frac{t}{d}}$
2 Determine the values for each parameter.	$m = 100\% + 7\%$ $= 1.07$ $N_0 = 2000$ $d = 9$
3 Substitute the parameter values into the general equation.	$N(t) = 2000 \times 1.07^{\frac{t}{9}}$, where t is the time in days.

(b) Determine the population growth rate per day. Express your answer as a percentage, correct to 2 decimal places.

1 Recall the growth equation.	$N(t) = 2000 \times 1.07^{\frac{t}{9}}$
2 Determine the values for two consecutive days, N_0 and N_1.	$N(0) = 2000$ $N(1) = 2000 \times 1.07^{\frac{1}{9}}$
3 Calculate the daily rate of growth, $\frac{N(1)}{N(0)}$.	Growth rate per day: $\frac{N(1)}{N(0)} = \frac{2000 \times 1.07^{\frac{1}{9}}}{2000}$ $= 1.07^{\frac{1}{9}}$ $= 1.0075\ (4\text{ d.p.})$

4 Interpret your answer, expressing it as a percentage, correct to 2 decimal places.

A rate of 1.0075 means that the population after one day is 1.0075 times the population of the day before.

This means that the population increases by 0.75% per day.

Additional information

Population and percentage

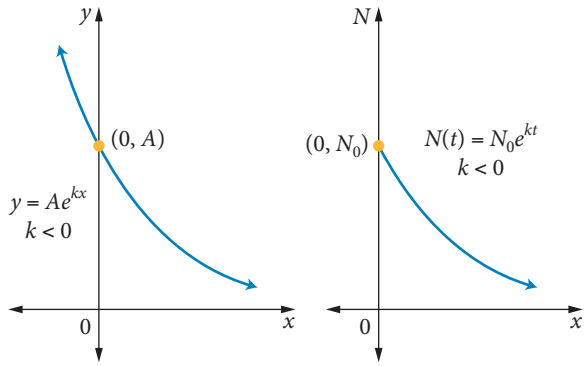
These interactive activities help consolidate the skills in Worked Example 14. Use a calculator where appropriate.

Exponential decay

Exponential decay can apply to many natural phenomena, such as the temperature of a cooling of a cup of tea or radioactive decay.

Exponential decay can be thought of in exactly the same way as exponential growth. The equation $N(t) = N_0 e^{kt}$ still applies, but with exponential decline, $k < 0$.

For $k < 0$, the first quadrant part of the exponential function gives just the lower portion of the graph.



The general formula $N(t) = N_0 \times (m)^{\frac{t}{d}}$ can be used for exponential decay, where N_0 is the initial size of the population and m is the proportion of the population that remains after the set period t.

One common example is radioactive decay, where the rate of decay is described as the half-life of the radioactive material. This measure indicates the period of time it takes for the radioactive material to reduce to half its original amount. In this case, $m = \frac{1}{2}$ since exactly half the material is left after the fixed half-life period, d.

One further application of exponential decline is the process of carbon dating, an interesting technique used for estimating the age of very old organic materials.

Additional information

Radioactive decay

These interactive activities explore radioactive decay.

Additional information

Carbon dating

Watch the video to see how exponential decline applies to the process of carbon dating. Then try the activities.

15 Radioactive decay

The half-life of thorium-234 is 24 days.

(a) Write a formula for the amount of thorium-234 remaining, in milligrams, after t days, from an initial sample of N_0 mg, in the following forms.

(i) $N(t) = N_0 e^{kt}$

THINKING	WORKING
1 Recall the general form of the equation.	$N(t) = N_0 e^{kt}$, where t is the time in days.
2 Substitute from the given information.	The half-life of thorium-234 is 24 days. $N(24) = 0.5N_0$
3 Solve for k.	$0.5N_0 = N_0 e^{k \times 24}$ $e^{24k} = 0.5$ $24k = \log_e(0.5)$ $k = \dfrac{\log_e(0.5)}{24}$ $k = -0.0289$ (3 sig. fig.)
4 Substitute the value of k into the formula.	$N(t) = N_0 e^{-0.0289t}$

(ii) $N(t) = N_0 \times (m)^{\frac{t}{d}}$

1 Recall the general form of the equation.	$N(t) = N_0 \times (m)^{\frac{t}{d}}$
2 Determine the values for each known parameter.	The half-life of thorium-234 is 24 days. $m = \dfrac{1}{2}$, $d = 24$
3 Substitute the parameter values. Simplify, if possible.	$N(t) = N_0 \times \left(\dfrac{1}{2}\right)^{\frac{t}{24}}$, where t is in days. So, $N(t) = N_0 \times 2^{-\frac{t}{24}}$

(b) Determine the quantity of thorium-234 remaining after 10 days from a sample of 50 mg.

1 Substitute for N_0 and t in one of the formulas, to calculate the value of $N(t)$.	$N(t) = N_0 e^{-0.0289t}$ where $N_0 = 50$, $t = 10$: $N(10) = 50e^{-0.0289 \times 10}$ $= 37.45\ldots$
2 Verify your result by substituting the values into the second formula.	$N(t) = N_0 \times 2^{-\frac{t}{24}}$ gives: $N(10) = 50 \times 2^{-\frac{10}{24}}$ $= 37.45\ldots$
3 Interpret and evaluate the reasonableness of your answer.	From the sample of 50 mg of thorium-234, with a half-life of 24 days, there will be 37.5 mg remaining after 10 days, i.e. more than half of the sample remains.

Calculating the time required

Exponential growth and decay formulas can be used to determine how much time is needed for the number or quantity to change to a required amount.

16 Solve for time

A company has \$1 000 000 to invest. It decides on a financial institution that promises that the value will grow according to the equation $A(t) = A_0 \times e^{0.0488t}$, where A_0 is the initial amount invested, t is the number of years since the money was first invested and $A(t)$ is the value of the investment at time t.

(a) Determine the value of the investment after 4.5 years, to the nearest dollar.

THINKING	WORKING
1 A_0 is the initial investment. Replace A_0 in the given equation.	$A(t) = A_0 \times e^{0.0488t}$ $A_0 = 1000000$ $A(t) = 1000000e^{0.0488t}$
2 Substitute for t, then calculate $A(t)$.	$A(4.5) = 1000000e^{0.0488\times 4.5}$ $= 1245578$
3 Interpret the answer.	The value of the investment after 4.5 years is 1 245 578.

(b) How long must the company maintain the investment, to the nearest month, if they want a 50% increase in the value of their investment?

1 Determine the value of the investment if it increases by 50%.	$A(t) = 1.5 \times A(0)$ $= 1.5 \times 1000000$ $= 1500000$
2 Substitute the required value for $A(t)$ and solve for t. Alternatively, substitute $1.5A_0$ into the original equation.	$1500000 = 1000000e^{0.0488t}$ $e^{0.0488t} = 1.5$ $0.0488t = \log_e(1.5)$ $t = \dfrac{\log_e(1.5)}{0.0488}$ $t = 8.3087\ldots$
3 Convert the t value to time in years and months.	$t = 8.3087\ldots$ Or $t = 8$ years and $0.3087\ldots \times 12 \approx 4$ months
4 Interpret the answer.	The company must maintain the investment for 8 years 4 months for a 50% increase in the value of the investment.

Extinction phenomena

Sometimes there are natural limits to the domain and range of a function. The domain is often time, so it is frequently defined as $t \geq 0$. The range in real-life situations may be limited by resources or once the population reaches zero. The growth of bacteria on an agar plate may be exponential, but when the plate is full and the bacteria run out of food they begin to poison themselves with their own waste and their numbers begin to fall.

Sometimes, as can be the case of extinction scenarios, the population may be fairly constant for a period of time, but then declines under pressure. This type of situation can be modelled using $P(t) = Ae^{kt} + c$, where c is the original estimate of the once stable population, $A < 0$, and k is the rate constant. If there is no intervention, then the population will drop to zero. If intervention is possible, then the population may stabilise and eventually begin to grow again.

Explore further

Extinction and stable populations

Explore population modelling using exponential functions.

17 Modelling using $P(t) = Ae^{kt} + c$

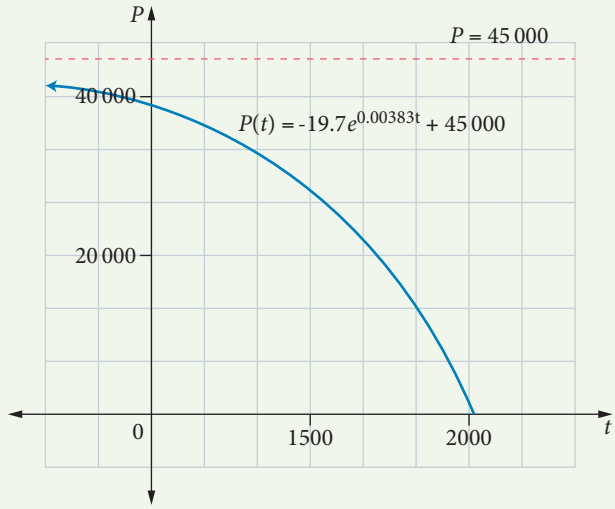
Scientists have been investigating the phenomenon of extinction for many years. In the Yangtze River in China there is a species of finless porpoise that has been struggling against loss of habitat and food, and increased shipping. A count in 2012 found only 1000 animals. The population had decreased from 2000 animals in 2006. Assume the population of the porpoise can be modelled by $P(t) = Ae^{kt} + 45\,000$, where t is the year.

(a) Determine values for A and k, correct to 3 significant figures, and then complete rule of $P(t)$.

THINKING	WORKING
1 Recall the general population equation $P(t)$.	$P(t) = Ae^{kt} + 45\,000$
2 Substitute the year and the population to create two equations in terms of A and k.	In $t = 2006$, $P(2006) = 2000$. $2000 = Ae^{2006k} + 45\,000$ $Ae^{2006k} = -43\,000 \quad [1]$ In $t = 2012$, $P(2012) = 1000$. $1000 = Ae^{2012k} + 45\,000$ $Ae^{2012k} = -44\,000 \quad [2]$
3 Solve simultaneous equations to determine the approximate values of A and k.	$[2] \div [1]$: $e^{6k} = \dfrac{-44\,000}{-43\,000}$ $e^{6k} = \dfrac{44}{43}$ $k = \dfrac{1}{6}\log_e\left(\dfrac{44}{43}\right)$ $k = 0.00383 \ (3 \text{ sig. fig.})$ Substitute k into [1]: $Ae^{2006 \times 0.00383\ldots} = -43\,000$ $A = -\dfrac{43\,000}{e^{2006 \times 0.00383\ldots}}$ $= -19.7 \ (3 \text{ sig. fig.})$
4 Determine the complete rule of $P(t)$.	$P(t) = -19.7e^{0.00383t} + 45\,000$

(b) Determine the stable population. Use a sketch to support your answer.

1 Sketch the graph, including the asymptote.



2 Interpret the value of the asymptote.

The stable population was around 45 000 porpoises.

(c) Determine when the population is predicted to reach zero.

1 Solve $P(t) = 0$ for t, or read the horizontal intercept from the graph.

$$0 = -19.7e^{0.00383t} + 45\,000$$
$$-45\,000 = -19.7e^{0.00383t}$$
$$e^{0.00383t} = \frac{45\,000}{19.7}$$
$$0.00383t = \log_e\left(\frac{45\,000}{19.7}\right)$$
$$t = \frac{\log_e\left(\frac{45\,000}{19.7}\right)}{0.00383}$$
$$t = 2019.27 \text{ (2 d.p.)}$$

2 Interpret the answer.

This model predicts that the population will reach zero in 2019.

(d) Determine and compare the rate of change in $t = 2006$ and $t = 2012$.

1 Determine an expression for the derivative $P'(t)$.

2 Substitute the individual values for t into the derivative equation $P'(t)$.

$$P'(t) = -19.7 \times 0.00383e^{0.00383t}$$
$$= -0.075\,451e^{0.00383t}$$
$$P'(2006) = -0.075\,451e^{0.00383\times 2006}$$
$$= -164 \text{ (3 sig. fig.)}$$
$$P'(2012) = -0.075\,451e^{0.00383\times 2012}$$
$$= -168 \text{ (3 sig. fig.)}$$

3 Interpret the answer.	The population is declining by 164 porpoises per year in 2006 and by 168 porpoises per year in 2012.
4 Compare the values.	The porpoise population is declining at an increasing rate.

EXERCISE 1.4 Applications of exponential functions

Worked Example 13

1 For each of the following tables of values, determine an exponential growth function of the form $N(t) = N_0 e^{kt}$. Support your answer with a model found from regression analysis. Round parameters correct to 3 significant figures, where appropriate.

(a)

t (weeks)	0	1	2
$N(t)$	23	54	128

(b)

t (days)	0	1	2
$N(t)$	165	516	1613

(c)

t (years)	0	1	2
$N(t)$ (thousands)	2.87	4.41	6.78

(d)

t (months)	0	1	2
$N(t)$ (thousands)	0.42	13	417

2 Over the centuries, modes of transportation have changed significantly from walking and horses to cars, trains and planes. Thinking in terms of your ability to travel, you can model the number of kilometres the average person will travel in their lifetime. For example, in 1850, the average person would travel a distance of 6.4 km per day. If you multiply by 365 (days in a year) and 50 (years in a lifetime) this gives 117 000 km travelled in the average lifetime.

The table below shows the comparison over the years:

Year	1850	1900	1950	2000	2050
Distance travelled (in thousands of km)	117	282	986	3524	11 895

(a) Use regression analysis to determine an exponential model in the form $d(t) = Am^t$, where d is the average distance travelled in thousands of kilometres and t is the number of years since 1850. Write the parameters correct to 5 significant figures.

(b) Estimate the average distance travelled in a lifetime, to the nearest hundred thousand kilometres, for the population in 2020.

3 Clare invests $1000. The following table shows the value of her investment over time. At any particular time, the value of her investment can be calculated by $I = I_0 e^{kt}$.

Years (t)	1	3	8
Investment value (I)	$1060	$1191	$1594

(a) The values for I_0 and k are:

A 1000 and 0.06 B 1060 and 0.06
C 1000 and 0.0583 D 1060 and 0.0583

(b) What is the percentage interest rate, per annum?

A 1.0583 B 1.06
C 5.83 D 6

4 A farmer has worked on organically enriching the soil. She introduced more organic matter and worms to a small section and began monitoring. She began by introducing 25 worms per square metre. Two months later, there were 50 worms per square metre and monthly monitoring showed that this trend of doubling every two months continued. Write the exponential equation for the size of the worm population N at time t months, determine the constants and rewrite the equation in its simplest form.

Worked Example

15

5 The half-life of strontium-90 is 28.79 years.

(a) Write a formula for the amount of strontium-90 remaining, in milligrams, after t years, from an initial sample of N_0 mg, in the following forms.

(i) $N(t) = N_0 e^{kt}$, rounding k to 3 significant figures.

(ii) $N(t) = N_0 \times (m)^{\frac{t}{d}}$, where m is the proportion left after d units of time.

(b) Determine the quantity of strontium-90 remaining after 5 years from a sample containing 20 mg.

16

6 A small business is set up by a local tradesman. In the first year his gross income is \$120 000 and the second year it is \$180 000. The proposed growth, for the first 10 years, can be modelled by $I(t) = Ae^{kt}$, where $I(t)$ is the gross annual income, t represents the number of years since he began the business, and A and k are real constants.

(a) Determine the value of A in whole dollars and find the value of k, correct to 3 decimal places.

(b) Determine the expected gross income for the tenth year, to the nearest dollar. Use the unrounded value of k.

(c) Once the annual turnover reaches \$1 000 000, the business owner believes that he will need to employ another person. This will decrease his profit margin, but enable his business to keep growing. How long before he needs to begin looking for an apprentice? Calculate the value of t, correct to 2 decimal places.

7 A student brought to school a bottle of frozen water from home. By the start of recess, the temperature of the water begins to rise above 0°C. After an additional 15 minutes, the temperature is 10°C. The warming can be modelled by an exponential equation of the form $T(t) = Ae^{kt} + 35$, where A and k are real numbers such that $k < 0$, $t \geq 0$ and is the time in minutes after recess begins, and T is the temperature in degrees Celsius of the water.

(a) Determine the temperature that the water will not exceed.

(b) Determine the value of A.

(c) Determine the value of k, correct to 3 significant figures.

(d) Write the equation to represent T.

(e) Once the temperature of the water reaches 20°C, it is no longer pleasant to drink. How long, to the nearest minute, can the student wait before drinking the water?

8 An initial population of 850 increases by 6% every half-year, where t is measured in years.

(a) Which of the following gives the population at any given time?

A $N(t) = 850 \times 1.06^{\frac{t}{2}}$ **B** $N(t) = 850 \times e^{0.53t}$

C $N(t) = 850 \times 1.06^{2t}$ **D** $N(t) = 850 \times 1.06^{-2t}$

(b) Explain the common error made be a student who wrote an equation with the power $\frac{t}{2}$.

9 The number of radioactive units in a sample initially containing 4000 units is given by $N = 4000e^{-\frac{t}{50}}$, where t is measured in years. Determine how many years have passed when the number of units is decreasing by 15 units per annum. Give your answer to the nearest year.

10 The world population doubles every T years depending on what year is used as the starting point. In 1500, the population was 0.5 billion and had doubled by 1804; thus the doubling period T was 304 years.

(a) In 1999, the total world population was 39 billion and the predicted doubling period was 39 years. Determine the projected world population for the year 2100 assuming that the doubling period remains at 39 years. Give your answer to the nearest billion.

(b) Sketch the graph of population over time for the model established in part **(a)**. Is this a realistic model? Evaluate the strengths and limitations of the model.

11 A zoo has set up a successful breeding program for an endangered species of marsupial. At the end of the first year, the zoo is able to release 6 animals back into a reserve. At the end of the third year, the zoo can release 20 animals. The anticipated release rate can be modelled by $R(t) = Ae^{kt}$, where $R(t)$ is the number of animals released each year and t represents the number of years that the breeding program has been running.

(a) Determine the values of A and k, correct to 3 decimal places.

(b) Determine the number of animals the zoo expects to release at the end of the tenth year.

(c) Once the zoo is able to release more than 100 animals, it will be able to begin releasing into another area. After how many years will this be?

12 The introduction of feral and domestic predators has had a large impact on the quoll populations in the wild. In one reserve, its population has settled at around 200 animals. It was thought that there were around 600 quoll in the area in 1950, and a count in 1990 found 300. The quoll population in this reserve can be modelled by the general equation $N(t) = Ae^{kt} + c$, where t is the number of years since 1850.

(a) Determine the equation for the population model, giving the parameters to 3 significant figures.

(b) How many quolls does this suggest were in the area in 1850?

Worked Example 14

13 An initial population of 400 increases by 80% every 6 months.

(a) Use the general equation $N(t) = N_0 \times m^{\frac{t}{d}}$, where m represents the population multiplier for every d units of time, to model the situation.

(b) Determine the population growth rate per month. Express your answer as a percentage, correct to 2 decimal places.

14 A cake has been removed from an oven and begins to cool. Its temperature can be modelled by $T = Ae^{kt} + 21$, where T is the core temperature of the cake in degrees Celsius and t represents the time in minutes since it was removed from the oven. After 5 minutes, the temperature of the cake is 133°C, and after 15 minutes the temperature of the cake is 76.5°C.

(a) Determine the equation of the asymptote for this function. Explain the significance of this asymptote.

(b) Establish the equation for this situation, writing the parameters to 3 significant figures.

(c) Determine the temperature of the cake when it was removed from the oven.

(d) The cake needs to be removed from the baking pan when it is between 80°C and 100°C. How long should it be allowed to stand before being removed from the pan? Give the range to the nearest minute.

(e) The cake cannot be decorated until it has reached a temperature that is lower than 30°C. Determine the minimum time, after the cake is removed from the oven, before it can be decorated, to the nearest whole minute.

15 Once viruses enter an organism, they encourage the host cells to reproduce them at a rapid rate. This can be modelled by $N(t) = N_0 e^{kt}$, where N_0 is the original number of virus particles, in thousands, and t is the number of hours after the infection first occurred. A sneeze has ensured that a new victim has 800 thousand virus cells enter her body, and in 6 hours there are 1500 thousand viable virus cells.

(a) Establish the equation for the number of virus particles in the victim's body, giving k correct to 3 significant figures.

(b) Determine the expression for $N'(t)$.

(c) Once a host is infected, and the number of virus particles begins to interfere with the body's function, the host begins to make antibodies. If the rate of antibody production matches the rate of virus production after 12 hours, what is this rate, rounded to the nearest thousand per hour?

(d) If the equation modelling the number of thousands of antibodies can be given by $A(t) = e^{m(t-6)}$, determine the equation for $A(t)$.

(e) When (to the nearest hour) does the number of antibodies first overtake the number of virus particles?

16 Carbon dating uses the carbon-14 isotope. Every living thing absorbs carbon-14 during its lifetime. Absorption stops when an organism dies. Carbon-14 decays with a half-life of 5730 years. The ratio of radioactive carbon-14 to the two other non-radioactive isotopes of carbon tells how many half-lives have passed. This can be modelled by $N(t) = N_0 e^{-\lambda t}$, where N is the percentage of carbon-14 remaining, and t is number of thousand years since the organism was alive.

(a) Determine the equation with λ correct to 3 significant figures.

(b) Determine the derivative of $N(t)$.

(c) Calculate each of the following, correct to 3 significant figures.

(i) $N'(4)$ (ii) $N'(10)$ (iii) $N'(40)$

(iv) $N'(50)$ (v) $N'(60)$ (vi) $N'(70)$

(d) Explain why scientists will not stand by carbon dating for objects that are more than 60 000 years old.

Worked Example

17

17 New management practices on a farm have increased the yield of wheat from 10 bushels per acre to 13 bushels per acre after the first year, and 15 bushels per acre after the second year. This can be modelled by $Y(t) = Ae^{kt} + c$, where $Y(t)$ represents the yield and t is the number of years since the new management practices were introduced.

(a) Determine the actual equation, giving k correct to 2 decimal places.

(b) Determine the expected yield after 5 years, correct to the nearest bushel per acre.

(c) Determine the maximum yield expected using these practices.

18 The growth pattern of the popular Australian plantation timber, often called the blue gum, is quite interesting. For the first 10 years, it grows very rapidly and is considered a softwood, but then its growth begins to slow down and it is considered a hardwood. The final height in plantation timber tends to settle at around 30 m, and any further growth is in thickness. The height can be modelled by the equation $H(t) = Ae^{kt} + c$, where t is the time in years since planting and $H(t)$ is the height in metres.

(a) Determine the actual equation for the height, given that the seedlings are 30 cm tall when planted and the average tree is 5 m after 2 years. Determine the value of k correct to 3 significant figures.

(b) Determine the average height of the blue gum trees after 10 years, correct to 1 decimal place.

(c) Determine an expression for $H'(t)$, rounding appropriately.

(d) Determine the rate of growth in metres per year when $t = 10$. Give your answer correct to 1 decimal place.

(e) What happens to the growth rate after 10 years? Support your explanation with a graph.

19 Radioactive dating can be used to estimate the age of rocks. Rocks on Earth contain trace amounts of uranium-235 when they first appear, but no lead-207. Over time, the uranium isotope radioactively decays to the lead isotope. It is said to have a half-life of 4.5 billion years. This means that after 4.5 billion years the number of lead-207 atoms is the same as the number of uranium-235 atoms. Because this is a ratio within any given rock sample, the exact number present initially does not need to be known, and the percentage present can be used. The equation that can be used to determine the relationship between the current percentage and time can be written as $N(t) = N_0 e^{-\lambda t}$, where t is the number of billion years since the rock was formed, and $N(t)$ is the percentage of ^{235}U atoms present. How old is a rock that has 80% ^{235}U and 20% ^{207}Pb? Give your answer in billions of years, correct to 2 decimal places.

20 Under normal conditions, the human core body temperature is around 37°C and the skin temperature is slightly lower at 36.8°C. When exposed to cold conditions, such as immersion in water, the body tries to conserve the core temperature to maintain healthy organ function. A young man fell overboard after his small craft hit a large wave. The water temperature was 15°C. Due to the rough conditions, it was 40 minutes before he was able to be rescued. Fortunately he was wearing a life jacket, which not only helped him keep afloat, but also helped to insulate his major organs. On checking his vital signs, his core body temperature was found to be 35.5°C, but his skin temperature was reduced to 21°C.

(a) Determine an equation of the form $T(t) = Ae^{kt} + c$ to model the man's core temperature. Give the value of k correct to 3 significant figures.

(b) Determine an equation of the form $T(t) = Ae^{kt} + c$ to model the man's skin temperature. Give the value of k correct to 3 significant figures.

(c) Hypothermia is a medical emergency, with patients suffering hallucinations, delirium and coma when the core body temperature drops below 32°C. How much more time did the young man have, to the nearest minute, before this temperature was reached?

(d) What would have been the temperature of his skin at this time? Give your answer correct to 1 decimal place.

1.5 Further differentiation

Review of the rules of differentiation

In this section you will differentiate expressions involving exponential functions combined with other functions, including polynomials.

In many cases, the product rule, quotient rule or chain rule should be applied.

The chain rule:

If $y = f(u)$, where $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

If $h(x) = f(g(x))$:

$$h'(x) = f'(g(x)) \times g'(x)$$

18 The chain rule

Differentiate $y = e^{(2x-7)}$.

THINKING

1 Define the rule as the composition of functions $y = e^u$.

2 Differentiate each expression to obtain $\frac{dy}{du}$ and $\frac{du}{dx}$.

3 Apply the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and simplify the expression in terms of x.

WORKING

$y = e^{(2x-7)^3}$

Let $y = e^u$, where $u = (2x-7)^3$.

$$\frac{dy}{du} = e^u$$
$$= e^{(2x-7)^3}$$

$$\frac{du}{dx} = 3(2x-7)^2 \times 2$$
$$= 6(2x-7)^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^{(2x-7)^3} \times 6(2x-7)^2$$
$$= 6(2x-7)^2 e^{(2x-7)^3}$$

The product rule:

If $y = uv$:

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$
$$= uv' + vu'$$

If $h(x) = f(x)g(x)$:

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$

19 The product rule

Differentiate $h(x) = xe^{2x}$.

THINKING	WORKING
1 Define the rule as the product of two functions $h(x) = f(x)g(x)$.	$h(x) = xe^{2x}$ $= f(x)g(x)$ where $f(x) = x$, $g(x) = e^{2x}$
2 Differentiate each function to obtain $f'(x)$ and $g'(x)$.	$f'(x) = 1$, $g'(x) = 2e^{2x}$
3 Apply the product rule $h'(x) = f(x)g'(x) + g(x)f'(x)$ and simplify the expression in terms of x.	$h'(x) = f(x)g'(x) + g(x)f'(x)$ $= x \times 2e^{2x} + e^{2x} \times 1$ $= 2xe^{2x} + e^{2x}$ $= e^{2x}(2x + 1)$

The quotient rule

If $y = \frac{u}{v}$:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

If $h(x) = \frac{f(x)}{g(x)}$:

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

20 The quotient rule

Differentiate $y = \frac{2x - 3}{e^{(x-3)^2}}$.

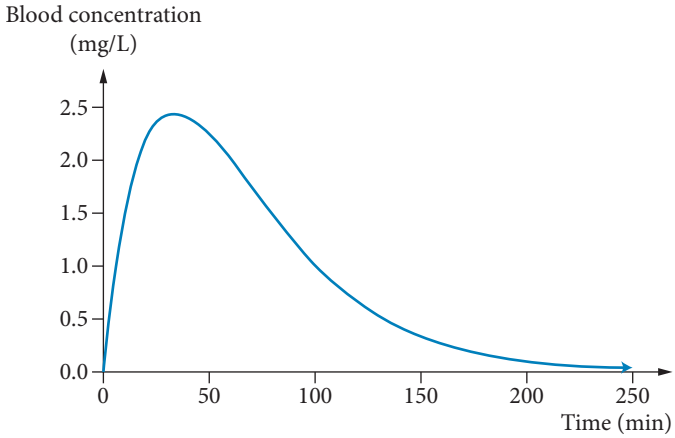
THINKING	WORKING
1 Define the rule as the quotient of two functions $y = \frac{u}{v}$.	$y = \frac{2x - 3}{e^{(x-3)^2}}$ $= \frac{u}{v}$ where $u = 2x - 3$ and $v = e^{(x-3)^2}$
2 Differentiate each function to obtain $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$\frac{du}{dx} = 2$ $\frac{dv}{dx} = 2(x - 3)e^{(x-3)^2}$
3 Apply the quotient rule, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $= \frac{e^{(x-3)^2} \times 2 - (2x - 3)\left(2(x - 3)e^{(x-3)^2}\right)}{\left(e^{(x-3)^2}\right)^2}$

4 Simplify the expression by factorising.

$$\frac{dy}{dx} = \frac{2e^{(x-3)^2}\left(1-(2x-3)(x-3)\right)}{\left(e^{(x-3)^2}\right)^2}$$
$$= \frac{2\left(1-2x^2+9x-9\right)}{e^{(x-3)^2}}$$
$$= \frac{-2\left(2x^2-9x+8\right)}{e^{(x-3)^2}}$$

Graphs of exponential functions combined with other functions

When exponential functions are combined with other functions, a large variety of graphs can be generated. These graphs may contain stationary points. Remember, at a stationary point the derivative of the function is equal to zero. For example, when medicine is delivered to a patient, there is a rise in the level of the active ingredient in the blood, and then it gradually lowers. This can be modelled by $A(t) = bte^{\lambda t}$, where b and λ are real constants, and $A(t)$ is the amount of drug in the blood at any time t minutes after delivery. The maximum level of the drug in the blood can be calculated by finding when the derivative is zero. By solving $A'(t) = 0$ for t and substituting it back into the original equation, the maximum concentration can be found.



21 Stationary points

For the function $f(x) = e^{x^2} - 3x + 2$, determine the coordinates of any stationary points, correct to 3 decimal places, and also determine their nature.

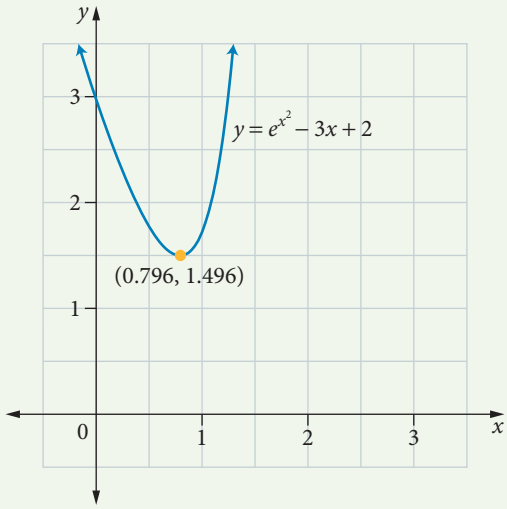
THINKING	WORKING
1 Differentiate the function.	$f(x) = e^{x^2} - 3x + 2$ $f'(x) = 2xe^{x^2} - 3$
2 Solve $f'(x) = 0$.	$f'(x) = 0$: $2xe^{x^2} - 3 = 0$ $2xe^{x^2} = 3$ From technology: $x = 0.796\,00$. $f(0.796) = e^{0.796\,00^2} - 3 \times 0.796\,00 + 2$ $= 1.496$ There is a stationary point at $(0.796, 1.496)$.

3 Choose values of x to the left and right of the stationary point to test the behaviour of the gradient at the stationary point.

4 Use technology to confirm the results.

$f'(0) = -3, f'(1) = 2e - 3 > 0$

The stationary point (0.796, 1.496) is a minimum.



t Technology worked example

Locating stationary points

22 Stationary points of combined functions

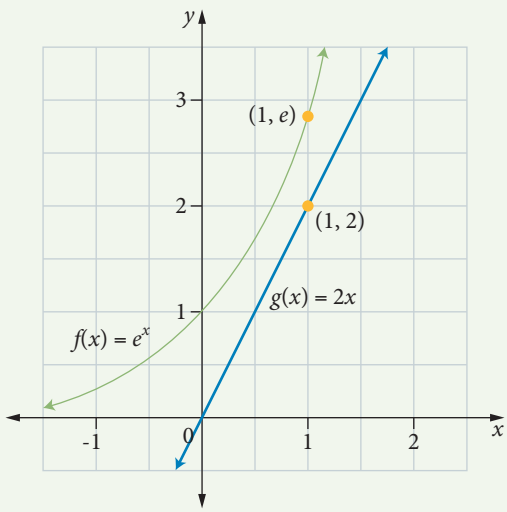
Consider the functions $f(x) = e^x$ and $g(x) = 2x$.

(a) Sketch the graphs of $f(x)$ and $g(x)$ on the same set of axes.

THINKING

Sketch the two functions.

WORKING



(b) Determine the minimum vertical distance between $f(x)$ and $g(x)$, and the value of x where it occurs.

1 Determine a function to give the vertical distance.	$f(x)$ is above $g(x)$, so let the vertical distance be: $h(x) = f(x) - g(x)$ $= e^x - 2x$
2 Differentiate the difference equation to obtain $h'(x)$.	$h'(x) = e^x - 2$
3 To determine the x-value of the minimum, solve $h'(x) = 0$ for x.	$e^x - 2 = 0$ $e^x = 2$ $x = \log_e(2)$
4 Substitute the x-value into the function $h(x)$ to determine the minimum distance between the functions.	$h(\log_e(2)) = e^{\log_e(2)} - 2\log_e(2)$ $= 2 - 2\log_e(2)$
5 Interpret the answer.	The minimum vertical distance is $2 - 2\log_e(2)$ units, and occurs at $x = \log_e(2)$.

EXERCISE 1.5 Further differentiation

Worked Example 18

1 Differentiate the following functions.

(a) $y = e^{x^2 - 3x}$ **(b)** $f(x) = \left(2x + e^{3x}\right)^4$ **(c)** $f(x) = e^{\sqrt{2x+1}}$ **(d)** $f(t) = e^{(2t-1)^3}$

2 Determine an expression for the derivative of $\dfrac{2}{e^{(x-3)^2}}$.

3 If $y = e^x\left(e^{5x} - 2e^{-x}\right)$, which expression represents $\dfrac{dy}{dx}$?

A $e^x\left(e^{5x} - 2e^{-x}\right)$ **B** $6e^{6x} - 2$ **C** $e^x\left(6e^{5x} + e^{-x}\right)$ **D** $6e^{6x}$

4 Consider the function $y = 3e^2\left(e^x + 4\right)^2$.

(a) Which expression gives the derivative?

A $6e^2\left(e^x + 4\right)$ **B** $6e^{x+2}\left(e^x + 4\right)$

C $6e^2\left(e^x + 4\right) + 3e^2\left(e^x + 4\right)^2$ **D** $6e^2\left(e^x + 4\right)\left(2e^x + 4\right)$

(b) Explain the common error made by a student who chose the first incorrect option.

5 Differentiate the following functions.

(a) $f(x) = \dfrac{1}{e^{x^3} + 2e^x}$ **(b)** $f(x) = \dfrac{1}{e^{x^2} - 3e^x}$

6 Determine an expression for the derivative of $r(t) = \left(2e^{-t} - e^{2t}\right)^3$.

Worked Example 19

7 Differentiate the following functions.

(a) $y = 5e^{2x}\left(3x^2 - 3\right)$ **(b)** $f(x) = 3xe^{3x}$ **(c)** $y = xe^{2x^3}$ **(d)** $y = 3e^x\left(e^x + 4\right)$

(e) $y = 2x^2e^{2x}$ **(f)** $y = e^{-2x}x^3$ **(g)** $n(x) = (5 - 3x)^2 e^{2x}$

8 Consider $f(t) = e^{t^2} - 50t + 17$ for $0 \le x \le 3$. 21

(a) Determine an expression for $f'(t)$.

(b) Determine the coordinates of the stationary point, correct to 2 decimal places.

(c) Calculate the value of each of the following, correct to 2 decimal places.

(i) $f'(1)$ (ii) $f'(2)$

(d) Determine the type of stationary point found in the given domain.

9 Consider $f(x) = \frac{1}{x}e^x$, where $x > 0$.

(a) Determine an expression for $f'(x)$.

(b) Determine the exact coordinates of any stationary points.

(c) Calculate each of the following, correct to 2 decimal places.

(i) $f'(0.8)$ (ii) $f'(1.2)$

(d) Determine the type of stationary point found in the given domain.

10 Consider $f(x) = 1 - e^{-x} - \frac{x}{20}$, for $0 \le x \le 20$.

(a) Determine an expression for $f'(x)$.

(b) Determine the exact coordinates of the stationary point.

(c) Calculate the value of each of the following, correct to 2 decimal places.

(i) $f'(0)$ (ii) $f'(20)$

(d) Determine the type of stationary point found in the given domain.

11 A particle is moving with velocity $v = \frac{t^2}{e^t}$ m/s, where t is time in seconds.

(a) Determine when the acceleration $\frac{dv}{dt}$ of the particle is zero at some time after the $t = 0$.

(b) What can you deduce about the velocity at this time?

12 For the function $y = (5x - 3)^3 e^{2x+5}$, determine an expression for $\frac{dy}{dx}$.

13 Differentiate the following functions. 20

(a) $y = \frac{x^3 + 1}{e^{2x}}$ (b) $f(x) = \frac{e^{x^2} - 1}{x - 3}$ (c) $y = \frac{4e^{5x}}{\sqrt{x-4}}$ (d) $f(x) = \frac{e^{3x}}{\sqrt{4x-1}}$

14 Differentiate the following functions.

(a) $f(x) = 7xe^{\sqrt{x}}$ (b) $f(x) = \frac{x^4}{e^{\sqrt{x}}}$

15 At a ski run the mountain slope can be approximated by an exponential function $g(x) = Ae^{-kx}$ for $0 \le x \le 100$, where g and x are measured in metres, vertically and horizontally respectively, from a fixed origin. In addition to this $g(0) = 50$ and $g(100) = 0.92$. 22

(a) Determine the actual equation for $g(x)$.

There is a ski lift to take skiers up the slope. This runs along a heavy cable that can be considered to form a straight line between two 3-metre-high towers located at $x = 0$ and $x = 100$.

(b) What are the two points at the start and finish of the ski lift?

(c) Determine the equation to the line $l(x)$ that models the path of the ski lift.

(d) Graph $g(x)$ and $l(x)$ on the same axes.

(e) Determine an equation for $h(x)$, the difference in vertical height of the ski lift from the slope.

(f) How far above the slope is a skier on the lift where $x = 30$? Give your answer correct to 2 decimal places.

(g) At what horizontal distance is the height above the ski slope at its maximum, and how high is the lift at this point? Give your answers correct to 2 decimal places.

16 The students in a particular year level at a school sit for a standardised test and their scores are recorded. The number of students achieving a particular score can be modelled by the equation $N(s) = 9e^{-\frac{1}{50}(s-28)^2}$, where $N(s)$ is the number of students achieving the score s.

(a) Determine the likely maximum possible score for the test.

(b) Determine the modal score and the number of students who achieved this score.

17 One model that estimates blood alcohol levels after the consumption of an alcoholic drink is $A(t) = 0.0045te^{-0.04t}$, where A is the percentage blood alcohol and t is the time in minutes since the consumption of the drink.

(a) Calculate the maximum blood alcohol level and the time after consumption when it occurs.

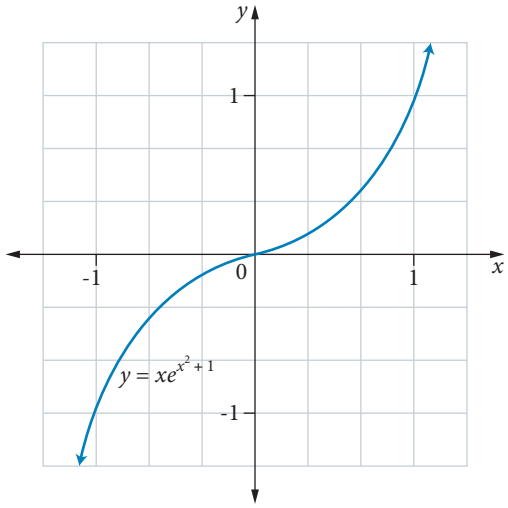
(b) Determine a rule for the blood alcohol of a person who waits 30 minutes before taking a drink, $A_2(t)$.

(c) Determine a piece-wise function, $C(t)$, that gives the cumulative blood alcohol for a person who has a drink and another after 30 minutes.

(d) Sketch $C(t)$.

(e) Use technology to find the maximum blood alcohol level and when it occurs after the second drink.

18 The tangent to $y = xe^{x^2+1}$ at point P with the x-coordinate $x = -1$ intersects the y-axis at point Q. The tangent to $y = xe^{x^2+1}$ at point R with the x-coordinate $x = 1$ cuts the y-axis at point T. Demonstrate that $PQRT$ is a parallelogram.



Summary

Euler's number

The number e is defined as $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ and is approximately equal to 2.7183.

Index laws

$e^m \times e^n = e^{(m+n)}$

$e^0 = 1$

$\left(e^m\right)^n = e^{(m \times n)}$

$\frac{1}{e^m} = e^{-m}$

$\frac{e^m}{e^n} = e^{(m-n)}$

$\sqrt[n]{e} = e^{\frac{1}{n}}$

Exponential equations

If $a^m = a^n$, then $m = n$ where $a > 0$.

$e^x = y$ is equivalent to $x = \log_e(y)$.

$a^x = e^{(\ln a)x}$

Exponential function $f(x) = e^x$

Properties of $f(x) = e^x$:

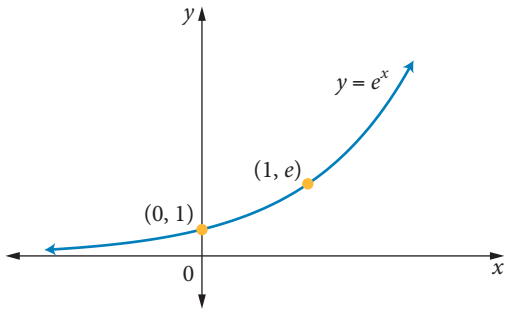
It is a strictly increasing function.

The coordinates of the y-intercept are $(0, 1)$.

The equation of the horizontal asymptote is $y = 0$, so there is no x-intercept.

The domain is $\mathbb{R}$.

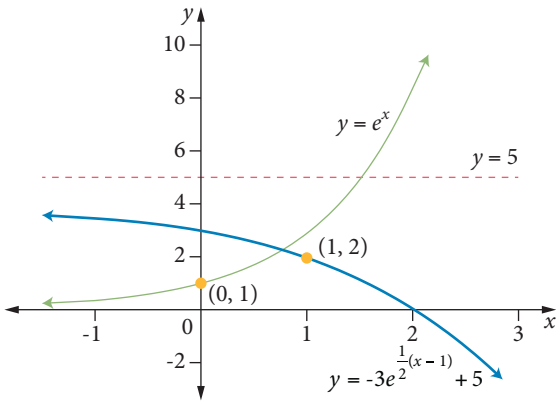
The range is $\mathbb{R}^+$.



Transformations of $y = e^x$

For $f(x) = ae^{b(x-c)} + d$, from the graph of $y = e^x$, this graph has been dilated by a factor of $|a|$ parallel to the y-axis and a factor of $\left|\frac{1}{b}\right|$ parallel to the x-axis and then translated horizontally c units and vertically d units.

If $a < 0$, there has been reflection in the x-axis; if $b < 0$, there has been reflection in the y-axis.



Differentiating e^x

$\lim\limits_{h\to 0}\dfrac{a^h-1}{h}$ gives the gradient of $y=a^x$ at $(0,1)$.

$\lim\limits_{h\to 0}\dfrac{\left(e^h-1\right)}{h}=1$

If $f(x)=e^x$, $f'(x)=e^x$.

If $f(x)=e^{mx+c}$, $f'(x)=me^{mx+c}$.

If $f(x)=ae^{g(x)}$, $f'(x)=a\times g'(x)e^{g(x)}$.

Rules for differentiation

Chain rule	If $y=f(u)$, where $u=g(x)$: $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$	If $h(x)=f(g(x))$: $h'(x)=f'(g(x))g'(x)$
Product rule	If $y=uv$: $\dfrac{dy}{dx}=u\times\dfrac{dv}{dx}+v\times\dfrac{du}{dx}$ $=uv'+vu'$	If $h(x)=f(x)g(x)$: $h'(x)=f(x)g'(x)+g(x)f'(x)$
Quotient rule	If $y=\dfrac{u}{v}$: $\dfrac{dy}{dx}=\dfrac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ $=\dfrac{vu'-uv'}{v^2}$	If $h(x)=\dfrac{f(x)}{g(x)}$: $h'(x)=\dfrac{g(x)f'(x)-f(x)g'(x)}{\left(g(x)\right)^2}$

Chapter review 1

1 Solve the following equations for x, giving your answers in exact form. Exercise 1.1

(a) $e^{2x} = e^5$

(b) $e^{x-7} = \frac{1}{e^3}$

(c) $e^x + 1 = 3$

(d) $e^{x+1} = 3$

2 Solve the following equations for x, giving your answers in exact form. Exercise 1.1

(a) $\left(e^x - 1\right)\left(e^x - 5\right) = 0$

(b) $e^{2x} - 3e^x + 2 = 0$

(c) $e^{4x} - 5e^{2x} + 4 = 0$

(d) $\frac{21}{e^x} = e^x - 4$

3 Given $e^{6x} = e^{2x+3}$, solve for x. Exercise 1.1

4 Choose the correct solution to each of the following equations. Exercise 1.1

(a) $e^{2x} + e^x = 2$

A $x = 0$ or $x = \log_e(2)$

B $x = 0$ or $x = \frac{1}{2}\log_e(2)$

C $x = \frac{1}{2}\log_e(2)$

D $x = 0$

(b) $e^{2x} + 4e^x + 3 = 0$

A $x = 0$ or $x = \log_e(3)$

B no solutions

C $x = \log_e(3)$

D $x = 0$

5 Determine the transformations from the graph of $y = e^x$ that have been applied to each of the following functions. Exercise 1.2

(a) $y = -3e^x + 1$

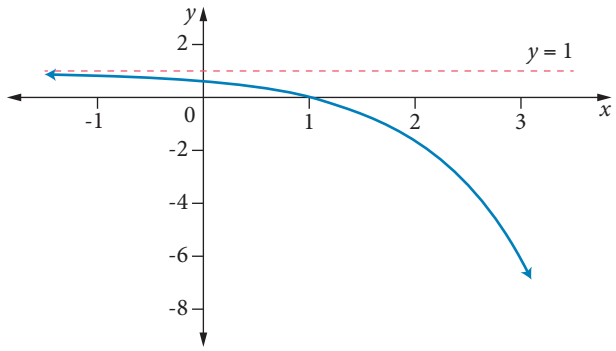
(b) $y = e^{2-x} - 5$

(c) $y = 2 - e^{2(x+3)}$

(d) $y = -2 \times e^{(1-5x)} - 3$

6 Choose the equation that represents each of the following graphs. Exercise 1.2

(a)



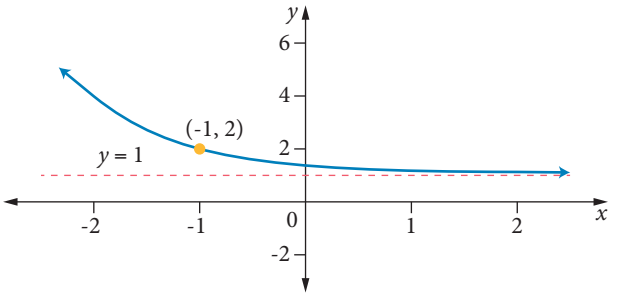
A $y = 1 - e^{x-1}$

B $y = 1 + e^{x+1}$

C $y = 1 + e^{1-x}$

D $y = 1 - e^{-x-1}$

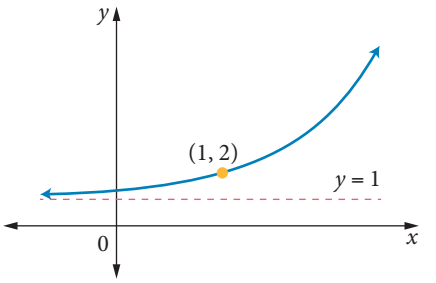
(b)



A $y = 1 + e^{x-1}$ **B** $y = 1 + e^{x+1}$

C $y = 1 + e^{1-x}$ **D** $y = 1 + e^{-x-1}$

(c)



A $y = 1 + e^{x-1}$ **B** $y = 1 + e^{x+1}$

C $y = 1 + e^{1-x}$ **D** $y = 1 + e^{-x-1}$

(d) $y = e^x$ is reflected in the x-axis, dilated by a factor of $\frac{1}{3}$ parallel to the x-axis and translated 2 units to the right.

A $y = -e^{3x-6}$ **B** $y = -e^{\left(\frac{x-2}{3}\right)}$

C $y = -\frac{1}{3}e^{(x-2)}$ **D** $y = \frac{1}{3}e^{(2-x)}$

(e) The horizontal asymptote is $y = -3$ and the y-intercept is $(0, -1)$.

A $y = 3e^x - 2$ **B** $y = 2e^{\frac{x}{2}} - 3$

C $y = e^{3x} - 3$ **D** $y = 2e^{3x} - 1$

Exercise 1.2 **7** Determine the equation of the asymptote of the graph of $y = 5 - e^{2x}$.

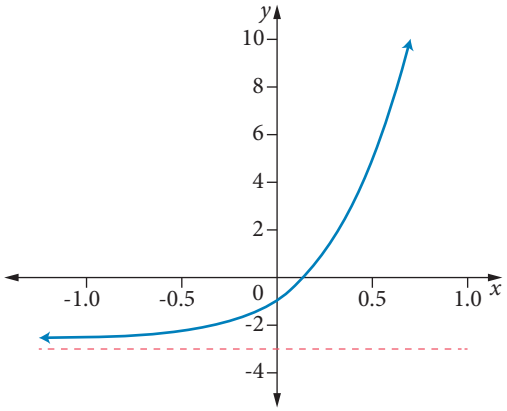
Exercise 1.2 **8** Determine the coordinates of the x-intercept of the graph of $y = 5e^x + 3$.

9 Choose the graph that could match each of the following rules.

Exercise 1.2

(a) $y = 2e^{-3x} - 3$

A



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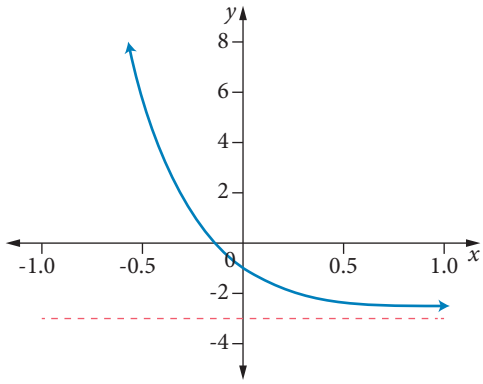

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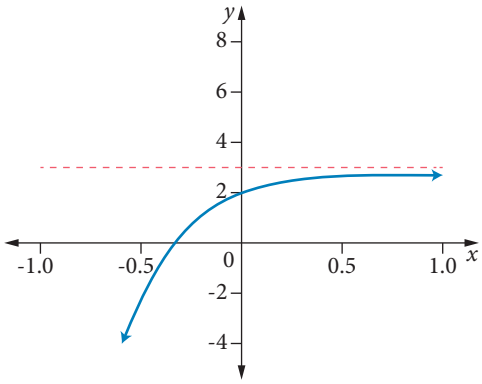

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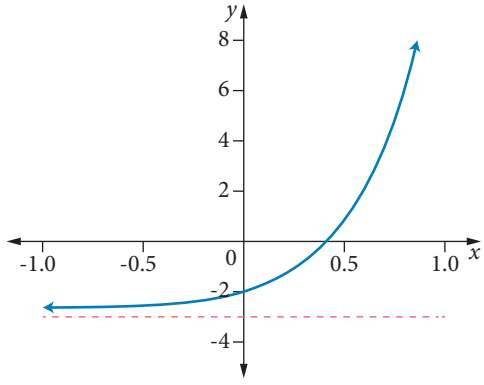
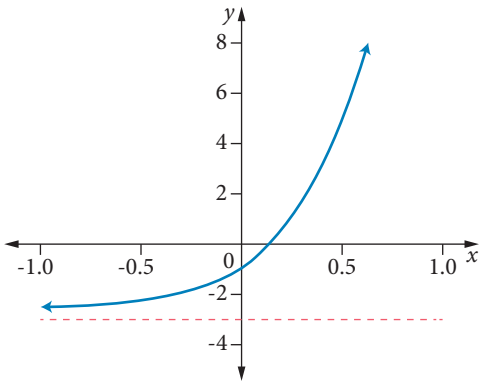


(b) $f(x) = 2 \times e^{3x} - 3$

A



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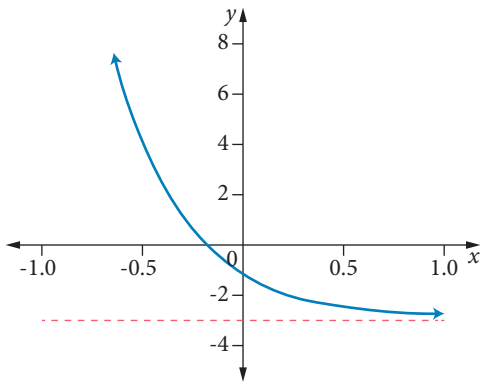

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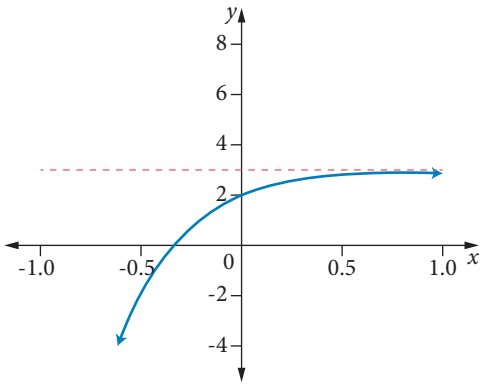

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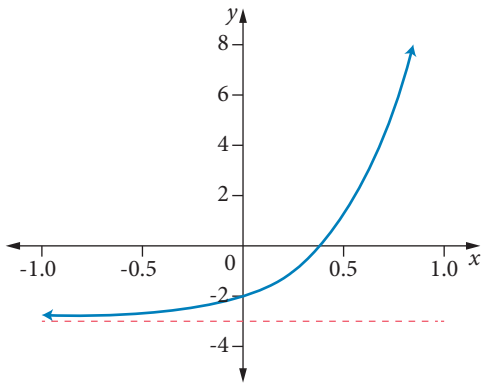


Exercise 1.3

10 Estimate $\lim_{h \to 0} \frac{a^h - 1}{h}$ for the given values of a, correct to 2 decimal places. Interpret your answers.

(a) $a = 10$

(b) $a = 0.1$

Exercise 1.3

11 Differentiate the following functions.

(a) $f(x) = 2e^x$

(b) $f(x) = e^{-2x}$

(c) $f(x) = e^{3x-1}$

(d) $f(x) = 3e^{\frac{x}{2}}$

Exercise 1.3

12 Differentiate the following functions.

(a) $y = x^2 + \frac{2}{x} + e^x$

(b) $y = 2e^{x^2}$

(c) $y = e^{x^3}$

(d) $y = e^{x^2+3x}$

Exercise 1.3

13 Choose the expression that gives the derivative of each of the given expressions.

(a) e^{2x+3}

A $(2x+3)e^{2x+3}$

B $2e^{2x}$

C $2e^{2x+3}$

D $2xe^{2x+3}$

(b) $5e^{x^2-3x}$

A $(2x-3)e^{x^2-3x}$

B $(2x-3)e^{2x-3}$

C $(10x-3)e^{x^2-3x}$

D $5(2x-3)e^{x^2-3x}$

(c) $\frac{e^{2x} - 5e^x + 3}{e^x}$

A $e^x + \frac{3}{e^x}$

B $e^x - 5 - \frac{3}{e^x}$

C $\frac{e^{2x} - 3}{e^x}$

D $e^x - 5 + \frac{3}{e^x}$

Exercise 1.3

14 Given that f is a differentiable function, determine the derivative of $f\left(e^{2x}\right)$.

Exercise 1.4

15 The cooling of a hot drink can be modelled as $T(t) = Ae^{kt} + c$, where T is the temperature of the drink in c, degrees Celsius, t the time in minutes since the drink was poured, and k influences the rate of cooling. A cup of coffee is made and delivered to a client at a temperature of 83°C. Determine the values of A, c and k, given the following information.

(a) The client stays in the cafeteria where the current temperature is 20°C, and it takes 20 minutes for the coffee to cool down to 50°C.

(b) The client moves out to the open grandstand where the air temperature is 13°C, and it takes 15 minutes for the coffee to cool down to 50°C.

16 The number of bacteria can be modelled by $N(t) = N_0 e^{kt}$, where t is the time in minutes. The initial number of bacteria is 4000, and after 1 hour there are 15 000. Determine the values of N_0 and k. Exercise 1.4

17 The decay of a medical radioactive isotope can be modelled by $A(t) = A_0 e^{-kt}$, where t is the number of days since it was introduced into the tumour and A is the amount that is remaining. The half-life of iodine-131 is 8 days. What percentage of iodine-131 remains after 32 days? Exercise 1.4

18 For the exponential equation $N(t) = N_0 e^{kx}$, $N(0) = 50$ and $N(1) = 29.43$. Determine the value of k, correct to 3 significant figures. Exercise 1.4

19 The number of virus cells that are viable within a host t hours after the initial infection can be described by $V(t) = Ae^{kt}$. After 10 virus cells were introduced to a contained laboratory environment, the following data was collected. Determine the value for k correct to 2 significant figures. Exercise 1.4

Hours (t)	0	1	2
Number of virus cells (V)	10	33	110

20 The annual income for a business that has been growing each year can be modelled as $P(t) = 250\,000e^{0.3t}$, where t is the number of years since 2000. When is the first time the annual income is at least double the income from 2000? Exercise 1.4

21 The number of bacteria on an agar plate can be modelled by $N(t) = 2500e^{0.03t}$, where t is the time in minutes after the experiment starts. Exercise 1.4

(a) How many bacteria were introduced to the agar plate?

(b) How many bacteria were there after 2 hours? Give your answer to the nearest thousand.

(c) Determine the equation that describes the rate of change in the bacterial population.

(d) How long (to the nearest minute) does it take for the number of bacteria to double?

(e) What is the rate of bacterial growth when $t = 30$?

(f) Determine the rate of population growth for the bacteria after 2 hours.

(g) What observation can be made about the rate of bacterial population growth?

Exercise 1.5 **22** Differentiate each of the following functions.

(a) $y = e^{(x-3)^2}$ **(b)** $y = \left(e^{2x} - x^2\right)^3$ **(c)** $y = \sqrt{e^x}$ **(d)** $y = e^{\frac{1}{x}}$

Exercise 1.5 **23** Determine the derivatives of each of the following functions.

(a) $f(x) = xe^x$ **(b)** $f(x) = \dfrac{x^2 - 1}{e^x}$

(c) $f(x) = \dfrac{e^x}{2x + 5}$ **(d)** $f(x) = (x+3)^2 e^{2x-1}$

Exercise 1.5 **24** What is the derivative of $2xe^{3x}$?

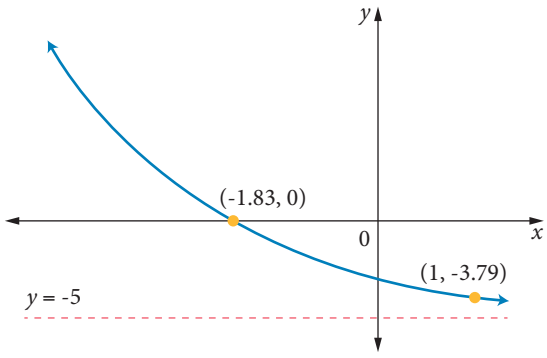
Exercise 1.5 **25** Determine the derivative of $e^{2x^3 - 7x}$.

Exercise 1.2 **26** For each of the following functions, sketch the graph, clearly showing the asymptote and axis intercepts. If there is no x-intercept, display an additional point. Determine the domain and range.

(a) $y = 2e^x$ **(b)** $y = e^{3x} + 1$

(c) $y = 5 - e^{x-1}$ **(d)** $y = e^{-x} - 2$

Exercise 1.2 **27** The rule for the following graph can be expressed as $y = Ae^{bx} + c$. Determine the equation for the graph, giving the values for A and b, correct to 3 significant figures.



Exercise 1.3 **28** Determine the equation for the tangent to $f(x) = 2e^{1-x}$, where $x = 1$.

Exercise 1.4 **29** A mass of radioactive material is decaying in such a way that, at time t years, its mass in grams is given by the formula $m = 360e^{-0.02t}$.

(a) Calculate the time taken, to the nearest year, for the radioactive material to become one-eighth of its original mass.

(b) Calculate the value of t such that the mass of the radioactive material will be decreasing at a rate of 5 g per year. Give your answer correct to 1 decimal place.

30 If the population of pygmy three-toed sloths is under pressure due to loss of habitat, then its population can often be modelled by $N(t) = Ae^{-kt}$, where t is the number of years since clearing began and $N(t)$ is the population. Agricultural colonisation began in a region of rainforest in the year 2000. It was thought that there were 5000 pygmy three-toed sloths in the area at this time, and by 2010, scientists believed the population was 1000. Exercise 1.4

(a) Determine the equation in terms of t to calculate the population, giving the value for k, correct to 3 significant figures.

(b) In what year does this model predict that the population of pygmy three-toed sloths will fall to 100?

31 Determine the derivative of $\frac{e^{2x} + e^x}{x+1}$. Exercise 1.5

32 Describe the stationary point for $y = xe^x$. Exercise 1.5

A There is no stationary point.

B a minimum at (-1, -0.37)

C a maximum at (-1, -0.37)

D a positive point of inflection at (-1, -0.37)

33 For the function $h(x) = e^{2x-x^2}$, determine the coordinates of the stationary point and its nature. Exercise 1.5

34 Consider the function $h(x) = e^{-x^2+2}$. Sarah argues that the average rate of change of $h(x)$ between $x = 1$ and $x = 2$ is greater than the instantaneous rate of change at $x = 1$. Do you agree with Sarah? Support your decision with all relevant calculations. Exercise 1.5

35 The concentration of a medication in the blood (mg/L) after a tablet has been taken can be modelled by $C(t) = 0.2te^{-0.03t}$, where t is the time in minutes since the tablet was taken. Exercise 1.5

(a) Sketch the graph of concentration versus time.

(b) Determine the maximum blood concentration, correct to 1 decimal place, and the time it occurs, correct to the nearest minute. Include relevant units in your answers.

(c) Repeat dosages cannot be taken until there is a relatively small blood concentration. If the blood concentration should be less than 0.2 mg/L but not less than 0.1 mg/L, what recommended frequency of medication is likely to appear on a prescription?

36 The concentration (mg/L) of drug in a patient's bloodstream t hours after an injection is administered is given by $A(t) = 80te^{-1.5t}$. Exercise 1.5

(a) Use calculus to determine when the concentration of the drug will be at its maximum. Give your answer in minutes.

(b) From a graph, estimate when the concentration is falling most rapidly.

37 Let $f(x) = e^x$ for $-1 \leq x \leq 2$, $g(x) = \sqrt{x}$ for $0 \leq x \leq 5$ and $h(x) = \frac{f(x)}{g(x)}$. Exercise 1.5

(a) Determine an expression for $h(x)$ and its maximal domain.

(b) Determine any local minima and local maxima of the function $h(x)$.

(c) On the same set of axes, sketch the graphs of $h(x)$ and $h'(x)$, showing any local maximum or minimum of $h(x)$ over the domain.

2

Derivatives of trigonometric functions

Recall 66

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2.5 Applications of trigonometric functions 116

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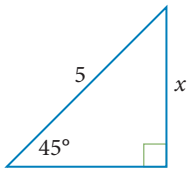
Chapter review 129

2 Recall

Use the trigonometric ratios in right-angled triangles to determine the length of an unknown side

1 Determine the length of each unknown side, correct to 1 decimal place.

(a)



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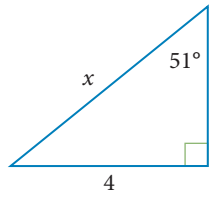

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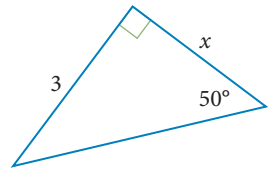


Determine the size of unknown angles using trigonometric ratios

2 Assuming that θ is an acute angle, solve each of the following equations, giving your answers in degrees, correct to 2 decimal places.

(a) $\cos(\theta) = \frac{3}{4}$ **(b)** $\tan(\theta) = \frac{8}{5}$ **(c)** $\sin(\theta) = \frac{9}{11}$

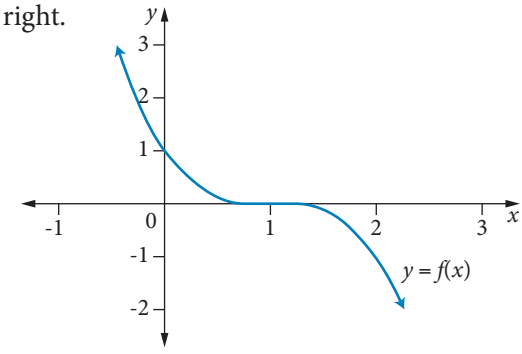
Determine the transformations of basic graphs, including dilations, reflections and translations

3 The graph of the function $y = f(x)$ is shown on the right. For each of the following transformed functions, sketch the graph, clearly showing the point of inflection.

(a) $y = 2f(x)$

(b) $y = -2f(x)$

(c) $y = -2f(x-1)$



Sketch graphs of simple trigonometric functions

4 For each of the following, sketch the graph for $0 \leq x \leq 2\pi$ and determine the period of the function.

(a) $y = \sin(x)$ **(b)** $y = \cos(x)$ **(c)** $y = \tan(x)$

Convert angles in degrees to radians, in terms of π

5 Convert each of the following angles to radians, in term of π.

(a) 75° **(b)** 50° **(c)** 200°

Convert angles in degrees to radians, in decimal form

6 Convert each of the following angles to radians, correct to 2 decimal places.

(a) 29° **(b)** 132° **(c)** 311°

Convert radians to degrees

7 Convert each of the following angles to degrees.

(a) $\frac{3\pi}{5}$ **(b)** $\frac{13\pi}{12}$ **(c)** $\frac{8\pi}{3}$

2.1 Radians and the unit circle

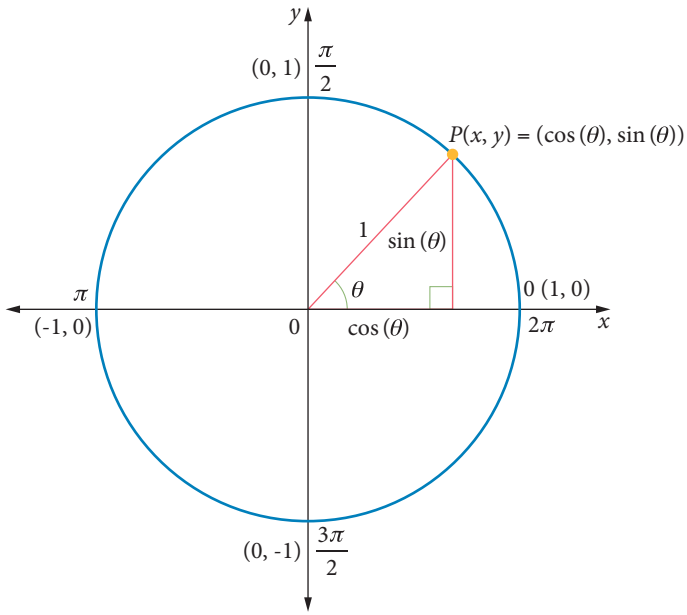
The sine and cosine functions

Trigonometric functions such as $\sin(\theta)$ and $\cos(\theta)$ are useful for modelling periodic motion, because they have values that increase and decrease by turns at regular intervals.

Common examples include the up-and-down motion of riding a Ferris wheel, the daily rising and setting of the sun and the daily alternations of high tide and low tide.

Periodic functions are based on the unit circle definitions of the trigonometric functions.

The unit circle has its centre at the origin $(0,0)$ and has a radius of 1 unit. It is described by the relation $x^2 + y^2 = 1$, where $-1 \leq x \leq 1$.



For any point $P(x, y)$ on the unit circle:

- The x-coordinate (horizontal distance) is defined as $\cos(\theta)$.
- The y-coordinate (vertical distance) is defined as $\sin(\theta)$.

Making connections

The unit circle

Explore the relationship between the unit circle and the sine and cosine functions.

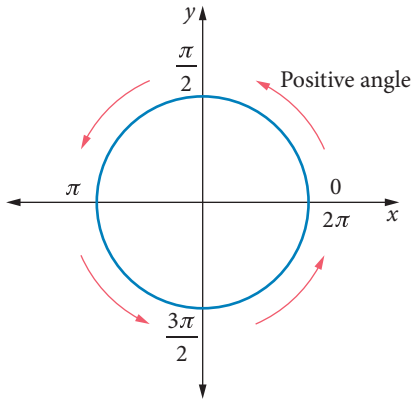
The Pythagorean identity

The relation $x^2 + y^2 = 1$ can be expressed as $\cos^2(\theta) + \sin^2(\theta) = 1$. This is called the Pythagorean identity for trigonometric functions.

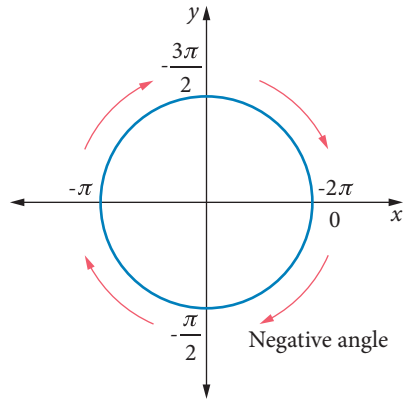
Positive and negative angles

Negative angles are used as a matter of convenience. For example, a spaceship in a circular orbit around Earth's equator could be travelling in either of two directions. You could say that it travelled through an angle of $-60° = -\frac{\pi}{3}$ to indicate that it went in the opposite direction from a spaceship that went through a $60° = \frac{\pi}{3}$ angle.

Angles measured anticlockwise starting from zero are positive

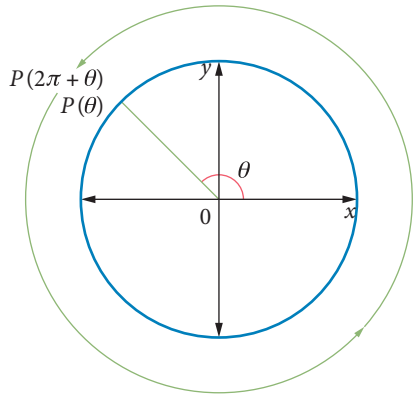


Angles measured clockwise starting from zero are negative



Periodicity

The circumference of the unit circle is 2π. Therefore, if a point P travels around the unit circle for a distance of 2π, it ends up where it started, and the points are said to be coterminal. That is, for any given point P, if 2π is added or subtracted, the coordinates of P remain unchanged. Even if you added 4π, you would still end up at point P. If any multiple of 2π is added or subtracted from the angle θ, it will always end up at the same point on the unit circle. Since the point P represents $(\cos(\theta), \sin(\theta))$, the value of cosine and sine also remain unchanged. The values of sine and cosine repeat themselves at intervals of 2π. That is, $\sin(\theta)$ and $\cos(\theta)$ are periodic functions with a period of 2π.



More generally, for $k \in \mathbb{Z}$:

$\cos(\theta + 2k\pi) = \cos(\theta)$

$\sin(\theta + 2k\pi) = \sin(\theta)$

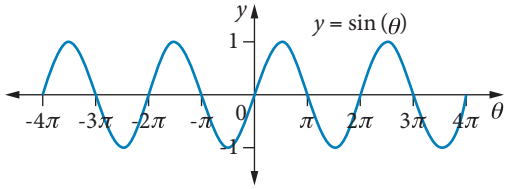
For example:

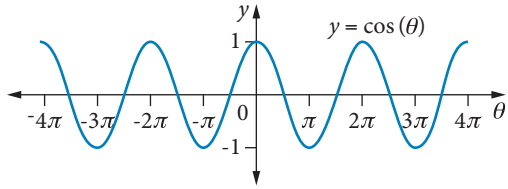
$$\sin\left(\frac{5\pi}{2}\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)$$

$$\cos(5\pi) = \cos(4\pi + \pi) = \cos(\pi)$$

$$\sin\left(-\frac{5\pi}{2}\right) = \sin\left(-2\pi - \frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right)$$

The periodic nature of sine and cosine is evident from the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$ shown below.





Additional information

The sine function

Watch the videos to view the construction of the sine wave.

Additional information

The cosine function

Watch the videos to view the construction of the graph of the cosine function.

The tangent function

The diagram on the next page shows the unit circle with a tangent drawn at point $A(1,0)$. The tangent is defined as the length of the line segment between the x-axis and the point of contact between the tangent line and the extension of line OP. The line OP is extended to cut the tangent at Q. By definition:

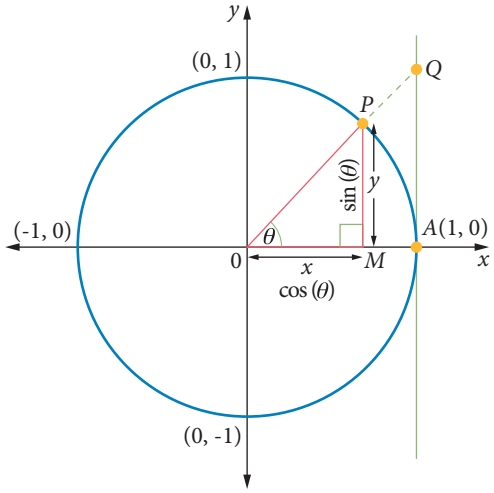
$$\tan(\theta) = \frac{AQ}{OA} = \frac{AQ}{1} = AQ = y\text{-coordinate of } Q$$

Making connections

The tangent function

Explore the relationship between the tangent function and the unit circle.

When $\theta = \frac{\pi}{2}$, then OP is parallel to and does not meet the tangent at A, hence AQ does not exist and the length of the line segment or $|AQ|$ is undefined.

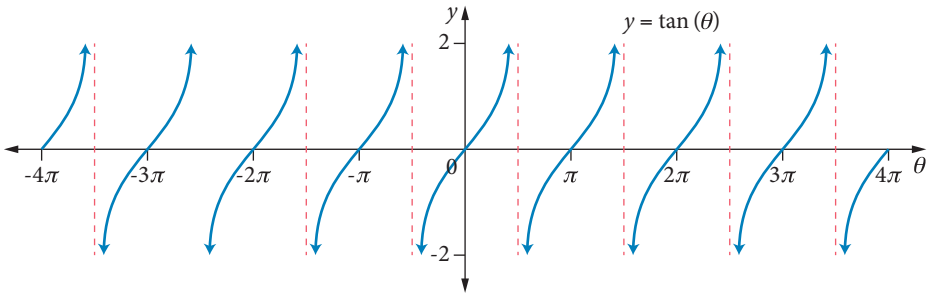


From the diagram on the right, note:

$$\tan(\theta) = \frac{MP}{OM}$$
$$= \frac{y}{x}$$
$$= \frac{\sin(\theta)}{\cos(\theta)}$$

This is true no matter which quadrant P is in, provided $\cos(\theta) \neq 0$. That is, $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$...

On a graph of the tangent function, these undefined values are represented by asymptotes. The values of the tangent function repeat themselves at intervals of π. That is, $\tan(\theta)$ is a periodic function with a period of π. The periodic nature of tangent is evident from the graph of $y = \tan(\theta)$ below.



More generally, for $k \in \mathbb{Z}$, $\tan(\theta + k\pi) = \tan(\theta)$.

Given x and y are coordinates in a unit circle:

- $\cos(\theta) = x$
- $\sin(\theta) = y$
- $\cos^2(\theta) + \sin^2(\theta) = 1$
- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

Additional information

The tangent function

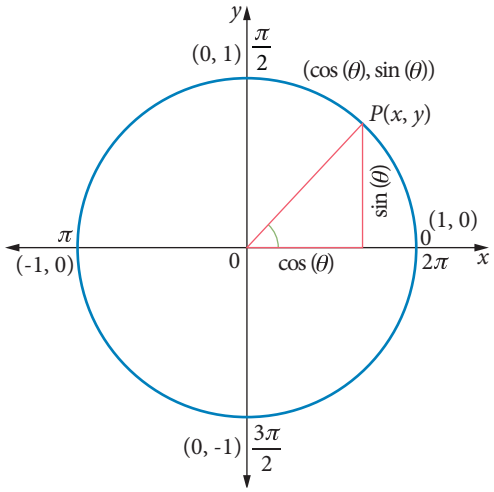
Construct the graph of the tangent function.

Exact values for 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π

The unit circle can be used to determine the exact values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ for $0° = 0$, $90° = \frac{\pi}{2}$, $180° = \pi$, $270° = \frac{3\pi}{2}$, $360° = 2\pi$.

For example, for the point on the unit circle's circumference where the angle of rotation is $\frac{3\pi}{2}$:

$$\sin\left(\frac{3\pi}{2}\right) = y\text{-coordinate}$$
$$= -1$$



For the point on the unit circle's circumference where the angle of rotation is $\frac{\pi}{2}$:

$$\tan\left(\frac{\pi}{2}\right) = \frac{y\text{-coordinate}}{x\text{-coordinate}}$$
$$= \frac{1}{0}$$

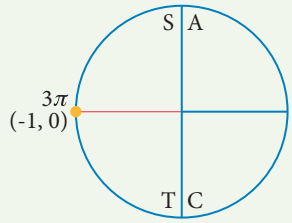
$\therefore \tan\left(\frac{\pi}{2}\right)$ is undefined.

$\sin(0) = 0$	$\sin\left(\frac{\pi}{2}\right) = 1$	$\sin(\pi) = 0$	$\sin\left(\frac{3\pi}{2}\right) = -1$	$\sin(2\pi) = 0$
$\cos(0) = 1$	$\cos\left(\frac{\pi}{2}\right) = 0$	$\cos(\pi) = -1$	$\cos\left(\frac{3\pi}{2}\right) = 0$	$\cos(2\pi) = 1$
$\tan(0) = 0$	$\tan\left(\frac{\pi}{2}\right)$ is undefined	$\tan(\pi) = 0$	$\tan\left(\frac{3\pi}{2}\right)$ is undefined	$\tan(2\pi) = 0$

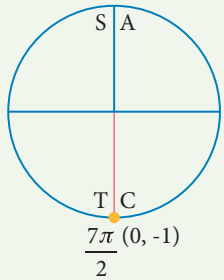
1 Determine exact values of trigonometric expressions

Determine the exact values of the following measures.

(a) $\cos(3\pi)$

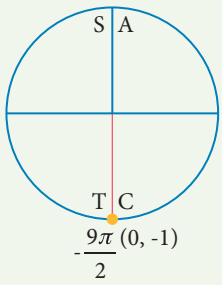
THINKING	WORKING
1 Draw a unit circle and determine the location of the angle.	
2 Recall that for the position on the unit circle (x, y), the value of the cosine is determined by the x-coordinate.	$\cos(3\pi) = -1$

(b) $\sin\left(\frac{7\pi}{2}\right)$

1 Draw a unit circle and determine the location of the angle.	
2 Recall that for the position on the unit circle (x, y), the value of the sine is determined by the y-coordinate.	$\sin\left(\frac{7\pi}{2}\right) = -1$

(c) $\tan\left(-\frac{9\pi}{2}\right)$

1 Draw a unit circle and determine the location of the angle.

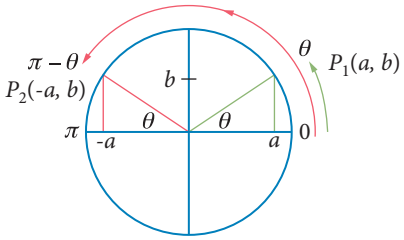


2 Recall that for the position on the unit circle (x, y), the value of tangent is determined by the ratio: $\tan(\theta) = \frac{y\text{-coordinate}}{x\text{-coordinate}}$

$\tan\left(-\frac{9\pi}{2}\right) = \frac{-1}{0}$

$\therefore \tan\left(-\frac{9\pi}{2}\right)$ is undefined.

The unit circle and symmetry properties



Consider a point, $P_1(a,b)$ in the first quadrant and a point, $P_2(-a,b)$ in the second quadrant as shown in the diagram.

In the first quadrant at point P_1:

$\cos(\theta) = x\text{-coordinate}$

$= a$

In the second quadrant at point P_2:

$\cos(\pi - \theta) = x\text{-coordinate}$

$= -a$

$= -\cos(\theta)$

Similar symmetry properties can be determined for sine, cosine and tangent in each of the four quadrants. The diagram summarises the symmetry properties that exist for each trigonometric function in the four quadrants for angles of θ, $\pi - \theta$, $\pi + \theta$, $2\pi - \theta$. The unit circle is symmetrical, hence the magnitudes of sine, cosine and tangent at the angles shown in each quadrant are the same but the sign varies. The four triangles shown are congruent. The first quadrant angle θ is often referred to as the reference angle.

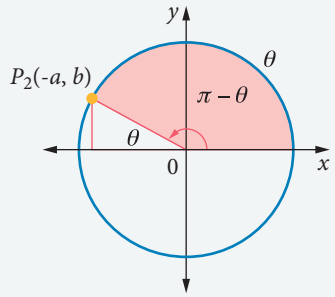
These symmetry properties continue beyond angles in the range $0 \le \theta \le 2\pi$. For example, $\cos(2\pi + \theta) = \cos(\theta)$, since adding 2π, results in collinear angles; that is, angles aligned to the same point on the unit circle.

Quadrant 2

$\cos(\pi - \theta) = -\cos(\theta)$

$\sin(\pi - \theta) = \sin(\theta)$

$\tan(\pi - \theta) = -\tan(\theta)$

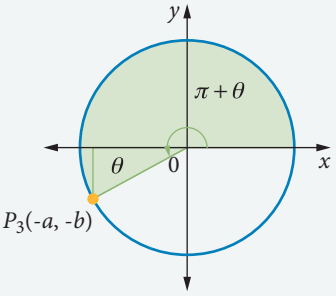


Quadrant 3

$\cos(\pi + \theta) = -\cos(\theta)$

$\sin(\pi + \theta) = -\sin(\theta)$

$\tan(\pi + \theta) = \tan(\theta)$

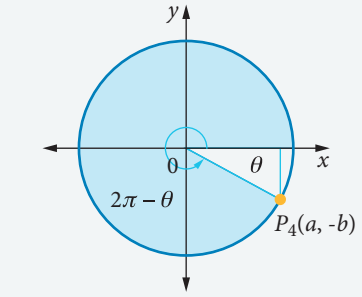


Quadrant 4

$\cos(2\pi - \theta) = \cos(\theta)$

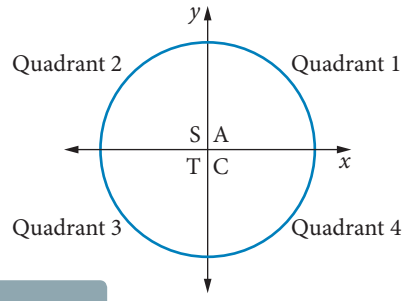
$\sin(2\pi - \theta) = -\sin(\theta)$

$\tan(2\pi - \theta) = -\tan(\theta)$



Signs of trigonometric functions

The symmetry properties for sine, cosine and tangent can be summarised for each of the four quadrants as follows.



- All values are positive in the first quadrant (A).
- Only the sine values are positive in the second quadrant (S).
- Only the tangent values are positive in the third quadrant (T).
- Only the cosine values are positive in the fourth quadrant (C).

Additional information

Symmetry properties of the sine curve

This video explores some symmetry properties of the unit circle involving the sine function.

Additional information

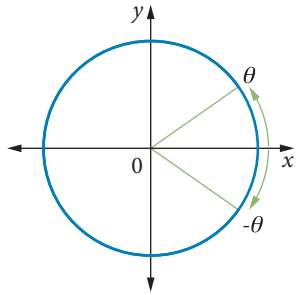
Symmetry properties of the cosine curve

This video explores some symmetry properties of the unit circle involving the cosine function.

The acronym ASTC, Always Stop To Check, can be used to remember in which quadrant sine, cosine and tangent are positive or negative.

Negative angles

The diagram shows $0 \leq \theta \leq \frac{\pi}{2}$, but the relationships are true for all values of θ.



$\cos(-\theta) = \cos(\theta)$

$\sin(-\theta) = -\sin(\theta)$

$\tan(-\theta) = -\tan(\theta)$

Complementary relationships

In the diagram, consider a point $P_1(a, b)$ in the first quadrant and a point $P_2(-b, a)$ in the second quadrant as shown in the diagram. The turning angle between P_1 and P_2 is 90°.

In the first quadrant at the point P_1:

$\cos(\theta) = x\text{-coordinate}$

$= a$

$\sin(\theta) = y\text{-coordinate}$

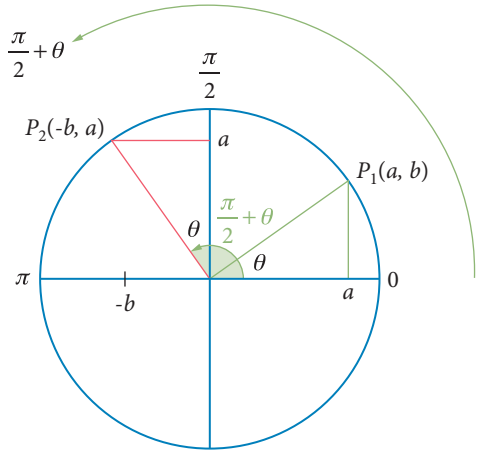
$= b$

In the second quadrant at the point P_2:

$\cos\left(\frac{\pi}{2} + \theta\right) = x\text{-coordinate}$

$= -b$

$= -\sin(\theta)$

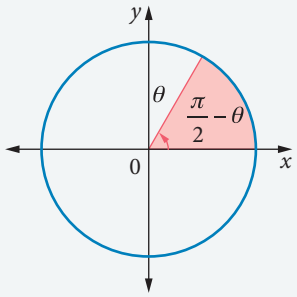


Similar complementary relationships can be determined for sine and cosine in each of the four quadrants. The complementary relationships for sine and cosine are summarised in the table below.

Quadrant 1

$\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$

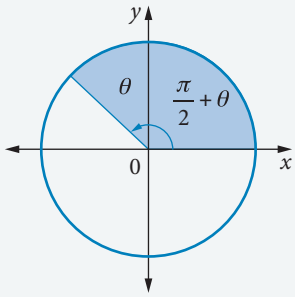
$\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$



Quadrant 2

$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin(\theta)$

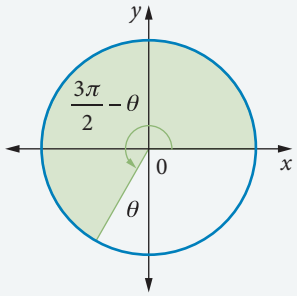
$\sin\left(\frac{\pi}{2}+\theta\right)=\cos(\theta)$



Quadrant 3

$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin(\theta)$

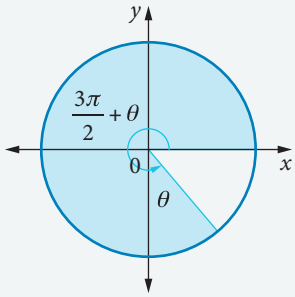
$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos(\theta)$



Quadrant 4

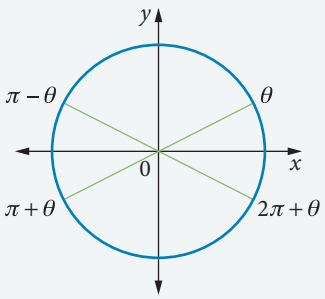
$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin(\theta)$

$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos(\theta)$



It is important to note that with complementary relationships, sine changes to cosine and cosine changes to sine.

Symmetry properties



$\cos(\pi-\theta)=-\cos(\theta)$

$\sin(\pi-\theta)=\sin(\theta)$

$\tan(\pi-\theta)=-\tan(\theta)$

$\cos(\pi+\theta)=-\cos(\theta)$

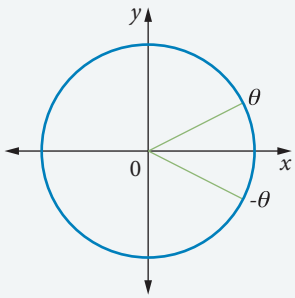
$\sin(\pi+\theta)=-\sin(\theta)$

$\tan(\pi+\theta)=\tan(\theta)$

$\cos(2\pi-\theta)=\cos(\theta)$

$\sin(2\pi-\theta)=-\sin(\theta)$

$\tan(2\pi-\theta)=-\tan(\theta)$

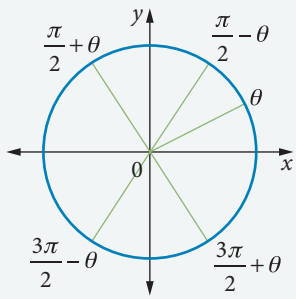


$\cos(-\theta)=\cos(\theta)$

$\sin(-\theta)=-\sin(\theta)$

$\tan(-\theta)=-\tan(\theta)$

Complementary properties



$\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$

$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin(\theta)$

$\sin\left(\frac{\pi}{2}+\theta\right)=\cos(\theta)$

$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin(\theta)$

$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos(\theta)$

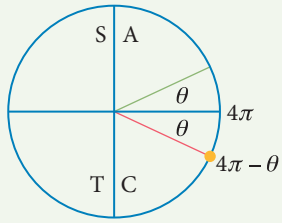
$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin(\theta)$

$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos(\theta)$

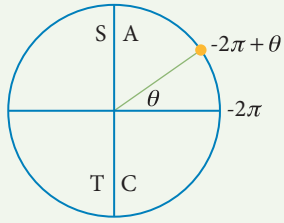
2 Simplify expressions using symmetry and complementary properties of the unit circle

Given $0 \leq \theta \leq \frac{\pi}{2}$, express the following in simplest form.

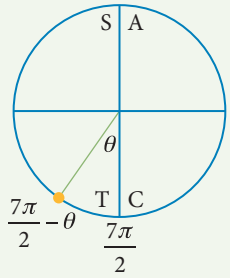
(a) $\sin(4\pi - \theta)$

THINKING	WORKING
1 Draw a diagram to determine the position of the angle and whether the sine ratio is positive or negative in the given quadrant. The reference angle is θ.	
	$4\pi - \theta$ is in the fourth quadrant; sine is negative.
2 Recall the symmetry properties of the unit circle to write the answer.	$\sin(4\pi - \theta) = -\sin(\theta)$

(b) $\cos(\theta - 2\pi)$

1 Rewrite the expression to make it easier to determine the quadrant of the angle.	$\cos(\theta - 2\pi) = \cos(-2\pi + \theta)$
2 Draw a diagram to determine the position of the angle and whether the cosine ratio is positive or negative in the given quadrant. The reference angle is θ.	
	$-2\pi + \theta$ is in the first quadrant.
3 Recall the symmetry properties of the unit circle to write the answer.	$\cos(\theta - 2\pi) = \cos(\theta)$

(c) $\sin\left(\frac{7\pi}{2} - \theta\right)$

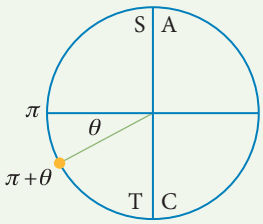
1 Draw a diagram to determine the position of the angle and whether the sine ratio is positive or negative in the given quadrant. The reference angle is θ.	
	$\frac{7\pi}{2} - \theta$ is in the third quadrant.

2 Recall the complementary relationships of the unit circle to write the answer.	$\sin\left(\frac{7\pi}{2}-\theta\right)=-\cos(\theta)$

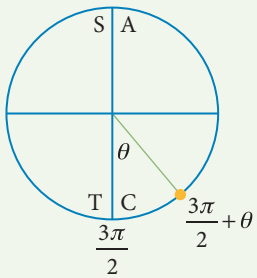
3 Determine values of trigonometric functions using symmetry properties

Given $\cos(\theta)=0.5$ and $\sin(\theta)=0.866$, determine the following measures.

(a) $\sin(\pi+\theta)$

THINKING	WORKING
1 Draw a diagram to determine the position of the angle and whether the sine ratio is positive or negative in the given quadrant.	 $\pi+\theta$ is in the third quadrant; sine is negative.
2 Recall the symmetry properties of the unit circle and the information given in the problem to write the answer.	$\sin(\pi+\theta)=-\sin(\theta)$ $=-0.866$

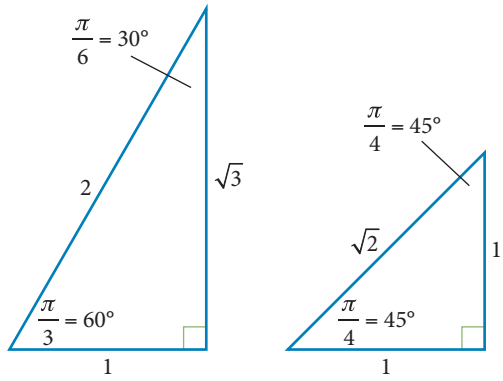
(b) $\cos\left(\frac{3\pi}{2}+\theta\right)$

1 Draw a diagram to determine the position of the angle and whether the cosine ratio is positive or negative in the given quadrant.	 $\frac{3\pi}{2}+\theta$ is in the fourth quadrant; cosine is positive.
2 Recall the complementary properties of the unit circle and the information given in the problem to write the answer.	$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin(\theta)$ $=0.866$

Exact values for first quadrant angles

The exact values of sine, cosine and tangent can be determined for the angles: $30° = \frac{\pi}{6}$, $60° = \frac{\pi}{3}$ and $45° = \frac{\pi}{4}$.

Consider these special triangles, which display trigonometric ratios of some common angles using exact values. The first pair of triangles is formed by the vertical bisection of an equilateral triangle of side length 2. The second pair of triangles is formed by the diagonal bisection of a unit square.

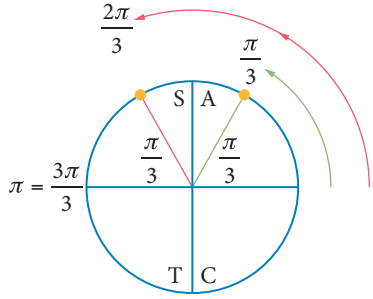


Exact values for some common angles:

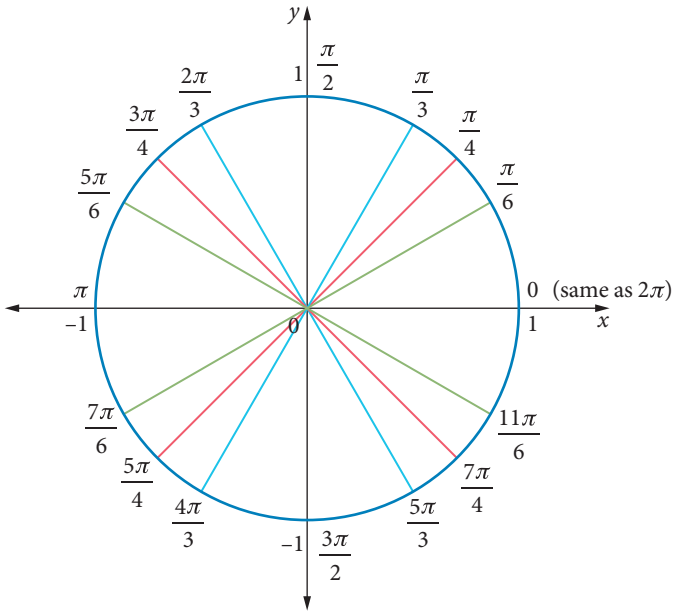
$\sin(0) = 0$	$\cos(0) = 1$	$\tan(0) = 0$
$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\tan\left(\frac{\pi}{4}\right) = 1$	$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$
$\sin\left(\frac{\pi}{2}\right) = 1$	$\cos\left(\frac{\pi}{2}\right) = 0$	$\tan\left(\frac{\pi}{2}\right)$ is undefined

Exact values for angles in any quadrant

The exact values of sine, cosine and tangent for angles that are integer multiples of $30° = \frac{\pi}{6}$, $60° = \frac{\pi}{3}$ and $45° = \frac{\pi}{4}$ can be readily determined using the symmetry properties of the unit circle. There is an equivalent angle, or reference angle, in the first quadrant for any other angle over the real numbers. For example, from the diagram on the right, the reference angle for $\frac{2\pi}{3}$ is $\frac{\pi}{3}$.



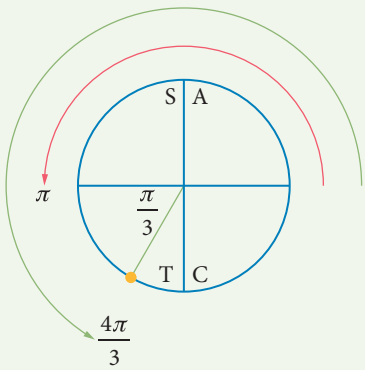
The diagram below shows the integer multiples of $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{4}$ from 0 to 2π.



4 Determine the exact values of trigonometric functions using symmetry properties

Determine the exact value of each of the following measures.

(a) $\cos\left(\frac{4\pi}{3}\right)$

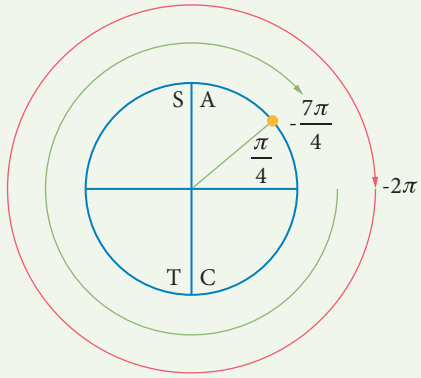
THINKING	WORKING
1 Draw a diagram to determine the position of the angle.	
2 Recall the symmetry property to determine the equivalent first quadrant angle (reference angle).	$\cos\left(\frac{4\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right)$ $= -\cos\left(\frac{\pi}{3}\right)$
3 Determine the exact value.	$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

(b) $\tan\left(-\frac{7\pi}{4}\right)$

1 Recall the symmetry property for the appropriate function and a negative angle.

$$\tan\left(-\frac{7\pi}{4}\right) = -\tan\left(\frac{7\pi}{4}\right)$$

2 Draw a diagram to determine the quadrant of the angle.



3 Use the symmetry property to determine the reference angle.

$$\tan\left(\frac{-7\pi}{4}\right) = \tan\left(-2\pi + \frac{\pi}{4}\right)$$
$$= \tan\left(\frac{\pi}{4}\right)$$

4 Determine the exact value.

$$\tan\left(-\frac{7\pi}{4}\right) = 1$$

Circles of radius *r*

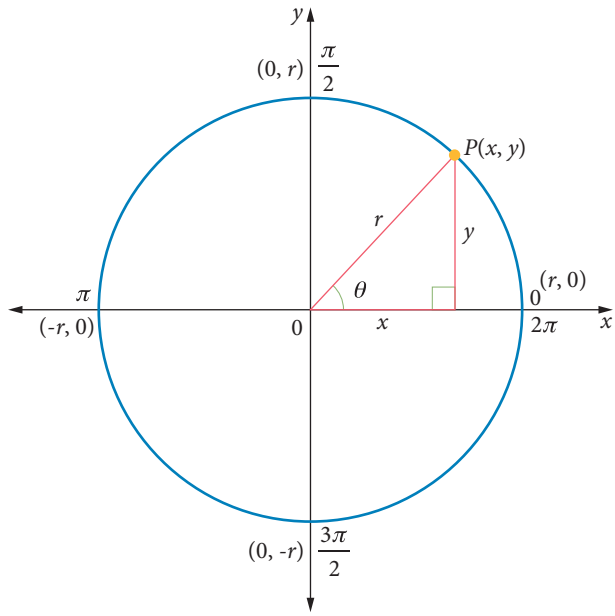
When one of the trigonometric ratios is given as a fraction, it is easier to draw a diagram using the numerator and denominator of the fraction as sides of the triangle, and place the triangle in a quadrant, according to its sign. Any circle with centre at the origin $(0,0)$ and radius of r units can be described by the relation $x^2 + y^2 = r^2$, where $-r \leq x \leq r$.

For any point $P(x, y)$ on the unit circle:

$\cos(\theta) = \frac{x}{r}$, $\sin(\theta) = \frac{y}{r}$, $\tan(\theta) = \frac{y}{x}$ (unchanged).

By Pythagoras' theorem, $x^2 + y^2 = r^2$, therefore:

$$\cos^2(\theta) + \sin^2(\theta) = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$$
$$= \frac{x^2 + y^2}{r^2}$$
$$= 1$$



5 Use a diagram or the Pythagorean identity to determine the value of a trigonometric function

Given $\sin(\theta) = -\frac{3}{5}$, $\pi \leq \theta \leq \frac{3\pi}{2}$, determine the value of each measure.

(a) $\cos(\theta)$

METHOD 1: USE THE PYTHAGOREAN IDENTITY

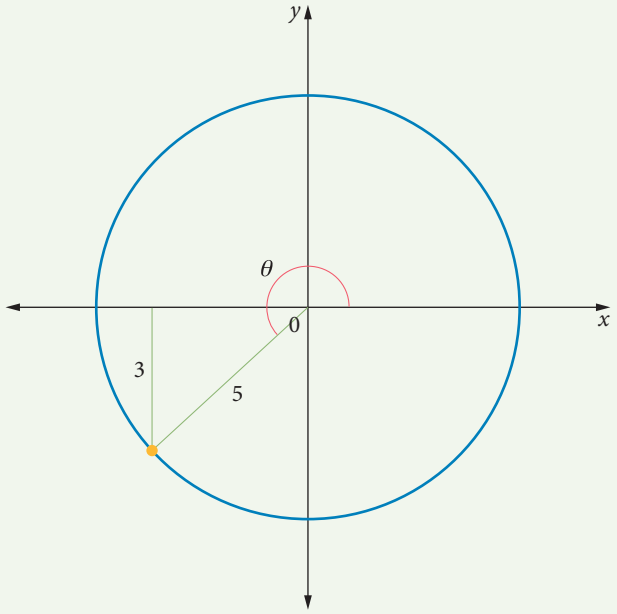
THINKING	WORKING
1 Recall the Pythagorean identity and substitute the given value.	$\cos^2(\theta) + \sin^2(\theta) = 1$ $\cos^2(\theta) + \left(-\frac{3}{5}\right)^2 = 1$
2 Solve for $\cos(\theta)$.	$\cos^2(\theta) + \frac{9}{25} = 1$ $\cos^2(\theta) = 1 - \frac{9}{25}$ $\cos^2(\theta) = \frac{16}{25}$ $\cos(\theta) = \pm\frac{4}{5}$
3 Check the original restrictions to determine whether the solution is positive or negative.	Given that $\pi \leq \theta \leq \frac{3\pi}{2}$, the angle is in the third quadrant, where the cosine ratio is negative. $\therefore \cos(\theta) = -\frac{4}{5}$.

METHOD 2: USE A DIAGRAM

1 Draw a diagram, including the given information in the appropriate quadrant.

$\pi \leq \theta \leq \frac{3\pi}{2}$, $\sin(\theta) = -\frac{3}{5}$

$\sin(\theta) = \frac{y}{r}$: let $y = -3$, $r = 5$



2 Recall Pythagoras' theorem and substitute values to determine the unknown side length of the right-angled triangle, and hence the unknown values of x, y or r.

$$\begin{aligned} 3^2 + a^2 &= 5^2 \\ a^2 &= 5^2 - 3^2 \\ &= 25 - 9 \\ a &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

The horizontal side length of the triangle is 4 units.

Hence, $x = -4$.

3 Use the trigonometric ratios to write the solution. Check whether the solution is positive or negative.

Given that $\pi \leq \theta \leq \frac{3\pi}{2}$, the angle is in the third quadrant, where the cosine ratio is negative.

$$\begin{aligned} \cos(\theta) &= \frac{x}{r} \\ &= -\frac{4}{5} \end{aligned}$$

(b) $\tan(\theta)$

1 Use the diagram to determine $\tan(\theta) = \frac{y}{x}$.

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \frac{-3}{-4} \\ &= \frac{3}{4} \end{aligned}$$

2 Alternatively, use the tangent identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

$$\begin{aligned} \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{-\frac{3}{5}}{-\frac{4}{5}} \\ &= \frac{3}{4} \end{aligned}$$

EXERCISE 2.1 Radians and the unit circle

Worked Example

1

1 Determine the exact values of the following measures.

(a) $\sin(4\pi)$ **(b)** $\cos\left(\frac{7\pi}{2}\right)$ **(c)** $\tan(-5\pi)$ **(d)** $\sin\left(-\frac{5\pi}{2}\right)$

2

2 Express each of the following measures in simplest form.

(a) $\cos(3\pi - \theta)$ **(b)** $\tan(\theta - 2\pi)$ **(c)** $\sin\left(\frac{5\pi}{2} - \theta\right)$ **(d)** $\cos\left(\theta - \frac{\pi}{2}\right)$

3

3 For $\cos(\theta) = 0.15$, $0 \leq \theta \leq \frac{\pi}{2}$ and $\tan(\alpha) = 0.05$, $0 \leq \alpha \leq \frac{\pi}{2}$, determine the following measures.

(a) $\cos(2\pi - \theta)$ **(b)** $\tan(\alpha - \pi)$ **(c)** $\sin\left(\frac{5\pi}{2} + \theta\right)$ **(d)** $\sin\left(\theta - \frac{3\pi}{2}\right)$

4 Simplify $\sin\left(\frac{\pi}{2}-x\right)+\sin(x-\pi)+\sin\left(x-\frac{3\pi}{2}\right)$.

Worked Example

5 Determine the exact value of each of the following measures.

4

(a) $\sin\left(\frac{5\pi}{6}\right)$ **(b)** $\cos\left(\frac{7\pi}{4}\right)$ **(c)** $\tan\left(-\frac{2\pi}{3}\right)$ **(d)** $\cos\left(-\frac{13\pi}{6}\right)$

(e) $\sin\left(\frac{11\pi}{6}\right)$ **(f)** $\tan\left(\frac{8\pi}{3}\right)$ **(g)** $\cos\left(-\frac{5\pi}{4}\right)$

6 Given the following information, calculate the value of x.

(a) $\frac{\pi}{2}\leq x\leq\pi$ and $\cos(x)=-\cos\left(\frac{\pi}{3}\right)$

(b) $\frac{\pi}{2}\leq x\leq\pi$ and $\cos(x)=-\cos\left(\frac{\pi}{4}\right)$

(c) $\pi\leq x\leq\frac{3\pi}{2}$ and $\sin(x)=-\sin\left(\frac{\pi}{3}\right)$

7 The height of the water in Panle Harbour changes with the tides. On a particular day the height of water in the harbour is given by $h=1.2+1.8\sin\left(\frac{\pi}{6}(t+4)\right)$, where h is the height of the tide in metres and t is the number of hours since midnight.

(a) Determine the height of the tide at 1 am.

(b) Determine the height of the tide at 3 pm.

8 The average number of people per day who visit the Panle Harbor shops is given by $N=3000+2000\sin\left(\frac{\pi}{2}(t+1)\right)$, where N is the number of people and t is the month. Given that January corresponds to $t=1$, calculate the average number of people visiting the Panle Harbour shops in:

(a) March

(b) December.

9 At Panle Harbour the average maximum temperature is given by $T=20-15\cos\left(\frac{\pi t}{6}\right)$, where T is the temperature in degrees Celsius and t is the time in months. Given that January corresponds to $t=1$, calculate the average maximum temperature in:

(a) June

(b) October.

10 Simplify each of the following expressions.

(a) $\tan\left(\frac{11\pi}{6}\right)-\sin\left(-\frac{11\pi}{4}\right)$ **(b)** $\cos\left(-\frac{\pi}{3}\right)\sin\left(\frac{5\pi}{4}\right)$

11 Given $\sin(x)=-\frac{4}{5}, x\in\left(\frac{3\pi}{2},2\pi\right)$, determine the value of each of the following measures.

5

(a) $\cos(x)$ **(b)** $\tan(x)$

12 Given $\sin(x) = -\frac{2}{3}$, where $\pi \leq x \leq \frac{3\pi}{2}$.

(a) What is the value of $\cos(x)$?

A $\frac{3}{\sqrt{5}}$ **B** $\frac{\sqrt{5}}{3}$ **C** $-\frac{2}{\sqrt{5}}$ **D** $-\frac{\sqrt{5}}{3}$

(b) Explain the common error made by a student who didn't consider the placement of the angle within the unit circle in part **(a)**.

13 Given $\cos(x) = -\frac{5}{12}$, $-\pi \leq x \leq -\frac{3\pi}{2}$, what is the value of $\sin(x)$?

A $\frac{\sqrt{119}}{12}$ **B** $-\frac{\sqrt{119}}{12}$ **C** $\frac{5}{\sqrt{61}}$ **D** $\frac{\sqrt{11}}{5}$

14 Given the following information, determine the value of x in terms of y.

(a) $\frac{\pi}{2} \leq x \leq \pi$, $0 \leq y \leq \frac{\pi}{2}$, and $\sin(x) = \cos(y)$ **(b)** $\pi \leq x \leq \frac{3\pi}{2}$, $0 \leq y \leq \frac{\pi}{2}$, and $\cos(x) = -\sin(y)$

15 Simplify each of the following expressions.

(a) $\cos(\pi - x)$, given $x \in \left(\pi, \frac{3\pi}{2}\right)$ **(b)** $\sin(2\pi - x)$, given $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

(c) $\cos\left(\frac{\pi}{2} - x\right)$, given $x \in \left(\frac{\pi}{2}, \pi\right)$ **(d)** $\sin\left(\frac{3\pi}{2} - x\right)$, given $x \in \left(0, \frac{\pi}{2}\right)$

16 For $0 \leq x \leq 2\pi$, determine the values of x for which $\sin(x) > \cos(x)$, where $\sin(x)$ and $\cos(x)$ have the same sign. Confirm your answer with an alternative explanation.

17 Given that $\sin(\theta) = -\frac{1}{\sqrt{5}}$, determine all possible values of $\sin(\theta) + \cos(\theta) + \tan(\theta)$.

Solving trigonometric equations

2.2

Simple trigonometric equations

Due to the periodic nature of the functions $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$, trigonometric equations of the form $\sin(x) = a$ and $\cos(x) = a$, where $-1 \leq a \leq 1$, and $\tan(x) = a$ for $a \in \mathbb{R}$ are over an unrestricted domain. When the domain is restricted to $[0, 2\pi]$, you can determine the size of the reference angle, then determine every solution from the unit circle or a graph of the function.

Additional information

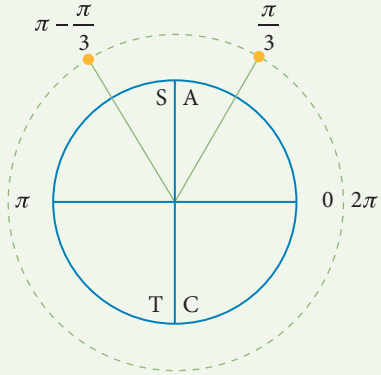
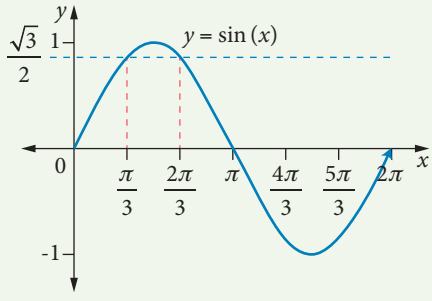
Solving trigonometric equations

Explore the technique to solve trigonometric equations using a graphical approach.

6 Solve simple equations involving circular functions

Determine all solutions to the following equations.

(a) $\sin(x) = \frac{\sqrt{3}}{2}$, $0 \leq x \leq 2\pi$

THINKING	WORKING
1 Determine the reference angle in the first quadrant.	Given that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, the reference angle is $\frac{\pi}{3}$.
2 Draw a unit circle to determine the number and location of the solutions.	The sine function is positive in quadrants 1 and 2. 
3 Determine the solutions and simplify.	$x = \frac{\pi}{3}$ or $x = \pi - \frac{\pi}{3}$ $x = \frac{\pi}{3}$ or $x = \frac{2\pi}{3}$
4 Evaluate the reasonableness of your solution by drawing a graph.	

(b) $2\cos(x)+1=0$, for $x \in [0, 4\pi]$

1 Transpose the equation to make the circular function the subject.

$$2\cos(x)+1=0$$
$$2\cos(x)=-1$$
$$\cos(x)=-\frac{1}{2}$$

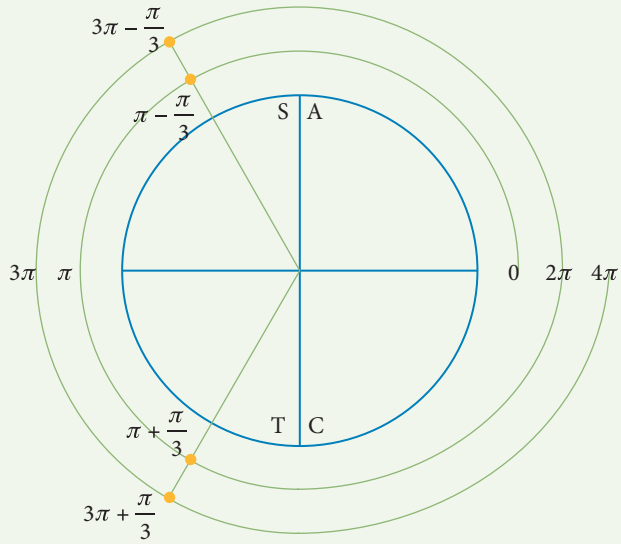
2 Determine the reference angle.

Given that $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$, the reference angle is $\frac{\pi}{3}$.

3 Draw a unit circle to determine the number and location of the solutions.

The cosine function is negative in quadrants 2 and 3.

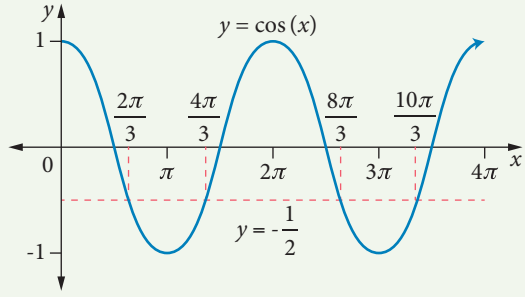
Over the domain $[0, 4\pi]$, there will be four solutions.



4 Determine the solutions.

$$x=\pi-\frac{\pi}{3}, \pi+\frac{\pi}{3}, 3\pi-\frac{\pi}{3}, 3\pi+\frac{\pi}{3}$$
$$x=\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$

5 Evaluate the reasonableness of your solution by drawing a graph.



Adjusting the domain

When the argument, or angle in the equation, is an expression, you need to alter the domain for x to determine a domain for the expression.

For example, if $0 \le x \le 2\pi$, the domain for $3x$ will be:
$3 \times 0 \le 3 \times x \le 3 \times 2\pi$
$0 \le 3x \le 6\pi$

The domain for $x - \frac{\pi}{2}$ will be:

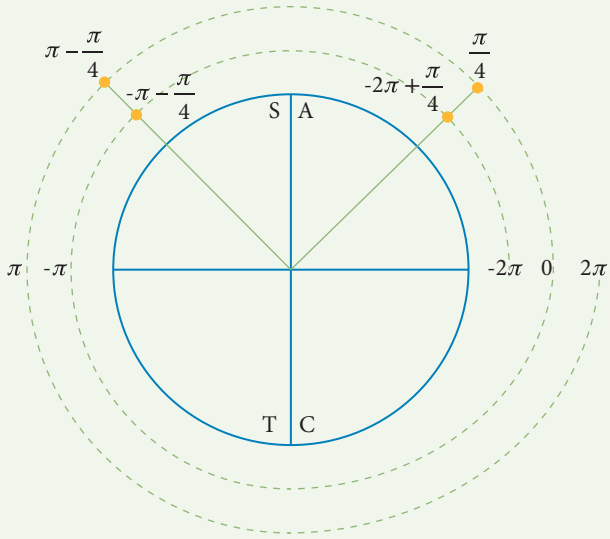
$$0 - \frac{\pi}{2} \leq x - \frac{\pi}{2} \leq 2\pi - \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq x - \frac{\pi}{2} \leq \frac{3\pi}{2}$$

7 Solve equations involving circular functions

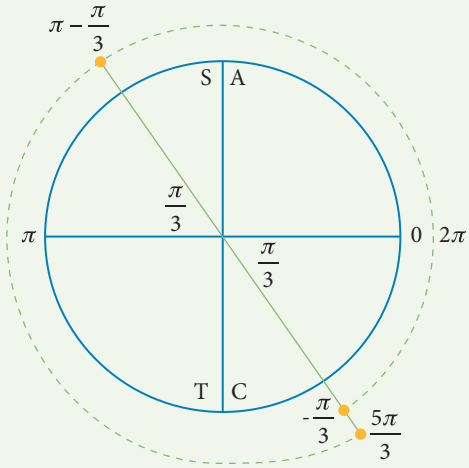
Determine all solutions, in the given domain for the following equations.

(a) $2\sin(2x) - \sqrt{2} = 0, -\pi \leq x \leq \pi$

THINKING	WORKING
1 Rearrange the equation to make the circular function the subject.	$2\sin(2x) - \sqrt{2} = 0$ $2\sin(2x) = \sqrt{2}$ $\sin(2x) = \frac{\sqrt{2}}{2}$
2 Adjust the domain to be consistent with the angle in the equation.	$-\pi \leq x \leq \pi$ $-2\pi \leq 2x \leq 2\pi$
3 Determine the reference angle.	Given that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, the reference angle is $\frac{\pi}{4}$.
4 Draw a unit circle to determine the number and location of the solutions.	The sine function is positive in quadrants 1 and 2. 
5 Determine all solutions to x within the adjusted domain.	$2x = -2\pi + \frac{\pi}{4}, -\pi - \frac{\pi}{4}, \frac{\pi}{4}, \pi - \frac{\pi}{4}$ $2x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
6 Solve for x and write all solutions in simplest form.	$x = -\frac{7\pi}{8}, -\frac{5\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}$

(b) $1 = -\frac{1}{\sqrt{3}}\tan\left(x - \frac{\pi}{3}\right), 0 \leq x \leq 2\pi$

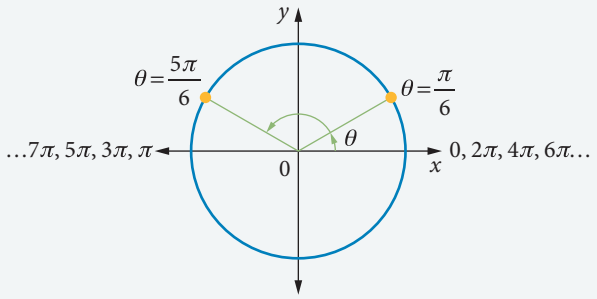
1 Rearrange the equation to make the circular function the subject.	$\tan\left(x - \frac{\pi}{3}\right) = -\sqrt{3}$
2 Adjust the domain to be consistent with the argument (angle in the equation).	$0 \leq x \leq 2\pi$ $0 - \frac{\pi}{3} \leq x - \frac{\pi}{3} \leq 2\pi - \frac{\pi}{3}$ $-\frac{\pi}{3} \leq x - \frac{\pi}{3} \leq \frac{5\pi}{3}$
3 Determine the reference angle.	Given that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, the reference angle is $\frac{\pi}{3}$.
4 Draw a unit circle to determine the number and location of the solutions.	The tangent function is negative in quadrants 2 and 4.
5 Determine all solutions to the argument within the adjusted domain.	$x - \frac{\pi}{3} = -\frac{\pi}{3}, \pi - \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$ $x - \frac{\pi}{3} = -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$
6 Solve for x and write all solutions in simplest form.	$x = -\frac{\pi}{3} + \frac{\pi}{3}, \frac{2\pi}{3} + \frac{\pi}{3}, \frac{5\pi}{3} + \frac{\pi}{3}$ $x = 0, \pi, 2\pi$



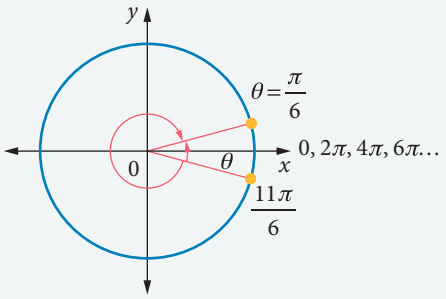
Unrestricted domain

The general solution to a trigonometric function involves calculating all solutions for one period of the function, then using periodicity to express the infinite solutions leading from each of these initial values.

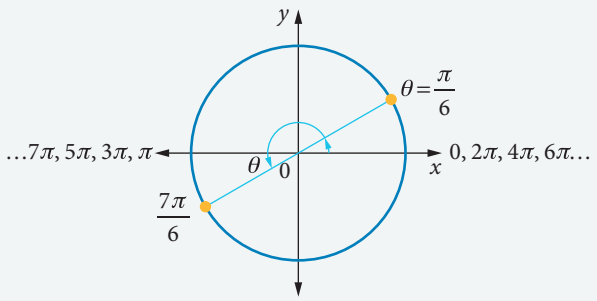
$\sin(x) = \frac{1}{2}$	
Quadrant 1	**Quadrant 2**
$x = \frac{\pi}{6}$	$x = \pi - \frac{\pi}{6}$
$x = 2\pi + \frac{\pi}{6}$	$x = 3\pi - \frac{\pi}{6}$
$x = 4\pi + \frac{\pi}{6}$	$x = 5\pi - \frac{\pi}{6}$



$\cos(x) = \frac{\sqrt{3}}{2}$	
Quadrant 1	**Quadrant 4**
$x = \frac{\pi}{6}$	$x = 2\pi - \frac{\pi}{6}$
$x = 2\pi + \frac{\pi}{6}$	$x = 4\pi - \frac{\pi}{6}$
$x = 4\pi + \frac{\pi}{6}$	$x = 6\pi - \frac{\pi}{6}$



$\tan(x) = \frac{1}{\sqrt{3}}$	
Quadrant 1	**Quadrant 3**
$x = \frac{\pi}{6}$	$x = \pi + \frac{\pi}{6}$
$x = 2\pi + \frac{\pi}{6}$	$x = 3\pi + \frac{\pi}{6}$
$x = 4\pi + \frac{\pi}{6}$	$x = 5\pi + \frac{\pi}{6}$



In general, consider $a > 0$.

- If $\sin(x) = a$, where $\sin^{-1}(a) = \theta$ represents the reference angle in the first quadrant: $x = 2n\pi + \theta$ or $x = (2n+1)\pi - \theta$ where $n \in \mathbb{Z}$.
- If $\cos(x) = a$, where $\cos^{-1}(a) = \theta$ represents the reference angle in the first quadrant: $x = 2n\pi \pm \theta$ where $n \in \mathbb{Z}$.
- If $\tan(x) = a$, where $\tan^{-1}(a) = \theta$ represents the reference angle in the first quadrant: $x = n\pi + \theta$ where $n \in \mathbb{Z}$.

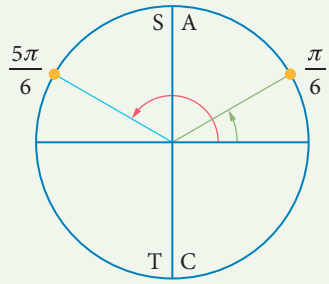
8 The general solution for simple equations

Solve each of the following equations.

(a) $\sin(x) = \frac{1}{2}$

THINKING	WORKING
1 Determine the quadrants in which the solution lies.	Sine is positive in quadrants 1 and 2.
2 Determine the reference angle in the first quadrant.	Given that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, the reference angle is $\frac{\pi}{6}$.

3 Recall the symmetry properties of the unit circle to determine the solutions in $0 \leq x \leq 2\pi$.



$x = \frac{\pi}{6}, \; x = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

4 Determine the general solution by writing an equation that adds integer multiples of 2π to each of the first two solutions.

In general:

$x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$, where $n \in \mathbb{Z}$

(b) $\cos(x) = -\frac{\sqrt{3}}{2}$

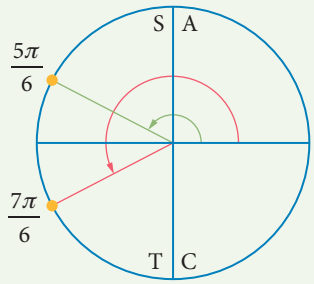
1 Determine the quadrants in which the solution lies.

The cosine function is negative in quadrants 2 and 3.

2 Determine the reference angle in the first quadrant.

Given that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, the reference angle is $\frac{\pi}{6}$.

3 Recall the symmetry properties of the unit circle to determine the solutions in $0 \leq x \leq 2\pi$.



$x = \pi - \frac{\pi}{6}, \quad x = \pi + \frac{\pi}{6}$

$= \frac{5\pi}{6} \qquad = \frac{7\pi}{6}$

4 Determine the general solution by writing an equation that adds integer multiples of 2π to each of the first two solutions.

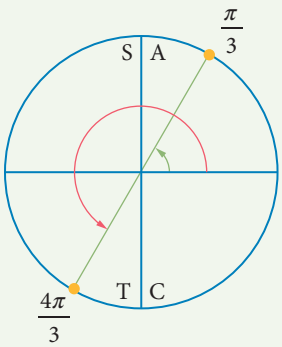
In general:

$x = \frac{5\pi}{6} + 2n\pi$ or $x = \frac{7\pi}{6} + 2n\pi$, where $n \in \mathbb{Z}$

5 Express as a neater, alternative solution.

Since $\frac{7\pi}{6}$ is equivalent to $-\frac{5\pi}{6}$, the general solution is: $x = 2n\pi \pm \frac{5\pi}{6}, n \in \mathbb{Z}$

(c) $\tan(x) = \sqrt{3}$

1 Determine the quadrants in which the solution lies.	The tangent function is positive in quadrants 1 and 3.
2 Determine the reference angle in the first quadrant.	$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ The reference angle is $\frac{\pi}{3}$.
3 Recall the symmetry properties of the unit circle to determine the solutions in $0 \leq x \leq 2\pi$.	 $x = \frac{\pi}{3}, \pi + \frac{\pi}{3}$ $= \frac{\pi}{3}, \frac{4\pi}{3}$
4 Determine the general solution by adding integer multiples of π to the original solution.	In general: $x = \frac{\pi}{3} + n\pi$ where $n \in \mathbb{Z}$

The general solution can be used when a limited number of specific solutions are required.

9 Determine and use the general solution

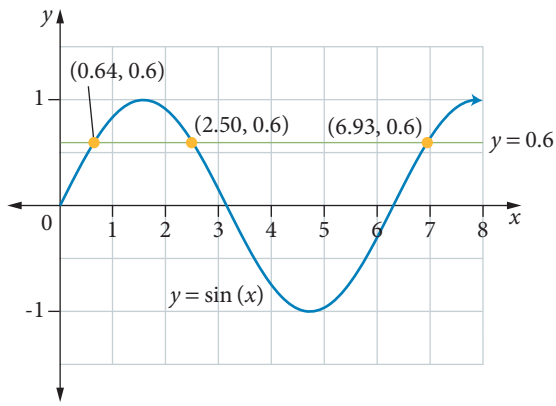
Determine the general solution and hence the first three positive solutions to the equation $\sin\left(2x - \frac{\pi}{3}\right) = -\cos\left(2x - \frac{\pi}{3}\right)$.

THINKING	WORKING
1 Divide both sides by the term on the right-hand side of the equation where $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.	$\sin\left(2x - \frac{\pi}{3}\right) = -\cos\left(2x - \frac{\pi}{3}\right)$ $\frac{\sin\left(2x - \frac{\pi}{3}\right)}{\cos\left(2x - \frac{\pi}{3}\right)} = -1$ $\tan\left(2x - \frac{\pi}{3}\right) = -1$
2 Determine the quadrants in which the solution lies.	The tangent function is negative in quadrants 2 and 4.
3 Determine the reference angle in the first quadrant.	Given that $\tan\left(\frac{\pi}{4}\right) = 1$, the reference angle is $\frac{\pi}{4}$.

4 Recall symmetry properties to determine the solution to the argument (the angle expression) for $0 \leq x \leq \pi$.	$2x - \frac{\pi}{3} = \frac{3\pi}{4}$
5 Write the general solution of the equation.	$2x - \frac{\pi}{3} = n\pi + \frac{3\pi}{4}$
6 Solve for x.	$2x = n\pi + \frac{3\pi}{4} + \frac{\pi}{3}$ $2x = n\pi + \frac{13\pi}{12}$ $x = \frac{n\pi}{2} + \frac{13\pi}{24}$ $x = \frac{12n\pi + 13\pi}{24}$ $x = \frac{(13 + 12n)\pi}{24}$
7 Substitute the values $n = -1$, $n = 0$ and $n = 1$ into the general solution.	Where $n = -1$: $x = \frac{13\pi - 12\pi}{24}$ $= \frac{\pi}{24}$ Where $n = 0$: $x = \frac{13\pi}{24}$ Where $n = 1$: $x = \frac{13\pi + 12\pi}{24}$ $= \frac{25\pi}{24}$
8 Provide the first three positive solutions.	$\therefore x = \frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}$

Practical situations

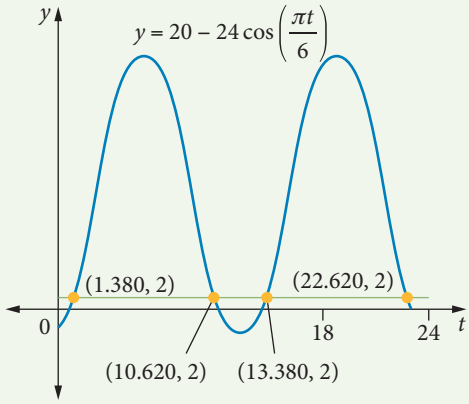
Using trigonometric equations in real situations will not lead to exact values very often. To solve a simple equation such as $\sin(x) = 0.6$, $0 \leq x \leq 8$, you can use a scientific calculator to determine the reference angle: $\sin^{-1}(0.6) \approx 0.6435$, remembering to give the value in radians. Further solutions would be $\pi - 0.6435 \approx 2.4981$, $2\pi + 0.6435 \approx 6.9267$ and so on. A graphics calculator, in graph mode gives all solutions within the required domain, as well as visual confirmation of the solution.



Technology worked example
Solve trigonometric equations

10 Solving practical problems

Deliveries can only be made to Mathtopia via the Bay when the depth of the water, y metres, is greater than 2 m. The water levels in the Bay of Mathtopia can be modelled by: $y = 20 - 24\cos\left(\frac{\pi t}{6}\right)$, $0 \leq t \leq 24$, where t is the time in hours after midnight. Give suitable intervals for deliveries to be made.

THINKING	WORKING
1 Create an algebraic equation to be solved.	$20 - 24\cos\left(\frac{\pi t}{6}\right) = 2$ for $0 \leq t \leq 24$
2 Using technology, plot appropriate graphs and read off the points of intersection.	 The coordinates of the point of intersection are $(1.380, 2)$, $(10.620, 2)$, $(13.380, 2)$ and $(22.620, 2)$.
3 Convert all t values into times.	1.380 hours after midnight: 1 h and 0.380×60 minutes $\approx$ 1 h 23 min The water level reaches 2 m at 1:23 am. 10.620 minutes after midnight: 10 h and 0.620×60 minutes $\approx$ 10 h 37 min The water level falls back to 2 m at 10:37 am. 13.380 minutes after midnight: 13 h and 0.380×60 minutes $\approx$ 13 h 23 min The water level reaches 2 m for the second time at 1:23 pm. 22.620 minutes after midnight: 22 h and 0.620×60 minutes $\approx$ 22 h 37 min The water level falls back to 2 m at 10:37 pm.
4 Interpret the graphical solution.	Thus, water levels are above 2 m and deliveries can be made between 1:23 am and 10:37 am or between 1:23 pm and 10:37 pm each day.

EXERCISE 2.2 Solving trigonometric equations

Worked Example

6

1 Determine all solutions to the following equations in the given domain.

(a) $\sin(x) = \frac{1}{\sqrt{2}}, 0 \leq x \leq 2\pi$ **(b)** $\cos(x) = -\frac{\sqrt{3}}{2}, 0 \leq x \leq 3\pi$

(c) $\sqrt{3}\tan(x) - 1 = 0, x \in [-\pi, \pi]$ **(d)** $4\sin(x) + 2 = 0, x \in [-2\pi, 2\pi]$

2 Determine all solutions in the domain $0 \leq x \leq 2\pi$, correct to 2 decimal places.

(a) $\sin(x) = -0.8$ **(b)** $\tan(x) = 3.4$

(c) $\cos(x) = 0.6$ **(d)** $\sin(x) = 0.44$

(e) $\tan(x) = -0.95$ **(f)** $\cos(x) = -0.31$

3 Determine all solutions in the given domain, correct to 2 decimal places.

(a) $3\sin\left(\frac{5x}{\pi}\right) = 1, x \in [0,5]$ **(b)** $17\cos\left(\frac{6x}{\pi} - 3\right) = 8, x \in [0,6]$

(c) $22.5\sin\left(\frac{12x}{\pi} + 4\right) = 20, x \in [0,3]$ **(d)** $84\cos\left(\frac{x}{15}\right) = 74, x \in [0,100]$

8

4 Solve each of the following equations.

(a) $\sin(x) = \frac{1}{\sqrt{2}}$ **(b)** $\tan(x) = -\sqrt{3}$ **(c)** $\cos(x) = \frac{1}{2}$ **(d)** $\sqrt{3}\tan(x) = 1$

5 Consider the equation $2\cos(x) = 1$.

(a) Determine the general solution.

(b) Using the general solution, determine the first four positive solutions in ascending order.

6 Consider the equation $\cos(2x) = -1$.

(a) Which is the general solution?

A $x = \frac{2\pi}{3} + 2n\pi$ or $x = \frac{4\pi}{3} + 2n\pi, n \in \mathbb{Z}$ **B** $x = \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$

C $x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$ **D** $x = \pi + 2n\pi, n \in \mathbb{Z}$

(b) Explain the common error made by a student who chose the first incorrect option.

7 A particle moves in a straight line such that its distance in metres x from a point O is given by the equation $x = 3 + 6\sin(4t)$, where t is the time in minutes.

(a) Determine the particle's initial distance from point O.

(b) Determine the time when the particle first reaches $x = 0$.

10

8 The average daily wind speed in kilometres per hour in Hervey Bay can be approximated by $s = 4.2\sin\left(\frac{\pi}{18}(t + 2.7)\right) + 10, 0 \leq t \leq 30$, where t is the time in days since 1 January.

(a) Determine the average wind speed on 1 January, correct to 1 decimal place.

(b) Determine the average wind speed on 3 January, correct to 1 decimal place.

(c) On what day(s) in January was the average wind speed 10 km/h?

(d) On what days of January was the wind speed 12 km/h?

Worked Example

9 Determine all solutions to the following equations. 7

(a) $\sin(2x) = \frac{1}{2}, 0 \le x \le 2\pi$

(b) $\cos(3x) = -\frac{\sqrt{3}}{2}, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(c) $3\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}, x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

(d) $2\cos\left(2x - \frac{\pi}{6}\right) + 1 = 0, -\pi \le x \le \pi$

10 How many solutions does the equation $\sin\left(2\left(\frac{\pi}{3} - x\right)\right) = \frac{\sqrt{3}}{2}$ have in the domain $0 \le x \le 2\pi$?

A 4 **B** 3 **C** 5 **D** 1

11 Solve the following equations.

(a) $\sin(3x) = \frac{\sqrt{3}}{2}$

(b) $\tan\left(x - \frac{\pi}{3}\right) = -1$

(c) $\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = -\frac{1}{2}$

12 Consider the equation $\sin\left(\frac{\pi}{4} - 3x\right) - \cos\left(\frac{\pi}{4} - 3x\right) = 0$. 9

(a) Determine the general solution.

(b) Using the general solution, determine all solutions for $0 \le x \le \pi$.

13 Solve the equation $2\cos^2(x) + \cos(x) - 1 = 0$.

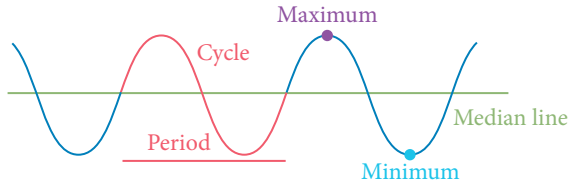
14 Solve the equation $\sin^2(x) + 5\cos^2(x) = 2$ for $0 \le x \le 2\pi$.

15 The power P in watts used by a microwave oven is the product of the voltage V in volts and the current I in amperes. The voltage can be modelled by the equation: $V = 175\cos(120\pi t)$ and the current can be modelled by the equation: $I = 12\cos(120\pi t)$, where t is the time in seconds. For $0 \le t \le \frac{1}{60}$, at what times does the microwave oven use more than 1750 watts of power? Give your answers correct to 3 decimal places.

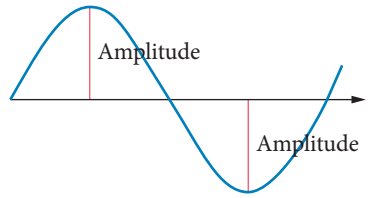
2.3 Graphs of trigonometric functions

Graphs of sine and cosine functions

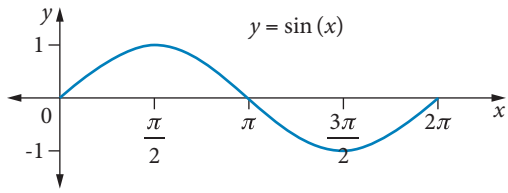
Trigonometric functions are periodic. That is, the same set of function values recurs after fixed intervals. The period is the time (or horizontal distance) it takes for the set of values to complete one cycle (one complete repetition of the pattern). The trigonometric functions $\sin(x)$ and $\cos(x)$ are periodic with a period of 2π. Values oscillate around a median value, which is halfway between the maximum and minimum values of the function.

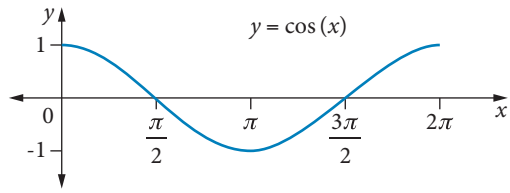


When the graph of the sine function is extended over domain $\mathbb{R}$, it forms a smooth and continuous curve that repeats itself after an interval of 2π units. The amplitude is defined as the distance between the median position (the horizontal central axis) of the graph and the maximum or minimum value.



One complete cycle of the graphs of $\sin(x)$ and $\cos(x)$ and their key features are shown:





Domain: 2π; range: $[-1,1]$; amplitude: 1; period: 2π

The median line is the x-axis $(y = 0)$.

The graph of $y = \cos(x)$ is produced from the graph of $y = \sin(x)$ by a translation of $\frac{\pi}{2}$ units to the left.

Graphs of trigonometric functions can be transformed just like other functions. For example, sound waves can be modelled by sine curves. When you listen to music on an MP3 player, you are listening to sound waves. The number of cycles the wave completes every second is the period of the sine wave, and determines what musical note you are hearing.

Additional information

The graphs of the sine and cosine functions

These videos explain the relationship between the unit circle and the graphs of sine and cosine.

Making connections

Transforming the sine and cosine functions

Explore the effect of each parameter on the graphs of $g(x) = A\sin\big(B(x + C)\big) + D$.

Transformations of the sine and cosine functions:

- $y = A\sin\big(B(x + C)\big) + D$ or $y = A\cos\big(B(x + C)\big) + D$, where $A \neq 0$ and $B > 0$
- Amplitude: $|A|$; dilation by a factor of $|A|$ parallel to the y-axis
- $A < 0$: reflection in the x-axis
- Period: $\frac{2\pi}{B}$; dilation by a factor of $\frac{1}{B}$ parallel to the x-axis, so there are B cycles between 0 and 2π
- C: phase shift of $-C$, so horizontal translation of C units in the negative x-direction
- D: vertical translation of D units in the positive y-direction

The median line is $y = D$.

Minimum y-value: $D - |A|$

Maximum y-value: $D + |A|$

If $B < 0$, you can simplify the equation before graphing.

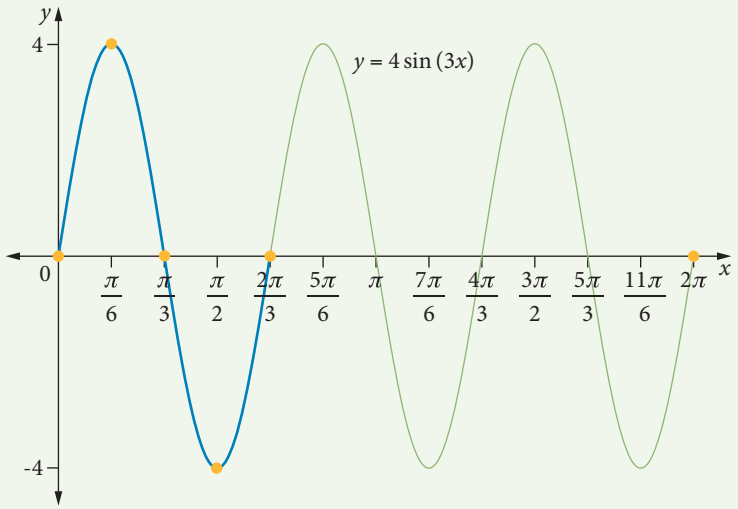
$f(x) = \sin(x)$ is an odd function, i.e. $f(-x) = -f(x)$. For example $\sin(-2x) = -\sin(2x)$, the graph of $f(x)$ is reflected in the x-axis.

$f(x) = \cos(x)$ is an even function, i.e. $f(-x) = f(x)$. For example $\cos(-2x) = \cos(2x)$, the graph of $f(x)$ remains unchanged.

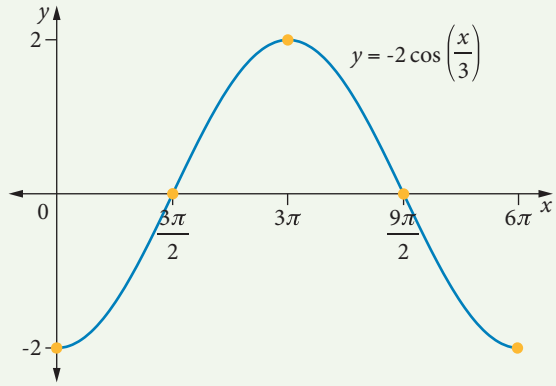
11 Sketch dilations of trigonometric functions

Determine the period and amplitude of each of the following, and how the graph of $y = \sin(x)$ has been transformed. Sketch the graph.

(a) $y = 4\sin(3x), 0 \leq x \leq 2\pi$

THINKING	WORKING
1 Recall the amplitude for $y = A\sin(Bx)$ is $\|A\|$. The period is $\frac{2\pi}{B}$.	$y = 4\sin(3x)$ Amplitude: 4 Period: $\frac{2\pi}{3}$
2 Determine the transformations.	The graph of $y = \sin(x)$ is dilated by a factor of 4 parallel to the y-axis and dilated by a factor of $\frac{1}{3}$ parallel to the x-axis.
3 Divide one period into four equal parts. Use this value to construct the scale on the x-axis.	$\frac{\text{period}}{4} = \frac{\frac{2\pi}{3}}{4}$ $= \frac{2\pi}{3} \times \frac{1}{4}$ $= \frac{\pi}{6}$
4 Mark the maximum and minimum values and the x-intercepts and draw a smooth curve. Extend the graph to the required domain.	

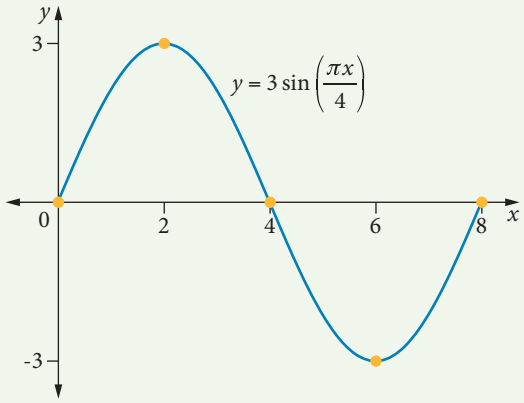
(b) $y = -2\cos\left(\frac{x}{3}\right), 0 \leq x \leq 6\pi$

1 Recall the amplitude for $y = A\cos(Bx)$ is $\lvert A\rvert$. The period is $\frac{2\pi}{B}$.	$y = -2\cos\left(\frac{x}{3}\right)$ Amplitude: 2 $A < 0$ Period: $\frac{2\pi}{\frac{1}{3}} = 6\pi$
2 Determine the transformations.	The graph of $y = \cos(x)$ is dilated by a factor of 2 parallel to the y-axis, reflected in the x-axis and dilated by a factor of 3 parallel to the x-axis.
3 Divide one period into four equal parts. Use this value to construct the scale on the x-axis.	$\frac{\text{period}}{4} = \frac{6\pi}{4}$ $= \frac{3\pi}{2}$
4 Mark the maximum and minimum values and the intercepts and draw a smooth curve. Label the endpoints.	

(c) $y = 3\sin\left(\frac{\pi x}{4}\right), 0 \leq x \leq 8$

1 Recall the amplitude for $y = A\sin(Bx)$ is $\lvert A\rvert$. The period is $\frac{2\pi}{B}$.	$y = 3\sin\left(\frac{\pi x}{4}\right)$ Amplitude: 3 Period: $\frac{2\pi}{\frac{\pi}{4}} = 8$
2 Determine the transformations.	The graph of $y = \sin(x)$ is dilated by a factor of 3 parallel to the y-axis and dilated by a factor of $\frac{4}{\pi}$ parallel to the x-axis.
3 Divide one period into four equal parts. Use this value to construct the scale on the x-axis.	$\frac{\text{period}}{4} = \frac{8}{4}$ $= 2$

4 Mark the maximum and minimum values and draw a smooth curve.



Graphs involving translations and dilations

One approach to graphing periodic functions that have undergone several transformations is to deal with reflections and dilations first, then translate to the final positon as the final step. It is usual to calculate the value of endpoints, but x-intercepts may have to be approximated.

12 Sketch transformed trigonometric functions including translations

Sketch the function $f(x) = 1 - 5\cos\left(4x + \frac{\pi}{2}\right)$, $0 \leq x \leq \pi$.

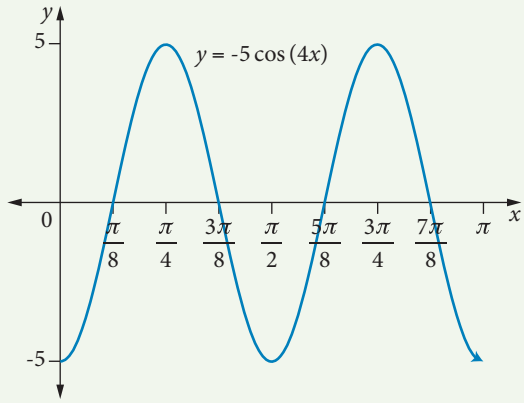
THINKING	WORKING
1 Rewrite the equation in the form $y = A\cos(B(x + C)) + D$.	$f(x) = 1 - 5\cos\left(4x + \frac{\pi}{2}\right)$ $= -5\cos\left(4\left(x + \frac{\pi}{8}\right)\right) + 1$
2 Determine the period and amplitude.	Amplitude: 5 Given that $A < 0$, the graph is reflected in the x-axis. Period: $\frac{2\pi}{4} = \frac{\pi}{2}$
3 Determine the coordinates of the y-intercept and the other endpoint.	y-intercept, $x = 0$: $f(0) = 1 - 5\cos\left(\frac{\pi}{2}\right)$ $= 1$ The coordinates of the y-intercept are $(0,1)$. $f(\pi) = 1 - 5\cos\left(4\pi - \frac{\pi}{2}\right)$ $= 1 - 0$ $= 1$ The coordinates of the endpoint are $(\pi,1)$.

4 Divide one period into four equal parts. Use this value to construct scale on the x-axis.

$$\frac{\text{period}}{4} = \frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$$

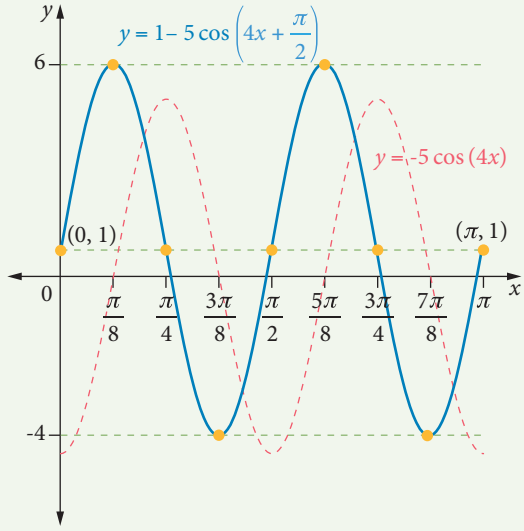
5 Sketch $f(x) = -5\cos(4x), 0 \leq x \leq \pi$.

Mark the maximum and minimum values and draw a smooth curve.



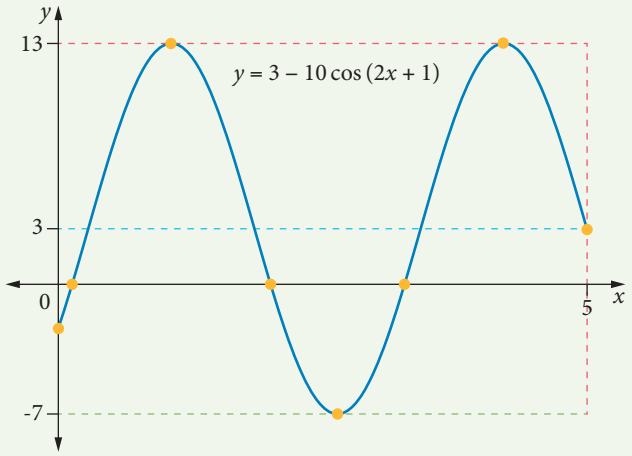
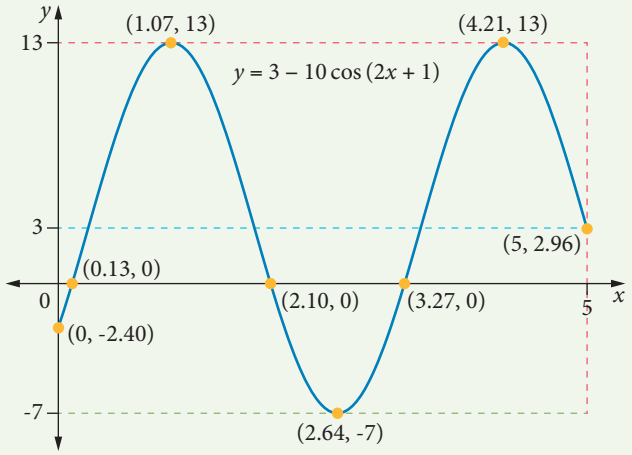
6 Translate the graph $\frac{\pi}{2}$ units in the negative x-direction and 1 unit in the positive y-direction to obtain the required graph.

Mark the median line ($y = 1$) and maximum and minimum values and draw a smooth curve.



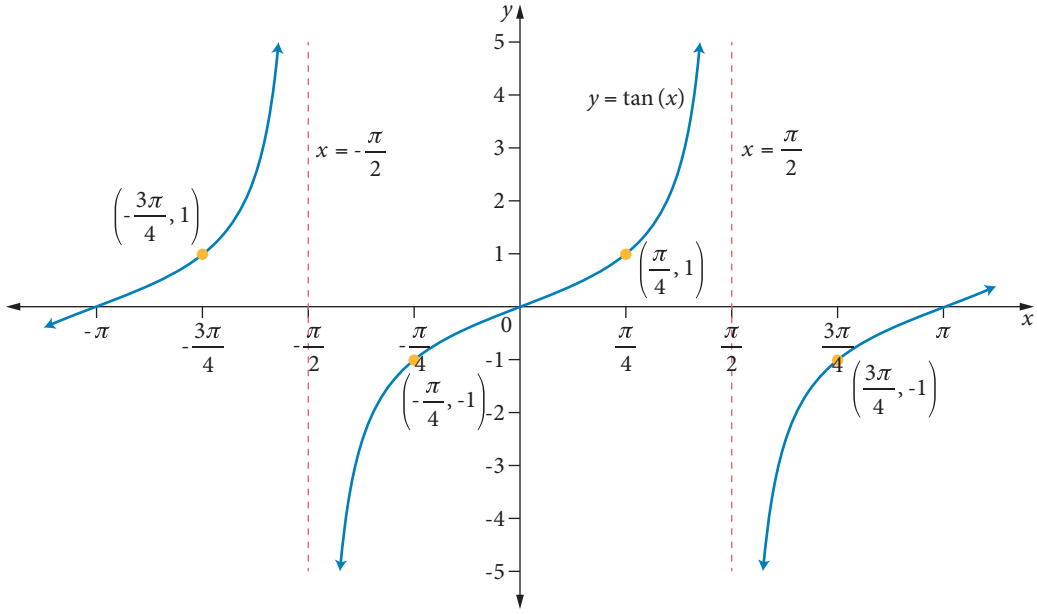
13 Graph transformed circular functions using technology

Graph the function $y = 3 - 10\cos(2x + 1)$, $0 \leq x \leq 5$, showing the median line, maximum and minimum values, axis intercepts and endpoints.

THINKING	WORKING
1 Calculate the median, maximum and minimum values of the function. Key in the equation and set up the domain and range.	The function is in the form $y = A\cos(B(x + C)) + D$, therefore, $A = -10$ and $D = 3$. Median: $y = 3$ Minimum: $y = 3 - 10 = -7$ Maximum: $y = 3 + 10 = 13$ Domain: $0 \leq x \leq 5$
2 Draw the graph, placing a dot at each of the required positions.	
3 Read off the required values from the calculator. Write the coordinates onto the graph.	y-intercept: $(0, -2.40)$ Other endpoint: $(5, 2.96)$ x-intercepts: $(0.13, 0)$, $(2.10, 0)$, $(3.27, 0)$ Maxima: $(1.07, 13)$, $(4.21, 13)$ Minimum: $(2.64, -7)$
	

The graph of the tan function

Like the sine and cosine functions, the tangent function is also periodic. The period of $\tan(x)$ is π. The tangent function does not have an amplitude. This is because the range of the function is all real numbers $\mathbb{R}$ and hence, the graph tends towards positive and negative infinity along the y-axis. Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, the graph of $\tan(x)$ is undefined for $\cos(x) = 0$. Therefore, for odd integer multiples of $\frac{\pi}{2}$, the graph of $\tan(x)$ has vertical asymptotes. These vertical asymptotes occur at a spacing of π, the period of the $\tan(x)$ graph. The graph of $\tan(x)$ with its key features is shown.



Features of the graph of $\tan(x)$:

Domain: $\mathbb{R} \setminus \left\{\ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \ldots\right\}$ or $\mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi\right\}, n \in \mathbb{Z}$.

Range: $\mathbb{R}$

Reference points:

Where $x = \ldots -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4} \ldots$, $y = 1$.

Where $x = \ldots -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4} \ldots$, $y = -1$.

Period: π

Asymptotes: $x = \ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \ldots$

Since the period is π, the asymptotes occur at intervals of π. You can add π to find the next asymptote or subtract π to find the preceding one.

x-intercepts: The x-intercepts occur a period apart at $x = \ldots -2\pi, -\pi, 0, \pi, 2\pi \ldots$

Additional information

The graph of the tangent function

View the construction of the tangent function.

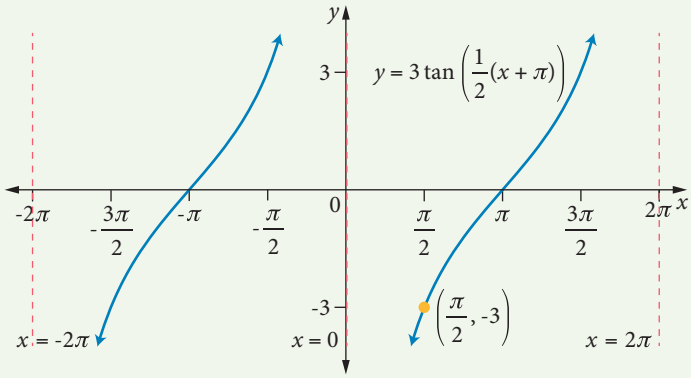
Making connections

Transforming the tangent function

Move the sliders to view the effect of a graph in the form $y = A\tan(Bx)$.

14 Sketch the graph of tangent functions

Sketch the graph of $y = 3\tan\left(\frac{1}{2}(x + \pi)\right)$ for $-2\pi \leq x \leq 2\pi$, showing the x-intercepts (points of inflection) and asymptotes.

THINKING	WORKING
1 Determine the period.	Period: $\frac{\pi}{\frac{1}{2}} = 2\pi$
2 Determine the coordinates of the x-intercepts.	x-intercept: $0 = 3\tan\left(\frac{1}{2}(x+\pi)\right)$ Because $\tan(0) = 0$, $0 = \frac{1}{2}(x+\pi)$ $x = -\pi$ The other x-intercepts can be found by adding or subtracting multiples of the period. $x = -\pi + 2\pi$ $= \pi$ Therefore, the x-intercepts are at $x = -\pi$, $x = \pi$.
3 Determine the equation of an asymptote, knowing that $\tan\left(\frac{\pi}{2}\right)$ is not defined.	One asymptote: $\frac{1}{2}(x+\pi) = \frac{\pi}{2}$ $x = \frac{\pi}{2} \times 2 - \pi$ $= 0$
4 Determine the equations of the rest of the asymptotes that lie within the given domain by adding or subtracting multiples of the period.	The equations of the asymptotes: $x = -2\pi$, $x = 0$, $x = 2\pi$
5 Determine a reference point. To solve for the x-value, locate a value a quarter of a period from an asymptote. Then substitute into the equation to solve for the corresponding y-value.	$2\pi \div 4 = \frac{\pi}{2}$ Where $x = \frac{\pi}{2}$: $y = 3\tan\left(\frac{1}{2}\left(\frac{\pi}{2}+\pi\right)\right)$ $= 3\tan\left(\frac{3\pi}{4}\right)$ $= -3$ The coordinates of a point that lie on the graph are $\left(\frac{\pi}{2}, -3\right)$.
6 Mark the asymptotes. Then sketch the graph over the domain. Use the reference point to help shape the graph. Label the asymptotes.	

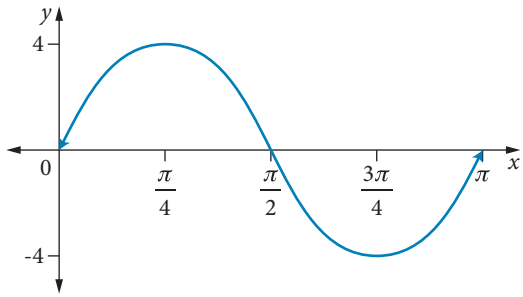
Determining the rule for a sine or cosine function

If you obtain data from observation or practical applications, you need to first determine the rule of the model. Typically look for information that can help you determine amplitude, period, phase and median value. Because of the periodic nature of trigonometric functions, when finding a rule for a graph, there are many possibilities. For example, a sine graph could also be a cosine graph that has been translated along the x-axis.

15 Determine the rule for trigonometric functions

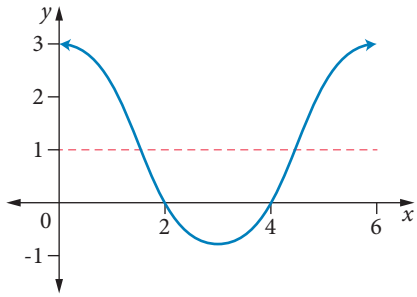
Determine a possible equation for each of the following graphs.

(a)



THINKING	WORKING
1 Choose a sine graph, since no translation will be required. Recall the general equation.	$y = A\sin(Bx)$
2 Use the period to determine the value of B. Use the amplitude to determine the value of A.	Period: π $\frac{2\pi}{B} = \pi$ $B = 2$ Amplitude: 4 $A = 4$
3 Write the equation.	$y = 4\sin(2x)$

(b)



1 Choose a cosine graph, since no horizontal translation will be required. Recall the general equation.	$y = A\cos(Bx) + D$
2 Use the period to determine the value of B. Use the amplitude to determine the value of A. Use the median line to determine the value of D.	Period: 6 $\frac{2\pi}{B} = 6$ $B = \frac{\pi}{3}$ Amplitude: 2 $A = 2$ Median: $y = 1$ $D = 1$
3 Write the equation.	$y = 2\cos\left(\frac{\pi x}{3}\right) + 1$

EXERCISE 2.3 Graphs of trigonometric functions

Worked Example

11

1 For each of the following, determine how the graph of $y = \sin(x)$ or $y = \cos(x)$ has been transformed and sketch the graph.

(a) $y = 2\sin(2x), 0 \le x \le 2\pi$

(b) $y = 5\cos(3x), 0 \le x \le \frac{2\pi}{3}$

(c) $y = -3\sin(4x), -\frac{\pi}{2} \le x \le \pi$

(d) $y = 4\sin\left(\frac{x}{2}\right), -4\pi \le x \le 4\pi$

(e) $y = 2\sin\left(\frac{\pi x}{3}\right), 0 \le x \le 6$

2 For each of the following, determine the period and amplitude and sketch the graph. Clearly label the axis intercepts and endpoints.

(a) $y = 2\cos(3x) + 1, 0 \le x \le \frac{2\pi}{3}$

(b) $y = 2 - 4\sin(-4x), \ 0 \le x \le \frac{\pi}{2}$

13

3 Sketch the function $y = 2 - 3\cos(2x - 1), 0 \le x \le 5$, showing the median line, maximum and minimum values, axis intercepts and endpoints correct to 2 decimal places.

4 Consider the functions $y = \cos(x)$ and $y = \sin\left(x + \frac{\pi}{2}\right)$.

(a) Sketch the graphs of both functions for $0 \le x \le 2\pi$ on the same axes. Explain your solution.

(b) Determine a cosine function that gives the same graph as each of the following.

(i) $y = \sin\left(x + \frac{\pi}{4}\right)$

(ii) $y = \sin\left(x - \frac{\pi}{3}\right)$

(c) Determine a sine function that gives the same graph as each of the following.

(i) $y = \cos\left(x - \frac{\pi}{6}\right)$

(ii) $y = \cos\left(x + \frac{\pi}{3}\right)$

5 Sketch the graph of $y = 3\tan(2x), 0 \le x \le 2\pi$, showing axes intercepts and asymptotes.

6 The graph of $y = \cos(x) - 1$ is reflected in the x-axis and translated $\frac{\pi}{3}$ units in the positive x-direction and 2 units in the negative y-direction.

(a) What is the equation of the image?

A $y = \cos\left(x - \frac{\pi}{3}\right) - 1$

B $y = -\cos\left(x - \frac{\pi}{3}\right) - 1$

C $y = -\cos\left(x + \frac{\pi}{3}\right) - 1$

D $y = \cos\left(x + \frac{\pi}{3}\right) + 1$

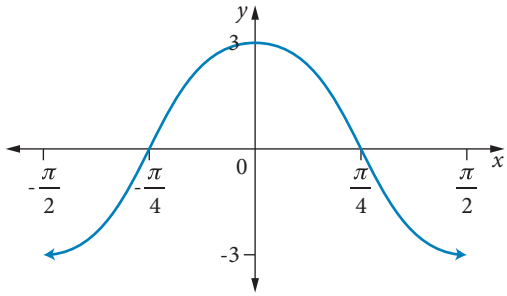
Worked Example

(b) Explain the common error made by a student who applied the incorrect sign to their translation.

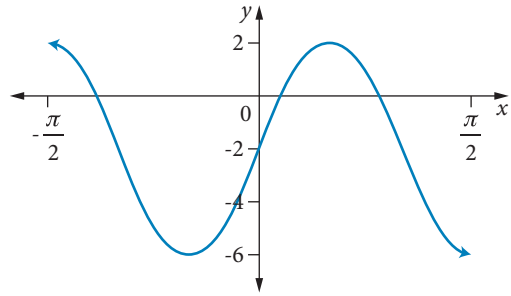
15

7 For each of the following, determine a possible equation.

(a)



(b)



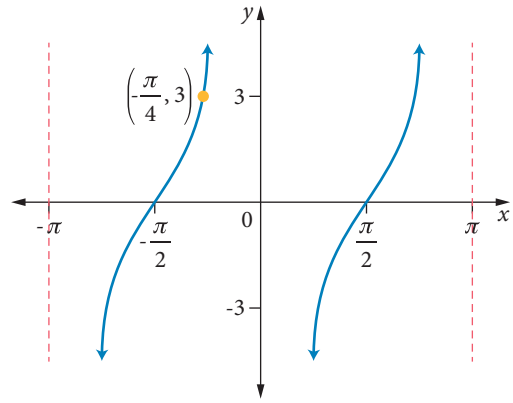
8 What is the equation of the graph shown?

A $y = \tan\left(3x + \frac{\pi}{2}\right)$

B $y = 3\tan\left(x - \frac{\pi}{2}\right)$

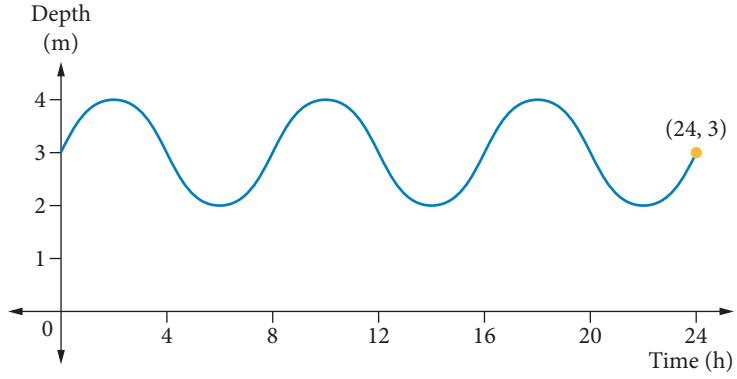
C $y = \tan\left(3x - \frac{\pi}{2}\right)$

D $y = \tan\left(3\left(x - \frac{\pi}{2}\right)\right)$



9 Determine the transformations required to transform the graph of $y = \sin(x)$ to $y = 4 - 3\sin\left(2x - \frac{\pi}{6}\right)$.

10 The depth of water at a particular point on the Brisbane River was measured at hourly intervals from midnight. The graph shows the results.



(a) Determine the amplitude and period.

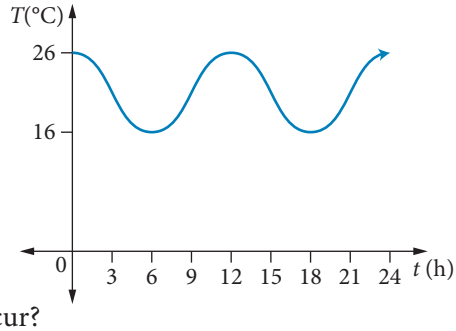
(b) Determine the maximum and minimum depths of the river.

(c) Determine the times when the water level was at a maximum.

(d) Determine the times when the water level was at a minimum.

(e) Determine the equation of the curve, in the form $y = A\sin(Bx) + D$.

11 The varying temperature T°C of a container over a 24-hour period is graphed on the right. The temperature can be modelled by a rule of the form: $T = A\cos(Bt) + D$.



(a) Determine the values of A, B and D.

(b) Determine the temperature at the following times. Give your answers correct to 1 decimal place.

(i) 8 am **(ii)** 1 pm

(c) What is the maximum daily temperature and when does it occur?

(d) What is the minimum daily temperature and when does it occur?

(e) At what times, to the nearest minute, is the temperature 20°C?

(f) At what times is the temperature higher than 20°C?

12 For each of the following, determine the period and amplitude and sketch the graph. Clearly label the axes, intercepts and endpoints.

(a) $y = 2\sin\left(x - \frac{\pi}{6}\right), -\pi \le x \le \pi$ **(b)** $y = 4\cos\left(x - \frac{\pi}{4}\right), -\frac{\pi}{4} \le x \le \frac{7\pi}{4}$

(c) $y = 3\cos\left(\frac{\pi}{2} - 3x\right), -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Worked Example 12

13 Sketch the function $y = 3 - 5\sin\left(\frac{\pi}{3} - 2x\right), 0 \le x \le \pi$.

Worked Example 14

14 Sketch the graph of each function, showing the y-intercept and endpoints (if they exist), asymptotes and points of inflection.

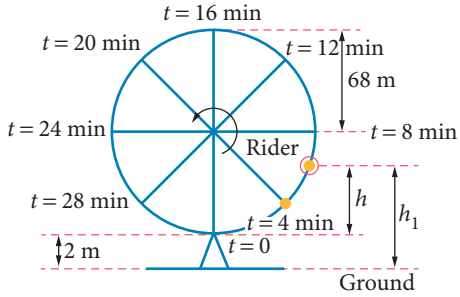
(a) $y = 2\tan\left(x - \frac{\pi}{6}\right), -\pi \le x \le \pi$ **(b)** $y = 3\tan\left(2x - \frac{\pi}{4}\right), 0 \le x \le \pi$

(c) $y = 2 - 3\tan\left(\frac{\pi}{4} - 2x\right), 0 \le x \le \frac{7\pi}{8}$

15 The graph of $y = 2\cos\left(3x - \frac{\pi}{2}\right) + 1$ was reflected in the x-axis and translated $\frac{\pi}{3}$ units in the negative direction of the x-axis. Determine the equation of the transformed graph. Sketch both graphs on the same set of axes for $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

16 The graph of $y = \tan\left(n(x - \varepsilon)\right)$ has one vertical asymptote at $x = \frac{5\pi}{4}$ and one x-intercept at $x = \frac{\pi}{4}$ over the domain $[0, 2\pi]$. Determine a possible pair of values of n and ε, then sketch the graph of $y = \tan\left(n(x - \varepsilon)\right)$ for $x \in [0, 2\pi]$.

17 London Eye is the world's biggest Ferris wheel, with a diameter of 136 metres. It makes one complete revolution in 32 minutes and is 2 metres off the ground at its lowest point.



(a) Assuming that the carriage is at the bottom of the Ferris wheel at $t = 0$, draw a graph of height above the ground (metres) against time (minutes), showing two complete cycles.

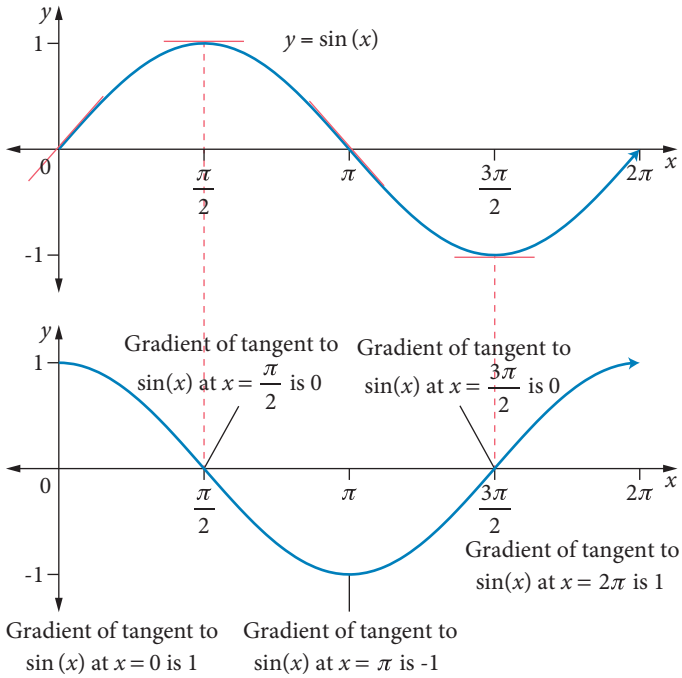
(b) Write a cosine function with a specified domain, to match your graph.

(c) For how long in each revolution is the rider above a height of 50 m? Give your answer to the nearest second.

2.4 Derivatives of trigonometric functions

The derivative of the sine function

The derivative of the sine function can be determined informally by drawing a derivative graph. The derivative is defined as the gradient of the tangent line at any point (x, y). The graph below shows $y = \sin(x)$ along with its tangents at various points.



Making connections

The derivative of the sine function

Analyse the graph of $y = A\sin(Bx)$ to determine the equation of the derivative.

From the top graph, the gradient of the tangents at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ is 0. The gradient of the tangent at $x = 0$ and $x = 2\pi$ is about 1 and by symmetry the gradient at $x = \pi$ is -1.

The graph below shows the derivative graph for $\sin(x)$ drawn using these tangents.

Additional information

The derivative of the sine function

This video explains how to obtain the derivative of the sine function.

From the sketch, it is reasonable to conclude that the derivative of $\sin(x)$ is $\cos(x)$.

Using first principles, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, the derivative for $f(x) = \sin(x)$ can be expressed as:

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

To manipulate the algebraic expression, knowledge of the addition theorem for the sine ratio (not covered in this course) is required: $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

Using the addition theorem:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin(x)\big(\cos(h) - 1\big) + \cos(x)\sin(h)}{h} \\ &= \lim_{h \to 0} \left(\frac{\sin(x)\cos(h) - 1}{h} \right) + \lim_{h \to 0} \left(\frac{\cos(x)\sin(h)}{h} \right) \\ &= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} \end{aligned}$$

To continue, you need to determine the following limits: $\lim_{h \to 0} \frac{\cos(h) - 1}{h}$ and $\lim_{h \to 0} \frac{\sin(h)}{h}$.

Using technology and values of h close to 0, the information in the table can be obtained.

h	$\frac{\cos(h)-1}{h}$	$\frac{\sin(h)}{h}$
0.1	-0.049 958 347 22	0.998 334 166 468
0.01	-0.004 999 958 333	0.999 983 333 417
0.001	-0.000 499 999 96	0.999 999 833 333

These results suggest: $\lim\limits_{h\to 0}\dfrac{\cos(h)-1}{h}=0$ and $\lim\limits_{h\to 0}\dfrac{\sin(h)}{h}=1$

returning to: $f'(x)=\sin(x)\lim\limits_{h\to 0}\dfrac{\cos(h)-1}{h}+\cos(x)\lim\limits_{h\to 0}\dfrac{\sin(h)}{h}$.

Therefore, $f'(x)=\sin(x)\times 0+\cos(x)$

$=\cos(x)$

The earlier conclusion that the derivative of $\sin(x)$ is $\cos(x)$ is confirmed.

The result $\lim\limits_{h\to 0}\dfrac{\sin(h)}{h}=1$ can be verified more formally using a geometric proof.

Consider a sector OAB of a unit circle with angle x as shown in the diagram. The line segment $\overline{BD}$ is perpendicular to $\overline{OA}$, and $\overline{CA}$ is a tangent at point A. Therefore, $OB=OA=1$, $BD=\sin(x)$ and $OD=\cos(x)$ and $CA=\tan(x)$.

From the diagram, you can see:

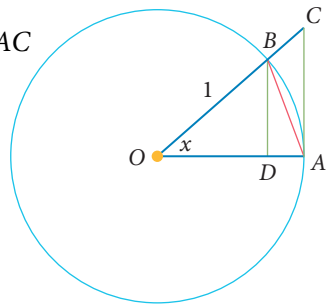
area of ΔOAB < area of sector ΔOAB < area of ΔOAC

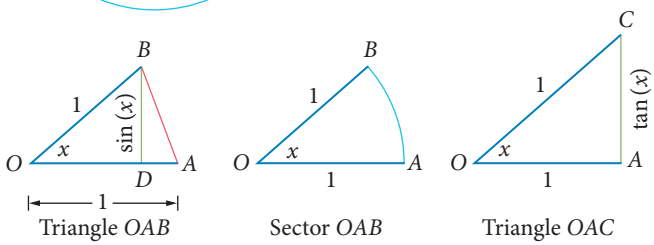
$$\frac{1}{2}(OA)(BD)<\frac{x}{2\pi}\times\pi(1)^2<\frac{1}{2}(OA)(CA)$$

$$\frac{1}{2}(1)\sin(x)<\frac{x}{2}<\frac{1}{2}(1)\tan(x)$$

$$\sin(x)<x<\tan(x)$$

$$\sin(x)<x<\frac{\sin(x)}{\cos(x)}$$





Dividing each expression by $\sin(x)$:

$$1<\frac{x}{\sin(x)}<\frac{1}{\cos(x)}$$

Taking the reciprocal of each expression:

$$1>\frac{\sin(x)}{x}>\cos(x)$$

$$\cos(x)<\frac{\sin(x)}{x}<1$$

As $x\to 0$, $\cos(x)\to 1$

$\therefore$ As $x\to 0$, $\dfrac{\sin(x)}{x}\to 1$

That is, $\lim\limits_{h\to 0}\dfrac{\sin(h)}{h}=1$.

This confirms the earlier observation of its value and that the derivative of $\sin(x)$ is $\cos(x)$.

The derivative of the cosine function

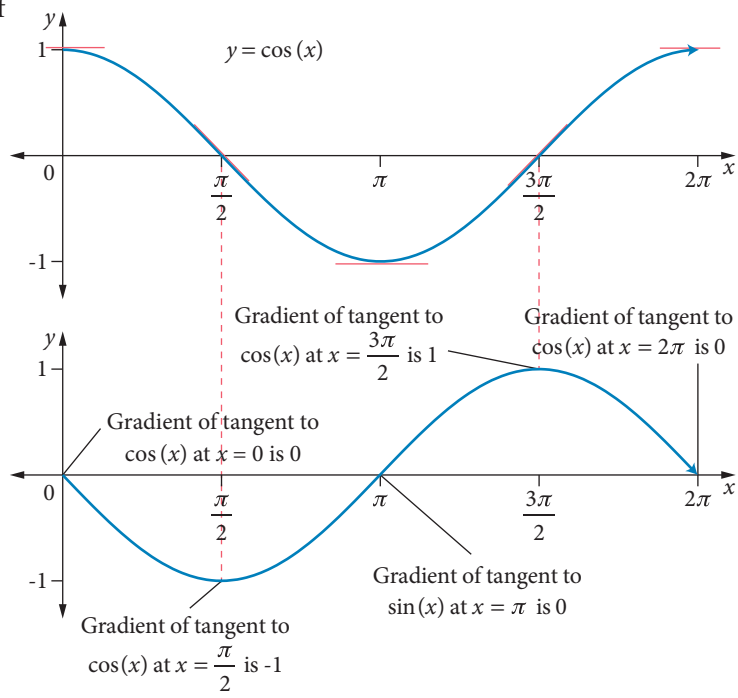
Like the derivative of the sine function, the derivative of the cosine function can be determined informally by drawing a derivative graph. The graph below shows $y=\cos(x)$ along with its tangents at various points.

C Making connections

The derivative of the cosine function

Analyse the graph of $y=A\cos(Bx)$ to determine the equation of the derivative.

From the graph, you can see the gradient of the tangents at $x = \frac{\pi}{2}$ is -1 and $x = \frac{3\pi}{2}$ is 1. The gradient of the tangent at $x = 0$, $x = \pi$ and $x = 2\pi$ is 0. The graph below shows the derivative graph for $\cos(x)$ drawn using these tangents.



From the lower graph, it is reasonable to conclude that the derivative of $\cos(x)$ is $-\sin(x)$. The derivative of $\cos(x)$ can also be determined from the derivative of $\sin(x)$ and the complementary properties of sine and cosine.

$$y = \cos(x)$$
$$= \sin\left(\frac{\pi}{2} - x\right)$$

Using the chain rule:

Let $u = \frac{\pi}{2} - x$ and $y = \sin(u)$.

$$\frac{du}{dx} = -1 \text{ and } \frac{dy}{du} = \cos(u)$$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$
$$= -\cos(u)$$
$$= -\cos\left(\frac{\pi}{2} - x\right)$$
$$= -\sin(x)$$

Additional information

The derivative of the cosine function

This video explains how to obtain the derivative of the cosine function.

The chain rule can be used to determine some useful rules.

$y = \sin(kx)$

Let $u = kx$ and $y = \sin(u)$.

$$\frac{du}{dx} = k \text{ and } \frac{dy}{du} = \cos(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= k\cos(u)$$
$$= k\cos(kx)$$

Similarly, from $y = \cos(kx)$:

Let $u = kx$ and $y = \cos(u)$.

$$\frac{du}{dx} = k \text{ and } \frac{dy}{du} = -\sin(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -k\sin(u)$$
$$= -k\sin(kx)$$

If $y = \sin(x)$, then $\frac{dy}{dx} = \cos(x)$.

If $y = \cos(x)$, then $\frac{dy}{dx} = -\sin(x)$.

$$\frac{d}{dx}(\sin(kx)) = k\cos(kx)$$
$$\frac{d}{dx}(\cos(kx)) = -k\sin(kx)$$

16 Differentiate sine and cosine functions

Differentiate each function.

(a) $y = \sin(2x)$

THINKING	WORKING
Differentiate using the rule: $\frac{d}{dx}(\sin(kx)) = k\cos(kx)$	$y = \sin(2x)$ $\frac{dy}{dx} = 2\cos(2x)$

(b) $y = 3\cos(5x)$

1 Differentiate using the rule: $\frac{d}{dx}(\cos(kx)) = -k\sin(kx)$	$y = 3\cos(5x)$ $\frac{dy}{dx} = 3 \times -5\sin(5x)$
2 Simplify.	$\frac{dy}{dx} = -15\sin(5x)$

(c) $y = 3\sin(-5x)$

1 Differentiate using the rule: $\frac{d}{dx}(\sin(kx)) = k\cos(kx)$	$y = 3\sin(-5x)$ $\frac{dy}{dx} = 3 \times -5\cos(-5x)$
2 Simplify. Use the properties of the unit circle to simplify further.	$\frac{dy}{dx} = 3 \times -5\cos(-5x)$ $= -15\cos(-5x)$ $= -15\cos(5x)$
3 Alternatively, simplify before differentiating by recalling the property $\sin(-x) = -\sin(x)$.	$y = 3\sin(-5x)$ $= -3\sin(5x)$ $\frac{dy}{dx} = -3 \times 5\cos(5x)$ $= -15\cos(5x)$

The chain rule

The chain rule can be used for differentiating composite functions. Where y is a function of u and u is a function of x: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

17 Differentiate sine and cosine functions using the chain rule

Differentiate each function, using the chain rule.

(a) $y = \sin\left(x^2 - 1\right)$

THINKING	WORKING
1 Express the rule as a function of u and differentiate.	$y = \sin\left(x^2 - 1\right)$ Let $u = x^2 - 1$ and $y = \sin(u)$. $\frac{du}{dx} = 2x$ and $\frac{dy}{du} = \cos(u)$
2 Use the chain rule to determine the derivative.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= \cos(u) \times 2x$ $= 2x\cos\left(x^2 - 1\right)$

(b) $y = \sin^3(x)$

1 Express the equation as a function of u and differentiate.	$y = \sin^3(x)$ $= \left(\sin(x)\right)^3$ Let $u = \sin(x)$ and $y = u^3$. $\frac{du}{dx} = \cos(x)$ and $\frac{dy}{du} = 3u^2$
2 Use the chain rule to determine the derivative.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= 3u^2\cos(x)$ $= 3\sin^2(x)\cos(x)$

(c) $y = e^{x + \cos(3x)}$

1 Express the equation as a function of u and differentiate.	$y = e^{x + \cos(3x)}$ Let $u = x + \cos(3x)$ and $y = e^u$. $\frac{du}{dx} = 1 - 3\sin(3x)$ and $\frac{dy}{du} = e^u$
2 Use the chain rule to determine the derivative.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= e^{x + \cos(3x)}\left(1 - 3\sin(3x)\right)$

(d) $y = \sqrt{\cos(3x+2)}$

1 Express the equation as a function of u and differentiate.	$y = \sqrt{\cos(3x+2)}$ Let $u = \cos(3x+2)$ and $y = \sqrt{u}$. $\frac{du}{dx} = -3\sin(3x+2)$ and $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$
2 Use the chain rule to determine the derivative.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= \frac{1}{2}u^{-\frac{1}{2}} \times -3\sin(3x+2)$ $= \frac{-3\sin(3x+2)}{2\sqrt{u}}$ $= \frac{-3\sin(3x+2)}{2\sqrt{\cos(3x+2)}}$

The derivative of the tangent function

The derivative of the tangent function can be determined using the quotient rule $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

$y = \tan(x)$

$= \frac{\sin(x)}{\cos(x)}$

Let $u = \sin(x)$ and $v = \cos(x)$.

$\frac{du}{dx} = \cos(x)$ and $\frac{dv}{dx} = -\sin(x)$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$

$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$

$= \frac{1}{\cos^2(x)}$

For $y = \tan(kx)$:

Let $u = kx$ and $y = \tan(u)$.

$\frac{du}{dx} = k$ and $\frac{dy}{du} = \frac{1}{\cos^2(u)}$

$\frac{dy}{dx} = \frac{k}{\cos^2(u)}$

$= \frac{k}{\cos^2(kx)}$

If $y = \tan(x)$, then $\frac{dy}{dx} = \frac{1}{\cos^2(x)}$.

$\frac{d}{dx}(\tan(kx)) = \frac{k}{\cos^2(kx)}$

18 Differentiate tangent functions

Differentiate $y = 2\tan(3x)$.

THINKING	WORKING
1 Differentiate using the rule: $\frac{d}{dx}(\tan(kx)) = \frac{k}{\cos^2(kx)}$	$y = 2\tan(3x)$
2 Simplify.	$\frac{dy}{dx} = 2 \times \frac{3}{\cos^2(3x)}$
	$= \frac{6}{\cos^2(3x)}$

The product rule

The product rule can be used for differentiating the product of two functions.

If $y = uv$: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

19 Differentiate trigonometric functions using the product rule

Differentiate each function, using the product rule.

(a) $y = x^2\cos(x)$

THINKING	WORKING
1 Recognise the function as the product of two factors in the form $y = uv$.	$y = x^2\cos(x)$ Let $u = x^2$ and $v = \cos(x)$. $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -\sin(x)$
2 Use the product rule to determine the derivative.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $= x^2 \times -\sin(x) + \cos(x) \times (2x)$ $= 2x\cos(x) - x^2\sin(x)$

(b) $y = e^x\tan(x)$

1 Recognise the function as the product of two factors in the form $y = uv$.	$y = e^x\tan(x)$ Let $u = e^x$ and $v = \tan(x)$. $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \frac{1}{\cos^2(x)}$

2 Use the product rule to determine the derivative.

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= e^x \times \frac{1}{\cos^2(x)} + \tan(x) \times e^x \\ &= \frac{e^x}{\cos^2(x)} + e^x \tan(x) \\ &= e^x\left(\frac{1}{\cos^2(x)} + \tan(x)\right)\end{aligned}$$

The quotient rule

The quotient rule can be used for differentiating the quotient of two functions.

If $y = \frac{u}{v}$: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

20 Differentiate trigonometric functions using the quotient rule

Differentiate the function $y = \frac{\sin(3x)}{\cos(2x)}$, using the quotient rule.

THINKING

1 Recognise the function as the quotient of the dividend u and divisor v.

2 Use the quotient rule to determine the derivative.

WORKING

$$y = \frac{\sin(3x)}{\cos(2x)}$$

Let $u = \sin(3x)$ and $v = \cos(2x)$.

$$\frac{du}{dx} = 3\cos(3x) \text{ and } \frac{dv}{dx} = -2\sin(2x)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\cos(2x) \times 3\cos(3x) - \sin(3x) \times -2\sin(2x)}{\cos^2(2x)} \\ &= \frac{3\cos(2x)\cos(3x) + 2\sin(3x)\sin(2x)}{\cos^2(2x)}\end{aligned}$$

EXERCISE 2.4 Derivatives of trigonometric functions

Worked Example

16

1 Differentiate each of the following functions.

(a) $f(x) = \sin(7x)$ (b) $h(x) = -\sin(4x)$ (c) $f(x) = 5\sin(-2x)$

(d) $g(x) = 4\sin\left(\frac{x}{2}\right)$ (e) $f(x) = 2\sin\left(-\frac{x}{2}\right)$ (f) $f(x) = \frac{1}{2}\sin(-5x)$

2 Differentiate each of the following functions.

(a) $f(x) = \cos(4x)$ (b) $g(x) = 2\cos(-5x)$ (c) $f(x) = -\cos(-3x)$

(d) $f(x) = 3\cos\left(\frac{x}{2}\right)$ (e) $h(x) = \cos\left(-\frac{x}{5}\right)$ (f) $f(x) = -\frac{1}{2}\cos(-4x)$

3 Determine an expression for $\frac{d}{dx}\left(\cos(3x) - 2\sin\left(\frac{x}{3}\right)\right)$.

17

4 Differentiate each of the following functions.

(a) $f(x) = \sin(2x+3)$ (b) $h(x) = 3\cos(2-5x)$ (c) $f(x) = \sin\left(x^2 - 3x\right)$

(d) $f(x) = 5\cos\left(\frac{3x-2}{4}\right)$ (e) $f(x) = e^{\cos(3x)}$ (f) $h(x) = \cos\left(e^{-x}\right)$

(g) $f(x) = \sin\left(e^x\right)$ (h) $g(x) = \sin^3(x) + \cos^2(x)$ (i) $f(x) = e^{\cos(x)}$

18

5 Differentiate each of the following functions.

(a) $k(x) = \tan(6x)$ (b) $f(x) = 2\tan(-x)$ (c) $h(x) = 3\tan(-2x)$

(d) $f(x) = 5\tan\left(\frac{x}{4}\right)$ (e) $k(x) = -\frac{2}{3}\tan\left(-\frac{x}{3}\right)$ (f) $f(x) = -2\tan\left(-\frac{2x}{3}\right)$

6 What is the derivative of $f(x) = e^{\sin(2x)}$?

A $2\cos(x)e^{\sin(2x)}$ B $\cos(2x)e^{\sin(2x)}$ C $-2\cos(2x)e^{\sin(2x)}$ D $2\cos(2x)e^{\sin(2x)}$

19

7 Determine an expression for the derivative of each function, using the product rule.

(a) $f(x) = -3x\sin(5x+2)$ (b) $f(x) = -x\sin(1-5x)$

(c) $k(x) = x\tan(x)$ (d) $f(x) = \cos(5x)\sin(3x)$

(e) $f(x) = x^2\sin(3x)$ (f) $f(x) = (2x-3)\cos(3x)$

(g) $h(x) = 3x\sin(1-5x)$

8 For the graph $y = 3\sin(2x)$, determine the value of the gradient at $x = \frac{\pi}{2}$.

9 For the graph $y = \cos\left(\frac{x}{3}\right)$, $-\pi \leq x \leq 3\pi$, determine the coordinates of the points, where the gradient is zero.

10 The amount of money $\$y$ in a savings account is given by $y = 1000 + 100\cos(t) - 200\sin(t)$, where t is the time in years after the initial deposit.

(a) Determine the expression for $\frac{dy}{dt}$.

(b) At what rate is the amount of money in the account changing at the end of the second year?

(c) Calculate the amount of money in the savings account at the end of the second year.

(d) Determine when the gradient of the function is zero, given the interval $[0,10]$. Give your answers correct to 2 decimal places.

11 Consider the function $h(x) = \dfrac{\sin(x)}{e^x}$.

(a) What is the gradient of the function at $x = 0$?

A 1 **B** $\dfrac{1}{e}$ **C** -1 **D** 0

(b) Explain the common error made by a student who calculated $h'(x)$ to be $\dfrac{\cos(x)}{e^x}$.

