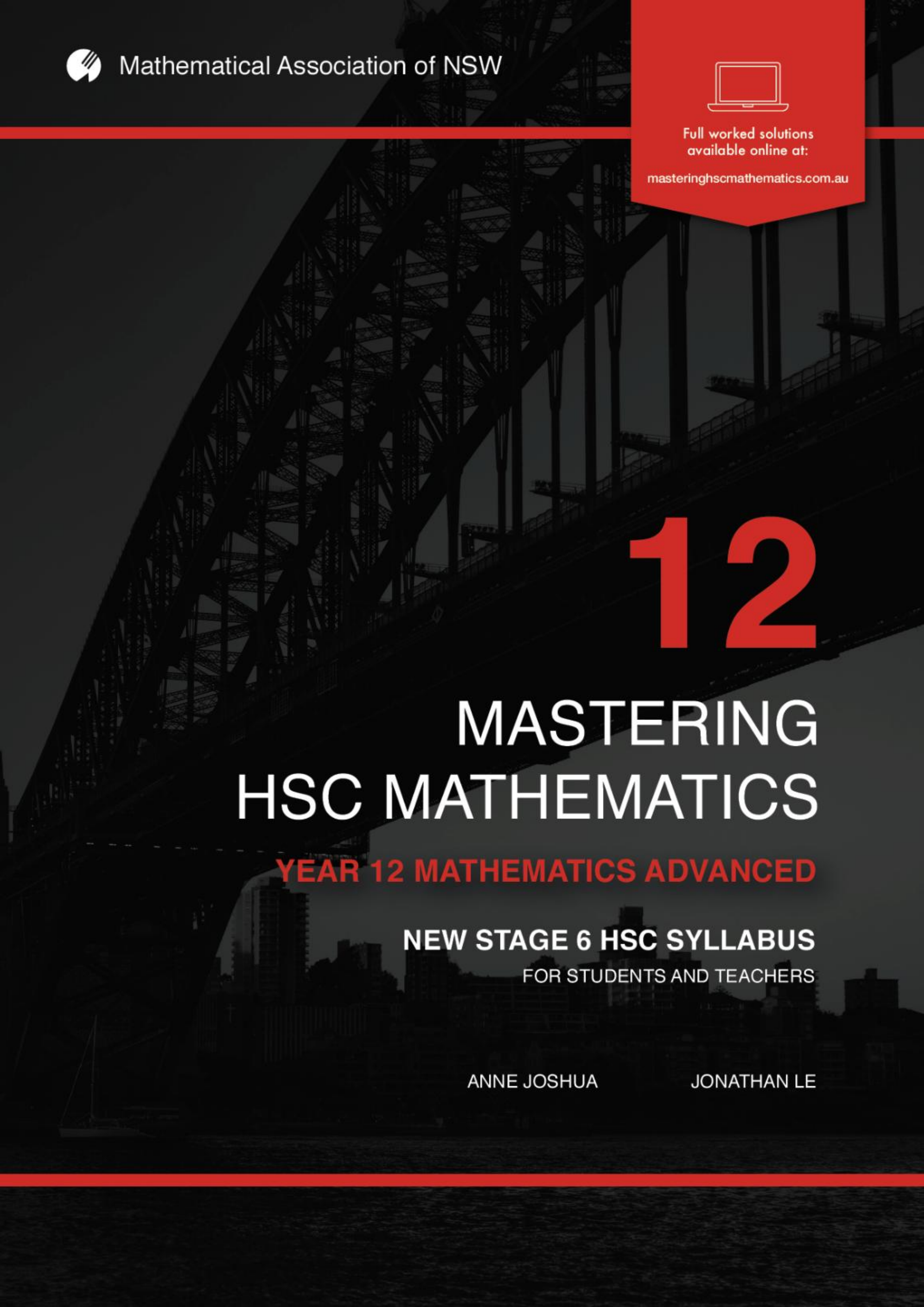


Mathematical Association of NSW

Full worked solutions available online at:

masteringhscmathematics.com.au

12

MASTERING HSC MATHEMATICS

YEAR 12 MATHEMATICS ADVANCED

NEW STAGE 6 HSC SYLLABUS

FOR STUDENTS AND TEACHERS

ANNE JOSHUA

JONATHAN LE

Mathematical Association of NSW

Mathematical Association of NSW
67/73 St Hilliers Road, Auburn, NSW 2144, Australia
Email: authors@masteringhscmathematics.com.au
Website: www.masteringhscmathematics.com.au

National Library of Australia
Mastering HSC Mathematics Advanced Year 12
ISBN: 978-0-9923745-6-3 Paperback

Published in Australia by the Mathematical Association of NSW

1st Edition published August 2019

Cover and interior design by Daisy Chen.
Printing by Round Printing.
Typeset in LaTeX by Jonathan Le, Elio A. Farina, José L. León, John Drake

Errors:
This is a second print-run and many of the errors identified in the first print have been corrected. However, whilst the authors have taken utmost care with the editing and checking of this book, the authors cannot guarantee that all errors have been found and corrected. If you identify an error, please contact the authors directly at authors@masteringhscmathematics.com.au

Features of this book

This book is suitable for all students studying the HSC Mathematics Advanced and HSC Mathematics Extension 1 course. It has been designed in a thoroughly organised manner to help students master each syllabus topic in the new Stage 6 HSC Mathematics Advanced course. This book will teach, consolidate, test and challenge students. It is an essential resource for all students and teachers.

In flavour with the new course, this book has the following features:

- Technology-based questions.
- Interpretation questions.
- Modelling and application problems.
- Verification questions.

Within each chapter, there are subsections divided as follows.

Fundamentals

The carefully constructed *fundamentals* section appears before the main body of questions. The purpose of this section is to

- test all key formulae, definitions, concepts and theory.
- test essential mathematical terms and language through cloze-passages.
- ensure that the student has knowledge of the essential prerequisites.
- provide a summary of basic requirements for the topic.

Questions

This is the main body of questions with the following features.

- Step-by-step questions to assist the student with more difficult problems.
- Carefully graded exercises.
- "Show"-type questions, both guides the student, and offers good exam preparation.
- Proofs and explanations to strengthen understanding and develop problem-solving skills.
- Application questions to demonstrate future uses of learned theory.
- Technology-based questions to teach and reinforce concepts.

Challenge

These are more difficult questions that provide

- a challenge for students wishing to test their mastery of the topic.
- rigour and higher-order thinking skills.
- extension and more in-depth treatment of the unit of work.

Chapter Review

This section appears at the end of every chapter, and offers the following.

- Revision and consolidation of the previous exercises.
- Questions that require a combination of ideas from previous exercises.

Investigations

These tasks are potential assignments and research projects. Teachers may use and adapt these to cover the new NESA requirements on investigative assessment tasks. This section provides for the student

- application and modelling scenarios.
- research tasks involving data collection and analysis.
- scaffolding of learning tasks.
- open-ended style problems for discussions.
- opportunity to use appropriate technology effectively in a range of contexts.
- opportunity for students to demonstrate critical thinking.

Answers

- Quick answers to questions.
- "Show" and "prove" answers can be found in the full worked solutions.

Full worked Solutions

- Can be found online for free, or a full-colour hard copy purchased for convenience.
- Provide complete worked solutions to all questions, except investigative tasks to maintain the open-ended nature of the tasks.
- Includes several alternative solutions to problems, where possible.

Anne Joshua

M.A. Dip.Ed. (Syd)
M.Sc. Mathematics Education (Oxon)

Jonathan Le

B.Sc. Pure Mathematics (Syd)

Acknowledgements

From Anne:
My husband Doug for his love, unconditional support and encouragement.
My daughter Amanda for her advice and helpful suggestions.
My entire wonderful family and gorgeous grandchildren for their love and understanding as they missed out a lot of Grannie Annie time while writing this book.

From Jonathan:
Thank you to my family and Elle for your love, advice, support and encouragement.
A huge thank you to Elio A. Farina for mentoring me through the works of producing a major publication, and for many hours spent teaching me the proper use and principles of LaTeX 2_ε.