12 Determine the exact value of $f'(\pi)$ if $f(x) = (x-1)\sin(2x-3)$.

Worked Example 20

13 Differentiate each of the following functions, using the quotient rule.

(a) $k(x) = \dfrac{\sin(2x)}{x+1}$ **(b)** $g(x) = \dfrac{\sin(-3x)}{\cos(2x)}$ **(c)** $h(x) = \dfrac{e^{3x}}{\cos(2x)}$

(d) $f(x) = \dfrac{1-\cos(x)}{1+\cos(x)}$ **(e)** $f(x) = \dfrac{e^{2x}}{\cos(2x)}$ **(f)** $g(x) = \dfrac{\sin^2(2x)}{2x}$

14 Determine an expression for the derivative of each function.

(a) $f(x) = \cos\left(\dfrac{5-x}{x+1}\right)$ **(b)** $h(x) = \dfrac{3}{\sin(x)}$

(c) $f(x) = \tan\left(e^{x^2+1}\right)$ **(d)** $g(x) = \sqrt{x}\tan(3x)$

15 Determine the equation of the tangent to $y = \sin(2x) + \cos^2(x)$ at $x = \dfrac{\pi}{4}$.

16 Consider the graph with rule $y = e^{\sin(3x)}$ for $x \in [1, 2]$. Determine the x-coordinate of the point on the curve at which the tangent to the curve is horizontal.

17 Consider the function $f(x) = \begin{cases} \sin(2x) & 0 \le x \le \dfrac{\pi}{2} \\ -\sin(2x) & \dfrac{\pi}{2} < x \le \pi \end{cases}$

(a) Sketch the graph of $f(x)$.

(b) For what values of x is $f(x)$ not differentiable?

(c) Determine the rule for the piecewise function $f'(x)$.

(d) Determine $f'\left(\dfrac{\pi}{3}\right)$.

(e) Determine the values of x for which the derivative is 0.

18 At the Great Fun Water Park there is a slide named 'Zipper Ride'. A section of this slide has been constructed using a scaled version of the model given by the equation $Z(x) = \sin(x) + \dfrac{1}{2}\sin(2x)$, $\dfrac{\pi}{3} \le x \le \dfrac{5\pi}{3}$. Sketch the graph of this ride and determine at what points along the ride a person will experience the steepest drop. Give your answers correct to 2 decimal places.

19 Create a piecewise function consisting of five line segments to approximate the graph of $y = \sin(x)$ for $[0, 2\pi]$.

2.5 Applications of trigonometric functions

Tangents to a graph

To determine the equation of the tangent to $y = f(x)$ at $x = a$, you must first determine $f(a)$ and $f'(a)$.

21 Determine the equation of a tangent

Determine the equation of the tangent to $f(x) = \cos(x)$ where $x = \frac{\pi}{3}$.

THINKING	WORKING
1 Substitute the given x-value and calculate y. Write the coordinates of the point.	$f(x) = \cos(x)$ $f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ $= \frac{1}{2}$ The coordinates of the point are $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.
2 Determine the equation of the derivative.	$f'(x) = -\sin(x)$
3 Substitute the given x-value to determine the gradient of the tangent.	$m = f'\left(\frac{\pi}{3}\right)$ $= -\sin\left(\frac{\pi}{3}\right)$ $= -\frac{\sqrt{3}}{2}$ The gradient of the tangent at the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is $-\frac{\sqrt{3}}{2}$.
4 Determine the equation of the tangent and simplify.	$y - y_1 = m(x - x_1)$ $y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$ $y = -\frac{\sqrt{3}}{2}x + \frac{\pi\sqrt{3}}{6} + \frac{1}{2}$
5 Interpret the answer.	The equation of the tangent to the curve $f(x) = \cos(x)$ at the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is $y = -\frac{\sqrt{3}}{2}x + \frac{\pi\sqrt{3}}{6} + \frac{1}{2}$.

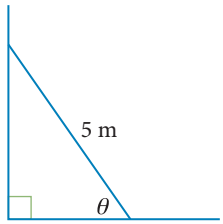
Rates of change involving angles

When more than one variable in a geometrical situation is given as a function of time, you may be able to use the chain rule to help solve rates-of-change problems.

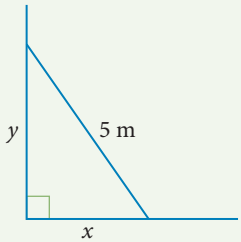
For functions involving angles, always work in radians.

22 Determine a rate of change

A 5 m ladder has its upper end leaning against a vertical wall and its lower end on the horizontal ground. The ladder makes an angle of θ radians with the horizontal ground. The lower end of the ladder is slipping away from the wall at a constant rate of 0.2 m/s. Consider the instant that the base of the ladder is 3 m from the wall.



(a) Determine the speed at which the top of the ladder is descending.

THINKING	WORKING
1 Draw a diagram and define any needed variables.	Let y metres be the vertical distance from the top of the ladder to the horizontal ground and x metres the horizontal distance of the bottom of the ladder from the wall. 
2 Determine which rate needs to be found and what relevant information is given.	Need to determine $\frac{dy}{dt}$, where $x = 3$, given $\frac{dx}{dt} = 0.2$
3 Use the chain rule to determine a relationship that will enable the rate to be calculated.	$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$
4 Use Pythagoras' theorem to determine a relationship between the variables x and y.	$y = \sqrt{5^2 - x^2}$ $= \sqrt{25 - x^2}$
5 Determine the derivative at the required x-value.	$\frac{dy}{dx} = \frac{1}{2}\left(25 - x^2\right)^{-\frac{1}{2}} \times -2x$ $= -\frac{x}{\sqrt{25 - x^2}}$ Where $x = 3$: $\frac{dy}{dx} = -\frac{3}{\sqrt{25 - 3^2}}$ $= -\frac{3}{4}$

6 Use the chain rule and the relevant information to determine the required rate.

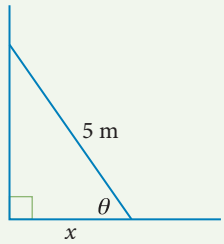
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$= -\frac{3}{4} \times 0.2$$
$$= -\frac{3}{20}$$

7 Interpret the answer.

The ladder is slipping down the wall at a rate of $\frac{3}{20}$ m/s.

(b) Determine the rate at which the angle θ is changing.

1 Draw a diagram and label it with the relevant information.



2 Determine which rate needs to be found and what relevant information is given.

Need to determine $\frac{d\theta}{dt}$, where $x = 3$, given $\frac{dx}{dt} = 0.2$

3 Use the chain rule to determine a relationship that will enable the rate to be calculated.

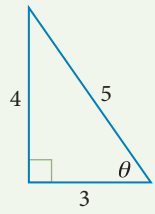
$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$$

4 Use the trigonometric ratios to determine the relationship between x and θ.

$$\cos(\theta) = \frac{x}{5}$$
$$x = 5\cos(\theta)$$

5 Determine the derivative at the required x-value.

$$\frac{dx}{d\theta} = -5\sin(\theta)$$
$$\therefore \frac{d\theta}{dx} = -\frac{1}{5\sin(\theta)}$$



Where $x = 3$, $\sin(\theta) = \frac{4}{5}$

$$\frac{d\theta}{dx} = -\frac{1}{5 \times \frac{4}{5}}$$
$$= -\frac{1}{4}$$

6 Use the chain rule and the relevant information to determine the required rate.	$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$ $= -\frac{1}{4} \times 0.2$ $= -\frac{1}{20}$
7 Interpret the answer.	The angle θ is decreasing at the rate of $\frac{1}{20}$ radians/s.

Sinusoidal shapes

Sine and cosine functions have many applications, from modelling the movement of tides to rock formations and the designing of roller coasters and Ferris wheels.

Periodic fluctuations in medical data can sometimes be modelled by a sine or cosine function.

23 Solve practical trigonometric problems

Thuy's blood pressure P measured in millimetres of mercury (mmHg) at t minutes after arriving at the doctor's surgery, can be approximately modelled by:

$P = 90 - 10\cos\left(\frac{\pi t}{15}\right), 0 \leq t \leq 30.$

(a) Determine the amplitude and period of the function.

THINKING	WORKING
Recall $y = A\cos\big(B(x+C)\big) + D$.	Amplitude: $\lvert A \rvert = 10$ The amplitude is 10 mmHg. $B = \frac{\pi}{15}$ period $= \frac{2\pi}{\frac{\pi}{15}}$ $= 30$ The period is 30 minutes.

(b) Sketch the graph of P for the first 30 minutes.

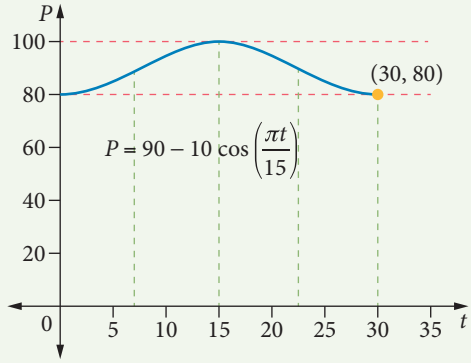
1 Determine the maximum and minimum values of the function.

$$\text{maximum} = 10 + 90 = 100$$
$$\text{minimum} = -10 + 90 = 80$$

2 Sketch the graph.

Mark in the minimum and maximum values of the function and draw the median line. Divide the period by 4 to help determine the t-scale. Sketch one cycle of the graph.

Median line: t-scale: $30 \div 4 = 7.5$



(c) When does Thuy's blood pressure first reach a reading of 85 mmHg?

1 Write an appropriate equation.

$$85 = 90 - 10\cos\left(\frac{\pi t}{15}\right)$$

2 Solve the equation for t, using the smallest value in the domain.

$$10\cos\left(\frac{\pi t}{15}\right) = 5$$
$$\cos\left(\frac{\pi t}{15}\right) = \frac{1}{2}$$
$$\frac{\pi t}{15} = \cos^{-1}\left(\frac{1}{2}\right)$$
$$\frac{\pi t}{15} = \frac{\pi}{3}$$
$$t = 5$$

3 Interpret the answer.

Thuy's blood pressure first reaches a reading of 85 mmHg 5 minutes after arriving at the surgery.

(d) For how long is Thuy's blood pressure higher than 85 mmHg?

1 Use the symmetry of the graph.

$90 - 10\cos\left(\frac{\pi t}{15}\right) = 85$ when $t = 5$ and $t = 25$

$25 - 5 = 20$

2 Interpret the answer.

Thuy's blood pressure is higher than 85 mmHg for 20 minutes.

(e) Determine the rate of change of Thuy's blood pressure after 5 minutes.

1 Determine the derivative of P and substitute the given time.

$$P = 90 - 10\cos\left(\frac{\pi t}{15}\right)$$

$$\frac{dP}{dt} = 10 \times \frac{\pi}{15}\sin\left(\frac{\pi t}{15}\right)$$

$$= \frac{2\pi}{3}\sin\left(\frac{\pi t}{15}\right)$$

When $t = 5$:

$$\frac{dP}{dt} = \frac{2\pi}{3}\sin\left(\frac{\pi}{15} \times 5\right)$$

$$= \frac{2\pi}{3}\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}\pi}{3}$$

2 Interpret the answer.

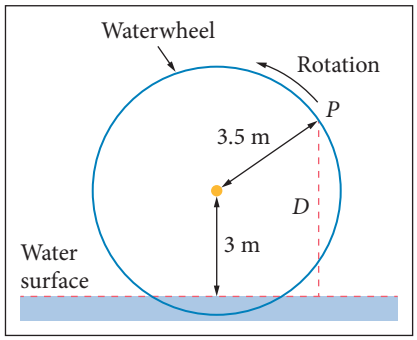
After 5 minutes, Thuy's blood pressure is rising at a rate of $\frac{\sqrt{3}\pi}{3}$ mmHg/min ≈ 1.81.

Circular motion

When a circle is rotated at a constant rate in a vertical plane, the height of any position on the circle can be written as a sine or cosine function of time.

24 Using circular functions to model

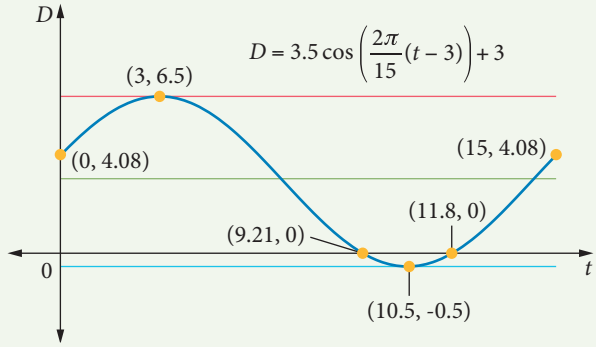
The diagram shows a waterwheel of radius 3.5 m, where P is a point on the edge of the wheel, and D is the height of point P above the surface of the water. The centre of the waterwheel is always 3 m above the surface of the water. It rotates at 4 revolutions per minute. Three seconds after the waterwheel starts turning, P is at its maximum height of 6.5 m above the surface of the water.



(a) Determine the maximum and minimum height of point P.

THINKING	WORKING
Determine the maximum and minimum height.	Maximum = 3 + 3.5 = 6.5 m Minimum = 3 − 3.5 = -0.5 m

(b) The height of point P at time t seconds can be modelled by a function of the form: $D = a\cos\big(b(t-c)\big)+d$. Sketch a graph of the function for one revolution.

1 Using $D = a\cos\big(b(t-c)\big)+d$, determine the value of a.	The amplitude a is the radius of the wheel. $a = 3.5\,\text{m}$
2 Determine the value of b by considering the time taken to complete one cycle (the period).	There are 4 revolutions in 60 seconds. Therefore, there is one revolution in $\frac{60}{4} = 15$ seconds. The period is 15 s. $\text{period} = \frac{2\pi}{b}$ $15 = \frac{2\pi}{b}$ $b = \frac{2\pi}{15}$
3 Determine the median height to determine the value of d.	The median line is at: $d = \frac{6.5+(-0.5)}{2}$ $= 3$
4 Determine the value of c. The function $y = \cos(x)$ starts on a maximum. Therefore, the graph has been translated.	At $t = 3$ the function has a maximum value of 6.5. $\therefore c = 3$
5 Write the equation of the model.	$D = 3.5\cos\left(\frac{2\pi}{15}(t-3)\right)+3$
6 Use technology to plot the graph. Write the value of endpoints and axis intercepts.	

(c) How high above or below the water's surface will point P be after 11 seconds? Give your answer correct to 2 decimal places.

1 Trace the value from the graph.	$D(11) = -0.42$
2 Interpret the answer.	After 11 seconds, point P is 0.42 m below the water.

(d) At what time does the point P first enter the water? Give your answer correct to 2 decimal places.

1 Read the first t-intercept from the graph.	The coordinates of the t-intercept are $(9.21, 0)$.
2 Interpret the answer.	Point P enters the water after 9.21 seconds.

EXERCISE 2.5 Applications of trigonometric functions

Worked Example 21

1 Determine the equation of the tangent to each of the curves at the corresponding point.

(a) $y = \sin(x), x = \frac{\pi}{6}$ (b) $y = 3\cos(2x), x = \frac{\pi}{2}$ (c) $y = \tan(2x), x = \frac{\pi}{2}$

2 Consider the function $f(x) = \tan(x)$.

(a) What is the equation of the tangent to the curve $f(x)$ at the point $x = \frac{\pi}{3}$?

A $y = 4x + \frac{\pi}{3} - 4\sqrt{3}$ B $y = 4x - \frac{4\pi}{3} - \sqrt{3}$ C $y = 4x + \sqrt{3} - \frac{4\pi}{3}$ D $y = 4x - \frac{4\pi}{3}$

(b) Explain the common errors made by a student who chose any of the incorrect options in part (a).

3 Consider the function $f(x) = \sin(x) + \cos(x), 0 \le x \le 2\pi$.

(a) Calculate each of the following.

(i) $f(0)$ (ii) the values of x such that $f(x) = 0$

(iii) the values of x such that $f'(x) = 0$

(b) Determine the coordinates of the stationary points of $f(x)$.

4 Determine the local maximum and minimum values of each function.

(a) $f(x) = 1 + 2\cos(x) - 2\cos^2(x), 0 \le x \le 2\pi$ (b) $f(x) = 2\cos(x) + 2\cos^2(x), 0 \le x \le 2\pi$

(c) $f(x) = \sin^2(x), 0 \le x \le 2\pi$

5 The position of a particle as it moves horizontally is described by the equation: $x = 2\sin(t) - \cos(t)$, $0 \le t \le 2\pi$, where x is the displacement in metres and t is the time in seconds.

(a) Determine the maximum distance to the left and right of the origin, correct to 2 decimal places.

(b) Determine the equation of the velocity, $v(t) = x'(t)$.

(c) Determine the rate of change of position at $t = \pi$ seconds. Give a meaning to this value.

6 The length in centimetres L of a tiger snake, t weeks after it is born, is modelled approximately by the function: $L = 1 + 5t + 2\sin\left(\frac{\pi t}{3}\right), 0 \le t \le 24$.

(a) Determine the length of the snake at birth.

(b) Determine the length of the snake at 24 weeks.

(c) Determine an expression for the rate of growth at any given time t.

(d) What is the minimum rate of growth and when does it occur? Give your answer correct to 2 decimal places.

7 The height h in metres above the ground of a person on a Ferris wheel t seconds after the ride has started is modelled by the function: $h = 6 - 4\cos\left(\frac{\pi t}{12}\right)$, $0 \leq t \leq 60$.

(a) Determine the initial height of the person from the ground.

(b) How long does the wheel take to complete a full rotation?

(c) What is the maximum and minimum height of the person from the ground?

(d) Sketch the graph of the function for the first 60 seconds.

(e) Calculate the average rate of change of height over the first 10 seconds.

(f) Calculate the rate of change of height at $t = 10$.

8 A particle moving in a straight line has a displacement function $s(t) = \sin(2\pi t) + 2\sin(\pi t)$, $t \geq 0$, where s is the displacement in centimetres and t is the time in seconds. In the first cycle of its motion, the particle is stationary for which values of t?

Worked Example

A $0, 1, 2$ B $\frac{1}{2}, 1, \frac{3}{2}$ C $\frac{1}{3}, 1, \frac{5}{3}$ D $0, \frac{2}{3}, \frac{4}{3}, 2$

23

9 Mary, the captain of the *Merry Ferry*, knows that the height h in metres of the tide in the harbour t hours after midnight can be modelled approximately by the function: $h = 6 + 3\cos\left(\frac{\pi t}{6}\right)$ for $0 \leq t \leq 24$.

(a) What is the height of the high tide and when does it occur?

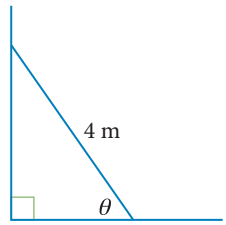
(b) What is the height of the low tide and when does it occur?

(c) Sketch the graph of the function.

(d) Mary knows that her ferry needs a depth of at least 7 metres to enter the harbour. Between what hours is she able to bring the ferry into the harbour?

22

10 A 4 m pole is leaning against a vertical wall. The lower end of the pole moves away from the wall at a constant rate of a m/s. Consider the instant that the bottom of the pole is 2 m from the wall and the pole makes an angle of θ with the horizontal ground.



(a) Determine the speed at which the top of the pole is descending.

(b) Determine the rate at which the angle θ is changing. Give your answer correct to 2 decimal places.

24

11 A Ferris wheel with a diameter of 10 m completes one rotation every 10 minutes. Riders board the Ferris wheel from a platform 2 m above the ground. The height h in metres above ground of a point P on the edge of the Ferris wheel at any time t minutes after the ride has started can be modelled by an equation of the form: $h = a\cos(bt) + c$, $0 \leq t \leq 15$.

(a) What are the lowest and highest heights of point P?

(b) Determine the equation of the model.

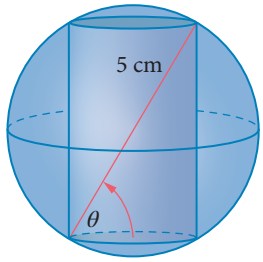
(c) Sketch the graph of h.

(d) How high above the ground will point P be at 10 minutes?

(e) At 11 minutes, will point P be going up or down?

(f) Determine the first time that point P is 10 m above the ground. Give your answer correct to the nearest second.

12 A cylinder is inscribed in a sphere as shown in the diagram. The radius of the sphere is 5 m. The angle between the diagonal of a rectangle representing the cylinder and the base of the cylinder is θ. Determine the dimensions of the cylinder with the greatest volume, to the nearest millimetre, and the maximum volume of the cylinder, to the nearest cubic centimetre.



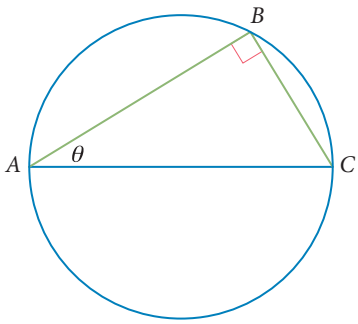
13 Brisbane's average monthly maximum and minimum temperatures, in degrees Celsius are given in the table below:

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Max	30	29	29	27	24	21	21	22	24	26	28	29
Min	22	21	20	18	15	12	10	11	14	16	19	21

(a) Use the statistics mode on your calculator to create sine functions to model each set of data, as temperature against day of the year. For example, the middle day of June would be found as follows: $31(\text{Jan}) + 28(\text{Feb}) + 31(\text{Mar}) + 30(\text{Apr}) + 31(\text{May}) + 15(\text{Jun}) = 166$. Plot both sets of raw data and the model graphs on the same set of axes.

(b) Evaluate the suitability of the sine graph to model the data. Describe the strengths and weaknesses of the model.

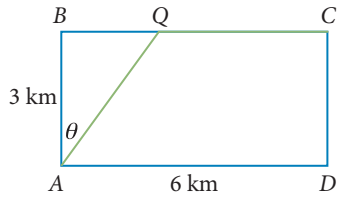
14 Chris swims at 2 km/h from A to B across a circular lake of radius 2 km, then walks at 3 km/h around the edge of the lake from B to C, as shown in the diagram.



(a) Determine an expression for the total time T hours that Chris takes to travel from A to B and then to C, in terms of θ.

(b) Calculate the maximum time T it would take Chris to travel from A to B and then to C. Give your answer correct to 1 decimal place.

15 Each morning Andrew goes for a walk along a rectangular field 3 km wide and 6 km long, as shown on the diagram. He starts at point A and walks at a constant speed of 4 km/h to point Q at an angle of θ from $\overline{AB}$, where θ is an acute angle. From point Q he walks at a constant speed of 5 km/h along the road to his work at point C.



(a) Determine an expression for the total time T in hours that Andrew takes to walk from A to Q and then to C, in terms of θ.

(b) Andrew allows 100 minutes to get to work. Determine whether it is possible for Andrew to arrive to work on time. Give the optimum angle in degrees, to the nearest degree, for his path.

2

Summary

Radians and degrees

To convert from: degrees to radians, multiply by $\frac{\pi}{180°}$.

To convert from: radians to degrees, multiply by $\frac{180°}{\pi}$.

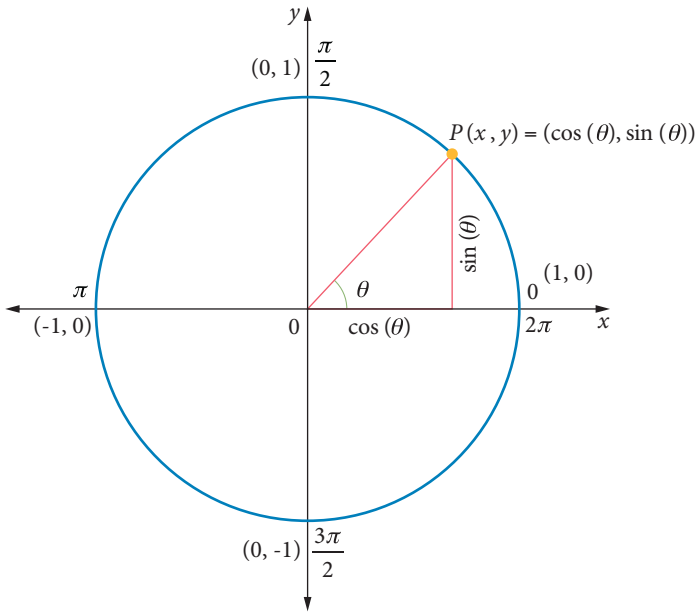
The unit circle

$\cos(\theta) = x$

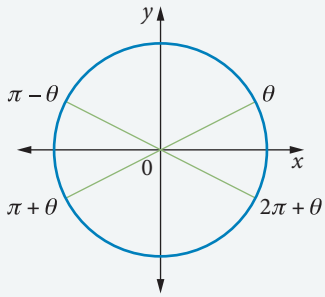
$\sin(\theta) = y$

$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

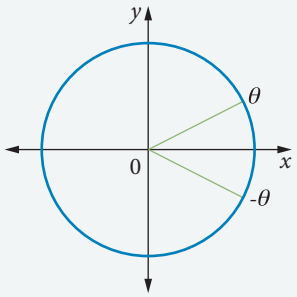
$\cos^2(\theta) + \sin^2(\theta) = 1$



Symmetry properties

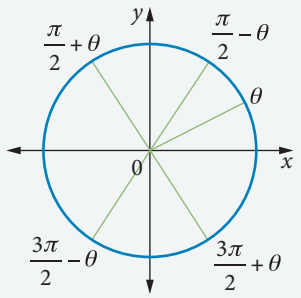


$\cos(\pi - \theta) = -\cos(\theta)$
$\sin(\pi - \theta) = \sin(\theta)$
$\tan(\pi - \theta) = -\tan(\theta)$
$\cos(\pi + \theta) = -\cos(\theta)$
$\sin(\pi + \theta) = -\sin(\theta)$
$\tan(\pi + \theta) = \tan(\theta)$
$\cos(2\pi - \theta) = \cos(\theta)$
$\sin(2\pi - \theta) = -\sin(\theta)$
$\tan(2\pi - \theta) = -\tan(\theta)$



$\cos(-\theta) = \cos(\theta)$
$\sin(-\theta) = -\sin(\theta)$
$\tan(-\theta) = -\tan(\theta)$

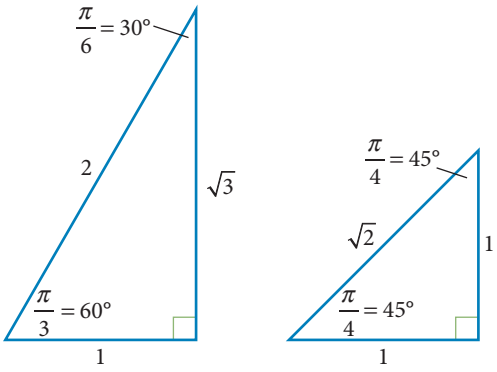
Complementary properties



$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$
$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$
$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$
$\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta)$

$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin(\theta)$
$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos(\theta)$
$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin(\theta)$
$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos(\theta)$

Reference triangles and exact values



$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$	$\tan\left(\frac{\pi}{4}\right) = 1$	$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

General solution of trigonometric equations

In general, consider $a > 0$.

- If $\sin(x) = a$, where $\sin^{-1}(a) = \theta$ represents the reference angle in the first quadrant: $x = 2n\pi + \theta$ or $x = (2n+1)\pi - \theta$ where $n \in \mathbb{Z}$.
- If $\cos(x) = a$, where $\cos^{-1}(a) = \theta$ represents the reference angle in the first quadrant: $x = 2n\pi \pm \theta$ where $n \in \mathbb{Z}$.
- If $\tan(x) = a$, where $\tan^{-1}(a) = \theta$ represents the reference angle in the first quadrant: $x = n\pi + \theta$ where $n \in \mathbb{Z}$.

Transformations of the sine and cosine functions

$y = A\sin\big(B(x+C)\big) + D$ or $y = A\cos\big(B(x+C)\big) + D$ for $B > 0$

Amplitude: $|A|$ (dilation by a factor of $|A|$ parallel to the y-axis)

Period: $\frac{2\pi}{B}$ (dilation by a factor of $\frac{1}{B}$ parallel to the x-axis, producing B cycles between 0 and 2π)

C: a horizontal translation of C units in the negative x-direction (a phase change)

D: a vertical translation of D units in the positive y-direction

The median line is $x = D$.

Minimum y-value: $-|A| + D$

Maximum y-value: $|A| + D$

Where A is negative, there is a reflection in the x-axis.

Graphing tangent functions

$y = A\tan\left(B\left(x + C\right)\right) + D$ for $B > 0$

Period: $\frac{\pi}{B}$

C: a horizontal translation of C units in the negative x-direction (a phase change)

D: a vertical translation of D units in the positive y-direction

Initial asymptote, solve $B\left(x + C\right) = \frac{\pi}{2}$, then add or subtract $\frac{\pi}{B}$ to find others.

Where A is negative, there is a reflection in the x-axis.

Differentiating trigonometric functions

$\lim_{h \to 0} \frac{\sin\left(h\right)}{h} = 1$

If $y = \sin\left(x\right)$, then $\frac{dy}{dx} = \cos\left(x\right)$.

If $y = \cos\left(x\right)$, then $\frac{dy}{dx} = -\sin\left(x\right)$.

If $y = \tan\left(x\right)$, then $\frac{dy}{dx} = \frac{1}{\cos^2\left(x\right)}$.

Also, $\frac{d}{dx}\left(\sin\left(kx\right)\right) = k\cos\left(kx\right)$, $\frac{d}{dx}\left(\cos\left(kx\right)\right) = -k\sin\left(kx\right)$, $\frac{d}{dx}\left(\tan\left(kx\right)\right) = \frac{k}{\cos^2\left(kx\right)}$

Rules for differentiation

The chain rule

Where y is a function of u and u is a function of x: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

The product rule

If $y = uv$: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

The quotient rule

If $y = \frac{u}{v}$: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Tangents to a graph

Substitute $x_1 = a$, $y_1 = f\left(a\right)$ and $m = f'\left(a\right)$ into $y - y_1 = m\left(x - x_1\right)$.

Rates of change

When more than one variable is given as a function of time, use the chain rule to help solve rates-of-change problems.

Modelling data

Periodic fluctuations in data can sometimes be modelled by a sine or cosine function.

When a circle is rotated at a constant rate in a vertical plane, the height of any position on the circle can be written as a sine or cosine function of time.

Chapter review 2

1 Calculate the exact value of $\sin\left(\frac{13\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)$. Exercise 2.1

2 Determine the exact value of each of the following. Exercise 2.1

(a) $\cos(5\pi)$ (b) $\tan\left(-\frac{5\pi}{2}\right)$

3 Calculate $\cos\left(\frac{\pi}{4}\right) + \sin\left(-\frac{5\pi}{6}\right)$. Exercise 2.1

4 Given that $f(x) = a\sin(x) + b\cos(x)$ and $f(x) = f(-x)$ for all $x \in \mathbb{R}$, which of the following statements is true? Exercise 2.1

A $a = -b$ B $a = 0$ C $b = 0$ D $a = b$

5 Solve the following equations. Exercise 2.2

(a) $\tan(x) = -1$ (b) $\sin(3x) = \frac{1}{2}$

6 Solve the equation $\cos^2(x) - 5\cos(x) = 0$ for the interval $0 \le x \le \pi$. Exercise 2.2

7 What is the solution of $\sin^3(x) - 4\sin(x) = 0$? Exercise 2.2

A $x = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$ B $x = \pi + 2n\pi, n \in \mathbb{Z}$

C $x = n\pi, n \in \mathbb{Z}$ D $x = \frac{\pi}{2} + (2n+1), n \in \mathbb{Z}$

8 Determine the number of solutions for the equation $\sin(3x) = -\cos(3x)$, $-\frac{2\pi}{3} \le x \le \frac{4\pi}{3}$. Exercise 2.2

9 What is the maximum value of $f(x) = \frac{7}{5 + 2\sin(x)}$? Exercise 2.3

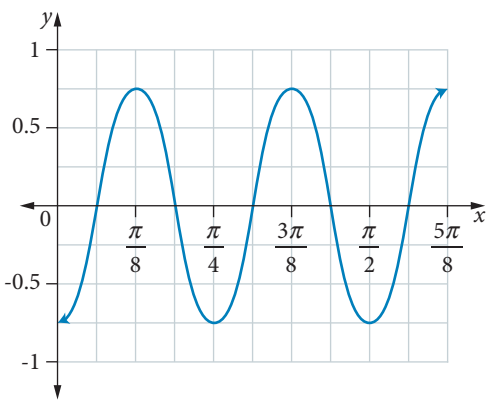
10 Describe how the graph of $y = \cos(x)$ is transformed to obtain the graphs of the following. Exercise 2.3

(a) $y = \frac{1}{2}\cos(3x)$ (b) $y = 1 - \cos(5x)$

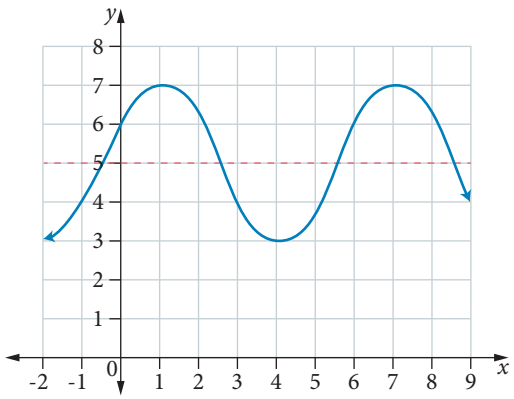
(c) $y = 2\cos\left(4x - \frac{\pi}{3}\right)$ (d) $-(y-1) = \cos(2\pi x)$

11 Determine possible cosine functions for each of the following graphs. Exercise 2.3

(a)



(b)



12 Sketch the graph of $y = 3\tan(5x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, labelling the x-intercepts and the asymptotes. Exercise 2.3

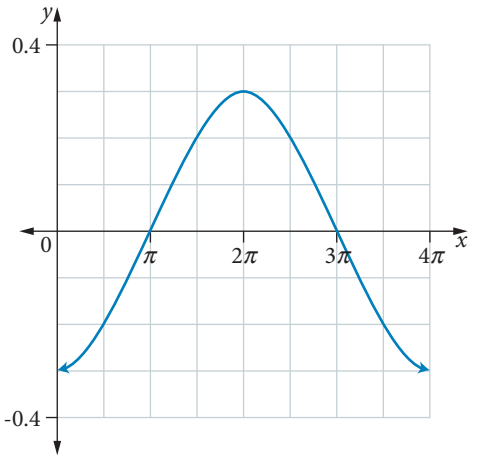
Exercise 2.3 **13** Choose a possible equation for each graph shown.

(a) **A** $y = 0.3\cos\left(\frac{x}{2}\right)$

B $y = -0.3\cos\left(\frac{x}{2}\right)$

C $y = 0.3\cos(2x)$

D $y = -0.3\cos(2x)$

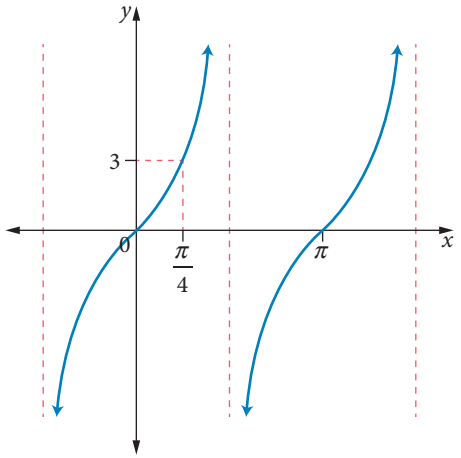


(b) **A** $y = \tan(3x)$

B $y = 3\tan(3x)$

C $y = \tan\left(\frac{x}{3}\right)$

D $y = 3\tan(x)$

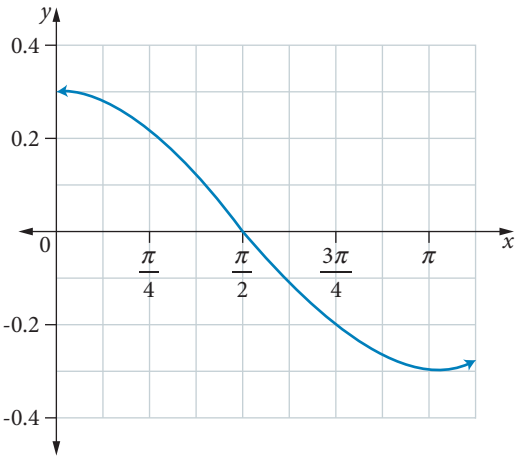


(c) **A** $y = 0.3\sin(x)$

B $y = 0.3\sin\left(x + \frac{\pi}{2}\right)$

C $y = 0.3\sin(x)$

D $y = 0.3\sin\left(x - \frac{\pi}{2}\right)$



Exercise 2.4 **14** Determine an expression for the derivative of each of the following.

(a) $y = 2\cos\left(4x - \frac{\pi}{6}\right)$ **(b)** $y = 3e^{x}\cos(3x + 2)$

Exercise 2.4 **15** The position of a particle as it moves horizontally is described by the equation: $x = \sin^2(t) - 2\cos^2(t)$, $-\pi \leq t \leq \pi$, where x is the distance in metres to the right of an origin and t is the time in seconds.

(a) Determine the equation of the velocity of the particle $\frac{dx}{dt}$.

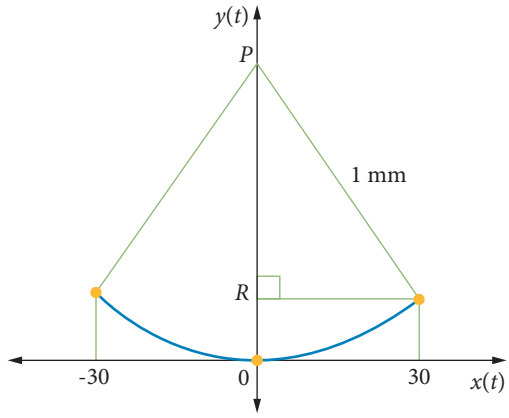
(b) Determine the maximum distance the particle travels to the left and to the right of the origin.

16 If $f(x) = \sin^2(3x)$, then which expression represents $f'(x)$? Exercise 2.4

A $-6\cos^2(3x)$ **B** $-6\sin(3x)\cos(3x)$ **C** $6\sin(3x)\cos(3x)$ **D** $3\sin(3x)\cos(3x)$

17 Determine the gradient of the tangent for the function, $y = \tan\left(\frac{5x}{4}\right)$ at the point $(\pi, -1)$. Exercise 2.4

18 A pendulum pictured in the diagram has a metal bob swinging freely. The length of the pendulum is 1 m and its horizontal position x, measured from the central position, varies between -30 cm and 30 cm. The position function $x(t)$ can be modelled by $x(t) = 30\sin(\pi t)$, where t is the time in seconds. Assume friction and air resistance are negligible. Exercise 2.5



(a) Determine the rule for $y(t)$, which gives the vertical position in centimetres of the pendulum's bob as a function of time.

(b) Determine the minimum and maximum values of $y(t)$ in centimetres, correct to 1 decimal place.

19 The authority that runs the reservoir that supplies water to the residents of a nearby town knows that in summer the demand D, in cubic metres of water, made by the residents of the town can be modelled by: $D = 120 + 60\sin\left(\frac{\pi t}{90}\right)$, $0 \le t \le 90$, where t is the number of days since the start of summer. Exercise 2.5

(a) Sketch the graph of D.

(b) What is the maximum and minimum water demand made by the residents during summer?

(c) What was the water demand on the tenth day of summer, correct to the nearest cubic metre?

(d) On which days in summer was the water demand 150 m^3?

(e) For how many days in summer was the water demand more than 150 m^3?

20 If $\sin(x) = \frac{2}{3}$ and x is in the second quadrant, determine the exact value of $\cos(x)$ and $\tan(x)$. Exercise 2.1

21 Determine all solutions to each of the following equations. Exercise 2.2

(a) $2\cos(2x) + \sqrt{2} = 0$, $-\pi \le x \le \pi$ (b) $1 = -\sqrt{3}\tan\left(x - \frac{\pi}{6}\right)$, $0 \le x \le 2\pi$

22 What are the points of intersection of the functions $y = 4\sin(x) + 1$ and $y = 2\sin(x) + 2$, $0 \le x \le 2\pi$? Exercise 2.2

A $\left(\frac{\pi}{6}, -3\right), \left(\frac{\pi}{2}, -3\right)$ **B** $\left(\frac{\pi}{6}, -3\right), \left(\frac{11\pi}{6}, -3\right)$

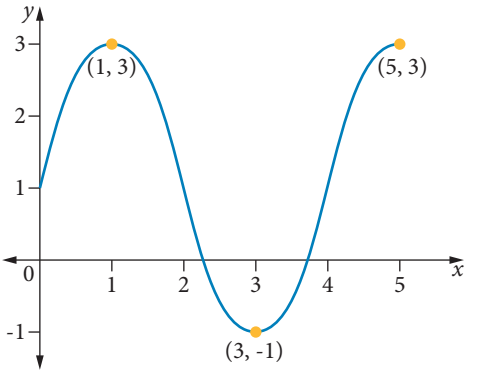
C $\left(\frac{\pi}{6}, 3\right), \left(\frac{5\pi}{6}, 3\right)$ **D** $\left(\frac{\pi}{6}, 3\right), \left(\frac{11\pi}{6}, -3\right)$

23 Sketch graphs of the following. Clearly label the axis intercepts and endpoints. Exercise 2.3

(a) $y = 4\sin(3x) + 2$, $0 \le x \le \frac{2\pi}{3}$ (b) $y = 5\sin\left(\frac{\pi}{3} - x\right)$, $-\pi \le x \le \pi$

Exercise 2.4

24 The diagram shows the graph of the function $f(x) = A\sin(Bx) + D$, for $0 \le x \le 5$.



(a) Determine the actual equation for the function.

(b) The line $y = k - \pi x$ is a tangent to the graph. Determine the exact value of k.

Exercise 2.4

25 For $f(x) = \sin^2(x) + \cos(x)$, $0 \le x \le 3$, determine the coordinates of the maximum point of the graph.

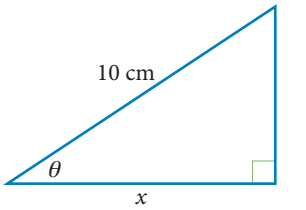
Exercise 2.4

26 What is the gradient of the function, $y = e^{2x}\cos\left(3x - \frac{\pi}{3}\right)$ at the point $\left(0, \frac{1}{2}\right)$?

A $\frac{\sqrt{3}+2}{2}$ **B** $\frac{3\sqrt{3}+2}{2}$ **C** $\frac{2}{3\sqrt{3}+2}$ **D** $\frac{2}{3\sqrt{3}-2}$

Exercise 2.5

27 Determine the maximum perimeter of a right-angled triangle with hypotenuse 10 cm and angle θ as shown. Give your answer as an exact value.



Exercise 2.5

28 A population of wombats in a particular habitat varies periodically. A study of the wombat population started in 1995. In the third year of study the population was at a minimum of 200, while in the eighth year it was a maximum of 500. The wombats in this habitat are considered an endangered species if their population drops below 300. It is believed that the population of wombats over a 14-year period can be modelled by a function of the form: $P = a\sin(b(t+c)) + d$.

(a) Given that $a > 0$, determine the rule for P.

(b) Sketch the graph of P.

(c) For how many years is the population of wombats above 400? Give your answer correct to 2 decimal places.

(d) For how many years is the population of wombats considered endangered? Give your answer correct to 2 decimal places.

Exercise 2.3

29 Consider the following functions: $f(x) = \frac{5x}{3} + 1$ and $g(x) = 3\cos\left(\frac{x}{2}\right) - 1$. The two functions are combined to create a composite function $h(x) = g(f(x))$. Determine the period and the range of $h(x)$.

Exercise 2.5

30 The depth D metres of the water at the entrance to a harbour at t hours after midnight on a particular day is given by: $D = 9 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$.

(a) Determine the exact depth of the water at 4 pm.

(b) Determine the times when the depth is 7.5 m or less.

(c) Boats that need a depth of h metres are allowed to enter the harbour only if the depth of the water at the entrance is at least h metres continuously for one hour. Calculate the greatest value of h for which this condition can be satisfied. Give your answer correct to 1 decimal place.

31 Each morning Lina would like to wake up at 7 am, leave for work at 8 am and return home at 6 pm. When the internal house temperature exceeds the external temperature by at least 2°C, she wakes up on time, leaves home and arrives at work at her preferred time. Otherwise, she is late waking up and leaving home and arrives at work 15 minutes late. On a particular day the internal temperature T_i °C of Lina's house t hours after midnight is given by: $T_i = 18 - 6\cos\left(\frac{\pi t}{12}\right)$. The external temperature T_e °C at t hours after midnight is given by: $T_e = 17 - 4\cos\left(\frac{\pi t}{12}\right)$. Exercise 2.5

(a) Determine the internal and external temperatures of the house at 6 am.

(b) Determine the maximum difference in temperature when the internal temperature is greater than the external temperature and when it is less than the external temperature.

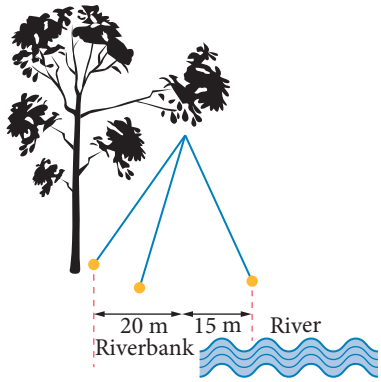
(c) Sketch the graphs of the temperatures T_i and T_e and $D = T_i - T_e$ on the same set of axes.

(d) On this particular morning, does Lina arrive at work on time?

(e) Determine the times when the internal temperature of the house is the same as the external temperature.

(f) Use calculus to determine when the difference between the internal and external temperature is the greatest.

32 One day while camping near the river's edge, Carzan decides to swing on a tyre swing. He moves back and forth alternately over land and water. His friend Jane decides to model his motion with a curve of the form: $D = a\sin\big(b(t+c)\big) + d$, where D is horizontal displacement from the water's edge, and is positive when Carzan is over water. Jane starts to time and measure his movement. She notices that after 2 seconds Carzan is above land at one end of the swing, 20 m from the riverbank. She also determines that after 5 seconds he reaches the other end of the swing, over water 15 m from the riverbank. Exercise 2.5



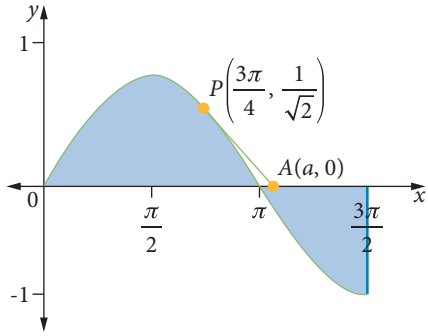
(a) Determine an equation expressing Carzan's distance D from the riverbank after t seconds.

(b) Sketch the graph of D for the first 16 seconds.

(c) Where was Carzan when Jane started timing him?

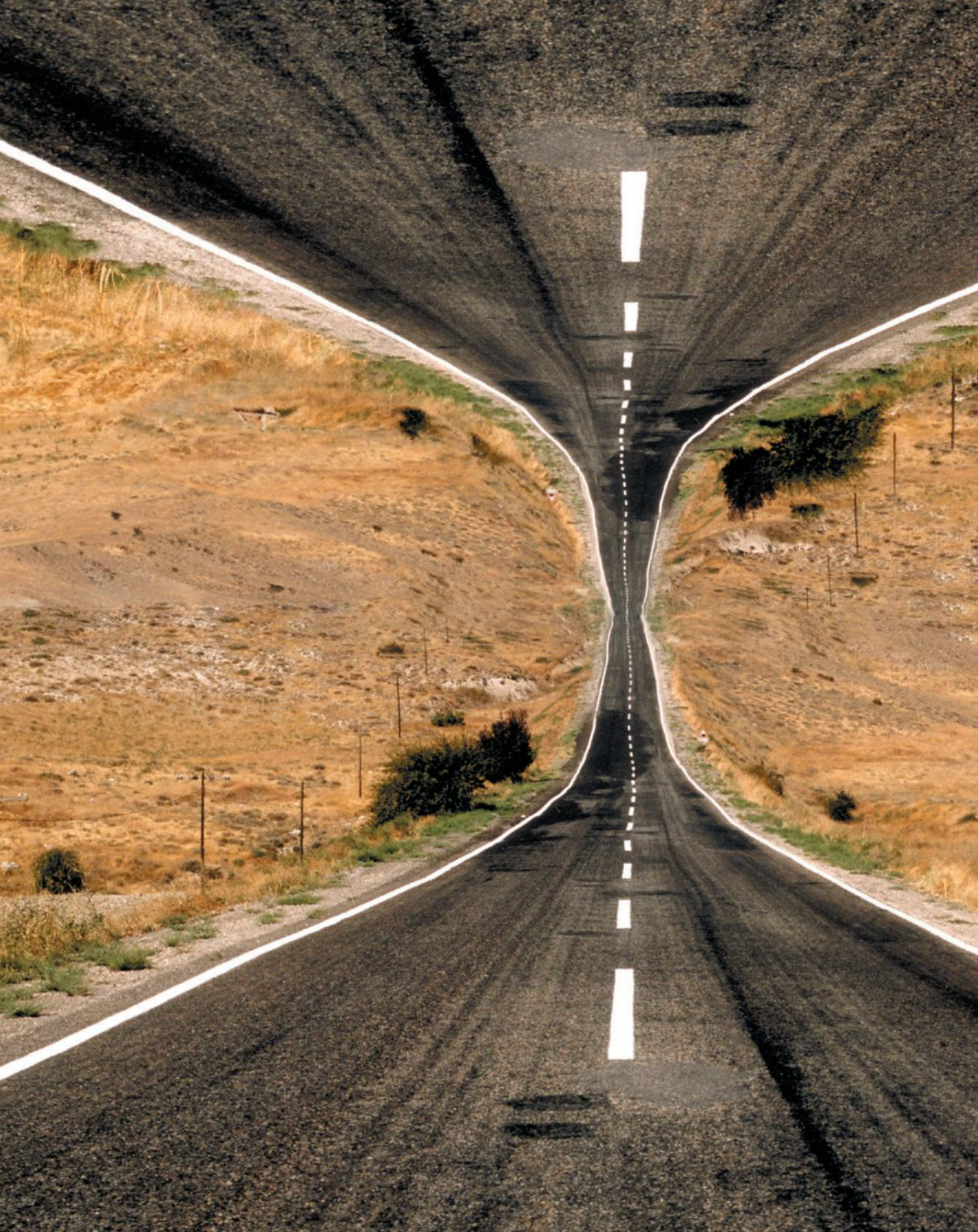
(d) Determine the earliest time t for which Carzan is directly over the riverbank. Give your answer correct to 2 decimal places.

33 Mike Amased is designing an amusement park that is to be built on flat ground. In his initial design he bases his design on the function $y = \sin(x)$, $0 \le x \le \frac{3\pi}{2}$. The shaded regions each represent a separate attraction. Exercise 2.5



(a) To move efficiently through the first two attractions, he includes a straight path PA, which is a tangent to the graph of $y = \sin(x)$ at $P\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$, as shown in the diagram. A has coordinates $(a, 0)$. Determine the value of a.

(b) Mike Amased adapts his plan for the amusement park. He uses a transformation of the plane defined as a dilation of a factor p parallel to the y-axis and a dilation of q parallel to the x-axis. Given $P'\left(\frac{3\pi}{8}, \frac{2}{\sqrt{2}}\right)$ is the image of P, determine the image equation under this transformation.



3

Logarithmic functions and their derivatives

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3

Recall

Recognise the index laws

1 Write each expression as a single power, if possible.

(a) $a^x \times b^y$ (b) $\dfrac{a^x}{a^y}$ (c) $\dfrac{1}{a^x}$

Use the index laws to simplify expressions

2 Simplify each of the following expressions using the index laws.

(a) $\left(3x^2y^{-3}\right)^2$ (b) $\sqrt{x} \times \sqrt[3]{x}$ (c) $3^{2m+1} \times 9^{m-2}$

Calculating numerical powers using the index laws

3 Calculate the value of each of the following expressions.

(a) 5^0 (b) 3^{-2} (c) $8^{\frac{2}{3}}$

Solve exponential equations using the index laws

4 Determine the exact solution for each of the following equations.

(a) $2^x = 8$ (b) $x^{\frac{2}{3}} = 9$ (c) $3^{2x-1} = 9^{-x+2}$

Sketch exponential graphs

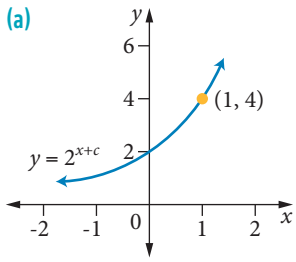
5 For each of the following functions, sketch the graph showing the x- and y-intercepts and the equation of the asymptote.

(a) $y = 2^x - 4$ (b) $y = 2^{x-1} - 1$

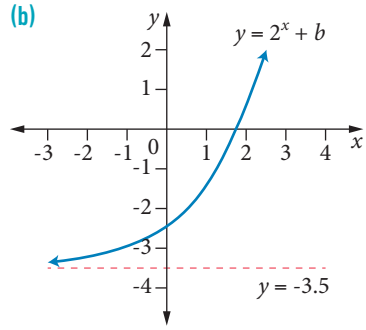
Determine the rule for an exponential graph

6 For each of the following graphs transformed from $y = 2^x$, determine the value of the unknown.

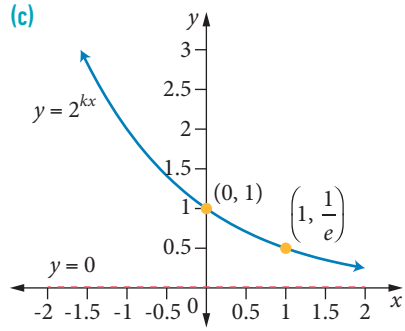
(a)



(b)



(c)



Logarithmic functions and their properties 3.1

Indices and logarithms

If a number is written in exponential (index) form, then the index can also be called a *logarithm*. Consider the statement written in exponential form:

Index
$2^3 = 8$
Base

This can also be written in logarithmic form:

$\log_2(8) = 3$ ← Index
Base

This is read as 'log to the base 2 of 8 is equal to 3'. Note that the base number is written as a subscript. Taking the logarithm of a number is the inverse process of raising a number to a power.

Additional information

Logarithmic functions

Watch the video for a further explanation of logarithmic functions.

Index
In general for $a > 0$, $a \neq 1$ and $y > 0$, $a^x = y \Leftrightarrow \log_a(y) = x$ ← Index
Base Base

Logarithms to base 10 and *e*

Logarithms to base 10 are also called *common logarithms*; the notation $\log_{10}(x)$ is used for these logarithms. Note that $\log(x)$ assumes that the base is 10. Logarithms to base *e* are also called *natural logarithms*. Both the notations $\log_e(x)$ and $\ln(x)$ are commonly used. Most calculators will calculate logarithms to base 10, base *e* and some others.

Logarithmic properties and laws

For each index law, there is a corresponding logarithmic law. The index laws can be used to establish these rules.

The logarithm of 1 (property): since $a^0 = 1$, then $\log_a(1) = 0$.

For example: $\log_{10}(1) = 0$.

Graphically this indicates that the coordinates of the x-intercept of the function $y = \log_a(x)$ are $(1, 0)$.

The logarithm of a number to its own base (property): since $a^1 = a$, then $\log_a(a) = 1$.

For example: $\log_2(2) = 1$.

1 Use the properties of logarithms

Simplify the following expressions.

(a) $3\log_5(1)$

THINKING	WORKING
The logarithm of 1 to any base is zero.	$3\log_5(1) = 3 \times 0$ $= 0$

(b) $5\log_4(4)$

The logarithm of a number to its own base is 1.	$5\log_4(4) = 5 \times 1$ $= 5$

Addition and subtraction of logarithms

Law for addition of logarithms

Let a, m, and n be positive numbers such that: $a^x = m \Leftrightarrow \log_a(m) = x$ and $a^y = n \Leftrightarrow \log_a(n) = y$	$a^x \times a^y = mn$ $\therefore a^{x+y} = mn \Leftrightarrow \log_a(mn) = x + y$ Also, since $x = \log_a(m)$ and $y = \log_a(n)$: $\log_a(mn) = \log_a(m) + \log_a(n)$	For example: $\log_{10}(5) + \log_{10}(3) = \log_{10}(5 \times 3)$ $= \log_{10}(15)$

In general: $\log_a(m) + \log_a(n) = \log_a(mn)$

Law for subtraction of logarithms

Let a, m and n be positive numbers such that: $a^x = m \Leftrightarrow \log_a(m) = x$ and $a^y = n \Leftrightarrow \log_a(n) = y$	$\frac{a^x}{a^y} = \frac{m}{n}$ $\therefore a^{x-y} = \frac{m}{n} \Leftrightarrow \log_a\left(\frac{m}{n}\right) = x - y$ Also, since $x = \log_a(m)$ and $y = \log_a(n)$: $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$	For example: $\log_3(10) - \log_3(2) = \log_a\left(\frac{10}{2}\right)$ $= \log_3(5)$

In general: $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$

2 Use the addition and subtraction laws for logarithms

Simplify the following expressions.

(a) $\log_2(3) + \log_2(5)$

THINKING	WORKING
1 Recall the law for addition of logarithms.	$\log_a(m) + \log_a(n) = \log_a(mn)$
2 Substitute the values and simplify.	$\log_2(3) + \log_2(5) = \log_2(3 \times 5)$ $= \log_2(15)$

(b) $\log_3(6) - \log_3(15)$

1 Recall the law for subtraction of logarithms.	$\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
2 Substitute the values and simplify.	$\log_3(6) - \log_3(15) = \log_3\left(\frac{6}{15}\right)$ $= \log_3\left(\frac{2}{5}\right)$

(c) $\log_{10}(5) + \log_{10}(4) - \log_{10}(2)$

1 Recall the law for addition of logarithms.	$\log_a(m) + \log_a(n) = \log_a(mn)$
2 Substitute the values and simplify.	$\log_{10}(5) + \log_{10}(4) - \log_{10}(2)$ $= \log_{10}(5 \times 4) - \log_{10}(2)$ $= \log_{10}(20) - \log_{10}(2)$
3 Recall the law for subtraction of logarithms.	$\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
4 Substitute the values and simplify.	$\log_{10}(20) - \log_{10}(2) = \log_{10}\left(\frac{20}{2}\right)$
5 Reduce the fraction and apply the property: $\log_a(a) = 1$	$\log_{10}\left(\frac{20}{2}\right) = \log_{10}(10)$ $= 1$

Law of logarithms and powers

Let a, m and n be positive numbers such that: $a^x = m \Leftrightarrow \log_a(m) = x$

If both sides of $a^x = m$ are raised to the power of n, the following is obtained:

$\left(a^x\right)^n = m^n$

$\therefore a^{xn} = m^n \Leftrightarrow \log_a\left(m^n\right) = xn$

Also, since $x = \log_a(m)$:

$\log_a\left(m^n\right) = n\log_a(m)$

For example:

$2\log_2(3) = \log_2\left(3^2\right)$

$= \log_2(9)$

In general: $\log_a\left(m^n\right) = n\log_a(m)$

That is, $\log_a\left(m^n\right)$ is equal to the product of the index n and the log of m.

Additional information

Logarithm laws

Watch the video for a further explanation of the logarithm laws.

From this law, some important properties can be obtained.

If $n = -1$ is substituted into $\log_a\left(m^n\right) = n\log_a(m)$

then: $\log_a\left(m^{-1}\right) = -\log_a(m)$ or $\log_a\left(\frac{1}{m}\right) = -\log_a(m)$

For example: $\log_3\left(2^{-1}\right) = -\log_3(2)$ or $\log_a\left(\frac{1}{2}\right) = -\log_a(2)$

If $m = a$ is substituted into $\log_a\left(m^n\right) = n\log_a(m)$

then: $\log_a\left(a^n\right) = n\log_a(a) = n$

Furthermore: $a^{\log_a(n)} = n$

For example: $\log_2\left(2^3\right) = 3$ or $2^{\log_2(3)} = 3$

In general: $\log_a\left(a^n\right) = n$ and $a^{\log_a(n)} = n$

3 Use the logarithm of a power law

Simplify each of the following expressions.

(a) $\log_2(32)$

THINKING	WORKING
Express the number as a power of 2 and use the property $\log_a\left(a^n\right) = n$.	$\log_2(32) = \log_2\left(2^5\right)$ $= 5 \times \log_2(2)$ $= 5 \times 1$ $= 5$

(b) $\log_3\left(\frac{1}{81}\right)$

Express the number as a power of 3 and use the property $\log_a\left(a^n\right) = n$.

$$\begin{aligned}\log_3\left(\frac{1}{81}\right) &= \log_3\left(\frac{1}{3^4}\right)\\ &= \log_3\left(3^{-4}\right)\\ &= -4 \times \log_3(3)\\ &= -4\end{aligned}$$

(c) $4\log_{16}(2)$

1 Recall the relevant formula.

$$\log_a\left(a^n\right) = n\log_a(a)$$

2 Substitute the values and simplify the expression using the property $\log_a(a) = 1$.

$$\begin{aligned}4\log_{16}(2) &= \log_{16}\left(2^4\right)\\ &= \log_{16}(16)\\ &= 1\end{aligned}$$

(d) $\log_e\left(\frac{1}{\sqrt{e}}\right)$

1 Express the number as a power of e.

$$\begin{aligned}\log_e\left(\frac{1}{\sqrt{e}}\right) &= \log_e\left(\frac{1}{e^{\frac{1}{2}}}\right)\\ &= \log_e\left(e^{-\frac{1}{2}}\right)\end{aligned}$$

2 Recall the property $\log_a\left(a^n\right) = n$ and use it to simplify.

$$\begin{aligned}&= -\frac{1}{2} \times \log_e(e)\\ &= -\frac{1}{2}\end{aligned}$$

If a number is greater than 1, its logarithm is positive.

If a number is between 0 and 1, its logarithm is negative.

4 Use inverses to simplify

Simplify: $2^{3\log_2(5)}$

THINKING	WORKING
1 Recall the law $\log_a\left(m^n\right) = n\log_a(m)$ and use it to rewrite the exponent.	$2^{3\log_2(5)} = 2^{\log_2\left(5^3\right)}$
2 Use the property $a^{\log_a(n)} = n$ and simplify the answer.	$2^{\log_2\left(5^3\right)} = 5^3$ $= 125$

Properties and laws of logarithms

- $\log_a(1) = 0$
- $\log_a(a) = 1$
- $\log_a(m) + \log_a(n) = \log_a(mn)$
- $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
- $\log_a\left(m^n\right) = n\log_a(m)$
- $\log_a\left(\frac{1}{m}\right) = \log_a\left(m^{-1}\right) = -\log_a(m)$
- $\log_a\left(a^n\right) = n$
- $a^{\log_a(n)} = n$
- $a^{p\log_a(m)} = a^{\log_a\left(m^p\right)} = m^p$

WARNING

It is a common misconception that these expressions are equivalent.

$\frac{\log_a(m)}{\log_a(n)} \neq \log_a(m) - \log_a(n)$ $\log_a(m+n) \neq \log_a(m) + \log_a(n)$

5 Use the properties and laws of logarithms

Simplify: $\frac{3}{2}\log_a(x) + \log_a\left(\sqrt{x}\right) - \log_a\left(x^2+1\right)$

THINKING	WORKING
1 Recall the relevant law.	$\log_a\left(m^n\right) = n\log_a(m)$
2 Substitute the values.	$\frac{3}{2}\log_a(x) + \log_a\left(\sqrt{x}\right) - \log_a\left(x^2+1\right)$ $= \log_a\left(x^{\frac{3}{2}}\right) + \log_a\left(x^{\frac{1}{2}}\right) - \log_a\left(x^2+1\right)$
3 Simplify by using the laws for addition and subtraction of logarithms.	$= \log_a\left(x^{\frac{3}{2}} \times x^{\frac{1}{2}}\right) - \log_a\left(x^2+1\right)$ $= \log_a\left(\frac{x^{\frac{3}{2}+\frac{1}{2}}}{x^2+1}\right)$
4 Use the index laws to simplify the numerator.	$= \log_a\left(\frac{x^2}{x^2+1}\right)$

EXERCISE 3.1 Logarithmic functions and their properties

Worked Example

1 Simplify each of the following expressions. 1

(a) $\log_4(1)$ (b) $2\log_4(1)$ (c) $\log_{10}(10)$ (d) $3\log_4(4)$

2 Which expression is equal to $\log_2(x) - \log_2(3x+2)$?

A $\log_2(4x+2)$ B $\log_2\left(3+\frac{3}{x}\right)$ C $\log_2(2x+2)$ D $\log_2\left(\frac{x}{3x+2}\right)$

3 Simplify each of the following expressions. 2

(a) $\log_5(3)+\log_5(5)$ (b) $\log_8(4)+\log_8(2)$ (c) $\log_3(5)-\log_3(4)$

(d) $\log_2(8)-\log_2(4)$ (e) $\log_{10}(6)-\log_{10}\left(\frac{3}{5}\right)$

(f) $\log_{10}(6)+\log_{10}(5)-\log_{10}(3)$ (g) $\log_2(12)-\log_2(2)-\log_2(3)$

4 Consider the expression $3\log_e(x)+2\log_e(x)+\log_e(4x)$.

(a) Which expression is equivalent to the original?

A $\log_e\left(2x^2+4x\right)$ B $\log_e\left(x^3+x^2+4x\right)$ C $\log_e\left(4x^6\right)$ D $6\log_e(4x)$

(b) Explain the common error made by a student who obtained an expression with a coefficient of 6 for part (a).

5 Simplify each of the following expressions. 3

(a) $\log_2\left(2^3\right)$ (b) $\log_3(27)$ (c) $\log_2\left(\frac{1}{2}\right)$

(d) $\log_{10}(0.01)$ (e) $3\log_8(2)$

(f) $2\log_e\left(\sqrt{e}\right)$ (g) $-2\log_2\left(\frac{1}{4}\right)$

6 Convert each of the following to logarithmic form.

(a) $3^4 = 81$ (b) $10^3 = 1000$ (c) $2^{-2} = \frac{1}{4}$ (d) $10^{-3} = 0.001$

7 Convert each of the following to exponential form.

(a) $\log_2(16) = 4$ (b) $\log_3(27) = 3$ (c) $\log_2\left(\frac{1}{8}\right) = -3$ (d) $\log_{10}(0.1) = -1$

8 Simplify each of the following expressions. 4

(a) $2^{\log_2(3)}$ (b) $e^{3\log_e(3)}$ (c) $\log_3\left(3^4\right)$

(d) $\log_e\left(\frac{1}{e}\right)$ (e) $a^{\log_a\left(x^2\right)}$ (f) $\log_b\left(b^{x+1}\right)$

9 Simplify each of the following expressions.

(a) $\log_2\left(\frac{4}{3}\right)+\log_2(6)$ (b) $\log_{10}(20)+\log_{10}(5)$

(c) $\log_2(24)-\log_2(3)$ (d) $4\log_{10}(2)-\log_{10}(16)$

(e) $\log_2(4)+2\log_2(3)-\log_2(9)$ (f) $2\log_2(3)+\log_2(6)-2\log_2\left(\frac{3}{2}\right)$

10 Given that $\log_{10}(2) = 0.3010$ and $\log_{10}(3) = 0.4771$, determine each of the following.

(a) $\log_{10}(6)$ **(b)** $\log_{10}\left(\frac{2}{3}\right)$ **(c)** $\log_{10}\left(\frac{4}{9}\right)$

11 Simplify each of the following expressions, writing answers as a single term.

(a) $\frac{\log_2(8)}{\log_2(16)}$ **(b)** $\frac{\log_3(27)}{\log_3(9)}$ **(c)** $(\log_2(4))(\log_2(8))$

(d) $\frac{\log_e\left(\sqrt{a}\right)}{\log_e(a)}$ **(e)** $2+\log_3(4)$ **(f)** $3+\log_5(2)$

Worked Example 5

12 Simplify each of the following expressions.

(a) $\log_2(x)+\log_2(4x-1)$ **(b)** $\log_2\left(\frac{4x+2}{3}\right)-\log_2\left(\frac{x}{3}\right)$

(c) $\log_2(a)+\log_2(b)-2\log_2(c)$ **(d)** $\log_e\left(e^x\right)-e^{(\log_e x)}+2\log_3\left(\sqrt{x}\right)$

(e) $4\log_e(x)-2\log_e(x+1)+\log_e(x+1)$

13 The function f has the rule $f(x) = 2\log_e(5x)$. If $f(3x) = \log_e(y)$, determine the rule for y.

14 The number N of millions of people in a particular city t years after the middle of 2010 can be approximately modelled by: $N(t) = 1.05^t$.

(a) What was the population in the middle of 2010?

(b) What was the population in 2015, to the nearest thousand?

(c) Write a logarithmic equation that gives the number of years it will take for the population to reach 2 million.

(d) Determine the year the population is predicted to reach 2 million.

15 Simplify the following logarithmic expressions.

(a) $\frac{1}{2}\log_a\left(a^2-1\right)-\log_a\sqrt{\frac{a^2-1}{2}}$

(b) $\log_b\left(\frac{ab^2}{c^3}\right)+\log_b\left(\frac{c^2}{ab}\right)$

(c) $\log_b\left(\frac{a}{b^2}\right)-\log_b\left(\frac{a^2}{b^{\frac{5}{2}}}\right)$

(d) $\log_e\left(\frac{x^3}{y^2z^2}\right)-\log_e\left(\frac{y^2}{x^2z^2}\right)-2\log_e\left(\frac{x^3z}{y}\right)-\frac{1}{2}\log_e\left(\frac{x^2}{z^8}\right)$

16 If $x = \log_2(a)$ and $y = \log_2(b)$, express each of the following in terms of x and y.

(a) $\log_2(ab)$ **(b)** $\log_2\left(\frac{a}{b}\right)$ **(c)** $\log_2\left(\frac{16a^2b^{-\frac{1}{2}}}{b^3}\right)$

Logarithmic equations

3.2

Solving logarithmic equations

Logarithmic equations arise when solving many problems, including exponential functions.

If $\log_a(x) = \log_a(y)$, then $x = y$.

If $\log_a(y) = x$, then $y = a^x$.

Given that y is equivalent to a^x, then $y > 0$, therefore $\log_a(y)$ is undefined for $y \leq 0$.

For $y = a^x$ to be a function, then $a > 0$.

Making connections

Logarithmic equations

Move the slider for a to view the values of y, where $y = (a)^x$, to consolidate your understanding about the restrictions of this function.

6 Solve simple logarithmic equations

Solve the following equations.

(a) $\log_3(x) = 4$

THINKING	WORKING
1 Recall the relevant property, and write the logarithmic expression in exponential form.	$\log_a(y) = x \Leftrightarrow a^x = y$ $\log_3(x) = 4 \Leftrightarrow x = 3^4$
2 Calculate the value of x.	$x = 81$

(b) $\log_e(x) = \log_e(6) + \log_e(2)$

1 Recall the law for adding logarithms.	$\log_a(x) + \log_a(y) = \log_a(xy)$
2 Simplify the expression.	$\log_e(x) = \log_e(6) + \log_e(2)$ $= \log_e(6 \times 2)$ $= \log_e(12)$
3 Use the implication: $\log_a(x) = \log_a(y) \Rightarrow x = y$	$x = 12$

(c) $\log_x(32) = \frac{5}{2}$

1 Recall the relevant property, and write the logarithmic expression in exponential form.	$\log_a(y) = x \Leftrightarrow a^x = y$ $\log_x(32) = \frac{5}{2} \Leftrightarrow x^{\frac{5}{2}} = 32$
2 Write the number in index form.	$x^{\frac{5}{2}} = 2^5$

3 Raise both sides of the equation to the reciprocal of the fractional power, simplify and calculate the value of x.	$\left(x^{\frac{5}{2}}\right)^{\frac{2}{5}} = \left(2^5\right)^{\frac{2}{5}}$ $x = 2^2$ $x = 4$

(d) $\log_{10}(3x+2) = 0$

1 Recall the relevant property and substitute the given values.	$\log_a(y) = x \Leftrightarrow a^x = y$ $\log_{10}(3x+2) = 0 \Leftrightarrow 3x+2 = 10^0$
2 Simplify and solve for x.	$3x+2 = 1$ $3x = -1$ $x = -\frac{1}{3}$
3 Evaluate the reasonableness of the solution by using the argument (expression for which the logarithm is taken) to determine a suitable domain.	Use the argument to determine a suitable domain: $3x+2 > 0$ $3x > -2$ $x > -\frac{2}{3}$ Therefore $x = -\frac{1}{3}$ is a reasonable solution.

(e) $\log_2(x) - \log_2(x-2) = 3$

1 Recall the law for subtracting logarithms.	$\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
2 Substitute the given values.	$\log_2(x) - \log_2(x-2) = 3$ $\log_2\left(\frac{x}{x-2}\right) = 3$
3 Write in exponential form.	$\frac{x}{x-2} = 2^3$
4 Simplify and solve for x.	$x = 8(x-2)$ $x = 8x - 16$ $7x = 16$ $x = \frac{16}{7}$ $x = 2\frac{2}{7}$
5 Evaluate the reasonableness of the solution by using the smallest argument to determine a suitable domain.	$x-2 > 0$ $x > 2$ Therefore $x = 2\frac{2}{7}$ is a reasonable solution.

WARNING

When solving logarithmic equations, always check the solutions by substituting them back into the *original equation*.

Using the implication: $\log_a(x) = \log_a(y) \Rightarrow x = y$

When both sides of an equation are logarithms of the same base, use $\log_a(x) = \log_a(y) \Rightarrow x = y$ to solve the equation.

7 Use the implication: $\log_a(x) = \log_a(y) \Rightarrow x = y$

Solve the equation: $\log_3(x+7) - \log_3(x-3) = \log_3(x+1)$

THINKING	WORKING
1 Recall the implication.	$\log_a(x) = \log_a(y) \Rightarrow x = y$
2 Simplify the left-hand side of the equation.	$\log_3(x+7) - \log_3(x-3) = \log_3(x+1)$ $\log_3\left(\frac{x+7}{x-3}\right) = \log_3(x+1)$ $\therefore \frac{x+7}{x-3} = x+1$
3 Solve the equation for x using quadratic methods.	$x+7 = (x-3)(x+1)$ $x+7 = x^2 - 2x - 3$ $x^2 - 3x - 10 = 0$ $(x+2)(x-5) = 0$ $x = -2$ or $x = 5$
4 Evaluate the reasonableness of the solution by using the smallest argument to determine a suitable domain.	Choosing the smallest argument: $x - 3 > 0$ $x > 3$ Therefore the only solution is $x = 5$. Check by substituting the value into the original equation: $\log_3(5+7) - \log_3(5-3) = \log_3(5+1)$ $\log_3(12) - \log_3(2) = \log_3(6)$ $\log_3\left(\frac{12}{2}\right) = \log_3(6)$ $\log_3(6) = \log_3(6)$ Given that LHS = RHS, then $x = 5$ is a solution.

Logarithmic equations that reduce to quadratics

In instances where there are three terms in a logarithmic equation, it is sometimes possible to substitute a variable for the logarithm and solve a quadratic equation to help solve the problem.

8 Solve logarithmic equations that reduce to a quadratic equation

Solve the equation: $(\log_e(x))^2 + \log_e(x) - 2 = 0$

THINKING	WORKING
1 Substitute a variable for the logarithm.	$(\log_e(x))^2 + \log_e(x) - 2 = 0$ Let $a = \log_e(x)$. $a^2 + a - 2 = 0$
2 Factorise the quadratic. Solve using the null factor law.	$(a+2)(a-1) = 0$ $a = -2$ or $a = 1$
3 Replace the logarithm for each solution and solve for x.	$\log_e(x) = -2$ or $\log_e(x) = 1$ $\log_e(x) = -2 \Leftrightarrow x = e^{-2}$ $\log_e(x) = 1 \Leftrightarrow x = e$ $\therefore x = e^{-2}$ or $x = e$
4 Evaluate the reasonableness of the solution by using the argument to determine a suitable domain.	Given $\log_e(x)$, then $x > 0$, both solutions are reasonable. Check by substituting the solutions into the equation: $\left(\log_e\left(e^{-2}\right)\right)^2 + \log_e\left(e^{-2}\right) - 2 = 0$ $(-2\log_e(e))^2 - 2\log_e(e) - 2 = 0$ $4 - 2 - 2 = 0$ $0 = 0$ Given that LHS = RHS, then $x = e^{-2}$ is a solution. $(\log_e(e))^2 + \log_e(e) - 2 = 0$ $(1)^2 + 1 - 2 = 0$ $1 + 1 - 2 = 0$ $0 = 0$ Given that LHS = RHS, then $x = e$ is a solution.

Using the equivalence: $\log_a(y) = x \Leftrightarrow a^x = y$

The equivalence $\log_a(y) = x \Leftrightarrow a^x = y$ may enable an equation to be written in a form that does not involve logarithms.

WARNING

Any restrictions on the variables of the logarithmic equation still apply to the equivalent equation.

9 Rewrite a logarithmic equation

For $\log_2(y) - 3\log_2(x+1) = 2$, write a relationship between x and y that does not involve logarithms. Make y the subject of the equation.

THINKING	WORKING
1 Recall the logarithmic law for powers and the logarithmic law for subtraction.	$\log_a(m^n) = n\log_a(m)$ $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$
2 Use the laws to simplify the expression.	$\log_2(y) - 3\log_2(x+1) = 2$ $\log_2(y) - \log_2(x+1)^3 = 2$ $\log_2\left(\frac{y}{(x+1)^3}\right) = 2$
3 Rewrite in exponential form and solve for y. Recall the restrictions from the original equation.	$\frac{y}{(x+1)^3} = 2^2$ $y = 4(x+1)^3$ where $x > -1$ and $y > 0$

Note: The restriction on the x-value in Worked Example 9 ensures the compliance of the y-value, so $x > -1 \Rightarrow y > 0$. In which case, the rule can be written: $y = 4(x+1)^3, x > -1$.

EXERCISE 3.2 Logarithmic equations

Worked Example 6

1 Solve each of the following equations for x.

(a) $\log_3(x) = 2$ (b) $\log_5(x) = -4$ (c) $\log_3\left(\frac{3}{x}\right) = 1$

(d) $\log_e(x^2) = 2$ (e) $\log_x(9) = 2$ (f) $2\log_x(9) = 4$

(g) $\log_x\left(\frac{1}{16}\right) = -4$ (h) $\log_x(64) = \frac{3}{2}$ (i) $\log_x(27) = \frac{3}{4}$

2 Solve each of the following equations for x.

(a) $\log_{10}(x) = \log_{10}(3) + \log_{10}(4)$ **(b)** $\log_3(x) - \log_3(6) = 2$ **(c)** $\log_2(5x-2) = 0$

(d) $3\log_2(x-2) = 6$ **(e)** $\log_2\left(2^{x+2}\right) = 4$ **(f)** $\log_x(2) = \frac{1}{3}$

(g) $4\ln(2x-5) = 8$ **(h)** $\ln\left(\frac{1}{x}\right) = 1$ **(i)** $\log_3\left(\frac{3x-2}{2x+1}\right) = 3$

3 Simplify each of the following equations and then solve for x.

(a) $2\log_2(x) + 3\log_2(x) = \log_2(32)$ **(b)** $\log_e(x) - \log_e(2x-1) = 0$

(c) $\log_e(x) - \log_e(x-3) = 2$ **(d)** $\ln(x) - \ln(2x-1) = 2$

4 Consider the equation: $\log_5(7x-1) - \log_5(x+1) = \log_5(2)$.

(a) What is the solution to the equation?

A $x = \frac{3}{5}$ **B** $x = -1$ **C** $x = \frac{1}{2}$ **D** no valid solution

(b) Explain which of the options in part **(a)** could be disregarded because of domain restrictions.

Worked Example

5 Solve each of the following equations for x.

(a) $2^{\log_2(x)} = 3$ **(b)** $2^{2\log_2(x)} = 16$ **(c)** $2^{3\log_2(x)} = 27$ **(d)** $e^{2\log_e(x)} = 64$

9

6 For each of the following equations, make y the subject. Write any restrictions that apply to the new equations.

(a) $2\log_2(x) - 3\log_2(y) = 2$ **(b)** $2\log_e(y) - 4\log_e(x) = 2$ **(c)** $\log_3(y) + \log_3(x) = \log_3(x+y)$

7 Given the equation $x = 2\ln(y) + 3$, solve for y in terms of x.

8 Given $\log_3(81) = x - y$ and $\log_2(32) = x + y$, determine x and y.

9 Given $\log_x(y) = a^4$ and $\log_y(x) = \frac{5}{a^3}$, determine a.

10 Solve each of the following equations for x.

(a) $(\log_e(x) - 1)(\log_e(x) - 2) = 0$ **(b)** $(\log_e(x) + 1)(\log_e(x^2) - 2) = 0$

7

11 Solve each of the following equations for x.

(a) $\log_2(x) + \log_2(x-1) = \log_2(x+3)$ **(b)** $\ln(2x+2) - \ln(x-1) = \ln(x+1)$

(c) $\log_2(4x^2 + x) = \log_2(2 - 2x^2)$ **(d)** $\log_e(4x) = 2\log_e(x)$

(e) $\log_e(5x) = 3\log_e(x)$ **(f)** $\ln(x-1) + \ln(x+2) = \ln(6x-8)$

12 If $(\log_2(x))^2 - \log_2(x) + 6 = 12$, what are the possible values of x?

A 8 only **B** 3 only **C** -2 or 3 **D** 8 or $\frac{1}{4}$

8

13 Solve each of the following equations for x.

(a) $(\ln(x))^2 + \ln(x^2) + 1 = 0$ **(b)** $2(\log_e(x))^2 = \log_e(x) + 1$ **(c)** $(\log_e(x))^2 = \log_e(x^5) - 6$

14 For the equation $\log_3(y) - \log_3(x) = \log_3(x+y)$, make y the subject, and determine the domain.

15 Solve the equation $\log_3(x) + \log_3(x-6) = \log_3(3) + 2$ for x.

16 The sales N_A and N_B in hundreds of units of two new products, A and B, are approximately given by the models: $N_A = 100 + 20\log_3(2t+1)$ and $N_B = 120 + 10\log_3(t+3)$, where t is the number of months after a product is introduced.

(a) Calculate the sales numbers for product A at each of these time frames.

(i) 1 month **(ii)** 4 months

(b) After how many months will sales of product A reach 15 000 units? Give your answer correct to 2 decimal places.

(c) Determine when the sales of the two products will be the same.

17 The number N, in kilolitres, of fresh water produced by two desalination plants in t hours, $t \geq 0.5$, can be modelled by: $N_A = 200 + 150\ln(t)$ for plant A and $N_B = 300 + 100\ln(t)$ for plant B.

(a) Determine the volume of fresh water that can be produced by each plant in 1 hour.

(b) How long would it take plant A to produce 1000 kL of fresh water? Give your answer in days, correct to 1 decimal place.

(c) After what amount of time, correct to 1 decimal place, would the two desalination plants produce the same amount of fresh water? What amount of water is this?

18 It is given that $a = \log_2(x)$ and $b = \log_2(y)$.

(a) Express $\log_2\left(4\sqrt{y}\right)$ in terms of b.

(b) It is also given that $\log_2\left(x^2 y\right) = 1$ and $\log_2\left(4\sqrt{y}\right) = 1$.

(i) Determine the values of a and b.

(ii) Determine the exact values of x and y.

19 Consider the function $f(x) = \log\left(x^2 - 7x + 10\right) + \log\left(9 - x^2\right)$, assuming common logarithms.

(a) Determine the maximal domain of $f(x)$.

(b) If the domain of $f(x)$ were restricted to integers only, determine the range.

(c) Use your knowledge of logarithms to consider possible values to be included in the range if the domain of $f(x)$ were not restricted. Roughly estimate the coordinates of the stationary point and discuss whether negative values are possible.

3.3 Logarithmic bases and exponential equations

Change of base

When using technology, either natural logarithms (base e) or common logarithms (base 10) may be required. To change the base of $\log_a(x)$ from a to b, follow the steps below.

Let $y = \log_a(x)$: $\therefore a^y = x$

Take the $\log_b$ of both sides:

$$\log_b\left(a^y\right) = \log_b(x)$$
$$y\log_b(a) = \log_b(x)$$
$$y = \frac{\log_b(x)}{\log_b(a)}$$
$$\therefore \log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

For $y = \log_a(x)$

The change-of-base rule shows:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Additional information

Change of base

Watch the video for a further explanation of the change-of-base rule.

10 Write logarithmic expressions using the change-of-base rule

Consider the expression $\log_2(12)$.

(a) Rewrite the expression using base 10.

THINKING	WORKING
1 Recall the change-of-base rule.	$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$
2 Substitute the given values.	$\log_2(12) = \frac{\log_{10}(12)}{\log_{10}(2)}$

(b) Calculate the value of the expression correct to 2 decimal places.

Use the log or ln key on your calculator.	$\frac{\log(12)}{\log(2)} = 3.58\,(2\text{ d.p.})$
	$\frac{\ln(12)}{\ln(2)} = 3.58\,(2\text{ d.p.})$

In the previous example, note that $\frac{\log(12)}{\log(2)} = 3.58\,(2\text{ d.p.})$, but any base can be used for approximations on your calculator.

Simplifying expressions using the change-of-base rule

Sometimes it is necessary to change the base of a logarithm to help to simplify or to combine logarithms with different bases.

Additional information

Simplifying expressions using change of base

Watch the video for further explanation of how to simplify expressions using the change-of-base rule.

11 Simplify expressions using the change-of-base rule

Simplify the expression $\log_3(x) + \log_9(y)$ by writing it as a single logarithm.

THINKING	WORKING
1 Recall the change-of-base rule to write the terms with the same base.	$\log_3(x) + \log_9(y)$ $= \log_3(x) + \dfrac{\log_3(y)}{\log_3(9)}$
2 Simplify the numerical expressions. Use the laws of logarithms to reduce to a single term.	$= \log_3(x) + \dfrac{\log_3(y)}{\log_3(3^2)}$ $= \log_3(x) + \dfrac{\log_3(y)}{2\log_3(3)}$ $= \log_3(x) + \dfrac{1}{2}\log_3(y)$ $= \log_3(x) + \log_3(\sqrt{y})$ $= \log_3(x\sqrt{y})$

WARNING

It is a common misconception that the following expressions are equal.

$$\frac{\log_2(x)}{\log_2(a)} \neq \frac{x}{a} \qquad \frac{\log_{10}(a \times m)}{\log_{10}(a \times n)} \neq \frac{\log_{10}(m)}{\log_{10}(n)}$$

Solution of exponential equations with the unknown in the exponent

Exponential equations such as $2^x = 3$ can be solved exactly for x by converting to logarithmic form:

$2^x = 3 \Leftrightarrow x = \log_2(3)$

With change of base, you can easily convert to common logs or natural logarithms:

$$x = \log_2(3) = \frac{\log_{10}(3)}{\log_{10}(2)} \text{ or } \frac{\ln(3)}{\ln(2)}$$

Keying either of these into your calculator will give a decimal approximation: $x \approx 1.58$

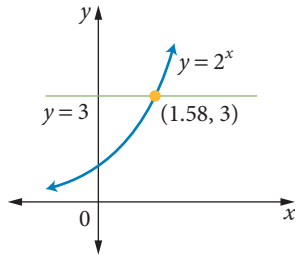
Alternatively, you can use base 10; for example, in the solving process:

$$\begin{aligned} 2^x &= 3 \\ \log_{10}(2^x) &= \log_{10}(3) \\ x\log_{10}(2) &= \log_{10}(3) \\ x &= \frac{\log_{10}(3)}{\log_{10}(2)} \end{aligned}$$

Technology can also be used to check the approximate answer using a graphical approach.

For example, plot the graphs of $y = 2^x$ and $y = 3$, then read the x-value at the point of intersection.

The point of intersection is where $x \approx 1.58$.



12 Solve exponential equations in exact form

Solve the equation $3^{x+1} = 5$. Write your answer as a single logarithm.

THINKING	WORKING
1 Convert to logarithmic form.	$3^{x+1} = 5$ $\log_3\left(3^{x+1}\right) = \log_3(5)$ $(x+1)\log_3(3) = \log_3(5)$ $x+1 = \log_3(5)$
2 Solve for x and simplify the expression using $\log_a(a) = 1$ and $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$.	$x = \log_3(5) - 1$ $= \log_3(5) - 1 \times \log_3(3)$ $= \log_3\left(\frac{5}{3}\right)$

Solving exponential inequalities

When solving an inequality, it is important to remember that multiplying or dividing an inequality by a negative number reverses the inequality sign.

With logarithms, it is important to remember that the logarithm of a number between 0 and 1 will be negative. For example, if you divide an inequality by $\log_{10}(0.6)$, you will need to reverse the inequality sign.

Additional information

Solving inequalities

Watch the two videos for an overview and explanation of solving inequalities.

Consider the solution of the inequality $2^x < 3$:

$$2^x < 3$$
$$\log_e\left(2^x\right) < \log_e(3)$$
$$x\log_e(2) < \log_e(3)$$
$$x < \frac{\log_e(3)}{\log_e(2)}$$

Since $\log_e(2)$ is positive, dividing both sides of the inequality by $\log_e(2)$ does not change the inequality sign.

Now consider the solution of the inequality $0.2^x < 3$:

$$\begin{aligned} 0.2^x &< 3 \\ \log_e\left(0.2^x\right) &< \log_e(3) \\ x\log_e(0.2) &< \log_e(3) \\ x &> \frac{\log_e(3)}{\log_e(0.2)} \end{aligned}$$

Since $\log_e(0.2)$ is negative, dividing both sides of the inequality by $\log_e(0.2) \approx -1.61$ changes the inequality sign.

13 Solve exponential inequalities

Solve the inequality $0.8^x \geq 0.3$, giving your answer in terms of natural logarithms.

THINKING	WORKING
1 Take $\log_e$ of both sides and use the logarithmic laws to simplify the left-hand side.	$0.8^x \geq 0.3$ $\log_e\left(0.8^x\right) \geq \log_e(0.3)$ $x\log_e(0.8) \geq \log_e(0.3)$
2 Change the sign of the inequality as the logarithm of a number less than 1 has a negative value.	$x \leq \frac{\log_e(0.3)}{\log_e(0.8)}$

Exponential equations that lead to a quadratic equation

If an exponential equation has three terms, it may be possible to make a substitution so that the equation is written as a quadratic.

14 Solve exponential equations that reduce to a quadratic equation

Solve $2^x - \frac{15}{2^x} - 2 = 0$ for x.

THINKING	WORKING
1 Multiply both sides of the equation by 2^x to eliminate 2^x from the denominator, and use the index laws to simplify.	$2^x - \frac{15}{2^x} - 2 = 0$ $2^x \times 2^x - \frac{15}{2^x} \times 2^x - 2 \times 2^x = 0 \times 2^x$ $2^{2x} - 15 - 2 \times 2^x = 0$
2 Use substitution for 2^x to rewrite the equation as a quadratic. Factorise and solve for a.	Let $a = 2^x$: $a^2 - 2a - 15 = 0$ $(a-5)(a+3) = 0$ $a = 5$ or $a = -3$

3 Substitute $a = 2^x$.	$2^x = 5$ or $2^x = -3$
4 Evaluate the reasonableness of the results and determine the solution.	$2^x = -3$ is not a reasonable solution since $a^x > 0$. Therefore the only solution is: $2^x = 5$ $\log_2(2^x) = \log_2(5)$ $x\log_2(2) = \log_2(5)$ $x = \log_2(5)$

You can use change of base to write a decimal approximation for the answer:

$$x = \log_2(5)$$
$$= \frac{\log_{10}(5)}{\log_{10}(2)} \text{ or } \frac{\ln(5)}{\ln(2)}$$
$$\approx 2.32$$

EXERCISE 3.3 Logarithmic bases and exponential equations

Worked Example

10

1 Consider the expression: $\log_2(20)$.
- (a) Write the expression using natural logarithms.
- (b) Write the expression using common logarithms.
- (c) Calculate the expression correct to 2 decimal places.

2 Simplify each of the following expressions using the change-of-base rule.

(a) $\frac{\log_2(9)}{\log_2(3)}$ (b) $\frac{\log_e(16)}{\log_e(2)}$ (c) $\frac{\log_3(25)}{\log_3(5)}$

3 For each of the following equations, solve for x.

(a) $\log_2(6) = \frac{\log_{10}(x)}{\log_{10}(2)}$ (b) $\log_3(7) = \frac{\log_{10}(x)}{\log_{10}(3)}$ (c) $3\log_e(4) = \frac{\log_5(x)}{\log_5(e)}$

4 For each of the following equations, determine the exact value of x.

(a) $2^x = 5$ (b) $3^x = 7$ (c) $e^{-x} = 9$ (d) $2^{-x} = 3$

5 If $2^{-x+1} = 3$, then what is the value of x?

A $\log_2\left(\frac{3}{2}\right)$ B $\log_2\left(\frac{2}{3}\right)$ C $\frac{5}{4}$ D 3

12

6 For each of the following equations, determine the exact value of x, written as a single logarithm.

(a) $2^{1-x} = 9$ (b) $2^{x+1} = 3$ (c) $e^{3x-1} = 5$ (d) $3^{2x-1} = 5$

7 For each of the following equations, determine the value of x, correct to 2 decimal places.

(a) $2^x = 50$ (b) $5^x = 3^{x+2}$ (c) $3^{1-x} = 22$ (d) $7^x = 7x$

8 For each of the following inequalities, solve for x. Write your answer in terms of natural logarithms. 13

(a) $0.6^x > 0.4$ **(b)** $0.2^x < 0.7$ **(c)** $5^x > 7$

9 Anime characters have large, exaggerated, emotive eyes. Anime creators take inspiration from the relationship between the eye length, L mm, of many animals and their body mass, m kg, which can be given by the model: $\log(L) - 0.98 = 0.2\log(m)$.

(a) What is the eye length of an animal with a mass of 20 kg? Give your answer correct to 1 decimal place.

(b) What is the mass of an animal of eye length 24 mm? Give your answer correct to the nearest kilogram.

(c) Write the rule for the model as an equation that does not involve logarithms.

10 Write each of the following expressions as a single logarithm. 11

(a) $\log_2(5) + \log_4(y)$ **(b)** $2\log_3(x^2) - \log_9(y)$

11 Consider the expression: $2 \times 5^x + 3 - 2 \times 5^{-x}$.

(a) What is the value of x for which the expression is equal to zero?

A $\log_5(2)$ **B** $\frac{1}{2}$ **C** -2 **D** $-\log_5(2)$

(b) Explain why there is only one solution to the equation in part **(a)**.

12 For each of the following equations, determine the exact value(s) of x. 14

(a) $3^x(3^x - 2) = 0$ **(b)** $3^{2x} - 3^x - 2 = 0$ **(c)** $e^{2x} = 30 - e^x$

(d) $e^x + 3e^{-x} - 4 = 0$ **(e)** $3^x - 2 \times 3^x + 1 = 0$

13 Solve the equation $\log_2(x) + \log_4(x^2) = \log_8(x) + \log_2(16)$ for x.

14 Solve $\log(2^x + 4^x) - \log(8) = \log\left(2^{x-1} - \frac{1}{4}\right)$ for x.

15 Carbon dating is used by scientists to date fossils. It is based on the principle that carbon-14 declines at a rate proportional to its mass. This relationship is given by the rule: $C = C_o e^{-kt}$, where C is the amount of the isotope remaining and t is the time in years. The half-life of carbon-14 is approximately 5730 years.

(a) Determine the exact value for k.

(b) An ancient skull is found and it is determined that it contains 10% of the carbon-14 found in a modern skull. How old is the skull that was found? Give your answer correct to the nearest thousand years.

(c) A fossil is found that has 30% carbon-14 compared to the living sample. How old is the fossil that was found? Give your answer to the nearest thousand years.

16 Solve $x^{\log(x)-1} = \frac{1}{\sqrt[4]{10}}$ for x.

17 For each of the following equations, one solution is $x = 0$. Determine the second solution as a power of e.

(a) $x^e = ex$ **(b)** $x^e = (ex)^2$

3.4 Graphs of logarithmic functions

Functions and inverses

Inverse functions allow us to change which variable is the dependent or independent variable.

For a function f to have an inverse function f^{-1}, f must have a 1:1 correspondence, so any vertical or horizontal line will intersect the graph, at most, once.

An inverse can be obtained in any of the following ways:

- interchanging x and y in each ordered pair
- reflecting the graph of $f(x)$ in the line $y = x$
- interchanging x and y in the equation and solving for y (making y the subject).

If a function f has an inverse function, then:

Domain of f^{-1} = Range of f and Range of f^{-1} = Domain of f

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

If the graphs of f and f^{-1} intersect, the point of intersection will occur on the line $y = x$.

Logarithmic functions and inverses

An exponential function is a 1:1 function, so it will have an inverse function.

Consider the function: $y = a^x$.

To determine the rule of the inverse, interchange x and y and solve for y: $x = a^y$.

Make y the subject by writing in logarithmic form: $\log_a(x) = y$.

Using function notation: $f^{-1}(x) = \log_a(x)$ is the inverse of the exponential function $f(x) = a^x$.

The graph of $y = \log_a(x)$, $a > 1$, can also be sketched by:

- reflecting the graph of $y = a^x$ in the line $y = x$ using 1:1 axes scales
- interchanging the x- and y-values in each pair of coordinates.

Also, recall the relationship: $\log_a\left(a^x\right) = x$ and $a^{\log_a(x)} = x$.

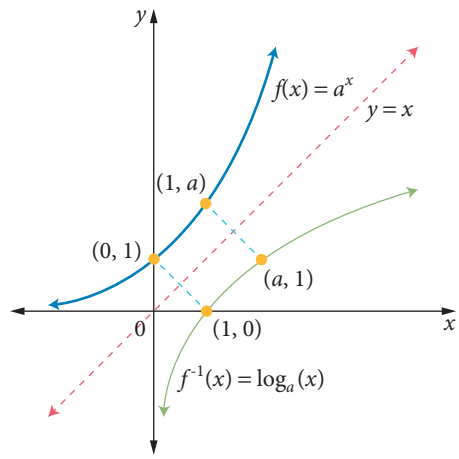
It is an example of the general rule: $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$.

That is, taking the logarithm of a number to a particular base is the inverse process to raising the same base to a power.

Explore further

Logarithmic functions and inverses

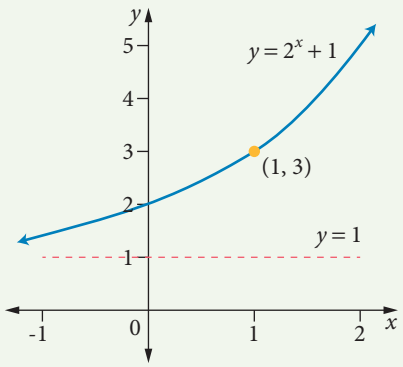
Explore the relationship between exponential functions and their inverse logarithmic functions.



15 Sketch logarithmic graphs from inverses

Consider $f(x) = 2^x + 1$.

(a) Sketch the graph of f.

THINKING	WORKING
1 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the value of y.	Where $x = 0$: $y = 2^0 + 1$ $= 1 + 1$ $= 2$ The coordiantes of the y-intercept are $(0, 2)$.
2 Determine the coordinates of a convenient point.	At $x = 1$: $y = 2^1 + 1$ $= 2 + 1$ $= 3$ The coordinates of another point are $(1, 3)$.
3 Determine the equation of the asymptote.	Given that $2^x > 0$ for $x \in \mathbb{R}$, then $y > 0 + 1$. The equation of the horizontal asymptote is $y = 1$.
4 Sketch the graph.	

(b) Determine the rule, domain and range of the inverse, $f^{-1}(x)$.

1 For the inverse function, interchange x and y.	$y = 2^x + 1$ Inverse: $x = 2^y + 1$
2 Make y the subject by writing in logarithmic form. Write the rule using function notation: f^{-1}.	$x - 1 = 2^y$ $y = \log_2(x - 1)$ $f^{-1}(x) = \log_2(x - 1)$
3 Domain of f^{-1} = Range of f Range of f^{-1} = Domain of f	Domain of $f^{-1} = (1, \infty)$ Range of $f^{-1} = \mathbb{R}$

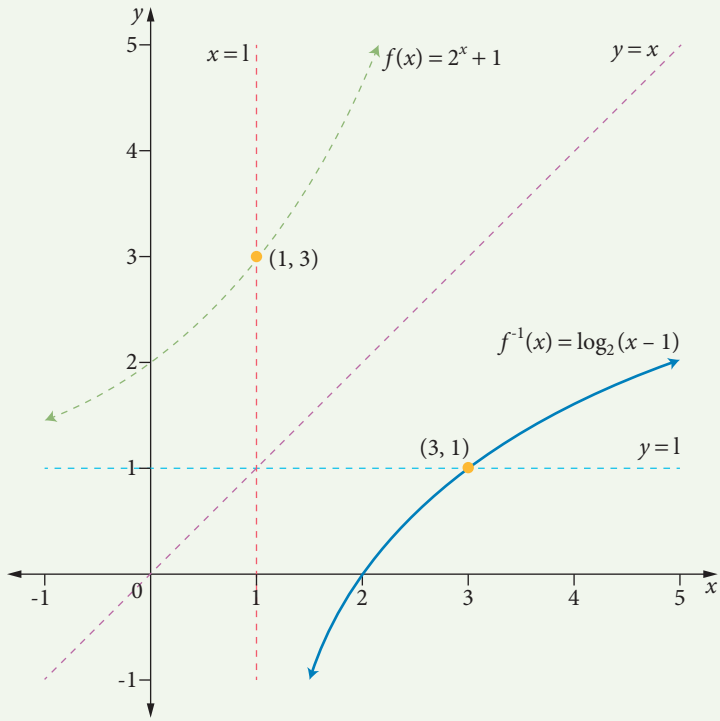
(c) Sketch the graph of the inverse $y = f^{-1}(x)$ on the same set of axes as $y = f(x)$.

1 Interchange x and y for all points determined for $f(x)$.

For the inverse function $f^{-1}(x)$:
$(0,2) \rightarrow (2,0)$
$(1,3) \rightarrow (3,1)$

Asymptote: $y = 1 \rightarrow x = 1$

2 Reflect the graph of $y = f(x)$ and its asymptote in the line $y = x$.

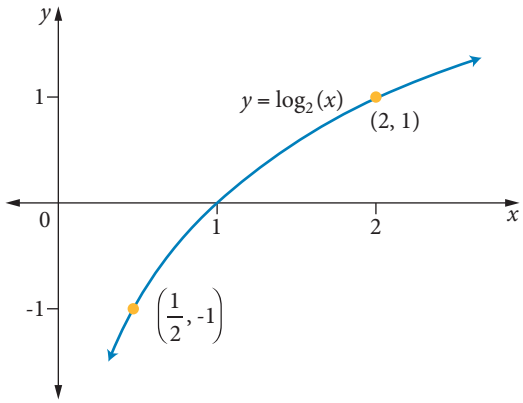


Graphs of logarithmic functions

Functions of the form $y = \log_a(x)$ are called logarithmic functions.

Graphs of logarithmic functions can be sketched by using key values. For example, to sketch $y = \log_2(x)$, use the following key values:

$x = 1$	$y = \log_2(1)$ $= 0$
$x = 2$	$y = \log_2(2)$ $= 1$
$x = \frac{1}{2}$	$y = \log_2\left(\frac{1}{2}\right)$ $= \log_2\left(2^{-1}\right)$ $= -1$



Additional information

The graph of a logarithmic function

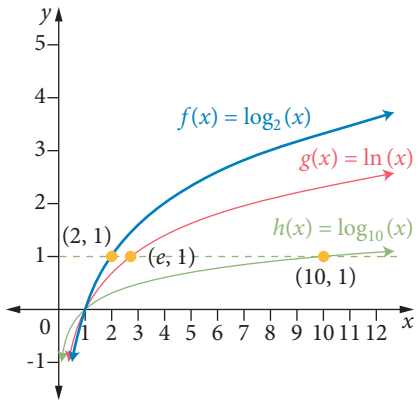
This video gives a further explanation of the graph of a logarithmic function.

Explore further

Graphs of logarithmic functions

Explore the properties of logarithmic functions that have been dilated, reflected and translated.

The graphs of $y = \log_a(x)$, where $a = 2$, $a = e$ and $a = 10$, have been sketched on the same axis to compare the effect of the size of the base.



Important features of $y = \log_a(x)$, where $a > 1$:

- As $x \to 0$, $y \to -\infty$, therefore $x = 0$ is a vertical asymptote.
- As x increases, y increases, hence it is an increasing function.
- The domain is $\mathbb{R}^+$; the range is $\mathbb{R}$.
- The graph includes the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$ and $(a, 1)$.
- The smaller the value of the base a, the steeper the graph.

Reflections

Graph of $y = \log_a(-x)$, where $a > 1$

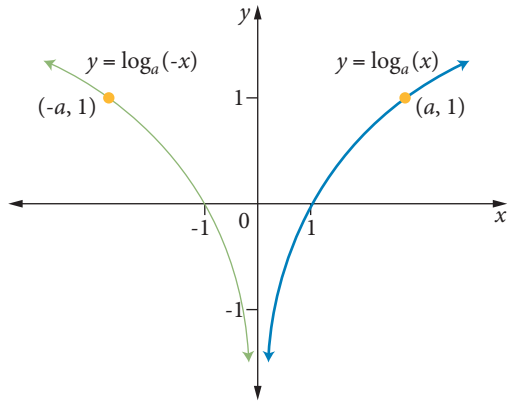
The graph of $y = \log_a(-x)$ can be obtained by reflecting the graph of $y = \log_a(x)$ in the y-axis.

The x-intercept is $(-1, 0)$.

Where $x = -a$, $y = \log_a(a) = 1$.

$x = 0$ is a vertical asymptote.

The domain is $\mathbb{R}^-$; the range is $\mathbb{R}$.



Graph of $y = -\log_a(x)$, where $a > 1$

The graph of $y = -\log_a(x)$ can be obtained by reflecting the graph of $y = \log_a(x)$ in the x-axis.

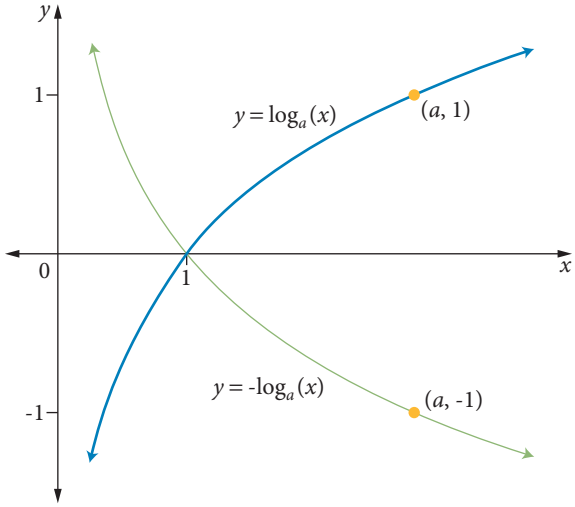
The x-intercept is $(1, 0)$.

Where $x = a$, $y = -\log_a(a) = -1$.

$x = 0$ is a vertical asymptote.

The domain is $\mathbb{R}^+$; the range is $\mathbb{R}$.

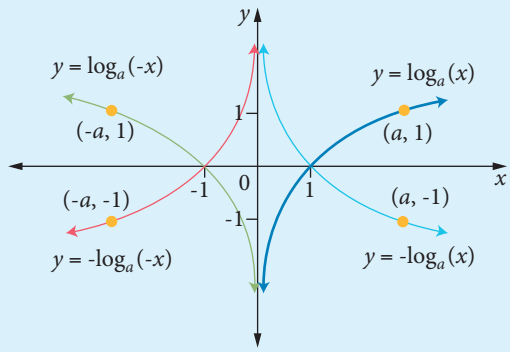
$y = -\log_a(x)$ can also be written as: $y = \log_a\left(x^{-1}\right)$ or $y = \log_a\left(\frac{1}{x}\right)$.



The graph of $y = \log_a(-x)$ can be obtained by reflecting $y = \log_a(x)$ in the y-axis.

The graph of $y = -\log_a(x)$ can be obtained by reflecting $y = \log_a(x)$ in the x-axis.

The graph of $y = -\log_a(-x)$ can be obtained by reflecting $y = \log_a(x)$ in both the x-axis and the y-axis.



Dilations of logarithmic graphs of the form $y = A\log_a(B_x)$

The graphs of $y = \log_2(x)$, $y = 2\log_2(x)$ and $y = \log_2(2x)$ are sketched below.

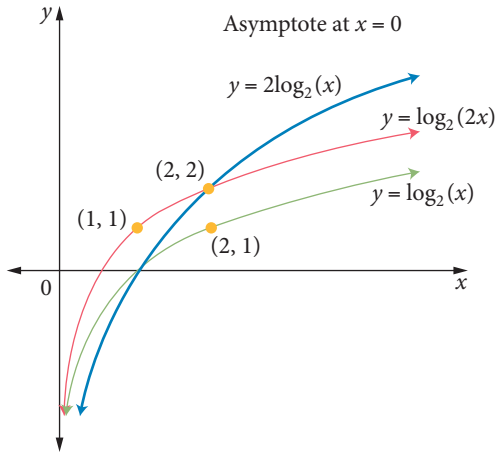
$y = 2\log_2(x)$ is obtained by dilating the graph of by a factor of 2 parallel to the y-axis.

For example, $(2,1) \rightarrow (2,2)$.

$y = \log_2(2x)$ is obtained by dilating the graph of $y = \log_2(x)$ by a factor of $\frac{1}{2}$ parallel to the x-axis. For example, $(2,1) \rightarrow (1,1)$.

A dilates the graph of $y = \log_a(x)$ by a factor of $|A|$ parallel to the y-axis.

B dilates the graph of $y = \log_a(x)$ by a factor of $\frac{1}{|B|}$ parallel to the x-axis.



Translations

Logarithmic graphs of the form $y = A\log_a(x + C) + D$ are transformed in the same way as other functions you have studied.

From the graph of $y = \log_a(x)$, the graph of $y = A\log_a(x + C) + D$ is translated C units in the negative direction of the x-axis; if $C < 0$, then the graph is translated C units in the positive direction of the x-axis (right).

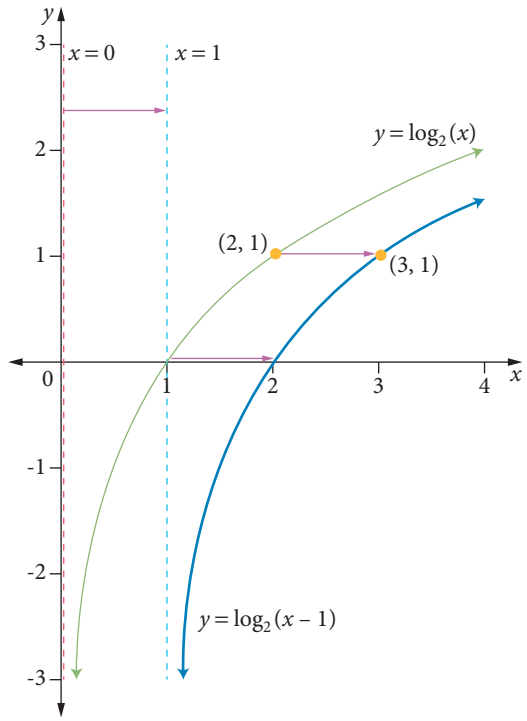
From the graph of $y = \log_a(x)$, the graph of $y = A\log_a(x + C) + D$ is translated D units in the positive direction of the y-axis; if $D < 0$, then the graph is translated D units is the negative direction of the y-axis (down).

$x = -C$ is a vertical asymptote.

The graphs of $y = \log_2(x)$ and $y = \log_2(x-1)$ are shown.

$y = \log_2(x-1)$ is obtained by translating the graph of $y = \log_2(x)$ by 1 unit in the positive direction of the x-axis. For example, $(2+1,1) \rightarrow (3,1)$.

The equation of the asymptote of $y = \log_2(x)$ is $x = 0$; the equation of the asymptote of $y = \log_2(x-1)$ is $x = 1$.



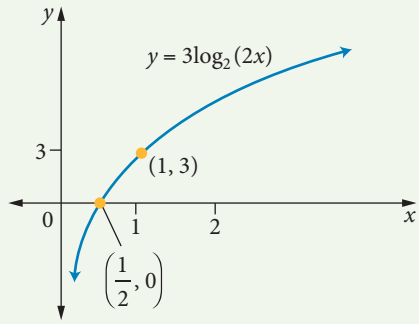
16 Sketch logarithmic graphs

Sketch the graph of each of the following functions.

(a) $y = 3\log_2(2x)$

THINKING	WORKING
1 Determine the coordinates of the x-intercept by letting $y = 0$ and solving for x.	$0 = 3\log_2(2x)$ $0 = \log_2(2x)$ $2^0 = 2x$ $1 = 2x$ $x = \frac{1}{2}$ The coordinates of the x-intercept are $\left(\frac{1}{2}, 0\right)$.
2 Determine the position of the asymptote.	The equation of the vertical asymptote is the y-axis; that is, $x = 0$.
3 Determine the coordinates of a second convenient point.	Where $x = 1$: $y = 3\log_2(2)$ $= 3$ The coordinates of another point are $(1, 3)$.

4 Sketch the graph.



(b) $y = 2\ln\left(\frac{1}{x}\right)$

1 Rewrite the equation using the logarithmic laws.

$$y = 2\ln\left(\frac{1}{x}\right)$$
$$= 2\ln\left(x^{-1}\right)$$
$$= -2\ln(x)$$

2 Determine the coordinates of the x-intercept by letting $y = 0$ and solving for x.

$$0 = -2\ln(x)$$
$$0 = \ln(x)$$
$$x = 1$$

The coordinates of the x-intercept are $(1, 0)$.

3 Determine the position of the asymptote.

The equation of the vertical asymptote is the y-axis: that is, $x = 0$.

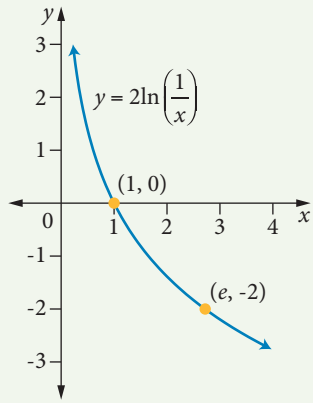
4 Determine the coordinates of a convenient point.

Where $x = e$:

$$y = -2\ln(e)$$
$$= -2$$

The coordinates of another point are $(e, -2)$.

5 Sketch the graph.



(c) $y = -\ln(x+2)$

1 Determine the coordinates of the x-intercept by letting $y = 0$ and solving for x.

$$0 = -\ln(x+2)$$
$$0 = \ln(x+2)$$
$$x + 2 = e^0$$
$$x + 2 = 1$$
$$x = -1$$

The coordinates of the x-intercept are $(-1, 0)$.

2 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the value of y.

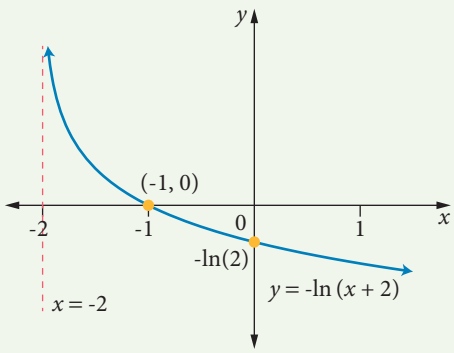
Where $x = 0$, $y = -\ln(2)$.

The coordinates of the y-intercept are $(0, -\ln(2))$.

3 Determine the equation of the asymptote.

The equation of the vertical asymptote is $x = -2$.

4 Sketch the graph.



(d) $y = 2\ln(x+1) - 3$

1 Determine the coordinates of the x-intercept by letting $y = 0$ and solving for x.

Where $y = 0$:

$$0 = 2\ln(x+1) - 3$$
$$\ln(x+1) = \frac{3}{2}$$
$$x + 1 = e^{\frac{3}{2}}$$
$$x = \sqrt{e^3} - 1$$

The coordinates of the x-intercept are $\left(\sqrt{e^3} - 1, 0\right)$.

2 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the value of y.

Where $x = 0$:

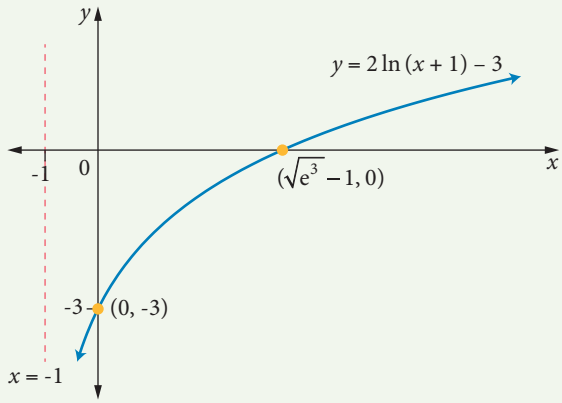
$$y = 2\ln(1) - 3$$
$$= -3$$

The coordinates of the y-intercept are $(0, -3)$.

3 Determine the equation of the asymptote.

The equation of the vertical asymptote is $x = -1$.

4 Sketch the graph.



(e) $y = 3 - 2\log_2(2 - x)$

1 Determine the coordinates of the x-intercept by letting $y = 0$ and solving for x.

Where $y = 0$:

$$0 = 3 - 2\log_2(2 - x)$$

$$\log_2(2 - x) = \frac{3}{2}$$

$$2 - x = 2^{\frac{3}{2}}$$

$$x = 2 - 2\sqrt{2}$$

The coordinates of the x-intercept are $\left(2 - 2\sqrt{2}, 0\right)$.

2 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the value of y.

Where $x = 0$:

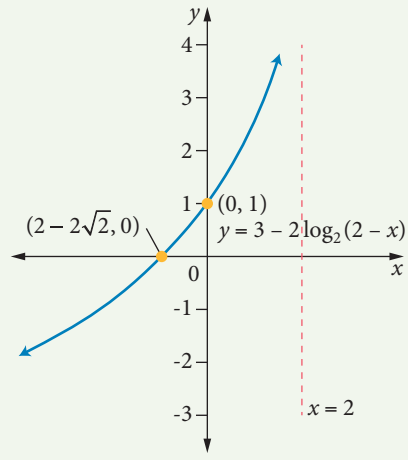
$$y = 3 - 2\log_2(2)$$
$$= 3 - 2$$
$$= 1$$

The coordinates of the y-intercept are $(0, 1)$.

3 Determine the equation of the asymptote.

The equation of the vertical asymptote is $x = 2$.

4 Sketch the graph.

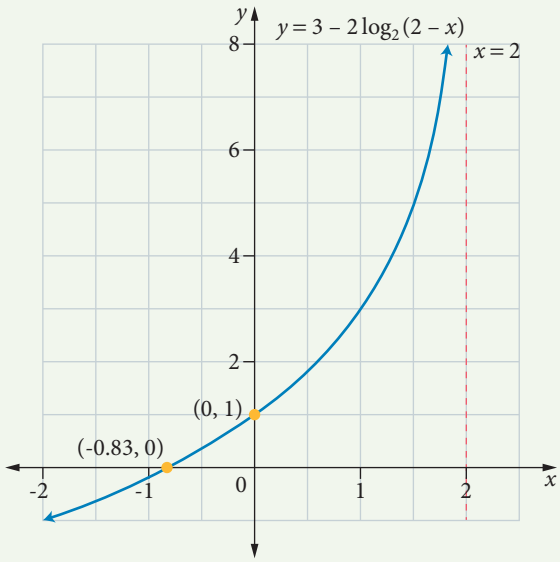


Logarithmic graphs using technology

To plot logarithmic graphs using technology, you may need to change the base to the natural (or common) logarithm first. You should identify the asymptote and add it to the calculator-drawn graph. Axis intercepts and other points can be read from the calculator, as decimals, rounded as required.

17 Plot logarithmic graphs using technology

Plot the graph of $y = 3 - 2\log_2(2 - x)$. Round axis intercepts correct to 2 decimal places.

THINKING	WORKING
1 Change the base to natural logarithms.	$y = 3 - 2\log_2(2 - x)$ $= 3 - \dfrac{2\ln(2 - x)}{\ln(2)}$
2 Determine the equation of the asymptote.	The equation of the vertical asymptote is $x = 2$.
3 Draw the graph and the asymptote, reading the coordinates of the axial intercepts from the graph.	
	The coordinates of the x-intercept are $(-0.83, 0)$ and the coordinates of the y-intercept are $(0, 1)$.

18 Determine the equation of a transformed logarithmic function

Determine the equations that result from each of the following transformations.

(a) The curve $y = \log_2(x)$ is dilated by a factor of 3 parallel to the y-axis and translated 2 units in the negative direction of the x-axis.

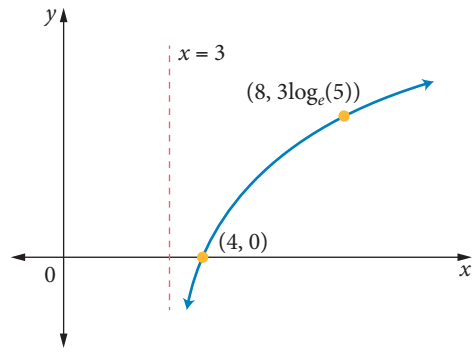
THINKING	WORKING
1 Dilate parallel to the y-axis.	$y = \log_2(x) \rightarrow y = 3\log_2(x)$
2 Translate in the negative x-direction.	$y = 3\log_2(x) \rightarrow y = 3\log_2(x + 2)$

(b) The curve $y = \log_e(x)$ is reflected in the y-axis, then translated 1 unit in the positive direction of the x-axis and 2 units in the positive y-direction.

1 Reflect in the y-axis.	$y = \log_e(x) \rightarrow y = \log_e(-x)$
2 Translate in the positive direction of the x-axis.	$y = \log_e(-x) \rightarrow y = \log_e(-(x-1))$
3 Translate in the positive direction of the y-axis.	$y = \log_e(-(x-1)) \rightarrow y = \log_e(-(x-1)) + 2$
4 Simplify the equation.	$y = \log_e(1-x) + 2$

19 Determine the rule for a logarithmic function

The rule for the function of the graph shown is in the form $y = A\log_e(x+c)$. Determine the values of A and c, hence state the rule of the graph.



THINKING	WORKING
1 Recall the form of the function.	$y = A\log_e(x+c)$
2 Substitute a given point into the equation. Alternatively, the asymptote gives the value of c.	At point $(4,0)$: $0 = A\log_e(4+c)$ $0 = \log_e(4+c)$ $4+c = 1$ $c = -3$ Therefore the rule is in the form $y = A\log_e(x-3)$.
3 Substitute the other given point into the equation.	For $(8, 3\log_e(5))$: $3\log_e(5) = A\log_e(8-3)$ $3\log_e(5) = A\log_e(5)$ $\therefore A = 3$
4 Give the values of the unknown variables and the rule of the function.	$A = 3, c = -3$ The rule of the function is $y = 3\log_e(x-3)$.

EXERCISE 3.4 Graphs of logarithmic functions

Worked Example 15

1 Consider the function $f(x) = 2^x + 1$.

(a) Sketch the graph of $f(x)$, showing the y-intercept and the asymptote.

(b) Determine the rule and the domain and range of the inverse f^{-1}, and sketch its graph on the same axes as f.

2 For each of the following functions, determine the rule and the domain and range of the inverse, f^{-1}.

(a) $f(x) = 2e^{x-1}$ **(b)** $f(x) = 2 - 3e^{x-1}$

(c) $f(x) = 2\ln(x-2)$ **(d)** $f(x) = 2\ln(x) - 2$

(e) $f(x) = 2 - \ln(x+1)$

3 Consider the function $f(x) = \ln(x+5)$.

(a) What is the expression for $f^{-1}(x)$?

A $e^x - 5$ **B** $e^x + 5$ **C** e^{x-5} **D** $\dfrac{1}{\ln(x+5)}$

(b) Explain the common error made by a student who misunderstood the notation $f^{-1}(x)$ to be the reciprocal function in part **(a)**.

4 If $f(x) = 3e^{2x}$, what is the expression for $f^{-1}(x)$?

A $\ln\left(\dfrac{\sqrt{x}}{3}\right)$ **B** $2\ln\left(\dfrac{x}{3}\right)$ **C** $\ln\left(\sqrt{\dfrac{x}{3}}\right)$ **D** $\ln\left(\dfrac{x}{6}\right)$

5 For each of the following, sketch the graph, showing the x-intercept and the asymptote. Plot a second convenient point.

(a) $y = \log_3(-x)$ **(b)** $y = 3\log_e(x)$ **(c)** $y = \log_2(2x)$

(d) $y = \log_2(-2x)$ **(e)** $y = -\log_2\left(\dfrac{x}{2}\right)$

6 For each of the following, sketch the graph, showing the x-intercept and the asymptote. Show the y-intercept if it exists, otherwise plot a second convenient point.

(a) $y = \log_2(x-3)$ **(b)** $y = \log_2(x+2)$ **(c)** $y = \ln(x) + 1$

(d) $y = \log_{10}(x) - 2$ **(e)** $y = \log_{10}(3-x)$ **(f)** $y = 2 - \log_e(x)$

7 Sketch the graph of $y = 2 - 5\ln(3-x)$, showing the asymptote and x- and y-intercepts correct to 2 decimal places. 17

8 Determine the equation that results from each of the following transformations. 18

(a) The curve $y = \log_{10}(x)$ is reflected in the x-axis, then translated 2 units up.

(b) The curve $y = \ln(x)$ is dilated by a factor of 2 parallel to the y-axis, then translated 2 units in the positive y-direction and 3 units in the negative x-direction.

(c) The curve $y = \ln(x)$ is dilated by a factor of $\dfrac{1}{2}$ parallel to the x-axis, reflected in the y-axis, then translated 2 units in the positive x-direction.

9 Use transformations to explain how the graph of each of the following functions is related to the graph of $y = \ln(x)$.

Worked Example

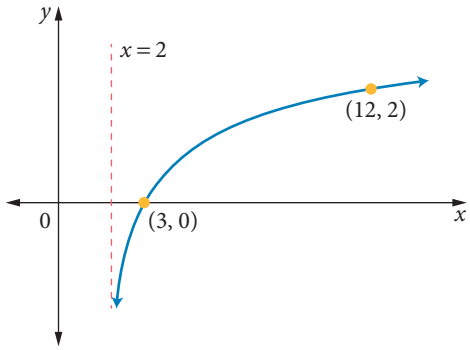
(a) $y = 3\ln(x-2)$ **(b)** $y = 2\ln(x+1)+3$

(c) $y = 5 - \ln(2-x)$ **(d)** $y = 3 - \ln(2x-4)$

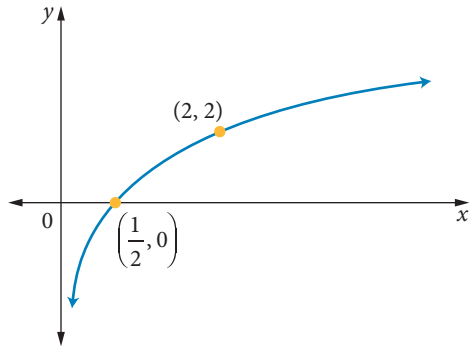
19

10 For each of the following, determine the value of the unknown variables and hence the rule for the graph.

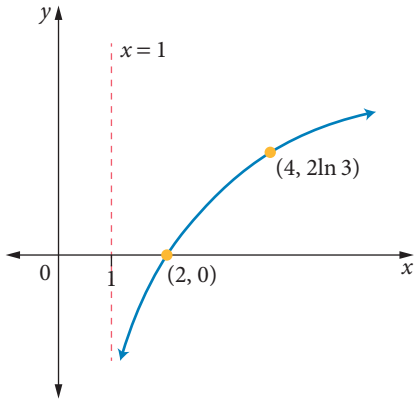
(a) $y = A\log_{10}(x+b)$



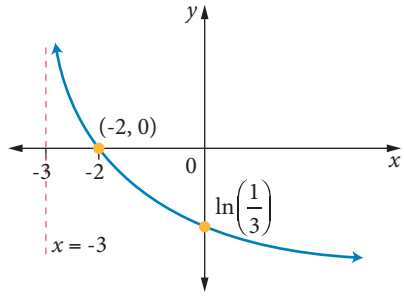
(b) $y = A\log_2(bx)$



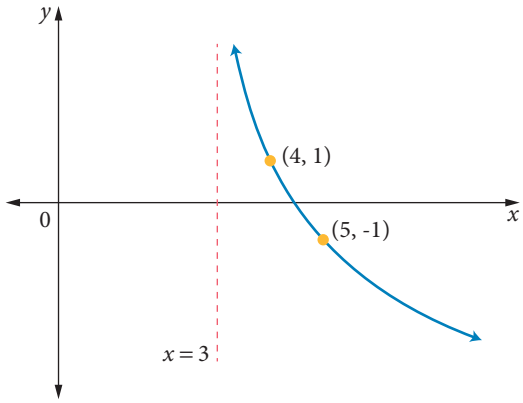
(c) $y = A\ln(x+b)$



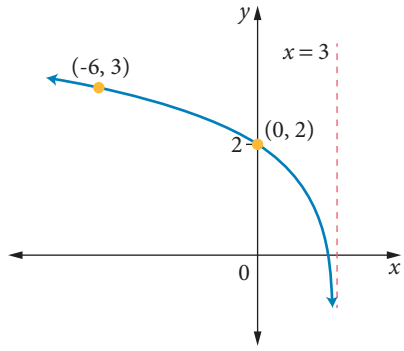
(d) $y = A\ln(b(x+c))$



(e) $y = A\log_2(x+b)+c$



(f) $y = \log_3(b-x)+c$



11 For what values of x does each of the following functions exist?

(a) $y = \ln(3x-1)$ **(b)** $y = \log_2(x(x-3))$ **(c)** $y = \ln((2-x)(x+3))$

Worked Example 16

12 For each of the following, sketch the graph, showing the x-intercept and the asymptote. Show the y-intercept if it exists, otherwise plot a second convenient point.

(a) $y = \log_2(x+2) - 3$ **(b)** $y = \log_2(x-1) + 2$

(c) $y = 2\log_2(2x) + 1$ **(d)** $y = \log_e(2x+2)$

(e) $y = 1 - \log_e(x+2)$

13 Consider the functions $f(x) = 2\log_2(2x)$ and $g(x) = 2^{\frac{x}{2}-1}$.

(a) Sketch the graph of f.

(b) Use your graph to determine the value(s) of x that satisfy the following restrictions.

(i) $2\log_2(2x) = 4$ **(ii)** $2\log_2(2x) > 8$ **(iii)** $2\log_2(2x) < 6$

(c) Demonstrate that the graphs of $f(x) = 2\log_2(2x)$ and $g(x) = 2^{\frac{x}{2}-1}$ are a pair of inverse functions by showing that $f(g(x)) = x$.

(d) Determine the points of intersection for $f(x) = 2\log_2(2x)$ and $g(x) = 2^{\frac{x}{2}-1}$ correct to 1 decimal place.

14 The graphs of $f(x) = 2^{x-1} - 3$ and $g(x) = \log_3(x+3) + k$ have the same y-intercept at point A. Determine the value of k and the coordinates of A.

15 The graphs of $f(x) = \left(\frac{1}{3}\right)^{x+3} - 1$ and $g(x) = \log_{\frac{1}{2}}(x+19) + p$ have the same x-intercept at point B. Determine the value of p and the coordinates of B.

16 Point A (10, 1) lies on the graph of $f(x) = \log_{10}(x)$. After a series of transformations, $f(x)$ became $g(x) = -2f(3x-1) + 1$.

(a) Determine the coordinates of the image point A' now on the graph of $g(x)$.

(b) Determine the expression for $g^{-1}(x)$.

(c) Demonstrate that $g\left(g^{-1}(x)\right) = x$.

17 A series of transformations is described below:

(i) reflection in the x-axis

(ii) reflection in the line $y = x$

(iii) translation 2 units to the right

(iv) dilation by a factor of 2 parallel to the y-axis.

Determine the image of each of the following functions after the four transformations have been made, in the order given. Also determine the domain and range of the final image.

(a) $y = -e^{3x}$ **(b)** $y = \ln(1-x)$

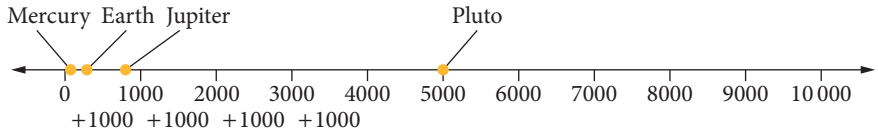
3.5 Logarithmic scales

The need for a logarithmic scale

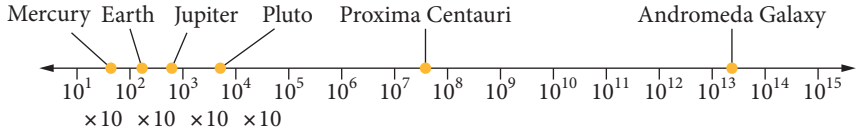
Consider the following table, which gives the distances of objects from the Sun in millions of kilometres.

Object	Mercury	Earth	Jupiter	Pluto	Proxima Centauri	Andromeda Galaxy
Distance from the Sun (in millions of km)	58	149	778	5000	4.1×10^{7}	2.4×10^{13}

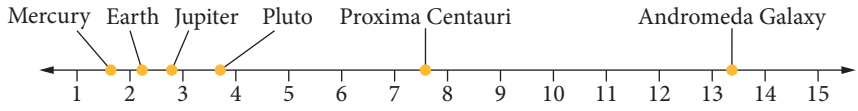
The linear scale below shows the data in the table. A linear scale is an additive scale. For example, if you look at the scale, each 1000 step is equally spaced out; that is, the distance from 0 to 1000 is the same as the distance from 7000 to 8000.



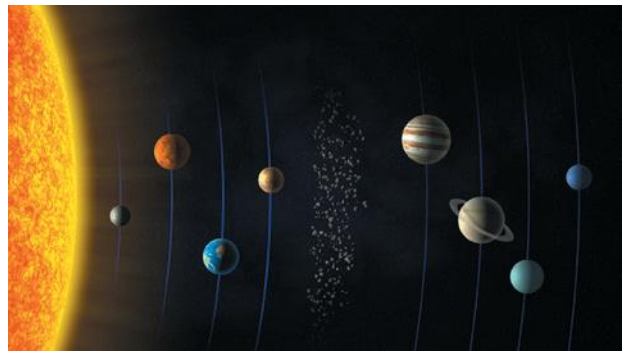
The linear scale graphs the objects close to the Sun quite well, but it does not work for distant objects like Proxima Centauri, which would be another 40 000 units further away on the scale. When numbers vary by large amounts, it is impractical to try to fit all of them on the same linear scale. The logarithmic scale below graphs all the objects more effectively.



The graph below shows only the logarithm (power of 10) of each number.



The logarithmic scale is a multiplicative scale.
It is graduated in powers of 10 instead of units of 1.
It shows Andromeda Galaxy as about 10^{13} and Earth as approximately 10^{2}; this means that the Andromeda Galaxy is about 10^{11} or 100 000 000 000 times further from the Sun than the Earth is. More precisely the logarithms of the distances can be given as follows:



Andromeda Galaxy: $\log_{10}\left(2.4\times10^{13}\right)=13.38$

Earth: $\log_{10}(149)=2.17$

Difference: $13.38-2.17=11.21$

This means that the Andromeda Galaxy is $10^{11.21}$ or 162 181 000 000 times further from the Sun than the Earth is.

Additional information

Logarithmic scale

Watch this video for a brief outline of why the logarithmic scale is used.

Commonly used logarithmic scales

Logarithmic scales are useful for measuring quantities that are widely spread out. Logarithmic scales include the decibel scale, which measures the loudness of sound; the Richter scale, which measures the strength of earthquakes; and the pH scale, which measures acidity levels.

Additional information

Applications of logarithms

Watch the video to see some applications of logarithms.

The Richter scale

The Richter scale assigns a single number (R) to quantify the magnitude of an earthquake. An increase of one on the Richter scale corresponds to an increase of ten times in size. For example, an earthquake measuring 6 on the Richter scale is 10 times as strong as one measuring 5 on the Richter scale. An earthquake of magnitude 7 is $10^2 = 100$ times as strong as an earthquake of magnitude 5. An earthquake of magnitude 8 is $10^3 = 1000$ times as strong as an earthquake of magnitude 5.

The most common logarithmic scale uses powers of 10.

… 0.001, 0.01, 0.1, 1, 10, 100, 1000 …

can be written as

$\ldots 10^{-3}, 10^{-2}, 10^{-1}, 10^{0}, 10^{1}, 10^{2}, 10^{3} \ldots$

Or showing the logarithmic values only: $\ldots -3\log_{10}(10), -2\log_{10}(10), -1\log_{10}(10), 0\log_{10}(10), 1\log_{10}(10), 2\log_{10}(10), 3\log_{10}(10)\ldots$

which can be simplified to: … -3, -2, -1, 0, 1, 2, 3 …

The formula for measuring an earthquake on the Richter scale is given by:

$$R = \log_{10}\left(\frac{I}{I_0}\right)$$

where R is magnitude of the earthquake on the Richter scale, I is intensity of the earthquake, I_0 is intensity of the threshold earthquake (can hardly be detected).

An increase of 1 on the Richter scale corresponds to an earthquake ten (10^1) times the magnitude. An increase of 2 on the Richter scale corresponds to an earthquake of one hundred (10^2) times the magnitude.

Additional information

The Richter scale

Watch this video for a brief explanation of how the intensity of earthquakes is measured.

Additional information

The impact of an earthquake on the Earth's surface

Try this activity to see the impact the size of the earthquake has on the Earth's surface. Move the slider on the left up or down and then click on the image and start the video animation.

20 Determine and compare the intensity of an earthquake

The 2004 Indian Ocean earthquake measured 9.15 on the Richter scale.

(a) Determine its intensity in terms of the threshold intensity I_0.

THINKING	WORKING
1 Recall the rule for the magnitude of an earthquake, and substitute the given R-value.	$R = \log_{10}\left(\frac{I}{I_0}\right)$ $9.15 = \log_{10}\left(\frac{I}{I_0}\right)$
2 Write the equation in index form.	$\frac{I}{I_0} = 10^{9.15}$
3 Make the intensity (I) the subject.	$I = 10^{9.15} I_0$
4 Write using scientific notation by keying in $10^{0.15}$ and rounding to 3 significant figures.	$I = 1.41 \times 10^9 I_0$
5 Interpret the answer.	The 2004 Indian Ocean earthquake measured 1.41×10^9 times the threshold of intensity.

(b) If the 2010 earthquake in New Zealand measured 7.1 on the Richter scale, how many times as intense was the Indian Ocean earthquake compared to the New Zealand one? Give your answer correct to the nearest whole number.

METHOD 1: USE LOGARITHMS

1 Define a variable for the intensity of each earthquake.	Let I_1 be the intensity of the Indian Ocean earthquake. Let I_2 be the intensity of the New Zealand earthquake.
2 Recall the rule for the magnitude of an earthquake and substitute the two intensities to create two equations.	$R = \log_{10}\left(\frac{I}{I_0}\right)$ Indian Ocean: $9.15 = \log_{10}\left(\frac{I_1}{I_0}\right)$ [1] New Zealand: $7.1 = \log_{10}\left(\frac{I_2}{I_0}\right)$ [2]
3 Subtract the smaller R-value from the larger. Simplify the logarithms using the law for subtraction.	[1] − [2] $9.15 - 7.1 = \log_{10}\left(\frac{I_1}{I_0}\right) - \log_{10}\left(\frac{I_2}{I_0}\right)$ $2.05 = \log_{10}\left(\frac{I_1}{I_2}\right)$

4 Write the equation in index form.	$\frac{I_1}{I_2} = 10^{2.05}$
5 Make the intensity of the Indian Ocean earthquake the subject and calculate the power of 10. Write your answer correct to the nearest whole number.	$I_1 = 10^{2.05} I_2$ $I_1 \approx 112 I_2$
6 Interpret the answer.	The Indian Ocean earthquake was approximately 112 times as intense as the New Zealand earthquake.

METHOD 2: USE AN EXPONENTIAL

1 Determine the difference of the two magnitudes.	$9.15 - 7.1 = 2.05$
2 Since it is a logarithmic scale, the comparative factor for intensity is given by $10^{\text{difference in magnitude}}$.	$10^{2.05} = 112$ (nearest whole number)
3 Interpret the answer.	The Indian Ocean earthquake was approximately 112 times as intense as the New Zealand earthquake.

The simpler method makes use of the formula developed below:

$$R_1 = \log_{10}\left(\frac{I_1}{I_0}\right) \text{ and } R_2 = \log_{10}\left(\frac{I_2}{I_0}\right)$$

$$\begin{aligned} R_2 - R_1 &= \log_{10}\left(\frac{I_2}{I_0}\right) - \log_{10}\left(\frac{I_1}{I_0}\right) \\ &= \log_{10}\left(\frac{I_2}{I_0} \div \frac{I_1}{I_0}\right) \\ &= \log_{10}\left(\frac{I_2}{I_1}\right) \\ R_2 &= R_1 + \log_{10}\left(\frac{I_2}{I_1}\right) \end{aligned}$$

21 Calculate the magnitude of an earthquake using the Richter scale

A major earthquake of magnitude 8.3 is 120 times as intense as a minor earthquake. Determine the magnitude of the minor earthquake. Give your answer correct to 1 decimal place.

THINKING	WORKING
1 Define a pronumeral for each variable. Write the information given in the problem in terms of these variables.	Let R_1 be the magnitude of the major earthquake and I_1 be the intensity of the major earthquake. Let R_2 be the magnitude of the minor earthquake and I_2 be the intensity of the minor earthquake.
2 Identify the given information in terms of the variables defined.	$R_1 = 8.3$ $I_1 = 120I_2$
3 Substitute the values into the rule for comparing the magnitude of two earthquakes.	$R_2 = R_1 + \log_{10}\left(\frac{I_2}{I_1}\right)$ $= 8.3 + \log_{10}\left(\frac{I_2}{120I_2}\right)$ $= 8.3 + \log_{10}\left(\frac{1}{120}\right)$ $= 8.3 - \log_{10}(120)$ $= 6.2\,(1\text{ d.p.})$
4 Interpret the answer.	The magnitude of the minor earthquake was 6.2.

Sound and decibels

Sounds can range from extremely quiet (e.g. a pin drop) to extremely loud (e.g. an aircraft taking off). To measure loudness, a sound's 'intensity' can be measured as its power per area, using standard units of watts per square metre. Although the personal experience of 'loudness' varies depending on the health of your ears, this measurement of sound intensity can measure 'loudness' in a universal way.

However, sound intensity in units of W/m^2 covers a very broad range of numbers, so it is more convenient to use a logarithmic scale instead. The scale of sound 'intensity level' is measured in decibels (dB).

The scale is set so that 0 dB is the threshold level of human hearing, a sound so quiet that it can barely be heard. Normal conversation has a level of 60–70 dB, while sounds above about 100 dB can damage human hearing if experienced for too long.

Sound is measured in decibels using the formula:

$$D = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where D is intensity level of the sound (in dB), I_0 is the intensity of the human hearing threshold sound (1.0×10^{-12} W/m^2), and I is the intensity of the sound being measured (in W/m^2).

Additional information

The intensity of sound

Watch this video for an introduction to measuring the intensity of sound in decibels.

22 Calculate the decibel level of a sound

If a sound is 200 times more intense than the threshold sound, determine its loudness to the nearest decibel.

THINKING	WORKING
1 Recall the rule for the loudness of a sound.	$D = 10\log_{10}\left(\frac{I}{I_0}\right)$
2 Substitute the given information to form an equation.	Substitute $I = 200I_0$: $D = 10\log_{10}\left(\frac{200I_0}{I_0}\right)$
3 Simplify and calculate the value of D.	$D = 10\log_{10}(200)$ $= 23.010\ldots$
4 Interpret the answer.	The loudness of the sound is 23 dB.

pH scale

The acidity of a chemical solution is related to its concentration of H^+ (hydrogen ions). When this measurement of concentration is converted to the logarithmic pH scale, it can be used to measure and classify chemical solutions more conveniently. A neutral solution has a pH of 7, an acid has a pH of less than 7, and a base has a pH of more than 7. An increase of 1 on the pH scale effectively corresponds to a decrease in H^+ concentration by a factor of ten, so a pH of 5 is ten times more acidic than a pH of 6.

The pH of a solution is defined as:

$$\text{pH} = -\log_{10}\left[H^+\right]$$

where $\left[H^+\right]$ is the effective concentration of hydrogen ions in moles per litre.

Hence, each unit increase in pH corresponds to a decrease in the activity of hydrogen ions $\left[H^+\right]$ by a factor of 10. The 'activity' of hydrogen ions is the effective concentration of H^+ ions in an ideal solution, a more complex concept that is not covered in this course.

23 Compare pH values

A solution with a pH of 3.0 has a hydrogen ion concentration that is one-tenth of a solution with what level of pH?

THINKING	WORKING
1 Each unit increase in pH corresponds to a decrease in $\left[H^+\right]$ by a factor of 10.	The other solution has $10 = 10^1$ the concentration. Hence, its pH will be lower by 1. $3.0 - 1 = 2.0$
2 Interpret the answer.	The solution with ten times the concentration of hydrogen ions has a pH of 2.0.

24 Calculate a pH value

Determine the pH of milk, whose concentration of hydrogen ions is 4×10^{-7} moles/litre. Give your answer correct to 1 decimal place.

THINKING	WORKING
1 Recall the formula for calculating pH.	$\text{pH} = -\log_{10}\left[\text{H}^+\right]$
2 Identify and substitute the given concentration of hydrogen ions into the rule for calculating the pH.	Given $\left[\text{H}^+\right] = 4 \times 10^{-7}$ $\text{pH} = -\log_{10}\left(4 \times 10^{-7}\right)$ $= 6.4\ (1\text{ d.p.})$
3 Interpret the answer.	The pH of the milk is 6.4.

EXERCISE 3.5 Logarithmic scales

Worked Example 20

1 Earthquake *A* measured 8.2 on the Richter scale.
   (a) Determine the intensity of the earthquake in terms of the threshold intensity. Give your answer as a power of 10.
   (b) If earthquake *B* measured 5.1 on the Richter scale, how many times as intense was earthquake *A* compared to earthquake *B*? Give your answer rounded to the nearest whole number.

2 If an earthquake has an intensity level 872 times that of a base level earthquake, determine its measurement on the Richter scale. Give your answer correct to 1 decimal place.

3 A 2008 earthquake in Rhodes, in Greece, measured 6.5 on the Richter scale, while the 1989 San Francisco earthquake measured 6.9. How many times as intense was the San Francisco earthquake as the one in Rhodes? Give your answer correct to 1 decimal place.

4 The 1906 San Francisco earthquake measured 8.25 on the Richter scale, while the 1989 Newcastle (NSW) earthquake measured 5.5 on the Richter scale. How many times as intense was the San Francisco earthquake as the one in Newcastle? Give your answer correct to the nearest whole number.

22 5 Determine the loudness of a television with an intensity of 3.20×10^{-5} W/m^2. Give your answer to the nearest decibel.

6 Prolonged exposure to sounds above 100 decibels can cause deafness. If the loudness of a pneumatic drill has an intensity of 1.58×10^{10} times the threshold intensity, what is its loudness in decibels? Give your answer to the nearest decibel.

7 If a sound measures 85 dB, how many times as intense as the threshold sound is it? Give your answer in scientific notation, correct to 2 decimal places.

8 A given acidic solution has a pH of 4. 23

(a) What is the pH of a solution that has a hydrogen ion concentration 100 times greater?

A 5 **B** 3 **C** 2 **D** 6

(b) Explain the common error made in part **(a)** by a student who misunderstood the value of pH in terms of the increasing or decreasing hydrogen concentration.

9 Solution A has a pH of 3 and solution B has a pH of 6. How many times greater is the hydrogen ion concentration in solution A than the hydrogen ion concentration in solution B?

A 100 **B** 3 **C** 2 **D** 1000

10 The hydrogen ion concentration for a liquid is 4.0×10^{-8} moles/litre. Calculate the pH value. Give your answer correct to 1 decimal place. 24

11 Determine the concentration of hydrogen ions of acid rain with a pH of 5.0.

12 The largest earthquake recorded occurred in Chile in 1960. It was about 18 times as intense as the 1906 San Francisco earthquake, which measured 8.25 on the Richter scale. What did the earthquake in Chile measure on the Richter scale? Give your answer correct to 2 decimal places. 21

13 Earthquake A had a magnitude of 8.2 on the Richter scale. Earthquake B was 2.3 times as intense. What was the magnitude of earthquake B on the Richter scale? Give your answer correct to 1 decimal place.

14 The loudness of Amanda's voice when she is singing has been measured to be 40 dB, while Kim's was measured to be 50 dB. How many times more intense is Kim's singing than Amanda's?

15 The loudness of a vacuum cleaner from a distance of 10 m is 70 dB, while the loudness of a motorcycle from the same distance is 3.2 times as intense. What is the loudness of the motorcycle? Give your answer to the nearest decibel.

16 Two solutions X and Y have pH values of 4.3 and 6.5, respectively. Determine the ratio of the hydrogen ion concentration of X to that of Y, correct to 3 significant figures.

17 You are given the formula $D = 10 \log_{10}\left(\frac{I}{I_0}\right)$.

(a) Express I in terms of D and I_0.

(b) Two sounds of intensity I_1 and I_2 register decibel levels of D_1 and D_2. Express $\frac{I_2}{I_1}$ as a power of 10 in terms of D_1 and D_2.

(c) If a sound registers 20 decibels more than another, how many times as intense is this sound?

18 The interval between two musical tones can be calculated from their sonic frequencies, using the formula:

$$C = 1200 \log_2\left(\frac{f_2}{f_1}\right)$$

where f_1 is the original tone's frequency in kHz, f_2 is the final tone's frequency in kHz, and C is the musical interval between the tones, measured in cents.

Here, a 'cent' is a fraction of a musical interval defined so that 100 cents is equal to one semitone. A semitone is the usual difference in tone between two adjacent keys on a piano, or between two adjacent frets on a guitar. An octave is equal to 12 semitones.

(a) What is the musical interval between a sound with frequency 44.1 kHz and a sound with frequency 32 kHz? Give your answer to the nearest:

(i) cent **(ii)** semitone

(b) If the number of cents in an interval is 500, what is the exact ratio of the final frequency compared to the original frequency?

(c) Write the formula $C = 1200 \log_2\left(\frac{f_2}{f_1}\right)$ using common logarithms.

19 The Palermo Impact Scale was developed to categorise the potential risks involved with Near Earth Objects. A scale value less than -2 means the event will not have any likely consequences, a value between -2 and 0 indicates the event requires careful monitoring, while a value greater than 0 indicates an event that is of some concern. The Palermo Impact Scale value (P) is calculated using the formula: $P = \log_{10}\left(\frac{p_i}{f_B T}\right)$, where p_i is the impact probability of the event in question, and T is the time until the potential event, measured in years.

The annual background impact frequency f_B is the annual probability of an impact event with energy E, in megatons, at least as large as the event in question. This annual background impact frequency is defined as: $f_B = 0.03E^{-\frac{4}{5}}$

(a) If the impact probability of an object is 0.2, the time until the potential event is 20 years and the energy of the impact is 200 megatons, then determine the measurement on the Palermo Impact Scale. Give your answer correct to 2 decimal places.

(b) If the Palermo Impact Scale value is -3.53, the time until the potential event is 50 years and the impact probability is 0.01, then determine the following, correct to 2 decimal places.

(i) background impact frequency

(ii) energy of the impact

(c) Write a formula for the energy, E, using index notation.

3.6 The derivative of the natural logarithm

Calculus and natural logarithms

The natural logarithm is used for the calculus of the logarithm. If you determined the gradient of the tangent at various points on the curve $y = \log_e(x)$ and sketched the gradient function, you would obtain the graph of $y = \frac{1}{x}$ for $x > 0$.

The derivative of $y = \log_e(x)$ or $y = \ln(x)$ can also be obtained by using the derivative of its inverse function, $y = e^x$, and the relationship $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Explore further

The derivative of the natural logarithm

Explore the graph of $f(x) = A\log_e(kx)$ and its derivative.

$y = \ln(x) \Leftrightarrow x = e^y$

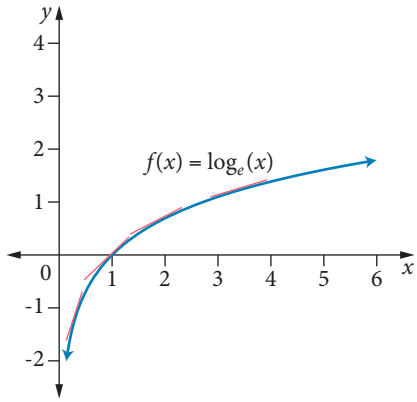
$\frac{dx}{dy} = e^y$

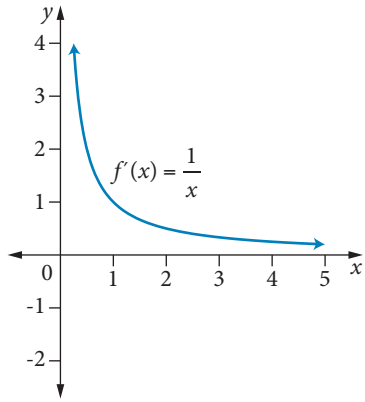
But, $x = e^y$, so:

$$\frac{dx}{dy} = x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \frac{d}{dx}(\ln(x)) = \frac{1}{x}$$





The derivative of $\log_e(kx)$

$y = \log_e(kx)$

Let $u = kx$ and $y = \log_e(u)$.

$\frac{du}{dx} = k$ and $\frac{dy}{du} = \frac{1}{u}$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u}k$$
$$= k\frac{1}{kx}$$
$$= \frac{1}{x}$$

Using the logarithmic laws:

$$y = \log_e(kx)$$
$$= \log_e(k) + \log_e(x)$$

$$\frac{dy}{dx} = 0 + \frac{1}{x}$$
$$= \frac{1}{x}$$

$$\therefore \frac{d}{dx}(\log_e(kx)) = \frac{1}{x}$$

25 Differentiate the natural logarithm

Determine the expression for each of the following derivatives.

(a) $\frac{d}{dx}\left(2\log_e(x)\right)$

THINKING	WORKING
1 Recall the formula for the derivative of the natural log.	$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$
2 Substitute the given values and determine the derivative.	$\frac{d}{dx}\left(2\log_e(x)\right)=2\times\frac{1}{x}$ $=\frac{2}{x}$

(b) $\frac{d}{dx}\left(\log_e(5x)\right)$

1 Recall the formula for the derivative of the natural log.	$\frac{d}{dx}\left(\log_e(kx)\right)=\frac{1}{x}$
2 Substitute the given values and determine the derivative.	$\frac{d}{dx}\left(\log_e(5x)\right)=\frac{1}{x}$

(c) $\frac{d}{dx}\left(\log_e\left(x^2\right), x>0\right.$

1 Recall the law.	$\log_a\left(m^n\right)=n\log_a(m)$
2 Substitute the given values.	$\frac{d}{dx}\left(\log_e\left(x^2\right)\right)=\frac{d}{dx}\left(2\log_e(x)\right)$
3 Recall the formula for the derivative of the natural log	$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$
4 Determine the derivative.	$\frac{d}{dx}\left(\log_e\left(x^2\right)\right)=\frac{d}{dx}\left(2\log_e(x)\right)$ $=2\times\frac{1}{x}$ $=\frac{2}{x}$

In part **(c)** of Worked Example 25, the log law could be used because the domain was specified as $x>0$. If the domain were not given, you would assume the maximal domain. In that case, $\log_e\left(x^2\right)$ and $2\log_e(x)$ are not equivalent since $\log_e\left(x^2\right)$ is defined for $x<0$ and $x>0$, whereas $2\log_e(x)$ is defined for $x>0$ only.

The derivative of $\log_e(ax+b)$

Given $\log_e(ax+b)$, $x > -\frac{b}{a}$:

Let $u = ax + b$ and $y = \log_e(u)$.

$\frac{du}{dx} = a$ and $\frac{dy}{du} = \frac{1}{u}$

Using the chain rule:

$$\frac{dy}{dx} = a \times \frac{1}{u}$$
$$= \frac{a}{ax+b}$$

Natural log laws:

$$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$$
$$\frac{d}{dx}(\log_e(kx)) = \frac{k}{kx} = \frac{1}{x}$$
$$\frac{d}{dx}(\log_e(ax+b)) = \frac{a}{ax+b}$$

26 Differentiate natural logarithms of the form $\log_e(ax+b)$

Determine the derivative of each of the following.

(a) $y = \log_e(4x+5)$

THINKING	WORKING
1 Recall the formula for the derivative of the natural log.	$\frac{d}{dx}(\log_e(ax+b)) = \frac{a}{ax+b}$
2 Substitute the given values and determine the derivative.	$y = \log_e(4x+5)$ $\frac{dy}{dx} = \frac{4}{4x+5}$

(b) $y = \log_e\left(\sqrt{2-3x}\right)$

1 Rewrite the equation using the logarithm laws.	$y = \log_e\left(\sqrt{2-3x}\right)$ $= \log_e\left((2-3x)^{\frac{1}{2}}\right)$ $= \frac{1}{2}\log_e(2-3x)$
2 Recall the formula for the derivative of the natural log	$\frac{d}{dx}(\log_e(ax+b)) = \frac{a}{ax+b}$
3 Substitute the given values and determine the derivative.	$\frac{dy}{dx} = \frac{-3}{2(2-3x)}$ $= \frac{3}{2(3x-2)}$

Differentiation patterns involving $\log_e(f(x))$

If $y = \ln(f(x)), f(x) > 0$

Let $y = \ln(u)$, where $u = f(x)$.

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times f'(x)$$
$$= \frac{f'(x)}{f(x)}$$

The derivative of $\log_e$ (function) has the form:

$$\frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{f'(x)}{f(x)}$$

given that the function is positive; that is, for $f(x) > 0$.

27 Derivatives involving $\log_e(f(x))$

For each of the following functions, determine the maximal domain and the derivative.

(a) $g(x) = \log_e(4 - 7x)$

THINKING	WORKING
1 Determine the largest set of x-values for which $4 - 7x > 0$.	$4 - 7x > 0$ $-7x > -4$ $x < \frac{4}{7}$ The maximal domain is $\left(-\infty, \frac{4}{7}\right)$.
2 Recall the formula for the derivative of the natural log.	$\frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{f'(x)}{f(x)}$
3 Substitute the given values.	$g(x) = \log_e(4 - 7x)$ $g'(x) = \frac{-7}{4 - 7x}$ $= \frac{7}{7x - 4}$
4 Interpret the answer.	$g'(x) = \frac{7}{7x - 4}, x \in \left(-\infty, \frac{4}{7}\right)$

(b) $g(x) = \log_e\left(x^2 + 1\right)$

1 Determine the largest set of x-values for which $x^2 + 1 > 0$.	$x^2 + 1$ is always positive. The maximal domain is $\mathbb{R}$.
2 Recall the formula for the derivative of the natural log.	$\frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{f'(x)}{f(x)}$
3 Substitute the given values.	$g(x) = \log_e\left(x^2 + 1\right)$ $g'(x) = \frac{2x}{x^2 + 1}$
4 Interpret the answer.	$g'(x) = \frac{2x}{x^2 + 1}, x \in \mathbb{R}$

(c) $g(x) = \log_e\left(e^x - 4\right)$

1 Solve $e^x - 4 > 0$ to determine the maximal domain.	$e^x - 4 > 0$ $e^x > 4$ $x > \ln(4)$ $x > \ln\left(2^2\right)$ $x > 2\ln(2)$ The maximal domain is $\left(2\ln(2), \infty\right)$.
2 Recall the formula for the derivative of the natural log.	$\frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{f'(x)}{f(x)}$
3 Substitute the given values.	$g(x) = \log_e\left(e^x - 4\right)$ $g'(x) = \frac{e^x}{e^x - 4}$
4 Interpret the answer.	$g'(x) = \frac{e^x}{e^x - 4}, x \in \left(2\ln(2), \infty\right)$

Using the product and quotient rules

Along with the chain rule, you can use the product and quotient rules for differentiating expressions with logarithms.

28 Use the product rule to differentiate the natural logarithm

Determine the derivative of $y = (2x+3)\log_e(2x)$.

THINKING	WORKING
1 Define and differentiate the factors u and v.	Let $u = 2x+3$ and $v = \log_e(2x)$. $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = \frac{1}{x}$
2 Recall the product rule, substitute the values and simplify.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $= (2x+3)\frac{1}{x} + \log_e(2x)\times(2)$ $= \frac{2x+3}{x} + 2\log_e(2x)$
3 Interpret the answer, including an appropriate domain.	$\frac{dy}{dx} = \frac{2x+3}{x} + 2\log_e(2x),\ x>0$

29 Use the quotient rule to differentiate the natural logarithm

Determine the derivative of each of the following.

(a) $y = \frac{\ln(x+2)}{x}$

THINKING	WORKING
1 Define and differentiate the dividend and divisor u and v.	Let $u = \ln(x+2)$ and $v = x$. $\frac{du}{dx} = \frac{1}{x+2}$ and $\frac{dv}{dx} = 1$
2 Recall the quotient rule, substitute the values and simplify.	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $= \frac{x\times\frac{1}{x+2} - \ln(x+2)\times 1}{x^2}$ $= \frac{x-(x+2)\ln(x+2)}{x^2(x+2)}$

(b) $y = \log_e\left(\frac{x^2}{2+x}\right)$

THINKING	WORKING
1 Express a in terms of x, and y in terms of a.	Let $a = \frac{x^2}{2+x}$ and $y = \log_e(a)$.
2 Differentiate u using the quotient rule, and simplify.	$a = \frac{x^2}{2+x}$ Let $u = x^2$ and $v = 2+x$. $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 1$ $\frac{da}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $= \frac{(2+x)(2x) - x^2(1)}{(2+x)^2}$ $= \frac{4x + x^2}{(2+x)^2}$ $= \frac{x(4+x)}{(2+x)^2}$
3 Differentiate y with respect to a. Substitute $a = \frac{x^2}{2+x}$ and simplify the equation.	$\frac{dy}{da} = \frac{1}{a}$ $= \frac{1}{\left(\frac{x^2}{2+x}\right)}$ $= \frac{2+x}{x^2}$
4 Recall the chain rule, substitute the values and simplify.	$\frac{dy}{dx} = \frac{da}{dx} \times \frac{dy}{da}$ $= \frac{x(4+x)}{(2+x)^2} \times \frac{2+x}{x^2}$ $= \frac{x+4}{x(x+2)}$

You may be able to simplify using the log laws before differentiating. If you do, make sure that the domain of any new expression is the same as the domain of the original. In Worked Example 29, $\log_e\left(\frac{x^2}{2+x}\right)$ could have been written as $\log_e\left(x^2\right) - \log_e(2+x)$ and $\frac{\ln(x+2)}{x}$ could have been written as $x^{-1}\ln(x+2)$, making the quotient rule unnecessary in each case.

EXERCISE 3.6 The derivative of the natural logarithm

Worked Example

25

1 Differentiate each expression.

(a) $\log_e(3x)$ (b) $\log_e(-5x)$ (c) $-3\ln(2x)$

(d) $\ln\left(-\frac{x}{2}\right)$ (e) $5\log_e(2)$ (f) $\log_e\left((2x)^2\right)$

(g) $\ln\left(\sqrt{x}\right)$ (h) $\ln\left(\frac{1}{x}\right)$ (i) $\ln\left(\frac{1}{x^2}\right)$

26

2 Differentiate each expression.

(a) $\log_e(2x-1)$ (b) $\log_e(5x+2)$ (c) $\log_e(1-4x)$ (d) $\log_e(1-2x)$

(e) $\ln\left(\sqrt{2x+1}\right)$ (f) $\ln\left(\frac{1}{x+2}\right)$ (g) $\ln\left(\frac{6}{3x-2}\right)$

27

3 For each of the following functions, determine the maximal domain and an expression for the derivative.

(a) $f(x)=\ln\left(4x^2\right)$ (b) $f(x)=\ln\left(2\sqrt{x}\right)$

(c) $f(x)=\log_e\left(x^2-2\right)$ (d) $f(x)=\ln\left(x^2+2\right)^2$

28

4 Differentiate each expression.

(a) $x\log_e(x)$ (b) $2e^x\log_e(3x+2)$ (c) $\log_e(x)\cos(2x-3)$

5 Determine the gradient of the curve $f(x)=\log_e\left(\frac{2-x}{\sqrt{x}}\right)$ at the point $(1,0)$.

6 Determine the derivative of each of the following with respect to x.

(a) $f(\ln(x))$ (b) $f(3\ln(2x))$

7 Consider the function $f(x)=\log_e\left((x+3)^2\right)$.

(a) Determine the domain of f.

(b) Can $\log_e\left((x+3)^2\right)$ be written as $2\log_e(x+3)$? Explain your answer.

(c) Consider the function $f(x)=\log_e\left((x+3)^2\right)$.

(i) Determine an expression for $f'(x)$.

(ii) Determine the domain of f'.

(iii) Determine the coordinates of the point on the graph of $f(x)$ where the gradient is 1.

8 Differentiate each expression.

(a) $\ln\left(\frac{1}{x^2+1}\right)$ (b) $\log_e\left(2e^{3x}\right)$ (c) $\cos\left(\log_e(x)\right)$

(d) $\sqrt{\log_e(x)}$ (e) $\left(\log_e(x)\right)^2$

29

9 Differentiate each expression.

(a) $\frac{\ln(x)}{x+2}$ (b) $\log_e\left(\frac{x+2}{x-1}\right)$

10 If $f(x) = \dfrac{\ln\left(\sqrt{2x}\right)}{\sqrt{x}}$, which of the following gives the value of $f'(1)$?

A $\ln\left(\sqrt{2}\right)$ **B** $-\dfrac{\ln(2)}{2}$ **C** $\dfrac{2-\ln(2)}{4}$ **D** $\dfrac{\ln(2)-2}{4}$

11 Consider the function $f(x) = \ln\left(\sqrt{\left(x^3+1\right)}\right)$.

(a) Which of the following gives the value of $f'(1)$?

A $\ln\left(\sqrt{2}\right)$ **B** 1 **C** $\dfrac{4}{3}$ **D** $\dfrac{3}{4}$

(b) Explain the common error made by a student who calculated $f'(1)$ as $\ln\left(\sqrt{\left(1^3+1\right)}\right)$ in part **(a)**.

12 Given $f(1) = 3$ and $f'(1) = 5$, determine the derivative of $y = f(x)\left(\log_e(x)\right)$ where $x = 1$.

13 Consider the functions $f(x) = \log_2(3x+2)$ and $g(x) = \log_e(x)\log_2(x)$.

(a) Write the rule for $f(x)$ using natural logarithms.

(b) Determine an expression for $f'(x)$.

(c) Determine an expression for $g'(x)$.

(d) Determine the equation of the tangent for $g(x)$ at $x = 2$.

14 Consider the functions $f(x) = \ln(x)$ and $g(x) = x^2 + 5x - 36$.

(a) Determine an expression for the function $h(x) = f(g(x))$ and its maximal domain.

(b) Determine the expression for $h'(x)$.

(c) Sketch the graph of $h(x)$, labelling asymptotes and exact values for any intercepts.

15 $f(x) = \dfrac{a\ln(x)-b}{a\ln(x)+b}$, where $a \neq 0$ and $b \neq 0$.

(a) Explain why the graph of $f(x)$ does not have stationary points.

(b) Determine the equations of the asymptotes of the graph of $f(x)$. Then determine the coordinates of the x-intercept and the domain of the function.

3.7 Applications of logarithms

Modelling with rates of change

Instantaneous and average rates of change may both be relevant in modelling situations. Always interpret the meaning of your answers to see if the values are reasonable.

30 Applications of logarithms and calculus

The number N of Tasmanian devils with a disease t weeks after starting a course of treatment is given by: $N(t) = 200 + 50\ln(10 - t)$, $0 \leq t \leq a$.

(a) Determine the number of Tasmanian devils that have the disease initially.

THINKING	WORKING
1 Recall the given rule and substitute $t = 0$.	$N(t) = 200 + 50\ln(10 - t)$ $N(0) = 200 + 50\ln(10 - 0)$ ≈ 315
2 Interpret the answer.	The number of Tasmanian devils that have the disease initially is 315.

(b) Determine the number of Tasmanian devils that have the disease 2 weeks after starting treatment.

1 Calculate the value of $N(2)$.	$N(2) = 200 + 50\ln(10 - 2)$ ≈ 304
2 Interpret the answer.	The number of Tasmanian devils that have the disease 2 weeks after starting treatment is 304.

(c) How many weeks after starting treatment would there be 200 Tasmanian devils with the disease?

1 Solve for t when $N(t) = 200$. Alternatively, graph $y = 200 + 50\ln(10 - t)$ and read off the value where $y = 200$.	$200 + 50\ln(10 - t) = 200$ $50\ln(10 - t) = 0$ $\ln(10 - t) = 0$ $10 - t = e^0$ $10 - t = 1$ $t = 9$
2 Interpret the answer.	After 9 weeks, there will be 200 Tasmanian devils with the disease.

(d) Assuming that the model is only valid until all the Tasmanian devils are cured, determine the value of a in the domain, correct to 2 decimal places.

1 Tasmanian devils are cured when there are no more Tasmanian devils with the disease. Solve for t when $N(t) = 0$. Alternatively, graph $y = 200 + 50\ln(10 - t)$ and read off the value of the x-intercept.	$200 + 50\ln(10 - t) = 0$ $50\ln(10 - t) = -200$ $\ln(10 - t) = -4$ $10 - t = e^{-4}$ $t = 10 - e^{-4}$ $= 9.981\ldots$ Therefore $a = 9.98$ (2 d.p.).
2 Interpret the answer.	All the Tasmanian devils are predicted to be cured after 10 weeks of treatment.

(e) Determine the average rate of change of $N(t)$ over the first 2 weeks, correct to 1 decimal place. Interpret your answer.

1 Recall the rule for average rate of change.	$\dfrac{f(b) - f(a)}{b - a}$
2 Substitute known values.	$\dfrac{N(2) - N(0)}{2}$ $= \dfrac{200 + 50\ln(10 - 2) - (200 + 50\ln(10 - 0))}{2}$ $= \dfrac{50\ln(8) - 50\ln(10)}{2}$ $= 25\ln(0.8)$ $= -5.57\ldots$
3 Interpret the answer, where a negative rate represents a decreasing population.	The number of Tasmanian devils with the disease is decreasing by an average rate of 5.6 per week over the first 2 weeks.

(f) What is the rate of change of the number of Tasmanian devils with the disease after 2 weeks?

1 Determine the derivative function of $N(t)$.	$N'(t) = \dfrac{-50}{10 - t}$
2 Calculate $N'(2)$.	$N'(2) = \dfrac{-50}{10 - 2}$ $= -6.25$
3 Interpret the answer, where a negative rate represents a decreasing population.	The number of Tasmanian devils with the disease is decreasing by 6.25 per week after 2 weeks.

Applications involving differentiation:

- The equation of the tangent at the point (x_1, y_1) with gradient m is $y - y_1 = m(x - x_1)$ where $x_1 = a$, $y_1 = f(a)$ and $m = f'(a)$.
- The average rate of change is the gradient of the line joining two points $(a,f(a))$ and $(b,f(b))$, given by $\frac{f(b) - f(a)}{b - a}$.
- Stationary points occur where $f'(x) = 0$.

Creating a model

When using a general equation to model numbers or growth, you need to determine values for the parameters.

WARNING

For logarithmic functions, $\ln(f(x))$, check that $f(x) > 0$.

31 Model and solve practical problems with logarithms

A marine biologist measured the length of a whale calf one month after birth to be 2.1 metres, and after 6 months to be 5.2 metres. She believes that the length, L metres, of the whale over the first 12 months can be modelled by the rule: $L(t) = 20\log_e\left(\frac{t}{a} + b\right)$, where a and b are positive and t is the time in months since birth.

(a) Determine the values of a and b and hence the rule correct to 2 decimal places.

THINKING	WORKING
1 Substitute pairs of values into the rule, to write each equation in exponential form.	$L(t) = 20\log_e\left(\frac{t}{a} + b\right)$ $(1, 2.1)$: $2.1 = 20\log_e\left(\frac{1}{a} + b\right)$ $\therefore \frac{1}{a} + b = e^{\frac{2.1}{20}}$ [1] $(6, 5.2)$: $5.2 = 20\log_e\left(\frac{6}{a} + b\right)$ $\therefore \frac{6}{a} + b = e^{\frac{5.2}{20}}$ [2]
2 Solve simultaneous equations to determine the values of the parameters.	[2] − [1]: $\frac{5}{a} = e^{\frac{5.2}{20}} - e^{\frac{2.1}{20}}$ $a = \frac{5}{e^{\frac{5.2}{20}} - e^{\frac{2.1}{20}}}$ $a = 26.850\ldots$ $b = e^{\frac{2.1}{20}} - \frac{1}{a}$ $= 1.073\ldots$
3 Substitute rounded values of the parameter into the rule.	$L(t) = 20\log_e\left(\frac{t}{26.85} + 1.07\right)$

(b) Predict the length of a whale calf at 5 months, correct to 1 decimal place.

1 Calculate the value of $L(5)$.

$$L(5) = 20\log_e\left(\frac{5}{26.85} + 1.07\right)$$
$$= 4.6 \text{ (1 d.p.)}$$

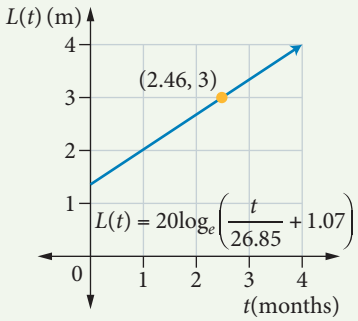
2 Interpret the answer.

At 5 months, the calf is predicted to be 4.6 metres.

(c) Predict the age of the whale calf when it is 3 metres in length. Give your answer correct to 1 decimal place.

1 Graph the function.

Read off the value for t when $L(t) = 3$.



2 Interpret the answer.

It is predicted that the calf would be 3 metres long when it is 2.5 months old.

EXERCISE 3.7 Applications of logarithms

Worked Example 30

1 The number N of horses with a disease t days after starting a course of treatment is given by: $N = 100 + 50\ln(e - t)$, $0 \leq t \leq a$.

(a) Determine the number of horses that have the disease initially.

(b) How many days after starting treatment are there 100 horses with the disease? Give your answer to the nearest day.

(c) Assuming that the model is only valid until all the horses are cured, determine the exact value of a.

(d) Determine the average rate of change of N over the first 2 days. Give your answer correct to 2 decimal places.

(e) Determine the exact rate of change of the number of horses with the disease after 2 days.

2 On 1 January 1990, there were 300 wombats on a nature reservation, while on 1 January 2014 the number of wombats had declined to 200. The number of wombats N on the reserve t years after 1 January 1990 is predicted to decline according to the rule: $N = a\log_e(t+1)+b$.

(a) Determine the exact values of a and b.

(b) How many wombats are predicted to be on the nature reservation on 1 January 2024?

(c) During which year will the number of wombats drop below 180?

3 Consider the function $f(x) = x^2\ln(x)$.

(a) Where does the graph of the function have stationary points?

A $x = 0$ only **B** $x = \frac{1}{\sqrt{e}}$ only **C** $x = 0$ and $x = 1$ **D** $x = 0$ and $x = \frac{1}{\sqrt{e}}$

(b) Explain the common error made by a student who used the null factor law to calculate the stationary points using $x^2 = 0$ or $\ln(x) = 0$.

4 The tangent to the graph of $y = \ln(x)$ where the gradient of the curve is 1 is given by which equation?

A $y = x - 1$

B $y = x + 1$

C $y = x$

D The tangent does not exist.

5 Consider the function $f(x) = 3\log_e(2x)$.

(a) Determine an expression for $f'(x)$.

(b) Determine the gradient of the curve at $x = 1$.

(c) Determine the equation of the tangent at $x = 1$.

(d) Determine the equation of the tangent to the curve f at the point where the tangent is parallel to the line $2x - y = 0$.

6 An aircraft takes off from an airport at sea level. Its altitude, h metres, at time t minutes is given by: $h(t) = 2000\ln(t+1)$.

(a) Determine an expression for the rate of change of the altitude.

(b) Determine the rate at which the aircraft is climbing after 3 minutes.

(c) After how long is the aircraft climbing at a rate of 320 m/min?

7 Consider the function $f(x) = x\log_e(x)$.

(a) Calculate $f(1)$.

(b) Solve $f(x) = 0$.

(c) Determine an expression for $f'(x)$.

(d) Sketch the graph of f, showing the exact value of the turning point.

8 A child's attention span during the early years of life is approximately described by the function: $S(x) = \dfrac{5}{0.5\ln(x) - x + 3.5}$ for $0.5 \leq x \leq 4$, where x is the age in years and S is a measure of the child's attention span in minutes.

(a) Determine the child's attention span at age 2, correct to 1 decimal place.

(b) What is the average rate of change of attention span from age 2 to 4? Give your answer correct to 2 decimal places.

(c) Determine an expression for the derivative of S with respect to x.

(d) Determine the age when a child's attention span is the shortest.

9 The percentage P of a colony of penguins who have a disease t days after the disease is introduced to the colony can be modelled by: $P(t) = 6\ln\left(t^2 + 1\right) - 0.1t^2$, $0 \leq t \leq 14$.

(a) Determine, to the nearest whole number, the percentage of penguins that have the disease after 2 days.

(b) Determine the average rate at which the disease was spreading from the end of day 2 to the end of day 5. Give your answer correct to 1 decimal place.

(c) Determine the rate of change of the percentage of penguins that have the disease after 5 days. Give your answer correct to 1 decimal place.

(d) After how many days does the largest percentage of penguins have the disease? What percentage of penguins has the disease at this time? Give your answers correct to 1 decimal place.

Worked Example 31

10 An ostrich farmer measured the weight of a chick one month after birth to be 4.9 kg, and after 6 months to be 19.1 kg. He believes that the growth of the chick over the first 12 months can be modelled by the rule: $W = 17\ln\left(\dfrac{t}{c} + d\right)$.

(a) Determine the values of c and d, correct to 2 decimal places, and sketch the graph.

(b) Determine the predicted weight of the chick after 8 months, correct to 1 decimal place.

11 The speed V km/s of a police car as it starts to chase a speeding motorist is given by: $V(t) = \log_e(t + a) + b$, where t is the time in seconds. The police car is initially stationary and the speed after 10 seconds is 0.03 km/s.

(a) Determine the values of a and b, correct to 2 decimal places.

(b) Calculate the speed of the police car in km/s, 20 seconds after it starts to chase the motorist. Give your answer correct to 2 decimal places.

(c) Determine when the speed of the police car will reach 200 km/h. Give your answer correct to 1 decimal place.

12 An aircraft takes off from an airport at sea level and its altitude h (in metres) at time t (in minutes) is given by $h(t) = 610\ln(t+1)$. Answer using whole numbers of appropriate units, where applicable.

(a) Plot the graph of $h(t)$ for $t \in [0, 20]$.

(b) Determine when the aircraft will reach an altitude of 3 km.

(c) Determine the rate of climb after 3 minutes.

(d) At what time will the rate of climb be 5.8 m/s?

(e) Determine the time, to the nearest second, at which the average rate of climb from the start to this instant is 5.8 m/s.

13 A space satellite is powered by batteries using a radioisotope. The power output, in watts, is modelled by an exponential function $P = P_0 e^{kt}$, where t is the time in days and P_0 is the initial power of the battery. The power output at launch is 60 watts. After 14 days, the power output has fallen to 56 watts. The satellite cannot function properly if the power output falls below 5 watts. For how many days will the satellite function properly?

14 Two drugs, A and B, must be given together to provide effective pain relief for up to 24 hours. Drug A is absorbed into the bloodstream over 12 hours and is cleared over the next 12 hours. Drug B is absorbed into the bloodstream over 6 hours and is cleared over the next 6 hours. The following equations model the situation:

$$A = \begin{cases} 50\ln(t+1), 0 \le t \le 12 \\ 50\ln(25-t), 12 < t < 24 \end{cases}$$

$$B = \begin{cases} 10\left(e^{\sqrt{t}} - 1\right), 0 \le t \le 6 \\ 10\left(e^{\sqrt{12-t}} - 1\right), 6 < t < 12 \end{cases}$$

where A and B give the amount of each drug in the bloodstream in milligrams after t hours.

(a) Sketch the graphs of A and B on the same set of axes. Hence determine the maximum amount of each drug absorbed into the bloodstream. Give your answers correct to 2 decimal places.

(b) Patients receive the maximum pain relief when the amounts of drug A and B are added together. Use addition of ordinates to sketch the graph of $A + B$ on the same set of axes as part **(a)**.

(c) How long after taking the drugs will a patient receive maximum pain relief?

(d) Determine the maximum combined amount of the two drugs in the bloodstream. Give your answer correct to 2 decimal places.

(e) Effective pain relief is reached when the concentration of each drug is more than 50 mg at the same time. Determine how long this period of effective pain relief lasts. Give your answer to the nearest minute.

Summary

Change to logarithmic form

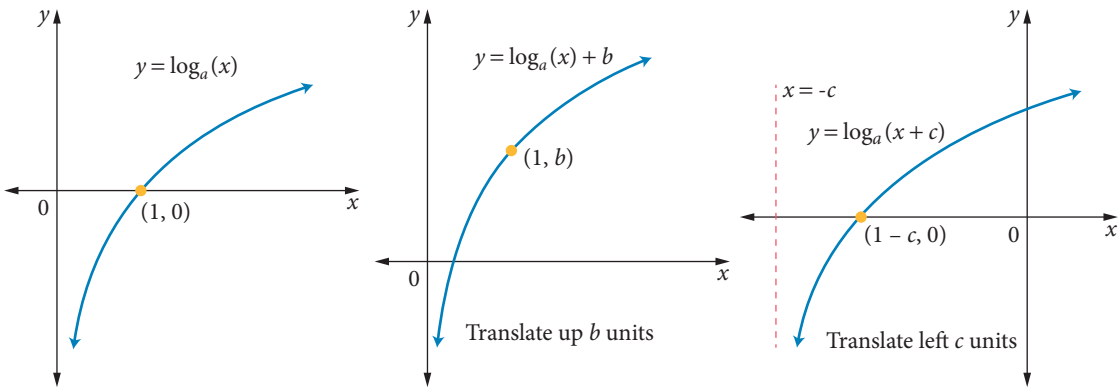
$a^x = y \Leftrightarrow \log_a(y) = x$

Properties and laws of logarithms

$\log_a(1) = 0$	$\log_a\left(\frac{1}{m}\right) = \log_a\left(m^{-1}\right) = -\log_a(m)$
$\log_a(a) = 1$	$\log_a\left(a^n\right) = n$
$\log_a(m) + \log_a(n) = \log_a(mn)$	$a^{\log_a(n)} = n$
$\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$	$a^{p\log_a(m)} = a^{\log_a\left(m^p\right)} = m^p$
$\log_a\left(m^n\right) = n\log_a(m)$	$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

Logarithmic graphs

Translations



Reflections

$y = -\log_a(x)$ is $y = \log_a(x)$ reflected in the x-axis.

$y = \log_a(-x)$ is $y = \log_a(x)$ reflected in the y-axis.

Dilations

$y = A\log_a(x)$, where $A > 0$, is $y = \log_a(x)$ dilated parallel to the y-axis by a factor of $|A|$.

$y = \log_a(Bx)$, where $B > 0$, is $y = \log_a(x)$ dilated parallel to the x-axis by a factor of $\left|\frac{1}{B}\right|$.

Inverses

$f^{-1}(x) = \log_a(x)$ is the inverse of the exponential function $f(x) = a^x$.

The graph of $y = \log_a(x)$ can be sketched by:

- reflecting the graph of $y = a^x$ in the line $y = x$
- interchanging the x- and y-values in each pair of coordinates.

The inverse of $f(x) = e^x$ is $f^{-1}(x) = \ln(x)$.

Domain f^{-1} = Range f and Range f^{-1} = Domain f

Logarithmic scales

Richter scale	Sound (decibels)	pH scale
$R = \log_{10}\left(\frac{I}{I_0}\right)$	$D = 10\log_{10}\left(\frac{I}{I_0}\right)$ $I_0 = 1.0 \times 10^{-12}$ W/m^2	$\text{pH} = -\log_{10}\left[\text{H}^+\right]$

Derivative of a logarithm

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\log_e(kx)\right) = \frac{k}{kx} = \frac{1}{x}$$

$$\frac{d}{dx}\left(\log_e(ax+b)\right) = \frac{a}{ax+b}$$

$$\frac{d}{dx}\left(\log_e(f(x))\right) = \frac{f'(x)}{f(x)}, \text{ where } f(x) > 0$$

Rules for differentiation

The chain rule	The product rule	The quotient rule
Where y is a function of u, and u is a function of x: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	If $y = uv$: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$	If $y = \frac{u}{v}$: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Tangents to a graph

Substitute $x_1 = a$, $y_1 = f(a)$ and $m = f'(a)$ into $y - y_1 = m(x - x_1)$.

Average rate of change

$$\frac{f(b) - f(a)}{b - a}$$

Chapter review 3

1 Simplify each expression. Exercise 3.1

(a) $4\ln\left(\frac{1}{e}\right)$ (b) $3^{\log_3(x^2)}$

2 Simplify each expression. Exercise 3.1

(a) $2\ln(x) - 2\ln(x+2) + \ln(x+2)$ (b) $\frac{\ln\left(\sqrt[3]{x}\right)}{\ln\left(\sqrt{x}\right)}$

(c) $\log_y\left(xy^3\right) - \log_y(x)$ (d) $\ln\left(9e^{2x}\right)$

3 If $2 = a^k$, express $\log_a\left(\frac{2}{a}\right)$ in terms of k. Exercise 3.1

4 Given $y = 2a^{3x} + b$, make x the subject of the equation. Exercise 3.1

5 Solve the following equations for x. Exercise 3.2

(a) $\log_2(y) = 3 + 2\log_2(x)$ (b) $2\ln(x) - \ln(x+1) = \ln\left(\frac{1}{2}\right)$

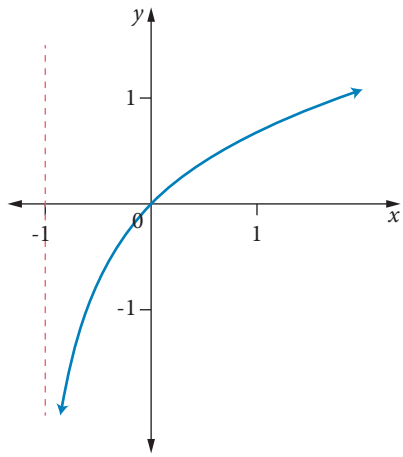
6 If $x = 3$ is a solution of the equation $\log_2(ax - 2) = 3$, determine the value of a. Exercise 3.2

7 If $2\log_e(2x) = 1 + \log_e(a)$, then what is the value of x? Exercise 3.2

8 Solve $5^{x-1} = 10$, correct to 2 decimal places. Exercise 3.3

9 Given the graph of the function $f(x) = 10^{x+1} - 2$ passes through the point $(a, 6)$, what is the value of a? Exercise 3.3

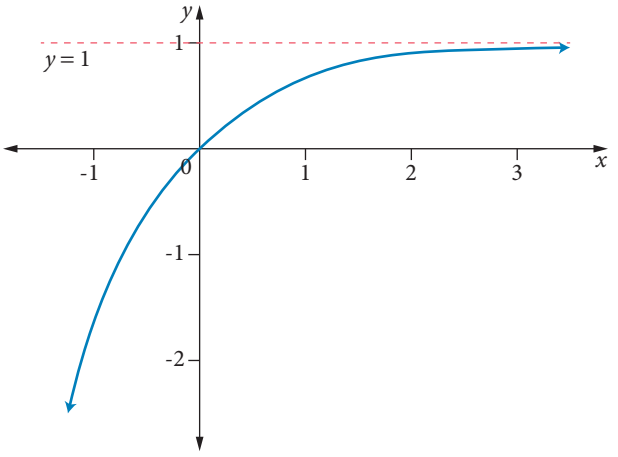
10 The rule for the graph shown is of the form $y = \log_e(x + b)$. Determine the value of b. Exercise 3.4



11 For $f(x) = \log_e(3x - 2) + 1$, determine the equation of the inverse function and its domain and range. Exercise 3.4

12 Determine the values of k and b, given that the graph of $y = \log_e(k(x - b))$ passes through the point $(6, 1)$ and has an asymptote at $x = 3$. Exercise 3.4

Exercise 3.4 **13** Which function most likely corresponds to the graph of $f(x)$, shown?



A $f(x) = 1 - e^{-x}$

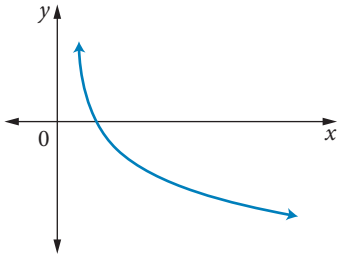
B $f(x) = 1 - e^{x}$

C $f(x) = e^{x} - 1$

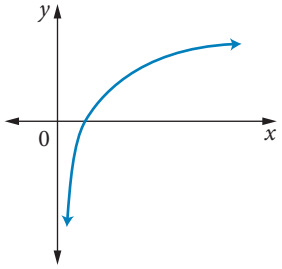
D $f(x) = \log_e(x+1)$

Exercise 3.4 **14** Which of the following could be the graph of $y = a\ln(bx + c)$, given that $a < 0$, $b > 0$ and $c < 0$?

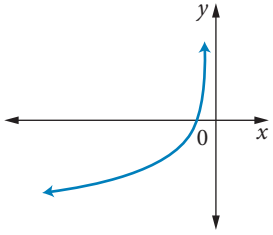
A



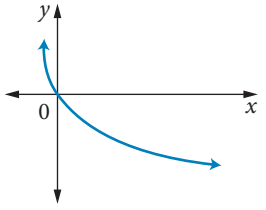
B



C



D



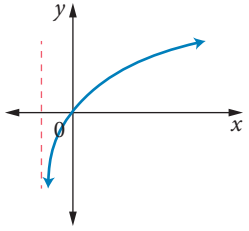
Exercise 3.4 **15** For which scenario will $f(x) = \log_e(x^4)$, $x \in [a, \infty)$ have an inverse?

A $a \leq 1$ **B** $a > 1$ **C** $a \geq -1$ **D** $a < 2$

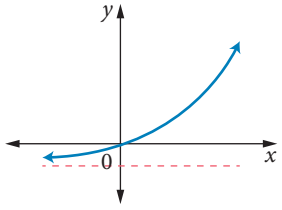
Exercise 3.4 **16** At what point does the graph of $f(x) = e^{x} + 2$ intersect the graph of $g(x) = e^{2x}$?

Exercise 3.4 **17** Which of the following could be the graph of $y = a\ln(x - b)$, given that $a < 0$ and $b > 0$?

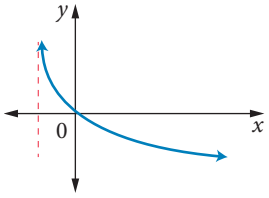
A



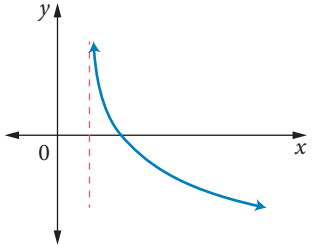
B



C



D



18 What is the maximal domain of the function $f(x) = \ln\left(x^2\right) + 1$? Exercise 3.4

19 In 1935, an earthquake with magnitude 6.9 on the Richter scale was recorded, while in 1982 an earthquake of magnitude 3.9 on the Richter scale was recorded. How many times greater was the magnitude of the 1935 earthquake than the 1982 earthquake? Exercise 3.5

20 If sound A registers 20 decibels more than sound B, then what is true about the intensity of sound A as compared to sound B? Exercise 3.5

A twice as intense **B** half as intense

C ten times as intense **D** 100 times as intense

21 What is the pH of a solution that has a hydrogen ion concentration that is 100 times greater than a solution with a pH of 3? Exercise 3.5

22 Differentiate $y = x\ln(2x - 3)$, and hence determine the gradient at the point where $x = 2$. Exercise 3.6

23 If $y = \log_e\left(\sin(2x)\right)$, then $\dfrac{dy}{dx}$ is equal to which of the following? Exercise 3.6

A $\tan(2x)$ **B** $\dfrac{\cos(2x)}{\sin(2x)}$ **C** $\dfrac{2\cos(2x)}{\sin(2x)}$ **D** $-\dfrac{2\cos(2x)}{\sin(2x)}$

24 Given $f(x) = \log_e(2x)$, determine $f'(1)$. Exercise 3.6

25 Determine the derivative of $f\left(\log_e(3x)\right)$ with respect to x. Exercise 3.6

26 Determine the gradient of the tangent to the curve $y = \log_e(x)$ at the point where $x = 2$. Exercise 3.7

27 Consider the quadratic equation $3x^2 - x\log(m) + \dfrac{4}{3} = 0$, where $m \in \mathbb{R}^+$. For what values of m does the equation have exactly one solution? Exercise 3.7

28 Solve the equation $e^{2x} - 6e^x - 16 = 0$ for x. Exercise 3.3

29 Given $e^x + e^y = 2$ and $e^x - e^y = 1$, determine the value of $x + y$. Exercise 3.3

30 Sketch the graph of $y = \log_2(x + 1) - 1$, showing axis intercepts and the equation of the asymptote. Exercise 3.4

31 The intensity level of sound D dB is defined as: $D = 10\log_{10}\left(\dfrac{I}{I_0}\right)$, where I is the intensity of the sound measured in watts per square metre $\left(\text{W/m}^2\right)$, and $I_0 = 1.0 \times 10^{-12}\ \text{W/m}^2$ is the threshold intensity of human hearing. Exercise 3.5

(a) If the sound of a jet plane taking off is 20 dB louder than a police siren, how many times more intense is the sound of a jet plane?

(b) If the sound of a motorcycle backfiring is measured as 90 dB, how much more intense is it than the threshold sound?

(c) The sound intensity of a food processor is $0.00001\ \text{W/m}^2$. Determine its decibel level.

(d) An ambulance siren has a decibel level of 124 dB. City traffic has a decibel level of 85 dB. How many times more intense is the sound of the ambulance siren than the sound of city traffic? Give your answer to the nearest 100.

(e) The sound of a truck is 12 times as intense as the sound of a car. If the decibel level of the car is 80 dB, what is the decibel level of the truck? Give your answer correct to the nearest unit.

Exercise 3.6 **32** For $y = \dfrac{\ln(x)}{x}$, determine the coordinates of the stationary point.

Exercise 3.6 **33** For the curve $y = \log_e\left(4x^2 - 5\right)$, determine the coordinates of the point where the gradient is 1.

Exercise 3.6 **34** The point $\left(\dfrac{1}{8}, -3\right)$ is on the graph of $f(x) = \ln(ax)$, and the point $(4, b)$ is on the graph of its inverse, f^{-1}. Determine the values of a and b.

Exercise 3.7 **35** A ski resort's records show that with an increase in visitor numbers, the number of snowboarding injuries also increased. Their records show that since 2005, the rate of the increase of injuries has slowed due to improvements in the safety standards of the slopes. The number of injuries N is approximately modelled by the rule: $N = 50\ln(3t + 1) + 100$, where t is the number of years after 2005.

(a) Sketch the graph of N for $0 \leq t \leq 10$.

(b) Determine the number of injuries in 2005.

(c) Determine the number of injuries predicted to occur in 2015.

(d) Determine the year in which the number of injuries is predicted to reach 200.

(e) Calculate the average rate of increase of snowboarding injuries from 2005 to 2010. Give your answer correct to 1 decimal place.

(f) Calculate the rate of increase of snowboarding injuries in 2010, correct to 1 decimal place.

Exercise 3.3 **36** For $x^a = e^2 x$, one solution is $x = 0$. Determine the second solution as a power of e.

Exercise 3.4 **37** Consider the functions f and g.

(a) $f(x) = e^{\frac{x-1}{2}} - 3$

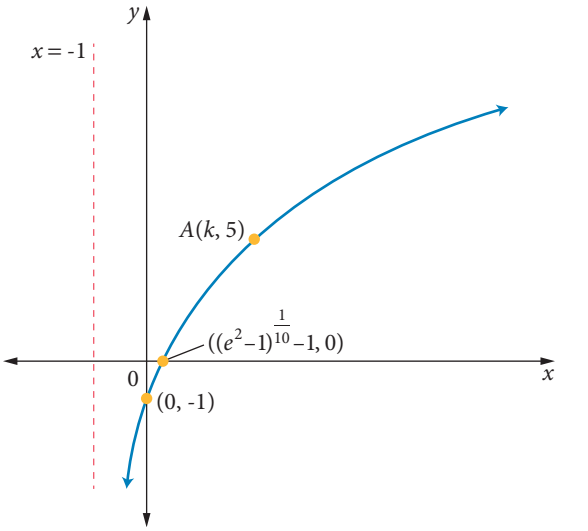
(i) Sketch the graph of f, showing the asymptote and labelling the x- and y-intercepts with their exact values.

(ii) Determine the rule and the domain and range of $f^{-1}(x)$.

(iii) Sketch the graph of f^{-1} on the same set of axes as f.

(iv) Determine the coordinates of the points of intersection of f and f^{-1}. Give your answer correct to 2 decimal places.

(b) The graph of $g(x) = a\ln(x + b) + c$ is shown on the right.



(i) Determine the values of a, b and c.

(ii) Determine the value of k, correct to 2 decimal places.

(iii) Determine the gradient of the tangent to the graph of g at the point A. Give your answer correct to 2 decimal places.

(c) Determine the point(s) of intersection for f and g. Give the coordinates correct to 2 decimal places.

38 Consider the function $f(x) = \ln\left(x^2 - 1\right)$. Exercise 3.4

(a) Determine the domain.

(b) Determine the coordinates of any stationary points.

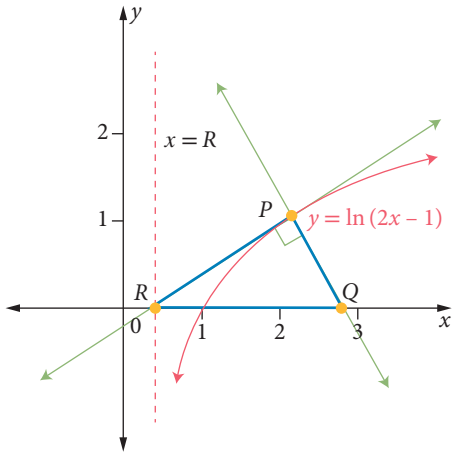
(c) Does $f^{-1}(x)$ exist? Justify your answer.

(d) The domain of $f(x)$ is restricted to form a new function $g(x) = \ln\left(x^2 - 1\right)$, $x \in (a, \infty)$.

(i) What is the smallest value of a?

(ii) Determine the inverse function $g^{-1}(x)$ and its domain and range.

39 Consider the curve $y = \ln(2x - 1)$. Let P be the point on the curve where $x = 2$. The tangent to the curve at P cuts the x-axis at R. A line, perpendicular to the curve at P, cuts the x-axis at Q. Determine the area of ΔPRQ. Reminder: the gradient of a line perpendicular to the tangent is $-\dfrac{1}{m_T}$ where m_T is the gradient of the tangent. Exercise 3.7



Mixed review

CHAPTERS 1–3

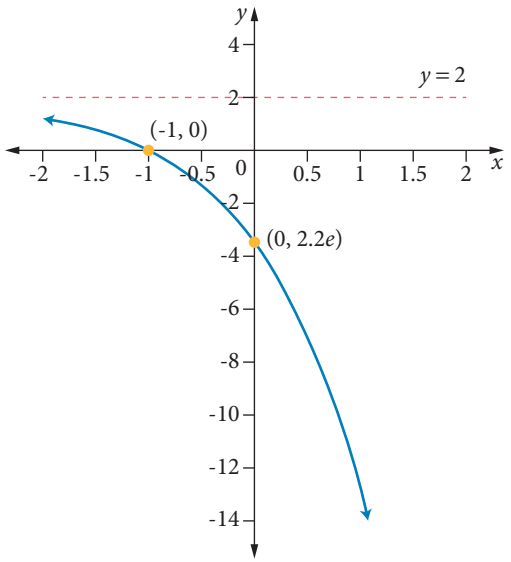
Exercise 1.1

1 Solve for x, giving your answer in exact form.

(a) $e^{3x} = e^{x-6}$ **(b)** $\sqrt{e} = e^{x-1}$ **(c)** $e^x = 7$

Exercise 1.2

2 Express the rule of the graph on the right in the form $f(x) = Ae^x + c$, and then simplify using index laws.



Exercise 1.3

3 Differentiate each of the following functions.

(a) $f(x) = e^{5x}$ **(b)** $f(t) = -5e^{2t}$ **(c)** $f(x) = e^{2x+1}$ **(d)** $f(x) = e^{x^2}$

Exercise 1.4

4 Under certain conditions, the population of bacteria on an agar dish doubles every 3 days. Initially there were 10 bacteria introduced to the environment. Write the exponential equation that describes the number of bacteria N at time t days, in its simplest form.

Exercise 2.1

5 Calculate the exact value of each of the following measures.

(a) $\cos\left(\frac{7\pi}{6}\right)$ **(b)** $\tan\left(-\frac{3\pi}{4}\right)$

Exercise 2.2

6 Determine all solutions to each of the following equations.

(a) $\tan(3x) = \sqrt{3}, -\pi \leq x \leq \pi$ **(b)** $\sin\left(3x - \frac{\pi}{6}\right) = \frac{1}{2}$

Exercise 2.4

7 Determine the derivative of each of the following functions.

(a) $y = \sin(3x)$ **(b)** $y = 2\cos(6x)$

(c) $y = 3\tan(5x)$ **(d)** $y = 4\sin(-3x)$

Exercise 2.5

8 Determine the equation of the tangent to the function $f(x) = \sin(x)$ at the point where $x = \frac{\pi}{6}$.

Exercise 3.1

9 Write $5^3 = 125$ in logarithmic form.

Exercise 3.3

10 Solve $5^x \geq 2$ for x, expressing your answer using natural logarithms.

11 Determine the pH (where $pH = -\log_{10}[H^+]$) of a liquid whose concentration of hydrogen ions is 3.2×10^{-6} moles/litre. Give your answer correct to 1 decimal place. Exercise 3.5

12 Determine the derivative of each of the following functions. Exercise 3.6

(a) $y = \log_e(2x - 3)$ **(b)** $y = \log_e\left(\sqrt{4 - 5x}\right)$

13 For each of the following functions, sketch the graph, clearly showing the asymptote and axis intercepts. If there is no x-intercept, plot an additional point. Identify the domain and range. Exercise 1.2

(a) $y = e^x - 2$ **(b)** $y = -2e^x + 1$ **(c)** $y = e^{2x+1}$ **(d)** $y = 1 - e^{x+2}$

14 Determine the derivative of each of the following functions. Exercise 1.5

(a) $y = \dfrac{e^{3x} + 2x - 2}{e^x}$ **(b)** $y = 2x\left(3x^2 - 2x + 1\right)^2$

15 Given $\sin(\theta) = -\dfrac{5}{13}$, $\pi \leq \theta \leq \dfrac{3\pi}{2}$, determine the value of each measure. Exercise 2.1

(a) $\cos(\theta)$ **(b)** $\tan(\theta)$

16 Consider the function $f(x) = 2 - 3\cos(2x + \pi)$, $0 \leq x \leq 2\pi$. Exercise 2.3

(a) Determine the amplitude and the period.

(b) Determine the coordinates of the y-intercept.

(c) Determine the coordinates of the endpoints and sketch the graph. You do not need to find the x-intercepts.

17 Consider the function $f(x) = 3^{x+2}$. Exercise 1.3

(a) Estimate the gradient of the function at $x = -1$ from first principles, correct to 3 decimal places.

(b) Determine the area of a triangle formed by the tangent to the function at $x = -1$ and the axes, correct to 3 decimal places.

18 Graph the function $y = \ln(\cos(x))$. Exercise 3.4

4

Anti-differentiation

Recall 208

4.1 Introduction to anti-differentiation 209

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Summary 249

Chapter review 250

4 Recall

Differentiate functions of the form $f(x) = x^n$

1 Determine the derivative of each of the following.

(a) $y = 4x^3 - 2x^2 - 3x + 2$ (b) $y = (x+1)^2 + x^2$ (c) $f(x) = 3x^6 + 4x - 5$

(d) $f(x) = 2x^{\frac{2}{3}}$ (e) $y = \sqrt{x} - x^{-3} + \dfrac{4}{x}$

Determine a derivative at a given value

2 Determine the value of each of the following.

(a) $\dfrac{dy}{dx}$ if $y = 4 - 3x^3 + x^2$ and $x = -2$ (b) $f'(3)$ if $f(x) = 3x - 4x^2$

(c) $f'(16)$ if $f(x) = x^{-\frac{1}{2}} - 2x^{-\frac{1}{4}}$

Differentiate functions of the the form $f(x) = (ax + b)^n$

3 Determine the derivative of each of the following.

(a) $y = (3x+2)^4$ (b) $y = (2x+5)^3$ (c) $y = (2x-1)^{\frac{1}{2}}$

(d) $y = 5(3x-4)^{-2}$ (e) $y = (2x+3)^{\frac{3}{2}}$

Differentiate exponential and logarithmic functions

4 Differentiate the following functions.

(a) $y = e^{5x}$ (b) $y = e^{\frac{x}{2}}$ (c) $f(x) = 3e^{5x^2 - x}$

(d) $f(x) = 7\log_e(2x-1),\ x > \dfrac{1}{2}$ (e) $y = \log_e\left(\dfrac{x^2}{3+x}\right),\ x > -3$

Differentiate trigonometric functions

5 Determine the derivative of each of the following.

(a) $\sin(3x)$ (b) $\cos(5x)$ (c) $y = 3\sin\left(\dfrac{2x}{3}\right) + 2\cos(\pi x)$

(d) $f(x) = \sin(3 - 5x)$ (e) $f(x) = 3\cos(11x - 2)$

Differentiate using the product and quotient rule

6 Use the product or quotient rule to differentiate the following functions.

(a) $y = x^2\sin(x)$ (b) $y = x^4 e^{2x}$ (c) $y = \dfrac{x^3}{\sin(x)}$

(d) $y = \dfrac{\log_e(x)}{e^{-x}}$ (e) $y = \dfrac{\sin(x)}{\cos(x)}$ (f) $y = \dfrac{\sqrt{x}}{x+5}$

Differentiate using the chain rule

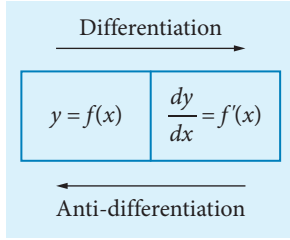
7 Use the chain rule to determine the derivative of each of the following.

(a) $y = \left(3x^2 + 2x\right)^4$ (b) $y = \sin\left(x^2\right)$ (c) $y = \sqrt{x^3 - 4x + 2}$

(d) $y = e^{x^2}$ (e) $y = \sin^2(x)$

4.1 Introduction to anti-differentiation

An anti-derivative is a function $F(x)$, such that $F'(x) = f(x)$.
Anti-differentiating can be considered the reverse process of differentiating. Hence anti-differentiation is the process of determining a function $f(x)$ from its derivative or gradient function $f'(x)$. The derivative of $3x^2$ is $6x$, so an anti-derivative of $6x$ is $3x^2$.



Consider the following functions and their derivatives:

$f(x)$	$f'(x)$
$3x^2$	$6x$
$3x^2 + 2$	$6x$
$3x^2 - 7$	$6x$

Additional information

Anti-differentiation as the reverse process of differentiation

In this activity, determine the expressions on the right that are anti-derivatives of the expressions on the left.

You can see that $3x^2$, $3x^2 + 2$, and $3x^2 - 7$ are all anti-derivatives of $6x$, differing only by the constant. Hence there is an infinite number of anti-derivatives of $6x$, all with the same gradient function. Therefore you write the anti-derivative of $6x$ as $3x^2 + c$, where c is an arbitrary constant, also known as the *constant of integration*. The notation you use to indicate the process of anti-differentiation is $\int 6x\,dx = 3x^2 + c$, where $c \in \mathbb{R}$. This is called the *indefinite integral* of $6x$ with respect to x.

Note that a constant coefficient can be written before or after the integral sign without changing the value of the integral; for example, $\int 6x\,dx = 6\int x\,dx$.

$\int f(x)dx$ is the indefinite integral or primitive of $f(x)$, and it indicates that you are finding the anti-derivative of the expression $f(x)$ with respect to x. The dx indicates that x is the variable with respect to which the anti-differentiation takes place.

Hence $\int f(x)dx = F(x) + c$

also, $\int f'(x)dx = f(x) + c$

or $\int \frac{dy}{dx}dx = y + c$

The derivative of the integral gives the original function, but the integral of the derivative does not (because of any constant term that may be present, c).

The two useful results in the box below can be used to simplify anti-differentiation. The first one allows placement of a constant coefficient either inside or outside the integral sign. The second one allows for anti-differentiation term by term of a series of terms.

Rules for indefinite integrals:

$$\int k f(x)dx = k\int f(x)dx$$

$$\int \big(f(x) \pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx$$

Additional information

Identify the components of an indefinite integral

This activity provides an explanation of the different components of an indefinite integral.

1 Use differentiation to determine an anti-derivative

Differentiate $f(x) = 4x^3 + 3x^2$ and hence determine the anti-derivative of $12x^2 + 6x$.

THINKING	WORKING
1 Determine the derivative.	$f'(x) = 3 \times 4x^{3-1} + 2 \times 3x^{2-1}$ $= 12x^2 + 6x$
2 Rewrite in the form $\int f'(x)dx = f(x) + c$.	$\int \left(12x^2 + 6x\right)dx = 4x^3 + 3x^2 + c$

2 Use differentiation to determine an anti-derivative with adjustment of a constant coefficient

Differentiate each of the following and hence determine the required anti-derivative.

(a) Differentiate $f(x) = 6x^4 - 2x^3$ and hence determine the anti-derivative of $4x^3 - x^2$.

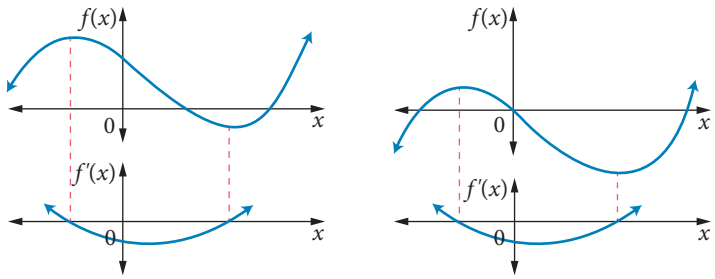
THINKING	WORKING
1 Determine the derivative.	$f'(x) = 24x^3 - 6x^2$
2 Express the derivative in terms of the required anti-derivative.	$f'(x) = 6\left(4x^3 - x^2\right)$
3 Express the anti-derivative required in terms of the derivative found where $\int f'(x)dx = f(x) + c$.	$\int \left(4x^3 - x^2\right)dx = \frac{1}{6} \times 6\int \left(4x^3 - x^2\right)dx$ $= \frac{1}{6}\left(6x^4 - 2x^3 + c_1\right)$
4 Express the anti-derivative in the required form.	$= x^4 - \frac{1}{3}x^3 + c$ where $c = \frac{c_1}{6}$

(b) Differentiate $f(x) = (3x - 2)^3$ and hence determine the anti-derivative of $\int 27(3x - 2)^2\, dx$.

1 Differentiate the function.	$f'(x) = 3(3x - 2)^2 \times 3$ $= 9(3x - 2)^2$
2 Express the derivative in terms of the required anti-derivative.	$f'(x) = \frac{1}{3} \times 27(3x - 2)^2$
3 Express the anti-derivative required in terms of the derivative found where $\int f'(x)dx = f(x) + c$.	$\frac{1}{3}\int 27(3x-2)^2\, dx = \int 9(3x-2)^2\, dx$ $\int 27(3x-2)^2\, dx = 3 \times \int 9(3x-2)^2\, dx$ $= 3 \times \left[(3x-2)^3 c_1\right]$
4 Express the anti-derivative in the required form.	$= 3(3x-2)^3 + c$, where $c = 3c_1$

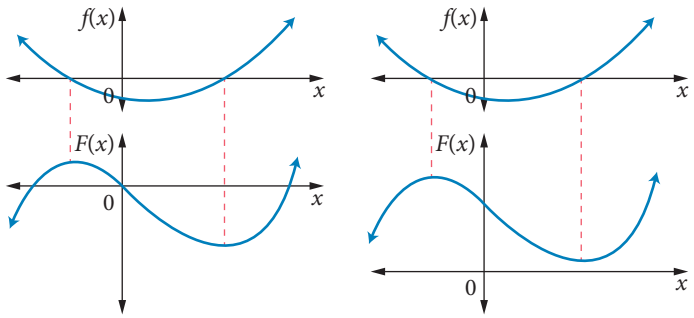
Graphing anti-derivatives

You will recall that the graph of a derivative function $y = f'(x)$ gives the value of the gradient of the original function $y = f(x)$ for each point on the graph. The graphs of each of the two functions below result in the same derivative graph because these functions have the same gradient for every value of x in the domain. You can obtain either of the graphs of $y = f(x)$ from the other by a simple translation up or down.

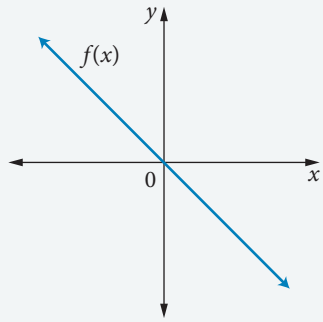
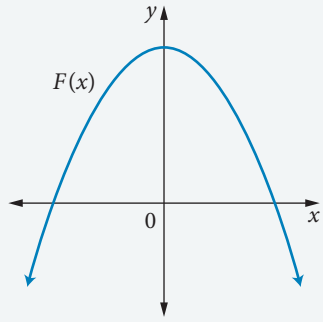


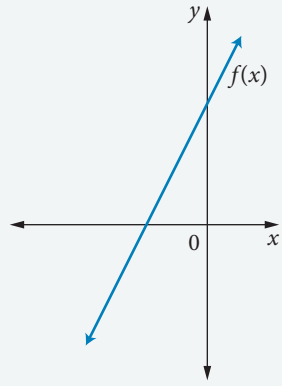
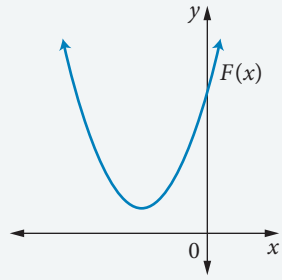
Where the function has a maximum or minimum, the graph of the derivative has an x-intercept, i.e. the function has a gradient of zero. Also, where the function has a positive gradient, the derivative graph is positive, and where the function has a negative gradient, the derivative graph is negative.

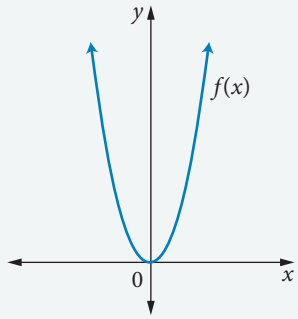
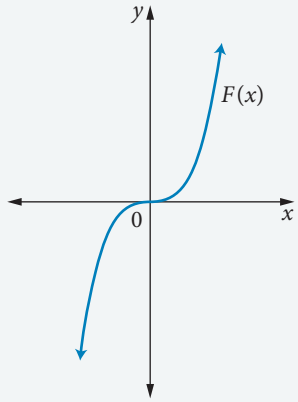
Finding an anti-derivative graph $y = F(x)$, given the graph of a function $y = f(x)$, is the reverse process of finding a derivative graph. You can look at the values of $f(x)$ and create a corresponding gradient for $F(x)$. From the graphs above, it is clear there will not be a unique result, so possible anti-derivative graphs are drawn.

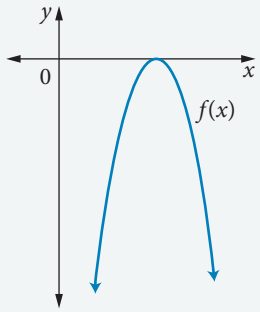
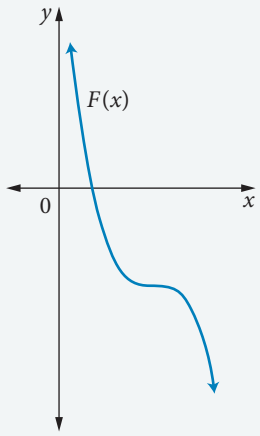


In this case, where the function has an x-intercept, the graph of the anti-derivative has a maximum or minimum. Where the function is positive, the graph of the anti-derivative is increasing, and where the function is negative, the graph of the anti-derivative is decreasing. Wherever you position the possible $y = F(x)$ graph, it belongs to the family of anti-derivative graphs for $y = f(x)$ obtained by translating the graph in a direction parallel to the y-axis. It is possible to determine the nature of the turning point on the graph of $y = F(x)$ by examining the sign of the original function around the x-intercept. Consider the graph above, in which $f(x)$ indicates the gradient of $F(x)$.

Maximum turning point				
Sign of $f(x)$	+	0	−	
Description of $f(x)$	Above the x-axis	x-intercept	Below the x-axis	
Shape of $F(x)$	/	–	\	
Description of $F(x)$	Increasing	Stationary point	Decreasing	

Minimum turning point				
Sign of $f(x)$	–	0	+	
Description of $f(x)$	Below the x-axis	x-intercept	Above the x-axis	
Shape of $F(x)$	\	–	/	
Description of $F(x)$	Decreasing	Stationary point	Increasing	

A stationary point of inflection where $f(x)$ is above the x-axis				
Sign of $f(x)$	+	0	+	
Description of $f(x)$	Above the x-axis	x-intercept	Above the x-axis	
Shape of $F(x)$	/	–	/	
Description of $F(x)$	Increasing	Stationary point	Increasing	

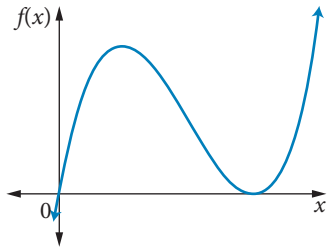
A stationary point of inflection where $f(x)$ is below the x-axis				
Sign of $f(x)$	–	0	–	
Description of $f(x)$	Below the x-axis	x-intercept	Below the x-axis	
Shape of $F(x)$	\	–	\	
Description of $F(x)$	Decreasing	Stationary point	Decreasing	

If the graph of the original function has a discontinuity, such as a hyperbola, you can deduce a possible anti-derivative graph for each part of the domain separately. The graph of $F(x)$ will have asymptotes at the same x-values as $f(x)$; more explicitly if a function has a vertical asymptote, then both the derivative and the anti-derivative are undefined for the value of x at the asymptote.

3 Sketch anti-derivatives

Sketch a possible graph of the anti-derivative of each of the following functions.

(a)



THINKING

1 Consider the x-intercepts of $f(x)$ and the corresponding features of $F(x)$.

WORKING

Near the origin, the gradient follows:

$f(x)$	–	0	+
$F(x)$	\	–	/

A minimum turning point on the graph of $F(x)$.

Near the minimum turning point, the gradient follows:

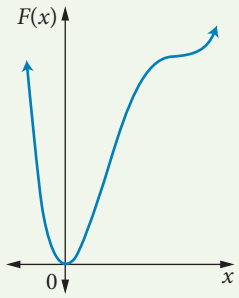
$f(x)$	+	0	+
$F(x)$	/	–	/

A stationary point of inflection exists on the graph of $F(x)$.

2 Consider the other features.

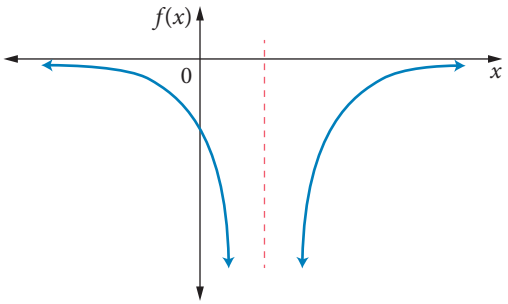
The maximum turning point indicates that the gradient increases to a maximum positive value and then starts to decrease (still positive). There is a point of inflection.

3 Sketch a possible graph of the anti-derivative.



This graph has been drawn passing through the origin, but any vertical translation of the shape is correct.

(b)



1 Consider the asymptotes.

The vertical asymptote will remain in the same position.

The horizontal asymptote indicates that the gradient approaches zero as $x \to \pm\infty$, but it might not be at $y = 0$.

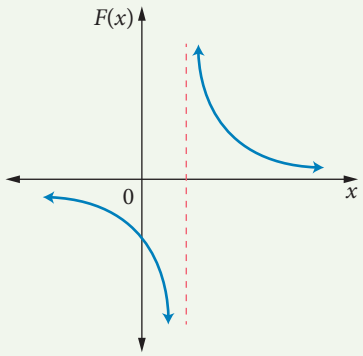
2 Consider the other features.

The derivative (gradient) is always negative.

Left-hand branch: the gradient is close to zero, then becomes increasingly negative.

Right-hand branch: The gradient is negative and becomes less negative, approaching zero.

3 Sketch a possible graph of the anti-derivative.



Any vertical translation of this graph is correct.

EXERCISE 4.1 Introduction to anti-differentiation

Worked Example 1

1 Differentiate each of the following functions and use the results to determine the given anti-derivatives.

(a) $f(x) = 2x^3 - 4x$, hence determine the anti-derivative of $6x^2 - 4$.

(b) $f(x) = 3x^2 - 2\sqrt{x}$, hence determine the anti-derivative of $6x - \frac{1}{\sqrt{x}}$.

(c) $f(x) = 5x^2 + 2x - 6$, hence determine the anti-derivative of $10x + 2$.

(d) $f(x) = 5x^2 - \frac{1}{x}$, hence determine $\int \left(10x + \frac{1}{x^2}\right) dx$.

Worked Example 2

2 Differentiate each of the following functions and use the results to determine the given anti-derivatives.

(a) $f(x) = 6x^3 - 10x$, hence determine the anti-derivative of $9x^2 - 5$.

(b) $f(x) = 5x^3 - x^5$, hence determine the anti-derivative of $3x^2 - x^4$.

(c) $f(x) = 3x^4 - 5x^3$, hence determine the anti-derivative of $4x^3 - 5x^2$.

(d) $f(x) = x^2 - \frac{1}{x}$, hence determine $\int \left(4x + \frac{2}{x^2}\right) dx$.

3 Differentiate each of the following functions and use the results to determine the given anti-derivatives.

(a) $f(x) = (3x - 2)^5$, hence determine $\int 15(3x - 2)^4 \, dx$.

(b) $f(x) = (6x - 1)^3$, hence determine $\int 36(6x - 1)^2 \, dx$.

4 Differentiate each of the following trigonometric functions and use the results to determine the given anti-derivatives.

(a) $f(x) = \sin(2x)$, hence determine $\int \cos(2x) dx$.

(b) $f(x) = 3\cos^3(x)$, hence determine $\int 27\cos^2(x)\sin(x) dx$.

(c) $f(x) = \sin^2(3x) - \cos^2(3x)$, hence determine $\int \sin(3x)\cos(3x) dx$.

5 Differentiate each of the following exponential functions and use the results to determine the given anti-derivatives.

(a) $f(x) = e^x$, hence determine $\int e^x \, dx$.

(b) $f(x) = e^{kx}$, hence determine $\int e^{kx} \, dx$.

(c) $f(x) = 3e^{5x}$, hence determine $\int 30e^{5x} \, dx$.

6 Differentiate each of the following logarithmic functions and use the results to determine the given anti-derivatives.

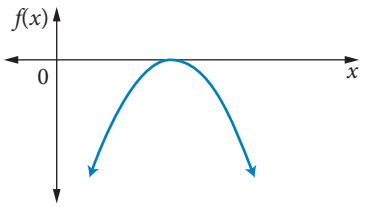
(a) $f(x) = \log_e(7x)$, hence determine $\int \frac{1}{2x} dx$.

(b) $f(x) = \log_e(2x + 1)$, hence determine $\int \frac{1}{2x + 1} dx$.

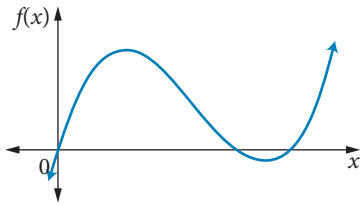
Worked Example 3

7 Sketch the graph of a possible anti-derivative function for each of the given functions, matching key values.

(a)

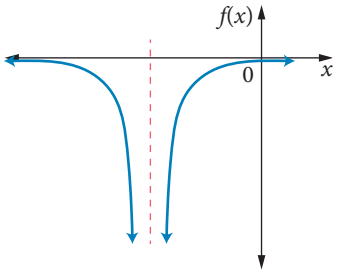


(b)

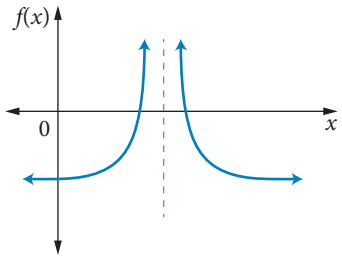


8 Sketch the graph of a possible anti-derivative function for each of the given functions, matching key values.

(a)



(b)



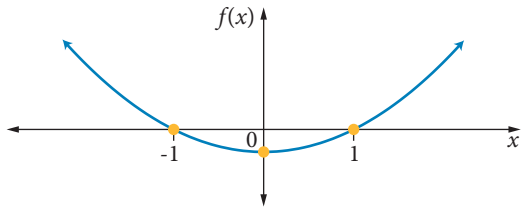
9 If the derivative of $x^2 \sin(2x)$ is $2x(x\cos(2x) + \sin(2x))$, then $\int \left(x^2 \cos(2x) + x\sin(2x)\right) dx$ is equal to which one of the following?

A $x^2 \sin(2x) + c$ **B** $2x(x\cos(2x) + \sin(2x)) + c$ **C** $\frac{1}{2}x^2 \sin(2x) + c$ **D** $2x^2 \sin(2x) + c$

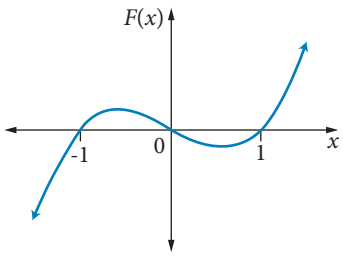
10 Which one of the following integrals cannot be determined by differentiating $x^3 e^x$?

A $\int x^2 e^x (x+3)\, dx$ **B** $\int e^x (x+3)\, dx$ **C** $\int \frac{1}{2} e^x \left(x^3 + 3x^2\right) dx$ **D** $\int 2x^3 e^x + 6x^2 e^x\, dx$

11 Which one of the following is a possible anti-derivative graph for the graph of $f(x)$ shown?



A



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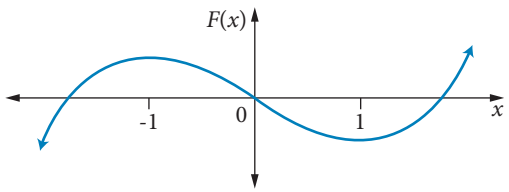

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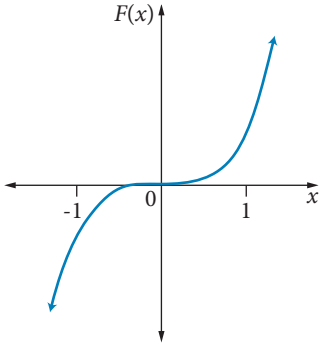

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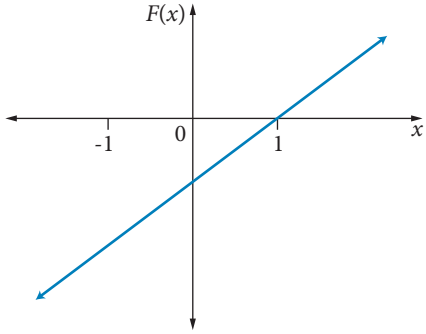
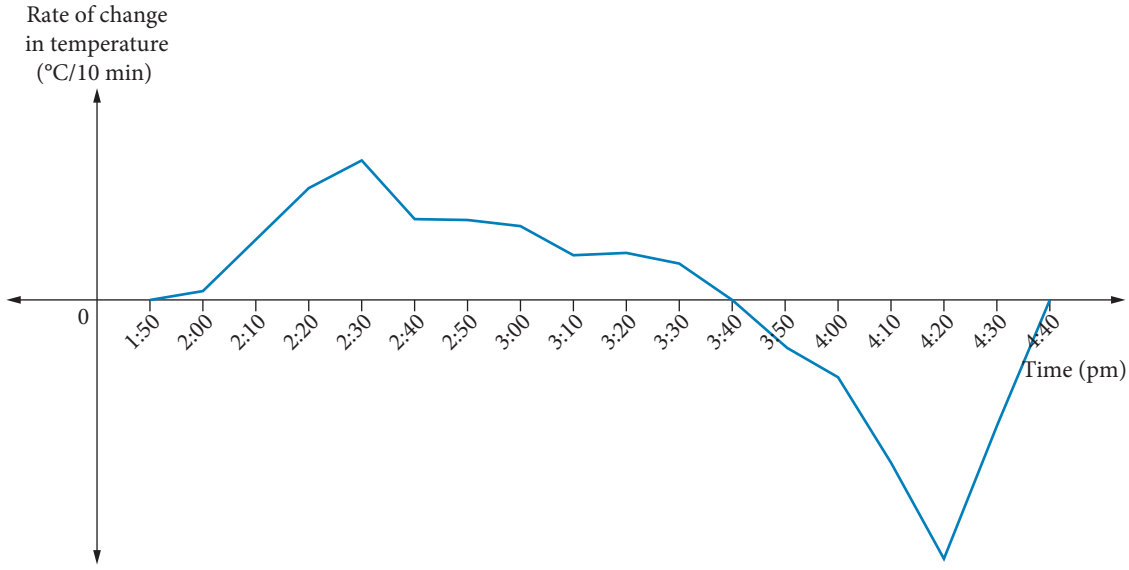


12 Verify that the derivative of $x^2(2x-1)^3$ can be expressed as $2x(5x-1)(2x-1)^2$. Hence, determine the value of the constant of integration c, given that $\int x(5x-1)(2x-1)^2\,dx = 20$ at $x=2$.

13 Differentiate $3(2x-1)^4$ and hence determine the equation of the curve $y=f(x)$ with a gradient function of $3(2x-1)^3$ that passes through the point $(1.5, 8)$.

14 A child began to feel unwell around 1:30 pm on Sunday, and by 1:50 pm she was very hot and feverish. The child's temperature was taken at 10-minute intervals for the next few hours. The graph shows the rate of change of temperature over time.



(a) At what approximate time was the child's temperature rising most rapidly?

(b) Between what times was the temperature rising?

(c) Describe what was happening to the temperature at approximately 3:40 pm.

(d) Sketch a possible graph of the child's temperature over the same period of time.

Anti-differentiation of power functions

4.2

In the previous section, you were introduced to anti-differentiation as the reverse operation of differentiation, noting that the indefinite integral has an arbitrary constant added. In this section, you will learn how to determine the anti-derivative (indefinite integral) of a power function and use given conditions to determine a particular value of the anti-derivative.

Consider the following derivatives:

$$\frac{d\left(x^2\right)}{dx} = 2x \qquad \frac{d\left(x^3\right)}{dx} = 3x^2$$

$$\int 2x\,dx = x^2 + c_1 \qquad \int 3x^2\,dx = x^3 + c_1$$

$$2\int x\,dx = x^2 + c_1 \qquad 3\int x^2\,dx = x^3 + c_1$$

$$\int x\,dx = \frac{1}{2}\left[x^2 + c_1\right] \qquad \int x^2\,dx = \frac{1}{3}\left[x^3 + c_1\right]$$

$$\int x\,dx = \frac{1}{2}x^2 + c \qquad \int x^2\,dx = \frac{1}{3}x^3 + c$$

where $c = \frac{1}{2}c_1$ where $c = \frac{1}{3}c_1$

You can check these by differentiating the result in each case.

Note that for the anti-derivative, the power is increased by one and the result is divided by the new power.

As you will have observed in the previous section, a constant coefficient may be written inside the anti-derivative sign or before it.

Additional information

Anti-differentiating powers of *x*

Practise using the pattern for anti-differentiating expressions of the form x^n.

In the general case:

$$\int x^n\,dx = \frac{1}{n+1}x^n + c, \quad n \neq -1$$

$$\int ax^n\,dx = a\int x^n\,dx, \quad n \neq -1$$

$$= \frac{ax^{n+1}}{n+1} + c$$

4 Anti-differentiate expressions of the form ax^n

Determine the following anti-derivatives.

(a) $\int 3x^4\,dx$

THINKING	WORKING
1 Express in the form $a\int x^n\,dx$.	$\int 3x^4\,dx = 3\int x^4\,dx$
2 Recall the formula.	$a\int x^n\,dx = \frac{ax^{n+1}}{n+1} + c, \quad n \neq -1$
3 Substitute the known values.	$= \frac{3x^5}{5} + c$
4 Interpret the answer.	The anti-derivative of $3x^4$; that is, $\int 3x^4\,dx$ is equal to $\frac{3x^5}{5} + c$.

(b) $\int 4x\sqrt{x}\,dx$

1 Express in the form $a\int x^n dx$.	$\int 4x\sqrt{x}\,dx = 4\int x^{\frac{3}{2}}dx$
2 Recall the formula.	$a\int x^n dx = \frac{ax^{n+1}}{n+1} + c, \quad n \neq -1$
3 Substitute the known values and simplify as needed.	$4\int x^{\frac{3}{2}}dx = \frac{4x^{\frac{5}{2}}}{\frac{5}{2}} + c$ $= \frac{8x^{\frac{5}{2}}}{5} + c$
4 Interpret the answer.	The anti-derivative of $4x\sqrt{x}$; that is, $\int 4x\sqrt{x}\,dx$ is equal to $\frac{8x^{\frac{5}{2}}}{5} + c$.

You will note that if $f(x) = 2x$, then $f'(x) = 2$, hence $\int 2\,dx = 2x + c$. Writing $\int 2\,dx$ as $\int 2x^0 dx$ and applying the rule for anti-differentiating x^n, you obtain:

$$2\int x^0 dx = 2 \times \frac{x^1}{1} + c$$
$$= 2x + c$$

$$\int a\,dx = ax + c$$

The anti-derivative of a sum or difference of a series of terms is equal to the sum or difference of the anti-derivative of each individual term.

$$\int \big(f(x) \pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx$$

5 Anti-differentiate term by term

Determine $\int \left(3x^2 - 3\sqrt{x} + 4\right)dx$.

THINKING	WORKING
1 Express terms in the form $a\int x^n dx$.	$\int \left(3x^2 - 3\sqrt{x} + 4\right)dx = 3\int x^2 dx - 3\int x^{\frac{1}{2}}dx + \int 4\,dx$ This step can be omitted, and the anti-differentiation can be done under the one integral sign.
2 Use the rule for anti-differentiating powers of x. Since c represents an arbitrary constant, you only need to add c once at the end.	$\int \left(3x^2 - 3\sqrt{x} + 4\right)dx = 3\left(\frac{x^3}{3}\right) - 3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + 4x + c$
3 Simplify the expression on the right-hand side of the equation.	$= x^3 - 3 \times \frac{2}{3}x^{\frac{3}{2}} + 4x + c$
4 Express the anti-differentiated expression.	$\int \left(3x^2 - 3\sqrt{x} + 4\right)dx = x^3 - 2x^{\frac{3}{2}} + 4x + c$

Notation

The following notations are used at various times to indicate the process of anti-differentiation.

$$\int f'(x)dx = f(x) + c$$

$$\int \frac{dy}{dx}dx = y + c$$

$$\int f(x)dx = F(x) + c$$

6 Simplify before anti-differentiating

Determine the anti-derivative of $\frac{(3x^2 - 2)(x^2 + 1)}{x^2}$.

THINKING	WORKING
1 Expand the brackets and collect like terms.	$\int \frac{(3x^2 - 2)(x^2 + 1)}{x^2}dx = \int \frac{3x^4 + x^2 - 2}{x^2}dx$
2 Rewrite each term with the denominator and express each one in the form ax^n.	$= \int \left(\frac{3x^4}{x^2} + \frac{x^2}{x^2} - \frac{2}{x^2}\right)dx$ $= \int \left(3x^2 + 1 - 2x^{-2}\right)dx$
3 Anti-differentiate each term using the rule and introduce the constant of integration.	$= 3 \times \frac{x^3}{3} + x - 2 \times \frac{x^{-1}}{-1} + c$
4 Simplify, expressing with positive powers, and write down the answer.	$= x^3 + x + 2x^{-1} + c$
5 Express the anti-differentiated expression.	$\int \frac{(3x^2 - 2)(x^2 + 1)}{x^2}dx = x^3 + x + \frac{2}{x} + c$

Applications of anti-differentiation

Given a rate of change, anti-differentiating can lead you to information about the quantity whose rate you are measuring. For example, velocity is a measure of the rate of change of position. In other words, the velocity v may be expressed as $v = \frac{dx}{dt}$, where x is the position of a particle at time t. Anti-differentiating with respect to t gives you an expression for the position.

7 Apply anti-differentiation

Determine an expression for the position x of an object if its velocity v is modelled by the function $v = \frac{3}{t^2} - 5$ at time t, for $t \in [2,5]$.

THINKING	WORKING
1 Express as an anti-derivative.	$x = \int v\,dt$ $= \int \left(\frac{3}{t^2} - 5\right)dt$

2 Write terms in the form $a\int x^n dx$.	$\int\left(\frac{3}{t^2}-5\right)dt = \int\left(3t^{-2}-5t^0\right)dt$
3 Use the rule to anti-differentiate each term, introduce the constant of integration and simplify the expression.	$= 3\times\frac{t^{-1}}{-1}-\frac{5t^1}{1}+c$ $= -\frac{3}{t}-5t+c$
4 Interpret the answer.	The position equation is given by $x = -\frac{3}{t}-5t+c$.

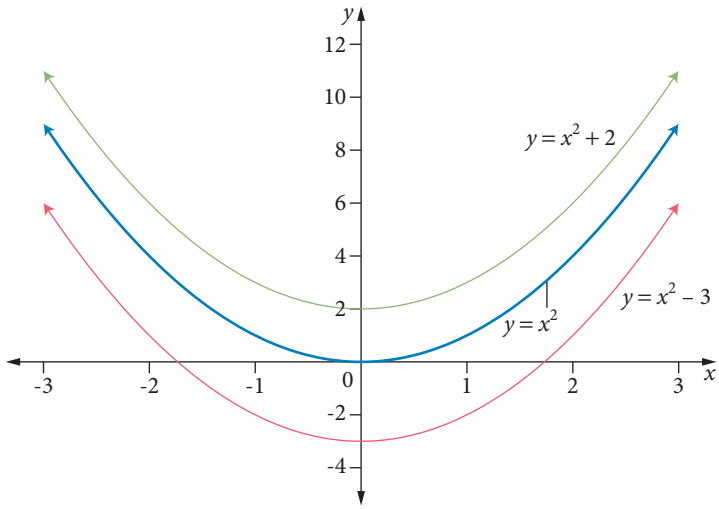
8 Determine the value of the constant of integration

If $\frac{dy}{dx} = \left(4-\frac{3}{x^2}\right)^2$, determine y in terms of x if $y = 13$ when $x = 1$.

THINKING	WORKING
1 Expand the expression and write each term in the form $a\int x^n dx$.	$\left(4-\frac{3}{x^2}\right)^2 = 16-\frac{24}{x^2}+\frac{9}{x^4}$ $= 16-24x^{-2}+9x^{-4}$
2 Express y as an anti-derivative of $\frac{dy}{dx}$.	$y = \int\frac{dy}{dx}dx$ $= \int\left(16-24x^{-2}+9x^{-4}\right)dx$
3 Anti-differentiate the expression, introduce the constant of integration and simplify.	$= 16x-24\times\frac{x^{-1}}{-1}+9\times\frac{x^{-3}}{-3}+c$ $= 16x+\frac{24}{x}-\frac{3}{x^3}+c$
4 Use the given conditions to determine the value of the constant.	When $x = 1$, $y = 13$: $13 = 16(1)+\frac{24}{1}-\frac{3}{1^3}+c$ $13 = 16+24-3+c$ $c = -24$
5 Interpret the answer.	The anti-derivative of $\frac{dy}{dx} = \left(4-\frac{3}{x^2}\right)^2$ that includes the point $(1, 13)$ is $y = 16x+\frac{24}{x}-\frac{3}{x^3}-24$.

Families of curves

If $f'(x) = 2x$, then $f(x) = x^2 + c$. A set of graphs for various values of the arbitrary constant c represents a family of curves, all of which have the same gradient function $2x$.



Each graph represents a translation of $y = x^2$ parallel to the y-axis. At any x-value, the curves will have the same gradient.

Additional information

Changing the value of the constant

Use the slider and observe the gradient of the tangents when the graph is translated up or down, representing different values of the constant.

EXERCISE 4.2 Anti-differentiation of power functions

Worked Example

1 Determine each of the following anti-derivatives. 4

(a) $\int x^6\,dx$ (b) $\int 12x^3\,dx$ (c) $\int 4x^2\,dx$ (d) $\int \frac{1}{3}x^5\,dx$

(e) $\int 3\sqrt{x}\,dx$ (f) $\int 2x^{\frac{2}{3}}\,dx$ (g) $\int \frac{3}{4}x^{\frac{2}{5}}\,dx$ (h) $\int 3x^{-2}\,dx$

2 Determine each of the following anti-derivatives. 5

(a) $\int \left(6x^2 - x + 2\right)dx$ (b) $\int \left(x^3 - 3x^2 + 2x\right)dx$ (c) $\int \left(6x^2 + 2\sqrt{x}\right)dx$

(d) $\int \left(4 - \frac{x^2}{2}\right)dx$ (e) $\int \left(4x - \frac{1}{\sqrt{x}} + x^3\right)dx$ (f) $\int \left(7 + \frac{2}{3w^2} + w^{\frac{2}{3}}\right)dw$

3 Determine each of the following anti-derivatives. 6

(a) $\int \frac{x^2 + x}{x}\,dx$ (b) $\int \frac{x^3 - 2x^2}{x^2}\,dx$ (c) $\int \frac{x^4 - 2x^2 + 3}{x^2}\,dx$

(d) $\int (x-1)\left(x^2 + 2\right)dx$ (e) $\int \frac{(x-2)(x+2)}{x^2}\,dx$ (f) $\int \frac{\left(x^2 - 3\right)\left(x^2 + 4\right)}{x^2}\,dx$

(g) $\int \frac{x^2 + x}{\sqrt{x}}\,dx$ (h) $\int \sqrt{x}\,(x-3)\,dx$

Worked Example

7

4 The velocity v of a particle is the rate of change of its position x at time t, so $v = \frac{dx}{dt}$. If $v = 3t^2 - 4t + 18$, determine an expression for the position of the particle at time t.

8

5 If the particle in question 4 is at a position $x = 2$ when $t = 0$, determine the value of each of the following.

(a) the constant of integration, c

(b) the position of the particle when $t = 3$

6 Determine $f(z)$, given:

(a) $f'(z) = z - 6z^2$

(b) $f'(z) = (z-2)(z+4)$

(c) $f'(z) = \frac{3z^5 - 4z}{z^3}$

(d) $f'(z) = \frac{-2}{z^2}$

7 For each of the following, determine y in terms of x for the given conditions.

(a) $y = \int \left(6x^2 - 2x\right) dx$, and $y = -10$ when $x = 2$

(b) $\frac{3}{x^2} - \frac{4}{x^3}$, and $y = 1$ when $x = 4$

8 The gradient of a curve at any point is given by $f'(x) = \left(1 + \sqrt{x}\right)^2$. Determine the equation of the curve if it passes through the point $\left(1, -\frac{1}{6}\right)$.

9 Determine an anti-derivative of each of the following.

(a) $\int \left(3 - \frac{x}{2}\right)^5 dx$

(b) $\int \sqrt{x}\,(4 - x)^2\, dx$

10 For each of the following, express y in terms of x.

(a) $\frac{dy}{dx} = 8x^3 + \frac{3}{x^2}$

(b) $\frac{dy}{dx} = 2x^{-3} + x^{-\frac{3}{2}}$

11 Differentiate $f(x) = \frac{2}{3}\left(1 + x^2\right)^{\frac{3}{2}}$ and use the result to determine $\int 2x\sqrt{1 + x^2}\, dx$.

12 The gradient of a curve varies directly as x^2. If the curve passes through the origin and the point $(3, 18)$, determine each of the following.

(a) the equation of the curve

(b) the value of the gradient at $x = -3$

(c) the value of y at $x = -3$

13 The gradient of a curve is given by $\frac{dy}{dx} = 3(2x - 1)^2$.

(a) Determine the equation of the family of curves that have this gradient.

(b) Determine the member of this family of curves that passes through each of the given points.

(i) $(0, 4)$

(ii) $(-1, 5)$

(c) Explain why the two curves from part (b) have no points of intersection.

14 The position x, velocity v, and acceleration a of a particle at time t are related by: $v = \frac{dx}{dt}$, $a = \frac{dv}{dt}$. If the acceleration of a particle is modelled by $a = 3t^2 - t - 2$, $t \geq 0$, determine the following.

(a) the velocity of the particle at $t = 10$ if the particle starts from rest

(b) the position of the particle at $t = 2$, if $x = \frac{1}{12}$ when $t = 1$

15 A tank of water is emptying at a rate given by $\frac{dV}{dt} = -2.8t$, where V is the volume in cubic metres m^3 at time t seconds.

(a) Determine an expression for the volume of water remaining in the tank in terms of time, if it is empty after 30 seconds.

(b) How long does it take for half of the initial volume to empty out? Give your answer correct to 1 decimal place.

16 Which one of the following is the anti-derivative of $x\left(3x^2 - 1\right)$?

A $\frac{x^2}{2}\left(\frac{3x^3}{3} - x\right) + c$

B $\left(3x^2 - 1\right) \times 1 + x(6x) + c$

C $3x^4 - x + c$

D $\frac{1}{2}x^2\left(\frac{3x^2}{2} - 1\right) + c$

17 Which one of the following statements is correct?

A $\int (x - 3)\,dx$ represents a family of parabolas with axes of symmetry at $x = 3$.

B $\int (x - 3)\,dx$ represents a family of parabolas with axes of symmetry at $x = 6$.

C $\int (x - 3)\,dx$ represents a family of parabolas with two x-intercepts.

D $\int (x - 3)\,dx$ could be equal to $\frac{x^2}{2} - 3x + 10x^2$.

18 Determine the following anti-derivatives.

(a) $\int \frac{\left(x + \frac{1}{x}\right)^2}{x^2} + 5\,dx$

(b) $\int \frac{x^3 - 2\sqrt{x} + x^{-2}}{x^2}\,dx$

19 Differentiate $f(x) = (x + 3)^{\frac{1}{2}} x^2$ and hence determine $\int \frac{5x^2 + 12x}{2\sqrt{x + 3}}\,dx$.

20 If $f'(x) = px^2 + \frac{2}{x^3}$, determine $f(x)$ if $f(1) = 2$ and $f(-1) = 4$.

21 Differentiate $(3x + 1)^4$ and hence determine the anti-derivative of $(3x + 1)^3$.

22 **(a)** Differentiate $(ax + b)^{n+1}$ and hence determine the anti-derivative of $\int (ax + b)^n\,dx$.

(b) Use the result found in part **(a)** to determine the following.

(i) $\int (6x + 1)^4\,dx$

(ii) $\int (3 - 2x)^3\,dx$

(iii) $\int (3x - 2)^{\frac{1}{2}}\,dx$

(iv) $f(x)$ if $f'(x) = 4(7x + 2)^3$ and $f(0) = -\frac{3}{7}$

4.3 Anti-differentiation of linear binomial power functions

In this section, you will determine the anti-derivative of the form $\int f(ax+b)\,dx$, where $ax+b$ is a linear expression. If $f(x)=(ax+b)^{n+1}$, the derivative can be found by applying the chain rule, hence: $f'(x)=a(n+1)(ax+b)^n$.

Therefore:

$$\int a(n+1)(ax+b)^n\,dx=(ax+b)^{n+1}+c_1$$

$$a(n+1)\int(ax+b)^n\,dx=(ax+b)^{n+1}+c_1$$

$$\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+\frac{c_1}{a(n+1)}$$

Hence $\int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c$, where $c=\frac{c_1}{a(n+1)}$.

Note: $n=-1$ results in division by zero, which is undefined. Expressions with $n=-1$ cannot be integrated using this rule.

$$\int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c,\ n\neq -1$$

Note: This rule is only valid if the expression being raised to a power is linear.

9 Anti-differentiate functions of the form $(ax+b)^n$, where $n\in\mathbb{Z}\setminus\{-1\}$

Determine the anti-derivative of $(2x-3)^4$.

THINKING	WORKING
1 Express as the required integral.	$\int(2x-3)^4\,dx$
2 Recall the rule.	$\int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c,\ n\neq -1$
3 Determine the anti-derivative.	$\int(2x-3)^4\,dx=\frac{(2x-3)^5}{2\times 5}+c$

10 Anti-differentiate functions of the form $(ax+b)^n$, where $n\in\mathbb{Q}\setminus\{-1\}$

Determine the anti-derivative of $(3x-5)^{\frac{1}{2}}$.

THINKING	WORKING
1 Express as the required integral.	$\int(3x-5)^{\frac{1}{2}}\,dx$
2 Recall the rule.	$\int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c,\ n\neq -1$
3 Determine the anti-derivative and simplify the expression.	$\int(3x-5)^{\frac{1}{2}}\,dx=\frac{(3x-5)^{\frac{3}{2}}}{3\times\frac{3}{2}}+c$ $=\frac{2(3x-5)^{\frac{3}{2}}}{9}+c$

11 Anti-differentiate functions where $n < 0$

Determine the anti-derivative of $\frac{4}{(3-5x)^3}$.

THINKING	WORKING
1 Express as the required integral in the form $(ax+b)^n$.	$\int \frac{4}{(3-5x)^3}dx = 4\int (3-5x)^{-3}\,dx$
2 Recall the rule.	$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$
3 Determine the anti-derivative and simplify the expression.	$\int 4(3-5x)^{-3}\,dx = 4 \times \frac{(3-5x)^{-2}}{-2 \times -5} + c$ $= \frac{2}{5(3-5x)^2} + c$

The set of graphs of $\int f'(x)dx = f(x) + c$ represents a family of curves. As you found in the previous section, additional information, such as a point on the curve, enables you to determine the value of c and therefore find the equation that represents a specific function.

12 Determine the equation of a curve from the gradient function at a known point

A curve has a gradient given by $\frac{dy}{dx} = (2x-1)^2$ and passes through the point $(2, 1)$. Determine the equation of the curve.

THINKING	WORKING
1 Determine the anti-derivative.	$y = \int (2x-1)^2 dx$ $= \frac{(2x-1)^3}{2 \times 3} + c$ $= \frac{(2x-1)^3}{6} + c$
2 Substitute the given point and solve for c, the constant of integration.	$1 = \frac{(2 \times 2 - 1)^3}{6} + c$ $1 - \frac{27}{6} = c$ $c = -\frac{7}{2}$
3 Write the equation of the curve.	$y = \frac{1}{6}(2x-1)^3 - \frac{7}{2}$

13 Anti-differentiate using the chain rule

The velocity v of a particle moving in a straight line is given by $v = 2(4+7t)^{-\frac{5}{2}}$, $t \geq 0$ at time t.

(a) Determine an expression for the position x of the particle at time t, given $v = \frac{dx}{dt}$.

THINKING	WORKING
1 Write the required integral.	$x = \int v\,dt$ $= 2\int (4+7t)^{-\frac{5}{2}}\,dt$
2 Recall the rule.	$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$
3 Determine the anti-derivative and simplify the expression.	$x(t) = \frac{2 \times (4+7t)^{-\frac{3}{2}}}{7 \times -\frac{3}{2}} + c$ $= -\frac{4}{21(4+7t)^{\frac{3}{2}}} + c$

(b) Determine the position at $t = 3$, if the initial position of the particle is $x = -\frac{1}{42}$.

1 Substitute the given initial condition.	$x(0) = \frac{1}{42}$
2 Determine the value of the constant of integration.	$-\frac{1}{42} = -\frac{4}{21(4)^{\frac{3}{2}}} + c$ $-\frac{1}{42} = -\frac{4}{21 \times 8} + c$ $-\frac{1}{42} = -\frac{1}{42} + c$ $c = 0$
3 Determine the expression for the position.	$x = -\frac{4}{21(4+7t)^{\frac{3}{2}}}$
4 Substitute the given time.	$t = 3$ $x = -\frac{4}{21(4+21)^{\frac{3}{2}}}$ $= -\frac{4}{21 \times 125}$ $= -\frac{4}{2625}$
5 Interpret the answer.	The position of the particle at $t = 3$ is $x = -\frac{4}{2625}$.

EXERCISE 4.3 Anti-differentiation of linear binomial power functions

Worked Example

1 Determine the anti-derivative of each of the following. 9

(a) $(2x+5)^4$ (b) $(3x-1)^3$ (c) $4(2x-1)^5$ (d) $3(1-2x)^3$

2 Determine the anti-derivative of each of the following. 10

(a) $(2x+1)^{\frac{1}{2}}$ (b) $(6x-5)^{-\frac{1}{2}}$ (c) $(2-11x)^{\frac{1}{2}}$ (d) $(1-10x)^{\frac{3}{4}}$

3 Determine the anti-derivative of each of the following. 11

(a) $\dfrac{2}{(6x+3)^2}$ (b) $\dfrac{10}{(5x-2)^2}$ (c) $\dfrac{-2}{(3-4x)^3}$

(d) $\dfrac{2}{3(1-2x)^4}$ (e) $\dfrac{1}{2(2x+3)^{\frac{3}{2}}}$ (f) $\dfrac{5}{\sqrt{1-3x}}$

4 Match a correct anti-derivative from the list (i)–(vi) with each of the functions (a)–(c).

(i) $(2x-1)^4$ (ii) $6(2x-1)^3$ (iii) $\dfrac{1}{2}(2x-1)^4$

(iv) $\dfrac{1}{4}(2x-1)^4$ (v) $\dfrac{1}{8}(2x-1)^4$ (vi) $\dfrac{1}{16}(2x-1)^4$

(a) $(2x-1)^3$ (b) $4(2x-1)^3$ (c) $\dfrac{1}{2}(2x-1)^3$

5 Determine an expression for y in terms of x for each of the following.

(a) $\dfrac{dy}{dx} = (5x-2)^3$ (b) $\dfrac{dy}{dx} = (2-3x)^{-\frac{3}{2}}$

6 Determine $f(x)$, given:

(a) $f'(x) = \dfrac{6}{\sqrt{1+2x}}$ (b) $f'(x) = 6(5x-7)^3$

7 Determine the equation of the curve that has a gradient given by $f'(x) = x^2 - \dfrac{1}{x^2}$ and passes through the point $(1, -2)$. 12

8 Determine the equation of the curve that has a gradient given by $\dfrac{dy}{dx} = (2x-5)^3$ and passes through the point $(2, 3)$.

9 Determine each of the following.

(a) $\displaystyle\int \frac{7}{3}(2x-1)^6\,dx$ (b) $\displaystyle\int 3(2-5x)^{-\frac{7}{2}}dx$

10 Determine $F(x)$ for each of the functions given.

(a) $f(x) = a(mx+b)^k$ (b) $f(x) = \dfrac{d}{(ax+b)^m}$

11 $9\int(4-5x)^{-3}dx$ is equal to which one of the following expressions?

Worked Example

A $\dfrac{9}{20(4-5x)^4}+c$ **B** $\dfrac{-9}{10(4-5x)^2}+c$ **C** $\dfrac{9}{10(4-5x)^2}+c$ **D** $\dfrac{15}{(4-5x)^4}+c$

13

12 The velocity v m/s of a particle travelling in a straight line is modelled by $v=2(3t-1)^3+6$ at time t seconds.

(a) Determine the initial velocity.

(b) Determine an expression for the position x m at time t.

(c) If the particle is at $x=0$ when $t=\dfrac{1}{3}$, determine the initial position.

13 The area A m^2 of a puddle after t seconds is changing at a rate such that $\dfrac{dA}{dt}=2\sqrt{1+4t}$. If the initial area of the puddle is 4 m^2, determine the area after 1 second. Give your answer correct to 1 decimal place.

14 **(a)** Expand and anti-differentiate $(3x-4)^3$ to demonstrate that $\int(ax+b)^n\,dx=\dfrac{(ax+b)^{n+1}}{a(n+1)}+c$.

(b) Given that $(3x-4)^4=81x^4-432x^3+864x^2-768x+256$, compare the two answers obtained in part **(a)**.

15 The gradient of a curve is given by $\dfrac{dy}{dx}=(ax+2)^3$. The tangent to this curve, drawn at the point $(1,28)$, has a y-intercept of $(0,1)$.

(a) Determine the equation of the curve.

(b) Graph the curve and tangent on the same axes, using technology, and verify that the point of intersection is $(1,28)$.

Anti-differentiation of exponential functions

4.4

If $f(x) = e^x$, then $f'(x) = e^x$.

Since $\int f'(x)dx = f(x) + c$, then $\int e^x dx = e^x + c$.

Similarly, if $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$, $k \in \mathbb{R} \setminus \{0\}$.

Therefore $\int ke^{kx} dx = e^{kx} + c_1$.

$k\int e^{kx} dx = e^{kx} + c_1$

$\int e^{kx} dx = \frac{1}{k}e^{kx} + \frac{c_1}{k}$

$\int e^{kx} dx = \frac{1}{k}e^{kx} + c$ where $c = \frac{c_1}{k}$

$$\int e^{kx} dx = \frac{1}{k}e^{kx} + c, \; k \in \mathbb{R} \setminus \{0\}$$

14 Anti-differentiate functions of the form e^{kx}

(a) Determine the anti-derivative of e^{-5x}.

THINKING	WORKING
1 Recall the rule.	$\int e^{kx} dx = \frac{1}{k}e^{kx} + c$
2 Anti-differentiate using the rule.	$\int e^{-5x} dx = \frac{1}{-5}e^{-5x} + c$ $= -\frac{1}{5}e^{-5x} + c$

(b) Determine the anti-derivative of $e^{2x} - 6x + 4$.

1 Recall the rule.	$\int e^{kx} dx = \frac{1}{k}e^{kx} + c$
2 Anti-differentiate each term and simplify the expression.	$\int \left(e^{2x} - 6x + 4\right) dx = \frac{1}{2}e^{2x} - \frac{6x^2}{2} + 4x + c$ $= \frac{1}{2}e^{2x} - 3x^2 + 4x + c$

Also,

$\frac{d}{dx}\left(e^{kx+b}\right) = ke^{kx+b}$

$k\int e^{kx+b} \, dx = e^{kx+b} + c$

$\int e^{kx+b} \, dx = \frac{1}{k}e^{kx+b} + c$

$$\int e^{kx+b} \, dx = \frac{1}{k}e^{kx+b} + c, \; k \in \mathbb{R} \setminus \{0\}$$

15 Anti-differentiate functions of the form e^{kx+b}

Determine the following anti-derivatives.

(a) $\int e^{4x+7}\,dx$

THINKING	WORKING
1 Recall the rule.	$\int e^{kx+b}\,dx = \frac{1}{k}e^{kx+b} + c$
2 Anti-differentiate using the rule.	$\int e^{4x+7}\,dx = \frac{1}{4}e^{4x+7} + c$

(b) $\int -6e^{3x-1}\,dx$

1 Recall the rule.	$\int e^{kx+b}\,dx = \frac{1}{k}e^{kx+b} + c$
2 Rewrite the expression with the constant in front of the integral.	$\int -6e^{3x-1}dx = -6\int e^{3x-1}\,dx$
3 Anti-differentiate using the rule.	$= -6 \times \frac{e^{3x-1}}{3} + c$

16 Simplify exponential functions before anti-differentiating

Determine $\int \frac{e^x\left(3 - e^{-2x}\right)}{e^{3x}}dx$.

THINKING	WORKING
1 Expand and simplify.	$\int \frac{e^x\left(3 - e^{-2x}\right)}{e^{3x}}dx = \int \frac{3e^x - e^{-x}}{e^{3x}}dx$ $= \int\left(\frac{3e^x}{e^{3x}} - \frac{e^{-x}}{e^{3x}}\right)dx$ $= \int\left(3e^{-2x} - e^{-4x}\right)dx$
2 Anti-differentiate each term and simplify the expression.	$= 3\frac{e^{-2x}}{-2} - \frac{e^{-4x}}{-4} + c$ $= -\frac{3}{2e^{2x}} + \frac{1}{4e^{4x}} + c$

EXERCISE 4.4 Anti-differentiation of exponential functions

Worked Example 14

1 Determine the following anti-derivatives.

(a) $\int e^{3x}\,dx$ (b) $\int e^{-x}dx$ (c) $\int e^{\frac{2x}{3}}\,dx$

(d) $\int e^{-\frac{x}{2}}\,dx$ (e) $\int 2e^{-\frac{t}{2}}dt$

2 Determine $f(x)$ if each of the following expressions represents $f'(x)$.

(a) $3e^{6x}$ (b) $\dfrac{1}{e^{4x}}$ (c) $6e^{-3x}$

(d) $\dfrac{5}{2}e^{5x}$ (e) $\dfrac{2e^{-6x}}{3}$ (f) $e^{\frac{x}{3}}$

3 Determine the anti-derivative of each of the following.

(a) $e^{3x}-6x^2-2$ (b) $4e^{2x}+2x-1$

(c) $2e^{-2x}+e+6$ (d) $5e^{x}-2e^{3x}+x$

Worked Example 15

4 Determine the anti-derivative of each of the following.

(a) e^{2x+1} (b) $3e^{3x-2}$ (c) $4e^{1-2x}$

(d) $6e^{2-3x}$ (e) $\int e^{\frac{2x}{3}+4}dx$ (f) $\int 3e^{2-5x}dx$

Worked Example 16

5 Determine the anti-derivative of each of the following.

(a) $\left(e^{2x}-1\right)^2$ (b) $\left(2-3e^{-x}\right)^2$ (c) $\dfrac{e^{x}\left(2e^{3x}-2\right)}{e^{2x}}$

(d) $\dfrac{2e^{2x}\left(3-e^{-x}\right)}{e^{-x}}$ (e) $\dfrac{e^{-x}\left(4e^{4x}-e^{x}\right)}{2e^{2x}}$ (f) $\dfrac{e^{3x}\left(6-3e^{2x}\right)}{e^{-x}}$

6 Given the gradient function $f'(x)$ of a curve, and a point on the curve, determine $f(x)$.

(a) $f'(x)=e^{2x}-\dfrac{1}{e^{2x}}$ and $f(0)=3$

(b) $f'(x)=e^{2x}\left(2-e^{-x}\right)$ and $f(1)=0$

7 Determine the equation of the curve that passes through the point $(0, 0.5)$ and whose gradient function is $\dfrac{dy}{dx}=e^{2x}-\dfrac{1}{e^{x}}$.

8 Which one of the following is a possible anti-derivative of e^2?

A $\dfrac{e^3}{3}$ B e^2x C $2e$ D $\dfrac{x}{e^2}$

9 Determine $F(x)$, given $f(x)=ae^{2x}-3$, $f(0)=1$ and $F(0)=-2$.

10 Determine $\int\left(e^{\frac{x}{2}}-\dfrac{1}{\sqrt{x}}\right)dx$.

11 $\dfrac{dy}{dx}=pe^{-x}-3$, where p is a real number. Where $x=0$, $\dfrac{dy}{dx}=1$ and $y=-12$. Determine the exact value of y when $x=3$.

12 Which one of these expressions is equal to $\int e^{3x}e^{2x-1}\,dx$?

A $\frac{e^{3x}}{3}\times\frac{e^{2x-1}}{2}+c$ **B** $\frac{1}{6}e^{5x-1}+c$ **C** $\frac{1}{5}e^{5x-1}+c$ **D** $e^{\frac{5x^2}{2}-x}+c$

13 Given $g'(x)=e^{-x}-2e^{1-2x}$ and $g(0)=e$, determine an expression for $g(x)$.

14 The velocity v m/s of a particle moving in a straight line is modelled by $v(t)=3\left(1-e^{-0.5t}\right)$, where t is the time taken in seconds.

(a) Determine the initial velocity.

(b) Determine an expression for the position x m of the particle in terms of time.

(c) If the initial position of the particle is $x=0$, determine its position after 4 seconds, correct to 2 decimal places.

15 Differentiate e^{x^2} and hence determine $\int 2xe^{x^2}\,dx$.

16 A curve has a gradient function $\frac{dy}{dx}=ae^{2x}$. When $x=0$, $y=1$ and when $x=2$, $y=4$.

(a) Verify that $a=\frac{6}{e^4-1}$.

(b) Express y in terms of x.

(c) Demonstrate that the curve passes through the point $(2,4)$.

17 A curve has the equation $y=xe^x$.

(a) Determine $\frac{dy}{dx}$.

(b) Hence determine $\int xe^x\,dx$.

(c) Determine the equation of the curve that passes through the point $(1,e)$ and whose gradient function is $\frac{dy}{dx}=xe^x$.

18 A curve has a gradient given by $\frac{dy}{dx}=\frac{1}{2}e^{kx}$, where k is a real number. The tangent to this curve at the point $\left(2,e^4\right)$ passes through the origin.

(a) Determine the value of k.

(b) Determine the equation of the curve.

19 One way of representing e^x is by the series $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$, where $n!=n(n-1)(n-2)(n-3)\ldots\times 1$.

For example:

$3!=3\times 2\times 1$

$=6$

(a) Anti-differentiate the series.

(b) Assuming $\int e^x\,dx=e^x$, i.e. the arbitrary constant is equal to zero, determine the value of c from part **(a)** that will make the two integrals equal.

Anti-differentiation of sine and cosine functions

4.5

You have already considered differentiation of the sine and cosine functions, therefore you can use these derivatives to determine their anti-derivatives.

If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

Since $\int f'(x)\,dx = f(x) + c$, then $\int \cos(x)\,dx = \sin(x) + c$.

If $f(x) = \sin(kx)$, then $f'(x) = k\cos(kx)$, hence $\int k\cos(kx)\,dx = \sin(kx) + c_1$.

$$k\int \cos(kx)\,dx = \sin(kx) + c_1$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + c, \text{ where } c = \frac{c_1}{k}$$

Similarly, if $f(x) = \cos(kx)$ then $f'(x) = -k\sin(kx)$, hence $\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + c$.

Note: When differentiating or anti-differentiating trigonometric functions, the argument is always in radians.

You can check the given rules by differentiating the right-hand side in each case.

Additional information

Anti-derivatives of the sine and cosine functions

Practise anti-differentiating the sine and cosine functions.

The anti-derivative of a sine or cosine function whose argument is a linear expression is given by:

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + c$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + c$$

An extension of the above results is as follows:

$$\int \sin(kx + m)\,dx = -\frac{1}{k}\cos(kx + m) + c$$

$$\int \cos(kx + m)\,dx = \frac{1}{k}\sin(kx + m) + c$$

17 Anti-differentiate sin (*kx*) and cos (*kx*)

Anti-differentiate the following expressions.

(a) $6\cos(2x)$

THINKING	WORKING
1 Express as an integral.	$\int 6\cos(2x)\,dx = 6\int \cos(2x)\,dx$
2 Recall the relevant anti-derivative rule.	$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + c$
3 Determine an anti-derivative using the rule and simplify the expression.	$= 6 \times \frac{1}{2}\sin(2x) + c$ $= 3\sin(2x) + c$

(b) $3\sin\left(\frac{x}{2}\right)$

1 Express as an integral.	$\int 3\sin\left(\frac{x}{2}\right)dx = 3\int \sin\left(\frac{x}{2}\right)dx$
2 Recall the relevant anti-derivative rule.	$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + c$
3 Determine an anti-derivative using the rule and simplify the expression.	$= -\frac{3}{\frac{1}{2}}\cos\left(\frac{x}{2}\right) + c$ $= -6\cos\left(\frac{x}{2}\right) + c$

18 Anti-differentiate sin (*kx* + *m*) and cos (*kx* + *m*)

Determine the anti-derivative of the following.

(a) $-6\sin(2x+1)$

THINKING	WORKING
1 Express as an integral.	$\int -6\sin(2x+1)\,dx = -6\int \sin(2x+1)\,dx$
2 Recall the relevant anti-derivative rule.	$\int \sin(kx+m)\,dx = -\frac{1}{k}\cos(kx+m) + c$
3 Determine an anti-derivative using the rule and simplify the expression.	$\int -6\sin(2x+1)\,dx = -6 \times -\frac{1}{2}\cos(2x+1) + c$ $= 3\cos(2x+1) + c$

(b) $2\cos(-5x+3)$

1 Express as an integral.	$\int 2\cos(-5x+3)\,dx = 2\int \cos(-5x+3)\,dx$
2 Recall the relevant anti-derivative rule.	$\int \cos(kx+m)\,dx = \frac{1}{k}\sin(kx+m) + c$
3 Determine an anti-derivative using the rule and simplify the expression.	$\int 2\cos(-5x+3)\,dx = \frac{2}{-5}\sin(-5x+3) + c$ $= -\frac{2}{5}\sin(-5x+3) + c$

19 Determine the constant in the anti-derivative of a trigonometric function

If $f'(x) = 3\cos\left(2x - \frac{\pi}{3}\right)$ and $f\left(\frac{\pi}{3}\right) = \frac{11\sqrt{3}}{4}$, determine an expression for $f(x)$.

THINKING	WORKING
1 Express as an integral.	$f(x) = \int f'(x)\,dx$ $= \int 3\cos\left(2x - \frac{\pi}{3}\right)dx$ $= 3\int \cos\left(2x - \frac{\pi}{3}\right)dx$

2 Recall the relevant anti-derivative rule.	$\int \cos(kx+m)\,dx = \frac{1}{k}\sin(kx+m)+c$
3 Determine an anti-derivative using the rule and simplify the expression.	$3\int \cos\left(2x-\frac{\pi}{3}\right)dx = 3\times\frac{1}{2}\sin\left(2x-\frac{\pi}{3}\right)+c$ $= \frac{3}{2}\sin\left(2x-\frac{\pi}{3}\right)+c$
4 Substitute the given values and solve for c.	For $f\left(\frac{\pi}{3}\right)=\frac{11\sqrt{3}}{4}$ $\frac{11\sqrt{3}}{4} = \frac{3}{2}\sin\left(\frac{2\pi}{3}-\frac{\pi}{3}\right)+c$ $\frac{11\sqrt{3}}{4} = \frac{3}{2}\sin\left(\frac{\pi}{3}\right)+c$ $c = \frac{11\sqrt{3}}{4}-\frac{3}{2}\sin\left(\frac{\pi}{3}\right)$ $= \frac{11\sqrt{3}}{4}-\frac{3\sqrt{3}}{4}$ $= 2\sqrt{3}$
5 Determine the expression for $f(x)$.	$f(x) = \frac{3}{2}\sin\left(2x-\frac{\pi}{3}\right)+2\sqrt{3}$

20 Anti-differentiate more complex trigonometric functions

$\cos(2\theta)$ can be written as $\cos(2\theta) = 1-2\sin^2(\theta)$. Hence determine $\int \sin^2(3x)\,dx$.

THINKING	WORKING
1 Rearrange the rule and substitute the given variables.	$\cos(2\theta) = 1-2\sin^2(\theta)$ $2\sin^2(\theta) = 1-\cos(2\theta)$ $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$ Let $\theta = 3x$: $\sin^2(3x) = \frac{1}{2}(1-\cos(2\times 3x))$ $= \frac{1}{2}(1-\cos(6x))$
2 Express as an integral.	$\int \sin^2(3x)\,dx = \frac{1}{2}\int 1-\cos(6x)\,dx$
3 Use the appropriate rule to anti-differentiate each term.	$= \frac{x}{2}-\frac{1}{12}\sin(6x)+c$

EXERCISE 4.5 Anti-differentiation of sine and cosine functions

Worked Example

17

1 Determine the anti-derivative of each of the following.

(a) $\cos(3x)$ (b) $\sin(4x)$ (c) $6\sin(3x)$ (d) $-2\cos(4x)$

(e) $4\sin\left(\frac{x}{2}\right)$ (f) $3\cos\left(\frac{x}{3}\right)$ (g) $-\frac{2}{3}\sin(2x)$ (h) $\frac{1}{2}\cos(2x)$

2 Determine the anti-derivative of each of the following.

(a) $3\cos(5x)$ (b) $2\sin\left(\frac{x}{3}\right)$ (c) $-\frac{1}{2}\sin\left(\frac{2x}{3}\right)$

(d) $-3\cos\left(\frac{3x}{2}\right)$ (e) $4\cos(3\theta)$ (f) $-2\sin\left(\frac{2\theta}{5}\right)$

18

3 Determine the anti-derivative of each of the following.

(a) $\int \sin(2x)\,dx$ (b) $\int \sin(2x+1)\,dx$ (c) $\int \sin(1-2x)\,dx$

(d) $\int \cos(2x)\,dx$ (e) $\int \cos(2x+1)\,dx$ (f) $\int \cos(1-2x)\,dx$

4 Determine the anti-derivative of each of the following.

(a) $2\cos(3x+1)$ (b) $3\sin(2x-3)$ (c) $-3\sin\left(\frac{2x}{3}-4\right)$

(d) $-2\cos\left(\frac{2x}{3}+1\right)$ (e) $\sin(2x)-\cos\left(\frac{2x}{3}\right)$ (f) $2\cos\left(\frac{3x}{2}\right)-3\sin(4x)$

(g) $2x-3\sin\left(\frac{3x}{4}\right)+e^{3x}$ (h) $3x^2-\sqrt{2}\cos\left(\frac{\sqrt{2}x}{2}\right)$

19

5 Determine $f(x)$, given the following information.

(a) $f'(x)=2\cos\left(3x-\frac{\pi}{2}\right)$ and $f\left(\frac{\pi}{2}\right)=-1$

(b) $f'(x)=-\frac{2}{\sqrt{3}}\sin\left(2x-\frac{\pi}{6}\right)$ and $f\left(\frac{\pi}{6}\right)=-2$

6 Determine $f(x)$, given the following information: $f'(x)=2\cos\left(5x-\frac{\pi}{6}\right)$ and $f\left(\frac{\pi}{6}\right)=2$.

7 If a curve has a gradient function $\frac{dy}{dx}=6x^2-\cos(2x)$, determine the equation of the curve if it passes through the point $\left(\frac{\pi}{2},0\right)$.

8 The tangent to a curve at the point where $x=\pi$ has a gradient of 1. The curve has a gradient function $\frac{dy}{dx}=a\sin\left(\frac{x}{6}\right)$ and passes through the point $(2\pi,5)$. Determine the equation of the curve.

9 Determine $f(x)$, given the following information: $f'(x)=-b\sin(2x)$, $f'\left(\frac{\pi}{4}\right)=-2$ and $f(0)=3$.

10 Determine $f(x)$, given the following information: $f'(x)=a\cos(3x)$, $f(0)=2$ and $f\left(\frac{\pi}{2}\right)=-2$.

11 If $\sin(2\theta)$ can be written as $2\sin(\theta)\cos(\theta)$, determine $\int 10\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)dx$.

12 Another way of writing $\cos(2\theta)$ is $2\cos^2(\theta)-1$. Use this identity to determine $\int 1-2\cos^2(4x)\,dx$.

13 Which one of the following is equal to $\int 2\cos\big(4(x+\pi)\big)\,dx$?

A $\frac{1}{2}\sin(4x+\pi)+c$

B $-\frac{1}{2}\sin\big(4(x+\pi)\big)+c$

C $8\sin\big(4(x+\pi)\big)+c$

D $\frac{1}{2}\sin\big(4(x+\pi)\big)+c$

14 Given that $\sin(2\theta)=2\sin(\theta)\cos(\theta)$, which one of the following is equal to $\int \sin(2x)\cos(2x)\,dx$?

A $-\frac{1}{8}\cos(4x)+c$ B $\frac{1}{8}\cos(4x)$ C $-\frac{1}{2}\cos(x)$ D $-\frac{1}{2}\cos(4x)$

15 If $y=e^x\left(\sin(x)+\cos(x)\right)$, determine $\frac{dy}{dx}$ and use the result to determine $\int e^x\cos(x)\,dx$.

16 Use the identity $\tan(x)=\frac{\sin(x)}{\cos(x)}$ to answer the following.

(a) Determine the derivative of $\tan(x)$.

(b) Rewrite $\tan^2(x)$ in terms of $\sin(x)$ and $\cos(x)$.

(c) Using $\sin^2(x)+\cos^2(x)=1$, verify that $\tan^2(x)=\frac{1}{\cos^2(x)}-1$.

(d) Hence determine $\int \tan^2(x)\,dx$.

17 If $f(x)=x\sin(x)$, determine the following.

(a) $f'(x)$

(b) $\int x\cos(x)\,dx$

(c) the equation of the curve that passes through the point $\left(\frac{\pi}{2},1\right)$ and has a gradient function $f'(x)=x\cos(x)$

18 The acceleration a m/s^2 of a particle at time t seconds, travelling in a straight line, is modelled by the function $a=e^{\frac{t}{2}}-\cos(2t)$.

(a) Given that $a=\frac{dv}{dt}$, and the particle starts from rest, determine an expression for the velocity v m/s.

(b) Given that $v=\frac{dx}{dt}$, and the particle starts at $x=0$, determine an expression for the position x m of the particle.

4.6 Anti-differentiation of linear reciprocal functions

In the earlier sections of this chapter, you used the rule $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \ n \neq -1$. If $n = -1$, then the right-hand side of the rule has zero in the denominator and is therefore undefined. This means that another approach is needed to anti-differentiate functions of the form $f(x) = \frac{k}{ax+b}$.

First, consider the derivative of $y = \log_e(x), \ x > 0$.

If $y = \log_e(x)$, then $x = e^y$.

By differentiating this expression with respect to y: $\frac{d}{dy}\left(e^y\right) = e^y$.

And given that $y = \log_e(x)$, then:

$$\frac{d}{dy}\left(e^y\right) = e^{\log_e(x)}$$
$$= x$$

Therefore:

$$\frac{dy}{dx} = x$$
$$\frac{dx}{dy} = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \log_e|x| + c, \ x \in \mathbb{R} \setminus \{0\}$$

The absolute value sign is used because the log function is not defined for a negative real argument.

Given $y = \int \frac{dy}{dx} dx$, it follows that $\log_e(x) = \int \frac{1}{x} dx$.

To extend this to the more general case, the chain rule is used to differentiate $y = \log_e(ax+b)$ where the argument $ax + b$ is a linear function. This does not apply to higher order polynomials.

$$\frac{dy}{dx} = \frac{a}{ax+b}$$
$$\frac{1}{a}\int \frac{dy}{dx} dx = \int \frac{1}{ax+b} dx$$
$$\int \frac{1}{ax+b} dx = \frac{1}{a}\log_e(ax+b), \ x > -\frac{b}{a}$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\log_e|ax+b| + c, \ x \in \mathbb{R} \setminus \left\{-\frac{b}{a}\right\}$$

21 Anti-differentiate functions of the form $f(x) = \frac{k}{x}$

(a) Determine the anti-derivative of $\int \frac{3}{x} dx$.

THINKING	WORKING
1 Take the coefficient outside the integral sign.	$\int \frac{3}{x} dx = 3\int \frac{1}{x} dx$
2 Recall the anti-derivative rule.	$\int \frac{1}{x} dx = \log_e\|x\| + c$
3 Determine the anti-derivative using the rule.	$\int \frac{3}{x} dx = 3\log_e\|x\| + c$

(b) Determine the anti-derivative of $\int \frac{-5}{x}\,dx,\ x > 0$.

1 Take the coefficient outside the integral sign.	$\int \frac{-5}{x}\,dx = -5\int \frac{1}{x}\,dx$
2 Recall the anti-derivative rule.	$\int \frac{1}{x}\,dx = \log_e \lvert x \rvert + c$
3 Determine the anti-derivative using the rule. Note that the absolute value sign is not needed, because you were given $x > 0$.	$\int \frac{-5}{x}\,dx = -5\log_e (x) + c$

22 Anti-differentiate functions of the form $f(x) = \frac{1}{kx}$

Determine the anti-derivative of $\int \frac{1}{4x}\,dx$.

METHOD 1

THINKING	WORKING
1 Take the coefficient outside the integral sign.	$\int \frac{1}{4x}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx$
2 Recall the anti-derivative rule.	$\int \frac{1}{x}\,dx = \log_e \lvert x \rvert + c$
3 Determine the anti-derivative using the rule.	$\int \frac{1}{4x}\,dx = \frac{1}{4}\log_e \lvert x \rvert + c_1$

METHOD 2

1 Recall the anti-derivative rule.	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$
2 Determine the anti-derivative using the rule and write the answer.	$a = 4,\ b = 0$ $\int \frac{1}{4x}\,dx = \frac{1}{4}\log_e \lvert 4x \rvert + c_2$

Note that these two answers appear to be different. The only difference, however, is in the arbitrary constant.

$\frac{1}{4}\log_e \lvert 4x \rvert = \frac{1}{4}\log_e (4) + \frac{1}{4}\log_e \lvert x \rvert$, using the laws of logarithms.

If $c_1 = \frac{1}{4}\log_e (4) + c_2$, then the two answers are the same.

23 Anti-differentiate functions of the form $f(x) = \frac{k}{ax+b}$

Determine the anti-derivative of $\frac{-6}{2x+1}$.

THINKING	WORKING
1 Recall the anti-derivative rule.	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e\lvert ax+b\rvert + c$
2 Take the coefficient in the numerator outside the integral sign.	$\int \frac{-6}{2x+1}\,dx = -6\int \frac{1}{2x+1}\,dx$
3 Determine the anti-derivative using the rule and simplify the expression.	$= -6 \times \frac{1}{2}\log_e\lvert 2x+1\rvert + c$ $= -3\log_e\lvert 2x+1\rvert + c$

24 Determine the constant of integration for linear reciprocal functions

A curve has a gradient function $\frac{dy}{dx} = 2x + \frac{7}{x}$, $x > 0$. Determine the equation of the curve if it passes through the point $(e, 0)$.

THINKING	WORKING
1 Anti-differentiate the gradient function term by term.	$y = \int\left(2x + \frac{7}{x}\right)dx$ $= \int 2x\,dx + \int \frac{7}{x}\,dx$ $= 2 \times \int x\,dx + 7 \times \int \frac{1}{x}\,dx$ $= x^2 + 7\log_e(x) + c$
2 Substitute the known point and solve for c, the constant of integration.	Given $(e, 0)$, let $x = e$ and $y = 0$: $0 = e^2 + 7 + c$ $c = -e^2 - 7$
3 Interpret the answer.	The equation of the curve is $y = x^2 + 7\log_e(x) - e^2 - 7$.

EXERCISE 4.6 Anti-differentiation of linear reciprocal functions

Worked Example

1 Determine each of the following anti-derivatives. 21

(a) $\int \frac{23}{x}\,dx$ (b) $\int \frac{-2}{x}\,dx$ (c) $\int \frac{10}{t}\,dt,\ t>0$

(d) $\int\left(x^3+\frac{7}{x}\right)dx$ (e) $\int\left(\sin(2\theta)+\frac{1}{\theta}\right)d\theta,\ 0<\theta<\pi$

2 Determine each of the following anti-derivatives. 22

(a) $\int \frac{1}{3x}\,dx$ (b) $\int -\frac{1}{5x}\,dx$ (c) $\int \frac{3}{7x}\,dx$

(d) $\int -\frac{2}{3t}\,dt,\ t>0$ (e) $\int\left(\frac{1}{6x}+e^{6x}+2\right)dx$ (f) $\int\left(\frac{1}{3x}+\frac{4}{5x}\right)dx$

3 Determine the anti-derivative of each of the following functions. 23

(a) $\frac{1}{3x-2}$ (b) $\frac{1}{3-11x}$ (c) $\frac{4}{2x-1}$ (d) $\frac{2}{7-10x}$

4 Determine $f(x)$ for each of the following.

(a) $f'(x)=\frac{2x^3}{9x^4},\ x>0$ (b) $f'(x)=\frac{3x^2+2}{x},\ x>0$

(c) $f'(x)=\frac{4x^3+2x}{x^2},\ x>0$ (d) $f'(x)=\frac{(2x-3)^2}{x^2},\ x>0$

5 Determine the equation of each of the following curves, given the gradient function and a point on the curve. 24

(a) $\frac{dy}{dx}=\frac{2}{x-3},\ x>3$; passing through $(4,7)$ (b) $\frac{dy}{dx}=\frac{1}{4x},\ x>0$; passing through $\left(e^2,\frac{1}{2}\right)$

(c) $\frac{dy}{dx}=\frac{1}{x}-2x,\ x\neq 0$; passing through $\left(\frac{1}{e},0\right)$ (d) $\frac{dy}{dx}=\frac{x-3}{x^2},\ x\neq 0$; passing through $(-e,1)$

6 Verify that $\int\left(\frac{1}{4+x}+\frac{1}{3-x}\right)dx=\log_e\left|\frac{4+x}{3-x}\right|+c$.

7 Verify that $\int\left(\frac{2}{x-1}+\frac{1}{2x+3}\right)dx=\log_e\left((x-1)^2\sqrt{2x+3}\right)+c,\ x>1$.

8 Determine whether each of these statements is true or false.

(a) $-\frac{1}{3x^2}$ is an anti-derivative of $\frac{1}{3x}$. (b) $\frac{1}{3}\log_e(x)$ is an anti-derivative of $\frac{1}{3x}$.

(c) $\frac{1}{3}\log_e(3x)$ is an anti-derivative of $\frac{1}{3x}$.

9 If $f'(x)=\frac{6x^2+6x-1}{2x},\ x\neq 0$ determine $f(x)$ if $f(1)=\frac{1}{2}$.

10 Determine $\int\frac{\left(2x^2-3\right)^2}{2x^3}\,dx,\ x>0$.

11 The velocity v of a particle moving in a straight line at time t is modelled by $v = \dfrac{2}{4t+1}$, $t \geq 0$.

Determine the position x of the particle at $t = 2$, if $x = 7$ when $t = 0$. Give your answer correct to 1 decimal place.

12 Which one of the following statements gives an invalid domain?

A $\int \dfrac{1}{8x}\,dx = \dfrac{1}{8}\log_e(x) + c$ when $x > 0$

B $\int \dfrac{1}{8x}\,dx = \dfrac{1}{8}\log_e(x) + c$ when $x \in (0, 8]$

C $\int \dfrac{1}{8x}\,dx = \dfrac{1}{8}\log_e(x) + c$ when $|x| > 0$

D $\int \dfrac{1}{8x}\,dx = \dfrac{1}{8}\log_e(8x) + c$ when $x > 0$

13 Which one of these statements is true?

A $\int \dfrac{k}{ax+b}\,dx = \dfrac{k}{a}\log_e|ax+b| + c$ only if $x > -\dfrac{b}{a}$.

B $\int \dfrac{k}{ax+b}\,dx = \dfrac{k}{a}\log_e|ax+b| + c$ for any value of x.

C $\int \dfrac{k}{ax+b}\,dx = \dfrac{k}{a}\log_e|ax+b| + c$ only if $x \neq -\dfrac{b}{a}$.

D $\int \dfrac{k}{ax+b}\,dx = \dfrac{k}{a}\log_e|x+b| + c$ for any value of x.

14 Consider the expression $\dfrac{1}{x-1} - \dfrac{1}{x+2}$, $x > 1$.

(a) Write the expression as a single fraction.

(b) Hence determine $\int \dfrac{1}{x^2+x-2}\,dx$

15 If $y = \log_e\left(\dfrac{x-3}{x+3}\right)$, $x > 3$, determine $\dfrac{dy}{dx}$ and hence determine $\int \dfrac{1}{x^2-9}\,dx$.

16 Consider the expression $\dfrac{4}{(x+1)} - \dfrac{4}{(x+1)^2}$.

(a) Verify that $\dfrac{4}{(x+1)} - \dfrac{4}{(x+1)^2} = \dfrac{4x}{(x+1)^2}$.

(b) Hence determine $\int \dfrac{4x}{(x+1)^2}\,dx$.

17 Determine $\int \dfrac{x-1}{x+1}\,dx$.

18 It is possible to write the expression $\dfrac{1}{x+1}$ as an infinite series $\dfrac{1}{x+1} = \dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{1}{x^3} - \dfrac{1}{x^4} + \dfrac{1}{x^5} - \ldots$, $x > 1$.

(a) Determine an approximation for $\dfrac{1}{3}$ correct to 3 decimal places by using the first eight terms of the series.

(b) Anti-differentiate both sides of the equation $\dfrac{1}{x+1} = \dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{1}{x^3} - \dfrac{1}{x^4} + \dfrac{1}{x^5} - \ldots$, setting the arbitrary constant to zero. Use the first five terms of the series.

(c) Determine the value of each integral for $x = 2$ correct to 3 decimal places and then comment on the results. Use the first five terms of the series.

Further anti-differentiation

4.7

In previous sections, rates of change were determined by differentiating functions with respect to time. The process of anti-differentiating allows you to use known rates of change to determine an original function.

25 Applications of anti-differentiation

A balloon is deflating at a rate of $\frac{1}{t+5}$ cm^3/min, where V cm^3 is the volume after t minutes. This means that $\frac{dV}{dt} = -\frac{1}{t+5}$.

(a) Determine an expression for the volume of the balloon after t minutes.

THINKING	WORKING
1 Express the required anti-derivative.	$V = \int -\frac{1}{t+5}\,dt$
2 Anti-differentiate.	$V = -\log_e(t+5) + c$

(b) If the initial volume of the balloon is 40 cm^3, determine the volume after 4 minutes. Leave your answer correct to one decimal place.

1 Identify the given information.	When $t = 0$, $V = 40$ cm^3
2 Substitute the given initial values and solve for c.	$V = -\log_e(t+5) + c$ $40 = -\log_e(0+5) + c$ $c = 40 + \log_e(5)$ $c = 41.6\,(1\text{ d.p.})$
3 Determine the volume at the required time.	$V = -\log_e(t+5) + 41.6$ $V(4) = -\log_e(4+5) + 41.6$ $= 39.4\,(1\text{ d.p.})$
4 Interpret the answer.	The volume after 4 minutes is 39.4 cm^3.

Integration by recognition

The concept of anti-differentiation was introduced using the concept of integration by recognition.

The same principle can be used to anti-differentiate more complex functions by first determining the derivative of a similar function.

If $\frac{d}{dx}f(x) = f'(x)$

Then $\int f'(x)\,dx = f(x) + c$

26 Integration by recognition

(a) Differentiate $f(x) = x\log_e(x),\ x > 0$.

THINKING	WORKING
1 Recognise the product of two functions in the form $y = uv$	Let $u = x$ and $v = \log_e(x)$ Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x}$
2 Use the product rule to determine the derivative.	$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$ $= x \times \frac{1}{x} + \log_e(x) \times 1$ $= 1 + \log_e(x)$

(b) Hence determine $\int \log_e(x)dx$.

1 Rearrange the answer from part **(a)** to the required form.	$\log_e(x) = f'(x) - 1$ $\int \log_e(x)dx = \int (f'(x) - 1)dx$ $= \int f'(x)dx - \int 1\ dx$
2 Anti-differentiate term by term.	$= f(x) - x + c$
3 Interpret the answer.	$\int \log_e(x)dx = x\log_e(x) - x + c$

EXERCISE 4.7 Further anti-differentiation

Worked Example 25

1 For each of the following, determine the equation of the curve with the given gradient function that passes through the given point.

(a) $f'(x) = 4x^2 - 6; (3, 8)$

(b) $f'(x) = 6x - \frac{1}{\sqrt{x}}; (4, 30)$

(c) $f'(x) = 10x + \frac{1}{x^2}; (1, 0)$

(d) $f'(x) = 2x - \frac{2}{x^3}; (2.5, 25)$

2 Water is pouring into a tank at a rate given by $\frac{dV}{dt} = e^{-3t} + (t + 2)^2$, where V cm^3 is the volume of water and t is the time in minutes. Determine the volume of water in the tank after 1 minute if the initial volume is $\frac{20}{3}$cm^3. Give your answer correct to 1 decimal place.

3 A curve $y = f(x)$ has a gradient function $\frac{dy}{dx} < 0$ for all values of x. Which one of the following statements is correct?

A The graph of $y = f(x)$ has all negative y-values.

B The graph of $y = f(x)$ has a negative gradient for all x-values.

C The graph of $y = f(x)$ has a stationary point of inflection.

D The graph of the gradient function has no maximum or minimum turning points.

4 A curve has a gradient given by $\frac{dy}{dx} = e^{\frac{x}{2}} - x^{-\frac{x}{2}}$ and passes through the point $(0, 3)$. Determine the equation of the curve.

5 Given $f'(x) = 2\cos(2x) - 3\sin\left(\frac{x}{2}\right)$ and $f\left(\frac{\pi}{2}\right) = 2$, determine the equation of the curve.

6 The gradient of a curve is given by $\frac{dy}{dx} = \frac{x}{\sqrt{x-1}}$.

(a) Determine the equation representing the family of curves that have this gradient function.

(b) Determine the member of this family of curves that passes through the point given.

(i) $(1, 4)$ **(ii)** $\left(5, \frac{1}{3}\right)$

7 Differentiate $\frac{\log_e(x)}{x}$ and hence determine $\int \frac{1 - \log_e(x)}{x^2}\,dx$.

8 A ball is thrown upward with a speed of 12 m/s. If x is the distance travelled by the ball, then the velocity of the ball is $v = \frac{dx}{dt}$ and its acceleration is $a = \frac{dv}{dt}$.

(a) Determine the velocity function, $v(t)$, of the ball if the acceleration due to gravity as a function of time is $g(t) = -9.8\ \text{m/s}^2$.

(b) Determine the position function, $x(t)$, of the ball if the acceleration due to gravity, as a function of time is $g(t) = -9.8\ \text{m/s}^2$.

(c) When the ball reaches the maximum height, its instantaneous velocity is zero. Calculate the time it takes the ball to reach this maximum height and the height of the ball above the release point. Give your answer correct to 1 decimal place.

(d) Determine the velocity and the position of the ball 1 second after the beginning of the motion.

9 A curve has a gradient given by $\frac{dy}{dx} = \frac{a}{(1-3x)^2}$. The curve passes through the points (0,-2) and (1,-5).

(a) Determine the equation of the curve.

(b) Calculate the value of y when $x = 3$.

(c) Calculate the value of x when $y = 8$.

26

10 Consider the function $f(x) = x\cos(x)$.

(a) Determine $f'(x)$.

(b) Hence determine $\int x\sin(x)\,dx$.

(c) Determine the rule of the function whose gradient function is $f'(x) = x\sin(x)$ and which passes through the point (0,2).

11 Differentiate $f(x) = x^3\log_e(2x)$ and hence determine $\int x^2\log_e(2x)\,dx$.

12 Use the identity $\cos(2A) = 2\cos^2(A) - 1$ to determine an anti-derivative of $\cos^2\left(\frac{\theta}{4}\right)$.

13 Consider the function $f(x) = \frac{1}{\tan(x)}$.

(a) Determine $f'(x)$.

(b) Verify that $1 + \frac{1}{\tan^2(x)} = \frac{1}{\sin^2(x)}$.

(c) Hence determine $\int \frac{1}{\tan^2(x)}\,dx$.

14 The exponential function e^x can be expressed as a series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$,

where $n! = n(n-1)(n-2)\ldots1$. For example

$4! = 4\times3\times2\times1$
$= 24$

(a) Write down the first four terms of the series for $e^{-0.2x}$.

(b) Hence determine $\int xe^{-0.2x}\,dx$, using the first four terms.

15 **(a)** Use the substitution $u = x^3 - 5$ to determine $\int\left(x^3 - 5\right)3x^2\,dx$.

(b) Expand $\left(x^3 - 5\right)3x^2$ and anti-differentiate to demonstrate that the two answers are the same when adjusted for a constant.

16 A spherical balloon is inflated in such a way that its radius, r cm, increases over time, t minutes, at a rate of $\frac{dr}{dt} = \frac{r}{5} + 15$.

(a) Assume initially $r = 0$ and demonstrate that $t = 5\log_e\left(\frac{1}{75}(75 + r)\right)$.

Hint: $\frac{dt}{dr} = \frac{1}{\frac{dr}{dt}}$

(b) Calculate the time it takes to inflate the balloon until it reaches a radius of 5 cm and calculate the volume of the balloon at this moment. Give your answer correct to 1 decimal place.

(c) The balloon can be inflated safely until it reaches a radius of 20 cm.

Is it safe to inflate the balloon for another 2 minutes? Show your reasoning. Give your answer correct to 1 decimal place.

Summary

4

Terminology

An anti-derivative is a function $F(x)$, such that $F'(x) = f(x)$.

Anti-differentiation is the reverse process of differentiation.

The indefinite integral is given by $\int f(x)dx = F(x) + c$ or $\int f'(x)dx = f(x) + c$.

c represents an arbitrary constant and is a real number. It is called the *constant of integration*.

The set of graphs of anti-derivatives $y = F(x) + c$ is a family of curves related by a vertical translation and obtained when the constant is unknown.

If given $f'(a) = b$ or $F(a) = b$, then the value of c can be determined, giving a specific curve.

If $\frac{dy}{dx} = f'(x)$, then $\int \frac{dy}{dx}dx = y + c$.

Properties of the indefinite integral

$\int af(x)dx = a\int f(x)dx$, where a is a real number.

$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

Rules for integration

$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c,\ n \neq -1$

$\int a\,dx = ax + c$, where a is any constant.

$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

$\int e^x dx = e^x + c$

$\int e^{kx} dx = \frac{1}{k}e^{kx} + c,\ k \in \mathbb{R} \setminus \{0\}$

$\int e^{kx+b} dx = \frac{1}{k}e^{kx+b} + c,\ k \in \mathbb{R} \setminus \{0\}$

$\int \sin(x)dx = -\cos(x) + c$

$\int \cos(x)dx = \sin(x) + c$

$\int \sin(kx)dx = -\frac{1}{k}\cos(kx) + c$

$\int \cos(kx)dx = \frac{1}{k}\sin(kx) + c$

$\int \sin(kx+m)dx = -\frac{1}{k}\cos(kx+m) + c$

$\int \cos(kx+m)dx = \frac{1}{k}\sin(kx+m) + c$

$\int \frac{1}{x}dx = \log_e|x| + c,\ x \in \mathbb{R} \setminus \{0\}$

$\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e|ax+b| + c,\ x \in \mathbb{R} \setminus \left\{-\frac{b}{a}\right\}$

Chapter review 4

Exercise 4.1

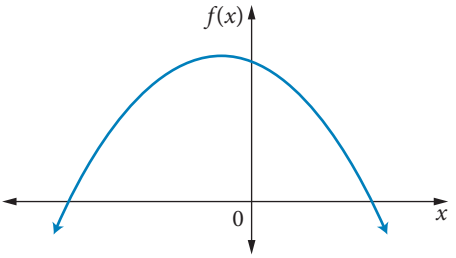
1 Differentiate the first function and hence determine the given anti-derivative.

(a) $f(x) = (3-5x)$; $\int (3-5x)^2\,dx$

(b) $f(x) = -(2x-7)^{-\frac{3}{2}}$; $\int (2x-7)^{-\frac{5}{2}}\,dx$

Exercise 4.1

2 Sketch an anti-derivative of the graph below, identifying key features.



Exercise 4.2

3 Determine an anti-derivative of each of the following.

(a) $7x^{\frac{3}{4}} - 6\sqrt{x}$

(b) $\left(x - \frac{2}{x}\right)^2$

(c) $\frac{3x^3 - 2x^2 + x}{x}$

Exercise 4.2

4 Determine an expression for y, given $\frac{dy}{dx} = x^n\left(x^{2n-1} - 3\right)$.

Exercise 4.2

5 Determine each of the following indefinite integrals.

(a) $\int \frac{x+1}{x^3}\,dx$

(b) $\int (y+3)(y-2)\,dy$

Exercise 4.3

6 Determine each of the following anti-derivatives.

(a) $\int (3x-4)^3\,dx$

(b) $\int \frac{-4}{(2x+1)^3}\,dx$

(c) $\int (3t-2)^{\frac{3}{2}}\,dt$

(d) $\int \frac{1}{\sqrt{2-x}}\,dx$

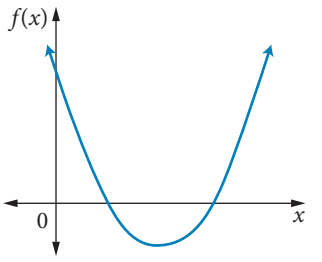
Exercise 4.3

7 Determine the equation of the curve that has a gradient function $f'(x) = (2x-1)$ and passes through the point $(1, 2)$.

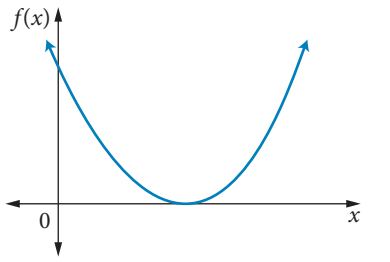
Exercise 4.1

8 Sketch a possible anti-derivative for each of the graphs shown.

(a)



(b)



Exercise 4.4

9 Determine $f(x)$ for each of the following.

(a) $f'(x) = e^{2x-3}$

(b) $f'(x) = 3e^{\frac{x}{2}}$

(c) $f'(x) = \left(e^x - 2\right)^4$

(d) $f'(x) = \frac{e^{2x}\left(3e^{-x} - e^{4x}\right)}{e^{-x}}$

10 Determine $\int 3\cos\left(\frac{2\theta}{3}\right)d\theta$. Exercise 4.5

11 Determine $F(x)$ for each of the following. Exercise 4.5

(a) $f(x) = -2\cos(4x)$ **(b)** $f(x) = \sin(1-2x)$

(c) $f(x) = 3\sqrt{x} - 2\sin(3x+2)$ **(d)** $f(x) = 2\sin\left(\frac{3x+1}{2}\right)$

12 If $g'(x) = 3\cos\left(\frac{x}{2}\right)$ and $g\left(\frac{\pi}{3}\right) = 6$, determine $g(x)$. Exercise 4.5

13 If $\frac{dy}{dx} = e^{2x} + e^{-x} + e^2$ and $y(0) = 0$, determine $y(x)$. Exercise 4.4

14 Determine each of the following anti-derivatives. Exercise 4.6

(a) $\int \frac{1}{12x}\,dx$ **(b)** $\int \frac{1}{2x-11}\,dx$

(c) $\int \frac{4}{\frac{1}{3}x-11}\,dx, \quad x > 33$ **(d)** $\int\left(\frac{5}{9x+1} + \frac{3}{4-2x}\right)dx$

15 Under which one of the following conditions is $\int \frac{1}{3-7x}\,dx = -\frac{1}{7}\log_e(3-7x) + c$? Exercise 4.6

A for $x \in \mathbb{R}$ **B** for $7x > 3$ **C** for $x < \frac{3}{7}$ **D** for $x \leq \frac{3}{7}$

16 $\int \frac{3}{(2-5x)^3}\,dx$ can be equal to which of the following? Exercise 4.3

A $\frac{30}{(2-5x)^4}$ **B** $\frac{-3}{10(2-5x)^2}$ **C** $\frac{3}{10(5x-2)^3}$ **D** $\frac{3}{10(2-5x)^2}$

17 $\frac{dy}{dx} = ae^{-2x} + e^{2x}$, where a is a real number. When $x = 0$, $\frac{dy}{dx} = 3$, and when $x = 2$, $y = 0$. Exercise 4.3
Determine the constant of integration.

18 Determine an anti-derivative of $\sqrt{x}\left(x^2 + \frac{1}{x}\right)$. Exercise 4.2

19 Use two different methods to determine y if $\frac{dy}{dx} = (x-2)^3$. Verify that the two answers are the same except for the value of the arbitrary constants. Exercise 4.3

20 Given $h'(x) = e^{3x-2} - 2e$ and $h(0) = \frac{1}{3}$, determine $h(x)$. Exercise 4.4

21 Consider the expression $x(2x+3)^3$. Exercise 4.7

(a) Determine the derivative in factorised form.

(b) Hence, determine the anti-derivative of $(2x+3)^2(8x+3)$.

22 If the derivative of $x\cos(ax)$ is $\cos(ax) - ax\sin(ax)$, where a is a real number, determine $\int x\sin(ax)\,dx$. Exercise 4.7

23 Determine the derivative of $x\sqrt{x+1}$, and hence verify that $\int \frac{x}{\sqrt{x+1}}\,dx = \frac{2}{3}(x-2)\sqrt{x+1} + c$. Exercise 4.7

Exercise 4.7 **24** Consider the function $f(x) = x(2x-3)^4$.

(a) Determine the derivative.

(b) Hence determine $\int 4x(2x-3)^3\,dx$.

(c) If $\int 4x(2x-3)^3\,dx = 3$ when $x = 2$, determine the value of the constant of integration.

Exercise 4.3 **25** The gradient of a curve is given by $\frac{dy}{dx} = (bx+3)^3$. The tangent to this curve drawn through the point $(1.5, 160)$ also passes through the point $(1, 52)$.

(a) Determine the gradient of the tangent.

(b) Determine the equation of the tangent.

(c) Determine the value of b in $\frac{dy}{dx} = (bx+3)^3$.

(d) Determine the equation of the curve.

Exercise 4.7 **26** Consider the function $f(x) = (ax+b)^{n+1}$.

(a) Determine $f'(x)$.

(b) Determine the derivative of $x(ax+b)^{n+1}$.

(c) Hence, determine $\int x(ax+b)^n\,dx$.

Exercise 4.5 **27** A fungus was first noticed growing in a lawn when the fungus covered an area of $900\,\text{cm}^2$. The rate of change of area covered by the fungus is given by $\frac{dA}{dt} = 200t + \frac{500\pi^2}{3}\cos\left(\frac{\pi t}{3}\right)$, where $A\,\text{cm}^2$ is the area of lawn covered by fungus, t months after the fungus was first noticed.

(a) When first noticed, at what rate was the fungus growing per month? Give your answer correct to the nearest cm^2/month.

(b) Express the fungus-infected area A in terms of t.

(c) At the start of which month does the growth rate exceed $2000\,\text{cm}^2/\text{month}$? Give your answer correct to the nearest month.

(d) When does the fungus cover $10\,000\,\text{m}^2$ of lawn? Give your answer correct to the nearest month.

Exercise 4.4 **28** The rate of change $C'(t)$ of the concentration in the bloodstream, in milligrams per litre per second, of a pain-killing drug, t hours after a tablet was administered is given by $C'(t) = \frac{1}{20}(3-t)e^{-t^2+6t-4}$.

(a) Given $y = \frac{1}{40}e^{-t^2+6t-4}$, determine $\frac{dy}{dt}$.

(b) Hence, determine $\int (3-t)e^{-t^2+6t-4}\,dt$.

(c) **(i)** Determine the rate at which the concentration of the drug is changing 4 hours after a tablet was administered.

(ii) Explain the negative sign in the answer to part **(i)**.

(d) Determine the time when the concentration of the drug in the bloodstream is the greatest.

(e) Assuming that none of the drug was present in the bloodstream when the tablet was administered, determine the rule expressing the concentration of the drug in the bloodstream in terms of t hours.

(f) What is the concentration of the drug present in the bloodstream 1 hour after the tablet was administered? Give your answer correct to 3 decimal places.

29 A manufacturing company has a marginal cost defined by the function $C_M(q) = \frac{1}{e^q} + q + 20$, where q is the number of units produced. In business, the marginal cost is the change in the total cost from producing 1 more unit. Exercise 4.4

(a) Determine the expression for the total cost $C(q)$ if the total cost for producing 1 unit is \$45. Make any approximations to the nearest whole number.

(b) Determine the total cost of producing 10 units. Give your answer correct to 2 decimal places.

(c) If the fixed cost is the cost incurred when no units are produced and the total variable cost is the total cost less the fixed cost, calculate the total variable cost for producing 10 units.

30 The acceleration of a car moving in a straight line is given by the equation $a(t) = -2t + 5$ in km/h^2. If x is the distance travelled by the car, then the velocity of the car is $v = \frac{dx}{dt}$ and its acceleration is $a = \frac{dv}{dt}$.

(a) Determine the velocity function $v(t)$ of the car if the initial velocity of the car is 60 km/h due east.

(b) Determine the position function $x(t)$ of the car if its initial position is zero.

(c) Determine the time it will take the car to become stationary if it continues to travel east. Give your answer correct to 1 decimal place.

31 Biologists use mathematical modelling to predict the population of mosquitoes on a tropical island. The first model tested predicts that the population of mosquitoes P is increasing at a rate given by $\frac{dP}{dt} = 2^{2-t} + 4$, where P is the number of mosquitoes in thousands, and t is the time in weeks. When the study commenced on 1 January, it was estimated that the island had a population of 400 000 mosquitoes. Exercise 4.7

(a) Sketch the graph of $\frac{dP}{dt}$ versus t.

(b) According to this model, what is the long-term rate of the population increase of the mosquitoes?

(c) Based on this model, determine, in exact form, the rule expressing the population size of mosquitoes at any time t.

(d) How many mosquitoes, to the nearest thousand, does this model predict will be on the island after 6 weeks?

(e) Owing to influences such as the breeding cycle of mosquitoes and the need for stagnant water in order to breed, the biologists predict that the model would only be valid until $\frac{dP}{dt} = \frac{t}{2}$. Determine the length of time, to the nearest whole week, for which the population model is valid.

5

Definite integrals

Recall 256

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5 Recall

Anti-differentiate integer powers of x using the result $\int (ax^n)\,dx = \frac{ax^{n+1}}{n+1} + c,\ n \neq -1$

1 Determine the following anti-derivatives.

(a) $\int (5x^2 - 3x + 2)\,dx$ (b) $\int (8x^3 - 3x^2 + x - 3)\,dx$

(c) $\int \left(2x^3 + 5x^2 + \frac{1}{x^2} - 1\right)dx$

Anti-differentiate fractional powers of x using the result $\int (ax^n)\,dx = \frac{ax^{n+1}}{n+1} + c,\ n \neq -1$

2 Determine the anti-derivative of each of the following.

(a) $2x^3 - 6\sqrt{x} + 1$ (b) $8x^3 - \frac{1}{x^2} + \frac{3}{\sqrt{x}}$

(c) $5x^2 + 3\sqrt{x}$ (d) $x^{\frac{1}{3}}\left(1 - \frac{3}{x^{\frac{5}{3}}}\right)$

Recognise $F(x)$ as the anti-derivative of $f(x)$, $f(x)$ as the anti-derivative of $f'(x)$ and y as the anti-derivative of $\frac{dy}{dx}$

3 Determine the anti-derivatives from the given information.

(a) Given $f(x) = 3x - 4$, determine $F(x)$.

(b) Given $f'(x) = x^2 - 8x$, determine $f(x)$.

(c) Given $\frac{dy}{dx} = 4x^3 - 2\sqrt{x}$, determine y in terms of x.

Use differentiation to find an anti-derivative

4 Determine the derivative of $(4x - 3)^3$ and hence determine $\int (4x - 3)^2\,dx$.

Anti-differentiate expressions using the result $\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c$

5 Match a correct anti-derivative from the list (i)–(v) with each of the functions (a)–(c)

(i) $(2x - 1)^4$ (ii) $\frac{1}{2}(2x - 1)^4$ (iii) $\frac{1}{4}(2x - 1)^4$ (iv) $\frac{1}{8}(2x - 1)^4$ (v) $\frac{1}{16}(2x - 1)^4$

(a) $(2x - 1)^3$ (b) $4(2x - 1)^3$ (c) $\frac{1}{2}(2x - 1)^3$

6 Determine the anti-derivative of each of the following.

(a) $\int (7x + 3)^4\,dx$ (b) $\int \frac{1}{(3x - 1)^5}\,dx$ (c) $\int \frac{3}{(6x - 5)^3}\,dx$ (d) $\int \frac{6}{\sqrt{3 - 2x}}\,dx$

Anti-differentiate exponential functions using $\int e^{kx+a}\,dx = \frac{1}{k}e^{kx+a} + c$

7 Determine each of the following anti-derivatives.

(a) $\int 3e^{2x}\,dx$ **(b)** $\int 3e^{2x+1}\,dx$ **(c)** $\int 8e^{1-2x}\,dx$ **(d)** $\int \frac{8}{e^{2x+3}}\,dx$

Anti-differentiate trigonometric functions using $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$ and $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

8 Determine each of the following anti-derivatives.

(a) $\int \sin(4x)\,dx$ **(b)** $\int \sin\left(\frac{x}{3}\right)dx$ **(c)** $\int 4\sin\left(\frac{2x}{5}\right)dx$ **(d)** $\int \cos(2x)\,dx$

(e) $\int 4\cos\left(\frac{3x}{2}\right)dx$ **(f)** $\int \cos(3x-1)\,dx$ **(g)** $\int \sin(2-5x)\,dx$

Anti-differentiate the reciprocal function using $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c,\ ax+b>0$

9 Determine each of the following anti-derivatives.

(a) $\int \frac{3}{x}dx, x>0$ **(b)** $\int \frac{2}{7x}dx, x>0$ **(c)** $\int \frac{1}{2x-1}dx, x>\frac{1}{2}$ **(d)** $\int \frac{4}{3x+2}dx, x>-\frac{2}{3}$

Use the additional information to determine a specific anti-derivative

10 **(a)** Determine the rule for the curve that has a gradient given by $f'(x) = 2x^3 - 2x$ and passes through the point $(2, 6)$.

(b) Determine $f(x)$, given $f'(x) = 6x^2 - 2x + 3$ and $f(2) = 24$.

(c) Determine the rule for the curve that has a gradient function given by $\frac{dy}{dx} = \cos(2x)$ and passes through the point $\left(\frac{\pi}{4}, -1\right)$.

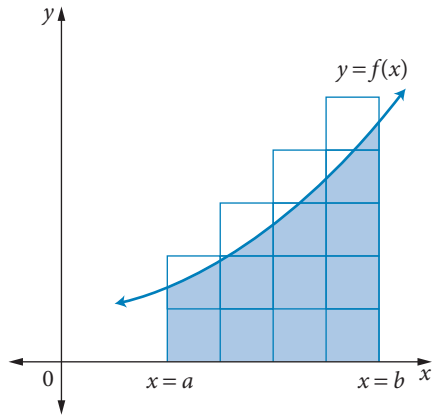
(d) Determine $f(x)$, given $f'(x) = 2e^{6x} + 1$ and $f(0) = \frac{4}{3}$.

(e) Determine the rule for the curve that has a gradient function given by $\frac{dy}{dx} = 3x^2 - 9\sqrt{x}$ and passes through the point $(4, 12)$.

5.1 Estimating area beneath curves

The area enclosed by the graph of $y = f(x)$, the x-axis, and the lines $x = b$ and $x = a$ can be approximated using a variety of techniques.

One way is to divide the area into unit squares and count the squares. Should only the whole squares be counted or should the partly enclosed squares also be taken into account?



Making connections

Approximating the area between a curve and the x-axis

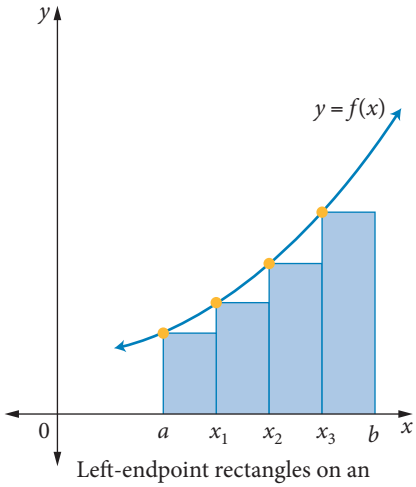
Move the sliders to estimate the area under curves and explore the properties of areas under curves.

Riemann sums

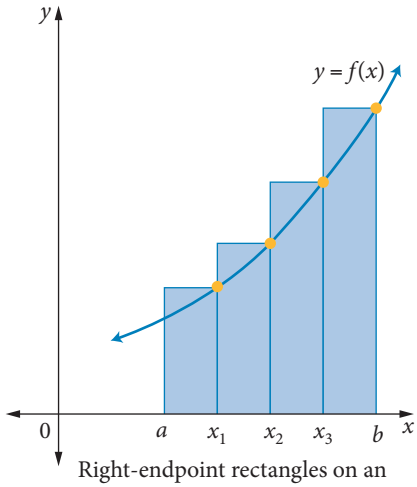
Another way is to draw vertical lines and construct rectangles to approximate the area bounded by the curve and the x-axis. To keep this process systematic, the rectangles should be of equal width and the rectangles are either the height of the left-hand end point or right-hand endpoint. This method is named after the founder, German mathematician Bernhard Riemann.

Consider the four functions on the right that sit above the x-axis; that is, $f(x) > 0$.

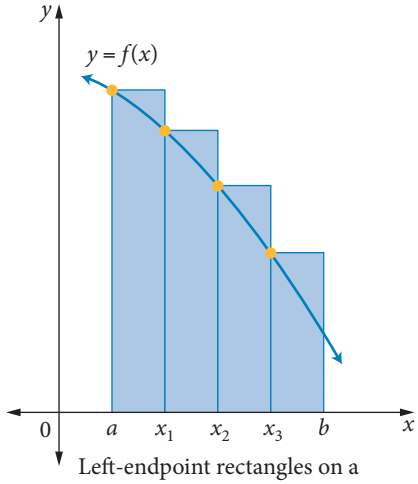
If a function is above the x-axis and increasing over a given interval, the left-endpoint rule will give an underestimate of the actual area while the right-endpoint rule will give an overestimate of the actual area. If the function is above the x-axis and decreasing in a given interval, the left-endpoint rule will give an overestimate of the actual area while the right-endpoint rule will give an underestimate of the actual area.



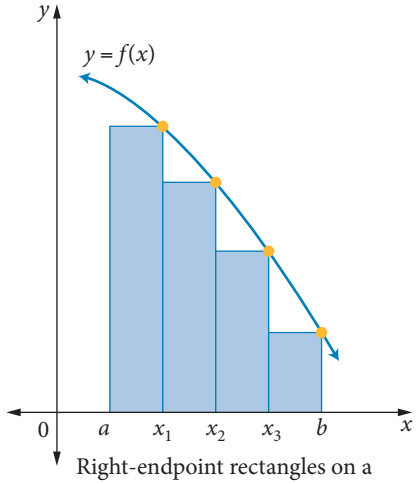
Left-endpoint rectangles on an increasing function underestimate the area between the curve and the x-axis.



Right-endpoint rectangles on an increasing function overestimate the area between the curve and the x-axis.

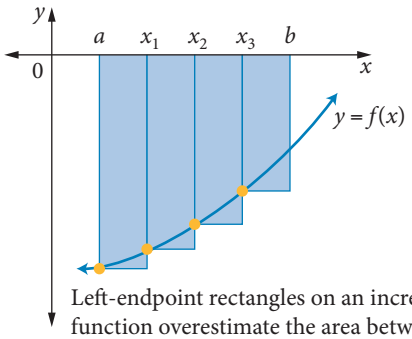


Left-endpoint rectangles on a decreasing function overestimate the area between the curve and the x-axis.

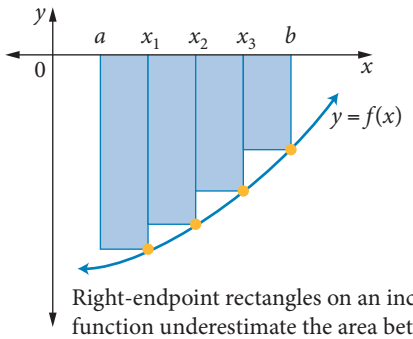


Right-endpoint rectangles on a decreasing function underestimate the area between the curve and the x-axis.

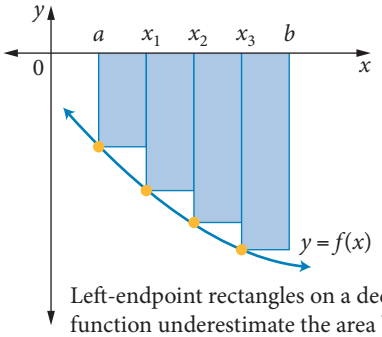
Consider the following estimates where $f(x) < 0$:



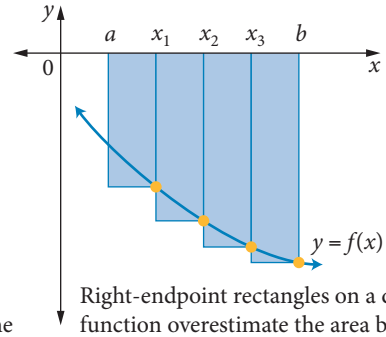
Left-endpoint rectangles on an increasing function overestimate the area between the curve and the x-axis.



Right-endpoint rectangles on an increasing function underestimate the area between the curve and the x-axis.



Left-endpoint rectangles on a decreasing function underestimate the area between the curve and the x-axis.



Right-endpoint rectangles on a decreasing function overestimate the area between the curve and the x-axis.

If a function is below the x-axis and increasing over a given interval, the left-endpoint rule will give an overestimate of the actual area while the right-endpoint rule will give an underestimate of the actual area. If the function is below the x-axis and decreasing in a given interval, the left-endpoint rule will give an underestimate of the actual area while the right-endpoint rule will give an overestimate of the actual area.

WARNING

It is a common misconception that the left-hand endpoint always underestimates the area under a curve and the right-hand endpoint method always overestimates the area under a curve. However, this assumption is only true for increasing functions above the x-axis or decreasing functions below the x-axis.

The approximate area bounded by the graph of $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ will be given by $A \approx \sum_{i=1}^{n} f(x_i)\delta x_i$, where $f(x)$ represents the height of each rectangle and δx_i its width.

If there are n intervals of equal width, then $\delta x = \frac{b-a}{n}$.

Therefore, the approximate total area using the left-endpoint rule:

$$S_L = \left(\frac{b-a}{n}\right)\left(f(a) + f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_{n-1})\right), \text{ where } a = x_0$$

The approximate total area using the right-endpoint rule:

$$S_R = \left(\frac{b-a}{n}\right)\left(f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_{n-1}) + f(b)\right), \text{ where } b = x_n$$

The left-endpoint and right-endpoint formulas can also be expressed:

$$S_L = \sum_{i=0}^{n-1}\left(\frac{b-a}{n}\right)f(x_i)$$

and

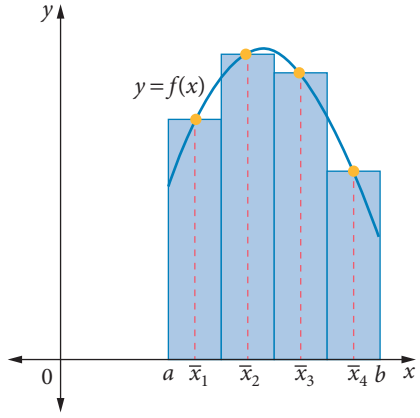
$$S_R = \sum_{i=1}^{n}\left(\frac{b-a}{n}\right)f(x_i)$$

A more accurate approximation of the area bounded by a given interval can be found by calculating the average of the left-endpoint and right-endpoint approximations:

$$\text{average approximate area} = \frac{\text{left-endpoint approximation} + \text{right-endpoint approximation}}{2}$$

$$= \frac{S_L + S_R}{2}$$

A third way of approximating area using rectangles is by using the midpoint rule. The midpoint rule is similar to the endpoint rules, but instead of using the lower or upper values to find the height of the rectangles, the middle value (midpoint) is used.



To determine the midpoint, denoted by $\overline{x}$ (read as 'x-bar'), calculate the average of the end points of the rectangle.
That is, the midpoint between x_i and x_{i-1} is:

$$\overline{x_i} = \frac{x_{i-1} + x_i}{2}$$

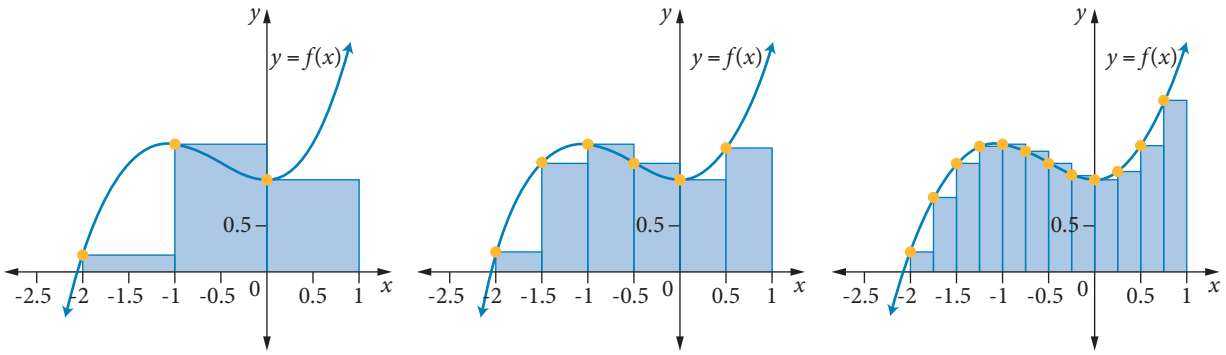
The approximate area is then calculated in a similar way to the endpoint rules, but this time using the midpoint values.
The approximate total area using the midpoint rule:

$$S_M = \left(\frac{b-a}{n}\right)\left(f\left(\overline{x_1}\right) + f\left(\overline{x_2}\right) + f\left(\overline{x_3}\right) + \ldots + f\left(\overline{x_n}\right)\right)$$

The midpoint formula can also be expressed:

$$S_M = \sum_{i=1}^{n}\left(\frac{b-a}{n}\right) f\left(\overline{x_i}\right)$$

Regardless of the method used, it is important to note that as the number of rectangles increases, the accuracy of the defined area also increases.



1 Area approximation using the left-endpoint rule

Use the total of the areas of the left-endpoint rectangles to approximate the area of the region bounded by the graph of $f(x) = x^2$, the x-axis, and the lines $x = 1$ and $x = 4$.

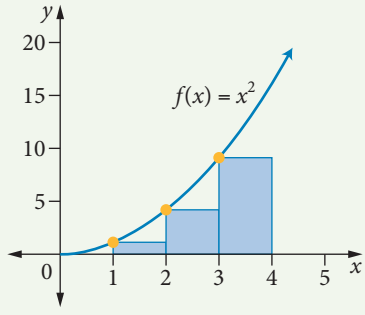
(a) Use three rectangles.

THINKING

1 Sketch the graph of $f(x)$ over the required domain and draw the required number of rectangles using the left endpoints, noting their width.

WORKING

Between $x = 1$ and $x = 4$, three rectangles each have a width of 1.



2 Determine the height of each rectangle by substituting x-values into $f(x)$.

$$f(1) = 1^2 = 1$$
$$f(2) = 2^2 = 4$$
$$f(3) = 3^2 = 9$$

3 Calculate the sum of the areas of the rectangles.

$$S_L = \sum_i f(x_i)\delta x = (1 + 4 + 9) \times 1 = 14$$

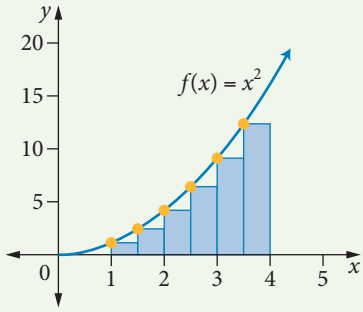
4 Interpret the answer.

The approximate area between the curve and the x-axis using the left-endpoint rule is 14 square units. This is an underestimate of the actual area.

(b) Use six rectangles.

1 Sketch the graph of $f(x)$ over the required domain and draw the required number of rectangles using the left-endpoints, noting their width.

Between $x = 1$ and $x = 4$, six rectangles each have a width of 0.5.



2 Determine the height of each additional rectangle by substituting x-values into $f(x)$.

$$f(1.5) = 1.5^2 = 2.25$$
$$f(2.5) = 2.5^2 = 6.25$$
$$f(3.5) = 3.5^2 = 12.25$$

3 Calculate the sum of the areas of the rectangles.

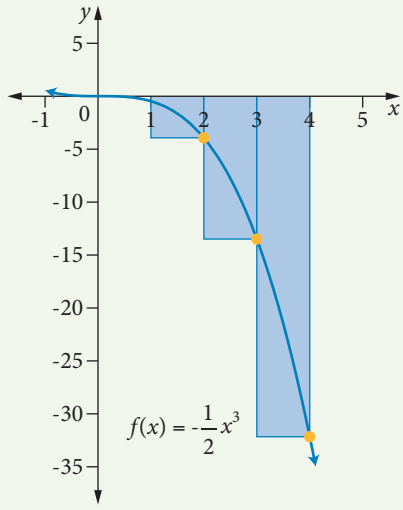
$$S_L = \sum_i f(x_i)\delta x = (1 + 2.25 + 4 + 6.25 + 9 + 12.25) \times 0.5 = 17.375$$

4 Interpret the answer.

The approximate area between the curve and the x-axis, using the left-endpoint rule, is 17.375 square units. This is an underestimate of the actual area.

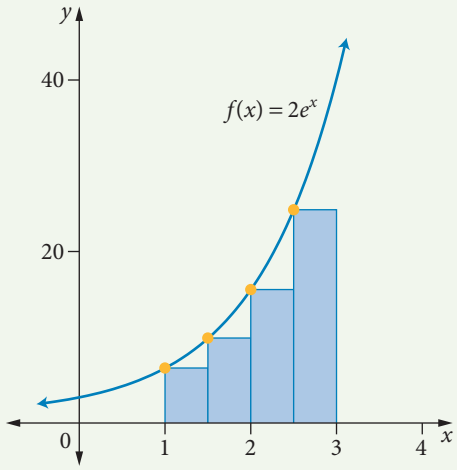
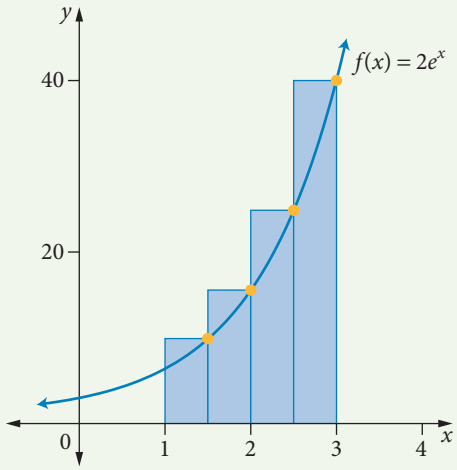
2 Area approximation using right-endpoint rule

Use the total of the areas of the right-endpoint rectangles to approximate the area of the region bounded by the curve $f(x) = -\frac{1}{2}x^3$, the x-axis, and the lines $x = 1$ and $x = 4$. Use three rectangles.

THINKING	WORKING
1 Sketch the graph of $f(x)$ over the required domain and draw the required number of rectangles using the right endpoints, noting their width.	Between $x = 1$ and $x = 4$, three rectangles each have a width of 1. 
2 Determine the height of each rectangle by substituting x-values into $f(x)$.	$f(2) = -\frac{1}{2}(2^3)$ $= -4$ $f(3) = -\frac{1}{2}(3^3)$ $= -13\frac{1}{2}$ $f(4) = -\frac{1}{2}(4^3)$ $= -32$
3 Calculate the sum of the areas of the rectangles.	$S_R = \sum_i f(x_i)\delta x$ $= \left(4 + 13\frac{1}{2} + 32\right) \times 1$ $= 49\frac{1}{2}$
4 Interpret the answer.	The approximate area between the curve and the x-axis, using the right-endpoint rule, is 49.5 square units. This is an overestimate of the actual area.

3 Determine the average approximate area

Use the totals of the areas of the lower rectangles and of the upper rectangles to determine the average approximate area of the region bounded the graph of $f(x) = 2e^x$, the x-axis, and the lines $x = 1$ and $x = 3$. Use intervals of width 0.5 units. Give your final answer correct to 2 decimal places.

THINKING	WORKING
1 Sketch the graph of $f(x)$ over the required domain and draw the required number of rectangles using the left endpoints.	
2 Determine the height of each left-endpoint rectangle by substituting the x-value into $f(x)$.	$f(1) = 2e^1 = 2e$ $\quad f(2) = 2e^2$ $f(1.5) = 2e^{1.5}$ $\quad f(2.5) = 2e^{2.5}$
3 Calculate the sum of the left-endpoint rectangles, S_L.	$S_L = \sum_i f(x_i)\delta x$ $= (2e \times 0.5) + (2e^{1.5} \times 0.5) + (2e^2 \times 0.5)$ $+ (2e^{2.5} \times 0.5)$ $= (2e + 2e^{1.5} + 2e^2 + 2e^{2.5}) \times 0.5$ $= 26.77\ldots$ square units
4 Interpret the value.	The approximate area between the curve and the x-axis, using the left-endpoint rule, is 26.77 square units. This is an underestimate of the actual area.
5 Sketch the graph of $f(x)$ over the required domain and draw the required number of rectangles using the right endpoints.	

6 Determine the height of each right-endpoint rectangle by substituting the x-value into $f(x)$.	$f(1.5) = 2e^{1.5}$ $f(2) = 2e^{2}$ $f(2.5) = 2e^{2.5}$ $f(3) = 2e^{3}$
7 Calculate the sum of the upper rectangles, S_U.	$S_R = \sum_i f(x_i)\delta x$ $= \left(2e^{1.5} \times 0.5\right) + \left(2e^{2} \times 0.5\right) + \left(2e^{2.5} \times 0.5\right)$ $+ \left(2e^{3} \times 0.5\right)$ $= \left(2e^{1.5} + 2e^{2} + 2e^{2.5} + 2e^{3}\right) \times 0.5$ $= 44.13\ldots$ square units
8 Interpret the value.	The approximate area between the curve and the x-axis, using the right-endpoint rule, is 44.14 square units. This is an overestimate of the actual area.
9 Calculate the average approximate area. Give your answer correct to 2 decimal places.	Average approximate area $= \dfrac{S_L + S_R}{2}$ $= \dfrac{26.771\ldots + 44.138\ldots}{2}$ $= 35.46\,(2 \text{ d.p.})$
10 Interpret the answer and evaluate its reasonableness.	The approximate area between the curve and the x-axis, using the average rule, is 35.46 square units. This is less than the overestimate of 44.14 square units and greater than underestimate of 26.77 square units.

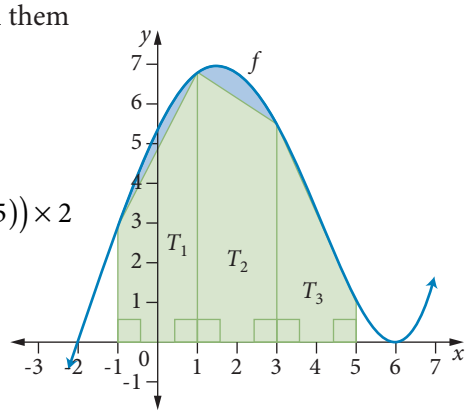
Trapezoidal rule

Another way of estimating the area between a curve and the x-axis in an interval is to divide the interval into trapeziums of equal width. Then the rule for the area of a trapezium can be used and the sum of the total areas gives an estimate of the area. In the diagram below, the three trapeziums T_1, T_2 and T_3 are 2 units wide. The area of a trapezium is given by the rule:

Area of a trapezium $= \dfrac{1}{2}$ (sum of parallel sides) $\times$ distance between them

Hence the total approximate area can be found:

$$\text{Area} = \sum\left(\frac{1}{2}\left(f(x_i) + f(x_{i+1})\right)\right)\delta x_i$$
$$= \frac{1}{2}\left(f(-1) + f(1)\right) \times 2 + \frac{1}{2}\left(f(1) + f(3)\right) \times 2 + \frac{1}{2}\left(f(3) + f(5)\right) \times 2$$
$$= f(-1) + f(1) + f(1) + f(3) + f(3) + f(5)$$
$$= f(-1) + 2f(1) + 2f(3) + f(5)$$



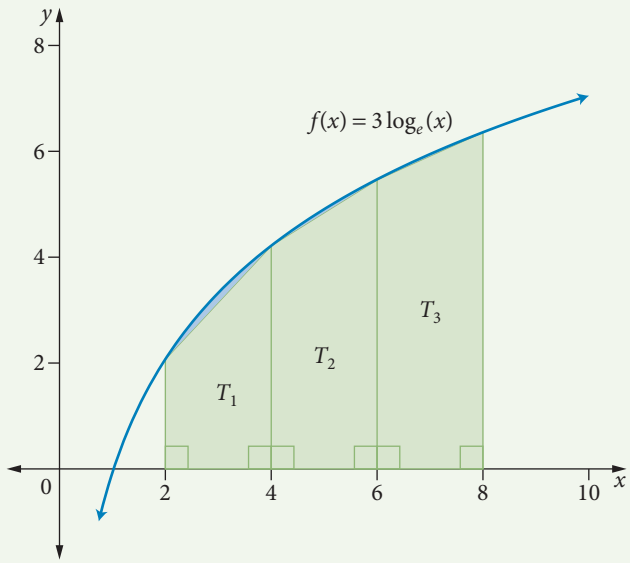
In general, the trapezoidal rule for the approximate area between the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by:

$$\text{Area} = \frac{1}{2} \times \frac{b-a}{n}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right)$$

where n is the number of trapeziums used, and $\frac{b-a}{n}$ is the interval width.

4 Use the trapezoidal rule

Determine an approximation for the area bounded by the graph of $y = 3\log_e(x)$, the x-axis and the lines $x = 2$ and $x = 8$, using the trapezoidal rule. Consider three trapeziums that are all 2 units wide. Give your answer correct to 2 decimal places.

THINKING	WORKING
1 Sketch the graph and divide the region as required.	
2 Determine the value of the function at each point used.	$f(2) = 3\log_e(2)$ $f(4) = 3\log_e(4)$ $f(6) = 3\log_e(6)$ $f(8) = 3\log_e(8)$
3 Substitute values into the trapezoidal rule.	$\text{Area} = \frac{1}{2} \times \frac{8-2}{3}\left(f(2) + 2f(4) + 2f(6) + f(8)\right)$ $= 3\log_e(2) + 6\log_e(4) + 6\log_e(6) + 3\log_e(8)$ $= 27.39\,(2\text{ d.p.})$
4 Interpret the answer.	The approximate area between the curve and the x-axis using the trapezoidal rule is 27.39 square units.

Technology worked example
Approximate the area under a curve

EXERCISE 5.1 Estimating area beneath curves

Worked Example

1 Use the sum of the areas of the left-endpoint rectangles to approximate the following areas, correct to 2 decimal places.

(a) the region bounded by the graph of $f(x) = x^2 + 2$, the x-axis and the lines $x = 1$ and $x = 4$, using three rectangles

(b) the region bounded by the graph of $f(x) = -x^2 + 4$, the x-axis, and the lines $x = 0$ and $x = 2$, using four rectangles

2 Use the sum of the areas of the right-endpoint rectangles to approximate the following areas, correct to 3 decimal places.

(a) the region bounded by the graph of $f(x) = -\sqrt{x}$, the x-axis, and the lines $x = 2$ and $x = 5$, using three rectangles

(b) the region bounded by the graph of $f(x) = \frac{1}{2}e^x$, the x-axis, and the lines $x = 1$ and $x = 3$, using four rectangles

3 Using the left-endpoint rule with rectangles 0.5 units wide, determine the approximate area of the region bounded by the graph of $f(x) = \frac{1}{2x}$, the x-axis and the lines $x = 1$ and $x = 3$. Give your answer correct to 3 decimal places.

4 Using the right-endpoint rule with rectangles 0.5 units wide, determine the approximate area of the region bounded by the curve $f(x) = \log_e(x)$, the x-axis and the lines $x = 2$ and $x = 3$. Give your answer correct to 3 decimal places.

5 Determine the average approximate areas to calculate the following, giving your answers correct to 2 decimal places.

(a) the area of the region bounded by the graph of $f(x) = \frac{1}{x^2}$, the x-axis, and the lines $x = 2$ and $x = 4$, using intervals 0.5 units wide

(b) the area of the region bounded by the graph of $f(x) = -\frac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = 2.5$, using intervals 0.5 units wide

6 Using the left-endpoint rule with intervals of $\frac{\pi}{4}$, demonstrate that the area bounded by the graph of $f(x) = \sin(x)$, the x-axis, and the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ is $\frac{\left(\sqrt{2}+1\right)\pi}{4}$ square units.

7 Which of the following statements is true for an approximation of the area bounded by the graph of $f(x) = -\sqrt{x}$, the x-axis, and the lines $x = 0$ and $x = 4$?

A The total area of the left-endpoint rectangles will overestimate the actual area.

B The total area of the left-endpoint rectangles will give the same estimate as the total area of the right-endpoint rectangles.

C The average of the total area of the left-endpoint rectangles and the total area of the right-endpoint rectangles will give the actual area.

D The actual area will lie between the total area of the left-endpoint rectangles and the total area of the right-endpoint rectangles.

8 A set of values is given for the function $y = f(x)$.

x	1	2	3	4	5
$f(x)$	1	0.79	0.69	0.63	0.58

(a) Which of the following statements is true for an approximation of the area bounded by the graph of $y = f(x)$, the x-axis, and the lines $x = 1$ and $x = 5$?

A The actual area beneath the curve is 2.9.

B The best approximation of the bounded area is 2.9.

C The total area of the right-endpoint rectangles is 3.11.

D The total area of the left-endpoint rectangles is 2.69.

(b) Use the midpoint method to approximate of the area bounded by the graph of $y = f(x)$, the x-axis, and the lines $x = 1$ and $x = 5$.

Worked Example 4

9 Use the trapezoidal rule to determine the approximate area bounded by the graph of $y = 4x^2 - 1$, the x-axis and the lines $x = 1$ and $x = 4$. Use intervals 0.5 units wide.

10 Consider the region bounded by the curve $f(x) = \frac{1}{2}x$, the x-axis, and the lines $x = 2$ and $x = 5$.

(a) Use the trapezoidal rule to determine the exact area of the region.

(b) Use the total area of the left-endpoint rectangles to approximate the area of the region using intervals of the given widths.

(i) 1 unit **(ii)** 0.5 units **(iii)** 0.1 units

(c) Use the sum of the areas of right-endpoint rectangles to approximate the area of the region using intervals of the given widths.

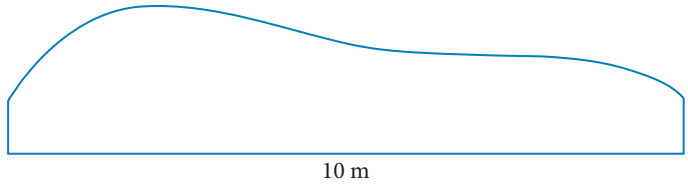
(i) 1 unit **(ii)** 0.5 units **(iii)** 0.1 units

(d) Comment on your results to part **(c)**.

11 Determine the approximate area bounded by the graph of $f(x) = e^{-2x}$, the x-axis and the lines $x = 0$ and $x = 3$, using the trapezoidal rule with intervals 1 unit wide. Give your answer correct to 3 decimal places.

12 The cosine curve is a symmetrical shape. Explain why the approximation of the area beneath the graph of $f(x) = 2\cos(x)$, using the left-endpoint rule in the interval from $x = 0$ to $x = \pi$, is not double the value of the approximation in the interval from $x = 0$ to $x = \frac{\pi}{2}$.

13 Hannah is a landscape gardener and wants to pave a part of a backyard as shown in the diagram. The length of the longest straight side of the backyard is 10 m. To calculate the approximate area of the piece of land, she divides the straight side of the land into rectangular strips 1 m wide. She then measures the length at 1 m intervals from the left and records the measurements.



Distance (m)	0	1	2	3	4	5	6	7	8	9	10
Length (m)	0.56	1.30	1.50	1.46	1.30	1.20	1.10	1.00	0.99	0.89	0.56

(a) Draw the left-endpoint rectangles and use these measurements to calculate the approximate total area of land. Give your answer correct to 2 decimal places.

(b) Draw the right-endpoint rectangles and use these measurements to calculate the approximate total area of land. Give your answer correct to 2 decimal places.

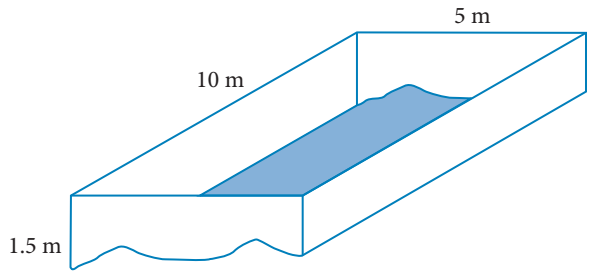
(c) Draw a trapezium on each rectangle and use these measurements to calculate the approximate total area of land. Give your answer correct to 2 decimal places.

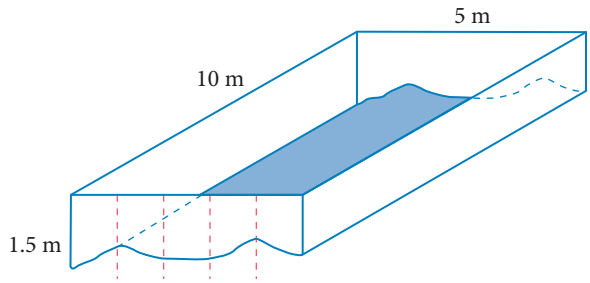
(d) Which one of the approximate areas calculated should Hannah use and why?

14 A farmer has constructed an underground reservoir 10 m long to collect rainwater. The reservoir is 5 m wide with a rectangular cross-section and an uneven floor as shown in the diagram.

To determine the approximate volume of the reservoir, the farmer has decided to divide one of the vertical surfaces of the reservoir into five strips, each 1 m wide, as shown in the diagram.

He measured the depth at the edge of each strip and recorded the measurements in the table below. For simplicity, assume the depth is even within each strip.





Distance from left (m)	0	1	2	3	4	5
Depth	1.5	1.3	1.4	1.4	1.2	1.3

(a) Draw the left-endpoint rectangles and use these measurements to calculate the sum of the volumes of the rectangular prisms to approximate the volume of water the reservoir could hold.

(b) Draw the right-endpoint rectangles and use the right-endpoint measurements to determine the approximate volume of water the reservoir could hold.

(c) Draw a trapezium on each strip and use these measurements to determine the approximate volume of water the reservoir could hold.

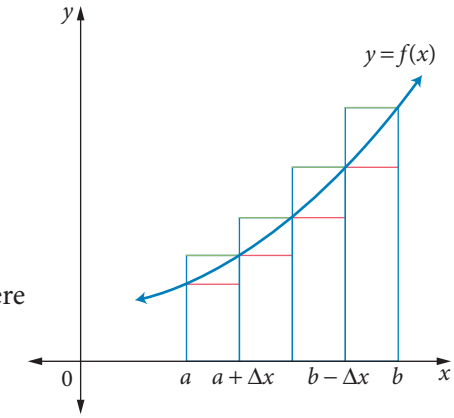
(d) Which one of the approximate volumes calculated would give the farmer the best approximation of the actual volume and why?

Definite integrals

5.2

The fundamental theorem of calculus

Consider the graph of $y = f(x)$, where $f(x)$ is continuous for all values of x between $x = a$ and $x = b$.



If a function is positive and increasing, the actual area A between the curve $f(x)$ and the x-axis, will be between the sum of the lower, left-endpoint rectangles (underestimate) and the upper right-endpoint rectangles (overestimate).

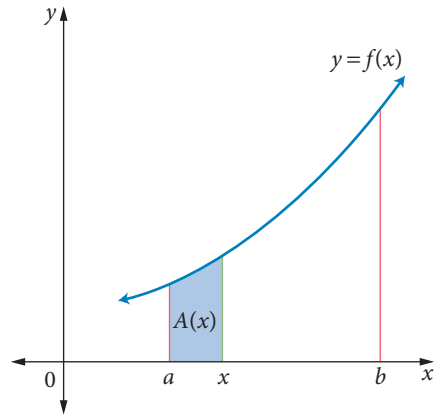
The sum of the lower rectangles is given by: $S_L = \sum_{x=a}^{b-\Delta x} f(x)\Delta x$, where $f(x)$ gives the height of each rectangle and Δx represents the width.

The sum of the upper rectangles is given by: $S_U = \sum_{x=a+\Delta x}^{b} f(x)\Delta x$, where $f(x)$ gives the height of each rectangle and Δx represents the width.

Hence $\sum_{x=a}^{b-\Delta x} f(x)\Delta x < A < \sum_{x=a+\Delta x}^{b} f(x)\Delta x$.

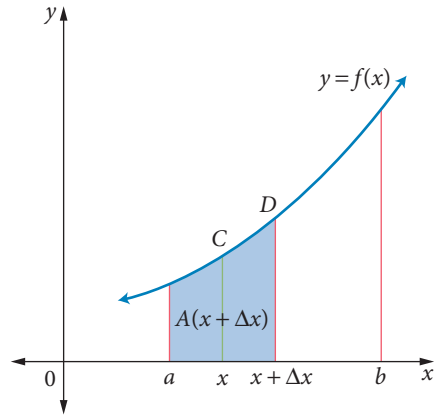
As you improve the approximation by taking more rectangles to calculate the area, $\Delta x \to 0$ and $S_L \to S_U$, hence $A = \lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x$.

This limit is written as $A = \int_a^b f(x)dx$ and is called the definite integral of $f(x)$ from a to b.

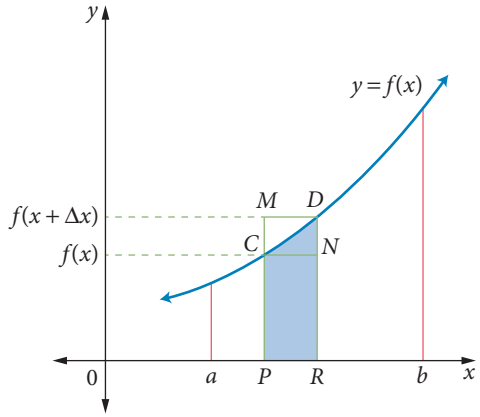


Define $A(x)$ as the area bounded by the graph of $y = f(x)$ and the x-axis between the x-values of a and x.

Consider the interval from x to $x + \Delta x$ where Δx represents a small increment in x. $A(x + \Delta x)$ will therefore represent the area bounded by the graph of $y = f(x)$ and the x-axis between the x-values of a and $x + \Delta x$.



The area beneath the graph between points C and D will therefore be given by $A(x+\Delta x)-A(x)$.



The area beneath the graph between points C and D will be between the area of the lower rectangle $PCNR$ (underestimate) and the area of the upper rectangle $PMDR$ (overestimate).

Hence, $f(x)\Delta x < A(x+\Delta x)-A(x) < f(x+\Delta x)\Delta x$

Dividing by Δx:

$$f(x) < \frac{A(x+\Delta x)-A(x)}{\Delta x} < f(x+\Delta x)$$

As $\Delta x \to 0$, point D gets closer and closer to point C, and the widths of the rectangles decrease. Hence both the smaller rectangle $PCNR$ and larger rectangle $PMDR$ approach the actual area beneath the graph, $PCDR$. As $\Delta x \to 0$, $f(x+\Delta x) \to f(x)$, hence the area beneath the graph, $PCDR$, is given by $\lim\limits_{\Delta \to 0} \dfrac{A(x+\Delta x)-A(x)}{\Delta x}$, which is $\dfrac{d}{dx}A(x)$, by the definition of derivative from first principles.

Hence, $\dfrac{d}{dx}A(x) = f(x)$.

Anti-differentiating both sides with respect to x:

$A(x) = F(x) + c$, where c is an arbitrary constant, whose value may be determined as follows.

- For $x = a$, $A(a) = 0$, therefore $0 = F(a) + c$.
- $c = -F(a)$, therefore $A(x) = F(x) - F(a)$.
- For $x = b$, $A(b) = F(b) - F(a)$.

Note: By definition, $A(b)$ represents the area bounded by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$.

Hence, $A = F(b) - F(a)$.

So, from $A = \int_a^b f(x)dx$ and $A = F(b) - F(a)$: $\int_a^b f(x)dx = F(b) - F(a)$

This result is known as the *fundamental theorem of calculus*. It states that the definite integral can be calculated in terms of an anti-derivative.

To determine the definite integral of $f(x)$ between $x = a$ and $x = b$, that is $\int_a^b f(x)dx$:

- Anti-differentiate $f(x)$ to determine $F(x)$.
- Determine $F(b) - F(a)$.
- $\int_a^b f(x)dx = \left[F(x)\right]_a^b$
  $= F(b) - F(a)$

Additional information

The definite integral

View the video for an introduction on how to use the definite integral to calculate the area below the curve.

Note that the arbitrary constant can be ignored as it appears in both $F(a)$ and $F(b)$, and is cancelled by the subtraction. Only an increasing function has been considered here, but the theorem is equally true for a decreasing function or one that is sometimes increasing and sometimes decreasing in the interval.

5 Determine a definite integral for powers of *x*

Determine $\int_1^3 \left(3x^2 - 4x + 1\right) dx$.

THINKING	WORKING
1 Anti-differentiate the given function to determine $F(x)$ and express in the form $\left[F(x)\right]_a^b$.	$\int_1^3 \left(3x^2 - 4x + 1\right) dx = \left[x^3 - 2x^2 + x\right]_1^3$
2 Calculate the value of the definite integral by substituting the upper and lower bounds into the anti-derivative.	$= (27 - 18 + 3) - (1 - 2 + 1)$ $= 12 - 0$ $= 12$

6 Determine definite integrals involving trigonometric functions

Determine $\int_0^{\frac{\pi}{2}} 3\sin(2x)\, dx$.

THINKING	WORKING
1 Anti-differentiate the given function to determine $F(x)$ and express in the form $\left[F(x)\right]_a^b$.	$\int_0^{\frac{\pi}{2}} 3\sin(2x)\, dx = \left[-\frac{3}{2}\cos(2x)\right]_0^{\frac{\pi}{2}}$
2 Calculate the value of the definite integral by substituting the upper and lower bounds into the anti-derivative.	$= \left(-\frac{3}{2}\cos(\pi)\right) - \left(-\frac{3}{2}\cos(0)\right)$ $= \left(-\frac{3}{2}\right)(-1) - \left(-\frac{3}{2}\right)(1)$ $= \frac{3}{2} + \frac{3}{2}$ $= 3$

7 Determine definite integrals involving exponential functions

Determine the exact value of $\int_0^1 3\left(e^{\frac{x}{2}} + 1\right)^2 dx$.

THINKING	WORKING
1 Expand the given function.	$\int_0^1 3\left(e^{\frac{x}{2}} + 1\right)^2 dx = 3\int_0^1 \left(e^x + 2e^{\frac{x}{2}} + 1\right) dx$
2 Anti-differentiate the given function to determine $F(x)$ and express in the form $\left[F(x)\right]_a^b$.	$\int_0^1 \left(e^{\frac{x}{2}} + 1\right)^2 dx = 3 \times \left[e^x + 4e^{\frac{x}{2}} + x\right]_0^1$
3 Calculate the value of the definite integral by substituting the upper and lower bounds into the anti-derivative.	$= 3\left[\left(e + 4e^{\frac{1}{2}} + 1\right) - (1 + 4 + 0)\right]$ $= 3\left(e + 4e^{\frac{1}{2}} - 4\right)$

Additional information

Calculating the definite integral

Watch the video and complete the activity to consolidate your understanding of the definite integral.

8 Determine definite integrals involving the reciprocal function

Determine $\int_0^2 \frac{2}{3x+5}dx$ correct to 2 decimal places.

THINKING	WORKING
1 Anti-differentiate the given function (the integrand) to determine $F(x)$ and express in the form $\left[F(x)\right]_a^b$.	$\int_0^2 \frac{2}{3x+5}dx = \frac{2}{3}\left[\log_e(3x+5)\right]_0^2$ where $3x+5>0$ $3x>-5$ $x>-\frac{5}{3}$
2 Calculate the value of the definite integral by substituting the upper and lower bounds into the anti-derivative.	$\frac{2}{3}\left[\log_e(3x+5)\right]_0^2 = \frac{2}{3}\left(\log_e(11)-\log_e(5)\right)$ $= \frac{2}{3}\log_e\left(\frac{11}{5}\right)$ $= 0.53$

WARNING

$\int_a^b f(x)dx$ is not defined if $f(x)$ is discontinuous in the interval $[a, b]$.

For example, $\int_{-1}^{1}\left(\frac{1}{x^2}\right)dx$ cannot be determined because the function $f(x)=\frac{1}{x^2}$ is undefined at $x=0$ and the graph is therefore discontinuous at $x=0$.

Technology worked example

Calculate a definite integral

Properties of definite integrals

These can be derived using $\int_a^b f(x)dx = F(b)-F(a)$. You will be asked to establish or verify some of these in the exercises.

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b \left(f(x)\pm g(x)\right)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \text{ if } b \text{ lies between } a \text{ and } c.$$

9 Use properties of definite integrals

Given $\int_3^5 f(x)\,dx = 6$, determine the following definite integrals.

(a) $\int_3^5 4f(x)\,dx$

THINKING	WORKING
1 Rewrite using the appropriate property.	$\int_3^5 4f(x)\,dx = 4\int_3^5 f(x)\,dx$
2 Substitute the value of the given definite integral and calculate the answer.	$= 4 \times 6$ $= 24$

(b) $\int_3^5 \left(f(x) - x\right)dx$

1 Rewrite using the appropriate property.	$\int_3^5 \left(f(x) - x\right)dx = \int_3^5 f(x)\,dx - \int_3^5 (x)\,dx$
2 Substitute the value of the given definite integral and determine any other integrals.	$= 6 - \left[\frac{1}{2}x^2\right]_3^5$
3 Calculate the answer.	$= 6 - \left(\frac{25}{2} - \frac{9}{2}\right)$ $= 6 - 8$ $= -2$

10 Calculate the value of the upper or lower bound of an integral

(a) Calculate the possible values of b if $\int_0^b \left(3x^2 - 4\right)dx = 0$.

THINKING	WORKING
1 Integrate the function.	$\int_0^b \left(3x^2 - 4\right)dx = \left[x^3 - 4x\right]_0^b$
2 Substitute the values of the upper and lower bounds into the integral and simplify.	$= \left(b^3 - 4b\right) - (0)$ $= b\left(b^2 - 4\right)$ $= b(b-2)(b+2)$
3 Equate the expression to the value defined in the question and solve for the unknown.	Therefore $b(b-2)(b+2) = 0$, Using the null factor law: $b = 0$, $b = -2$ or $b = 2$
4 Interpret the answer.	Therefore for $\int_0^b \left(3x^2 - 4\right)dx = 0$, $b = 0$ or $b = \pm 2$.

(b) Calculate the values of a if $\int_a^4 (2x-5)\,dx < 0$.

1 Integrate the function.

$$\int_a^4 (2x-5)\,dx = \left[x^2 - 5x\right]_a^4$$

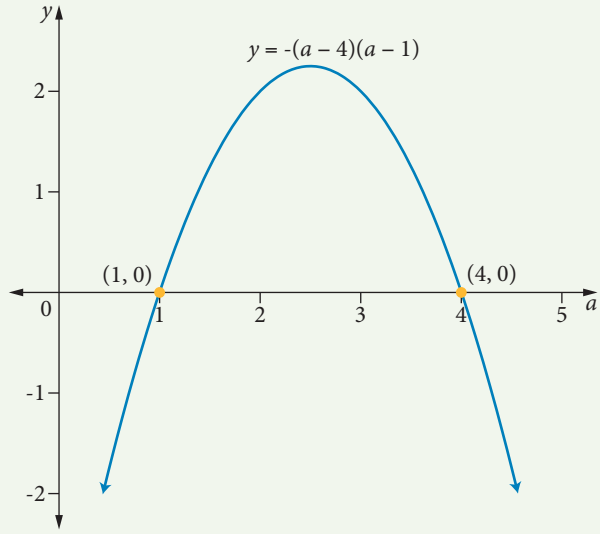
2 Substitute the values of the upper and lower bounds into the integral and simplify.

$$= \left(4^2 - 5\times 4\right) - \left(a^2 - 5a\right)$$
$$= 16 - 20 - a^2 + 5a$$
$$= -a^2 + 5a - 4$$
$$= -\left(a^2 - 5a + 4\right)$$
$$= -(a-4)(a-1)$$

3 Equate the expression to the value defined in the question and solve for the unknown.

Therefore $-(a-4)(a-1) < 0$,

Consider the graph of $y = -(a-4)(a-1)$.



$-(a-4)(a-1) < 0$ for $a < 1$ or $a > 4$

4 Interpret the answer.

Therefore for $\int_a^4 (2x-5)\,dx < 0$,

$a < 1$ or $a > 4$.

Technology worked example

Calculate the value of the upper bound of an integral

EXERCISE 5.2 Definite integrals

Worked Example

1 Determine each of the following. 5

(a) $\int_1^3 \left(3x^2 - 6x + 2\right) dx$ (b) $\int_{-1}^2 \left(x^2 - 3x + 1\right) dx$ (c) $\int_1^2 \frac{1}{3x^2} dx$

(d) $\int_1^2 (2x - 3)^4 dx$ (e) $\int_{-1}^2 (3 - 2x)^3 dx$ (f) $\int_0^4 \left(\sqrt{x} + 3\right)^2 dx$

2 Determine each of the following. 6

(a) $\int_0^{\frac{\pi}{2}} 2\cos(3x) dx$ (b) $\int_0^{\pi} 3\sin\left(\frac{x}{2}\right) dx$

(c) $\int_0^{\frac{\pi}{4}} 2\sin 3\left(\theta - \frac{\pi}{4}\right) d\theta$ (d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{1}{2}\sin(2\pi - x) dx$

3 Determine each of the following. 7

(a) $\int_{-2}^1 3e^{-2x} dx$ (b) $\int_0^{\pi} \left(e^{2x} - \sin(x)\right) dx$ (c) $\int_0^2 \left(e^x + 2e^{-x}\right)^2 dx$

4 Determine each of the following. 8

(a) $\int_1^3 \frac{2}{x} dx$ (b) $\int_{-3}^{-1} \frac{2}{x} dx$ (c) $\int_0^1 \frac{1}{2x + 7} dx$ (d) $\int_0^2 \frac{1}{5 - 2x} dx$

5 Determine each of the following definite integrals, giving your answers in exact form.

(a) $\int_0^{\frac{\pi}{2}} \left(\sin(x) - \cos(2x)\right) dx$ (b) $\int_{\frac{\pi}{2}}^{\pi} \left(2\cos(x) - \frac{1}{2}\sin\left(\frac{x}{2}\right)\right) dx$

(c) $\int_{-1}^1 \frac{2}{(2 + 6x)^3} dx$ (d) $\int_{-2}^{-1} \left(e^x + \frac{1}{x^2}\right) dx$

6 Determine each of the following definite integrals, giving your answers correct to 2 decimal places.

(a) $\int_{-1}^1 \left(\sin(x) - e^{3x}\right) dx$ (b) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \left(\cos(x) - \sin(2x)\right) dx$

(c) $\int_{-2}^0 \frac{3}{7 - 2x} dx$ (d) $\int_{-1}^1 \left(2\sin\left(\frac{x}{2}\right) - e^{-2x}\right) dx$

7 Determine each of the following definite integrals, giving your answers correct to 2 decimal places.

(a) $\int_0^1 \left(e^x - e^{-x}\right)^2 dx$ (b) $\int_{\pi}^{\frac{3\pi}{2}} \left(\cos(x) + \sin(x)\right)^2 dx$

(c) $\int_1^3 \left(x + \frac{2}{x}\right)^2 dx$ (d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(2\cos\left(\frac{x}{2}\right) - e^{-x}\right)^2 dx$

8 Given $\int_2^3 f(x) dx = 5$, determine each of the following definite integrals. 9

(a) $\int_2^3 3f(x) dx$ (b) $\int_2^3 \left(f(x) - 6x^2\right) dx$

9 Given $\int_1^5 f(x) dx = 8$, determine each of the following definite integrals.

(a) $-3\int_1^5 2f(x) dx$ (b) $\int_1^5 \left(f(x) - 6\right) dx$ (c) $\int_5^1 3f(x) dx$ (d) $\int_5^1 \left(f(x) - x\right) dx$

10 Determine the definite integral for $\int_1^2 \left(1 - \frac{1}{x^2}\right)^2 dx$

11 $\int_0^5 \frac{1}{x-6}\,dx$ is equal to which one of the following?

A $\left[-\frac{1}{(x-6)^2}\right]_0^5$ **B** $\log_e(5-6)-\log_e(0-6)$

C $\log_e|5-6|-\log_e|0-6|$ **D** $\int_0^5 \frac{1}{x-6}\,dx$ is undefined.

12 If $F(x)$ is an anti-derivative of $f(x)$, then which expression is equal to $\int_{-b}^{a} f(x)\,dx$?

A $F(a)-F(b)$ **B** $F(a)-F(-b)$ **C** $F(b)-F(a)$ **D** $F(a)+F(b)$

13 **(a)** Verify that $\frac{3x+3}{(x-1)(x+2)}=\frac{2}{x-1}+\frac{1}{x+2}$

(b) Use your answer to part **(a)** to determine the exact value of $\int_2^3 \frac{3x+3}{x^2+x-2}\,dx$.

14 Consider the property $\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$ for definite integrals.

(a) Verify that $\int_{-1}^{4} e^{2x+1}\,dx=-\int_4^{-1} e^{2x+1}\,dx$.

(b) Prove the general result $\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$.

Worked Example

15 Use a single definite integral to determine the value of $\int_1^4 \frac{1}{x^2}\,dx+\int_4^{10} \frac{1}{x^2}\,dx$.

10

16 Given that $\int_{-1}^{m}(mx)\,dx=m$, determine the value of m.

17 For each definite integral, determine the value of b correct to 2 decimal places.

(a) $\int_2^b e^{2x}\,dx=7$ **(b)** $\int_b^3\left(x^2-3x\right)dx=12$ **(c)** $\int_1^b -e^{-x}\,dx=-0.318$ **(d)** $\int_1^b\left(x^2-\frac{1}{x^2}\right)dx=4$

18 Determine an expression for $\int_{-m}^{m}(m-4x)\,dx$ in terms of m.

19 Determine exact values of m for the following definite integrals.

(a) $\int_2^m(4x+3)\,dx=-15$ **(b)** $\int_m^{\frac{\pi}{2}} 3\sin(2x)\,dx=\frac{6+3\sqrt{3}}{4}, m\in[0,2\pi]$

20 Consider the function $f(x)=\frac{1}{3}\left(x^2-1\right)^{\frac{3}{2}}$.

(a) Differentiate the function to determine an expression for $f'(x)$.

(b) Hence determine $\int_1^2 5x\sqrt{x^2-1}\,dx$.

21 If $\int_a^3\left(2-2x-x^2\right)dx=-6$, calculate the exact values of a.

22 **(a)** Determine $\int 3\sin\left(2\left(x-\frac{\pi}{2}\right)\right)dx$.

(b) Determine the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 3\sin\left(2\left(x-\frac{\pi}{2}\right)\right)dx$.

(c) Determine exact values for a, such that $\int_{\frac{\pi}{6}}^{a} 3\sin\left(2\left(x-\frac{\pi}{2}\right)\right)dx=-\frac{3}{4}$, where $a\in[0,2\pi]$.

(d) Determine an exact value for b such that $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} b\sin\left(2\left(x-\frac{\pi}{2}\right)\right)dx=\frac{5}{8}$.

5.3 Calculating area under curves

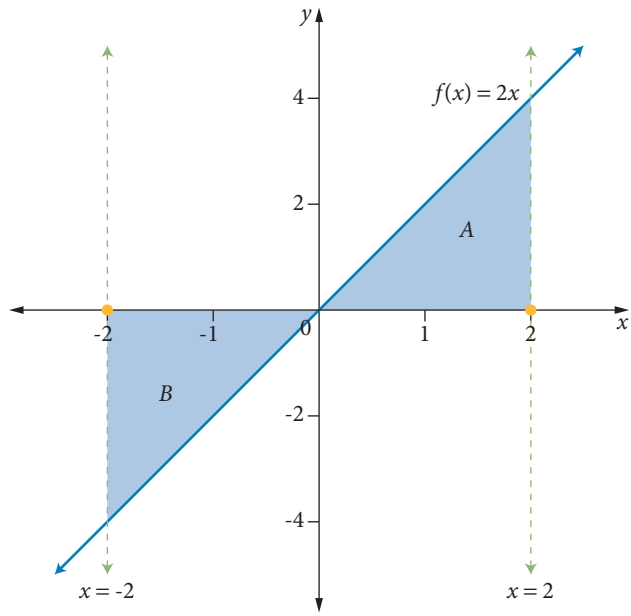
As evidenced by the fundamental theorem of calculus, discussed in the previous section, the exact area bounded by the graph of $f(x)$ and the x-axis between $x = a$ and $x = b$ can be found by determining the definite integral $\int_a^b f(x)dx = F(b) - F(a)$, when the function is continuous and non-negative over this interval.

Consider the area bounded (enclosed) by the graph of $f(x) = 2x$, the x-axis, and the lines $x = -2$ and $x = 2$.

The area of each triangle can be determined as follows:

$$\begin{aligned}\text{Area} &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 2 \times 4 \\ &= 4 \text{ square units}\end{aligned}$$

Hence the area bounded by the graph of $f(x) = 2x$, the x-axis, and the lines $x = 2$ and $x = -2$ is $4 + 4 = 8$ square units.



Explore further

Signed and unsigned areas

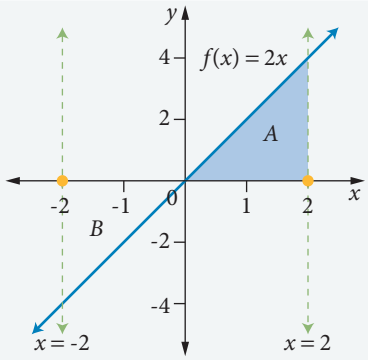
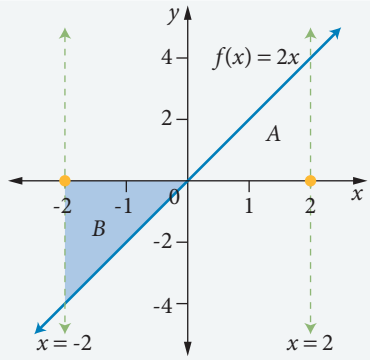
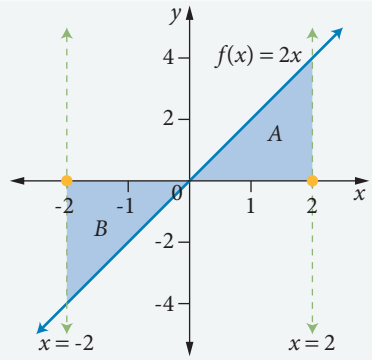
Explore signed and unsigned areas under curves where the interval may be above or below the x-axis.

Additional information

Area under the graph of a non-negative function

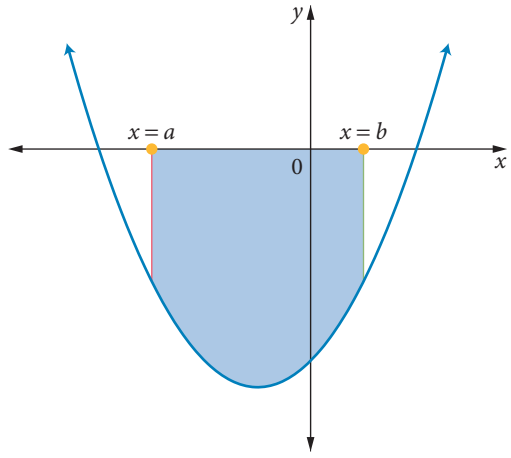
This video reviews the definite integral determining the area under the graph of a non-negative function.

Consider now the following three definite integrals.

$\int_0^2 (2x)dx = \left[x^2\right]_0^2$ $= 4 - 0$ $= 4$	$\int_{-2}^0 (2x)dx = \left[x^2\right]_{-2}^0$ $= 0 - 4$ $= -4$	$\int_{-2}^2 (2x)dx = \left[x^2\right]_{-2}^2$ $= 4 - 4$ $= 0$
This gives the area above the x-axis, labelled A.	This can be read as, an area of 4 square units below the x-axis. When determining an integral in an interval where $f(x)$ is below the x-axis, you will obtain a negative value.	This takes the sign of the areas into consideration, taking the area above the x-axis as positive and the area below the x-axis as negative, and calculating the total of the values.
		

Signed area: The signed area is given by $\int_a^b f(x)dx$, without regard to whether the graph of $f(x)$ sits above or below the x-axis.

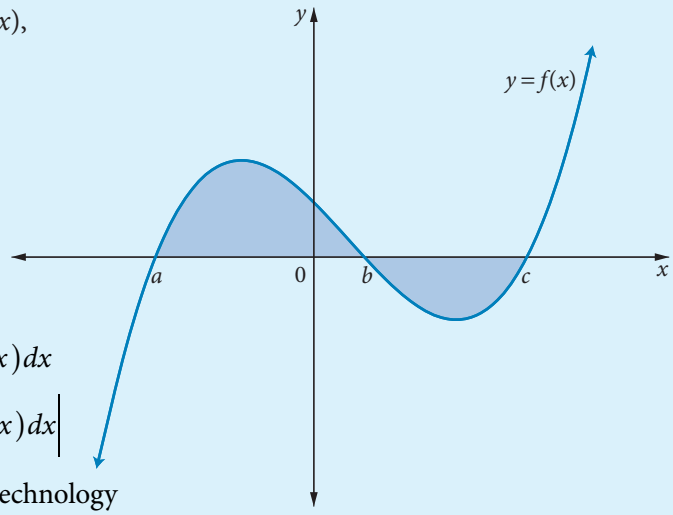
Unsigned area: To determine the actual or unsigned area, a sketch must be drawn to determine if any of the required area lies below the x-axis. Each region above and below the x-axis is then integrated individually. The magnitude of each region is then added to obtain the actual or unsigned area. When simply asked to calculate 'area', the actual or unsigned area is required.



The actual or unsigned area shaded above may be determined by a variety of methods. The required region is below the x-axis, therefore the integral $\int_a^b f(x)dx$ will return a negative value. Hence, the unsigned or actual area may be given by:

$$\begin{aligned}\text{Area} &= -\int_a^b f(x)dx \\ &= \int_b^a f(x)dx \\ &= \left|\int_a^b f(x)dx\right|\end{aligned}$$

Consider the area enclosed by the curve $y = f(x)$, between $x = a$ and $x = c$, as shown.



The actual or unsigned area bounded by the graph of $f(x)$ and the x-axis between $x = a$ and $x = c$, where $a < b < c$, and b is an x-intercept for the graph $f(x)$, is found by determining any one of the following definite integrals:

$$\int_a^b f(x)\,dx - \int_b^c f(x)\,dx = \int_a^b f(x)\,dx + \int_c^b f(x)\,dx$$
$$= \int_a^b f(x)\,dx + \left|\int_b^c f(x)\,dx\right|$$
$$= \int_a^c |f(x)|\,dx \quad \text{using technology}$$

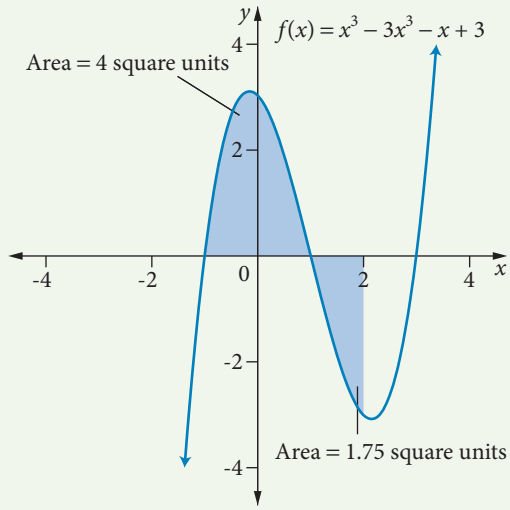
The signed area is given by $\int_a^c f(x)\,dx$.

11 Determine a signed area

Determine the exact signed area bounded by the graph of $f(x) = x^3 - 3x^2 - x + 3$ and the x-axis, between $x = -1$ and $x = 2$.

THINKING	WORKING
1 Identify the required definite integral.	$\int_{-1}^{2}\left(x^3 - 3x^2 - x + 3\right)dx$
2 Anti-differentiate $f(x)$, and express in the form $\left[F(x)\right]_a^b$.	$= \left[\frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x\right]_{-1}^{2}$
3 Substitute the limits a and b into $F(b) - F(a)$ to calculate the signed area.	$= \left(\frac{1}{4}(16) - 8 - \frac{1}{2}(4) + 6\right) - \left(\frac{1}{4}(1) - (-1) - \frac{1}{2}(1) - 3\right)$ $= (4 - 8 - 2 + 6) - \left(\frac{1}{4} + 1 - \frac{1}{2} - 3\right)$ $= 0 - \left(\frac{1}{4} - 2 - \frac{2}{4}\right)$ $= 2\frac{1}{4}$ square units
4 Interpret the answer.	The area above the x-axis is $2\frac{1}{4}$ square units greater than the area below the x-axis.

5 Evaluate the reasonableness of the solution by graphing the function.



The signed area is calculated as:
$4 + (-1.75) = 2.25$ square units. Given that the difference is positive, a greater area is bound by the curve and the x-axis above the x-axis.

12 Determine an unsigned area

Determine the exact unsigned area bounded by the graph of $f(x) = (x + 4)(x - 2)$, the x-axis, and the lines $x = 1$ and $x = 3$.

THINKING

1 Sketch the graph of $f(x)$, labelling x-intercepts and shading the required region.

WORKING

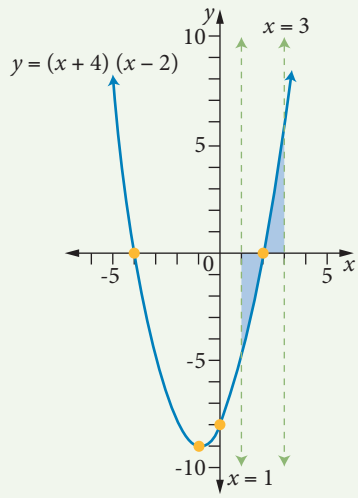
The x-intercepts are when $y = 0$:

The coordinates of the x-intercepts are $(-4, 0)$ and $(2, 0)$.

The y-intercept occurs when $x = 0$:

$$y = (0 + 4)(0 - 2)$$
$$= 4 \times -2$$
$$= -8$$

The coordinates of the y-intercept are $(0, -8)$.



2 Expand $f(x)$.	$f(x) = (x+4)(x-2)$ $= x^2 + 2x - 8$
3 Determine the definite integral for the region above the x-axis.	$\int_2^3 (x^2 + 2x - 8)\,dx = \left[\frac{x^3}{3} + x^2 - 8x\right]_2^3$ $= (9 + 9 - 24) - \left(\frac{8}{3} + 4 - 16\right)$ $= -6 - \frac{8}{3} + 12$ $= 3\frac{1}{3}$
4 Interpret the value.	The area above the x-axis is $3\frac{1}{3}$ square units.
5 Determine the definite integral for the region below the x-axis.	$\int_1^2 (x^2 + 2x - 8)\,dx = \left[\frac{x^3}{3} + x^2 - 8x\right]_1^2$ $= \left(\frac{8}{3} + 4 - 16\right) - \left(\frac{1}{3} + 1 - 8\right)$ $= \frac{8}{3} - 12 - \frac{1}{3} + 7$ $= 2\frac{2}{3} - 5 - \frac{1}{3}$ $= -2\frac{2}{3}$
6 Interpret the value.	The area below the x-axis is $2\frac{2}{3}$ square units.
7 Add the magnitude of the integrals.	Area $= 3\frac{1}{3} + 2\frac{2}{3}$ $= 6$ square units
8 Interpret the answer.	The area bound by the curve and the x-axis between $x = 1$ and $x = 3$ is 6 square units.
9 Evaluate the reasonableness of the answer by creating and calculating an equivalent integral using technology.	Area $= \int_2^1 (x^2 + 2x - 8)\,dx + \int_2^3 (x^2 + 2x - 8)\,dx$ $= 6$ square units

Note that the area of the entire region $A + B + C$ can be calculated by the single formula:

Total area $= \int_1^5 \left|x^2 - 5x + 6\right| dx$

This is the most efficient way to calculate the total area using technology.

Additional information

Areas above and below the *x*-axis

This video demonstrates how to determine the area between a curve and the x-axis when the required region lies partly below and partly above the x-axis.

Technology worked example

Determine an actual or unsigned area

EXERCISE 5.3 Calculating area under curves

Worked Example

11

1 For each of the following, calculate the signed area bounded by the curve and the x-axis, between the given values.

(a) $f(x) = x^3 - 4x$ between $x = -1$ and $x = 2$

(b) $f(x) = x^3 + 2x^2 - x - 2$ between $x = -2$ and $x = 1$

(c) $f(x) = x^2 - 6x + 4$ between $x = -1$ and $x = 2$

(d) $f(x) = 4 - e^{2x}$ between $x = 0$ and $x = 1$

(e) $f(x) = 2\sin(3x)$ between $x = 0$ and $x = \frac{\pi}{2}$

(f) $f(x) = 3\cos(2x)$ between $x = 0$ and $x = \frac{\pi}{2}$

2 Each of the graphs of the following functions is either completely above or completely below the x-axis in the required interval. Determine the unsigned area bounded by the curve, the x-axis and the given values.

(a) $f(x) = \frac{1}{2x - 5}$ between $x = 3$ and $x = 6$

(b) $f(x) = -\sqrt{2x + 12}$ between $x = 0$ and $x = 2$

(c) $f(x) = 2e^{3x} - 7$ between $x = -1$ and $x = 0$

(d) $f(x) = \frac{1}{(3x + 1)^2}$ between $x = -2$ and $x = -1$

12

3 For each of the following, calculate the unsigned (actual) area bounded by the curve and the x-axis between the given values.

(a) $f(x) = x^3 - 4x$ between $x = -1$ and $x = 2$

(b) $f(x) = x^3 + 2x^2 - x - 2$ between $x = -2$ and $x = 1$

(c) $f(x) = x^2 - 6x + 5$ between $x = -1$ and $x = 2$

(d) $f(x) = 4 - 2x$ between $x = 1$ and $x = 4$

(e) $f(x) = 2\sin(3x)$ between $x = 0$ and $x = \frac{\pi}{2}$

(f) $f(x) = 3\cos(2x)$ between $x = 0$ and $x = \frac{\pi}{2}$

4 Consider the area bounded by the graph of $f(x) = x^2 - 6x + 4$, the x-axis, and the lines $x = -1$ and $x = 2$. Calculate the following, correct to 3 decimal places where relevant.

(a) the signed area

(b) the unsigned area

5 Consider the area bounded by the graph of $f(x) = 3x^3 - 4x^2 - 2$, the x-axis, and the lines $x = -1$ and $x = 2$. Calculate the following, correct to 3 decimal places.