Images:
Alexandre Godreau, Alvaro Pinot, Danuel Straub, David Jorre, Gerson Repreza, Jason Leung, Joakim Honksal, Lucas Gallone, Ricardo Gomez, Sylvie Tittel, Sam Wermut

Table of Contents

1

TRANSFORMATIONS

- Translations and dilations
- Combining transformations
- Curve sketching
- Using graphs to solve equations and inequalities

Exercise 1A

Translations and dilations

Fundamentals

Fundamentals 1
Let $f(x)$ be a function and b, $c > 0$. The graph of

(a) $y = f(x) + c$ is translated c units up/down (circle one) from the original function.

(b) $y = f(x) - c$ is translated c units up/down (circle one) from the original function.

(c) $y = f(x - b)$ is translated b units left/right (circle one) from the original function.

(d) $y = f(x + b)$ is translated b units left/right (circle one) from the original function.

Fundamentals 2
Let $f(x)$ be a function and a, $k > 0$. The graph of

(a) $y = kf(x)$ is 'stretched'/'squashed' (circle one) v________ if $k > 1$.

(b) $y = kf(x)$ is 'stretched'/'squashed' (circle one) v________ if $k < 1$.

(c) $y = f(ax)$ is 'stretched'/'squashed' (circle one) h________ if $a > 1$.

(d) $y = f(ax)$ is 'stretched'/'squashed' (circle one) h________ if $a < 1$.

Fundamentals 3
When $y = f(ax)$ is dilated horizontally by a factor of ___. This means that if $a > 1$, the graph is 'squashed'/'stretched' (circle one) whereas if $a < 1$, the graph is is 'squashed'/'stretched' (circle one).

Question 1 Given the curve $y = x^2$, write down the translation that is required to obtain the graphs of the following.

(a) $y = x^2 + 2$ (b) $y = x^2 - 3$ (c) $y = (x + 1)^2$ (d) $y = (x - 2)^2$

Question 2 Let $f(x) = x^2$. Sketch the graph of the following, labelling intercepts with the coordinate axes.

(a) $y = f(x) + 1$ (b) $y = f(x + 1)$ (c) $y = f(x) - 4$ (d) $y = f(x - 2)$

Question 3 Given the curve $y = e^x$, write down the translation that is required to obtain the graphs of the following.

(a) $y = e^x - 1$ (b) $y = e^{x-2}$ (c) $y = e^x + 1$ (d) $y = e^{x+1}$

Question 4 Starting with $f(x)$, write down the translation that is required to obtain $g(x)$.

(a) $f(x) = \frac{1}{x},\ g(x) = \frac{1}{x-1}$

(b) $f(x) = \frac{1}{x},\ g(x) = \frac{1}{x} + 1$

(c) $f(x) = |x|,\ g(x) = |x-3|$

(d) $f(x) = |x|,\ g(x) = |x| - 3$

(e) $f(x) = -x^3,\ g(x) = -(x-2)^3$

(f) $f(x) = -x^3,\ g(x) = -x^3 - 3$

(g) $f(x) = \ln(x),\ g(x) = \ln(x-2)$

(h) $f(x) = \ln(x),\ g(x) = \ln(x) - 2$

Question 5 Write down the new equation obtained when the following translation is applied to $f(x)$.

(a) $f(x) = 2x$, translate left 2 units

(b) $f(x) = 2x$, translate down 3 units

(c) $f(x) = \frac{1}{x-1}$, translate left 2 units

(d) $f(x) = \frac{1}{x-1}$, translate down 3 units

(e) $f(x) = 2x^2$, translate right 4 units

(f) $f(x) = 2x^2$, translate up 2 units

(g) $f(x) = 3 - 2x$, translate up 2 units

(h) $f(x) = 3 - 2x$, translate left 3 units

Question 6 [Replacing x with $(x - a)$]

Write down the new equation obtained when the following translation is applied to the graph.

(a) $2x + 3y + 1 = 0$, translated right by 2 units.

(b) $x^2 + y^2 = 4$, translated left by 4 units.

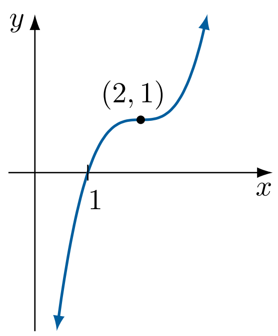
(c) $xy = 1$, translated right by 3 units.

(d) $x^2 + xy = 1$, translated left by 1 unit.

Question 7 The points $P(0, -2)$ and $Q(-4, -6)$ lie on the graph of $y = f(x)$. The graph of $y = f(x+2) + 3$ is then drawn. State the new locations of P and Q

Question 8 The graphs of $y = f(x)$ are drawn below. Describe the transformation below, write down the new equation and sketch the new curve.

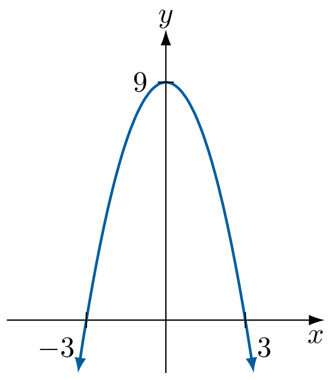
(a) $f(x) = (x-2)^3 + 1$



(i) $y = f(x+1)$

(ii) $y = f(x) + 1$

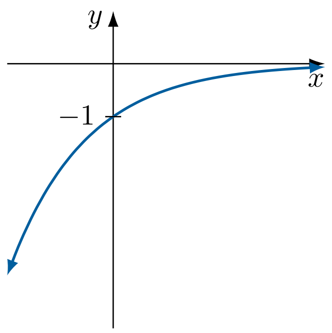
(b) $f(x) = (3 - x)(3 + x)$



(i) $y = f(x - 1)$

(ii) $y = f(x) - 1$

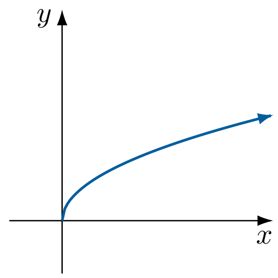
(c) $f(x) = -2^{-x}$



(i) $y = f(x + 2)$

(ii) $y = f(x) + 2$

Question 9 The graph of $y = \sqrt{x}$ is drawn below.



Draw the graphs of the following equations.

(a) $y = \sqrt{x} + 2$

(b) $y = \sqrt{x - 2}$

(c) $y = \sqrt{x + 2} + 1$

Question 10 By considering an appropriate 'original equation' and translating it accordingly, draw a sketch of the following equations.

(a) $y = |x - 1|$

(b) $y = |x| - 2$

(c) $y = \dfrac{1}{x - 2}$

(d) $y = \dfrac{1}{x} + 1$

(e) $y = (x - 2)^2 + 3$

(f) $y = |x + 2| - 4$

Question 11 [Be careful with minus signs]

Write down the equation of the new equations if the following curves are translated to the right by two units.

(a) $f(x) = 3 - 2x$ (b) $f(x) = 3 - 4x - x^2$ (c) $f(x) = 2^{-x}$

Question 12 Sketch the following graphs by applying a translation(s) to either $y = \pm e^x$ or $y = \pm e^{-x}$. For each answer, draw the 'original curve' using dotted lines.

(a) $y = e^{x-2}$ (b) $y = e^x + 1$ (c) $y = -e^{x+2}$

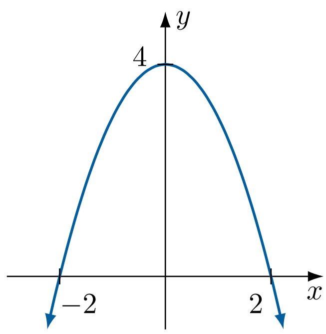
(d) $y = -e^x - 3$ (e) $y = -e^{-x} + 1$ (f) $y = -e^{x+1} + 1$

Question 13 Sketch the following graphs by applying a translation(s) to $y = \ln(x)$. For each answer, draw the 'original curve' using dotted lines.

(a) $y = \ln(x - 1)$ (b) $y = \ln(x + 2)$

(c) $y = \ln(x) + 2$ (d) $y = \ln(x) - 1$

Question 14 The diagram below shows the graph of $y = f(x)$.