(a) the signed area

(b) the unsigned area

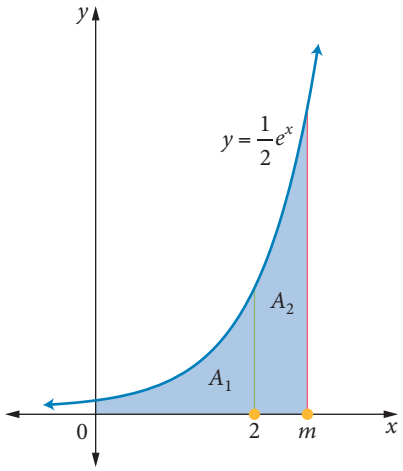
6 Determine the area of the region enclosed by the graph of $y = 2\sin\left(x - \frac{\pi}{2}\right)$, the x-axis, and the lines $x = 0$ and $x = \frac{3\pi}{2}$.

7 Determine the value of a such that the area enclosed by the graph of $y = a\cos\left(\frac{x}{3}\right)$ and the x-axis between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ is 0.374 square units. Give your answer correct to 2 decimal places.

8 Determine the exact area enclosed by the curve $f(x) = x^2(x+2)(x-4)$ and the x-axis. Give your answer correct to 1 decimal place.

9 Determine the area enclosed by the graph of $f(x) = -\frac{1}{3-x}$ and the x-axis between $x = -2$ and $x = 2$.

10 Determine the value of m for which $A_1 = A_2$ for the region bounded by the x-axis, $x = 0$, $x = m$, and the graph of $y = \frac{1}{2}e^x$ as shown. Give your answer correct to 2 decimal places.



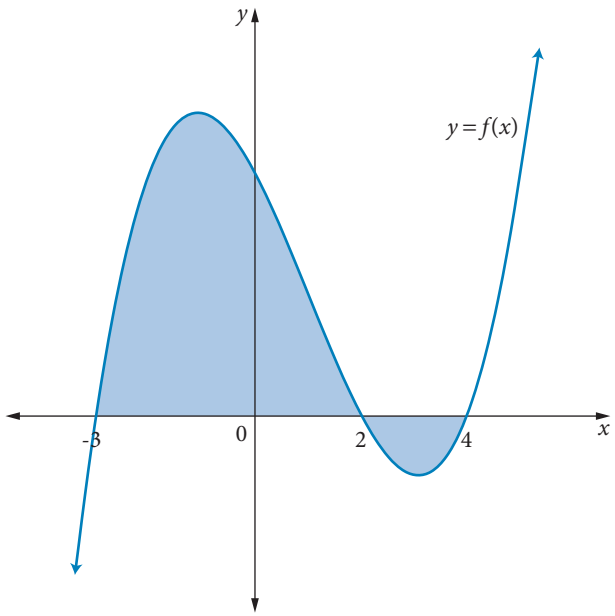
11 The signed area between the graph of $y = f(x)$ and the x-axis between $x = -3$ and $x = 4$ is given by which of the following?

A $\int_{-3}^{2} f(x)\,dx + \int_{4}^{2} f(x)\,dx$

B $\int_{-3}^{4} f(x)\,dx$

C $\int_{-3}^{2} f(x)\,dx - \int_{2}^{4} f(x)\,dx$

D $\int_{2}^{4} f(x)\,dx - \int_{-3}^{2} f(x)\,dx$

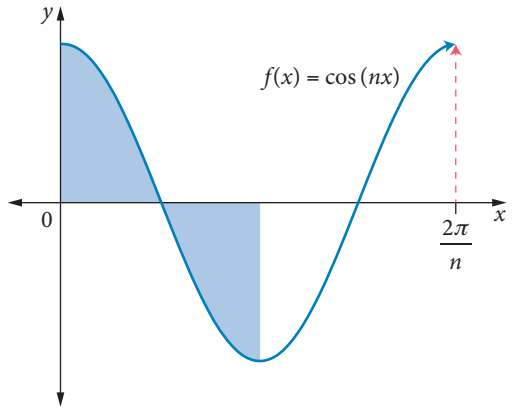


12 The graph shown represents the function $f(x) = \cos(nx)$ in the interval $\left[0, \frac{2\pi}{n}\right]$.

What is the unsigned shaded area?

A 0 **B** $-2n$

C 2 **D** $\frac{2}{n}$

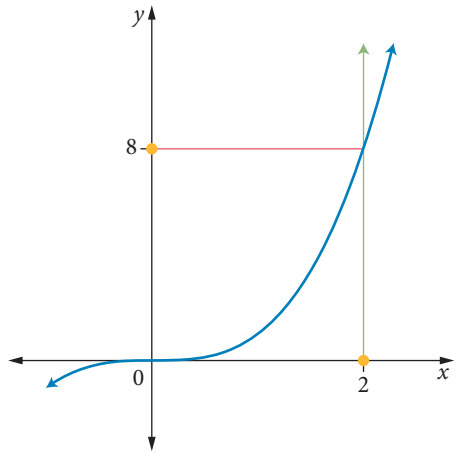


13 Determine the two values of b that satisfy $\int_b^2 \left(3 - x^3\right) dx = 0$, where $b \leq 2$. Explain what each answer represents.

14 Consider the graph of $f(x) = x^3$ between $x = 0$ and $x = 2$.

(a) Determine the area bounded by the graph of $f(x) = x^3$ and the x-axis between $x = 0$ and $x = 2$.

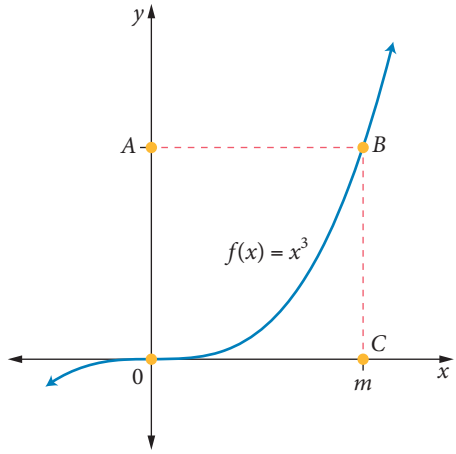
(b) Hence, determine the area bounded by the graph of $f(x) = x^3$ and the y-axis between $y = 0$ and $y = 8$.



15 Consider the graph shown.

(a) Verify that the area bounded by the graph of $f(x) = x^3$ and the y-axis between $y = 0$ and $y = m^3$ is equal to $\frac{3m^4}{4}$ square units.

(b) The graph of $f(x) = x^3$ divides the rectangle $OABC$ into two regions whose area is in the ratio p:q. If p denotes the area bounded by the y-axis and q denotes the area bounded by the x-axis, determine the values of p and q.



16 Consider the function $f(x) = \sin(4x)$.

(a) Determine the period of the function.

(b) Determine the area bounded by the graph of the function, the x-axis in the interval $\left[0, \frac{\pi}{4}\right]$.

(c) Without further integration, determine the total area between the curve and the x-axis in the interval $[6\pi, 12\pi]$.

(d) Use your answer to part **(c)** to determine the area between the graph of $g(x) = \cos(4x)$ and the x-axis in the interval $[6\pi, 12\pi]$.

17 Consider the function $f(x) = 3|x - 3|$.

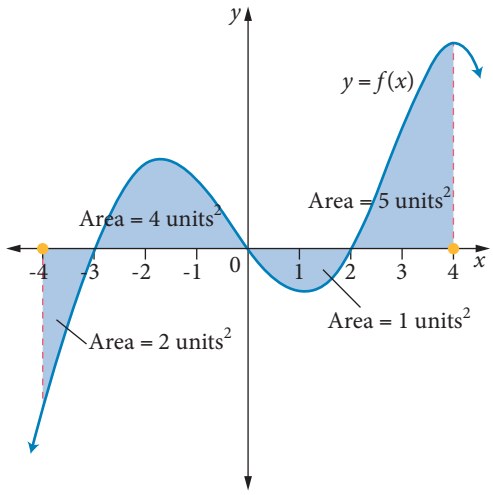
(a) Use calculus to determine the area between the graph of the function, the x-axis and the lines $x = 0$ and $x = 6$.

(b) Evaluate the reasonableness of your answer to part **(a)**, using the area of a triangle.

18 **(a)** Differentiate the function $f(x) = x \log_e(x)$.

(b) Use your answer to part **(a)** to determine the exact area between the graph of $y = \log_e(x)$, the x-axis and the lines $x = 1$ and $x = 3$.

19 The area enclosed by the graph of $y = f(x)$, the x-axis and the lines $x = -4$ and $x = 4$ is shown on the right.



(a) Calculate the value of $\int_{-4}^{2} f(x)\,dx$.

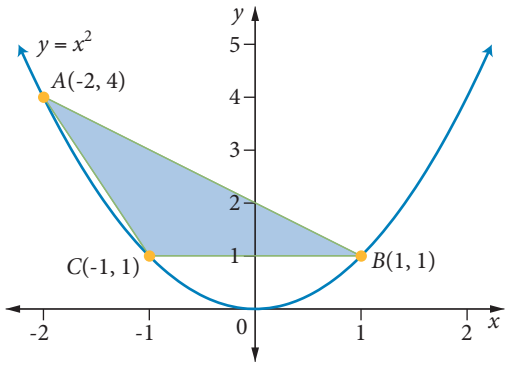
(b) Determine the area enclosed by the graph of $y = f(x)$ and the x-axis between $x = -4$ and $x = 2$.

(c) Calculate the value of $\int_{0}^{4} (-f(x))\,dx$.

(d) Calculate the value of $\int_{0}^{4} f(-x)\,dx$.

(e) Calculate the value of $\int_{-2}^{1} f'(x)\,dx$.

20 Let $f : \mathbb{R} \to \mathbb{R}$ be the quadratic function defined by the equation $f(x) = x^2$. A triangle ABC is constructed inside the parabola with vertices $A(-2,4)$, $B(1,1)$ and $C(-1,1)$.



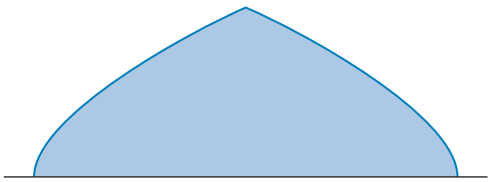
(a) Determine the equations of the line segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$.

(b) Determine the area of ΔABC using integration.

(c) Determine the area of ΔABC using geometry.

(d) Comment on the answers obtained in part **(a)** and part **(b)**.

21 Jenny is an architect who uses different shapes in her designs. She has just designed the roof of a house shaped as shown in the diagram on the right.



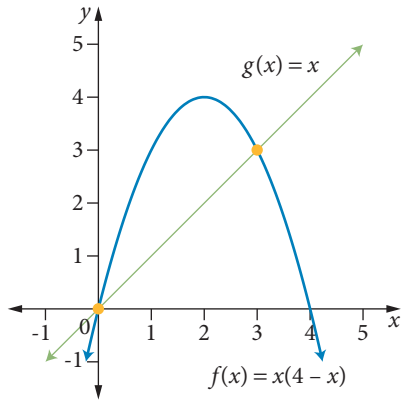
(a) Jenny used two functions to construct this roof: $f(x) = \sqrt{x}$ for $0 \le x \le 2.5$ and $g(x) = \sqrt{-x + 5}$ for $2.5 \le x \le 5$. Sketch the two functions on a Cartesian plane and calculate the exact area of the shaded region.

(b) Jenny has decided to use this shape for other features of the building she is designing. She used the functions $f_{1,2}(x) = \dfrac{\pm\sqrt{x}}{2}$ for x between 0 and 2.5, and $g_{1,2}(x) = \dfrac{\pm\sqrt{-x + 5}}{2}$ for x between 2.5 and 5. Sketch the four functions on the same set of axes and determine the area covered by the enclosed shape.

(c) Compare the two shaded areas from parts **(a)** and **(b)** and give a reason for these answers.

5.4 The area between curves

Consider the area bounded by the graphs of $f(x) = x(4 - x)$ and $g(x) = x$, as shown.



The two graphs meet at the origin. The other point of intersection can be found:

$x(4 - x) = x$

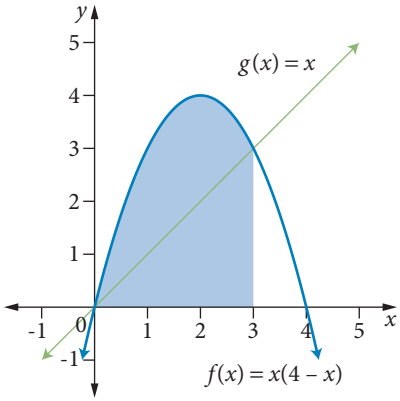
$4x - x^2 = x$

$x^2 - 3x = 0$

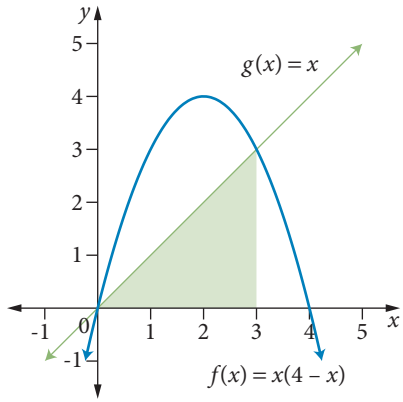
$x(x - 3) = 0$

$x = 0$ or $x = 3$

First consider the area bound by $f(x)$ and the x-axis, that is $\int_0^3 f(x)\,dx$:



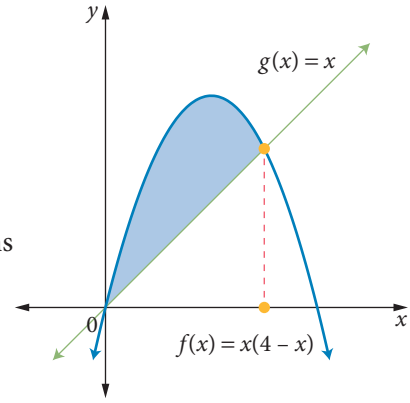
Then consider the area bound by $g(x)$ and the x-axis; that is $\int_0^3 g(x)\,dx$:



Then you can calculate the area bound by the curves:

$$\int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx = \int_0^3 \big(f(x) - g(x)\big)\,dx$$

If you subtract the area beneath the straight line specified by $g(x) = x$ from the area beneath the parabola specified by $f(x) = x(4 - x)$, you obtain the area bounded, or enclosed, by the graphs of the two functions and their points of intersection. Hence the area enclosed by the upper function $f(x)$ and the lower function $g(x)$ can be calculated:



$$\text{Area} = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx$$

The upper function simply refers to $f(x)$ being further from the x-axis than $g(x)$, which means $f(x) \geq g(x)$ in this case. This area can also be determined from $\int_0^3 \big(f(x) - g(x)\big)\,dx$.

Additional information

Area between two curves

Watch the video demonstrating how to determine the area between two curves, then complete the activity.

Making connections

Area between curves

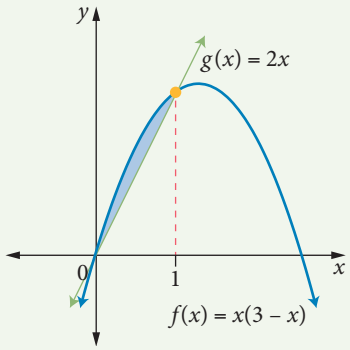
Explore the relationship between the areas of curves bound by the x-axis and the areas of curves bound by other curves.

If $f(x)$ and $g(x)$ represent two functions, where $f(x) \geq g(x)$ for all x in the interval $[a, b]$, the area enclosed by $f(x)$ and $g(x)$ between $x = a$ and $x = b$ is given by:

$$\int_a^b \big(f(x) - g(x)\big)\,dx \text{ or } \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

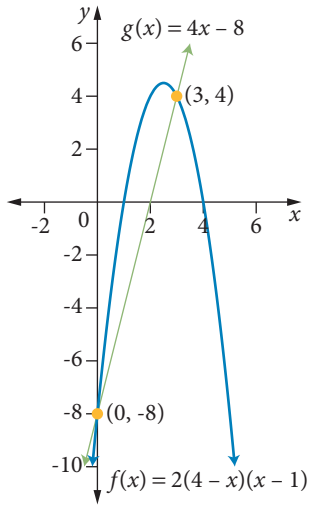
13 Area enclosed by two curves

Determine the area enclosed by the graphs of $f(x) = x(3 - x)$ and $g(x) = 2x$.

THINKING	WORKING
1 Solve $f(x) = g(x)$ to determine the x-values of the points of intersection.	$x(3 - x) = 2x$ $3x - x^2 = 2x$ $x - x^2 = 0$ $x(1 - x) = 0$ $x = 0$ or $x = 1$ The x-values of the points of intersection are at $x = 0$ and $x = 1$.
2 Sketch the graphs and identify the required region.	
3 Express the integral in the form: $\int_a^b \big(f(x) - g(x)\big)\,dx$, where $f(x)$ is the upper function, and simplify the integrand.	Area $= \int_0^1 \big(x(3 - x)\big)\,dx - \int_0^1 (2x)\,dx$ $= \int_0^1 \big(x(3 - x) - 2x\big)\,dx$ $= \int_0^1 \big(3x - x^2 - 2x\big)\,dx$ $= \int_0^1 \big(x - x^2\big)\,dx$

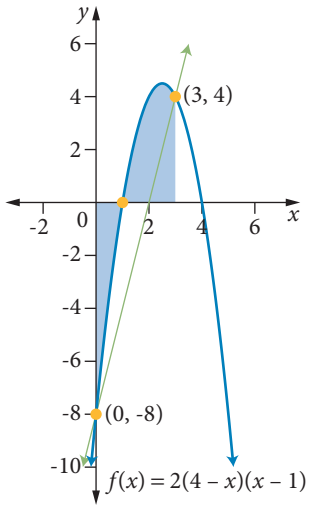
4 Calculate the value of the definite integral.	$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$ $= \left(\frac{1}{2} - \frac{1}{3}\right) - 0$
5 Interpret the answer.	The area bounded by the curves $f(x) = x(3 - x)$ and $g(x) = 2x$ is $\frac{1}{6}$ square units.

Consider the graph to determine the area enclosed by the graphs of $f(x) = 2(4 - x)(x - 1)$ and $g(x) = 4x - 8$, where part of the enclosed region is below the x-axis and part of the enclosed region is above the x-axis.



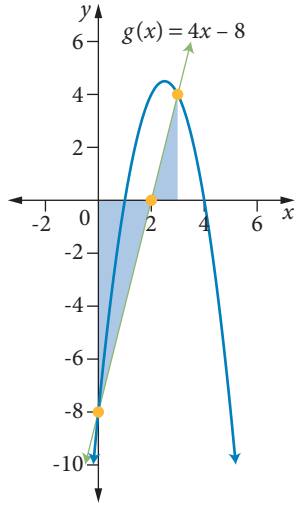
First consider the area bounded by $f(x) = 2(4 - x)(x - 1)$ and the x-axis between $x = 0$ and $x = 3$:

$$\text{Area} = -\int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx$$



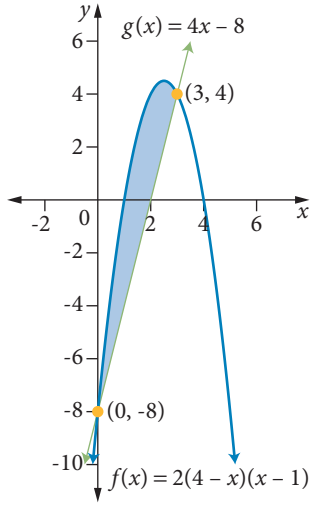
Then consider the area bounded by $g(x) = 4x - 8$ and the x-axis between $x = 0$ and $x = 3$:

$$\text{Area} = -\int_0^2 g(x)\,dx + \int_2^3 g(x)\,dx$$



Then the area bounded by the curves $f(x) = 2(4 - x)(x - 1)$ and $g(x) = 4x - 8$ can be calculated:

$$-\int_0^1 \big(g(x) - f(x)\big)\,dx + \int_1^2 f(x)\,dx - \int_1^2 g(x)\,dx + \int_2^3 \big(f(x) - g(x)\big)\,dx$$
$$= \int_0^1 \big(f(x) - g(x)\big)\,dx + \int_1^2 \big(f(x) - g(x)\big)\,dx + \int_2^3 \big(f(x) - g(x)\big)\,dx$$
$$= \int_0^3 \big(f(x) - g(x)\big)\,dx$$



Regardless of their positions on the Cartesian plane, the total area enclosed by two curves can be calculated using $\int_b^a \big(f(x) - g(x)\big)\,dx$.

Additional information

Area between graphs of two non-positive functions

This video demonstrates the calculation of the area between two non-positive functions.

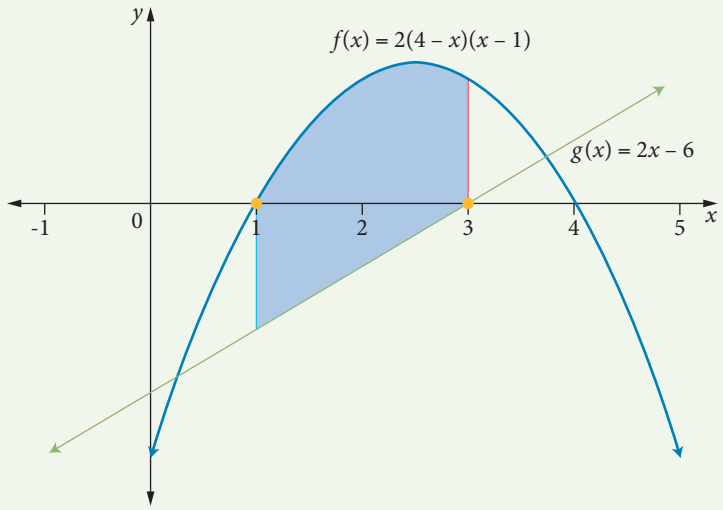
14 Area between two curves in a given interval

Determine the area enclosed by the graphs of $f(x) = 2(4 - x)(x - 1)$ and $g(x) = 2x - 6$, in the interval $[1, 3]$.

THINKING

1 Sketch the graphs and identify the required region.

WORKING



2 Express the integral in the form: $\int_a^b \big(f(x) - g(x)\big)\,dx$, where $f(x)$ is the upper function, and simplify the integrand.

$$\int_1^3 \big(2(4 - x)(x - 1) - (2x - 6)\big)\,dx$$
$$= \int_1^3 \big(-2x^2 + 10x - 8 - 2x + 6\big)\,dx$$
$$= \int_1^3 \big(-2x^2 + 8x - 2\big)\,dx$$

3 Calculate the value of the definite integral.	$= \left[-\frac{2x^3}{3} + 4x^2 - 2x \right]_1^3$ $= \left(-\frac{54}{3} + 36 - 6 \right) - \left(-\frac{2}{3} + 4 - 2 \right)$ $= -18 + 30 + \frac{2}{3} - 2$ $= \frac{32}{3}$
4 Interpret the answer.	The area bounded by the curves $f(x) = 2(4 - x)(x - 1)$ and $g(x) = 2x - 6$, in the interval $[1, 3]$ is $10\frac{2}{3}$ square units.

WARNING

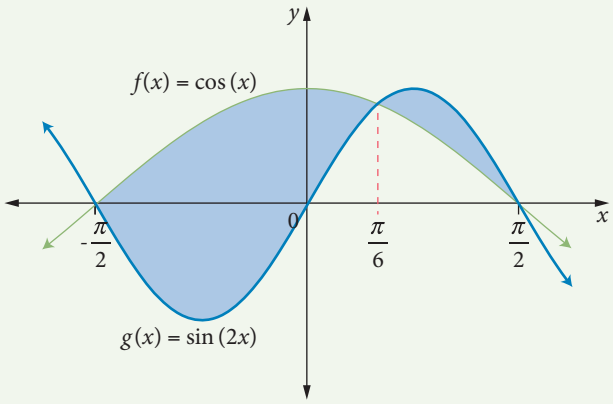
Whether working by hand or using technology, it is ideal to identify the upper function by sketching or plotting the graphs of the functions involved. If the two curves intersect within a given interval, it's possible that the upper function will become the lower function and change over the interval. Therefore, to determine the enclosed area, the required calculations for each region between the points of intersection must be separated.

15 Area between curves with an intersection in the required interval

Determine the area bounded by the graphs of $f(x) = \cos(x)$ and $g(x) = \sin(2x)$, which can also be written as $g(x) = 2\sin(x)\cos(x)$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$.

THINKING	WORKING
1 Let $f(x) = g(x)$, and solve to determine the x-values of the points of intersection in the given domain.	$\sin(2x) = \cos(x)$ $\sin(2x) - \cos(x) = 0$ $2\sin(x)\cos(x) - \cos(x) = 0$ $\cos(x)(2\sin(x) - 1) = 0$ $\cos(x) = 0$ $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ or $2\sin(x) - 1 = 0$ $\sin(x) = \frac{1}{2}$ $x = \frac{\pi}{6}$

2 Sketch the curves and identify the required regions.



3 Since the equation of the upper curve changes, integrate regions on each side of the point of intersection individually.

In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{6}\right]$ consider $f(x) = \cos(x)$ to be the upper function.

$$\begin{aligned} A_1 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\cos(x) - \sin(2x)\right) dx \\ &= \left[\sin(x) + \frac{1}{2}\cos(2x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \\ &= \left(\sin\left(\frac{\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{\pi}{3}\right)\right) - \left(\sin\left(-\frac{\pi}{2}\right) + \frac{1}{2}\cos(-\pi)\right) \\ &= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(-1 - \frac{1}{2}\right) \\ &= 2\frac{1}{4} \end{aligned}$$

4 In the interval $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ consider $g(x) = \sin(2x)$ to be the upper function.

$$\begin{aligned} A_2 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin(2x) - \cos(x)\right) dx \\ &= \left[-\frac{1}{2}\cos(2x) - \sin(x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(-\frac{1}{2}\cos(\pi) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\frac{1}{2}\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{6}\right)\right) \\ &= \left(\frac{1}{2} - 1\right) - \left(-\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{1}{4} \end{aligned}$$

5 Add the areas of the separate regions.

$$\begin{aligned} \text{Area} &= A_1 + A_2 \\ &= 2\frac{1}{4} + \frac{1}{4} \\ &= 2\frac{1}{2} \text{ square units} \end{aligned}$$

6 Interpret the answer.

The area bound by the curves $f(x) = \cos(x)$ and $g(x) = \sin(2x)$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is $2\frac{1}{2}$ square units.

Technology worked example

Determine the area between two curves

EXERCISE 5.4 The area between curves

Worked Example

13

1 Determine the area enclosed by the graphs of each of the following pairs of functions.

(a) $f(x) = x(5 - x)$ and $g(x) = x$

(b) $f(x) = x(x - 2)$ and $g(x) = 2x$

(c) $f(x) = x(x + 4)$ and $g(x) = 2x$

(d) $f(x) = x(4 - x)$ and $g(x) = 3x$

2 Determine the area bounded by the following graphs.

(a) $f(x) = 4\sqrt{x}$ and $g(x) = 2x$

(b) $f(x) = \dfrac{10}{x}$ and $g(x) = -x + 7$

14

3 For each of the following, determine the area bounded by the graphs of the given functions over the required interval.

(a) $f(x) = 2(3 - x)(x - 2)$ and $g(x) = x - 4$, in the interval $[2, 3]$

(b) $f(x) = 3(1 - x)(x + 2)$ and $g(x) = x + 1$, in the interval $[-2, 0]$

(c) $f(x) = (x - 2)(x + 2)(x + 3)$ and $g(x) = x - 2$, in the interval $[-1, 1]$

(d) $f(x) = (x - 3)(x + 4)(x - 1)$ and $g(x) = 2x - 7$, in the interval $[-3, -1]$

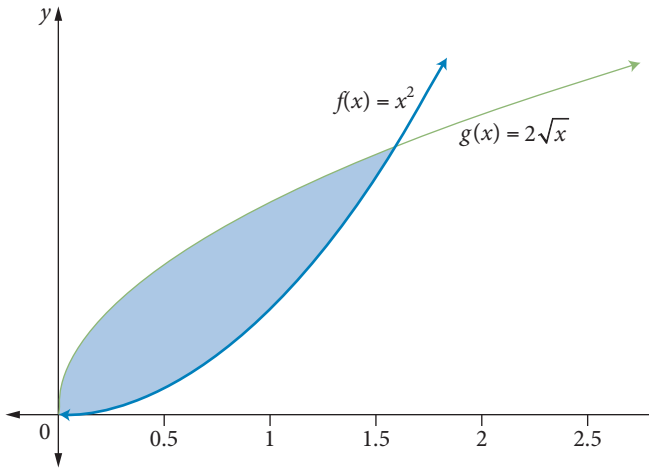
(e) $f(x) = (x + 4)(x - 6)$ and $g(x) = (x + 1)(4 - x)$, in the interval $[0, 3]$

(f) $f(x) = (x + 2)(x - 5)$ and $g(x) = (x - 4)(3 - x)$, in the interval $[1, 3]$

(g) $f(x) = \sin\left(\dfrac{x}{2}\right)$ and $g(x) = \cos(x)$, in the interval $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$

(h) $f(x) = \sin(x)$ and $g(x) = \cos(2x)$, in the interval $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$

4 Determine the exact area of the shaded region.

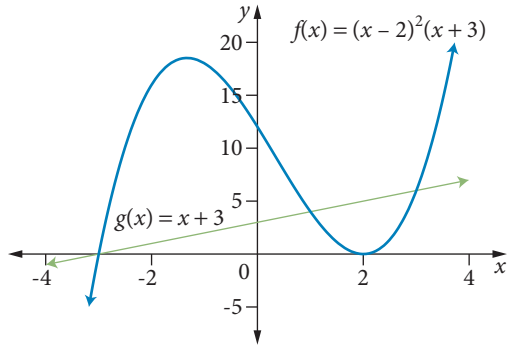


15

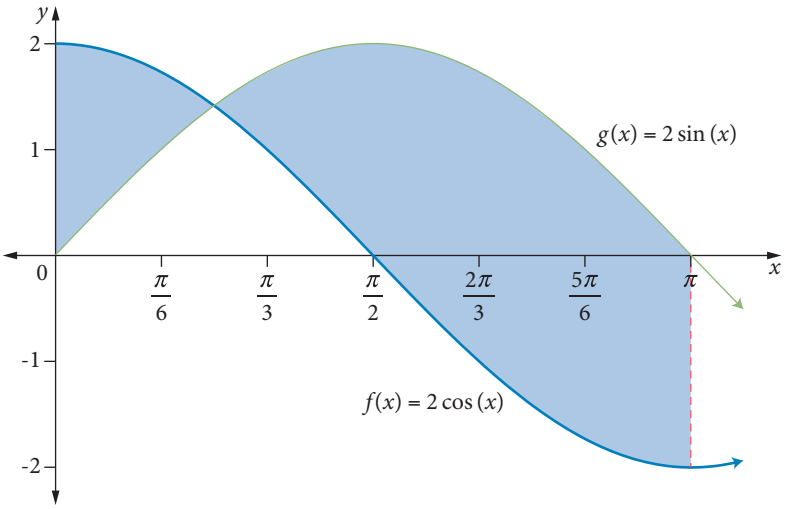
5 Consider the graph of $f(x) = (x - 2)^2(x + 3)$ and $g(x) = x + 3$ shown on the right.

(a) Determine the x-values of the points of intersection.

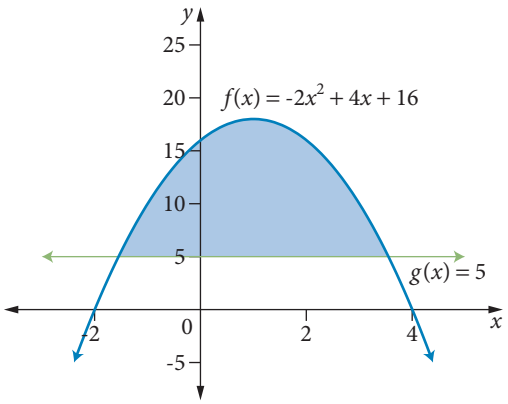
(b) Determine the exact area enclosed by the graphs.



6 Determine the area bounded by the graphs of $f(x) = 2\cos(x)$ and $g(x) = 2\sin(x)$ shown below.



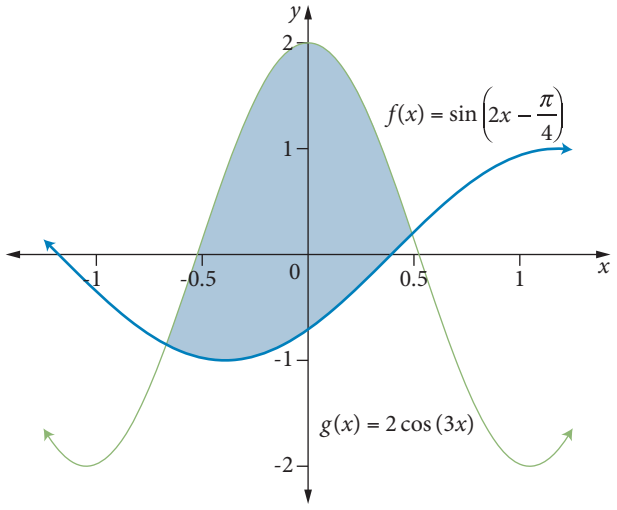
7 Determine the area bounded by the graphs of $f(x)$ and $g(x)$ shown. Give your answer correct to 2 decimal places.



8 Determine the area bounded by the graphs of $f(x) = 3x(x-2)$ and $g(x) = 4 - x^2$. Give your answers correct to 2 decimal places.

9 Calculate the area bounded by the graphs of $f(x) = e^{1-x}$, $g(x) = e^{2x}$ and the y-axis. Give your answer correct to 3 decimal places.

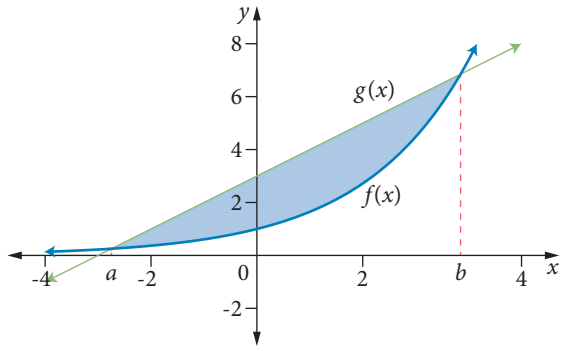
10 Calculate the area bounded by the graphs of $f(x)$ and $g(x)$, as shown. Give your answer correct to 2 decimal places.



11 Determine the area bounded by $f(x) = (x+3)(x-1)(x-3)$ and $g(x) = x^2 - x - 2$, correct to 2 decimal places.

12 Determine the area bounded by the graphs of $f(x) = 2\sin\left(\frac{x}{2}\right)$ and $g(x) = 3\cos(2x-1)$, between $x = 0$ and $x = \pi$, correct to 2 decimal places.

13 Which integral can be used to determine the shaded area shown in the graph?



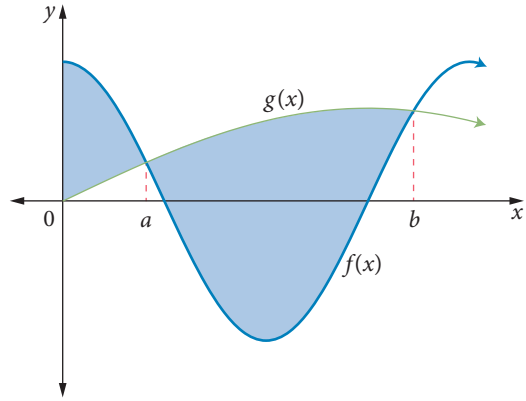
A $\int_a^b \big(f(x) - g(x)\big)\,dx$

B $\int_b^a \big(g(x) - f(x)\big)\,dx$

C $\int_b^a \big(f(x) - g(x)\big)\,dx$

D $\int_a^0 \big(g(x) - f(x)\big)\,dx + \int_0^b \big(f(x) - g(x)\big)\,dx$

14 Which integral can be used to determine the shaded area shown in the graph?



A $\int_0^b \big(f(x) - g(x)\big)\,dx$

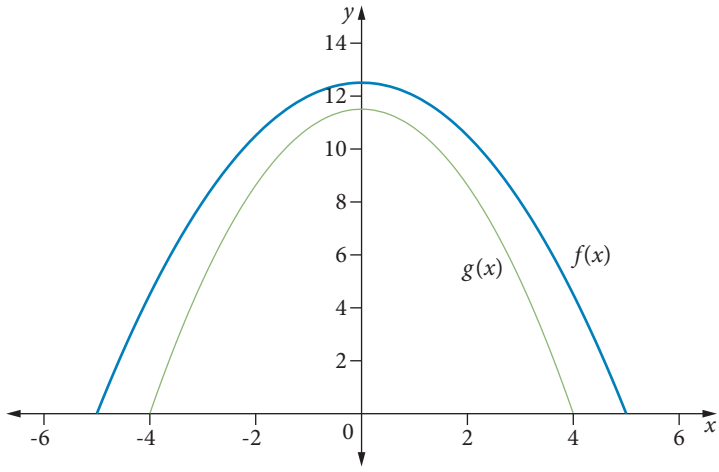
B $\int_0^a \big(f(x) - g(x)\big)\,dx + \int_a^b \big(g(x) - f(x)\big)\,dx$

C $\int_0^a \big(g(x) - f(x)\big)\,dx + \int_a^b \big(f(x) - g(x)\big)\,dx$

D $\int_0^a \big(f(x) + g(x)\big)\,dx - \int_a^b \big(g(x) - f(x)\big)\,dx$

15 Determine the exact area within the interval $[0, \pi]$ totally enclosed by the two graphs of $f(x) = \sin(x)$ and $g(x) = \cos(2x)$.

16 The outer arch of a brick railway bridge can be modelled by part of the graph of $f(x) = 0.5\left(25 - x^2\right)$, and the inner arch can be modelled by part of the graph of $g(x) = a\left(16 - x^2\right)$ as shown below.



(a) Demonstrate that if the thickness of the arch is 1 metre at the top of the bridge, then the value of a is $\frac{23}{32}$.

(b) Determine the area of the outer surface of the bricks on the arch at the entrance to the bridge.

(c) A section of paint 0.5 metres wide has flaked off on the right-hand side of the bridge, starting 2 metres from the centre of the bridge. What area of the brick surface needs to be repainted? Give your answer correct to 2 decimal places.

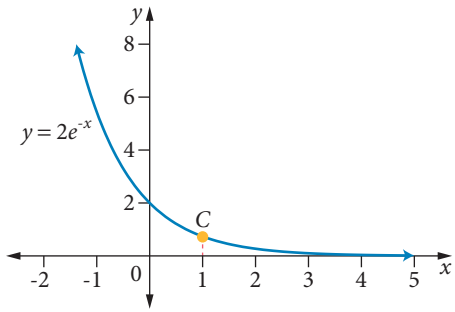
17 Demonstrate that the area bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$ is equal to one-third of the area of the unit square with its corners at the points $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

18 Consider the function $y = x^2 + 3$.

(a) Determine the equation of the tangent at the point where $x = 2$.

(b) Determine the area enclosed by the curve, the tangent and the coordinate axes, correct to 2 decimal places.

19 Determine the area bounded by the graph of $f(x) = 2e^{-x}$, the normal to the curve drawn through point C shown, and the y-axis.



20 Two walkways in a park are modelled by the functions $f(x) = e^x x^2 - 1$ and $g(x) = -x^4 + 2$.

(a) Plot the two functions on the same set of axes. Label all key features correct to 2 decimal places.

(b) The area between the two curves forms the surface area of a pond. Calculate the volume of water contained in this pond if the depth of pond is 50 cm. All measurements on the graph are in metres. Give your answer correct to 2 decimal places.

(c) There is a fence 6 m below the y-intercept of $f(x)$. The area between the two walkways and the fence is a playground. Determine the area of the playground correct to 2 decimal places.

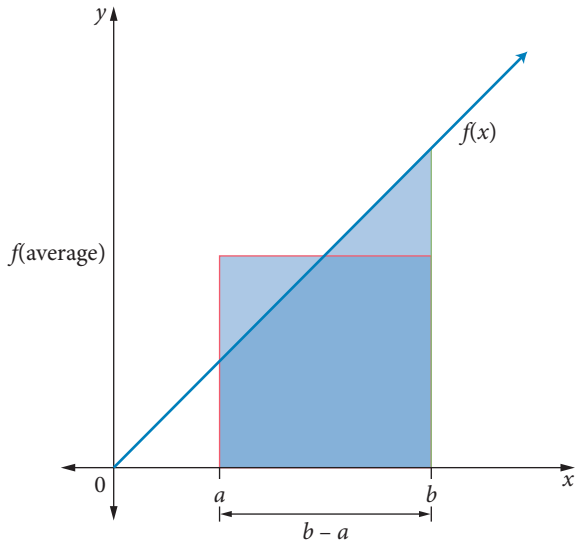
5.5 Kinematics and applications of integration

Average value of a function

The average value of a function $y = f(x)$ over the interval $[a, b]$ is given by: $f_{\text{average}} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

This formula can be transposed and rewritten as: $(b-a)\,f_{\text{average}} = \int_a^b f(x)\,dx$.

If you consider $(b-a)$ as the width of a rectangle and f_{average} as the height, then the average value of a function is the height of the rectangle that has the same width as the interval, and an area equal to the area under the function over that interval.



16 Determine the average value of a function

Determine the average value of the function $f(x) = \frac{1}{2}x^3$ over the interval $[1, 3]$.

THINKING

1 Recall the average value formula.

2 Substitute the given function and interval values into the formula.

3 Anti-differentiate the function and calculate the average value.

4 Interpret the answer.

5 Evaluate the reasonableness of the solution by drawing a graph.

WORKING

$$f_{\text{average}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

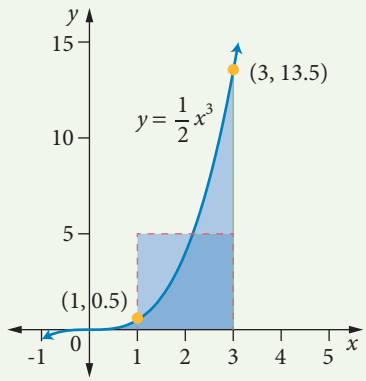
$$f_{\text{average}} = \frac{1}{3-1}\int_1^3 \left(\frac{1}{2}x^3\right)dx$$

$$= \frac{1}{2}\left[\frac{x^4}{8}\right]_1^3$$

$$= \frac{1}{2}\left(\frac{81}{8} - \frac{1}{8}\right)$$

$$= 5$$

The average value of the function $f(x) = \frac{1}{2}x$ over the interval $[1, 3]$ is 5.



Determining total change

In many practical applications, you know the rate of change and you need to calculate the total or net change over a given interval.

The net change of a function is the difference between the values at the beginning and the values at the end of an interval. Applications of this include the following.

The integral of a rate of change is equal to the total or net change.

$\int_a^b F'(x)\,dx = F(b) - F(a)$

Note: $F'(x) = f(x)$

- $\int_{t_1}^{t_2} P'(t)\,dt = P(t_2) - P(t_1)$: net change in population between times t_1 and t_2.
- $\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$: where $C'(x)$ represents the marginal cost of producing x units, and $C(x_2) - C(x_1)$ represents the change in cost that results when the production level increases from x_1 units to x_2 units.
- $\int_{t_1}^{t_2} v(t)\,dt = x(t_2) - x(t_1)$: net change in distance, or displacement, for the time from t_1 to t_2.
- $\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$: net change in velocity from t_1 to t_2, where the acceleration a is the rate of change of the velocity.

17 Determine total change in population

The rate of change of a population of rabbits at time t is found to be given by $P'(t) = 3e^{\frac{t}{2}} - 4t$, where t is the time in weeks after 1 January.

(a) Determine the rate of change of the rabbit population after 6 weeks.

THINKING	WORKING
1 Substitute the given value into the rate expression.	$P'(6) = 3e^{\frac{6}{2}} - 4(6)$ $= 3e^3 - 24$ $= 36.26$
2 Interpret the answer.	The rabbit population is increasing at a rate of approximately 36 rabbits per week after 6 weeks.

(b) Determine the change in rabbit population between week 3 and week 6.

1 Express the total change as an integral.	$\int_3^6 \left(3e^{\frac{t}{2}} - 4t\right) dt$
2 Anti-differentiate.	$= \left[6e^{\frac{t}{2}} - 2t^2\right]_3^6$
3 Interpret the expression.	The rabbit population at time t weeks after 1 January is given by $P(t) = 6e^{\frac{t}{2}} - 2t^2$.

4 Calculate the value of the definite integral.	$\left[6e^{\frac{t}{2}} - 2t^2\right]_3^6 = \left(6e^3 - 72\right) - \left(6e^{\frac{3}{2}} - 18\right)$ $= 6e^3 - 6e^{\frac{3}{2}} - 54$ $= 22.18\,(2\text{ d.p.})$
5 Interpret the answer.	Between weeks 3 and 6 there is an increase of approximately 22 rabbits.

Kinematics applications of integration

Kinematics is the study of motion without regard to the causes of motion. Let x represent the position of a particle. The rate of change of position with respect to time is the velocity v of the particle. Hence $v = x'(t)$. Since v is a function of t, this can be expressed as $v(t) = \frac{dx}{dt}$.

By the total net change theorem, $\int_{t_1}^{t_2} x'(t)\,dt = x(t_2) - x(t_1)$. Hence $\int_{t_1}^{t_2} v(t)\,dt$ gives the change in position, or displacement, from the initial position of the particle in the time interval $t_2 - t_1$.

In studying straight-line motion, movement to the right is considered positive and movement to the left is considered negative.

18 Determine the change in position

A particle moves in a straight line so that its velocity in metres per second at time t seconds is given by $v(t) = t^2 - 4t + 3$. Determine the change in position between $t = 2$ and $t = 3$.

THINKING	WORKING
1 Express the total change as an integral.	$\int_2^3 \left(t^2 - 4t + 3\right)dt$
2 Anti-differentiate to determine the equation of the position function.	$= \left[\frac{t^3}{3} - 2t^2 + 3t\right]_2^3$
3 Calculate the value of the definite integral.	$= (9 - 18 + 9) - \left(\frac{8}{3} - 8 + 6\right)$ $= 0 - \frac{2}{3}$ $= -\frac{2}{3}$
4 Interpret the answer.	Between $t = 2$ and $t = 3$, the particle moved $\frac{2}{3}$ metres in a negative direction.

WARNING

If the velocity of a particle is given by $v(t) = t^2 - 6t + 5$, this does not indicate that the particle is moving in a parabolic path. If you substitute a t-value into $v(t) = t^2 - 6t + 5$, then you will obtain the velocity at that instant.

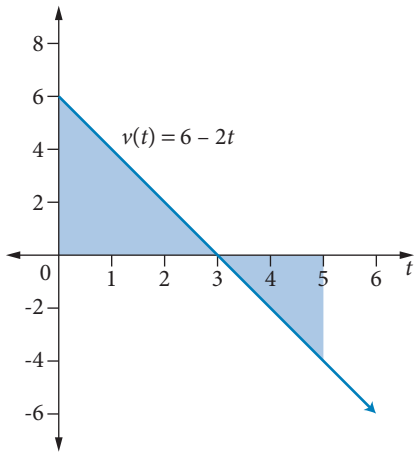
Explore further

Displacement, velocity and acceleration graphs

Explore the relationship between displacement, velocity and acceleration of a particle using graphing technology.

Determining total distance travelled

Consider the first 5 seconds of motion of an object whose velocity is given by $v(t) = 6 - 2t$.



The change in position between $t = 0$ and $t = 3$ is given by:

$$\begin{aligned}\int_0^3 (6-2t)\,dt &= \left[6t - t^2\right]_0^3 \\ &= 18 - 9 \\ &= 9\end{aligned}$$

The change in position between $t = 3$ and $t = 5$ is given by:

$$\begin{aligned}\int_3^5 (6-2t)\,dt &= \left[6t - t^2\right]_3^5 \\ &= (30 - 25) - (18 - 9) \\ &= -4\end{aligned}$$

In the first 3 seconds, the object travels a distance of 9 metres in a positive direction. Between 3 and 5 seconds, the object travels a distance of 4 metres in the opposite, or negative, direction. Hence the object has a displacement of 5 metres, given by $\int_0^5 (6-2t)\,dt = 5$, and has travelled a total distance of 13 metres, given by $\int_0^3 (6-2t)\,dt + \left|\int_3^5 (6-2t)\,dt\right| = \int_0^5 |6-2t|\,dt$.

$\int_{t_1}^{t_2} v(t)\,dt = x(t_2) - x(t_1)$ represents the net change in distance, or displacement, for the time from t_1 to t_2. Hence the signed area under the velocity–time graph between t_1 and t_2 represents the object's displacement between these two times.

$\int_{t_1}^{t_2} |v(t)|\,dt$ represents the total distance travelled between times t_1 and t_2.

19 Determine the change in position and distance travelled

A particle moves in a straight line so that its velocity in metres per second after t seconds is given by $v(t) = t^2 - 6t + 5$.

(a) Determine the change in position between $t = 0$ and $t = 3$.

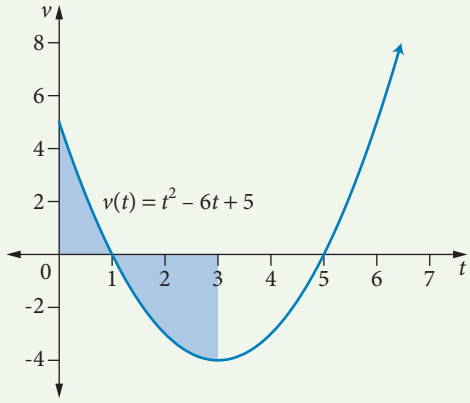
THINKING	WORKING
1 Express the change as an integral.	$\int_0^3 \left(t^2 - 6t + 5\right)dt$
2 Anti-differentiate the velocity function to obtain the position function and calculate the value of the definite integral.	$= \left[\frac{t^3}{3} - 3t^2 + 5t\right]_0^3$ $= \left(\frac{3^3}{3} - 3 \times (3)^2 + 5(3)\right) - (0)$ $= (9 - 27 + 15) - 0$ $= -3$
3 Interpret the answer.	The particle is 3 metres to the left of its initial position.

(b) Determine the total distance travelled in the first 3 seconds of motion.

1 Express the distance travelled as an integral.

$\int_0^3 \left| t^2 - 6t + 5 \right| dt$

2 Sketch the graph of $v(t)$ and shade the required area.



3 Determine the area bounded by the curve and the t-axis between the given time values.

$$\begin{aligned}\text{Area} &= \int_0^1 \left(t^2 - 6t + 5\right) dt - \int_1^3 \left(t^2 - 6t + 5\right) dt \\ &= \left[\frac{t^3}{3} - 3t^2 + 5t\right]_0^1 - \left[\frac{t^3}{3} - 3t^2 + 5t\right]_1^3 \\ &= \left(\left(\frac{1}{3} - 3 + 5\right) - 0\right) - \left((9 - 27 + 15) - \left(\frac{1}{3} - 3 + 5\right)\right) \\ &= 2\frac{1}{3} - \left(-5\frac{1}{3}\right) \\ &= 7\frac{2}{3}\end{aligned}$$

4 Interpret the answer.

The particle has travelled a distance of $7\frac{2}{3}$ metres.

Determine position, given initial values

Velocity is the rate of change of position with respect to time, hence $v = \frac{dx}{dt} = x'(t)$.

$\int_{t_1}^{t_2} x'(t)\,dt = x(t_2) - x(t_1)$ gives the change in position between t_1 and t_2.

Acceleration measures the rate of change of velocity with respect to time, hence $a = \frac{dv}{dt} = v'(t)$.

$\int_{t_1}^{t_2} v'(t)\,dt = v(t_2) - v(t_1)$ gives the net change in velocity between t_1 and t_2.

To determine the position of an object, given the initial values of velocity and acceleration:

STEP 1: Express velocity v as a function of time t.

STEP 2: Then calculate the net change in position.

20 Determine position, given initial values of velocity and acceleration

An object is dropped and falls with a constant vertical acceleration of $9.8\,\text{m/s}^2$. How far does the object fall in the first 4 seconds?

THINKING	WORKING
1 Recall the formula to express the velocity as a function of time.	$v = \int_{t_1}^{t_2} v'(t)\,dt$
2 Substitute the known values.	$v = \int_0^t 9.8\,dt$ $= [9.8t]_0^t$ $= 9.8t$
3 Determine the change in position using $\int_{t_1}^{t_2} x'(t)\,dt = x(t_2) - x(t_1)$.	$\int_0^4 9.8t\,dt = \left[4.9t^2\right]_0^4$ $= 4.9 \times 4^2 - 4.9 \times 0^2$ $= 4.9 \times 16 - 0$ $= 78.4\,\text{m}$
4 Interpret the answer.	The object falls a total distance of 78.4 metres in the first 4 seconds.

21 Determine the position of a projected object

A rocket is fired vertically from ground level with an initial velocity of 35 m/s. The rocket undergoes a constant vertical deceleration due to gravity of $9.8\,\text{m/s}^2$. Determine the height of the rocket after 30 seconds.

METHOD 1

THINKING	WORKING
1 Recall the formula to express the velocity as a function of time.	$v = v_0 + \int_{t_1}^{t_2} v'(t)\,dt$
2 Substitute the known values.	$v = 35 + \int_0^t -9.8\,dt$ $= 35 + [-9.8t]_0^t$ $= 35 - 9.8t$
3 Determine the change in position using $\int_{t_1}^{t_2} x'(t)\,dt = x(t_2) - x(t_1)$.	$\int_0^3 (35 - 9.8t)\,dt = \left[35t - 4.9t^2\right]_0^3$ $= \left(35(3) - 4.9(3)^2\right) - 0$ $= 60.9\,\text{m}$
4 Interpret the answer.	After 30 seconds of travel, the rocket is 60.9 metres above the ground.

METHOD 2

1 Recall the formula to express the velocity as a function of time.	$v = \int v'(t)\,dt$
2 Substitute the known values.	$v = \int -9.8\,dt$ $= -9.8t + c_1$
3 Substitute initial values of t and v to determine the value of the constant of integration.	$t = 0,\ v = 35$ $35 = -9.8(0) + c_1$ $c_1 = 35$
4 Write the velocity equation.	$x'(t) = -9.8t + 35$
5 Determine the change in position using $\int_{t_1}^{t_2} x'(t)\,dt = x(t_2) - x(t_1)$.	$\int_0^3 (35 - 9.8t)\,dt = \left[35t - 4.9t^2\right]_0^3$ $= \left(35(3) - 4.9(3)^2\right) - 0$ $= 60.9$ m
6 Interpret the answer.	After 30 seconds of travel, the rocket is 60.9 metres above the ground.

EXERCISE 5.5 Kinematics and applications of integration

Worked Example 16

1 Determine, in exact form, the average value of each function over the given interval.

(a) $f(x) = 6x^2$; $[1, 4]$

(b) $f(x) = \sqrt{x}$; $[0, 9]$

(c) $f(x) = (2x - 1)^3$; $[-1, 2]$

(d) $f(x) = e^{2x}$; $[1, 3]$

2 The temperature T in degrees Celsius, t hours after 8:00 am in the City of Bendigo in November, was modelled by the function $T(t) = 23 + 4\sin\left(\frac{12\pi}{t}\right)$. Determine the average temperature predicted by this model between 10:00 am and 3:00 pm. Give your answer correct to 1 decimal place.

3 The amount D in milligrams of a drug in the bloodstream, t hours after being administered, is given by $D = 0.5e^{-0.2t}$.

(a) What is the initial dose of the drug?

(b) What is the average amount of the drug present in the bloodstream during the first 3 hours after the drug is administered? Give your answer correct to 2 decimal places.

4 The velocity v, in metres per second, of a falling object t seconds after it is released is given by $v(t) = 5 + 9.8t$. Determine the average velocity for $2 \leq t \leq 5$.

5 An object moves in a straight line so that its velocity, in metres per second, after t seconds is given by $v(t) = 3t^2 + 4t - 1$. 18

(a) Determine the initial velocity of the object.

(b) Determine the change in position of the object in the first 3 seconds.

(c) Determine the change in position of the object from $t = 2$ to $t = 4$ seconds.

6 A valve on a water tank is opened at time $t = 0$ minutes. Water flows out of the tank at a rate given by $v = v_0 + \int_{t_1}^{t_2} v'(t)\,dt$ litres per minute. If $V'(t) = -200e^{-\frac{t}{20}}$, determine, to the nearest litre, the volume of water poured out of the tank for the given time period.

(a) Determine the volume of water poured out of the tank in the first 5 minutes. Give your answer to the nearest litre.

(b) Determine the volume of water poured out of the tank in the second 5-minute period. Give your answer to the nearest litre.

7 The velocity of a mass oscillating on the end of a spring is given by $v(t) = \cos(t)$ m/s. Determine the distance travelled by the mass in the first $\frac{\pi}{2}$ seconds.

8 A company discharges toxic waste into a storage vat at a rate of $\frac{dV}{dt} = 60t(2 - t)$ L/min over a period of 2 minutes. Calculate the total amount of toxic waste discharged. 17

9 A particle moves in a straight line so that its velocity, in metres per second, after t seconds is given by $v(t) = 2t^2 - 8t + 6$. 19

(a) Determine the change in position between $t = 0$ and $t = 4$.

(b) Determine the total distance travelled in the initial 4 seconds.

10 A particle moves in a straight line, so that its velocity, in metres per second, at time t is given by $v(t) = 2\sin\left(\frac{t}{3}\right) - t^2$. Give your answer correct to 2 decimal places.

(a) Determine the displacement, in metres, in the first 2 seconds of motion.

(b) Determine the total distance travelled in the first 2 seconds of motion.

11 An object is dropped and falls with a constant vertical acceleration of 9.8 m/s^2. Determine how far the object falls in the first 3 seconds. 20

12 A rocket is fired vertically from ground level with an initial velocity of 20 m/s. The rocket undergoes a constant vertical deceleration due to gravity of 9.8 m/s^2. Determine the height of the rocket after 4 seconds. Give your answer correct to 1 decimal place. 21

13 A body starts from the origin and moves in a straight line with a velocity given by $x'(t) = (12 - 2t)$ m/s. After how many seconds is the object 9 m to the right of its starting position? Give your answer correct to 2 decimal places.

14 The rate of increase of the weight in kilograms per week of a Hereford cow is represented by $w'(t)$. The weight gained between the beginning of week 4 and the end of week 8 is given by which expression?

A $\int_4^8 w(t)\,dt$ B $\int_3^8 w(t)\,dt$ C $\int_3^8 w'(t)\,dt$ D $\int_4^8 w'(t)\,dt$

15 A particle initially moving at 8 m/s is subject to a constant retardation of 4 m/s^2. Determine the distance travelled before the particle comes to rest.

16 A particle moves in a straight line so that its velocity, in metres per second, at time t is given by $v(t) = 2\cos\left(\frac{t}{2}\right) + 1$.

(a) Verify that the change in position of the particle between $t = \frac{\pi}{2}$ and $t = 2\pi$ seconds is $\left(\frac{3\pi}{2} - 2\sqrt{2}\text{ m}\right)$.

(b) Determine the total distance travelled between $t = \frac{\pi}{2}$ and $t = 2\pi$ seconds. Give your answer correct to 2 decimal places.

17 After t seconds, the acceleration of an object moving in a straight line is given by $(10 - 4t)\,\text{m/s}^2$. Initially the displacement of the object is -3 m from a fixed origin (i.e. 3 m left of the origin) and has a velocity of -8 m/s.

(a) Determine the velocity v of the object at any time t.

(b) Determine the velocity of the object at the instant when its acceleration is zero. Give your answer correct to 1 decimal place.

(c) Determine the position of the object at the instant when its acceleration is zero. Give your answer correct to 2 decimal places.

(d) Determine the times when the object is instantaneously at rest.

(e) Determine the position of the object at the times when the object is instantaneously at rest.

(f) Determine the distance travelled in the first 4 seconds.

18 A car is travelling at 30 m/s when the driver notices a fallen branch on the road ahead. He immediately slows down in order to navigate safely around the branch before gradually speeding up to resume his original velocity. For the period of time when his velocity is less than 30 m/s, his acceleration in m/s^2 is given by $a(t) = \frac{3}{4}t - 5$. The car passes the fallen branch 7 seconds after the driver first becomes aware of it.

(a) Determine the velocity of the car when it passes the fallen branch.

(b) Calculate the distance travelled by the car until it reaches the fallen branch, correct to 3 decimal places.

(c) How long will it take the car, from the moment it passes the fallen branch, to reach its initial velocity of 30 m/s? Give your answer correct to 2 decimal places.

(d) Sketch a velocity–time graph for this motion.

19 The velocity, in metres per second, of a particle moving in a straight line with variable acceleration is described by the equation $v(t) = -4t^2 + 12t + 27$, where t is the time in seconds measured from the beginning of the particle's motion, and the motion starts at the origin.

(a) Determine the displacement of the particle when its acceleration is zero.

(b) Determine the position of the particle when its velocity is zero. Give your answer correct to 1 decimal place.

(c) Determine the acceleration of the particle at $t = 4.5$ and interpret its meaning.

20 The acceleration, in m/s^2, of an object moving in a straight line is given by the equation $a(t) = -5e^{0.5t}$, where t is the time in seconds. Initially the object is 120 m to the right of the origin and is moving to the right at 20 m/s. Give your answers correct to 1 decimal place, where necessary.

(a) Determine an expression for the velocity v of the object at any time t.

(b) Determine an expression for the displacement of the object, in x metres at any time t.

(c) Determine the time when the object becomes stationary and the distance travelled from the beginning of the motion at that time.

21 The velocity, in metres per second, of a cyclist travelling in a straight line is given by the equation $v(t) = 7 - t + \sqrt{5t + 4}$, where t is the time in seconds, until the cyclist becomes stationary.

(a) Determine the initial velocity and position x of the cyclist if her position at $t = 1$ is $x = 12.9$.

(b) Determine the time when the cyclist comes to rest and the distance travelled from the beginning of the motion. Give your answer correct to 1 decimal place.

(c) Determine the time when the cyclist reaches a maximum speed and the value of the maximum speed. Give your answer correct to 2 decimal places. Explain if this is a local maximum or an absolute maximum.

22 A toy rocket is launched straight up with a speed of 10 m/s. The acceleration acting on the rocket is the acceleration due to gravity, $g = -9.8\,\text{m/s}^2$.

(a) Determine the velocity function, $v(t)$, of the rocket if the acceleration due to gravity as a function of time is $g(t) = -9.8$ m/s.

(b) Determine the position function $x(t)$ of the toy rocket.

(c) Determine the time it takes the toy rocket to reach its maximum height and the value of this maximum height. Give your answer correct to 2 decimal places.

(d) What is the velocity and the position of the toy rocket 2 seconds after the beginning of the motion?

23 Ryan is a 1.9-metre-tall professional stuntman. He is going to bungee jump from a platform 46 metres above sea level. A strong cord is safely attached to his body harness, clipping at his ankles. Answer the following questions, assuming air resistance is negligible. Give your answers correct to 1 decimal place, where necessary.

(a) Determine the maximum length of the cord that holds Ryan such that, when it first becomes taut, the top of Ryan's head just misses the water flowing in the river below, and the time at which this occurs.

(b) For safety reasons, the length of the cord, when taut, is 35 m. Determine the time at which the cord first becomes taut and Ryan's velocity at this moment.

(c) Calculate the distance travelled and the change in velocity between $t = 1$ s and $t = 2$ s.

24 Viv is standing on the top of a 40-metre tower, conducting experiments on objects falling under the constant acceleration due to gravity. Take the downward acceleration due to gravity as $9.8\,\text{m/s}^2$, and give all answers correct to 1 decimal place.

(a) Viv drops an object from the top of the tower. How far above the ground is the object after 2 seconds?

(b) Viv now throws an object upwards with a speed of u m/s.

(i) Verify that the velocity, in m/s, of the object is $v(t) = u - 9.8t$.

(ii) The object strikes the ground after 4 seconds. Determine the initial speed of the object.

(iii) What distance had the object travelled during its voyage to the ground?

(c) Viv now throws an object downwards and it strikes the ground 2 seconds later. With what speed did Viv throw the object?

(d) Viv now goes to the top of a tall building. She fires two objects at the same instant, one upwards and one downwards, both with the same speed. The object fired upwards strikes the ground after 4 seconds and the object fired downwards strikes the ground after 2 seconds.

(i) Determine the speed with which Viv fired the two objects.

(ii) Determine the height of the building.

5

Summary

Area estimates

Left endpoint

The left-endpoint approximation takes the y-value on the left-hand side of the rectangle as the height of the rectangle.

Left-endpoint area estimate: $S_L = \frac{b-a}{n}\left(f(a)+f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_{n-1})\right)$

$S_L = \sum_{i=0}^{n-1}\left(\frac{b-a}{n}\right)f(x_i)$, where $a = x_0$

Right endpoint

The right-endpoint approximation takes the y-value on the right-hand side of the rectangle as the height of the rectangle.

Right-endpoint area estimate: $S_U = \frac{b-a}{n}\left(f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_{n-1})+f(b)\right)$

$S_U = \sum_{i=1}^{n}\left(\frac{b-a}{n}\right)f(x_i)$, where $b = x_n$

$$\text{average approximate area} = \frac{\text{left-endpoint approximation} + \text{right-endpoint approximation}}{2}$$

$$= \frac{S_L + S_U}{2}$$

Midpoint

The midpoint approximation takes the y-value from the middle of the rectangles as the height of the rectangle.

Midpoint area estimate: $S_M = \left(\frac{b-a}{n}\right)\left(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+\ldots+f(\bar{x}_n)\right)$

$S_M = \sum_{i=1}^{n}\left(\frac{b-a}{n}\right)f(\bar{x}_i)$

Trapezoidal rule

Another way of estimating the area between a curve and the x-axis in an interval is to divide the interval into trapeziums of equal width. The rule for the area of a trapezium can then be used and the total calculated to give an estimate of the area.

Area of a trapezium $= \frac{1}{2}$ (sum of parallel sides) $\times$ distance between them

The fundamental theorem of calculus

The definite integral of $f(x)$ between $x = a$ and $x = b$ can be written as $\int_a^b f(x)dx$.

The definite integral is calculated by substituting $x = b$ (upper limit or terminal) and $x = a$ (lower limit or terminal) into the expression $F(b) - F(a)$, where $F(x)$ is the anti-derivative of $f(x)$.

In summary: $\int_a^b f(x)dx = \left[F(x)\right]_a^b$
$= F(b) - F(a)$

Properties of definite integrals

$\int_b^a f(x)dx = -\int_a^b f(x)dx$

$\int_a^b kf(x)dx = k\int_a^b f(x)dx$

$\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

$\int_a^a f(x)dx = 0$

If b lies between a and c, then $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.

The area bounded by a curve and the x-axis

The area above the x-axis bounded by a curve between the upper terminal $x = b$ and lower terminal $x = a$ is given by:

$\text{Area} = \int_a^b f(x)dx$

The area below the x-axis bounded by a curve and the x-axis between $x = b$ and $x = a$ is given by:

$-\int_a^b f(x)dx = \int_b^a f(x)dx$
$= \left|\int_a^b f(x)dx\right|$

Signed area: Calculate the value of $\int_a^b f(x)dx$, without regard to whether $f(x)$ sits above or below the x-axis. If the result is negative, this indicates that over the interval $[a, b]$ a greater area lies below the x-axis than above the x-axis over.

Unsigned area: To determine the actual or unsigned area, a sketch must be drawn to determine if any of the required area lies below the x-axis. The magnitude of each region is added to obtain the actual or unsigned area. When the goal is simply to calculate the 'area', the actual or unsigned area is required.

The area between two curves

If $f(x) \geq g(x)$ for all values of x in the interval $[a, b]$; that is, $f(x)$ is above $g(x)$, the area enclosed by $f(x)$ and $g(x)$ between the lower limit $x = a$ and the upper limit $x = b$ is given by: $\int_a^b \left(f(x) - g(x)\right)dx$.

If $f(x)$ and $g(x)$ intersect within the region $[a, b]$, the function that is the upper function changes over the interval. Therefore, to determine the enclosed area, separate calculations are required for each part of the region.

Average value of a function

The average value of a function $y = f(x)$ over an interval $[a, b]$ is given by: $f_{\text{average}} = \dfrac{1}{b-a}\int_a^b f(x)dx$.

Determining total change

The integral of a rate of change is equal to the net or total change.

$\int_a^b F'(x)dx = F(b) - F(a)$

Note: $F'(x) = f(x)$

$\int_{t_1}^{t_2} P'(t)dt = P(t_2) - P(t_1)$ gives the net change in population between times t_1 and t_2, where $P'(t)$ represents the rate of growth of the population at time t.

$\int_{x_1}^{x_2} C'(x)dx = C(x_2) - C(x_1)$, where $C'(x)$ represents the marginal cost of producing x units, and $C(x_2) - C(x_1)$ represents the change in cost that results when the production level increases from x_1 units to x_2 units.

Kinematics applications

Let x represent the position of a particle. The rate of change of position is the velocity v of the particle. Hence $v = x'(t)$. Since v is a function of t, this can be expressed as:

$$v(t) = \frac{dx}{dt}$$
$$= x'(t)$$

$\int_{t_1}^{t_2} x'(t)dt = x(t_2) - x(t_1)$ gives net change in distance, or displacement, for the time from t_1 to t_2.

$\int_{t_1}^{t_2} |v(t)|dt$ represents the total distance travelled between times t_1 and t_2.

Acceleration measures the rate of change of velocity, hence:

$$a = \frac{dv}{dt}$$
$$= v'(t)$$

$\int_{t_1}^{t_2} v'(t)dt = v(t_2) - v(t_1)$ gives the net change in velocity between t_1 and t_2, where the acceleration $v'(t)$ is the rate of change of velocity.

To determine the position of an object given the initial values of velocity and acceleration, first express velocity v as a function of time t, then find the net change in position.

Gravity causes a downward acceleration of $9.8\ \text{m/s}^2$; therefore, $g = -9.8\ \text{m/s}^2$.

Chapter review 5

1 Consider the area enclosed by the curve $f(x) = \sin(x)$ and the x-axis between $x = \frac{\pi}{4}$ and $x = \pi$. Determine the approximate enclosed area using the following criteria. Give your answers in exact form. Exercise 5.1

(a) three rectangles with the left-endpoint method

(b) three rectangles with the right-endpoint method

(c) the average of the left-endpoint and right-endpoint methods

2 Calculate the value of each of the following definite integrals. Give your answers in exact form. Exercise 5.2

(a) $\int_{-1}^{3}\left(2x^2 - 3x + 1\right)dx$ **(b)** $\int_{\frac{\pi}{2}}^{\pi}\left(-2\cos\left(\frac{x}{3}\right)\right)dx$ **(c)** $\int_{\frac{\pi}{3}}^{\pi}\left(3\sin 2\left(x + \frac{\pi}{6}\right)\right)dx$

(d) $\int_{-1}^{1}(3x - 4)^3\,dx$ **(e)** $\int_{0}^{\frac{\pi}{2}}\left(\sin(3x) - x^2\right)dx$ **(f)** $\int_{1}^{2}\frac{1}{x^2}\,dx$

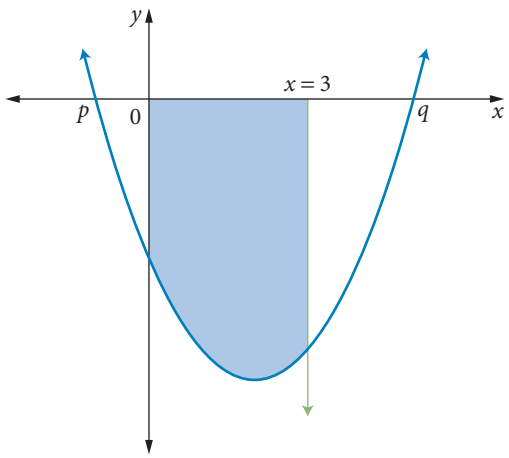
3 Determine $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(f(x) - \sin(2x)\right)dx$, given $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x)\,dx = 2$. Exercise 5.2

4 Consider the area enclosed by the graph of $f(x) = x(2 - x)$ and the x-axis between $x = 0$ and $x = 3$. Determine the following. Exercise 5.3

(a) the signed area **(b)** the unsigned area

5 Determine the exact area enclosed by the graph of $y = e^{2x}$, the coordinate axes and the line $x = 3$. Exercise 5.3

6 The graph of $f(x) = x^2 + bx - 5$ is shown. If the shaded area is 24 square units, determine the values of b, p and q. Exercise 5.3



7 Determine the exact area enclosed by the curves $y_1 = x(8 - x)$ and $y_2 = 2x$. Exercise 5.4

8 Determine the exact area bounded by the graphs of $f(x) = x(x - 1)(x + 4)$ and $g(x) = x - 7$ between $x = 0$ and $x = 3$. Exercise 5.4

Exercise 5.5 **9** The rice consumed in a province is given by $R(t) = 1.7 + 1.09^t$, where $R(t)$ represents millions of tonnes per year, and t is the time in years since the beginning of 1980.

(a) Write an integral that would give the total consumption of rice from the beginning of 1985 until the end of 1988.

(b) Calculate the total consumption of rice from the beginning of 1985 until the end of 1988, correct to 2 decimal places.

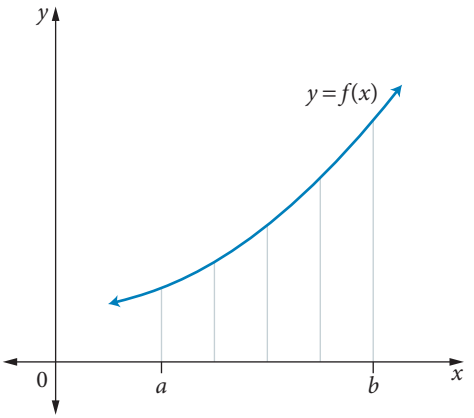
Exercise 5.5 **10** A particle moves in a straight line so that its velocity at time t, where $t \geq 0$, is given by $v(t) = t^2 - 6t + 5$ m/s.

(a) Determine the change in position between $t = 0$ and $t = 3$ seconds.

(b) Determine the exact total distance travelled in the first 3 seconds.

Exercise 5.5 **11** A rock is dropped from the top of a 45-metre-high cliff and falls with a constant vertical acceleration of 9.8 m/s^2. After 2 seconds, how high is the rock above the ground?

Exercise 5.1 **12** Which one of the following expressions can be used to approximate the area bounded by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$, as shown, using the right-endpoint method with n intervals?



A $\frac{b-a}{n}\left(f(x_1) + f(x_2) + ... + f(x_n) + f(b)\right)$

B $\frac{b-a}{n}\left(f(a) + f(x_1) + f(x_2) + ... + f(x_n) + f(b)\right)$

C $\frac{b-a}{n}\left(f(a) + f(x_1) + f(x_2) + ... + f(x_n)\right)$

D $\frac{b-a}{n}\left(x_1 + x_2 + ... + x_n + b\right)$

Exercise 5.1 **13** Which expression represents the area under the curve $y = f(x)$ between $x = a$ and $x = b$, approximated using the right-endpoint method with intervals of width 1 unit? Assume $b - a$ is a whole number value and the curve is above the x-axis in the region $[a, b]$.

A $\sum_{a}^{b} f(x)$ **B** $\sum_{a}^{b-1} f(x)$ **C** $\sum_{a+1}^{b} f(x)$ **D** $\sum_{a+1}^{b+1} f(x)$

Exercise 5.1 **14** Determine the approximate area under the curve with equation $y = e^{-x} + \sin(x)$ between $x = 0$ and $x = \pi$, using the left-endpoint method with intervals of width $\frac{\pi}{2}$.

Exercise 5.2 **15** Given $\int_{-3}^{2} f(x)\,dx = -3$, calculate the value of $\int_{-3}^{2}\left(f(x) - 2x^2\right)dx$.

Exercise 5.2 **16** If $\int_{1}^{3} f(x)\,dx = 5$, calculate the value of $\int_{3}^{1}\left(4f(x) - 2\right)dx$.

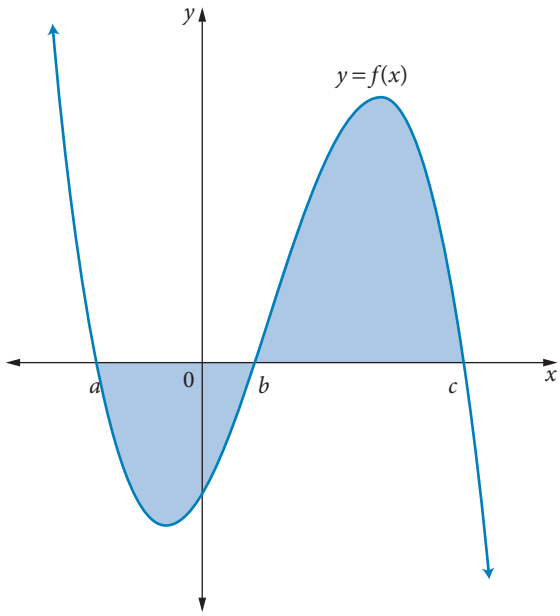
Exercise 5.2 **17** Given that $\int_{-1}^{2}(ax + 2)^2\,dx = 57$, determine the value of a, where $a > 0$.

Exercise 5.2 **18** Determine the function $f(x)$, such that $f'(x) = 2\cos\left(\frac{x}{3}\right)$ and $f\left(\frac{\pi}{2}\right) = 1$.

19 Determine an expression that, when integrated, will give the signed area bounded by the graph of $f(x) = \sin\left(\frac{2x}{3}\right)$ and the x-axis between $x = 0$ and $x = \frac{9\pi}{4}$.

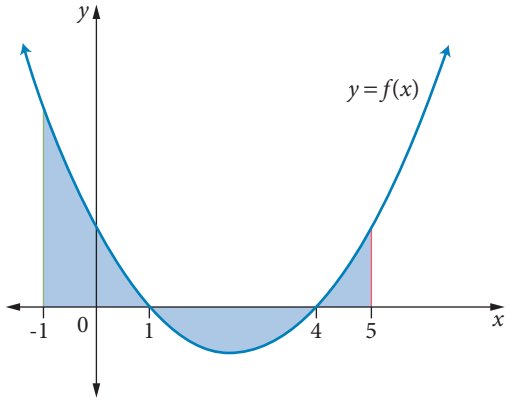
20 Determine an expression that, when integrated, will give the unsigned area as shaded in the graph.

Exercise 5.3



21 Which one of the expressions below represents the total shaded region in the diagram?

Exercise 5.3



A $\int_1^5 f(x)\,dx$

B $\int_4^5 f(x)\,dx + \int_1^4 f(x)\,dx + \int_{-1}^1 f(x)\,dx$

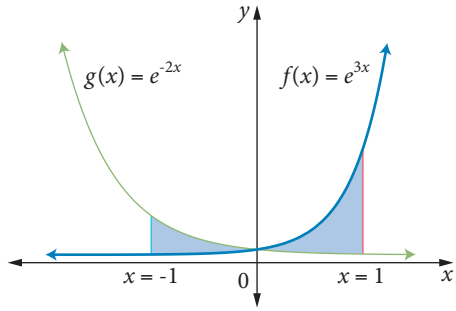
C $\int_4^5 f(x)\,dx - \int_4^1 f(x)\,dx + \int_{-1}^1 f(x)\,dx$

D $\int_4^5 f(x)\,dx + \int_4^1 f(x)\,dx + \int_{-1}^1 f(x)\,dx$

22 Determine the area of the region within the interval $[0, \pi]$ bounded by the curves $f(x) = 2\sin(x)$ and $g(x) = \dfrac{4-x}{2}$ correct to 2 decimal places.

Exercise 5.4

23 Determine an expression that, when integrated, will give the area of the region bounded by the graphs of $f(x) = e^{3x}$ and $g(x) = e^{-2x}$ between $x = -1$ and $x = 1$ as shaded on the right.



24 A medical research facility produces 50 grams of the isotope luijium-235 for a research project. The rate of decay of luijium-235 is given by $N'(t) = -0.35e^{-0.007t}$, where N is the amount of luijium-235 present and t is the time in days since the isotope was produced. Determine the amount of luijium-235 remaining after 15 days, correct to 2 decimal places.

Exercise 5.5

25 A particle moves in a straight line such that its velocity is given by $x'(t) = 6t$. At time $t = 0$, the position of the particle is $x = 2$. What is the position of the particle at time $t = 1$?

Exercise 5.5

Exercise 5.5

26 A rock falls from the top of a cliff 43 metres high. In falling, the rock is subject to a constant vertical acceleration of $9.8\,\text{m/s}^2$. How long does it take, in seconds, for the rock to hit the ground? Give your answer correct to 1 decimal place.

Exercise 5.1

27 Consider the area bounded by the graph of $f(x) = 0.2e^{0.5x}$ and the x-axis between $x = 0$ and $x = 2$.

(a) Use the left-endpoint rule with four intervals to approximate the required area correct to 3 decimal places.

(b) Use the right-endpoint rule with four intervals to approximate the required area correct to 3 decimal places.

(c) Use the sum of the areas of the lower rectangles and upper rectangles to approximate the required area correct to 3 decimal places.

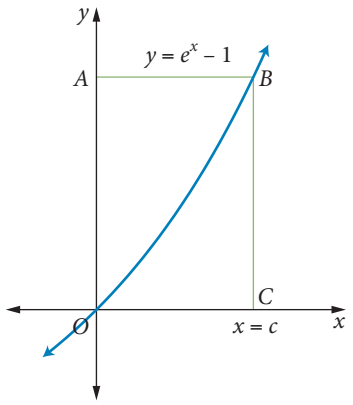
(d) Using the trapezoidal rule, approximate the area using the following intervals, correct to 3 decimal places.

(i) two trapeziums **(ii)** four trapeziums

(e) Use integration to determine the area bounded by $f(x)$ and the x-axis between $x = 0$ and $x = 2$.

Exercise 5.4

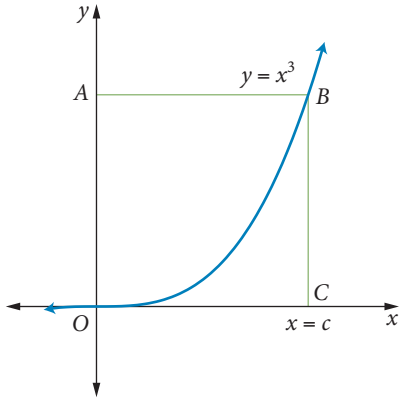
28 The graph of $y = e^x - 1$ divides the rectangle $OABC$ into two regions, one bounded by the x-axis and one bounded by the y-axis.



(a) Determine, in terms of c, the area of the region within the rectangle bounded by the x-axis.

(b) Using the result from part **(a)**, verify that the area of the region within the rectangle bounded by the y-axis is $ce^c - e^c + 1$.

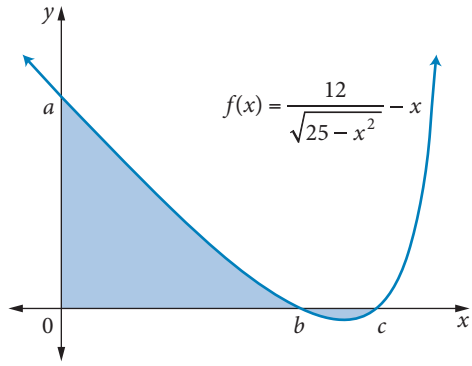
(c) The curve with equation $y = x^3$ divides the rectangle $OABC$ into two regions, one bounded by the y-axis and one bounded by the x-axis. Demonstrate that the areas of these two regions are in the ratio of 3:1.



(d) The curve $y = x^n$ divides the rectangle $OABC$ into two regions, one bounded by the x-axis and one bounded by the y-axis. Verify that the region within the rectangle bounded by the x-axis has an area of $\dfrac{c^{n+1}}{n+1}$.

(e) Hence verify that the curve $y = x^n$ divides the rectangle $OABC$ into two regions, with areas in the ratio n:1.

29 The graph of the function $f:[0,5) \to \mathbb{R}$, where $f(x) = \dfrac{12}{\sqrt{25-x^2}} - x$, is shown. The y-intercept is a, and the x-intercepts are b and c. Exercise 5.3



(a) Determine the exact values of a, b and c.

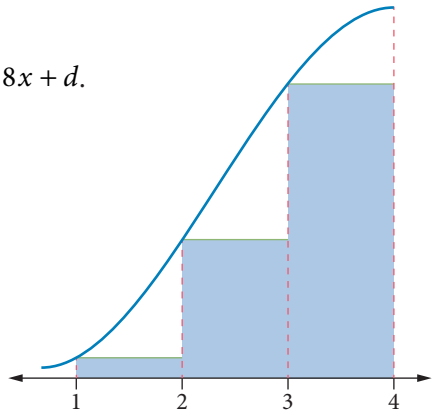
(b) Determine the signed area bounded by the curve and the x-axis between $x = 0$ and $x = 4$. Give your answer correct to 2 decimal places.

(c) Interpret the result for part **(b)** in terms of the area enclosed by the curve and the x-axis.

(d) Determine the total area bounded by the curve and the x-axis. Give your answer correct to 2 decimal places.

(e) Calculate the value of $\int_0^3 \left(f(x)\right)^2 dx$, correct to 2 decimal places.

30 The giant waterslide shown in the diagram is modelled by the function $f:\left[\dfrac{2}{3}, 4\right] \to \mathbb{R}$, defined by $f(x) = -x^3 + 7x^2 - 8x + d$. Exercise 5.1



(a) Determine the value of d if the total area of the shaded rectangles is 23 square units. Then apply it to the equation of the function $f(x)$.

(b) Use the equation found in part **(a)** to estimate the area under the waterslide using three trapeziums of width 1 unit.

(c) Use integration to calculate the area under the waterslide (between 1 and 4). Which of the two approximate areas from parts **(a)** and **(b)** is closer to the actual area?

31 The velocity, in m/s, of a particle moving in a straight line with variable acceleration is described by the equation $v(t) = t^3 + t^2 - t - 1$, where t is the time in seconds from the beginning of the particle's motion, $t \geq 0$. For the following questions, give your answers correct to 2 decimal places, where necessary. Exercise 5.5

(a) Determine the displacement of the particle when its acceleration is zero, given that the particle starts its movement at the origin.

(b) Determine the position of the particle when its velocity is zero.

(c) Determine the acceleration of the particle when its velocity is zero, and interpret its meaning.

32 Let $f(x) = \sin(x)$ and $g(x) = \sin\left(\dfrac{x}{2}\right)$. Exercise 5.4

(a) Sketch the two functions on the same set of axes in the interval $[0, 2\pi]$, and verify that the points of intersection between the two functions occur at $x = 0$, $x = \dfrac{2\pi}{3}$ and $x = 2\pi$. Name all key features.

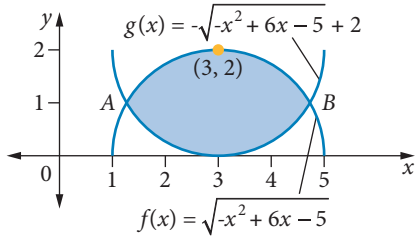
(b) Using integration, verify that the area between $f(x)$ and the x-axis equals the area between $g(x)$ and the x-axis. Explain why this is the case.

(c) Verify that the area between the two curves is 5 square units.

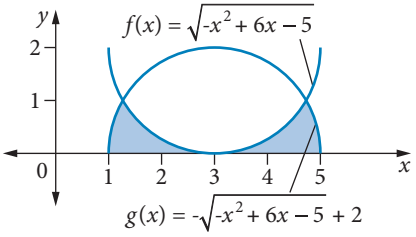
33 Consider the semicircle defined by the equation $f(x) = \sqrt{-x^2 + mx - 5}$.

(a) Determine the value of m if the derivative at $x = 3$ is zero. Hence, write the equation of the function $f(x)$ and sketch its graph.

(b) A second semicircle is drawn as shown in the diagram below. Determine the exact value of the shaded area between the two curves if the equation of the second semicircle is $g(x) = -\sqrt{-x^2 + 6x - 5} + 2$.



(c) Determine the exact value of the shaded area between the two curves (as shown in the diagram) using integration.



(d) Verify that the sum of the two areas from parts **(b)** and **(c)** equals 2π. Explain why this is the case.

Exercise 5.5

34 An object is dropped from the roof of a building with five floors. The height of the building is 20 m. Assume air resistance is negligible. The object falls from the fifth floor onto a balcony below on the third floor. Consider that all levels are equal. Give all your answers correct to 2 decimal places, where appropriate. Assume that the vertical acceleration due to gravity is 9.8 m/s^2 downwards.

(a) Determine the time at which the object reaches the third floor and the corresponding velocity.

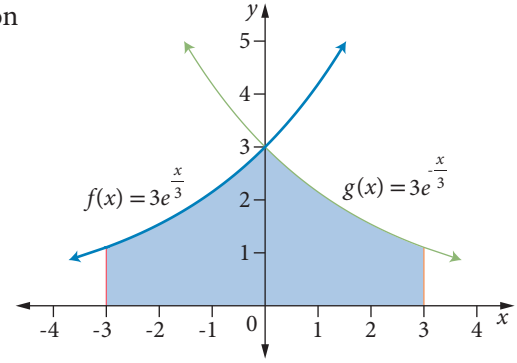
(b) A second object is dropped from the top of the building, but this object reaches the ground.

(i) Determine the time at which the object reaches the ground and the corresponding velocity.

(ii) Determine the velocity of the object as it reaches the ground if it was thrown down with an initial velocity of 10 m/s.

35 Merlin is designing the entrance for the new attraction 'Merlin's Cave', at Merlin's Magic-land theme park. His initial plan is to make the entrance to the cave 6 m wide, formed beneath two intersecting exponential functions: $f(x) = 3e^{\frac{x}{3}}$ and $g(x) = 3e^{-\frac{x}{3}}$. Merlin plans to have a curtain made to exactly cover the cave entrance.

Exercise 5.4

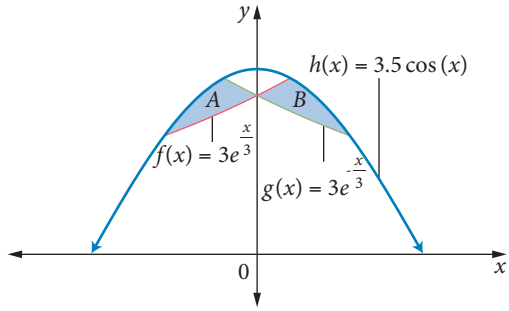


(a) Determine three separate possible integrals Merlin could use to calculate the area to be covered by the curtain.

(b) Calculate the area of the curtain. Give your answer correct to 2 decimal places.

(c) Merlin is considering painting a 2-metre-wide region above the entrance purple, framed by the curves $f(x)$ and $g(x)$. If he proceeds with this plan, calculate the area that will be painted purple. Give your answer correct to 2 decimal places.

(d) Merlin does not like the idea of straight sides on the entrance to his cave, so he decides to introduce the function $h(x) = 3.5\cos(x)$ into the entrance design. If the entrance is the region bounded by the ground and his three curves, calculate the area of curtain Merlin will now need to cover the entrance. Give your answer correct to 2 decimal places.



(e) Merlin is satisfied with the new design for the entrance to his cave. He decides to make the entrance more spectacular by overlaying gold foil over the entrance in the regions bounded by all three curves. Determine the area of gold foil Merlin needs to lay above the entrance to his cave. Give your answer correct to 2 decimal places.

36 A particle moves in a straight line with a velocity given by $v(t) = \dfrac{2}{\sqrt{1+t^2}}$. Velocity is in m/s and time is in seconds.

Exercise 5.5

(a) Express the acceleration of the particle in terms of t.

(b) Determine the initial velocity and initial acceleration of the particle.

(c) Describe the velocity as $t \to \infty$.

(d) Determine the distance travelled by the particle in the first 5 seconds of motion. Give your answer correct to 2 decimal places.

Exam review

UNIT 3

Paper 1: Technology-free

Exercise 1.5

1 Solve each of the following for x in exact form.

(a) $\left(e^x - 1\right)\left(e^x - 7\right) = 0$

(b) $e^{2x} - 7e^x + 12 = 0$

(c) $e^{2x} - e^x - 20 = 0$

(d) $e^{2x} + 5e^x = -6$

Exercise 2.1

2 Given $\cos(\theta) = 0.94$ and $\sin(\theta) = 0.34$, determine the following measures.

(a) $\sin(\pi + \theta)$

(b) $\sin\left(\frac{3\pi}{2} + \theta\right)$

Exercise 2.4

3 Determine an expression for the derivative of each of the following functions.

(a) $y = \sin\left(x^3 - 2\right)$

(b) $y = \cos^4(x)$

(c) $y = e^{2x + \sin(4x)}$

(d) $y = \sqrt{\sin(2x - 3)}$

Exercise 3.2

4 For $\log_3(b) - 2\log_3(a + 3) = 4$, write a relationship between a and b that does not involve logarithms. Make b the subject of the equation.

Exercise 3.3

5 Solve each of the following equations for x.

(a) $\log_x(5) = 2$

(b) $\log_2(x + 2) - 3 = \log_2(2x - 1)$

(c) $\left(\log_e(x)\right)^2 = 3\log_e(x)$

(d) $3^x = 7$

(e) $4^x = 3^{x-2}$ (Express logarithms using base 4.)

Exercise 4.3

6 Determine each of the following anti-derivatives.

(a) $\int \left(24x^3 - 6x^2\right) dx$

(b) $\int \left(2\cos(3\theta) - 3\sin(2\theta)\right) d\theta$

(c) $\int \frac{4}{\sqrt{5 - 12t}}\, dt$

(d) $\int 7(2x + 9)^{-1}\, dx$

Exercise 4.4

7 Determine $f(x)$ for each of the following.

(a) $f'(x) = \frac{3x^3 - x^2 + 5}{x^2}$, $f(1) = \frac{1}{2}$

(b) $f'(x) = \left(e^{-x} + e\right)^2$, $f(0) = \frac{1}{2}$

Exercise 5.2

8 Calculate the value of each of the following definite integrals.

(a) $\int_0^1 \left(e^x + e^{-x}\right)^2 dx$

(b) $\int_0^{\pi} \left(\sin(2x) - \sin(x)\right) dx$

(c) $\int_1^2 \left(2 + \frac{1}{x^2}\right)^2 dx$

(d) $\int_0^1 \frac{2}{3x + 4}\, dx$

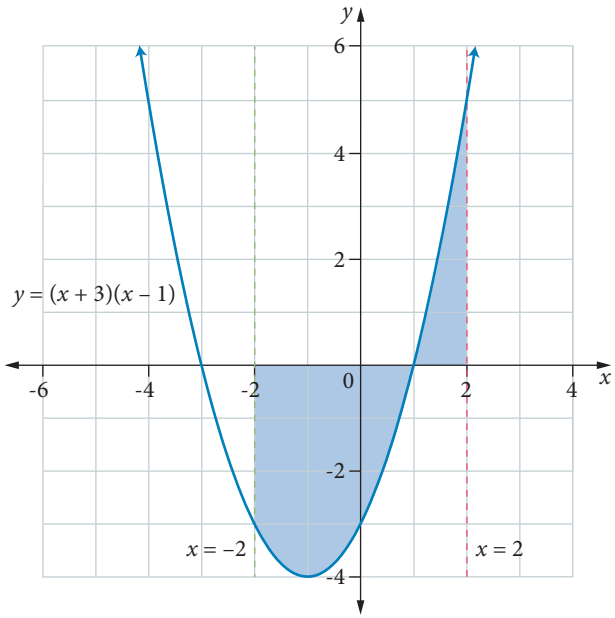
Exercise 5.2

9 Given $\int_1^2 f(x)\, dx = 3$, calculate the value of each of the following definite integrals.

(a) $\int_1^2 \left(f(x) + 3x^2\right) dx$

(b) $\int_1^2 7f(x)\, dx + \int_2^1 3f(x)\, dx$

10 Determine the exact unsigned area bounded by the graph of $f(x) = (x+3)(x-1)$, the x-axis, and the lines $x = -2$ and $x = 2$. Exercise 5.3



11 Determine the equation of the tangent to $f(x) = 2e^{3-x}$ where $x = 3$. Exercise 1.3

12 Determine the area enclosed by the graphs of $f(x) = (x-4)(x+1)$ and $g(x) = 8 - 2x$, in the interval $[-1, 2]$. Exercise 5.4

13 Graph the function $y = \sin(2x) + \sin^2(2x)$, $0 \le x \le \frac{3\pi}{2}$, showing all key values. Exercise 2.5

14 Throughout this question, assume all constants of integration are equal to zero. Exercise 4.2

(a) Differentiate xe^x and hence determine $\int xe^x\,dx$.

(b) Differentiate x^2e^x.

(c) Use your answers to parts **(a)** and **(b)** to determine $\int x^2e^x\,dx$.

Paper 2: Technology-active

Exercise 1.4

1 The growth of yeast in a yoghurt culture may be represented by $M(t) = M_0 \times e^{kt}$, where t is the number of hours since the culture was introduced to the milk and M represents the mass of the culture in grams. Given that 5 g is added initially and after 5 hours the mass of yeast is 37 g, answer the following questions.

(a) What is the value of M_0?

(b) Determine the value of k correct to 3 significant figures.

(c) Write the equation for $M(t)$.

(d) Determine an expression for $M'(t)$.

(e) After 2 hours, what is the mass of the culture and at what rate is the mass increasing? Give your answers correct to 3 significant figures.

Exercise 3.3

2 Solve each of the following equations for x.

(a) $\log_e(3x-2) - \log_e(x) = 0$

(b) $3\ln(2x-3) = 6$

(c) $\ln\left(\frac{2}{x}\right) = 3$

(d) $e^{\ln(x^2)} = 4$

(e) $\log_2(5) = \frac{\log_3(x)}{\log_3(2)}$

Exercise 4.4

3 If $\frac{dx}{dt} = 5(3-2t)^{-\frac{3}{2}}$, $t \geq 0$

(a) Determine $x(t)$ if $x(0) = \frac{5}{\sqrt{3}}$.

(b) Write down the set of possible values for t, and calculate the value of $x(1)$.

Exercise 3.7

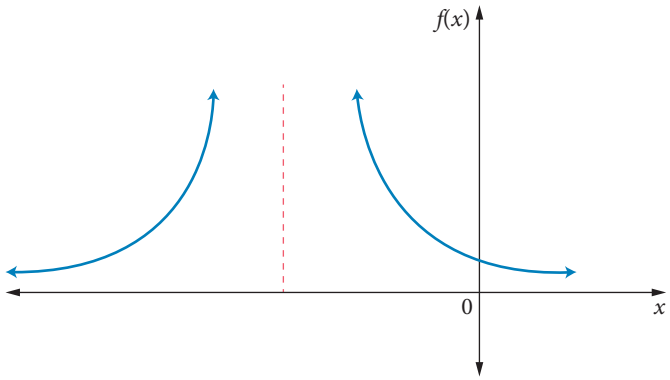
4 The cooling curve of liquid can be modelled by the equation: $T = T_S + (T_0 - T_S)e^{-kt}$, where T is the temperature of liquid at any time t hours after 12 noon, T_0 is the initial temperature of liquid and T_S is the room temperature. Jack poured boiling water (100°C) into the pot at 12 noon. By 1 pm, the water had cooled to 80°C. The room temperature was kept constant at 20°C. How long will it take for the water temperature to cool to 60°C? Give your answer correct to 2 decimal places.

Exercise 4.5

5 **(a)** Simplify the expression $1 - \sin^2(5\theta) + \cos^2(5\theta)$.

(b) Using the identity $\cos(2A) = 2\cos^2(A) - 1$, determine $\int \left(1 - \sin^2(5\theta) + \cos^2(5\theta)\right) d\theta$.

6 The graph shown is $f(x)$. Sketch a possible graph of the anti-derivative. Exercise 4.1



7 Use the total of the areas of the lower rectangles and of the upper rectangles to determine the average approximate area of the region bounded by the graph of $f(x) = x^3 + 1$, the x-axis, and the lines $x = 1$ and $x = 2.5$. Use intervals of width 0.5 units. Do not round your answers until the final step. Give your final answer correct to 2 decimal places. Exercise 5.1

8 In a day's production, the cost in dollars per pair of sandals is given by $\frac{dC}{dn} = 50 - 3e^{0.02n}$, where n is the number of pairs of sandals manufactured for the day. Exercise 5.5

(a) What is the cost per item if 100 pairs of sandals are manufactured?

(b) Express the cost C as a function of n. Assume that the cost of producing zero pairs is \$0.

(c) Determine the total cost of producing 100 pairs of sandals. Give your answer to the nearest dollar.

(d) Determine the change in cost between producing 100 pairs of sandals and 150 pairs of sandals.

9 A wooden curtain rod is supported horizontally at both ends. A small sag develops throughout the rod, with maximum deflection from the horizontal in the middle of the rod. Let the two fixed ends of the rod be O and A, 2 metres apart. The rate of deflection of the rod is given by $\frac{dy}{dx} = 0.001\left(3x^2 - x^3 - 2\right)$, where y metres is the deflection from the horizontal at a point x metres from O. Exercise 5.5

(a) Express the deflection y in terms of x.

(b) Verify that the deflection at A is 0 m.

(c) Calculate the sag at a point 0.8 m from point O. Give your answer to the nearest tenth of a millimetre.

(d) Determine the angle that the rod makes with the horizontal at point A. Give your answer in degrees correct to 3 decimal places.

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6

Further applications of differentiation

Recall 322

6.1 The derivative and curve sketching 323

6.2 Motion and derivatives 334

6.3 The second derivative and curve sketching 341

6.4 Maximum and minimum problems 352

Summary 362

Chapter review 364

6 Recall

Sketch the graph of a cubic function

1 Sketch the graph of each of the following functions.

(a) $y = (x+2)(x-2)(x-4)$ (b) $y = -x^3 - 4x^2 - x + 6$ (c) $y = (x-2)^3 - 1$

Solve a system of linear equations

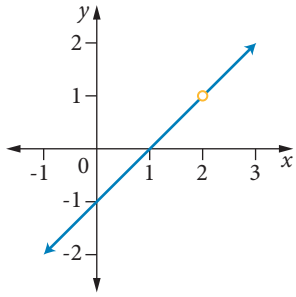
2 Solve the following simultaneous linear equations algebraically.

(a) $x + y = 4;\ 3x + 2y = 11$ (b) $2x - 2y = -6;\ x + 4y = 7$ (c) $5x - y = 17;\ 3x - 2y = -1$

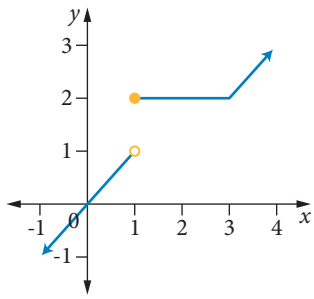
Determine a point of discontinuity, given a curve

3 Determine at which x-value each function has a discontinuity.

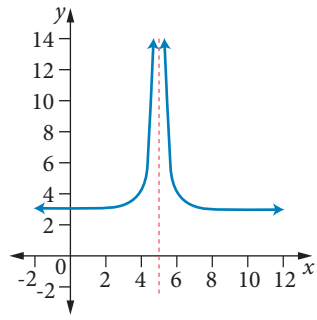
(a)



(b)



(c)



Derivatives of linear combinations of power functions

4 Calculate the derivatives of the following functions.

(a) $y = \dfrac{2x^3 - 5x^2}{x},\ x \neq 0$ (b) $f(x) = \sqrt{x} - \dfrac{2}{x},\ x > 0$ (c) $y = -x(x-2)^2$

Apply the derivative to basic curve sketching

5 For the following functions, use calculus to locate any stationary points and determine their nature.

(a) $f(x) = x^3 + 3x^2 - 4$ (b) $f(x) = x^2(5 - 2x)$ (c) $f(x) = -\dfrac{2x^3}{3} - 3x^2 + 2$

Apply the derivative in the study of motion in a straight line

6 A particle moves in a horizontal line such that its position x metres relative to a fixed origin O at time t seconds is given by $x(t) = -3(t-1)^2 + 1,\ t \geq 0$.

(a) Sketch the position–time graph showing all key features.

(b) Determine the particle's maximum displacement in the positive direction.

(c) Determine the velocity of the particle after 2 seconds.

Graphs of non-polynomial functions

7 Graph each of the following functions.

(a) $y = 2\sin(2x) + 3,\ 0 \leq x \leq 2\pi$ (b) $y = \dfrac{1}{x-2}$ (c) $y = \sqrt{16 - x^2}$

Calculate derivatives using the chain, product and quotient rules

8 Calculate the derivatives of the following functions.

(a) $y = \sqrt{2x^2 - 3x}$ (b) $y = (x^2 - 5)\cos(x)$ (c) $y = \dfrac{e^{5x}}{x^4 + 2}$

The derivative and curve sketching

6.1

Key features of a curve

When sketching a curve, it is important to clearly show its key features. Such key features include the general shape of the curve, axial intercepts, stationary points and asymptotes.

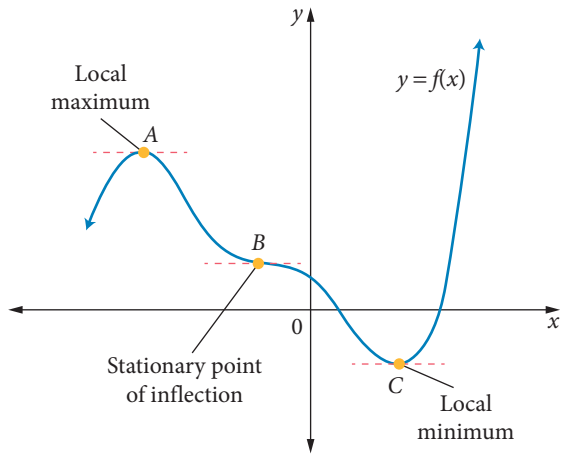
C Making connections

Revise the shapes of polynomial functions

Move the sliders to revise the general shapes of quadratic, cubic and quartic functions.

Stationary points

Consider the function $y = f(x)$. A point $P(a, f(a))$ on the graph of f is classified as a *stationary point* if the gradient is zero; that is, if $f'(a) = 0$. At P, the curve is neither rising nor falling; it is stationary. The graph of $y = f(x)$ shown illustrates the three types of stationary points: local maximum stationary points, local minimum stationary points and stationary points of inflection.

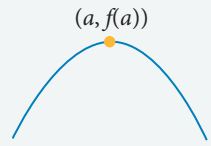


At points A, B and C, the gradient is zero; that is, $f'(x) = 0$. Hence a tangent to the curve $y = f(x)$ at points A, B and C would have zero gradient and so would be parallel to the x-axis.

A *local maximum stationary point* at $x = a$ on the graph of $y = f(x)$ has the following properties.

- $f(a) \geq f(x)$ for all x in the neighbourhood of a.
- $f'(a) = 0$ and $f'(x) > 0$ for $x < a$, and $f'(x) < 0$ for $x > a$ in the neighbourhood of a.

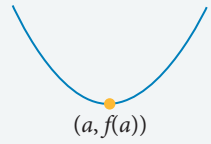
The gradient is zero at $x = a$, and changes sign as it passes from the left to the right of $x = a$. A local maximum stationary point is called a *turning point* because the curve turns through the stationary point.



A *local minimum stationary point* at $x = a$ on the graph of $y = f(x)$ has the following properties.

- $f(a) \leq f(x)$ for all x in the neighbourhood of a.
- $f'(a) = 0$ and $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > a$ in the neighbourhood of a.

The gradient is zero at $x = a$, and changes sign as it passes from the left to the right of $x = a$. A local minimum stationary point is called a *turning point* because the curve turns through the stationary point.



A *stationary point of inflection* at $x = a$ on the graph of $y = f(x)$ has the following properties.

Either:

- $f(a) > f(x)$ for $x < a$ and $f(a) < f(x)$ for $x > a$ in the neighbourhood of a
- $f'(a) = 0$ and $f'(x) > 0$ in the neighbourhood of a

or

- $f(a) < f(x)$ for $x < a$ and $f(a) > f(x)$ for $x > a$ in the neighbourhood of a
- $f'(a) = 0$ and $f'(x) < 0$ in the neighbourhood of a.

The gradient is zero at $x = a$ and either positive or negative elsewhere in the neighbourhood of a.

The gradient does not change sign and the curve does not turn through the point. So a stationary point of inflection is not a turning point.

Additional information

Maximum and minimum points using the derivative

Check your understanding of the local maximum and minimum (extremum) points and the value of the derivative.

Additional information

Critical points

Check your understanding of the critical points of a graph.

These ideas can be applied to the sketching of polynomial functions.

1 Sketch the graph of a cubic polynomial function

Consider the function $f(x) = (x + 4)(x + 1)^2$.

(a) Use calculus to locate any stationary points and determine their nature.

THINKING	WORKING
1 Recognise the function as a product of two factors $u(x)$ and $v(x)$ and determine the derivative of each.	$f(x) = (x + 4)(x + 1)^2$ Let $u(x) = x + 4$ and $v(x) = (x + 1)^2$. $u'(x) = 1$ and $v'(x) = 2(x + 1)$
2 Use the product rule to determine $f'(x)$. (Alternatively, you could expand the rule and then differentiate.)	$f'(x) = u(x)v'(x) + v(x)u'(x)$ $= (x + 4)(2(x + 1)) + (x + 1)^2(1)$ $= (x + 1)(2(x + 4) + (x + 1))$ $= (x + 1)(3x + 9)$ $= 3(x + 3)(x + 1)$
3 Let $f'(x) = 0$ and solve for x to determine the values of x where the gradient is 0.	$3(x + 3)(x + 1) = 0$ $x + 3 = 0$ $\quad$ $x + 1 = 0$ $x = -3$ $\quad$ $x = -1$ Therefore $f'(x) = 0$ where $x = -3$ or $x = -1$.

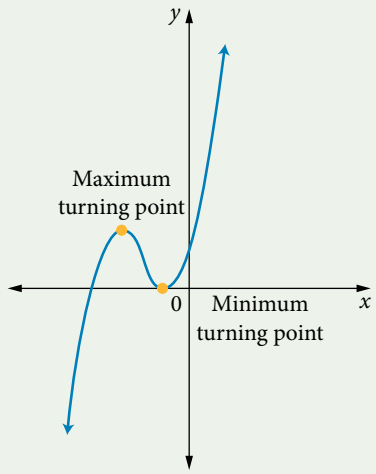
4 Substitute the values of x for which $f'(x) = 0$ into $f(x)$ to determine the coordinates of each stationary point.

$f(-3) = (-3+4)(-3+1)^2$ $\quad$ $f(-1) = (-1+4)(-1+1)^2$
$= 1 \times 4$ $\quad$ $= 3 \times 0$
$= 4$ $\quad$ $= 0$

The coordinates of the stationary points are $(-3, 4)$ and $(-1, 0)$.

5 Consider the shape of the function.

Since f is a positive cubic function it will have the following shape.



6 Determine the type of each stationary point.

Therefore, the gradient of the graph is 0 at $(-3, 4)$, which is a local maximum, and $(-1, 0)$, which is a local minimum.

(b) Sketch the graph of $y = f(x)$, showing all key features.

1 Determine the coordinates of the y-intercept by substituting $x = 0$ and calculating the value of y.

$f(0) = (0+4)(0+1)^2$
$= 4$

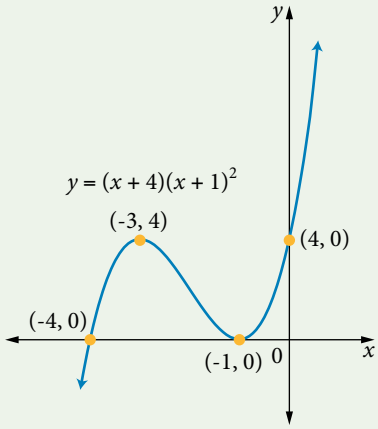
The coordinates of the y-intercept are $(0, 4)$.

2 Determine the coordinates of the x-intercept(s) by letting $f(x) = 0$ and solving for x.

$(x+4)(x+1)^2 = 0$ where $x = -4$ or $x = -1$.

The coordinates of the x-intercepts are $(-4, 0)$ and $(-1, 0)$.

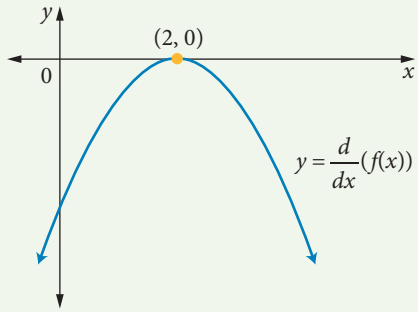
3 Sketch the graph of $y = f(x)$, showing all key features.



2 Sketch the graph of a cubic polynomial function in expanded form

Consider the function $f(x) = -x^3 + 6x^2 - 12x + 9$.

(a) Use calculus to locate any stationary points and determine their nature.

THINKING	WORKING
1 Determine the derivative, $f'(x)$ using the power rule.	$f(x) = -x^3 + 6x^2 - 12x + 9$ $f'(x) = -3x^2 + 12x - 12$
2 Let $f'(x) = 0$ and solve for x to determine the values of x where the gradient is 0.	$-3x^2 + 12x - 12 = 0$ $-3(x^2 - 4x + 4) = 0$ $-3(x-2)^2 = 0$ $f'(x) = 0$ where $x = 2$.
3 Substitute the values of x for which $f'(x) = 0$ into $f(x)$ to determine the coordinates of the stationary point.	$f(2) = -2^3 + 6(2)^2 - 12(2) + 9$ $= -8 + 24 - 24 + 9$ $= 1$ The coordinates of the stationary point are $(2, 1)$.
4 Consider the shape of the derivative function.	The graph of $f'(x) = -3(x-2)^2$ is a negative parabola that touches the x-axis at $x = 2$.  From the graph of $y = f'(x)$: $x < 2$: $f'(x) < 0$ $x = 2$: $f'(x) = 0$ $x > 2$: $f'(x) < 0$ $f'(x) = -3(x-2)^2 \leq 0$ for $x \in \mathbb{R}$ The gradient is negative on both sides of the stationary point; that is, it does not change sign.
5 Classify the stationary point.	$(2, 1)$ is a stationary point of inflection.

(b) Sketch the graph of $y = f(x)$, showing all key features.

1 Determine the coordinates of the y-intercept by substituting $x = 0$ and calculating the value of y.

$f(x) = -x^3 + 6x^2 - 12x + 9$
$f(0) = 9$
The coordinates of the y-intercept are $(0, 9)$.

2 Determine the coordinates of the x-intercept(s) by letting $f(x) = 0$ and solving for x.

By trial and error (considering the factors of 9):

$$\begin{aligned} f(3) &= -(3)^3 + 6(3)^2 - 12(3) + 9 \\ &= -27 + 54 - 36 + 9 \\ &= 0 \end{aligned}$$

$f(3) = 0$ and therefore $(x - 3)$ is a factor.

By inspection:

$$-x^3 + 6x^2 - 12x + 9 = -(x - 3)\left(x^2 + kx + 3\right)$$

$$\begin{aligned} 3x^2 - kx^2 &= 6x^2 \\ (3 - k)x^2 &= 6x^2 \\ 3 - k &= 6 \\ k &= -3 \end{aligned}$$

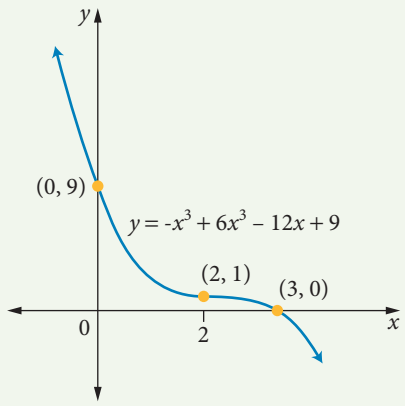
For $x^2 - 3x + 3$, calculate the discriminant Δ to determine whether any real factors exist.

$$\begin{aligned} b^2 - 4ac &= 9 - 4(1)(3) \\ &= -3 \end{aligned}$$

Therefore, the quadratic factor has no real factors.

There is a single x-intercept at $(3, 0)$.

3 Sketch the graph of $y = f(x)$, showing all key features.



When using your calculator to sketch the graph, draw the graph first, then read off all key values.

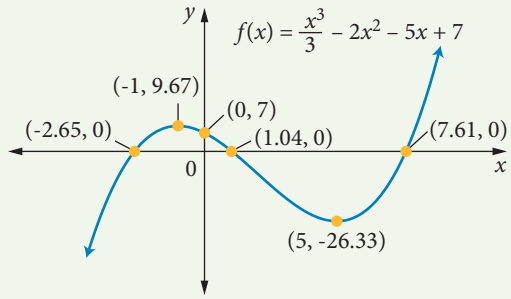
Technology worked example

Determine the key features of polynomials

3 Use technology to sketch the graph of a cubic polynomial function

Consider the polynomial function $f(x) = \frac{x^3}{3} - 2x^2 - 5x + 7$.

(a) Sketch the graph of $f(x)$, showing all key points correct to 2 decimal places.

THINKING	WORKING
Plot the graph on your calculator. Determine the roots (x-intercepts). Trace the coordinates of the y-intercepts. Determine the coordinates of the maximum and minimum points.	

(b) Use the graph to determine the values of x for which $f(x)$ is an increasing and decreasing function.

1 Identify the section/s of the graph with a positive gradient.	$f(x)$ is increasing for $x < -1$ and $x > 5$.
2 Identify the section/s of the graph with a negative gradient.	$f(x)$ is decreasing for $-1 < x < 5$.

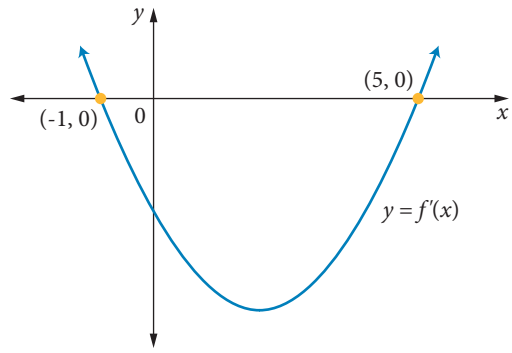
In Worked Example 3, since f is a positive cubic, it is clear that the first stationary point will be a local maximum and the second will be a local minimum. Alternatively, you can justify the nature of the stationary points by graphing $y = f'(x)$.

$$f(x) = \frac{x^3}{3} - 2x^2 - 5x + 7$$

Therefore:

$$f'(x) = x^2 - 4x - 5$$
$$= (x + 1)(x - 5)$$

The graph shown of $y = (x + 1)(x - 5)$ is a parabola with x-intercepts at $x = -1$ and $x = 5$.



From the graph of $y = f'(x)$, you can conclude the following.

- $f'(x) > 0$ where $x < -1$ or $x > 5$.
- $f'(x) < 0$ where $-1 < x < 5$.
- As the gradient changes sign from positive to negative through $x = -1$, $(-1, 9.67)$ is a local maximum.
- As the gradient changes sign from negative to positive through $x = 5$, $(5, -26.33)$ is a local minimum.

Global maximum and minimum values

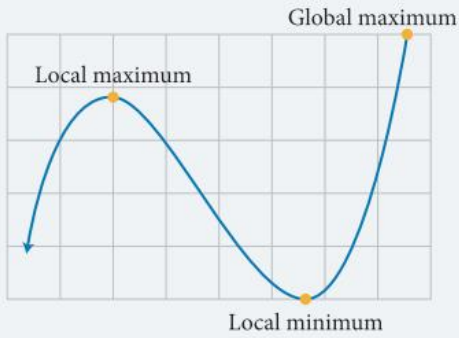
Local (relative) maximum and minimum turning points do not necessarily give the maximum or minimum value of a function. For cubic (and all odd) functions with unrestricted domain, there is no global maximum or minimum value as the values approach $-\infty$ at one extreme and $+\infty$ at the other. If a function f has a restricted domain, the maximum or minimum value of f can occur at either an endpoint of the domain or a stationary point. The maximum and minimum values of a function are referred to as the *absolute or global maximum* and *global minimum* respectively.

A maximum point for the entire domain is called an absolute or global maximum.

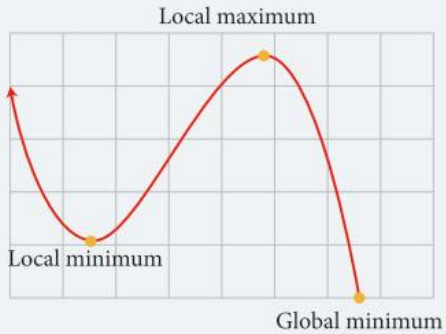
A minimum point for the entire domain is called an absolute or global minimum.

For example, consider the shapes of the graphs shown below.

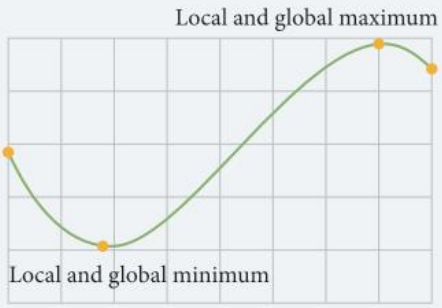
The global maximum is an endpoint; the local minimum and maximum occur at the stationary points. There is no global minimum as the graph continues indefinitely, as shown by the arrowhead.



Global minimum is at the right endpoint; the local minimum and local maximum occur at the stationary points. There is no global maximum as the graph continues indefinitely, as shown by the arrowhead.



Global minimum is at the local minimum and global maximum is at the local maximum.



Global minimum is at the left endpoint and global maximum is at the right endpoint.

WARNING

Graphs with vertical asymptotes or other discontinuities may have trends towards $-\infty$ or $+\infty$ at the discontinuity, or the discontinuity may contain a maximum or minimum value.

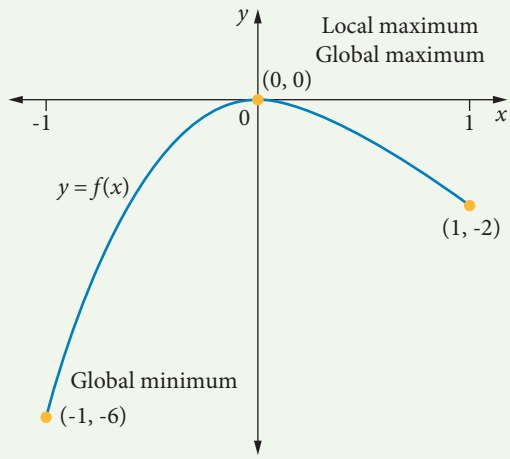
Additional information

Maximum and minimum values

Watch this interactive video and try the activities to check your understanding of the global maximum and minimum points over a given interval.

4 Determine the global maximum and minimum values of a function

Determine the global maximum and the global minimum values of $f(x) = 2x^3 - 4x^2$, $-1 \leq x \leq 1$.

THINKING	WORKING
1 Substitute the x-values of the end points of the given domain into $f(x)$ to determine the coordinates of the end points.	$f(-1) = 2(-1)^3 - 4(-1)^2$ $= -2 - 4$ $= -6$ The coordinates of the left end point are $(-1, -6)$. $f(1) = 2(1)^3 - 4(1)^2$ $= 2 - 4$ $= -2$ The coordinates of the right end point are $(1, -2)$.
2 Let $f'(x) = 0$ and solve for x to determine the values of x where the gradient is 0, taking into account the domain of the function given.	$f(x) = 2x^3 - 4x^2$, $-1 \leq x \leq 1$ $f'(x) = 6x^2 - 8x$ $6x^2 - 8x = 0$ $2x(3x - 4) = 0$ $f'(x) = 0$ where $x = 0$ or $x = \frac{4}{3}$ As $-1 \leq x \leq 1$, reject $x = \frac{4}{3}$.
3 Substitute the value/s of x into $f(x)$ to determine the coordinates of the stationary points.	$f(0) = 0$ The coordinates of the only stationary point are $(0, 0)$.
4 Classify the stationary point.	Since f is a positive cubic function, $(0, 0)$ is a local maximum.
5 Sketch the graph, and identify the global maximum and minimum values of f.	 The global minimum value of -6 occurs at $x = -1$. The global maximum value of 0 occurs at $x = 0$.

EXERCISE 6.1 The derivative and curve sketching

Worked Example

1 Consider the function $f(x) = x(x-1)^2$. Use calculus to locate any stationary points and determine their nature. Then sketch the graph of $y = f(x)$, showing all key features. 1

2 Consider the function $f(x) = -2x(x+2)^2$. Use calculus to locate any stationary points and determine their nature. Then sketch the graph of $y = f(x)$, showing all key features.

3 Consider the function $y = (x-2)(x-3)^2$.

(a) Where does the graph of the function have a local maximum?

A $x = 3$ B $x = -\frac{5}{2}$ C $x = \frac{7}{3}$ D $x = \frac{5}{2}$

(b) Explain the common error made by a student who chose the third incorrect option.

4 Consider the function $f(x) = -x^3 - 6x^2 - 9x - 4$. Use calculus to locate any stationary points and determine their nature. Then sketch the graph of $y = f(x)$, showing all key features.

5 Consider the function f defined by $f(x) = -x^3 - 12x^2 - 48x - 56$. Use calculus to locate any stationary points and determine their nature. Then sketch the graph of $y = f(x)$, showing all key features. 2

6 Consider the function $g(x) = \frac{1}{3}x^3 + 4x^2 + 7x - 9$. 3

(a) Use calculus to determine the exact coordinates of the stationary points on the graph of $y = g(x)$.

(b) Sketch the graph of $y = g(x)$, labelling the exact coordinates of the stationary points and the intercepts with the axes, correct to 2 decimal places where necessary.

(c) Use the graph to determine the values of x for which $g(x)$ is an increasing function and the values for which it is a decreasing function.

7 Consider the function g defined by $g(x) = -\frac{1}{3}x^3 - x^2 + 8x - 2$.

(a) Use calculus to determine the exact coordinates of the stationary points on the graph of $y = g(x)$.

(b) Sketch the graph of $y = g(x)$, labelling the exact coordinates of the stationary points and the intercepts with the axes, correct to 2 decimal places where necessary.

8 Consider the function $f(x) = (x^2 - 4)(x^2 - 16)$.

(a) Use calculus to determine the exact coordinates of the stationary points on the graph of $y = f(x)$.

(b) Sketch the graph of $y = f(x)$, giving the exact coordinates of the stationary points and the intercepts with the axes.

9 Sketch the graph of $y = (x+1)(2-x)^3$, showing the axis intercepts and the coordinates of any stationary points, correct to 2 decimal places.

10 At what value of x does the global maximum of the function $f(x) = 2x^3 + x^2 - 4x + 5$, for $-2 \leq x \leq 1$, occur?

A $x = -2$ B $x = -\frac{2}{3}$ C $x = -1$ D $x = \frac{2}{3}$

11 For the function $f(x) = x^3 + 4x - 1$, use $f'(x)$ to explain why the graph of $y = f(x)$ has no stationary points.

12 Determine the value of c, given that the graph of $y = -x^3 - 5x^2 + cx + 9$ has a stationary point at $x = -3$.

Worked Example

4

13 Consider the function $g(x) = x^3 - x^2, -\frac{1}{4} \leq x \leq 2$.

(a) Use calculus to locate any stationary points and determine their nature.

(b) Determine the global maximum value of g.

(c) Determine the global minimum value of g.

(d) Determine the minimum gradient and the exact coordinates of the point where this occurs.

14 Consider the function $f(x) = x^3 + x^2 - x - 1$. Use calculus to locate the coordinates of any stationary points and determine their nature. Then sketch the graph of $y = f(x)$, showing all key features.

15 Consider the function $g(x) = 4x^3 + x^2 - 4x - 1$, where $-1 \leq x \leq 2$.

(a) Determine the coordinates and nature of the stationary points.

(b) Sketch the graph and hence determine the global maximum and minimum values of the function.

16 Consider the function $f(x) = x^3 + bx^2 + cx + 28$. The graph of $y = f(x)$ has a stationary point at $(2, 0)$.

(a) Determine the values of b and c.

(b) Sketch the graph of $y = f(x)$, giving the coordinates of the stationary points and the intercepts with the axes.

17 Consider the function $g(x) = \frac{1}{8}(x-1)^3(8-3x) + 1$.

(a) Determine an expression for $g'(x)$ in factorised form.

(b) Determine the exact set of values of x for which $g'(x) \leq 0$.

(c) The graph of $y = g(x)$ has stationary points at $(1, m)$ and $\left(\frac{9}{4}, n\right)$.

(d) Determine the values of x for which $g(x) = (g(x))^2$, correct to 2 decimal places.

18 Determine the rule and sketch the graph of a cubic function $y = g(x)$, given that $g(3) = 3$, $g'(3) = 0$, $g(0) = 4$, $g'(0) = 0$, $g'(x) > 0$ for $x < 0$ and $x > 3$, and $g'(x) < 0$ for $0 < x < 3$. Give the x-intercept correct to 2 decimal places.

19 Consider the function $f(x) = (2x-3)^3$.

(a) Sketch the function $f(x)$ and the corresponding gradient function, labelling all key features.

(b) The function $f(x)$ is multiplied by $x - 1$ and becomes $g(x) = (2x-3)^3(x-1)$. Plot the graph of the function $g(x)$.

(c) Determine the coordinates of the global minimum point and the global maximum point of the function $g(x)$ over the domain $[0.9, 1.7]$. Sketch the corresponding graph. Label all key points correct to 2 decimal points.

20 Consider the function $f(x) = x^3 + bx^2 + cx + 1$.

(a) Determine the relationship between b and c when $y = f(x)$ has exactly one stationary point.

(b) Let $c = 0$.

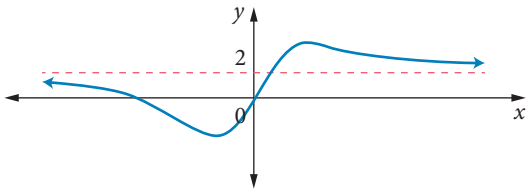
(i) Use calculus to determine the coordinates of any stationary points.

(ii) Determine the exact set of values of b for which the curve has a stationary point with a negative y-coordinate.

(iii) Determine the values of b for which the curve has a local maximum at $(0, 1)$.

(iv) What is the nature of the stationary point at $(0, 1)$ when $b = 0$?

21 Consider the function $f(x) = \dfrac{4x^2 + bx}{2x^2 - k}$, where b and k are non-zero numbers. The graph of $y = f(x)$ is shown below.



(a) If the graph of $y = f(x)$ has no vertical asymptotes, determine the possible values of k.

(b) If $g(x) = \dfrac{4x^2 - bx}{2x^2 - k}$, describe how the graph of $y = g(x)$ can be obtained from the graph of $y = f(x)$. Hence, sketch the graph of $y = g(x)$.

(c) Determine the number of stationary points on the graph of $y = \left(f(x)\right)^2$.

22 Consider the function $f(x) = -\dfrac{1}{3}x^3 + ax - 1$ where $a > 0$.

(a) Determine the values of x for which the function is decreasing.

(b) For $a = 10$, calculate the coordinates of the stationary points of $f(x)$, correct to 1 decimal place, and determine their nature.

(c) Sketch the function $f(x) = -\dfrac{1}{3}x^3 + 10x - 1$ and the corresponding gradient function, labelling all key features, and clearly showing the intervals where the function $f(x)$ is decreasing.

6.2 Motion and derivatives

The second derivative

If $y = f(x)$, then the first derivative is $\frac{dy}{dx} = f'(x)$ and the second derivative is $\frac{d^2y}{dx^2} = f''(x)$.

The second derivative, or the derivative of a derivative, gives the rate of change of the first derivative function.

Consider the function $f(x) = x^3 - 4x^2 - 5$. The first derivative, or the *derived or gradient function* is $f'(x) = 3x^2 - 8x$. You can differentiate again to find the equation of the *second derivative* of f; that is, the derivative of the first derivative. Thus, the second derivative, denoted by $f''(x)$ is, $f''(x) = 6x - 8$.

5 Determine the second derivative of a function

Determine the second derivative of each of the following functions.

(a) $f(x) = 2x^3 - 3x^2 + 4x - 1$

THINKING	WORKING
1 Differentiate the function $f(x)$ using the power rule to give $f'(x)$.	$f(x) = 2x^3 - 3x^2 + 4x - 1$ $f'(x) = 6x^2 - 6x + 4$
2 Differentiate $f'(x)$ to give $f''(x)$.	$f''(x) = 12x - 6$

(b) $f(x) = \sin(x)$

1 Differentiate the function $f(x)$ to give $f'(x)$.	$f(x) = \sin(x)$ $f'(x) = \cos(x)$
2 Differentiate $f'(x)$ to give $f''(x)$.	$f''(x) = -\sin(x)$

When the function is a compound, or a product or quotient of two functions, you may need to use the chain, product or quotient rule to determine the first and second derivatives.

6 Determine the second derivative of more complex functions

Determine an expression for the second derivative of each of the following functions.

(a) $y = \frac{x-1}{\sqrt{x}}$

THINKING	WORKING
1 Divide each term in the numerator by $\sqrt{x}$. Express each term in index notation.	$y = \frac{x-1}{\sqrt{x}}$ $= \frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}$ $= x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

2 Differentiate the function using the power rule to determine $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} - \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)\\ &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}\end{aligned}$$

3 Differentiate $\frac{dy}{dx}$ to give $\frac{d^2y}{dx^2}$.
Express the final answer in surd form, as a single fraction.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{1}{2}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) + \frac{1}{2}\left(-\frac{3}{2}x^{-\frac{5}{2}}\right)\\ &= -\frac{1}{4}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}\\ &= -\frac{1}{4x\sqrt{x}} - \frac{3}{4x^2\sqrt{x}}\\ &= -\frac{x+3}{4x^2\sqrt{x}}\end{aligned}$$

(b) $y = \sqrt{2x-1}$

1 Recognise the function as a composition of two functions.

$$\begin{aligned}y &= \sqrt{2x-1}\\ &= (2x-1)^{\frac{1}{2}}\end{aligned}$$

Let $u = 2x - 1$ and so $y = u^{\frac{1}{2}}$.

$\frac{du}{dx} = 2$ and $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

2 Use the chain rule to determine the derivative.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{2}u^{-\frac{1}{2}} \times 2\\ &= \frac{1}{\sqrt{u}}\\ &= \frac{1}{\sqrt{2x-1}}\end{aligned}$$

3 To determine $\frac{d^2y}{dx^2}$, differentiate $\frac{dy}{dx}$ using the chain rule. Introduce another variable, if necessary, to show the steps.

Let $\frac{dy}{dx} = w$, so $\frac{d^2y}{dx^2} = \frac{dw}{dx}$

Let $u = 2x - 1$ and so $w = u^{-\frac{1}{2}}$.

$\frac{du}{dx} = 2$ and $\frac{dw}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{1}{2}u^{-\frac{3}{2}} \times 2\\ &= -\frac{1}{u^{\frac{3}{2}}}\\ &= -\frac{1}{(2x-1)^{\frac{3}{2}}}\end{aligned}$$

4 Convert the expression for the second derivative to surd form to match the form of the question.

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{1}{(2x-1)(2x-1)^{\frac{1}{2}}}\\ &= -\frac{1}{(2x-1)\sqrt{2x-1}}\end{aligned}$$

Acceleration

The acceleration $a(t)$ of a particle is its rate of change of velocity with respect to time:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

Consider a particle moving in a straight line such that at time t its position from a fixed origin is $x(t)$. The convention is to describe positions to the right as positive.

The first derivative with respect to time, $x'(t) = \frac{dx}{dt}$, gives the velocity, $v(t)$, of the particle. So velocity is the rate of change of position with respect to time.

The second derivative of position with respect to time, $x''(t) = \frac{d^2x}{dt^2}$, gives the acceleration $a(t)$ of the particle. So $a(t)$, or alternatively $v'(t)$, is the rate of change of velocity with respect to time.

7 Determine the acceleration function of a particle

Determine the acceleration function of particles whose positions, x metres, from a fixed origin at time t seconds are given by the following functions.

(a) $x = -5t^2 + 12t + 2$

THINKING	WORKING
1 Differentiate the position function to determine the velocity function $\frac{dx}{dt}$.	$x = -5t^2 + 12t + 2$ $v = \frac{dx}{dt}$ $= -10t + 12$
2 Differentiate the velocity function to determine the acceleration function $\frac{d^2x}{dt^2}$.	$a = \frac{d^2x}{dt^2}$ $= \frac{dv}{dt}$ $= -10$

(b) $x = \frac{1}{4t+3}$

1 Recognise the composition of functions in the form $x = u^{-1}$.	$x = \frac{1}{4t+3}$ $= (4t+3)^{-1}$ Let $x = u^{-1}$, where $u = 4t + 3$. $\frac{dx}{du} = -u^{-2}$ and $\frac{du}{dt} = 4$
2 Use the chain rule to determine the velocity function, $\frac{dx}{dt} = \frac{dx}{du} \times \frac{du}{dt}$, and simplify.	$v = \frac{dx}{dt}$ $= \frac{dx}{du} \times \frac{du}{dt}$ $= -u^{-2} \times 4$ $= -\frac{4}{u^2}$ $= -\frac{4}{(4t+3)^2}$

3 Differentiate $\frac{dx}{dt}$ using the chain rule to determine the acceleration function $\frac{d^2x}{dt^2}$.	Let $u = 4t + 3$, so $\frac{du}{dt} = 4$. $v = -\frac{4}{u^2}$, $\therefore \frac{dv}{du} = \frac{8}{u^3}$ $a = \frac{d^2x}{dt^2}$ $= \frac{dv}{dt}$ $a = \frac{dv}{du} \times \frac{du}{dt}$ $= 4 \times \frac{8}{u^3}$ $= 4 \times \frac{8}{(4t+3)^3}$ $= \frac{32}{(4t+3)^3}$

Determining position, velocity and acceleration

The position and behaviour of a particle at any time can be found by determining $x(t)$, $v(t) = x'(t)$ and $a(t) = x''(t)$ for particular values of t. If a particle's acceleration is *positive* and the velocity of the particle is increasing, one of two things is happening.

- The particle is moving in a positive direction and its speed is increasing.
- The particle is moving in a negative direction and its speed is decreasing.

If a particle's acceleration is *negative* and the velocity of the particle is decreasing, one of two things is happening.

- The particle is moving in a positive direction and its speed is decreasing.
- The particle is moving in a negative direction and its speed is increasing; for example, a change from $v = -3$ m/s to $v = -5$ m/s over 1 second.

WARNING

A negative velocity with a negative acceleration means the particle is accelerating, increasing speed, but moving in a negative direction.

8 Determine expressions for position, velocity and acceleration

The position, x metres, of a particle travelling in a straight line from a fixed origin after t seconds is given by $x(t) = -2t^3 + 12t^2 + 8t$ for $t \geq 0$.

(a) Determine an expression for the particle's velocity, in m/s, at any time.

THINKING	WORKING
1 Determine the velocity function, $x'(t)$.	$x(t) = -2t^3 + 12t^2 + 8t$ $v(t) = x'(t)$ $= -6t^2 + 24t + 8$

2 Interpret the answer.	The velocity of the particle at time t seconds, is given by $\left(-6t^2 + 24t + 8\right)$ m/s.

(b) Determine an expression for the particle's acceleration, in m/s^2, at any time.

1 Differentiate the velocity function, $x'(t)$, to determine the acceleration function, $x''(t)$.	$x'(t) = -6t^2 + 24t + 8$ $a(t) = x''(t)$ $= -12t + 24$
2 Interpret the answer.	The acceleration of the particle at time t seconds, is given by $\left(-12t + 24\right)$ m/s^2.

(c) Determine the particle's initial acceleration.

1 Calculate $x''(0)$.	$x''(t) = -12t + 24$ $x''(0) = -12(0) + 24$ $= 24$
2 Interpret the answer.	The particle's initial acceleration (at $t = 0$) is 24 m/s^2.

(d) Explain what is happening when the particle's acceleration is zero.

1 Let $x''(t) = 0$ and solve for t.	If $x''(t) = 0$: $-12t + 24 = 0$ $t = 2$
2 Interpret the value.	Given that $x'(t) = -6t^2 + 24t + 8$, the maximum velocity of the particle (the point at which $v'(t) = 0$) occurs after 2 seconds.
3 Substitute $t = 2$ into the velocity function to calculate the velocity at the time $x'(2)$.	$x'(2) = -6(2)^2 + 24(2) + 8$ $= -24 + 48 + 8$ $= 32$
4 Interpret the value.	The maximum velocity of the particle is 32 m/s and occurs 2 seconds into the journey.
5 Substitute $t = 2$ into the position or displacement function to calculate $x(2)$.	$x(2) = -2(2)^3 + 12(2)^2 + 8(2)$ $= -16 + 48 + 16$ $= 48$
6 Interpret the results.	The acceleration is zero after 2 seconds, at which time the particle is 48 metres to the right of the origin travelling at a maximum velocity of 32 m/s.

EXERCISE 6.2 Motion and derivatives

Worked Example

1 Determine an expression for $\frac{d^2y}{dx^2}$ for the following functions. 5

(a) $y = 2x^3 + 3x - 1$ (b) $y = (x-1)^2$ (c) $y = x^2 + 6x - \frac{1}{x}$

(d) $y = (2x+1)^4$ (e) $y = (x-2)(x^2-1)$

2 For $f(x) = -x^3 - 5x + 1$, calculate $f''(-1)$.

3 For the following functions, determine expressions for $f'(x)$ and $f''(x)$ and their corresponding maximal domains.

(a) $f(x) = 5x^2 - 8x + 3$ (b) $f(x) = \frac{x^2 + 4x - 2}{x}$ (c) $f(x) = \sqrt{x-2}$ (d) $f(x) = x + \sqrt{x}$

4 For each of the following position functions $x(t)$, $t \geq 0$, determine the expression for the acceleration, $a(t)$, of a particle whose position is x metres from a fixed origin at time t seconds. 7

(a) $x = 2t^2 - 3t + 4$ (b) $x = \sqrt{t+1}$ (c) $x = (t-1)^3$

5 A particle moves in a straight line so that its position, x metres, at time t seconds from a fixed origin is given by $x = 2t - t^3$, $t \geq 0$.

(a) After 2 seconds, which of the following describes the particle?

A travelling in the negative direction, speeding up

B travelling in the negative direction, slowing down

C travelling in the positive direction, speeding up

D travelling in the positive direction, slowing down

(b) Explain the common error made by a student who misunderstood the concept of a negative velocity.

6 The position, x metres, of a particle travelling in a straight line from a fixed origin at time t seconds is given by $x = t^2 - 2\sqrt{t}$, $t \geq 0$. Determine the acceleration of the body after 1 second.

7 The position, x metres, of a particle travelling in a straight line from a fixed origin at time t seconds is given by $x(t) = 3t^3 - 9t^2 + 12t - 2$, $0 \leq t \leq 3$. 8

(a) Determine the particle's acceleration at $t = 1$.

(b) Determine the particle's velocity at $t = 1$.

(c) Justify the particle's velocity being a minimum at $t = 1$.

(d) Determine, correct to 2 decimal places, the time(s) when the numerical component of the particle's velocity and the numerical component of the acceleration are equal.

8 The position, x metres, of a particle travelling in a straight line from a fixed origin at time t seconds is given by $x(t) = 2t^3 - 12t^2 + 8t$, $t \geq 0$.

(a) Determine an expression for the particle's velocity $v(t)$, in m/s, at any time.

(b) Determine an expression for the particle's acceleration $a(t)$, in m/s^2, at any time.

(c) Calculate the particle's initial acceleration.

(d) Determine the time when the particle's acceleration is zero and the position of the particle at that instant.

(e) Determine the minimum velocity and the time it occurs.

9 A particle moves in a straight line so that its position, x metres, at time t seconds is given by $x = \sqrt{t+2}$, $t \geq 0$.

(a) Justify the particle's velocity being always positive.

(b) Determine the particle's exact initial acceleration.

(c) Justify the particle's acceleration being always negative.

Worked Example

6

10 Determine an expression for $\frac{d^2y}{dx^2}$ for each of the following functions.

(a) $y = \frac{x}{x+1}$ **(b)** $y = \sqrt{3x-1}$ **(c)** $y = \frac{\sqrt{x}}{x+4}$

11 Determine the equation of the tangent to the curve $y = x^3 + 3x^2 - 5x - 3$ at the point where $\frac{d^2y}{dx^2} = 0$.

12 For $y = \sqrt{x}$, determine the value of x for which $y + \frac{d^2y}{dx^2} = 0$.

13 A particle moves in a straight line so that its position, x metres, at time t seconds is given by $x = \frac{2}{\sqrt{1+t^2}}$, $t \geq 0$.

(a) Determine the particle's initial velocity.

(b) Determine the particle's initial acceleration.

14 For $\frac{d}{dx}(g(x)) = h(x)$ and $\frac{d}{dx}(h(x)) = g(x^2)$, $\frac{d^2}{dx^2}(g(x^3))$ is equal to which of the following?

A $h(x^3)$ **B** $g(x^6)$ **C** $9x^4g(x^6) + 6xh(x^3)$ **D** $3x^2h(x^3)$

15 The position of a particle from a fixed origin O is given by $x(t) = (t-1)(t-3)(t-6)$, $0 \leq t \leq 7$, where x is the distance in centimetres, to the right is positive, and t is time in seconds.

(a) Describe the particle's behaviour, in terms of position, speed and acceleration, at the start and finish.

(b) Determine the period of time, in the form $a \leq t \leq b$, when the particle is:

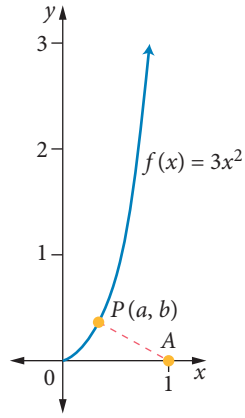
(i) to the left of the origin and moving away from the origin

(ii) to the right of the origin and moving towards the origin.

16 Let $P(a, b)$ be a point on the curve of $f(x) = 3x^2$, $0 \leq x \leq 1$, as shown in the diagram.

(a) If the function $r(a)$ gives the distance AP, where A is the point $(1, 0)$, determine expressions for $r(a)$, $r'(a)$ and $r''(a)$.

(b) Determine the range of the function $r(a)$ in exact value form.



The second derivative and curve sketching 6.3

Concavity and the second derivative

Given a function f, the second derivative, $f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$, is the rate of change of the gradient function, $f'(x) = \frac{dy}{dx}$. The second derivative indicates how the first derivative of the function f changes. Hence the second derivative can be used to provide additional information about the shape and behaviour of the graph of $y = f(x)$, in particular, the location of points of inflection.

Explore further

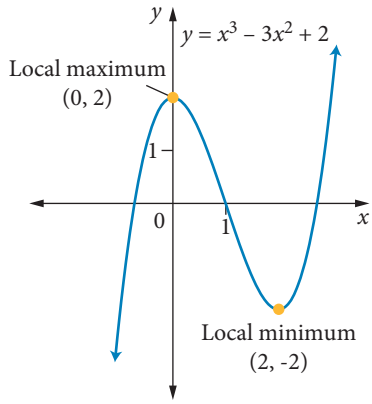
Concavity and the second derivative

Explore the graph of the second derivatives and the corresponding properties of the original functions.

Consider the curve shown on the right with equation $y = x^3 - 3x^2 + 2$.

$$\frac{dy}{dx} = 3x^2 - 6x$$
$$= 3x(x-2)$$

Therefore $\frac{dy}{dx} = 0$ at $x = 0$ and $x = 2$.

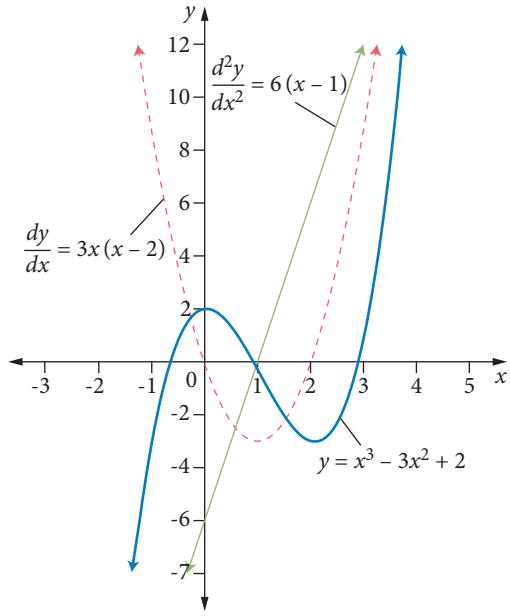


By firstly considering the coefficient of x^3 in the original function, which is positive, you can determine the shape of the cubic function; in this case the graph is positive and therefore the maximum occurs to the left of the minimum. The graph has a local maximum at $(0, 2)$ and a local minimum at $(2, -2)$.

Consider the second derivative.

$$\frac{d^2y}{dx^2} = 6x - 6$$
$$= 6(x-1)$$

Note the following results:



RESULT 1: $\frac{d^2y}{dx^2} < 0$, where $x < 1$; it follows that for $x < 1$, the rate of change of the gradient function is negative.

RESULT 2: $\frac{d^2y}{dx^2} = 0$, where $x = 1$; for $x = 1$, the rate of change of the gradient function is zero.

RESULT 3: $\frac{d^2y}{dx^2} > 0$, where $x > 1$; for $x > 1$, the rate of change of the gradient function is positive.

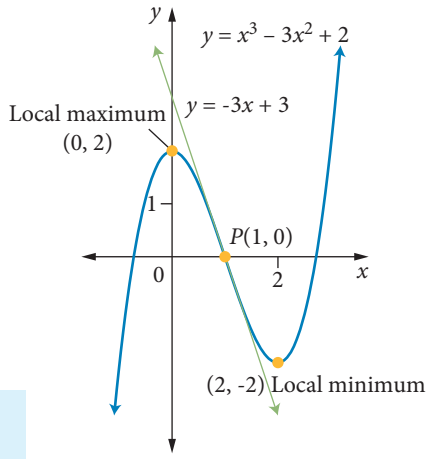
Referring to the cubic graph above, for a large negative value of x, the graph has a large positive gradient. As x increases, the gradient decreases until it reaches zero at the local maximum $(0, 2)$. The gradient continues to decrease, becoming increasingly negative until it reaches a value of -3 at $x = 1$. After this point, the gradient increases, reaching zero at the local minimum $(2, -2)$. The gradient then becomes increasingly positive as x-values increase further to the right.

Accordingly, it is said that for $x < 1$, the above graph is *concave down*.

For $x > 1$, the graph is *concave up*.

At $x = 1$, where the concavity changes, there is a *point of inflection* (POI). Thus the point $P(1, 0)$ is a point of inflection. Note that because the curvature changes from concave downwards to concave upwards at P, the tangent at P cuts the curve at P.

Concave up can be visualised as cup-shaped while concave down is dome-shaped.



If $f''(x) > 0$ (or in Leibniz notation, $\frac{d^2y}{dx^2} > 0$), then the curve $y = f(x)$ is concave up.

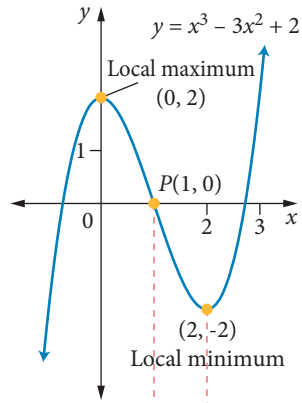
If $f''(x) < 0$ (or in Leibniz notation, $\frac{d^2y}{dx^2} < 0$), then the curve $y = f(x)$ is concave down.

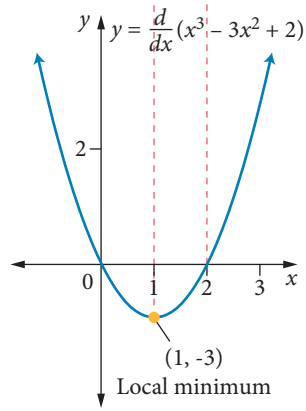
If $f''(x) = 0$ (or $\frac{d^2y}{dx^2} = 0$) at $x = a$ and if $f''(x)$ (or $\frac{d^2y}{dx^2}$) changes sign (the concavity changes) at $x = a$, then the point where $x = a$ is a point of inflection.

You have already met the idea of a point of inflection when studying the graph of the cubic $y = x^3$. The point of zero gradient at $(0, 0)$ is a *stationary point of inflection*.

If $\frac{d^2y}{dx^2} = 0$ at a point $(a, f(a))$ on the graph of $y = f(x)$, it does not necessarily follow that $(a, f(a))$ is a point of inflection. This is because the concavity may not change at this point. For example, for the curve with equation $y = x^4$, $\frac{d^2y}{dx^2} = 0$ at $x = 0$, but $(0, 0)$ is a local minimum.

If there is a point of inflection at $x = a$, then the gradient function $\frac{dy}{dx}$ is a minimum or maximum at this point. This can be seen in the graph of $y = x^3 - 3x^2 + 2$ and the graph of its derivative $\frac{dy}{dx}$ shown below. Here, the point $(1, 0)$ is a point of inflection. This is because the gradient function, $\frac{dy}{dx}$, has a local minimum of -3 at $x = 1$, and so the most negative gradient on the curve is -3 at $(1, 0)$.





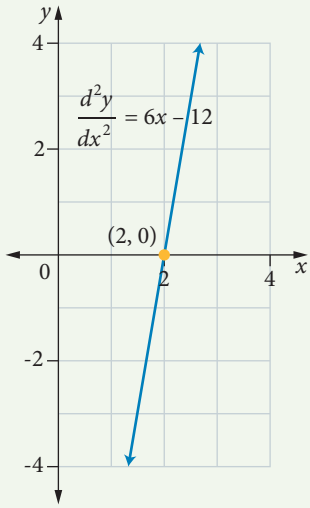
Additional information

The first and second derivative of a given function

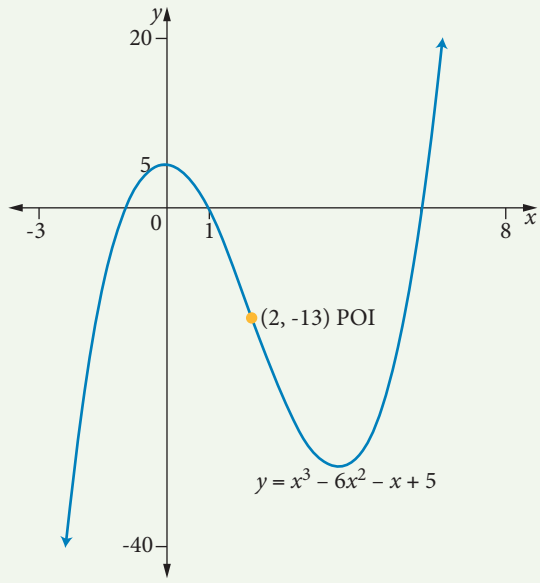
Watch a short video and try the activities to check your understanding of the relationship between a function, its first derivative and its second derivative.

9 Locate a point of inflection

Determine the coordinates of the point of inflection of the function $y = x^3 - 6x^2 - x + 5$.

THINKING	WORKING
1 Differentiate the function to give the gradient function $\frac{dy}{dx}$.	$y = x^3 - 6x^2 - x + 5$ $\frac{dy}{dx} = 3x^2 - 12x - 1$
2 Differentiate $\frac{dy}{dx}$ to give $\frac{d^2y}{dx^2}$.	$\frac{d^2y}{dx^2} = 6x - 12$
3 Let $\frac{d^2y}{dx^2} = 0$ and solve for x.	$\frac{d^2y}{dx^2} = 0$ $6x - 12 = 0$ $6x = 12$ $x = 2$
4 Interpret the value.	At $x = 2$ the rate of change of the gradient function is 0.
5 Determine whether $\frac{d^2y}{dx^2}$ changes sign at $x = 2$.	 Where $x < 2$, $\frac{d^2y}{dx^2} < 0$. Where $x > 2$, $\frac{d^2y}{dx^2} > 0$. Hence there is a point of inflection at $x = 2$.
6 Substitute the x-value into y to determine the y-coordinate of the point of inflection.	Substitute $x = 2$ into y. $y = x^3 - 6x^2 - x + 5$ $= 2^3 - 6(2)^2 - 2 + 5$ $= 8 - 24 + 3$ $= -13$
7 Interpret the answer.	The coordinates of the point of inflection are $(2, -13)$.

8 Evaluate the reasonableness of the solution by graphing the function.



The second derivative test

In previous sections, by considering the sign of the gradient in the neighbourhood of a stationary point, you were able to determine whether the stationary point was a local maximum, a local minimum or a stationary point of inflection. The sign of the second derivative provides another method of determining the behaviour and shape of a curve near a stationary point.

Additional information

Second-order derivatives

Try the activities to calculate the second derivative of polynomials.

If $f'(a) = 0$ and $f''(a) > 0$, then the curve $y = f(x)$ has a local minimum stationary point at $(a, f(a))$.

If $f'(a) = 0$ and $f''(a) < 0$, then the curve $y = f(x)$ has a local maximum stationary point at $(a, f(a))$.

WARNING

If $f'(a) = 0$ and $f''(a) = 0$, there is insufficient information to determine the nature of the stationary point. It may be a local maximum, a local minimum or a stationary point of inflection.

10 Use second derivative analysis to sketch a quartic graph

Consider the function $f(x) = x^2(6 - x^2)$.

(a) Use second derivative analysis to locate the coordinates of any stationary points and determine their nature.

THINKING	WORKING
1 Expand $f(x)$.	$f(x) = x^2(6 - x^2)$ $= 6x^2 - x^4$
2 Determine $f'(x)$ and factorise if possible.	$f'(x) = 12x - 4x^3$ $= 4x(3 - x^2)$

3 To determine the x-coordinate(s) of the stationary points, let $f'(x) = 0$ and solve for x.

Let $f'(x) = 0$:

$$4x\left(3 - x^2\right) = 0$$

$$4x\left(\sqrt{3} + x\right)\left(\sqrt{3} - x\right) = 0$$

The stationary points are at $x = -\sqrt{3}$, $x = 0$ and $x = \sqrt{3}$.

4 Substitute the x-values of the stationary points into $f(x)$ to determine the coordinates of each stationary point.

At $x = -\sqrt{3}$:

$$f\left(-\sqrt{3}\right) = \left(-\sqrt{3}\right)^2\left(6 - \left(-\sqrt{3}\right)^2\right)$$

$$= 3 \times (6 - 3)$$

$$= 9$$

At $x = 0$:

$$f(0) = 0$$

At $x = \sqrt{3}$:

$$f\left(\sqrt{3}\right) = \left(\sqrt{3}\right)^2\left(6 - \left(\sqrt{3}\right)^2\right)$$

$$= 3 \times (6 - 3)$$

$$= 9$$

The coordinates of the stationary points are $\left(-\sqrt{3}, 9\right)$, $(0, 0)$ and $\left(\sqrt{3}, 9\right)$.

5 Determine the derivative of the gradient function (the second derivative) $f''(x)$.

$$f'(x) = 12x - 4x^3$$

$$f''(x) = 12 - 12x^2$$

$$= 12\left(1 - x^2\right)$$

6 Use the second derivative test to classify each stationary point.

$$f''\left(-\sqrt{3}\right) = 12\left(1 - \left(-\sqrt{3}\right)^2\right)$$

$$= 12 \times -2$$

$$= -24$$

Given that $f''\left(-\sqrt{3}\right) < 0$, then $\left(-\sqrt{3}, 9\right)$ is a local maximum.

$$f''(0) = 12\left(1 - (0)^2\right)$$

$$= 12 \times 1$$

$$= 12$$

Given that $f''(x) > 0$, then $(0, 0)$ is a local minimum.

$$f''\left(\sqrt{3}\right) = 12\left(1 - \left(\sqrt{3}\right)^2\right)$$

$$= 12 \times -2$$

$$= -24 (< 0)$$

Given that $f''\left(\sqrt{3}\right) < 0$, then $\left(\sqrt{3}, 9\right)$ is a local maximum.

(b) Determine the coordinates of any point(s) of inflection.

1 Let $f''(x) = 0$ and solve for x.

$$12\left(1 - x^2\right) = 0$$
$$12(1 + x)(1 - x) = 0$$

The points of inflection occur at $x = -1$ and $x = 1$.

2 Determine the y-coordinates of the points of inflection.

At $x = -1$:

$$y = x^2\left(6 - x^2\right)$$
$$= (-1)^2\left(6 - (-1)^2\right)$$
$$= 5$$

At $x = 1$:

$$y = x^2\left(6 - x^2\right)$$
$$= (1)^2\left(6 - (1)^2\right)$$
$$= 5$$

3 Interpret the answer.

The coordinates of the points of inflection are $(-1, 5)$ and $(1, 5)$.

(c) Sketch the graph of $y = f(x)$, marking in all key features.

1 Substitute $x = 0$ and calculate the value of y to determine the coordinates of the y-intercept.

$$f(0) = (0)^2\left(6 - (0)^2\right)$$
$$= 0$$

The coordinates of the y-intercept are $(0, 0)$.

2 Let $f(x) = 0$ and solve for x to determine the coordinates of the x-intercept(s).

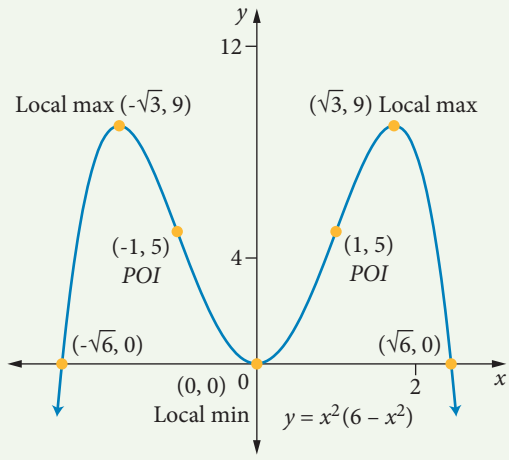
Let $f(x) = 0$:

$$x^2\left(6 - x^2\right) = 0$$
$$x^2\left(6 + \sqrt{x}\right)\left(6 - \sqrt{x}\right) = 0$$

$x = 0$, $x = -\sqrt{6}$ or $x = \sqrt{6}$.

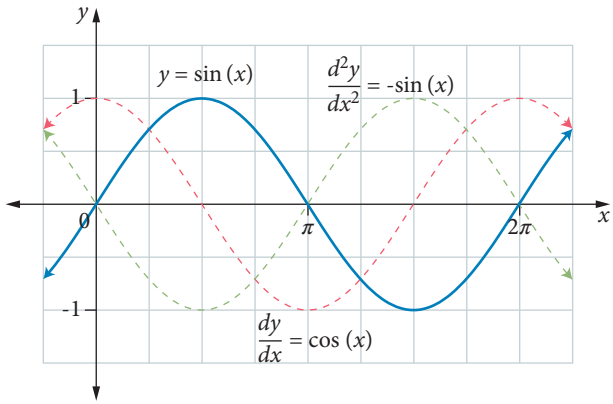
The coordinates of the x-intercepts are $\left(-\sqrt{6}, 0\right)$, $(0, 0)$ and $\left(\sqrt{6}, 0\right)$.

3 Sketch the graph of $y = f(x)$, marking in all key features.



Second derivative analysis and trigonometry

For the basic sine graph, $y = \sin(x)$, the gradient graph is given by the basic cosine graph, $\frac{dy}{dx} = \cos(x)$. If you go a step further, $\frac{d^2y}{dx^2} = -\sin(x)$, which shows that the original graph is concave down where the graph is above the x-axis, and concave up where it is below the x-axis. As expected, the concavity changes at the x-intercepts.



11 Use second derivative analysis to sketch a trigonometric graph

Consider the function $f(x) = \sin(x) + \sqrt{3}\cos(x)$, $0 \le x \le 2\pi$.

(a) Use second derivative analysis to locate the coordinates of any stationary points and determine their nature.

THINKING	WORKING
1 Determine the gradient function $f'(x)$.	$f(x) = \sin(x) + \sqrt{3}\cos(x)$ $f'(x) = \cos(x) - \sqrt{3}\sin(x)$
2 Let $f'(x) = 0$ and solve for x.	$\cos(x) - \sqrt{3}\sin(x) = 0$ $\sqrt{3}\sin(x) = \cos(x)$ $\tan(x) = \frac{1}{\sqrt{3}}$ The stationary points occur at $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.
3 Substitute each x-value into $f(x)$ to determine the coordinates of each stationary point.	$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) + \sqrt{3}\cos\left(\frac{\pi}{6}\right)$ $= \frac{1}{2} + \sqrt{3} \times \frac{\sqrt{3}}{2}$ $= \frac{1}{2} + \frac{3}{2}$ $= 2$ $f\left(\frac{7\pi}{6}\right) = \sin\left(\frac{7\pi}{6}\right) + \sqrt{3}\cos\left(\frac{7\pi}{6}\right)$ $= -\frac{1}{2} + \sqrt{3} \times \left(-\frac{\sqrt{3}}{2}\right)$ $= -\frac{1}{2} - \frac{3}{2}$ $= -2$ The coordinates of the stationary points are $\left(\frac{\pi}{6}, 2\right)$ and $\left(\frac{7\pi}{6}, -2\right)$.

4 Determine the derivative of the gradient function, $f''(x)$.

Note that $f''(x) = -f(x)$, so the original graph is concave down where the graph is above the x-axis, and concave up when it is below the x-axis.

$$f'(x) = \cos(x) - \sqrt{3}\sin(x)$$
$$f''(x) = -\sin(x) - \sqrt{3}\cos(x)$$

5 Use the second derivative test to classify each stationary point.

$$f''\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) - \sqrt{3}\cos\left(\frac{\pi}{6}\right)$$
$$= -\frac{1}{2} - \sqrt{3} \times \frac{\sqrt{3}}{2}$$
$$= -2$$

Given that $f''\left(\frac{\pi}{6}\right) < 0$, then $\left(\frac{\pi}{6}, 2\right)$ is a local maximum.

$$f''\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{7\pi}{6}\right) - \sqrt{3}\cos\left(\frac{7\pi}{6}\right)$$
$$= -\left(-\frac{1}{2}\right) - \sqrt{3} \times \left(-\frac{\sqrt{3}}{2}\right)$$
$$= 2$$

Given that $f''\left(\frac{7\pi}{6}\right) > 0$, then $\left(\frac{7\pi}{6}, -2\right)$ is a local minimum.

(b) Determine the coordinates of any point(s) of inflection.

1 Let $f''(x) = 0$ and solve for x.

$$-\sin(x) - \sqrt{3}\cos(x) = 0$$
$$-\sin(x) = \sqrt{3}\cos(x)$$
$$\tan(x) = -\sqrt{3}$$

The solutions to $f''(x) = 0$ over $0 \le x \le 2\pi$ are $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.

2 Determine the y-coordinates of the points of inflection.

$$f\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) + \sqrt{3}\cos\left(\frac{2\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2} + \sqrt{3} \times \left(-\frac{1}{2}\right)$$
$$= 0$$

$$f\left(\frac{5\pi}{3}\right) = \sin\left(\frac{5\pi}{3}\right) + \sqrt{3}\cos\left(\frac{5\pi}{3}\right)$$
$$= -\frac{\sqrt{3}}{2} + \sqrt{3} \times \frac{1}{2}$$
$$= 0$$

The coordinates of the points of inflection are the x-intercepts $\left(\frac{2\pi}{3}, 0\right)$ and $\left(\frac{5\pi}{3}, 0\right)$.

(c) Sketch the graph of $y = f(x)$, marking in all key features.

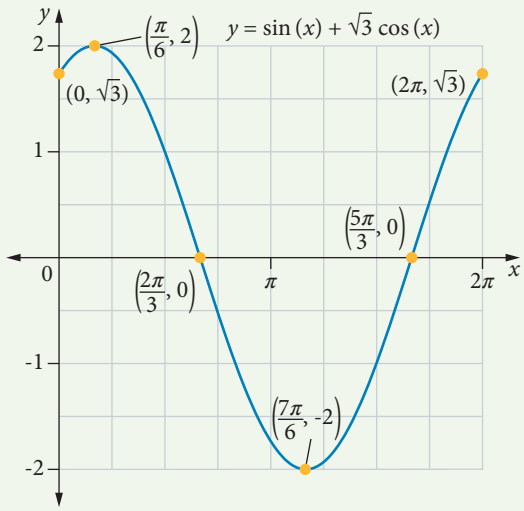
1 Substitute the endpoints $x = 0$ and $x = 2\pi$ into $y = f(x)$ and determine the coordinates of the endpoints.

$$f(0) = \sin(0) + \sqrt{3}\cos(0)$$
$$= \sqrt{3}$$

The coordinates of the y-intercept are $\left(0, \sqrt{3}\right)$.

The coordinates of the other endpoint are $\left(2\pi, \sqrt{3}\right)$.

2 Sketch the graph of $y = f(x)$, marking in all key features.



EXERCISE 6.3 The second derivative and curve sketching

Worked Example 9

1 Determine the coordinates of the point of inflection on the graph of $y = x^3 - 3x^2 - 2x - 2$.

2 Determine the coordinates of the point of inflection on the graph of $y = -2x^3 + 12x^2 + 2x - 1$.

3 Demonstrate that the graph with equation $y = -3x^2 - x + 5$ does not have a point of inflection.

4 Determine the set of values of x for which the graph with equation $y = 4x^2 - x^3$ is concave downwards.

5 The graph of the function $f(x) = \frac{3}{2}x^5 - 10x^3$ is concave up for which of the following values of x?

A $x > 0$

B $-\sqrt{2} < x < 0$ and $x > \sqrt{2}$

C $x > \sqrt{2}$

D $-2 < x < 2$

6 Consider the function $f(x) = -x^3 + 3x + 2$.

(a) Use second derivative analysis to locate any stationary points and determine their nature.

(b) Determine the coordinates of the point of inflection.

(c) Determine the coordinates of the y-intercept.

(d) Determine the coordinates of the x-intercepts and hence sketch the graph of $y = f(x)$, marking in all key features.

7 Consider the function $f(x) = x^4 - 4x$.

(a) Use second derivative analysis to locate any stationary points and determine their nature.

(b) Demonstrate that the graph $y = f(x)$ does not have a point of inflection.

(c) Determine the coordinates of the axis intercepts. Then plot the graph of $y = f(x)$, marking in all key features.

8 Consider the function $y = 5x^4 - x^5$.

(a) Where does the graph of the function have a point of inflection?

A $(0, 0)$ only

B $(3, 162)$ only

C $(0, 0)$ and $(3, 162)$

D $(0, 0)$ and $(4, 256)$

(b) Explain the common error made by a student who failed to eliminate the second solution to $\frac{d^2y}{dx^2} = 0$ in part **(a)**.

9 Consider the function $y = 2x(3x + 1)(2 - x)^2$.

(a) Determine the coordinates of the y-intercept.

(b) Determine the exact coordinates of the x-intercepts.

(c) Calculate the coordinates, correct to 2 decimal places, of the stationary points and determine their nature.

(d) Determine the coordinates of the points of inflection, correct to 2 decimal places. Then sketch the graph of $y = f(x)$, marking in all key features.

10 Consider the function $f(x) = x^3 - \frac{12}{x}$, $x \in \{[-4, 0) \cup (0, 4]\}$.

(a) Determine expressions for the first and second derivatives of the function.

(b) Determine the coordinates of any points of inflection of $f(x)$ with the given domain.

Worked Example

10 11 Consider the function $f(x) = x^2(x^2 - 3)$.

(a) Determine the coordinates of any stationary points and their nature.

(b) Determine the coordinates of the points of inflection.

(c) Determine the coordinates of the axis intercepts. Then sketch the graph of $y = f(x)$, marking in all key features.

12 Consider the function $f(x) = x^4 - 2x^2 - 8$.

(a) Use second derivative analysis to locate any stationary points and determine their nature.

(b) Determine the exact coordinates of any point(s) of inflection.

(c) Determine the coordinates of the axis intercepts and hence sketch the graph of $y = f(x)$, marking in all key features.

13 Consider the function $f(x) = \cos(x) + \sqrt{3}\sin(x)$, $0 \leq x \leq 2\pi$. 11

(a) Use second derivative analysis to locate the coordinates of any stationary points and determine their nature.

(b) Determine the coordinates of any points of inflection.

(c) Sketch the graph of $y = f(x)$, marking in all key features.

14 Consider the function $g(x) = \dfrac{1}{x^2 + k}$, $k > 0$.

(a) Use second derivative analysis to locate any stationary points and determine their nature.

(b) Determine the coordinates of any points of inflection.

(c) Determine the domain and range of g.

(d) Sketch the graph of $y = g(x)$, marking in all key features.

15 Consider the function $f(x) = x + \dfrac{4}{x}$.

(a) Use second derivative analysis to locate the coordinates of any stationary points and determine their nature.

(b) Demonstrate that the curve $y = f(x)$ does not have a point of inflection.

(c) Determine the equations of any asymptotes.

(d) Sketch the graph of $y = f(x)$, marking in all key features.

6.4 Maximum and minimum problems

Second derivative analysis with optimisation problems

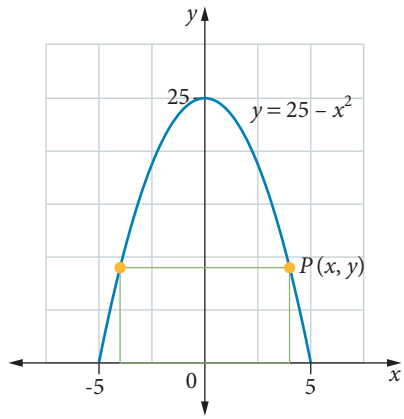
Solving a maximum or minimum problem involves determining the value of the independent variable for which a maximum or minimum value occurs, and hence determining the maximum or minimum value of a quantity (dependent variable).

The first derivative is used to locate stationary points, then the second derivative confirms that the point gives either the required maximum value ($f''(x) < 0$) or minimum value ($f''(x) > 0$).

12 Determine the maximum area of a rectangle under a parabola

A rectangle is drawn from the x-axis to the parabola $y = 25 - x^2$, as shown.

Use the second derivative analysis to determine the maximum area of the rectangle. Determine the coordinates of the first quadrant point P for which this occurs.



THINKING	WORKING
1 Define the variables.	Let x be the horizontal distance of P from the origin so $0 \leq x \leq 5$. Let y be the vertical distance of P from the origin where $y = 25 - x^2$.
2 Determine the relationship between the variable to be optimised and a single known variable.	Area of rectangle: $A = lw$ $A(x) = 2x\left(25 - x^2\right)$
3 Calculate the first and second derivatives.	$A(x) = 50x - 2x^3$ $A'(x) = 50 - 6x^2$ $A''(x) = -12x$

4 Determine the location of the stationary points within the domain and determine their nature.

Let $A'(x) = 0$:

$$50 - 6x^2 = 0$$

$$x^2 = \frac{25}{3}$$

$$x = \frac{5}{\sqrt{3}}, (x \geq 0)$$

$$A''\left(\frac{5}{\sqrt{3}}\right) = -12 \times \frac{5}{\sqrt{3}}$$

$$= -\frac{60}{\sqrt{3}}$$

Given that $A''\left(\frac{5}{\sqrt{3}}\right) < 0$, then a maximum area occurs at $x = \frac{5}{\sqrt{3}}$.

5 Determine the maximum area of the rectangle under the curve.

$$A\left(\frac{5}{\sqrt{3}}\right) = 50 \times \frac{5}{\sqrt{3}} - 2\left(\frac{5}{\sqrt{3}}\right)^3$$

$$= \frac{250}{\sqrt{3}} - \frac{250}{3\sqrt{3}}$$

$$= \frac{750 - 250}{3\sqrt{3}}$$

$$= \frac{500}{3\sqrt{3}}$$

The maximum area of the rectangle under the curve is $\frac{500}{3\sqrt{3}}$ units^2.

6 Substitute the known x-value into y to determine the coordinates of point P.

When $x = \frac{5}{\sqrt{3}}$,

$$A\left(\frac{5}{\sqrt{3}}\right) = 25 - \left(\frac{5}{\sqrt{3}}\right)^2$$

$$= 25 - \frac{25}{3}$$

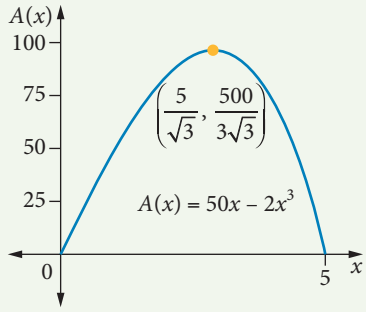
$$= \frac{50}{3}$$

The coordinates of point P on the curve are $\left(\frac{5}{\sqrt{3}}, \frac{50}{3}\right)$.

7 Interpret the answer.

The rectangle has a maximum area of $\frac{500}{3\sqrt{3}}$ square units when P is $\left(\frac{5}{\sqrt{3}}, \frac{50}{3}\right)$.

8 Where possible, evaluate the reasonableness of your result by drawing a graph.



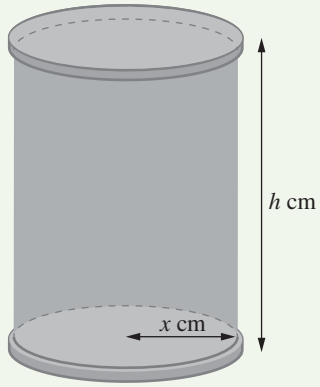
Optimisation with technology

When solving optimisation problems with technology, a graph can assist the process early on, with calculations confirming the results.

13 Relationship between height and radius of a cylinder to minimise surface area

Ben buys baked beans in a cylindrical can with a volume of $440\,\text{cm}^3$.

(a) If x cm is the radius of the circular cross-section of the cylindrical can, determine an expression in terms of x for the surface area, $A\,\text{cm}^2$, of the thin tin sheet required to make the can. (Assume no extra tin sheet is needed at the joins.)

THINKING	WORKING
1 Draw a labelled diagram and define the variables.	Let h cm be the height of the can and $V\,\text{cm}^3$ be its volume. 
2 Recall formulas connecting the variables. Substitute the given value. Eliminate the non-requested variable.	$V = \pi r^2 h$, where $V = 440$, $r = x$ $\pi x^2 h = 440$ $h = \frac{440}{\pi x^2}$ Surface area: $A = 2\pi r^2 + 2\pi rh$, where $r = x$ and $h = \frac{440}{\pi x^2}$ $A = 2\pi x^2 + 2\pi x \times \frac{440}{\pi x^2}$ $= 2\pi x^2 + \frac{880}{x}$

(b) Use second derivative analysis to determine the minimum area of tin sheet required and determine the dimensions of the can in this case. Give all answers correct to 2 decimal places.

1 Determine the gradient function $\frac{dA}{dx}$.

$$\frac{dA}{dx} = 4\pi x - \frac{880}{x^2}$$

2 Let $\frac{dA}{dx} = 0$ and solve for x to determine where the area is a maximum or minimum.

$$4\pi x - \frac{880}{x^2} = 0$$
$$4\pi x^3 - 880 = 0$$
$$4\pi x^3 = 880$$
$$x^3 = \frac{880}{4\pi}$$
$$x^3 = \frac{220}{\pi}$$
$$x = \sqrt[3]{\frac{220}{\pi}}$$
$$x = 4.12 \text{ (2d.p.)}$$

3 Interpret the value.

A stationary point exists at $x \approx 4.12$.

4 Determine the rate of change of the gradient function $\frac{d^2A}{dx^2}$.

$$\frac{d^2A}{dx^2} = 4\pi + \frac{1760}{x^3}$$

5 Determine the sign of $\frac{d^2A}{dx^2}$ at the stationary point and draw a conclusion on its nature.

Where $x > 0$, $\frac{d^2A}{dx^2} > 0$

Therefore $A(x)$ is a minimum where $x = 4.12$.

6 Substitute the x-value to determine the height of the can.

$$h(x) = \frac{440}{\pi x^2}$$
$$h\left(\sqrt[3]{\frac{220}{\pi}}\right) = 8.24 \text{ (2d.p.)}$$

The minimum area of tin sheeting occurs when the radius of the can is approximately 4.12 cm and the height is 8.24 cm.

7 Substitute the x-value to determine the minimum area.

$$A(x) = 2\pi x^2 + \frac{880}{x}$$
$$A\left(\sqrt[3]{\frac{220}{\pi}}\right) = 320.25 \text{ (2d.p.)}$$

The can with a minimum amount of tin sheet has surface area $320.25\,\text{cm}^2$.

8 Interpret the answer.

The minimum surface area of the can is $320.25\,\text{cm}^2$ and occurs when the radius of the can is approximately 4.12 cm and the height is 8.24 cm.

(c) Determine the relationship between the height and radius of the can in this case.

1 Determine the ratio between h and r.	Height: 8.24 cm; radius: 4.12 cm $\frac{h}{r} = \frac{8.24}{4.12} = 2$
2 Interpret the answer.	The height is twice the radius.

Endpoint maxima/minima

In many cases of optimisation, the extreme values of the domain give a value of zero for the quantity, so need not be considered when determining maximum values. When a domain is restricted in less obvious ways, the endpoints need to be checked to determine whether the maximum or minimum value of the function occurs there.

14 Solve endpoint maximum/minimum problems

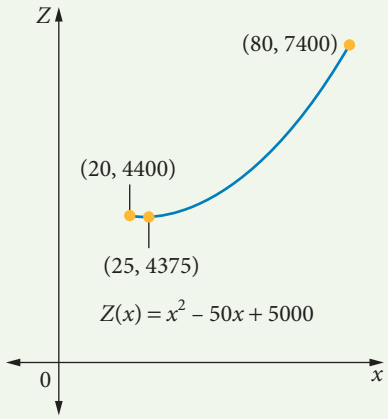
Given $x + y = 100$, $20 \leq x \leq 80$, use second derivative analysis to maximise the value of Z where $Z = x^2 + 50y$.

THINKING	WORKING
1 Use substitution to write the variable to be maximised in terms of a single variable.	$x + y = 100$, so $y = 100 - x$ $Z = x^2 + 50y$ $Z(x) = x^2 + 50(100 - x)$ $= x^2 - 50x + 5000$
2 Determine the first and second derivatives.	$Z'(x) = 2x - 50$ $Z''(x) = 2$
3 Let $Z'(x) = 0$ and solve for x to determine the location of the stationary point.	$2x - 50 = 0$ $2x = 50$ $x = 25$ $Z(25) = 25^2 - 50 \times 25 + 5000$ $= 625 - 1250 + 5000$ $= 4375$ Since the second derivative is positive, the stationary point $(25, 4375)$ is a local minimum.
4 Calculate the coordinates of the endpoint values.	$Z(20) = 20^2 - 50 \times 20 + 5000$ $= 400 - 1000 + 5000$ $= 4400$ The coordinates of the left-hand endpoint are $(20, 4400)$. $Z(80) = 80^2 - 50 \times 80 + 5000$ $= 6400 - 4000 + 5000$ $= 7400$ The coordinates of the right-hand endpoint are $(80, 7400)$.

5 Determine the maximum value within the restricted domain.

Sketch a basic graph to support the conclusion.

The maximum value of Z is 7400, where $x = 80$.

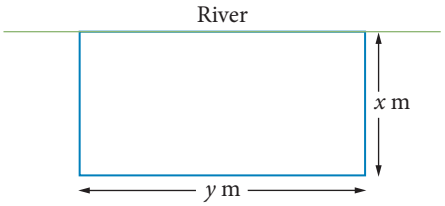


EXERCISE 6.4 Maximum and minimum problems

Worked Example 12

1 A rectangle has two of its vertices on the x-axis and the other two lie above the x-axis on the parabola $y = 9 - x^2$. Determine the dimensions of the rectangle for it to have maximum area.

2 A rectangular field borders a straight river. It is enclosed on three sides (excluding the riverbank) by 60 metres of fencing. The side of the field perpendicular to the river is x metres and its length is y metres.



(a) Express y in terms of x.

(b) Use second derivative analysis to determine the maximum area of the field.

3 A rectangular field borders a straight river. It is enclosed on three sides (excluding the riverbank) by L m of fencing. The width of the field that runs perpendicular to the river is x m, and its length, which runs parallel to the river, is y m.

(a) Determine an expression for the length y in terms of L and x.

(b) Determine an expression in factorised form for the area $A(x)$ in terms of L and x.

(c) Use second derivative analysis to determine the maximum area of the field in terms of L.

(d) Hence determine the dimensions of the field of maximum area in terms of L.

4 The sum of two numbers is S.

(a) Determine expressions for the numbers, in terms of S, if their product is a maximum.

(b) Choose the correct statement describing the maximum product.

A could be positive, negative or zero
B must be positive or zero
C must be positive
D must be positive or negative

(c) Explain the error made by a student who assumed the product could be negative.

5 The sum of two non-negative numbers is 30.

(a) Use second derivative analysis to determine the minimum value for the sum of their squares.

(b) Calculate the maximum value for the sum of their squares.

6 A rectangular paddock of dimensions x m by y m has a fixed perimeter of P m.

(a) Express y in terms of x and P.

(b) The area of the rectangular paddock is A m^2. Determine an expression for $A(x)$.

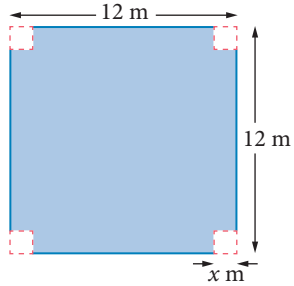
(c) Determine the domain of $A(x)$.

(d) Determine the expressions for $A'(x)$ and $A''(x)$.

(e) Determine the value of x such that $\frac{dA}{dx} = 0$ and explain why this value of x corresponds to a maximum.

(f) In terms of P, determine the maximum area of the paddock and specify its dimensions. What shape is the paddock?

7 A square sheet of metal measures 12 m by 12 m. A square of side length x m is cut from each corner and the resulting piece of metal is then folded and welded to form an open-topped box. Let V cubic metres be the volume of the box.



(a) Express V in terms of x in factorised form.

(b) Determine the domain of $V(x)$.

(c) Use second derivative analysis to determine the value of x for which the box has a maximum volume. Hence determine the maximum volume and the dimensions of the box in this case and sketch the graph of $V(x)$.

8 Use second derivative analysis to determine the coordinates of the point on the graph of $f(x) = \frac{1}{x}$, $x > 0$, that is closest to the origin.

9 Consider all cylinders whose height and base circumference sum to 30 metres. What radius of the cylinder, in metres, gives the maximum volume of the cylinder?

A 10 **B** 20 **C** $\frac{30}{\pi^2}$ **D** $\frac{10}{\pi}$

10 A rectangular piece of metal measures 6 m by 12 m. A square of side length x m is cut from each corner and the resulting piece of metal is then folded and welded to form an open-topped box. Let V m^3 be the volume of the box.

(a) Determine an expression for $V(x)$ in factorised form, including the domain.

(b) Use second derivative analysis to determine the exact value of x for which the box has a maximum volume.

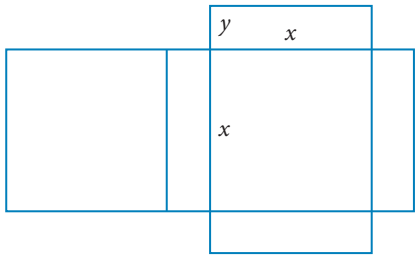
(c) Calculate the maximum volume of the box.

11 The sum of two non-negative numbers, x and y, is 12.

(a) Determine the maximum value of $x^3 + y^2$.

(b) Determine the minimum value of $x^3 + y^2$, correct to 2 decimal places.

12 The surface area of a shoebox in the form of a square-based prism is 5400 cm^2. The dimensions of the box are $x \times x \times y$ as shown in the diagram.



(a) Write an expression for the volume of the box, $V(x)$.

(b) Determine the maximum volume and the dimensions of the box that produce the maximum volume.

(c) Determine the domain of $V(x)$ and sketch its graph.

Worked Example

13 A company plans to melt a block of metal to form a solid cylinder of volume 10 m^3. The plan involves making a cylinder whose external surface area is a minimum. The external surface area, A m^2, of a cylinder whose base radius is r metres and height h metres is given by $A = 2\pi rh + 2\pi r^2$.

(a) Determine an expression for $A(r)$.

(b) Determine the exact value of r for which $A'(r) = 0$.

(c) Use second derivative analysis to determine that the value of r found in part **(b)** gives the minimum external surface area.

(d) Determine the minimum external surface area possible for this cylinder. Give your answer correct to 1 decimal place.

(e) Use index laws to determine the relationship between the height and radius of the cylinder in this case.

14 Given $2x + y = 40$, $5 \leq x \leq 15$, use second derivative analysis to determine the maximum value of Z where $Z = 2x^2 + 12y$.

15 Determine the dimensions of the rectangle with the greatest area that can be inscribed in the semicircle with curve section given by $y = \sqrt{4 - x^2}$.

16 The selling price per item, S dollars, is $50 - 0.03x$, where x is the number of items produced in a week. The weekly cost, C dollars, of producing and selling x items is $6x + 2000$. Determine how many items should be produced and sold each week to maximise profit.

17 A metal wire of length 80 cm is cut into two pieces. One piece is shaped into a rectangle, twice as long as it is wide, and the other is shaped into a circle. Let x be the length used for the circle.

(a) Determine an expression for the area enclosed by the circular piece of metal wire, $A(x)$.

(b) Determine the rule, $B(x)$, giving the area enclosed by the rectangular piece of wire.

(c) The total area enclosed by the two pieces of metal wire is T square metres. Determine an expression for $T(x)$.

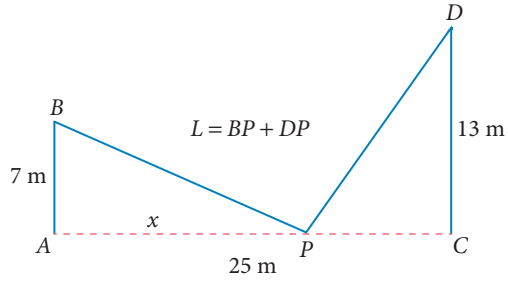
(d) Determine the exact value of x that minimises the total area enclosed by the two pieces of metal wire.

(e) Determine the area of each piece such that the area is a minimum.

(f) When the total enclosed area is a minimum, determine the exact ratio of the area enclosed by the circle to the area enclosed by the rectangle.

(g) How should the metal wire be divided to maximise the total area enclosed?

18 Two vertical posts, 25 m apart and with heights of 7 m and 13 m, are secured by a rope that connects the top of one post to a point on the ground between the posts and then to the top of the other post. The length of rope is L m and the distance from the 7 m high post to the point on the ground between the two posts is x m.



(a) Write an expression for $L(x)$.

(b) Determine the location of the point on the ground so that the amount of rope used is a minimum.

(c) Determine, correct to 1 decimal place, the minimum length of rope used.

19 A company produces x litres of juice per minute. To produce the juice, it costs the company \$2 per minute and an extra \$1 per litre. The revenue per minute is 6 times the square root of the number of litres produced in that time.

(a) Write an expression for $P(x)$, the profit of the company per minute.

(b) Determine the number of litres the company has to produce per hour for maximum profit and calculate the maximum profit.

(c) Sketch the graph of $P(x)$ and determine the domain of the function if the company is to make a profit.

20 The pollution level, L units, at a point on a straight road between two factories 5 kilometres apart is modelled by the function $L(x) = \frac{m}{x+1} + \frac{n}{11-x}$, $0 \leq x \leq 10$, where x km is the distance of the point from one of the factories. The pronumerals m and n are related to the output of pollutants of the two factories. Let $m = 4$ and $n = 16$.

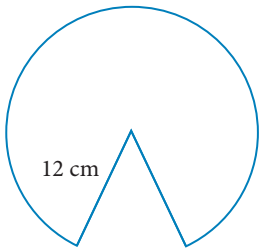
(a) Use calculus to determine the value of x for which the pollution level is a minimum.

(b) Determine the minimum pollution level at this point between the two factories.

(c) Determine the exact set of values of x for which $L \leq 4$.

(d) If $m = n$, determine the value of m for which the minimum value of L is 3.

21 A company manufactures conical party hats. Each hat is made by cutting, from a piece of paper, a small sector of a circle of radius 12 cm and then folding together the edge radii of the large sector to create the cone. If the large sector is a fraction x of the original circular paper, where $0.5 < x < 1$, the volume $V(x)\text{cm}^3$ of the cone is given by $V(x) = 576\pi x^2\sqrt{1-x^2}$.



(a) Determine an expression for $V'(x)$.

(b) Determine the exact value of x for which the volume of the cone is a maximum.

(c) Plot the graph of $V(x)$ to determine the maximum volume, correct to 1 decimal place.

(d) The company realises that they can make two cones from the circular piece of paper by using both sectors. Given that the first sector is a fraction x of the original circular paper, where $0.5 < x < 1$, the second sector is a fraction, $1 - x$, of the paper. The total volume, $T(x)\text{cm}^3$, of the two cones formed is given by: $T(x) = 576\pi\left(x^2\sqrt{1-x^2} + (1-x)^2\sqrt{2x-x^2}\right)$. Sketch the graph of $y = T(x)$ for $0 < x < 1$. Use your graph to determine the maximum volume, correct to 1 decimal place. Interpret the value of x that gives the maximum volume.

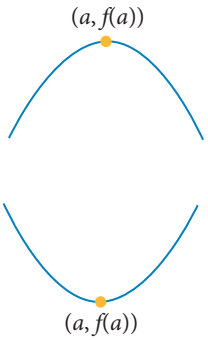
6

Summary

The derivative and curve sketching

A *local maximum stationary point* at $x = a$ on the graph of $y = f(x)$ has the following properties.

- $f'(a) = 0$ and $f'(x) > 0$, for $x < a$.
- $f'(a) < 0$, for $x > a$ in the neighbourhood of $x = a$.

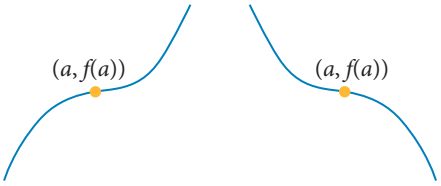


A *local minimum stationary point* at $x = a$ on the graph of $y = f(x)$ has the following properties.

- $f'(a) = 0$ and $f'(a) < 0$, for $x < a$.
- $f'(a) > 0$, for $x > a$ in the neighbourhood of $x = a$.

A *stationary point of inflection* at $x = a$ on the graph of $y = f(x)$ has the following properties.

- $f'(a) = 0$ and $f'(a) > 0$ or $f'(a) < 0$ in the neighbourhood of $x = a$.
- A function f is increasing at $x = a$ if $f'(a) > 0$.
- A function f is decreasing at $x = a$ if $f'(a) < 0$.



Global maxima and minima

A maximum point for the entire domain is called an *absolute or global maximum*.

A minimum point for the entire domain is called an *absolute or global minimum*.

The global maximum and global minimum values of a polynomial function f for $x \in [a,b]$ can be found by applying the following method.

STEP 1: Determine the x-coordinates of all stationary points of f for $x \in (a,b)$.

STEP 2: Calculate $(f(a), f(b))$ and the function values of all the stationary points for $x \in (a,b)$.

STEP 3: Select the largest value of $f(x)$ from STEP 2. This is the global maximum of f.

STEP 4: Select the smallest value of $f(x)$ from STEP 2. This is the global minimum of f.

Motion and derivatives

If $y = f(x)$, the first derivative is $\frac{dy}{dx} = f'(x)$ and the second derivative is $\frac{d^2y}{dx^2} = f''(x)$.

The derivative of position with respect to time, $x'(t) = \frac{dx}{dt}$, gives the *velocity*, $v(t)$, of the particle.

The second derivative of position with respect to time, $x''(t) = \frac{d^2x}{dt^2}$, gives the *acceleration*, $a(t)$, of the particle. So $a(t)$, or alternatively $v'(t)$, is the rate of change of velocity with respect to time.

The second derivative and curve sketching

If $f''(x) > 0$, then the curve $y = f(x)$ is concave up.

If $f''(x) < 0$, then the curve $y = f(x)$ is concave down.

If $f''(x) = 0$ at $x = a$ and if $f''(x)$ changes sign (the concavity changes) at $x = a$, then the point where $x = a$ is a point of inflection.

If $f'(a) = 0$ and $f''(a) > 0$, then the curve $y = f(x)$ has a local minimum at $(a, f(a))$.

If $f'(a) = 0$ and $f''(a) < 0$, then the curve $y = f(x)$ has a local maximum at $(a, f(a))$.

If $f'(a) = 0$ and $f''(a) = 0$, another method should be used to determine the nature of the stationary point.

Maximum and minimum problems

Determining the value of the independent variable for which a maximum or minimum value occurs.

Hence determining the maximum or minimum value of a quantity (dependent variable).

The first derivative is used to locate stationary points.

The second derivative confirms that the point gives either the required maximum or minimum value.

Chapter review 6

Exercise 6.1

1 Consider the function $f(x) = -x(x-6)^2$.

(a) Use calculus to locate any stationary points and determine their nature.

(b) Sketch the graph of $y = f(x)$, showing all key features.

Exercise 6.1

2 For the function $f(x) = (x+3)^2(x-2)$, what is the subset of $\mathbb{R}$ for which the gradient of f is negative?

Exercise 6.1

3 When does the cubic polynomial $g(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants, have two stationary points?

A $3ac < b^2$

B $3ac > b^2$

C $3ac = b^2$

D $4ac > b^2$

Exercise 6.1

4 If $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 5$, $0 \le x \le 9$, determine the absolute maximum value of f and the value of x when this occurs.

Exercise 6.2

5 Determine expressions for the acceleration, $a(t)$, of each of the following particles whose positions in x metres from a fixed origin at time t seconds, $t \ge 0$, are given by the following equations.

(a) $x = -5t^2 + 5t - 2$ **(b)** $x = \sqrt{2t-3}$ **(c)** $x = (2-t)^3$

Exercise 6.2

6 A particle moves in a straight line so that its position, x metres, at time t seconds is given by $x(t) = t^2 + 4t + 4$, $t \ge 0$. What is the acceleration in m/s^2 of the particle at time $t = 4$?

A 0

B 2

C 4

D 8

Exercise 6.2

7 The position x metres from a fixed origin O at time t seconds of a particle travelling in a straight line is given by $x(t) = t^3 - 5t^2 + 3t - 5$ for $0 \le t \le 7$.

(a) Determine the initial position of the particle.

(b) Determine the time when the particle's acceleration is zero.

(c) Determine the particle's minimum velocity.

(d) Determine the exact time when the numerical component of the particle's velocity and the numerical component of the acceleration are equal. Interpret your results.

(e) Determine the exact times when the particle changes direction.

8 A particle moves in a straight line so that its position x metres at time t seconds from a fixed origin is given by $x = 3 + 5t - t^3$, $t \geq 0$. After 3 seconds, which of the following describes the particle? Exercise 6.2

- **A** travelling in the positive direction, slowing down
- **B** travelling in the negative direction, slowing down
- **C** travelling in the positive direction, speeding up
- **D** travelling in the negative direction, speeding up

9 If $g(x) = -3x^4 + x^2 + 5$, calculate $g''(1)$. Exercise 6.3

10 Determine the equation of the tangent to the curve $y = -2x^3 + 6x^2 + 3x - 7$ at the point where $\frac{d^2y}{dx^2} = 0$. Exercise 6.3

11 Determine the set of x values for which the graph with equation $y = x^3 - 8x^2 + 4x - 5$ is concave upwards. Exercise 6.3

12 Determine the values of x for which the graph of $f(x) = \frac{-3}{x-2}$ is concave downwards. Exercise 6.3

13 If $g''(x) = x(x+2)(x-3)^2$, then the graph of g has inflection points when x is equal to which of the following? Exercise 6.3

- **A** -2 only
- **B** -2 and 0 only
- **C** -2 and 3 only
- **D** -2, 0 and 3

14 Determine the value of b and c if the graph of $y = x^3 + bx^2 + cx - 4$ has a point of inflection at $(1, -5)$. Exercise 6.3

15 Determine the expression for the second derivative of each of the following functions. Exercise 6.3

(a) $f(x) = -x^4 + 3x^2 - x + 7$ **(b)** $y = \frac{x+1}{x-1}$

16 A rectangular field is to be fenced so that the resulting area is k square units. Use second derivative analysis to determine the dimensions of the field for which the perimeter is a minimum. Exercise 6.4

17 Use second derivative analysis to determine the dimensions of the cylindrical can of greatest volume that can be made from $600\pi\,\text{cm}^2$ of sheet metal. Exercise 6.4

18 Given $y = 2x - 8$, determine the minimum value of the product xy. Exercise 6.4

19 Consider the function $f(x) = x^3 + 3x^2 + 3x + 28$. Exercise 6.1

(a) Use second derivative analysis to determine the coordinates and nature of any stationary points.

(b) Sketch the graph of $y = f(x)$, showing all key features.

20 Determine an expression for $\frac{d^2y}{dx^2}$ for each of the following functions. Exercise 6.2

(a) $y = \frac{2x}{x-3}$ **(b)** $y = \sqrt{5-2x}$

Exercise 6.2 **21** A particle moves in a straight line so that its position x metres at time t seconds is given by $x = t\sqrt{t+1}$, $t \geq 0$.

(a) Determine the particle's initial velocity.

(b) Determine the particle's initial acceleration.

Exercise 6.3 **22** Consider the function $g(x) = x^4 - x^2$.

(a) Use second derivative analysis to locate any stationary points and determine their nature.

(b) Determine the exact coordinates of any points of inflection.

(c) Determine the coordinates of any axis intercepts and then plot the graph of g, marking all key features.

Exercise 6.3 **23** Consider the function $f(x) = ax^3 + bx^2 + cx + d$.

(a) Determine the values of a, b, c and d if $f(0) = -\frac{1}{2}$, $f'(0) = 5$, $f'(4) = -63$ and $f(1) = 6$. Hence determine the expression for $f(x)$.

(b) Determine the coordinates of the stationary points, using the second derivative analysis to determine their nature.

(c) Determine the coordinates of the point of inflection, and the intervals for which $f(x)$ is increasing and decreasing, then sketch the graph of $f(x)$.

Exercise 6.3 **24** Consider the function $f(x) = x^3 - 12x + d$.

(a) Let $d = 0$.

(i) Use second derivative analysis to locate any stationary points and determine their nature.

(ii) Sketch the graph of $y = f(x)$, showing the features identified so far as well as the axis intercepts.

(b) Let d be any number.

(i) Determine the set of values of d for which the graph of $y = f(x)$ has three distinct x-intercepts.

(ii) Determine the set of values of d for which the graph of $y = f(x)$ meets the x-axis in precisely two distinct points.

(iii) For each value of d found in part **(ii)**, determine the coordinates of the two distinct x-intercepts.

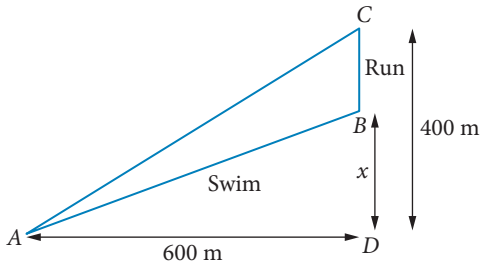
Exercise 6.4 **25** A piece of wire of length p metres is to be cut with one piece forming an equilateral triangle and the other forming a circle. Let the length of wire used to form an equilateral triangle be $3x$ metres.

(a) Determine the set of possible values that x can take.

(b) Determine an expression for the total area $A(x)$, in square metres, of the regions enclosed by the wire.

(c) Use second derivative analysis to determine the value of x for which the total area is a minimum.

26 Jacob participates in a school sporting event where he has to swim from a point 600 m away from the beach and then run to a designated place as shown in the diagram. Jacob swims at an average rate of 1 m/s and runs at a rate of 4 m/s. Exercise 6.4

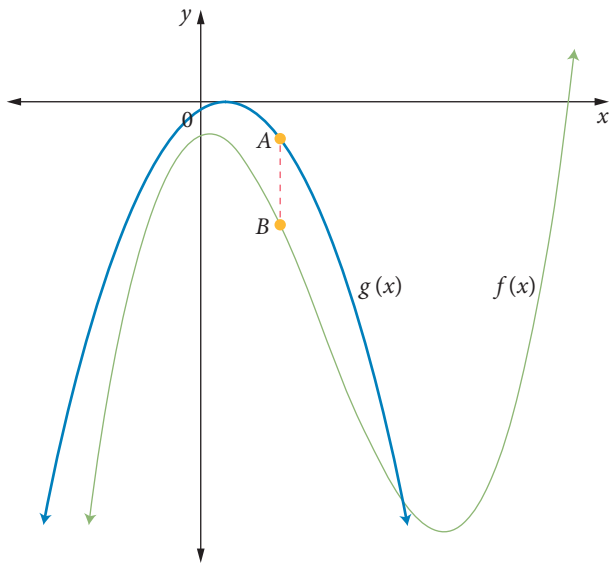


(a) If the distance BD is x m, write an expression for the time (in seconds) taken to cover the distance from point A to point C, passing through B, in terms of x.

(b) How far from the designated point C should Jacob come out from the water in order to minimise his time and what is this minimum time? Give your answers correct to 1 decimal place.

(c) Determine the domain of the function $t(x)$, sketch its graph and explain the meaning of the shape of the graph for this practical situation.

27 Consider the functions $f(x) = x^3 - 9x^2 + 3x - 7$ and $g(x) = -5x^2 + 7x - 2$ as shown in the diagram. Exercise 6.4



(a) Write an expression in terms of x for the vertical distance AB shown.

(b) Determine the coordinates of points A and B such that the vertical distance AB is a local minimum (but not a global minimum) and calculate this distance. Write your results correct to 1 decimal place.

7

Sine and cosine rules

Recall 370
7.1 Trigonometry review 372
7.2 The area of a triangle 383
7.3 The sine rule 389
7.4 The cosine rule 399
Summary 409
Chapter review 412

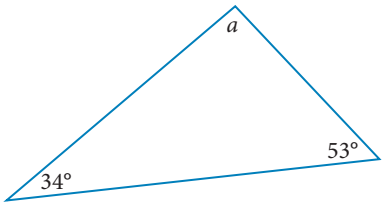
7

Recall

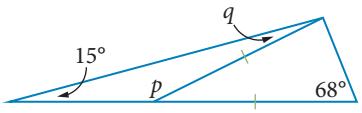
Use degree measurements for angles

1 Determine the size of the unknown angles in each of the following diagrams.

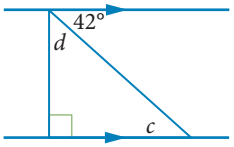
(a)



(b)



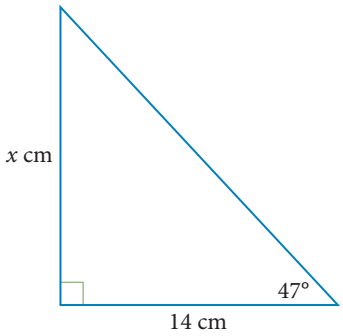
(c)



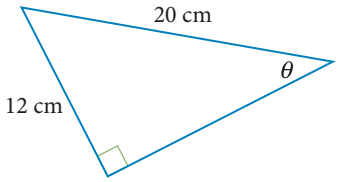
Identify the trigonometric ratios in right-angled triangles

2 For each of the following right-angled triangles, determine the trigonometric equation that should be used to solve for the unknown in each case.

(a)



(b)



Solve equations involving trigonometric ratios

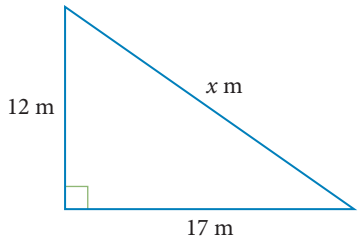
3 Solve each of the following, giving your answer correct to 2 decimal places. Ensure that your technology is in the appropriate angle mode.

(a) $\sin(20°) = \frac{x}{9}$ (b) $\cos(20°) = \frac{3.4}{x}$ (c) $\tan(\theta) = \frac{3}{4}$, where $\theta < 90°$

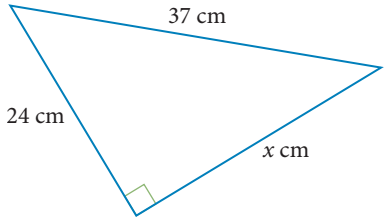
Use Pythagoras' theorem to solve right-angled triangles

4 Calculate the unknown side length in each of the following right-angled triangles. Give your answers correct to 2 decimal places.

(a)



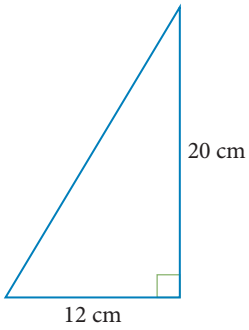
(b)



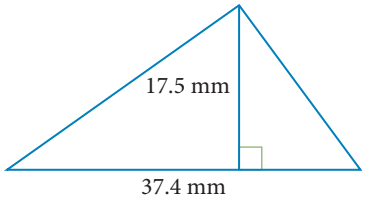
Calculate the area of a triangle given its base length and height

5 Calculate the exact area of each of the following triangles.

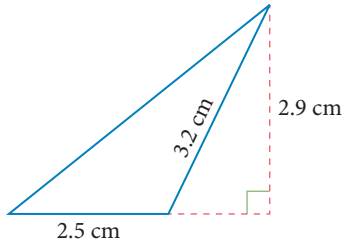
(a)



(b)



(c)



Convert angles in degrees to radians, in terms of π

6 Convert each of the following to radians, in terms of π.

(a) $20°$ **(b)** $135°$ **(c)** $390°$

Convert radians to degrees

7 Convert each of the following to degrees.

(a) $\frac{\pi}{5}$ **(b)** $\frac{5\pi}{6}$ **(c)** $\frac{7\pi}{3}$

Simplify trigonometric expressions using symmetry properties

8 Simplify each of the following expressions.

(a) $\cos(180° - \theta)$ **(b)** $\tan(\pi + \theta)$ **(c)** $\sin(2\pi - \theta)$

Use special triangles to calculate exact trigonometric ratios

9 Calculate the exact value of each of the following expressions.

(a) $\tan(60°)$ **(b)** $\sin(135°)$ **(c)** $\cos(150°)$

7.1 Trigonometry review

Solving for sides in right-angled triangles

To solve for any side of a right-angled triangle, you need to identify two sides in terms of one of the acute angles, either opposite (O), adjacent (A) or hypotenuse (H). One of the sides should be known, and the other should be the side you are required to solve. Determine which rule to use with the help of the mnemonic SOH-CAH-TOA:

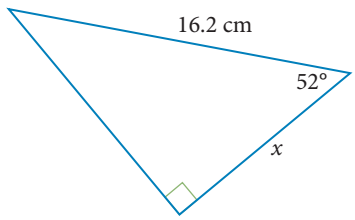
$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}, \cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$

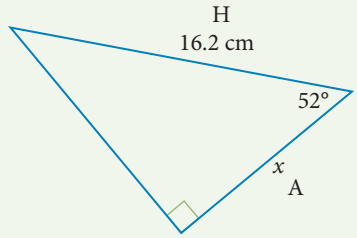
Substitute the given values, using a pronumeral for the unknown and solve for the unknown quantity.

1 Calculate the lengths of unknown sides

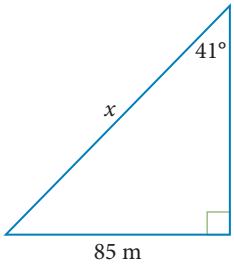
Calculate the length of the side labelled x in each of the following, correct to 2 decimal places.

(a)

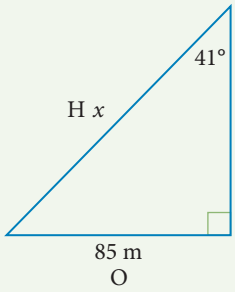


THINKING	WORKING
1 Label the sides to be used in the ratio.	
2 Recall an appropriate trigonometric ratio.	$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$
3 Substitute the given values, using a pronumeral for the unknown and solve for the unknown.	$\frac{x}{16.2} = \cos(52°)$ $x = 16.2\cos(52°)$ $x = 9.97\text{ cm}\ (2\text{ d.p.})$
4 Evaluate the reasonableness of the solution.	The answer is smaller than the hypotenuse, as expected.

(b)



1 Label the sides to be used in the ratio.



2 Recall an appropriate trigonometric ratio.

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

3 Substitute the given values, using a pronumeral for the unknown and solve for the unknown.

$$\sin(41°) = \frac{85}{x}$$
$$x\sin(41°) = 85$$
$$x = \frac{85}{\sin(41°)}$$
$$x = 129.56\,\text{m}\ (2\ \text{d.p.})$$

4 Evaluate the reasonableness of the answer.

The answer is larger than the given side, which is reasonable because it is the hypotenuse.

Solving for angles in right-angled triangles

To solve for an unknown angle in a right-angled triangle, you need to identify two known sides in terms of the acute angle to be solved.

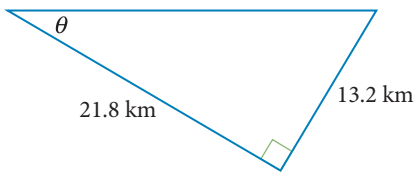
Additional information

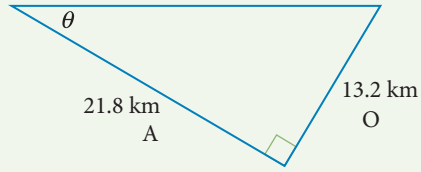
Calculating the lengths of missing sides

Use this activity for additional practice at solving for sides of right-angled triangles.

2 Calculate the size of an unknown angle

Calculate the size of the angle θ in the following triangle, to the nearest degree.



THINKING	WORKING
1 Label the sides to be used.	
2 Recall an appropriate trigonometric ratio.	$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$
3 Substitute the given values and solve for θ.	$\tan(\theta) = \frac{13.2}{21.8}$ $\theta = \tan^{-1}\left(\frac{13.2}{21.8}\right)$ $\theta = 31°$ (nearest degree)
4 Evaluate the reasonableness of the answer.	The angle is less than 45° because it is opposite the smaller of the two sides.

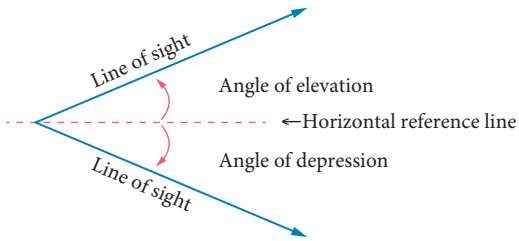
Additional information

Determine the size of unknown angles in right-angled triangles

Practise calculating the size of unknown angles in right-angled triangles.

Angles of elevation and depression

An *angle of elevation* is defined as an angle between the line of sight and a horizontal reference line in which the line of sight is above the horizontal. An *angle of depression* is defined as an angle between the line of sight and a horizontal reference line in which the line of sight is below the horizontal.



Additional information

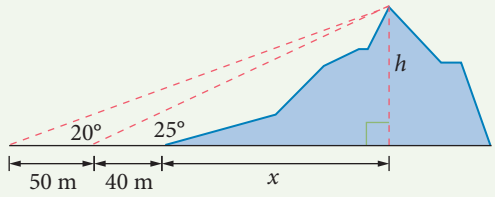
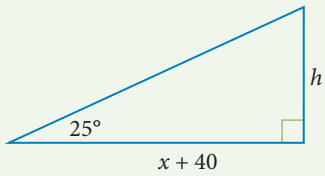
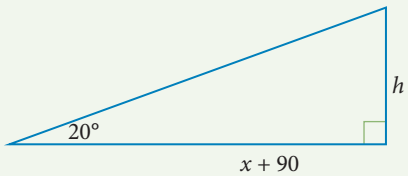
Angles of elevation

Review your understanding of the angle of elevation.

Practical situations involving angles of elevation or depression may involve compound diagrams of triangles within other triangles.

3 Solve right-angled triangles with angles of elevation or depression

A person standing 40 m from the base of a waterfall measures the angle of elevation from the ground to the top of the waterfall as 25°. The person then walks a further 50 m away from the waterfall and measures the new angle of elevation as 20°. Determine the height of the waterfall, correct to the nearest metre.

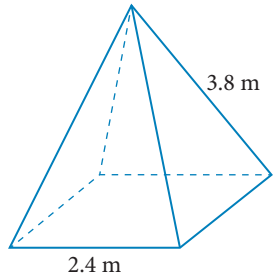
THINKING	WORKING
1 Draw a labelled diagram to define the variables for solving.	
2 Draw the smaller right-angled triangle and label the sides and angles of interest. Write a trigonometric equation and make the unknown h the subject.	 $\frac{h}{x+40} = \tan(25°)$ $h = (x+40)\tan(25°)$ [1]
3 Draw the larger right-angled triangle and label the sides and angles of interest. Write a trigonometric equation and make the unknown h the subject.	 $\frac{h}{x+40+50} = \tan(20°)$ $\frac{h}{x+90} = \tan(20°)$ $h = (x+90)\tan(20°)$ [2]
4 Solve [1] and [2] simultaneously to determine the height of the waterfall.	Equating expressions for h: $(x+40)\tan(25°) = (x+90)\tan(20°)$ $x\tan(25°) + 40\tan(25°) = x\tan(20°) + 90\tan(20°)$ $x(\tan(25°) - \tan(20°)) = 90\tan(20°) - 40\tan(25°)$ $x = \frac{90\tan(20°) - 40\tan(25°)}{\tan(25°) - \tan(20°)}$ $x = 137.8\ldots$ Substituting x into [1]: $h = (137.8\ldots + 40)\tan(25°)$ $= 177.8\ldots\tan(25°)$ $= 82.9\ldots$
5 Interpret the answer.	The height of the waterfall is approximately 83 m.

Right-angled triangles in three dimensions

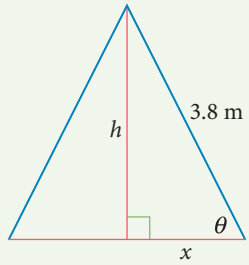
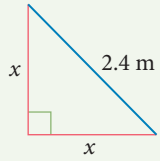
To solve problems in three dimensions, a sound strategy is to break the three-dimensional diagram into a series of two-dimensional diagrams.

4 Solve right-angled triangles in three dimensions

A square-based right pyramid has a base of side length 2.4 m and a sloping edge of length 3.8 m.



(a) Determine the angle that the sloping edge makes with the base, to the nearest degree.

THINKING	WORKING
1 Draw the triangle, showing the position of the angle.	
2 Since there is not enough information to solve the unknown, draw a diagram involving one of the other unknowns. Write a trigonometric equation and solve to determine the diagonal length across the base of the pyramid.	 Using Pythagoras' theorem: $x^2 + x^2 = 2.4^2$ $2x^2 = 2.4^2$ $x^2 = \frac{2.4^2}{2}$ $\therefore x = \frac{2.4}{\sqrt{2}}$ $= \frac{2.4\sqrt{2}}{2}$ $= 1.2\sqrt{2}$
3 Solve the required angle.	$\cos(\theta) = \frac{x}{3.8}$ $= \frac{1.2\sqrt{2}}{3.8}$ $\theta = \cos^{-1}\left(\frac{1.2\sqrt{2}}{3.8}\right)$ $= 63.474\ldots°$ $= 63°$ (to the nearest whole angle)
4 Interpret the answer.	The sloping edge makes an angle of 63° with the base.

(b) Determine the height of the pyramid.

1 Use the diagram involving the height. Write a trigonometric equation and solve, using the unrounded value for the angle.	$\frac{h}{3.8} = \sin(\theta)$ $h = 3.8\sin(63.4\ldots°)$ $h = 3.4$ (1 d.p.)
2 Alternatively, use Pythagoras' theorem with the lengths.	Confirming: $h = \sqrt{3.8^2 - \left(1.2\sqrt{2}\right)^2}$ $= \sqrt{14.44 - 2.88}$ $= \sqrt{11.56}$ $= 3.4$ (1 d.p.)
3 Interpret the answer.	The height of the pyramid is 3.4 m. This is shorter than the sloping edge, as expected.

The unit circle and periodicity

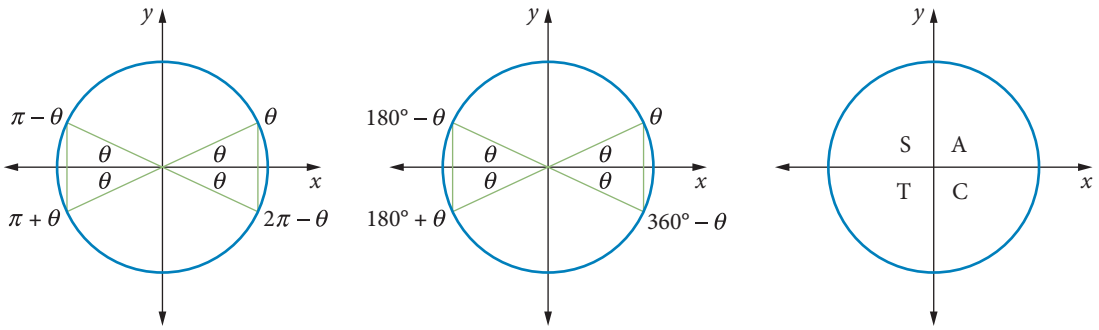
The summary below is a reminder of the periodic nature of the trigonometric functions and the symmetry properties that exist in the four quadrants for angles in degrees and radians.

C Making connections

Angle symmetry

Move the slider to view the symmetry properties of angles in the unit circle.

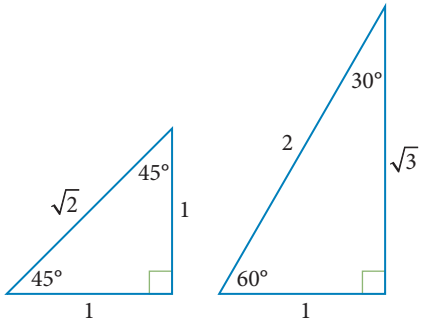
The first quadrant angle θ is the reference angle.



The third image is a reminder that all ratios are positive in the first quadrant, but only sine is positive in the second, tangent in the third and cosine in the fourth.

Special triangles and exact ratios

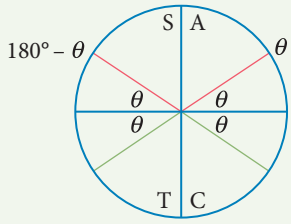
The symmetry of the equilateral triangle and the square lead to right-angled triangles with known ratios. You should be familiar with the trigonometric ratios for 30°, 45° and 60°, and their equivalent angles in radians: $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ respectively.



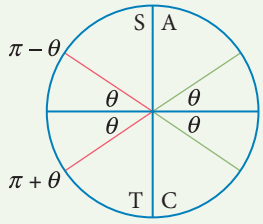
5 Solve equations using trigonometric functions and the unit circle

Determine all solutions to the following equations.

(a) $\sin(x) = \frac{1}{2}, 0° \leq x \leq 360°$

THINKING	WORKING
1 Determine the reference angle. Identify the related angles with the correct sign.	$\sin(30°) = \frac{1}{2}$, so the reference angle is 30°. The sine function is positive in the first and second quadrants. 
2 Write the solutions and simplify.	$x = 30°$ and $x = 180° - 30°$ Therefore, the solutions in the domain are $x = 30°$ and $x = 150°$.

(b) $\cos(x) = -\frac{\sqrt{2}}{2}, 0 \leq x \leq 2\pi$

1 Determine the reference angle. Identify the related angles with the correct sign.	$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, so the reference angle is $\frac{\pi}{4}$. The cosine function is negative in the second and third quadrants. 
2 Write the solutions and simplify.	$x = \pi - \frac{\pi}{4}$ and $\pi + \frac{\pi}{4}$ Therefore, the solutions in the domain are $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$.

WARNING

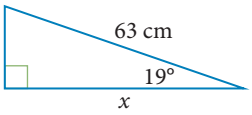
Make sure your calculator is in the correct mode for either degrees or radians.

EXERCISE 7.1 Trigonometry review

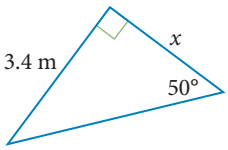
Worked Example 1

1 Calculate the length of x in each triangle correct to 2 decimal places.

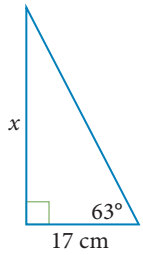
(a)



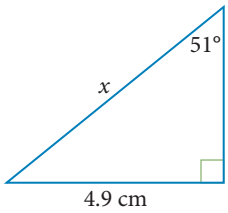
(b)



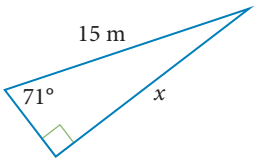
(c)



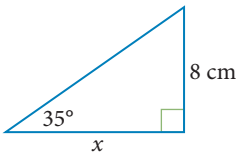
(d)



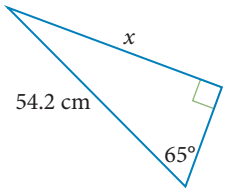
(e)



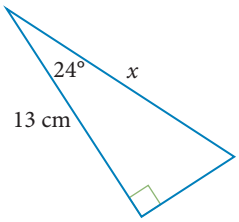
(f)



(g)

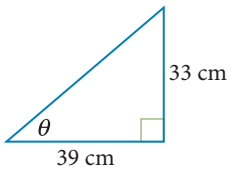


(h)

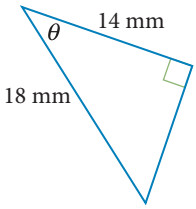


2 Calculate the size of angle θ for each triangle, to the nearest degree.

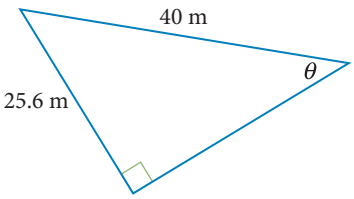
(a)



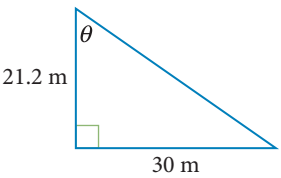
(b)



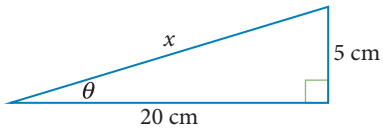
(c)



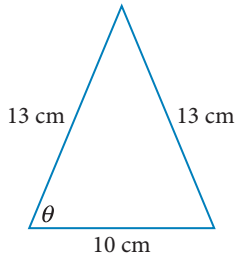
(d)



3 Solve the triangle for both pronumerals. Give the length correct to 2 decimal places and the angle to the nearest degree.

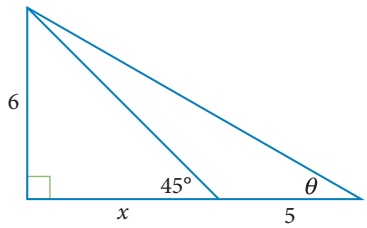


4 Calculate θ to the nearest degree.

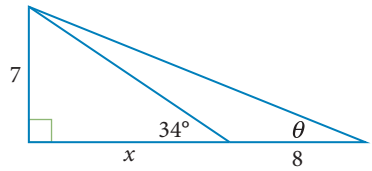


5 For each of the following diagrams, determine the values of θ and x, correct to 1 decimal place where necessary.

(a)



(b)



Worked Example

5

6 Determine all solutions to the following equations in the given domain.

(a) $\sin(x) = \frac{1}{\sqrt{2}}, 0° \leq x \leq 360°$

(b) $\cos(x) = -\frac{1}{2}, 0 \leq x \leq 2\pi$

(c) $\tan(x) = -\sqrt{3}, 0° \leq x \leq 360°$

(d) $\cos(x) = \frac{\sqrt{3}}{2}, 0 \leq x \leq 2\pi$

(e) $\sin(x) = -\frac{1}{2}, -\pi \leq x \leq \pi$

(f) $\cos(x) = \frac{1}{\sqrt{2}}, -180° \leq x \leq 180°$

7 **(a)** Given that $\cos(1) = 0.54$, what is the value of $\cos(1 - \pi)$?

A 0.54 **B** -0.54 **C** 0.46 **D** -0.46

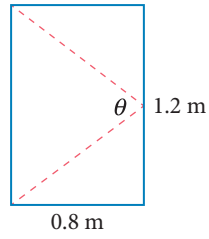
(b) Explain the common error made by a student who misunderstood the quadrant placement of the angle $(1 - \pi)$.

8 Which option is the same as $\sin(160°) - \cos(160°)$?

A $\sin(20°) + \cos(20°)$ **B** $\sin(20°) - \cos(20°)$

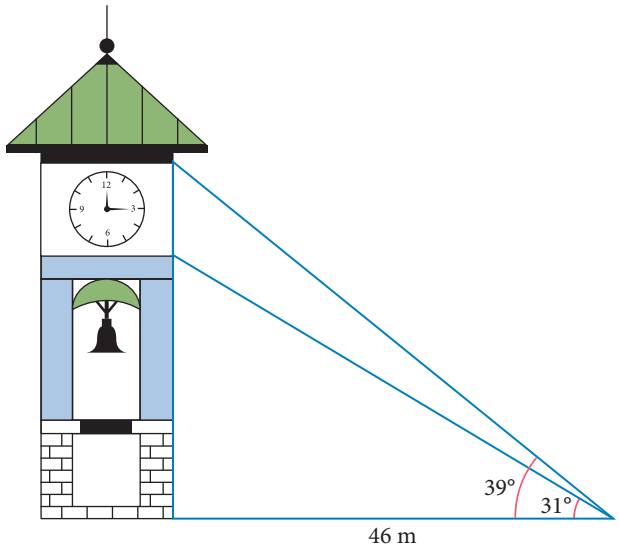
C $-\sin(20°) + \cos(20°)$ **D** $-\sin(20°) - \cos(20°)$

9 Gina strengthens a 1.2 m-tall rectangular metal gate by adding two diagonal struts (shown as dashed lines). She welds the struts together halfway along the vertical side of the gate. Calculate the angle θ to the nearest degree.

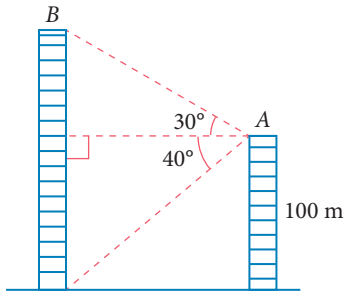


10 Carter is swimming 70 m due north of the shore when he is caught in a rip. He conserves energy and drifts 100 m east parallel to the shore before being rescued. What is Carter's direction from his starting point when he is rescued? Give your answer to the nearest degree.

11 A surveyor standing 46 m from the base of a building measures the angle to the top of the building's clock and finds it to be 39°. The surveyor then measures the angle to the base of the clock, and finds that it is 31°. Determine the height of the clock from the base to the top, correct to 1 decimal place.



12 From the top of a 100 m-high building (A), the angle of depression to the bottom of building B is 40°. The angle of elevation from the top of building A to the top of building B is 30°. Calculate the height of building B in metres, correct to 1 decimal place.

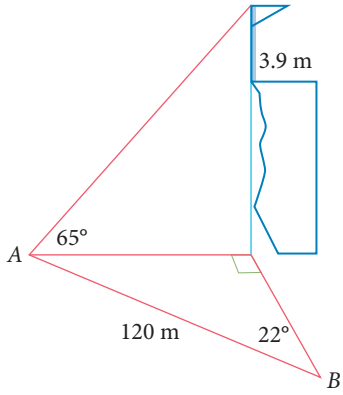


13 Huynh is in a hot air balloon, 328.2 m above the ground. She spots a parked truck at an angle of depression of 38°. The balloon rises another 50 m and the angle of depression to the truck is now 42°.

(a) Determine, to the nearest metre, the direct distance of Huynh from the truck before the balloon rises.

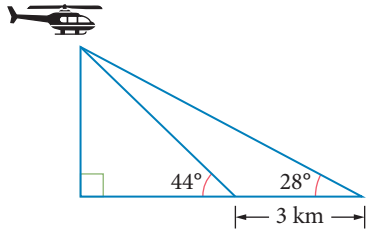
(b) Determine, to the nearest metre, the direct distance of Huynh from the truck after the balloon rises.

14 Suri navigates her way 120 m from campsite B to campsite A, travelling in the direction N22°W. She then measures the angle of elevation to the top of a flagpole, 3.9 m above a cliff, to be 65°. Determine the height of the cliff, correct to 2 decimal places.

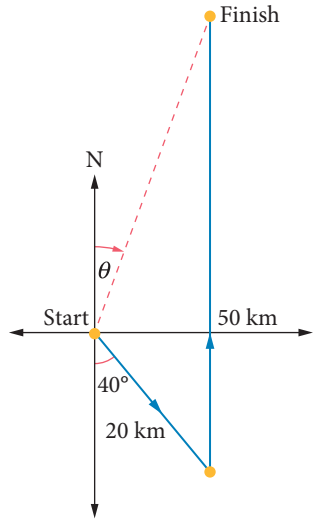


3

15 A helicopter is sighted at the same time by two observers, Len and Alaine, on the ground. Len and Alaine are 3 km apart on the same side of the helicopter. They find the angles of elevation to be 44° and 28° respectively. Determine the height of the helicopter in kilometres, correct to 1 decimal place.



16 During a car rally, a competitor drives 20 km from the start in the direction S40°E, then due north for 50 km. Determine the competitor's final displacement.

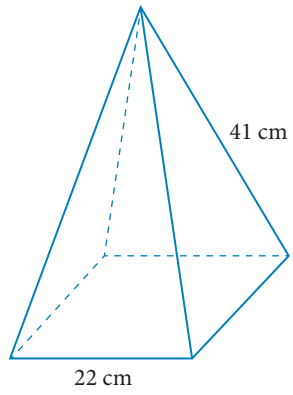


4

17 A pyramid stands on a square base of edge length 22 cm. The sloping edges are each 41 cm.

(a) Determine the angle that the sloping edge makes with the base, to the nearest degree.

(b) Determine the height of the pyramid, correct to 2 decimal places.



18 Patrick flies a light plane from Cloncurry to Longreach, a distance of 495 km on a bearing of 129°T, then continues due east to Rockhampton, a further 639 km. Patrick returns directly to Cloncurry. Determine Patrick's return journey in terms of distance and direction. Explain any assumptions you had to make.

19 The lines $y = 3x - 4$ and $2x + y = 5$ intersect on the Cartesian plane. Determine the angle between the lines, to the nearest degree, under the following conditions.

(a) The scale on both axes is the same.

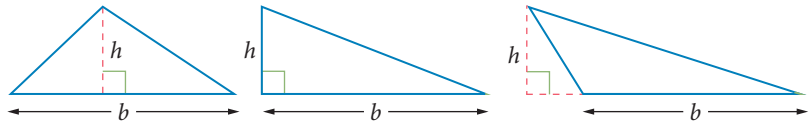
(b) The scale on the vertical axis gives 2 units for the same distance as 1 unit on the vertical axis.

The area of a triangle

7.2

Area given base and height

The area of a triangle, given the base b and the perpendicular height (or altitude) h, is given by the formula: $A = \frac{1}{2}bh$.



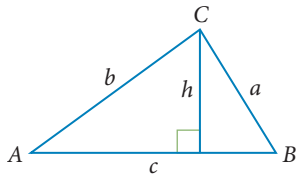
Area given two sides and the included angle

When two sides of a triangle and the included angle are known, the shape of the triangle is fixed, according to the SAS congruence test. Hence it should be possible to determine the area of the triangle. For the ΔABC shown below, the naming convention has been used, where side a is opposite angle A, side b is opposite angle B and side c is opposite angle C. The base is c and the height is $h = b\sin(A)$ because:

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin(A) = \frac{h}{b}$$

$$\therefore h = b \times \sin(A)$$



Substituting these into the formula for the area of a triangle gives:

$$\text{Area of } \Delta ABC = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times c \times (b \times \sin(A))$$

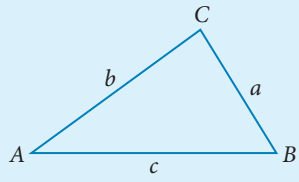
$$= \frac{1}{2}bc\sin(A)$$

For ΔABC, the area can be found from any two sides and the included angle.

$$\text{Area of } \Delta ABC = \frac{1}{2}ab\sin(C)$$

$$= \frac{1}{2}bc\sin(A)$$

$$= \frac{1}{2}ac\sin(B)$$



Note: Here the side lengths c and b and the included angle A were used. In fact, any of the sides could be used as the base.

Additional information

The formula for the area of a triangle

View the video for an explanation of the formula for the area of a triangle.

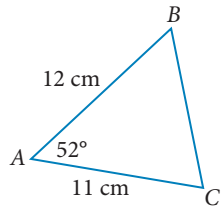
Additional information

Determine the area of a non-right-angled triangle

Try this activity to practise calculating the area of a triangle.

6 Calculate the area of a triangle

Calculate the area of the triangle, correct to 2 decimal places.



THINKING	WORKING
1 Identify the given information in terms of pronumerals.	$A = 52°$, $b = 11\,\text{cm}$, $c = 12\,\text{cm}$
2 Recall the appropriate rule to calculate the unknown.	$\text{Area} = \frac{1}{2}bc\sin(A)$ $= \frac{1}{2} \times 11 \times 12 \times \sin(52°)$ $= 52.01\,(2\text{ d.p.})$
3 Interpret the answer.	The area of the triangle is $52.01\,\text{cm}^2$.

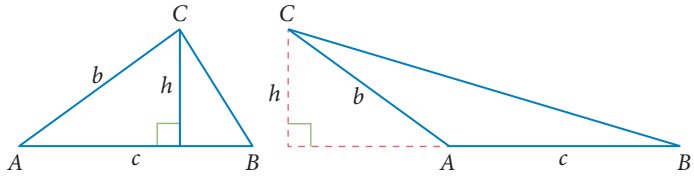
Making connections

Calculating the area of a non-right-angled triangle

Move the sliders to determine the area of the triangles formed.

Determine the size of an unknown angle when area is known

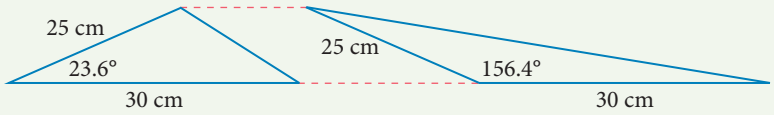
Knowing the area of a triangle and two given sides, it seems reasonable that the size of the included angle could be determined. In fact, for any given set of values there will be two possible triangles that can be drawn, unless the triangle is right-angled. The illustration shows triangles with two sides the same and identical areas.



The heights of the triangles are the same, equal to $b\sin(A)$ in each case, because for $A < 90°$, $\sin(A) = \sin(180° - A)$. Hence the value of $\sin(A)$ is the same for both triangles, even though $\angle CAB$ is acute in one triangle and obtuse in the other. From your study of the unit circle definitions for trigonometric ratios, you know that $\sin(180° - \theta) = \sin(\theta)$, so the angles shown here as A will be supplementary.

7 Calculate the included angle from the area and two side lengths

For a triangle of area 150 cm^2 and side lengths 30 cm and 25 cm, determine the possible values of the included angle of the triangle, correct to 1 decimal place. Draw both possible triangles.

THINKING	WORKING
1 Identify the given information in terms of pronumerals.	Area $= 150$ cm^2; let $a = 30$ cm, $b = 25$ cm
2 Recall the appropriate rule to calculate the sine value.	$\text{Area} = \frac{1}{2}ab\sin(C)$ $150 = \frac{1}{2} \times 30 \times 25 \times \sin(C)$ $150 = 375\sin(C)$ $\sin(C) = \frac{150}{375}$ $\sin(C) = 0.4$
3 Use inverse sine for the acute angle.	One solution is: $C = \sin^{-1}(0.4)$ $= 23.6°\ (1\text{ d.p.})$
4 Use supplementary angles to give the obtuse angle.	The other solution is $180 - 23.6 = 156.4°$. The included angle could be 23.6° or 156.4°.
5 Draw both triangles, ensuring they have the same height.	

Area from the side lengths

When three sides of a triangle are known, the shape of the triangle is fixed, according to the SSS congruence test. Hence it should be possible to determine the area of the triangle. You will be able to use the cosine rule (covered later in this chapter) in conjunction with one of the area formulas such as $\text{Area} = \frac{1}{2}bc\sin(A)$.

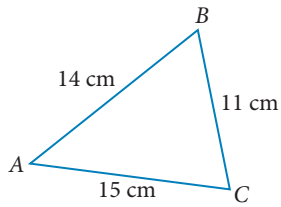
A Greek mathematician came up with a formula for finding the area of a triangle around 200–100 BCE. While the QCAA syllabus does not require knowledge of this formula, it is used here to consolidate concepts of the shape of a triangle in relation to its area.

Heron's formula is: Area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the side lengths of ΔABC and s is the semi-perimeter, where $s = \frac{1}{2}(a+b+c)$.

Heron's formula: Area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$

8 Use Heron's formula to calculate the area of a triangle

Calculate the area of ΔABC using Heron's formula.



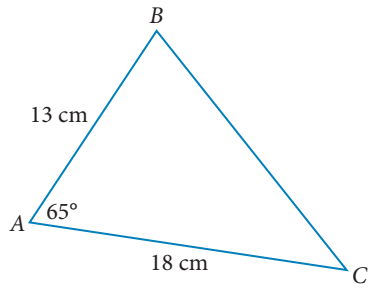
THINKING	WORKING
1 Identify the known pronumerals.	$a = 11\text{ cm}, b = 15\text{ cm}, c = 143\text{ cm}$
2 Recall the rule to calculate the semi-perimeter.	$s = \frac{1}{2} \times (11 + 15 + 14)$ $= 20\text{ cm}$
3 Recall Heron's rule to calculate the area.	Area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$ $= \sqrt{20(20-11)(20-15)(20-14)}$ $= \sqrt{20 \times 9 \times 5 \times 6}$ $= \sqrt{5400}$ $= 73.484\ldots$
4 Interpret the answer.	The area of the triangle is 73.48 cm^2.

EXERCISE 7.2 The area of a triangle

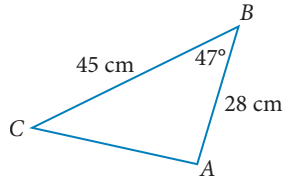
Worked Example 6

1 Calculate the area, correct to 2 decimal places, of each of the following triangles.

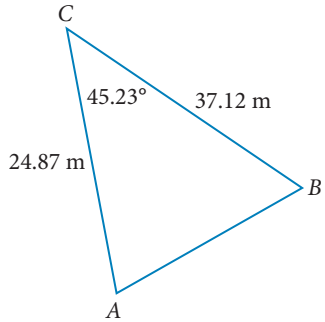
(a)



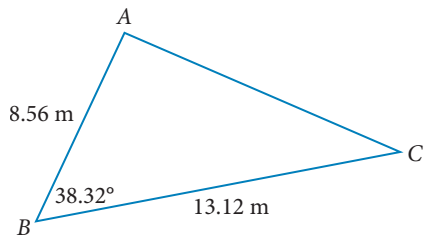
(b)



(c)

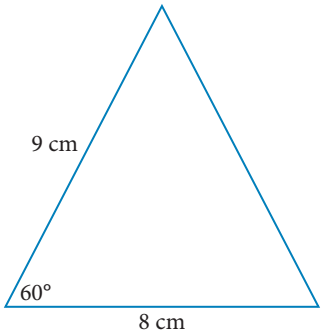


(d)

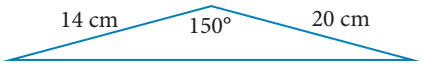


2 Calculate the exact area of each of the following triangles.

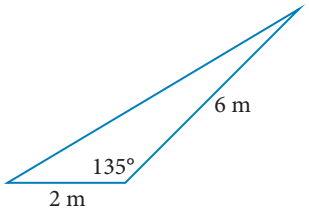
(a)



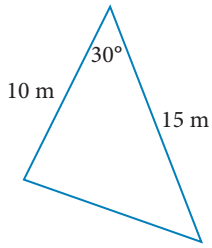
(b)



(c)



(d)



3 A triangle has two sides of lengths 25 cm and 46 cm, with an included angle of 34°. Determine the area of the triangle, correct to the nearest square centimetre.

4 An acute-angled triangular paddock has an area of 5000 m^2. If two of the fences measure 100 m and 200 m, determine the angle at which these two fences meet.

5 A parallelogram has side lengths of 24 cm and 36 cm. The angle between one pair of adjacent sides is 25°. Determine the area of the parallelogram, correct to 2 decimal places.

6 Emergency crews are trying to contain an oil spill by surrounding it with three booms that form a triangle. The booms are each 300 cm long. Calculate the area that is being contained, correct to the nearest square metre.

7 In ΔABC, $a = 8$ cm, $b = 5.2$ cm, $c = 4$ cm and $\angle ABC = 56°$.

(a) Which of the following is the area of the triangle, in square centimetres?

A 8.62 **B** 13.26 **C** 17.24 **D** 26.53

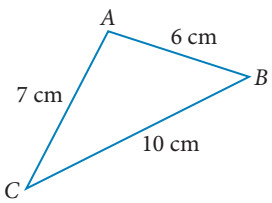
(b) Which of the incorrect options would be chosen by a student who placed the included angle in the wrong position?

8 In ΔABC, if $a = 16$ cm, $b = 13.3$ cm, $c = 8$ cm and $\angle ABC = 56°$, calculate the area of the triangle in square centimetres, correct to 2 decimal places.

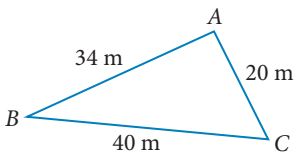
Worked Example 8

9 Calculate the area of ΔABC using Heron's formula, correct to 2 decimal places.

(a)



(b)



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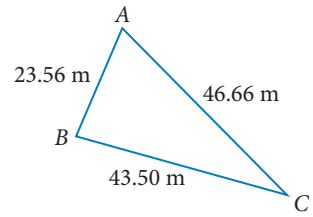
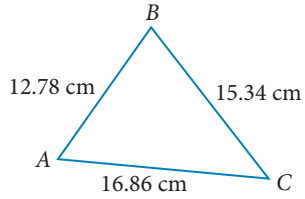


(d)



10 Determine the area of a triangle with side lengths 23 cm, 34 cm and 35 cm, correct to 2 decimal places.

11 Rikeshi and Jacqueline have just bought a triangular block of land and the property developer who sold them the block told them that its area is approximately 370 m^2. The three boundaries measure 24 m, 32 m and 34 m. Determine the area of the block of land, correct to the nearest square metre.

12 Calculate the area of the Bermuda Triangle if the sides of the triangle have approximate lengths of 1368 km, 1489 km and 2092 km. Give the answer correct to the nearest square kilometre. Comment on any assumptions you made in using your chosen formula.

13 In ΔABC, $AB = 15$ m, $BC = 12$ m and $AC = 14$ m. Which of the following is closest to the area of the triangle, in square metres?

A 17.43 **B** 20.5 **C** 78.93 **D** 6229.44

Worked Example

7

14 For a triangle of area 3.16 m^2 and side lengths 2.1 m and 3.5 m, determine the possible values of the included angle of the triangle, correct to 1 decimal place. Draw both possible triangles.

15 A triangular plot of land has an area of 750 m^2. If two of the fenced sides are 50 m and 85 m, calculate the angle at which these two fences could meet, correct to the nearest degree.

16 A triangular block of land has frontages of 59 m and 85 m on two streets and a third side of length 110 m. Determine the angle at which the streets meet, correct to the nearest degree.

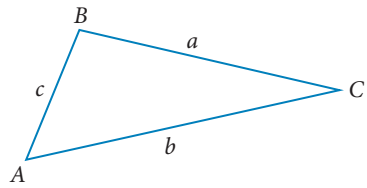
17 Two triangles both have side lengths of 8 cm and 10 cm. The area of one of the triangles is 1.5 times the area of the other. Draw two triangles that meet this description, showing the included angle in each, correct to 1 decimal place.

The sine rule

7.3

The sine rule

Right-angled triangles demonstrate a specific case of the longest side of a triangle being opposite the largest angle, the right angle. The sine rule goes much further, connecting each side with the angle opposite, using the naming convention, for convenience.



For a ΔABC, the sine rule states that the sides of a triangle are proportional to the sines of the opposite angles.

The sine rule: $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$

Additional information

The sine rule

This activity explains the sine rule.

Proof of the sine rule

In the previous section, you saw that the area of ΔABC could be expressed using any two sides and the included angle, so each of the three expressions for area are equal.

$$\frac{1}{2}ab\sin(C) = \frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B)$$

Taking the first two expressions:

$$\frac{1}{2}ab\sin(C) = \frac{1}{2}bc\sin(A)$$
$$a\sin(C) = c\sin(A)$$
$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

Similarly, $\dfrac{1}{2}bc\sin(A) = \dfrac{1}{2}ac\sin(B) \Rightarrow \dfrac{b}{\sin(B)} = \dfrac{a}{\sin(A)}$

Combining these two results gives the sine rule:

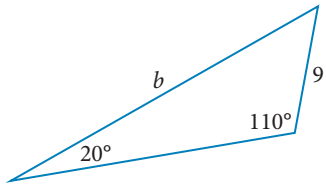
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \text{ or alternatively } \frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

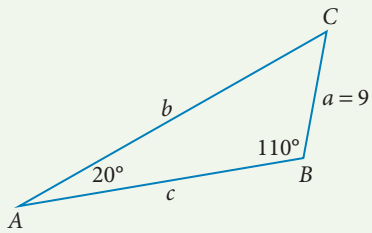
Two angles and one side are known

When two angles and any side are known, the triangle has a fixed shape, by the AAS (or ASA) congruence test. The third angle can be found, if needed, using the angle sum of a triangle. Hence, the sine rule can be used to calculate either of the unknown sides.

9 Solve non-right-angled triangles using the sine rule

Determine the value of the unknown pronumeral, correct to 1 decimal place.



THINKING	WORKING
1 Redraw the triangle and label the vertices and the sides.	
2 Identify the pairs of information.	$A = 20°$, $a = 9$ and $B = 110°$, $b = ?$
3 Recall the sine rule using the correct variables. Substitute the identified information and solve for the unknown.	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$ $\frac{b}{\sin(110°)} = \frac{9}{\sin(20°)}$ $b = \frac{9\sin(110°)}{\sin(20°)}$ $= 24.7 (1 \text{ d.p.})$
4 Evaluate the reasonableness of the solution.	The side opposite the largest angle is longer than the given side, as expected.

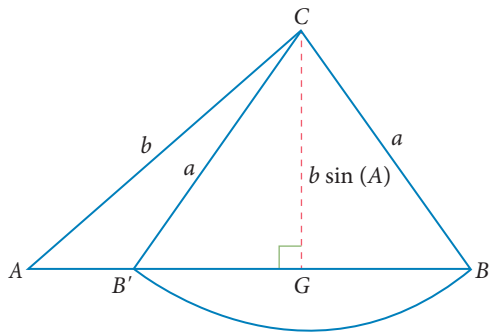
Two sides and a non-included angle are known

If two sides and a non-included angle are given, sometimes two different triangles can be drawn.

The *ambiguous case* should be considered for a ΔABC where the two sides a and b and the non-included angle A are known and A is acute and $b > a > b\sin(A)$. The ambiguous case is shown in the diagram. The two triangles are ABC and $AB'C$.

WARNING

There is no SSA congruence test.



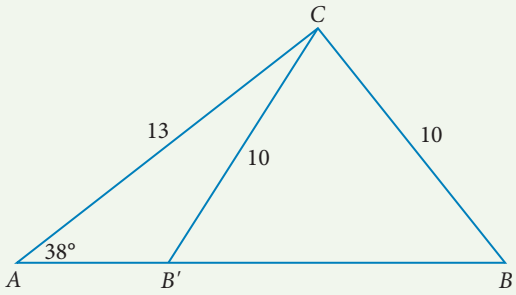
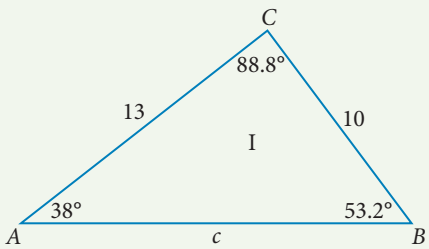
Additional information

Ambiguous case of the sine rule

This activity explores the ambiguous case of the sine rule.

10 Solve problems involving the ambiguous case of the sine rule

Given the following information, determine the missing information of ABC, correct to 1 decimal place: $A = 38°$, $a = 10$ and $b = 13$.

THINKING	WORKING
1 Draw a labelled diagram.	
2 Identify pairs of information.	$A = 38°$, $a = 10$, and $B' = ?$, $b = 13$ and $B = ?$
3 Recall the sine rule to determine the size of angle B.	$\frac{b}{\sin(B)} = \frac{a}{\sin(A)}$ $\frac{13}{\sin(B)} = \frac{10}{\sin(38°)}$ $\sin(B) = \frac{13\sin(38°)}{10}$ $\sin(B) = 0.80\ldots$ $B = \sin^{-1}(0.80\ldots)$ $B = 53.2°\,(1\text{ d.p.})$
4 Use the symmetry properties of the sine function to determine the size of angle B'.	$B' = 180 - B$ $= 180 - 53.2$ $= 126.8°$
5 Evaluate the reasonableness of your solution for the obtuse angle.	$B' + A \approx 126.8 + 38$ $\approx 164.8°$ $< 180°$ $\therefore B'$ is a feasible angle.
6 Determine $\angle BCA$ in ΔABC.	$C = 180 - 38 - 53.2$ $= 88.8°\,(1\text{ d.p.})$
7 Draw the ΔABC and label all known values.	

8 Determine the length of the remaining side.

List the solved measurements for this triangle.

$$\frac{c}{\sin(C)} = \frac{a}{\sin(A)}$$
$$\frac{c}{\sin(88.8°)} = \frac{10}{\sin(38°)}$$
$$c = \frac{10 \times \sin(88.8°)}{\sin(38°)}$$
$$c = 16.2 (1 \text{ d.p.})$$

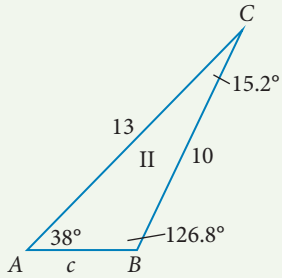
For triangle I:

$B = 53.2°$, $C = 88.8°$, $c = 16.2$

9 Determine $\angle B'CA$ in $\Delta AB'C$.

$C = 180 - 38 - 126.8$
$= 15.2°$

10 Draw the second possibility for ΔABC and label all known values.



11 Determine the length of the remaining side.

$$\frac{c}{\sin(C)} = \frac{a}{\sin(A)}$$
$$\frac{c}{\sin(15.2°)} = \frac{10}{\sin(38°)}$$
$$c = \frac{10 \times \sin(15.2°)}{\sin(38°)}$$
$$c = 4.3 (1 \text{ d.p.})$$

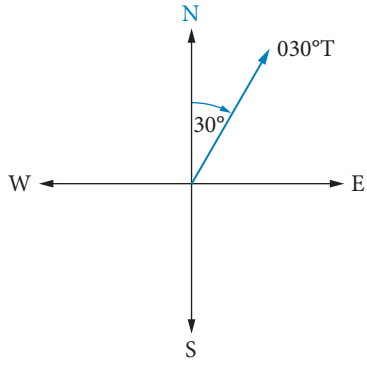
12 List the solved measurements for this triangle.

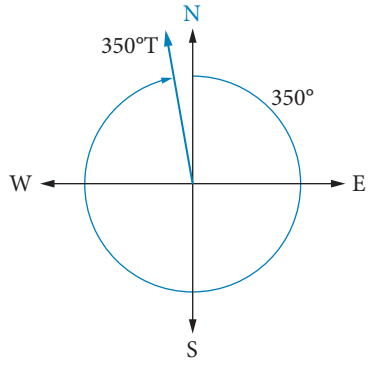
For triangle II:

$B = 126.8°$, $C = 15.2°$, $c = 4.3$

Application to distance and bearings

Problems involving distance and directions may be solved more efficiently using non-right-angled triangles. For questions where the direction is given as a bearing, remember that bearings are always measured clockwise, starting from north, and are written as three-digit angles, sometimes followed by the letter T (for True).





For example, 030°T describes a bearing 30° clockwise from north, or N30°E, and 350° or 350°T describe a bearing 350° clockwise from north, or N10°W.

Additional information

True bearings

This activity demonstrates the use of true bearings.

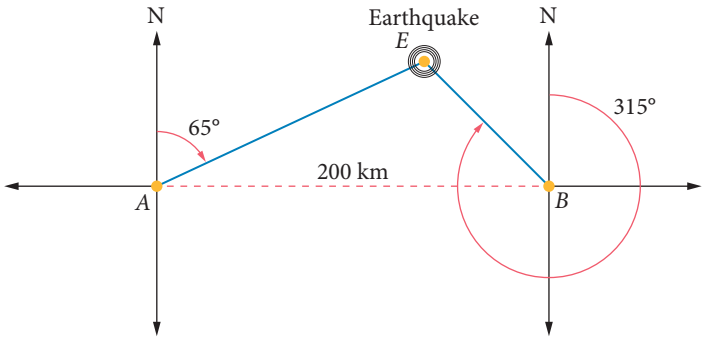
Making connections

Compass directions and true bearings

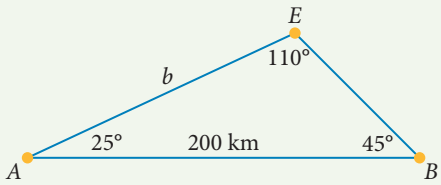
Explore the relationship between compass directions and true bearings with the activity.

11 Application to distance and bearings

Two earthquake detection stations, A and B, are situated 200 km apart, with station B due east of station A. Station A detects an earthquake on a bearing of 065°T. Station B detects the same earthquake on a bearing of 315°T.

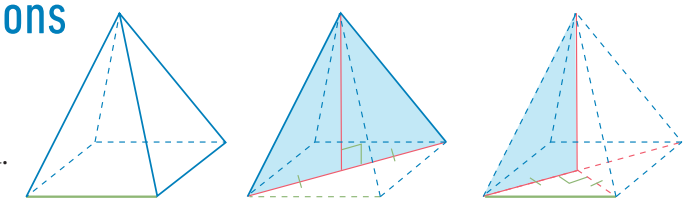


How far is the earthquake from station A? Express your answer correct to 1 decimal place.

THINKING	WORKING
1 Calculate angles within the triangle.	 $A = 90 - 65$ $= 25°$ $B = 315 - 270$ $= 45°$ $E = 180 - (25 + 45)$ $= 110°$
2 Recall the sine rule and substitute the given values. Solve to determine the length of side b.	$\frac{b}{\sin(B)} = \frac{e}{\sin(E)}$ $\frac{b}{\sin(45°)} = \frac{200}{\sin(110°)}$ $b = \frac{200\sin(45°)}{\sin(110°)}$ $b = 150.5 \text{ km} \ (1 \text{ d.p.})$
3 Interpret the answer.	The earthquake is 150.5 km from station A.

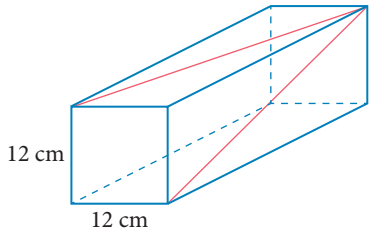
Applications to three dimensions

The sine rule is sometimes the most efficient way to solve three-dimensional problems where non-right-angled triangles are involved.



12 Application to three dimensions

A box in the shape of a square-based prism has two pieces of ribbon attached to the vertices as shown.

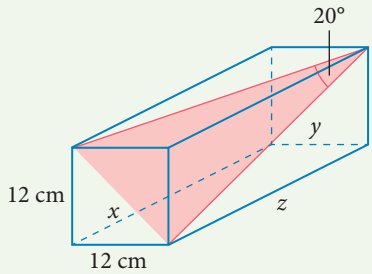


Determine the length of the box, correct to 1 decimal place, if the angle between the ribbons is 20°.

THINKING

1 Draw a labelled diagram.

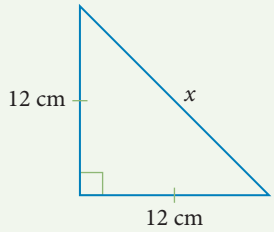
WORKING



2 Calculate the base and angles within the sloping triangle.

From the square base:

$$12^2 + 12^2 = x^2$$
$$x^2 = 2 \times 12^2$$
$$x = 12\sqrt{2}$$



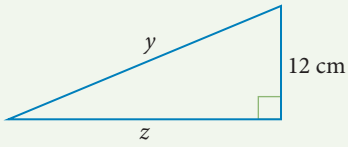
From symmetry, the unknown angles are both:

$$\frac{180 - 20}{2} = 80°$$

3 Recall the sine rule and solve for y.

$$\frac{x}{\sin(X)} = \frac{y}{\sin(Y)}$$
$$\frac{x}{\sin(20°)} = \frac{y}{\sin(80°)}$$
$$y = \frac{12\sqrt{2} \times \sin(80°)}{\sin(20°)}$$
$$y = 48.8\ldots$$

4 Recall Pythagoras' theorem and solve for z.



$$z = \sqrt{y^2 - 12^2}$$
$$= \sqrt{48.8\ldots^2 - 12^2}$$
$$= 47.4\,(1\text{ d.p.})$$

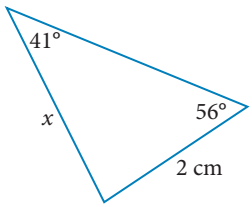
5 Interpret the result.

The length of the box is 47.4 cm.

EXERCISE 7.3 The sine rule

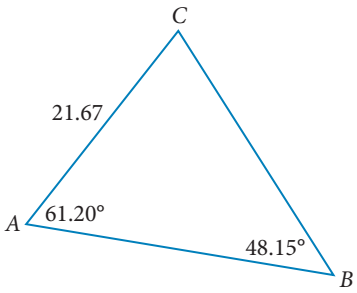
Worked Example 9

1 Determine the value of x, correct to 1 decimal place.



2 Solve the following triangles, calculating all unknown side lengths and angles, correct to 2 decimal places.

(a)



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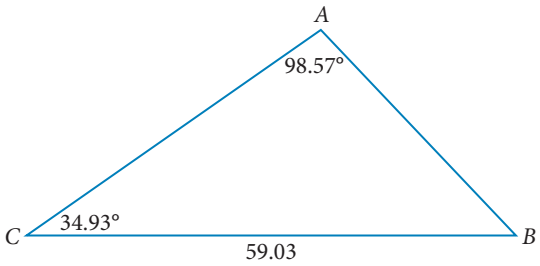
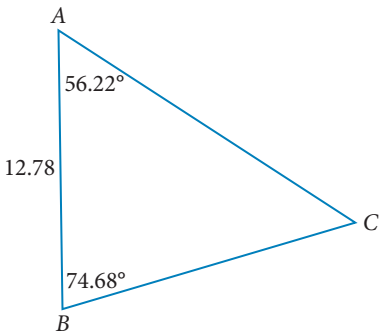
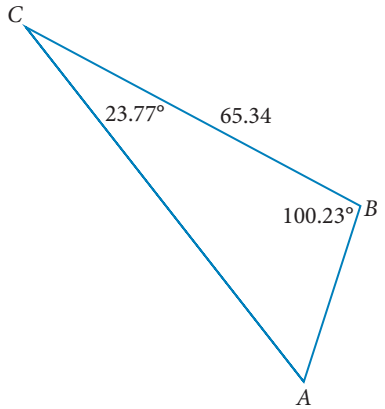


(c)

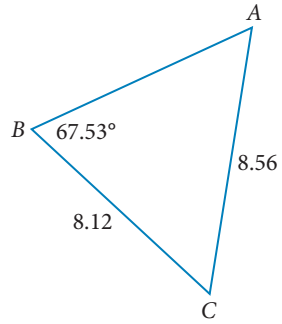


(d)

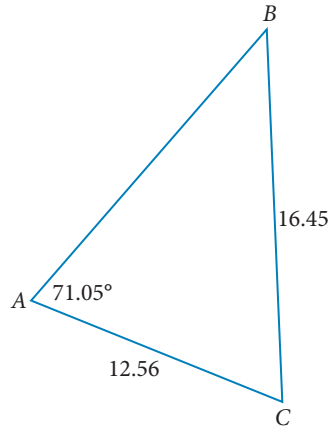


3 Solve the following triangles, given that any unknown angles are acute. Calculate unknown side lengths and angles, correct to 2 decimal places.

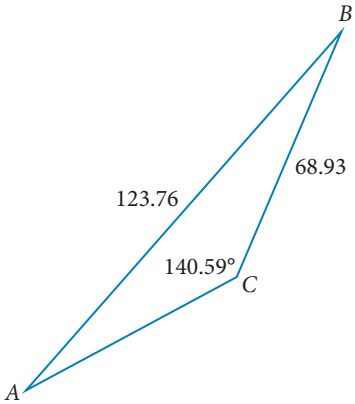
(a)



(b)



(c)



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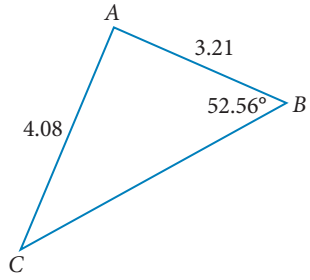


4 In ΔABC, $a = 2$, $A = 20°$ and $b = 2$. Which of the following is closest to angle B?

A 23° **B** 20° **C** 11° **D** 10°

5 In ΔABC, $a = 15$, $\angle BAC = 40°$ and $\angle ABC = 74°$.

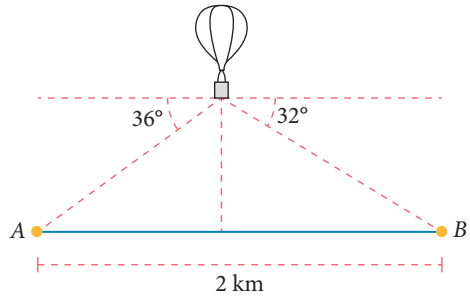
(a) Which of the following is closest to the length of side c?

A 15.78 **B** 10.55 **C** 22.43 **D** 21.32

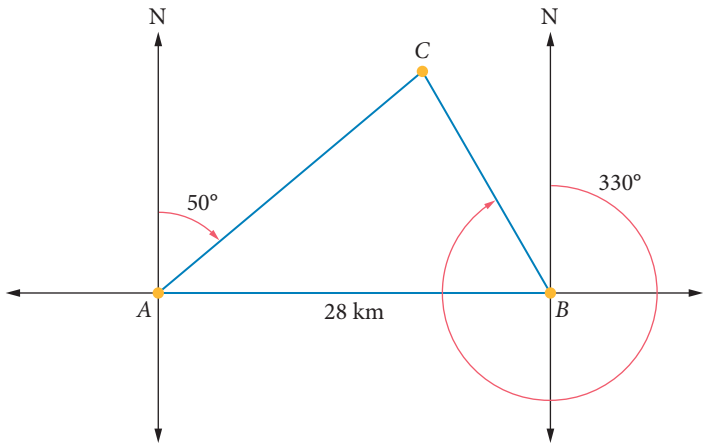
(b) Which of the incorrect options gives the unmarked side of the triangle?

6 A triangular block of land has two angles measuring 46.23° and 55.95° and the boundary opposite the unknown angle is 34 m long. Determine the measures of all unknown sides and angles, correct to 2 decimal places.

7 A balloonist is directly above a straight road 2 km long that joins two towns A and B. She finds that the town closer to her is at an angle of depression of 36°, and the farther town is at an angle of depression of 32°. How high above the ground is the balloon? Give your answer correct to the nearest metre.



8 Three friends, Alison, Brogan and Cassie, use a satellite navigation system to locate each other. Alison is 28 km due west of Brogan. Cassie is on a bearing of 050°T from Alison, and on a bearing of 330°T from Brogan.

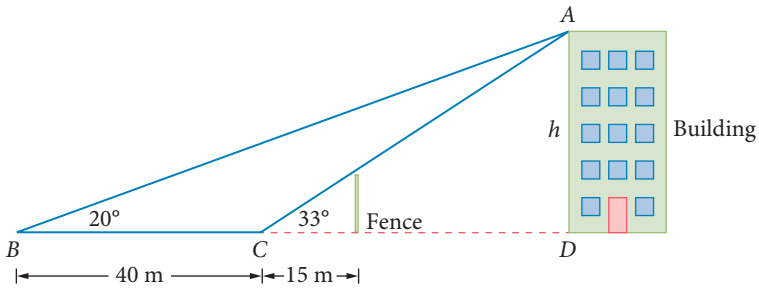


(a) Determine the angles A, B and C in the triangle depicting the locations.

(b) How far apart are Cassie and Alison, correct to 2 decimal places?

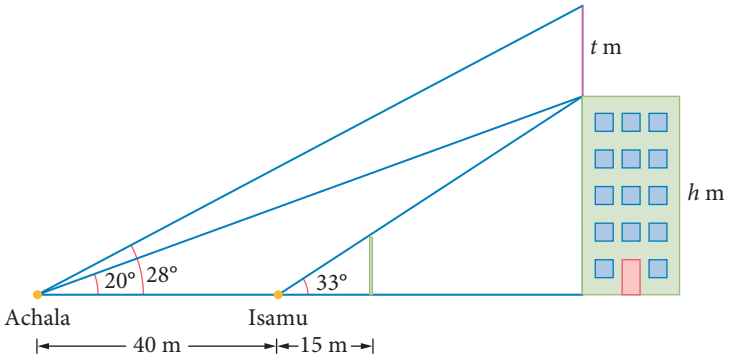
(c) How far apart are Cassie and Brogan, correct to 2 decimal places?

9 Building surveyors Isamu and Achala need to measure the height of a derelict building that has been fenced off for safety reasons. Isamu stands 15 m from the fence to measure the angle of elevation from ground level of the top of the building, and calculates it to be 33° from the ground. Achala stands 40 m farther away and calculates the angle to be 20°.

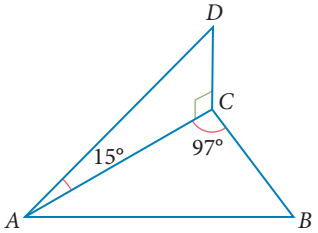


(a) Calculate the height h of the building, correct to the nearest metre.

(b) Achala also wants to determine the height of a radio tower t, on top of the building. From her standing point she measures the angle of elevation of the top of the radio tower to be 28°. Calculate the height of the radio tower, correct to the nearest metre.



10 In the diagram, ΔABC represents a triangular paddock with $AB = 80$ m, $AC = 65$ m and $\angle ACB = 97°$. At point C there is a tree of height CD. The angle of elevation of the top of the tree, D, when viewed from A, is 15°. Calculate the following, correct to 2 decimal places.



(a) the height of the tree

(b) the length of BC

(c) the area of the paddock ABC

(d) the angle of elevation of the top of the tree when viewed from B

Worked Example

10

11 Solve ΔABC, given the following information, calculating all unknown side lengths and angles, correct to 2 decimal places.

(a) $A = 23.42°$, $a = 34.56$, $b = 46.76$

(b) $A = 36.20°$, $a = 15.61$, $b = 23.45$

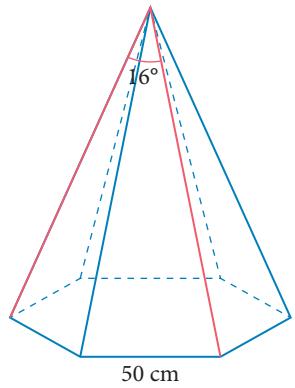
(c) $B = 56.38°$, $b = 25.37$, $c = 26.68$

(d) $C = 12.85°$, $a = 45.09$, $c = 26.68$

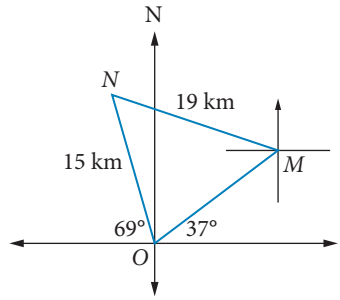
12 ΔABC has side lengths $a = 23.56$ cm and $b = 45.12$ cm and $\angle ABC = 56.43°$. Determine the area of ΔABC, correct to 2 decimal places.

12

13 A symmetrical hexagon-based pyramid has two non-adjacent sloping edges meeting at 16°, as shown. Determine the height of the prism, correct to 1 decimal place, if the edge length of the base is 50 cm.



14 Oscar sets up base camp O while two friends, Neil and Marcus, hike to landmarks N and M along the directions shown. GPS data indicates that Neil hikes 15 km, and is 19 km from Marcus's position. How far is Marcus from base camp, correct to 2 decimal places?



15 In ΔABC, $\sin(A) = \frac{1}{5}$, $\sin(B) = \frac{1}{3}$ and $a = 6$.

(a) Determine the value of b.

(b) Given that A is an acute angle, calculate C to the nearest degree.

16 A lookout at a fire station detects two fires. Fire A is 16 km from the station on a bearing of 053°T. Fire B is on a bearing of 074°T from the station and is 19 km from fire A.

(a) How far is fire B from the station, correct to 2 decimal places?

(b) Calculate the bearing of fire A from fire B, to the nearest degree.

The cosine rule

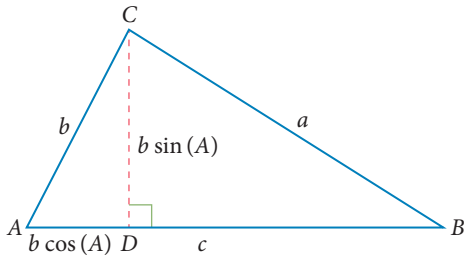
7.4

Proof of the cosine rule

For ΔABC, the cosine rule states that the square of the length of any side of a triangle equals the sum of the squares of the length of the other sides minus twice their product multiplied by the cosine of their included angle.

$a^2 = b^2 + c^2 - 2bc\cos(A)$

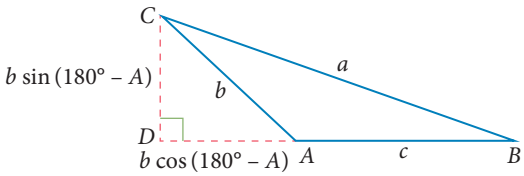
The formula can be derived using Pythagoras' theorem with the right-angled triangle CDB, where $0° < A < 90°$:



$$\begin{aligned} a^2 &= \left(b\sin(A)\right)^2 + \left(c - b\cos(A)\right)^2 \\ &= b^2\sin^2(A) + c^2 - 2bc\cos(A) + b^2\cos^2(A) \\ &= b^2\left(\sin^2(A) + \cos^2(A)\right) + c^2 - 2bc\cos(A) \\ &= b^2 + c^2 - 2bc\cos(A) \end{aligned}$$

The final step of the simplifying process used the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$.

The formula can be confirmed using Pythagoras' theorem with the right-angled triangle CDB:



$$\begin{aligned} a^2 &= \left(b\sin(180° - A)\right)^2 + \left(c + b\cos(180° - A)\right)^2 \\ &= b^2\sin^2(A) + c^2 + 2bc\cos(180° - A) + b^2\cos^2(180° - A) \\ &= b^2\sin^2(A) + c^2 - 2bc\cos(A) + b^2\cos^2(A) \\ &= b^2 + c^2 - 2bc\cos(A) \end{aligned}$$

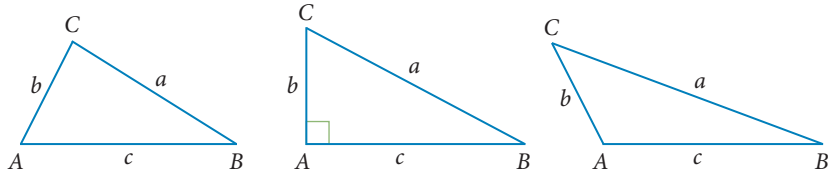
Two sides and an included angle

Additional information

The cosine rule

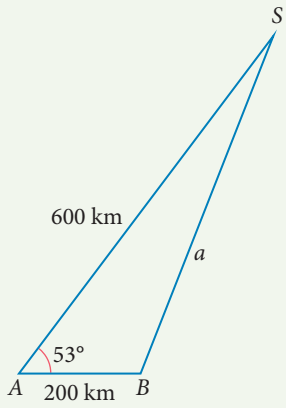
This activity explains the cosine rule.

When two sides and the included angle are known, the SAS congruence test ensures that the triangle has a fixed shape. Using the cosine rule, $a^2 = b^2 + c^2 - 2bc\cos(A)$, to calculate the side opposite the known angle is very similar to calculating the hypotenuse using Pythagoras' theorem. If the angle A is acute, the side a will be shorter than a hypotenuse between the same sides, because $2bc\cos(A)$ is subtracted before taking the square root. If the angle A is obtuse, the side a will be longer than a hypotenuse between the same sides, since $\cos(A)$ is negative, so $-2bc\cos(A)$ effectively increases the value before the square root is taken.



13 Solve a side using the cosine rule

The path of a satellite orbiting the Earth causes it to pass directly over two tracking stations A and B, which are 200 km apart. When the satellite is on one side of the two stations, the angle of elevation at A is 53°, and the direct distance to the satellite is 600 km. How far is the satellite from station B, to the nearest kilometre?

THINKING	WORKING
1 Draw a diagram using appropriate pronumerals.	
2 Recall the cosine rule and substitute the required version of the subject.	$a^2 = b^2 + s^2 - 2bs\cos(A)$, where $b = 600$, $s = 200$, $A = 53°$
3 Substitute the given values and solve for the unknown.	$a^2 = 600^2 + 200^2 - 2 \times 600 \times 200\cos(53°)$ $a^2 = 255\,564.3\ldots$ $\therefore a = \sqrt{255\,564.3\ldots}$ $a = 505.5\ldots$
4 Interpret the answer.	The satellite is approximately 506 km from station B.

All sides given

When three sides are known, the SSS congruence test ensures that the triangle has a fixed shape. The cosine rule can also be expressed as $\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$ if an angle is required. Since any of the three angles could be found in this way, take care identifying the side opposite the required angle.

WARNING

There will be no ambiguous case when using the inverse cosine function.

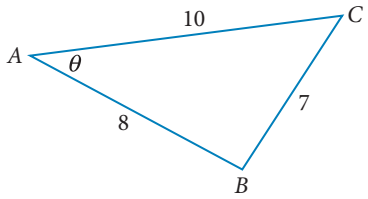
If $\cos(A) > 0$, then $0° < A < 90°$.

If $\cos(A) < 0$, then $90° < A < 180°$.

14 Determine the size of an unknown angle using the cosine rule

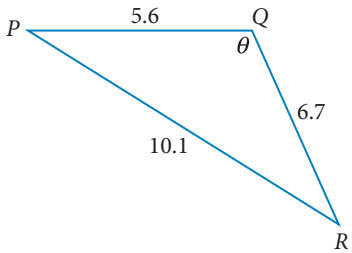
Determine the value of θ in each triangle, correct to 1 decimal place.

(a)



THINKING	WORKING
1 Recall the angle version of the cosine rule, and substitute the required version of the subject.	$\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$, where $a = 7, b = 10, c = 8, A = \theta$
2 Substitute the given values and solve for θ.	$\cos(\theta) = \dfrac{10^2 + 8^2 - 7^2}{2(10)(8)}$ $\cos(\theta) = \dfrac{115}{160}$ $\theta = \cos^{-1}\left(\dfrac{115}{160}\right)$ $= 44.0°\,(1\text{ d.p.})$

(b)



1 Recall the angle version of the cosine rule, and substitute the required version of the subject.	$\cos(Q) = \dfrac{p^2 + r^2 - q^2}{2pr}$, where $p = 6.7, q = 10.1, r = 5.6, Q = \theta$
2 Substitute the given values and solve for θ. The angle will be obtuse since the cosine is negative.	$\cos(\theta) = \dfrac{6.7^2 + 5.6^2 - 10.1^2}{2(6.7)(5.6)}$ $\cos(\theta) = -\dfrac{23}{67}$ $\theta = \cos^{-1}\left(-\dfrac{23}{67}\right)$ $= 110.1°\,(1\text{ d.p.})$

The cosine rule:

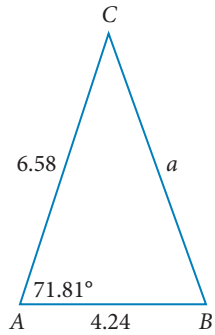
$a^2 = b^2 + c^2 - 2bc\cos(A)$	$b^2 = a^2 + c^2 - 2ac\cos(B)$	$c^2 = a^2 + b^2 - 2ab\cos(C)$
$\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$	$\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$	$\cos(C) = \dfrac{a^2 + b^2 - c^2}{2ab}$

Solving triangles

The cosine rule is applicable in circumstances where the sine rule cannot be used. Together they can be used, given sufficient information, to solve any triangle.

15 Solve a triangle using the cosine and sine rules

Solve ΔABC.



THINKING	WORKING
1 Identify the given information.	$A = 71.81°$, $b = 6.58$, $c = 4.24$
2 Recall the appropriate rule to calculate one unknown. Keep extra decimal places for subsequent calculations.	$a^2 = b^2 + c^2 - 2bc\cos(A)$ $= 6.58^2 + 4.24^2 - 2 \times 6.58 \times 4.24 \times \cos(71.81°)$ $= 43.855\ldots$ $\therefore a = \sqrt{43.855\ldots}$ $= 6.622\ldots$
3 Calculate the second unknown angle. Since a is the longest side, there is no possibility of an obtuse angle at B.	$\frac{\sin(B)}{b} = \frac{\sin(A)}{a}$ $\frac{\sin(B)}{6.58} = \frac{\sin(71.81°)}{6.622\ldots}$ $\sin(B) = \frac{\sin(71.81°)}{6.622\ldots} \times 6.58$ $\sin(B) = 0.943\ldots$ $\therefore B = \sin^{-1}(0.94\ldots)$ $B = 70.73°\ (2\text{ d.p.})$
4 Calculate the size of the third unknown angle.	From the angle sum of a triangle: $C = 180° - (71.81° + 70.73°)$ $= 37.46°$
5 Write the solutions.	$a = 6.62$, $B = 70.73°$, $C = 37.46°$

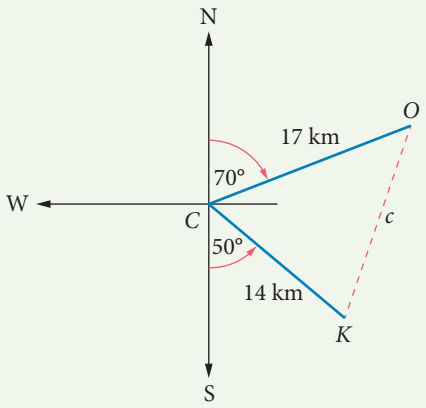
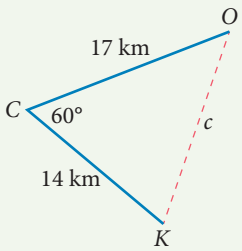
Applications of the cosine rule

Any practical situations where a triangle can be formed could potentially be solved using the cosine rule. Remember that if the SSS or SAS congruence tests hold for a triangle, the cosine rule can be applied directly.

16 Calculate distances and bearings

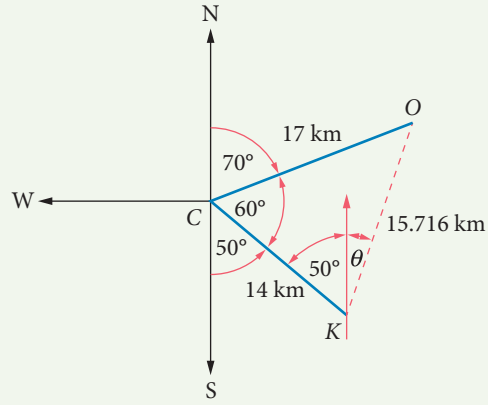
Two hikers leave the same base camp C for different destinations. Olga travels 17 km on a bearing of 070°T before setting up a new camp for the night. Katrina hikes 14 km on a bearing of S50°E before pitching her tent.

(a) How far apart are the hikers' new campsites? Give your answer to 1 decimal place.

THINKING	WORKING
1 Draw a diagram using appropriate pronumerals for locations and include the given information. Note the cosine rule uses two side lengths and the angle between. Calculate that angle.	 Calculate angle C in ΔOCK. $C = 180 - (70 + 50)$ $= 60°$
2 Recall the cosine rule and write a version that includes the given pronumerals. Substitute the given values and solve for c, rounding as directed.	 $c^2 = o^2 + k^2 - 2ok\cos(C)$ $= 14^2 + 17^2 - 2 \times 14 \times 17\cos(60°)$ $= 247$ $\therefore c = \sqrt{247}$ $= 15.7$ (1 d.p.)
3 Interpret the solution.	The hikers' new campsites are 15.7 km apart.

(b) What is the true bearing of Olga from Katrina's position? Give your answer to the nearest degree.

1 Draw a north–south line segment, and mark the required bearing (θ) and the additional information.



Note the use of alternate angles.

2 Recall the sine rule and use it to calculate the required angle at K.

Since the diagram is reasonably to scale, the obtuse-angle solution need not be considered.

$$\frac{\sin(K)}{17} = \frac{\sin(60°)}{\sqrt{247}}$$

$$\sin(K) = \frac{17\sin(60°)}{\sqrt{247}}$$

$$= 0.936\ldots$$

$$K = \sin^{-1}(0.936\ldots)$$

$$= 70° \text{ (nearest degree)}$$

3 Calculate the angle needed for the bearing.

$$\theta = 70 - 50$$

$$= 20°$$

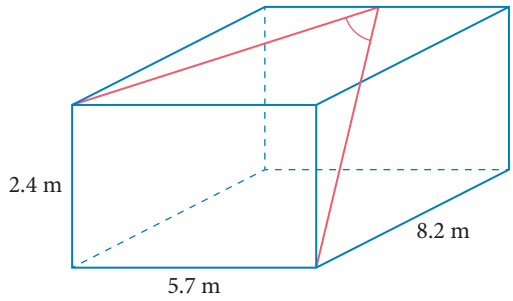
4 Interpret the solution.

The true bearing of O from K is 020°T.

17 Application to three dimensions

A box-shaped room has two pieces of ribbon attached to the midpoint of the wall–ceiling edge, stretched tightly to meet the top and bottom vertices on the opposite wall, as shown.

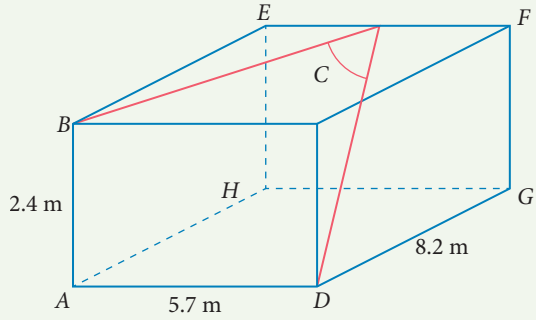
Determine the angle between the ribbons, correct to 1 decimal place, if the dimensions of the room are 5.7 m, 8.2 m and 2.4 m.



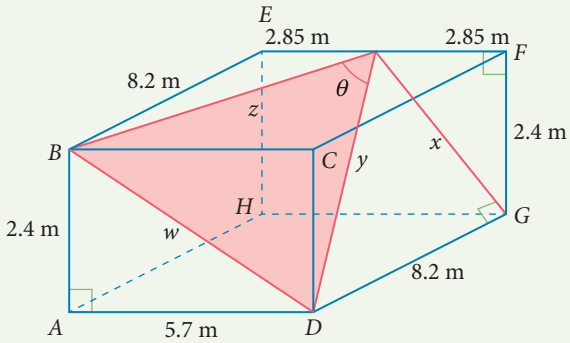
THINKING

1 Label the vertices.

WORKING



2 Solve right-angled triangles to determine the side lengths of the sloping triangle.



$$x^2 = (FG)^2 + \left(\frac{1}{2}EF\right)^2$$
$$= 2.85^2 + 2.4^2$$
$$= 13.8825$$

$$y^2 = x^2 + (DG)^2$$
$$= 13.8825^2 + 8.2^2$$
$$= 81.1225$$
$$\therefore y = \sqrt{81.1225}$$

$$z^2 = \left(\frac{1}{2}EF\right)^2 + (BE)^2$$
$$= 2.85^2 + 8.2^2$$
$$= 75.3625$$
$$\therefore z = \sqrt{75.3625}$$

$$w^2 = (AB)^2 + (AD)^2$$
$$= 2.4^2 + 5.7^2$$
$$= 38.25$$
$$\therefore w = \sqrt{38.25}$$

3 Recall the cosine rule and use it to solve for θ.

$$\cos(\theta) = \frac{y^2 + z^2 - w^2}{2yz}$$
$$= \frac{81.1225 + 75.3625 - 38.25}{2\sqrt{81.1225}\sqrt{75.3625}}$$
$$= 0.756\ldots$$
$$\theta = \cos^{-1}(0.756\ldots)$$
$$= 40.9° \text{ (1 d.p.)}$$

4 Interpret the answer.

The angle between the ribbons is 40.9°.

EXERCISE 7.4 The cosine rule

Worked Example

13

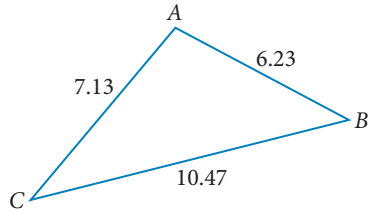
1 The path of a hot air balloon is being tracked by two observers on the ground, who are 8 km apart. From observer A, the direct distance to the hot air balloon is 500 m, and the angle of elevation is measured to be 60°. How far, to the nearest metre, is the balloon from observer B?

2 An observer on the ground at point A watches a rocket ascend. The observer is 800 m to the left of the launch point P. A second observer on the ground at point B is 700 m to the right of the launch point. When the rocket is at a distance of 1200 m from A, the angle of elevation of the rocket is 35°. What is the distance of B from the rocket at this time? Give your answer correct to the nearest metre.

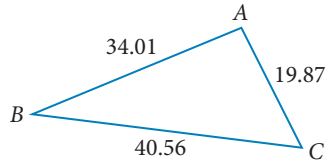
14

3 Calculate the magnitude of all unknown angles in each triangle, correct to 2 decimal places.

(a)



(b)



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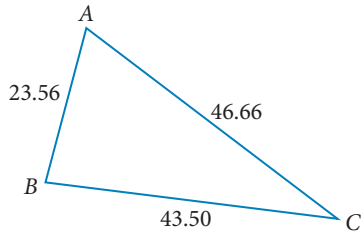
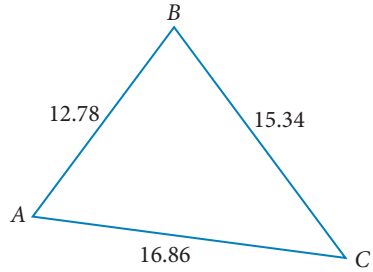


(d)



4 ΔABC has side lengths $a = 7$ m, $b = 6$ m and $c = 11$ m. Determine the magnitude of the largest angle in the triangle, correct to the nearest degree.

5 In ΔABC, $a = 5$, $b = 6$ and $c = 7$.

(a) Which of the following could be $\angle ACB$?

A 44.42° **B** 57.12° **C** 78.46° **D** 78.46° or 101.54°

(b) Explain why an ambiguous case is not possible to a student who selected an incorrect response in part **(a)**.

6 Calculate, to the nearest degree, all angles of a triangle with sides in the ratio 5:8:11.

7 A small triangular grazing paddock is advertised as having an approximate area of 100 m^2. If the three boundaries of the paddock are 12 m, 16 m and 17 m, determine which of the following statements is correct.

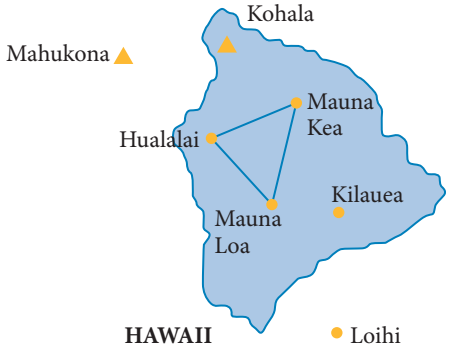
A The advertised area is correct.

B The actual area is approximately 8 m^2 smaller than advertised.

C The actual area is approximately 95 m^2.

D The actual area is half the area advertised.

8 The distance between landmarks on the island of Hawaii can change slightly after an earthquake. The diagram on the right shows a triangle with one vertex on a distance-measuring station on Mauna Loa and the other two vertices on stations on Hualalai and Mauna Kea.



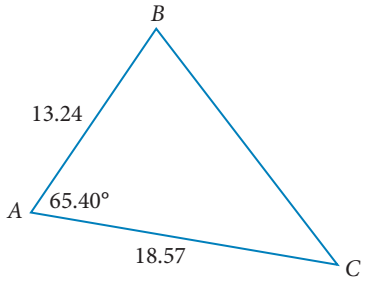
(a) The straight line distance from Mauna Kea to Mauna Loa was measured to be 36.177 km, and the distance from Mauna Kea to Hualalai was 45.464 km. Also, the angle between these two lines was measured to be 58.57°. Calculate the straight-line distance between Mauna Loa and Hualalai, correct to 3 decimal places.

(b) After an earthquake, the straight-line distance from Mauna Loa to Hualalai was found to be 40.636 km. Determine the new angle between the two lines from Mauna Kea to the other volcanoes, correct to 2 decimal places, assuming the other lengths remain the same.

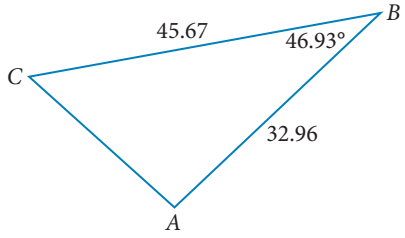
Worked Example 15

9 Solve the following triangles. Calculate unknown side lengths and angles correct to 2 decimal places.

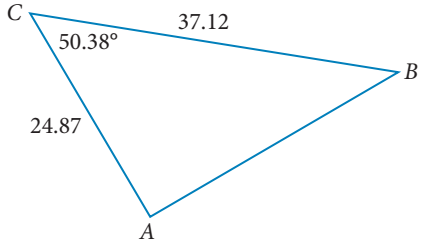
(a)



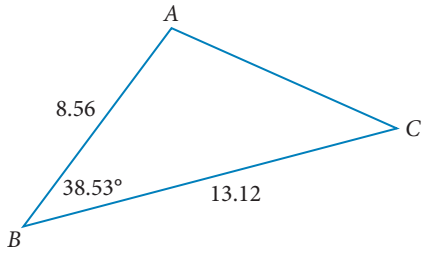
(b)



(c)

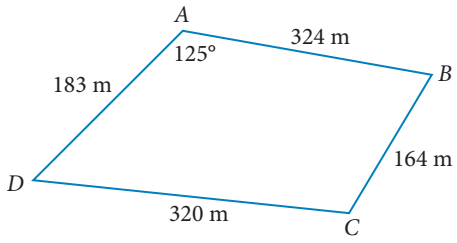


(d)



10 Raphaela has drawn a triangle and labelled the vertices A, B and C. She has measured side AB to be 15 cm and side BC to be 17 cm. She also knows that angle $\angle ABC$ measures 60°. Determine the unknown side length, correct to the nearest centimetre, and the unknown angles, correct to the nearest degree.

11 Kashiare owns a paddock. The lengths of the fences are 324 m, 164 m, 320 m and 183 m as shown in the diagram. The angle between fence AB and fence AD is 125°. Calculate the area of Kashiare's paddock, to the nearest ten square metres.



16

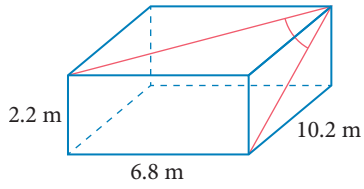
12 Two hikers leave the same base camp A for different destinations. Breeana (B) travels 13 km on a bearing 050°T before setting up a new camp for the night. Campbell (C) hikes 9 km on a bearing S20°E before pitching her tent.

(a) What is the distance between the new campsites? Give your answer to 1 decimal place.

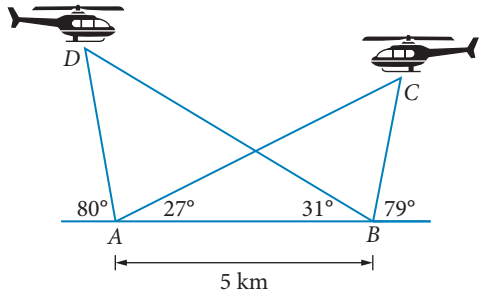
(b) What is the bearing of Breeana from Campbell's position? Give your answer to the nearest degree.

17

13 A box-shaped room has two pieces of ribbon attached to one corner of the ceiling, streched tightly to meet the diagonally opposite corner of the ceiling and the diagonally opposite corner of one wall, as shown. Determine the angle between the ribbons, correct to 1 decimal place, if the dimensions of the room are 10.2 m, 6.8 m and 2.2 m.



14 Two helicopters C and D are hovering directly over a level freeway. The diagram shows the angle of elevation of the helicopters from two observers at A and B who are 5 km apart.



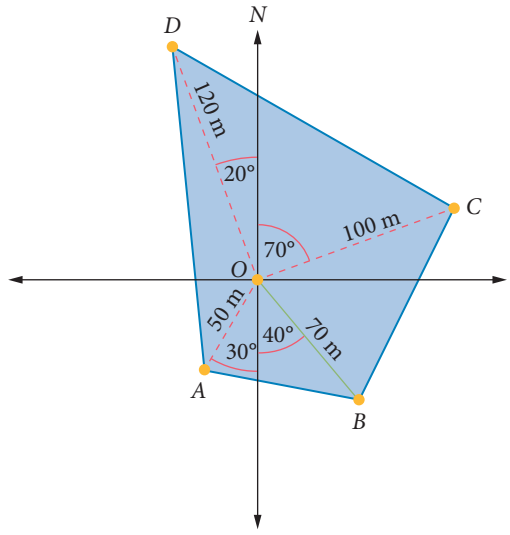
(a) Determine the direct distance of observer A from helicopter D. Give your answer correct to 2 decimal places.

(b) Determine the direct distance of observer B from helicopter C. Give your answer correct to 2 decimal places.

(c) Which helicopter is higher, and by how much? Give your answer to the nearest 10 metres.

(d) Determine the direct distance between the helicopters. Give your answer to the nearest kilometre.

15 Organisers are setting up an orienteering course in school grounds and they place checkpoints at A, B, C and D to define the area of the course. The bearing and distance of each checkpoint from a central command point O are shown.



(a) Determine the perimeter of the course to the nearest metre.

(b) If checkpoint A moved along the same bearing so that the perimeter of the course is 500 m, should the distance AO be less than 60 m or more than 60 m? Assume that all other checkpoints remain fixed.

(c) Calculate the original area within the course boundaries (that is, the area of shape $ABCD$) to the nearest square metre.

(d) Checkpoint A is to be moved along the same bearing so that the area of the course will become 15 000 m^2. Assuming that all other checkpoints remain fixed, determine the new length OA (correct to 1 decimal place).

Summary

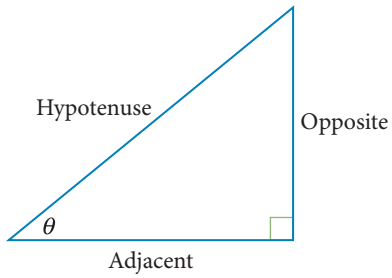
7

Right-angled triangles

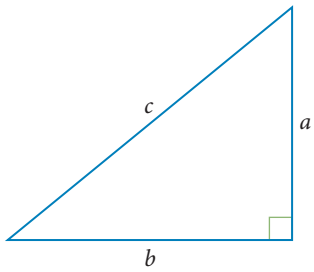
$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

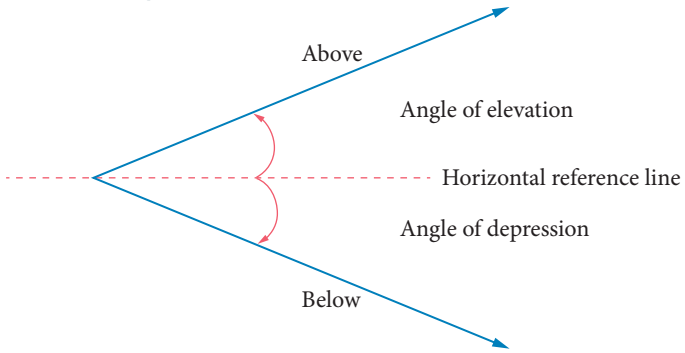
$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$



Pythagoras' theorem: $c^2 = a^2 + b^2$

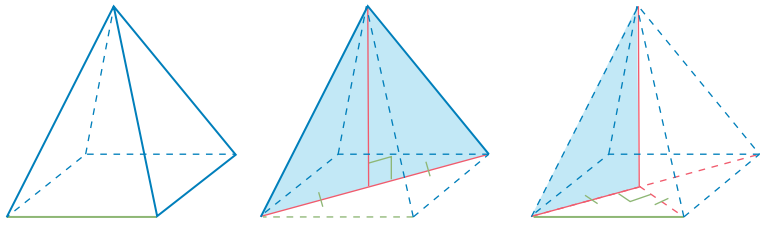


Angles of elevation and depression



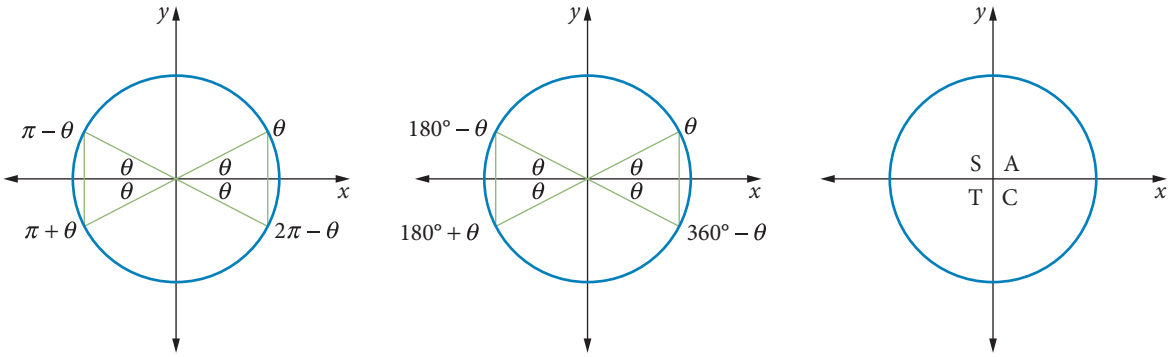
Three dimensions

Produce two-dimensional shapes from the three-dimensional figure.



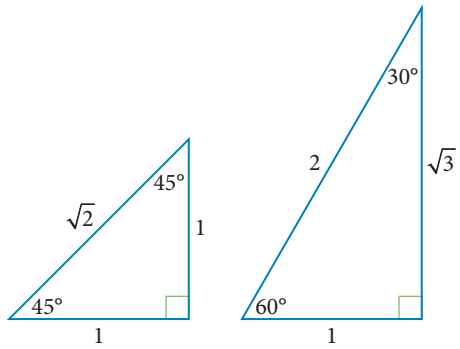
The unit circle and periodicity

$\pi \leftrightarrow 180°$



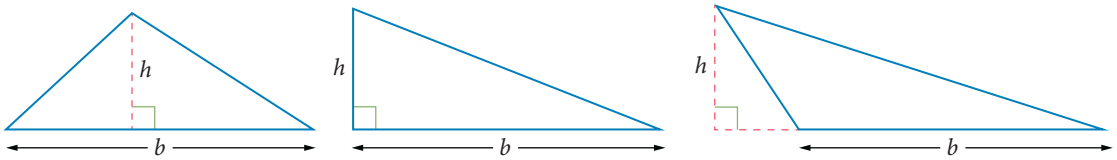
Special triangles and exact ratios

$30° \leftrightarrow \frac{\pi}{6}$, $45° \leftrightarrow \frac{\pi}{4}$, $60° \leftrightarrow \frac{\pi}{3}$

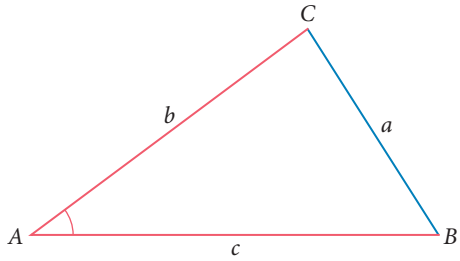


Area of a triangle

Given the base and height of a triangle: Area $= \frac{1}{2}bh$



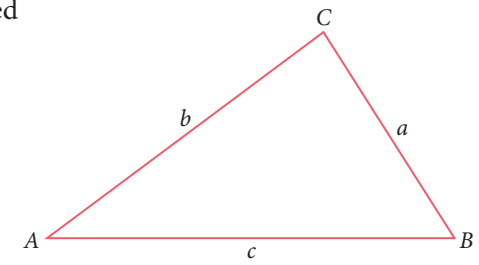
Given two sides and the included angle: Area $= \frac{1}{2}bc\sin(A)$



When given three sides of a triangle that is not a right-angled triangle, Heron's formula can be used.

Area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$

where a, b and c are the side lengths of ΔABC and s is the semi-perimeter and is given by $s = \frac{1}{2}(a+b+c)$.



The sine rule

Use the sine rule when calculating the length of a second side, given two angles and one side.

Calculate the third angle, if necessary, using the angle sum of a triangle.

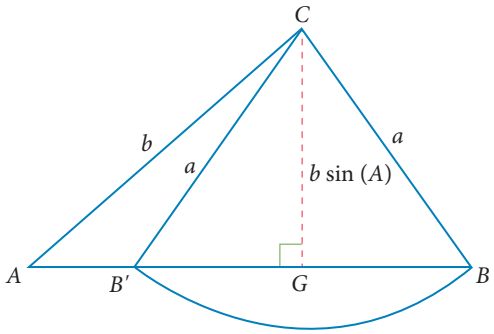
The AAS (or ASA) congruence test applies.

$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ (use two ratios only)

Use the sine rule when calculating the size of a second angle, given two sides and a non-included angle.

$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$ (use two ratios only).

There is no SSA congruence test, so an angle opposite the larger of the given sides could be acute or obtuse, leading to the ambiguous case.

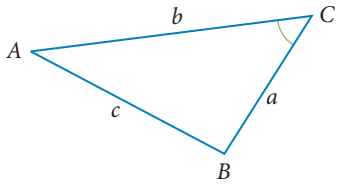


The cosine rule

Use the cosine rule when calculating the third side (c), given two sides and the included angle.

The SAS congruence test applies.

$c^2 = a^2 + b^2 - 2ab\cos(C)$

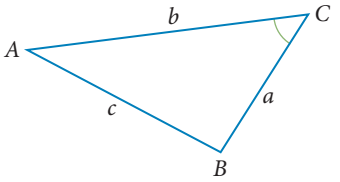


Use the cosine rule when calculating any angle, given three sides.

The SSS congruence test applies.

$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$

If the required angle is obtuse, its cosine will be negative.



Bearings

Bearings are always measured clockwise, starting from north, and are written as three-digit angles, sometimes followed by the letter T (for True).

Chapter review 7

Exercise 7.1

1 Convert each of the following angles.

(a) Convert to degrees, correct to 2 decimal places where necessary.

(i) $\frac{3\pi}{4}$ **(ii)** 1.26^{c}

(b) Convert 124° to radians, in terms of π.

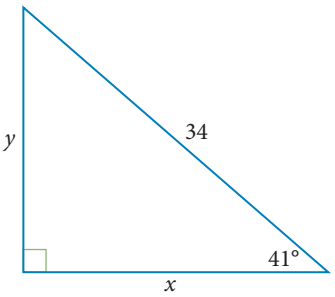
(c) Convert 47° to radians, correct to 3 decimal places.

Exercise 7.1

2 At 3 pm a telecommunications tower $50\sqrt{3}$ m high produces a shadow of length 50 m. Determine the angle that the Sun's rays make with the horizontal at this time.

Exercise 7.1

3 The values of x and y respectively in the given triangle are closest to which one of the following pairs of values?



A 25.66 and 22.31

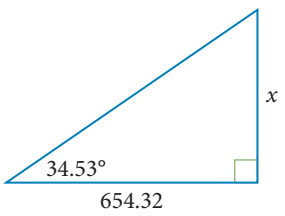
B 22.31 and 25.66

C 29.56 and 25.66

D 22.31 and 29.56

Exercise 7.1

4 Determine the length of side x, correct to 2 decimal places.



Exercise 7.1

5 A right-angled triangle has a side of length 62 cm and a hypotenuse of 110 cm. Calculate the angle opposite the 62 cm side length, correct to 2 decimal places.

Exercise 7.1

6 Singh is standing 20 m from the base of a building. He measures the angle to the top of the building and finds it to be 40°. He then measures the angle to the top of a radio mast on top of the building, finding that it is 52°. Determine the height of the radio mast, correct to 2 decimal places.

Exercise 7.1

7 From the top of a 45 m skyscraper the angle of depression of an object at P on the wall of a smaller building opposite is 40°. The width of the street is 18 m.

(a) Determine the height of the object at P, correct to the nearest metre.

(b) Determine the angle of elevation of P from the foot of the skyscraper, correct to the nearest degree.

Exercise 7.2

8 In order to try to control a bushfire in the Avon Wilderness, firefighters plan to form containment lines in a triangular shape around the perimeter of the fire. Two containment lines starting from Huggetts Crossing will be 33 km and 36 km long and the angle between them is 30°. Determine the area enclosed by the containment lines.

9 In ΔABC, if $a = 12$ cm, $b = 14$ cm, $c = 17$ cm and $\angle ABC = 65°$, what is the area of the triangle in square centimetres? Exercise 7.2

A 152 **B** 36 **C** 108 **D** 76

10 Milosh needs to plant a canola crop in a triangular paddock. One side of the paddock measures 475 m, another measures 612 m and the angle between these two sides is 132°. Exercise 7.2

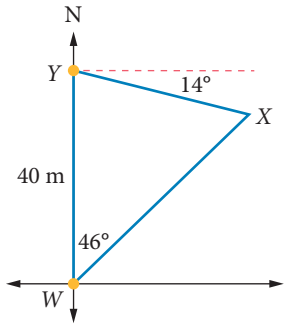
(a) Calculate the area in which the canola is to be planted, correct to the nearest square metre.

(b) Milosh's mother, Anika, has finished preparing an extra area for planting such that the side of the paddock measuring 612 m is now extended to 643 m. How much extra area now needs to be planted, correct to the nearest square metre?

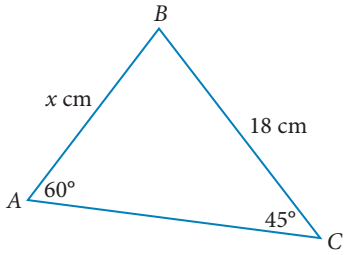
11 Ethan wants to calculate the area of a triangular lot. He determines that two sides are 70 m and 30 m and the angle between them is 4°. What is the area of the lot, correct to 1 decimal place? Exercise 7.2

12 In ΔABC, $a = 3.3$ m, $b = 3.4$ m and $c = 3.5$ m. Calculate the largest angle and hence the area. Give your answers correct to 2 decimal places. Exercise 7.4

13 The sewerage system for a particular property has three inspection points: W, X and Y. Point Y is 40 m due north of W, and point X is in a direction S76°E from Y and N46°E from W. How far apart are points W and X, correct to 2 decimal places? Exercise 7.3



14 Calculate the exact value of the unknown side length, x cm. Exercise 7.3

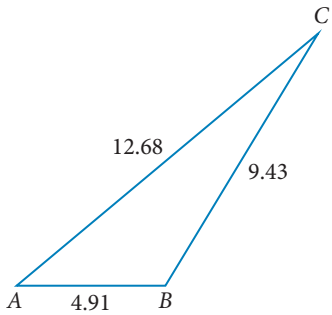


15 In ΔABC, if $a = 15$, $\angle ABC = 35°$ and $b = 21$, then $\angle BAC$ could be which of the following angles? Exercise 7.3

A 53° **B** 24° only **C** 24° or 156° **D** 92°

16 In ΔABC, $c = 15$, $\angle ABC = 56°$ and $\angle BCA = 67°$. Calculate the length of side a, correct to 2 decimal places. Exercise 7.3

17 Solve ΔABC by calculating each angle, correct to 2 decimal places. Exercise 7.4

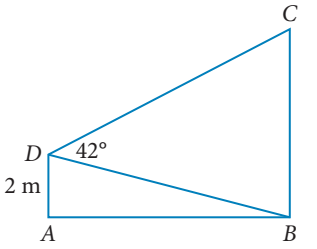


Exercise 7.4 **18** A triangular block of land has two boundary fences of length 50 m and 80 m and the angle that the fences make with each other is 60°. Determine the exact length of the third boundary fence.

Exercise 7.4 **19** In ΔABC, if $BC = 12$, $AC = 13$ and $AB = 14$, then $\angle ABC$ is equal to which one of the following angles?

A 67.98° **B** 59.41° **C** 52.62° **D** 120.59°

Exercise 7.1 **20** Simon is looking up at the top of a flagpole. His eyes are 2 m above the ground. The vertical flagpole subtends an angle of 42° at Simon's eyes as shown in the diagram. The angle of depression of the foot of the flagpole from Simon's eyes is 12°.

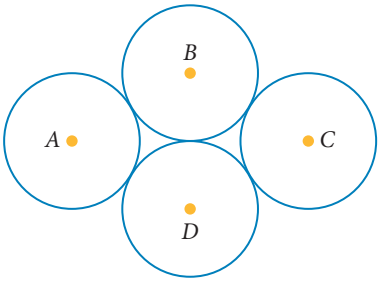


(a) Determine the height of the flagpole, correct to 2 decimal places.

(b) Determine the direct distance from Simon's eyes to the top of the flagpole, correct to 2 decimal places.

(c) Determine the angle of depression from the top of the flagpole to the point A on the ground at Simon's feet, correct to 2 decimal places.

Exercise 7.1 **21** Four circular discs, each of radius 5 cm, with centres A, B, C and D, are arranged so that they touch as shown in the diagram.



(a) Calculate the lengths of BC and AC.

(b) Determine the area of ΔABD.

Exercise 7.3 **22** Given $A = 19.23°$, $a = 22.46$, b = 36.12, solve ΔABC by calculating the unknown side length and angles, correct to 2 decimal places.

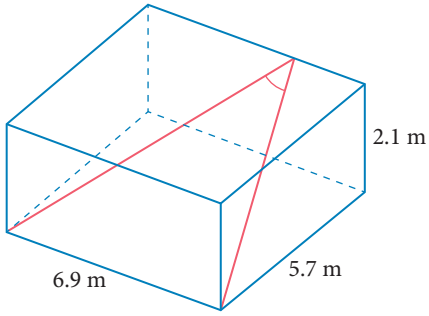
Exercise 7.3 **23** The Moon's orbit around the Earth is elliptical. At perigee (its closest approach) the Moon comes as close as 363 104 km. At apogee (the farthest away it gets) the Moon is 406 696 km from the Earth. The distance from the Earth to the Moon can be found by measuring the angle of elevation of the Moon at two locations at precisely the same time. On 26 April 1976, the angle of elevation of the Moon was measured at the German towns of Bochum and Donaueschingen, which are 404 km apart. At 11:35 am the angle of elevation was 52.70° at Bochum and 52.75° at Donaueschingen.

(a) Calculate the distance from Bochum to the Moon on this day.

(b) Compare the distance calculated in part **(a)** with the average distance from the Earth to the Moon. State any assumptions you had to make.

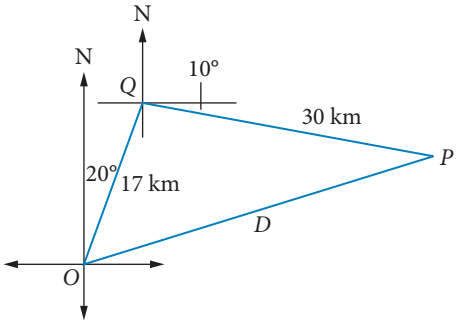
24 A box-shaped room has two pieces of ribbon attached to an edge of the ceiling, two-thirds of the way along from the corner. The pieces are stretched tightly to meet the corners of the floor diagonally opposite, as shown. Determine the angle between the ribbons, correct to 1 decimal place, if the dimensions of the room are 6.9 m, 5.7 m and 2.1 m.

Exercise 7.4



25

Exercise 7.4

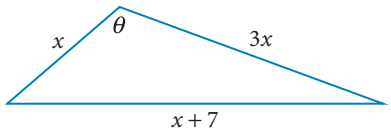


(a) Determine the distance D to the nearest kilometre.

(b) Determine the bearing of point P from point O to the nearest degree.

26 Each of the two equal angles in an isosceles triangle is four times as large as the remaining angle. The perimeter of the triangle is 30 cm. Determine the area of the triangle, correct to 2 decimal places.

27 The area of the triangle shown on the right is $4\sqrt{11}$ cm^2. The length of the shortest side is x cm, the length of the longest side is $(x + 7)$ cm and the length of the third side is $3x$ cm. Use technology to determine all possible values of x, correct to 1 decimal place.



8

Binomial distributions

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8 Recall

Distinguish between discrete and continuous data

1 Indicate whether the following random variables are associated with discrete or continuous scales.

(a) the number of snakes living on properties in north Queensland

(b) the birth mass of each student in your class

Calculate conditional probability from the sample space

2 Calculate the following conditional probabilities directly from the sample space.

(a) A coin is tossed three times. Determine the probability that exactly one head is obtained, given that the first toss resulted in a tail.

(b) A fair six-sided die labelled 1 to 6 is rolled twice. Determine the probability that the difference between the two numbers is 2, given the first number rolled was a two.

(c) A spinner evenly divided into three sections and labelled 1 to 3 is spun three times. Determine the probability that the sum of the three numbers is greater than seven, given the first number spun was a three.

Values for the variable in a discrete probability distribution

3 You have a jar that contains three red marbles, five green marbles and two blue marbles. Let X be the number of blue marbles drawn.

(a) You draw three marbles from the jar. What values can X take, given the following conditions?

(i) The marbles are drawn individually and replaced after each draw.

(ii) The marbles are drawn simultaneously.

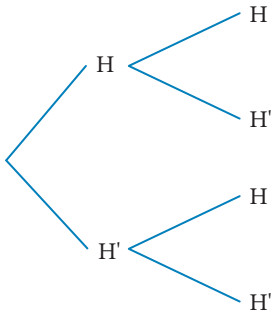
(b) What values can X take, given that the marbles are drawn simultaneously and:

(i) nine marbles are selected? (ii) ten marbles are selected?

Draw a tree diagram to assist in displaying a probability distribution

4 Kat has a record of 3 out of 4 shots at a target being successful. She is about to take two shots.

(a) Complete the following tree diagram by selecting the correct probability for each branch. H stands for 'hit'.



(b) Complete the probability distribution table for the number of hits.

x	0	1	2
$P(X = x)$			

(c) Draw a graph of the probability distribution.

Recognise a discrete probability distribution

5 Determine whether or not each table represents a discrete probability distribution. Explain your choice.

(a)

x	3	4	5	6	4
$P(X=x)$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$

(b)

x	-3	-2	-1
$P(X=x)$	$\frac{5}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$

(c)

x	1	2	3	4	5
$P(X=x)$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

Calculate unknown probabilities in a discrete probability distribution

6 For each of the following examples, determine the value of k that enables the table to represent a discrete probability distribution.

(a)

x	8	9	10	11
$P(X=x)$	$\frac{1}{12}$	k	$\frac{1}{6}$	$\frac{2}{9}$

(b)

x	0	1	2	3
$P(X=x)$	$\frac{1}{4}$	$4k$	$\frac{1}{2}$	k

Calculate the expected value from a probability table

7 Calculate $E(X)$ for each of the following probability distributions.

(a)

x	0	1	2	3	4
$P(X=x)$	0.25	0.12	0.2	0.1	0.33

(b)

x	-2	-1	0	1	2
$P(X=x)$	0.5	0.05	0.25	0.1	0.1

(c)

x	20	32	37	55	58
$P(X=x)$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$

Determine the number of combinations

8 Determine the number of ways the given number of items can be selected, assuming order is not important, from the group of items.

(a) six items from eight **(b)** ${}^{10}C_4$ **(c)** $\binom{7}{2}$

8.1 Discrete probability distributions review

Probability distribution tables

For a simple experiment such as rolling a single die, the probability distribution can be uniform, since each outcome is equally likely.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

For theoretical probabilities with a small number of possible outcomes, fractions should be used to ensure exact values. For probabilities involving estimated probability from accumulated data, decimals or percentages make more sense. To complete a probability distribution table from a multiple event, constructing a sample space is highly recommended.

1 Probability distribution tables from multiple events

Two spinners, each numbered from 1 to 4, are spun simultaneously.

(a) Construct a probability distribution table for the sum of the two numbers, X.

THINKING

1 Assuming the numbers 1 to 4 are equally likely, construct a sample space for the outcome, as ordered pairs.

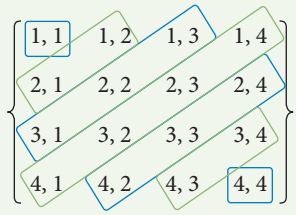
2 Count the ways of getting each of the possible totals, and write the probability $P(X) = \frac{n(X)}{n(\varepsilon)}$ as a fraction of the total number.

3 Write the probabilities into a table.

WORKING

Sample space

$$\begin{Bmatrix} 1,1 & 1,2 & 1,3 & 1,4 \\ 2,1 & 2,2 & 2,3 & 2,4 \\ 3,1 & 3,2 & 3,3 & 3,4 \\ 4,1 & 4,2 & 4,3 & 4,4 \end{Bmatrix}$$



$P(2) = \frac{1}{16}$ $P(3) = \frac{2}{16} = \frac{1}{8}$ $P(4) = \frac{3}{16}$ $P(5) = \frac{4}{16} = \frac{1}{4}$

$P(6) = \frac{3}{16}$ $P(7) = \frac{2}{16} = \frac{1}{8}$ $P(8) = \frac{1}{16}$

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

(b) Determine the probability of a sum between 5 and 7 inclusive.

1 Add the individual probabilities, or count the number of ways from the sample space.	$P(5 \le X \le 7) = P(5) + P(6) + P(7)$ $= \frac{4+3+2}{16}$ $= \frac{9}{16}$
2 Interpret the result.	The probability of a sum between 5 and 7 inclusive is $\frac{9}{16}$.

(c) Determine the probability of a sum of at least 3.

1 Use complementary events to find the value.	$P(\text{sum of } X \ge 3) = 1 - P(2)$ $= 1 - \frac{1}{16}$ $= \frac{15}{16}$
2 Interpret the result.	The probability of a sum of at least 3 is $\frac{15}{16}$.

Expected value, variance and standard deviation

The expected value or mean, $E(X)$ or μ is an important measure of central tendency. From a probability distribution table: $E(X) = \sum x_i p_i$

The variance, denoted by σ^2 ('sigma squared') or by $Var(X)$, gives an indication of how far the values of a data set are spread out from the mean.

$$\begin{aligned} Var(X) &= \sigma^2 \\ &= E(X - \mu)^2 \\ &= \sum (X - \mu)^2 p_i \\ &= E\left(X^2\right) - \left(E(X)\right)^2 \end{aligned}$$

The standard deviation of X, σ is used to compare relative positions within a distribution or between related distributions.

$$\begin{aligned} \text{The standard deviation of } X &= \sqrt{Var(X)} \\ &= \sqrt{\sigma^2} \\ &= \sigma \end{aligned}$$

Expected value: $E(X) = \sum x_i p_i$

Variance: $Var(X) = \sum p_i \left(x_i - \mu\right)^2 = E\left(X^2\right) - \left(E(X)\right)^2$

Standard deviation: $\sigma(X) = \sqrt{Var(X)}$

2 The mean, variance and standard deviation of a discrete probability distribution

Consider the following probability distribution.

x	0	1	2	3	4	5
$P(X = x)$	0.1	0.05	0.3	0.25	0.2	0.1

(a) Calculate the expected value (mean).

THINKING	WORKING
1 Recall and use $E(X) = \sum x_i p_i$.	$E(X) = 0 \times 0.1 + 1 \times 0.05 + 2 \times 0.3 + 3 \times 0.25 + 4 \times 0.2 + 5 \times 0.1$ $= 0 + 0.05 + 0.6 + 0.75 + 0.8 + 0.5$ $= 2.7$
2 Interpret the result.	The long-term mean, or expected value, is 2.7.

(b) Consider the probability distribution replacing the x-values with the x^2 values and use it to calculate the variance.

x^2	0	1	4	9	16	25
$P(X = x)$	0.1	0.05	0.3	0.25	0.2	0.1

METHOD 1

1 Recall the formula	$E\left(X^2\right) = \sum x^2 p$
2 Substitute the values and calculate.	$E\left(X^2\right) = 0 \times 0.1 + 1 \times 0.05 + 4 \times 0.3 + 9 \times 0.25 + 16 \times 0.2 + 25 \times 0.1$ $= 0 + 0.05 + 1.2 + 2.25 + 3.2 + 2.5$ $= 9.2$
3 Calculate $Var(X)$ from $E\left(X^2\right)$ and $E(X)$.	$Var(X) = E\left(X^2\right) - \left(E(X)\right)^2$ $= 9.2 - 2.7^2$ $= 1.91$
4 Interpret the result.	The variance is 1.91.

METHOD 2

1 Recall the formula	$Var(X) = \sum p_i \left(x_i - \mu\right)^2$
2 Substitute the values and calculate.	$Var(X) = 0.1 \times (0 - 2.7)^2 + 0.05 \times (1 - 2.7)^2 + \ldots 0.1 \times (5 - 2.7)^2$ $= 1.91$
3 Interpret the result.	The variance is 1.91.

(c) Calculate the standard deviation, correct to 2 decimal places.

1	Take the square root of the variance.	$\sigma = \sqrt{Var(X)}$ $= \sqrt{1.91}$ $= 1.38\ (2\text{ d.p.})$
2	Interpret the result.	The standard deviation is 1.38.

Technology worked example
Expected value, variance and standard deviation

Mean and spread of transformed data

The significance of the mean and variance values becomes clearer when you see what happens to these values following a linear transformation, so $X \rightarrow aX + b$.

Consider the set of data:

x	2	3	7
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$E(X) = 2 \times \frac{1}{3} + 3 \times \frac{1}{3} + 7 \times \frac{1}{3}$$
$$= \frac{1}{3} \times (2 + 3 + 7)$$
$$= \frac{2}{3} + \frac{3}{3} + \frac{7}{3}$$
$$= \frac{12}{3}$$
$$= 4$$

$$Var(X) = E\left(X^2\right) - \left[E(X)\right]^2$$
$$= 2^2 \times \frac{1}{3} + 3^2 \times \frac{1}{3} + 7^2 \times \frac{1}{3} - 4^2$$
$$= \frac{4}{3} + \frac{9}{3} + \frac{49}{3} - 16$$
$$= \frac{62}{3} - \frac{48}{3}$$
$$= \frac{14}{3}$$

$$\sigma = \sqrt{Var(X)}$$
$$= \sqrt{\frac{14}{3}}$$
$$\approx 2.160$$

$$E(aX + b) = aE(X) + b$$
$$Var(aX + b) = a^2 Var(X)$$
$$\sigma(aX + b) = |a|\sigma(X)$$

Transformation	Expected value	Variance	Standard deviation
$X \rightarrow X + 50$ 52, 53, 57	$E(X + 50)$ $= 52 \times \frac{1}{3} + 53 \times \frac{1}{3} + 57 \times \frac{1}{3}$ $= \frac{1}{3}(2 + 3 + 7) + \frac{1}{3} \times (3 \times 50)$ $= E(X) + 50$ $= 50 + 4$ $= 54$	$Var(X + 50)$ $= E\left((X + 50)^2\right) - \left[E(X + 50)\right]^2$ $= \frac{1}{3}\left(52^2 + 53^2 + 57^2\right) - \left(E(X) + 50\right)^2$ $= \frac{1}{3}\left(52^2 + 53^2 + 57^2\right) - 54^2$ $= \frac{14}{3}$ $= Var(X)$	$\sigma(X + 50)$ $= \sqrt{Var(X + 50)}$ $= \sqrt{Var(X)}$ $= \sqrt{\frac{14}{3}}$ ≈ 2.160

Transformation	Expected value	Variance	Standard deviation
$X \to 10X$ 20, 30, 70	$E(10X)$ $= 20 \times \frac{1}{3} + 30 \times \frac{1}{3} + 70 \times \frac{1}{3}$ $= \frac{1}{3}(20 + 30 + 70)$ $= \frac{1}{3} \times 10 \times (2 + 3 + 7)$ $= 10 \times E(X)$ $= 10 \times 4$ $= 40$	$Var(10X)$ $= E\left((10X)^2\right) - \left[E(10X)\right]^2$ $= \frac{1}{3} \times \left(20^2 + 30^2 + 70^2\right) - \left(E(10X)\right)^2$ $= \frac{1}{3} \times 10^2 \times \left(2^2 + 3^2 + 7^2\right) - 10^2 \times E(X)^2$ $= 10^2 \times \left[\frac{1}{3}\left(2^2 + 3^2 + 7^2\right) - E(X)^2\right]$ $= 10^2 \times \frac{14}{3}$ $= \frac{1400}{3}$ $= 10^2 Var(X)$	$\sigma(10X)$ $= \sqrt{Var(10X)}$ $= \sqrt{10^2 Var(X)}$ $= 10\sqrt{Var(X)}$ $= 100\sigma$ $= 10 \times \sqrt{\frac{14}{3}}$ ≈ 21.602
$X \to 5X + 1$ 11, 16, 36	$E(5X + 1)$ $= 11 \times \frac{1}{3} + 16 \times \frac{1}{3} + 36 \times \frac{1}{3}$ $= \frac{1}{3}(10 + 15 + 35) + \frac{1}{3} \times 3$ $= \frac{1}{3} \times 5 \times (2 + 3 + 7) + 1$ $= 5 \times E(X) + 1$ $= 5 \times 4 + 1$ $= 21$	$Var(5X + 1)$ $= E\left((5X + 1)^2\right) - \left[E(5X + 1)\right]^2$ $= \frac{1}{3} \times \left(11^2 + 16^2 + 36^2\right) - \left(5 \times E(X) + 1\right)^2$ $= \frac{1}{3} \times \left(11^2 + 16^2 + 36^2\right) - 21^2$ $= \frac{1673}{3} - 441$ $= \frac{350}{3}$ $= 25 \times \frac{14}{3}$ $= 5^2 \times Var(X)$	$\sigma(5X + 1)$ $= \sqrt{Var(5X + 1)}$ $= \sqrt{5^2 Var(X)}$ $= 5\sqrt{Var(X)}$ $= 5\sigma$ $= 5 \times \sqrt{\frac{14}{3}}$ ≈ 10.801

3 Determine the mean, variance and standard deviation of transformed data

Consider the following probability distribution and the transformed distribution.

x	1	3	7
$P(X = x)$	0.5	0.3	0.2

y	42	46	54
$P(Y = y)$	0.5	0.3	0.2

(a) Determine the relationship between X and Y.

THINKING

1 Assume the relationship is linear between the points and express the relationship in the form $Y = aX + b$.

WORKING

$y - y_1 = m(x - x_1)$

$Y - 42 = \frac{46 - 42}{3 - 1}(Y - 1)$

$Y - 42 = \frac{4}{2}(X - 1)$

$Y = 2(X - 1) + 42$

$Y = 2X - 2 + 42$

$Y = 2X + 40$

2 Verify the relationship by substituting the third point.	Consider the point (7, 54): $Y = 2 \times 7 + 40$ $= 14 + 40$ $= 54$ This confirms the relationship between the variables.
3 Interpret the result.	The relationship between the two variables is $Y = 2X + 40$.

(b) Determine the expected value of X and hence the expected value of Y.

1 Calculate $E(X)$.	$E(X) = 1 \times 0.5 + 3 \times 0.3 + 7 \times 0.2$ $= 0.5 + 0.9 + 1.4$ $= 2.8$
2 Interpret the value.	The expected value of X is 2.8.
3 Assume the relationship is linear between the points and express in the relationship in the form: $Y = aX + b$	$y - y_1 = m(x - x_1)$ $Y - 42 = \frac{46 - 42}{3 - 1}(Y - 1)$ $Y - 42 = \frac{4}{2}(X - 1)$ $Y = 2(X - 1) + 42$ $Y = 2X - 2 + 42$ $Y = 2X + 40$
4 Use $E(aX + b) = aE(X) + b$.	$E(Y) = E(2X + 40)$ $= 2E(X) + 40$ $= 2 \times 2.8 + 40$ $= 45.6$
5 Interpret the result.	The expected value of Y is 45.6.

(c) Determine the variance of X and hence the variance of Y.

1 Calculate $E(X^2)$.	$E(X^2) = 1^2 \times 0.5 + 9^2 \times 0.3 + 7^2 \times 0.2$ $= 1 \times 0.5 + 9 \times 0.3 + 49 \times 0.2$ $= 0.5 + 2.7 + 9.8$ $= 13$
2 Calculate $Var(X)$ from $E(X^2)$ and $E(X)$.	$Var(X) = E(X^2) - (E(X))^2$ $= 13 - 2.8^2$ $= 5.16$
3 Interpret the value.	The variance of X is 5.16.

4 Use $Var(aX+b)=a^2Var(X)$.	$Var(Y)=Var(2X+40)$ $=2^2\,Var(X)$ $=4\times 5.16$ $=20.64$
5 Interpret the result.	The variance of Y is 20.64.

(d) Determine the standard deviation of X and hence the standard deviation of Y. Calculate your results correct to 2 decimal places.

1 Take the square root of the variance.	$\sigma=\sqrt{Var(X)}$ $=\sqrt{5.16}$ $=2.271\ldots$
2 Interpret the value.	The standard deviation of X is 2.27.
3 Use $\sigma(aX+b)=\lvert a\rvert\sigma(X)$.	$\sigma(2X+40)=2\sigma(X)$ $\sigma(Y)=2\times 2.271\ldots$ $=4.54\,(2\text{ d.p.})$
4 Interpret the result.	The standard deviation of Y is 4.54.

EXERCISE 8.1 Discrete probability distributions review

Worked Example 1

1 Two fair dice are rolled. Let X be the difference between the two dice.

(a) Construct the probability distribution of X. Express the probabilities as fractions in simplest form.

x	0	1	2	3	4	5
$P(X=x)$						

(b) Determine the following probabilities.

(i) The difference is between 1 and 3 inclusive.

(ii) The difference is no more than 4.

2 A spinner with the numbers 1 to 5 is spun and a coin is flipped. The result of each trial is given a numerical value of:

- double the number if an odd number is spun and the coin toss lands on heads
- the number squared if an even number is spun and the coin toss lands on tails
- 0 otherwise.

(a) Construct a probability distribution table for X. Express the probabilities as decimals.

(b) Determine the following probabilities.

(i) The recorded value is a square number.

(ii) The recorded value is at least 10.

(iii) The recorded value is even.

Worked Example 2

3 Consider the following probability distribution.

x	7	9	10	15
$P(X = x)$	0.5	0.1	0.23	0.17

Determine the following values.

(a) $E(X)$ **(b)** $Var(X)$

(c) the standard deviation of X, correct to 2 decimal places

4 Giles is about to take two shots at goal. From past performance his chance of scoring a goal is $\frac{2}{5}$ each time he takes a shot.

(a) Construct a probability distribution table for X, the number of goals scored out of two attempts.

(b) Determine the following values as decimals, correct to 2 decimal places where necessary.

(i) $E(X)$ **(ii)** $Var(X)$ **(iii)** the standard deviation of X

5 Variable X has the probability distribution:

x	-4	-3	-2	-1
$P(X = x)$	0.2	0.1	0.4	0.3

(a) What is the standard deviation of X?

A $\sqrt{1.16}$ **B** -1.16 **C** $\sqrt{10.84}$ **D** 1.16

(b) Explain the common error made by a student who chose the second incorrect option.

6 Two spinners are spun. One spinner X has the following probability distribution.

x	1	2	3	4
$P(X = x)$	0.4	0.3	0.2	0.1

The other spinner Y has the following probability distribution.

y	1	2	3	4	5
$P(Y = y)$	0.1	0.2	0.4	0.2	0.1

Calculate $E(X) + E(Y)$ and $E(X + Y)$ separately to demonstrate that $E(X + Y) = E(X) + E(Y)$.

7 Consider the probability distributions. What is the transformation that has occurred?

x	0	1	2	3
$P(X=x)$	$\frac{7}{18}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{3}$

y	32	31.8	31.6	31.4
$P(Y=y)$	$\frac{7}{18}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{3}$

A $Y = 5X + 32$ **B** $Y = 0.2X - 32$ **C** $Y = -0.2X + 32$ **D** $Y = 32 + 0.2X$

8 Transform the probability distribution according to the given rules.

x	1	3	5	8	9
$P(X=x)$	0.2	0.5	0.1	0.05	0.15

(a) $Y = 0.5X + 40$ **(b)** $Y = 30X - 18$ **(c)** $Y = 100 - 0.1X$ **(d)** $Y = 2X - 5$

9 A standard 52-card deck is reduced to 18 by removing all number cards below 10 as well as the black tens. Two cards are drawn without replacement.

(a) Construct a tree diagram to determine the sample space in terms of the suits: H (hearts), D (diamonds), S (spades) and C (clubs). Write probabilities on the branches of the tree diagram. Your answer does not need to include the probabilities for the paired outcomes.

(b) Construct a probability distribution table for X, the number of clubs drawn.

(c) Determine the probability that both cards are red.

(d) Determine the probability of drawing at least one red card from the pack.

(e) Determine the probability that both cards are clubs, given that both are black.

10 In a game, two spinners numbered 1 to 5 are spun. If the numbers are the same, then X is the number displayed on both spinners. If one number is odd and the other is even, then X is the difference between the numbers. Otherwise, X is the sum of the numbers.

(a) Draw a table to show the probability distribution of X. Express the probabilities as fractions.

(b) Calculate the following probabilities.

(i) $P(X < 8)$

(ii) $P(X = 5 | X \text{ is odd})$

(iii) $P(X = 3 | \text{same number on both spinners})$

Worked Example

3

11 Consider the probability distribution.

x	1	2	5	8
$P(X=x)$	0.2	0.4	0.1	0.3

(a) Determine the expected value, variance and standard deviation, correct to 2 decimal places where necessary.

(b) Use your answers from part **(a)** to determine the expected value, variance and standard deviation of each of the following distributions.

(i)

y	10	20	50	80
$P(Y=y)$	0.2	0.4	0.1	0.3

(ii)

z	61	62	65	68
$P(Z=z)$	0.2	0.4	0.1	0.3

(iii)

w	2	7	22	37
$P(W=w)$	0.2	0.4	0.1	0.3

12 By first transforming X to simpler values, determine the expected value and variance of the following probability distribution.

x	17.01	17.02	17.03	17.04
$P(X=x)$	0.35	0.45	0.15	0.05

13 The probability distribution of X is given by:

$$P(X=x)=\begin{cases} k(9-2x) & \text{if } x \in \{0,1,2,3,4\} \\ 0 & \text{for all other values of } x \end{cases}$$

Determine the following values.

(a) k, as a fraction

(b) $E(X)$, as a decimal

(c) $Var(X)$, as a decimal

(d) σ, correct to 4 decimal places

14 A die is biased so that the probability of obtaining each number is given by the function:

$$P(X=x)=\begin{cases} k(15-2x) & \text{if } x \in \{1,2,3,4,5,6\} \\ 0 & \text{for all other values of } x \end{cases}$$

The die is rolled twice and Y is the difference between the two results.

(a) Draw a table to show the probability distribution of Y. Express the probabilities as fractions in simplest form.

(b) Calculate $E(Y)$, correct to 2 decimal places.

(c) Determine a single swap of values on the faces of the die such that the value of $E(Y)$ will increase. Explain the reason for your choice and demonstrate, by calculations, that the swap was successful.

15 A tennis tournament is to be played with a match won when one player is two sets ahead, or five sets have been played. By how much will the expected number of sets increase if the players are equally matched compared to a match where the win ratio for the pair is 7:3?

8.2 Bernoulli trials and the binomial distribution

Bernoulli trials

When a coin is flipped, there are two possible outcomes: heads and tails. When a standard die is rolled, there are six possible outcomes: 1, 2, 3, 4, 5 and 6. If you are interested in whether or not you roll a particular number, such as a 6, there are only two outcomes possible to this scenario: 'getting a 6' and 'not getting a 6'. The same argument can be applied to many practical situations: an archer hits the target or misses the target; the goal kicker in rugby scores a goal or misses; a medical test indicates the presence or absence of a disease. In all of these cases, you need to consider *success* and *failure*. Since there are only two possible mutually exclusive outcomes when viewed in this sense, if the probability of success is p, then the probability of failure will be $1 - p$.

Associated with Bernoulli trials are Bernoulli random variables. If X represents a Bernoulli random variable, and a probability of p is associated with success, where $0 < p < 1$, then:

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

The probability distribution table for this situation is shown below.

x	0	1
$P(X = x)$	$1 - p$	p

The expected value is given by:

$$\begin{aligned} E(X) &= 0 \times (1 - p) + 1 \times p \\ &= p \end{aligned}$$

The variance is given by:

$$\begin{aligned} Var(X) &= E\left(X^2\right) - \left(E(X)\right)^2 \\ &= 0^2 \times (1 - p) + 1^2 p - p^2 \\ &= p - p^2 \\ &= p(1 - p) \end{aligned}$$

The variance is maximised when the probability of success and failure is equal; that is, $p = 0.5$. Conversely, if success (or failure) is nearly inevitable, then nearly all the trials will have that result, so the variance will be low.

For a Bernoulli random variable X:

$E(X) = p$

$Var(X) = p(1 - p)$

The binomial distribution

Bernoulli trials can guide you to understand the most important of the discrete probability distributions: the binomial distribution. Suppose you are conducting an experiment, consisting of n trials, under the following conditions:

- n is determined before the experiment begins.
- All n trials are identical Bernoulli trials, with the probability of success represented by p, and the probability of failure represented by $q = 1 - p$.
- All the trials are independent; that is, the outcome from any one trial has no effect on the outcome of any other trial.

In these cases there is a *binomial random variable*, which in turn has a *binomial probability distribution*.

There is a shorthand notation to indicate a binomial distribution.

$X \sim B(n, p)$ is read: 'X is distributed as a binomial variable with parameters n and p'.

Consider the experiment of drawing three cards, one at a time with replacement, from a normal pack of cards. If you are interested in the number of hearts selected, X, then X is a binomial variable. Consider drawing a heart to be a success.

In terms of notation, $n = 3$ (the number of trials) and $p = \frac{1}{4}$ (the probability of success on any particular trial, as $P(\text{heart}) = \frac{1}{4}$).

So, $X \sim B\left(3, \frac{1}{4}\right)$.

$X \sim B(n, p)$ indicates a random variable X that has a binomial distribution with n identical trials and a probability of success of p.

n and p are called the parameters of the distribution.

WARNING

If you were drawing the cards *without replacement*, then X would not be a binomial variable because the probability of success p would change with each draw.

Now determine the probability of obtaining exactly two hearts in a selection of three cards. There are three (that is, ${}^3C_2$) ways to obtain exactly two hearts: HHN, HNH, NHH, where H represents 'getting a heart' and N represents 'not getting a heart'. In each case, the probability is $\frac{1}{4} \times \frac{1}{4} \times \frac{3}{4}$, although the order of the fractions varies according to the order in which the cards are selected.

$$P(\text{exactly two hearts}) = 3 \times \left(\frac{1}{4}\right)^2 \times \frac{3}{4}$$
$$= \frac{9}{64}$$

It is easy to work simple examples like this, but when finding the probability of drawing 13 hearts from 30 draws, it is not as easy to list all the possible outcomes. This scenario represents a number of combinations. You can use ${}^{30}C_{13}$ to determine the total number of possible outcomes. With binomial distribution, use the alternative notation for combinations; that is, $\binom{n}{r}$ or, in this case, $\binom{30}{13}$.

Consider the probability of obtaining any one of these outcomes, such as 13H followed by 17N. The probability of drawing exactly 13 hearts remains the same regardless of the order in which those hearts are drawn (since every draw is independent of the others): $\left(\frac{1}{4}\right)^{13} \times \left(\frac{3}{4}\right)^{17}$. Therefore, the probability of drawing exactly 13 hearts from 30 cards, drawn in a particular order multiplied by the number of possible ways in which the cards could be selected can be shown:

$$P(\text{exactly 13 hearts from 30 draws}) = \binom{30}{13} \times \left(\frac{1}{4}\right)^{13} \times \left(\frac{3}{4}\right)^{17}.$$

A general expression for the probability that X takes a particular value, x:

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

Recalling that the probability of failure is sometimes written as q, where $p + q = 1$, this can also be written as:

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Using technology to calculate, the expression is equal to 0.013 441 444 88, which can be rounded to 0.0134.

4 Calculate a binomial probability

A variable x follows the distribution $X \sim B(10, 0.6)$.

Calculate $P(X = 5)$ for this distribution, expressing your answer correct to 4 decimal places.

THINKING	WORKING
1 Identify n (the number of trials), x (the number of successes), p (the probability of success), and $1 - p$ (the probability of failure).	$n = 10, x = 5, p = 0.6, 1 - p = 0.4$
2 Substitute into the binomial probability formula and calculate.	$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ $P(X = 5) = \binom{10}{5} (0.6)^5 (0.4)^5$ $= 0.2007 \text{ (4 d.p.)}$
3 Interpret the result.	In 10 trials where the probability of success is 0.6, the probability of achieving 5 successes is 0.2007.

The binomial expansion

You can think of a binomial distribution in terms of the expansion of $(q + p)^n$, where $q + p = 1$, and so $(q + p)^n = 1$.

$$(q+p)^n = q^n + nq^{n-1}p + \binom{n}{2} q^{n-2}p^2 + \ldots + p^n$$

The terms, in order, give the probabilities of 0, 1, 2, …, n successes.

Technology worked example
Binomial probability

Technology worked example
Cumulative density functions

Cumulative density function

Probabilities of individual values are easily calculated from a formula. If you need to determine the probability of a range of values you can either add the individual values using the formula repeatedly, or use the cumulative density function on a graphics calculator.

5 Calculate the probability of achieving a given interval

You draw 30 cards from a standard pack, with replacement. Determine the probability that the number of hearts drawn is between 10 and 13 inclusive, correct to 3 decimal places.

THINKING	WORKING
1 Identify n, x, p, and $1-p$.	$n = 30,\ 10 \le x \le 13,\ p = 0.25,\ 1-p = 0.75$
2 Use the formula repeatedly: $P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$	$P(10 \le X \le 13) = P(X=10) + P(X=11) + P(X=12) + P(X=13)$ $= \binom{30}{10}(0.25)^{10}(0.75)^{20} + \binom{30}{11}(0.25)^{11}(0.75)^{19}$ $+ \binom{30}{12}(0.25)^{12}(0.75)^{18} + \binom{30}{13}(0.25)^{13}(0.75)^{17}$ $= 0.188\ (3\text{ d.p.})$
3 Alternatively, use the cumulative density function on your calculator.	$P(10 \le X \le 13) = P(X \le 13) - P(X \le 9)$ $= 0.99182 - 0.80341$ $= 0.188\ (3\text{ d.p.})$
4 Interpret, the answer.	From 30 trials, the probability that between 10 and 13 hearts are drawn is 0.188.

Complementary events

If an event complementary to the event in question has fewer terms, use the probability of the complementary events to simplify the calculation.

Complementary events:

$P(X \ge a) = 1 - P(X < a)$

$P(X \ge 1) = 1 - P(X = 0)$

6 Use complementary events

A particular medical test correctly identifies the presence of an illness 98% of the time. If 10 people are tested, calculate the following probabilities, correct to 3 decimal places.

(a) Determine the probability that no one is incorrectly diagnosed.

THINKING	WORKING
1 Identify the variables.	Let success be a correct diagnosis, $p = 0.98$ 10 people gives $n = 10$ No incorrect diagnosis, gives 10 correct diagnoses or $x = 10$
2 Substitute into the binomial probability formula, $P(X=10) = p^2$, and calculate.	$P(X=10) = \binom{10}{10}(0.98)^{10}(0.02)^0$ $= (0.98)^{10}$ $= 0.817\ (3\text{ d.p.})$
3 Interpret the answer.	The probability that no one is given an incorrect diagnosis is 0.817.

(b) Determine the probability that at least one person is incorrectly diagnosed.

1 Use the complementary event to calculate the complement.	$P(X < 10) = 1 - P(X = 10)$ $= 1 - 0.817$ $= 0.183$
2 Interpret the result.	The probability that at least one person was given an incorrect diagnosis is 0.183.

7 Choose appropriate methods

Accurate Andy is a darts player of some renown. His favourite shot is the treble-twenty. From past experience, he will be successful with this shot 80% of the time. In each of the following situations, Andy has five shots. In each case, calculate your answer correct to 4 decimal places, where necessary.

(a) Determine the probability that Andy makes exactly two treble-twenties.

THINKING	WORKING
1 Identify n, x, p and q.	$n = 5, x = 2, p = 0.8, q = 0.2$
2 Substitute into the binomial probability formula and calculate.	$P(X = x) = \binom{n}{x} p^x q^{n-x}$ $P(X = 2) = \binom{5}{2} (0.8)^2 (0.2)^3$ $= 0.0512\,(4\text{ d.p.})$
3 Interpret the answer.	In five shots, the probability that Andy makes exactly two treble-twenties is 0.0512.

(b) Determine the probability that Andy makes more than three treble-twenties.

1 Write a mathematical statement to describe the required situation.	$P(X > 3) = P(X = 4) + P(X = 5)$ $= {}^5C_4 (0.8)^4 (0.2)^1 + (0.8)^5$ $= 0.7373\,(4\text{ d.p.})$
2 Interpret the answer.	In five shots, the probability that Andy makes more than three treble-twenties is 0.7373.

(c) Determine the probability that Andy makes treble-twenties on his first and fifth attempts only.

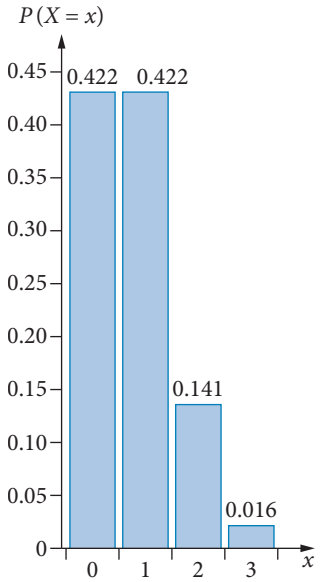
1 Use the multiplication rule for the specific order of success and failure.	$P(\text{SFFFS}) = 0.8 \times 0.2 \times 0.2 \times 0.2 \times 0.8$ $= 0.0051\,(4\text{ d.p.})$
2 Interpret the answer.	In five shots, the probability that Andy makes a treble-twenty on the first and fifth shots only is 0.0051.

The graph of the binomial distribution

Consider the experiment where you draw three cards, with replacement, from a standard pack of 52 playing cards. Hearts are considered to be a success, so the distribution is given as $X \sim B(3, 0.25)$. The following table shows the distribution of the random variable.

x	0	1	2	3
$P(X = x)$	0.422	0.422	0.141	0.016

Here, the sum of the probabilities is 1.001; this is due to the rounding of the probabilities and can therefore be regarded as 1. A graph of this distribution is shown below.

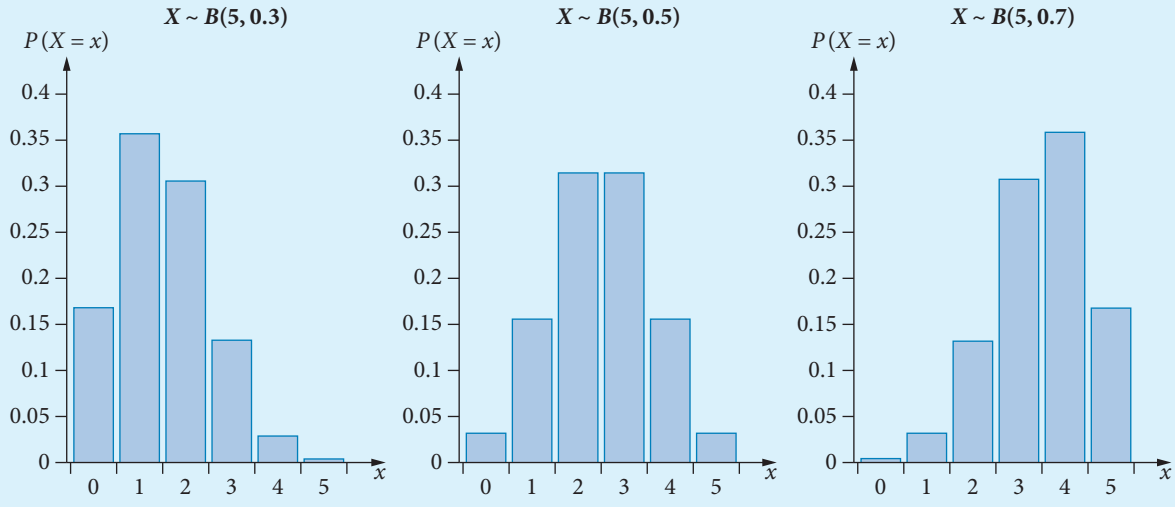


Technology worked example
Graph the binomial distribution

Making connections
The shape and position of the binomial distribution
Move the sliders to view the effect the values n and p have on the graph of the distribution.

For graphs of the binomial distribution, where $X \sim B(n, p)$:

- If $p < 0.5$, the graph is skewed to the right or positively skewed.
- If $p = 0.5$, the graph is approximately symmetrical.
- If $p > 0.5$, the graph is skewed to the left or negatively skewed.



The median of the distribution

The median value gives an indication of how the data is skewed. For a discrete probability distribution, the median occurs where the cumulative probability distribution first exceeds 0.5.

8 Locate the median

Use technology to determine the median of the distribution $X \sim B(7, 0.8)$.

THINKING	WORKING
1 Since $p > 0.5$, calculate cumulative frequencies for values to the right of $\frac{n}{2}$. Write the calculator values for before and after $p = 0.5$ is reached.	$X \sim B(7, 0.8)$ $P(X \leq 5) = 0.42\ldots$ $P(X \leq 6) = 0.79\ldots$
2 Interpret the answer.	$x = 6$ is when the cumulative frequency passes 0.5; the median is 6.

EXERCISE 8.2 Bernoulli trials and the binomial distribution

Worked Example

4

1 Determine the stated probability for the following binomial distributions. Give your answers correct to 4 decimal places.

(a) $P(X = 4)$ if $X \sim B(12, 0.7)$

(b) $P(X = 6)$ if $X \sim B(20, 0.45)$

(c) $P(X = 2)$ if $X \sim B(8, 0.3)$

(d) $P(X = 30)$ if $X \sim B(50, 0.6)$

2 A spinner is divided into eight equal sections, of which three are shaded. Determine the probability, correct to 4 decimal places, of the spinner landing on a shaded section exactly five times out of ten spins.

6

3 A standard six-sided die is rolled 50 times. Determine the probability, correct to 5 decimal places, of the following outcomes.

(a) Exactly 30 even numbers are rolled.

(b) At least one six is rolled.

4 Peter, a keen gardener, knows from past experience that only 60% of his tulip bulbs will flower. He plants 20 bulbs. Determine the probability of the following outcomes. Give your answers correct to 5 decimal places.

(a) Exactly 15 bulbs flower.

(b) More than 15 bulbs flower.

5 Determine whether or not each of the following situations is suitable for binomial modelling. For those that are not, explain why.

(a) You are playing a game where you are required to roll a six on the die before you can start. The number of rolls required for the eight players is recorded.

(b) A die is biased such that $P(4) = \frac{1}{3}$. The die is rolled 40 times and the number of 4s is recorded.

Worked Example 8

6 Use technology to determine the median of each of the following distributions.

(a) $X \sim B(6, 0.2)$ **(b)** $X \sim B(9, 0.15)$ **(c)** $X \sim B(12, 0.4)$

(d) $X \sim B(10, 0.65)$ **(e)** $X \sim B(15, 0.82)$

7 If 100 cards are drawn with replacement from a normal pack of cards, determine the probability of the following events. Give your answers correct to 4 decimal places.

(a) Exactly 25 hearts are drawn. **(b)** Exactly 40 cards are clubs.

(c) Exactly 55 red cards are drawn. **(d)** Exactly 60 cards are black.

8 The probability that Robin, an archer of some skill, will make a bull's-eye with any particular arrow is $\frac{4}{5}$. He shoots a flight of ten arrows.

(a) Which of the following expressions represents the probability that he misses the bull's-eye with exactly two of his arrows?

A $\binom{10}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^8$

B $\binom{10}{8}\left(\frac{4}{5}\right)^8\left(\frac{1}{5}\right)^2$

C $\binom{8}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^8$

D $\binom{8}{2}\left(\frac{4}{5}\right)^8\left(\frac{1}{5}\right)^2$

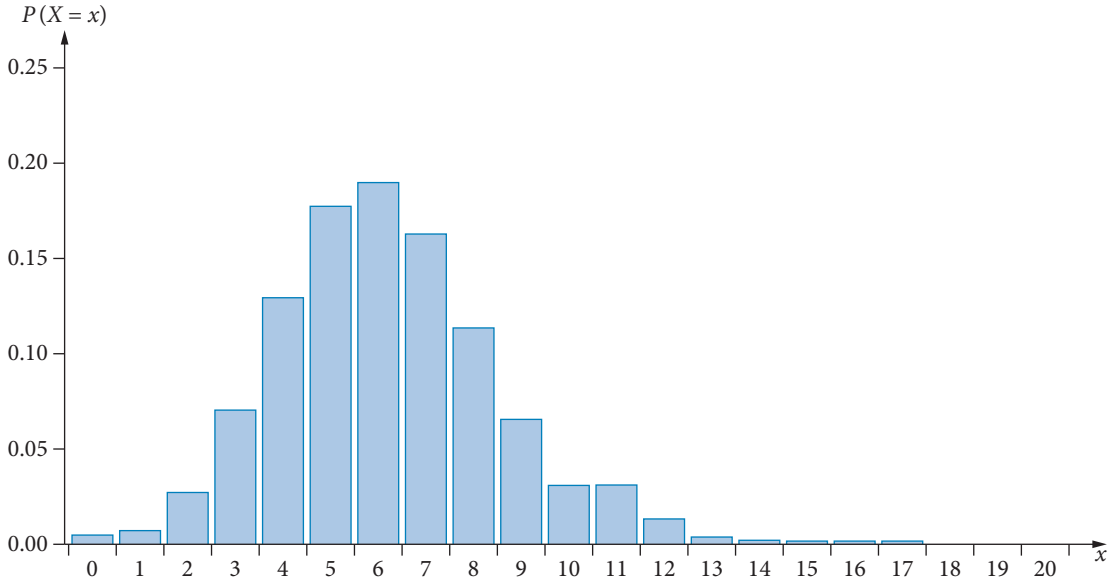
(b) Explain the common error made by a student who chose the first incorrect option.

9 In a northern Queensland community, 62% of all births are girls. Calculate the probability, correct to 4 decimal places, that of eight births exactly half are girls.

10 In the town of Sinistraville, 35% of the population is left-handed. Determine the probability, correct to 4 decimal places, that exactly 35 people out of a random sample of 100 will be left-handed.

11 A coin is biased in such a way that $P(\text{Heads}) = 3 \times P(\text{Tails})$. Determine the probability, correct to 3 decimal places, that exactly 6 tosses out of 10 land heads up.

12 For the graph shown, choose the best representation for its binomial distribution.



A $X \sim B(12, 0.5)$ **B** $X \sim B(20, 0.3)$ **C** $X \sim B(12, 0.3)$ **D** $X \sim B(20, 0.7)$

13 Cards are dealt from a normal pack of cards without replacement. Let X be the number of hearts in your hand and assume five cards are dealt. Explain why this experiment could not be described as $X \sim B\left(5, \frac{1}{4}\right)$.

14 Boxes of matches are supposed to contain 47 matches. Production records indicate that 80% of boxes contain 47 matches. A batch of 20 boxes is sampled. If more than four boxes do not contain exactly 47 matches, production is stopped.

(a) Determine the probability, correct to 4 decimal places, that of the 20 boxes selected, the number of boxes that do not contain 47 matches is:

(i) 0 **(ii)** 1 **(iii)** 2

(iv) 3 **(v)** 4

(b) Use your answers from part **(a)** to determine the probability that the number of boxes that do not have 47 is no more than 4.

(c) Use a technology-based cumulative probability distribution function to verify the answer from part **(b)**.

(d) Determine the probability that production is stopped.

Worked Example

5

15 A coin is tossed 20 times. Determine the probability of each of the following outcomes, correct to 3 decimal places.

(a) between 8 and 12 heads, inclusive **(b)** between 10 and 15 heads, inclusive

(c) between 12 and 19 heads, inclusive

16 On the way to work, Philomena must pass through six sets of traffic lights. The lights are independent of each other and the probability that Philomena must stop at any particular set is 0.7. Determine the probabilities that Philomena has to stop as described below. Give your answers correct to 4 decimal places.

(a) at exactly five sets of lights

(b) at fewer than two sets of lights

(c) at more than five sets of lights

(d) at the first three sets of lights

(e) at the second and sixth sets of lights only

(f) at four sets of lights, including the first two

17 The local baker knows that, on average, 8% of the loaves of bread baked each day become slightly burnt and she cannot sell them at the full price. Assume the likelihood of a particular loaf being slightly burnt is independent of any other loaf being slightly burnt. To break even, the baker has to sell at least 40% of the loaves at full price. Calculate the probability, correct to 4 decimal places, that she would make a profit when 12 loaves are baked.

18 A recording company receives a large number of new songs from various artists. On average, only 4% of the new songs become popular hits. One of the producers decides to select a random sample of 35 new songs.

(a) Determine the most likely number of songs from this sample that could become popular hits. Explain your answer using appropriate calculations.

(b) The producer decides to select 70 songs at random from the new songs received. If the probability of a new song becoming a popular hit has not changed, what would be the most likely number of popular hits from this sample?

19 DVDs from a local second-hand store have a 0.15 chance of being faulty. A boxed set of five DVDs is purchased. Assume independence exists between the DVDs. Determine the probability, correct to 3 decimal places, that the boxed set contains fewer than four faulty DVDs if you know it contains at least one faulty DVD.

20 An archer hits a target with 75% of his attempts.

(a) How many attempts must be made for the probability of missing the target every time to drop below 5%?

(b) Determine the number of attempts to be made if the probability of missing at most one target is less than 10%.

8.3 Statistics of binomial probability distributions

Mean and variance of a Bernoulli random variable

For an individual Bernoulli trial, the mean number of successes is simply the probability of success, p, and the variance is $p(1-p)$.

For a Bernoulli distribution:
mean $= p$
variance $= p(1-p)$

9 Calculate Bernoulli statistics

A die is rolled and you record whether the outcome is a multiple of 3 or not. Calculate the following Bernoulli statistics.

(a) the probability of success

THINKING	WORKING
1 Determine the number of successful outcomes.	Let $p = P(\text{multiple of 3})$. Sample space $= \{1, 2, 3, 4, 5, 6\}$ Success $= \{3, 6\}$
2 Calculate the probability of success.	$p = \frac{2}{6}$ $= \frac{1}{3}$
3 Interpret the result.	The probability of success, rolling a multiple of 3, is $\frac{1}{3}$. For a large number of Bernoulli trials, success would be expected in approximately $\frac{1}{3}$ of trials.

(b) the variance

1 Recall the formula for variance, substitute the values and calculate.	$\text{Variance} = p(1-p)$ $= \frac{1}{3} \times \left(1 - \frac{1}{3}\right)$ $= \frac{1}{3} \times \frac{2}{3}$ $= \frac{2}{9}$
2 Interpret the result.	The variance in the expected success rate is $\frac{2}{9}$.

Mean and variance of a binomial distribution

Consider a random variable $X \sim B(5, 0.4)$. The probability distribution for X is shown in the following table.

x	0	1	2	3	4	5
$P(X = x)$	0.0776	0.2592	0.3456	0.2304	0.0768	0.01024

You can determine the expected value of X, $E(X)$.

$$\begin{aligned} E(X) &= \sum_{i=0}^{5} x_i p_i \\ &= 0 \times 0.077\,76 + 1 \times 0.2592 + 2 \times 0.3456 + 3 \times 0.2304 + 4 \times 0.0768 + 5 \times 0.010\,24 \\ &= 0 + 0.2592 + 0.6912 + 0.6912 + 0.3072 + 0.0512 \\ &= 2 \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) &= \sum_{i=0}^{5} x_i^2 p_i \\ &= 0 \times 0.077\,76 + 1 \times 0.2592 + 4 \times 0.3456 + 9 \times 0.2304 + 16 \times 0.0768 + 25 \times 0.010\,24 \\ &= 0 + 0.2592 + 1.3824 + 2.0736 + 1.2288 + 0.256 \\ &= 5.2 \end{aligned}$$

Therefore:

$$\begin{aligned} Var(X) &= E\left(X^2\right) - \left(E(X)\right)^2 \\ &= 5.2 - 2^2 \\ &= 1.2 \end{aligned}$$

When working with the binomial distribution, there are short cuts you can take to find both the expected value and the variance (and, therefore, the standard deviation).

You can check these results for $X \sim B(5, 0.4)$.

Using the rules:

$$\begin{aligned} E(X) &= np \\ &= 5 \times 0.4 \\ &= 2 \end{aligned}$$

$$\begin{aligned} Var(X) &= np(1 - p) \\ &= 5 \times 0.4 \times 0.6 \\ &= 1.2 \end{aligned}$$

The results match those obtained from the probability distribution calculation method.

For $X \sim B(n, p)$:

$E(X) = \mu = np$

$Var(X) = np(1 - p) = npq$

$\sigma(X) = \sqrt{np(1 - p)}$

10 Calculate binomial statistics

Calculate the following statistics for $X \sim B(15, 0.3)$. If necessary, give your answers correct to 2 decimal places.

(a) $E(X)$

THINKING	WORKING
1 Identify the values of the parameters.	$X \sim B(15, 0.3)$, so $n = 15$ and $p = 0.3$
2 Recall the expected value formula, substitute the known values and calculate the value of $E(X)$.	$E(X) = np$ $= 15 \times 0.3$ $= 4.5$
3 Interpret the result.	The expected value of the binomial distribution is 4.5. That is, in 15 trials, the expected number of successes experienced is 4.5.

(b) $Var(X)$

1 Recall the formula to calculate variance. Substitute the value of $E(X)$ into the formula and calculate.	$Var(X) = np(1 - p)$ $= 4.5 \times (1 - 0.3)$ $= 3.15$
2 Interpret the answer.	The variance of the distribution is 3.15.

(c) $\sigma(X)$

1 Recall the formula for standard deviation.	$\sigma(X) = \sqrt{Var(X)}$
2 Substitute your value for $Var(X)$ into the formula and calculate.	$\sigma(X) = \sqrt{np(1 - p)}$ $= \sqrt{3.15}$ $= 1.77\ (2\text{ d.p.})$
3 Interpret the answer.	The standard deviation of the distribution is 1.77.

Solving for a parameter

When the number of trials and the expected value are known, you can solve an equation to calculate p, the probability of success.

11 Solve for a binomial parameter

If $X \sim B(20, p)$ and $E(X) = 5$, calculate the value of p.

THINKING	WORKING
1 Identify the given information.	$E(X) = 5, n = 20$
2 Recall the expected value formula.	$E(X) = np$
3 Substitute the known values and solve for p.	$20 \times p = 5$ $p = \frac{5}{20}$ $= 0.25$
4 Interpret the answer.	In 20 trials of a binomial distribution where the expected number of successes is 5, the probability of success is 0.25.

When the mean (expected value) and variance are known, you can solve simultaneous equations to calculate p, the probability of success, and n, the number of trials.

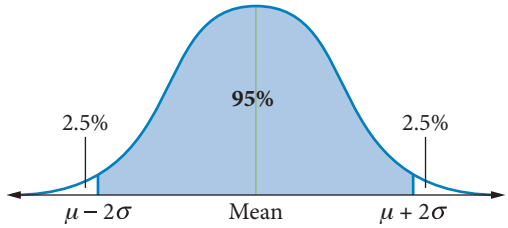
12 Solve simultaneously to calculate the binomial parameters

If $X \sim B(n, p)$ with $\mu = 9$ and $\sigma^2 = 6.3$, determine the values of n and p.

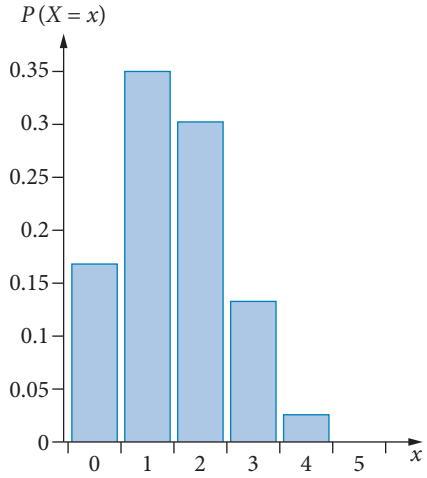
THINKING	WORKING
1 Identify the given information.	$\mu = np, \mu = 9$ $\sigma^2 = np(1-p), \sigma^2 = 6.3$
2 Substitute the known value of np into σ^2 and solve for p.	$np(1-p) = 6.3$ $9(1-p) = 6.3$ $(1-p) = 0.7$ $p = 0.3$
3 Substitute the known value of p into the formula for μ and solve for n.	$np = 9$ $n \times 0.3 = 9$ $n = \frac{9}{0.3}$ $= 30$
4 Determine the solution.	$n = 30, p = 0.3$, so $X \sim B(30, 0.3)$

Distribution about the mean

In normal distributions, roughly 95% of values lie within 2 standard deviations of the mean, μ.



Binomial distributions fit this model well when the number of trials is large and the value of p is close to 0.5. To make the model fit as well as possible, remember that the probabilities for any discrete probability distribution are represented by rectangles that cover half to the right and half to the left of the consecutive values.



13 Calculate binomial probabilities

A tree farmer is planting an experimental plot of 100 gum trees. The farmer knows that the probability of a tree surviving to maturity is 0.45. Let X be the random variable specifying the number of trees surviving to maturity.

(a) Determine the number of trees expected to survive to maturity.

THINKING	WORKING
1 Identify n and p.	$n = 100$, $p = 0.45$
2 Apply the rule $E(X) = np$.	$E(X) = np$ $= 100 \times 0.45$ $= 45$
3 Interpret the answer.	The farmer would expect 45 trees to survive to maturity.

(b) Estimate a range of values about the expected number that would contain 95% of the possible outcomes.

1 Recall the formula for variance, substitute the values and calculate.	$Var(X) = np(1-p)$ $= 45 \times 0.55$ $= 24.75$
2 Hence, calculate the value of $\sigma(X)$.	$\sigma(X) = \sqrt{Var(X)}$ $= \sqrt{24.75}$ $= 4.97$
3 Determine the interval for $\mu + 2\sigma$ and $\mu - 2\sigma$.	$\mu + 2\sigma = 45 + 2 \times 4.97$ $= 54.94$ $\mu - 2\sigma = 45 - 2 \times 4.97$ $= 35.06$
4 Interpret the answer.	The farmer would expect that, when planting 100 trees, in 95% of cases, the number of trees surviving to maturity would be between 35 and 54 inclusive.

Also, note that since $p = 0.45$ rather than 0.5, the model will not fit this binomial distribution perfectly.

EXERCISE 8.3 Statistics of binomial probability distributions

Worked Example 9

1 Determine the exact values of the mean and the variance for each of the following Bernoulli distributions.

(a) A fair die is rolled and you record whether the outcome is a 1 or not.

(b) Two coins are tossed and you record whether the outcome is two heads or not.

(c) A goal shooter is successful 86% of the time.

(d) Two dice are rolled and you record whether or not the sum of the dice is at least 11.

Worked Example 10

2 Determine the values of $E(X)$, $Var(X)$ and $\sigma(X)$ for each of the following binomial distributions. Give your answers correct to 2 decimal places where necessary.

(a) $X \sim B(20, 0.7)$ **(b)** $X \sim B(100, 0.55)$ **(c)** $X \sim B\left(50, \frac{2}{3}\right)$

(d) $X \sim B(10, 0.5)$ **(e)** $X \sim B(25, 0.8)$ **(f)** $X \sim B\left(40, \frac{2}{5}\right)$

11

3 For each of the following, calculate the exact value of the unknown parameter, n or p.

(a) $X \sim B(15, p)$, $E(X) = 5$

(b) $X \sim B(30, p)$, $E(X) = 3$

(c) $X \sim B(n, 0.2)$, $E(X) = 7$

(d) $X \sim B\left(n, \frac{2}{5}\right)$, $E(X) = 10$

12

4 For each of the following, calculate the values of n and p.

(a) $X \sim B(n, p)$, $\mu = 4.8$ and $\sigma^2 = 2.88$

(b) $X \sim B(n, p)$, $\mu = 2$ and $\sigma^2 = 1.8$

(c) $X \sim B(n, p)$, $\mu = 4.5$ and $\sigma^2 = 3.825$

(d) $X \sim B(n, p)$, $\mu = 11$ and $\sigma^2 = 8.58$

(e) $X \sim B(n, p)$, $\mu = 65$ and $\sigma^2 = 22.75$

(f) $X \sim B(n, p)$, $\mu = 41.25$ and $\sigma^2 = 29.906\,25$

5 Consider the binomial distribution $X \sim B(30, 0.4)$.

(a) Which of the following pairs of values is correct?

A $E(X) = 12$, $Var(X) = \sqrt{7.2}$

B $E(X) = 12$, $\sigma^2 = 7.2$

C $E(X) = 18$, $\sigma^2 = 7.2$

D $E(X) = 18$, $\sigma = \sqrt{7.2}$

(b) Explain the common error made by a student who chose the first incorrect option.

6 Given $X \sim B(n, p)$, $\mu = 10$ and $\sigma^2 = 8$, which of the following pairs of values is correct?

A $n = 80$, $p = 0.8$

B $n = 50$, $p = 0.8$

C $n = 80$, $p = 0.2$

D $n = 50$, $p = 0.2$

7 A spinner is divided into four equal sections, one of which is coloured blue. If the spinner is spun eight times, determine the probability, correct to 3 decimal places, of obtaining less than the expected number of blue outcomes.

8 You are told that for the variable $X \sim B(40, p)$, $\sigma^2 = 9.6$. Determine all possible values of p and evaluate the reasonableness of your solution.

13

9 The birthing unit staff at the local hospital know from past records that 53% of all births are female.

(a) Determine the likely number of females in the next 60 births, rounded to the nearest whole number.

(b) Estimate the whole number range within which the number of females would be found 95% of the time.

10 A normal die is rolled 100 times. Estimate the range within which you would expect the number of 3s obtained to lie, in 95% of cases.

11 Astrid is planning a visit to her grandmother's house. She calculates that she must pass through 10 sets of traffic lights. Assuming that all the sets of lights are independent and that Astrid's chance of being stopped at any particular set of lights is $\frac{3}{5}$, estimate the range for the number of times within which Astrid would expect to be stopped 95% of the time.

12 Given $X \sim B(n, p)$, $E(X) = 48$ and $\sigma = 2\sqrt{3}$, determine the exact values of n, p, $\mu - 2\sigma$ and $\mu + 2\sigma$.

13 Calculate $P(X = 3)$, correct to 2 decimal places, given that $E(X) = 3$ and $Var(X) = 0.75$.

14 **(a)** Consider $X \sim B(8, 0.25)$.

(i) Construct a table showing the probability distribution. Where necessary, give the probabilities to 4 decimal places.

(ii) Use the rule $\mu = np$ to calculate $E(X)$.

(iii) Use the rule $\sigma = \sqrt{(Var(X))}$ to calculate $\sigma(X)$, correct to 2 decimal places.

(iv) Use your answers from parts **(ii)** and **(iii)** to estimate the range within which 95% of outcomes will lie.

(v) According to the values recorded in the probability distribution table, what is the actual percentage of values that fall within the 95% interval from part **(iv)**?

(b) Now consider $Y \sim B(8, 0.5)$.

(i) Estimate a similar 95% interval.

(ii) Construct a probability distribution table.

(iii) According to the values recorded in the probability distribution table, what is the actual percentage of values that fall within the 95% interval from part **(i)**?

(c) Now consider $Z \sim B(8, 0.65)$.

(i) Estimate a similar 95% interval.

(ii) Construct a probability distribution table.

(iii) According to the values recorded in the probability distribution table, what is the actual percentage of values that fall within the 95% interval from part **(i)**?

(d) Comment on the strengths and weaknesses of the model used to estimate the 95% interval.

15 Determine the maximum standard deviation for a Bernoulli distribution. Support your explanation with a graph.

16 A researcher has calculated the mean and the variance for a sample of a given random variable that has a binomial distribution.

(a) If the mean of the data set is 42 and the variance is 6, determine the number of trials (n) and the probability of success (p).

(b) A new set of data was collected and the researcher noticed that the mean and variance of the new set of data were double the corresponding values of the first set of data. Compare the two sets of data using appropriate calculations involving the values of n and p.

(c) The researcher realised that she had omitted the results for 10 trials from the first set of data. When she calculated the mean, after adding the 10 additional results, she noticed that the mean was unchanged at $\mu = 42$. What variance should the researcher expect for this set of data? Give your answer correct to 2 decimal places.

(d) Another researcher has conducted the same experiment and recorded the data for 56 trials. He calculated the probability of success for this set of data to be $p = \frac{6}{7}$. What are the values of the mean and variance the researcher would expect to calculate for this set of data? Give your answers correct to 2 decimal places.

(e) Write a statement to describe the effect on the mean and variance as the number of trials increases.

8

Summary

Discrete random variables

For a discrete random variable:

- The number of outcomes is countable.
- $0 \leq P(X = x) \leq 1$ for all values of x.
- $\sum P(X = x) = 1$.

Statistics of discrete random variables

Expected value: $E(X) = \mu = \sum x_i p_i$

Variance: $Var(X) = E\left(X^2\right) - \left(E(X)\right)^2$

Standard deviation: $\sigma(X) = \sqrt{Var(X)}$

Statistics of transformed data

$E(aX + b) = aE(X) + b$

$Var(aX + b) = a^2 Var(X)$

$\sigma(aX + b) = |a|\sigma(X)$

Estimated probability of an interval

For normal distributions, about 95% of results lie within two standard deviations of the mean. This is also true of binomial distributions where $p = 0.5$. This interval can be represented as $\mu \pm 2\sigma$.

Bernoulli trials

In a Bernoulli trial, the experiment has exactly two outcomes, usually described as success and failure.

Statistics of Bernoulli distributions

The expected probability of success in a Bernoulli trial is the theoretical probability of success, p.

The variance in the probability of success is $p(1 - p)$, where $1 - p$ is the probability of failure.

Binomial distribution

A binomial variable x is the number of successes from a series of independent Bernoulli trials.

$X \sim B(n, p)$ indicates a variable X with a binomial distribution, where the parameters are n, the number of trials, and p, the probability of success per trial.

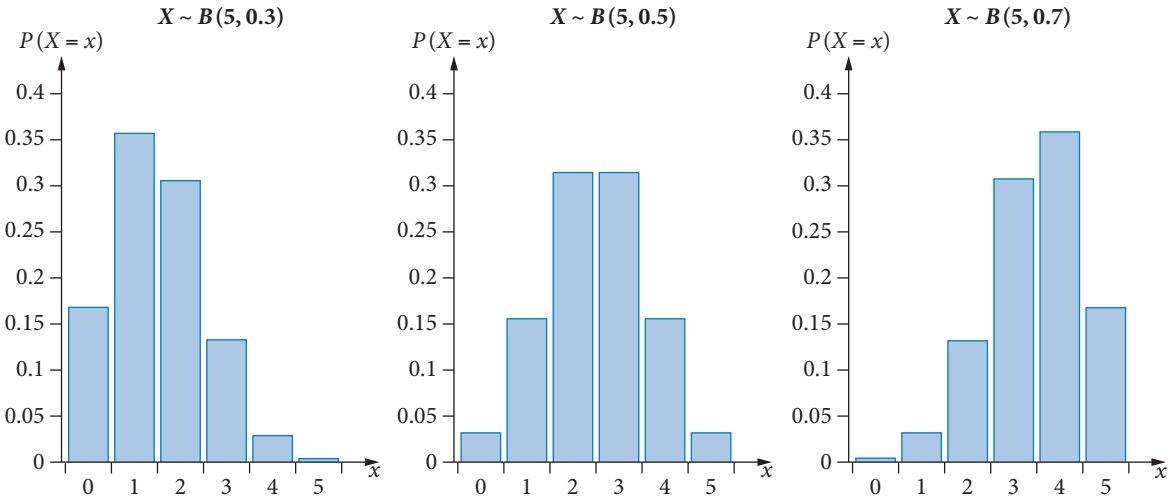
$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

Graph of a binomial distribution

$p < 0.5$ skews the graph to the right, also called positive skew.

$p = 0.5$ produces a graph symmetrical about the mode.

$p > 0.5$ skews the graph to the left, also called negative skew.



Statistics of binomial distributions

Expected value: $E(X) = np$

Variance: $Var(X) = np(1 - p)$

Chapter review 8

Exercise 8.1

1 A die is rolled and a spinner numbered from 7 to 10 is spun. Let X be the difference between the numbers on the spinner and die.

(a) Construct a table to show the probability distribution of X. Express the probabilities as fractions in simplest form.

(b) Determine the following probabilities.

(i) The difference is between 4 and 6 inclusive.

(ii) The difference is at least 3.

Exercise 8.1

2 Two fair dice are rolled. One is numbered 1, 1, 2, 2, 3, 3 and the other 2, 2, 4, 4, 5, 6. Let X be the sum of the numbers on the upper faces of the two dice.

(a) Construct the probability distribution table for X. Express the probabilities as fractions in simplest form.

(b) Determine the exact mean sum of the pair of numbers on the dice.

Exercise 8.1

3 Consider the following probability distribution.

x	20	40	60	80	100
$P(X=x)$	0.129	0.215	0.346	0.180	0.130

Determine the following values.

(a) $E(X)$ **(b)** $Var(X)$

Exercise 8.1

4 Consider the following probability distribution. Determine the standard deviation of X, correct to 2 decimal places.

x	16	17	18	19
$P(X=x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$

Exercise 8.1

5 Transform the probability distribution according to the rule $Y = 0.7X - 5$.

x	80	90	100	110	120
$P(X=x)$	0.01	0.27	0.25	0.16	0.31

Exercise 8.2

6 Two coins are flipped 40 times. Determine the probability, correct to 3 decimal places, of the following outcomes.

(a) two heads exactly ten times

(b) a double (heads or tails) at least half of the time

7 When Jacob and Lachlan play croquet, Lachlan wins 55% of the time. If they play the best of 15 games, what is the chance that Jacob will win? Give your answer correct to 2 decimal places. Exercise 8.2

8 Use a probability distribution to determine the median of $X \sim B(5, 0.45)$. Exercise 8.2

9 Use technology to determine the median of $X \sim B(60, 0.84)$. Exercise 8.2

10 Calculate the stated probability for the following binomial distributions. Express your answers as fractions in simplest form. Exercise 8.2

(a) If $X \sim B\left(5, \frac{3}{5}\right)$, calculate $P(X = 3)$.

(b) If $X \sim B\left(6, \frac{7}{10}\right)$, calculate $P(X = 4)$.