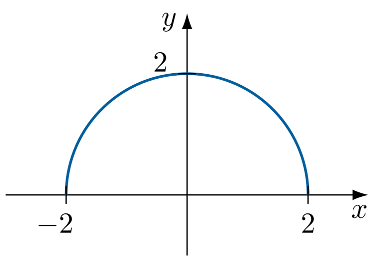


Describe the transformation and sketch the graph of the following.

(a) $y = 2f(x)$ (b) $y = \frac{1}{2}f(x)$

(c) $y = f(2x)$ (d) $y = f\left(\frac{x}{2}\right)$

Question 15 The diagram below shows the graph of $y = f(x)$.

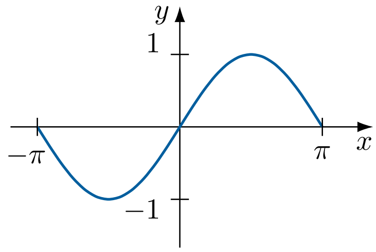


Describe the transformation and sketch the graph of the following.

(a) $y = 2f(2x)$ (b) $y = \frac{1}{2}f\left(\frac{x}{2}\right)$

(c) $y = \frac{1}{2}f(2x)$ (d) $y = 2f\left(\frac{x}{2}\right)$

Question 16 The diagram below shows the graph of $y = \sin x$ in the domain $x \in [-\pi, \pi]$



Sketch the following graphs.

(a) $y = 2\sin x$

(b) $y = \dfrac{1}{2}\sin\left(\dfrac{x}{2}\right)$

(c) $y = \dfrac{1}{2}\sin(2x)$

(d) $y = 2\sin\left(\dfrac{x}{2}\right)$

Challenge Problems

Problem 1 [If the coefficient of x is not 1, translations become complicated]

Consider the graphs of $f(x) = (2x+1)^2$ and $g(x) = (2x-5)^2$. Bob claims that $y = g(x)$ is the graph of $y = f(x)$ but translated to the right by 6 units.

(a) Translate $f(x)$ to the right by 6 units, and determine whether Bob is correct or incorrect.

(b) Write down the equation of the graph of $y = f(x)$ if it is shifted to the right by a units, where $a > 0$.

(c) Hence, determine how many units to the right $f(x)$ needs to be translated to obtain the graph of $g(x)$.

Problem 2 For each of the following, in what direction was $f(x)$ translated to yield $g(x)$, and by how many units?

(a) $y = \sqrt{2x-3},\ y = \sqrt{2x+7}$

(b) $y = (3x+5)^2,\ y = (3x-1)^2$

(c) $y = e^{-x},\ y = e^{-x+1}$

(d) $y = e^{-2x},\ y = e^{-2x+4}$

Problem 3 [Transformations may not be unique]

(a) Show that horizontally dilating $y = \ln x$ is equivalent to translating it vertically.

(b) Show that horizontally translating $y = e^x$ is equivalent to dilating it vertically.

(c) Show that vertically dilating $y = x^2$ is equivalent to horizontally dilating it.

Exercise 1B

Combining transformations

Fundamentals

Fundamentals 1

Consider a curve $f(x)$, and let a, $c > 0$.

(a) If $f(x)$ is translated to the right by c units, we replace x with ______, so the new curve is ______.

(b) If this new curve is then squashed horizontally by a factor of a, we replace x with ___, so the new curve is ______.

(c) In this case, if we swap the order of the transformations, the result is the same/different (circle one).

Fundamentals 2

Consider a curve $f(x)$, and let $c > 0$.

(a) If $f(x)$ is translated to the left by c units, we replace x with ______, so the new curve is ______.

(b) If this new curve is then reflected horizontally, we replace x with ___, so the new curve is ______.

(c) In this case, if we swap the order of the transformations, the result is the same/different (circle one).

Fundamentals 3

Consider a curve $f(x)$, and let $a > 0$.

(a) If $f(x)$ is squashed horizontally by a factor of a, we replace x with ___, so the new curve is ______.

(b) If this new curve is then reflected horizontally, we replace x with ___, so the new curve is ______.

(c) In this case, if we swap the order of the transformations, the result is the same/different (circle one).

Fundamentals 4

In general, when applying two types of horizontal transformations, order matters/does not matter (circle one), whereas applying two types of vertical transformations, order matters/does not matter (circle one).

Question 1 [Easy way to think of transformations]

Complete the following.

(a) To translate a curve to the right by ___ units, replace x with $x - 2$.

For example $y = x^2$ now becomes $y =$ ______.

(b) To translate a curve to the left by 1 unit, replace x with ______.

For example, $y = \ln(x)$ now becomes $y =$ ______.

(c) To stretch a curve horizontally by a factor of 2, replace x with ______.

For example, $y = \sin(x)$ now becomes $y =$ ______.

(d) To squash a curve horizontally by a factor of ___ replace x with $2x$.

For example, $y = \cos(x)$ now becomes $y =$ ______.

Question 2 Consider the graph of $y = x^2$.

(a) Write down the equation of the graph when it is translated right by 1 unit.

(b) Write down the new equation if the graph is then dilated horizontally by a factor of $\frac{1}{2}$.

(c) Complete the following set of steps that were followed in the previous parts.

$$x^2 \xrightarrow[\text{translate}]{} (x - \underline{\quad})^2 \xrightarrow[\text{dilate}]{} (\underline{\quad} - 1)^2$$

(d) Complete the following set of steps if we instead first dilate and then translate.

$$x^2 \xrightarrow[\text{dilate}]{} (\underline{\qquad})^2 \xrightarrow[\text{translate}]{} (\underline{\qquad})^2$$

(e) From the above, did the order matter or did you end up with the same curve doing it either way?

Question 3 Consider the graph of $y = \ln(x)$. Find the equation of the final graph if $y = \ln(x)$ is

(a) translated right by 1 unit, then dilated horizontally by $\frac{1}{2}$ units.

(b) dilated horizontally by $\frac{1}{2}$ units, then translated right by 1 unit.

(c) translated left by 2 units, then dilated horizontally by 3 units.

(d) dilated horizontally by 3 units, then translated left by 2 units.

Question 4 Consider the graph of $y = \frac{1}{x}$. Find the equation of the final graph if $y = \frac{1}{x}$ is

(a) translated left by 2 units, then dilated horizontally by $\frac{1}{3}$ units.

(b) dilated horizontally by $\frac{1}{3}$ units, then translated left by 2 units.

(c) translated right by 5 units, then dilated horizontally by 2 units.

(d) dilated horizontally by 2 units, then translated right by 5 units.

Question 5 [If you swap the order you have to change the quantities]

A teacher gives Bob and Mary the equation $y = \sqrt{x}$ to transform. Bob is instructed to first translate left by 1 unit and then squash horizontally by a factor of 2. Mary is instructed to first squash it horizontally by a factor of 2 and then translate it horizontally by a different amount.

(a) Write down the equation that Bob should end up with.

(b) Write down the equation that Mary has after squashing it only.

(c) Find the value of a so that $\sqrt{2(x+a)} = \sqrt{2x+1}$.

(d) Hence, by how many units should Mary translate her curve so it becomes the same as Bob's?

Question 6 Find the equation of the final curve in the form $y = f(ax + b)$ if $y = f(x)$ is

(a) translated right by 2 units, then squashed horizontally by a factor of 3.

(b) translated left by 3 units, then stretched horizontally by a factor of 2.

Question 7 Find the equation of the final curve in the form $y = f(ax + b)$ if $y = f(x)$ is

(a) squashed horizontally by 2 units, then translated left by 1 unit.

(b) stretched horizontally by 3 units, then translated right by 6 units.

Question 8 For the following forms, determine two ways that a dilation and translation can be used to obtain the required form.

For example the form $f(2x - 4)$ can be obtained using either of the following

$$f(x) \xrightarrow[\text{translate right by 4}]{} f(x-4) \xrightarrow[\text{squash factor of 2}]{} f(2x-4)$$

$$f(x) \xrightarrow[\text{squash factor of 2}]{} f(2x) \xrightarrow[\text{translate right by 2}]{} f\big(2(x-2)\big) = f(2x-4)$$

(a) $f(2x + 4)$

(b) $f(3x - 2)$

(c) $f(2x - 1)$

(d) $f(3x + 1)$

Question 9 [Easy way to think of reflections]

To reflect a curve across the y-axis, simply replace x with $-x$. Use this to write down the new equation if the following graphs are reflected across the y-axis.