11 A coin, biased so that $P(\text{H}) = 2 \times P(\text{T})$, is flipped eight times. The probability that a coin flipper gets exactly three heads is best represented by which of the following expressions? Exercise 8.2

A $\binom{8}{3}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3$

B $\binom{8}{5}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$

C $\binom{8}{5}\left(\frac{2}{5}\right)^3 + \left(\frac{8}{3}\right)\left(\frac{1}{3}\right)^5$

D $\binom{8}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^5$

12 A die is rolled and you record a success for any value greater than 2. Determine the values of the mean and the variance of the Bernoulli distribution (the number of successes in a single trial). Exercise 8.3

13 For each of the following, calculate the exact value of the unknown parameter, n or p. Exercise 8.3

(a) $X \sim B(28, p)$, $E(X) = 2$

(b) $X \sim B(n, 0.15)$, $E(X) = 7.5$

14 For the variable $Y \sim B(n, p)$, it is known that $E(Y) = 32$ and $Var(Y) = 6.4$. Calculate the number of trials, n, and the probability of success, p. Exercise 8.3

15 When Yehudi and Carlos play racquetball, the probability that Yehudi wins a point is 0.35. Exercise 8.3

(a) How many points would you expect Yehudi to win from 15 points? Give your answer to the nearest whole number of points.

(b) Comment on your reaction if Yehudi won 10 from 15 points.

16 If $Y \sim B(100, 0.2)$, what is the value of μ? Exercise 8.3

A 0.8 **B** 20 **C** 80 **D** 16

17 For a particular binomial random variable Y, it is known that $Var(Y) = 14.4$. If 60 trials are conducted, what could be the probability of success, p? Exercise 8.3

A 0.6 only **B** 0.24 only **C** 0.6 or 0.4 **D** 0.24 or 0.76

18 A quality control officer at a manufacturing company has found that, on average, 10% of the products are faulty. Exercise 8.3

(a) If a batch of 20 products is chosen, draw a graph to represent the probabilities of each possible number of faulty products in the batch.

(b) Determine the average number of faulty products for a sample of 20 products and its standard deviation, correct to 2 decimal places.

Exercise 8.1

19 A standard 52-card deck is reduced to 32 by removing all of the picture cards (king, queen, jack), all of the aces and the black twos and threes. Two cards are drawn without replacement.

(a) Construct a probability distribution table for the number of hearts drawn, X.

(b) Determine the probability that both cards are black.

(c) Determine the probability of drawing at least one black card from the pack.

(d) Determine the probability that both cards are diamonds, given that both are red.

Exercise 8.2

20 On his shift at the emergency ward, Michael takes 15 patients to have X-rays. Of these patients, 20% are children. Determine the following probabilities, correct to 4 decimal places.

(a) Michael takes exactly three children for an X-ray.

(b) Michael takes fewer than three children for an X-ray.

(c) Michael takes three children for an X-ray, including the first two patients.

Exercise 8.3

21 A coin is biased in such a way that $P(\text{H}) = 3 \times P(\text{T})$. The coin is tossed 100 times. Let X stand for the number of tails obtained.

(a) Use the mean and standard deviation to estimate a range of values of X within which the number of tails would fall in 95% of cases.

(b) Calculate the actual probability of the range of X values occurring, as a percentage, correct to 1 decimal place.

Exercise 8.2

22 E. T. 'Hoops' Hooper has an 85% success rate from the free-throw line in basketball. At each training session, he shoots 50 times from the line.

(a) What is the number of successful free-throws Hoops should expect?

A 15 **B** 57 or 58 **C** 35 **D** 42 or 43

(b) Within what range of successes would Hoops expect to be about 95% of the time?

A 40 to 45 **B** 31 to 54 **C** 38 to 47 **D** 36 to 49

Exercise 8.3

23 A die labelled 1 to 6 is loaded so that $P(\text{odd}) = 3 \times P(\text{even})$.

(a) If rolling an odd number is considered to be a success, determine $P(\text{success})$.

(b) The die is rolled 30 times.

(i) Write this information in the form $X \sim B(n, p)$.

(ii) What is the expected number of odd numbers that will occur in the 30 rolls?

(iii) Discuss the likelihood of rolling 28 odd numbers in the 30 rolls.

(c) The die is used in a game that requires two odd numbers to be rolled in a row before a player can start. The single die is rolled twice to get the pair of numbers. Draw a table to show the probability distribution of the number of odd numbers in the two rolls.

24 A game is set up with the probability distribution of X given by: Exercise 8.1

$$P(X = x) = \begin{cases} k(6-x) & \text{if } x \in \{0,1,2,3,4\} \\ 0 & \text{for all other values of } x \end{cases}$$

In the game, scoring 0 or 4 pays \$5 and scoring 1 pays \$20.

(a) For the game to be fair, what should be the charge to play?

(b) Determine the standard deviation of the payout amount, to the nearest cent.

(c) Determine the percentage of payouts that fall within the interval $\mu \pm 2\sigma$. Comment on the result in terms of the estimated 95% interval for many distributions.

25 In Michelle's game of chance, the player rolls a die. If a prime number results, Michelle pays the player \$20. If a six results, the die is rolled again. If a second six results, Michelle pays the player \$50. If 200 games are to be played in an evening, and Michelle needs to cover costs of \$500 for hire of the venue and refreshments as well as raise \$1000 for charity, what is the minimum whole dollar amount she should charge to play the game? Exercise 8.1

26 A market stall convenor purchases mechanical toys from three different suppliers: 80% of the toys from supplier A are perfect, while 70% of the toys from supplier B and only 40% of the toys from supplier C are perfect. She needs 15 toys for her stall each week. If the convenor buys equal numbers from each supplier, what is the probability that fewer than 3 are faulty? Give your answer correct to 3 decimal places. Exercise 8.2

27 A cereal manufacturer is running a promotion on the single-serving boxes of cereal stating that, 'One in five cereal boxes contains a free gift!' Carol intends to buy one cereal box every day for one week. For the following questions, give your answers correct to 3 decimal places, where necessary. Exercise 8.2

(a) If the variable X represents the number of cereal boxes with a free gift, calculate the probability that Carol will win exactly one free gift from the seven cereal boxes she intends to buy.

(b) What is the probability that Carol will win no free gifts from the seven cereal boxes she intends to buy?

(c) If Carol wins one free gift on the first day, calculate the probability that she will win exactly one more free gift in the next six days.

(d) If Carol does not win a free gift in the first five days, calculate the probability that she will win more than one free gift in the next two days.

(e) Carol thinks that if she buys one cereal box per day for 10 days, her chances of winning more than two free gifts will increase. Explain why Carol is correct in her thinking, using appropriate calculations.

(f) Carol wants to be 95% sure that she will win at least one free gift. For how many days does she have to buy one cereal box per day for this chance?

Mixed review

CHAPTERS 6–8

Exercise 6.1

1 Consider the function f defined by $f(x) = -2(x-4)(x-1)$.

(a) Use calculus to locate any stationary points and determine their nature.

(b) Sketch the graph of $y = f(x)$, showing all key features.

Exercise 6.2

2 Determine expressions for the acceleration $a(t)$ of particles given by each of the following functions, whose positions are x metres from a fixed origin at time t seconds.

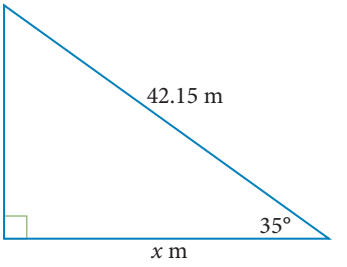
(a) $x(t) = 2t^2 - 3t + 4$ **(b)** $x(t) = \dfrac{1}{2t+5}$

3 Determine the coordinates of the point of inflection on the graph of $y = -2x^3 - 12x^2 - 3x - 2$.

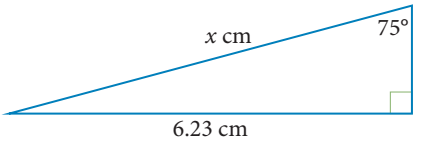
Exercise 7.1

4 Calculate the length of side x, correct to 2 decimal places, in each of the following triangles.

(a)



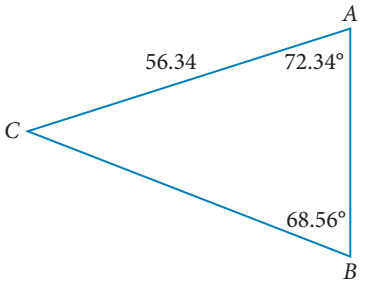
(b)



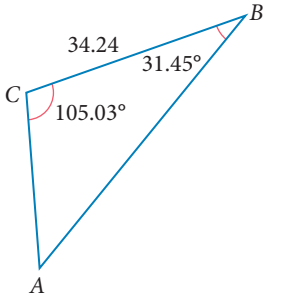
Exercise 7.3

5 Solve ΔABC by calculating, correct to 2 decimal places, the unknown side lengths and angle.

(a)



(b)

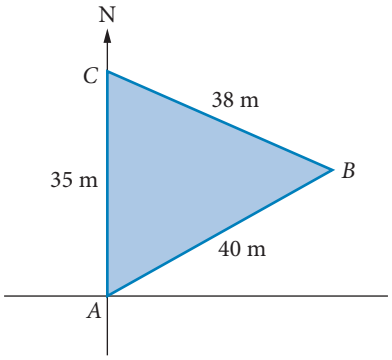


Exercise 7.2

6 A block of land is advertised as having an approximate area of 400 m^2. If the three boundaries of the block are 22 m, 36 m and 19 m, determine the size of the smallest angle, correct to 2 decimal places, and hence the area, to the nearest square metre.

Exercise 7.4

7 Three fences surround a triangular paddock as shown. Determine the true bearing of corner A from corner B, to the nearest degree.



8 You have a jar that contains four white marbles, five red marbles and one blue marble. You draw two marbles, with replacement, from the jar. Let X be the number of blue marbles drawn. Exercise 8.1

(a) Construct the probability distribution table for X. Express the probabilities in decimal form.

(b) What is the probability of drawing at least one blue marble from the jar?

9 A particular medical test correctly identifies whether or not a person has an illness 97% of the time. Determine the following probabilities, correct to 3 decimal places, given that 15 people are tested. Exercise 8.2

(a) None is incorrectly diagnosed.

(b) At least one is incorrectly diagnosed.

10 For $X \sim B(n, p)$, $\mu = 12.5$ and $\sigma^2 = 9.375$. Calculate the values of n and p. Exercise 8.3

11 Determine an expression for the second derivative of $y = \sqrt{x^2 + 1}$. Exercise 6.3

12 Given $A = 28°$, $a = 9$ and $b = 12$, solve ΔABC. Give your answers correct to 1 decimal place. Exercise 7.3

13 A family has six children. Assume the probability of any child being a boy is 0.5 and that the gender of each child is independent of the others. Calculate the following probabilities, giving your answer correct to 4 decimal places, where necessary. Exercise 8.2

(a) the first two children born were male

(b) three boys and three girls, in any order

(c) more girls than boys

(d) at least one boy, but more girls than boys

(e) no consecutive births of the same gender

14 A rectangle is drawn with one side along the x-axis through the origin, and the opposite side with vertices on the graph of $y = \cos(x)$. Use second derivative analysis to determine the maximum area of the rectangle, correct to 2 decimal places. Exercise 6.3

15 At an egg-packaging manufacturing company, researchers are testing a new type of material for egg cartons to reduce cracking and breakage during transport. Each testing container can hold one egg only, and no containers are tested without an egg inside. If the egg does not crack or break during testing, the outcome is considered a 'success'. Exercise 8.3

(a) From the tests done so far, on average, five containers per batch end up with an intact egg at the end of the tests. If each batch consists of fifteen containers, what is the probability of 'success'?

(b) The researcher has concluded that 95% of batches would have between two and eight intact eggs. If the distribution of the data is approximately normal, explain, using mathematical reasoning, the researcher's conclusion.

(c) Discuss how the 95% interval would change if the researcher chose to test batches of ten containers each.

9

Continuous random variables and the normal distribution

Recall 458

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9.2 The normal distribution 473

9.3 The standard normal distribution 485

9.4 Inverse of standard normal distributions 493

Summary 502

Chapter review 504

9

Recall

Distinguish between continuous and discrete random variables

1 Determine whether the following variables are associated with a continuous or discrete scale.

(a) the mass of each individual packet of potato chips in a carton

(b) the number of goals scored in a netball match by each individual player

(c) the height of each student in every Mathematical Methods class in Australia

Calculate definite integrals

2 Calculate the exact value for each definite integral.

(a) $\int_1^5 2x\,dx$ (b) $\int_3^5 \frac{1}{x^2}\,dx$

(c) $\int_{-4}^{3} 5x(2x+3)\,dx$

Determine the maximum value of a function within a specified domain

3 For each of the following functions, determine the maximum value, and the value of x at which it occurs.

(a) $y = -\frac{(x-4)^2}{4} + 2$, where $x \in [0, 6]$

(b) $y = x^3 - 4x$, where $x \in [-2, 1]$

(c) $y = x^3 + 2$, where $x \in [-1, 1]$

Calculate conditional probabilities

4 From the universal set: $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, a number x is drawn at random. Calculate the following probabilities.

(a) $P(x < 7 | x \text{ is even})$ (b) $P(x > 3 | x < 7)$ (c) $P(x < 8 | x > 4)$

Calculate values associated with a binomial distribution

5 If $X \sim B(10, 0.8)$, calculate the following values.

(a) μ, the expected value

(b) the probability of failure, $q = 1 - p$

(c) σ, the standard deviation, correct to 3 decimal places

Calculate probabilities associated with a binomial distribution

6 Given $X \sim B(10, 0.8)$, calculate the following correct to 3 decimal places.

(a) $P(X = 6)$ (b) $P(X \leq 3)$ (c) $P(5 \leq X \leq 7)$

Continuous random variables

9.1

Probability density functions

When a variable can take an infinite number of values in a given interval (e.g. when you measure it rather than count it), the variable is called a *continuous random variable*. Quantities that can be modelled using continuous random variables include height, weight, time and mass.

A continuous random variable is defined by its *probability density function* (*pdf*), which is usually represented by $f(x)$. A continuous random variable is defined over the interval $(-\infty, \infty)$. In practice, however, often the probability associated with much of this interval is 0. Therefore, $f(x)$ is usually defined as a piece-wise function.

An understanding of continuous random variables can be gained through the following example. Dirk travels to work by train every day. Over a 10-week period (50 work days) he measures the time in minutes that the train is late arriving at the station.

Minutes late	0–<2	2–<4	4–<6	6–<8	8–<10
Frequency	5	15	17	10	3

Dirk wants to calculate the probability of his train being less than 4 minutes late:

Let X be the number of minutes the train is late,

$$P(X < 4) = \frac{n(0 \le X < 2) + n(2 \le X < 4)}{n(0 \le X < 10)}$$
$$= \frac{5 + 15}{50}$$
$$= \frac{20}{50}$$
$$= \frac{2}{5}$$

This is written as: $P(\text{train is less than 4 minutes late}) = \frac{2}{5}$ and read as: The probability that Dirk's train is less than 4 minutes late is $\frac{2}{5} = 0.4$; that is, 40% of the time, Dirk's train arrives within 4 minutes after the scheduled time.

By converting the frequency into probabilities, the probability distribution looks like this:

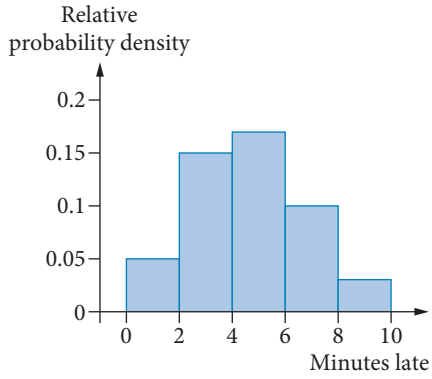
Minutes late	0–<2	2–<4	4–<6	6–<8	8–<10
Probability	$\frac{5}{50} = 0.1$	$\frac{15}{50} = 0.3$	$\frac{17}{50} = 0.34$	$\frac{10}{50} = 0.2$	$\frac{3}{50} = 0.06$

Note that the individual probabilities add to 1.

To form a probability density function, you need relative probability values such that the area for each rectangle matches the probability for that range of values. The relative probability values are obtained by dividing the probabilities by the bin width (the width of the columns), which in this case is 2.

Minutes late	0–<2	2–<4	4–<6	6–<8	8–<10
Probability	0.1	0.3	0.34	0.2	0.06
Relative probability density	$\frac{0.1}{2} = 0.05$	$\frac{0.3}{2} = 0.15$	$\frac{0.34}{2} = 0.17$	$\frac{0.2}{2} = 0.1$	$\frac{0.06}{2} = 0.03$

If you calculate the area of each of the rectangles in this histogram, you will find that their sum is 1. This is because the area represents the total probability.



$$\begin{aligned} A &= A_1 + A_2 + A_3 + A_4 + A_5 \\ &= l_1 w + l_2 w + l_3 w + l_4 w + l_5 w \\ &= w\left(l_1 + l_2 + l_3 + l_4 + l_5\right) \\ &= 2 \times (0.05 + 0.15 + 0.17 + 0.1 + 0.03) \\ &= 2 \times 0.5 \\ &= 1 \end{aligned}$$

By calculating the area of the graph between 0 and 4:

$$\begin{aligned} A &= A_1 + A_2 \\ &= l_1 w + l_2 w \\ &= 0.05 \times 2 + 0.15 \times 2 \\ &= 0.1 + 0.3 \\ &= 0.4 \end{aligned}$$

If X is a continuous random variable, then $P(X = x) = 0$, for all possible values of x.

The area of the histogram from 0 to 4 represents the probability of the Dirk's train arriving within 4 minutes of the scheduled time. The probability of any individual value is zero, since a rectangle of zero width has an area of zero.

As $P(X = x) = 0$, different probability expressions involving inequalities can have the same meaning.

WARNING

$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < x < b)$

Using the definite integral

When a continuous probability density function is written as a rule, you can calculate the area under the curve, and hence the probability, by integrating the function between the appropriate values.

For a function $f(x)$ to be a probability density function:

$f(x) \geq 0$ for all values of x

$\int_{-\infty}^{\infty} f(x)dx = 1$ (The area enclosed by the graph $y = f(x)$ and the x-axis is equal to 1.)

WARNING

$f(x)$ is not the probability. You need to integrate between two limit values to obtain a probability. That is, the probability of $X < 1$ is given by $\int_0^1 f(x)dx$ not by $f(1)$.

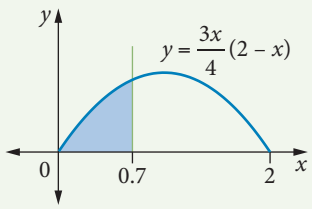
Technology worked example
Continuous random variables

1 Calculate probability by integration

A particular continuous random variable has the following probability density function:

$$f(x) = \begin{cases} \frac{3x}{4}(2-x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $P(X < 0.7)$, correct to 3 decimal places.

THINKING	WORKING
1 Identify the bounds of integration to use.	The lower bound is 0 and the upper bound is 0.7.
2 Expand the quadratic expression.	$\frac{3x}{4}(2-x) = \frac{3}{2}x - \frac{3}{4}x^2$
3 Determine the required integral. Note: If you are using technology, the integration can be calculated without expanding the expression.	$\int_0^{0.7}\left(\frac{3}{2}x - \frac{3}{4}x^2\right)dx = \left[\frac{3}{4}x^2 - \frac{1}{4}x^3\right]_0^{0.7}$ $= \left(\frac{3}{4}(0.7)^2 - \frac{1}{4}(0.7)^3\right) - (0)$ $= 0.28175$ $= 0.282\,(3\text{ d.p.})$
4 Interpret the answer.	$P(X < 0.7) = 0.282$
5 Evaluate the reasonableness of your result by examining the area under the curve.	The area under the curve for the entire domain will be 1, so the area representing the probability should appear to be the same portion of the whole as the probability value. 

2 Calculate probability when the interval involves infinity

A particular continuous random variable has the following probability density function:

$$f(x) = \begin{cases} \frac{80(x-40)}{x^3} & x \ge 40 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the exact value of $P(X > 80)$.

METHOD 1

THINKING	WORKING
1 Identify the bounds of integration to use.	The lower bound is 80 and the upper bound is ∞.

2 Write the expression as individual terms.

$$\frac{80(x-40)}{x^3} = \frac{80}{x^2} - \frac{3200}{x^3}$$
$$= 80x^{-2} - 3200x^{-3}$$

3 Calculate the required integral.

Note that terms with x on the denominator approach 0 as $x \to \infty$.

Note: To calculate using a graphics calculator, or to check your result, you will have to insert a large number in place of infinity. To check if the number is large enough, try an even larger number and see if the answer is the same, within the required accuracy.

$$\int_{80}^{\infty}\left(\frac{80}{x^2} - \frac{3200}{x^3}\right)dx = \int_{80}^{\infty}\left(80x^{-2} - 3200x^{-3}\right)dx$$
$$= \left[-80x^{-1} + 1600x^{-2}\right]_{80}^{\infty}$$
$$= \left[-\frac{80}{x} + \frac{1600}{x^2}\right]_{80}^{\infty}$$
$$= [-0+0] - \left[-\frac{80}{80} + \frac{1600}{80^2}\right]$$
$$= 0 - \left[-\frac{3}{4}\right]$$
$$= \frac{3}{4}$$

4 Interpret the answer.

$$P(X > 80) = \frac{3}{4}$$

METHOD 2

1 Identify complementary events and hence the bounds of the integration.

$P(X > 80) = 1 - P(X \leq 80)$

The lower bound is 40 and the upper bound is 80.

2 Write the expression as individual terms.

$$\frac{80(x-40)}{x^3} = \frac{80}{x^2} - \frac{3200}{x^3}$$
$$= 80x^{-2} - 3200x^{-3}$$

3 Calculate the required integral.

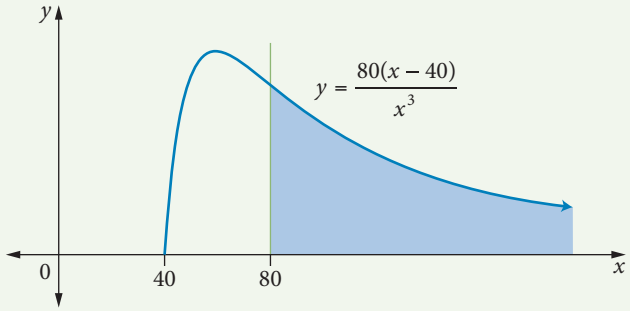
$$P(X > 80) = 1 - \int_{40}^{80}\left(\frac{80}{x^2} - \frac{3200}{x^3}\right)dx$$
$$= 1 - \int_{40}^{80}\left(80x^{-2} - 3200x^{-3}\right)dx$$
$$= 1 - \left[-80x^{-1} + 1600x^{-2}\right]_{40}^{80}$$
$$= 1 - \left[-\frac{80}{x} + \frac{1600}{x^2}\right]_{40}^{80}$$
$$= 1 - \left(\left[-\frac{80}{80} + \frac{1600}{80^2}\right] - \left[-\frac{80}{40} + \frac{1600}{40^2}\right]\right)$$
$$= 1 - \left(-\frac{3}{4} + 1\right)$$
$$= \frac{3}{4}$$

4 Interpret the answer.

$$P(X > 80) = \frac{3}{4}$$

(b) Evaluate the reasonableness of your solution.

Draw a graph and shade the given area to determine if the proportion of the graph shaded is an accurate representation of the probability determined.



$P(X > 80) = \frac{3}{4}$ (Approximately 75% of the area under the curve exists for $X > 80$.)

Linear piece-wise functions

When a linear piece-wise model is used, more than one equation may be required to give a suitable shape.

An inspection of the graph will reveal that the area under the graph of a probability density function is indeed 1. Note that a probability density function can take values greater than 1. Keep in mind that it is the area bounded by the curve and the x-axis that must be 1.

3 Calculate probability from a piece-wise function with two equations

A particular continuous random variable has the following probability density function:

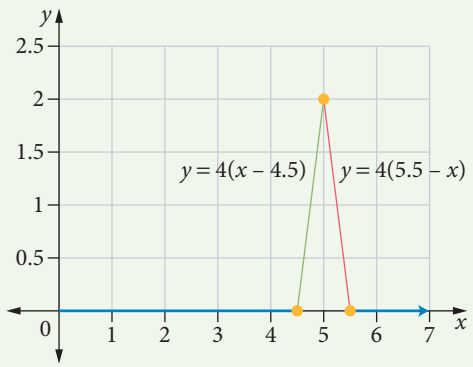
$$f(x) = \begin{cases} 4(x - 4.5) & 4.5 < x \leq 5.0 \\ 4(5.5 - x) & 5.0 < x \leq 5.5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Draw a graph of the probability density function and verify that it does in fact represent a probability density function.

THINKING

1 Sketch a graph of the function.

WORKING



2 Write the piece-wise function in expanded form.

$$f(x) = \begin{cases} 4x - 18 & 4.5 < x \leq 5.0 \\ 22 - 4x & 5.0 < x \leq 5.5 \\ 0 & \text{otherwise} \end{cases}$$

3 Determine the area bounded by the curve and the x-axis.

$$\begin{aligned} A &= \int_{4.5}^{5} 4x - 18\,dx + \int_{5}^{5.5} 22 - 4x\,dx \\ &= \left[\frac{4x^2}{2} - 18x\right]_{4.5}^{5} + \left[22x - \frac{4x^2}{2}\right]_{5}^{5.5} \\ &= \left[2x^2 - 18x\right]_{4.5}^{5} + \left[22x - 2x^2\right]_{5}^{5.5} \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1 \end{aligned}$$

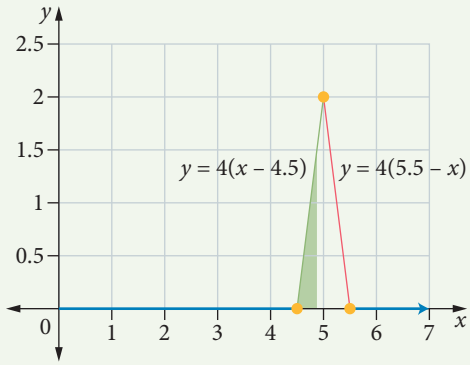
4 Alternatively, because the area under the curve is a triangle, calculate the area using $A = \frac{1}{2}bh$.

$$\begin{aligned} b &= 5.5 - 4.5 \\ &= 1 \\ h &= 2 \\ A &= \frac{1}{2} \times b \times h \\ &= \frac{1}{2} \times 1 \times 2 \\ &= 1 \end{aligned}$$

(b) Use technology to calculate the following probabilities.

(i) $P(X < 4.7)$

1 Determine the area of the graph to be calculated.



2 Identify the rule for the required region and the upper and lower bounds.

Area under the curve $f(x) = 4(x - 4.5)$ between $x = 4.5$ and $x = 4.7$.

3 Use technology to determine the area under the curve.

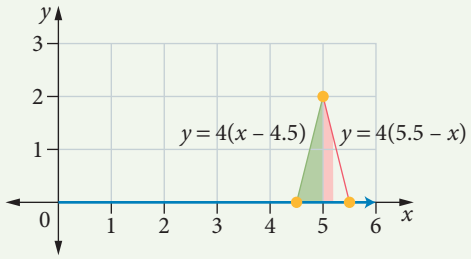
$$\begin{aligned} P(X < 4.7) &= \int_{4.5}^{4.7} 4(x - 4.5)\,dx \\ &= \left[2x^2 - 18x\right]_{4.5}^{4.7} \\ &= 0.08 \end{aligned}$$

4 Interpret the answer.

$P(X < 4.7) = P(4.5 < X < 4.7) = 0.08$

(ii) $P(X < 5.2)$

1 Determine the area of the graph to be calculated.



2 Identify the rule for the required region and the upper and lower bounds.

Area under the curve $f(x) = 4(x - 4.5)$ between $x = 4.5$ and $x = 5$ and the area under the curve $f(x) = 4(5.5 - x)$ between $x = 5$ and $x = 5.2$.

3 Choose the rule for each of two regions.

Choose the upper and lower bounds for each integration.

Use technology to calculate each integral.

$$P(X < 5) = \int_{4.5}^{5} 4(x - 4.5)\,dx$$
$$= 0.5$$

$$P(5 < X < 5.2) = \int_{5}^{5.2} 4(5.5 - x)\,dx$$
$$= 0.32$$

4 Add the areas for the final result.

$$P(X < 5.2) = P(X < 5) + P(5 < X < 5.2)$$
$$= 0.5 + 0.32$$
$$= 0.82$$

WARNING

If any of your integrals give values not in the range $[0, 1]$, then an error has been made.

The statistics of continuous random variables

Recall that for any discrete probability distribution, the expected value (mean) is found by adding together the sum of the products of the score and the individual probability values $(\Sigma x_i p_i)$ and that the variance can be found from $E(X^2) - [E(X)]^2$. The integral is the continuous distribution equivalent of Σ.

For a continuous probability density function, f:

- The mean is $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$
- The variance is $\sigma^2 = \left(\int_{-\infty}^{\infty} x^2 f(x)\,dx\right) - \mu^2$
- The standard deviation is the square root of the variance.

WARNING

To calculate the mean and variance, determine the interval where the function is non-zero. If $[a, b]$ is the non-zero continuous interval, the formulas become:

$$\mu = \int_{a}^{b} x f(x)\,dx$$
$$\sigma^2 = \left(\int_{a}^{b} x^2 f(x)\,dx\right) - \mu^2$$

4 Calculate the mean and standard deviation of a continuous density function

For a particular Infant Welfare Centre, the probability density function of the ages of the children, x years, brought to the centre is given by:

$$f(x) = \begin{cases} \frac{2}{9}x(3-x) & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the following, correct to 3 decimal places, as necessary.

(a) the mean age

THINKING	WORKING
1 Determine the expression for $xf(x)$.	$xf(x) = x\frac{2}{9}x(3-x)$ $= \frac{2}{3}x^2 - \frac{2}{9}x^3$
2 Calculate the mean by calculating the integral over the entire non-zero continuous domain $\mu = \int_{-\infty}^{\infty} xf(x)dx$. Note: If technology is used, there is no need to expand the expression.	$\mu = \int_0^3 \left(\frac{2}{3}x^2 - \frac{2}{9}x^3\right)dx$ $= \left[\frac{2x^3}{9} - \frac{2x^4}{36}\right]_0^3$ $= \left(\frac{2\times(3)^3}{9} - \frac{2\times(3)^4}{12}\right) - (0)$ $= \left(\frac{2\times 27}{9} - \frac{2\times 81}{36}\right)$ $= 6 - \frac{9}{2}$ $= \frac{3}{2}$ $= 1.5$
3 Interpret the answer.	The mean age is 1.5 years.

(b) the standard deviation of the ages

1 Determine the expression for $x^2 f(x)$.	$x^2 f(x) = x^2 \times \frac{2x}{9}(3-x)$ $= \frac{2x^3}{9}(3-x)$

2 Calculate the value of the integral over the entire non-zero continuous domain $\int_{-\infty}^{\infty} x^2 f(x)dx$. Note: If technology is used, there is no need to expand the expression.	$\int_0^3 \left(\frac{2x^3}{9}(3-x)\right)dx = \int_0^3 \left(\frac{2x^3}{3} - \frac{2x^4}{9}\right)dx$ $= \left[\frac{2x^4}{12} - \frac{2x^5}{45}\right]_0^3$ $= \left(\frac{1\times 3^4}{6} - \frac{2\times 3^5}{45}\right) - (0)$ $= \frac{1\times 3\times 27}{6} - \frac{2\times 9\times 27}{45}$ $= \frac{27}{2} - \frac{54}{5}$ $= \frac{27}{10}$ $= 2.7$
3 Recall the variance formula and calculate the variance.	$\sigma^2 = \left(\int_{-\infty}^{\infty} x^2 f(x)dx\right) - \mu^2$ $\sigma^2 = 2.7 - (1.5)^2$ $= 0.45$
4 Determine the standard deviation, where $\sigma = \sqrt{\sigma^2}$, rounding to the specified level of accuracy.	$\sigma = \sqrt{\frac{9}{20}}$ $= 0.671\ (3\text{ d.p.})$
5 Interpret the answer.	The standard deviation of the ages is 0.671 years.

It may be wise to check that your answers are plausible. For instance, has the mean value fallen inside the interval for which the probability is defined? Also, as a rule of thumb, the range of the distribution will be about five times the standard deviation.

For the Infant Welfare Centre in Worked Example 4, you can see the mean is about the middle of the defined interval and $5 \times 0.671 = 3.355$, where the range is actually 3.

Cumulative distribution functions

You can also find the median of a continuous pdf. As should be expected, the median, m, is the value such that $P(X < m) = 0.5$; that is, the point where 50% of the distribution is below and 50% is above. This means you need to solve an equation resulting from $\int_a^m f(x)dx = 0.5$, or $F(x) = 0.5$, where the non-zero part of the piece-wise function starts at a.

The cumulative distribution function in this case would be $F(x) = \int_a^x f(x)dx$.

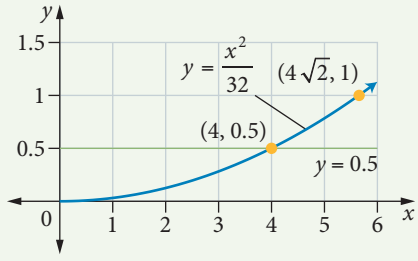
It is important to note that $0 \leq F(x) \leq 1$ for the interval $[a,b]$ where the original function is continuous and non-zero. The $y = F(x)$ and $y = 0.5$ can be graphed on the same set of axes as an alternative way of calculating the median.

5 Calculate the median of a continuous density function

A particular continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} \frac{x}{16} & 0 \le x \le 4\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

Calculate the median.

THINKING	WORKING
1 Recall the cumulative distribution function.	$F(x) = \int_a^x f(x)dx$
2 Construct an equation in this form.	Let the median be m. $\int_0^m f(x)dx = 0.5$
3 Integrate and simplify.	$\int_0^m \frac{x}{16}dx = \left[\frac{x^2}{32}\right]_0^m$ $= \frac{m^2}{32} - 0$ $= \frac{m^2}{32}$
4 Solve the equation for m and state the solution, where $a \le m \le b$. Interpret the result.	$\frac{m^2}{32} = 0.5$ $m^2 = 16$ $m = \pm\sqrt{16}$ $= \pm 4$ Since $0 \le m \le 4\sqrt{2}$, $m = 4$.
5 Interpret the result.	The median is 4.
6 Evaluate the reasonableness of your answer by drawing the graph of the cumulative distribution function and determine the point of intersection with the graph of $y = 0.5$.	$F(x) = \begin{cases} \frac{x^2}{32} & 0 \le x \le 4\sqrt{2} \\ 0 & \text{elsewhere} \end{cases}$  The value of the median is the same as found earlier. This is represented by the x-value of the point of intersection.

EXERCISE 9.1 Continuous random variables

Worked Example

1 Consider the continuous random variable with probability density function: 1

$$f(x) = \begin{cases} \frac{3}{13}\left(x^2 + 4\right) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the following probabilities, correct to 4 decimal places.

(a) $P(X < 0.75)$ **(b)** $P(X < 0.9)$ **(c)** $P(X > 0.25)$ **(d)** $P(X > 0.65)$

2 $f(x) = \begin{cases} -3(x-1)^2 + 2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

Nigel performs the following calculation:

$$\begin{aligned} P(X) &= \int_0^1 \left(-3x^2 + 6x - 1\right) dx \\ &= \left[-x^3 + 3x^2 - x\right]_0^1 \\ &= \left(-(1)^3 + 3(1)^2 - (1)\right) - (0) \\ &= 1 \end{aligned}$$

Nigel then concludes that the function represents a probability density function. Explain why Nigel is incorrect.

3 Consider the continuous random variable with probability density function: 2

$$f(x) = \begin{cases} \frac{15\,000(x-50)}{x^4} & x \ge 50 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the exact values of the following probabilities.

(a) $P(X < 60)$ **(b)** $P(X < 75)$ **(c)** $P(X > 65)$ **(d)** $P(X > 99)$

4 A particular continuous random variable has the following probability density function: 3

$$f(x) = \begin{cases} \frac{1}{20}x - \frac{1}{10} & 2 \le x \le 6 \\ \frac{2}{5} - \frac{1}{30}x & 6 < x \le 12 \\ 0 & \text{otherwise} \end{cases}$$

Use technology to calculate the following probabilities, correct to 4 decimal places.

(a) $P(X < 3.5)$ **(b)** $P(X > 9.8)$ **(c)** $P(X < 9.2)$ **(d)** $P(X > 4.6)$

5 Calculate the mean of the continuous random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{15\,000(x-50)}{x^4} & x \geq 50 \\ 0 & \text{otherwise} \end{cases}$$

6 If $f(x) = \begin{cases} \dfrac{x}{45} + k & 0 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$ defines a probability density function, what is the value of k?

A 0 **B** $-\dfrac{1}{10}$ **C** $\dfrac{1}{10}$ **D** $\dfrac{1}{2}$

7 Consider the continuous random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{2x}{25} & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Determine the following probabilities.

(a) $P(1.5 < X < 2.5)$ **(b)** $P(2 < X < 4.5)$ **(c)** $P(1.75 < X < 3.15)$

8 If $f(x) = \begin{cases} \dfrac{\pi}{12}\sin\left(\dfrac{\pi x}{6}\right) & 0 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$, what is $P(X < 3)$?

A 0.25 **B** 0.5 **C** 0.55 **D** 0.6

9 Consider the uniform continuous random variable with probability density function:

$$f(x) = \begin{cases} k & 0 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the exact value of k. **(b)** Calculate $P(X < 3)$. **(c)** Calculate $P(X < 5)$.

10 Determine the exact value of the median of the continuous random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{\sin(x)}{2} & 0 \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Worked Example 4

11 Consider the continuous random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{3}{13}\left(x^2 + 4\right) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the following values, correct to 4 decimal places.

(a) the mean **(b)** the standard deviation

12 Calculate the median, correct to 4 decimal places, of the continuous random variable with probability density function:

$$f(x) = \begin{cases} \frac{3}{13}\left(x^2 + 4\right) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

13 Calculate the median of the continuous random variable with probability density function:

$$f(x) = \begin{cases} \frac{15\,000(x - 50)}{x^4} & x \ge 50 \\ 0 & \text{otherwise} \end{cases}$$

14 In the javelin competition at the primary school's athletics carnival, it is found that the distance that the javelin is thrown, s metres, is a continuous random variable with probability density function:

$$f(s) = \begin{cases} \frac{1}{486}\left(81 - s^2\right) & 0 \le s \le 9 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the following, giving your answers correct to 3 decimal places where necessary.

(a) the mean distance the javelin is thrown

(b) the standard deviation for the distance the javelin is thrown

(c) the median distance the javelin is thrown

15 Let the time, X minutes, of a telephone conversation be represented by the following continuous random variable with probability density function:

$$f(x) = \begin{cases} Ce^{-\frac{x}{10}} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of C, as a fraction.

(b) Determine the following probabilities, correct to 4 decimal places.

(i) $P(X < 2.5)$ **(ii)** $P(X > 6.4)$ **(iii)** $P(2 < X < 7)$

(c) Use technology to calculate $E(X)$ and $Var(X)$.

(d) Calculate and interpret $P(X < E(X))$ as a percentage, correct to 2 decimal places.

16 A physical therapist uses a practical test to determine the reaction times of her patients. From experience, she has determined that the reaction time, t seconds, can be modelled by the following continuous random variable with probability density function:

$$f(t) = \begin{cases} k\left(t - \frac{1}{4}\right)\left(\frac{5}{4} - t\right)^2 & \frac{1}{4} \le x \le \frac{5}{4} \\ 0 & \text{otherwise} \end{cases}$$

Calculate the following, giving your answers correct to 3 decimal places where necessary.

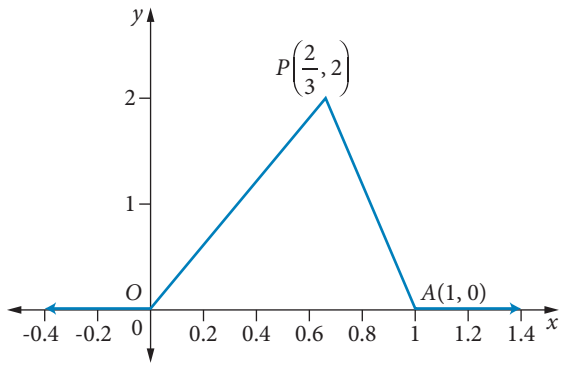
(a) the value of k

(b) the mean and median reaction times

(c) the proportion of patients who react in the following periods of time (use technology for this question)

(i) less than one second. (ii) between $\frac{2}{5}$ and $\frac{4}{5}$ of a second (iii) longer than $\frac{3}{4}$ of a second

17 Consider the function f whose graph is shown below.



(a) Demonstrate that the function could be used to represent a probability density function. Explain your answer using mathematical reasoning.

(b) Specify the rule of f.

(c) Calculate $P\left(X < \frac{2}{3}\right)$ and $P\left(X > \frac{2}{3}\right)$ using the areas of the triangles.

(d) Calculate $P\left(X < \frac{2}{3}\right)$ and $P\left(X > \frac{2}{3}\right)$ using integration. Comment on the results from parts (c) and (d).

(e) Calculate $P\left(\frac{1}{2} < X < \frac{5}{6}\right)$, using integration.

18 Consider the probabilities that a train is late, given by the table below.

Minutes late	0–<2	2–<4	4–<6	6–<8	8–<10
Probability	0.1	0.3	0.34	0.2	0.06

The probability distribution function is to be modelled by a quadratic equation using the same limits as the table.

(a) Determine an appropriate function in piece-wise form.

(b) Analyse the suitability of your model by comparing the probability values predicted by the model with the actual probability values.

(c) Determine the model's prediction for the probability that the train is less than 3 minutes late.

The normal distribution

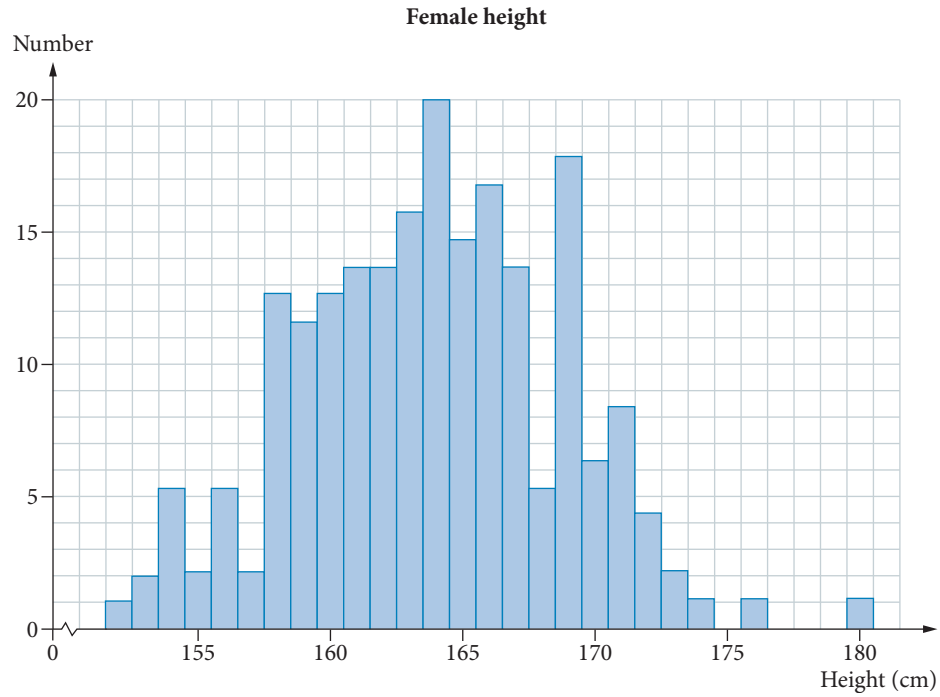
9.2

Modelling the probability distribution for height

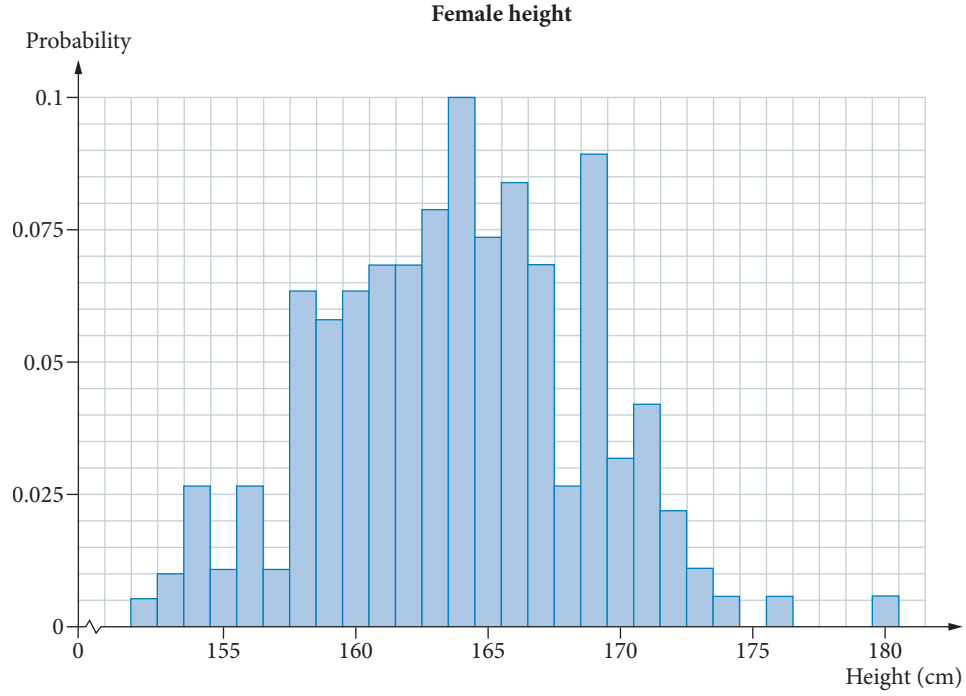
The following table shows the heights of 200 randomly selected Australian women, measured in centimetres to 1 decimal place accuracy.

154.3	157.7	152.0	158.3	170.4	164.5	161.5	160.3	165.2	155.0
163.7	161.4	161.0	164.9	168.1	163.1	161.6	158.9	163.6	161.4
160.7	164.9	164.0	162.9	171.2	160.8	169.1	158.8	160.4	165.2
159.7	152.8	167.5	159.9	159.2	156.0	162.6	158.6	162.4	161.7
166.3	172.6	169.5	165.5	163.3	170.8	159.9	158.1	169.6	169.3
168.9	165.5	160.7	165.3	155.3	164.0	167.9	169.2	166.2	169.2
158.2	170.5	164.5	158.5	171.2	165.6	158.6	164.1	161.0	162.1
163.4	167.5	163.4	164.3	156.5	161.5	157.1	163.4	171.6	166.1
166.8	156.5	154.5	159.6	169.3	167.9	157.6	167.9	163.0	160.0
171.7	158.4	158.1	156.8	162.3	165.3	161.1	156.3	164.9	157.9
167.1	169.5	165.9	159.8	168.5	166.0	171.0	153.8	158.2	163.7
165.6	165.3	158.8	163.9	167.3	168.8	161.3	171.2	158.9	161.5
159.2	163.7	175.8	164.3	167.0	168.7	174.1	163.1	164.0	162.1
152.8	159.9	170.6	163.3	173.1	163.2	163.3	180.1	167.2	164.2
159.5	159.6	168.6	163.4	159.3	163.5	158.2	167.3	160.3	163.5
166.7	165.4	163.3	161.2	166.2	167.2	160.0	162.0	168.2	169.8
169.1	168.8	165.6	163.8	168.6	167.4	168.5	166.8	166.2	158.3
165.7	163.9	163.6	169.7	164.8	172.4	156.3	162.0	161.2	166.2
172.4	166.3	164.4	153.5	163.2	169.2	166.2	168.5	170.5	167.3
162.1	162.0	165.3	154.3	166.0	160.7	162.3	164.1	169.7	159.3

The data presented in histogram form:



The total area of the rectangles in the histogram is 200, the size of the sample, so the first thing that needs to be done is to make sure the sum of the areas of the rectangles equals 1, so it meets that condition of a probability distribution. To do this, divide the vertical scale values by 200. The bin width (the width of the columns in the histogram) is set to 1, which means that the area of each bin represents the probability of a woman having a height in that 1 cm range. More importantly, the total area of the rectangles is now 1, so the situation is like a probability density function.

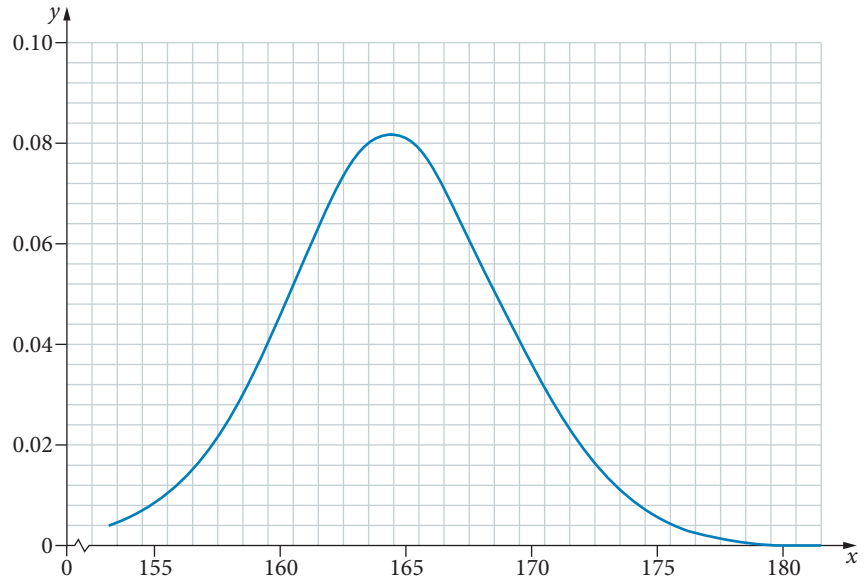


From the raw data, the mean and standard deviation can be calculated:

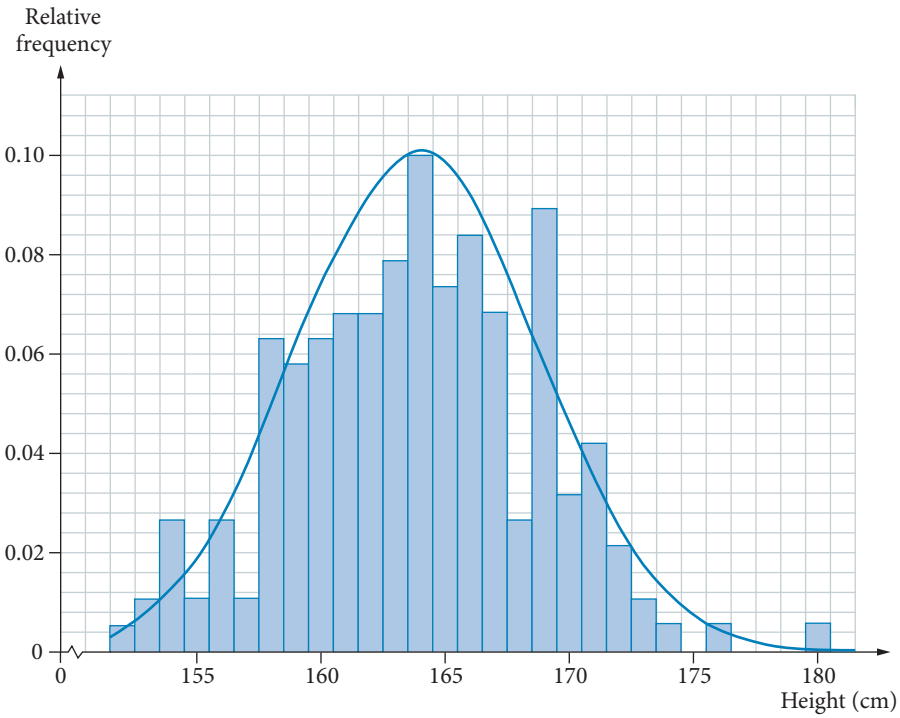
mean = 163.8085

standard deviation = 4.858855

From the general shape of the histogram, you can determine the approximate shape of the model to be similar to the symmetrical bell-shaped curve.

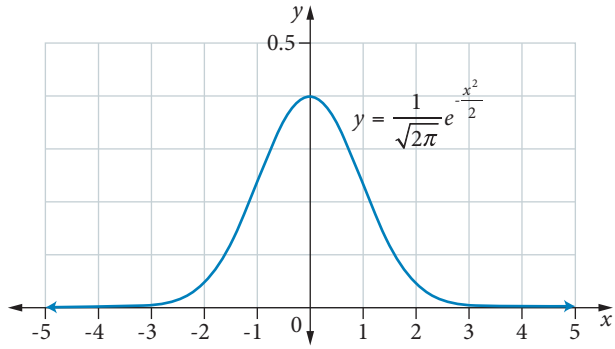


You can put the two images together to see how they might fit.



The graph matches the general shape of the curve quite well, although like all models it is certainly not perfect. The way the graph trails off suggests an exponential function of some kind, but it trails off on both sides. Mathematicians have found the function with domain $\mathbb{R}$ and rule $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ gives the same general shape.

Although the graph looks like it meets the x-axis, it is actually asymptotic to the x-axis in both directions. That is, as $x \to \infty, f(x) \to 0$ and as $x \to -\infty, f(x) \to 0$, where $f(x) > 0$ for all $x \in \mathbb{R}$.



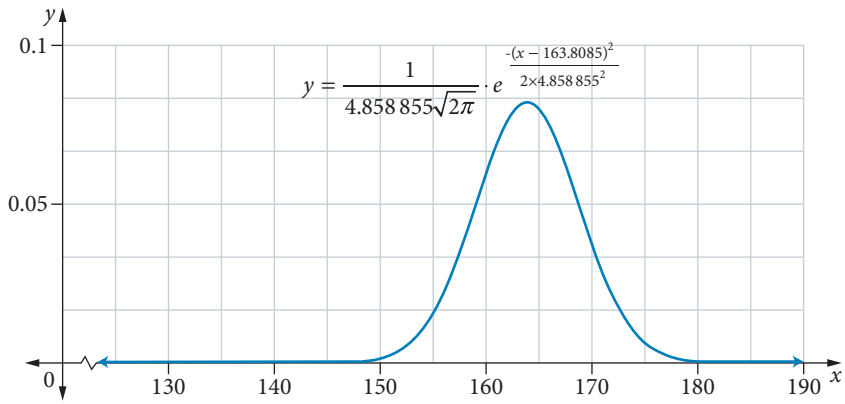
Use your calculator to check that the area bounded by the graph of $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ over $\mathbb{R}$ is 1.

$\int_{-100}^{100} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dx = 1$ is appropriate for a calculator that does not allow the infinity symbol.

You have a probability distribution function (pdf), but you still need to translate it and dilate it to match the data.

From the calculations above, with a mean of 163.808 5 and a standard deviation of 4.858 855,

$f(x) = \frac{1}{4.858855\sqrt{2\pi}} \cdot e^{-\frac{(x-163.8085)^2}{2\times 4.858855^2}}$ and $\int_{140}^{190} \frac{1}{4.858855\sqrt{2\pi}} \cdot e^{-\frac{(x-163.8085)^2}{2\times 4.858855^2}}\, dx \approx 1$.



The general rule for the normal curve, where μ is the median and σ is the standard deviation, is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Explore further

Normal distribution

Explore the effect on probabilities and the normal curve if the mean, standard deviation and boundary values are changed.

Normal distribution probabilities

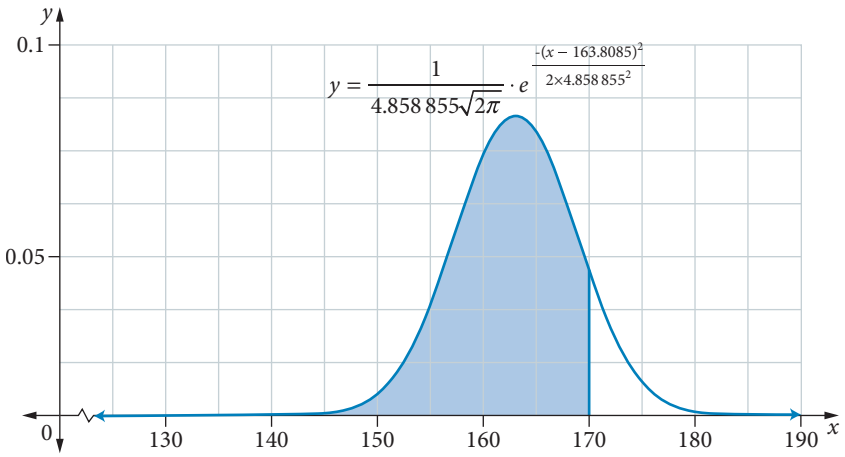
The distribution represented by the pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ over the domain $\mathbb{R}$, where μ is the mean of the distribution and σ is the standard deviation of the distribution, is known as the *normal distribution*. The special case where $\mu = 0$ and $\sigma = 1$ is called the standard normal distribution, because all other normal distributions can be obtained from it by translation and dilation. You are not required to memorise this formula. One of the functions built into your technology will be the normal distribution. To use it, you will need to know the mean μ and standard deviation σ.

Note that this distribution is defined over $\mathbb{R}$; that is, from $-\infty$ to ∞. In practical situations that can be modelled by a normal distribution, there will usually be an interval $[a, b]$ for which the value of the pdf can 'reasonably' be interpreted in that situation.

For the heights of the adult Australian women at the start of this section, $\mu = 163.82$ and $\sigma = 4.859$. To calculate the probability of the height of a randomly selected woman being less than 170 cm, you would calculate $\int_0^{170} f(x)\, dx$. The lower bound has been chosen as 0. You know it will be much higher than this. However, it does not affect your calculations, and while you cannot be sure of the minimum height, you do know that each woman's height will be a positive value.

To use the normal distribution model, your graphics calculator requires an upper and lower bound to be entered, as well as the mean and standard deviation. Here, the bounds match the limits for integration:

lower bound: 0; upper bound: 170; σ: 4.859 and μ: 163.81 gives $P(X < 170) = 0.899$ (3 d.p.)



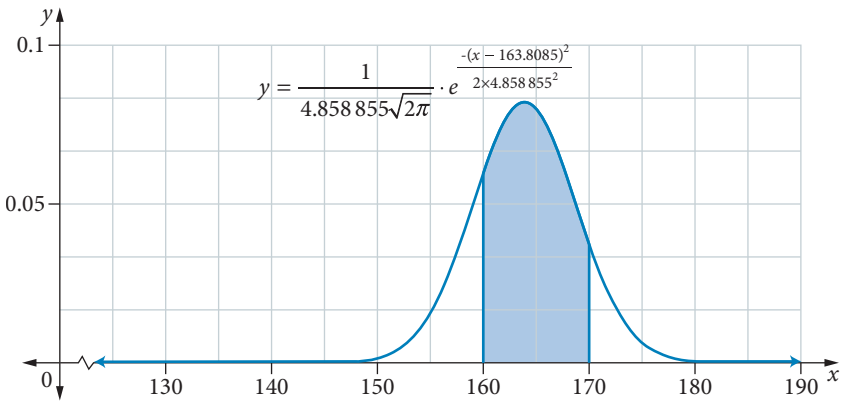
Technology worked example
Normal distribution

If you wanted to know the proportion of women who were over 170 cm tall, the lower bound would be 170 and the upper bound could be any value higher than the tallest woman could be. Alternatively, use complementary events:

$$\begin{aligned} P(X > 170) &= 1 - P(X < 170) \\ &= 1 - 0.899 \\ &= 0.101 \text{ (3 d.p.)} \end{aligned}$$

If you wanted to know the proportion of women who were between 160 and 170 cm tall, the lower and upper bounds are 160 and 170 respectively.

$P(160 < X < 170) = 0.682$ (3 d.p.)



A continuous random variable that has a normal distribution with mean μ and variance σ^2 is written as $X \sim N(\mu, \sigma^2)$.

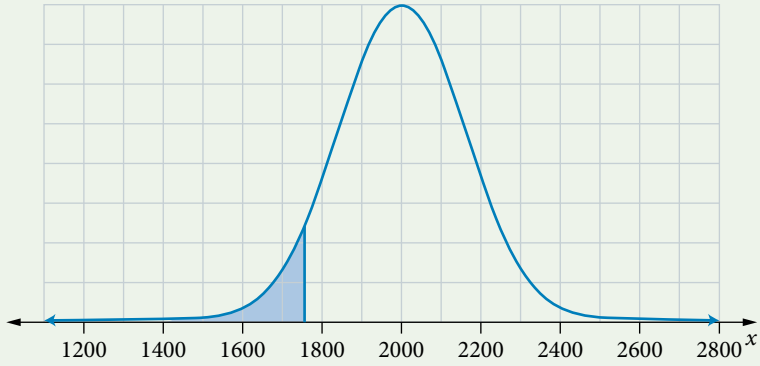
WARNING

The formal description of the normal distribution $N(\mu, \sigma^2)$ uses the variance σ^2 as a parameter. However, questions often give the standard deviation, σ, and technology uses the standard deviation and not the variance.

6 Use technology to calculate normal distribution probabilities

The annual rainfall, X mm, in the town of Aquaville is such that $X \sim N(2000, 40\,000)$. Calculate the probability for each of the following, correct to 4 decimal places.

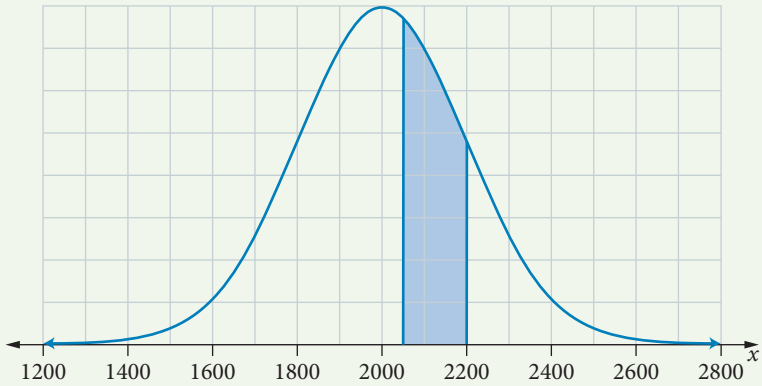
(a) The annual rainfall was less than 1750 mm.

THINKING	WORKING
1 Identify μ, σ, the lower bound and the upper bound. Negative rainfall is not possible, so you will use 0 as the lower bound in this example.	$\mu = 2000$; $\sigma = \sqrt{40\,000}$ $= 200$ lower bound = 0; upper bound = 1750
2 Use the normal distribution facility on your calculator, then round as directed.	$P(X < 1750) = 0.1056$ (4 d.p.)
3 Interpret the result.	The probability that the rainfall is less than 1750 mm is 0.1056.
4 Evaluate the reasonableness of your solution by drawing a graph.	

(b) $P(2050 < X < 2200)$

1 Identify μ, σ, the lower bound and the upper bound.	$\mu = 2000$; $\sigma = 200$ lower bound = 2050; upper bound = 2200
2 Calculate the probability of the interval and round your answer to the required level of accuracy.	$P(2050 < X < 2200) = 0.2426$
3 Interpret the result.	The probability that the rainfall is between 2050 and 2200 mm is 0.2426.

4 Evaluate the reasonableness of your solution by drawing a graph.



(c) $P(X > 2500)$

1 Identify μ, σ, the lower bound and the upper bound.

$\mu = 2000$; $\sigma = 200$

lower bound = 2500; upper bound = 10 000

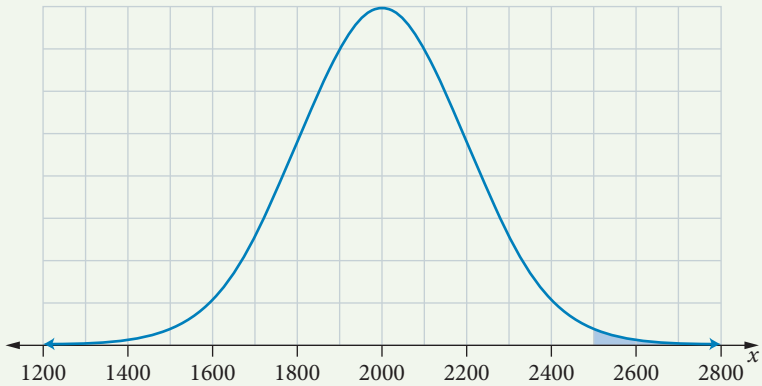
2 Calculate the probability of the interval and round your answer to the required level of accuracy.

$P(X > 2500) = 0.0062$

3 Interpret the result.

The probability that the rainfall will exceed 2500 mm is 0.0062.

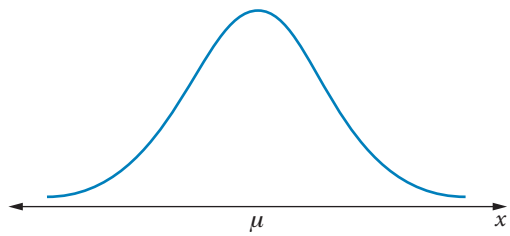
4 Evaluate the reasonableness of your solution by drawing a graph.



Features of the normal curve

The normal distribution is the most useful of all the probability distributions for continuous random variables. As you have seen, its graph is characterised by a symmetrical bell shape. This shape can be used as a model for the data collected from many naturally occurring variables, such as the height of a population, the intelligence quotient (IQ) of a population, or the distribution of errors in a production process.

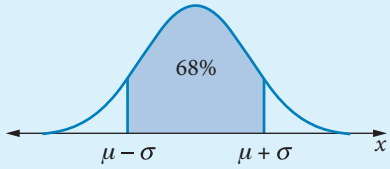
The symmetrical nature of the graph means that the mean and median coincide for the normal distribution and this value becomes the axis of symmetry. It should also be noted that, in theory, the graph extends infinitely in both the positive and negative directions, although it is always located above the x-axis. Thus, the normally distributed variable can assume any value.



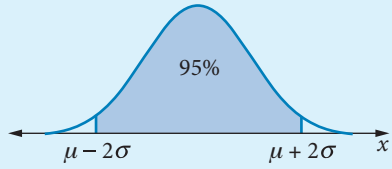
A useful feature of the normal distribution is related to the percentage of observations expected to fall within a certain number of standard deviations of the mean. The following example helps you explore this.

For normal distribution graphs:

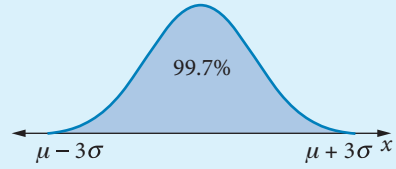
- Increasing μ shifts the graph to the right.
- Increasing σ makes the curve flatter.
- The area under the graph is always one square unit.



Approximately 68% of the observations will lie within one standard deviation of the mean.



Approximately 95% of the observations will lie within two standard deviations of the mean. It can be said that it is *very likely* that any particular observation will fall within this range.



Approximately 99.7% of the observations will lie within three standard deviations of the mean. It can be said that it is *almost certain* that a particular observation will lie within this range.

For instance, if you knew that the height of a particular population was normally distributed with a mean of 170 cm and a standard deviation of 6 cm, you would expect approximately:

- 68% in the range $\mu \pm \sigma$, i.e. 170 ± 6, which gives the range 164 to 176 cm
- 95% in the range $\mu \pm 2\sigma$, i.e. 170 ± 12, which gives the range 158 to 182 cm
- 99.7% in the range $\mu \pm 3\sigma$, i.e. 170 ± 18, which gives the range 152 to 188 cm.

7 Use the 68%, 95%, 99.7% rules

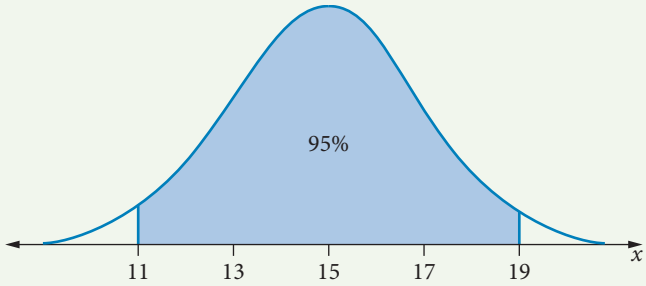
The time taken in seconds, X, for all competitors to finish a 100 m race at the school athletics carnival was found to follow a normal distribution where $X \sim N(15, 4)$.

(a) Determine the time range in which you would expect to find the middle 68% of competitors.

THINKING	WORKING
1 Identify the mean and the standard deviation.	$\mu = 15$ $\sigma^2 = 4$ $\sigma = \sqrt{4}$ $= 2$
2 The middle 68% describes the values within one standard deviation of the mean, i.e. calculate $\mu \pm \sigma$.	$\mu - \sigma = 15 - 2$ $= 13$ $\mu + \sigma = 15 + 2$ $= 17$
3 Interpret the interval.	You would expect the middle 68% of participants to complete the race in 13 to 17 seconds.

(b) Determine the percentage of students you would expect to take more than 19 seconds.

1 Identify the value in terms of the mean and the standard deviation. Locate the value on the bell-shaped curve, with accompanying percentages.



$\mu = 15$; $\sigma = 2$

Therefore, $19 = \mu + 2\sigma$

2 Use the symmetry of the curve to answer the question. In this case, you are using only one tail, so halve the percentage.

100% – 95% = 5%, so you would expect 5% outside the range $\mu \pm 2\sigma$.

5% ÷ 2 = 2.5%

You would expect 2.5% of competitors to take more than 19 seconds to complete the race.

EXERCISE 9.2 The normal distribution

Worked Example 6

1 The annual rainfall, X mm, in the town of Dry Gulch is such that $X \sim N(500, 10\,000)$.

(a) Calculate each of the following probabilities, correct to 4 decimal places.

(i) $P(X < 490\text{ mm})$ **(ii)** $P(X < 450\text{ mm})$ **(iii)** $P(X < 250\text{ mm})$

(b) Determine the probability that in a particular year the rainfall was within each of the following ranges. Give your answers correct to 4 decimal places.

(i) $P(250 < X < 400)$ **(ii)** $P(450 < X < 650)$ **(iii)** $P(250 < X < 650)$

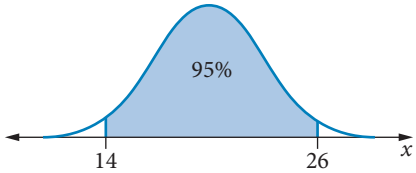
2 $X \sim N(10, 4)$

(a) Determine the range of x-values in which you would expect to find the middle 68%.

(b) Determine the range of x-values in which you would expect to find the middle 95%.

(c) Determine the range of x-values in which you would expect to find the middle 99.7%.

3 The graph represents a continuous random variable that has a normal distribution.



(a) The distribution is best represented by which of the following?

A $N(14, 26)$ **B** $N(20, 3)$

C $N(26, 12)$ **D** $N(20, 9)$

(b) Explain the common error made by a student who chose the second incorrect option.

4 Determine the following probabilities, correct to 4 decimal places.

(a) Given $X \sim N(50, 100)$, calculate $P(62 < X < 70)$.

(b) Given $X \sim N(65, 81)$, calculate $P(53 < X < 72)$.

(c) Given $X \sim N(2, 0.25)$, calculate $P(1.6 < X < 2.7)$.

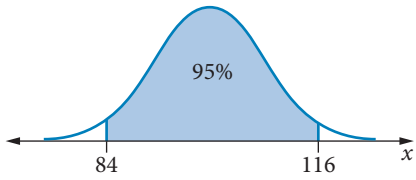
(d) Given $X \sim N(5, 0.09)$, calculate $P(4.5 < X < 5.6)$.

(e) Given $X \sim N(16, 16)$, calculate $P(17 < X < 22)$.

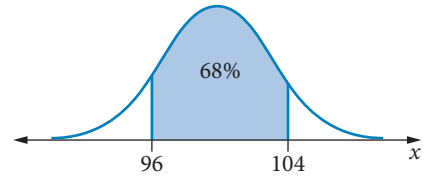
(f) Given $X \sim N(25, 25)$, calculate $P(19 < X < 24)$.

5 Given that X is normally distributed with a mean of 100 and a variance of 16, which of the following graphs correctly represents the distribution?

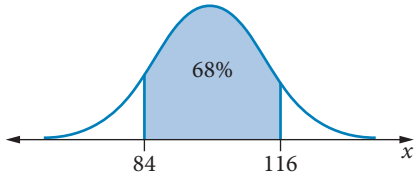
A



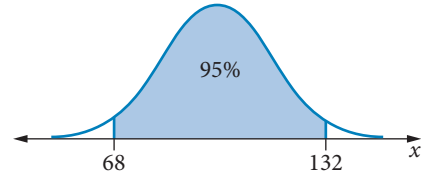
B



C



D



6 The marks, X, obtained by students in an examination were normally distributed with a mean of 85 and a standard deviation of 4. If the top 2.5% of students received a prize, what was the minimum score to receive a prize?

7 Determine the following probabilities, correct to 4 decimal places.

(a) Given $X \sim N(5, 1)$, calculate $P(X < 6.5)$.

(b) Given $X \sim N(12, 9)$, calculate $P(X < 13.2)$.

(c) Given $X \sim N(10, 4)$, calculate $P(X > 7.5)$.

(d) Given $X \sim N(4, 1)$, calculate $P(X > 3.5)$.

8 The daily maximum temperature, X°C, during December at the Solaris resort follows a normal distribution with a mean of 29°C and a variance of 12. Determine the probability that on any particular day during December, the maximum temperature meets the following conditions. Give your answers correct to 4 decimal places.

(a) The maximum temperature is above 36°C.

(b) The maximum temperature is between 25°C and 35°C.

(c) If quoted temperatures are rounded correct to the nearest degree, calculate the following probabilities assuming rounded values.

(i) The maximum temperature is above 36°C.

(ii) The maximum temperature is between 25°C and 35°C inclusive.

9 Statzone potato chips are packed by two different machines. Machine A fills the packets following a normal distribution with a mean of 100 g and a standard deviation of 3 g. Machine B fills the packets following the normal distribution $N(104, 16)$.

(a) Between what values would you expect to find the middle 68% of packets for:

(i) machine A? **(ii)** machine B?

(b) Between what values would you expect to find the middle 95% of packets for:

(i) machine A? **(ii)** machine B?

(c) Between what values would you expect to find the middle 99.7% of packets for:

(i) machine A? **(ii)** machine B?

(d) If you bought a packet of the chips, which machine would you prefer to have packed it? Explain your answer in terms of average quantity and consistency.

Worked Example 7

10 The time taken by all competitors to finish the 50 m freestyle at the school swimming carnival, X s, was found to follow a normal distribution where $X \sim N(45, 9)$. Calculate the following values.

(a) the time range in which you would expect to find the middle 95% of the results

(b) the percentage of students you would expect to take more than 48 seconds

(c) the percentage of students you would expect to take less than 36 seconds

11 The height, X cm, of a population is known to be distributed as $X \sim N(170, 81)$.

(a) What percentage of the population is expected to be found in the range 152–188 cm?

(b) What percentage of the population is expected to be taller than 197 cm?

(c) What percentage of the population is expected to be shorter than 170 cm?

(d) What percentage of the population is expected to have heights in the range 161–197 cm?

12 The mass, M g, of a batch of commemorative coins is such that $M \sim N(50, 9)$. Each coin is checked before packaging, and a coin will be rejected if its mass is less than 47 g. What percentage of coins should you expect to be rejected?

13 Packets of Greatstart breakfast cereal are labelled as having a mass of 500 g. However, the machine that fills the packets can be modelled by normal distribution with a mean of 510 g and a standard deviation of 5 g.

(a) Determine the percentage of packets that have a mass less than 500 g. Give your answer correct to 2 decimal places.

(b) Calculate $P(X < 500 | X < 510)$, correct to 4 decimal places.

(c) Use a trial and error process to determine the two whole number values between which k falls, such that $P(X < k) = 0.9$.

14 Seventy students enrolled in a university course had their height measured in metres, correct to 2 decimal places where necessary. The table below shows the heights of the 70 students.

1.5	1.57	1.63	1.65	1.68	1.69	1.7	1.7	1.7	1.7
1.7	1.7	1.7	1.7	1.7	1.7	1.72	1.72	1.73	1.73
1.73	1.73	1.73	1.73	1.75	1.75	1.75	1.75	1.75	1.75
1.76	1.76	1.76	1.76	1.77	1.78	1.78	1.78	1.78	1.78
1.78	1.78	1.78	1.78	1.78	1.78	1.8	1.8	1.8	1.8
1.8	1.8	1.8	1.82	1.83	1.83	1.83	1.83	1.83	1.84
1.85	1.85	1.85	1.85	1.85	1.85	1.85	1.88	1.93	1.95

(a) Group the results using 1.5–<1.55 … as the intervals and draw a histogram to represent the data.

(b) Comment on the shape of the histogram and whether it would be reasonable to suggest that the sample follows a normal distribution.

(c) Use technology to determine the mean and standard deviation (sample) of the original ungrouped data. Give your answers correct to 3 decimal places.

(d) Determine the percentage of results that fall within each of the following. Give your answers correct to 1 decimal place.

(i) one standard deviation of the mean

(ii) two standard deviations of the mean

(iii) three standard deviations of the mean

(e) Would it be reasonable to suggest that the sample follows a normal distribution?

15 A business analyst has weighed some packets of biscuits and calculated the difference between the actual weights of the packets and the weights stated on the labels. Packets weighing more than the label weight are recorded as positive values, and packets that weigh less than the label weight are recorded as negative values. The data is normally distributed with a mean of 0.5 g above the label weight and a standard deviation of 0.3 g.

(a) Calculate the probability that the difference between the actual weight and the label weight is not greater than 1 g. Give your answer correct to 4 decimal places.

(b) Use appropriate calculations to show that a packet of biscuits is more likely to be more than 0.4 g above the label weight than to be within 0.4 g of the label weight.

(c) Calculate the probability that the actual weight of a packet of biscuits is the same as the label weight.

16 Two golfers, Charlie Birdyman and Lester Hooksoften, play a round of golf. In golf, the lower the score the better. Charlie scores 70 and Lester scores 75. Charlie's scores C follow a normal distribution where $C \sim N(72, 4)$, and Lester's scores L are such that $L \sim N(80, 9)$.

(a) Which golfer scored better in this round compared to his normal performance? Explain your answer.

(b) What score would Lester need to have a comparative performance equal to Charlie's?

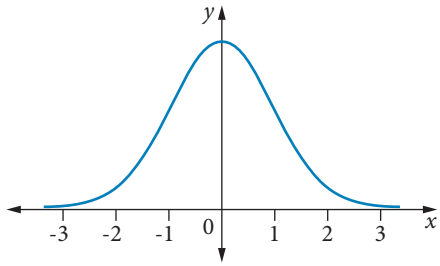
17 A certain variety of apple has mass, X g, such that $X \sim N(120, 400)$. Apples of this variety are classified as 'large' if they have a mass greater than 145 g. What percentage of 'large' apples have a mass less than 165 g? Give your answer to the nearest whole percentage.

The standard normal distribution

9.3

Transformations of normal distributions

Although it is quite easy to use technology to determine values associated with the normal distribution, it is often useful, especially when comparing distributions, to use what is called the *standard normal distribution*. This is a normal distribution that has a mean of 0 and a variance of 1, and probability density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ with domain $\mathbb{R}$. Due to its importance, a special letter Z is reserved for the random variable of the standard normal distribution.



The standard normal distribution is such that $Z \sim N(0, 1)$.

If $X \sim N(\mu, \sigma^2)$ then $z = \frac{x - \mu}{\sigma}$ and $x = \mu + \sigma z$.

The probability density function for $X \sim N(\mu, \sigma^2)$ can be obtained from that of $Z \sim N(0, 1)$ by the transformation $x = \mu + \sigma z$. Thus, to transform a normal distribution into the standard normal distribution, the transformation $z = \frac{x - \mu}{\sigma}$ is applied. That is, the mean, μ, is subtracted from the observed value of X, and the result is divided by the standard deviation, σ.

Prior to the widespread use of digital technology around the mid-1990s, tables of values were used to calculate probabilities associated with normal distributions. This was one of the reasons it was so important to be able to change the distribution to the standard normal, as it was the only distribution for which tables of values were readily available. In subjects such as this, it is now used mainly for comparisons.

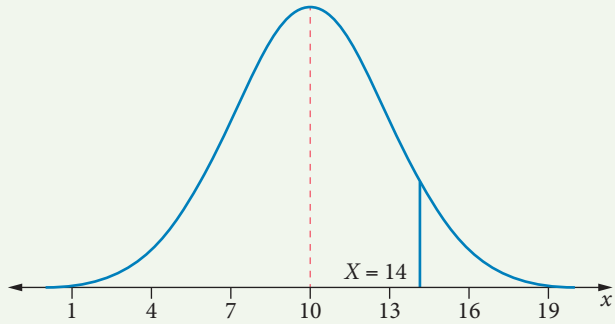
However, in the wider community, z-scores, or z-values as they are sometimes known, have some important applications. One of these is in the field of paediatric health. Children grow at different rates, so it is difficult to use the more standard statistical tests when assessing children for serious health defects or conditions. For these applications to be useful, large statistical samples are needed for children of various ages, heights and weights. Then a particular child can be compared with other children of the same age, height, weight or some combination of these factors.

For example, it may be suspected that a child has an inappropriately large dilation of a heart ventricle. The simple fact that the dilation increases over time is not necessarily a problem, because this function would be expected to increase as a child grows. However, if the z-score increases over time, this is evidence that there is indeed a problem because the ventricle is dilating at a disproportionate rate.

8 Calculate a *z*-score

Calculate the *z*-score that would be used to represent the given value of x. Give your answer correct to 2 decimal places.

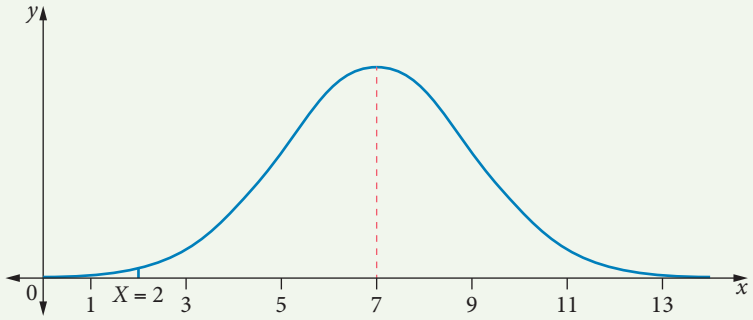
(a) $x = 14$, where $X \sim N(10, 9)$

THINKING	WORKING
1 Identify μ and σ.	$\mu = 10$ $\sigma = \sqrt{9}$ $= 3$
2 Use the formula $z = \frac{x - \mu}{\sigma}$ to convert the x-value to the equivalent z-score.	$x = 14$ $z = \frac{14 - 10}{3}$ $= \frac{4}{3}$ $= 1.33\ (2\text{ d.p.})$
3 Evaluate the reasonableness of your result by drawing a graph.	
4 Interpret the result.	The observed value of X, $x = 14$, is 1.33 standard deviations greater than the mean.

(b) $x = 2$, where $X \sim N(7, 4)$

1 Identify μ and σ.	$\mu = 7$ $\sigma = \sqrt{4}$ $= 2$
2 Use the formula $z = \frac{x - \mu}{\sigma}$ to convert the x-value to the equivalent z-score.	$x = 2$ $z = \frac{2 - 7}{2}$ $= \frac{-5}{2}$ $= -2.5$

3 Evaluate the reasonableness of your result by drawing a graph.



4 Interpret the result.

The observed value of X, $x = 2$, is 2.5 standard deviations below the mean.

9 Compare *z*-scores

Juan has been applying for scholarships. He scored 45 on a test where the results followed the distribution $X \sim N(40, 4)$. He scored 85 on another test, which followed the distribution $Y \sim N(75, 25)$. On which test did Juan do better?

THINKING

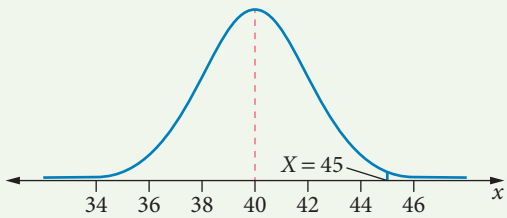
1 Calculate the z-score for the first test.

WORKING

First test: $X \sim N(40, 4)$, so $\mu = 40$; $\sigma = 2$

$$z = \frac{X - \mu}{\sigma}$$
$$= \frac{45 - 40}{2}$$
$$= 2.5$$

2 Interpret the value using a graph to evaluate the reasonableness of your statement.



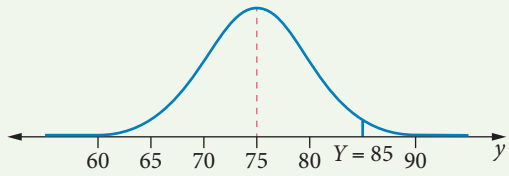
On the first test, Juan's score is 2.5 standard deviations above the mean score of 40 for the cohort.

3 Calculate the z-score for the second test.

Second test: $Y \sim N(75, 25)$, so $\mu = 75$; $\sigma = 5$

$$z = \frac{Y - \mu}{\sigma}$$
$$= \frac{85 - 75}{5}$$
$$= 2$$

4 Interpret the value using a graph to evaluate the reasonableness of your solution.



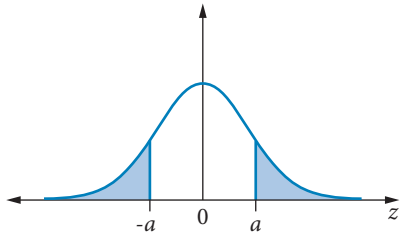
On the second test, Juan's score is two standard deviations above the mean score of 75 for the cohort.

5 Compare the z-scores to draw a conclusion.

Compared to his cohort, Juan did better on the first test. His result was 2.5 standard deviations above the mean, compared to only two standard deviations on test 2.

Symmetry of the normal distribution

The symmetry of the normal distribution helps with some calculations. This is most easily demonstrated using the standard normal curve, where $\mu = 0$ and $\sigma^2 = 1$ (and, therefore, $\sigma = 1$).



Making connections

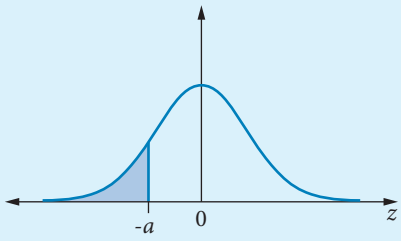
Symmetry properties of the normal distribution

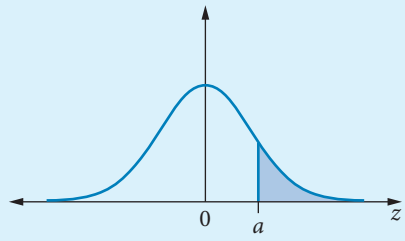
Explore the symmetry properties of the normal distribution curve.

In the diagram above, the two shaded areas are equal. For this to be the case, the z-scores must have opposite values. This result is useful as it means that $P(Z < -a) = P(Z > a)$. Also note from this diagram that $P(Z > a) = 1 - P(Z < a)$, since the total probability is 1.

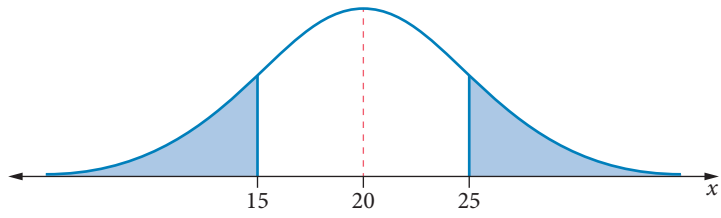
$P(Z < -a) = P(Z > a) = 1 - P(Z < a)$

A negative z-score indicates that the observed value is less than the mean.



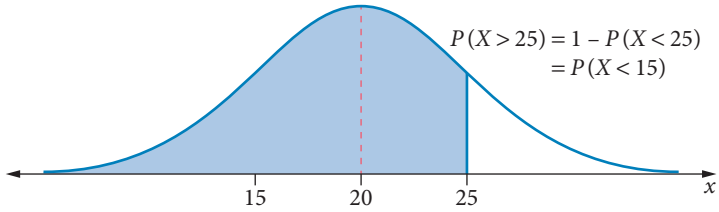


This result can be extended to any variable X that follows a normal distribution. Assume $\mu = 20$. The value 5 units to the left of the mean is 15, and the value 5 units to the right of the mean is 25.

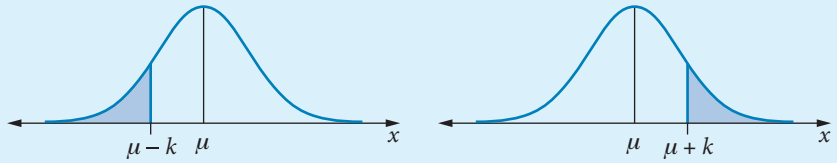


This means, in this case, that $P(X < 15) = P(X > 25)$.

This can also be expressed as $P(X < 15) = 1 - P(X < 25)$.

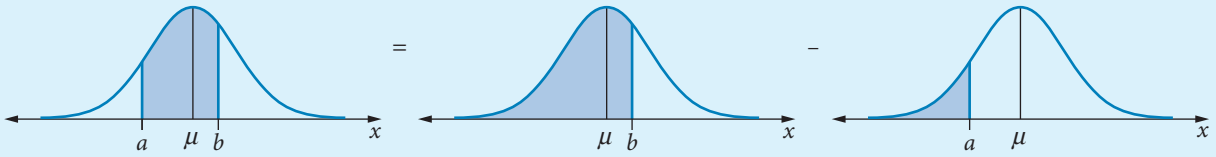


$P(X < (\mu - k)) = P(X > (\mu + k))$ and $P(X < (\mu - k)) = 1 - P(X < (\mu + k))$



Depending on the information given, you may be able to determine the probability of a given interval by subtracting areas.

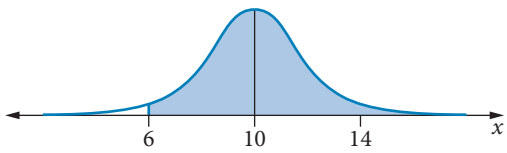
$P(a < X < b) = P(X < b) - P(X < a)$



10 Use the symmetry of the normal distribution

A normal distribution graph is shown.

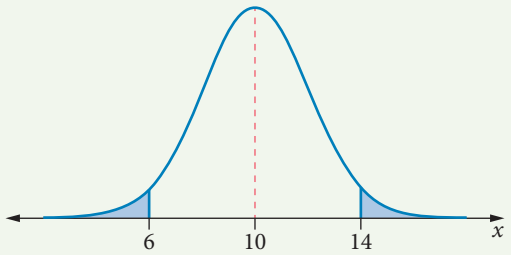
If $P(X > 14) = 0.22$, what is $P(X > 6)$?



THINKING

1 Draw a graph of the given information.

WORKING



2 Identify the relevant symmetry aspects of the diagram in terms of probabilities.

$P(X > 14) = 0.22$, so $P(X < 6) = 0.22$

3 Determine the complementary event.

$$\begin{aligned} P(X > 6) &= 1 - P(X < 6) \\ &= 1 - P(X > 14) \\ &= 1 - 0.22 \\ &= 0.78 \end{aligned}$$

EXERCISE 9.3 The standard normal distribution

Worked Example

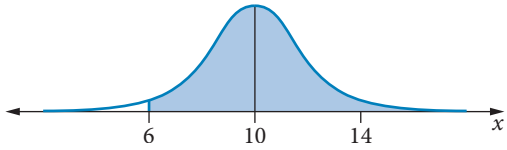
8 10

1 If $X \sim N(15, 9)$, calculate the exact z-scores corresponding to the following x-values.

(a) $x = 18$ (b) $x = 21$ (c) $x = 22$

(d) $x = 16$ (e) $x = 12$ (f) $x = 7.5$

2 A normal distribution graph is shown. If $P(X > 14) = 0.35$, calculate $P(X > 6)$.



3 For a particular normal distribution, $P(X > a) = 0.71$ and $P(X < b) = 0.44$. Calculate $P(a < X < b)$.

4 For a particular normal distribution, $P(X < a) = 0.214$ and $P(X < b) = 0.496$, where $a < b$. Determine the following probabilities.

(a) $P(X > a)$ (b) $P(X > b)$ (c) $P(a < X < b)$

5 For $Z \sim N(0, 1)$ it is known that $P(Z < 0.85) = 0.8023$. Determine the following probabilities.

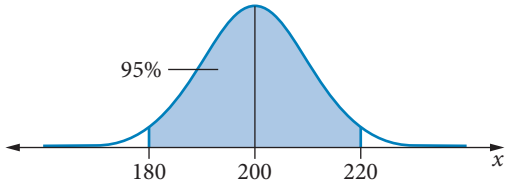
(a) $P(Z > 0.85)$ (b) $P(Z < -0.85)$

(c) $P(Z > -0.85)$ (d) $P(-0.85 < Z < 0.85)$

6 For $Z \sim N(0,1)$, it is known that $P(-1.5 < Z < 1.5) = 0.8664$. Determine the following probabilities.

(a) $P(Z < -1.5)$ (b) $P(Z > -1.5)$ (c) $P(Z < -1.5)$

7 Given the probability distribution graph for X, which follows a normal distribution, which of the following expressions best represents the shaded area?



A $P(Z < 2)$ B $P(Z < 2) + P(Z < -2)$

C $P(-2 < Z < 2)$ D $P(180 < Z < 220)$

8 X is a random variable that follows a normal distribution. The values of a and b are such that $P(a < X < b) = 0.6$, where this represents the middle 60% of values. Which of the following represents a common error for this situation?

A $P(X > b) = 0.2$ B $P(X < a) = 0.4$

C $P(X > a) = 0.8$ D $P(X < b) = 0.8$

9 Verify the following statements using $X \sim N(20,9)$, with technology. Give your answers as percentages, correct to 2 decimal places.

(a) About 68% of the results lie within one standard deviation of the mean.

(b) About 95% of the results lie within two standard deviations of the mean.

(c) About 99.7% of results lie within three standard deviations of the mean.

10 Anita's daily charges for gas consumption in her home form a normal distribution with an average daily cost of $7.65 and a variance of $1.44, where the random variable C represents the daily cost for the gas used, in dollars.

(a) What is the probability that on any day, Anita's cost is more than $4.25? Give your answer correct to 4 decimal places.

(b) Determine the number of standard deviations above or below the mean for each of the following daily costs. Give your answers correct to 2 decimal places.

(i) $8.05 **(ii)** $6.65

Worked Example 9

11 Felipe has been applying for scholarships. He got a score of 42 on a test that followed the distribution $X \sim N(40,4)$. He scored 82 on another test that followed the distribution $Y \sim N(75,25)$. On which test did Filipe have a better score?

12 Consider the probability distributions $A \sim N(56, 39.69)$ and $B \sim N(12, 2.89)$. Give your answers correct to 1 decimal place.

(a) Determine equivalent values from the distribution of A for each of the following values from the distribution of B.

(i) $b = 17.1$ **(ii)** $b = 10.5$ **(iii)** $b = 15.2$

(b) Determine equivalent values from the distribution of B for each of the following values from the distribution of A.

(i) $a = 43.4$ **(ii)** $a = 70.8$ **(iii)** $a = 51.9$

13 Use $Z = \dfrac{X - \mu}{\sigma}$ to demonstrate each of the following.

(a) $P(-1 < Z < 1) \approx 68\%$ gives $P((\mu - \sigma) < X < (\mu + \sigma)) \approx 68\%$

(b) $P(-2 < Z < 2) \approx 95\%$ gives $P((\mu - 2\sigma) < X < (\mu + 2\sigma)) \approx 95\%$

(c) $P(-3 < Z < 3) \approx 99.7\%$ gives $P((\mu - 3\sigma) < X < (\mu + 3\sigma)) \approx 99.7\%$

14 The distribution of weights for all 2-year-old girls with a height of 80.5 cm is such that the mean weight is 10.3 kg with a standard deviation of 0.8 kg. z-scores are used as a definition for some forms of malnutrition. For this height, moderate acute protein-energy malnutrition is defined as having a z-score in the range $[-3.0, -2.0]$, and severe acute protein-energy malnutrition is defined as having a z-score less than -3.0.

(a) Adah weighs 8.3 kg. Which diagnosis would be made for Adah, moderate acute protein-energy malnutrition, severe acute protein-energy malnutrition or neither?

(b) Jamilah has been diagnosed with severe acute protein-energy malnutrition. What is Jamilah's maximum possible weight, correct to 1 decimal place?

(c) Xhosa is not diagnosed with either form of acute protein-energy malnutrition. What is Xhosa's minimum weight?

15 The percentage marks obtained by a group of students in a mathematics examination are represented by a random variable M. The marks are normally distributed with a mean of 72 and a variance of 121. All marks are rounded to the nearest per cent.

(a) Calculate the probability that a student obtained a mark of at least 50% in this examination, correct to 4 decimal places. Then calculate the number of standard deviations this cut-off value is below the mean, correct to 3 decimal places.

(b) Determine the probability that a student obtained a mark of less than 45%, correct to 4 decimal places. Then, calculate the value m for which $P(M > m)$ has the same probability. Interpret the result.

(c) To obtain an A^+, a student has to be in the top 10% of the cohort of students who have undertaken this examination. Given $P(Z > 1.28) = 0.1$, calculate the minimum mark a student needs in this examination to be awarded an A^+. Explain any assumptions you had to make.

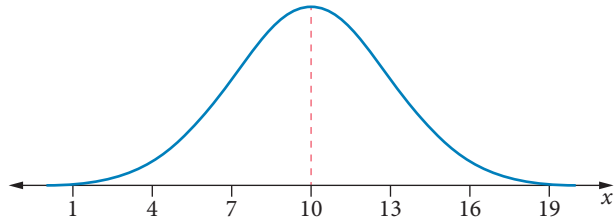
(d) The marks in the previous year's mathematics examination were normally distributed with a mean of 70 and a variance of 144. Would a student who obtained a mark of 86% have been awarded an A^+ grade? Use appropriate calculations in your explanation.

Inverse of standard normal distributions

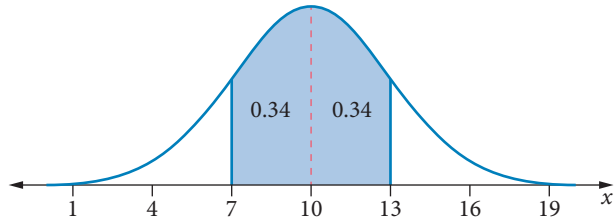
9.4

The inverse normal distribution

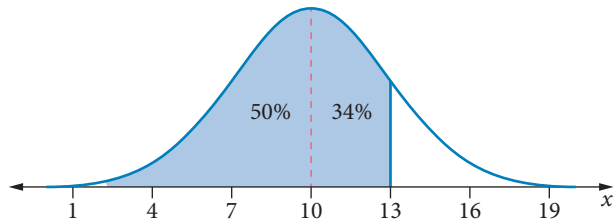
Consider the variable X, which has a normal distribution with known mean and variance $X \sim N(10, 9)$.



To calculate k such that $P(X < k) = 0.85$, the process is the reverse, or inverse, of calculating the probability. Given that 68% of data lies within one standard deviation of the mean, then $P(10 < X < 13) = 0.34$.



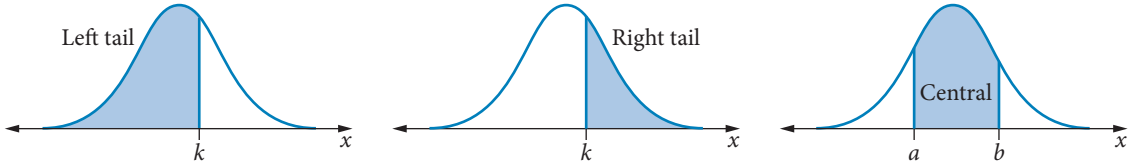
Using symmetry, you can also see that 50% of the area lies to the left of the mean (and 50% to the right). Therefore, k where $P(X < k) = 0.85$ can be approximated as $k \approx 13$ because $P(X < 13) = 0.84$.



You could estimate the value of k to be about 13 but it is impossible to get an exact result. To get accurate values, you will use what is called the *inverse normal distribution*.

Technology worked example
Inverse normal distribution

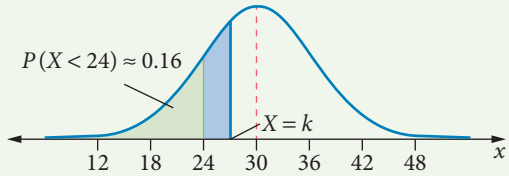
Your calculator can solve for data values for any normal distribution, given the area under the curve for the left tail, the right tail or a central portion, symmetrical about the mean. With the left- and right-tail areas, it does not matter whether the mean is included in the area; however, it should be apparent from the probability whether or not this is the case.



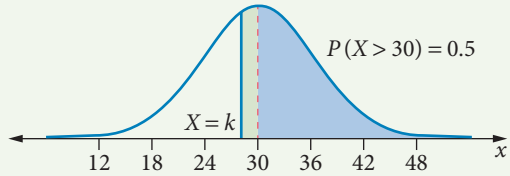
11 Use the inverse normal distribution

For the variable $X \sim N(30, 40)$, determine the unknown values, correct to 1 decimal place.

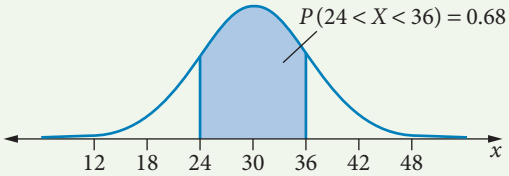
(a) $P(X < k) = 0.31$

THINKING	WORKING
1 Identify the mean, the standard deviation and the area under the curve.	$\mu = 30$; $\sigma = \sqrt{40}$; area $= 0.31$
2 Use symmetry to approximate a value of k.	Using an approximation for $\sigma = \sqrt{40} \approx 6$ and $P(X < 30) = 0.5$. $P(\mu - \sigma < X < \mu) = 0.34$ Therefore $P(X < 24) = 0.5 - 0.34$ $= 0.16$ Therefore you would expect $24 < k < 30$.
3 Evaluate the reasonableness of your estimation by drawing a diagram.	
4 Use the inverse normal function to determine the required value of x, given here as k.	$k = 26.9$ (1 d.p.) for $P(X < k) = 0.31$
5 Interpret the answer.	k is to the left of the mean, as expected. The area under the normal distribution curve to the left of 26.9 is 0.31.

(b) $P(X > k) = 0.57$

1 Identify the mean, the standard deviation and the area under the curve.	$\mu = 30$; $\sigma = \sqrt{40}$; area $= 0.57$
2 Use symmetry to approximate a value of k.	$P(X > k) = 0.57$ means $k < 30$.
3 Evaluate the reasonableness of your estimation by drawing a diagram.	
4 Use the inverse normal function to determine the required value of x, given here as k.	$k = 28.9$ (1 d.p.) for $P(X > k) = 1 - P(X < k)$ $= 0.57$
5 Interpret the answer.	k is to the left of the mean, as expected. The area under the normal distribution curve to the right of 28.9 is 0.57.

(c) $P(a < X < b) = 0.63$, where the middle 63% is required

1 Identify the mean, the standard deviation and the area under the curve.	$\mu = 30$; $\sigma = \sqrt{40}$; area = 0.63
2 Use symmetry to approximate a value of k.	Given $P(\mu - \sigma < X < \mu + \sigma) = 0.68$ and $P(a < X < b) = 0.63$, the smaller area indicates that $a \approx 24$ and $b \approx 36$.
3 Evaluate the reasonableness of your estimation by drawing a diagram.	
4 Use the inverse normal function to determine the required values of x, given here as a and b.	$a = 24.3$ (1 d.p.) $b = 35.7$ (1 d.p.)
5 Interpret the answer.	$P(24.3 < X < 35.7) = 0.63$ The area under the normal distribution curve between $a = 24.3$ and $b = 35.7$ is 0.63. a and b are equal distances from the mean ($30 - 24.3 = 35.7 - 30$), meaning the area is representative of the middle 63% of data, as required.

For Worked Example 11 part **(c)**, the values had to be given as equally spaced from the mean, otherwise the set of possible solutions would be infinite. For example, the values for this part could have been solved using: $P(a < X < b) = 0.63 \Leftrightarrow P(X < b) = 0.5 + \frac{0.63}{2}$ and then solving for a from symmetry about the mean.

Quantiles

The left tail is most commonly used in research involving statistics, with the 97th percentile referring to the value below which 97% of values lie. Hence, just the extreme 3% of values at either end of the scale lie outside of the 3rd to 97th percentile range.

The 97th percentile could be referred to as the 0.97 quantile, because both terms refer to a value below which a certain proportion of the data lies. The *percentile* expresses this value as a percentage, while the *quantile* expresses it as a decimal.

Instead of saying you wanted to determine the value of k such that $P(X < k) = 0.85$, you could ask for the 85th percentile or the 0.85 quantile.

Quartiles are the three values that divide a data set into four equal sections. The first and third quartiles are often called the lower and upper quartile respectively, while the second quartile, like the 50th, is simply the median.

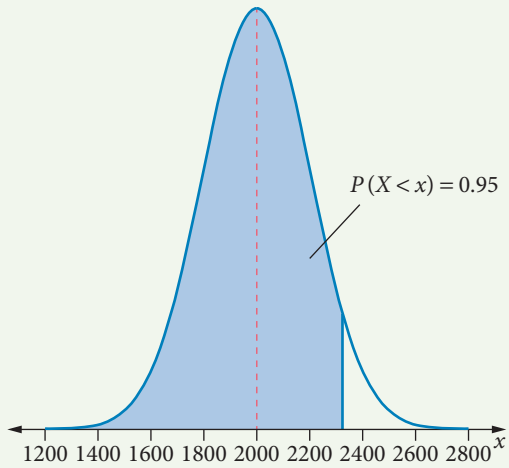
Deciles are appropriate for some data analysis, with nine values dividing the data into ten equal portions. The 6th decile is the same as the 60th percentile, both having 0.6 of the data below them.

Percentile and other *quantiles* are terms used to express the value below which a certain proportion of the distribution lies.

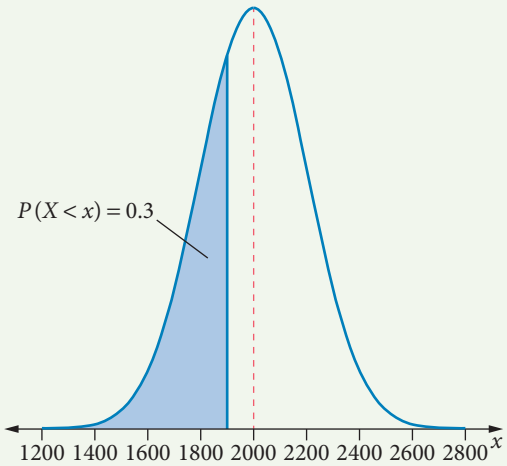
12 Evaluate percentiles and other quantiles

The annual rainfall, X mm, in the town of Aquaville is distributed normally with an average rainfall of 2000 mm and a variance of 40 000 mm. Calculate the following, correct to the nearest millimetre.

(a) the 95th percentile

THINKING	WORKING
1 Identify the given information.	$X \sim N(2000, 40\,000)$
2 Identify the mean, the standard deviation and the area under the curve. Percentiles are always left tail.	$\mu = 2000$ $\sigma = \sqrt{40\,000}$ $= 200$ area = 0.95, left tail
3 Use symmetry properties to estimate the value of x.	Using symmetry, where $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95$: $P(\mu < X < \mu + 2\sigma) = 0.5 \times 0.95$ $= 0.475$ Which gives: $P(X < \mu + 2\sigma) = 0.5 + 0.475$ $= 0.975$ The 97.5th percentile is $\mu + 2\sigma = 2000 + 2 \times 200$ $= 2000 + 400$ $= 2400$ Therefore, $P(X < x) = 0.95$, then $x < 2400$.
4 Draw a diagram to evaluate the reasonableness of your estimation.	
5 Use the inverse normal function to determine the value of x to the required accuracy and interpret the answer.	The 95th percentile is 2329 mm. 95% of the time, the annual rainfall will be less than 2329 mm.

(b) the 0.3 quantile

1 Identify the given information.	$X \sim N(2000, 40\,000)$
2 Identify the mean, the standard deviation and the area under the curve. Quantiles are always left tail.	$\mu = 2000$ $\sigma = 200$ area = 0.3, left tail
3 Use symmetry properties to estimate the value of x.	Using symmetry, where $P(\mu - \sigma < X < \mu + 2\sigma) = 0.68$: $P(X < \mu - \sigma) = 0.5 - 0.5 \times 0.68$ $= 0.5 - 0.34$ $= 0.16$ $\therefore P(X < 1800) = 0.16$ Given that $P(X < \mu) = 0.50$, then $P(X < 2000) = 0.5$, and $1800 < x < 2000$.
4 Draw a diagram to evaluate the reasonableness of your estimation.	
5 Use the inverse normal function to determine the value of x to the required accuracy and interpret the answer.	The 0.3 quantile is 1895 mm. 30% of the time, the annual rainfall will be less than 1895 mm.

Solving for unknown parameters using *z*-scores

It is possible to determine the value of the mean or standard deviation if you know the tail or central probability portion for a particular datum, given the other parameter. Using a graphics calculator, you can determine the z-score corresponding to the area under the curve.

The formula $z = \dfrac{x - \mu}{\sigma}$ or its equivalent, $x = \mu + \sigma z$ can be used to solve for the unknown.

13 Evaluate the mean using the inverse normal distribution

The height, H cm, of a group of people is distributed normally, with a variance of 25 cm. In addition, it is known that $P(H < 180) = 0.99865$. Calculate the mean.

THINKING	WORKING
1 Identify the mean and standard deviation for H in the form $H \sim N(\mu, \sigma)$.	$H \sim N(\mu, 25)$
2 Use the inverse normal function to determine the z-score from the known area.	For the standard normal distribution: $\mu = 0$; $\sigma = 1$ $P(Z < z) = 0.99865$, left tail Gives: $z = 3$
3 Interpret the value.	$P(H < 180) = 0.99865$ represents the area $P(X < \mu + 3\sigma) = 0.99865$.
4 Determine the standard deviation for the given distribution.	$\sigma = \sqrt{25}$ $= 5$
5 Recall the formula $x = \mu + \sigma z$, substitute the values and solve for μ.	For $h = 180$, $\sigma = 5$ and $z = 3$: $h = \mu + \sigma z$ $180 = \mu + 5 \times 3$ $\mu = 180 - 15$ $= 165$
6 Interpret the result.	The mean height of the group of people is 165 cm.

14 Evaluate the standard deviation using the inverse normal distribution

The height, H cm, of a group of people is distributed normally, with a mean height of 175 cm. In addition, it is known that $P(H < 190) = 0.99$. Calculate the standard deviation, correct to 2 decimal places.

THINKING	WORKING
1 Identify the mean and standard deviation for H in the form $H \sim N(\mu, \sigma)$.	$H \sim N(175, \sigma)$
2 Use the inverse normal function to determine the z-score from the known area.	For the standard normal distribution: $\mu = 0$; $\sigma = 1$; area = 0.99, left tail Gives: $z = 2.3263\ldots$
3 Interpret the value.	$P(H < 190) = 0.99$ represents the area $P(H < \mu + 2.3263\ldots \times \sigma) = 0.99$.
4 Determine the mean for the given distribution.	$\mu = 175$

5 Recall the formula $x = \mu + \sigma z$, substitute the values and solve for σ.	For $h = 190$, $\mu = 175$ and $z = 2.3263\ldots$ $h = \mu + \sigma z$ $190 = 175 + \sigma \times 2.3263\ldots$ $\sigma = \dfrac{190 - 175}{2.3263\ldots}$ $\sigma = 6.45\,(2\text{ d.p.})$
6 Interpret the result.	The standard deviation of the heights is 6.45 cm.

EXERCISE 9.4 Inverse of standard normal distributions

Worked Example 11

1 The heights, X m, of a group of people are known to be normally distributed with $X \sim N(1.7, 0.0225)$. For the following questions, give your answers correct to 2 decimal places and include the required units.

(a) Determine the height under which you would expect to find 97.5% of the population.

(b) Calculate k such that $P(X > k) = 0.7$.

(c) Determine the end values for the middle 0.35 of the heights.

Worked Example 12

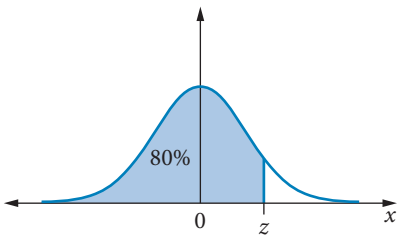
2 The weights, X kg, of a group of people are known to be normally distributed with $X \sim N(60, 90.25)$. Calculate the following, correct to 2 decimal places.

(a) the 4th decile

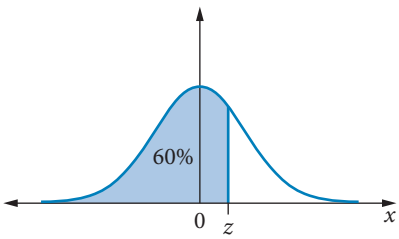
(b) the 85th percentile

3 For each of the following diagrams, consider $Z \sim N(0, 1)$. Calculate z, correct to 3 decimal places.

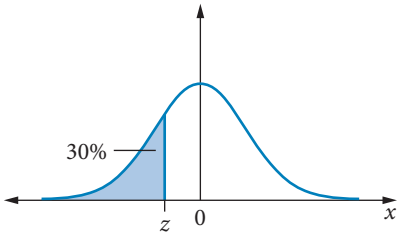
(a)



(b)



(c)



4 For the standard normal distribution, calculate z for each of the following. Give your answers correct to 3 decimal places.

(a) the 4th decile

(b) the 82nd percentile

(c) the 0.68 quantile

(d) 7th percentile

5 X is normally distributed with $\mu = 50$, $\sigma = 4$ and $P(X < a) = 0.95$.

(a) Which of the following values is closest to a?

A 56.58 **B** 1.6449 **C** 57.84 **D** 43.42

(b) Which option incorrectly uses the right-tail area as 0.95?

(c) Which option incorrectly calculates a as the upper bound of the central area of 0.95?

(d) What common error has been made by a student who chose the incorrect option not featured in part **(b)** or part **(c)**?

6 Using $Z \sim N(0, 1)$, calculate the value of k, correct to 2 decimal places.

(a) $P(-k < Z < k) = 0.85$ **(b)** $P(Z < -k \cup Z > k) = 0.05$

7 The scores X on an aptitude test are such that $X \sim N(75, 35)$.

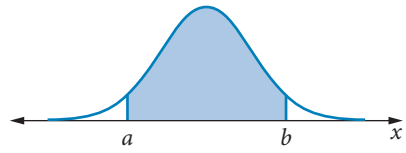
(a) Determine the end values of the interquartile range of the scores, the middle half, to the nearest whole numbers.

(b) Calculate the first decile of X, correct to 1 decimal place. Interpret the result.

(c) Calculate the 80th percentile of X, correct to 2 decimal places. Interpret the result.

8 The diagram on the right represents the distribution of X, where $X \sim N(25, 8)$. The shaded area represents the middle 80% of observations.

Which of the following best represent the values of a and b?



A $a = 22.89$, $b = 28.62$ **B** $a = 14.75$, $b = 35.25$

C $a = 21.38$, $b = 28.62$ **D** $a = 23.72$, $b = 26.68$

9 The scores X in a national mathematics competition are such that $X \sim N(80, 20)$.

(a) Determine the integer score needed to earn a High Distinction certificate, if these are awarded to the top 3% of students.

(b) Distinction certificates are awarded to students who score at or above the 87th percentile. Calculate the minimum integer score needed to earn a Distinction certificate.

(c) What is the minimum integer score needed to obtain a Credit certificate, if the cut-off is the 0.65 quantile?

10 The mass of honey, X kg, taken from a hive follows a normal distribution with a mean of 2.1 kg and a standard deviation of $\sqrt{0.6}$ kg. For each of the following, calculate the value of k, correct to 3 decimal places.

(a) $P(X < k) = 0.92$ **(b)** $P(X > k) = 0.62$ **(c)** $P(X < k) = 0.45$ **(d)** $P(X > k) = 0.16$

11 For $Z \sim N(0, 1)$, calculate t, correct to 4 decimal places.

(a) $P(0 < Z < t) = 0.1234$ **(b)** $P(Z < t) = 0.6789$ **(c)** $P(Z > t) = 0.2357$

12 For $Z \sim N(0, 1)$, evaluate the following.

(a) $P(Z < t)$, given $P(0 < Z < t) = 0.1234$ **(b)** $P(0 < Z < t)$, given $P(Z < t) = 0.6789$

(c) $P(0 < Z < t)$, given $P(Z > t) = 0.2357$

Worked Example 13

13 Determine the mean for each given scenario. Give each answer as a whole number.

(a) The heights, H cm, of a group of people are such that $H \sim N(\mu, 25)$. In addition, $P(H < 160) = 0.158\,655$.

(b) The heights, H cm, of a group of people are such that $H \sim N(\mu, 36)$. In addition, $P(H < 180) = 0.952\,21$.

(c) The arm lengths, L cm, of a group of people are such that $L \sim N(\mu, 16)$. In addition, $P(L < 75) = 0.105\,65$.

14 Determine the whole number standard deviation for each given scenario. 14

(a) The heights, H cm, of a group of students are such that $H \sim N\left(165, \sigma^2\right)$. In addition, $P(H < 160) = 0.158\,655$.

(b) The heights, H cm, of a group of students are such that $H \sim N\left(160, \sigma^2\right)$. In addition, $P(H < 165) = 0.797\,672$.

(c) The arm lengths, L cm, of a population of students are such that $L \sim N\left(80, \sigma^2\right)$. In addition, $P(L < 85) = 0.952\,21$.

15 The monthly rainfall at the Solaris resort follows a normal distribution with a standard deviation of 12 mm. If the rainfall is less than 25 mm in 15% of all months, determine the mean monthly rainfall. Give your answer correct to 1 decimal place.

16 For $Z \sim N(0, 1)$, $P(t < Z < 2.1) = 0.4680$. Calculate t, correct to 4 decimal places.

17 A driving simulator was designed to check the reaction time of drivers. The random variable R, representing teenagers' reaction times in milliseconds, is normally distributed with a mean of 250 and a variance of 2500.

(a) Calculate the probability that a teenager had a reaction time of between 320 and 350 milliseconds. Give your answer correct to 4 decimal places.

(b) What is the reaction time k such that the probability that a teenager has a reaction time greater than k is 0.0580? Give your answer correct to the nearest millisecond.

(c) The same test was used to check the reaction time of drivers more than 20 years old. The random variable R, representing drivers' reaction times in milliseconds, is also normally distributed, with a mean of 280 and a variance of σ^2. Calculate the standard deviation for this set of data, if the probability that any one driver more than 20 years old has a reaction time greater than 300 milliseconds is 0.3580. Give your answer correct to the nearest millisecond.

18 A school recorded the 400-metre running times for a group of students. These times, in seconds, are represented by the random variable X, which is normally distributed with a variance of 31.36. If $P(X < 57) = 0.2961$, calculate the following, correct to 1 decimal place.

(a) the times that will place a student in the top 15% of runners

(b) the interquartile range

19 The probability distributions for two variables are defined as follows: $X \sim N\left(120, \sigma^2\right)$ and $Y \sim N\left(\mu, 2\sigma^2\right)$ and $P(X < 124) = P(Y > 124)$.

(a) Calculate μ, correct to 1 decimal place.

(b) If 124 is the 9th decile in the distribution of X, determine the standard deviation for both distributions, correct to 2 decimal places. What is the closest percentile for this value in the distribution of Y?

9

Summary

Continuous random variables

A continuous random variable can assume any value in the particular interval over which it is defined.

$P(X = x)$ is zero for all values of x.

$P(a < X < b) = \int_a^b f(x)dx$

For a function $f(x)$ to be a probability density function:

- $f(x) \geq 0$ for all values of x and $\int_{-\infty}^{\infty} f(x)dx = 1$
- The median m is such that $P(X < m) = 0.5$.
- The mean is $\mu = \int_{-\infty}^{\infty} xf(x)dx$.
- The variance is $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$.

Normal distribution

About 68% of results lie within one standard deviation of the mean.

About 95% of results lie within two standard deviations of the mean.

About 99.7% of results lie within three standard deviations of the mean.

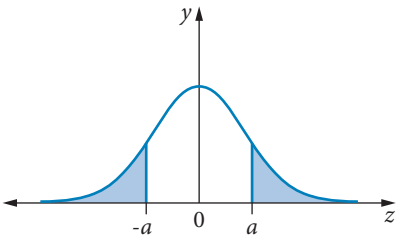
A variable X that follows a normal distribution can be written as: $X \sim N(\mu, \sigma^2)$, where μ is the mean and σ^2 is the variance.

Standard normal distribution

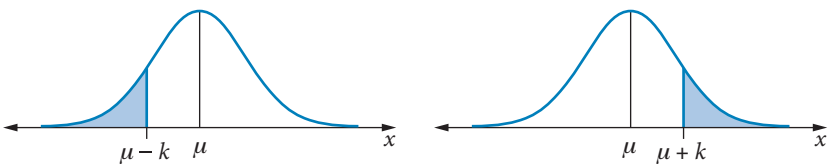
A variable Z that has a standard normal distribution is such that $Z \sim N(0, 1)$.

If $X \sim N(\mu, \sigma^2)$, $z = \dfrac{x - \mu}{\sigma}$ and $x = \mu + z\sigma$

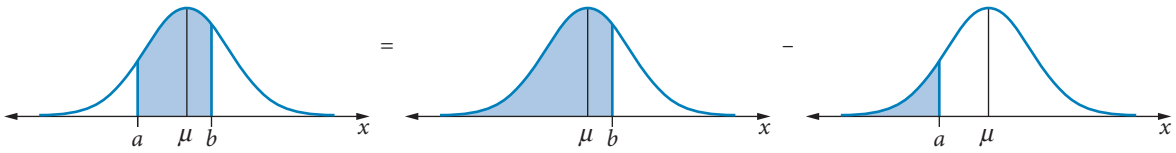
Symmetry and areas



$P(Z < -a) = P(Z > a)$ and $P(Z > a) = 1 - P(Z < a)$



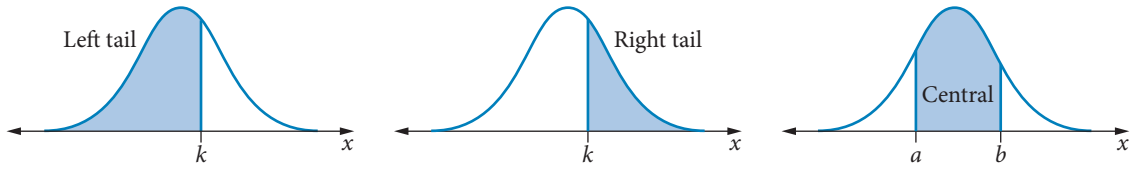
$P(X < \mu - k) = P(X > \mu + k)$ and $P(X < \mu - k) = 1 - P(X < \mu + k)$



$P(a < X < b) = P(X < b) - P(X < a)$

The inverse normal function

A graphics calculator will give data values corresponding to upper bounds for left-tail areas, lower bounds for right-tail areas or both bounds for central areas.



Quantiles

Percentiles, quartiles, deciles and other quantiles express the value below which a certain proportion of the distribution lies. Quantiles correspond to left tail areas.

Chapter review 9

Exercise 9.1

1 Determine the following values for the uniform continuous variable with probability density function $f(x) = \begin{cases} k & 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$.

(a) k **(b)** $P(X < 3)$ **(c)** $P(X < 5)$ **(d)** $P(2 < X < 9)$

Exercise 9.1

2 Calculate the exact values of following probabilities for the continuous random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{40(x-20)}{x^3} & x \ge 20 \\ 0 & \text{otherwise} \end{cases}$$

(a) $P(X < 25)$ **(b)** $P(X > 30)$

Exercise 9.1

3 Use technology to calculate the following probabilities, correct to 4 decimal places, for a particular continuous random variable that has the following probability density function:

$$f(x) = \begin{cases} \dfrac{1}{24}x - \dfrac{1}{24} & 1 \le x \le 7 \\ \dfrac{9}{8} - \dfrac{1}{8}x & 7 < x \le 9 \\ 0 & \text{otherwise} \end{cases}$$

(a) $P(X < 5.5)$ **(b)** $P(X > 7.6)$ **(c)** $P(X > 3.2)$

Exercise 9.1

4 Explain whether or not the following piece-wise function represents a probability density function.

$$f(x) = \begin{cases} x^2 + 4x - \dfrac{10}{3} & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

5 If $f(x) = \begin{cases} \dfrac{2x}{15+k} & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$ defines a probability density function, what is the value of k?

Exercise 9.2

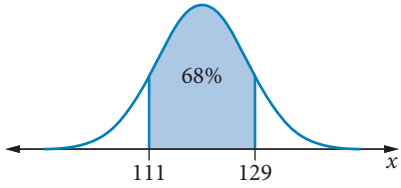
6 Honey is packed in jars with a labelled mass of 500 g. The actual amount X g of honey in each jar is such that $X \sim N(500, 25)$.

(a) Determine the probability that a jar chosen at random contains less than 495 g. Give your answer correct to 2 decimal places.

(b) Suppose jars containing less than 490 g cannot be sold. In a batch of 1000, how many jars would you expect to be rejected?

7 Describe the distribution shown in the graph in the form $N\left(\mu, \sigma^2\right)$

Exercise 9.2



8 The graph on the right shows two normal distributions drawn on the same set of axes. Graph 1 has a marker on it.

Compared to graph 1, graph 2 has which of the following?

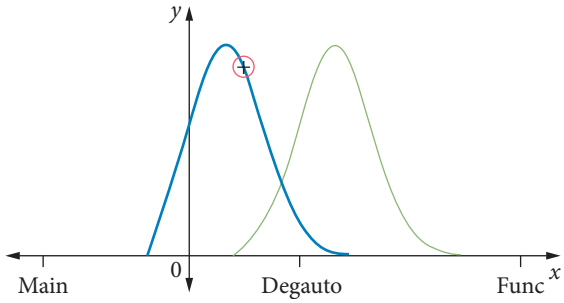
A the same variation but a smaller mean

B the same variation but a larger mean

C the same mean but a smaller variation

D the same mean but a larger variation

Exercise 9.2



9 The graph shows two normal distributions drawn on the same set of axes. Graph 1 has a marker on it.

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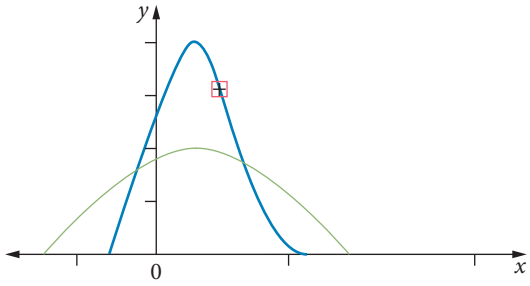
Exercise 9.2



10 For $X \sim N(50, 16)$, calculate each probability, correct to 4 decimal places.

Exercise 9.2

(a) $P(X > 48)$ **(b)** $P(X < 60)$ **(c)** $P(43 < X < 61)$

11 Calculate z for each x-value in the following distributions.

Exercise 9.3

(a) $X \sim N(12, 16)$

(i) $x = 8$ **(ii)** $x = 10$ **(iii)** $x = 15$ **(iv)** $x = 20$

(b) $X \sim N(28, 16)$

(i) $x = 24$ **(ii)** $x = 35$ **(iii)** $x = 17$

12 For a particular normal distribution, $P(X < a) = 0.322$ and $P(X < b) = 0.515$. Calculate the following probabilities.

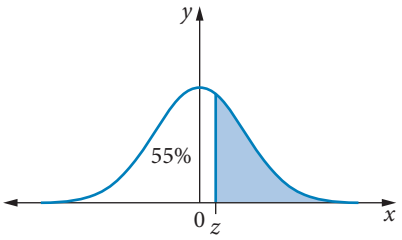
Exercise 9.3

(a) $P(X > a)$ **(b)** $P(X > b)$ **(c)** $P(a < X < b)$

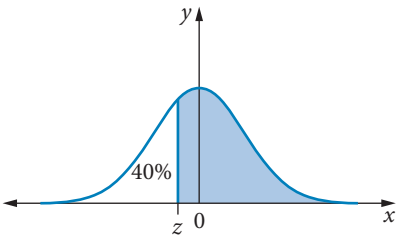
Exercise 9.4

13 For each of the following diagrams showing the standard normal probability distribution, determine the value of z. Give your answers correct to 3 decimal places.

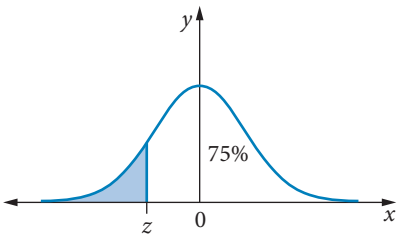
(a)



(b)



(c)



Exercise 9.4

14 For $X \sim N(30, 12)$ determine the following values, correct to 2 decimal places.

(a) the values between which you would expect to find the middle 60% of observations

(b) the 95th percentile

Exercise 9.4

15 For $X \sim N(120, 40)$, calculate k such that the following probability statements are correct. Give your answers correct to 2 decimal places.

(a) $P(X > k) = 0.15$ **(b)** $P(X < k) = 0.25$

Exercise 9.4

16 Consider the marks X in an examination were such that $X \sim N(85, 49)$.

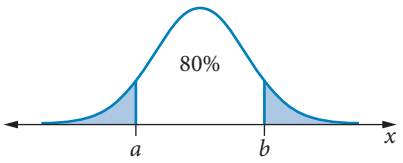
(a) The bottom 15% of students failed the examination. Determine the minimum integer mark required to pass.

(b) Determine the percentage of students who obtained full marks, given that the test was out of 100. Give your answer correct to 2 decimal places.

(c) Students with a score within 1.5 standard deviations above the mean received a credit. Determine the end values of the range of whole number scores that this represents.

Exercise 9.4

17 The graph below represents $X \sim N(40, 13)$. Determine the values for a and b, correct to 2 decimal places.



18 For the continuous variable with probability density function shown, determine the exact value of the median. Exercise 9.1

$$f(x) = \begin{cases} \dfrac{x^2}{12} & 0 \le x \le \sqrt[3]{36} \\ 0 & \text{otherwise} \end{cases}$$

19 An arrow is fired at a target. The distance, X m, from the centre of the target that the arrow pierces follows the function $f(x) = \begin{cases} \dfrac{3}{\pi\left(1+x^2\right)} & 0 \le x \le \sqrt{3} \\ 0 & \text{otherwise} \end{cases}$. Exercise 9.1

(a) Explain why it is reasonable to say that this represents a probability density function.

(b) Use technology to calculate the following, correct to 4 decimal places.

(i) $P(X < 1)$ **(ii)** $P(X > 1.25)$ **(iii)** $P(0.25 < X < 1.1)$

(c) Calculate the following parameters, correct to 4 decimal places.

(i) $E(X)$ **(ii)** $Var(X)$

20 For a mathematics assignment, Xandra and Zack have decided to record the amount of time they spend completing their daily homework. Xandra's times are represented by the continuous random variable X, in hours, with the probability density function: Exercise 9.1

$$f(x) = \begin{cases} -\dfrac{1}{9}\left(x^2 - 4x\right) & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Zack's times are represented by the continuous random variable Z, in hours, with the probability density function:

$$f(z) = \begin{cases} \dfrac{2}{57}\left(z^2 - 9z + 20\right) & 0 \le z \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the mean time each student spends completing homework on a given day and compare the results.

(b) Xandra has a school concert to attend and will try to complete her homework within 1.5 hours. Calculate the probability that Xandra will complete her homework in the required time.

21 Given that $X \sim N(10, 9)$, in what range would you expect to find the middle 95% of the observations? Exercise 9.2

A 9.5 to 10.5 **B** 0 to 28 **C** 10 to 19 **D** 4 to 16

Exercise 9.2

22 **(a)** In Australia, approximately 49% of the population has type-O blood. A random sample of 100 people in Australia were chosen for a survey. Give all probabilities correct to 4 decimal places.

(i) Calculate the probability that between 39 and 59 (both inclusive) people of those surveyed will have type-O blood, using a binomial distribution formula.

(ii) Calculate the probability that between 39 and 59 (both inclusive) people will have type-O blood for this sample, using a normal approximation to the binomial distribution. Compare this value to the value calculated in part **(i)**.

(b) In Japan, an average of 37% of the population have type-O blood. A random sample of 100 people in Japan were chosen for a survey. Give all probabilities correct to 4 decimal places.

(i) Calculate the probability that between 39 and 59 people of those surveyed in Japan will have type-O blood, using a binomial distribution formula.

(ii) Calculate the probability that between 39 and 59 people of those surveyed in Japan will have type-O blood, using a normal approximation to the binomial distribution. Compare this value to the value calculated in part **(i)**.

Exercise 9.3

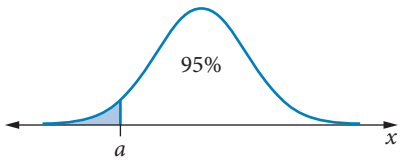
23 Consider the probability distributions $A \sim N(230, 38.44)$ and $B \sim N(16.5, 7.84)$. Give answers correct to 1 decimal place.

(a) Determine the equivalent value from the distribution of A when $b = 12.9$.

(b) Determine the equivalent value from the distribution of B when $a = 235.4$.

Exercise 9.4

24 The diagram represents $X \sim N\left(\mu, \sigma^2\right)$. Which expression is approximately equivalent to a?



A $\mu - 1.645\sigma^2$

B $\mu + 1.645\sigma$

C $\mu - 2\sigma$

D $\mu - 1.645\sigma$

Exercise 9.4

25 When fishing, fish of less than a given length have to be thrown back into the water. Red snappers have a minimum allowed length of 30 cm, and barramundi have a minimum allowed length of 55 cm. Let the variable R represent the lengths of red snappers caught, in centimetres, and let the variable B represent the lengths of barramundi caught, in centimetres. Observations of red snapper that have been caught show that R is normally distributed with mean 36 cm and a variance of 9 cm. Observations of barramundi that have been caught show that B is normally distributed with a variance of 16 cm.

(a) Calculate the mean length of barramundi caught if 2.5% of the barramundi caught are less than 54 cm long. Give your answer correct to the nearest whole number.

(b) Sketch the two probability density functions for the random variables R and B on the same set of axes.

(c) Calculate the z-scores for the minimum allowed lengths for both variables R and B, and interpret this in terms of which of the two kinds of fish is more likely to be too small to keep.

(d) Calculate the following probabilities, correct to 2 significant figures.

(i) A red snapper that is caught will be longer than 47 cm.

(ii) A barramundi that is caught will be less than 58 cm long.

26 The time H, in minutes, it takes Helena to complete one type of exercise at the local gym follows the probability density function: Exercise 9.2

$$f(h)=\begin{cases} -ah+\frac{3}{5} & 0\le h\le 2 \\ \frac{2}{5}a & 2<h\le 5, \text{ where } a \text{ is a real constant.} \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the value of a and substitute this value into $f(h)$.

(b) The time A, in minutes, it takes Ami to complete the same exercise as Helena is normally distributed with mean 2.5 and variance 0.36.

(i) Determine the probability that both Helena and Ami will finish the exercise in between 2 and 5 minutes. Give your answer correct to 2 decimal places.

(ii) Calculate Helena's mean time to finish the exercise, correct to 2 decimal places.

(iii) Determine Helena's median time, correct to 2 decimal places. Explain the meaning of this result.

(iv) Determine the probability that Ami will complete the exercise in less than Helena's median time. Give your answer correct to 4 decimal places.

27 The mass, X g, of a packet of potato chips is such that $X \sim N(150, 40)$. Packets with mass less than 143 g are rejected as too small. Packets with a mass of more than 162 g are rejected as too large. Exercise 9.2

(a) Determine the probability of rejection, correct to 4 decimal places.

(b) In a batch of 500 packets, what is the probability of fewer than 66 packets being rejected? Give your answer correct to 3 decimal places.

(c) Between which masses would you expect to find the middle 90% of production? Give your answer correct to 1 decimal place.

28 The local council is trying to decide what to do about the collection of recyclable materials. Each household has a bin that is collected on a fortnightly basis. Each truck that returns to the council depot will contain an amount X of materials that are not recyclable, where X is measured in units of 100 kg. It is known that the random variable X has the following probability density function: Exercise 9.1

$$f(x)=\begin{cases} kx(4-x)^2 & 0\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the value of k, as a fraction.

(b) Determine the mean and standard deviation of X, in kilograms.

(c) Determine the probability that a truck chosen at random contains at least 3 units of non-recyclable material. Write your answer as a fraction.

(d) The council has two choices as to how to proceed with the collection of recyclable material. The first is to continue to sell truckloads of unsorted material. If a truck contains less than 3 units of non-recyclable material, they will be able to sell the materials for $500, but if it contains more than that, they will only receive $300. Determine the mean value of a truck full of unsorted material to the nearest cent.

10

Interval estimates for proportions

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Summary 548
Chapter review 551

10

Recall

Distinguish between continuous and discrete scales

1 Determine if the following variables are associated with continuous or discrete scales.

(a) the number of phone calls received in one day

(b) the temperature of a pool at any one time

(c) the number of lollies eaten at a party

(d) the amount of rain that falls in one year

Describe the key features of a discrete probability distribution

2 Answer the following questions in relation to probability.

(a) In what interval must a probability lie?

(b) What value must the sum of all probabilities of a certain event take?

(c) Recall the formulas for calculating the mean, variance and standard deviation for a discrete probability distribution.

Calculate the expected value, variance and standard deviation of a discrete probability distribution

3 For the following discrete probability distributions, calculate the expected value, correct to 1 decimal place. Then calculate the variance and standard deviation, each correct to 2 decimal places.

(a)

x	5	3	2	1
$P(X = x)$	0.02	0.51	0.17	0.3

(b)

x	2	4	8	16	32
$P(X = x)$	0.25	0.3	0.2	0.15	0.1

(c)

x	3	6	9	12	15
$P(X = x)$	0.01	0.35	0.4	0.2	0.04

Identify independent events

4 Determine whether the following events are independent or dependent.

(a) the outcome of two rolls of a fair die

(b) from 8 employees, one is chosen as the team leader and then from the remaining employees, an assistant is chosen

(c) drawing a 10 from a deck of cards, replacing it and then drawing a king

Calculate values associated with a binomial distribution

5 Calculate the expected value, the variance and the standard deviation of the following binomial distributions. Give your answers correct to 2 decimal places, where applicable.

(a) $X \sim B(10, 0.2)$ (b) $X \sim B(100, 0.65)$ (c) $X \sim B(50, 0.5)$

Calculate the number of ways selections can be made

6 Calculate the number of ways the following selections can be made.

(a) ${}^{10}C_4$ (b) 10 items selected from 15 (c) 3 items selected from 30

Identify the common features of the normal distribution and apply them

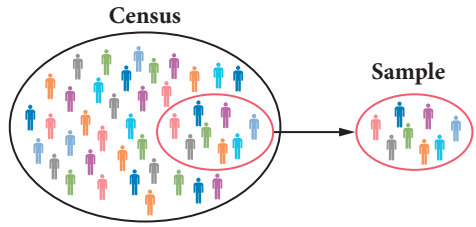
7 Identify the following features of the normal distribution.

(a) What percentage is captured by ± 1, ± 2 and ± 3 standard deviation from the mean?

(b) The average IQ is 100 with a standard deviation of 15, and is normally distributed. What range of scores would fall into one, two and three standard deviations from the mean?

10.1 Random sampling

You make sense of the world around you by studying particular features or characteristics that occur within populations. To study a whole population, whether it be comprised of people or objects, can be impractical, costly and time-consuming; sometimes it isn't possible to study an entire population at all.



This issue can be overcome by obtaining a sample and inferring characteristics about the population based on this sample. In other words, you study the 'part' in order to predict information about the 'whole'. This is the basis of statistical inference.

A *census* is a statistical technique whereby information is gathered from an entire population.

A *sample* is a statistical technique whereby information is gathered from part of a population. An inference about the whole population can then be drawn from the sample data.

Additional information

Census and survey

This video shows examples of when a survey or a census might be more appropriate.

In your previous studies, you would have analysed sets of data, summarised them and then presented the information. This is known as *descriptive statistics*. In this chapter, you will be using *inferential statistics*. While descriptive statistics summarise and present information, inferential statistics use the statistics from a random sample to draw conclusions about the population. Recall that a sample is considered to be random if every unit of the population has an equally likely chance of being selected as part of the sample.

Statistical inference, using the laws of probability, is the process of drawing conclusions about the whole population based on statistics gathered from a *random* sample of people or objects.

Surveys

Using data obtained from a sample requires some accountability and this starts with assessing its reliability. Reliability is related to the consistency of results. In other words, if an experiment were repeated, would the results be similar or totally different? Consistency in results can assist in drawing appropriate conclusions about the population. Understanding the different types of sampling will allow you to use the appropriate methods to obtain a random sample.

Types of sampling

Sampling is the process by which members of a population are selected for a sample. You can set up checks around sampling to make sure that the sample is indeed random and therefore the obtained data can be used effectively.

Non-probability sampling

Non-probability sampling uses techniques that select a sample based on some sort of judgement or convenience and there is no guarantee that each unit in the population has an equally likely chance of

being selected. Therefore, it can be difficult to predict the likelihood of any one person or object being selected, limiting the notion of randomness. Non-probability selection methods include:

- *convenience sampling* – a method of data collection from population members who are most conveniently able to participate in a survey
- *purposive sampling* – survey participants are selected purely based on the judgement of the researcher
- *snowball sampling* – survey participants are encouraged to use social networks and contacts to recruit additional participants.

Non-probability sampling has the advantage of being a relatively easy and cost-efficient method of sampling for the researcher, but the non-random nature of the technique often leads to bias and can affect the accuracy of any inferences drawn about the population.

Probability sampling

Probability sampling uses random selection. Random selection ensures that each unit in the population has an equally likely chance of being selected. The sample statistics obtained using random samples have a predictable theoretical relationship with the population parameters. Bias can be controlled or, at the very least, minimised. Sample selection methods include simple random sampling, systematic sampling, stratified sampling and cluster sampling.

A random sample is the foundation that statisticians rely on to provide measurable information about the population, with limited bias. If a sample is not random, then the mathematical theorems, which support the statistical procedures and analysis, break down.

One of the more common types of random samples is called a simple random sample. There are different techniques to obtain a simple random sample. For smaller samples, writing names on pieces of paper and drawing them out of a hat or box is sufficient. However, in real-world situations involving large sample sizes, this is not particularly efficient.

Additional information

Systematic sampling

These videos give an introduction to the systematic sampling method.

Additional information

Stratified sampling

These videos give examples of stratified sampling.

Statistical methods can only be applied to random samples. If a sample is not random, modified approaches need to be applied.

An easy way to generate a simple random sample is by using technology. Computers or calculators typically apply an algorithm to generate what are called pseudo-random numbers. These are not really random in the same way as a physically generated set of random numbers, but the algorithms used provide very good approximations of sets of random numbers that are needed for most purposes. So, in practice, they are called random numbers. There are several options to choose from depending on the type or number generated and the range of values required.

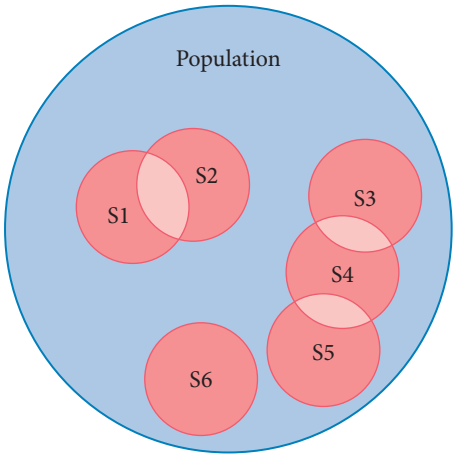
Statistical terminology

On the next page is a diagram depicting the relationship between the population and numerous samples labelled sample 1 (S1), sample 2 (S2), etc.

The relationship between the sample and population plays a central role in understanding how and why you can effectively use information from a sample to infer characteristics about the population. With this relationship comes specific terminology.

Technology worked example

Generating random numbers



The definitions below are the most common terms that are associated with populations and sampling.

- A *unit* is a single member of a finite population. It may be a person, animal, company or object.
- The *population* is the set of all the units that make up a particular group.
- A *census* measures characteristics from the entire population.
- A *sample* is a subset of units taken from the entire population.
- A *parameter* is a numerical measure that describes a characteristic of a population.
- A *statistic* is a numerical measure that describes a characteristic of a sample.

1 Identify population and sample elements

A simple random sample of undergraduate students from the University of Queensland is surveyed to estimate the proportion of undergraduates at this university who are living at home during the current term.

(a) Identify the population.

THINKING	WORKING
The population is the set of all the units that make up a particular group.	The population is all of the undergraduate students at the University of Queensland.

(b) Identify the parameter.

The parameter is the numerical measure that describes a characteristic of a population.	The parameter is the true proportion of University of Queensland undergraduates who live at home.

(c) Identify the sample.

The sample is the subset of units taken from the entire population.	The sample is the group of undergraduate students from the University of Queensland that was chosen to be surveyed.

(d) Identify the statistic.

The statistic is the numerical measure that describes a characteristic of a sample.	The statistic is the proportion of the sampled undergraduates who live at home.

Bias in sampling

Bias within a sample can produce results that are not valid. Validity is the degree to which a sample represents what it is supposed to represent. While the processes of random sampling can limit bias, these techniques alone do not guarantee a sample free of bias. Understanding how bias can occur is essential so that steps may be taken to reduce its effect.

A study or sample displays *bias* if it systematically favours or eliminates certain outcomes.

There are two main types of bias that can occur in random sampling: systematic bias and sampling bias. Some bias can be obvious and deliberate but bias can also occur naturally and be hidden within the constructs of the chosen sample or the sampling process. Not all types of bias are so easily identified or avoided.

Systematic bias

To obtain a random sample, you need to draw from a defined population. The *sample frame* can be identified as the set of units, be it person or object, from which a sample has been taken. Systematic bias occurs from decisions made in the preparation process. It is not about the method of selection, rather it is about from where the selection is being drawn. The most common systematic bias that occurs is a *sample frame error*.

Sample frame error occurs when the wrong sub-population is used to select a sample.

Consider the National Disability Insurance Scheme (NDIS). In order for the initial costing to be submitted, the government needed to know how many Australians would require funding under this scheme. How could the government determine how many Australians are living with some form of disability? Imagine that the government gathered data from Centrelink, detailing all people receiving a disability pension, as a means of calculating the cost of the scheme. Would they have access to every single person with a disability? The following enquiries could be made to help answer this question.

- Are children taken into account?
- What if a person is not eligible (due to means testing) for a disability pension?
- How accurate is the information in these files?
- What about people who have a disability but do not know that they can claim a pension?
- What about people who claim a disability pension falsely?
- What about people who have not been diagnosed and, therefore, have no way of claiming a disability pension?

Here you can see that not everyone who can be sampled is targeted, which does not allow a fair representation of the population of interest. This error undermines the integrity of the sample. To rectify this, a good understanding of the population, obtained by in-depth research, is required.

Sampling bias

Probability sampling techniques limit the presence of bias. As noted previously, even within these processes, bias can still occur. The following table provides a brief outline of the types of bias that can arise within probability samples.

Type of bias	Definition	Example
Undercoverage	This occurs when a random sample is taken that is not truly representative of the population you wish to target.	A company randomly selects stay-at-home women to research a particular cleaning product. This assumes that stay-at-home women are the only ones who clean and excludes stay-at-home men, working women and working men.
Nonresponse	This occurs when a selected member cannot or will not cooperate. Nonresponse to nongovernment surveys can reach as high as 60% and even with careful planning and several callbacks, this figure only drops slightly.	A large hospital sends out surveys designed to talk about patients' recent experience within the hospital. Patients may be unable or unwilling to participate. This leads to a smaller, more select sample being used that can skew the results.
Response	This occurs when a selected member of the sample may lie or give misleading answers. This may be intentional for research of a sensitive nature, such as illicit drug taking or antisocial behaviour.	A survey asks if you have ever committed a crime.

2 Identify sources of bias

A researcher is attempting to determine the frequency of illegal downloading of movies by school-age children. She deploys a team of researchers to conduct face-to-face interviews with a random selection of students at three independent metropolitan schools, to ask students directly about their downloading habits. Identify any potential sources of bias using this approach.

THINKING	WORKING
1 Consider the population and the coverage of the sample.	Although students in the three surveyed schools are selected at random, the fact that the selected schools were all independent and all metropolitan does not provide adequate coverage to make an inference about the population (all school-age children). Undercoverage bias exists – the targeted population is not adequately covered.
2 Consider the likely responses of the participants. In any survey involving human participants, there is likely to be a level of nonresponse bias, as some people may refuse to answer the questions.	There is likely to be a response bias, as the question being asked (referring to illegal downloads) is likely to result in a number of false replies. Nonresponse bias exists, as the survey involves human participants who may not respond.
3 Summarise your response.	Potential sources of bias are undercoverage bias, nonresponse bias and response bias.

EXERCISE 10.1 Random sampling

Worked Example

1

1 Identify the population, the sample, the parameter and the statistic for the following scenarios.

(a) The students' union at the Queensland University of Technology (QUT) is interested in the average time it takes a student at the university to complete a degree. They randomly select records of the previous year's graduate students to determine how long they studied for their degrees.

(b) A survey was conducted on 400 randomly selected members of the Queensland Association of Mathematics Teachers (QAMT). Of the 400 members surveyed, 280 identified that they would prefer to receive QAMT publications via email.

(c) A coin is flipped 100 times and the number of heads is 47.

2

2 For each of the following scenarios, determine the type of bias that could occur. (There may be more than one.)

(a) A fruit producer wants to know what proportion of households buys a certain fruit. A simple random sample of 1000 residential addresses is drawn and interviewers are sent to these addresses. The interviewers are employed to interview during working hours.

(b) Smithsville Council conducts a survey regarding the laws on leashing dogs in public. They use the information from their dog registry office to contact people.

(c) A simple random sample is conducted in a shopping centre. Each selected member is asked if they have ever shoplifted.

3 Which of the following techniques is *not* an example of a simple random sample?

A drawing names out of a hat

B numbering the population and using a spreadsheet or calculator to draw random numbers

C giving a survey to the first five people to walk into a room

D allocating a lucky door prize ticket to every dinner attendee and drawing a ticket from a box

4 What is the portion of the population that is selected for analysis called?

A a sample

B a parameter

C a statistic

D a census

5 The Department of Education and Communities decides to randomly test high school students in a particular district for Attention Deficit Hyperactivity Disorder (ADHD). There are three high schools in the district, each with years 7 to 12. The department randomly samples 250 students by generating random numbers and selecting students from a list of all students in the district. Determine whether this is a simple random sample.

6 For each of the following scenarios, determine whether or not the sampling method would produce a random sample, and whether there is likely to be bias in the sampling method. If so, determine what type of bias might affect the results.

(a) The effectiveness of a new brand of hydration drink is tested to determine if it assists with rehydrating the body after a cardio workout. Volunteers from a random sample of gyms within a specific area are targeted. Each person is given a free supply of the drink and is checked by a doctor and sports physiologist.

(b) A survey is conducted regarding the need for government funding within existing nursing homes. The interviewer surveys randomly selected people in shopping centres.

(c) An events company conducts a telephone survey regarding corporate Christmas parties. They obtain a listing of companies and their telephone numbers from the internet and randomly choose 3000 numbers.

7 After consultation with the health department and government guidelines, a school is seeking opinions regarding a proposed new tuckshop menu. Describe how you could obtain a random sample of 150 people. Include your sampling process and identify possible bias.

8 Use the random number functions on a graphics calculator or spreadsheet to assist with this question.

(a) Create a random sample of 100 integers ranging in value from 1 to 10 inclusive, and plot the results in a dot plot.

(b) Calculate the mean and standard deviation.

9 Consider a random list of n integers between 1 and 10 inclusive. By generating several different random lists for different values of n, and investigating the mean of these lists, explain what tends to happen to the mean of the lists as n becomes very large. (You will need to analyse multiple lists to observe this effect. Hint: Begin by generating lists of size $n = 10$ and compare with list sizes $n = 100$ and $n = 1000$).

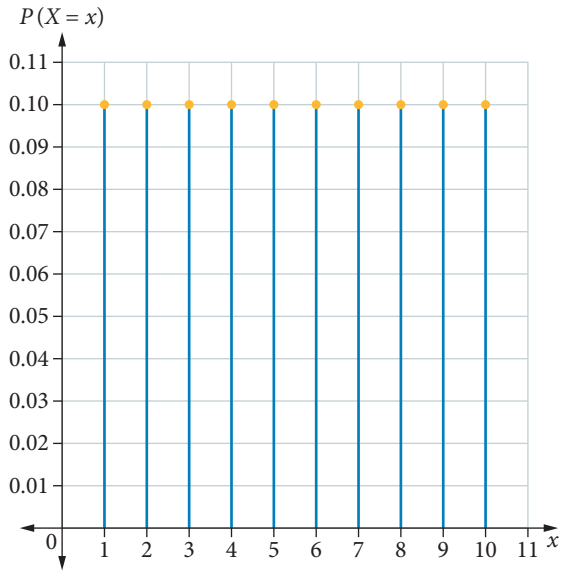
10.2 The central limit theorem

While the central limit theorem is not mentioned explicitly in the Mathematical Methods syllabus, the theorem is important for a thorough understanding of inferential statistics.

Sampling distributions and variability

In the previous section, you used your calculator to generate samples of random numbers and calculated various properties of these samples such as the mean. This process is an example of a simulation. Each simulation of random numbers would have resulted in slightly different values for the calculated mean, which is to be expected. When generating multiple sets of random numbers, it would be very unusual to have exactly the same mean for every set. The same theory holds for real-life surveying. A researcher would expect to have some variation in the survey responses each time a sample is taken, even if those samples are from the same parent population.

Consider again the simulation of random numbers between 1 and 10. In the previous section, you used technology to generate these numbers – another way of doing this would be to roll a fair, ten-sided die a large number of times. When such a die is rolled, the theoretical probability distribution looks like this:



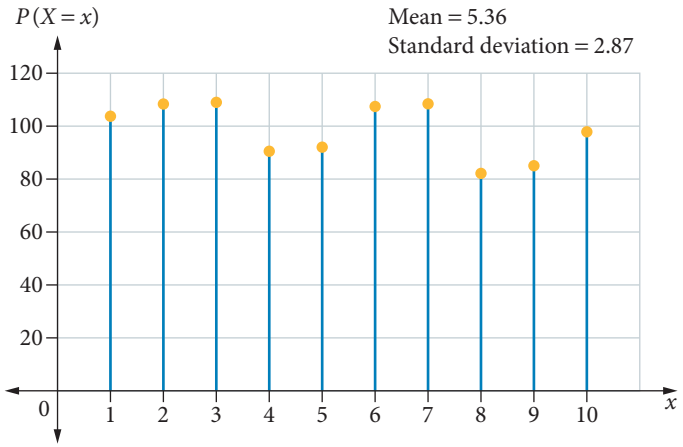
In theory, it is expected that the results will follow a discrete uniform distribution, as every number between 1 and 10 has an equally likely chance of being rolled. Using the formulas for the expected value and standard deviation of a uniform distribution for $a \leq x \leq b$:

$$E(X) = \frac{b+a}{2} = \frac{10+1}{2} = \frac{11}{2} = 5.5$$

and

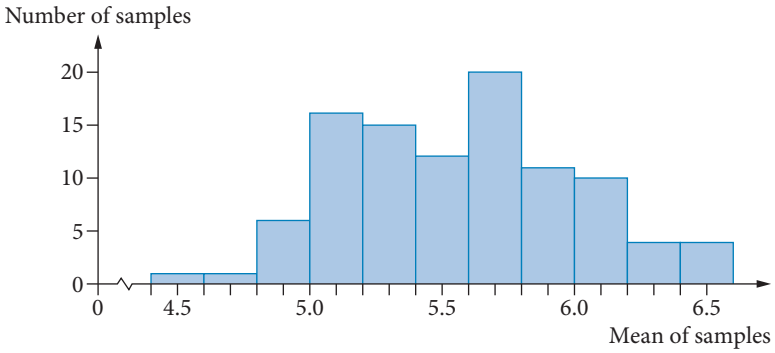
$$\sigma = \frac{b-a}{\sqrt{12}} = \frac{10-1}{\sqrt{12}} \approx 2.60$$

The graph below displays the results from 1000 observations of rolling a fair ten-sided die. This can be considered as a very large sample of a random variable from a discrete uniform distribution. The sample mean and standard deviation were calculated.

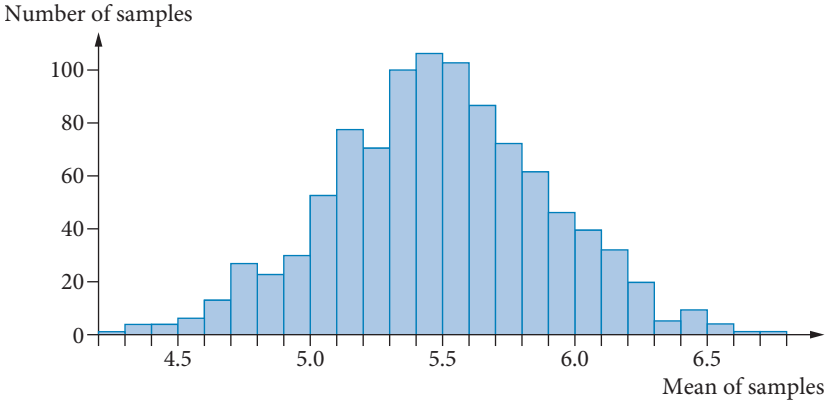


Note that the shape of this sample distribution is close, but not identical, to the discrete uniform distribution. This would be expected in most instances. Such a random sample would have a mean of approximately 5.5 and the expected number of times of each value occurring is close to one-tenth of the sample size, in this instance, 100. This sample's mean is 5.36 and the standard deviation is 2.87.

Now consider what happens if you take 100 samples of size 50 and plot the mean of each sample. In this case, a histogram is used to plot the means of the samples; this is so that similar-sized means are grouped within one interval, providing a better visual overview of the pattern than plotting each individual sample mean separately.



Repeating the same process 1000 times gives you a better indication of the shape of the distribution of the sample means.

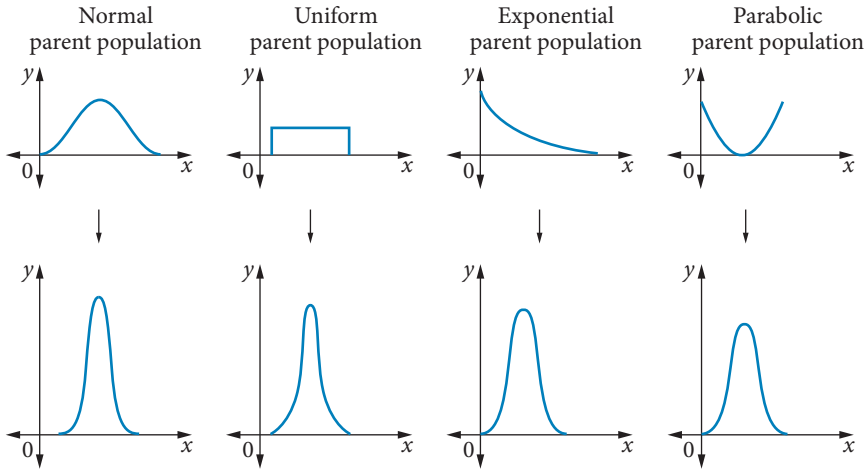


Here it can be seen that the distribution is becoming more bell-shaped and symmetrical, much like a normal distribution. From the histogram it can be seen that the mean is very close to 5.5, which reflects the probability of the discrete uniform distribution. This simulation demonstrates one of the most important theorems in statistics: the central limit theorem (CLT). Informally, the CLT says that, irrespective of the nature of the parent distribution, the sampling distribution for a given statistic is essentially normally distributed.

As previously mentioned, in most cases the population parameters are unknown, therefore the CLT is used in reverse. Taking samples of size n from any distribution, you can calculate the mean for each sample and know that the collection of sample means will be distributed approximately normally. This implies that the average of the sample means will be approximately equal to the population mean.

The central limit theorem

In the previous example, you observed how the distribution of sample means resembles a normal distribution when samples are taken from a uniform parent population. This pattern holds true for all parent populations, regardless of their shape, provided that the sample size n is sufficiently large.



Sampling distributions of $\overline{X}$ for $n = 30$

This result, known as the central limit theorem, is very important in inferential statistics as it makes it possible to draw inferences about an unknown population mean, regardless of the shape of the population.

The central limit theorem states that when the sample size n is sufficiently large, the distribution of the sample mean is approximately normal. For any parent population with mean μ and standard deviation σ, the distribution of the sample mean $\overline{X}$ tends to a normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$.

WARNING

If the parent population is normally distributed, then the distribution of the sample mean $\overline{X}$ will also be normally distributed for any sample size n. However, for parent populations with other shapes (such as uniform, exponential or parabolic), the distribution of the sample mean $\overline{X}$ will tend towards a normal shape only as n becomes sufficiently large.

The central limit theorem applies when the sample size n is 'sufficiently large'. For most parent populations, a sample size of $n \geq 30$ is sufficient to apply the theorem. However, there are slightly different requirements when the parent population is a binomial distribution, as you will see later in this chapter.

Making connections

The central limit theorem

Investigate distributions of sample means for populations of various shapes and various sizes of n.

Using the central limit theorem

Worked Example 3 demonstrates the power of the central limit theorem and the importance of sampling in inferential statistics. In this specific case, the population parameters are known – normally this would not be the case and inferential statistics could be used to estimate these parameters with a given degree of confidence.

3 Use the central limit theorem

The heights of a certain population of males are normally distributed with a mean of 178 cm and a standard deviation of 7.5 cm. Estimate the probability that the average height $\bar{X}$ of 60 randomly selected males from the population will be less than 180 cm. Give you answer correct to 4 decimal places.

THINKING	WORKING
1 Identify the given information. As you are dealing with height, the minimum used is 0, although in practice it would probably be much higher than this.	Lower bound: 0 Upper bound: 180 Mean: 178 Standard deviation of the sample means: $\frac{\sigma}{\sqrt{n}} = \frac{7.5}{\sqrt{60}}$
2 The shape of the parent distribution is unknown. However, since $n \geq 30$, the central limit theorem applies. The sample means will be normally distributed described by $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. Substitute the known values.	$\bar{X} \sim N\left(178, \frac{7.5^2}{60}\right)$
3 Use technology to calculate the answer.	$P\left(\bar{X} < 180\right) = 0.9806 \left(4 \text{ d.p.}\right)$
4 Interpret the answer.	The probability that the sample mean will be less than 180 cm is 0.9806.

It is possible to use the central limit theorem in situations where the parent population is clearly not normally distributed.

4 Use the central limit theorem for non-normal parent populations

A fair eight-sided die, with three faces marked 1, two faces marked 3, and one face each marked 5, 7 and 8 is rolled 100 times. The mean of the 100 scores is recorded. Calculate an approximation for the probability that the mean of the numbers obtained is greater than 4 (correct to 4 decimal places).

THINKING	WORKING
1 Define X.	Let X be the event that the mean of a sample results is greater than 4.
2 Construct the probability distribution for X, the score on a single roll.	$\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline x & 1 & 3 & 5 & 7 & 8 \\ \hline P(X=x) & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \hline \end{array}$
3 Calculate the mean $E(X)$ of the distribution.	$\mu = 1\times\frac{3}{8}+3\times\frac{1}{4}+5\times\frac{1}{8}+7\times\frac{1}{8}+8\times\frac{1}{8}$ $=\frac{3}{8}+\frac{3}{4}+\frac{5}{8}+\frac{7}{8}+1$ $=\frac{29}{8}$
4 Calculate the variance $\sigma^2 = E\left(X^2\right)-\left[E(X)\right]^2$ of the distribution.	$E\left(X^2\right)=1\times\frac{3}{8}+9\times\frac{1}{4}+25\times\frac{1}{8}+49\times\frac{1}{8}+64\times\frac{1}{8}$ $=\frac{3}{8}+\frac{9}{4}+\frac{25}{8}+\frac{49}{8}+\frac{64}{8}$ $=\frac{159}{8}$ $\sigma^2 = E\left(X^2\right)-\left[E(X)\right]^2$ $=\frac{159}{8}-\left(\frac{29}{8}\right)^2$ $=\frac{431}{64}$
5 Calculate the variance of the sample means.	$\frac{\sigma^2}{n}=\frac{\frac{431}{64}}{100}$ $=\frac{431}{6400}$
6 Determine the approximate normal distribution that will be used.	$\overline{X}\sim N\left(\mu,\frac{\sigma^2}{n}\right)$ $\overline{X}\sim N\left(\frac{29}{8},\frac{431}{6400}\right)$

7 Use technology to calculate the answer.	Lower bound: 4 Upper bound: 8 Mean: $\mu = E(X) = \frac{29}{8}$ Standard deviation of the sample means: $\frac{\sigma}{\sqrt{n}} = \sqrt{\frac{\sigma^2}{n}} = \sqrt{\frac{431}{6400}}$ Using technology, $P(4 < \bar{X} < 8) = 0.0742$ (4 d.p.)
8 Interpret the answer.	The probability that the mean of a trial results in a number greater than 4 is approximately 0.0742.

EXERCISE 10.2 The central limit theorem

1 In experiment A, samples of $n = 10$ are taken from a random number generator programmed to output whole numbers between 1 and 10. In experiment B, samples of $n = 100$ are taken from the same random number generator. The means of these samples are calculated. Which of the following statements is correct?

- A The mean from experiment A is likely to be closer to the theoretical mean of 5.5 than the mean from experiment B.
- B The mean from experiment B is likely to be closer to the theoretical mean of 5.5 than the mean from experiment A.
- C There is a 100% chance that the means from both samples will equal the theoretical mean of 5.5.
- D There is a 0% chance that the means from both samples will equal the theoretical mean of 5.5.

2 Determine which of the following distributions of sample means is likely to approximate a normal distribution.

- A a distribution from a normal parent population using sample size $n = 10$
- B a distribution from a uniform parent population using sample size $n = 10$
- C a distribution from an unknown parent population using sample size $n = 15$
- D a distribution from an irregular-shaped parent population using sample size $n = 15$

Worked Example 3

3 The heights of a large sample of males taken from a population have a mean of 179 cm and a standard deviation of 7.9 cm. Estimate the probability that the average height $\bar{X}$ of 70 randomly selected males from the population will be less than 178 cm. Give your answer correct to 4 decimal places.

4 A fair coin is flipped. A value of 1 is assigned to a head and a value of 0 is assigned to a tail. Thus, the score has a mean value of $\frac{1}{2}$ and a variance of $\frac{1}{4}$. If the coin is flipped 60 times, calculate the probability that the total score is less than 33. Give your answer correct to 4 decimal places.

5 A continuous random variable X is uniformly distributed between -1 and 1. It has a mean of 0 and a variance of $\frac{1}{3}$. Calculate the probability that the mean of an independent sample of size 30 from X is greater than 0.1. Give your answer correct to 4 decimal places.

Worked Example 4

6 A fair eight-sided die, with two faces marked 1, three faces marked 3, and one face each marked 5, 7 and 8 is rolled 100 times. The mean of the 100 scores is recorded. Calculate an approximation for the probability of the mean satisfying each of the following criteria. Give your answers correct to 4 decimal places.

(a) the mean is greater than 3.5

(b) the mean is less than 4

(c) the mean is between 3.5 and 4.5

7 When a fair, six-sided die is rolled, its score follows a uniform distribution.

(a) Calculate the mean and standard deviation for this uniform distribution. Give your answers correct to 1 decimal place.

(b) The die is rolled 50 times. Calculate the probability that the average score is less than 3.7. Give your answer correct to 4 decimal places.

8 Three fair coins are flipped at the same time.

(a) Construct a probability distribution table for X, the number of heads showing.

(b) Calculate the mean and variance of this distribution.

(c) If the three coins are flipped together 40 times, determine an approximate value for the probability that the average number of heads is less than 1.4. Give your answer correct to 4 decimal places.

(d) Alexander says that when the three coins are flipped together 40 times, there is a probability of 0.5125 that the total number of heads is 70 or more. Determine whether this statement is correct.

(e) Calculate the actual probability that the total number of heads is greater than 70. Give your answer correct to 4 decimal places.

9 At a particular factory, the lengths of small metal spikes produced by a machine have an unknown distribution with mean of 3.025 cm and standard deviation of 0.195 cm. A sample of 100 spikes is taken. Answer the following questions, giving your answers correct to 3 decimal places.

(a) Estimate the probability that the mean length of this sample is less than 3.01 cm.

(b) Estimate the probability that the mean length of this sample is greater than 3.03 cm.

(c) Estimate the probability that the mean length of this sample will be between 3.02 cm and 3.04 cm.

(d) Calculate each of the following probabilities under the given circumstance.

(i) the mean length of the sample is less than 3.01 cm, if the sample size was 200

(ii) the mean length of the sample is less than 3.01 cm, if the sample size was 300

(e) The manager of the factory is attempting to determine the sample size required for there to be a 10% probability that the mean length of the sample is less than 3.01 cm. Based on your previous calculations, estimate the sample size required to satisfy this requirement. Validate your estimation mathematically.

10.3 Proportions

In the previous section, statistical questions of a quantitative nature were discussed. However, statistical questioning can take on the form of a survey where the occurrence of a particular characteristic or trait is counted. For example, asking 'Are you left-handed?' to a class will yield answers that are either 'yes' or 'no'. If you are interested in the proportion of students who are left-handed, then you will count the number of students who answered 'yes'.

In this section, the idea of the relationship between discrete data values obtained from simple random samples, in the form of categorical data, is explored.

Sample proportions

Medical research indicates that 4% of the Australian population carry the gene for cystic fibrosis, one of the most common life-threatening genetic conditions in Australia. One could ask, 'How accurate is 4% with respect to the population?' Realistically, the exact population parameter is unknown. Unless you are willing to perform a census, your information and subsequent calculations will come from a random sample. However, it can be shown that a proportion calculated from a simple random sample can provide a reasonable estimate of the population parameter.

Notation and terminology

Suppose that a random sample of 4000 people is tested for the cystic fibrosis gene. From this sample, it is found that 140 people are carrying the gene. This gives a proportion of $\frac{140}{4000} = 0.035$. As this proportion has been obtained from the random sample, it is called the *sample proportion*. The table below illustrates definitions and terminology associated with sample proportions from the example above.

Notation	Definition	Example
x	Observation: For any one sample, the number of people or things that possess a certain characteristic is called the observation.	The observation is the 140 people who carry the gene.
n	Sample size: This is the number of people or things that make up the sample.	The sample size is 4000.
$\hat{p} = \frac{x}{n}$	Sample proportion: This is the number of people or things with a characteristic that occurs within the sample (observation) over the number in the sample.	140 out of 4000 carry the gene. $\hat{p} = \frac{140}{4000} = 0.035$
p	Population proportion: This is the true proportion of the people or things with a characteristic that occurs within the whole population.	The population proportion is unknown in this case, as no census has been taken. It can be estimated using the sample proportion.

The sample proportion $\hat{p} = \frac{x}{n}$ is not the population proportion; rather it is the best estimate of the population proportion from a single sample. This leads to the sample proportion sometimes being referred to as the *point estimate*.

5 Identify and calculate proportion elements

Consider a sample of 2000 seventeen-year-olds in Queensland. From this sample, 210 have their provisional driver's licence.

(a) Identify the population proportion.

THINKING	WORKING
The population proportion, which is unknown.	p is the population proportion of all seventeen-year-olds in Queensland who have their provisional licence.

(b) Identify the sample size.

The sample size is n, the number of people in the sample.	$n = 2000$

(c) Identify the observation value.

The observation value is x, the number of people in the sample who have their provisional driver's licence.	$x = 210$

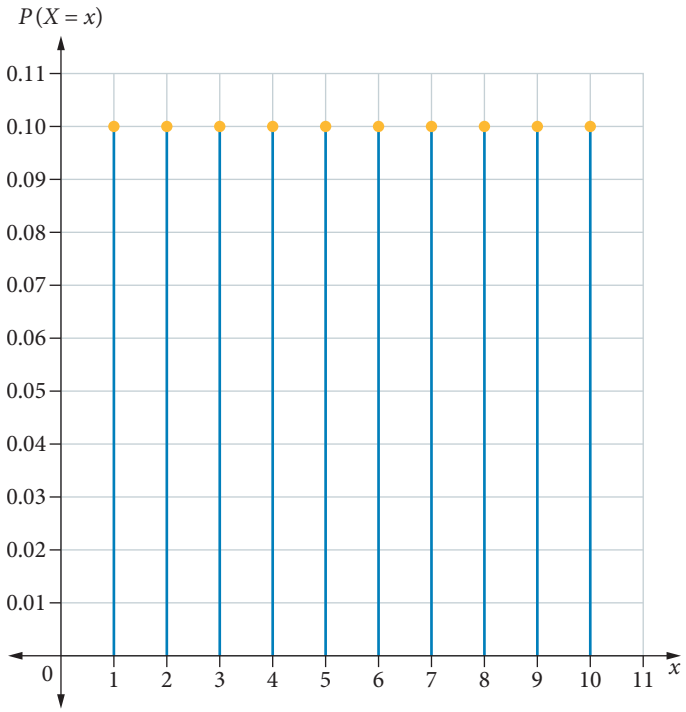
(d) Identify the sample proportion.

The sample proportion is given by $\hat{p} = \frac{x}{n}$.	$\hat{p} = \frac{x}{n}$ $= \frac{210}{2000}$ $= 0.105$ ≈ 0.11 That is, of the sample selected, 0.11 or approximately 11% of the seventeen-year-olds had their provisional licence.

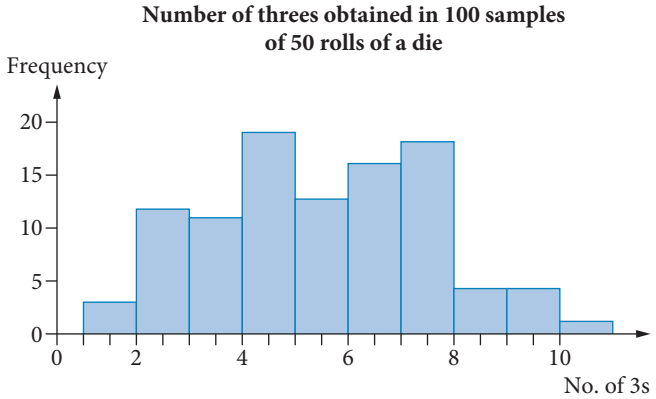
The central limit theorem and proportions

Earlier in this chapter, the variability of results between random samples taken from a single population was explored. Recall that any statistic from all of the possible samples is a random variable and each of these random variables has a probability distribution.

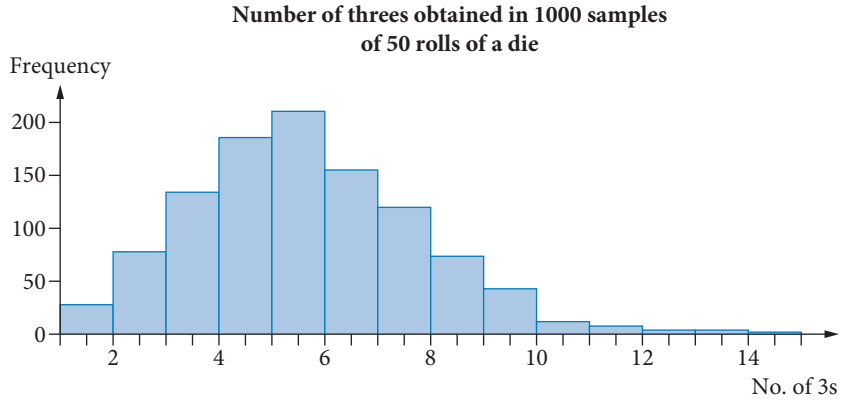
Recall that the probability distribution of a fair ten-sided die is a discrete, uniform distribution as shown below:



Rather than considering the sampling distribution of the means, consider what happens if a particular number occurs, in this case the number 3. In other words, this simulation is the distribution of a Bernoulli random variable, where rolling a 3 can be considered a 'success' and not rolling a 3 can be considered a 'failure'. This simulation will observe 100 samples of size 50. Again a histogram is used to plot the occurrences of the number 3, because it accurately shows the spread of the sample distribution.



Note that the distribution of the occurrence of the number 3 does not resemble a uniform distribution. Repeating the same process 1000 times gives a better indication of the shape of the distribution.



Here it can be seen that the distribution is becoming more bell-shaped and symmetrical, like a normal distribution.

The central limit theorem applies equally to the proportion of a specific random variable in a sampling distribution, such as the number of 3s obtained in samples of size of 50, as it does to the sampling distribution of means.

Explore further

Repeated random sampling

Analyse the shape of histograms for an increasing number of trials.

Taking samples of size n, from any distribution, a specific proportion of a random variable can be calculated and it will be approximately normally distributed. This implies that the proportion of a specific random variable in a sample will be approximately equal to the proportion of the same specific random variable in the population.

Sampling distribution for sample proportions

You will now briefly explore the relationship between Bernoulli random variables and binomial distributions, and how these relate to sample proportions.

Recall from your work on discrete random variables that a Bernoulli random variable is a discrete random variable that has two outcomes: one being a 'success' with an associated probability of p and the other being a 'failure' with the associated probability of $1 - p$. Within a binomial setting, if n units are chosen at random, then the variable X (the number of successes in a sample) has a binomial distribution $X \sim B(n, p)$.

The sample proportion $\hat{p} = \dfrac{X}{n}$ is the binomial count divided by the sample size, which gives a value between 0 and 1. This means that the set of $\hat{p}$ values are related to a random variable called $\hat{P}$ and follow a binomial distribution. Notice the differences in notation. The values x and $\hat{p}$ are used when observing a single random sample. However, when considering the whole set of values that can be obtained from *all* samples, X and $\hat{P}$ are used.

The standard deviation of a binomial distribution can be estimated from the sample proportion $\hat{p}$ of one sample, rather than many. This cannot be done easily for other distributions.

As previously discussed, both the mean and the standard deviation are defined using the population parameter p; however, the value of p is unknown, hence the exact mean and standard deviation cannot be calculated. To overcome this problem, use the sample proportion $\hat{p}$ as an estimate for both the expected value and standard deviation, and replace p with $\hat{p}$. When $\hat{p}$ replaces p in the standard deviation, the term standard error $S\left(\hat{P}\right)$ is often used. This is so that a distinction can be made between the use of a parameter and a statistic.

The distribution of $\hat{P}$ for the sample proportions is defined as $\hat{P} \sim N\left(\hat{p}, \sqrt{\dfrac{\hat{p}\left(1-\hat{p}\right)}{n}}\right)$

where the expected value $E\left(\hat{P}\right) \approx \hat{p}$ and the standard deviation $SD \approx \sqrt{\dfrac{\hat{p}\left(1-\hat{p}\right)}{n}} = S\left(\hat{P}\right)$.

6 Identify the expected value and standard error

A social media survey found that 134 out of 265 respondents had accessed some form of social media within the past year.

(a) Calculate the expected value for the proportion of people who access social media, correct to 2 decimal places.

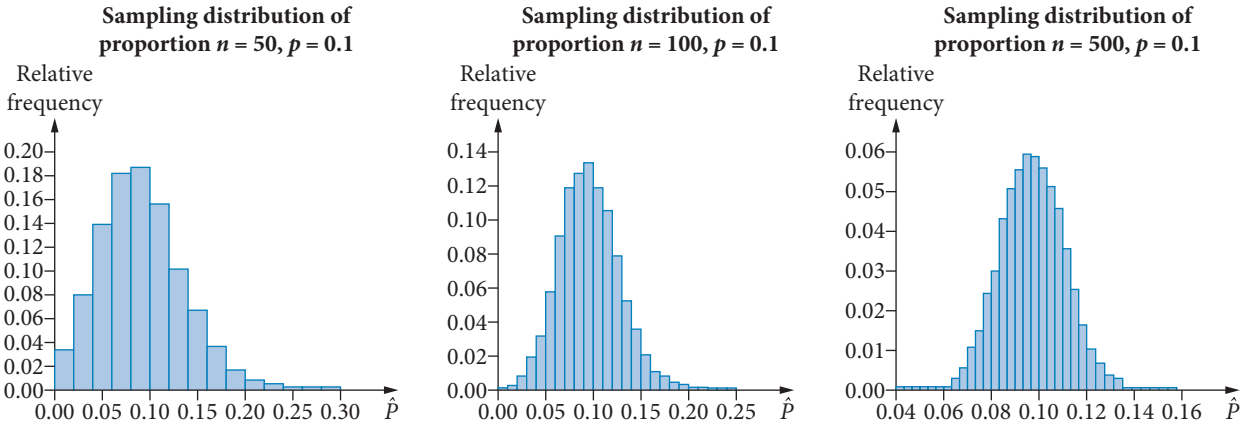
THINKING	WORKING
1 Recall the formula for the expected value.	$E\left(\hat{P}\right) \approx \hat{p}$ $\approx \dfrac{x}{n}$
2 Identify and substitute the given values into the formula.	For $x = 134$ and $n = 265$ $E\left(\hat{P}\right) \approx \dfrac{134}{265}$ ≈ 0.51
3 Interpret the answer.	The expected value $E\left(\hat{P}\right) \approx 0.51$.

(b) Calculate the standard error, correct to 3 decimal places.

1 Recall the formula for the standard error.	$S\left(\hat{P}\right) = \sqrt{\dfrac{\hat{p}\left(1-\hat{p}\right)}{n}}$
2 Substitute the given values into the formula.	$S\left(\hat{P}\right) = \sqrt{\dfrac{\hat{p}\left(1-\hat{p}\right)}{n}}$ $= \sqrt{\dfrac{0.51 \times 0.49}{265}}$ ≈ 0.031
3 Interpret the answer.	The standard error $S\left(\hat{P}\right) = 0.031$.

Sample size

As a result of the central limit theorem, you know that when a sufficiently large random sample of size n has x 'successes', the sampling distribution of the sample proportion $\hat{p}$ will be approximately normal with mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$.



On the left-hand graph, for $p = 0.1$ and $n = 50$, the distribution may look bell-shaped but it is noticeably skewed. However, as the value for n increases, the shape of the distribution becomes more symmetrical about p. This means that even if the value of p is not close to 0.5 (i.e. the population is unbalanced), the larger the sample, the more the sampling distribution of $\hat{p}$ resembles a normal distribution.

So what value of n is considered sufficiently large? The conditions in the box allow you to determine if the central limit theorem can be applied for a binomial distribution. Later in this chapter, you will see how this affects confidence intervals. There is one exception to the conditions and that is when the population already has a normal distribution.

The following two conditions must be met for the central limit theorem to apply to the binomial distribution:

$n\hat{p} \geq 5$

$n(1-\hat{p}) \geq 5$

You would also need to approximate the population proportion p with the sample proportion $\hat{p}$.

7 Assess normality

A survey of 25 people has obtained a sample proportion of 0.2. Determine whether the distribution can be approximated by a normal distribution.

THINKING	WORKING
1 Define the parameters.	$n = 25$ and $\hat{p} = 0.2$
2 Test these parameters for each condition.	$n\hat{p} \geq 5$ and $n(1-\hat{p}) \geq 5$ $25 \times 0.2 = 5$ $\quad$ $25 \times 0.8 = 20$ $5 \geq 5$ $\quad$ $20 \geq 5$
3 Determine validity and interpret your answer.	Both conditions are met so the distribution can be approximated by a normal distribution.

EXERCISE 10.3 Proportions

Worked Example

1 For the following scenarios, identify the population proportion, sample size, observation value and sample proportion. Give your answers correct to 2 decimal places where necessary. 5

(a) A report by an IT data company in the Northern Territory surveyed 800 email users and reported that 42% of them have received a fake email claiming they have won a large amount of money.

(b) A database search shows that approximately 215 000 Tasmanians own a Holden. A survey of 2500 Holden owners in Tasmania showed that 1893 prefer a medium- to large-sized car over a small model.

2 A survey has found that 64 of 104 Year 12 students are not sure what to do after school. Calculate the expected value and standard error for the population proportion of Year 12 students who are not sure what to do after school. Give your answers correct to 3 decimal places. 6

3 The results of a survey of 334 people have obtained a sample proportion of 0.98. Determine whether the given values allow the distribution to be approximated by a normal distribution. 7

4 Calculate the expected value and standard error for the following proportion elements.

(a) $x = 507$ and $n = 1300$ (b) $x = 415$ and $n = 500$ (c) $x = 50$ and $n = 1000$

5 Determine whether the following values satisfy the conditions for normality.

(a) $\hat{p} = 0.55$ and $n = 1000$ (b) $\hat{p} = 0.9$ and $n = 10$ (c) $\hat{p} = 0.1$ and $n = 50$

6 Which of the following sample proportions would be most likely to generate a normal-shaped sampling distribution using a sample size of $n = 20$?

A $\hat{p} = 0.1$ B $\hat{p} = 0.4$ C $\hat{p} = 0.7$ D $\hat{p} = 0.99$

7 Which of the following sample sizes would be most likely to generate a skewed sampling distribution for the sample proportion $\hat{p} = 0.9$?

A $n = 10$ B $n = 20$ C $n = 50$ D $n = 500$

8 For each of the following, given the sample proportion and observation value, calculate the sample size. Give your answers correct to the nearest whole number.

(a) $\hat{p} = 0.7$ and $x = 91$ (b) $\hat{p} = 0.15$ and $x = 300$ (c) $\hat{p} = 0.83$ and $x = 15$

9 For each of the following, given the sample proportion and the sample size, calculate the observation value.

(a) $\hat{p} = 0.34$ and $n = 2500$ (b) $\hat{p} = 0.02$ and $n = 100$ (c) $\hat{p} = 0.98$ and $n = 5000$

10 For the following cases, calculate the expected value, correct to 2 decimal places. Then calculate the standard error, correct to 3 decimal places, and determine whether it is reasonable to estimate the sampling distribution using the normal distribution.

(a) Jess is running for school captain. From a simple random sample of 900 students, the proportion of students who favour Jess is 0.75.

(b) A simple random sample of 50 smokers found that 5 people were prepared to quit smoking within the week.

(c) Last year, 67 000 children were hospitalised. From these admissions, 42 600 were boys.

11 For a particular survey, a researcher is limited to taking random samples of size $n = 20$. Determine the range of values for the sample proportion $\hat{p}$ for which the researcher would be justified in assuming a normal sampling distribution for proportions.

12 Calculate the minimum sample size required such that the sampling distribution for proportions would approximate the normal distribution under the following conditions.

(a) $\hat{p} = 0.1$ **(b)** $\hat{p} = 0.02$

13 According to an ABC Public Service survey of 500 employers, 78% of employers agreed that the demand for jobs has increased. This survey was repeated many times, with random samples being taken from the same population and the sample proportions recorded.

(a) Describe the proportions from each survey, where 0.78 was the proportion of employers believing that the demand for jobs had increased.

(b) Determine whether a sample proportion value of 0 would be possible. What could this mean?

(c) Determine whether a sample proportion value lower than 0.45 would be possible. What could cause this?

(d) Is the sample proportion of 0.78 likely to be close to the true population proportion? Justify your answer.

(e) In what scenario might the sample proportion value not be representative of the true population proportion?

14 According to the Australian Health Commission, a random sample of 1500 patient records showed that since the introduction of heparin injections for all patients, only 7% of patients developed blood clots. This research was repeated many times with random samples being taken from the same population, and the sample proportions recorded.

(a) Explain why the sample proportion varies for each sample.

(b) Determine whether a sample proportion value of 1 is possible, and explain what this would mean.

(c) If a sample proportion appeared abnormally high, such as 0.23, explain what this could indicate.

(d) A proportion of 7% is fairly small. Explain why it is appropriate to use the normal distribution to approximate the samples.

(e) In what scenario might the sample proportion value not be representative of the true population proportion?

15 A fitness club surveyed a sample of 120 of its members on their level of satisfaction with the club's facilities. The survey results showed that half of the members were satisfied, one-third were indifferent and one-sixth were dissatisfied.

(a) Calculate the observation value for each type of response.

(b) Calculate the point estimate for each type of response, correct to 2 decimal places, where necessary.

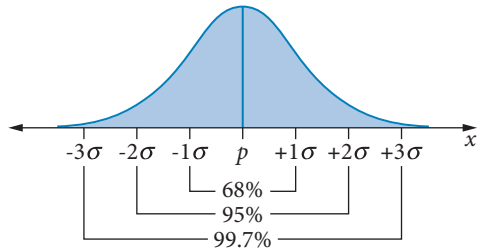
(c) Determine how the point estimate of the dissatisfied members would change if the sample size were doubled but the results stayed in the same proportion.

(d) How would the point estimate of the dissatisfied members change if the sample size stayed the same but the proportion of dissatisfied customers changed to one-fifth of the sample?

10.4 Estimating with confidence

In the previous section, you learnt that the distribution of sample proportions $\hat{P}$ can be approximated by a normal distribution, allowing you to estimate the population parameter p by the sample proportion $\hat{P}$. Recall that the middle 68%, 95% or 99.7% of any normal distribution is within approximately one, two or three standard deviations of the mean, respectively. In this section, the degree of confidence in this estimation will be explored.

For sample proportions, the standard deviation is the standard error, $S\left(\hat{P}\right)$.



- 68% of sample proportions lie within one standard error of the population proportion.
- 95% of sample proportions lie within two (or more accurately 1.96) standard errors of the population proportion.
- 99.7% of sample proportions lie within three standard errors of the population proportion.

This is known as the 68–95–99.7 rule.

Note that different or more precise confidence intervals can be calculated using techniques demonstrated later in this section.

Confidence intervals

For a single sample, the sample proportion gives the best estimate of the population proportion. A 95% confidence interval is a range of values either side of the sample proportion for which you can be 95% confident that the population proportion will lie.

A *confidence interval* is a specific interval based on the sample proportion $\hat{p}$. When the population parameters are unknown, the true population proportion *may* be located within this interval. The interval acts as an indicator of the accuracy for the estimate of the population proportion, and allows you to make a statement of the probability that this interval contains the population proportion.

Constructing a confidence interval of a sample proportion is similar to constructing one for a sample mean, and uses probability ranges and their corresponding standard deviations.

You will need to assess the appropriateness of estimating samples using a normal distribution by using $n\hat{p} \geq 5$ and $n\left(1-\hat{p}\right) \geq 5$, and once this is confirmed, you can calculate the confidence interval.

Consider again the example study of cystic fibrosis carriers. The researcher chooses a random sample of 4000 people and determines that the proportion of cystic fibrosis carriers from this sample is $\hat{p} = 0.035$. The true population proportion p could lie above or below this sample mean.

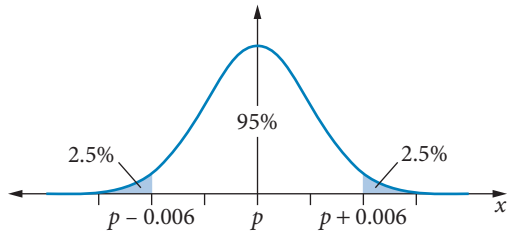
The distribution of $\hat{P}$ can be approximated by a normal distribution with $E\left(\hat{P}\right) \approx 0.035$ and

$$\begin{aligned} S\left(\hat{P}\right) &\approx \sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}} \\ &\approx \sqrt{\frac{0.035(1-0.035)}{4000}} \\ &\approx 0.003 \end{aligned}$$

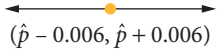
For the middle 95% of the distribution, you will find that a likely value of p is within ±0.006 (which is two standard deviations either side of $\hat{p}$; that is $\pm 2 \times S(\hat{P})$) of the population parameter p. Another way of looking at this is by saying that the true value of p could lie within the interval $0.035 \pm 2 \times 0.003$; that is:

$$E(\hat{P}) - 2S(\hat{P}) \leq p \leq E(\hat{P}) + 2S(\hat{P})$$
$$0.035 - (2 \times 0.003) \leq p \leq 0.035 + (2 \times 0.003)$$
$$0.035 - 0.006 \leq p \leq 0.035 + 0.006$$
$$0.029 \leq p \leq 0.041$$

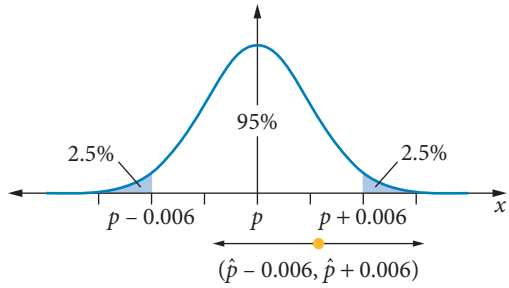
The distribution of sample proportions for this study can be shown in the graph on the right.



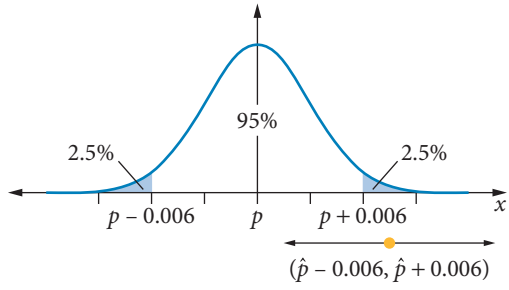
In this diagram, the actual population proportion p is unknown; however, 95% of sample proportions should lie within 0.006 units either side of p. Consider a sample proportion that was calculated in this study. This sample proportion can be represented by plotting a dot with arrows on either side to show the confidence interval:



Because it is known that approximately 95% of sample proportions will lie within two standard deviations of the population proportion p, there is a 95% probability that any particular sample proportion will lie in the region as shown in the graph on the right.



Importantly, you would expect confidence intervals calculated in this way for any given sample proportion to include the true population proportion 95% of the time; that is, 95% of intervals produced this way will include the population proportion. However, by chance alone, the sample proportion may lie outside the unshaded region of the graph; that is, further than two standard deviations from the population proportion, as shown in the graph on the right.



In this case, the confidence interval does *not* include the population proportion p. While this may be unfortunate for a researcher who is relying on a single sample to estimate the population proportion, it is purely a reflection of the inherent uncertainty involved in inferential statistics. In fact, if 100 such random samples are taken, with a confidence interval of 95%, you would expect 5 such samples to occur by chance alone.

Confidence levels

A confidence interval requires you to define the level or degree to which you wish to express your confidence about p. Selecting the level of confidence is a decision that is made by the researcher in relation to the context. In practice, confidence intervals are most commonly calculated at a 95% level. However, various levels such as 90%, 98% and 99% are also used.

There is a trade-off between the confidence level and the size of the interval. The higher the level of confidence, the broader the interval.

Students who study both Mathematical Methods and Specialist Mathematics will note that there are many similarities between sampling of proportions and sampling of means. However, one key difference is the way that the standard error is calculated for the sample distributions. For the distribution of sample means as studied in Specialist Mathematics, the standard error is calculated as $S\left(\overline{X}\right) \approx \frac{s}{\sqrt{n}}$. For the distribution of sample proportions as studied in Mathematical Methods, the standard error is calculated as $S\left(\hat{P}\right) = \sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$.

Interestingly, this implies that the width of the confidence interval for sample means will be fixed, whereas the width of the confidence interval for sample proportions will change depending on the value of the sample proportion itself.

WARNING

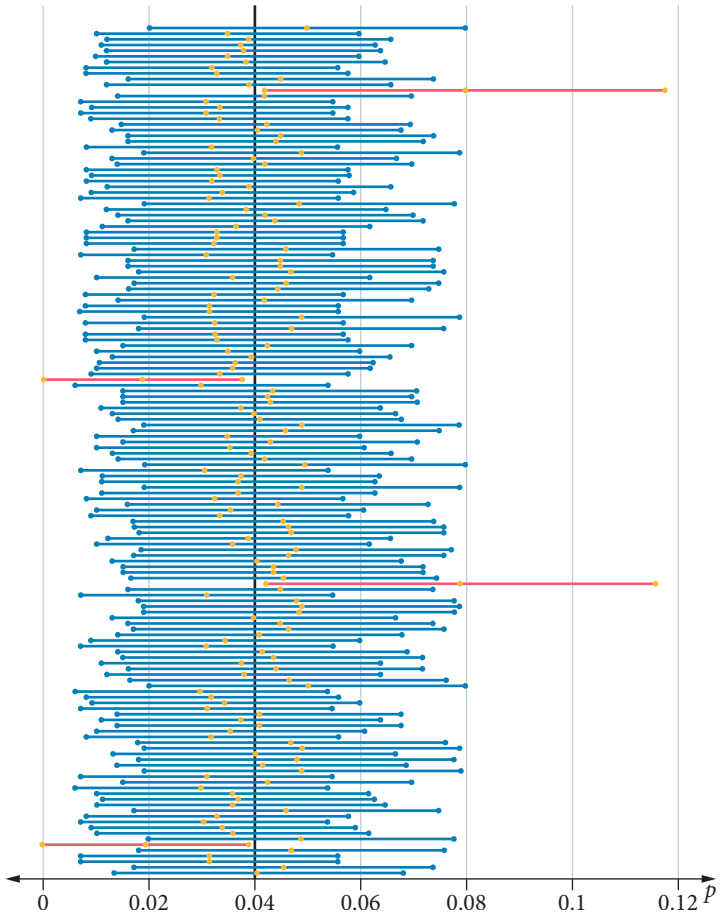
Remember, your sample proportion is a single value obtained from a single sample from which you created a single interval. You know that the sample proportion varies from sample to sample, so it would be safe to assume that your intervals will also vary from sample to sample.

Consider the cystic fibrosis study. If a researcher took another random sample of people, they might obtain a sample proportion of $\hat{p} = 0.04$. This would then produce a 95% confidence interval of $(0.026, 0.054)$.

The graph on the right represents 150 sample proportions of the occurrence of the cystic fibrosis gene and their corresponding confidence intervals.

It is assumed, for the sake of the discussion, that the true population proportion is 0.04. You will observe that the centre of each interval is the value of the sample proportion. An important observation is that the assumed population parameter p does not vary; it is fixed.

From the graph, you can observe that four of the 95% confidence intervals, noted by the red lines, do not contain the actual population proportion. You do not know if your interval actually contains p because you do not know the true value of p.



A 95% confidence level means that, with repeated samples, 95% of the confidence intervals will contain the population proportion.

Explore further

Simulating variations in confidence intervals

Explore the change in mean and confidence interval with changing samples.

8 Calculate an approximate confidence interval manually

A recent survey indicated that 70 out of 100 people preferred to shop online. Calculate an approximate 95% confidence interval.

THINKING	WORKING
1 Recall the required statistics.	$E\left(\hat{P}\right) \approx \hat{p}$ $S\left(\hat{P}\right) = \sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$
2 Identify and substitute the values into the formulas and calculate, correct to 3 decimal places where necessary.	$E\left(\hat{P}\right) \approx 0.7$ $S\left(\hat{P}\right) = \sqrt{\frac{0.7 \times 0.3}{100}}$ $= 0.046$
3 Assess the normality.	$n\hat{p} \geq 5$ and $n\left(1-\hat{p}\right) \geq 5$ $100 \times 0.7 = 70$ $100 \times 0.3 = 30$ Normality applies.
4 Define the required interval. Recall that approximately 95% of all scores will fall within two standard deviations of the mean.	$\left(\hat{p} \pm \left(2 \times S\left(\hat{P}\right)\right)\right)$
5 Substitute the values into the formula for the interval and calculate.	$\left(\hat{p} - 2S\left(\hat{P}\right), \hat{p} + 2S\left(\hat{P}\right)\right)$ $\left(0.7 - \left(2 \times 0.046\right), 0.7 + \left(2 \times 0.046\right)\right)$ $\left(0.7 - 0.092, 0.7 + 0.092\right)$ $\left(0.608, 0.792\right)$
6 Write the confidence interval.	The confidence interval is $(0.608, 0.792)$.
7 Interpret the answer, giving the interval correct to 2 decimal places.	With repeated samples, 95% of the confidence intervals will contain the population proportion. The confidence interval for this sample is (0.608, 0.792).

Confidence intervals and *z*-scores

Technology worked example
Calculate a confidence interval

When working with a normal distribution, finding probabilities within the common intervals of 68%, 95% and 99.7% from the mean is fairly straightforward. Should you wish to find other intervals, another calculation would be required. As presented in Chapter 9, you can standardise a distribution and work with *z*-scores, rather than standard deviations. Taking this approach will also require the sample proportion to be standardised.

Using the standard normal distribution, $Z \sim N(0,1)$, the sample proportion $\hat{p}$ has an approximate standardised value of $\dfrac{\hat{p}-p}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}}$.

A standardised normal distribution requires the use of z-scores. Prior to the development of advanced technology, tables of standardised values and corresponding z-scores were used. However, you can now calculate the value of a z-score using technology.

Calculating z-scores using technology

To calculate z-scores the inverse normal function or equivalent is used. For a given probability, mean and standard deviation, the inverse normal function calculates the z-score for which the area to the left of this score is equal to the probability.

While technology can be used to calculate z-scores and confidence intervals, it is essential that you understand the fundamental structure of the confidence interval. The z-scores that correspond to a 95% confidence level are approximately equal to -1.96 and 1.96. (Note: This is more accurate than the approximation of $z = 2$ stated earlier in this section.) Using these, a 95% confidence interval is defined:

$$P\left(-1.96 < \frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} < 1.96\right) \approx 0.95$$

Technology worked example
Calculate confidence intervals using z-scores

Rearrange this expression by multiplying each side of the inequality by the denominator to obtain the required confidence interval:

$$P\left(-1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < \hat{p}-p < 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) \approx 0.95$$

and

$$P\left(\hat{p}-1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p}+1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) \approx 0.95$$

This expression gives an interval centred on p.

For any given probability, the confidence interval can be calculated using the formulas shown in the box, where the appropriate z-score is calculated from the standard normal distribution.

A confidence interval can be defined as

$$\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) \text{ or } \hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where z is the defined z-score that corresponds to the appropriate confidence level.

9 Calculate confidence intervals using *z*-scores

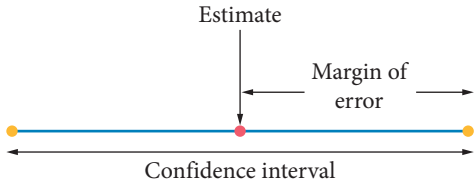
A social media survey shows that 134 out of 265 respondents accessed some form of social media within the past month. Calculate the confidence interval corresponding to the 95% confidence level.

THINKING	WORKING
1 Determine the sample proportion estimate.	$\hat{p} = \frac{x}{n}$ $= \frac{134}{265}$ $E\left(\hat{P}\right) \approx \hat{p}$ ≈ 0.51
2 Calculate the standard error correct to 3 decimal places.	$S\left(\hat{P}\right) = \sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$ $= \sqrt{\frac{\frac{134}{265}\left(1-\frac{134}{265}\right)}{265}}$ ≈ 0.031
3 Assess normality using the exact values of $\hat{p}$.	$n\hat{p} = 265 \times \frac{134}{265}$ $= 134$ $n\left(1-\hat{p}\right) = 265 \times \left(1 - \frac{134}{265}\right)$ $= 131$ Given that $n\hat{p} \geq 5$ and $n\left(1-\hat{p}\right) \geq 5$, it is appropriate to estimate confidence intervals using a normal distribution.
4 Use technology to define the required *z*-score. Aim to be as accurate as possible.	A 95% confidence level equates to $z = 1.96$.
5 Recall the required formula and substitute the required values. Give your answer correct to 3 decimal places.	$\hat{p} \pm zS\left(\hat{P}\right)$ $0.51 \pm 1.96 \times 0.031$ 0.51 ± 0.061
6 Write the corresponding interval.	A 95% confidence interval is $(0.449, 0.571)$.
7 Interpret the result. Give your interval correct to 2 decimal places.	With repeated samples, 95% of the confidence intervals will contain the population proportion. The confidence interval for this sample is (0.449, 0.571).

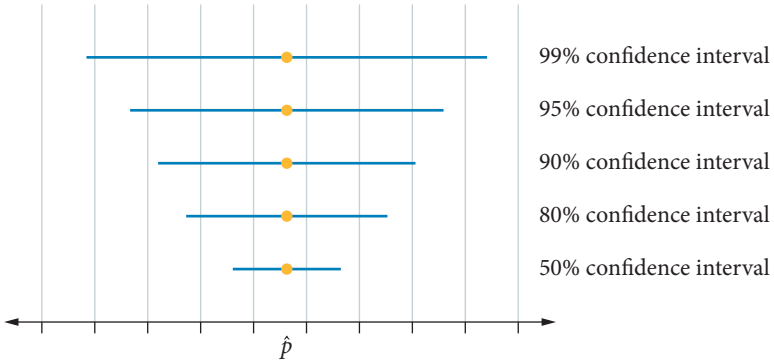
Margin of error and confidence intervals

Different z-scores will vary the level of confidence, which in turn varies the length of the interval. Understanding the effect of varying the level of confidence is essential. Recall that a confidence interval is built around the sample proportion. It does not reflect the variability of the unknown population mean; it provides a range of values that may include the unknown population mean. Other measures such as the level of confidence, the standard error and the margin of error all affect the precision of the results.

Begin by looking at the margin of error. The margin of error is defined as the distance between the sample mean and the end point of the interval. It is sometimes referred to as the radius of the confidence interval.



When the confidence level varies, the confidence interval will also vary, which increases or decreases the margin of error. The larger the margin of error, the wider the interval.



The margin of error is defined as $E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$, where the z-score reflects the level of confidence.

10 Calculate the margin of error for various confidence levels

A social media survey shows that 134 of 256 people accessed some form of social media within the past month. Calculate the margin of error for the given proportion for a 98% confidence interval, correct to 3 decimal places. Assume that normality applies.

THINKING	WORKING
1 Define the required formula for the margin of error.	$E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$
2 Calculate the sample proportion, correct to 2 decimal places.	$\hat{p} = \dfrac{134}{256}$ $= 0.52$
3 Determine the required z-score using technology.	98% confidence level equates to $z = 2.326$.
4 Substitute the required values into the formula for the margin of error.	$E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ $= 2.326 \times \sqrt{\dfrac{0.52 \times 0.48}{256}}$ ≈ 0.073

Interpreting the margin of error

In Worked Example 10, it was found that approximately 52% of people have accessed social media within the past month with a margin of error of approximately 7.3%. This means that, after repeated sampling, the overall percentage of people who accessed social media within the past month would remain within ±7.3% of the population proportion 98% of the time. In other words, you are 98% confident that between 44.7% and 59.3% of people have accessed social media within the past month.

Recall the previous definition of the confidence interval:

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) \text{ or } \hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

This interval will be now redefined using the margin of error.

A confidence interval can be expressed as $\hat{p} \pm E$, where

$$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

11 Calculate the margin of error and expected value from a confidence interval

A given confidence interval is (0.506, 0.594).

(a) Calculate the margin of error.

THINKING	WORKING
1 The margin of error is the length from $\hat{p}$ to the endpoint.	(0.506, 0.594)
2 Calculate the total length of the confidence interval.	Length $= 0.594 - 0.506$ $= 0.088$ The length of the interval is 0.88.
3 Halve the interval length.	$0.088 \div 2 = 0.044$
4 Write the answer.	The margin of error is 0.044.

(b) Calculate the expected value.

1 Recall the formula for expected value.	$E\left(\hat{P}\right) \approx \hat{p}$
2 Take one of the two given endpoints and add/subtract the margin of error. Give your answer correct to 2 decimal places.	$0.594 - 0.044 = 0.55$
3 Write the answer.	$E\left(\hat{P}\right) \approx \hat{p} = 0.55$

Sample size for a desired margin of error

Understanding the relationship between the level of confidence, the interval and the margin of error is essential. In statistical inference, the idea of precision is linked to the margin of error in the confidence interval.

The level of confidence is designed to produce an interval that may capture the true population proportion, but the higher the level, the wider the interval. While this may give a better chance of capturing the population proportion, a higher margin of error is produced. This produces a less precise, larger range of values. For example, two popular baby books indicate when a baby will begin teething. One book states that teething will begin anywhere between 5 and 8 months. The other book states that teething will begin between 3 and 9 months. While both books may be correct, the more precise range is given as 5 to 8 months.

Statistical research is rarely undertaken with the view of 'Let's see what happens'. Research is planned, to see if data supports an idea or conclusion, and is calculated with a high confidence level and a small margin of error. From the above explanation, it looks as though you need to trade off between the level of confidence and a low margin of error. Thankfully, you can adjust the size of the sample to achieve the level of confidence that you want and a desired margin of error.

As a 'rule of thumb', the optimal value of the margin of error is anywhere between 1% and 4% and the value for z is defined by the desired level of confidence. As mentioned, researchers generally aim to minimise the margin of error. Consider again the formula for calculating margin of error, and think about the various ways of minimising the value produced by this equation:

$$E = z\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$$

The determined confidence level will directly affect the value of z in this equation. This leaves $\hat{p}$ and n as the only other variables that will affect the margin of error. The maximum value for the $\hat{p}\left(1-\hat{p}\right)$ part of the equation will occur when $\hat{p} = 0.5$. In other words, when all other factors are held constant, the margin of error will be greatest when $\hat{p} = 0.5$. But the sample proportion $\hat{p}$ is not something that can be manipulated by the researcher; in fact, $\hat{p}$ is what the researcher is trying to use to estimate the population proportion, p.

This leaves the sample size, n, as the only variable available to decrease the margin of error, and fortunately, this variable *is* able to be controlled by the researcher. Because n is in the denominator of the margin of error equation, a larger sample size will result in a smaller margin of error.

Therefore, by assuming that $\hat{p} = 0.5$ to maximise the value of $\hat{p}\left(1-\hat{p}\right)$, and then by rearranging the margin of error equation to solve for n, it is possible to calculate the sample size required to limit your margin of error to an acceptable specified value.

Start with the margin of error formula: $E = z\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$

Rearranging and solving for n, you obtain the formula $n = \left(\frac{z}{E}\right)^2 \hat{p}\left(1-\hat{p}\right)$ for the desired sample size.

The sample size required, when defined by a specific confidence level z and margin of error E, is given by: $n = \left(\frac{z}{E}\right)^2 \hat{p}\left(1-\hat{p}\right)$.

As the value of $\hat{p} = 0.5$ ensures that your margin of error will be less than or equal to the specified value, the formula can be simplified to: $n = 0.25\left(\frac{z}{E}\right)^2$ or $n = \left(\frac{z}{2E}\right)^2$.

WARNING

Note that the substitution of $\hat{p} = 0.5$ into the equation does not mean that the value of $\hat{p}$ will necessarily be 0.5. Rather, it ensures that an appropriate sample size can be calculated, which will safeguard against all possible values of $\hat{p}$ in the study. If the true value for the population proportion p is less than or greater than 0.5, then a smaller margin of error will be generated when using the calculated sample size.

12 Calculate a sample size for a given margin of error

A survey is conducted to determine the proportion of people who support a particular candidate for an upcoming election. Calculate the size of the sample required with 95% confidence and a margin of error of 2.5%.

THINKING	WORKING
1 Define the required values.	$E = 0.025$ 95% confidence level, $z = 1.96$
2 Recall the required formula, substitute the defined values and calculate the sample size.	$n = \left(\frac{z}{2E}\right)^2$ $= \left(\frac{1.96}{2 \times 0.025}\right)^2$ $= 1536.64$
3 Round to the nearest whole number and give the solution.	A sample of 1537 people would be required.

EXERCISE 10.4 Estimating with confidence

Worked Example 10

1 A random sample of 500 people indicated that 53% of them used public transport to travel to work. Calculate the margin of error for the following confidence intervals. Assume that normality applies. Give your answers as percentages correct to 1 decimal place.

(a) a 90% confidence interval

(b) a 99% confidence interval

2 A random sample of 1500 people who were all employed on a full-time basis were asked, if they won $500 000 in the lottery, would they quit work? Of this sample, 300 said that they would. Calculate the margin of error for a 90% confidence interval. Assume that normality applies. Give your answer as a percentage correct to 1 decimal place.

3 For the following confidence intervals, calculate the margin of error, correct to 3 decimal places. Then calculate the expected value correct to 2 decimal places. 11

(a) confidence interval $(0.554, 0.647)$

(b) confidence interval $(0.405, 0.495)$

(c) confidence interval $(0.748, 0.792)$

4 Which of the following techniques could a researcher use to reduce the margin of error in a study involving proportion estimates?

A Increase the sample size, n.

B Decrease the sample size, n.

C Set the proportion estimate $\hat{p}$ as close as possible to 0.5.

D Set the proportion estimate $\hat{p}$ as far away as possible from 0.5.

5 The results of a recent clinical trial calculated that the proportion of patients successfully responding to a particular drug is $\hat{p} = 0.4$. Wishing to recruit for the next phase, clinicians are aiming for a 95% confidence interval that should have a margin of error no greater than 0.03. What sample size will they need for the next phase trial? 12

6 Jake and Elwood are arguing about confidence intervals. Jake claims that a confidence interval of 90% is wider than a confidence interval of 95% for the same data set. Elwood believes the opposite is true. Determine who is correct, and justify your answer.

7 A survey is to be conducted at a local university proposing an increase in fees to support a new resource centre. It is decided that 100 students will be surveyed.

(a) From this sample size, construct a table showing the margins of error for 95% confidence intervals for the following values of $\hat{p}$: 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 0.9.

(b) For a sample of 100, determine whether it would be appropriate to use a confidence interval based on $\hat{p} = 0.03$. Justify your answer.

8 The military would like to record the proportion of soldiers who hit a specific target when conducting parachuting jumps. Real-life parachute jumps are dangerous and expensive. A margin of error of 1.5% on a 95% confidence interval is satisfactory; however, a margin of error of 1% is more desirable. If the military were to use a margin of error of 1% rather than 1.5%, how many more jumps would be required?

9 An inspector tests 400 items in a manufacturing plant. She finds that 50 are defective. Using this information, construct an approximate 95% confidence interval. Give your answers correct to 3 decimal places. 8

10 A secondary school survey finds that 78% of 1500 students would like to wear plain clothes to school instead of a uniform. Using this information, construct an approximate 68% confidence interval. Give your answers correct to 3 decimal places.

11 A recent survey showed that 770 out of 1000 people think that there should be tougher penalties for speeding. Calculate the following confidence intervals for the true proportion of people who think that there should be tougher penalties for speeding. Give your answers correct to 3 decimal places. 9

(a) the 90% confidence interval

(b) the 98% confidence interval

12 Results from a survey indicated that 13 out of 50 people preferred dark chocolate to milk chocolate. After assessing normality, calculate the following confidence intervals for the true proportion of people who preferred dark chocolate to milk chocolate. Give your answers correct to 3 decimal places.

- **(a)** the 95% confidence interval
- **(b)** the 99% confidence interval

13 A sample of 450 residents from a rural community were surveyed, and it was found that 180 came into town to shop at least once a month. After assessing normality, construct a 95% confidence interval to estimate the true proportion of rural residents who come into town to shop at least once a month. Give your answer correct to 3 decimal places.

14 A recent poll found that of the 1685 people surveyed, 650 stated that they regularly read in bed at night.

- **(a)** Determine whether normality applies to the information.
- **(b)** Construct a 99% confidence interval for the proportion of all people who regularly read in bed at night, correct to 3 decimal places.
- **(c)** Do the results support the conclusion that less than half of the population regularly read in bed at night?
- **(d)** Calculate how large a sample would be required to obtain a margin of error of 1.5% in a 99% confidence interval for the proportion of people who regularly read in bed at night.

15 A biological company testing a new growth hormone on 600 trees is confident that between 55.35% and 64.65% of the trees responded well to the treatment. Calculate the level of confidence used, correct to the nearest whole number. Assume that normality applies.

16 A survey of 1500 Brisbane secondary school students reports that 48% feel they have done enough study for an upcoming mathematics exam.

- **(a)** Identify all proportion elements.
- **(b)** Calculate the standard error. Give your answer correct to 4 decimal places.
- **(c)** Assess normality and determine if a confidence interval can be constructed.
- **(d)** If possible, construct a 98% confidence interval on the true proportion of students who feel that they have done enough study. Give your answer correct to 3 decimal places.
- **(e)** Calculate the margin of error as a percentage.
- **(f)** Suppose you wish to refine the estimate of the population proportion and decide to cut the width of the interval in half. From this new information, answer the following questions.
  - **(i)** Determine whether doubling the sample size would cut the width of the interval in half. Justify your answer.
  - **(ii)** Determine the sample size required to cut the interval width in half.
  - **(iii)** Determine the sample size needed to decrease the interval to a point where 50% is not included in a 98% confidence interval centred about the original sample proportion $\hat{p} = 0.48$.

17 Janette, the manager of a school cafeteria, intends to change the lunch menu; however, before proceeding with the change she surveys 125 students to see if the new menu is more appealing than the current menu. Janette has collated the results and found that 70 students like the new menu more than the current menu.

(a) Assess the normality for this sample and determine if a confidence interval can be constructed.

(b) Janette asked Brooke, the assistant manager of the school cafeteria, to check her results. Brooke noticed that Janette had miscounted and that there were in fact 8 more in favour of the new menu. Determine two 95% confidence intervals, one for Janette's results and one for Brooke's. Interpret your findings.

(c) Determine which of the two sets of data has a higher margin of error. Explain your answer using mathematical reasoning.

(d) Janette wants an even smaller margin of error for the actual results for the 95% confidence interval. What should she do to further decrease the margin of error?

18 A new brand of dishwashing liquid is delivered in 750 mL bottles. The bottles are automatically filled and random samples are selected and checked so that the standard volume is maintained. On a given day, a sample of 100 bottles has the volume of liquid measured and 94% of the bottles are found to contain between 745 mL and 755 mL.

(a) Assuming normality for this sample, determine the sample proportion of the bottles that contains less than 745 mL or more than 755 mL. Give your answer correct to 2 decimal places.

(b) Calculate the margin of error for a 95% confidence interval of the bottles containing between 745 mL and 755 mL. Give your answer correct to 4 decimal places.

(c) Determine the 95% confidence interval and (with a 95% confidence) the number of bottles that would contain between 745 mL and 755 mL in a sample of 100 bottles.

(d) Determine what sample size should be chosen for a margin of error of 0.02.

(e) Determine how the margin of error $E = 0.02$ would change the 95% confidence level. Include an interpretation for both the margin of error and the confidence interval for this situation.

10

Summary

Types of sampling

Non-probability sampling

Convenience sampling: a method of data collection from population members who are most conveniently able to participate in a survey.

Purposive sampling: survey participants are selected purely based on the judgement of the researcher.

Snowball sampling: survey participants are encouraged to use social networks and contacts to recruit additional participants.

Probability sampling

Random sampling: each unit in the population has an equally likely chance of being selected. The sample statistics obtained using random samples have a predictable theoretical relationship with the population parameters. Bias can be controlled or, at the very least, minimised.

A simple random sample can be obtained through methods such as writing names on a piece of paper and drawing them out of a hat. For larger samples, technology can be used to generate pseudo-random numbers.

Statistical terminology

A unit is a single member of a finite population. It may be a person, animal, company or object.

The population is the set of all the units that make up a particular group.

A census measures characteristics from the entire population.

A sample is a subset of units taken from the entire population.

A statistic is a numerical measure that describes a characteristic of a sample.

Bias in sampling

A study or sample displays bias if it systematically favours or eliminates certain outcomes.

Systematic bias occurs from decisions made in the preparation of the statistical process.

Sample frame error occurs when the wrong sub-population is used to select a sample.

Undercoverage occurs when a random sample is taken that is not truly representative of the population you wish to target.

Nonresponse bias occurs when a selected member cannot or will not cooperate.

Response bias occurs when a selected member of the sample may lie or give misleading answers.

The central limit theorem

The central limit theorem (CLT) states that when the sample size n is sufficiently large, the distribution of the sample mean is approximately normal. For any parent population with mean μ and standard deviation σ, the distribution of the sample mean $\overline{X}$ tends to a normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$.

The following two conditions must be met for the central limit theorem to apply to the binomial distribution.

- $n\hat{p} \geq 5$
- $n(1-\hat{p}) \geq 5$

Sample proportions

Notation and terminology

Notation	Definition
x	Observation: For any one sample, this is the number of people or things that possess a certain characteristic.
n	Sample size: This is the number of people or things that make up the sample.
$\hat{p} = \frac{x}{n}$	Sample proportion: This is the number of people or things with a characteristic that occurs within the sample (observation) over the number in the sample.
p	Population proportion: This is the true proportion of the people or things with a characteristic that occurs within the whole population.

The sample proportion $\hat{p} = \frac{x}{n}$ is not the population proportion; rather it is the best estimate of the population proportion from a single sample. This leads to the sample proportion sometimes being referred to as the point estimate.

The standard deviation of a binomial distribution can be estimated from the sample proportion $\hat{p}$ of one sample, rather than many. This cannot be done easily for other distributions.

The distribution of $\hat{P}$ for the sample proportions is defined as $\hat{P} \sim N\left(\hat{p}, \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$, where the expected value $E(\hat{P}) \approx \hat{p}$ and the standard deviation $SD \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = S(\hat{P})$.

Confidence intervals

For normal distributions:

- A 68% range occurs within approximately one standard deviation of the population proportion.
- A 95% range occurs within approximately two standard deviations of the population proportion.
- A 99.7% range occurs within approximately three standard deviations of the population proportion.

A 95% confidence level means that the probability that the confidence interval will capture the true population proportion p in repeated samples is 0.95 or 95%.

Confidence intervals and z-scores

Using the standard normal distribution, $Z \sim N(0, 1)$, the sample proportion $\hat{p}$ has an approximate standardised value of $\dfrac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$.

A confidence interval can be defined as

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right) \text{ or } \hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where z is the defined z-score that corresponds to the appropriate confidence level.

Margin of error and confidence intervals

The margin of error is defined as $E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$, where the z-score reflects the level of confidence.

A confidence interval can be expressed as $\hat{p} \pm E$, where $E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.

Sample size for a desired margin of error

The sample size required, when defined by a specific confidence level z and margin of error E is given by $n = \left(\dfrac{z}{E}\right)^2 \hat{p}(1-\hat{p})$.

As the value of $\hat{p} = 0.5$ ensures that your margin of error will be less than or equal to the specified value, the formula can be simplified to $n = 0.25\left(\dfrac{z}{E}\right)^2$ or $n = \left(\dfrac{z}{2E}\right)^2$.

Chapter review 10

1 For each of the following cases, identify the population, the sample, the parameter and the statistic. Exercise 10.1

(a) In 2013, a survey of 5000 patients showed that the average waiting time in the emergency department at public hospitals was approximately 6 hours.

(b) Every Monday, a quality control officer tests 50 packets of a particular brand of coloured, coated chocolate buttons to confirm that the colours are evenly dispersed in each packet.

2 For each of the following cases, identify any source of potential bias. (There may be more than one answer.) Exercise 10.1

(a) A record of all car number plates is taken by tollbooths on all Sydney motorways. Each motorist is sent a questionnaire regarding funding for future motorways.

(b) A motor dealership emails a survey regarding customers' thoughts on the recent service of their vehicle.

3 The average score for a geography exam out of 100 is 65 marks, with a standard deviation of 9 marks. Estimate the probability that the average scores of 35 randomly selected students will be more than 68 marks. Give your answer correct to 4 decimal places. Exercise 10.2

4 For each of the following cases, identify the population proportion, sample size, observation value and sample proportion. Exercise 10.3

(a) A fitness magazine stated that from a survey of 545 people, 324 felt that they did not use their gym membership to its fullest potential.

(b) The results of an Australian-wide survey found that out of 10 000 records, one in three households owns more than one motor vehicle.

5 Consider each of the following cases involving sample proportions. Exercise 10.3

(a) $\hat{p} = 0.27$ and $n = 350$

(i) Calculate the expected value. Then calculate the standard error, correct to 3 decimal places.

(ii) Assess the normality.

(b) $x = 9$ and $n = 10$

(i) Calculate the expected value. Then calculate the standard error, correct to 3 decimal places.

(ii) Assess the normality.

6 Calculate the following values from the given information. Give your answers correct to the nearest whole number. Exercise 10.3

(a) Given $\hat{p} = 0.88$ and $x = 30$, calculate the sample size.

(b) Given $\hat{p} = 0.05$ and $x = 400$, calculate the sample size.

(c) Given $\hat{p} = 0.45$ and $n = 1000$, calculate the observation value.

(d) Given $\hat{p} = 0.9$ and $n = 730$, calculate the observation value.

7 A statistician flipped a fair coin 1000 times. From this, a count of 512 heads was obtained. Determine the number of times the coin should be flipped for 99% confidence with a margin of error of 1%. Exercise 10.4

Exercise 10.4

8 A researcher surveys 250 people who have been arrested for driving under the influence (DUI). Of those surveyed, 36 people have had previous DUI charges. Calculate the margin of error for a 90% confidence interval, correct to 1 decimal place.

Exercise 10.4

9 If you increase the sample size and assume all other values stay the same, explain what happens to the width of a confidence interval.

Exercise 10.1

10 Define the difference between a sample and a census, and why a company would choose to analyse a sample rather than a census.

Exercise 10.4

11 Explain the importance of sample size when conducting a statistical investigation.

Exercise 10.4

12 Explain the strengths and weaknesses of using simple random sampling as a statistical technique.

Exercise 10.4

13 A 95% confidence interval is used to obtain an estimate for an unknown population proportion. Determine the sample size that will guarantee a margin of error of 3.5%.

Exercise 10.4

14 An entomologist samples 75 randomly chosen locations within a forest for a particular type of insect. He successfully finds the insect in 13 locations. Calculate the following confidence intervals for the true proportion of all possible locations of the particular insect. Give your answers correct to 3 decimal places. Assume the normality criteria are met.

(a) 90% confidence interval

(b) 95% confidence interval

(c) 98% confidence interval

(d) 99% confidence interval

Exercise 10.3

15 As part of a statistics project, Jiang needs to determine the effects of sample size on the sample proportion and standard error. He surveys students in groups of varying sizes and asks students if they are considering completing a postgraduate degree after university.

(a) For each of the following observations, calculate the sample proportion and standard error.

(i) 6 out of 40 students are considering completing a postgraduate degree.

(ii) 15 out of 100 students are considering completing a postgraduate degree.

(iii) 54 out of 360 students are considering completing a postgraduate degree.

(iv) 75 out of 500 students are considering completing a postgraduate degree.

(b) Discuss what Jiang observed from performing these calculations.

Exercise 10.4

16 Brothers Ahmed and Farid share a digital music device that can play songs in a random selection. There are a total of 3570 songs on the device, some loaded by Ahmed and some by Farid. When in random play mode, 18 of the first 42 songs were loaded by Farid. Confirming normality, construct a 99% confidence interval for the proportion of songs that were loaded by Ahmed.

Exercise 10.4

17 A random sample of 250 Year 7 students found that 100 students preferred to work through mathematical problems in pencil, whereas 150 students preferred to use pen.

(a) Confirming normality, construct a 95% confidence interval for the proportion of Year 7 students that prefer to work through mathematical problems in pen.

(b) From these results, a teacher has stated that more than half of the students in Year 7 prefer to work mathematical problems out in pen. Determine whether this statement is supported by your results. Justify your answer.

18 A random sample of 600 students from a large high school found that 445 students would have chosen at least one different subject if they had been given more information. Exercise 10.4

(a) Confirming normality, construct a 99% confidence interval for the proportion of students at this high school who would have chosen at least one different subject. Give your answer correct to 3 decimal places.

(b) Express your results from part **(a)** in terms of the percentage of students who would have chosen different subjects.

(c) The head of secondary studies is concerned with this information and wishes to make more information available to students. The secondary school has a population of 2100. Convert the confidence interval (found in part **(a)**) to a confidence interval for the number of students who would have chosen different subjects.

19 A major research company surveyed 1500 people regarding their concern about the environment. The results of this survey show that 1254 people have high concerns about the environment, 187 have some concern, while the rest have no concern. Exercise 10.4

(a) Calculate the point estimate of the population proportion of people who have no concern about the environment. Give your answer correct to 3 decimal places.

(b) Calculate the margin of error for 90%, 95% and 98% confidence intervals for the people who are concerned about the environment. Give your answers correct to 4 decimal places.

(c) Calculate the three confidence intervals.

20 In a trial for a new clinical treatment, 930 subjects showed improvement from the treatment. The research team declared, 'We are 95% confident that this treatment will improve the conditions of 59.54% to 64.46% of the patients'. Exercise 10.4

(a) Demonstrate $\hat{p} = \dfrac{CI_U + CI_L}{2}$, where CI_L is the lower limit of the confidence interval and CI_U is the upper limit of the confidence interval. Hence, calculate the point estimate for this sample and the margin of error.

(b) Determine the number of subjects in the sample for this point estimate.

(c) Demonstrate $E = \dfrac{CI_U - CI_L}{2}$, where CI_L is the lower limit of the confidence interval and CI_U is the upper limit of the confidence interval. Hence or otherwise, calculate the z-score for this level of confidence.

21 At the end of each financial year, companies estimate their profits for the following financial year. Out of 258 companies, 168 reported that they beat their estimated profit, 39 matched their estimates, while 51 fell short of their estimates. Exercise 10.4

(a) Calculate the point estimate for the companies that beat their estimated profit and the ones who matched their estimates. Give your answers correct to 3 decimal places.

(b) Calculate a 98% confidence interval for the companies that fell short of their estimated profits.

(c) Determine the margin of error for the companies that fell short of their estimated profits. Give your answer correct to 4 decimal places.

(d) Determine the size of the sample required for a margin of error of 0.05.

(e) What confidence interval would this margin of error produce? Compare it with the confidence interval $(0.140, 0.256)$.

Exam review

UNIT 4

Paper 1: Technology-free

Exercise 6.3

1 If $f(x) = x^4 + 6x^3 - 2x^2 + x - 3$, determine the value of $f''(-2)$.

Exercise 6.4

2 A rectangular garden is to be formed next to an existing fence. It is enclosed on three sides (excluding the fence side) by 32 m of border trim. The width of the garden that runs perpendicular to the fence is x m, and its length, which runs parallel to the fence, is y m.

(a) Determine an expression for the length y in terms of x.

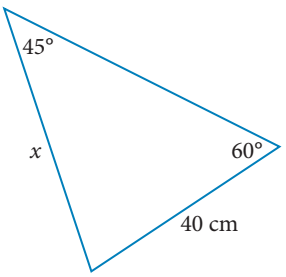
(b) Determine an expression in factorised form for the area $A(x)$.

(c) Use second derivative analysis to determine the maximum area of the garden.

(d) Hence determine the dimensions of the garden of maximum area.

Exercise 7.3

3 Determine the exact value of x.



Exercise 7.4

4 A Brahminy kite (bird of prey) is being photographed by two ornithologists on the ground, who are 2 km apart. From one ornithologist, the direct distance to the Brahminy kite is 100 m and the angle of elevation is 30°. Determine the exact distance, in metres, of the Brahminy kite from the other ornithologist.

Exercise 8.2

5 A coin is biased in such a way that $P(\text{Heads}) = 4 \times P(\text{Tails})$. Determine the probability that exactly 5 tosses out of 8 land heads up. Write your answer as a product of prime factors.

Exercise 8.3

6 Three coins are tossed and you count the outcome of two heads and a tail as a success. Determine the exact values of the mean and the variance of the Bernoulli distribution (the number of successes in a single trial).

Exercise 9.1

7 If $f(x)$ represents a probability density function, determine the exact value of $P(X < 2)$ given:

$$f(x) = \begin{cases} \frac{\pi}{16}\sin\left(\frac{\pi x}{8}\right) & 0 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 9.3

8 For a particular normal distribution, $P(X < a) = 0.18$ and $P(X < b) = 0.45$. Determine the following probabilities.

(a) $P(X > a)$ **(b)** $P(X > b)$ **(c)** $P(a < X < b)$

9 In the following cases, identify the source of bias. Exercise 10.1

(a) A council department mails out a survey regarding opinion on the demolition of a community centre.

(b) A TAFE campus surveys students at lunchtime to determine whether they have engaged in plagiarism in the past.

10 An education journal states that in a survey of 600 first-year teachers, only 300 felt that they had received adequate mentoring so that they were fully prepared to teach their assigned classes. Identify the population proportion, sample size, observation value and sample proportion. Give your answer to 2 decimal places, where necessary. Exercise 10.3

11 Consider the function $f(x) = \frac{1}{3}x^3 - x^2 - 15x$. Exercise 6.1

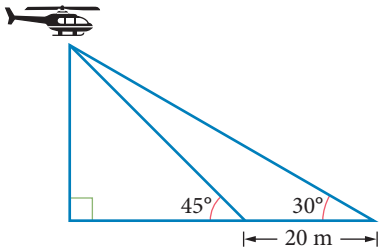
(a) Determine an expression for $f'(x)$.

(b) Calculate $f'(-2)$.

(c) Determine the gradient of the graph of the function at the points where it meets the x-axis.

(d) Determine the set of values of x for which $f'(x) \leq 0$.

12 A helicopter is sighted at the same time by two observers, Fran and Cathy, on the ground. Fran and Cathy are 20 m apart on the same side of the helicopter, as shown. They find the angles of elevation to be 45° and 30°. Determine the exact height of the helicopter in metres. Exercise 7.1



13 In a game, two spinners numbered 1 to 6 are spun. If the numbers are both even, then X is the larger number of the two. If the numbers are both odd, then X is the smaller number of the two. If one number is odd and the other is even, then X is the difference between the numbers. Exercise 8.1

(a) Draw a table to show the probability distribution of X. Express the probabilities as fractions.

(b) Calculate the following probabilities.

(i) $P(X < 6)$

(ii) $P(X = 2 | X \text{ is even})$

(iii) $P(X = 6 | \text{same number on both spinners})$

14 Determine estimates for the standard deviation for each given scenario. Exercise 9.4

(a) The height H cm of a group of students is such that $H \sim N\left(140, \sigma^2\right)$. In addition, $P(H < 124) = 0.025$.

(b) The height H cm of a group of students is such that $H \sim N\left(172, \sigma^2\right)$. In addition, $P(H < 183) = 0.84$.

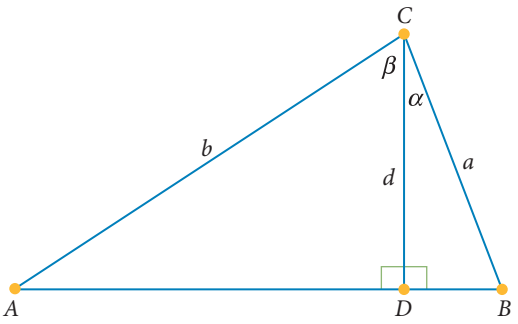
(c) The length L cm of babies born in May in a large hospital is such that $L \sim N\left(50, \sigma^2\right)$. In addition, $P(L > 43.1) = 0.9985$.

Exercise 10.4

15 A political poll is conducted by Polling Company A, which determines that a certain party's support is given by the 95% confidence interval of (14.6%, 15.4%). At the same time, a poll is conducted by rival Polling Company B, which shows that support for the same party is given by the 95% confidence interval of (14.9%, 15.1%). Determine approximately how much larger was the sample used by Polling Company B compared to Polling Company A.

Exercise 7.2

16 Consider the ΔABC.



(a) Write expressions for d, in terms of a and α, then in terms of b and β.

(b) Determine the area of ΔADC in terms of b, d and β.

(c) Determine the area of ΔDBC in terms of a, d and α.

(d) Determine the area of ΔABC in terms of a, b and $\alpha + \beta$.

(e) Write an expression for $\sin(\alpha + \beta)$ in terms of $\sin(\alpha)$, $\cos(\alpha)$, $\sin(\beta)$ and $\cos(\beta)$.

(f) Determine the exact value of $\sin(105°)$.

Paper 2: Technology-active

Exercise 6.1

1 Consider the function $f(x) = -2x^3 + 4x^2 + 5x - 6$.

(a) Use calculus to determine the exact values of x where the stationary points occur on the graph of $y = f(x)$.

(b) Sketch the graph of $y = f(x)$, labelling the stationary points and the intercepts with the axes, correct to 2 decimal places where necessary.

(c) Use the graph to determine the values of x for which $f(x)$ is an increasing and decreasing function.

Exercise 6.2

2 The position x cm from a fixed origin at time t s of a particle travelling in a straight line is given by $x(t) = t^3 - 8t^2 + 10t + 7$ for $0 \leq t \leq 3$.

(a) Describe the particle's behaviour at the start.

(b) Determine the particle's acceleration at $t = 1$.

(c) Determine the particle's velocity at $t = 1$.

(d) Determine, correct to 2 decimal places, the time(s) when the particle changes direction.

Exercise 7.2

3 A triangular paddock has an area of 8420 m^2. If two of the fences measure 260 m and 310 m, determine the acute angle at which these two fences meet, correct to 2 decimal places.

Exercise 7.4

4 A bushwalker walks from point U, 8 km on a bearing of 252°T, to point S, then a further 20 km on a bearing of 112°T to point T. How far must she walk to return to point U? Give your answer to the nearest 10 metres.

5 Consider the following probability distribution. Exercise 8.1

x	12	13	14	15
$P(X = x)$	0.02	0.23	0.44	0.31

Determine the following values.

(a) $E(X)$ **(b)** $Var(X)$

(c) the standard deviation of X, correct to 2 decimal places

6 For the binomial distribution $X \sim B(45, 0.65)$, determine the values of $E(X)$, $Var(X)$ and $\sigma(X)$. Give your answers correct to 2 decimal places, where necessary. Exercise 8.3

7 Determine the following probabilities, correct to 4 decimal places. Exercise 9.2

(a) Given $X \sim N(21.3, 8.77)$, calculate $P(22.6 < X < 23.5)$.

(b) Given $X \sim N(6.2, 5.4)$, calculate $P(X > 5)$.

8 The height X cm of a group of children is known to be normally distributed with $X \sim N(123.1, 9.68)$. For the following questions, give your answers correct to 1 decimal place and include the required units. Exercise 9.4

(a) What is the height under which you would expect to find 80% of the population?

(b) Calculate k such that $P(X > k) = 0.9$.

(c) Determine the end values for the middle 20% of the heights.

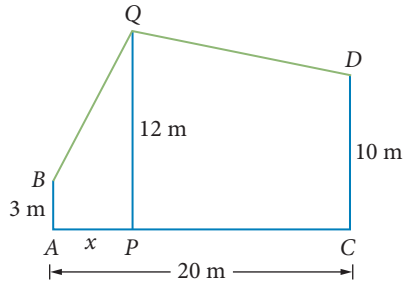
9 A survey of 500 people indicates a sample proportion of $\hat{p} = 0.65$. Exercise 10.3

(a) Calculate the standard error, correct to 3 decimal places.

(b) Assess the normality.

10 A 95% confidence interval is used to obtain an estimate for an unknown population proportion. Determine the sample size that will guarantee a margin of error of 6%. Exercise 10.4

11 Two vertical posts, 20 m apart and with heights of 3 m and 10 m, are secured by a rope that connects the tops of the posts to the top of a third post of height 12 m, placed between the two. The length of rope is L m and the distance from the 3 m-high post to the middle post is x m. Exercise 6.4



(a) Determine an expression for $L(x)$.

(b) Determine the location of the middle post, correct to 2 decimal places, so that a minimum amount of rope is used.

(c) Calculate, correct to 1 decimal place, the minimum length of rope used.

12 Solve ΔABC, given $A = 35°$, $a = 12$ and $b = 15$. Give your answers correct to 1 decimal place. Exercise 7.3

13 Harrison takes 15 shots at goal with a 70% chance of scoring for each shot. Determine the probability of each of the following outcomes, correct to 3 decimal places. Exercise 8.2

(a) between 10 and 12 goals, inclusive

(b) between 5 and 10 goals, inclusive

Exercise 9.1

14 Andrea is waiting for a taxi at a popular venue at a busy time of the day. The continuous random variable T, in hours, that represents the time spent by people waiting for a taxi at this venue is given by the following probability density function:

$$f(t) = \begin{cases} \frac{5}{2}t\sqrt{t} & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the mean time Andrea should expect to wait to catch a taxi. Give your answer correct to the nearest minute.

(b) Calculate the probability of each of the following, correct to 2 decimal places.

(i) Andrea will have to wait more than 15 minutes to catch a taxi.

(ii) Andrea's wait for the taxi will be between 10 and 40 minutes.

Exercise 10.4

15 A biosecurity expert inspects 80 randomly chosen items of imported furniture for the presence of pests. She detects traces of the exotic wood borer beetle in 18 of the items. Calculate the following confidence intervals for the true proportion of instances of the wood borer beetle in this type of imported furniture. Give your answers correct to 3 decimal places and assume the normality criteria are met.

(a) 90% confidence interval

(b) 95% confidence interval

Exercise 6.3

16 The function $f(x)$ has second derivative $f''(x) = 6x + k$. The graph of the function has a y-intercept at $(0, 14)$, a point of inflection at $x = \frac{5}{3}$ and a turning point at $x = 4$. Determine the equation of the function and sketch its graph, showing axis intercepts, turning points and point of horizontal inflection. Give values correct to 2 decimal places, where necessary.

Exercise 9.3

17 In a Year 12 Mathematical Methods class, the mean height of the girls is 165 cm, with a standard deviation of 5.4 cm, whereas the mean height of the boys is 175 cm, with a standard deviation of 6.2 cm. Two-thirds of the girls and three-quarters of the boys in the class study chemistry. If a boy and a girl are selected at random from the class, determine the probability that at least one of them is a chemistry student taller than 170 cm. Give your answer correct to 2 decimal places.

Exam review

UNITS 3 & 4

Paper 1: Technology-free

1 Solve $e^{2-x} = 5$ for x in exact form. Exercise 1.1

2 What transformation(s) of $y = e^x$ would be necessary to draw the graphs of each of the following? Exercise 1.2

(a) $y = e^{-x}$ (b) $y = 3e^x - 7$

3 Determine $\frac{dy}{dx}$ for each of the following functions. Exercise 1.3

(a) $y = 2e^x - \frac{1}{e^x}$ (b) $y = \sqrt{e^{2x}}$ (c) $y = \frac{4}{e^{x+1}}$

4 The tail motion of a particular species of fish can be modelled by the equation $D = \sin(15\pi t)$, $t \geq 0$, where D represents the position of the fish's tail at time t seconds, as measured in centimetres to the left (negative) or right (positive) of a straight line along its length. Exercise 2.2

(a) At 2.9 seconds, is the tail to the left or right of the straight line along its length?

(b) How far to the left or right is the tail at 2.9 seconds?

(c) In the domain $0 \leq t \leq \frac{2}{15}$, at what times is the tail position at $D = 0.5$?

5 Differentiate each of the following functions. Exercise 2.4

(a) $y = x^3 \sin(x)$ (b) $y = e^{3x} \tan(2x)$

6 Determine the derivative of each of the following functions. Exercise 3.6

(a) $y = \log_e\left(3x^4 + 2x\right)$ (b) $y = (5x - 2)\log_e(3x)$

7 Determine each of the following anti-derivatives. Exercise 4.5

(a) $\int \left(2x^3 - 3x + 2\sqrt{x}\right) dx$ (b) $\int \left(4x - \cos\left(2x + \frac{\pi}{3}\right)\right) dx$

(c) $\int \frac{2}{(3-4x)^5} dx$ (d) $\int \frac{e^{4x} - 1 + e^{-3x}}{e^{2x}} dx$

8 Determine y as a function of x for each of the following. Exercise 4.2

(a) $\frac{dy}{dx} = \frac{-4}{3+7x}$ and $y = 2$ when $x = -\frac{4}{7}$

(b) $\frac{dy}{dx} = \frac{(x-5)^2}{2}$ and $y = 0$ when $x = 0$

(c) $\frac{dy}{dx} = \frac{1}{\sqrt{2+14x}}$ and $y = 0$ when $x = 1$

Exercise 4.4

9 Determine $f(x)$ for each of the following.

(a) $f'(x) = -6\sin\left(3(x-\pi)\right),\ f(0) = 2$

(b) $f'(x) = e^{2x}e^{1-3x},\ f(0) = 2e$

(c) $f'(x) = \dfrac{2}{5-2x},\ f(0) = \log_e(25),\ x < \dfrac{5}{2}$

(d) $f'(x) = \cos\left(2x + \dfrac{\pi}{6}\right),\ f(0) = \dfrac{9}{4}$

Exercise 4.3

10 Determine the equation of the curve that has a gradient given by $\dfrac{dy}{dx} = a(2x-5)^3$ and passes through the point $(2, 0)$. The gradient of the tangent at this point is -4.

Exercise 5.2

11 Calculate the value of the following definite integrals.

(a) $\int_0^{\frac{\pi}{2}} 2\cos\left(\dfrac{x}{2}\right)dx$

(b) $\int_0^{\frac{\pi}{2}} \left(\cos(2x) - 6x^2\right)dx$

(c) $\int_3^7 \sqrt{2x-5}\,dx$

(d) $\int_e^{2e} \dfrac{3}{2x}dx$

Exercise 6.1

12 Determine the global maximum and the global minimum values for the function $f(x) = -x^3 + 12x,\ -1 \le x \le 3$.

Exercise 6.2

13 The position x metres of a particle travelling in a straight line from a fixed origin at time t seconds is given by $x(t) = -t^3 + 3t^2 + 18t$ for $t \ge 0$.

(a) Determine an expression for the particle's velocity, in m/s, at time t seconds.

(b) Determine an expression for the particle's acceleration, in m/s^2, at time t seconds.

(c) Determine the particle's initial acceleration.

(d) Determine when the particle's acceleration is zero and where the particle is at this instant.

(e) Determine the maximum velocity and the time at which it occurs.

Exercise 10.1

14 The NSW Nurses Cooperative selects a random sample of 2000 nurses and asks if they have worked overtime at least once in the previous two weeks. Identify the population, the sample, the parameter and the statistic.

Exercise 10.3

15 A simple random sample of 730 people over eighteen years of age were surveyed in Perth, and as a result a Perth newspaper stated that two-thirds of Perth residents over the age of eighteen years own a smartphone. Using this information, identify the population proportion, sample size, observation value and sample proportion.

Exercise 2.3

16 Sketch the graph of $y = 5\tan\left(\dfrac{1}{2}(x-\pi)\right)$ for $-2\pi \le x \le 2\pi$, labelling the x-intercepts and asymptotes.

Exercise 3.3

17 Given that $\dfrac{e^x}{e^y} = 3$ and that $\ln(2x+y) = \ln(3) + \ln(5)$, determine the values of x and y.

Exercise 4.3

18 A curve has a gradient function $\dfrac{dy}{dx} = a(4x-5)^3$. The equation of the tangent to the curve at the point $(1, 1)$ is $y = -3x + 4$. Determine the equation of the curve.

19 Differentiate the function $f(x) = x^3(3x-2)^2$ and hence determine $\int 9x^2(3x-2)(5x-2)\,dx$. Exercise 4.1

20 Consider the relation $y^2 = 16(4-3x)$. Exercise 4.7

(a) Write the relation as two separate functions and determine the anti-derivative of each one.

(b) Verify that the sum of the two anti-derivatives is independent of x.

21 Determine the area enclosed by the curves $f(x) = x(x+4)(x-2)$ and $g(x) = x^2 + 2x - 8$. Exercise 5.4

22 Consider the function $f(x) = x^4 - 2x^2$. Exercise 6.3

(a) Use second derivative analysis to locate any stationary points and determine their nature.

(b) Determine the coordinates of any points of inflection.

(c) Sketch the graph of $y = f(x)$, showing all key features.

23 Sketch the graph of $y = \left(x^2 - 15\right)e^x$ in as much detail as possible. Exercise 1.5

24 If $f'(x) = ax + b$, determine expressions for the following. Exercise 4.7

(a) $\int f'(x)\,dx$

(b) $\int \left(f'(x)\right)^3 dx$

(c) $\int \dfrac{1}{f'(x)}\,dx$

(d) $\int 3\sin\left(2f'(x)\right)dx$

Paper 2: Technology-active

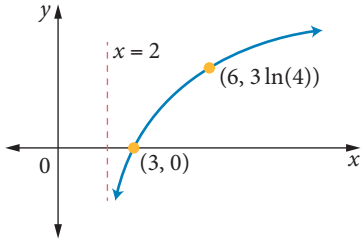
1 What transformation(s) of $y = e^x$ would be necessary to draw the graphs of each of the following? Exercise 1.2

(a) $y = -e^{\frac{x}{2}} + 1$

(b) $y = e^{x-3} + 2$

2 A water tank is filling at a rate of $\dfrac{dV}{dt} = 4 + e^{-0.8t}$ where $V\,\text{m}^3$ is the volume of water in the tank and t is the time in minutes. If the tank initially holds $10\,\text{m}^3$, determine the volume of water in the tank after 4 minutes. Give your answer correct to 1 decimal place. Exercise 4.2

3 The rule for the graph of the following function is of the form $y = A\log_e(x-b)$. Determine the values of A and b. Exercise 3.4



Exercise 4.3

4 Determine the following anti-derivatives.

(a) $\int \frac{4}{3x-7}\,dx$

(b) $\int -3\cos(1-\pi x)\,dx$

(c) $\int \frac{x^2(3x-5)}{x^3}\,dx$

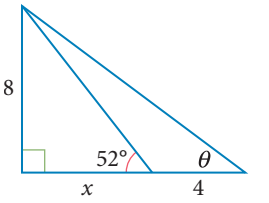
(d) $\int \frac{2}{(3-5t)^{\frac{3}{2}}}\,dt$

Exercise 5.1

5 Use the left-endpoint rule to approximate the area of the region bounded by the graph of $f(x)=\frac{1}{2}x^2$, the x-axis, and the lines $x=1$ and $x=4$. Use three rectangles.

Exercise 7.1

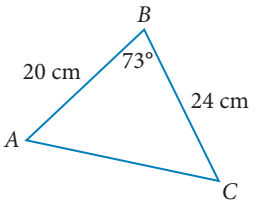
6 For the following diagram, calculate the value of each pronumeral, correct to 1 decimal place.



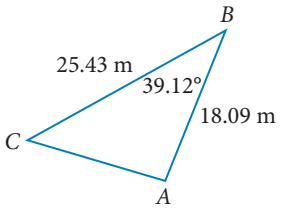
Exercise 7.2

7 Calculate the area, correct to 2 decimal places, of each of the following triangles.

(a)



(b)



Exercise 8.1

8 The variable X has the following probability distribution.

x	4	5	6	7	8	9
$P(X=x)$	0.2	0.15	0.1	0.3	0.2	0.05

Determine the standard deviation, correct to 4 decimal places.

Exercise 8.2

9 The probability of any particular egg being cracked in a carton containing a dozen eggs is 0.075. Determine the probability that exactly three eggs are cracked, giving your answer correct to 3 decimal places.

Exercise 9.4

10 Using $Z \sim N(0,1)$, calculate k, correct to 3 decimal places.

(a) $P(-k<Z<k)=0.88$

(b) $P(Z<-k \cup Z>k)=0.1$

Exercise 10.3

11 A recent survey of 1500 Year 12 students showed that 990 were intending to apply for university. Calculate the expected value for the proportion of people applying for university, correct to 2 decimal places. Then calculate the standard error, correct to 4 decimal places.

12 An avocado farmer is planting an experimental plot of 200 avocado trees of a new variety. The farmer knows that the probability of a tree surviving to maturity is 0.55. Exercise 8.3

(a) Determine the number of trees expected to survive to maturity.

(b) Determine a range of values between which you could predict the number of trees in 95% of cases.

13 Consider the continuous variable with probability density function:

$$f(x) = \begin{cases} \dfrac{x+2}{16} & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the following, correct to 4 significant figures. Exercise 9.1

(a) the mean

(b) the standard deviation

14 Television ratings show that 994 out of a sample of 1329 Australians watched a particular movie. Calculate a 95% confidence interval, correct to 2 decimal places. Exercise 10.4

15 Books@Review publishing company has just published a new mathematics textbook. In order to determine the selling price of the textbook, the sales manager of the company has recorded the prices of a random sample of 132 comparable textbooks available on the market and concluded that 75% of the textbooks cost more than \$65.75. Exercise 10.3

(a) Calculate the expected value and the standard error for the textbooks that cost more than \$65.75.

(b) Determine a 99% confidence interval for the proportion of textbooks that cost less than \$65.75.

(c) Determine a 99% confidence interval for the textbooks that cost more than \$65.75 and interpret its meaning.

(d) Determine the margin of error for the two price brackets. What do you notice about the two values? Explain your answer.

16 Use the substitution $u = \sin(\theta)$ and the identity $\int f(u)du = \int f(u)\dfrac{du}{d\theta}d\theta$ to determine $\int \sin^2(\theta)\cos(\theta)d\theta$. Exercise 4.5

17 The acceleration a m/s^2 of an object moving in a straight line at time t seconds is given by $a = t - e^{\frac{t}{2}}$. Exercise 4.2

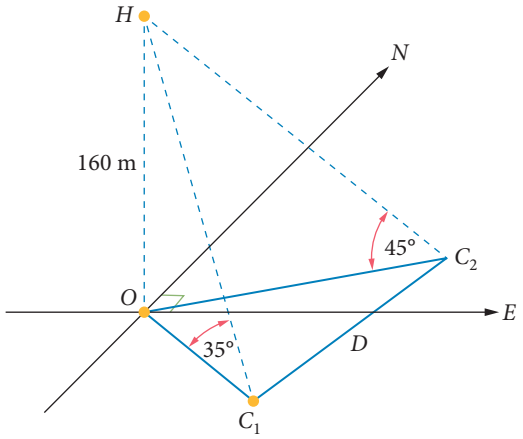
(a) If the object has an initial velocity of 20 m/s, determine an expression for the velocity of the object in terms of time.

(b) Determine the time when the velocity is zero, correct to 2 decimal places.

(c) Describe the motion of the object in the next few seconds after the velocity is zero.

Exercise 7.4

18 A police helicopter (H) is hovering 160 m above point O on the ground and sights two floodwater rescue crews, C_1 and C_2, at angles of depression 35° and 45°, respectively. The three-dimensional situation is shown below, but for clarity, the corresponding angles of elevation are shown. If the ground bearings from O to the rescue crews are 070°T, and 130°T respectively, determine the distance D between the rescue crews (to the nearest metre).



Exercise 9.2

19 Ivana sells freshly cut Christmas trees. She has 50 trees on her property ready to be cut and delivered. Prices depend on the height of the tree. The table below shows the heights of the trees, in metres.

1.61	1.66	1.68	1.68	1.71	1.71	1.71	1.72	1.73	1.73
1.73	1.76	1.76	1.77	1.77	1.77	1.77	1.79	1.79	1.79
1.81	1.81	1.81	1.81	1.82	1.83	1.83	1.83	1.83	1.84
1.86	1.86	1.86	1.86	1.86	1.86	1.87	1.87	1.88	1.91
1.91	1.91	1.93	1.93	1.94	1.94	1.96	1.96	1.96	2.01

(a) Calculate the mean height of the Christmas trees.

(b) Display this data in a frequency table using 1.6–<1.65 as the first interval.

(c) Display this data on a histogram and comment on its shape.

(d) Calculate the standard deviation for this set of data using the formula $\sigma = \sqrt{E\left(X^2\right) - \left[E(X)\right]^2}$. Give your answer correct to 4 decimal places.

(e) What percentage of trees does the normal model predict to be in the category 1.7–<1.75? How does this compare to the actual data? What assumption did you make in answering this question?

(f) What percentage of trees does the normal model predict to be in the interval 1.82–<1.92? Why is it difficult to compare this result with the actual sample values?

20 A random sample of 800 shareholders were surveyed about various issues regarding the share market. One of the results of this survey is that 49% of the respondents bought new shares in the previous year. Exercise 10.4

(a) Assess normality for this sample.

(b) Determine the number of shareholders that you can say with 98% confidence bought new shares in the previous year.

(c) Out of all the respondents, 15% bought shares for the first time in the previous year. Can you be 90% confident that 125 people bought shares for the first time in the previous year, assuming normality of the data?

(d) What sample size would be needed to produce a margin of error of 1.5% for the shareholders who bought shares for the first time the previous year?

Answers

Chapter 1: Derivatives of exponential functions

RECALL

1 (a) m^4n^6 (b) $2a^3$ (c) $\dfrac{a^{15}b^5}{32c^{10}}$

2 (a) $3x^{\frac{1}{2}} + x^{\frac{3}{2}}$ (b) $x^{-3} - 5x^{-1}$ (c) $4x^{-\frac{1}{2}} + x^{-\frac{5}{2}}$

3 (a) $3x(x+3)$ (b) $(x+3)(x-3)$

(c) $(x+9)(x-2)$ (d) $(3x-5)(x-2)$

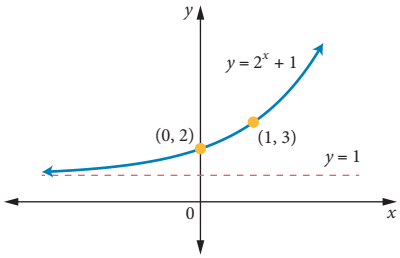
4 (a) $x = -3$ or $x = 2$ (b) $x = -6$ or $x = 1$

(c) $x = \dfrac{1-\sqrt{73}}{4}$ or $x = \dfrac{1+\sqrt{73}}{4}$

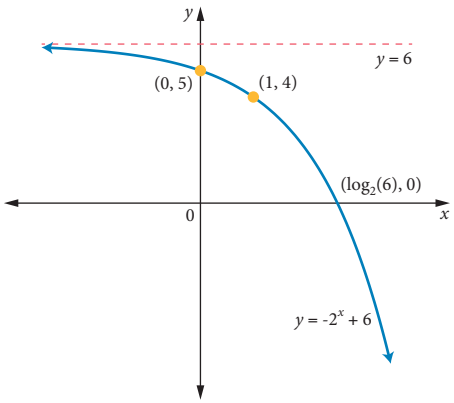
5 (a) $x = \log_5(37)$ (b) $x = \log_5(20)$

(c) $x = \log_5(12)$ or $x = 1$

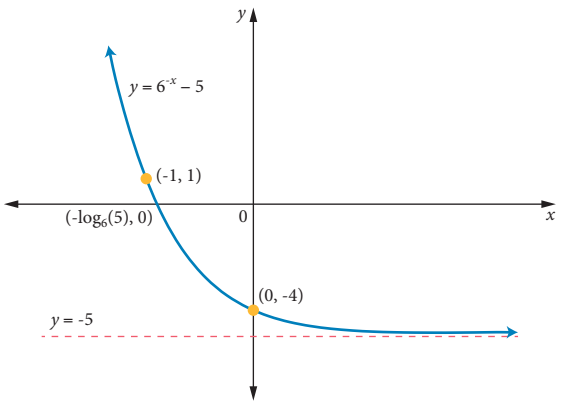
6 (a)



(b)



(c)



7 (a) $\dfrac{dy}{dx} = 15x^2 - 6x + 2$ (b) $\dfrac{dy}{dx} = -\dfrac{6}{x^3}$

(c) $\dfrac{dy}{dx} = 5 - \dfrac{1}{2\sqrt{x}}$

8 (a) $\dfrac{dy}{dx} = 15(5x+1)^2$ (b) $\dfrac{dy}{dx} = -\dfrac{2(2x+3)}{(x^2+3x-7)^3}$

(c) $\dfrac{dy}{dx} = \dfrac{3x^2 - 6x}{2\sqrt{x^3 - 3x^2}}$

9 (a) $\dfrac{dy}{dx} = 8x^3 + 21x^2 + 6x$ or $x(8x^2 + 21x + 6)$

(b) $\dfrac{dy}{dx} = -\dfrac{5}{(3x-1)^2}$ (c) $\dfrac{dy}{dx} = \dfrac{x(4+x)}{(2-x)^4}$

EXERCISE 1.1

1 (a) 54.60 (b) 1.93 (c) 1.95 (d) 1.72

(e) 10.39 (f) 8.92 (g) 1.40 (h) -2.58

2 (a) $x = 3$ (b) $x = 3$ (c) $x = -4$ (d) $x = 0$

(e) $x = -\dfrac{3}{2}$ (f) $x = -\dfrac{1}{2}$ (g) $x = \dfrac{3}{2}$

(h) $x = \dfrac{1}{2}$ or $x = -\dfrac{1}{2}$

3 (a) $x = \log_e(5)$ (b) $x = \dfrac{1}{2}\log_e(13)$ (c) $x = \log_e(12)$

(d) $x = \dfrac{\log_e(3) - 1}{3}$ (e) no solution (f) $x = 3\log_e(7)$

4 (a) $x = 1.10$ (b) $x = 1.04$ (c) $x = 1.40$ (d) $x = -0.51$

5 (a) 50 (b) 166 (c) 2 hours 19 minutes

6 (a) $1500 = N_0 \times e^{-k}$ and $800 = N_0 \times e^{-5k}$

(b) $N_0 = 1755$, $k = 0.1572$ (c) less than 31 years

7 (a) $R(t) = 310t$

(b) $B(t) = 15000e^{0.0038t} - 310t$

(c) 62 months

8 (a) $x = 0$ or $x = \log_e(2)$

(b) $x = \log_e(4)$ or $x = \log_e\left(\dfrac{3}{2}\right)$ (c) $x = \log_e(3)$

9 (a) A

(b) $\log_e(-5) \neq -\log_e(5)$; in fact, $\log_e(-5)$ is not defined.

10 B

11 (a) $x = 0$ or $x = \log_e(4)$ (b) $x = \log_e(2)$ or $x = \log_e(4)$

(c) $x = \log_e(2)$ (d) $x = 0$

(e) $x = \log_e\left(\dfrac{4}{3}\right)$ or $x = \log_e(3)$

12 (a) $x = 1.10$ or $x = 1.39$ (b) $x = 1.79$

(c) $x = 0.92$ or $x = 1.79$ (d) no solutions

13 (a) $e^{-x} + e^{x} = \dfrac{1 + e^{2x}}{e^{x}}$ (b) $x = 0$

14 If $k = \sqrt{24}$, there will be one solution: $x = \dfrac{1}{2}\log_e(6)$.

If $k > \sqrt{24}$, there will be two solutions; for example, $k = 5$: $x = \log_e(2)$ or $x = \log_e(3)$.

If $k < \sqrt{24}$, there will be no solutions.

15 $x = \log_e\left(\sqrt{21} - 1\right)$

EXERCISE 1.2

1 (a) reflection in the x-axis

(b) factor of 2 dilation parallel to the y-axis; translation 1 unit up

(c) reflection in the y-axis; translation 3 units down

(d) reflection in the x-axis; factor of 0.5 dilation parallel to the x-axis; factor of 10 dilation parallel to the y-axis

(e) reflection in the x-axis; translation 6 units up

(f) factor of 4 dilation parallel to the y-axis; translation 1 unit right

(g) reflection in the y-axis; translation 5 units down

(h) reflection in the y-axis; factor of 0.5 dilation parallel to the x-axis; factor of 2 dilation parallel to the y-axis; translations 2.5 units right, 3 units up

(i) factor of 0.25 dilation parallel to the y-axis; translation 2 units left

(j) factor of 2 dilation parallel to the x-axis; factor of 2 dilation parallel to the y-axis; translations 1 unit right, 1 unit up

2 (a) $y = -e^{\frac{x}{2}}$ (b) $y = e^{x-2} - 1$ (c) $y = -e^{2x} + 3$

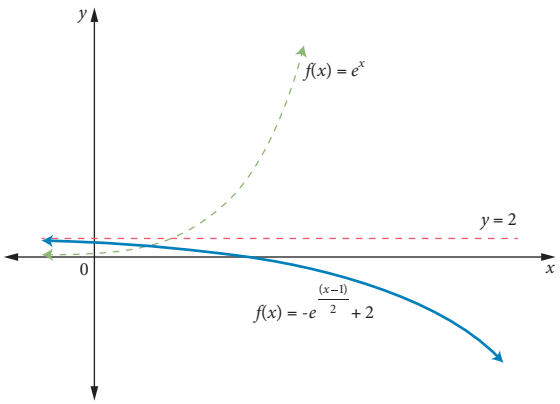
3 (a) B (b) D (c) D

4 (a) C

(b) $y = e^{x-2}$ is $y = e^{x}$ moved to the right (not left) by 2 units.

5 (a) $f(x) = -e^{\frac{x-1}{2}} + 2$

(b)



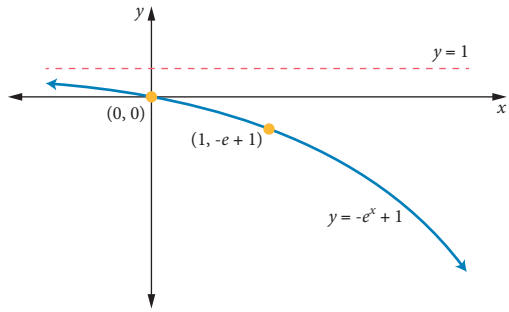
6 $y = -2e \times e^{x} + 4$; $y = -2e^{x+1} + 4$

7 $y = \dfrac{1}{e-1}e^{-x} + 4 - \dfrac{1}{e-1}$

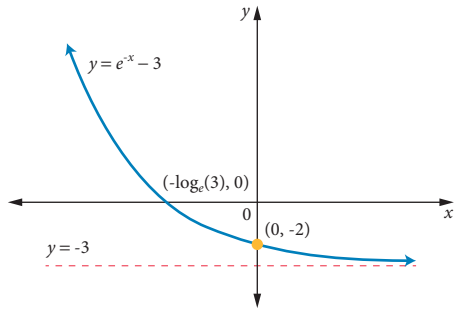
8 $y = e^{1.98x} - 4$

9 $y = e^{-x+3} - 1$

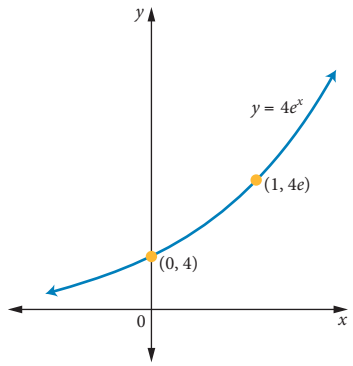
10 (a) Domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 1)$



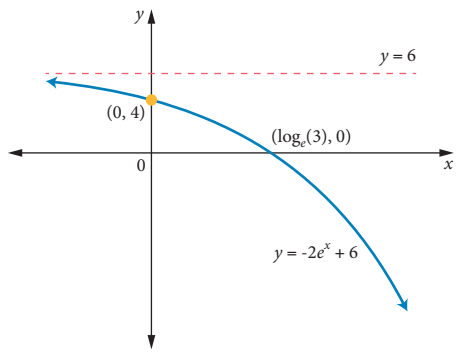
(b) Domain: $x \in \mathbb{R}$; range: $y \in (-3, \infty)$



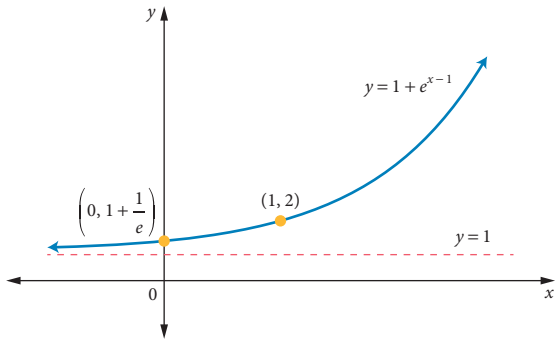
(c) Domain: $x \in \mathbb{R}$; range: $y \in (0, \infty)$



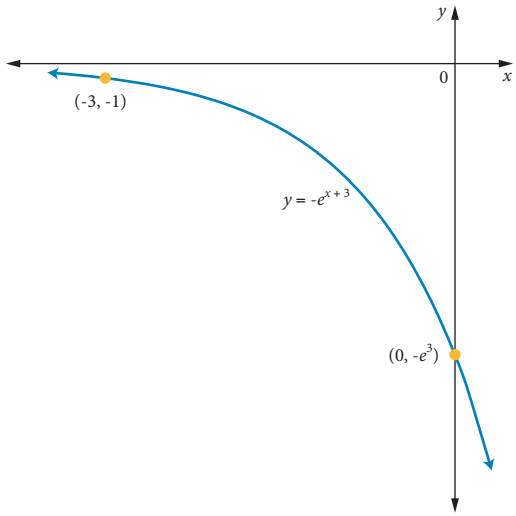
(d) Domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 6)$



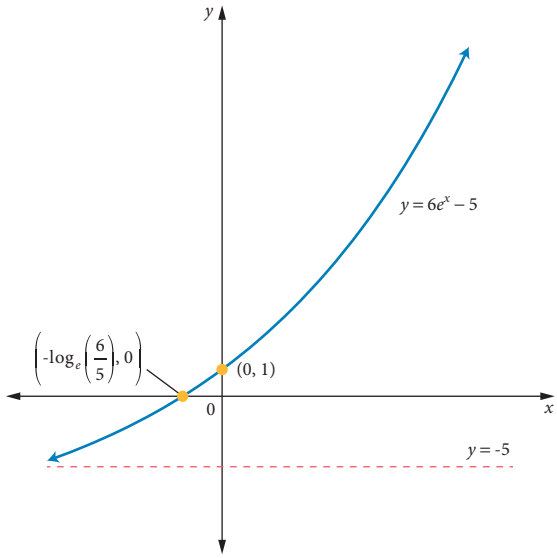
11 (a) Domain: $x \in \mathbb{R}$; range: $y \in (1, \infty)$



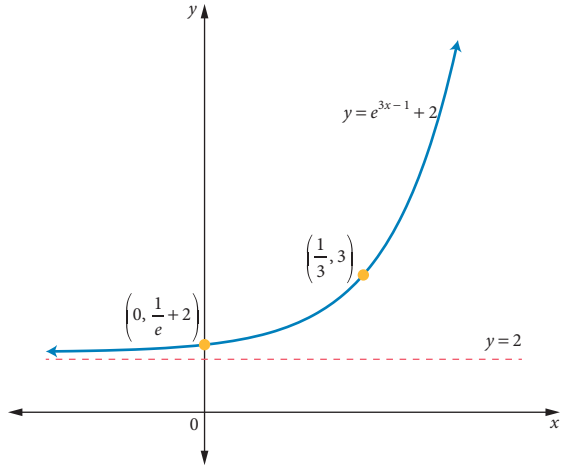
(b) Domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 0)$



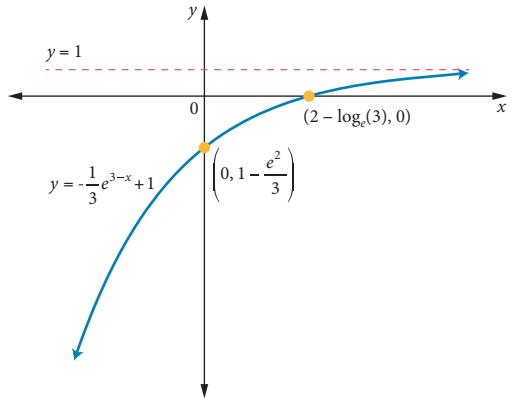
(c) Domain: $x \in \mathbb{R}$; range: $y \in (-5, \infty)$



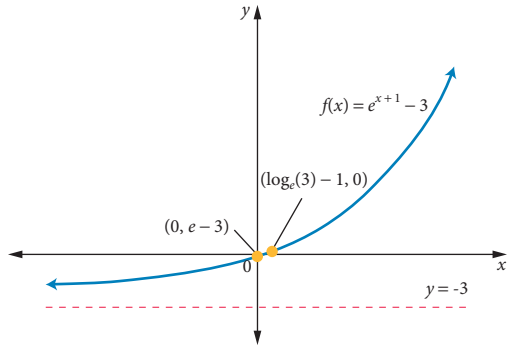
(d) Domain: $x \in \mathbb{R}$; range: $y \in (2, \infty)$



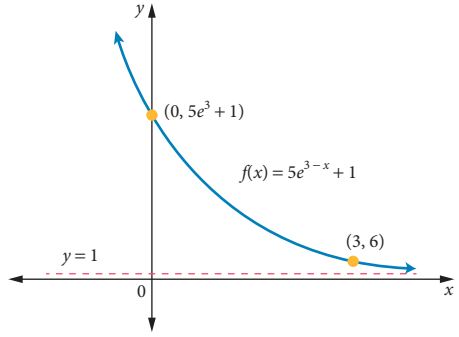
(e) Domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 1)$



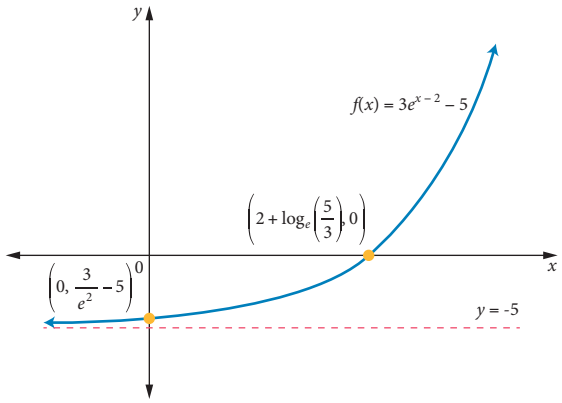
12 (a) Domain: $x \in \mathbb{R}$; range: $y \in (-3, \infty)$



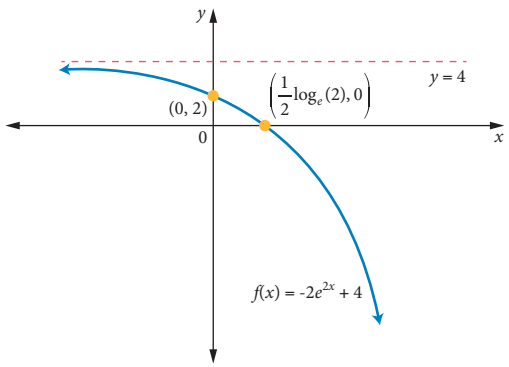
(b) Domain: $x \in \mathbb{R}$; range: $y \in (1, \infty)$



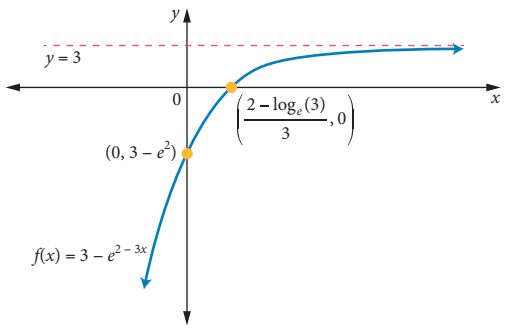
(c) Domain: $x \in \mathbb{R}$; range: $y \in (-5, \infty)$



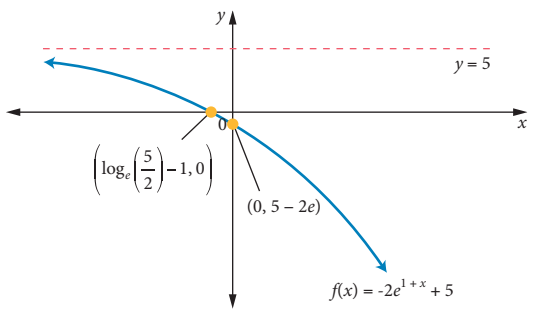
(d) Domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 4)$



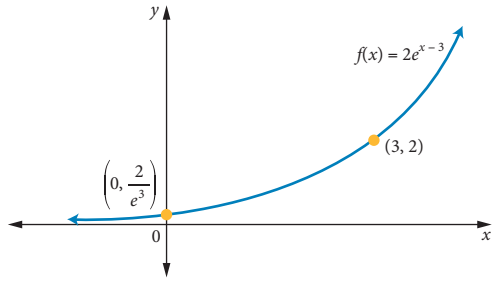
(e) Domain is $x \in \mathbb{R}$; range: $y \in (-\infty, 3)$



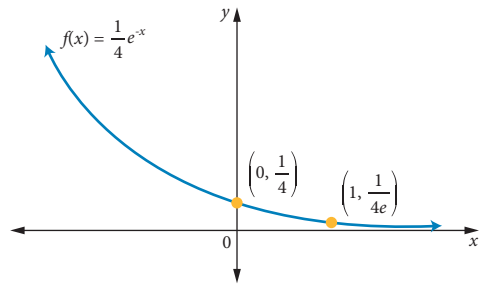
(f) Domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 5)$



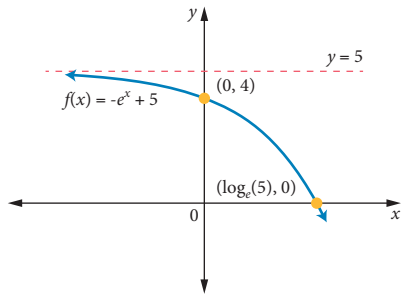
13 (a)



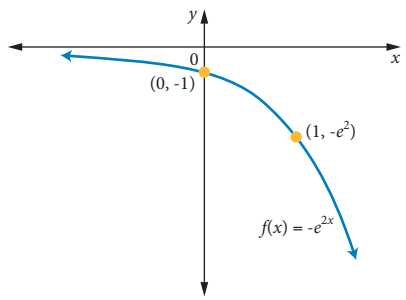
(b)



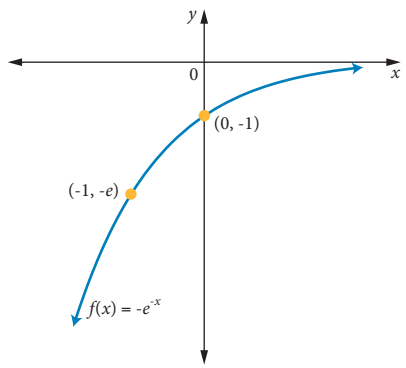
(c)



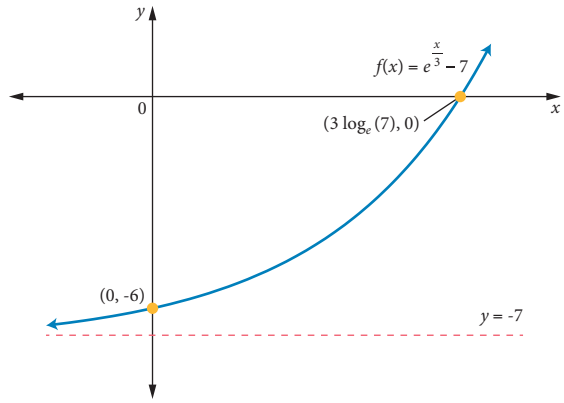
(d)



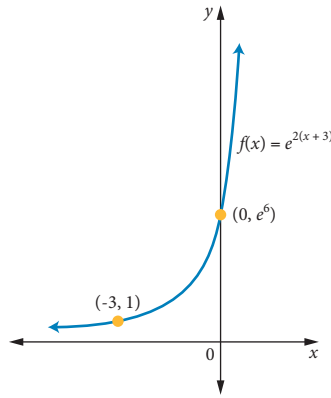
(e)



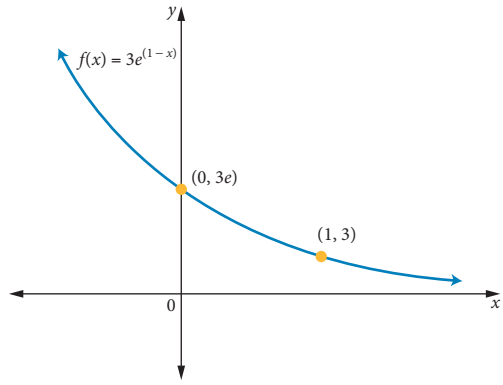
(f)



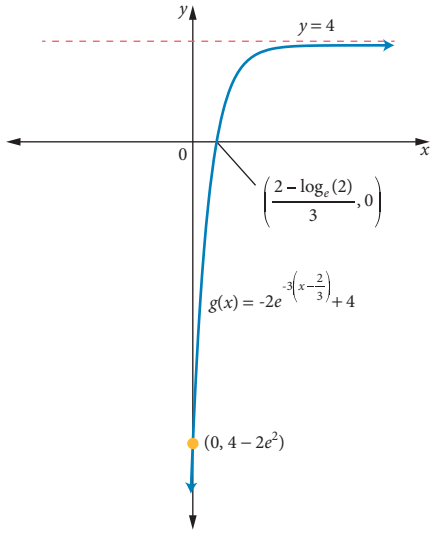
(g)



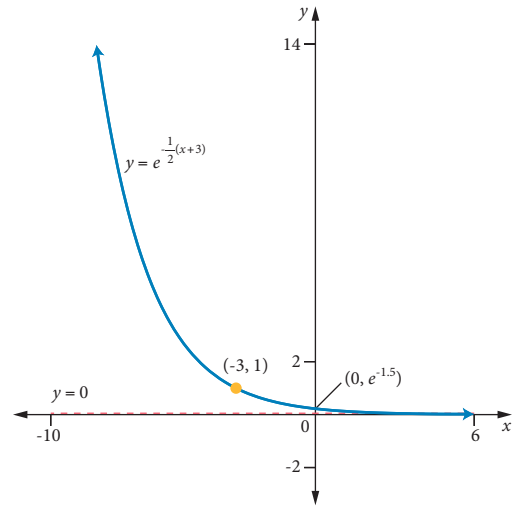
(h)



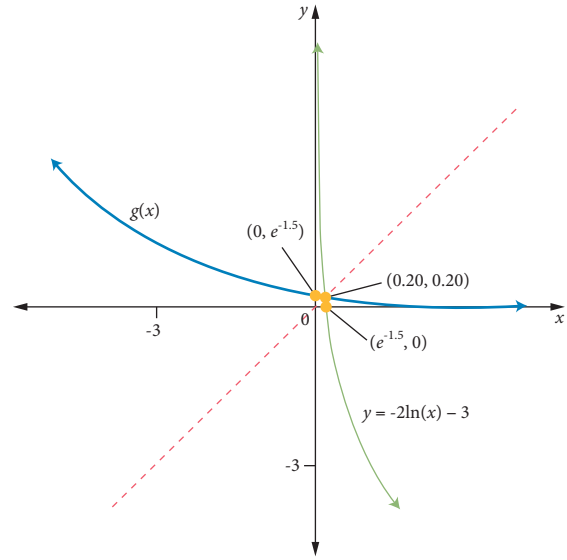
14 Domain: $x \in \mathbb{R}$; range: $y < 4$



15 (a) Domain: $x \in \mathbb{R}$; range: $y > 0$



(b) Domain: $x > 0$; range $y \in \mathbb{R}$



EXERCISE 1.3

1 (a) $\lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10$; gradient of $y = 3^x$ at $(0, 1)$ is 1.10; $3^x = e^{1.10x}$

(b) $\lim_{h \to 0} \frac{0.33^h - 1}{h} \approx -1.11$; gradient of $y = 0.33^x$ at $(0, 1)$ is -1.11; $0.33^x = e^{-1.11x}$

(c) $\lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$; gradient of $y = 2^x$ at $(0, 1)$ is 0.69; $2^x = e^{0.69x}$

(d) $\lim_{h \to 0} \frac{0.5^h - 1}{h} \approx -0.69$; gradient of $y = 0.5^x$ at $(0, 1)$ is -0.69; $0.5^x = e^{-0.69x}$

(e) $\lim_{h \to 0} \frac{7^h - 1}{h} \approx 1.95$; gradient of $y = 7^x$ at $(0, 1)$ is 1.95; $7^x = e^{1.95x}$

(f) $\lim_{h \to 0} \frac{4^h - 1}{h} \approx 1.39$; gradient of $y = 4^x$ at $(0, 1)$ is 1.39; $4^x = e^{1.39x}$

2 (a) $\frac{dy}{dx} = 2e^{2x}$ (b) $\frac{dy}{dx} = \frac{1}{4}e^{\frac{1}{4}x}$

(c) $f'(x) = \frac{2}{3}e^{\frac{2x}{3}}$ (d) $\frac{dy}{dx} = -3e^{-x}$

(e) $h'(x) = 15e^{3x}$ (f) $\frac{dy}{dx} = 15e^{-5x}$

(g) $\frac{dy}{dx} = e^{0.25x}$ (h) $\frac{dy}{dx} = -6e^{-3x}$

3 (a) $f'(x) = 3x^2 + 2e^{2x}$ (b) $\frac{dy}{dx} = -2e^{-x} - 10e^{2x}$

(c) $g'(x) = 3x^2 - 10x + 6e^{2x}$ (d) $\frac{dy}{dx} = -6e^{-3x} + 20e^{4x}$

(e) $\frac{dy}{dx} = 3e^{x}$

4 (a) $f'(x) = 5e^{5x+2}$ (b) $f'(x) = (2x-4)e^{x^2-4x}$

(c) $f'(x) = 16e^{8x-7}$ (d) $f'(x) = e^{\frac{x}{5}+3}$

(e) $f'(x) = 2xe^{x^2-7}$ (f) $f'(x) = 2(x+1)e^{7-2x-x^2}$

(g) $f'(x) = (x^2+1)e^{x^3+3x}$

(h) $f'(x) = 2(2x-3)e^{4-3x+x^2}$

(i) $f'(x) = -4(x-3)e^{(x-3)^2}$

5 B

6 (a) A

(b) If $y = e^u$, $\frac{dy}{dx} = u'e^u$ not $\frac{dy}{dx} = u'e^{u'}$.

7 (a) $\frac{dy}{dx} = 3e^{3x} + \frac{2}{e^x}$ (b) $\frac{dy}{dx} = 10e^{2x} - \frac{3}{e^{3x}} + 6x$

(c) $\frac{dy}{dx} = e^{3x}$

8 (a) $f'(x) = 6e^{2x+5}$ (b) $f(1) = 3e^7$

(c) $f'(1) = 6e^7$ (d) $y = 6e^7x - 3e^7$

9 (a) $\frac{dy}{dx} = \frac{2(e^{4x}+1)}{e^{2x}}$ (b) $\frac{dy}{dx} = -\frac{6}{e^{2x-5}}$

(c) $\frac{dy}{dx} = \frac{1-2x}{e^{x^2-x-1}}$

10 (a) $f'(x) = 2(x-3)e^{(x-3)^2}$

(b) $f'(x) = \frac{3}{2\sqrt{3x-1}}e^{\sqrt{3x-1}}$

(c) $f'(x) = 3e^x(e^x+4)^2$

(d) $f'(x) = 16(2x+5)^3 e^{(2x+5)^4}$

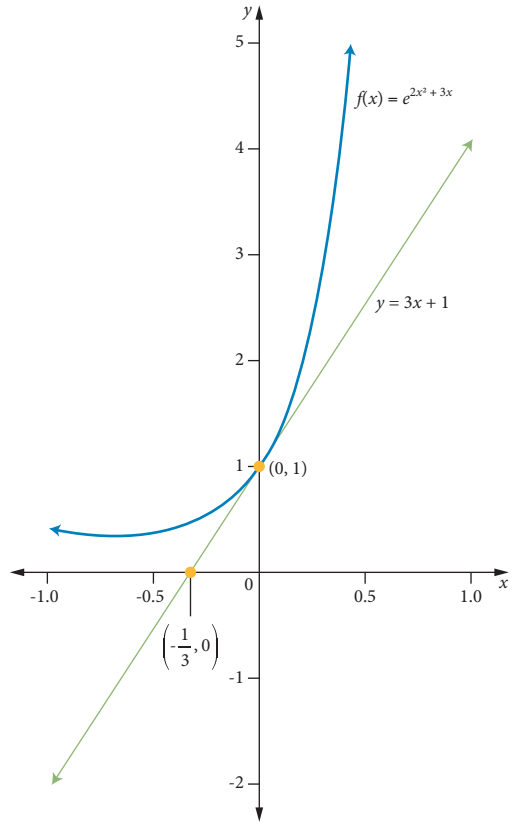
(e) $f'(x) = \frac{-4(x-3)}{e^{(x-3)^2}}$

11 (a) $y = \frac{-x-3}{e^2}$ (b) $y = e$

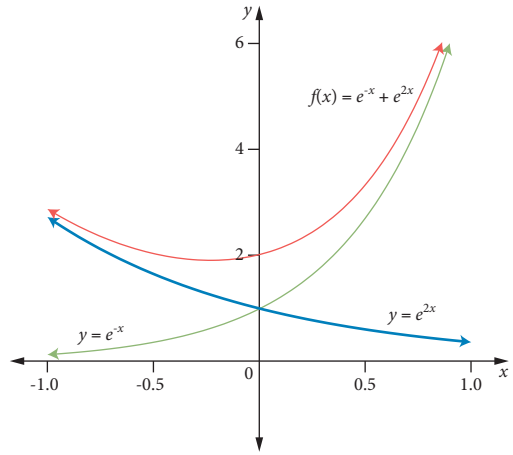
12 The equation of the tangent, $y = ex$, has no constant term, so it must pass through the origin.

13 (a) $y = 3x + 1$

(b)

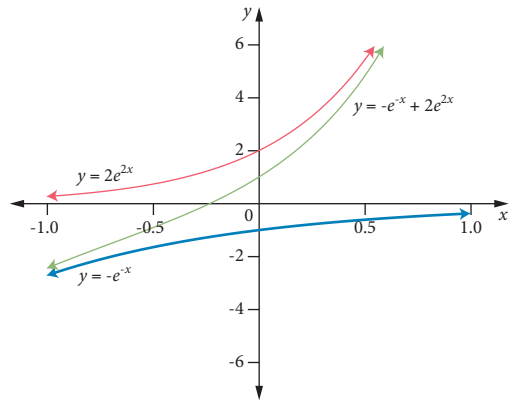


14 (a)



(b) $f'(x) = -e^{-x} + 2e^{2x}$

(c)



(d) $x = -0.23$

(e) The point (-0.23, 1.89) represents a minimum value of the function.

15 $y = -1.26x + 1.53$ 16 $a = 2$

17 (a) $y = 1.79x + 1$ (b) $y = 384.64x - 943.92$

18 $f'(x) = 6x^2e^{x^3} - \dfrac{3e^{\sqrt{x}}}{\sqrt{x}}$;

$f''(x) = e^{x^3}\left(12x + 18x^4\right) + \dfrac{3e^{\sqrt{x}}\left(1 - \sqrt{x}\right)}{2x\sqrt{x}}$

EXERCISE 1.4

1 (a) $N(t) = 23e^{0.853t}$, where t is in weeks; regression analysis gives $N(t) = 23.0e^{0.858t}$

(b) $N(t) = 165e^{1.14t}$, where t is in days; regression analysis gives $N(t) = 165.0e^{1.14t}$

(c) $N(t) = 2.87e^{0.430t}$, where t is in months and N is in thousands; regression analysis gives $N(t) = 2.87e^{0.430t}$

(d) $N(t) = 0.42e^{3.44t}$, where t is in months and N is in thousands; regression analysis gives $N(t) = 0.418e^{3.45t}$

2 (a) $d(t) = 101.09 \times 1.0238^t$ (b) 5 500 000 km

3 (a) C (b) D

4 $N(t) = 25 \times 2^{\frac{t}{2}}$

5 (a) (i) $N(t) = N_0e^{-0.0241t}$, where t is in years

(ii) $N(t) = N_0 \times \left(\dfrac{1}{2}\right)^{\frac{t}{28.79}}$, where t is in years

(b) 17.7 mg

6 (a) $A = 80\,000$ and $k = 0.405$ (b) $4 613 203

(c) 6.23 years

7 (a) 35°C (b) $A = -35$ (c) $k = -0.0224$

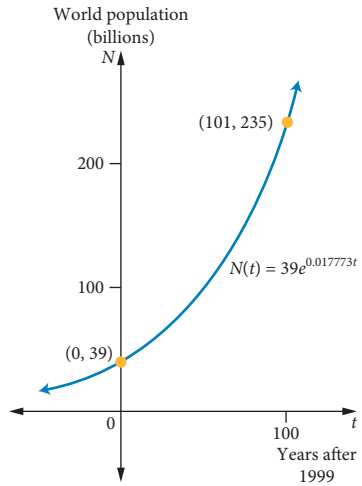
(d) $T(t) = -35e^{-0.0224t} + 35$ (e) in the first 37 minutes

8 (a) C (b) $t \div \dfrac{1}{2} = 2t$, not $\dfrac{t}{2}$

9 84 years

10 (a) 235 billion

(b) The model is not realistic for the sustained period shown in the graph.



11 (a) $A = 3.286$, $k = 0.602$

(b) 1352 animals (c) 6 years

12 (a) $N(t) = 12\,800e^{-0.0347t} + 200$ (b) 13 000

13 (a) $N(t) = 400 \times 1.8^{\frac{t}{6}}$, where t is the time in months

(b) 10.29% per month

14 (a) $T = 21$; the cake cools towards room temperature, 21°C.

(b) $T = 159e^{-0.0702t} + 21$ (c) 180°C

(d) 10 to 14 minutes (e) 41 minutes

15 (a) $N(t) = 800e^{0.105t}$ (b) $N'(t) = 84e^{0.105t}$

(c) 296 000 antibodies per hour

(d) $A(t) = e^{0.956(t-6)}$ (e) 15 hours

16 (a) $N(t) = 100e^{-0.121t}$ (b) $N'(t) = -12.1e^{-0.121t}$

(c) (i) $N'(4) = -7.46$ (ii) $N'(10) = -3.61$

(iii) $N'(40) = -0.0957$ (iv) $N'(50) = -0.0285$

(v) $N'(60) = -0.00851$ (vi) $N'(70) = -0.00254$

(d) Carbon dating looks for the change in ratio of carbon-14 to other compounds. $N'(60) = -0.00851$, so after 60 000 years the percentage is decreasing by less than 0.009% p.a., which is too small a change to be significant for the measurements that can be taken.

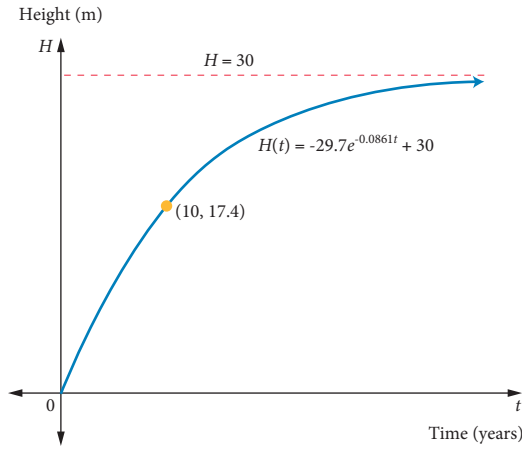
17 (a) $Y(t) = -9e^{-0.41t} + 19$ (b) 18 bushels per acre

(c) 19 bushels per acre

18 (a) $H(t) = -29.7e^{-0.0861t} + 30$ (b) 17.4 m

(c) $H'(t) = 2.56e^{-0.0861t}$ (d) 1.1 m per year

(e) The rate at which the height increases reduces gradually.



19 1.45 billion years

20 (a) $T(t) = 22e^{-0.00177t} + 15$ (b) $T(t) = 21.8e^{-0.0323t} + 15$

(c) 106 minutes (d) 15.2°C

EXERCISE 1.5

1 (a) $\dfrac{dy}{dx} = (2x - 3)e^{x^2 - 3x}$

(b) $f'(x) = 4\left(2 + 3e^{3x}\right)\left(2x + e^{3x}\right)^3$

(c) $f'(x) = \dfrac{e^{\sqrt{2x+1}}}{\sqrt{2x+1}}$ (d) $f'(t) = 6(2t-1)^2e^{(2t-1)^3}$

2 $\dfrac{-4(x-3)}{e^{(x-3)^2}}$ **3** D

4 (a) B

(b) The student has found $\dfrac{dy}{du} = 6e^2u$ and has not multiplied by $\dfrac{du}{dx} = e^x$.

5 (a) $f'(x) = -\dfrac{\left(3x^2e^{x^3} + 2e^x\right)}{\left(e^{x^3} + 2e^x\right)^2}$ (b) $f'(x) = \dfrac{3e^x - 2xe^{x^2}}{\left(e^{x^2} - 3e^x\right)^2}$

6 $r'(t) = -6\left(2e^{-t} - e^{2t}\right)^2\left(e^{-t} + e^{2t}\right)$

7 (a) $\dfrac{dy}{dx} = 30e^{2x}\left(x^2 + x - 1\right)$ (b) $f'(x) = 3e^{3x}(3x+1)$

(c) $\dfrac{dy}{dx} = e^{2x^3}\left(1 + 6x^3\right)$ (d) $\dfrac{dy}{dx} = 6e^x\left(e^x + 2\right)$

(e) $\dfrac{dy}{dx} = 4xe^{2x}(1+x)$ (f) $\dfrac{dy}{dx} = x^2e^{-2x}(3-2x)$

(g) $n'(x) = 2(2-3x)(5-3x)e^{2x}$

8 (a) $f'(t) = 2te^{t^2} - 50$ (b) (1.65, -50.28)

(c) (i) $f'(1) = -44.56$ (ii) $f'(2) = 168.39$

(d) minimum

9 (a) $f'(x) = \dfrac{e^x(x-1)}{x^2}$ (b) (1, e)

(c) (i) $f'(0.8) = -0.70$ (ii) $f'(1.2) = 0.46$

(d) minimum

10 (a) $f'(x) = e^{-x} - \dfrac{1}{20}$

(b) $\left(\log_e(20), \dfrac{19 - \log_e(20)}{20}\right)$

(c) (i) $f'(0) = 0.95$ (ii) $f'(20) = -0.05$

(d) maximum

11 (a) after 2 seconds

(b) The velocity is a maximum of $\dfrac{4}{e^2}$ m/s.

12 $\dfrac{dy}{dx} = e^{2x+5}(10x+9)(5x-3)^2$

13 (a) $\dfrac{dy}{dx} = \dfrac{-2x^3 + 3x^2 - 2}{e^{2x}}$

(b) $f'(x) = \dfrac{e^{x^2}\left(2x^2 - 6x - 1\right) + 1}{(x-3)^2}$

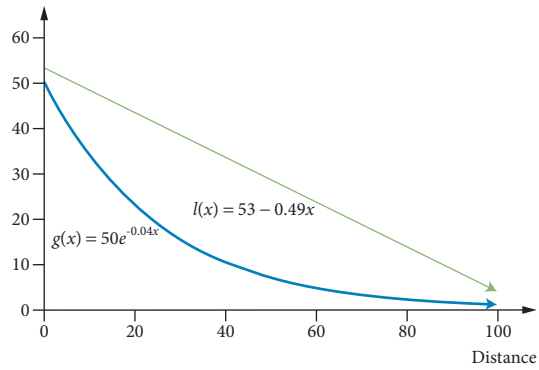
(c) $\dfrac{dy}{dx} = \dfrac{2e^{5x}(10x-41)}{(x-4)^{\frac{3}{2}}}$ (d) $f'(x) = \dfrac{(12x-5)e^{3x}}{(4x-1)^{\frac{3}{2}}}$

14 (a) $f'(x) = 7e^{\sqrt{x}}\left(\dfrac{\sqrt{x}}{2} + 1\right)$ (b) $f'(x) = \dfrac{x^3\left(8 - \sqrt{x}\right)}{2e^{\sqrt{x}}}$

15 (a) $g(x) = 50e^{-0.04x}$ (b) (0, 53) and (100, 3.92)

(c) $l(x) = 53 - 0.49x$

(d) Altitude



(e) $h(x) = 53 - 0.49x - 50e^{-0.04x}$

(f) 23.24 m

(g) Horizontal distance: 35.16 m;
height above the slope: 23.52 m

16 (a) The maximum score was probably 50.

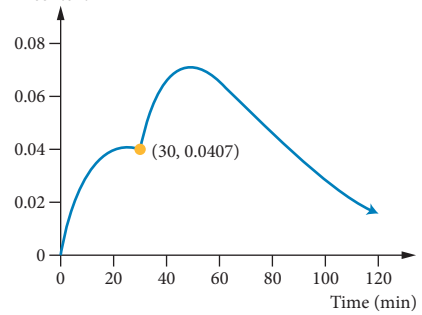
(b) Nine students scored 28.

17 (a) 0.0414%, after 25 minutes

(b) $A_2(t) = 0.0045(t-30)e^{-0.04(t-30)}$

(c) $C(t) = \begin{cases} 0.0045te^{-0.04t} & 0 \le t < 30 \\ 0.0045te^{-0.04t} + 0.0045(t-30)e^{-0.04(t-30)} & 30 \le t \end{cases}$

(d) Blood alcohol content



(e) 0.0711%, 18 minutes

18 The gradient of both tangents PQ and TR is $3e^2$, so PQ is parallel to TR. Gradient of both QR and PT is $-e^2$, hence QR and PT are parallel. Hence $PQRT$ is a parallelogram, since both pairs of opposite sides are parallel. Alternatively, you could show that PQ and TR are both parallel and equal in length.

CHAPTER REVIEW 1

1 (a) $x = \dfrac{5}{2}$ (b) $x = 4$

(c) $x = \log_e(2)$ (d) $x = \log_e(3) - 1$

2 (a) $x = 0$ or $x = \log_e(5)$ (b) $x = 0$ or $x = \log_e(2)$

(c) $x = 0$ or $x = \log_e(2)$ (d) $x = \log_e(7)$

3 $x = \dfrac{3}{4}$

4 (a) D (b) B

5 **(a)** reflection in the x-axis; dilation of factor 3 parallel to the y-axis; translation 1 unit up

(b) reflection in the y-axis; translations 2 units right and 5 units down

(c) reflection in the x-axis; dilation of factor $\frac{1}{2}$ parallel to the x-axis; translations 3 units left and 2 units up

(d) reflections in both axes; dilations: factor of 2 parallel to the y-axis and $\frac{1}{5}$ parallel to the x-axis; translations $\frac{1}{5}$ unit right and 3 units down

6 **(a)** A **(b)** D **(c)** A **(d)** A **(e)** B

7 $y = 5$

8 There is no x-intercept.

9 **(a)** B **(b)** A

10 **(a)** $\lim_{h \to 0} \frac{10^h - 1}{h} = 2.30$; the gradient of $y = 10^x$ at $(0, 1)$ is 2.30; $10^x = e^{2.30x}$

(b) $\lim_{h \to 0} \frac{0.1^h - 1}{h} = -2.30$; the gradient of $y = 0.1^x$ at $(0, 1)$ is -2.30; $0.1^x = e^{-2.30x}$

11 **(a)** $f'(x) = 2e^x$ **(b)** $f'(x) = -2e^{-2x}$

(c) $f'(x) = 3e^{3x-1}$ **(d)** $f'(x) = \frac{3}{2}e^{\frac{x}{2}}$

12 **(a)** $\frac{dy}{dx} = 2x - \frac{2}{x^2} + e^x$ **(b)** $\frac{dy}{dx} = 4xe^{x^2}$

(c) $\frac{dy}{dx} = 3x^2 e^{x^3}$ **(d)** $\frac{dy}{dx} = (2x + 3)e^{x^2+3x}$

13 **(a)** C **(b)** D **(c)** C

14 $2e^{2x} f'\left(e^{2x}\right)$

15 **(a)** $A = 63,\ c = 20,\ k = \frac{1}{20}\log_e\left(\frac{10}{21}\right)$

(b) $A = 70,\ c = 13,\ k = \frac{1}{15}\log_e\left(\frac{37}{70}\right)$

16 $N_0 = 4000,\ k = \frac{1}{60}\log_e\left(\frac{15}{4}\right)$

17 6.25% **18** $k = -0.530$ **19** $k = 1.2$ **20** 2003

21 **(a)** 2500 **(b)** 91 000 **(c)** $N'(t) = 75e^{0.03t}$

(d) 23 minutes **(e)** 184 bacteria per minute

(f) 2745 bacteria per minute

(g) The rate of population growth is increasing exponentially.

22 **(a)** $\frac{dy}{dx} = 2(x - 3)e^{(x-3)^2}$ **(b)** $\frac{dy}{dx} = 6\left(e^{2x} - x\right)\left(e^{2x} - x^2\right)^2$

(c) $\frac{dy}{dx} = \frac{1}{2}\sqrt{e^x}$ **(d)** $\frac{dy}{dx} = -\frac{e^{\frac{1}{x}}}{x^2}$

23 **(a)** $f'(x) = e^x + xe^x$ **(b)** $f'(x) = \frac{1 + 2x - x^2}{e^x}$

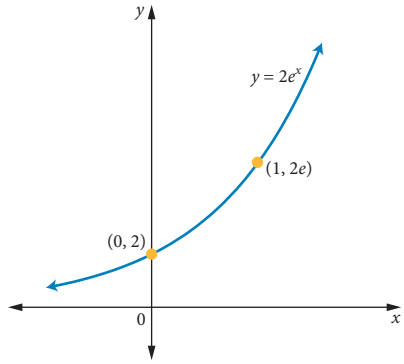
(c) $f'(x) = \frac{e^x(2x + 3)}{(2x + 5)^2}$

(d) $f'(x) = 2e^{2x-1}(x + 3)(x + 4)$

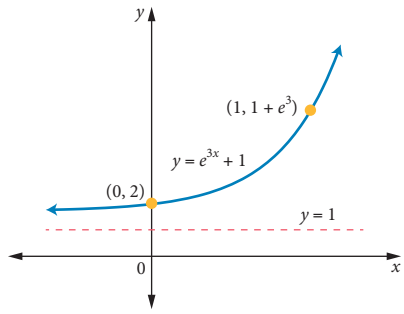
24 $2e^{3x}(1 + 3x)$

25 $\left(6x^2 - 7\right)e^{2x^3-7x}$

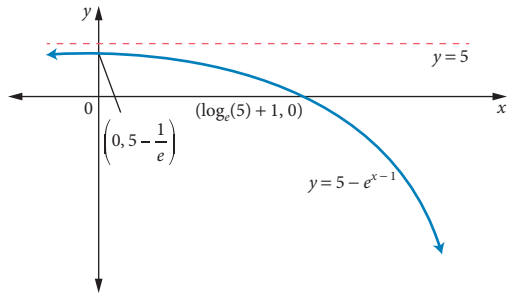
26 **(a)** Domain: $x \in \mathbb{R}$; range: $y \in (0, \infty)$



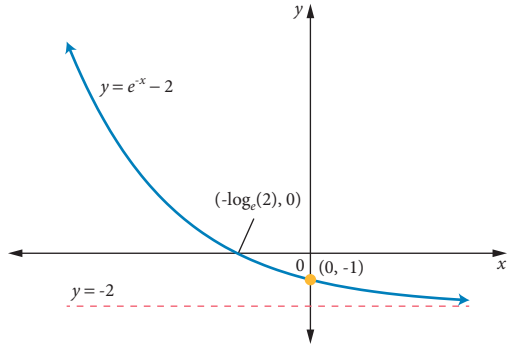
(b) Domain: $x \in \mathbb{R}$; range: $y \in (1, \infty)$



(c) Domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 5)$



(d) Domain: $x \in \mathbb{R}$; range: $y \in (-2, \infty)$



27 $y = 2.00e^{-0.501x} - 5$

28 $y = -2x + 4$

29 **(a)** 104 years **(b)** $t = 18.2$

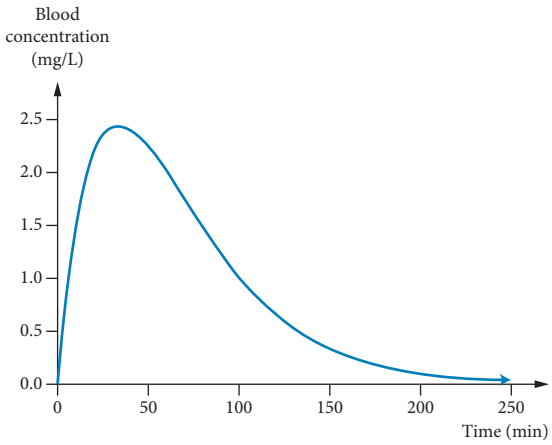
30 **(a)** $N(t) = 5000e^{-0.161t}$ **(b)** 2024

31 $\dfrac{dy}{dx} = \dfrac{2xe^{2x} + e^{2x} + xe^{x}}{(x+1)^2}$ **32** B

33 maximum: $(1, e)$

34 -2.58 > -5.44: Sarah is correct because a negative number with a small magnitude is greater than a negative number with a large magnitude.

35 (a)



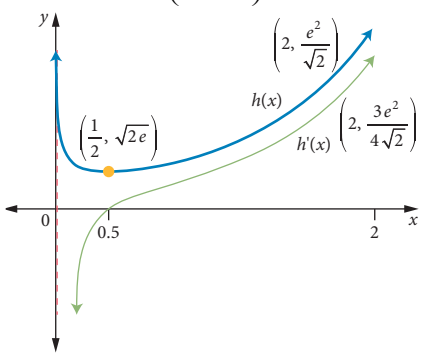
(b) 2.5 mg/L; 33 minutes **(c)** 3 hours

36 (a) 40 minutes **(b)** 80 minutes

37 (a) $h(x) = \dfrac{e^x}{\sqrt{x}}$, $0 < x \leq 2$

(b) local minimum: $\left(\dfrac{1}{2}, \sqrt{2e}\right)$

(c)



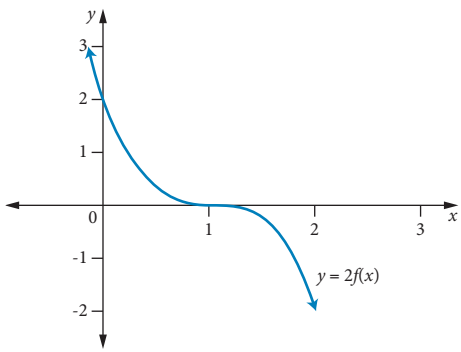
Chapter 2: Derivatives of trigonometric functions

RECALL

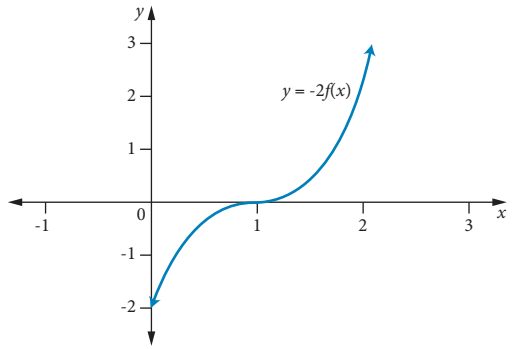
1 (a) $x = 3.5$ **(b)** $x = 5.1$ **(c)** $x = 2.5$

2 (a) $\theta = 41.41°$ **(b)** $\theta = 57.99°$ **(c)** $\theta = 54.90°$

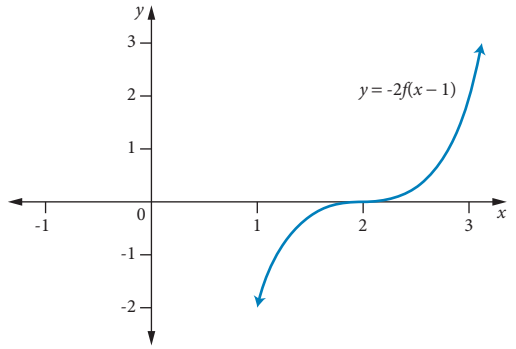
3 (a)



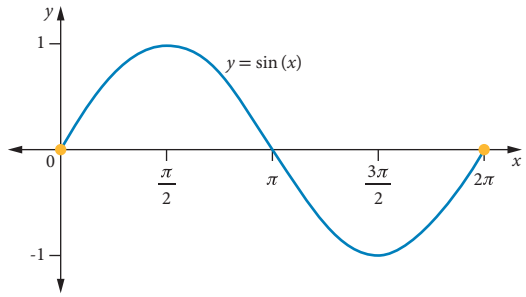
(b)



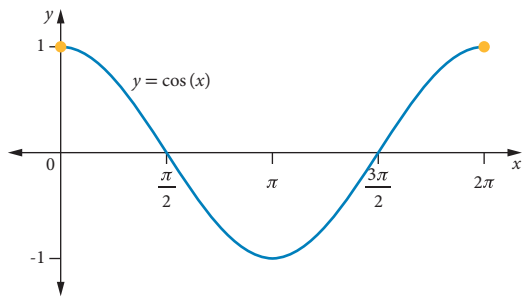
(c)



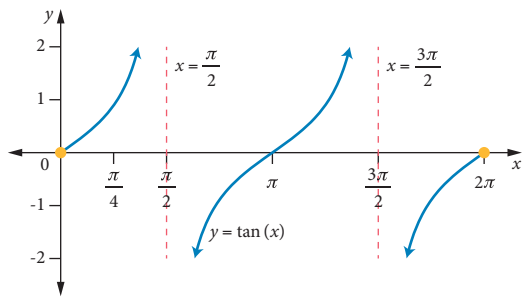
4 (a) Period: 2π



(b) Period: 2π



(c) Period: π



5 (a) $\frac{5\pi}{12}$ (b) $\frac{5\pi}{18}$ (c) $\frac{10\pi}{9}$

6 (a) 0.51 (b) 2.30 (c) 5.43

7 (a) 108° (b) 195° (c) 480°

EXERCISE 2.1

1 (a) 0 (b) 0 (c) 0 (d) -1

2 (a) $-\cos(\theta)$ (b) $\tan(\theta)$ (c) $\cos(\theta)$ (d) $\sin(\theta)$

3 (a) 0.15 (b) 0.05 (c) 0.15 (d) 0.15

4 $2\cos(x) - \sin(x)$

5 (a) $\frac{1}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\sqrt{3}$ (d) $\frac{\sqrt{3}}{2}$

(e) $-\frac{1}{2}$ (f) $-\sqrt{3}$ (g) $-\frac{\sqrt{2}}{2}$

6 (a) $x = \frac{2\pi}{3}$ (b) $x = \frac{3\pi}{4}$ (c) $x = \frac{4\pi}{3}$

7 (a) 2.1 m (b) 0.3 m

8 (a) 3000 (b) 5000

9 (a) 35°C (b) 12.5°C

10 (a) $\frac{3\sqrt{2} - 2\sqrt{3}}{6}$ (b) $-\frac{1}{2\sqrt{2}}$ or $-\frac{\sqrt{2}}{4}$

11 (a) $\cos(x) = \frac{3}{5}$ (b) $\tan(x) = -\frac{4}{3}$

12 (a) D

(b) In the third quadrant, sine and cosine are both negative.

13 A

14 (a) $x = \frac{\pi}{2} + y$ (b) $x = \frac{3\pi}{2} - y$

15 (a) $-\cos(x)$ (b) $-\sin(x)$

(c) $\sin(x)$ (d) $-\cos(x)$

16 $\frac{\pi}{4} < x < \frac{\pi}{2}$ and $\pi < x < \frac{5\pi}{4}$

17 $\frac{\sqrt{5} - 6}{2\sqrt{5}}$ or $\frac{2 - \sqrt{5}}{2\sqrt{5}}$

EXERCISE 2.2

1 (a) $x = \frac{\pi}{4}, \frac{3\pi}{4}$ (b) $x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}$

(c) $x = -\frac{5\pi}{6}, \frac{\pi}{6}$ (d) $x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

2 (a) $x = 4.07, x = 5.36$ (b) $x = 1.28, x = 4.43$

(c) $x = 0.93, x = 5.36$ (d) $x = 0.46, x = 2.69$

(e) $x = 2.38, x = 5.52$ (f) $x = 1.89, x = 4.40$

3 (a) $x = 0.21, x = 1.76, x = 4.16$

(b) $x = 1.00, x = 4.29, x = 5.43$

(c) $x = 0.88, x = 1.13, x = 2.53, x = 2.78$

(d) $x = 7.39, x = 86.85$

4 (a) $x = 2n\pi + \frac{\pi}{4}$ or $x = (2n+1)\pi - \frac{\pi}{4}$, where $n \in \mathbb{Z}$

(b) $x = n\pi + \frac{2\pi}{3}$; alternatively, $x = n\pi - \frac{\pi}{3}$ $n \in \mathbb{Z}$

(c) $x = 2n\pi \pm \frac{\pi}{3}$, $n \in \mathbb{Z}$

(d) $x = n\pi + \frac{\pi}{6}$, $n \in \mathbb{Z}$

5 (a) $x = 2n\pi \pm \frac{\pi}{3}$, $n \in \mathbb{Z}$

(b) $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

6 (a) B (b) A

7 (a) 3 m (b) $\frac{7\pi}{24}$ minutes

8 (a) 11.9 km/h (b) 13.1 km/h

(c) 16 January (d) 1 January and 13 January

9 (a) $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ (b) $x = -\frac{7\pi}{18}, -\frac{5\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}$

(c) $x = \frac{5\pi}{12}, \frac{17\pi}{12}$ (d) $x = -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$

10 C

11 (a) $x = \frac{(6n+1)\pi}{9}$ or $x = \frac{2(3n+1)\pi}{9}$, $n \in \mathbb{Z}$

(b) $x = n\pi + \frac{\pi}{12}$, $n \in \mathbb{Z}$

(c) $x = n\pi + \frac{\pi}{12}$ or $x = n\pi + \frac{3\pi}{4}$, $n \in \mathbb{Z}$

12 (a) $x = \frac{n\pi}{3}$, $n \in \mathbb{Z}$ (b) $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

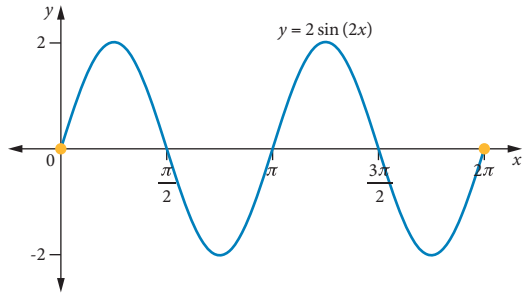
13 $x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$, where $n \in \mathbb{Z}$

14 $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

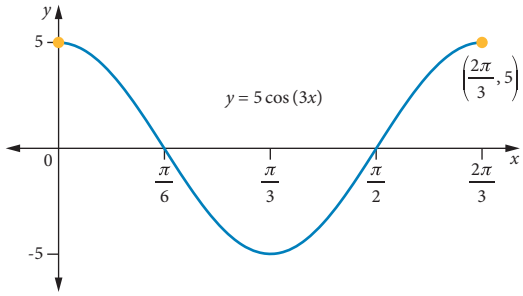
15 $0 \le t \le 0.001, 0.007 < t < 0.009, 0.016 < t \le \frac{1}{60}$

EXERCISE 2.3

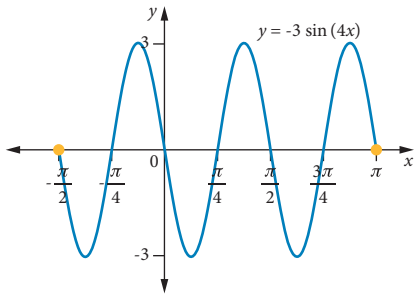
1 (a) $y = \sin(x)$ is dilated by a factor of 2 parallel to the y-axis and by a factor of $\frac{1}{2}$ parallel to the x-axis.



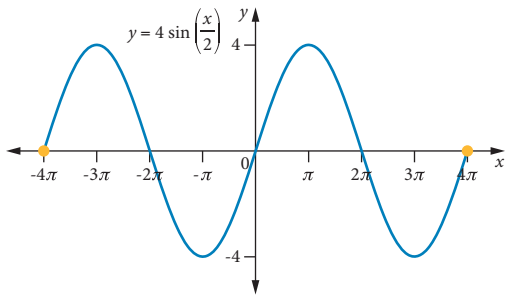
(b) $y = \cos(x)$ is dilated by a factor of 5 parallel to the y-axis and by a factor of $\frac{1}{3}$ parallel to the x-axis.



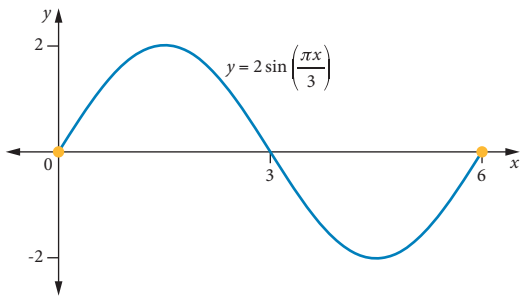
(c) $y = \sin(x)$ is reflected in the x-axis, dilated by a factor of 3 parallel to the y-axis and dilated by a factor of $\frac{1}{4}$ parallel to the x-axis.



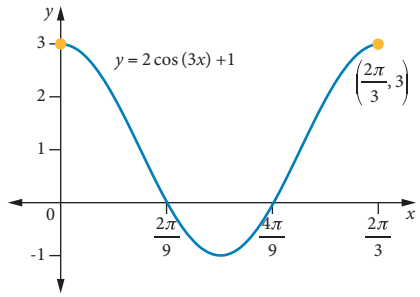
(d) $y = \sin(x)$ is dilated by a factor of 4 parallel to the y-axis and by a factor of 2 parallel to the x-axis.



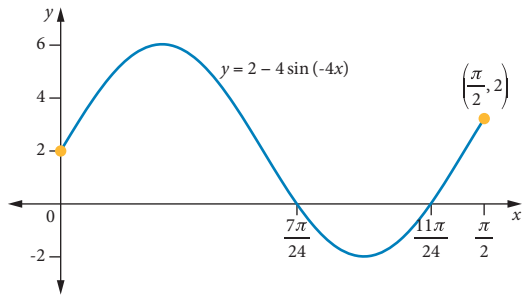
(e) $y = \sin(x)$ is dilated by a factor of 2 parallel to the y-axis and by a factor of $\frac{3}{\pi}$ parallel to the x-axis.



2 (a) period $= \frac{2\pi}{3}$, amplitude $= 2$



(b) period $= \frac{\pi}{2}$, amplitude $= 4$



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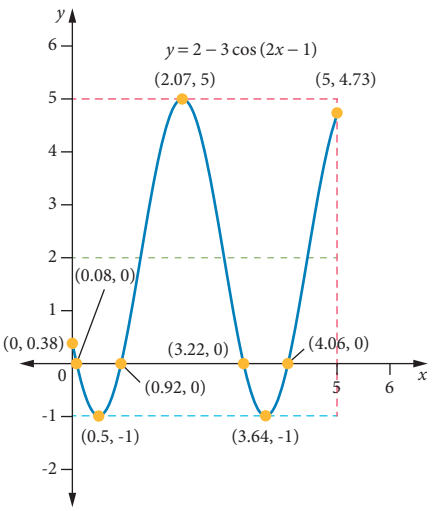
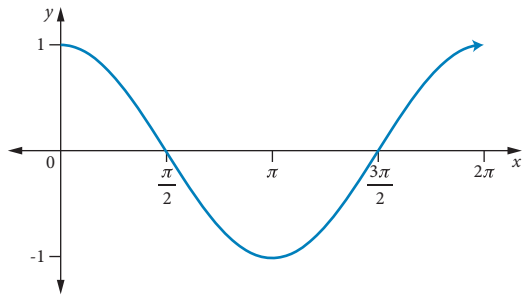


4 (a) The two functions represent the same graph due to the relationship between sine and cosine with complementary angles.



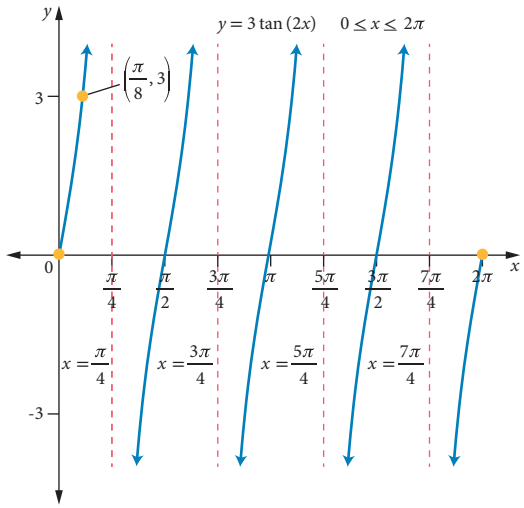
(b) **(i)** Possible equation: $y = \cos\left(x - \frac{\pi}{4}\right)$

(ii) Possible equation: $y = \cos\left(x - \frac{5\pi}{6}\right)$

(c) **(i)** Possible equation: $y = \sin\left(x + \frac{\pi}{3}\right)$

(ii) Possible equation: $y = \sin\left(x + \frac{5\pi}{6}\right)$

5



6 **(a)** B

(b) A phase shift to the right means the amount is subtracted, not added.

7 **(a)** $y = 3\cos(2x)$ **(b)** $y = 4\sin(3x) - 2$

8 B

9 reflection in the x-axis, dilation of factor 3 parallel to the y-axis, dilation of factor $\frac{1}{2}$ parallel to the x-axis and translations $\frac{\pi}{12}$ units in the positive x-direction and 4 units in the positive y-direction.

10 **(a)** amplitude = 1, period = 8

(b) maximum: 4 m; minimum: 2 m

(c) 2 am, 10 am and 6 pm

(d) 6 am, 2 pm and 10 pm

(e) $y = \sin\left(\frac{\pi x}{4}\right) + 3$

11 **(a)** $A = 5$, $B = \frac{\pi}{6}$, $D = 21$

(b) **(i)** 18.5°C **(ii)** 25.3°C

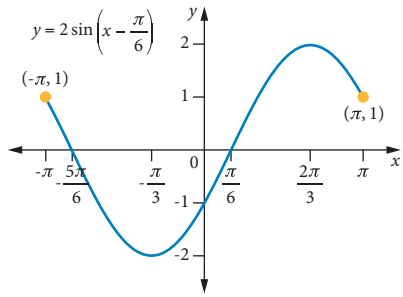
(c) 26°C; midnight and noon

(d) 16°C; 6 am and 6 pm

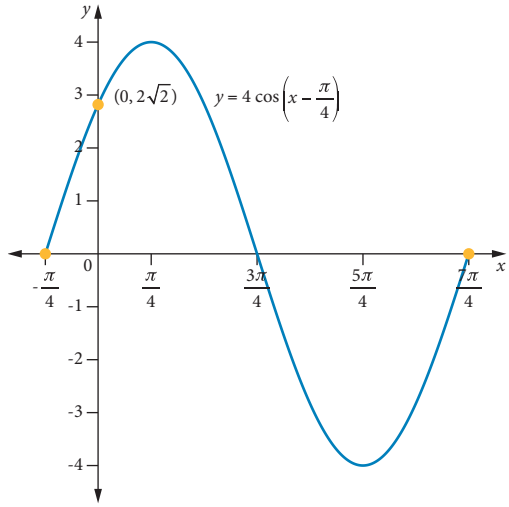
(e) 3:23 am, 8:37 am, 3:23 pm and 8:37 pm

(f) between midnight and 3:23 am; between 8:37 am and 3:23 pm; between 8:37 pm and midnight

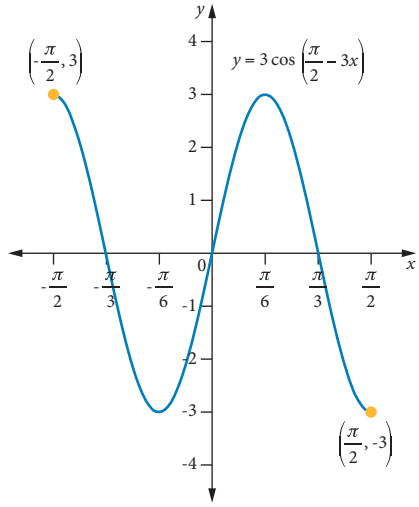
12 **(a)** period = 2π, amplitude = 2



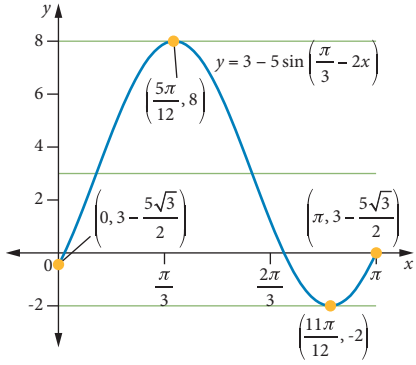
(b) period = 2π, amplitude = 4



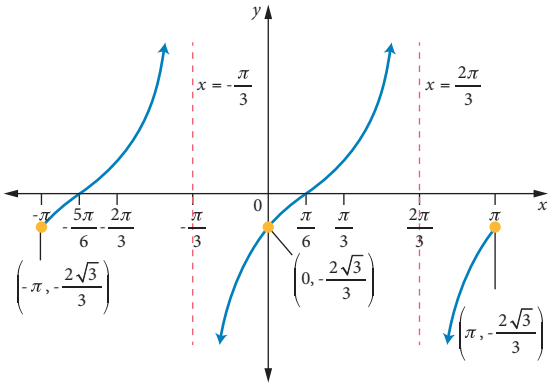
(c) period = $\frac{2\pi}{3}$, amplitude = 3



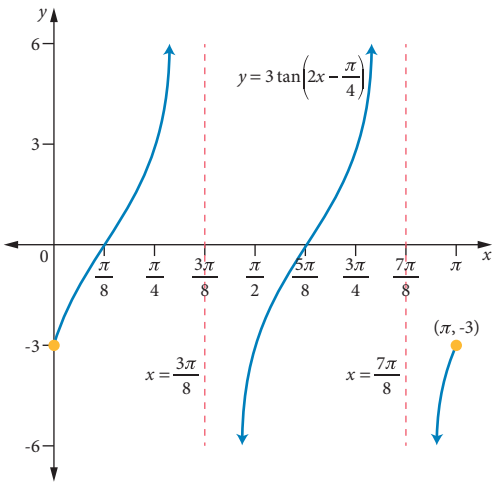
13



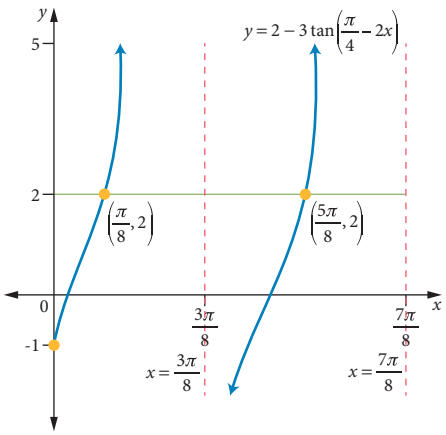
14 (a)



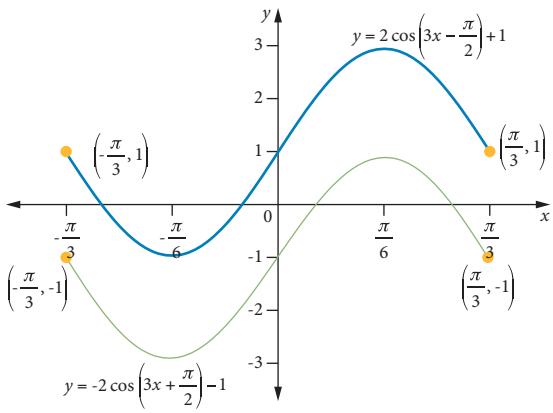
(b)



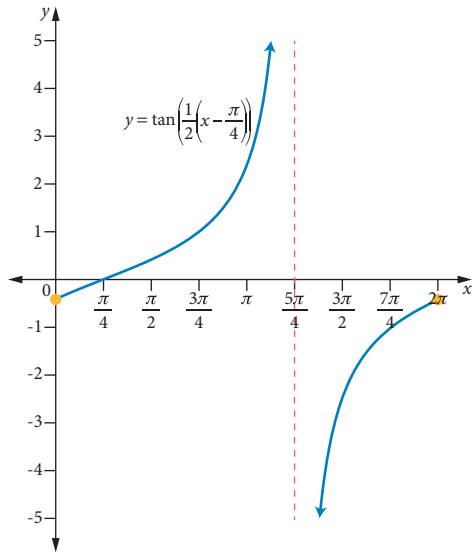
(c)



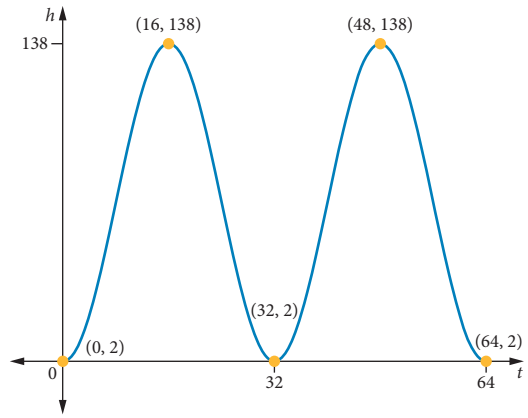
15 Equation of the transformed graph: $y = -2\cos\left(3x + \frac{\pi}{2}\right) - 1$



16 $n = 0.5, \varepsilon = \frac{\pi}{4}$



17 (a)



(b) $h(t) = -68\cos\left(\frac{\pi t}{16}\right) + 70,\ 0 \le t \le 64$

(c) 19 min 2 s

EXERCISE 2.4

1 (a) $f'(x) = 7\cos(7x)$ (b) $h'(x) = -4\cos(4x)$

(c) $f'(x) = -10\cos(2x)$ (d) $g'(x) = 2\cos\left(\frac{x}{2}\right)$

(e) $f'(x) = -\cos\left(\frac{x}{2}\right)$ (f) $f'(x) = -\frac{5}{2}\cos(5x)$

2 (a) $f'(x) = -4\sin(4x)$ (b) $g'(x) = -10\sin(5x)$

(c) $f'(x) = 3\sin(3x)$ (d) $f'(x) = -\frac{3}{2}\sin\left(\frac{x}{2}\right)$

(e) $h'(x) = -\frac{1}{5}\sin\left(\frac{x}{5}\right)$ (f) $f'(x) = 2\sin(4x)$

3 $-3\sin(3x) - \frac{2}{3}\cos\left(\frac{x}{3}\right)$

4 (a) $f'(x) = 2\cos(2x + 3)$ (b) $h'(x) = 15\sin(2 - 5x)$

(c) $f'(x) = (2x - 3)\cos(x^2 - 3x)$

(d) $f'(x) = -\frac{15}{4}\sin\left(\frac{3x - 2}{4}\right)$ (e) $f'(x) = -3\sin(3x)e^{\cos(3x)}$

(f) $h'(x) = e^{-x}\sin\left(e^{-x}\right)$ (g) $f'(x) = e^{x}\cos\left(e^{x}\right)$

(h) $g'(x) = \sin(x)\cos(x)\left(3\sin(x) - 2\right)$

(i) $f'(x) = -e^{\cos(x)}\sin(x)$

5 (a) $k'(x) = \dfrac{6}{\cos^2(6x)}$ (b) $f'(x) = -\dfrac{2}{\cos^2(x)}$

(c) $h'(x) = -\dfrac{6}{\cos^2(2x)}$ (d) $f'(x) = \dfrac{5}{4\cos^2\left(\frac{x}{4}\right)}$

(e) $k'(x) = \dfrac{2}{9\cos^2\left(\frac{x}{3}\right)}$ (f) $f'(x) = \dfrac{4}{3\cos^2\left(\frac{2x}{3}\right)}$

6 D

7 (a) $f'(x) = -15x\cos(5x+2) - 3\sin(5x+2)$

(b) $f'(x) = 5x\cos(1-5x) - \sin(1-5x)$

(c) $k'(x) = \dfrac{x + \sin(x)\cos(x)}{\cos^2(x)}$

(d) $f'(x) = 3\cos(5x)\cos(3x) - 5\sin(5x)\sin(3x)$

(e) $f'(x) = 3x^2\cos(3x) + 2x\sin(3x)$

(f) $f'(x) = 2\cos(3x) - 3(2x-3)\sin(3x)$

(g) $h'(x) = -15x\cos(5x-1) - 3\sin(5x-1)$

8 -6

9 $(0, 1)$ and $(3\pi, -1)$

10 (a) $\dfrac{dy}{dt} = -100\sin(t) - 200\cos(t)$

(b) decreasing at \$7.70 per year

(c) \$776.53

(d) 2.03, 5.18 and 8.32 years

11 (a) A (b) $e^0 = 1$, $e^0 \neq e$

12 $f'(\pi) = 2(\pi - 1)\cos(3) - \sin(3)$

13 (a) $k'(x) = \dfrac{2(x+1)\cos(2x) - \sin(2x)}{(x+1)^2}$

(b) $g'(x) = \dfrac{-3\cos(2x)\cos(3x) - 2\sin(3x)\sin(2x)}{\cos^2(2x)}$

(c) $h'(x) = \dfrac{e^{3x}\left(3\cos(2x) + 2\sin(2x)\right)}{\cos^2(2x)}$

(d) $f'(x) = \dfrac{2\sin(x)}{\left(1+\cos(x)\right)^2}$

(e) $f'(x) = \dfrac{2e^{2x}\cos(2x) + 2e^{2x}\sin(2x)}{\cos^2(2x)}$

(f) $g'(x) = \dfrac{4x\sin(2x)\cos(2x) - \sin^2(2x)}{2x^2}$

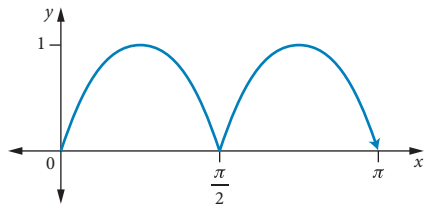
14 (a) $f'(x) = \dfrac{6}{(x+1)^2}\sin\left(\dfrac{5-x}{x+1}\right)$

(b) $h'(x) = -\dfrac{3\cos(x)}{\sin^2(x)}$ (c) $f'(x) = \dfrac{2xe^{x^2+1}}{\cos^2\left(e^{x^2+1}\right)}$

(d) $g'(x) = \dfrac{6x + \cos(3x)\sin(3x)}{2\sqrt{x}\cos^2(3x)}$

15 $y = -x + \dfrac{\pi}{4} + \dfrac{3}{2}$ 16 $x = \dfrac{\pi}{2}$

17 (a)

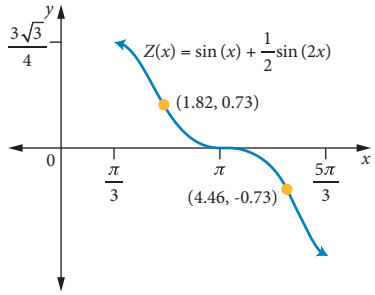


(b) $x = 0$, $x = \dfrac{\pi}{2}$, $x = \pi$

(c) $f'(x) = \begin{cases} 2\cos(2x), & 0 < x < \dfrac{\pi}{2} \\ -2\cos(2x), & \dfrac{\pi}{2} < x < \pi \end{cases}$

(d) $f'\left(\dfrac{\pi}{3}\right) = -1$ (e) $x = \dfrac{\pi}{4}$, $x = \dfrac{3\pi}{4}$

18 (1.82, 0.73) and (4.46, -0.73)



19 $y = \begin{cases} x, & 0 \leq x < 1 \\ 1, & 1 \leq x < \pi - 1 \\ -x + \pi, & \pi - 1 \leq x < \pi + 1 \\ -1, & \pi + 1 \leq x < 2\pi - 1 \\ x - 2\pi, & 2\pi - 1 \leq x \leq 2\pi \end{cases}$

EXERCISE 2.5

1 (a) $y = \dfrac{\sqrt{3}x}{2} - \dfrac{\sqrt{3}\pi}{12} + \dfrac{1}{2}$ (b) $y = -3$ (c) $y = 2x - \pi$

2 (a) C

(b) Options B and D show algebraic errors when rearranging the equation to make y the subject. Option A shows the incorrect placement of the values $\dfrac{\pi}{3}$ and $\sqrt{3}$ into the formula $y - y_1 = m(x - x_1)$.

3 (a) (i) $f(0) = 1$ (ii) $x = \dfrac{3\pi}{4}$, $x = \dfrac{7\pi}{4}$

(iii) $x = \dfrac{\pi}{4}$, $x = \dfrac{5\pi}{4}$

(b) $\left(\dfrac{\pi}{4}, \sqrt{2}\right)$, $\left(\dfrac{5\pi}{4}, -\sqrt{2}\right)$

4 (a) local maxima: $\dfrac{3}{2}$ at $x = \dfrac{\pi}{3}$, $x = \dfrac{5\pi}{3}$;
local minima: -3 at $x = \pi$; 1 at $x = 0$, $x = 2\pi$

(b) local maxima: 4 at $x = 0$, $x = 2\pi$; 0 at $x = \pi$;
local minima: $-\dfrac{1}{2}$ at $x = \dfrac{2\pi}{3}$, $x = \dfrac{4\pi}{3}$

(c) local maxima: 1 at $x = \dfrac{\pi}{2}$, $x = \dfrac{3\pi}{2}$;
local minima: 0 at $x = 0$, $x = \pi$, $x = 2\pi$

5 **(a)** 2.24 m to the right and 2.24 m to the left

(b) $v(t) = 2\cos(t) + \sin(t)$

(c) -2 m/s; the particle is travelling at 2 m/s to the left.

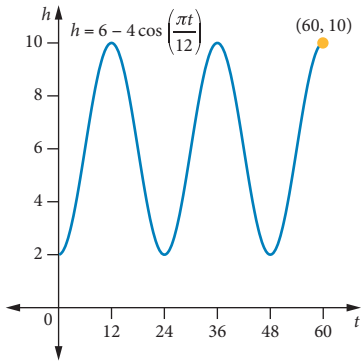
6 **(a)** 1 cm **(b)** 121 cm **(c)** $5 + \frac{2\pi}{3}\cos\left(\frac{\pi t}{3}\right)$

(d) 2.91 cm/week occurs at: 3, 9, 15 and 21 weeks.

7 **(a)** 2 m **(b)** 24 seconds

(c) maximum: 10 m; minimum: 2 m

(d)



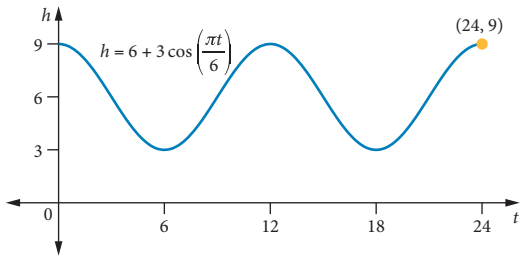
(e) $\frac{2+\sqrt{3}}{5}$ m/s **(f)** $\frac{\pi}{6}$ m/s

8 C

9 **(a)** High tide occurs at midnight and noon at a height of 9 m.

(b) Low tide occurs at 6 am and 6 pm at a height of 3 m.

(c)

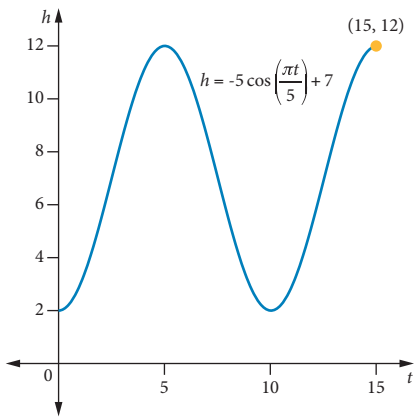


(d) between midnight and 2:21 am; between 9:39 am and 2:21 pm; between 9:39 pm and midnight

10 **(a)** $\frac{a}{\sqrt{3}}$ m/s **(b)** -0.29a radians/s

11 **(a)** 2 m and 12 m **(b)** $h = -5\cos\left(\frac{\pi t}{5}\right) + 7$

(c)

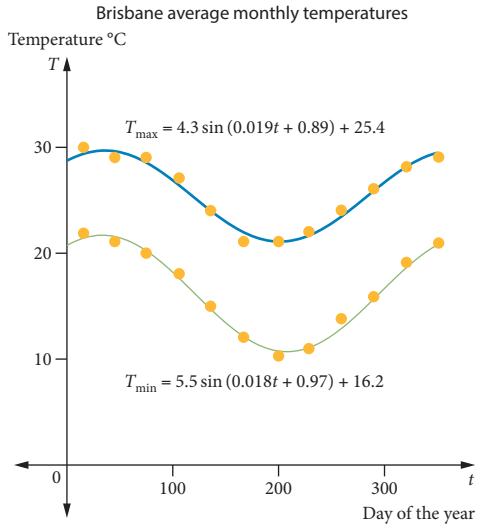


(d) 2 m **(e)** going up **(f)** 3 min 31 s

12 greatest volume: 302 cm^3; $r = 4.1$ cm, $h = 5.8$ cm

13 **(a)** $T_{max} = 4.3\sin(0.019t + 0.89) + 25.4$;

$T_{min} = 5.5\sin(0.018t + 0.97) + 16.2$



(b) Strengths: A periodic function suits the periodic nature of the data. The slow rate of change during summer and winter and the rapid rate of change in spring and autumn suit the sinusoidal shape.

Weaknesses: Since there are only 12 average values, estimating for any of the 365 days of the year would give much greater variation than the model suggests. You might think from the graph that Brisbane's temperatures do not go above 30°C or below 10°C.

14 **(a)** $T = 2\cos(\theta) + \frac{4\theta}{3}$ **(b)** 2.5 hours

15 **(a)** $T = \left(\frac{3}{4\cos(\theta)} + \frac{6 - 3\tan(\theta)}{5}\right)$

(b) With $\theta = 0.927$, or about 53°, Andrew will arrive at work on time, with about 1 minute to spare.

CHAPTER REVIEW 2

1 $\frac{1-\sqrt{2}}{2}$

2 **(a)** $\cos(5\pi) = -1$ **(b)** undefined

3 $\frac{\sqrt{2}-1}{2}$

4 B

5 **(a)** $x = -\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$

(b) $x = \frac{\pi(12n+1)}{18}$ or $x = \frac{\pi(12n+5)}{18}, n \in \mathbb{Z}$

6 $x = \frac{\pi}{2}$ **7** C **8** 6 **9** $\frac{7}{3}$

10 **(a)** dilated by a factor of $\frac{1}{2}$ parallel to the y-axis and a factor of $\frac{1}{3}$ parallel to the x-axis

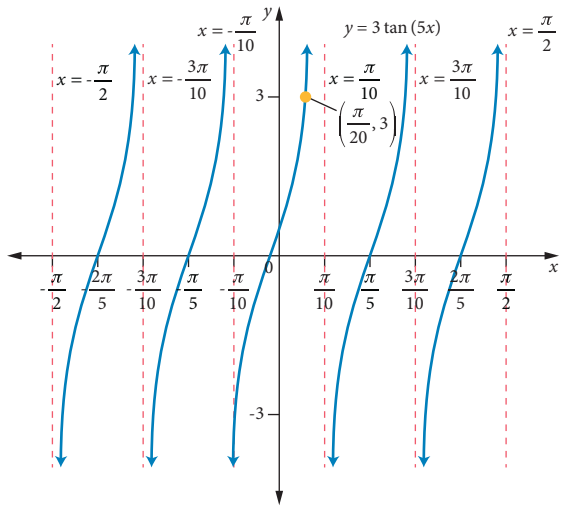
(b) reflected in the x-axis, dilated by a factor of $\frac{1}{5}$ parallel to the x-axis and translated 1 unit in the positive direction of the y-axis

(c) dilated by a factor of 2 parallel to the y-axis and by a factor of $\frac{1}{4}$ parallel to the x-axis and translated by $\frac{\pi}{12}$ in the positive direction of the x-axis

(d) reflected in the x-axis, dilated by a factor of $\frac{1}{2\pi}$ parallel to the x-axis and translated by 1 unit in the positive direction of the y-axis

11 (a) $y = -\frac{3}{4}\cos(8x)$ (b) $y = 2\cos\left(\frac{\pi}{3}(x-1)\right)+5$

12



13 (a) B (b) D (c) B

14 (a) $\frac{dy}{dx} = -8\sin\left(4x - \frac{\pi}{6}\right)$

(b) $\frac{dy}{dx} = 3e^x\left(\cos(3x+2) - 3\sin(3x+2)\right)$

15 (a) $\frac{dx}{dt} = 6\sin(t)\cos(t)$ (b) right: 1 m; left: 2 m

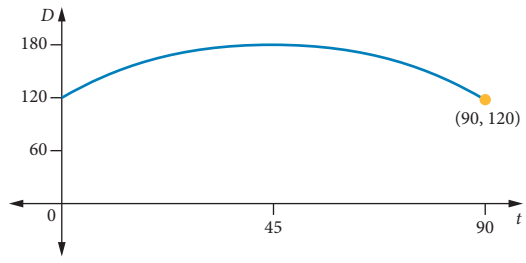
16 C

17 $\frac{5}{2}$

18 (a) $y(t) = 100 - \sqrt{10\,000 - 900\sin^2(\pi t)}$

(b) minimum: 0 cm; maximum: 4.6 cm

19 (a)



(b) maximum: 180 m^3; minimum: 120 m^3

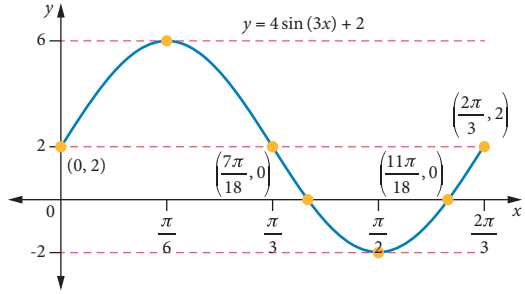
(c) 141 m^3 (d) day 15 and day 75 (e) 60 days

20 $\cos(x) = -\frac{\sqrt{5}}{3}$, $\tan(x) = -\frac{2\sqrt{5}}{5}$

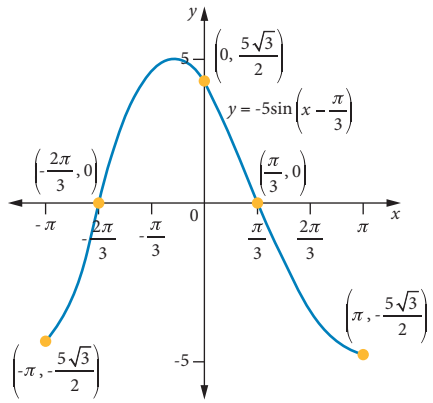
21 (a) $x = -\frac{5\pi}{8}, -\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}$ (b) $x = 0, \pi, 2\pi$

22 C

23 (a)



(b)



24 (a) $f(x) = 2\sin\left(\frac{\pi x}{2}\right)+1$ (b) $k = 2\pi + 1$

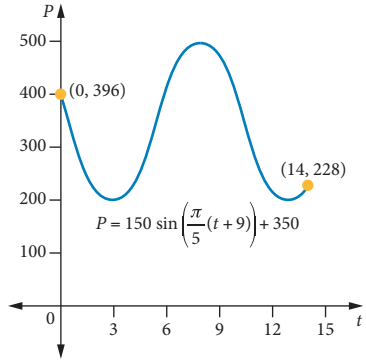
25 $\left(\frac{\pi}{3}, \frac{5}{4}\right)$ **26** B

27 $\left(10\sqrt{2}+10\right)$ cm

28 (a) $P = 150\sin\left(\frac{\pi}{5}\left(t+\frac{9}{2}\right)\right)+350$ or

$P = 150\sin\left(\frac{\pi}{5}\left(t-\frac{11}{2}\right)\right)+350$

(b)



(c) 3.92 years (d) 6.88 years

29 period: $\frac{12\pi}{5}$; range: $[-4, 2]$

30 (a) $\frac{3\sqrt{3}+18}{2}$ m

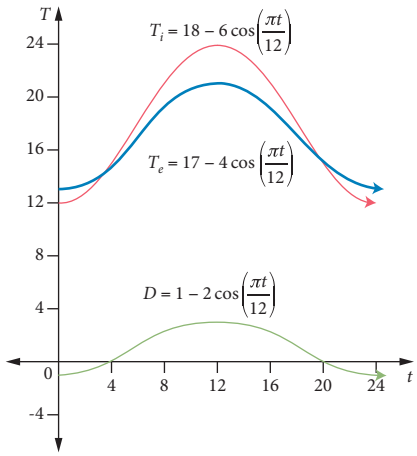
(b) from 7 am to 11 am and from 7 pm to 11 pm

(c) 11.9 m

31 (a) internal: 18°C; external: 17°C

(b) Internal temperature will be a maximum of 3°C greater and a maximum of 1°C less than the external temperature.

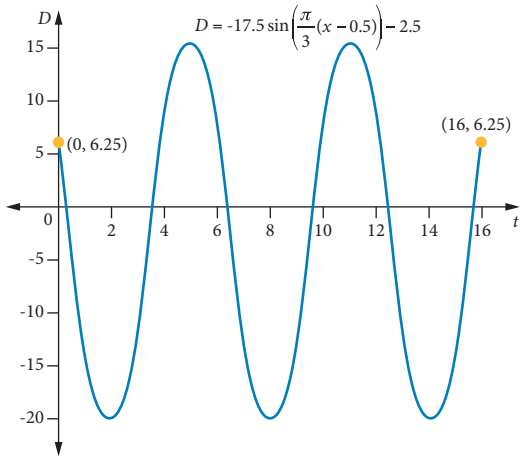
(c)



(d) no **(e)** 4 am and 8 pm **(f)** noon

32 (a) $D = -17.5\sin\left(\frac{\pi}{3}(t - 0.5)\right) - 2.5$

(b)



(c) over water, 6.25 m from the edge of the river

(d) 0.36 seconds

33 (a) $a = 1 + \frac{3\pi}{4}$ **(b)** $y = 2\sin(2x)$

Chapter 3: Logarithmic functions and their derivatives

RECALL

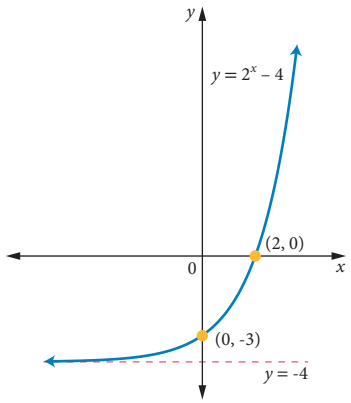
1 (a) not possible **(b)** a^{x-y} **(c)** a^{-x}

2 (a) $\frac{9x^4}{y^6}$ **(b)** $\sqrt[6]{x^5}$ **(c)** 3^{4m-3}

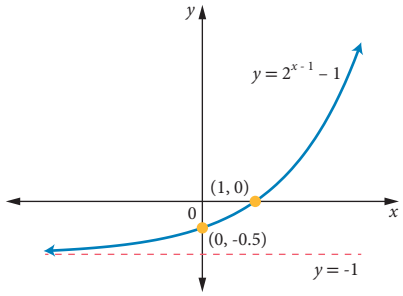
3 (a) 1 **(b)** $\frac{1}{9}$ **(c)** 4

4 (a) $x = 3$ **(b)** $x = 27$ **(c)** $x = \frac{5}{4}$

5 (a)



(b)



6 (a) $c = 1$ **(b)** $b = -3.5$ **(c)** $k = -1$

EXERCISE 3.1

1 (a) 0 **(b)** 0 **(c)** 1 **(d)** 3

2 D

3 (a) $\log_5(15)$ **(b)** 1 **(c)** $\log_3\left(\frac{5}{4}\right)$ **(d)** 1

(e) 1 **(f)** 1 **(g)** 1

4 (a) C

(b) $6\log_e(4x) = \log_e\left((4x)^6\right)$ and $(4x)^6 \neq 4x^6$

5 (a) 3 **(b)** 3 **(c)** -1 **(d)** -2

(e) 1 **(f)** 1 **(g)** 4

6 (a) $\log_3(81) = 4$ **(b)** $\log_{10}(1000) = 3$

(c) $\log_2\left(\frac{1}{4}\right) = -2$ **(d)** $\log_{10}(0.001) = -3$

7 (a) $2^4 = 16$ **(b)** $3^3 = 27$

(c) $2^{-3} = \frac{1}{8}$ **(d)** $10^{-1} = 0.1$

8 (a) 3 **(b)** 27 **(c)** 4

(d) -1 **(e)** x^2 **(f)** $x + 1$

9 (a) 3 **(b)** 2 **(c)** 3

(d) 0 **(e)** 2 **(f)** $\log_2(24) = 3 + \log_2(3)$

10 (a) 0.7781 **(b)** -0.1761 **(c)** -0.3522

11 (a) $\frac{3}{4}$ **(b)** $\frac{3}{2}$ **(c)** 6

(d) $\frac{1}{2}$ **(e)** $\log_3(36)$ **(f)** $\log_5(250)$

12 (a) $\log_2\left(4x^2 - x\right)$ (b) $\log_2\left(\frac{4x+2}{x}\right)$ (c) $\log_2\left(\frac{ab}{c^2}\right)$

(d) $\log_3(x)$ (e) $\log_e\left(\frac{x^4}{x+1}\right)$

13 $y = 225x^2$

14 (a) 1 million (b) 1 276 000

(c) $t = \log_{1.05}(2)$ (d) 2024

15 (a) $\log_a\left(\sqrt{2}\right)$ (b) $1 - \log_b(c)$

(c) $\frac{1}{2} - \log_b(a)$ (d) $2\log_e\left(\frac{z}{xy}\right)$

16 (a) $\log_2(ab) = x + y$ (b) $\log_2\left(\frac{a}{b}\right) = x - y$

(c) $\log_2\left(\frac{16a^2b^{-\frac{1}{2}}}{b^3}\right) = 4 + 2x - \frac{7y}{2}$

EXERCISE 3.2

1 (a) $x = 9$ (b) $x = \frac{1}{625}$ (c) $x = 1$

(d) $x = \pm e$ (e) $x = 3$ (f) $x = 3$

(g) $x = 2$ (h) $x = 16$ (i) $x = 81$

2 (a) $x = 12$ (b) $x = 54$ (c) $x = \frac{3}{5}$

(d) $x = 6$ (e) $x = 2$ (f) $x = 8$

(g) $x = \frac{e^2 + 5}{2}$ (h) $x = \frac{1}{e}$ (i) $x = -\frac{29}{51}$

3 (a) $x = 2$ (b) $x = 1$ (c) $x = \frac{3e^2}{e^2 - 1}$

(d) $x = \frac{e^2}{2e^2 - 1}$

4 (a) A

(b) Option B, because neither $\log_5(0)$ nor $\log_5(-8)$ is defined.

5 (a) $x = 3$ (b) $x = 4$ (c) $x = 3$ (d) $x = 8$

6 (a) $y = \sqrt[3]{\frac{x^2}{4}}$; $x > 0$ (and $y > 0$)

(b) $y = ex^2$; $x > 0$ (and $y > 0$)

(c) $y = \frac{x}{x-1}$, where $x > 1$ (and $y > 0$)

7 $y = e^{\frac{x-3}{2}}$

8 $x = \frac{9}{2}$ and $y = \frac{1}{2}$

9 $a = \frac{1}{5}$

10 (a) $x = e$ or $x = e^2$ (b) $x = \frac{1}{e}$ or $x = e$

11 (a) $x = 3$ (b) $x = 3$ (c) $x = -\frac{2}{3}$ or $x = \frac{1}{2}$

(d) $x = 4$ (e) $x = \sqrt{5}$ (f) $x = 2$ or $x = 3$

12 D

13 (a) $x = \frac{1}{e}$ (b) $x = e$ or $x = \frac{1}{\sqrt{e}}$ (c) $x = e^2$ or $x = e^3$

14 $y = \frac{x^2}{1-x}$, where $0 < x < 1$

15 $x = 9$

16 (a) (i) 12 000 (ii) 14 000

(b) 7.29 months (c) $3\frac{1}{4}$ months

17 (a) plant *A*: 200 kL; plant *B*: 300 kL

(b) 8.6 days (c) 7.4 hours, 500 kL

18 (a) $\log_2\left(4\sqrt{y}\right) = 2 + \frac{b}{2}$

(b) (i) $a = \frac{3}{2}$ and $b = -2$ (ii) $x = 2\sqrt{2}$ and $y = \frac{1}{4}$

19 (a) $-3 < x < 2$

(b) $\{\log(32), \log(90), \log(124), \log(140)\}$

(c) maximum (a, b), where $-3 < a < -1$ and $2 < b < 3$; negative values will occur very close to the extremes of the domain; possible range: $(-\infty, b]$, where $2 < b < 3$

EXERCISE 3.3

1 (a) $\frac{\ln(20)}{\ln(2)}$ (b) $\frac{\log(20)}{\log(2)}$ (c) 4.32

2 (a) 2 (b) 4 (c) 2

3 (a) $x = 6$ (b) $x = 7$ (c) $x = 64$

4 (a) $x = \log_2(5)$ (b) $x = \log_3(7)$ (c) $x = -\log_e(9)$

(d) $x = -\log_2(3)$

5 B

6 (a) $x = \log_2\left(\frac{2}{9}\right)$ (b) $x = \log_2\left(\frac{3}{2}\right)$

(c) $x = \log_e\left(\sqrt[3]{5e}\right)$ (d) $x = \log_e\left(\sqrt{15}\right)$

7 (a) $x = 5.64$ (b) $x = 4.30$

(c) $x = -1.81$ (d) $x = 0.22$ or $x = 1$

8 (a) $x < \frac{\ln(0.4)}{\ln(0.6)}$ (b) $x > \frac{\ln(0.7)}{\ln(0.2)}$ (c) $x > \frac{\ln(7)}{\ln(5)}$

9 (a) 17.4 mm (b) 100 kg (c) $L = 10^{0.98}m^{0.2}$

10 (a) $\log_2\left(5\sqrt{y}\right)$ (b) $\log_3\left(\frac{x^4}{\sqrt{y}}\right)$

11 (a) D

(b) One of the factors in the factorised quadratic expression yielded a negative value for the power. Since powers with positive bases must be positive, only one of the factors yielded a viable solution.

12 (a) $x = \log_3(2)$ (b) $x = \log_3(2)$ (c) $x = \log_e(5)$

(d) $x = \log_e(3)$ or $x = 0$ (e) $x = 0$

13 $x = \sqrt[5]{2^{12}}$

14 $x = 0$ or $x = 1$

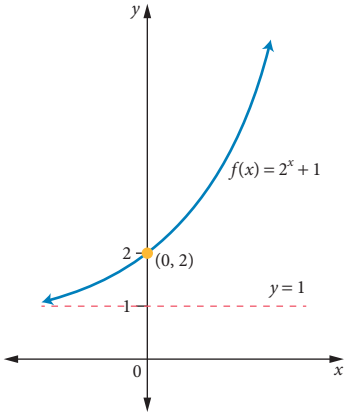
15 **(a)** $k = \dfrac{\ln(2)}{5730}$ **(b)** 19 000 years old **(c)** 10 000 years old

16 $x = \sqrt{10}$

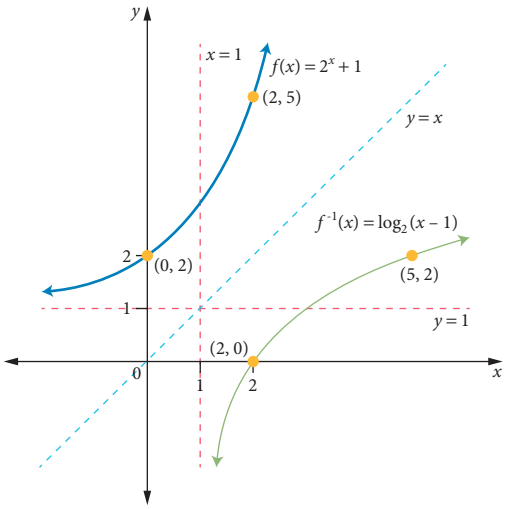
17 **(a)** $x = e^{\frac{1}{e-1}}$ **(b)** $x = e^{\frac{2}{e-2}}$

EXERCISE 3.4

1 **(a)**



(b) $f^{-1}(x) = \log_2(x-1)$; Dom f^{-1}: $x > 1$ and Ran f^{-1}: $\mathbb{R}$



2 **(a)** $f^{-1}(x) = \ln\left(\dfrac{x}{2}\right) + 1$; Dom f^{-1}: $x > 0$ and Ran f^{-1}: $\mathbb{R}$

(b) $f^{-1}(x) = \ln\left(\dfrac{2-x}{3}\right) + 1$; Dom f^{-1}: $-\infty < x < 2$ and Ran f^{-1}: $\mathbb{R}$

(c) $f^{-1}(x) = e^{\frac{x}{2}} + 2$; Dom f^{-1}: $\mathbb{R}$ and Ran f^{-1}: $y > 2$

(d) $f^{-1}(x) = e^{\frac{x+2}{2}}$; Dom f^{-1}: $\mathbb{R}$ and Ran f^{-1}: $\mathbb{R}^+$

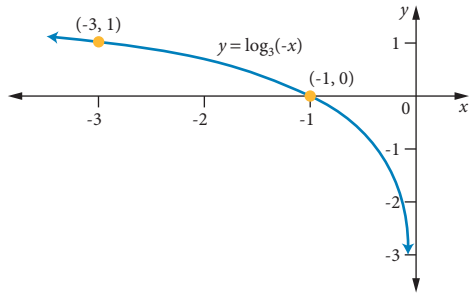
(e) $f^{-1}(x) = e^{2-x} - 1$; Dom f^{-1}: $\mathbb{R}$ and Ran f^{-1}: $y < -1$

3 **(a)** A

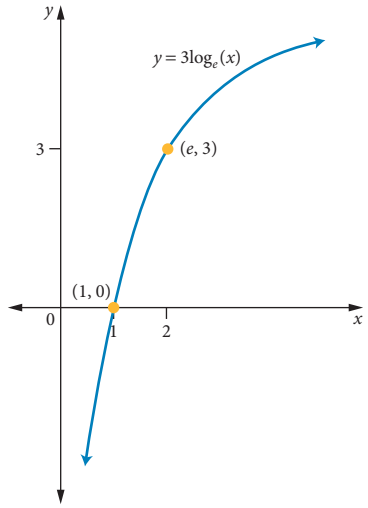
(b) The student mistook $\dfrac{1}{f(x)}$ for $f^{-1}(x)$.

4 C

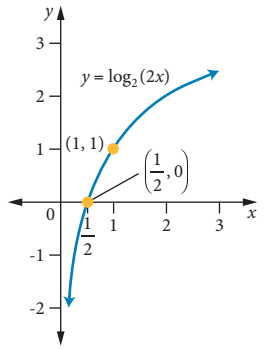
5 **(a)**



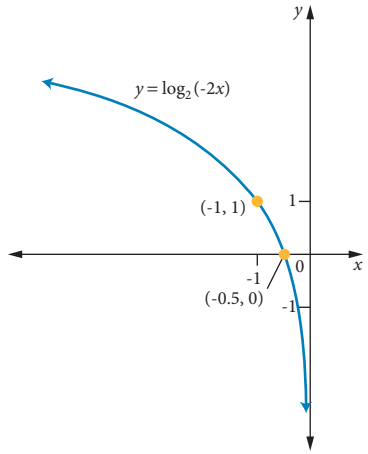
(b)



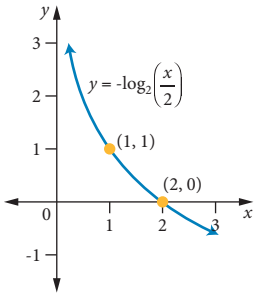
(c)



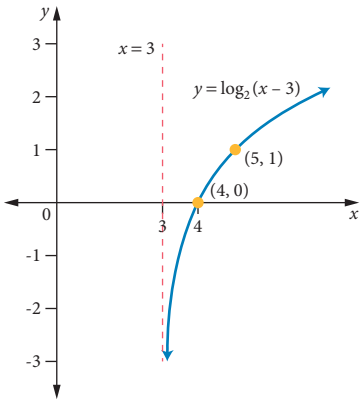
(d)



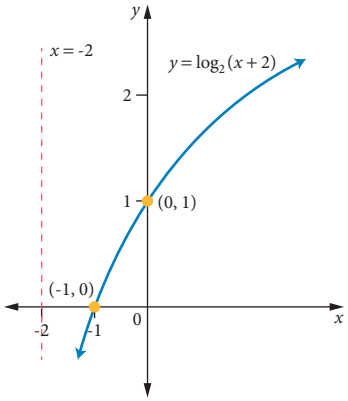
(e)



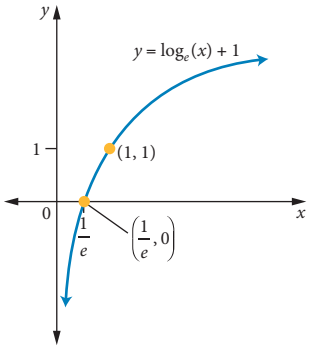
6 (a)



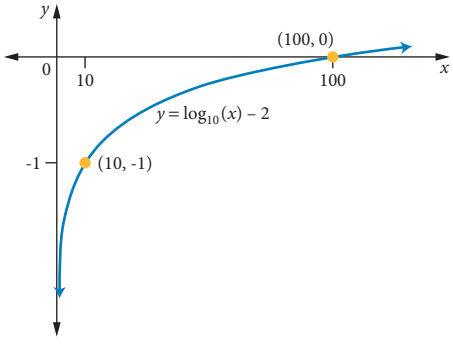
(b)



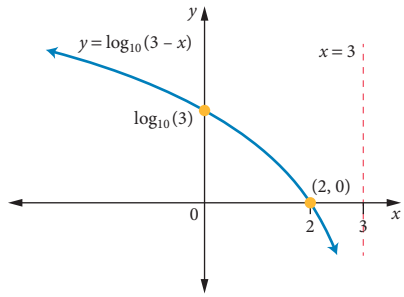
(c)



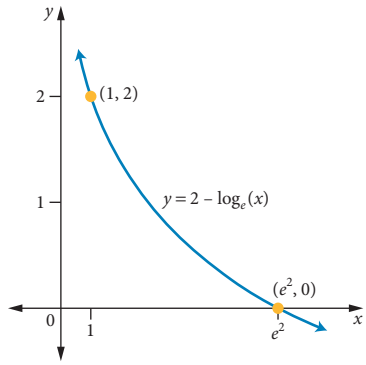
(d)



(e)



(f)



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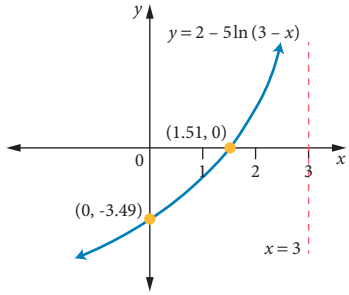


8 (a) $y = -\log_{10}(x) + 2$ **(b)** $y = 2\ln(x + 3) + 2$

(c) $y = \ln(4 - 2x)$

9 (a) dilated by a factor of 3 parallel to the y-axis and translated 2 units in the positive direction of the x-axis

(b) dilated by a factor of 2 parallel to the y-axis, translated 1 unit in the negative direction of the x-axis and 3 units in the positive direction of the y-axis

(c) reflected in both axes, then translated horizontally 2 units in the positive direction of the x-axis and vertically 5 units in the positive direction of the y-axis

(d) reflected in the x-axis, dilated by a factor of $\frac{1}{2}$ parallel to the x-axis, translated 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

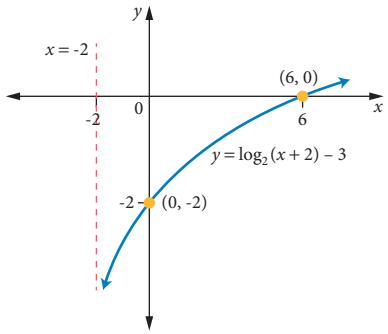
10 (a) $y = 2\log_{10}(x - 2)$ **(b)** $y = \log_2(2x)$

(c) $y = 2\ln(x - 1)$ **(d)** $y = -\ln(x + 3)$

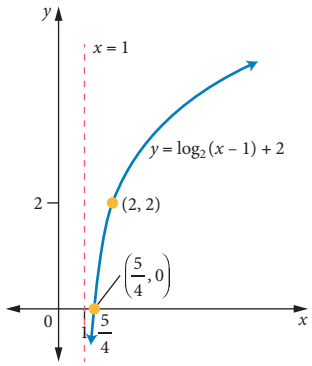
(e) $y = 1 - 2\log_2(x - 3)$ **(f)** $y = \log_3(3 - x) + 1$

11 **(a)** $x > \frac{1}{3}$ **(b)** $x < 0$ or $x > 3$ **(c)** $-3 < x < 2$

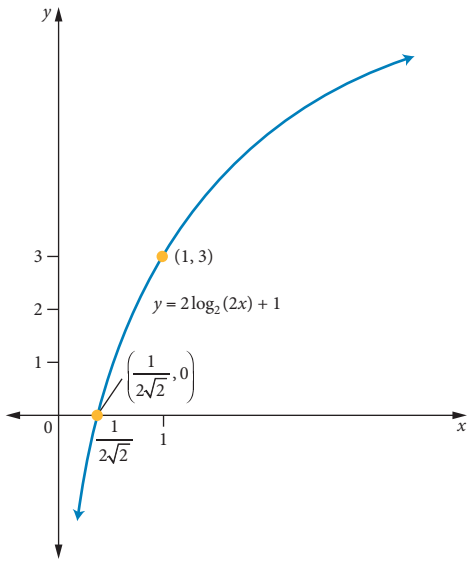
12 **(a)**



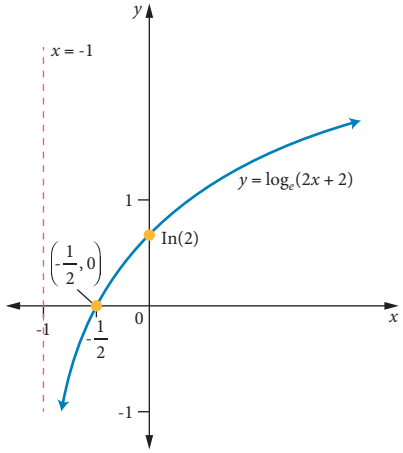
(b)



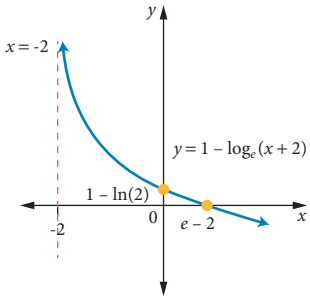
(c)



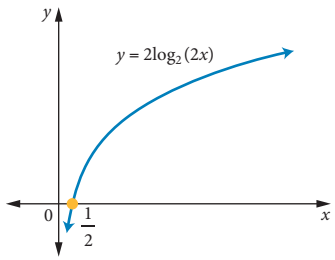
(d)



(e)



13 **(a)**



(b) **(i)** $x = 2$ **(ii)** $x > 8$ **(iii)** $0 < x < 4$

(c)
$$\begin{aligned} f(g(x)) &= f\left(2^{\frac{x}{2}-1}\right) \\ &= 2\log_2\left(2 \times \left(2^{\frac{x}{2}-1}\right)\right) \\ &= 2\log_2\left(2^{\frac{x}{2}-1+1}\right) \\ &= 2\log_2\left(2^{\frac{x}{2}}\right) \\ &= 2 \times \frac{x}{2}\log_2(2) \\ &= x \end{aligned}$$

(d) $(0.6, 0.6)$ and $(8, 8)$

14 $k = -\frac{7}{2}$; $A\left(0, -\frac{5}{2}\right)$

15 $p = 4$; $B(-3, 0)$

16 **(a)** $\left(\frac{11}{3}, -1\right)$ **(b)** $g^{-1}(x) = \frac{1}{3}\left(10^{\frac{1-x}{2}} + 1\right)$

(c)
$$\begin{aligned} g\left(g^{-1}(x)\right) &= -2\log_{10}\left(3\left(\frac{1}{3}\left(10^{\frac{1-x}{2}} + 1\right)\right) - 1\right) + 1 \\ &= -2\log_{10}\left(10^{\frac{1-x}{2}} + 1\right) - 1 \\ &= -2\left(\frac{1-x}{2}\right) + 1 \\ &= x - 1 + 1 \\ &= x \end{aligned}$$

17 **(a)** $y = \ln\left(\sqrt[3]{(x-2)^2}\right)$; domain: $x > 2$; range: $\mathbb{R}$

(b) $y = 2 - 2e^{2-x}$; domain: $\mathbb{R}$; range: $y < 2$

EXERCISE 3.5

1 **(a)** $I = 10^{8.2} I_0$ **(b)** 1259

2 $R = 2.9$ **3** 2.5 **4** 562

5 75 dB **6** 102 dB **7** 3.16×10^8 times

8 **(a)** C

(b) For pH, the higher the value of $\left[H^+\right]$, the lower the pH value.

9 D **10** pH = 7.4

11 10^{-5} moles/litre **12** 9.51

13 8.6 **14** 10 times

15 75 dB **16** 158

17 **(a)** $I = 10^{\frac{D}{10}} \times I_0$ **(b)** $\frac{I_2}{I_1} = 10^{\frac{D_2 - D_1}{10}}$ **(c)** 100 times

18 **(a)** **(i)** 555 cents **(ii)** 6 semitones

(b) $2^{\frac{5}{12}}$

(c) $C = \frac{1200 \log_{10}\left(\frac{f_2}{f_1}\right)}{\log_{10}(2)}$

19 **(a)** 1.36

(b) **(i)** 0.68 per year **(ii)** 0.02 megatons

(c) $E = \left(\frac{3 \times 10^{2-P} T}{p_i}\right)^{\frac{5}{4}}$

EXERCISE 3.6

1 **(a)** $\frac{1}{x}$ **(b)** $\frac{1}{x}$ **(c)** $-\frac{3}{x}$

(d) $\frac{1}{x}$ **(e)** 0 **(f)** $\frac{2}{x}$

(g) $\frac{1}{2x}$ **(h)** $-\frac{1}{x}$ **(i)** $-\frac{2}{x}$

2 **(a)** $\frac{2}{2x-1}$ **(b)** $\frac{5}{5x+2}$ **(c)** $\frac{4}{4x-1}$ **(d)** $\frac{2}{2x-1}$

(e) $\frac{1}{2x+1}$ **(f)** $-\frac{1}{x+2}$ **(g)** $\frac{3}{2-3x}$

3 **(a)** domain: $x < 0 \cup x > 0$; $f'(x) = \frac{2}{x}$

(b) domain: $x > 0$; $f'(x) = \frac{1}{2x}$

(c) domain: $x < -\sqrt{2} \cup x > \sqrt{2}$; $f'(x) = \frac{2x}{x^2 - 2}$

(d) domain: $\mathbb{R}$; $f'(x) = \frac{4x}{x^2 + 2}$

4 **(a)** $\log_e(x) + 1$ **(b)** $2e^x \log_e(3x+2) + \frac{6e^x}{3x+2}$

(c) $\frac{\cos(2x-3)}{x} - 2\log_e(x)\sin(2x-3)$

5 $-\frac{3}{2}$

6 **(a)** $\frac{f'(\ln(x))}{x}$ **(b)** $\frac{3f'(3\ln(2x))}{x}$

7 **(a)** $\mathbb{R} \setminus \{-3\}$

(b) No; $\log_e\left((x+3)^2\right)$ and $2\log_e(x+3)$ have different domains.

(c) **(i)** $f'(x) = \frac{2}{x+3}$ **(ii)** $\mathbb{R} \setminus \{-3\}$

(iii) $\left(-1, \log_e(4)\right)$

8 **(a)** $-\frac{2x}{x^2+1}$ **(b)** 3 **(c)** $-\frac{\sin\left(\log_e(x)\right)}{x}$

(d) $\frac{1}{2x\sqrt{\log_e(x)}}$ **(e)** $\frac{2\log_e(x)}{x}$

9 **(a)** $\frac{x + 2 - x\ln(x)}{x(x+2)^2}$ **(b)** $-\frac{3}{(x-1)(x+2)}$

10 C

11 **(a)** D

(b) The student has calculated $f(1)$ instead of $f'(1)$.

12 $\frac{dy}{dx} = 3$

13 **(a)** $f(x) = \frac{\log_e(3x+2)}{\log_e(2)}$

(b) $f'(x) = \frac{3}{(\log_e(2))(3x+2)}$

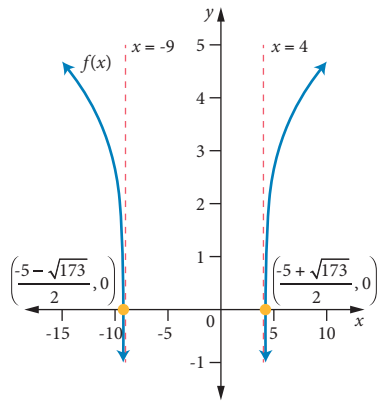
(c) $g'(x) = \frac{2\log_e(x)}{x\log_e(2)}$

(d) $y = x - 2 + \log_e(2)$

14 **(a)** $h(x) = \ln\left(x^2 + 5x - 36\right)$; $x \in (-\infty, -9) \cup (4, \infty)$

(b) $h'(x) = \frac{2x+5}{x^2+5x-36}$

(c)



15 **(a)** $f'(x) = \frac{2ab}{x\left(a\ln(x) + b\right)^2}$; the derivative is a fraction with numerator $2ab$. Since neither a nor b can equal 0, there are no solutions to the equation $f'(x) = 0$, and hence no stationary points.

(b) vertical asymptotes: $x = 0$ and $x = e^{-\frac{b}{a}}$; x-intercept: $\left(e^{\frac{b}{a}}, 0\right)$; domain: $x \in (0, \infty) \setminus \left\{e^{-\frac{b}{a}}\right\}$

EXERCISE 3.7

1 **(a)** 150 horses **(b)** 2 days **(c)** $a = e - e^{-2}$

(d) -33.27 horses per day **(e)** $\frac{50}{2-e} \approx -64$ horses per day

2 **(a)** $a = -\frac{50}{\log_e(5)}$; $b = 300$

(b) 190 wombats **(c)** 2036

3 **(a)** B

(b) The student did not consider that the domain is restricted because $\ln(0)$ is not defined.

4 A

5 **(a)** $f'(x) = \frac{3}{x}$ **(b)** 3

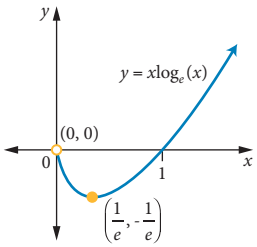
(c) $y = 3x - 3 + \log_e(8)$ **(d)** $y = 2x - 3 + \log_e(27)$

6 **(a)** $\frac{2000}{t+1}$ m/min **(b)** 500 m/min **(c)** 5.25 min

7 **(a)** $f(1) = 0$ **(b)** $x = 1$

(c) $f'(x) = \log_e(x) + 1$

(d)



8 **(a)** 2.7 minutes

(b) 11.59 minutes per year

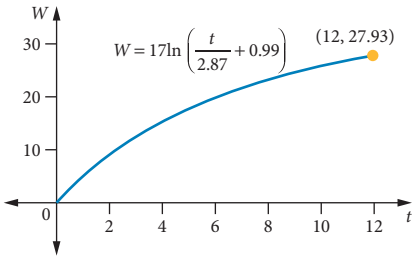
(c) $S'(x) = \frac{10(2x-1)}{x(\ln(x) - 2x + 7)^2}$

(d) 6 months

9 **(a)** 9% **(b)** 2.6% of penguins per day

(c) 1.3% of penguins per day **(d)** 7.7 days; 18.7%

10 **(a)** $c = 2.87$, $d = 0.99$

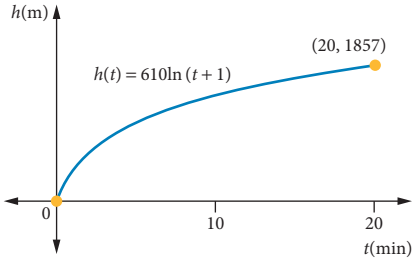


(b) 22.6 kg

11 **(a)** $a = 328.36$ and $b = -5.79$

(b) 0.06 km/s **(c)** 17.3 s

12 **(a)**

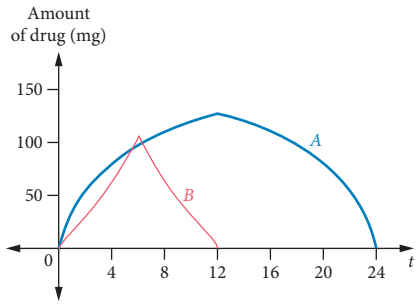


(b) 136 minutes or 2 h 16 min **(c)** 153 m/min

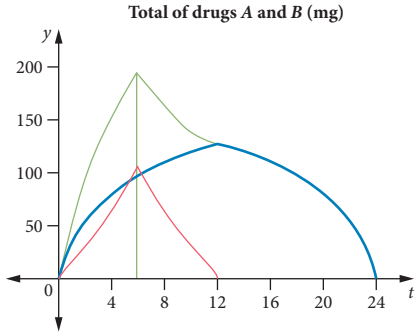
(d) 45 seconds **(e)** 1 min 49 s

13 504 days

14 **(a)** maximum drug *A*: 128.25 mg;
maximum drug *B*: 105.82 mg



(b)



(c) A patient will receive maximum pain relief 6 hours after taking the drugs.