(a) $y = e^x$ (b) $y = \ln(x)$ (c) $y = \sin(x)$

(d) $y = \sqrt{x}$ (e) $y = 2x - 1$ (f) $y = 3 - 2x$

(g) $y = \dfrac{x}{x-1}$ (h) $y = f(x-1)$ (i) $y = f(2x+1)$

Question 10 Consider the graph of $y = \ln(x)$.

(a) Write down the equation of the graph when it is translated right by 1 unit.

(b) Write down the new equation if the graph is then reflected across the y-axis.

(c) Complete the following set of steps that were followed in the previous parts.

$$\ln(x) \xrightarrow[\text{translate}]{} \ln(x-1) \xrightarrow[\text{reflect}]{} \ln(\underline{\qquad})$$

(d) Complete the following set of steps if we instead first reflect and then translate.

$$\ln(x) \xrightarrow[\text{reflect}]{} \ln(-x) \xrightarrow[\text{translate}]{} \ln(\underline{\qquad})$$

(e) From the above, did the order matter or did you end up with the same curve doing it either way?

Question 11 Consider the graph of $y = \sqrt{x}$. Find the equation of the final graph if $y = \sqrt{x}$ is

(a) translated right by 3 units, then reflected across the y-axis.

(b) reflected across the y-axis, then translated right by 3 units.

(c) translated left by 2 units, then reflected across the y-axis.

(d) reflected across the y-axis, then translated left by 2 units.

Question 12 Find the equation of the final graph if $y = f(x)$ is transformed in the following order.

(a) Translated right by 1 unit, stretched horizontally by a factor of 2, reflected across the y-axis.

(b) Translated right by 1 unit, reflected across the y-axis, stretched horizontally by a factor of 2.

(c) Reflected across the y-axis, stretched horizontally by a factor of 2, translated right by 1 unit.

(d) Reflected across the y-axis, translated right by 1 unit, stretched horizontally by a factor of 2.

(e) Stretched horizontally by a factor of 2, translated right by 1 unit, reflected across the y-axis.

(f) Stretched horizontally by a factor of 2, reflected across the y-axis, translated right by 1 unit.

Question 13 Bob is given a function $y = f(x)$ and is instructed to turn it into $y = f(1-2x)$. Describe a possible sequence of transformations so that Bob can obtain $y = f(1-2x)$ from $y = f(x)$, if the sequence must begin with a

(a) translation. (b) reflection. (c) dilation.

Question 14 Consider the graph of $y = e^x$.

(a) Write down the equation of the graph when it is dilated vertically by a factor of 2.

(b) Write down the equation of the new graph if the graph is then translated up by 1 unit.

(c) Complete the following set of steps that were followed in the previous parts.

$$e^x \xrightarrow[\text{dilate}]{} 2e^x \xrightarrow[\text{translate}]{} \underline{\hspace{2em}}$$

(d) Complete the following set of steps if we instead first translate and then dilate.

$$e^x \xrightarrow[\text{translate}]{} e^x + 1 \xrightarrow[\text{dilate}]{} \underline{\hspace{2em}}$$

(e) From the above, did the order matter or did you end up with the same curve doing it either way?

Question 15 Bob is asked to sketch $y = 2\sqrt{x} + 1$ using a vertical translation and vertical dilation, in an order of his choosing. Although it is possible to do it either way, which order will be easier for Bob?

Question 16 Consider the graph of $y = e^x$.

(a) Write down the equation of the graph when it is reflected across the x-axis.

(b) Write down the equation of the new graph if the graph is then translated up by 1 unit.

(c) Complete the following set of steps that were followed in the previous parts.

$$e^x \xrightarrow[\text{reflect}]{} -e^x \xrightarrow[\text{translate}]{} \underline{\hspace{2em}}$$

(d) Complete the following set of steps if we instead first translate and then reflect.

$$e^x \xrightarrow[\text{translate}]{} e^x + 1 \xrightarrow[\text{reflect}]{} \underline{\hspace{2em}}$$

(e) From the above, did the order matter or did you end up with the same curve doing it either way?

Question 17 Bob is asked to sketch $y = 2 - 3\sqrt{x}$ using a vertical translation, vertical dilation, and a reflection, in an order of his choosing. Although it is possible to do it in more than one way, which order will be easier for Bob?

Question 18 Sketch each of the following.

(a) $y = 2\sqrt{x+3} - 5$ (b) $y = -2\sqrt{x-3} + 5$ (c) $y = -\sqrt{3-x} + 5$

(d) $y = 2\sqrt{3-x}$ (e) $y = \sqrt{2x-6}$ (f) $y = \sqrt{6-2x}$

Question 19 Sketch each of the following.

(a) $y = 2e^x$ (b) $y = -e^{-x}$ (c) $y = -3e^{\frac{x}{2}}$

(d) $y = e^{x-2} - 1$ (e) $y = \dfrac{1}{2}e^{3-2x}$ (f) $y = 2 - e^{x+1}$

Question 20 Sketch each of the following.

(a) $y = 2\ln x$ (b) $y = 2\ln(-x)$ (c) $y = \dfrac{1}{2}\ln(x)$

(d) $y = -\ln(2x)$ (e) $y = 2\ln(2x-5) + 3$ (f) $y = 3 - 2\ln(5-2x)$

Question 21 Sketch each of the following.

(a) $y = \dfrac{2}{x-1}$ (b) $y = -\dfrac{1}{3-x}$ (c) $y = \dfrac{1}{3-x} + 1$

(d) $y = -\dfrac{1}{3-2x}$ (e) $y = -\dfrac{1}{x+1} + 2$ (f) $y = -\dfrac{1}{4-2x} + 2$

Challenge Problems

Problem 1 [Symmetry property]

(a) Let $f(x) = x^2$. Sketch the graph of $y = f(2-x)$.

(b) Let $f(x) = \ln x$. Sketch the graph of $y = f(4-x)$.

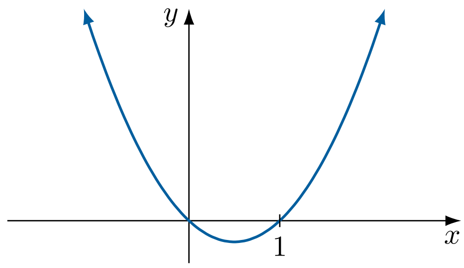
(c) Show that the graph of $y = f(2a-x)$ is the graph of $y = f(x)$ reflected across the vertical line $x = a$.

Problem 2 Bob is given the graph of $y = 1 - 2f(x)$. Give three possible sequences of transformations so that he recovers the original graph $y = f(x)$.

Problem 3 Mary is given the graph of $y = f(3x-6)$. Give two possible sequences of transformations so that she recovers the original graph $y = f(x)$.

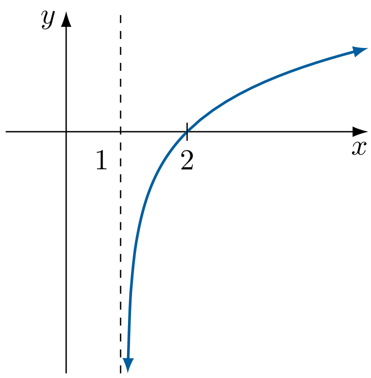
Problem 4 Mary is given the graph of $y = f(3-2x)$. Give three possible sequences of transformations so that she recovers the original graph $y = f(x)$.

Problem 5 The diagram below shows the graph of $y = f(x)$.



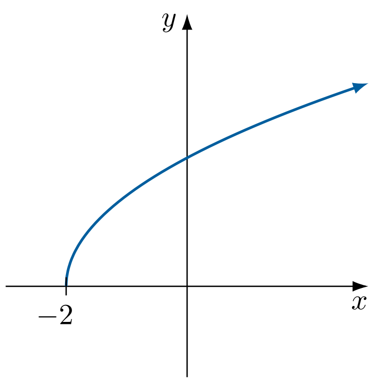
Describe a suitable sequence of transformations, and hence sketch the graph of $y = f(2x - 1)$.