(d) 203.12 mg **(e)** 5 h 35 min

CHAPTER REVIEW 3

1 **(a)** -4 **(b)** x^2

2 **(a)** $\ln\left(\frac{x^2}{x+2}\right)$ **(b)** $\frac{2}{3}$

(c) 3 **(d)** $2\ln(3) + 2x$

3 $\log_a\left(\frac{2}{a}\right) = k - 1$ **4** $x = \frac{1}{3}\log_a\left(\frac{y-b}{2}\right)$

5 **(a)** $x = \sqrt{\frac{y}{8}}$ **(b)** $x = 1$

6 $a = \frac{10}{3}$ **7** $x = \frac{\sqrt{ae}}{2}$

8 $x = 2.43$ **9** $a = \log_{10}\left(\frac{4}{5}\right)$

10 $b = 1$

11 $f^{-1}(x) = \frac{e^{x-1} + 2}{3}$; domain $f^{-1}(x) = \mathbb{R}$ and range $f^{-1}(x) = \left(\frac{2}{3}, \infty\right)$

12 $k = \frac{e}{3}$; $b = 3$ **13** A

14 A **15** B

16 $(\ln(2), 4)$ **17** D

18 $\mathbb{R} \setminus \{0\}$ **19** 1000 times

20 D **21** 1

22 $\frac{dy}{dx} = \ln(2x-3) + \frac{2x}{2x-3}$; gradient = 4

23 C

24 $f'(1) = 1$

25 $\dfrac{f'(\log_e(3x))}{x}$

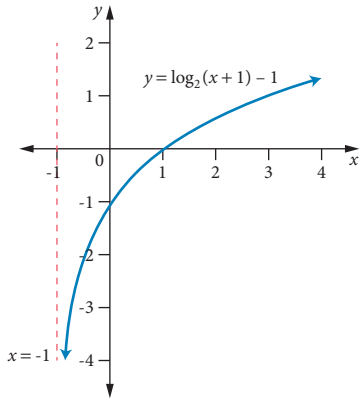
26 $\dfrac{1}{2}$

27 $m = 10\,000$ or 0.0001

28 $x = \log_e(8)$

29 $\ln\left(\dfrac{3}{4}\right)$

30



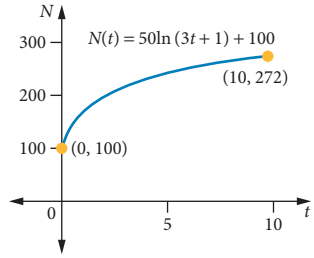
31 (a) 100 times (b) 10^9 times (c) 70 dB
(d) 7900 times (e) 91 dB

32 $\left(e, \dfrac{1}{e}\right)$

33 $\left(\dfrac{5}{2}, \log_e(20)\right)$

34 $a = \dfrac{8}{e^3}; b = \dfrac{e^7}{8}$

35 (a)



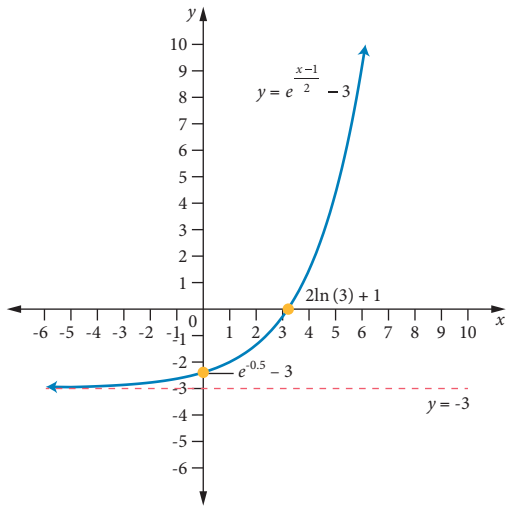
(b) 100 injuries (c) 272 injuries (d) 2008

(e) 27.7 injuries per year

(f) 9.4 injuries per year (instantaneous rate of change) or 10.4 injuries per year (actual increase from one year to the next)

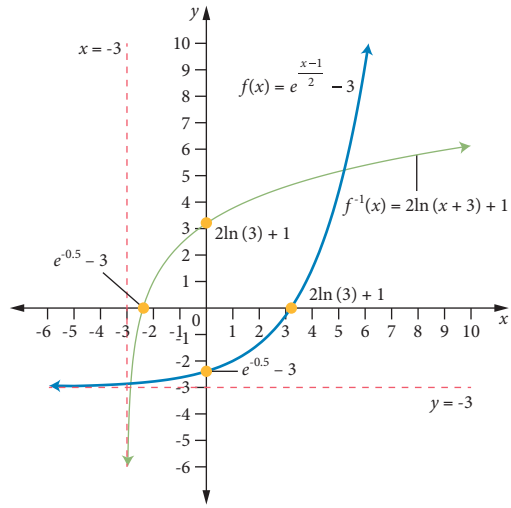
36 $x = e^{\frac{2}{a-1}}$

37 (a) (i)



(ii) $f^{-1}(x) = 2\ln(x+3) + 1$; domain $f^{-1}(x)$: $(-3, \infty)$; range $f^{-1}(x)$: $\mathbb{R}$

(iii)



(iv) $(-2.85, -2.85)$ and $(5.21, 5.21)$

(b) (i) $a = \dfrac{10}{\ln(e^2 - 1)}$, $b = 1$, $c = -1$

(ii) $k = 204$

(iii) 1.77

(c) $(-0.24, -2.46)$ and $(6.06, 9.54)$

38 (a) $x < -1 \cup x > 1$

(b) There are no stationary points.

(c) $f(x)$ is not a one-to-one function; therefore the inverse function does not exist.

(d) (i) 1

(ii) $g^{-1}(x) = \sqrt{e^x + 1}$; domain: $x \in \mathbb{R}$; range: $y > 1$

39 $\dfrac{13(\ln(3))^2}{12}$ square units

Mixed review: Chapters 1–3

1 (a) $x = -3$ (b) $x = \dfrac{3}{2}$ (c) $x = \log_e(7)$

2 $f(x) = -2e \times e^x + 2$, $f(x) = 2 - 2e^{x+1}$

3 (a) $f'(x) = 5e^5$ (b) $f'(t) = -10e^{2t}$
(c) $f'(x) = 2e^{2x+1}$ (d) $f(x) = 2xe^{x^2}$

4 $N(t) = 10 \times 2^{\frac{t}{3}}$

5 (a) $-\dfrac{\sqrt{3}}{2}$ (b) 1

6 (a) $x = -\dfrac{8\pi}{9}, -\dfrac{5\pi}{9}, -\dfrac{2\pi}{9}, \dfrac{\pi}{9}, \dfrac{4\pi}{9}, \dfrac{7\pi}{9}$

(b) $x = \dfrac{\pi}{9} + \dfrac{2n\pi}{3}$ or $x = \dfrac{\pi}{3} + \dfrac{2n\pi}{3}$, $n \in \mathbb{Z}$

7 **(a)** $\frac{dy}{dx} = 3\cos(3x)$ **(b)** $\frac{dy}{dx} = -12\sin(6x)$

(c) $\frac{dy}{dx} = \frac{15}{\cos^2(5x)}$ **(d)** $\frac{dy}{dx} = -12\cos(3x)$

8 $y = \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}\pi}{12} + \frac{1}{2}$

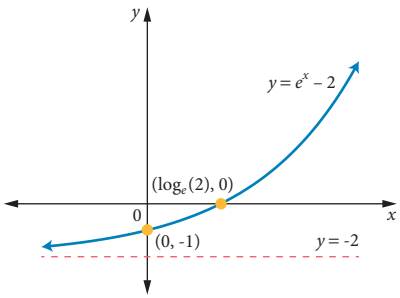
9 $\log_5(125) = 3$

10 $x \geq \frac{\ln(2)}{\ln(5)}$

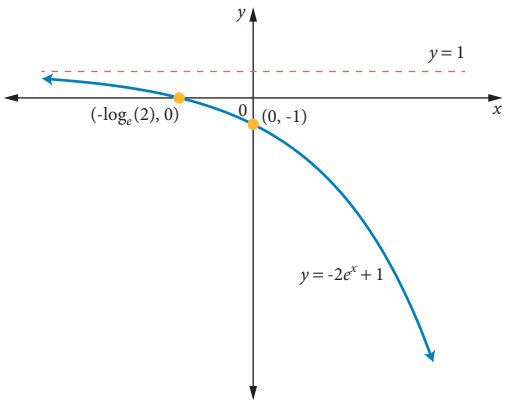
11 pH = 5.5

12 **(a)** $\frac{dy}{dx} = \frac{2}{2x-3}$ **(b)** $\frac{dy}{dx} = \frac{5}{2(5x-4)}$

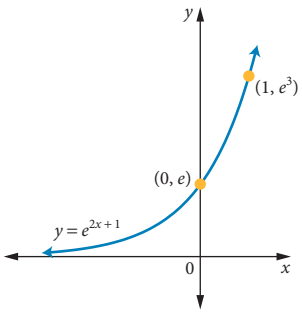
13 **(a)** domain: $x \in \mathbb{R}$; range: $y \in (-2, \infty)$



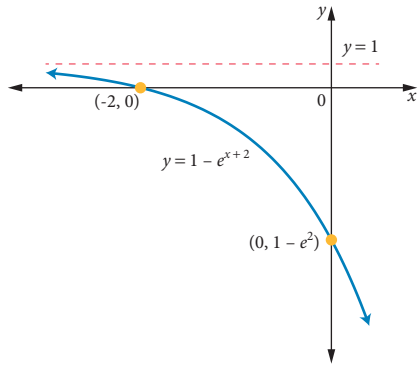
(b) domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 1)$



(c) domain: $x \in \mathbb{R}$; range: $y \in (0, \infty)$



(d) domain: $x \in \mathbb{R}$; range: $y \in (-\infty, 1)$



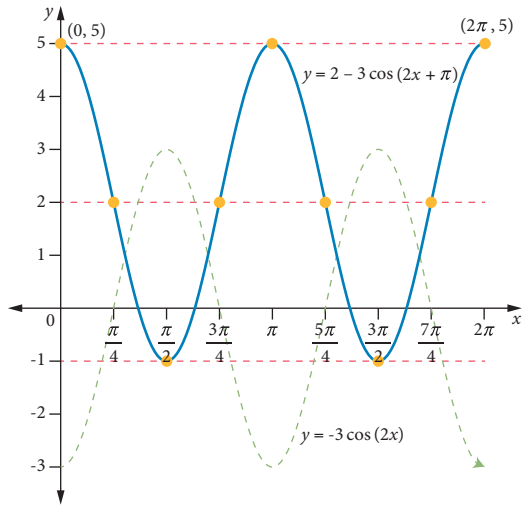
14 **(a)** $\frac{dy}{dx} = \frac{2e^{3x} - 2x + 4}{e^x}$

(b) $\frac{dy}{dx} = 2(3x^2 - 2x + 1)(15x^2 - 6x + 1)$

15 **(a)** $\cos(\theta) = -\frac{12}{13}$ **(b)** $\tan(\theta) = \frac{5}{12}$

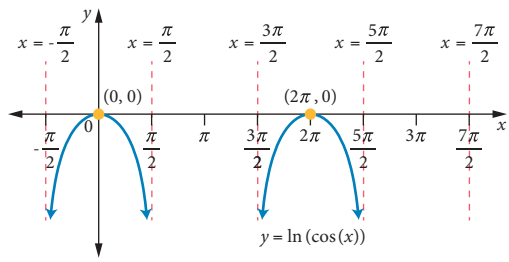
16 **(a)** amplitude: 3; period: π **(b)** $(0, 5)$

(c)



17 **(a)** 3.296 **(b)** 6.013 square units

18



Chapter 4: Anti-differentiation

RECALL

1 **(a)** $12x^2 - 4x - 3$ **(b)** $4x + 2$ **(c)** $18x^5 + 4$

(d) $\frac{4}{3}x^{-\frac{1}{3}}$ **(e)** $\frac{1}{2\sqrt{x}} + \frac{3}{x^4} - \frac{4}{x^2}$

2 **(a)** -40 **(b)** -21 **(c)** $\frac{1}{128}$

3 (a) $12(3x+2)^3$ (b) $6(2x+5)^2$ (c) $(2x-1)^{-\frac{1}{2}}$

(d) $-\dfrac{30}{(3x-4)^3}$ (e) $3(2x+3)^{\frac{1}{2}}$

4 (a) $5e^{5x}$ (b) $\dfrac{1}{2}e^{\frac{x}{2}}$ (c) $3(10x-1)e^{5x^2-x}$

(d) $\dfrac{14}{2x-1}$ (e) $\dfrac{6+x}{x(3+x)}$

5 (a) $3\cos(3x)$ (b) $-5\sin(5x)$ (c) $2\cos\left(\dfrac{2x}{3}\right)-2\pi\sin(\pi x)$

(d) $-5\cos(3-5x)$ (e) $-33\sin(11x-2)$

6 (a) $\dfrac{dy}{dx}=2x\sin(x)+x^2\cos(x)$ (b) $2x^3e^{2x}(2+x)$

(c) $\dfrac{3x^2\sin(x)-x^3\cos(x)}{\sin^2(x)}$ (d) $e^x\left(\log_e(x)+\dfrac{1}{x}\right)$

(e) $\dfrac{1}{\cos^2(x)}$ (f) $\dfrac{1}{2\sqrt{x}(x+5)}+\dfrac{\sqrt{x}}{(x+5)^4}$

7 (a) $8x^3(3x+1)(3x+2)^3$ (b) $2x\cos(x^2)$

(c) $\dfrac{3x^2-4}{2\sqrt{x^3-4x+2}}$ (d) $2xe^{x^2}$

(e) $2\sin(x)\cos(x)$

EXERCISE 4.1

1 (a) $2x^3-4x+c$ (b) $3x^2-2\sqrt{x}+c$

(c) $5x^2+2x-6+c$ (d) $5x^2-\dfrac{1}{x}+c$

2 (a) $3x^3-5x+c$ (b) $x^3-\dfrac{1}{5}x^5+c$

(c) $x^4-\dfrac{5}{3}x^3+c$ (d) $2x^2-\dfrac{2}{x}+c$

3 (a) $(3x-2)^5+c$ (b) $2(6x-1)^3+c$

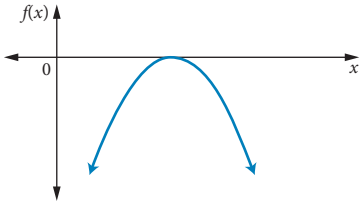
4 (a) $\dfrac{1}{2}\sin(2x)+c$ (b) $-9\cos^3(x)+c$

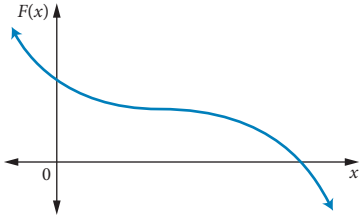
(c) $\dfrac{1}{12}\sin^2(3x)-\dfrac{1}{12}\cos^2(3x)+c$

5 (a) e^x+c (b) $\dfrac{1}{k}e^{kx}+c$ (c) $6e^{5x}+c$

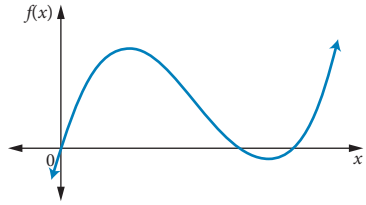
6 (a) $\dfrac{1}{2}\log_e(7x)+c$ (b) $\dfrac{1}{2}\log_e(2x+1)+c$

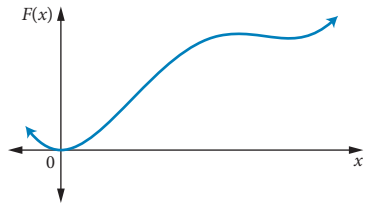
7 (a) Note that the placement of the curve in relation to the x-axis is arbitrary.



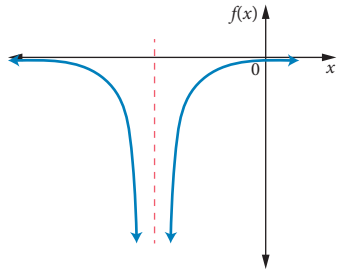


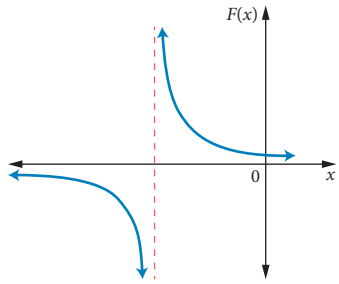
(b)



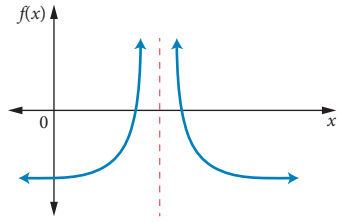


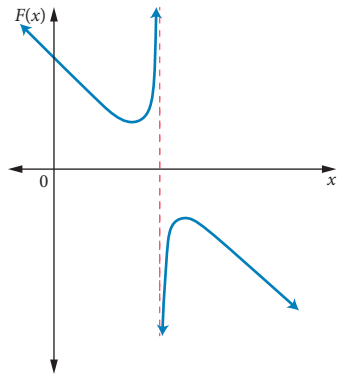
8 (a)





(b)





9 C **10** B

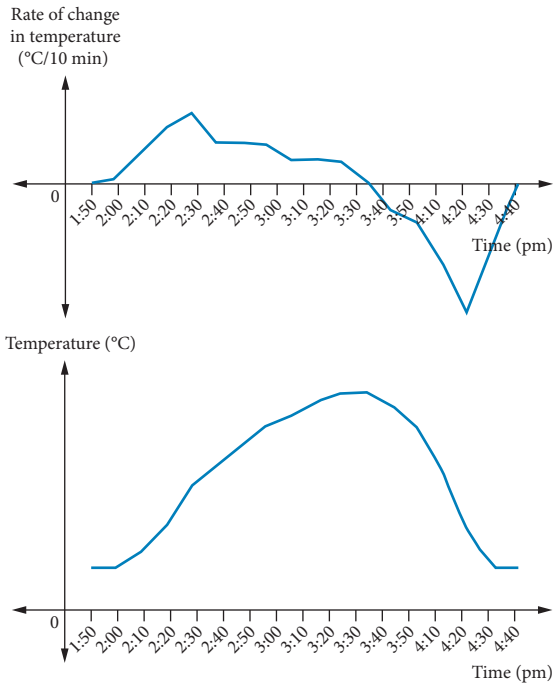
11 B **12** $c=-34$

13 $f(x)=\dfrac{3}{8}(2x-1)^4+2$

14 **(a)** 2:30 pm **(b)** between 1:50 pm and 3:40 pm

(c) The temperature stops increasing and starts to decrease.

(d)



EXERCISE 4.2

1 **(a)** $\frac{x^7}{7}+c$ **(b)** $3x^4+c$ **(c)** $\frac{4x^3}{3}+c$ **(d)** $\frac{x^6}{18}+c$

(e) $2x^{\frac{3}{2}}+c$ **(f)** $\frac{6x^{\frac{5}{3}}}{5}+c$ **(g)** $\frac{15x^{\frac{7}{5}}}{28}+c$

(h) $-\frac{3}{x}+c$

2 **(a)** $2x^3-\frac{1}{2}x^2+2x+c$ **(b)** $\frac{1}{4}x^4-x^3+x^2+c$

(c) $2x^3+\frac{4x\sqrt{x}}{3}+c$ **(d)** $4x-\frac{1}{6}x^3+c$

(e) $2x^2-2x^{\frac{1}{2}}+\frac{x^4}{4}+c$ **(f)** $7w-\frac{2}{3w}+\frac{3w^{\frac{5}{3}}}{5}+c$

3 **(a)** $\frac{x^2}{2}+x+c$ **(b)** $\frac{1}{2}x^2-2x+c$

(c) $\frac{x^3}{3}-2x-\frac{3}{x}+c$ **(d)** $\frac{x^4}{4}-\frac{x^3}{3}+x^2-2x+c$

(e) $x+\frac{4}{x}+c$ **(f)** $\frac{x^3}{3}+x+\frac{12}{x}+c$

(g) $\frac{2x^{\frac{5}{2}}}{5}+\frac{2x^{\frac{3}{2}}}{3}+c$ **(h)** $\frac{2}{5}x^{\frac{5}{2}}-2x^{\frac{3}{2}}+c$

4 $x=t^3-2t^2+18t+c$

5 **(a)** 2 **(b)** 65

6 **(a)** $\frac{z^2}{2}-2z^3+c$ **(b)** $\frac{z^3}{3}+z^2-8z+c$

(c) $z^3+\frac{4}{z}+c$ **(d)** $\frac{2}{z}+c$

7 **(a)** $y=2x^3-x^2-22$ **(b)** $y=-\frac{3}{x}+\frac{2}{x^2}+\frac{13}{8}$

8 $f(x)=x+\frac{4x^{\frac{3}{2}}}{3}+\frac{x^2}{2}-3$

9 **(a)** $-\frac{(x-6)^6}{192}+c$ **(b)** $\frac{2x^{\frac{7}{2}}}{7}-\frac{16x^{\frac{5}{2}}}{5}+\frac{32x^{\frac{3}{2}}}{3}+c$

10 **(a)** $y=2x^4-\frac{3}{x}+c$ **(b)** $y=-\frac{1}{x^2}-\frac{2}{\sqrt{x}}+c$

11 $f'(x)=\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}+c$; $\int 2x\sqrt{1+x^2}\,dx=\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}+c$

12 **(a)** $y=\frac{2x^3}{3}$ **(b)** 18 **(c)** -18

13 **(a)** $y=4x^3-6x^2+3x+c$

(b) **(i)** $y=4x^3-6x^2+3x+4$

(ii) $y=4x^3-6x^2+3x+18$

(c) Curves related by a vertical translation have no intersection points.

14 **(a)** 930 **(b)** $-\frac{1}{3}$

15 **(a)** $V=-1.4t^2+1260$ **(b)** 21.2 seconds

16 D **17** A

18 **(a)** $x-\frac{2}{x}-\frac{1}{3x^3}+5x+c$ **(b)** $\frac{x^2}{2}+\frac{4}{\sqrt{x}}-\frac{1}{3x^3}+c$

19 $\int\frac{5x^2+12x}{2\sqrt{x+3}}dx=(x+3)^{\frac{1}{2}}x^2+c$

20 $-x^3-\frac{1}{x^2}+4$

21 $\frac{(3x+1)^4}{12}+c$

22 **(a)** $\frac{(ax+b)^{n+1}}{a(n+1)}+c$

(b) **(i)** $\frac{(6x+1)^5}{30}+c$ **(ii)** $-\frac{(3-2x)^4}{8}+c$

(iii) $\frac{2(3x-2)^{\frac{3}{2}}}{9}+c$ **(iv)** $\frac{(7x+2)^4-19}{7}$

EXERCISE 4.3

1 **(a)** $\frac{1}{10}(2x+5)^5+c$ **(b)** $\frac{1}{12}(3x-1)^4+c$

(c) $\frac{1}{3}(2x-1)^6+c$ **(d)** $-\frac{3}{8}(1-2x)^4+c$

2 **(a)** $\frac{1}{3}(2x+1)^{\frac{3}{2}}+c$ **(b)** $\frac{(6x-5)^{\frac{1}{2}}}{3}+c$

(c) $-\frac{2(2-11x)^{\frac{3}{2}}}{33}+c$ **(d)** $-\frac{2(1-10x)^{\frac{7}{4}}}{35}+c$

3 **(a)** $-\frac{1}{9(2x+1)}+c$ **(b)** $-\frac{2}{5x-2}+c$ or $\frac{2}{2-5x}+c$

(c) $-\frac{1}{4(3-4x)^2}+c$ **(d)** $\frac{1}{9(1-2x)^3}+c$

(e) $-\frac{1}{2\sqrt{2x+1}}+c$ **(f)** $-\frac{10(1-3x)^{\frac{1}{2}}}{3}+c$

4 (a) v (b) iii (c) vi

5 (a) $y = \frac{1}{20}(5x-2)^4 + c$ (b) $y = \frac{2}{3\sqrt{2-3x}} + c$

6 (a) $f(x) = 6\sqrt{1+2x} + c$ (b) $\frac{3}{10}(5x-7)^4 + c$

7 $f(x) = \frac{x^3}{3} + \frac{1}{x} - \frac{10}{3}$

8 $y = \frac{1}{8}(2x-5)^4 + \frac{23}{8}$

9 (a) $\frac{(2x-1)^7}{6} + c$ (b) $\frac{6}{25(2-5x)^{\frac{5}{2}}} + c$

10 (a) $F(x) = \frac{a}{m(k+1)}(mx+b)^{k+1} + c$

(b) $F(x) = \frac{d}{a(1-m)}(ax+b)^{1-m} + c$

11 C

12 (a) 4 m/s (b) $x = \frac{(3t-1)^4}{6} + 6t + c$ (c) $-\frac{11}{6}$ m

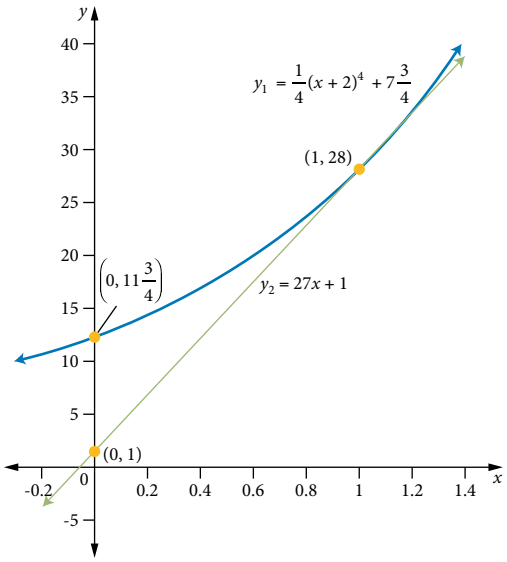
13 7.4 m^2

14 (a) $\frac{1}{12}(3x-4)^4 + c_1; \frac{27x^4}{4} - 36x^3 + 72x - 64x + c_2$

(b) The two answers are the same when an adjustment is made to the constants. $c_2 = \frac{64}{3} + c_1$

15 (a) $y = \frac{1}{4}(x+2)^4 + 7\frac{3}{4}$

(b) The coordinates of the points of intersection of the graphs are (1, 28).



EXERCISE 4.4

1 (a) $\frac{1}{3}e^{3x} + c$ (b) $-e^{-x} + c$ (c) $\frac{3e^{\frac{2x}{3}}}{2} + c$

(d) $-2e^{-\frac{x}{2}} + c$ (e) $-\frac{4}{e^{\frac{t}{2}}} + c$

2 (a) $\frac{1}{2}e^{6x} + c$ (b) $-\frac{1}{4e^{4x}} + c$ (c) $-\frac{2}{e^{3x}} + c$

(d) $\frac{1}{2}e^{5x} + c$ (e) $-\frac{1}{9e^{6x}} + c$ (f) $3e^{\frac{x}{3}} + c$

3 (a) $\frac{1}{3}e^{3x} - 2x^3 - 2x + c$ (b) $2e^{2x} + x^2 - x + c$

(c) $-\frac{1}{e^{2x}} + (e+6)x + c$ (d) $5e^x - \frac{2}{3}e^{3x} + \frac{1}{2}x^2 + c$

4 (a) $\frac{1}{2}e^{2x+1} + c$ (b) $e^{3x-2} + c$ (c) $-2e^{1-2x} + c$

(d) $-2e^{2-3x} + c$ (e) $\frac{3}{2}e^{\frac{2x}{3}+4} + c$ (f) $-\frac{3}{5}e^{2-5x} + c$

5 (a) $\frac{1}{4}e^{4x} - e^{2x} + x + c$ (b) $4x + \frac{12}{e^x} - \frac{9}{2e^{2x}} + c$

(c) $e^{2x} + \frac{2}{e^x} + c$ (d) $2e^{3x} - e^{2x} + c$

(e) $2e^x + \frac{1}{4e^{2x}} + c$ (f) $\frac{3}{2}e^{4x} - \frac{1}{2}e^{6x} + c$

6 (a) $f(x) = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} + 2$ (b) $f(x) = e^{2x} - e^x + e - e^2$

7 $y = \frac{1}{2}e^{2x} + \frac{1}{e^x} - 1$

8 B

9 $F(x) = 2e^{2x} - 3x - 4$

10 $2e^{\frac{x}{2}} - 2\sqrt{x} + c$

11 $y = \frac{-4}{e^3} - 17$

12 C

13 $g(x) = -e^{-x} + e^{1-2x} + 1$

14 (a) 0 m/s (b) $x(t) = 3\left(t + 2e^{-0.5t}\right) + c$ (c) 6.81 m

15 $e^{x^2} + c$

16 (a) $\frac{dy}{dx} = ae^{2x}$

$y = \int ae^{2x}\,dx$

$= \frac{ae^{2x}}{2} + c$

The curve passes through (0,1).

$1 = \frac{ae^{2(0)}}{2} + c$

$1 = \frac{a}{2} + c \quad [1]$

The curve passes through (2,4).

$4 = \frac{ae^{2(2)}}{2} + c$

$4 = \frac{a^4}{2} + c \quad [2]$

[2] − [1] gives

$4 - 1 = \frac{ae^4 - a}{2}$

$3 = \frac{a\left(e^4 - 1\right)}{2}$

$a = \frac{6}{e^4 - 1}$

(b) $y = \frac{1}{e^4 - 1}\left(3e^{2x} + e^4 - 4\right)$

(c) $y = \frac{1}{e^4 - 1}\left(3e^{2x} + e^4 - 4\right)$

$x = 2$

$y = \frac{1}{e^4 - 1}\left(3e^{2(2)} + e^4 - 4\right)$

$= \frac{1}{e^4 - 1}\left(4e^4 - 4\right)$

$= \frac{4\left(e^4 - 1\right)}{e^4 - 1}$

$= 4$

The point $(2, 4)$ is on the curve.

17 (a) $xe^x + e^x$ (b) $xe^x - e^x + c$ (c) $y = xe^x - e^x + e$

18 (a) 2 (b) $y = \frac{1}{4}e^{2x} + \frac{3}{4}e^4$

19 (a) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + c$ (b) 1

EXERCISE 4.5

1 (a) $\frac{1}{3}\sin(3x) + c$ (b) $-\frac{1}{4}\cos(4x) + c$

(c) $-2\cos(3x) + c$ (d) $-\frac{1}{2}\sin(4x) + c$

(e) $-8\cos\left(\frac{x}{2}\right) + c$ (f) $9\sin\left(\frac{x}{3}\right) + c$

(g) $\frac{1}{3}\cos(2x) + c$ (h) $\frac{1}{4}\sin(2x) + c$

2 (a) $\frac{3\sin(5x)}{5} + c$ (b) $-6\cos\left(\frac{x}{3}\right) + c$ (c) $\frac{3}{4}\cos\left(\frac{2x}{3}\right) + c$

(d) $-2\sin\left(\frac{3x}{2}\right) + c$ (e) $\frac{4}{3}\sin(3\theta) + c$ (f) $5\cos\left(\frac{2\theta}{5}\right) + c$

3 (a) $-\frac{1}{2}\cos(2x) + c$ (b) $-\frac{1}{2}\cos(2x + 1) + c$

(c) $\frac{1}{2}\cos(1 - 2x) + c$ (d) $\frac{1}{2}\sin(2x) + c$

(e) $\frac{1}{2}\sin(2x + 1) + c$ (f) $-\frac{1}{2}\sin(1 - 2x) + c$

4 (a) $\frac{2}{3}\sin(3x + 1) + c$ (b) $-\frac{3}{2}\cos(2x - 3) + c$

(c) $\frac{9}{2}\cos\left(\frac{2x}{3} - 4\right) + c$ (d) $-3\sin\left(\frac{2x}{3} + 1\right) + c$

(e) $-\frac{1}{2}\cos(2x) - \frac{3}{2}\sin\left(\frac{2x}{3}\right) + c$

(f) $\frac{4}{3}\sin\left(\frac{3x}{2}\right) + \frac{3}{4}\cos(4x) + c$

(g) $x^2 + 4\cos\left(\frac{3x}{4}\right) + \frac{1}{3}e^{3x} + c$

(h) $x^3 - 2\sin\left(\frac{\sqrt{2}x}{2}\right) + c$

5 (a) $f(x) = \frac{2}{3}\sin\left(3x - \frac{\pi}{2}\right) - 1$

(b) $f(x) = \frac{1}{\sqrt{3}}\cos\left(2x - \frac{\pi}{6}\right) - \frac{5}{2}$

6 $f(x) = \frac{2}{5}\sin\left(5x - \frac{\pi}{6}\right) + \frac{10 - \sqrt{3}}{5}$

7 $y = 2x^3 - \frac{1}{2}\sin(2x) - \frac{\pi^3}{4}$ 8 $y = -12\cos\left(\frac{x}{6}\right) + 11$

9 $f(x) = \cos(2x) + 2$ 10 $f(x) = 4\sin(3x) + 2$

11 $-5\cos(x) + c$ 12 $-\frac{1}{8}\sin(8x) + c$

13 D 14 A

15 $\frac{1}{2}e^x\left(\sin(x) + \cos(x)\right) + c$

16 (a) $\frac{1}{\cos^2(x)}$ (b) $\tan^2(x) = \frac{\sin^2(x)}{\cos^2(x)}$

(c) $\sin^2(x) + \cos^2(x) = 1$

$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$

$\left(\frac{\sin(x)}{\cos(x)}\right)^2 + 1 = \frac{1}{\cos^2(x)}$

$\tan^2(x) + 1 = \frac{1}{\cos^2(x)}$

$\tan^2(x) = \frac{1}{\cos^2(x)} - 1$

(d) $\tan(x) - x + c$

17 (a) $x\cos(x) + \sin(x)$ (b) $x\sin(x) + \cos(x) + c$

(c) $f(x) = x\sin(x) + \cos(x) + 1 - \frac{\pi}{2}$

18 (a) $v = 2e^{\frac{1}{2}t} - \frac{1}{2}\sin(2t) - 2$

(b) $x = 4e^{\frac{1}{2}t} + \frac{1}{4}\cos(2t) - 2t - \frac{17}{4}$

EXERCISE 4.6

1 (a) $23\log_e|x| + c$ (b) $-2\log_e|x| + c$ (c) $10\log_e(t) + c$

(d) $\frac{x^4}{4} + 7\log_e|x| + c$ (e) $-\frac{1}{2}\cos(2\theta) + \log_e(\theta) + c$

2 (a) $\frac{1}{3}\log_e|x| + c$ (b) $-\frac{1}{5}\log_e|x| + c$

(c) $\frac{3}{7}\log_e|x| + c$ (d) $-\frac{2}{3}\log_e(t) + c$

(e) $\frac{1}{6}\log_e|x| + \frac{1}{6}e^{6x} + 2x + c$ (f) $\frac{1}{3}\log_e|x| + \frac{4}{5}\log_e|x| + c$

3 (a) $\frac{1}{3}\log_e|3x - 2| + c$ (b) $-\frac{1}{11}\log_e|3 - 11x| + c$

(c) $2\log_e|2x - 1| + c$ (d) $-\frac{1}{5}\log_e|7 - 10x| + c$

4 (a) $\frac{2}{9}\log_e(x) + c$ (b) $\frac{3x^2}{2} + 2\log_e(x) + c$

(c) $2x^2 + 2\log_e(x) + c$ (d) $4x - 12\log_e(x) - \frac{9}{x} + c$

5 (a) $y = 2\log_e(x) + 7$ (b) $y = \frac{1}{4}\log_e(x)$

(c) $y = \log_e|x| - x^2 + 1 + \frac{1}{e^2}$ (d) $y = \log_e|x| + 3x^{-1} + \frac{3}{e}$

6 $\int\left(\frac{1}{4+x}+\frac{1}{3-x}\right)dx = \int\left(\frac{1}{4+x}\right)dx + \int\left(\frac{1}{3-x}\right)dx$
$= \log_e|4+x| - \log_e|3-x| + c$
$= \log_e\left|\frac{4+x}{3-x}\right| + c$

7 $\int\left(\frac{2}{x-1}+\frac{1}{2x+3}\right)dx = 2\log_e(x-1) + \frac{1}{2}\log_e(2x+3) + c$
$= \log_e\left((x-1)^2\right) + \log_e\left(\sqrt{2x+3}\right) + c$
$= \log_e\left((x-1)^2\sqrt{2x+3}\right) + c$

8 (a) false (b) true (c) true

9 $f(x) = \frac{3x^2}{2} + 3x - \frac{1}{2}\log_e|x| - 4$

10 $x^2 - 6\log_e(x) - \frac{9}{4}x^{-2} + c$

11 8.1 **12** C **13** C

14 (a) $\frac{3}{(x-1)(x+2)}$ (b) $\frac{1}{3}\log_e\left(\frac{x-1}{x+2}\right) + c$

15 $\frac{dy}{dx} = \frac{6}{x^2-9}$; $\frac{1}{6}\log_e\left(\frac{x-3}{x+3}\right) + c$

16 (a) $\frac{4}{(x+1)} - \frac{4}{(x+1)^2} = \frac{4(x+1)-4}{(x+1)^2}$
$= \frac{4x+4-4}{(x+1)^2}$
$= \frac{4x}{(x+1)^2}$

(b) $4\log_e|x+1| + \frac{4}{(x+1)} + c$

17 $x - 2\log_e|x+1| + c$

18 (a) 0.332

(b) $\int\frac{1}{x+1}dx = \log_e(x+1)$

$\int\left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \frac{1}{x^5} - \ldots\right)dx = \log_e(x) + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \ldots$

(c) $\log_e(2+1) = 1.099$

$\log_e(2) + \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \ldots \approx 1.094$

They are approximately equal, but using more terms of the series would be a more accurate approximation.

EXERCISE 4.7

1 (a) $f(x) = \frac{4x^3}{3} - 6x - 10$ (b) $f(x) = 3x^2 - 2\sqrt{x} - 14$

(c) $f(x) = 5x^2 - \frac{1}{x} - 4$ (d) $f(x) = x^2 + \frac{1}{x^2} + 18.59$

2 13.3 cm^3 **3** B **4** $y = 2e^{\frac{x}{2}} - 2\sqrt{x} + 1$

5 $f(x) = \sin(2x) + 6\cos\left(\frac{x}{2}\right) + \left(2 - 3\sqrt{2}\right)$

6 (a) $y = \frac{2\sqrt{x-1}}{3}(x+2) + c$

(b) (i) $y = \frac{2\sqrt{x-1}}{3}(x+2) + 4$ (ii) $y = \frac{2\sqrt{x-1}}{3}(x+2) - 9$

7 $\frac{\log_e(x)}{x} + c$

8 (a) $v(t) = -9.8t + 12$ (b) $x(t) = -4.9t^2 + 12t$

(c) After 1.2 seconds, the ball reaches its maximum height of 7.3 metres above the release point.

(d) After 1 second, the ball is 7.1 metres above the release point and its velocity is 2.2 m/s.

9 (a) $y = \frac{2}{1-3x} - 4$ (b) $y = -\frac{17}{4}$ (c) $x = \frac{5}{18}$

10 (a) $-x\sin(x) + \cos(x)$

(b) $\int x\sin(x)\,dx = -x\cos(x) + \sin(x) + c$

(c) $f(x) = \sin(x) - x\cos(x) + 2$

11 $\frac{1}{3}\left(x^3\log_e(2x) - \frac{x^3}{3}\right) + c$

12 $\frac{1}{2}\left(2\sin\left(\frac{\theta}{2}\right) + \theta\right) + c$

13 (a) $-\frac{1}{\sin^2(x)}$ (b) $1 + \frac{1}{\tan^2(x)} = 1 + \frac{\cos^2(x)}{\sin^2(x)}$
$= \frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)}$
$= \frac{1}{\sin^2(x)}$

(c) $-\frac{1}{\tan(x)} - x + c$

14 (a) $1 - 0.2x + \frac{0.04x^2}{2!} - \frac{0.008x^3}{3!}$

(b) $\frac{x^2}{2} - \frac{x^3}{15} + \frac{x^4}{200} - \frac{x^5}{3750} + c$

15 (a) $\frac{\left(x^3-5\right)^2}{2} + c$

(b) $\int\left(3x^5 - 15x^2\right)dx = \frac{3x^6}{6} - \frac{15x^3}{3} + c_1$
$= \frac{x^6}{2} - 5x^3 + c_1$

Also $\frac{\left(x^3-5\right)^2}{2} + c = \frac{x^6}{2} - \frac{10x^3}{2} + \frac{25}{2} + c$
$= \frac{x^6}{2} - 5x^3 + \frac{25}{2} + c$

The answers are equal for $c_1 = \frac{25}{2} + c$.

16 (a) $\frac{dr}{dt} = \frac{r}{5} + 15$
$= \frac{r+75}{5}$

$\frac{dt}{dr} = \frac{1}{\left(\frac{r+75}{5}\right)}$
$= \frac{5}{r+75}$

$$t = \int \frac{dt}{dr}\,dr$$
$$= \int \frac{5}{r+75}\,dr$$
$$= 5\int (r+75)^{-1}\,dr$$
$$= 5\log_e(75+r) + c$$

Given that $t = 0$ when $r = 0$:

$$0 = 5\log_e(75+0) + c$$
$$c = -5\log_e(75)$$

$$t = 5\log_e(75+r) - 5\log_e(75)$$
$$= 5\left[\log_e(75+r) - \log_e(75)\right]$$
$$= 5\log_e\left(\frac{1}{75}(75+r)\right)$$

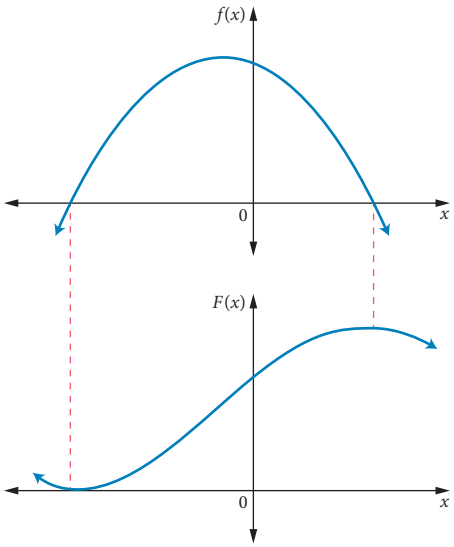
(b) $t = 19.4$ seconds, $V_{\text{sphere}} = 523.6\,\text{cm}^3$

(c) It is not safe to inflate the balloon for another 2 minutes because the safe radius is reached after another 53 seconds. After another 2 minutes, the radius would theoretically be 44.3 cm but in reality, the balloon would have burst already.

CHAPTER REVIEW 4

1 (a) $-\frac{1}{15}(3-5x)^3 + c$ (b) $-\frac{1}{3}(2x-7)^{-\frac{3}{2}} + c$

2



(or any vertical translation of $F(x)$ shown)

3 (a) $4x^{\frac{7}{4}} - 4x^{\frac{3}{2}} + c$ (b) $\frac{1}{3}x^3 - 4x - \frac{4}{x} + c$

(c) $x^3 - x^2 + x + c$

4 $y = \frac{x^{3n}}{3n} - \frac{3x^{n+1}}{n+1} + c$

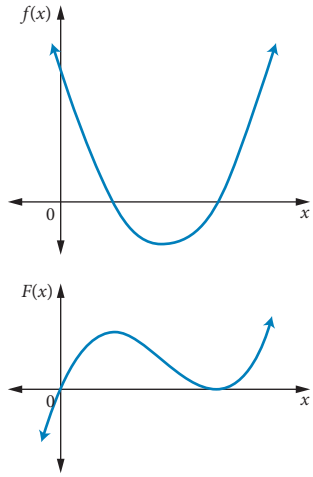
5 (a) $-\frac{1}{x} - \frac{1}{2x^2} + c$ (b) $\frac{1}{3}y^3 + \frac{1}{2}y^2 - 6y + c$

6 (a) $\frac{(3x-4)^4}{12} + c$ (b) $\frac{1}{(2x+1)^2} + c$

(c) $\frac{2}{15}(3t-2)^{\frac{5}{2}} + c$ (d) $-2\sqrt{2-x} + c$

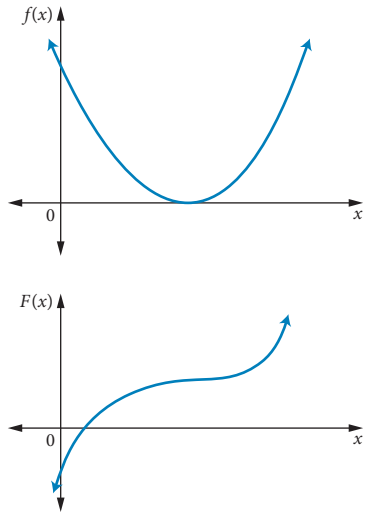
7 $f(x) = \frac{(2x-1)^5 + 19}{10}$

8 (a)



(or any vertical translation of $F(x)$ shown)

(b)



(or any vertical translation of $F(x)$ shown)

9 (a) $\frac{1}{2}e^{2x-3} + c$ (b) $6e^{\frac{x}{2}} + c$

(c) $\frac{1}{2}e^{2x} - 4e^x + 4x + c$ (d) $\frac{3}{2}e^{2x} - \frac{1}{7}e^{7x} + c$

10 $\frac{9}{2}\sin\left(\frac{2\theta}{3}\right) + c$

11 (a) $-\frac{1}{2}\sin(4x) + c$ (b) $\frac{1}{2}\cos(1-2x) + c$

(c) $2x^{\frac{3}{2}} + \frac{2}{3}\cos(3x+2) + c$ (d) $-\frac{4}{3}\cos\left(\frac{3x+1}{2}\right) + c$

12 $g(x) = 6\sin\left(\frac{x}{2}\right) + 3$

13 $y(x) = \frac{1}{2}e^{2x} - e^{-x} + e^2x + \frac{1}{2}$

14 (a) $\frac{1}{12}\log_e|x|+c$ (b) $\frac{1}{2}\log_e|2x-11|+c$

(c) $12\log_e\left(\frac{1}{3}x-11\right)+c$

(d) $\frac{5}{9}\log_e|9x+1|-\frac{3}{2}\log_e|4-2x|+c$

15 C **16** D

17 $c=e^{-4}-\frac{1}{2}e^4$

18 $\frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{1}{2}}+c$

19 $y=\frac{1}{4}(x-2)^4+c_1$

$y=\frac{x^4}{4}-2x^3+6x^2-8x+c_2$

They are equal when $c_1=4+c_2$.

20 $\frac{e^{3x-2}}{3}+2e^{-x}-\frac{1}{3}\left(5+e^{-2}\right)$

21 (a) $(2x+3)^2(8x+3)$ (b) $x(2x+3)^3+c$

22 $\frac{1}{a^2}\sin(ax)-\frac{x}{a}\cos(ax)+c$

23 Let $y=x\sqrt{x+1}$

$$\frac{dy}{dx}=\sqrt{x+1}\times 1+x\times\frac{1}{2\sqrt{x+1}}$$
$$=\sqrt{x+1}+\frac{x}{2\sqrt{x+1}}$$

$$\int\left(\frac{x}{2\sqrt{x+1}}+\sqrt{x+1}\right)dx=x\sqrt{x+1}$$
$$\int\frac{x}{2\sqrt{x+1}}dx+\int\sqrt{x+1}\,dx=x\sqrt{x+1}$$
$$\int\frac{x}{2\sqrt{x+1}}dx=x\sqrt{x+1}-\int\sqrt{x+1}\,dx$$
$$=x\sqrt{x+1}-\int(x+1)^{\frac{1}{2}}dx$$
$$=x\sqrt{x+1}-\frac{2}{3}(x+1)^{\frac{3}{2}}+c$$
$$=x\sqrt{x+1}-\frac{2}{3}(x+1)(x+1)^{\frac{1}{2}}+c$$
$$=\sqrt{x+1}\left(x-\frac{2}{3}(x+1)\right)+c$$
$$\int\frac{x}{2\sqrt{x+1}}dx=\sqrt{x+1}\left(\frac{x-2}{3}\right)+c$$
$$\int\frac{x}{\sqrt{x+1}}dx=\frac{2}{3}(x-2)\sqrt{x+1}+c$$

24 (a) $(2x-3)^3(10x-3)$

(b) $\frac{x}{2}(2x-3)^4-\frac{1}{20}(2x-3)^5+c$

(c) $c=2\frac{1}{20}$

25 (a) 216 (b) $y=216x-164$

(c) $b=2$ (d) $y=\frac{1}{8}(2x+3)^4-2$

26 (a) $a(n+1)(ax+b)^n$

(b) $ax(n+1)(ax+b)^n+(ax+b)^{n+1}$

(c) $\frac{1}{a(n+1)}\left(x(ax+b)^{n+1}-\frac{(ax+b)^{n+2}}{a(n+2)}\right)+c$

27 (a) 1645 cm^2/month

(b) $A=100t^2+500\pi\sin\left(\frac{\pi t}{3}\right)+900$

(c) 6 months

(d) after 11 months

28 (a) $\frac{dy}{dt}=\frac{1}{20}(3-t)e^{-t^2+6t-4}$ (b) $\frac{1}{2}e^{-t^2+6t-4}+c$

(c) (i) 2.73 mg/L per hour

(ii) The concentration of the drug in the bloodstream is decreasing at a rate of 2.73 mg/L per hour.

(d) at $t=3$ hours

(e) $C(t)=\frac{1}{40}\left(e^{-t^2+6t-4}-e^{-4}\right)$

(f) 0.067 mg/L

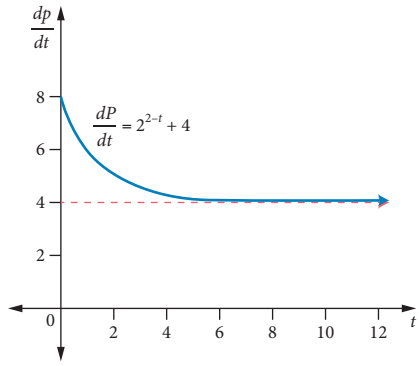
29 (a) $C(q)=-\frac{1}{e^q}+\frac{q^2}{2}+20q+25$

(b) $C(10)=\$275$ (c) \$251

30 (a) $v(t)=-t^2+5t+60$ (b) $x(t)=-\frac{1}{3}t^3+\frac{5}{2}t^2+60t$

(c) 10.6 hours

31 (a)



(b) 4000 mosquitoes/week

(c) $P(t)=400+\frac{4}{\log_e(2)}+4t-\frac{4}{2^t\log_e(2)}$

(d) 430 000 (e) 8 weeks

Chapter 5: Definite integrals

RECALL

1 (a) $\frac{5x^3}{3}-\frac{3x^2}{2}+2x+c$ (b) $2x^4-x^3+\frac{x^2}{2}-3x+c$

(c) $\frac{x^4}{2}+\frac{5x^3}{3}-\frac{1}{x}-x+c$

2 (a) $\frac{x^4}{2}-4x^{\frac{3}{2}}+x+c$ (b) $2x^4+\frac{1}{x}+6\sqrt{x}+c$

(c) $\frac{5x^3}{3}+2x\sqrt{x}+c$ (d) $\frac{3x^{\frac{4}{3}}}{4}+\frac{9}{x^{\frac{1}{3}}}+c$

3 (a) $\frac{3x^2}{2}-4x+c$ (b) $\frac{x^3}{3}-4x^2+c$

(c) $y=x^4-\frac{4x\sqrt{x}}{3}+c$

4 $\int(4x-3)^2dx=\frac{1}{12}(4x-3)^3+c$

5 (a) iv: $\frac{1}{8}(2x-1)^4$ (b) ii: $\frac{1}{2}(2x-1)^4$ (c) v: $\frac{1}{16}(2x-1)^4$

6 (a) $\frac{(7x+3)^5}{35}+c$ (b) $-\frac{1}{12(3x-1)^4}+c$

(c) $-\frac{1}{4(6x-5)^2}+c$ (d) $-6(3-2x)^{\frac{1}{2}}+c$

7 (a) $\frac{3}{2}e^{2x}+c$ (b) $\frac{3}{2}e^{2x+1}+c$

(c) $-4e^{1-2x}+c$ (d) $-\frac{4}{e^{2x+3}}+c$

8 (a) $-\frac{1}{4}\cos(4x)+c$ (b) $-3\cos\left(\frac{x}{3}\right)+c$

(c) $-10\cos\left(\frac{2x}{5}\right)+c$ (d) $\frac{1}{2}\sin(2x)+c$

(e) $\frac{8}{3}\sin\left(\frac{3x}{2}\right)+c$ (f) $\frac{1}{3}\sin(3x-1)+c$

(g) $\frac{1}{5}\cos(2-5x)+c$

9 (a) $3\log_e(x)+c$ (b) $\frac{2}{7}\log_e(x)+c,\ x>0$

(c) $\frac{1}{2}\log_e(2x-1)+c,\ x>\frac{1}{2}$

(d) $\frac{4}{3}\log_e(3x+2)+c,\ x>-\frac{2}{3}$

10 (a) $f(x)=\frac{1}{2}x^4-x^2+2$ (b) $f(x)=2x^3-x^2+3x+6$

(c) $y=\frac{1}{2}\sin(2x)-\frac{3}{2}$ (d) $f(x)=\frac{1}{3}e^{6x}+x+1$

(e) $y=x^3-6x\sqrt{x}-4$

EXERCISE 5.1

1 (a) 20 square units (b) 6.25 square units

2 (a) 5.968 square units (b) 11.035 square units

3 0.642 square units

4 0.959 square units

5 (a) 0.25 square units (b) 0.93 square units

6 $\frac{(\sqrt{2}+1)\pi}{4}$ square units

7 D

8 (a) B (b) 2.9 square units

9 81.5 square units

10 (a) $5\frac{1}{4}$ square units

(b) (i) 4.5 square units (ii) 4.88 square units

(iii) 5.18 square units (2 d.p.)

(c) (i) 6 square units

(ii) 5.63 square units (2 d.p.)

(iii) 5.33 square units (2 d.p.)

(d) For the given increasing function, the total area of the lower rectangles underestimates the actual area, and the total area of the upper rectangles overestimates the actual area. As the number of intervals increases, both the left-endpoint and right-endpoint approximations achieve results closer to the actual area.

11 0.655 square units

12 The left-endpoint approximation overestimates the area in $\left[0,\frac{\pi}{2}\right]$ and underestimates the area in $\left[\frac{\pi}{2},\pi\right]$.

13 (a) $11.30\,m^2$ (b) $11.30\,m^2$

(c) $11.30\,m^2$

(d) Hannah could use any of the three approximations.

14 (a) $68\,m^3$ (b) $66\,m^3$ (c) $67\,m^3$ (d) $67\,m^3$

EXERCISE 5.2

1 (a) 6 (b) $1\frac{1}{2}$ (c) $\frac{1}{6}$

(d) $\frac{1}{5}$ (e) 78 (f) 76

2 (a) $-\frac{2}{3}$ (b) 6

(c) $\frac{-\sqrt{2}-2}{3}$ (d) $\frac{\sqrt{2}}{4}$

3 (a) $\frac{3e^4}{2}-\frac{3e^{-2}}{2}$ (b) $\frac{e^{2\pi}-5}{2}$ (c) $\frac{e^4}{2}+\frac{19}{2}-\frac{2}{e^4}$

4 (a) $2\log_e(3)$ (b) $-2\log_e(3)$

(c) $\frac{1}{2}\log_e\left(\frac{9}{7}\right)$ (d) $\frac{1}{2}\log_e(5)$

5 (a) 1 (b) $-\frac{4+\sqrt{2}}{2}$

(c) $\frac{3}{128}$ (d) $\frac{1}{2}+\frac{1}{e}-\frac{1}{e^2}$

6 (a) -6.68 (b) -2.21 (c) 0.68 (d) -3.63

7 (a) 1.63 (b) 2.57 (c) 19.33 (d) 1.41

8 (a) 15 (b) -33

9 (a) -48 (b) -16 (c) -24 (d) 4

10 $\frac{7}{24}$ **11** C **12** B

13 (a)
$$\frac{2}{x-1}+\frac{1}{x+2}=\frac{2(x+2)}{(x-1)(x+2)}+\frac{1(x-1)}{(x-1)(x+2)}$$
$$=\frac{2(x+2)+(x-1)}{(x-1)(x+2)}$$
$$=\frac{2x+4+x-1}{(x-1)(x+2)}$$
$$=\frac{3x+3}{(x-1)(x+2)}$$
$$\therefore \frac{3x+3}{(x-1)(x+2)}=\frac{2}{x-1}+\frac{1}{x+2}$$

(b) $\log_e\left(\frac{5}{4}\right)$

14 (a)
$$\int_{-1}^{4} e^{2x+1}\,dx=\left[\frac{1}{2}e^{2x+1}\right]_{-1}^{4}$$
$$=\frac{1}{2}\left(e^9-e^{-1}\right)$$
$$-\int_{4}^{-1} e^{2x+1}\,dx=-\left[\frac{1}{2}e^{2x+1}\right]_{4}^{-1}$$
$$=-\frac{1}{2}\left(e^{-1}-e^9\right)$$
$$=\frac{1}{2}\left(e^9-e^{-1}\right)$$
$$=\int_{-1}^{4} e^{2x+1}\,dx$$

Therefore $\int_{-1}^{4} e^{2x+1}\,dx=-\int_{4}^{-1} e^{2x+1}\,dx$ as required.

(b)
$$\int_b^a f(x)\,dx=F(a)-F(b)$$
$$-\int_a^b f(x)\,dx=-\left(F(b)-F(a)\right)$$
$$=F(a)-F(b)$$
$$=\int_b^a f(x)\,dx$$
$$\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$$

15 $\frac{9}{10}$

16 $0, \pm\sqrt{3}$

17 (a) 2.11 (b) -2.63 (c) 3.00 (d) 0.19, 2.45

18 $2m^2$

19 (a) $-1, -\frac{1}{2}$ (b) $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

20 (a) $x\sqrt{x^2-1}$ (b) $5\sqrt{3}$

21 $-3, \pm\sqrt{6}$

22 (a) $-\frac{3}{2}\cos(2x-\pi)+c$ (b) $-\frac{3}{2}$

(c) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (d) $-\frac{5}{2}$

EXERCISE 5.3

1 (a) $-2\frac{1}{4}$ square units (b) $-2\frac{1}{4}$ square units

(c) 6 square units (d) $4\frac{1}{2}-\frac{e^2}{2}$ square units

(e) $\frac{2}{3}$ square units (f) 0 square units

2 (a) $\frac{1}{2}\log_e(7)$ square units (b) $\frac{1}{3}\left(64-12^{\frac{3}{2}}\right)$ square units

(c) $\frac{19+2e^{-3}}{3}$ square units (d) $\frac{1}{10}$ square units

3 (a) $5\frac{3}{4}$ square units (b) $3\frac{1}{12}$ square units

(c) $12\frac{1}{3}$ square units (d) 5 square units

(e) 2 square units (f) 3 square units

4 (a) 6 square units (b) 11.574 square units

5 (a) -6.750 square units (b) 8.909 square units

6 6 square units **7** ± 1.50

8 100.8 square units **9** $\log_e(5)$ square units

10 2.62 **11** B **12** D

13 0.685, for the areas above and below the x-axis to be equal; 2, the trivial answer giving an area of zero

14 (a) 4 square units (b) 12 square units

15 (a) $f(m)=m^3$

Therefore, the rectangle enclosing the whole region will have dimensions $m\times m^3$ and an area of m^4.

The required area = Area of rectangle $OABC$ − Area under the curve in the rectangle

$$A=m^4-\int_0^m x^3dx$$
$$=m^4-\left[\frac{x^4}{4}\right]_0^m$$
$$=m^4-\left(\frac{m^4}{4}-0\right)$$
$$=\frac{3m^4}{4}$$

(b) $p=3, q=1$

16 (a) $\frac{\pi}{2}$ (b) $\frac{1}{2}$ square unit

(c) 12 square units (d) 12 square units

17 (a) 27 square units (b) $2\times\left(\frac{1}{2}\times 3\times 9\right)=27$

18 (a) $\log_e(x)+1$ (b) $3\log_e(3)-2$

19 (a) 1 (b) 7 square units

(c) $\int_0^4\left(-f(x)\right)dx=-4$ (d) $\int_0^4 f(-x)\,dx=2$

(e) -8

20 (a) $\overline{AB}: y=-x+2$; $\overline{AC}: y=-3x-2$; $\overline{BC}: y=1$

(b) 3 square units (c) 3 square units

(d) The area of the triangle does not change, regardless of the method used.

21 (a) $\frac{5\sqrt{10}}{3}$ square units (b) $\frac{5\sqrt{10}}{3}$ square units

(c) The combined area between the functions and the x-axis equals the original shaded area because of the reduction factor of $\frac{1}{2}$ in the y-axis and the reflections in the x-axis.

EXERCISE 5.4

1 (a) $10\frac{2}{3}$ square units (b) $10\frac{2}{3}$ square units

(c) $1\frac{1}{3}$ square units (d) $\frac{1}{6}$ square units

2 (a) $\frac{16}{3}$ square units

(b) $31\frac{1}{2}+10\log_e\left(\frac{2}{5}\right)$ square units

3 (a) $1\frac{5}{6}$ square units (b) 10 square units

(c) 18 square units (d) 78 square units

(e) $88\frac{1}{2}$ square units (f) $18\frac{2}{3}$ square units

(g) $\left(2\sqrt{2}+2\right)$ square units

(h) $\left(1+\sqrt{2}\right)$ square units

4 $\frac{4}{3}$

5 (a) $x=-3$, $x=1$, $x=3$ (b) $49\frac{1}{3}$ square units

6 $4\sqrt{2}$ square units 7 44.19 square units

8 10.42 square units 9 0.297 square units

10 2.02 square units 11 47.46 square units

12 8.14 square units 13 C

14 B 15 $\frac{3\sqrt{3}}{2}$ square units

16 (a) Given that

$f(0)=0.5(25-0)$

$=12.5$

$g(x)$ must have a y-intercept at $(0,11.5)$.

$11.5=a(16-0)$

$16a=11.5$

$a=\frac{11.5}{16}$

$a=\frac{23}{32}$

(b) 22 m^2 (c) 1.06 m^2

17 The curves intersect where:

$x^2=\sqrt{x}$

$x^4=x$

$x\left(x^3-1\right)=0$

$x=0,1$

The enclosed area is:

$$\int_0^1\left(\sqrt{x}-x^2\right)dx=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{x^3}{3}\right]_0^1$$

$$=\frac{2}{3}-\frac{1}{3}$$

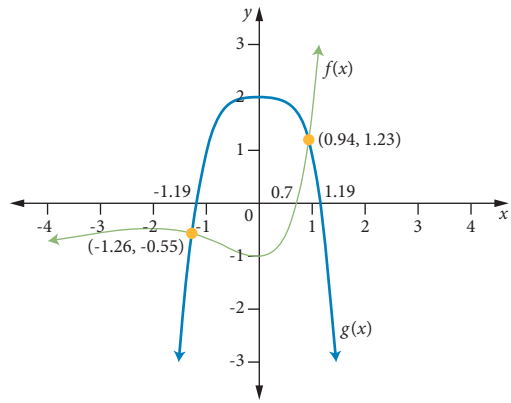
$$=\frac{1}{3}$$

The enclosed area is one-third of the area of the unit square as required.

18 (a) $y=4x-1$ (b) 2.54 square units

19 $\frac{e^2+8e-16}{4e}$ square units

20 (a)



(b) 2.49 m^3 (c) 19.96 m^2

EXERCISE 5.5

1 (a) 42 (b) 2

(c) 0 (d) $\frac{1}{4}\left(e^6-e^2\right)$

2 23.2°C

3 (a) 0.5 mg (b) 0.38 mg

4 39.3 m/s

5 (a) 1 m/s in the negative direction

(b) 42 m to the right (c) 78 m to the right

6 (a) 885 litres (b) 689 litres

7 1 m

8 80 L

9 (a) $2\frac{2}{3}$ m (b) 8 m

10 (a) 1.38 m to the left (b) 1.48 m

11 44.1 m

12 1.6 m

13 $t=0.80$ s or $t=11.20$ s

14 C

15 8 m

16 (a) $x=\int_{\frac{\pi}{2}}^{2\pi}\left(2\cos\left(\frac{t}{2}\right)+1\right)dt$

$$=\left[4\sin\left(\frac{t}{2}\right)+t\right]_{\frac{\pi}{2}}^{2\pi}$$

$$=\left(4\sin(\pi)+2\pi\right)-\left(4\sin\left(\frac{\pi}{4}\right)+\frac{\pi}{2}\right)$$

$$=2\pi-4\left(\frac{\sqrt{2}}{2}\right)-\frac{\pi}{2}$$

$$=\frac{3\pi}{2}-2\sqrt{2}$$

(b) 4.62 m

17 **(a)** $v = \left(10t - 2t^2 - 8\right)$ m/s **(b)** $v = 4.5$ m/s

(c) $2\frac{1}{6}$ m to the left of the origin **(d)** $t = 1$ s, $t = 4$ s

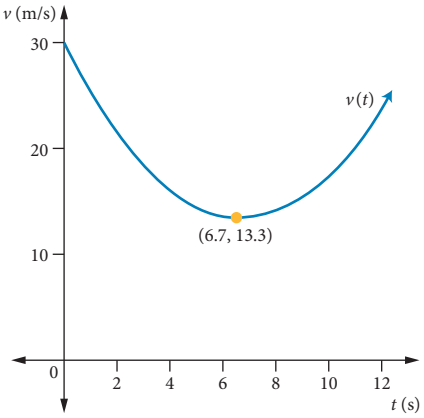
(e) $6\frac{2}{3}$ m left of the origin, $2\frac{1}{3}$ m right of the origin

(f) $12\frac{2}{3}$ m

18 **(a)** 13.375 m/s **(b)** 130.375 m

(c) 6.33 seconds

(d)



19 **(a)** 49.5 m

(b) 121.5 m to the right of the starting point

(c) -24 m/s^2 Since this is when velocity is 0 and position is to the right, the particle has slowed down to a stop and is about to accelerate in the negative direction.

20 **(a)** $v(t) = -10e^{0.5t} + 30$

(b) $x(t) = -20e^{0.5t} + 30t + 140$ **(c)** 2.2 s, 25.9 m

21 **(a)** 9 m/s, 3.9 m **(b)** 16.2 s, 85.6 m

(c) $v = 9.05$ m/s This point is a global maximum, because the cyclist accelerates to $t = 0.45$ s and then decelerates until she stops.

22 **(a)** $v(t) = -9.8t + 10$ **(b)** $x(t) = -4.9t^2 + 10t$

(c) 1.02 s, 5.10 m

(d) 9.6 m/s, 0.4 m above the ground

23 **(a)** 44.1 m at $t = 3$ s **(b)** 2.7 s, 26.2 m/s

(c) 14.7 m, 9.8 m/s in the downwards direction

24 **(a)** 20.4 m

(b) **(i)** $v = v_0 + \int_0^t a\,dt$

$= v_0 + \int_0^t -9.8\,dt$

$= u + \left[-9.8\right]_0^t$

$= u - 9.8t$

(ii) 9.6 m/s **(iii)** 49.4 m

(c) 10.2 m/s

(d) **(i)** 9.8 m/s **(iii)** 39.2 m

CHAPTER REVIEW 5

1 **(a)** $\frac{\pi\left(1+\sqrt{2}\right)}{4}$ square units **(b)** $\frac{\pi\left(2+\sqrt{2}\right)}{8}$ square units

(c) $\frac{\pi}{16}\left(4+3\sqrt{2}\right)$ square units

2 **(a)** $10\frac{2}{3}$ **(b)** $3-3\sqrt{3}$ **(c)** $-2\frac{1}{4}$

(d) -200 **(e)** $\frac{8-\pi^3}{24}$ **(f)** $\frac{1}{2}$

3 $1\frac{1}{4}$

4 **(a)** 0 square units **(b)** $\frac{8}{3}$ square units

5 $\frac{1}{2}\left(e^6 - 1\right)$ square units **6** $b = -4$, $p = -1$, $q = 5$

7 36 square units **8** $45\frac{3}{4}$ square units

9 **(a)** $R(t) = \int_5^9 \left(1.7 + 1.09^t\right) dt$ **(b)** 14.15 tonnes

10 **(a)** 3 m to the left **(b)** $7\frac{2}{3}$ m

11 25.4 m above the ground **12** A

13 C **14** 3.47 square units

15 $-14\frac{1}{3}$ **16** -16

17 $a = 3$ **18** $f(x) = 6\sin\left(\frac{x}{3}\right) - 2$

19 $\int_0^{\frac{9\pi}{4}} \sin\left(\frac{2x}{3}\right) dx$ **20** $\int_b^c f(x)\,dx - \int_a^b f(x)\,dx$

21 D **22** 4.29 square units

23 $\int_{-1}^{0} \left(e^{-2x} - e^{3x}\right) dx + \int_0^1 \left(e^{3x} - e^{-2x}\right) dx$

24 45.02 g **25** $x = 4$

26 3.0 seconds

27 **(a)** 0.605 square units **(b)** 0.777 square units

(c) 0.691 square units

(d) **(i)** 0.702 square units **(ii)** 0.691 square units

(e) 0.687 square units

28 **(a)** $e^c - c - 1$

(b)

Area bounded by y-axis = Area$_{\text{rect } OABC}$ − Area bounded by x-axis

$A = c \times \left(e^c - 1\right) - \left(e^c - c - 1\right)$

$= ce^c - c - e^c + c + 1$

$= ce^c - e^c + 1$

(c) Region bounded by the x-axis:

$\int_0^c x^3\,dx = \left[\frac{1}{4}x^4\right]_0^c$

$= \frac{c^4}{4}$

Area of rectangle $OABC$:

$A = c \times c^3$

$= c^4$

Region bounded by the y-axis:

$$A = c^4 - \frac{c^4}{4}$$
$$= \frac{3c^4}{4}$$

Ratio of areas: $\frac{3c^4}{4} : \frac{c^4}{4}$

The ratio of areas of the two regions is 3:1 as required.

(d) $A = \int_0^c x^n \, dx$

$$= \left[\frac{1}{n+1}x^{n+1}\right]_0^c$$
$$= \frac{c^{n+1}}{n+1}$$

(e) Area of rectangle $OABC$: $A = c \times c^n = c^{n+1}$

Area bounded by y-axis:

$$A = c^{n+1} - \frac{c^{n+1}}{n+1}$$
$$= \frac{1}{n+1}\left[(n+1)c^{n+1} - c^{n+1}\right]$$
$$= \frac{c^{n+1}}{n+1}[n+1-1]$$
$$= \frac{nc^{n+1}}{n+1}$$

The ratio of the region bounded by the y-axis to the region bounded by the x-axis:

$$\frac{nc^{n+1}}{n+1} : \frac{c^{n+1}}{n+1} = n : 1$$

29 (a) $a = 2\frac{2}{5}, b = 3, c = 4$

(b) 3.13 square units

(c) The area above the x-axis is 3.13 square units greater than the area below the x-axis.

(d) 3.32 square units

(e) 4.96

30 (a) $d = 3$, $f(x) = -x^3 + 7x^2 - 8x + 3$

(b) 32 square units

(c) 32.25 square units. The trapezoidal rule gives the closer approximation.

31 (a) 0.37 m to the left of the origin

(b) 0.9 m to the left of the origin

(c) 4 m/s^2 At this time, the particle is changing its direction of motion from negative to positive.

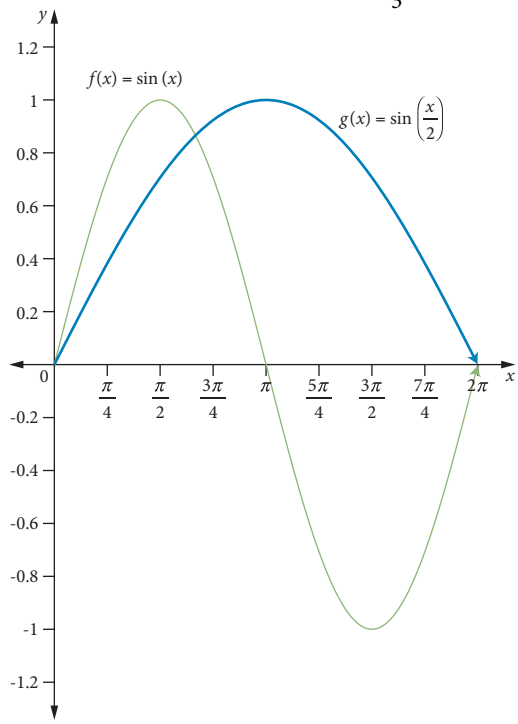
32 (a) $f(x) = \sin(x)$:

y-intercept at $(0,0)$, x-intercepts at $(0,0)$, $(\pi,0)$ and $(2\pi,0)$, maximum at $\left(\frac{\pi}{2},0\right)$, minimum at $\left(\frac{3\pi}{2},0\right)$

$g(x) = \sin\left(\frac{x}{2}\right)$:

y-intercept at $(0,0)$, maximum at $(\pi,1)$.

Points of intersection are at $x = 0$, $x = \frac{2\pi}{3}$ and $x = 2\pi$.



(b) $\text{Area}_{f(x)} = 2\int_0^{\pi} \sin(x)\,dx$

$$= 2\left[-\cos(x)\right]_0^{\pi}$$
$$= 2\left(-\cos(\pi) + \cos(0)\right)$$
$$= 2(1+1)$$
$$= 4 \text{ square units}$$

$$\text{Area}_{g(x)} = \int_0^{2\pi} \sin\left(\frac{x}{2}\right)dx$$
$$= 2\left[-\cos\left(\frac{x}{2}\right)\right]_0^{2\pi}$$
$$= 2\left(-\cos(\pi) + \cos\right.$$
$$= 2(1+1)$$
$$= 4 \text{ square units}$$

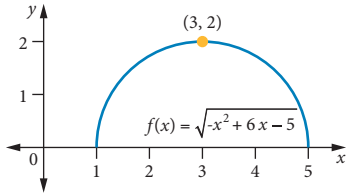
The two areas are equal.

The function $g(x)$ is a dilation of the function $f(x)$ of factor 2 parallel to the x-axis. The value of the area of $g(x)$ is increased by the same factor.

(c) $\text{Area} = \int_0^{\frac{2\pi}{3}} \left[\sin(x) - \sin\left(\frac{x}{2}\right)\right]dx + \int_{\frac{2\pi}{3}}^{2\pi} \left[\sin\left(\frac{x}{2}\right) - \sin(x)\right]dx$

$$= \left[-\cos(x) + 2\cos\left(\frac{x}{2}\right)\right]_0^{\frac{2\pi}{3}} + \left[-2\cos\left(\frac{x}{2}\right) + \cos(x)\right]_{\frac{2\pi}{3}}^{2\pi}$$
$$= \left(-\cos\left(\frac{2\pi}{3}\right) + 2\cos\left(\frac{\pi}{3}\right) + \cos(0) - 2\cos(0)\right)$$
$$+ \left(-2\cos(\pi) + \cos(2\pi) + 2\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{2\pi}{3}\right)\right)$$
$$= \left(\frac{1}{2} + 1 + 1 - 2\right) + \left(2 + 1 + 1 + \frac{1}{2}\right)$$
$$= \frac{1}{2} + \frac{9}{2}$$
$$= 5 \text{ square units}$$

33 **(a)** $m = 6$, $f(x) = \sqrt{-x^2 + 6x - 5}$



(b) $\frac{8\pi}{3} - 2\sqrt{3}$ square units

(c) $2\sqrt{3} - \frac{2\pi}{3}$ square units

(d) $\text{Total area}_{\text{shaded}} = \frac{8\pi}{3} - 2\sqrt{3} + 2\sqrt{3} - \frac{2\pi}{3}$

$= \frac{6\pi}{3}$

$= 2\pi$ square units

The total shaded area is the area of the semicircle. The area of a circle is: $A = \pi r^2$. For the semicircle given of radius 2 units, its area is:

$A = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi \times 2^2$

$= 2\pi$ square units

34 **(a)** 1.56 s, 15.34 m/s

(b) **(i)** 2 s, 19.8 m/s **(ii)** 22.2 m/s

35 **(a)** Option 1 $= \int_0^3 3e^{-\frac{x}{3}}\,dx + \int_{-3}^0 3e^{\frac{x}{3}}\,dx$

Option 2 $= \int_0^3 3e^{-\frac{x}{3}}\,dx - \int_0^{-3} 3e^{\frac{x}{3}}\,dx$

Option 3 $= \int_{-3}^0 3e^{\frac{x}{3}}\,dx - \int_3^0 3e^{\frac{x}{3}}\,dx$

or $2\int_0^3 3e^{-\frac{x}{3}}dx$ or $2\int_{-3}^0 3e^{\frac{x}{3}}dx$

(b) 11.38 m^2 **(c)** 2.02 m^2

(d) 6.18 m^2 **(e)** 0.64 m^2

36 **(a)** $a(t) = \frac{-2t}{\left(1+t^2\right)\sqrt{1+t^2}}$ m/s^2 **(b)** 2 m/s, 0 m/s^2

(c) When $t \to \infty$, $v \to 0$. **(d)** 4.62 m

Exam review: Unit 3

PAPER 1: TECHNOLOGY-FREE

1 **(a)** $x = 0$ or $x = \log_e(7)$

(b) $x = \log_e(3)$ or $x = \log_e(4)$

(c) $x = \log_e(5)$ **(d)** no solutions

2 **(a)** -0.34 **(b)** -0.94

3 **(a)** $\frac{dy}{dx} = 3x^2\cos\left(x^3 - 2\right)$ **(b)** $\frac{dy}{dx} = -4\cos^3(x)\sin(x)$

(c) $\frac{dy}{dx} = e^{2x+\sin(4x)}\left(2 + 4\cos(4x)\right)$

(d) $\frac{dy}{dx} = \frac{\cos(2x-3)}{\sqrt{\sin(2x-3)}}$

4 $b = 81(a+3)^2$

5 **(a)** $x = \sqrt{5}$ **(b)** $x = \frac{2}{3}$ **(c)** $x = 1$ or $x = e^3$

(d) $x = \log_3(7)$ **(e)** $x = \frac{2\log_4(3)}{\log_4(3) - 1}$

6 **(a)** $6x^4 - 2x^3 + c$ **(b)** $\frac{2}{3}\sin(3\theta) + \frac{3}{2}\cos(2\theta) + c$

(c) $-\frac{2}{3}(5 - 12t)^{\frac{1}{2}} + c$ **(d)** $\frac{7}{2}\log_e|2x+9| + c$

7 **(a)** $f(x) = \frac{3x^2}{2} - x - \frac{5}{x} + 5$

(b) $f(x) = -\frac{1}{2}e^{-2x} - 2e^{1-x} + e^2x + 1 + 2e$

8 **(a)** $\frac{1}{2}e^2 + 2 - \frac{1}{2e^2}$ **(b)** -2

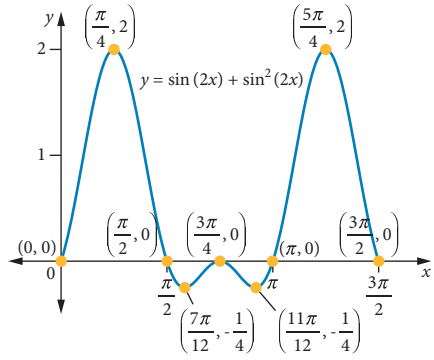
(c) $6\frac{7}{24}$ **(d)** $\frac{2}{3}\log_e\left(\frac{7}{4}\right)$

9 **(a)** 10 **(b)** 12

10 $11\frac{1}{3}$ square units **11** $y = -2x + 8$

12 $34\frac{1}{2}$ square units

13



14 **(a)** $xe^x - e^x$ **(b)** $x^2e^x + 2xe^x$ **(c)** $x^2e^x - 2\left(e^x - xe^x\right)$

PAPER 2: TECHNOLOGY-ACTIVE

1 **(a)** $M_0 = 5$ **(b)** $k = 0.400$ **(c)** $M(t) = 5e^{0.400t}$

(d) $M'(t) = 2e^{0.400t}$ **(e)** 11.1 g, 4.45 g/h

2 **(a)** $x = 1$ **(b)** $x = \frac{e^2+3}{2}$ **(c)** $x = \frac{2}{e^3}$

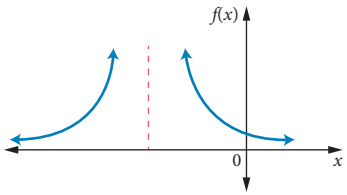
(d) $x = -2$ or $x = 2$ **(e)** $x = 5$

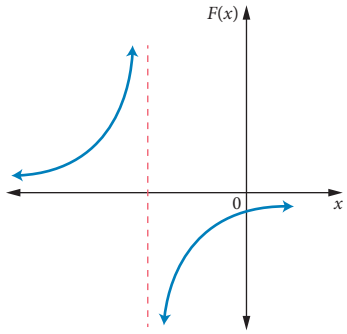
3 **(a)** $x(t) = \frac{5}{(3-2t)^{\frac{1}{2}}}$ **(b)** $0 \le t < \frac{3}{2}$, $x(1) = 5$

4 2.41 hours

5 **(a)** $2\cos^2(5\theta)$ **(b)** $\frac{1}{10}\sin(10\theta) + \theta + c$

6





7 11.34 square units

8 **(a)** \$27.83 **(b)** $C = 50\left(n + 3 - 3e^{0.02n}\right)$

(c) \$4042 **(d)** \$596

9 **(a)** $y = 0.001\left(x^3 - \frac{1}{4}x^4 - 2x\right)$

(b) From part **(a)**, $y = 0.001\left(x^3 - \frac{x^4}{4} - 2x\right)$

At A, $x = 2$:

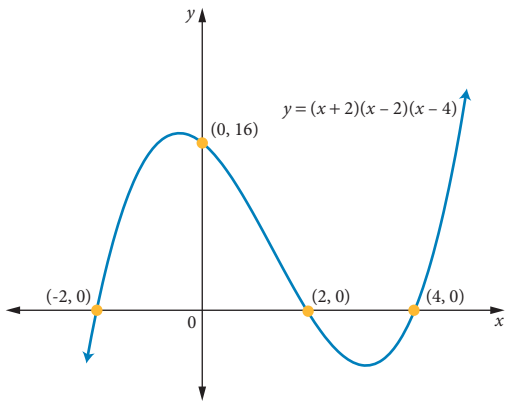
$$y = 0.001\left(2^3 - \frac{1}{4}2^4 - 2(2)\right)$$
$$= 0.001(8 - 4 - 4)$$
$$= 0$$

(c) 1.2 mm **(d)** 0.115°

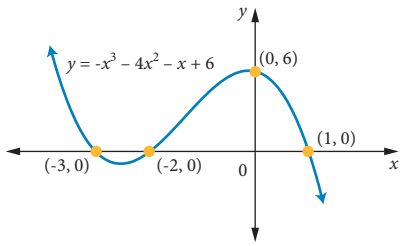
Chapter 6: Further applications of differentiation

RECALL

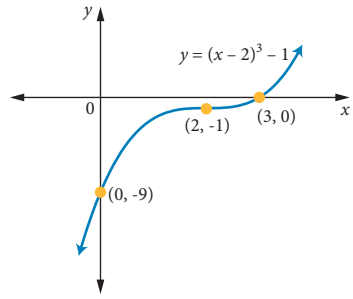
1 **(a)**



(b)



(c)



2 **(a)** $x = 3, y = 1$ **(b)** $x = -1, y = 2$ **(c)** $x = 5, y = 8$

3 **(a)** $x = 2$ **(b)** $x = 1$ **(c)** $x = 5$

4 **(a)** $\frac{dy}{dx} = 4x - 5, x \neq 0$ **(b)** $f'(x) = \frac{1}{2\sqrt{x}} + \frac{2}{x^2}$

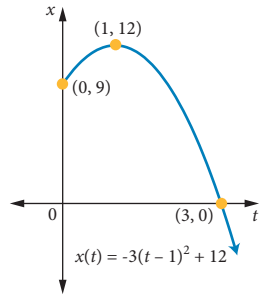
(c) $\frac{dy}{dx} = -3x^2 + 8x - 4$

5 **(a)** minimum: $(0, -4)$; maximum: $(-2, 0)$

(b) minimum: $(0, 0)$; maximum: $\left(\frac{5}{3}, \frac{125}{27}\right)$

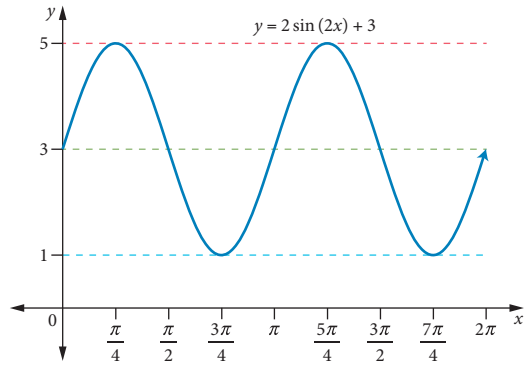
(c) minimum: $(-3, -7)$; maximum: $(0, 2)$

6 **(a)**

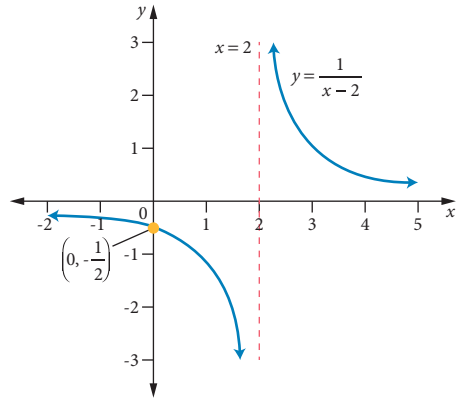


(b) 12 m **(c)** 6 m/s in the negative direction

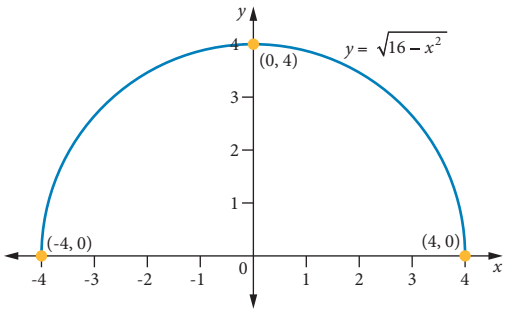
7 **(a)**



(b)



(c)



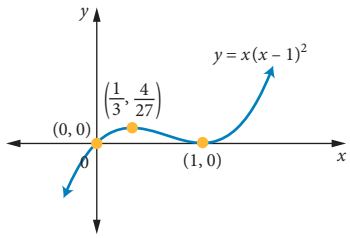
8 (a) $\dfrac{dy}{dx} = \dfrac{4x-3}{2\sqrt{2x^2-3x}}$

(b) $\dfrac{dy}{dx} = 2x\cos(x) - (x^2-5)\sin(x)$

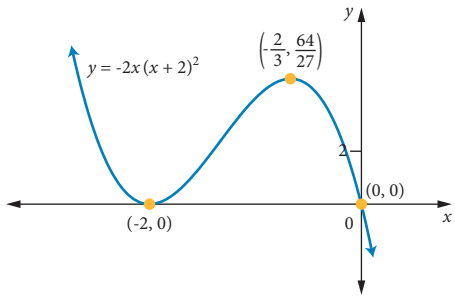
(c) $\dfrac{dy}{dx} = \dfrac{e^{5x}(5x^4-4x^3+10)}{(x^4+2)^2}$

EXERCISE 6.1

1



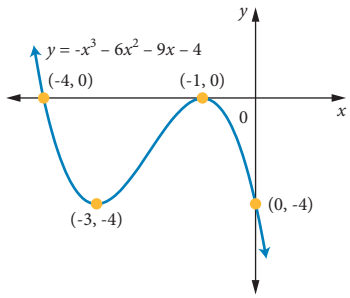
2



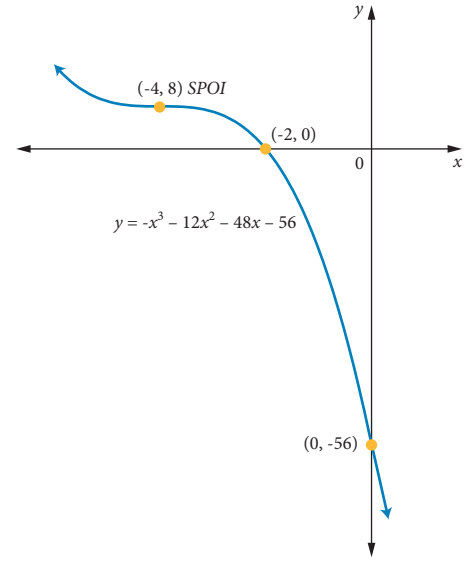
3 (a) C

(b) For cubics, there is no axis of symmetry between the x-intercepts.

4

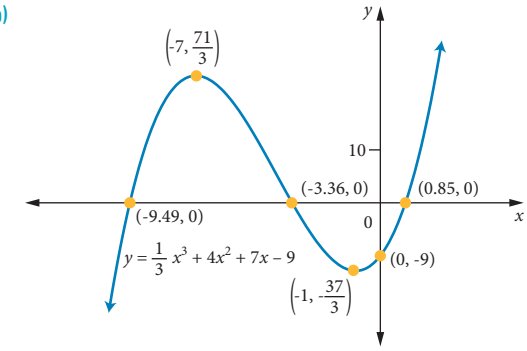


5



6 (a) $\left(-7, \dfrac{71}{3}\right)$ and $\left(-1, -\dfrac{37}{3}\right)$

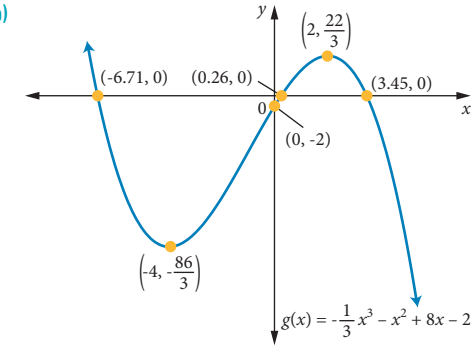
(b)



(c) increasing: $x < -7$ and $x > -1$; decreasing: $-7 < x < -1$

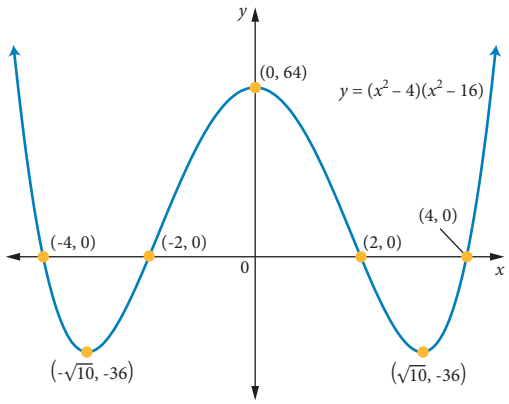
7 (a) $\left(-4, -\dfrac{86}{3}\right)$ and $\left(2, \dfrac{22}{3}\right)$

(b)

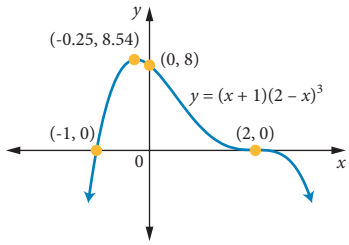


8 (a) $\left(-\sqrt{10}, -36\right)$, $(0, 64)$ and $\left(\sqrt{10}, -36\right)$

(b)



9



10 C

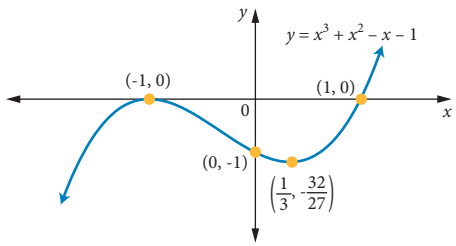
11 $f'(x) = 3x^2 + 4$; $3x^2 + 4$ is always positive, so the graph has no stationary points.

12 $c = -3$

13 **(a)** $(0, 0)$ is a local maximum and $\left(\frac{2}{3}, -\frac{4}{27}\right)$ is a local minimum.

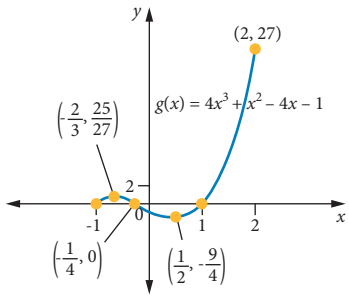
(b) 4 **(c)** $-\frac{4}{27}$ **(d)** $-\frac{1}{3}; \left(\frac{1}{3}, -\frac{2}{27}\right)$

14



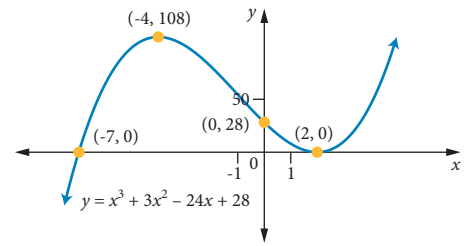
15 **(a)** local maximum: $\left(-\frac{2}{3}, \frac{25}{27}\right)$; local minimum: $\left(\frac{1}{2}, -\frac{9}{4}\right)$

(b) global minimum: $-\frac{9}{4}$; global maximum: 27



16 **(a)** $b = 3$, $c = -24$

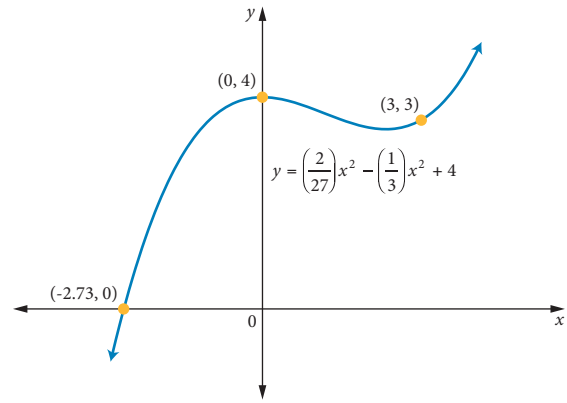
(b)



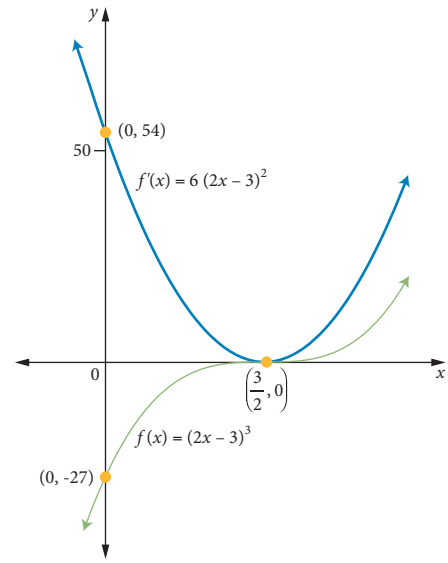
17 **(a)** $g'(x) = \frac{3}{8}(x-1)^2(9-4x)$ **(b)** $x = 1$ and $x \geq \frac{9}{4}$

(c) $m = 1$, $n = 1.31$ **(d)** 0, 1, 2.67, 3

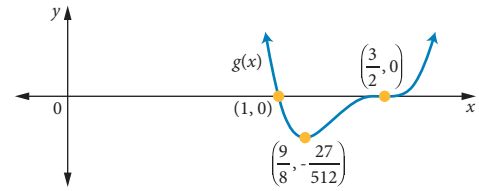
18



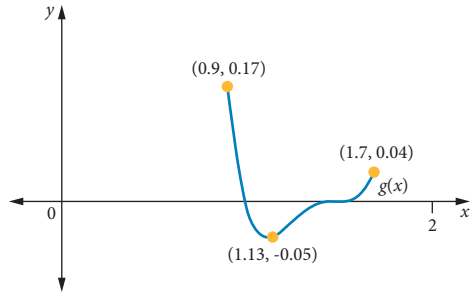
19 **(a)**



(b) Not shown on graph: y-intercept $= (0, 27)$



(c) global maximum: (0.90, 0.17); global minimum: (1.13, -0.05)



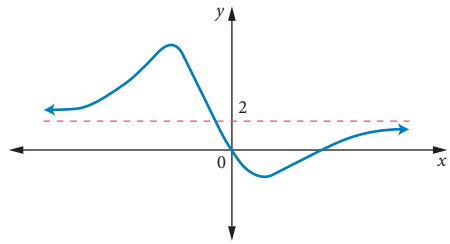
20 (a) $b^2 = 3c$

(b) (i) $(0, 1)$ and $\left(-\frac{2b}{3}, \frac{4b^3}{27} + 1\right)$ (ii) $b < \frac{-3}{\sqrt[3]{4}}$

(iii) $b < 0$ (iv) a stationary point of inflection

21 (a) $k < 0$

(b) reflection in the y-axis

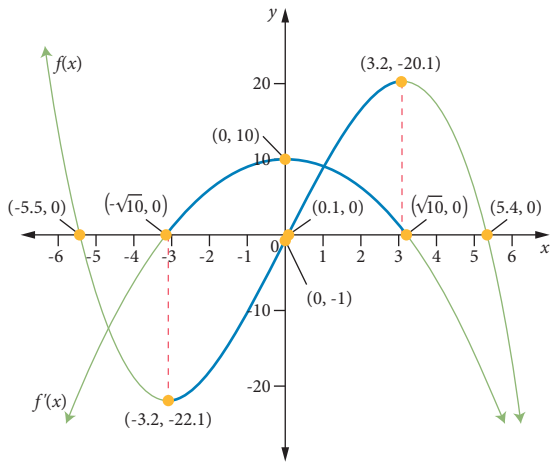


(c) four

22 (a) $x < -\sqrt{a}$ and $x > \sqrt{a}$

(b) minimum: (-3.2, 22.1); maximum: (3.2, 20.1)

(c)



EXERCISE 6.2

1 (a) $\frac{d^2y}{dx^2} = 12x$ (b) $\frac{d^2y}{dx^2} = 2$

(c) $\frac{d^2y}{dx^2} = 2 - \frac{2}{x^3}$ (d) $\frac{d^2y}{dx^2} = 48(2x+1)^2$

(e) $\frac{d^2y}{dx^2} = 2(3x-2)$

2 $f''(-1) = 6$

3 (a) $f'(x) = 10x - 8$, domain: $\mathbb{R}$; $f''(x) = 10$, domain: $\mathbb{R}$

(b) $f'(x) = 1 + \frac{2}{x^2}$, domain: $\mathbb{R} \setminus \{0\}$; $f''(x) = -\frac{4}{x^3}$, domain: $\mathbb{R} \setminus \{0\}$

(c) $f'(x) = \frac{1}{2\sqrt{x-2}}, x > 2$; $f''(x) = -\frac{1}{4(x-2)\sqrt{x-2}}, x > 2$

(d) $f'(x) = 1 + \frac{1}{2\sqrt{x}}, x > 0$; $f''(x) = -\frac{1}{4x\sqrt{x}}, x > 0$

4 (a) $a(t) = 4$ (b) $a(t) = -\frac{1}{4(t+1)\sqrt{t+1}}$

(c) $a(t) = 6(t-1)$

5 (a) A

(b) Negative acceleration describes slowing down only when the starting velocity is positive.

6 2.5 m/s^2

7 (a) 0 m/s^2 (b) 3 m/s

(c) $v(t) = 9t^2 - 18t + 12$ is a positive quadratic with a minimum turning point. $a(1) = 0$, so the minimum velocity is at $t = 1$.

(d) 1.18 s and 2.82 s

8 (a) $v(t) = 6t^2 - 24t + 8$ (b) $a(t) = 12t - 24$

(c) -24 m/s^2

(d) 2 seconds; 16 m to the left of the origin

(e) -16 m/s; after 2 seconds

9 (a) $v = \frac{1}{2\sqrt{t+2}}$; $\sqrt{t+2} > 0$, so $v > 0$ for all values of $t \geq 0$

(b) $-\frac{\sqrt{2}}{16}$ m/s^2

(c) $\sqrt{t+2} > 0$ for all values, and since $t \geq 0$ then $t + 2 > 0$. So, $a = -\frac{1}{4(t+2)\sqrt{t+2}}$ is negative, since all factors are positive.

10 (a) $-\frac{2}{(x+1)^3}$ (b) $-\frac{9}{4(3x-1)\sqrt{3x-1}}$

(c) $\frac{3x^2 - 24x - 16}{4x\sqrt{x}(x+4)^3}$

11 $y = -8x - 4$ **12** $x = \frac{1}{2}$

13 (a) 0 m/s (b) -2 m/s^2

14 C

15 (a) At the start, the particle is 18 cm to the left of the origin travelling at 27 cm/s towards the origin, slowing down at a rate of 20 cm/s^2. At the finish, the particle is 24 cm to the right of the origin travelling at 34 cm/s away from the origin, speeding up at a rate of 22 cm/s^2.

(b) (i) $3 < t < \frac{10 + \sqrt{19}}{3}$ (ii) $\frac{10 - \sqrt{19}}{3} < t < 3$

16 **(a)** $r(a)=\sqrt{1-2a+a^2+9a^4}$, $0\le a\le 1$;

$$r'(a)=\frac{18a^3+a-1}{\sqrt{9a^4+a^2-2a+1}};$$

$$r''(a)=\frac{\left(54a^2+1\right)\left(9a^4+a^2-2a+1\right)-\left(18a^3+a-1\right)^2}{\left(9a^4+a^2-2a+1\right)\sqrt{9a^4+a^2-2a+1}}$$

(b) $\frac{\sqrt{5}}{3}\le r\le 3$

EXERCISE 6.3

1 $(1,-6)$ **2** $(2,35)$

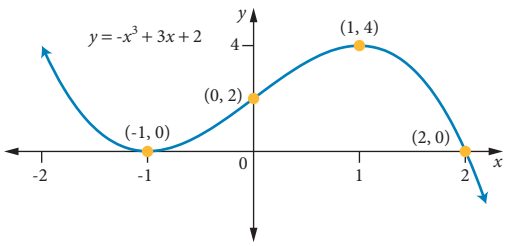
3 $\frac{d^2y}{dx^2}=-6$ Since the second derivative is constant, it cannot change sign, so there are no points of inflection.

4 $x>\frac{4}{3}$ **5** B

6 **(a)** local minimum: $(-1,0)$; local maximum: $(1,4)$

(b) $(0,2)$ **(c)** $(0,2)$

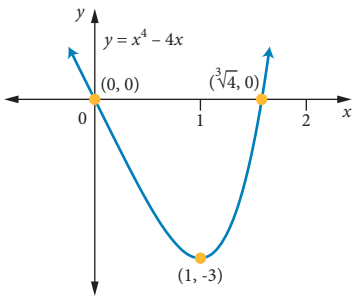
(d)



7 **(a)** minimum: $(1,-3)$

(b) The second derivative does not change sign over its domain. Hence, the graph of f does not have a point of inflection.

(c)



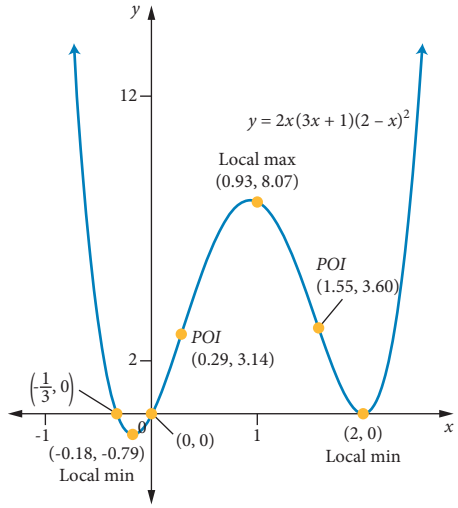
8 **(a)** B

(b) When the second derivative is zero, there may be a local maximum, a local minimum or a stationary point of inflection.

9 **(a)** $(0,0)$ **(b)** $\left(-\frac{1}{3},0\right)$, $(0,0)$, $(2,0)$

(c) local minima: $(-0.18,-0.79)$ and $(2,0)$; local maximum: $(0.93,8.07)$

(d)



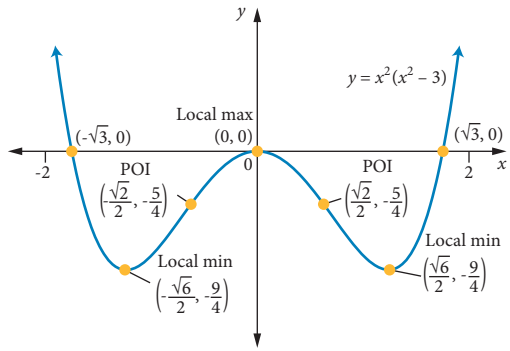
10 **(a)** $f'(x)=3x^2+\frac{12}{x^2}$, $f''(x)=6x-\frac{24}{x^3}$

(b) $\left(-\sqrt{2},4\sqrt{2}\right)$, $\left(\sqrt{2},-4\sqrt{2}\right)$

11 **(a)** local minima: $\left(-\frac{\sqrt{6}}{2},-\frac{9}{4}\right)$ and $\left(\frac{\sqrt{6}}{2},-\frac{9}{4}\right)$; local maximum: $(0,0)$

(b) $\left(-\frac{\sqrt{2}}{2},-\frac{5}{4}\right)$, $\left(\frac{\sqrt{2}}{2},-\frac{5}{4}\right)$

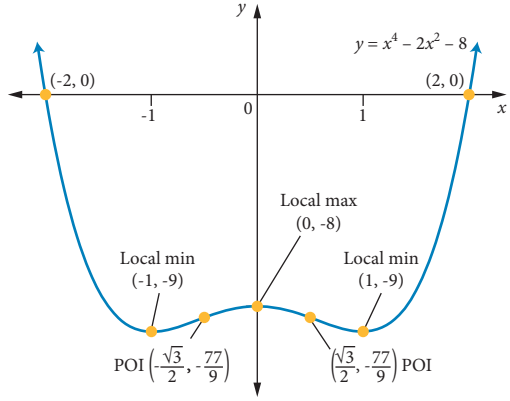
(c)



12 **(a)** local minima: $(-1,-9)$ and $(1,-9)$; local maximum: $(0,-8)$

(b) $\left(-\frac{\sqrt{3}}{3},-\frac{77}{9}\right)$ and $\left(\frac{\sqrt{3}}{3},-\frac{77}{9}\right)$

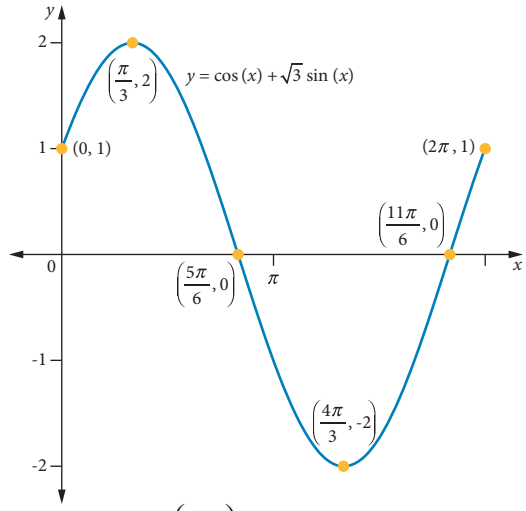
(c)



13 **(a)** local maximum: $\left(\frac{\pi}{3},2\right)$; local minimum: $\left(\frac{4\pi}{3},-2\right)$

(b) $\left(\frac{5\pi}{6}, 0\right)$ and $\left(\frac{11\pi}{6}, 0\right)$

(c)

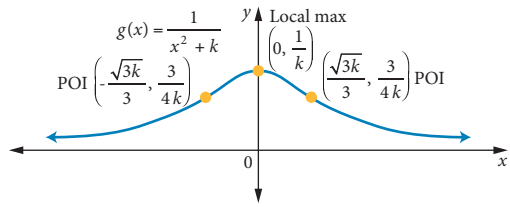


14 (a) local maximum: $\left(0, \frac{1}{k}\right)$

(b) $\left(-\sqrt{\frac{k}{3}}, \frac{3}{4k}\right), \left(\sqrt{\frac{k}{3}}, \frac{3}{4k}\right)$

(c) domain: $(-\infty, \infty)$; range: $\left(0, \frac{1}{k}\right]$

(d)

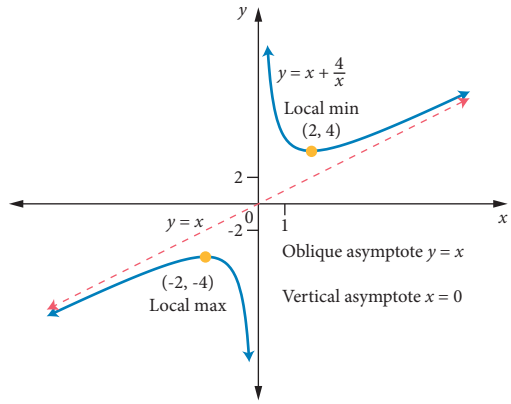


15 (a) local maximum: $(-2, -4)$; local minimum: $(2, 4)$

(b) $f''(x) = \frac{8}{x^3}$ No value of x can make this equal zero, so there are no points of inflection.

(c) $x = 0, y = x$

(d)



EXERCISE 6.4

1 horizontal: $2\sqrt{3}$ units; vertical: 6 units

2 (a) $y = 60 - 2x$ (b) 450 m^2

3 (a) $y = L - 2x$ (b) $A(x) = x(L - 2x)$ (c) $\frac{L^2}{8}$ m^2

(d) perpendicular to the river: $\frac{L}{4}$ m; parallel to the river: $\frac{L}{2}$ m

4 (a) Both numbers are $\frac{S}{2}$. (b) B

(c) The student has not considered the possibility that S could be zero.

5 (a) 450 (b) 900

6 (a) $y = \frac{P}{2} - x$ (b) $A(x) = \frac{Px}{2} - x^2$

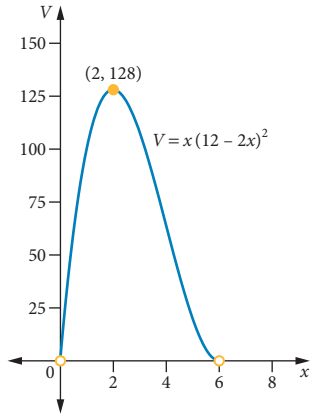
(c) $0 < x < \frac{P}{2}$ (d) $A'(x) = \frac{P}{2} - 2x$; $A''(x) = -2$

(e) $x = \frac{P}{4}$ The second derivative is negative, so the stationary point gives the maximum value.

(f) area: $\frac{P^2}{16}$ m^2; $\frac{P}{4}$ m $\times$ $\frac{P}{4}$ m; square

7 (a) $V(x) = x(12 - 2x)^2$ (b) $0 < x < 6$

(c) maximum volume: 128 m^3; dimensions: 8 m $\times$ 8 m $\times$ 2 m



8 (1, 1) 9 D

10 (a) $V(x) = 4x(6 - x)(3 - x)$, $0 < x < 3$

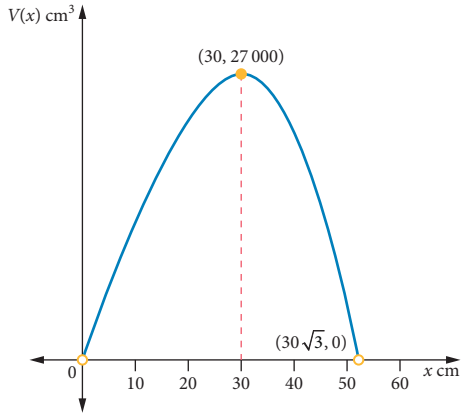
(b) $x = (3 - \sqrt{3})$ m (c) $24\sqrt{3}$ m^3

11 (a) 1728 (b) 105.87

12 (a) $V(x) = 1350x - \frac{1}{2}x^3$

(b) $V_{max} = 27\,000$ cm^3; $x = 30$ cm and $y = 30$ cm

(c) domain: $0 < x < 30\sqrt{3}$



13 (a) $A(r) = \frac{20}{r} + 2\pi r^2$

(b) $r = \sqrt[3]{\frac{5}{\pi}}$ m (c) $A''\left(\sqrt[3]{\frac{5}{\pi}}\right) = 12\pi > 0$

(d) $25.7\,\text{m}^2$ (e) $h = 2r$

14 570

15 horizontally: $2\sqrt{2}$ units, vertically: $\sqrt{2}$ units

16 733 items

17 (a) $A(x) = \frac{x^2}{4\pi}$ (b) $B(x) = \frac{(80-x)^2}{18}$

(c) $T(x) = \frac{x^2}{4\pi} + \frac{(80-x)^2}{18}$ (d) $x = \frac{160\pi}{9+2\pi}$ m

(e) circle: $\frac{6400\pi}{(9+2\pi)^2}\,\text{m}^2$; rectangle: $\frac{28\,800}{(9+2\pi)^2}\,\text{m}^2$

(f) $200\pi : 9$

(g) All of the wire should be devoted to the circle. The description 'cut into two pieces' does not apply.

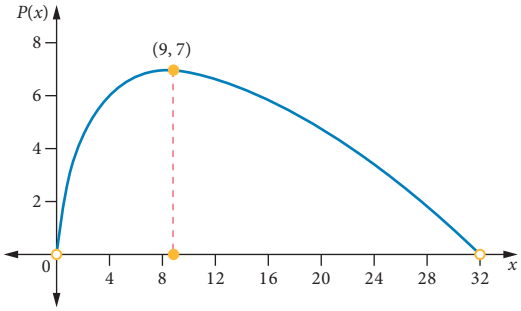
18 (a) $L(x) = \sqrt{x^2+49} + \sqrt{x^2-50x+794}$

(b) 8.75 m from the 7 m pole (c) 32.0 m

19 (a) $P(x) = 6\sqrt{x} - x - 2$

(b) \$420 per hour; 540 L

(c) domain: $0.125 < x < 31.875$

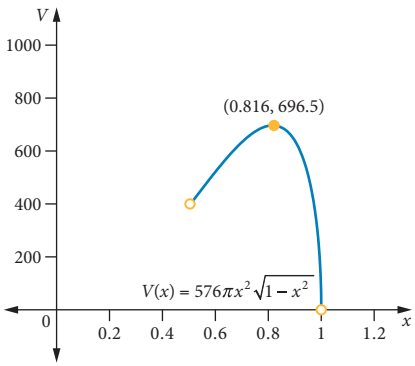


20 (a) $x = 3$ km (b) 3 units

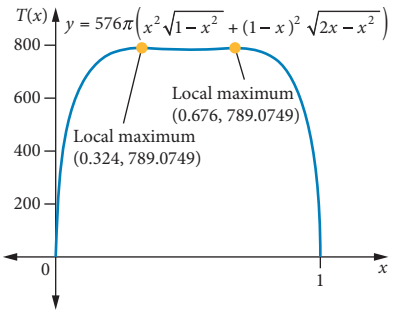
(c) $x \in \left[\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right]$ (d) $m = 9$

21 (a) $V'(x) = \frac{576\pi x\left(2-3x^2\right)}{\sqrt{1-x^2}}$

(b) $x = \frac{\sqrt{6}}{3}$ cm (c) $696.5\,\text{cm}^3$



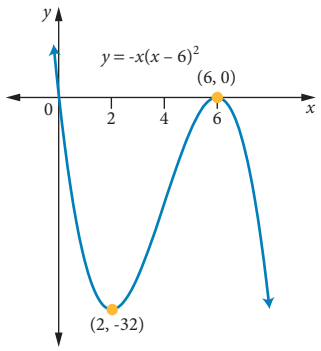
(d) $789.1\,\text{cm}^3$ The major sector should use about 68% of the circle, leaving 32% for the minor sector.



CHAPTER REVIEW 6

1 (a) local maximum: $(6,0)$: local minimum: $(2,-32)$

(b)



2 $x \in \left(-3, \frac{1}{3}\right)$ 3 A

4 global maximum: 22, where $x = 9$

5 (a) $a(t) = -10$ (b) $a(t) = -\frac{1}{(2t-3)\sqrt{2t-3}}$

(c) $a(t) = -6(t-2)$

6 B

7 (a) 5 metres to the left of O (b) after $\frac{5}{3}$ seconds

(c) $-\frac{16}{3}$ m/s

(d) After 1 second, the velocity is -4 m/s and the acceleration is $-4\,\text{m/s}^2$. After $4\frac{1}{3}$ seconds, the velocity is 16 m/s and the acceleration is $16\,\text{m/s}^2$.

(e) after $\frac{1}{3}$ second and after 3 seconds

8 D

9 $g''(-1) = -34$ 10 $y = 9x - 9$

11 $x \in \left(\frac{8}{3}, \infty\right)$ 12 $x > 2$

13 B 14 $b = -3$, $c = 1$

15 (a) $f''(x) = -12x^2 + 6$ (b) $\frac{d^2y}{dx^2} = \frac{4}{(x-1)^3}$

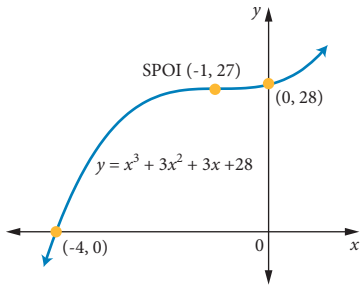
16 square with side length $\sqrt{k}$ units

17 radius: 10 cm; height: 20 cm

18 -8

19 **(a)** $(-1, 27)$: stationary point of inflection

(b)



20 **(a)** $\dfrac{d^2y}{dx^2} = \dfrac{12}{(x-3)^3}$ **(b)** $\dfrac{d^2y}{dx^2} = \dfrac{1}{(2x-5)\sqrt{5-2x}}$

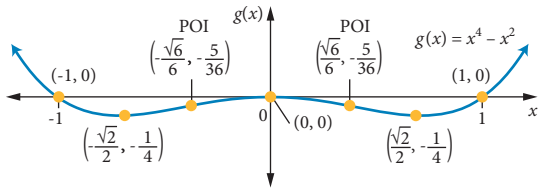
21 **(a)** 1 m/s **(b)** $1\,\text{m/s}^2$

22 **(a)** local minima: $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{1}{4}\right)$ and $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{1}{4}\right)$:

local maximum: $(0,0)$

(b) $\left(-\dfrac{\sqrt{6}}{6}, -\dfrac{5}{36}\right)$ and $\left(\dfrac{\sqrt{6}}{6}, -\dfrac{5}{36}\right)$

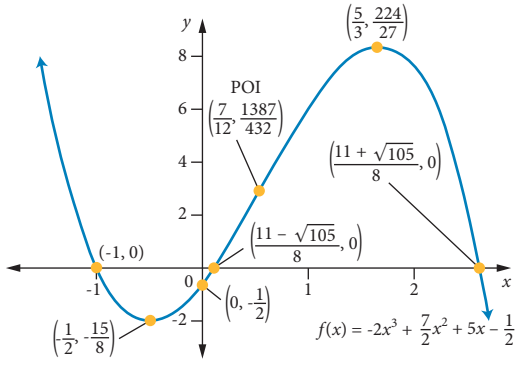
(c)



23 **(a)** $a = -2, b = \dfrac{7}{2}, c = 5, d = -\dfrac{1}{2}$; $f(x) = -2x^3 + \dfrac{7}{2}x^2 + 5x - \dfrac{1}{2}$

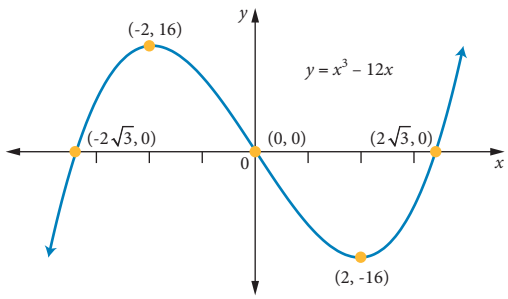
(b) local minimum: $\left(-\dfrac{1}{2}, -\dfrac{15}{8}\right)$; local maximum: $\left(\dfrac{5}{3}, \dfrac{224}{27}\right)$

(c) $f(x)$ is increasing: $-\dfrac{1}{2} < x < \dfrac{5}{3}$; decreasing: $x < -\dfrac{1}{2}$ and $x > \dfrac{5}{3}$



24 **(a)** **(i)** local maximum: $(-2, 16)$; local minimum: $(2, -16)$

(ii)



(b) **(i)** $-16 < d < 16$ **(ii)** $d = \pm 16$

(iii) $d = 16$: $(-4, 0)$ and $(2, 0)$; $d = -16$: $(-2, 0)$ and $(4, 0)$

25 **(a)** $0 < x < \dfrac{p}{3}$ **(b)** $A(x) = \dfrac{\sqrt{3}x^2}{4} + \dfrac{(p-3x)^2}{4\pi}$

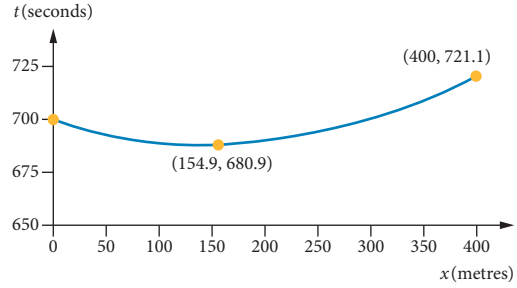
(c) $x = \dfrac{3p}{\sqrt{3}\pi + 9}$ m

26 **(a)** $t(x) = \dfrac{\sqrt{600^2 + x^2}}{4} + \dfrac{400 - x}{4}$

(b) 245.1 m south of the designated point C; minimum time: 11 min 20.9 s

(c) domain: $0 \le x \le 400$

The total time spent covering the distances AB and BC is a minimum of 680.9 s when the exit point is 154.9 m north of D. If the exit point was D, the time would be 700 s, and if the exit point was C, the time would be 721.1 s.



27 **(a)** $AB = -x^3 + 4x^2 + 4x + 5$

(b) $A(-0.4, -5.9)$, $B(-0.4, -10.0)$; 4.1 units

Chapter 7: Sine and cosine rules

RECALL

1 **(a)** $a = 93°$ **(b)** $p = 136°, q = 29°$

(c) $c = 42°, d = 48°$

2 **(a)** $\tan(47°) = \dfrac{x}{14}$ **(b)** $\sin(\theta) = \dfrac{12}{20}$

3 **(a)** $x = 3.08$ **(b)** $x = 3.62$ **(c)** $\theta = 36.87°$

4 **(a)** $x = 20.81$ m **(b)** $x = 28.16$ m

5 **(a)** $120\,\text{cm}^2$ **(b)** $327.25\,\text{mm}^2$ **(c)** $3.625\,\text{cm}^2$

6 **(a)** $\dfrac{\pi}{9}$ **(b)** $\dfrac{3\pi}{4}$ **(c)** $\dfrac{13\pi}{6}$

7 **(a)** 36° **(b)** 150° **(c)** 420°

8 **(a)** $-\cos(\theta)$ **(b)** $\tan(\theta)$ **(c)** $-\sin(\theta)$

9 **(a)** $\sqrt{3}$ **(b)** $\dfrac{\sqrt{2}}{2}$ **(c)** $-\dfrac{\sqrt{3}}{2}$

EXERCISE 7.1

1 **(a)** $x = 59.57$ cm **(b)** $x = 2.85$ m **(c)** $x = 33.36$ cm

(d) $x = 6.31$ cm **(e)** $x = 14.18$ m **(f)** $x = 11.43$ cm

(g) $x = 49.12$ cm **(h)** $x = 14.23$ cm

2 (a) $\theta = 40°$ (b) $\theta = 39°$ (c) $\theta = 40°$ (d) $\theta = 55°$

3 $x = 20.62$ cm, $\theta = 14°$ 4 $\theta = 67°$

5 (a) $x = 6$ units, $\theta = 28.6°$ (b) $x = 10.4$ units, $\theta = 20.9°$

6 (a) $x = 45°, x = 135°$ (b) $x = \frac{2\pi}{3}, x = \frac{4\pi}{3}$

(c) $x = 120°, x = 300°$ (d) $x = \frac{\pi}{6}, x = \frac{11\pi}{6}$

(e) $x = -\frac{5\pi}{6}, x = -\frac{\pi}{6}$ (f) $x = 45°, x = -45°$

7 (a) B (b) $\cos(\theta) = 0.54 \Rightarrow \cos(-\theta) = 0.54$

8 A 9 $\theta = 74°$ 10 N55°E

11 9.6 m 12 168.8 m

13 (a) 533 m (b) 565 m

14 92.50 m 15 3.5 km 16 36.99 km, N20°E

17 (a) 68° (b) 37.93 cm

18 1070 km, N73°W Assume that the curvature of the Earth does not have a big effect on the results.

19 (a) 45° (b) 79°

EXERCISE 7.2

1 (a) 106.04 cm^2 (b) 460.75 cm^2 (c) 327.70 m^2 (d) 34.82 m^2

2 (a) $18\sqrt{3}$ cm^2 (b) 70 cm^2 (c) $3\sqrt{2}$ m^2 (d) 37.5 m^2

3 322 cm^2 4 30°

5 365.14 cm^2 6 38 971 m^2

7 (a) B (b) A or C

8 53.06 cm^2

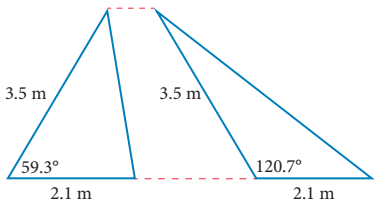
9 (a) 20.66 cm^2 (b) 339.82 m^2 (c) 507.96 m^2 (d) 93.76 cm^2

10 373.71 cm^2 11 368 m^2

12 1015 929 km^2 Assume the curvature of the Earth does not have a significant effect on the area calculation.

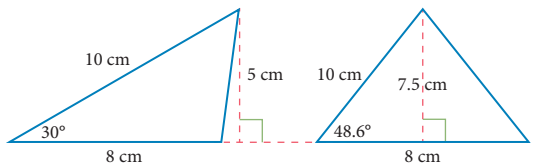
13 C

14



15 21° or 159° 16 98°

17 A possible solution:



EXERCISE 7.3

1 $x = 2.5$ cm

2 (a) $a = 25.49, C = 70.65°, c = 27.45$

(b) $B = 46.50°, b = 43.30, c = 34.18$

(c) $a = 14.05, b = 16.31, C = 49.10°$

(d) $A = 56.00°, b = 77.56, c = 31.77$

3 (a) $A = 61.23°, C = 51.24°, c = 7.22$

(b) $B = 46.23°, C = 62.72°, c = 15.46$

(c) $A = 20.71°, B = 18.70°, b = 62.50$

(d) $A = 88.78°, a = 5.14, C = 38.66°$

4 B

5 (a) D (b) C

6 77.82°, 25 m, 29 m 7 672 m

8 (a) $A = 40°, B = 60°, C = 80°$ (b) 24.62 km

(c) 18.28 km

9 (a) 33 m (b) 15 m

10 (a) 17.42 m (b) 39.38 m (c) 1270.42 m^2 (d) 23.86°

11 (a) $B = 147.47°, C = 9.11°, c = 13.77$; or $B = 32.53°$, $C = 124.05°, c = 72.04$

(b) $B = 117.47°, C = 26.33°, c = 11.72$; or $B = 62.53°$, $C = 81.27°, c = 26.12$

(c) $A = 4.75°, C = 118.87°, a = 2.52$; or $A = 62.49°$, $C = 61.13°, a = 27.02$

(d) $A = 16.18°, B = 150.97°, b = 78.51$; or $A = 163.82°$, $B = 3.33°, b = 9.40$

12 526.62 cm^2 13 307.1 cm 14 16.51 km

15 (a) $b = 10$ (b) $C = 8°$ or $C = 149°$

16 (a) 33.05 km (b) 272°T

EXERCISE 7.4

1 7762 m 2 861 m

3 (a) $A = 102.99°, B = 41.57°, C = 35.44°$

(b) $A = 93.97°, B = 29.26°, C = 56.77°$

(c) $A = 67.54°, B = 82.43°, C = 30.03°$

(d) $A = 60.49°, B = 73.04°, C = 46.47°$

4 115°

5 (a) C

(b) When three sides are known, there will be just one solution. If the angle were obtuse, the cosine would be negative.

6 24.62°, 41.80°, 113.58° 7 B

8 (a) 40.748 km (b) 58.38°

9 (a) $a = 17.76, B = 71.93°, C = 42.67°$

(b) $A = 86.96°, b = 33.41, C = 46.11°$

(c) $A = 87.52°$, $B = 42.10°$, $c = 28.62$

(d) $A = 78.17°$, $b = 8.35$, $C = 63.30°$

10 $A = 66°$, $C = 54°$, $b = 16$ cm

11 42 070 m^2

12 (a) 18.2 km (b) 022°T

13 35.6°

14 (a) 3.41 km (b) 2.88 km (c) D; 530 m (d) 6 km

15 (a) 484 m (b) more than 60 m

(c) 13 232 m^2 (d) 72.4 m

CHAPTER REVIEW 7

1 (a) (i) 135° (ii) 72.19°

(b) $\frac{31\pi}{45}$ (c) 0.820^c

2 60° 3 A 4 $x = 450.21$ units

5 34.31° 6 8.82 m

7 (a) 30 m (b) 59°

8 297 km^2 9 D

10 (a) 108 016 m^2 (b) 5471 m^2

11 73.2 m^2 12 62.96°, 5.00 m^2 13 45.77 m

14 $x = 6\sqrt{6}$ cm 15 B 16 $a = 13.67$ units

17 $A = 39.58°$, $B = 121.04°$, $C = 19.38°$

18 70 m 19 B

20 (a) 7.43 m (b) 10.86 m (c) 38.30°

21 (a) $BD = 10$ cm, $AC = 10\sqrt{3}$ cm (b) $25\sqrt{3}$ cm^2

22 $B = 31.98°$, $C = 128.79°$, $c = 53.15$; or $B = 148.02°$, $C = 12.75°$, $c = 15.05$

23 (a) 368 509 km

(b) closer than average distance; closer to perigee than to apogee

24 57.9°

25 (a) $D = 37$ km (b) 073° T

26 27.93 cm^2

27 $x = 3$ cm or $x = 6.8$ cm

Chapter 8: Binomial distributions

RECALL

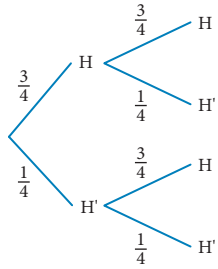
1 (a) discrete (b) continuous

2 (a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$

3 (a) (i) 0, 1, 2 and 3 (ii) 0, 1 and 2

(b) (i) 1, 2 (ii) 2

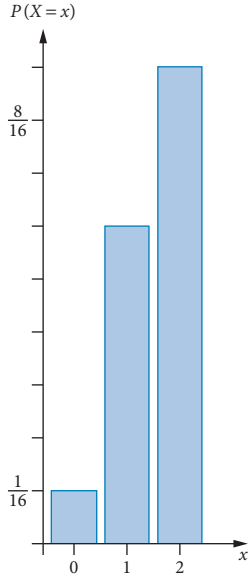
4 (a)



(b)

x	0	1	2
$P(X = x)$	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{9}{16}$

(c)



5 (a) No; the sum of the probabilities is less than 1.

(b) No; $-\frac{1}{6}$ is not a legitimate probability.

(c) Yes; the probability sum is 1 and all probabilities are between 0 and 1.

6 (a) $k = \frac{19}{36}$ (b) $k = \frac{1}{20}$

7 (a) $E(X) = 2.14$ (b) $E(X) = -0.75$ (c) $E(X) = 45\frac{1}{2}$

8 (a) 28 ways (b) 210 ways (c) 21 ways

EXERCISE 8.1

1 (a)

x	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

(b) (i) $\frac{2}{3}$ (ii) $\frac{17}{18}$

2 (a)

x	0	2	4	6	10	16
$P(X = x)$	0.5	0.1	0.1	0.1	0.1	0.1

(b) (i) 0.2 (ii) 0.2 (iii) 1

3 (a) $E(X) = 9.25$ (b) $Var(X) = 8.2875$ (c) $\sigma = 2.88$

4 **(a)**

x	0	1	2
$P(X=x)$	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{4}{25}$

(b) **(i)** $E(X)=0.8$ **(ii)** $Var(X)=0.48$ **(iii)** $\sigma=0.69$

5 **(a)** A **(b)** $6-(-2.2)^2=6-4.84$, not $6+4.84$

6 $E(X)=2$, $E(Y)=3$, $E(X+Y)=5$

7 C

8 **(a)**

y	40.5	41.5	42.5	44	44.5
$P(Y=y)$	0.2	0.5	0.1	0.05	0.15

(b)

y	12	72	132	222	252
$P(Y=y)$	0.2	0.5	0.1	0.05	0.15

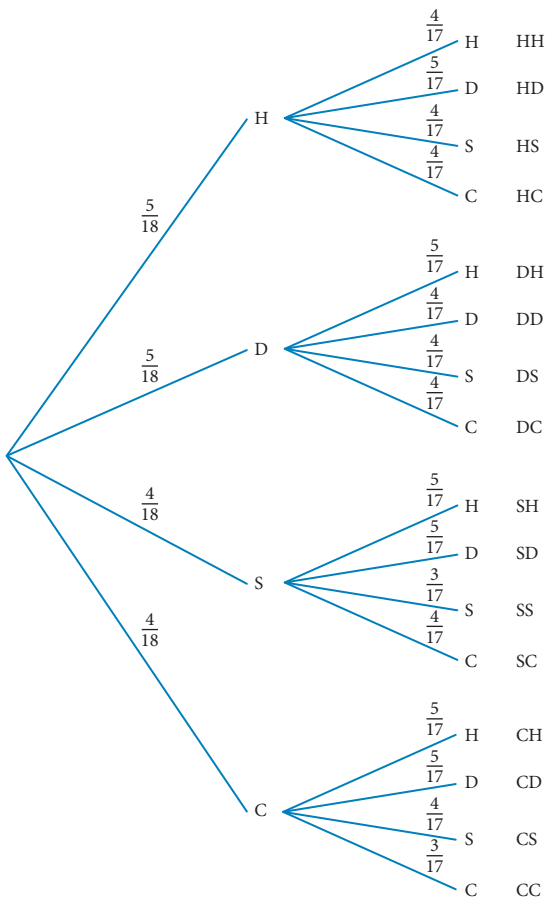
(c)

y	99.9	99.7	99.5	99.2	99.1
$P(Y=y)$	0.2	0.5	0.1	0.05	0.15

(d)

y	-3	1	5	11	13
$P(Y=y)$	0.2	0.5	0.1	0.05	0.15

9 **(a)**



(b)

x	0	1	2
$P(X=x)$	$\frac{91}{153}$	$\frac{56}{153}$	$\frac{2}{51}$

(c) $\frac{5}{17}$ **(d)** $\frac{125}{153}$ **(e)** $\frac{3}{14}$

10 **(a)**

x	1	2	3	4	5	6	8
$P(X=x)$	$\frac{9}{25}$	$\frac{1}{25}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{25}$	$\frac{4}{25}$	$\frac{2}{25}$

(b) **(i)** $\frac{23}{25}$ **(ii)** $\frac{1}{15}$ **(iii)** $\frac{1}{5}$

11 **(a)** expected value: 3.9; variance: 8.29; standard deviation: 2.88

(b) **(i)** expected value: 39; variance: 829; standard deviation: 28.8

(ii) expected value: 63.9; variance: 8.29; standard deviation: 2.88

(iii) expected value: 16.5; variance: 207.25; standard deviation: 14.4

12 expected value: 17.019; variance: 0.000 069

13 **(a)** $k=\frac{1}{25}$ **(b)** $E(X)=1.2$

(c) $Var(X)=1.36$ **(d)** $\sigma=1.1662$

14 **(a)**

y	0	1	2	3	4	5
$P(Y=y)$	$\frac{227}{1152}$	$\frac{355}{1152}$	$\frac{65}{288}$	$\frac{173}{1152}$	$\frac{49}{576}$	$\frac{13}{384}$

(b) $E(Y)=1.72$

(c) Possible answer: swap 6 with 2; $E(Y)=2.13$

15 0.2336 sets

EXERCISE 8.2

1 **(a)** $P(X=4)=0.0078$ **(b)** $P(X=6)=0.0746$

(c) $P(X=2)=0.2965$ **(d)** $P(X=30)=0.1146$

2 0.1782

3 **(a)** 0.04186 **(b)** 0.99989

4 **(a)** 0.07465 **(b)** 0.05095

5 **(a)** Not suitable, because the number of trials was not determined beforehand.

(b) suitable

6 **(a)** 1 **(b)** 1 **(c)** 5 **(d)** 7 **(e)** 12

7 **(a)** 0.0918 **(b)** 0.0004 **(c)** 0.0485 **(d)** 0.0108

8 **(a)** B

(b) The student has mixed up the number of successes with the number of failures when matching the powers to the probabilities.

9 0.2157 **10** 0.0834 **11** 0.146

12 B

13 The probability of success is not constant.

14 (a) (i) 0.0115 (ii) 0.0576 (iii) 0.1369
(iv) 0.2054 (v) 0.2182
(b) 0.6296 (c) $P(X \leq 4) = 0.6296$ (d) 0.3704

15 (a) 0.737 (b) 0.582 (c) 0.252

16 (a) 0.3025 (b) 0.0109 (c) 0.1176
(d) 0.343 (e) 0.0040 (f) 0.1297

17 0.9984

18 (a) 1 (b) 2

19 0.996

20 (a) 3 (b) 15

EXERCISE 8.3

1 (a) mean: $\frac{1}{6}$; variance: $\frac{5}{36}$ (b) mean: $\frac{1}{4}$; variance: $\frac{3}{16}$
(c) mean: 0.86; variance: 0.12 (d) mean: 0.08; variance: 0.08

2 (a) $E(X) = 14$, $Var(X) = 4.2$, $\sigma(X) = 2.05$
(b) $E(X) = 55$, $Var(X) = 24.75$, $\sigma(X) = 4.97$
(c) $E(X) = 33.33$, $Var(X) = 11.11$, $\sigma(X) = 3.33$
(d) $E(X) = 5$, $Var(X) = 2.5$, $\sigma(X) = 1.58$
(e) $E(X) = 20$, $Var(X) = 4$, $\sigma(X) = 2$
(f) $E(X) = 16$, $Var(X) = 9.6$, $\sigma(X) = 3.10$

3 (a) $p = \frac{1}{3}$ (b) $p = 0.1$ (c) $n = 35$ (d) $n = 25$

4 (a) $n = 12$, $p = 0.4$ (b) $n = 20$, $p = 0.1$
(c) $n = 30$, $p = 0.15$ (d) $n = 50$, $p = 0.22$
(e) $n = 100$, $p = 0.65$ (f) $n = 150$, $p = 0.275$

5 (a) B
(b) The student has mixed up standard deviation and variance. $\sigma^2 = 7.2$ and $\sigma = \sqrt{7.2}$

6 D **7** 0.367

8 $p = 0.4$ or 0.6 The solution is reasonable because the possible values for p are symmetrical about $p = 0.5$.

9 (a) 32 (b) 25 to 39

10 10 to 24 **11** 3 to 9

12 $n = 64$, $p = 0.75$, $\mu - 2\sigma = 48 - 4\sqrt{3}$ and $\mu + 2\sigma = 48 + 4\sqrt{3}$

13 $P(X = 3) = 0.42$

14 (a) (i)

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.1001	0.2670	0.3115	0.2076	0.0865	0.0231	0.0038	0.0004	0.0001

(ii) $E(X) = 2$ (iii) $\sigma(X) = 1.22$
(iv) 0 to 4 (v) 97.27%
(b) (i) 2 to 6
(ii)

y	0	1	2	3	4	5	6	7	8
$P(Y = y)$	0.0039	0.0313	0.1094	0.2188	0.2734	0.2188	0.1094	0.0313	0.0039

(iii) 92.98%
(c) (i) 3 to 7
(ii)

z	0	1	2	3	4	5	6	7	8
$P(Z = z)$	0.0002	0.0033	0.0217	0.0808	0.1875	0.2786	0.2587	0.1373	0.0319

(iii) 94.29%
(d) All three distributions were close to having 95% of their values within two standard deviations of the mean. As the number of values increased, you would expect to see results closer to the prediction. The boundary adjustments necessary due to the variable being discrete contributed to some of the discrepancy in the probability values. Otherwise, you would expect the distribution where $p = 0.5$ to have given a closer estimate than the other two.

15 $\frac{1}{2}$

16 (a) $n = 49$, $p = \frac{6}{7}$
(b) The probability of success does not change, $p = \frac{6}{7}$, but the number of data collected doubles to $n = 98$.
(c) 12.10 (d) $E(X) = 48$, $Var(X) = 6.86$
(e) If the mean is the same but the number of trials is larger, p decreases, which, in turn, produces an increase in the variance of the sample. If the probability of success is the same but the number of trials is larger, this produces an increase in both the mean and the variance of the sample.

CHAPTER REVIEW 8

1 (a)

x	1	2	3	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$

(b) (i) $\frac{1}{2}$ (ii) $\frac{7}{8}$

2 (a)

x	3	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{18}$

(b) $5\frac{5}{6}$

3 (a) $E(X) = 59.34$ (b) $Var(X) = 571.9644$

4 0.86

5

y	51	58	65	72	79
$P(Y = y)$	0.01	0.27	0.25	0.16	0.31

6 (a) 0.144 (b) 0.563

7 0.35 **8** 2

9 51

10 (a) $P(X = 3) = \frac{216}{625}$ (b) $P(X = 4) = \frac{64\,827}{200\,000}$

11 D

12 mean: $\frac{2}{3}$; variance: $\frac{2}{9}$

13 **(a)** $p = \frac{1}{14}$ **(b)** $n = 50$

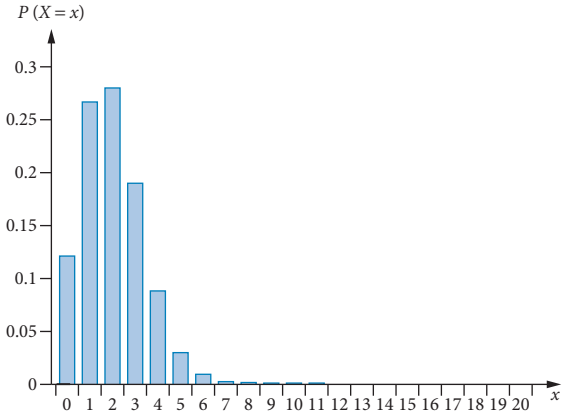
14 $n = 40$, $p = 0.8$

15 **(a)** 5 points

(b) Quite surprised, because the number is well above the expected number.

16 B **17** C

18 **(a)**



(b) average: 2; standard deviation: 1.34

19 **(a)**

x	0	1	2
$P(X = x)$	$\frac{253}{496}$	$\frac{207}{496}$	$\frac{9}{124}$

(b) $\frac{91}{496}$ **(c)** $\frac{343}{496}$ **(d)** $\frac{4}{17}$

20 **(a)** 0.2501 **(b)** 0.3980 **(c)** 0.0071

21 **(a)** 17 to 33 **(b)** 95.1%

22 **(a)** D **(b)** C

23 **(a)** $P(\text{success}) = 0.75$

(b) **(i)** $X \sim B(30, 0.75)$

(ii) 22 or 23

(iii) Unlikely, because it is outside the range for which 95% of results occur.

(c)

y	0	1	2
$P(Y = y)$	0.0625	0.375	0.5625

24 **(a)** \$7 **(b)** \$7.81

(c) 100% With only three values in the distribution, 100% is as close as possible to 95%.

25 \$19

26 0.037

27 **(a)** 0.367 **(b)** 0.210

(c) 0.393 **(d)** 0.04

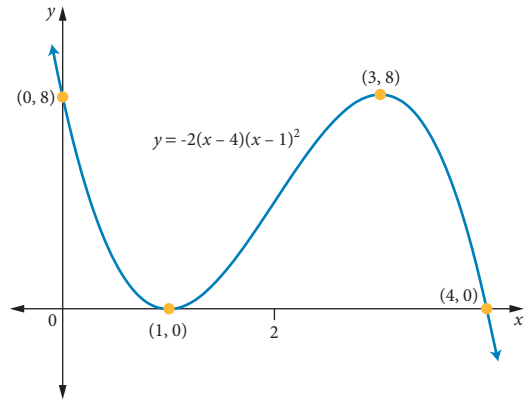
(e) $P(X > 2(10 \text{ days})) = 0.322 > P(X > 2(7 \text{ days})) = 0.148$

(f) 14

Mixed review: Chapters 6–8

1 **(a)** local minimum: (1, 0); local maximum: (3, 8)

(b)



2 **(a)** $a(t) = 4$ **(b)** $a(t) = \frac{8}{(2t+5)^3}$

3 (-2, -28)

4 **(a)** 34.53 **(b)** 6.45

5 **(a)** $a = 57.68$ units, $C = 39.10°$, $c = 38.17$ units

(b) $A = 43.52°$, $b = 25.94$ units, $c = 48.02$ units

6 26.38°, 176 m^2

7 240°T

8 **(a)**

x	0	1	2
$P(X = x)$	0.81	0.18	0.01

(b) 0.19

9 **(a)** 0.633 **(b)** 0.367

10 $n = 50$, $p = 0.25$

11 $\frac{d^2y}{dx^2} = \frac{1}{(x^2+1)\sqrt{x^2+1}}$

12 $B = 38.8°$, $c = 17.6$, $C = 113.2°$; or $B = 141.2°$, $c = 3.6$, $C = 10.8°$

13 **(a)** 0.25 **(b)** 0.3125 **(c)** 0.3438

(d) 0.3281 **(e)** 0.0313

14 1.12 square units

15 **(a)** $\frac{1}{3}$

(b) $\mu = 5$, $\sigma = \sqrt{\frac{10}{3}}$, $\mu - 2\sigma = 1.35$, $\mu + 2\sigma = 8.65$ Discrete values for this range become two to eight eggs intact.

(c) one to six eggs intact

Chapter 9: Continuous random variables and the normal distribution

RECALL

1 (a) continuous (b) discrete (c) continuous

2 (a) 24 (b) $\frac{2}{15}$ (c) $250\frac{5}{6}$

3 (a) maximum: 2; $x = 4$

(b) maximum: $\frac{16\sqrt{3}}{9}$; $x = -\frac{2\sqrt{3}}{3}$

(c) maximum: 3; $x = 1$

4 (a) $\frac{3}{5}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2}$

5 (a) $\mu = 8$ (b) $q = 0.2$ (c) $\sigma = 1.265$

6 (a) 0.088 (b) 0.001 (c) 0.316

EXERCISE 9.1

1 (a) 0.7248 (b) 0.8868 (c) 0.7680 (d) 0.3789

2 The given function is below the x-axis some of the time, so it is not a valid probability density function.

3 (a) $\frac{2}{27}$ (b) $\frac{7}{27}$ (c) $\frac{1900}{2197}$ (d) $\frac{492\,500}{970\,299}$

4 (a) 0.0563 (b) 0.0807 (c) 0.8693 (d) 0.8310

5 150 6 C

7 (a) 0.16 (b) 0.65 (c) 0.2744

8 B

9 (a) $k = \frac{1}{6}$ (b) $\frac{1}{2}$ (c) $\frac{5}{6}$

10 $\frac{\pi}{2}$

11 (a) 0.5192 (b) 0.2903

12 0.5293 13 100

14 (a) 3.375 m (b) 2.193 m (c) 3.126 m

15 (a) $C = \frac{1}{10}$

(b) (i) 0.2212 (ii) 0.5273 (iii) 0.3221

(c) $E(X) = 10$, $Var(X) = 100$

(d) 63.21% of all calls are shorter than the average call time of 10 minutes.

16 (a) $k = 12$ (b) mean: 0.65 s; median: 0.636 s

(c) (i) 0.949 (ii) 0.649 (iii) 0.313

17 (a) $f(x)$ is on or above the x-axis, hence $f(x) \geq 0$ and the area under the graph is 1, so $\int_{-\infty}^{\infty} f(x)dx = 1$.

(b) $f(x) = \begin{cases} 3x & 0 \leq x \leq \frac{2}{3} \\ -6x + 6 & \frac{2}{3} < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Alternative answers could include 0, $\frac{2}{3}$, 1 in any of the adjacent regions.

(c) $P\left(X < \frac{2}{3}\right) = \frac{2}{3}$, $P\left(X > \frac{2}{3}\right) = \frac{1}{3}$

(d) $P\left(X < \frac{2}{3}\right) = \frac{2}{3}$ and $P\left(X > \frac{2}{3}\right) = \frac{1}{3}$

Regardless of the method used (triangles or integration), the answers should be the same.

(e) $P\left(\frac{1}{2} < X < \frac{5}{6}\right) = \frac{13}{24}$

18 (a) A possible function is

$$f(x) = \begin{cases} -0.006x(x - 10) & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(b)

Minutes late	0–<2	2–<4	4–<6	6–<8	8–<10
Probability	0.1	0.3	0.34	0.2	0.06
Calculated probability	0.104	0.248	0.296	0.248	0.104

The model is symmetrical where the original had lower values to the right.

The model shares the probability more evenly, so has a lower peak, and higher values at the extreme.

(c) 0.216

EXERCISE 9.2

1 (a) (i) 0.4602 (ii) 0.3085 (iii) 0.0062

(b) (i) 0.1524 (ii) 0.6247 (iii) 0.9270

2 (a) 8 to 12 (b) 6 to 14 (c) 4 to 16

3 (a) D

(b) The student has given $N(\mu, \sigma)$ instead of $N(\mu, \sigma^2)$.

4 (a) 0.0923 (b) 0.6904 (c) 0.7074

(d) 0.9295 (e) 0.3345 (f) 0.3057

5 B 6 93

7 (a) 0.9332 (b) 0.6554 (c) 0.8944 (d) 0.6915

8 (a) 0.0217 (b) 0.8343

(c) (i) 0.0152 (ii) 0.8727

9 (a) (i) 97 g and 103 g (ii) 100 g and 108 g

(b) (i) 94 g and 106 g (ii) 96 g and 112 g

(c) (i) 91 g and 109 g (ii) 92 g and 116 g

(d) Machine B is preferable, producing on average a mass 4 g higher. You might prefer machine A if you want less variability.

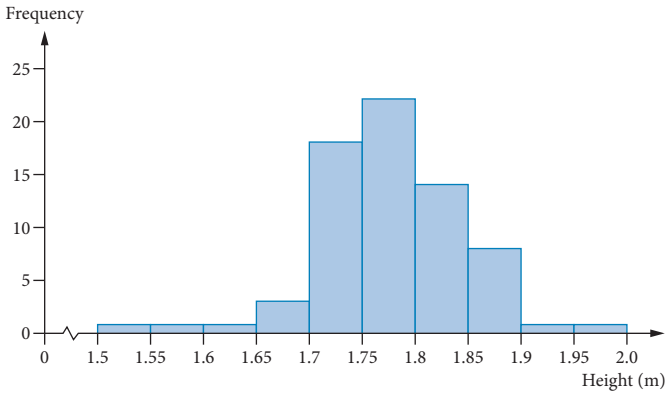
10 (a) 39 to 51 seconds (b) 16% (c) 0.15%

11 (a) 95% (b) 0.15% (c) 50% (d) 83.85%

12 16%

13 **(a)** 2.28% **(b)** 0.0455 **(c)** $516 < k < 517$

14 **(a)**



(b) The histogram displays a distinct bell shape, although it is not perfectly symmetrical. Therefore, it would seem reasonable to suggest that the sample does follow a normal distribution.

(c) mean: 1.765 m; standard deviation: 0.074 m

(d) **(i)** 75.7% **(ii)** 94.3% **(iii)** 98.6%

(e) The portion within one standard deviation of the mean is too high, at 75.7% instead of 68%. While the distribution exhibits the shape of a bell curve, it would not be correct to conclude that the data is normally distributed. It might be suitable to use the normal model to estimate further results, only if no other model is available.

15 **(a)** 0.9522

(b) The probability of being more than 0.4 g above the label weight, 0.6306, is greater than the probability of being within 0.4 g of the label weight, 0.3681.

(c) 0

16 **(a)** Lester's score is better than Charlie's when each is compared to his own overall performance. Lester does this performance or better 7% of the time, whereas Charlie does this performance or better 23% of the time. Also, Lester's score is a greater number of standard deviations below the mean, 1.67 compared to Charlie's score.

(b) 77

17 88%

EXERCISE 9.3

1 **(a)** $z = 1$ **(b)** $z = 2$ **(c)** $z = 2\frac{1}{3}$

(d) $z = \frac{1}{3}$ **(e)** $z = -1$ **(f)** $z = -2.5$

2 0.65 **3** 0.15

4 **(a)** $P(X > a) = 0.786$ **(b)** $P(X > b) = 0.504$

(c) $P(a < X < b) = 0.282$

5 **(a)** $P(Z > 0.85) = 0.1977$ **(b)** $P(Z < -0.85) = 0.1977$

(c) $P(Z > -0.85) = 0.8023$

(d) $P(-0.85 < Z < 0.85) = 0.6046$

6 **(a)** $P(Z < -1.5) = 0.0668$ **(b)** $P(Z > -1.5) = 0.9332$

(c) $P(Z < 1.5) = 0.9332$

7 C **8** B

9 **(a)** 68.27% of the results lie within one standard deviation of the mean.

(b) 95.45% of the results lie within two standard deviations of the mean.

(c) 99.73% of the results lie within three standard deviations of the mean.

10 **(a)** 0.9977

(b) **(i)** 0.33 standard deviations above the mean

(ii) 0.83 standard deviations below the mean

11 Felipe did better on the second test.

12 **(a)** **(i)** $a = 74.9$ **(ii)** $a = 50.4$ **(iii)** $a = 67.9$

(b) **(i)** $b = 8.6$ **(ii)** $b = 16.0$ **(iii)** $b = 10.9$

13 **(a)**
$$P\left(-1 < \frac{X-\mu}{\sigma} < 1\right) \approx 68\%$$
$$P(-\sigma < X - \mu < \sigma) \approx 68\%$$
$$P(\mu - \sigma < X < \mu + \sigma) \approx 68\%$$

(b)
$$P\left(-2 < \frac{X-\mu}{\sigma} < 2\right) \approx 95\%$$
$$P(-2\sigma < X - \mu < 2\sigma) \approx 95\%$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 95\%$$

(c)
$$P\left(-3 < \frac{X-\mu}{\sigma} < 3\right) \approx 99.7\%$$
$$P(-3\sigma < X - \mu < 3\sigma) \approx 99.7\%$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 99.7\%$$

14 **(a)** moderate acute protein-energy malnutrition

(b) 7.9 kg **(c)** 8.7 kg

15 **(a)** 0.9796; 2.045 standard deviations below the mean

(b) 0.0062, $m = 99.5$ There is the same probability of scoring 100% as there is of scoring below 45%.

(c) The minimum mark for A^+ is 86% if it is possible to distinguish between students with the same rounded score. If everyone with 86% is included, more than 10% will receive an A^+. If this is not allowed, and scores are rounded, then 87% would be the minimum score for A^+, with less than 10% having this score.

(d) A mark of 86% would have been awarded an A^+ in the previous year's examination.

EXERCISE 9.4

1 **(a)** 1.99 m **(b)** $k = 1.62$ m **(c)** 1.63 m, 1.77 m

2 **(a)** 57.59 kg **(b)** 69.85 kg

3 **(a)** $z = 0.842$ **(b)** $z = 0.253$ **(c)** $z = -0.524$

4 **(a)** $z = -0.253$ **(b)** $z = 0.915$

(c) $z = 0.468$ **(d)** $z = -1.476$

5 (a) A (b) D (c) C

(d) The student has used the standard normal distribution.

6 (a) $k = 1.44$ (b) $k = 1.96$

7 (a) 71 to 79

(b) 67.4

(c) 79.98

8 C

9 (a) 89 (b) 86 (c) 82

10 (a) $k = 3.188$ kg (b) $k = 1.863$ kg

(c) $k = 2.003$ kg (d) $k = 2.870$ kg

11 (a) $t = 0.3144$ (b) $t = 0.4646$ (c) $t = 0.7202$

12 (a) 0.6234 (b) 0.1789 (c) 0.2643

13 (a) 165 cm (b) 170 cm (c) 80 cm

14 (a) 5 cm (b) 6 cm (c) 3 cm

15 37.4 mm **16** $t = 0.0354$

17 (a) 0.0580 (b) 329 ms (c) 55 ms

18 (a) less than 54.2 s (b) 7.6 s

19 (a) $\mu = 129.7$

(b) standard deviation for X: $\sigma = 3.12$; standard deviation for Y: $\sqrt{2}\sigma = 4.41$; 124 is the 10th percentile for Y.

CHAPTER REVIEW 9

1 (a) $k = 0.1$ (b) $P(X < 3) = 0.3$

(c) $P(X < 5) = 0.5$ (d) $P(2 < X < 9) = 0.7$

2 (a) $\frac{1}{25}$ (b) $\frac{8}{9}$

3 (a) 0.4219 (b) 0.1225 (c) 0.8992

4 $f(x)$ does not represent a probability density function because $f(x)$ is not ≥ 0 for all values of the domain.

5 $k = -6$

6 (a) 0.16 (b) 23 jars

7 $N(120, 9^2)$ or $N(120, 81)$

8 B **9** D

10 (a) 0.6915 (b) 0.9938 (c) 0.9570

11 (a) (i) $z = -1$ (ii) $z = -0.5$ (iii) $z = 0.75$ (iv) $z = 2$

(b) (i) $z = -1$ (ii) $z = \frac{7}{4}$ (iii) $z = -\frac{11}{4}$

12 (a) 0.678 (b) 0.485 (c) 0.193

13 (a) $z = 0.126$ (b) $z = -0.253$ (c) $z = -0.674$

14 (a) 27.08 and 32.92 (b) 35.70

15 (a) $k = 126.55$ (b) $k = 115.73$

16 (a) 78 (b) 1.61% (c) 85 to 95

17 $a = 35.38$, $b = 44.62$

18 $\sqrt[3]{18}$

19 (a) It is reasonable because $f(x) \geq 0$ for the values in the domain and the integral equals 1.

(b) (i) $P(X < 1) = 0.75$ (ii) $P(X > 1.25) = 0.1443$

(iii) $P(0.25 < X < 1.1) = 0.5615$

(c) (i) $E(X) = 0.6619$ (ii) $Var(X) = 0.2159$

20 (a) Xandra: 1.75 hours; Zack: 1.03 hours; Xandra spends, on average, 43 minutes more completing homework than Zack.

(b) 0.375

21 D

22 (a) (i) 0.9648

(ii) Normal distribution: 0.9643. This value is 0.0005 lower than the binomial distribution estimate.

(b) (i) 0.3750

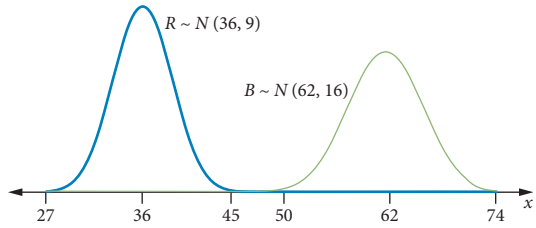
(ii) Normal distribution: 0.3780. This value is 0.0030 higher than the binomial distribution value.

23 (a) $a = 222.0$ (b) $b = 18.9$

24 D

25 (a) 62 cm

(b)



(c) $z_R = -2$, $z_B = -1.75$ The proportion of red snapper that must be thrown back is smaller than the proportion of barramundi that must be thrown back.

(d) (i) 0.00012 (ii) 0.16

26 (a) $a = \frac{1}{4}$, $f(h) = \begin{cases} -\frac{1}{4}h + \frac{3}{5} & 0 \leq h \leq 2 \\ \frac{1}{10} & 2 < h \leq 5 \\ 0 & \text{otherwise} \end{cases}$

(b) (i) 0.24 (ii) 1.58 minutes

(iii) 1.07 minutes; Helena completes half of the exercises in less than 1.07 minutes and the other half of the exercises takes between 1.07 minutes and 5 minutes.

(iv) 0.0086

27 (a) 0.1631 (b) 0.024 (c) 139.6 g and 160.4 g

28 (a) $k = \frac{3}{64}$

(b) mean: 160 kg; standard deviation: 80 kg

(c) $\frac{13}{256}$ (d) \$489.84

Chapter 10: Interval estimates for proportions

RECALL

1 (a) discrete (b) continuous
(c) discrete (d) continuous

2 (a) $0 \leq P(X) \leq 1$ (b) Sum$[P]\,(X = x) = 1$
(c) mean: $E(X) = \sum x_i p_i$
variance: $Var(X) = E\left(X^2\right) - \left[E(X)\right]^2$
standard deviation: $\sigma = \sqrt{Var(X)}$

3 (a) $E(X) = 2.3$, $Var(X) = 0.92$, $\sigma(X) = 0.96$
(b) $E(X) = 8.9$, $Var(X) = 80.19$, $\sigma(X) = 8.95$
(c) $E(X) = 8.7$, $Var(X) = 6.68$, $\sigma(X) = 2.58$

4 (a) independent (b) dependent (c) independent

5 (a) $E(X) = 2$, $Var(X) = 1.6$, $\sigma(X) = 1.26$
(b) $E(X) = 65$, $Var(X) = 22.75$, $\sigma(X) = 4.77$
(c) $E(X) = 25$, $Var(X) = 12.5$, $\sigma(X) = 3.54$

6 (a) ${}^{10}C_4 = 210$ (b) ${}^{15}C_{10} = 3003$ (c) ${}^{30}C_3 = 4060$

7 (a) 68% is captured by ±1 standard deviation from the mean. 95% is captured by ±2 standard deviations from the mean. 99.7% is captured by ±3 standard deviations from the mean.
(b) One standard deviation: 85 to 115. Two standard deviations: 70 to 130. Three standard deviations: 55 to 145.

EXERCISE 10.1

1 (a) Population: all students who have graduated from QUT. Sample: the selection of the previous year's graduate students from QUT. Parameter: the average time it takes for QUT students to complete their degree. Statistic: the average time for the selection of last year's graduates to complete their degree.
(b) Population: all QAMT members. Sample: the 400 selected QAMT members. Parameter: the unknown proportion of QAMT members who would prefer to receive publications via email. Statistic: the 280 out of 400 (or 70%) of the sample.
(c) Population: all coin flips with this particular coin. Sample: the 100 coin flips. Parameter: the percentage of all coin flips that would result in heads, assume 50% if the coin is fair. Statistic: the 47% that resulted in heads from the 100 coin flips in this trial.

2 (a) undercoverage bias, nonresponse bias, response bias
(b) sample frame error, undercoverage bias, nonresponse bias
(c) response bias, nonresponse bias

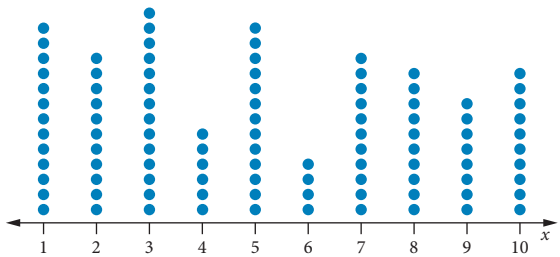
3 C 4 A

5 Yes, because the students were chosen at random.

6 (a) This sampling method would not produce a random sample. There is a sample frame error and undercoverage bias. To limit the bias, a correct sample frame would be required to target the correct population. Everyone in the selected gyms should have an equally likely chance of being selected.
(b) This sampling method would not produce a random sample. By limiting the target population to people in shopping centres, there is undercoverage bias. The results could be skewed by excluding people who cannot go to shopping centres. Electoral rolls could be used and people could be interviewed by phone, email and mail in order to limit the bias.
(c) This sampling method would produce a random sample of businesses with an internet presence. There could be a nonresponse bias as some companies may refuse to participate in the survey.

7 Allocate a unique identification number to every student, their parents and all teachers. Generate a random sample of numbers and matching these numbers, survey those people. If a number is selected twice, disregard that number and go to the next one. There is a possibility of certain groups being underrepresented due to the nature of random samples. Another method could be to consider a stratified sample and then within that, generate random samples. There is also the risk of nonresponse from this survey.

8 (a) Since all values from 1 to 10 are equally likely to occur, it would be rare to see any obvious peaks or troughs in the graph. The shape is likely to be more or less rectangular. Your answer will depend on the values of your random numbers. An example is shown.



(b) Your answer will depend on your random list of integers.

9 The higher the value of n, the closer the sample mean is likely to be to the theoretical expected value of the sample mean, EV = 5.5.

EXERCISE 10.2

1 B 2 A 3 0.1448
4 0.7807 5 0.1714
6 (a) 0.9393 (b) 0.6972 (c) 0.9344
7 (a) mean: 3.5, standard deviation: 1.7 (b) 0.7962

8 **(a)**

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(b) mean: $\frac{3}{2} = 1.5$, variance: $\frac{3}{4} = 0.75$ **(c)** 0.2326

(d) The expected number for one group of 3 is 1.4, so for 40 groups it will be 60. The probability that the number of heads is greater than 60 would be 0.5; therefore, the probability that the number of heads is greater than 70 would be less than 0.5, so Alexander's statement must be incorrect.

(e) 0.0339

9 **(a)** 0.221 **(b)** 0.399 **(c)** 0.380

(d) **(i)** 0.138 **(ii)** 0.091

(iii) A sample size of $n = 280$ yields a probability of $P(\overline{X} < 3.01) \approx 0.099 \approx 9.9\%$. This is very close to the target probability of 10%.

EXERCISE 10.3

1 **(a)** The population proportion is the proportion of all Northern Territory email users who have received a fake email. The sample size is 800. The observation value is 336. The sample proportion is 0.42.

(b) The population proportion is the proportion of Tasmanian Holden owners who prefer a medium- to large-sized car. The sample size is 2500. The observation value is 1893. The sample proportion is 0.76.

2 $E(\hat{P}) \approx 0.615$, $S(\hat{P}) = 0.048$

3 Conditions for normality are met.

4 **(a)** $E(\hat{P}) \approx 0.39$ (2 d.p.), $S(\hat{P}) \approx 0.01$ (2 d.p.)

(b) $E(\hat{P}) \approx 0.83$ (2 d.p.), $S(\hat{P}) \approx 0.017$ (3 d.p.)

(c) $E(\hat{P}) \approx 0.05$ (2 d.p.), $S(\hat{P}) \approx 0.007$ (3 d.p.)

5 **(a)** yes **(b)** no **(c)** yes

6 B **7** A

8 **(a)** 130 **(b)** 2000 **(c)** 18

9 **(a)** 850 **(b)** 2 **(c)** 4900

10 **(a)** $E(\hat{P}) \approx 0.75$ and $S(\hat{P}) \approx 0.014$. The distribution can be approximated by a normal distribution.

(b) $E(\hat{P}) \approx 0.1$ and $S(\hat{P}) \approx 0.042$. The distribution can be approximated by a normal distribution.

(c) $E(\hat{P}) \approx 0.64$ and $S(\hat{P}) \approx 0.002$. The distribution can be approximated by a normal distribution.

11 $0.25 \leq \hat{p} \leq 0.75$

12 **(a)** $n \geq 50$ **(b)** $n \geq 50$

13 **(a)** The sample proportion can be different for each survey. However, there may be times when the sample proportion value will be the same as the average value of sample proportions.

(b) A sample proportion of zero would be virtually impossible. It could mean that there was an error with the sampling process or that bias had occurred.

(c) It is virtually impossible. A large difference in sample proportion values could indicate a problem with the sampling process or bias.

(d) With such a large sample size, the variability from one sample to another is very small, hence it is expected that most values will be close to 0.78.

(e) The sample proportion value may not be representative of the true population proportion if a simple random sample is not used, if the sample is too small $(n < 30)$ with a proportion close to 0 or 1, or if there is a bias skewing the results.

14 **(a)** The sample proportion varies because a new sample is chosen each time. If the sample proportion were the same every time, then it could indicate bias, incorrect sampling methods or a possible sampling frame error. However, there may be times where the sample proportion value will be the same.

(b) A sample proportion of 1 would be virtually impossible since n is so large. A sample proportion of 1 means that 100% of a sample experienced blood clots after taking heparin.

(c) It could indicate a problem with the sampling process or bias. It could be that those particular patients have not had a heparin shot and that there is an error within the records.

(d) A large value of n compensates for the small sample proportion value of $\hat{p}$, where $n\hat{p} \geq 5$ and $n(1 - \hat{p}) \geq 5$.

(e) If a simple random sample is not used, if the sample is too small with a proportion close to 0 or 1, or if there is a bias skewing the results

15 **(a)** $x_S = 60$, $x_I = 40$ and $x_D = 20$

(b) $\hat{p}_S = 0.5$, $\hat{p}_I = 0.33$, $\hat{p}_D = 0.17$

(c) The point estimate does not change, because the proportion of observations doesn't change.

(d) The point estimate becomes greater because the proportion of results is greater, which produces a higher observation value.

EXERCISE 10.4

1 **(a)** 0.037 **(b)** 0.057

2 0.017

3 **(a)** $E = 0.047$, $E(\hat{P}) = 0.60$ **(b)** $E = 0.045$, $E(\hat{P}) = 0.45$

(c) $E = 0.022$, $E(\hat{P}) = 0.77$

4 A **5** 1067

6 The higher the percentage of the confidence interval, the wider the spread of values that must be included. Therefore, a confidence interval of 95% is wider than a confidence interval of 90%. Elwood is correct.

7 (a)

$\hat{p}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
E	0.059	0.078	0.090	0.096	0.098	0.096	0.090	0.078	0.059

(b) no, because $n\hat{p} < 5$

8 5336

9 $(0.092, 0.158)$

10 $(0.769, 0.791)$

11 (a) $(0.748, 0.792)$ (b) $(0.739, 0.801)$

12 (a) $(0.138, 0.382)$ (b) $(0.100, 0.420)$

13 $(0.355, 0.445)$

14 (a) Yes, normality applies. (b) $(0.355, 0.417)$

(c) yes (d) 7373

15 98% confidence level

16 (a) p = all Brisbane secondary school students preparing for the upcoming exam; $n = 1500$; $x = 720$; $\hat{p} = 0.48$

(b) 0.0129

(c) Conditions for normality are met; therefore, a confidence interval can be constructed.

(d) $(0.450, 0.510)$ (e) 3%

(f) (i) No, the formula has $\sqrt{n}$ in the denominator, so doubling n will reduce the value of E by a factor of $\frac{1}{\sqrt{2}} \approx 0.7$.

(ii) 6000 people

(iii) sample size larger than 3376 people

17 (a) Normality applies, hence a confidence interval can be constructed.

(b) Janette's results: $(0.473, 0.647)$; Brooke's results: $(0.539, 0.709)$ The confidence interval is larger for Brooke's set of data because the sample proportion is greater for Brooke's set than for Janette's set.

(c) The margin of error for Janette's set is slightly larger than the one for Brooke's set because the standard error of Janette's set is slightly larger than that of Brooke's set.

(d) Janette should increase the sample size.

18 (a) $\hat{p} = 0.06$ (b) $E = 0.0465$

(c) 95% confidence level = $(0.8935, 0.9865)$, number of bottles = $(90, 98)$

(d) 542 bottles

(e) The 95% confidence interval is $(0.92, 0.96)$ or $(92, 96)$ bottles. A margin of error of 0.02 means that the difference between the sample mean and the true population mean will be within 2 bottles of dishwashing liquid at least 95% of the time. A 95% confidence interval of $(92, 96)$ means that you can be 95% confident that the population mean number of dishwashing liquid bottles containing between 745 mL and 755 mL will be between 92 and 96 bottles.

CHAPTER REVIEW 10

1 (a) Population: the waiting times of every person in 2013 who attended the emergency department in a public hospital. Sample: 5000 patients selected. Parameter: the true average waiting time in the emergency department at public hospitals. Statistic: 6 hours

(b) Population: every packet of coloured, coated chocolate buttons that is manufactured. Sample: 50 selected packets. Parameter: the proportion of each colour in every packet. Statistic: the proportion of each colour calculated from the 50 selected packets

2 (a) nonresponse bias, undercoverage

(b) nonresponse bias

3 0.0243

4 (a) Population proportion: the proportion of people who feel that they do not use their gym membership to its fullest potential. Sample size: $n = 545$. Observation value: $x = 324$. Sample proportion: $\hat{p} = 0.59$

(b) Population proportion: the proportion of all households who own more than one vehicle. Sample size: $n = 10\,000$. Observation value: $x = 3333$. Sample proportion: $\hat{p} = 0.33$

5 (a) (i) $E(\hat{P}) = 0.27$, $S(\hat{P}) = 0.024$

(ii) Conditions for normality are met.

(b) (i) $E(\hat{P}) = 0.9$, $S(\hat{P}) = 0.095$

(ii) Conditions for normality are not satisfied.

6 (a) about 34 (b) 8000 (c) 450 (d) 657

7 16 589 times 8 3.7%

9 The width of the confidence interval decreases.

10 A census gathers data from the whole population, whereas a sample is part of the population. A census can be time-consuming and very expensive.

11 The sample size must be large enough so that it accurately represents the population. Choosing a sample size that is too small may not give an accurate representation of the population distribution. Too large a sample size is wasteful and sometimes impossible to complete.

12 The simple random sample requires less knowledge about the population than other techniques, but it does have two major drawbacks. One is the fact that if the population is large, a great deal of time must be spent listing and numbering the members. The other is that a simple random sample will not adequately represent many population attributes (characteristics) unless the sample is relatively large.

13 784

14 (a) $(0.101, 0.245)$ (b) $(0.087, 0.259)$
(c) $(0.071, 0.275)$ (d) $(0.061, 0.286)$

15 (a) (i) $\hat{p} = 0.15, S(\hat{P}) = 0.056$ (ii) $\hat{p} = 0.15, S(\hat{P}) = 0.036$
(iii) $\hat{p} = 0.15, S(\hat{P}) = 0.019$ (iv) $\hat{p} = 0.15, S(\hat{P}) = 0.016$
(b) Even as the sample proportion stays the same, if you increase the sample size, the standard error decreases.

16 $(0.374, 0.768)$

17 (a) $(0.539, 0.661)$
(b) 'More than half' indicates 'greater than 0.5'. The 95% confidence interval of $(0.539, 0.661)$ has all values greater than 0.5, so the teacher's statement is supported.

18 (a) $(0.696, 0.788)$
(b) Between 69.6% and 78.8% of students would have chosen different subjects.
(c) $(1462, 1655)$

19 (a) $x = 59, \hat{p} = 0.039$
(b) $E_{90\%} = 0.0157, E_{95\%} = 0.0187, E_{98\%} = 0.0222$
(c) The 90% confidence interval is $(0.8203, 0.8517)$. The 95% confidence interval is $(0.8173, 0.8547)$. The 98% confidence interval is $(0.8138, 0.8582)$.

20 (a) $\hat{p} = 0.62$ and $E = 0.246$ (b) 1500 (c) $z = 1.96$

21 (a) companies who beat their estimated profit: $\hat{p} = 0.651$; companies who matched their estimated profit: $\hat{p} = 0.151$
(b) $(0.140, 0.256)$ (c) 0.0577 (d) 344
(e) The confidence interval $(0.148, 0.248)$ is smaller than the confidence interval $(0.140, 0.256)$ because the margin of error allowed is smaller.

Exam review: Unit 4

PAPER 1: TECHNOLOGY-FREE

1 $f''(-2) = -28$

2 (a) $y = 32 - 2x$ (b) $A(x) = 2x(16 - x)$
(c) $128\,\text{m}^2$ (d) width: 8 m; length: 16 m

3 $x = 20\sqrt{6}$ cm

4 $100\sqrt{401 - 20\sqrt{3}}$ m

5 $\dfrac{7 \times 2^{13}}{5^8}$

6 mean: $\dfrac{3}{8}$; variance: $\dfrac{15}{64}$

7 $P(X < 2) = \dfrac{2 - \sqrt{2}}{4}$

8 (a) $P(X > a) = 0.82$ (b) $P(X > b) = 0.55$
(c) $P(a < X < b) = 0.27$

9 (a) nonresponse bias
(b) undercoverage bias, response bias

10 Population proportion: the proportion of first-year teachers who feel that they have received adequate mentoring to teach their assigned classes. Sample size: $n = 600$. Observation value: $x = 300$. Sample proportion: $\hat{p} = 0.50$

11 (a) $f'(x) = x^2 - 2x - 15$ (b) $f'(-2) = -7$
(c) $(0, 0)$: -15; $\left(\dfrac{3 + 3\sqrt{21}}{2}, 0\right)$: $\dfrac{3}{2}\left(21 + \sqrt{21}\right)$;
$\left(\dfrac{3 - 3\sqrt{21}}{2}, 0\right)$: $\dfrac{3}{2}\left(21 - \sqrt{21}\right)$
(d) $-3 \leq x \leq 5$

12 $\dfrac{20}{\sqrt{3} - 1}$ m or $10\left(\sqrt{3} + 1\right)$ m

13 (a)

x	1	2	3	4	5	6
$P(X = x)$	$\frac{5}{12}$	$\frac{1}{36}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{5}{36}$

(b) (i) $\dfrac{31}{36}$ (ii) $\dfrac{1}{9}$ (iii) $\dfrac{1}{6}$

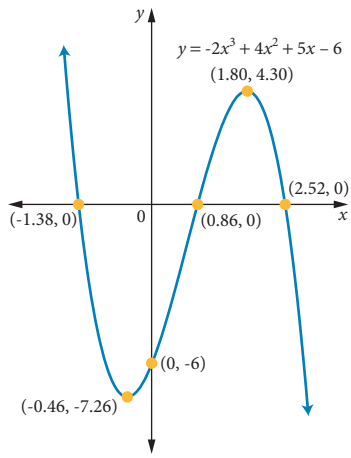
14 (a) 8 cm (b) 11 cm (c) 2.3 cm

15 The number of people surveyed by Company B is 16 times greater than the number surveyed by Company A.

16 (a) $d = a\cos(\alpha)$; $d = b\cos(\beta)$ (b) area $= \dfrac{1}{2}bd\sin(\beta)$
(c) area $= \dfrac{1}{2}ad\sin(\alpha)$ (d) area $= \dfrac{1}{2}ab\sin(\alpha + \beta)$
(e) $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
(f) $\dfrac{\sqrt{3} + 1}{2\sqrt{2}}$ or $\dfrac{\sqrt{6} + \sqrt{2}}{4}$

PAPER 2: TECHNOLOGY-ACTIVE

1 (a) $x = \dfrac{4 - \sqrt{46}}{6}$, $x = \dfrac{4 + \sqrt{46}}{6}$
(b)



(c) increasing: $-0.46 < x < 1.80$; decreasing: $x < -0.46$ and $x > 1.80$

2 (a) The particle is initially 7 cm to the right of the origin, slowing down from 10 cm/s as it travels away from the origin.
(b) $-10\,\text{cm/s}^2$ (c) -3 cm/s (d) 0.72 s

3 12.06°

4 14.79 km

5 (a) $E(X) = 14.04$ (b) $Var(X) = 0.6184$ (c) $\sigma = 0.79$

6 $E(X) = 29.25$, $Var(X) = 10.24$, $\sigma(X) = 3.20$

7 (a) 0.1016 (b) 0.6972

8 (a) 125.7 cm (b) $k = 119.1$ cm

(c) 122.3 cm, 123.9 cm

9 (a) $S(\hat{P}) = 0.021$

(b) Both criteria are met; therefore, conditions for normality are satisfied.

10 267

11 (a) $L(x) = \sqrt{x^2 + 81} + \sqrt{x^2 - 40x + 404}$

(b) 16.36 m from the 3 m post (c) 22.8 m

12 $c = 20.7$, $B = 45.8°$, $C = 99.2°$; or $c = 3.9$, $B = 134.2°$, $C = 10.8°$

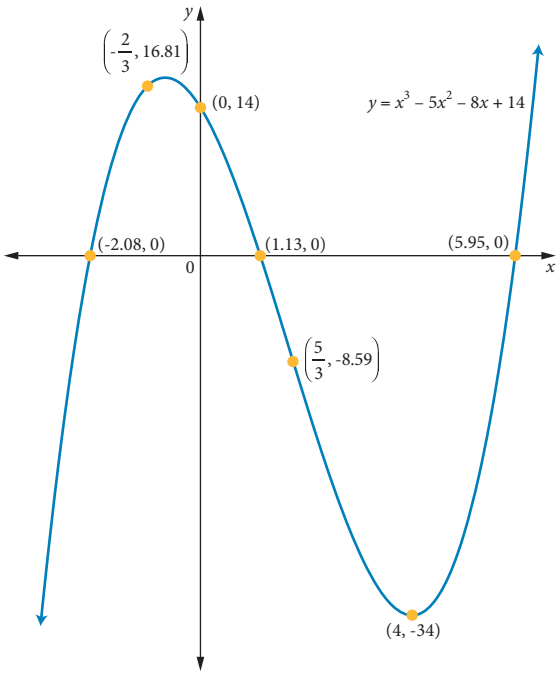
13 (a) 0.737 (b) 0.484

14 (a) 43 minutes

(b) (i) 0.97 (ii) 0.35

15 (a) (0.148, 0.302) (b) (0.133, 0.317)

16 $f(x) = x^3 - 5x^2 - 8x + 14$



17 0.64

Exam review: Units 3 & 4

PAPER 1: TECHNOLOGY-FREE

1 $x = 2 - \log_e(5)$

2 (a) reflection in the y-axis

(b) dilation of factor 3 parallel to the y-axis and a translation of 7 units down

3 (a) $\frac{dy}{dx} = 2e^x + \frac{1}{e^x}$ (b) $\frac{dy}{dx} = e^x$ (c) $\frac{dy}{dx} = -\frac{4}{e^{x+1}}$

4 (a) left (b) 1 cm (c) $\frac{1}{90}$ s and $\frac{1}{18}$ s

5 (a) $\frac{dy}{dx} = x^2(3\sin(x) + x\cos(x))$

(b) $\frac{dy}{dx} = e^{3x}\left(3\tan(2x) + \frac{2}{\cos^2(2x)}\right)$

6 (a) $\frac{dy}{dx} = \frac{12x^3 + 2}{3x^4 + 2x}$ (b) $\frac{dy}{dx} = 5\log_e(3x) + \frac{5x - 2}{x}$

7 (a) $\frac{x^4}{2} - \frac{3x^2}{2} + \frac{4x^{\frac{3}{2}}}{3} + c$ (b) $2x^2 - \frac{1}{2}\sin\left(2x + \frac{\pi}{3}\right) + c$

(c) $\frac{1}{8(3 - 4x)^4} + c$ (d) $\frac{e^{2x}}{2} + \frac{e^{-2x}}{2} - \frac{e^{-5x}}{5} + c$

8 (a) $y = -\frac{4}{7}\log_e|3 + 7x| + 2$ (b) $y = \frac{x^3}{6} - \frac{5x^2}{2} + \frac{25x}{2}$

(c) $y = \frac{(2 + 14x)^{\frac{1}{2}}}{7} - \frac{4}{7}$

9 (a) $f(x) = 2\cos(3(x - \pi)) + 4$

(b) $f(x) = -e^{1-x} + 3e$

(c) $-\log_e(5 - 2x) + 3\log_e(5)$ or $\log_e\left(\frac{5^3}{5 - 2x}\right)$

(d) $\frac{1}{2}\sin\left(2x + \frac{\pi}{6}\right) + 2$

10 $y = \frac{1}{2}(2x - 5)^4 - \frac{1}{2}$

11 (a) $2\sqrt{2}$ (b) $-\frac{\pi^3}{4}$ (c) $\frac{26}{3}$ (d) $\frac{3}{2}\log_e(2)$

12 global minimum: -11; global maximum: 16

13 (a) $v(t) = -3t^2 + 6t + 18$ (b) $a(t) = -6t + 6$

(c) 6 m/s^2

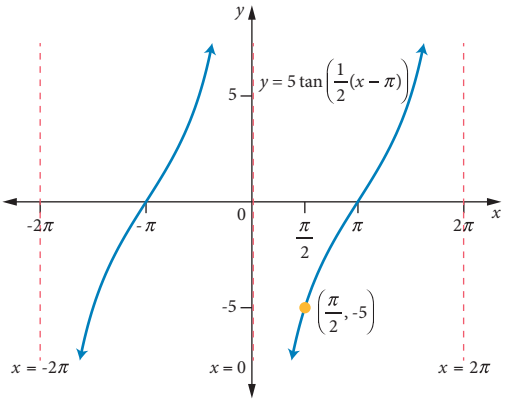
(d) after 1 second; 20 metres to the right of the origin

(e) 21 m/s after 1 second

14 Population: all NSW Nurses Cooperative members. Sample: 2000 selected nurses. Parameter: an unknown proportion of the NSW Nurses Cooperative who have worked overtime at least once in the previous two weeks. Statistic: the proportion of the 200 nurses who worked overtime at least once in the previous two weeks

15 Population proportion: the proportion of all Perth residents over the age of 18 years who own a smartphone. Sample size: $n = 730$. Observation value: 487. Sample proportion: $\hat{p} = 0.67$ (to 2 d.p.)

16



17 $x = 5 + \frac{1}{3}\log_e(3),\ y = 5 - \frac{2}{3}\log_e(3)$

18 $y = \frac{3(4x-5)^4}{16} + \frac{13}{16}$

19 $3x^3(3x-2)^2 + c$

20 (a) $-\frac{8}{3}(4-3x)^{\frac{3}{2}} + c_1;\ \frac{8}{3}(4-3x)^{\frac{3}{2}} + c_2$

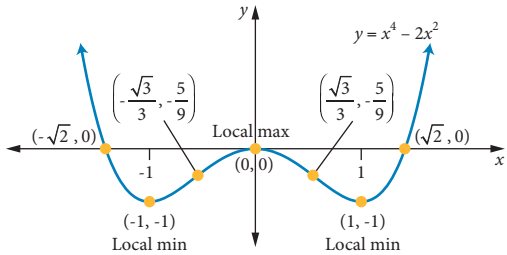
(b) $-\frac{8}{3}(4-3x)^{\frac{3}{2}} + c_1 + \frac{8}{3}(4-3x)^{\frac{3}{2}} + c_2 = c_1 + c_2$, which is a constant and therefore independent of x.

21 $73\frac{5}{6}$ square units

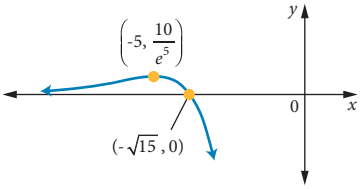
22 (a) local minima: $(-1,-1)$, $(1,-1)$; local maximum: $(0,0)$

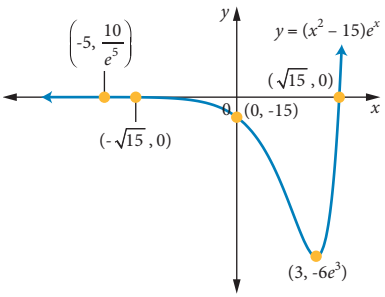
(b) $\left(-\frac{1}{\sqrt{3}}, -\frac{5}{9}\right), \left(\frac{1}{\sqrt{3}}, -\frac{5}{9}\right)$

(c)



23 Featuring the maximum, rescaled





24 (a) $\frac{1}{2}ax^2 + bx + c$ (b) $\frac{1}{4a}(ax+b)^4 + c$

(c) $\frac{1}{a}\log_e|ax+b| + c$ (d) $-\frac{3}{2a}\cos(2(ax+b)) + c$

PAPER 2: TECHNOLOGY-ACTIVE

1 (a) reflection in the x-axis, a dilation of factor 2 parallel to the x-axis, and a translation of 1 unit up

(b) translations of 3 units to the right and 2 units up

2 $27.2\,\text{m}^3$

3 $A = 3,\ b = 2$

4 (a) $\frac{4}{3}\log_e|3x-7| + c$ (b) $\frac{3}{\pi}\sin(1-\pi x) + c$

(c) $3x - 5\log_e|x| + c$ (d) $\frac{4}{5(3-5t)^{\frac{1}{2}}} + c$

5 7 square units

6 $x = 6.3$ units, $\theta = 38.0°$

7 (a) $229.51\,\text{cm}^2$ (b) $145.13\,\text{m}^2$

8 1.5524 **9** 0.046

10 (a) $k = 1.555$ (b) $k = 1.645$

11 $E(\hat{P}) \approx 0.66,\ S(\hat{P}) = 0.0122$

12 (a) 110 trees (b) 41 to 69

13 (a) 2.333 (b) 1.106

14 $(0.72, 0.77)$

15 (a) $\hat{p} = 0.75$ and $S(\hat{P}) = 0.0377$

(b) $(0.1529, 0.3471)$ (c) $(0.6529, 0.8471)$

(d) The two variables have the same margin of error, $E = 0.0971$. The two margins of error are equal because, although the two variables have different values for $\hat{p}$ and $(1-\hat{p})$, the product of the two values is the same: $0.25 \times 0.75 = 0.75 \times 0.25$.

16 $\frac{\sin^3(\theta)}{3} + c$

17 (a) $v(t) = \frac{t^2}{2} - 2e^{\frac{t}{2}} + 22$ (b) $t = 5.99$ seconds

(c) The object accelerates in the negative direction.

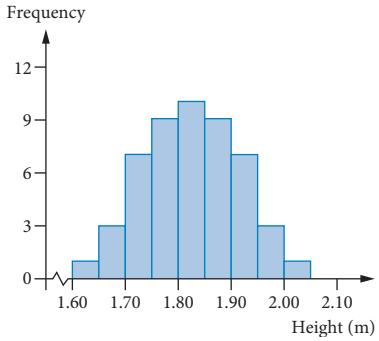
18 $D = 203\,\text{m}$

19 (a) 1.82 m

(b)

Height (m)	Frequency
1.6–<1.65	1
1.65–<1.7	3
1.7–<1.75	7
1.75–<1.8	9
1.8–<1.85	10
1.85–<1.9	9
1.9–<1.95	7
1.95–<2	3
2–<2.05	1

(c) The heights of the 50 trees are normally distributed.



(d) 0.0884 m

(e) actual: 14%; normal model: 13% using $P(1.7 < X < 1.75)$ and 12% using $P(1.695 - < 1.745)$

(f) 37% The closest you could get from the table would be the total of categories 1.8–<1.85 and 1.85–<1.9. If you use the raw data, it has been rounded to the nearest centimetre, so values of 1.82 and 1.92 may or may not be correctly included.

20 (a) The conditions for normality are satisfied.

(b) (360, 424)

(c) Yes, because 125 is in the 90% confidence level (104, 136) for the shareholders who bought shares for the first time in the previous year.

(d) 1534