Problem 6 The diagram below shows the graph of $y = f(x)$.



Describe a suitable sequence of transformations, and hence sketch the graph of $y = f(1 - x)$.

Problem 7 The diagram below shows the graph of $y = f(x)$.



Describe a suitable sequence of transformations, and hence sketch the graph of $y = f(4 - 2x)$.

Exercise 1C

Curve sketching

Fundamentals

Fundamentals 1

The following are useful features to find when producing a sketch of a curve. In general not all of the following are needed to produce the sketch but the more features you have, the more certain you will be about the shape of the curve.

(a) ___ and ___-intercepts

(b) The d_____ of the curve

(c) Behaviour of the curve as $x \to \pm$___.

(d) Any vertical or horizontal a_______

(e) Behaviour of the curve on either end of the vertical a_______

(f) If there are horizontal asymptotes, whether it approaches it from a_____ or b_____.

(g) If the domain is restricted, then the behaviour of the curve at either side of the d_____.

(h) Any possible symmetry i.e. is the curve e_____ or o_____?

(i) If easy to find, then also the r_____ of the curve.

Question 1 Let $f(x) = \dfrac{2}{2x-1}$

(a) Find any intercepts with the coordinate axes.

(b) State the equation of the vertical and horizontal asymptote.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Determine the behaviour of the curve on either end of the vertical asymptote.

(e) Hence, sketch the graph of $y = f(x)$.

Question 2 Let $f(x) = \dfrac{2x+1}{x-1}$.

(a) Find any intercepts with the coordinate axes.

(b) State the equation of the vertical and horizontal asymptote.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Determine whether the function is even, odd, or neither.

(e) Hence, sketch the graph of $y = f(x)$.

Question 3 Let $f(x) = \dfrac{1}{x^2}$.

(a) State the equation of the vertical and horizontal asymptote.

(b) Describe the behaviour of the curve as $x \to \infty$.

(c) Show that the function is even.

(d) Explain why the graph is always above the x-axis.

(e) Hence, sketch the graph of $y = f(x)$.

Question 4 Let $f(x) = \dfrac{1}{x^2 + 1}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Explain briefly why the graph is always above the x-axis.

(e) Determine whether the function is even, odd, or neither.

(f) Hence, sketch the graph of $y = f(x)$.

Question 5 Let $f(x) = \dfrac{1}{x^2 - 4}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Describe the behaviour of the curve near the vertical asymptotes.

(e) Determine whether the function is even, odd, or neither.

(f) Hence, sketch the graph of $y = f(x)$.

Question 6 Let $f(x) = \dfrac{x}{x^2 + 4}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) State where the curve is above and below the x-axis.

(e) Determine whether the function is even, odd, or neither.

(f) Hence, sketch the graph of $y = f(x)$.

Question 7 Let $f(x) = \dfrac{x^2}{x^2 - 9}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Determine whether the function is even, odd, or neither.

(e) Hence, sketch the graph of $y = f(x)$.

Question 8 Let $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Determine whether the function is even, odd, or neither.

(e) Hence, sketch the graph of $y = f(x)$.

Question 9 Let $f(x) = \sqrt{x^2 - 1}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the domain of $f(x)$.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Determine whether the function is even, odd, or neither.

(e) Hence, sketch the graph of $y = f(x)$.

Question 10 Let $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Find the domain of $f(x)$.

(d) Determine whether the function is even, odd, or neither.

(e) Hence, sketch the graph of $y = f(x)$.

Question 11 Let $f(x) = \left(\dfrac{x - 1}{x + 1}\right)^2$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) State where the curve is above or below the x-axis.

(e) Determine whether the function is even, odd, or neither.

(f) Hence, sketch the graph of $y = f(x)$.

Question 12 Sketch the following curves.

(a) $y = \dfrac{1}{x^2 + x - 2}$

(b) $y = \dfrac{x}{x^2 + x - 6}$

Question 13 [Trick question]

Sketch the graph of $y = \sqrt{x-1} + \sqrt{1-x}$.
Hint: What's the domain of the graph?

Challenge Problems

Problem 1 Let $f(x) = \ln(x^2 - 1)$.

(a) Find the domain of $f(x)$.
(b) Show that $f(x)$ is even.
(c) State the vertical asymptotes of $f(x)$.
(d) Describe the behaviour of the curve near the vertical asymptotes.
(e) Describe the behaviour of the curve as $x \to \infty$.
(f) Hence, sketch the curve.

Problem 2 Let $f(x) = \ln(x^2 + 1)$.

(a) Find the domain of $f(x)$.
(b) Show that $f(x)$ is even.
(c) Describe the behaviour of the curve as $x \to \infty$.
(d) Hence, sketch the curve.

Problem 3 Sketch the graphs of the following equations.

(a) $y = \dfrac{1}{\sqrt{x} - 1}$

(b) $y = \dfrac{1}{\sqrt{x} + 1}$

(c) $y = \dfrac{\sqrt{x}}{x + 1}$

Problem 4 Sketch the graphs of the following equations.

(a) $y = \dfrac{1}{e^x - 1}$

(b) $y = \dfrac{1}{e^x + 1}$

(c) $y = \dfrac{e^x + 4}{e^x - 4}$

Exercise 1D

Using graphs to solve equations and inequalities

Fundamentals

Fundamentals 1

The solutions to $f(x) > 0$ can be found by sketching $y =$ ______ and observing the x-coordinates where the curve is a______ the ___-axis.

Fundamentals 2

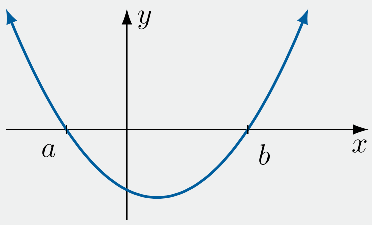
(a) To solve a quadratic inequality in the form $ax^2 + bx + c \geq 0$ or $ax^2 + bx + c \leq 0$, first sketch the graph of $y =$ ________.

(b) Then, depending on the direction of the inequality, shade the region that is either a______ or b______ the ___-axis.

(c) The set of ___-values that are shaded is the solution set.

Fundamentals 3

The diagram below shows the sketch of $y = (x - a)(x - b)$.



Write down the inequality that corresponds to

(a) $(x - a)(x - b) \geq 0$

(b) $(x - a)(x - b) < 0$

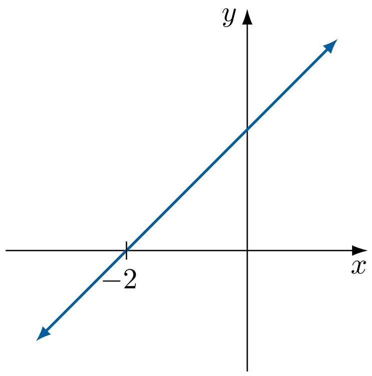
Fundamentals 4

(a) To find the intersection points of $y = f(x)$ and $y = g(x)$, we solve the two equations s________.

(b) The solutions of $f(x) = g(x)$ correspond to the ___-coordinates of where the two graphs intersect.

(c) Hence, the number of solutions to $f(x) - g(x) = 0$ can be found by instead sketching $y =$ ______ and $y =$ ______, then counting how many times they intersect.

Question 1 The diagram below shows the graph of $y = x+2$. Use your graph to solve the following.



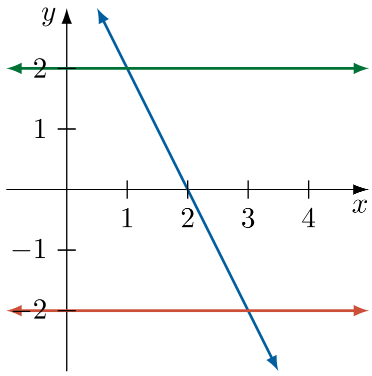
(a) $x + 2 = 0$ (b) $x + 2 < 0$ (c) $x + 2 > 0$

Question 2 By drawing a sketch, or otherwise, solve the following inequalities.

(a) $2x - 4 \leq 0$ (b) $1 - 3x \geq 0$

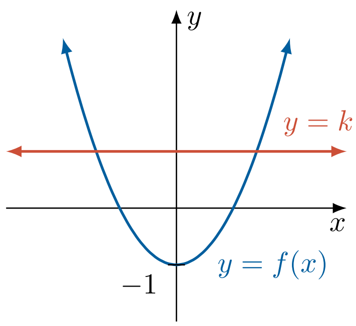
(c) $2 - \frac{1}{2}x < 0$ (d) $\frac{2x}{3} + \frac{1}{2} > 0$

Question 3 The graphs of $y = -2x + 4$ and $y = \pm 2$ are drawn below.



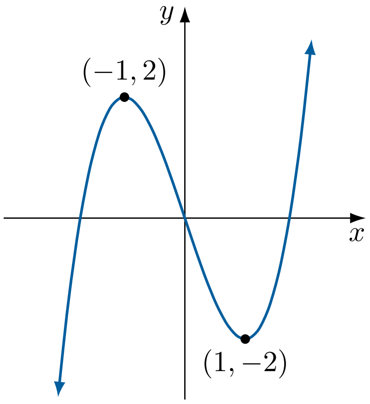
Use this diagram to write down the solution of $-2 \leq 4 - 2x \leq 2$.

Question 4 The diagram below shows the graph of a function $y = f(x)$, and a horizontal line $y = k$. Find the value(s) of k such that $f(x) = k$ has



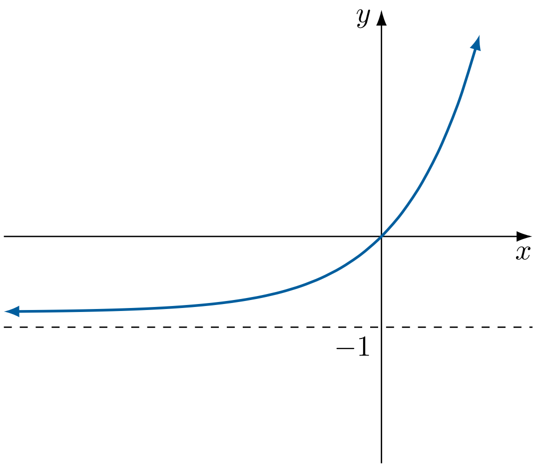
(a) one solution. (b) two solutions. (c) no solutions.

Question 5 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



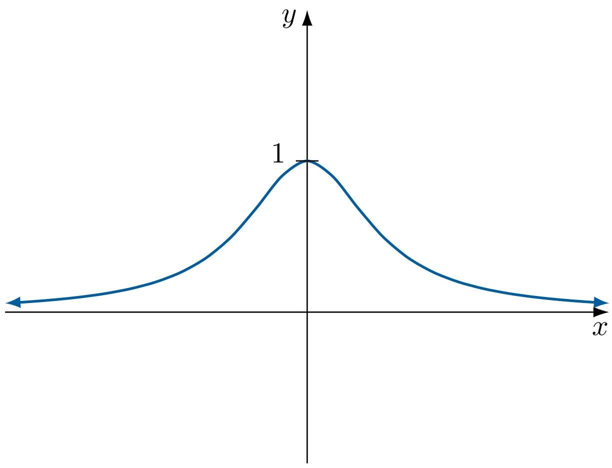
(a) one solution. (b) two solutions. (c) three solutions.

Question 6 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



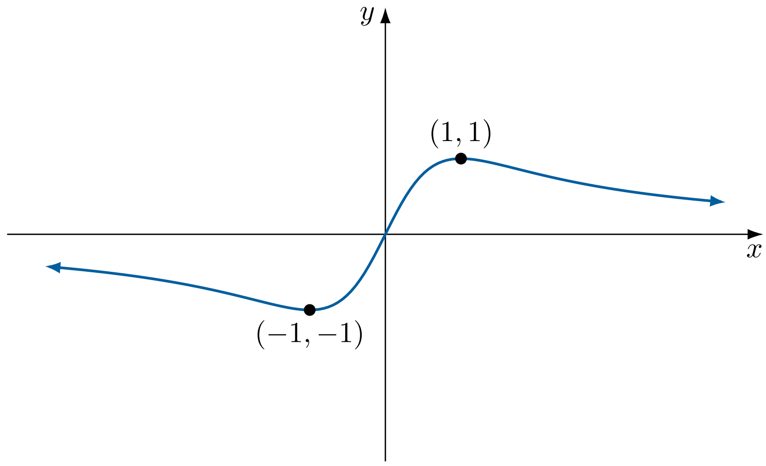
(a) one solution. (b) two solutions. (c) no solutions.

Question 7 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



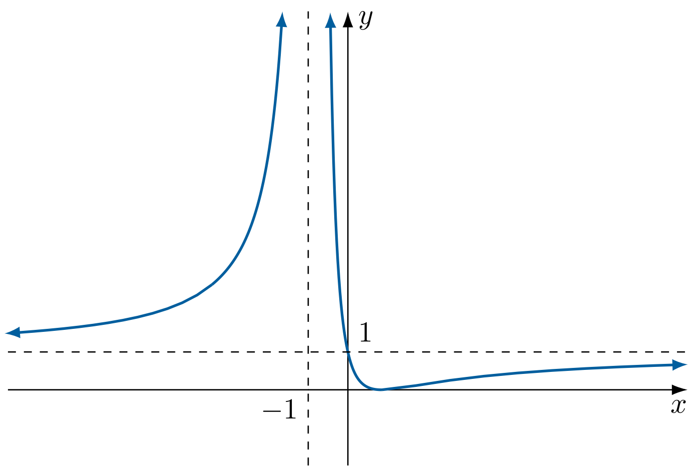
(a) one solution. (b) two solutions. (c) no solutions.

Question 8 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



(a) one solution. (b) two solutions. (c) no solutions.

Question 9 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



(a) one solution. (b) two solutions. (c) no solutions.

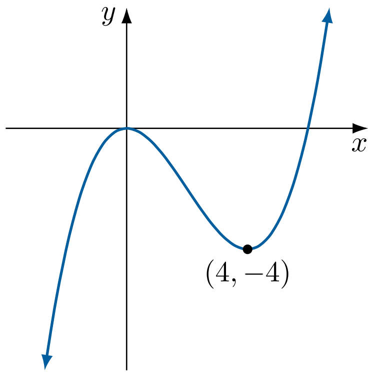
Question 10

(a) Sketch $y = x^2 - 4x + 5$, labelling the vertex.

(b) Use your diagram to find the value(s) of k such that $x^2 - 4x + 5 - k = 0$ has two solutions.

(c) Verify your answer by finding the discriminant of $x^2 - 4x + (5 - k) = 0$.

Question 11 The graph of $y = f(x)$ is sketched below.



(a) For what values of k does the equation $f(x) = k$ have

(i) 1 solution? (ii) 2 solutions? (iii) 3 solutions?

(b) For what values of k does the equation $2f(x) = k$ have

(i) 1 solution? (ii) 2 solutions? (iii) 3 solutions?

(c) For what values of k does the equation $f(x+1) = k$ have

(i) 1 solution? (ii) 2 solutions? (iii) 3 solutions?

Question 12

(a) Sketch the graph of $y = (x-2)(x+1)$, labelling the x-intercepts. You do not need to find the coordinates of the vertex.

(b) Hence, state the value(s) of x for which

(i) $(x-2)(x+1) = 0$. (ii) $(x-2)(x+1) < 0$. (iii) $(x-2)(x+1) > 0$.

Question 13 Use a similar technique to the above question to solve the following inequalities.

(a) $(x-3)(x+2) < 0$ (b) $(x+4)(x-5) \geq 0$ (c) $(2-x)(x+1) \leq 0$

(d) $(1-2x)(1+x) > 0$ (e) $(x-2)(2x-4) \leq 0$ (f) $(-x-2)(x+2) > 0$

Question 14 Solve the following inequalities by sketching an appropriate graph.

(a) $x^2 - 1 \geq 0$ (b) $4 - x^2 > 0$ (c) $x^2 - 5x + 4 \leq 0$

(d) $6 - x - x^2 < 0$ (e) $2x^2 + 7x + 5 < 0$ (f) $1 - x - 6x^2 \leq 0$

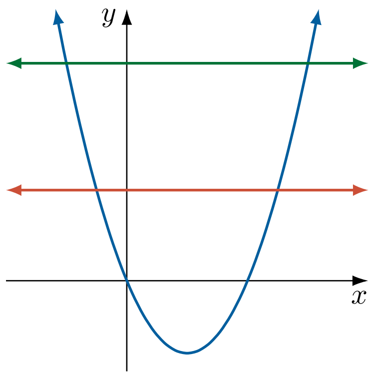
Question 15 [Trick questions]

Solve the following inequalities by sketching an appropriate graph.

(a) $x^2 + 1 > 0$ (b) $x^2 + 4 < 0$ (c) $x^2 \leq 0$

(d) $x^2 + 2x + 2 > 0$ (e) $-x^2 + 2x - 2 \geq 0$ (f) $4x^2 - 4x + 1 \leq 0$

Question 16 The diagram below shows the graph of $y = x^2 - 4x$, $y = 5$ and $y = 12$.



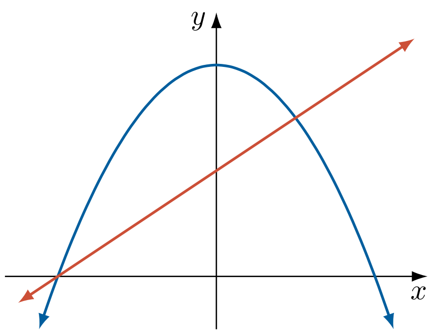
(a) Find the x-intercepts of the parabola, and the intersections with $y = 5$ and $y = 12$.

(b) Hence, use your diagram to solve the following inequalities.

(i) $x^2 - 4x > 0$ (ii) $x^2 - 4x \geq 12$

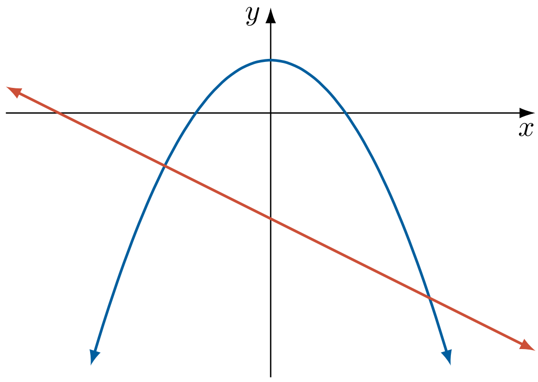
(iii) $x^2 - 4x \leq 5$ (iv) $5 < x^2 - 4x < 12$

Question 17 The diagram below shows the graph of $y = 4 - x^2$ and $y = x + 2$.



(a) Find where the two graphs intersect. (b) Hence, solve $4 - x^2 > x + 2$.

Question 18 The diagram below shows a sketch of $y = 2 - x^2$ and $y = -x - 4$.



(a) Find where the two graphs intersect.

(b) Hence, solve $2 - x^2 < -x - 4$.

Question 19 Draw appropriate graphs to solve the following quadratic inequalities.

(a) $(2 - x)(x + 4) < 0$

(b) $x^2 - 3x + 2 \geq 0$

(c) $x^2 < -2x$

(d) $x^2 \leq x + 6$

(e) $12x - x^2 \geq 0$

(f) $x^2 - 4 \geq -2x + 4$

Question 20

(a) Sketch the graph of $y = x^2 - 2x - 8$

(b) Sketch the graph of $y = |x^2 - 2x - 8|$.

(c) Use your diagram to explain briefly why $|x^2 - 2x - 8| = k$ can never have exactly one solution, for any value of k.

(d) Hence, find the value(s) of k such that $|x^2 - 2x - 8| = k$ has

(i) no solutions.

(ii) two solutions.

(iii) three solutions.

(iv) four solutions.

Question 21 Use graphing software to sketch the pairs of functions below, and hence state how many solutions there are to the equation $f(x) = g(x)$.

(a) $f(x) = |x - 4|$, $g(x) = \frac{1}{2}x$

(b) $f(x) = |x + 2| =$, $g(x) = -2x$

(c) $f(x) = x^3 - x$, $g(x) = \cos x$

(d) $f(x) = e^{-x^2}$, $g(x) = x^2$

Question 22 By sketching $y =$ LHS and $y =$ RHS, state the number of solutions to the following equations.

(a) $x = 4 - x^2$

(b) $1 - x^2 = x^2 - 4$

(c) $x = x^3 + 2$

(d) $x = \sqrt{x}$

(e) $x^2 + 1 = \frac{1}{x}$

(f) $1 - x^2 = 3 - 2x$

Question 23 State two appropriate curves that could be used to determine the number of solutions to the following equations.

(a) $x - \cos x = 0$ (b) $x^2 - \ln x = 0$ (c) $e^x - x - 2 = 0$

(d) $x - \sqrt{x} + 1 = 0$ (e) $\ln x - 1 - x = 0$ (f) $1 - \dfrac{1}{x-1} - x^2 = 0$

Question 24 Determine the number of solutions to the following equations, by drawing appropriate sketches.

(a) $e^x - 1 + x^2 = 0$ (b) $e^x + \cos x - 2 = 0$ (c) $x^2 - \cos x - 1 = 0$

Question 25 Determine the number of solutions to the following equations by drawing appropriate sketches.

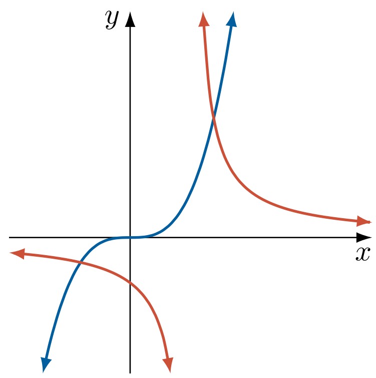
(a) $x(x^2 - 1) = 1$ (b) $(x-1)(x^2+1) = 1$ (c) $x^3 - x + 1 = 0$

(d) $x^3 + 2x - 3 = 0$ (e) $x^3 - x^2 - 2 = 0$ (f) $x^4 - x + 1 = 0$

Question 26 Show that the equation $\cos x = e^x$ has infinitely many solutions.

Question 27 [Finding the number of solutions of polynomial equations]

The diagram shows a sketch of $y = x^3$ and $y = \dfrac{1}{x-1}$ on the same set of axes.



(a) How many solutions does $x^3 = \dfrac{1}{x-1}$ have?

(b) How many solutions does $x^3(x-1) = 1$ have?

(c) Hence, how many solutions does $x^4 - x^3 - 1 = 0$ have?

Challenge Problems

Problem 1

(a) Draw a sketch of $y = x^3$ and $y = k - x$ for when $k > 0$ and when $k < 0$.

(b) Hence, show that if $k > 0$ then $x^3 + x - k = 0$ has one positive solution, but if $k < 0$ then $x^3 + x - k = 0$ has one negative solution.

Problem 2 Show that the equation $x^4 - kx - 1 = 0$ will always have two real solutions, and that there will always be one positive and one negative solution.

Problem 3 The equation $x^2 + \ln x = k$ has one real root $x = \alpha$ for all real values of k. Describe the behaviour of α for varying values of k.

Problem 4 Consider the equation $x^3 - kx + 1 = 0$.

(a) Show that if the equation has only one real root, then the root must be negative.

(b) Show that if the equation has a double root, then the double root will be positive.

(c) Show that if the equation has three distinct real roots, then two must be positive and one must be negative.

(d) As $k \to \infty$, one root approaches infinity and another approaches negative infinity. What happens to the third root?

(e) Describe the behaviour of the real root as $k \to -\infty$.

Chapter 1 Review

Transformations

Review

Question 1 In each of the following, $f(x)$ was transformed a certain way for it to become $g(x)$. Describe the transformation.

(a) $f(x) = \sqrt{x},\ g(x) = \sqrt{x+1}$

(b) $f(x) = \sqrt{x},\ g(x) = \sqrt{x} + 2$

(c) $f(x) = \ln(x+1),\ g(x) = \ln(2x+1)$

(d) $f(x) = \dfrac{2}{x-1},\ g(x) = \dfrac{1}{x-1}$

(e) $f(x) = \sin(x),\ g(x) = \sin\left(\dfrac{x}{2}\right)$

(f) $f(x) = e^x + 1,\ g(x) = e^x - 2$

(g) $f(x) = (x+1)^2,\ g(x) = (x-3)^2$

(h) $f(x) = 4x^2 - 1,\ g(x) = x^2 - 1$

Question 2 Write down the equation of the new curve when the following curves have been transformed in the following ways.

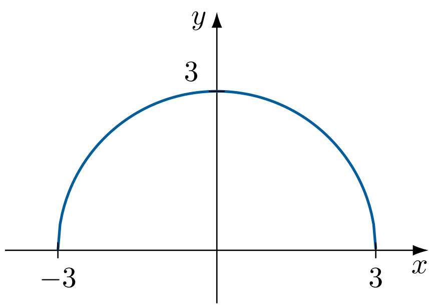
(a) $f(x) = (x-2)^3$, translate left 3 units.

(b) $f(x) = (x-2)^3$, translate right 2 units.

(c) $f(x) = 3x + 2$, translate left 3 units.

(d) $f(x) = 3x + 2$, translate right 4 units.

Question 3 The diagram below shows the graph of $y = f(x)$.



Describe the transformation and sketch the graph of the following.

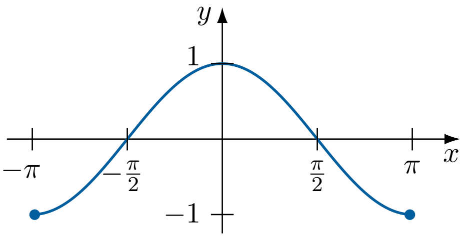
(a) $y = 3f(3x)$

(b) $y = \dfrac{1}{3}f\left(\dfrac{x}{3}\right)$

(c) $y = \dfrac{1}{3}f(3x)$

(d) $y = 3f\left(\dfrac{x}{3}\right)$

Question 4 The diagram below shows the graph of $y = \cos x$ in the domain $x \in [-\pi, \pi]$. By applying the appropriate dilations, sketch the following graphs.



(a) $y = 2\cos(2x)$

(b) $y = \dfrac{1}{2}\cos(2x)$

(c) $y = 2\cos\left(\dfrac{x}{2}\right)$

(d) $y = \dfrac{1}{2}\cos\left(\dfrac{x}{2}\right)$

Question 5 Consider the graph of $y = \sqrt{x}$. Find the equation of the graph if $y = \sqrt{x}$ is

(a) translated left by 2 units, then stretched horizontally by a factor of 3.

(b) stretched horizontally by a factor of 3, then translated left by 2 units.

(c) translated right by 1 unit, then squashed horizontally by a factor of 2.

(d) squashed horizontally by a factor of 2, then translated right by 1 unit.

Question 6 The curve $y = f(x)$ is translated and dilated, not necessarily in that particular order, to obtain the final curve $y = f(ax + b)$. Find $f(ax + b)$.

(a) translated left by 1 unit, squashed horizontally by a factor of 2.

(b) squashed horizontally by a factor of 2, translated left by 1 unit

(c) translated right by 2 units, stretched horizontally by a factor of 3.

(d) stretched horizontally by a factor of 3, translated right by 2 units,

Question 7 The curve $y = f(x)$ is translated and reflected, not necessarily in that particular order, to obtain the final curve $y = f(ax + b)$. Find $f(ax + b)$.

(a) translated right by 2 units, then reflected across the y-axis.

(b) reflected across the y-axis, then translated right by 2 units.

(c) translated left by 3 units, then reflected across the y-axis.

(d) reflected across the y-axis, then translated left by 3 units.

Question 8 Describe a suitable sequence of transformations that can turn $f(x)$ into $g(x)$ below.

(a) $f(x) = \sin(x),\ g(x) = \sin\left(2x + \dfrac{\pi}{3}\right)$

(b) $f(x) = \ln(x),\ g(x) = \ln(3x - 6)$

(c) $f(x) = \sqrt{x},\ g(x) = \sqrt{3 - x}$

(d) $f(x) = \tan(x),\ g(x) = \tan\left(\dfrac{\pi}{4} - x\right)$

Question 9 Describe a suitable sequence of transformations that can turn $f(x)$ into $g(x)$ below.

(a) $f(x) = \cos(x),\ g(x) = \cos\left(\dfrac{\pi}{3} - 2x\right)$

(b) $f(x) = \ln(x),\ g(x) = \ln\left(2 - \dfrac{x}{3}\right)$

Question 10 Let $f(x) = \dfrac{x-1}{x+1}$

(a) Find any intercepts with the coordinate axes.

(b) State the equation of the vertical and horizontal asymptote.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Determine whether the function is even, odd, or neither.

(e) Hence, sketch the graph of $y = f(x)$.

Question 11 Let $f(x) = \dfrac{x^2}{x^2+9}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) State where the curve is above and below the x-axis.

(e) Determine whether the function is even, odd, or neither.

(f) Hence, sketch the graph of $y = f(x)$.

Question 12 Let $f(x) = \dfrac{x^2+1}{x^2-4}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Determine whether the function is even, odd, or neither.

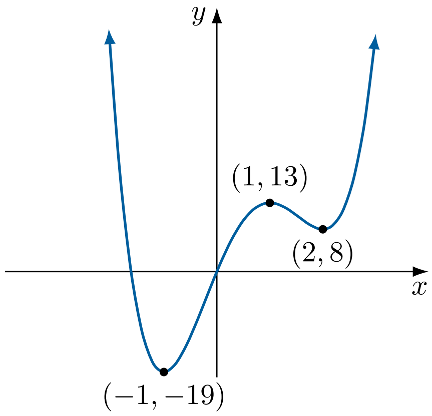
(e) Hence, sketch the graph of $y = f(x)$.

Question 13 Sketch the following, labelling all important features.

(a) $y = \dfrac{\sqrt{x}+1}{\sqrt{x}-1}$

(b) $y = \dfrac{e^x - 2}{e^x + 2}$

Question 14 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



(a) one solution.

(b) two solutions.

(c) three solutions.

(d) four solutions.

Question 15 Solve the following quadratic inequalities.

(a) $(x+3)(x-2) \geq 0$

(b) $(2-3x)(x+4) > 0$

(c) $x^2 + 9x + 20 \leq 0$

(d) $6 + x - x^2 \geq 0$

(e) $4x^2 + 12x - 7 > 0$

(f) $2 + 3x - 9x^2 > 0$

Question 16 Solve the following quadratic inequalities.

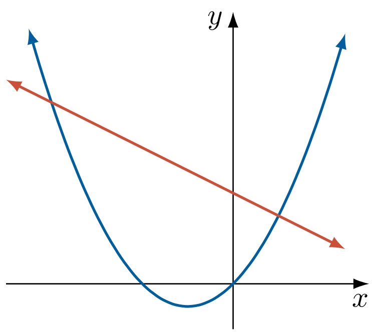
(a) $x^2 + 4x + 5 > 0$

(b) $x^2 + 6x + 10 < 0$

(c) $4x^2 + 4x + 1 \leq 0$

Question 17 By drawing an appropriate graph, solve the inequality $-5 \leq x^2 - 6x \leq 7$.

Question 18 The diagram below shows a sketch of $y = x^2 + 2x$ and $y = 4 - x$.



(a) Find where the two graphs intersect.

(b) Hence, solve $x^2 + 2x > 4 - x$.

Question 19 Determine the number of solutions to the following equations by drawing appropriate sketches.

(a) $x^2 - \sin x = 0$ (b) $2x - \sin x - 1 = 0$ (c) $e^x + x - 2 = 0$

(d) $e^{-x} + x^2 - 2 = 0$ (e) $x^2 - \ln x - 4 = 0$ (f) $\ln x + x + 4 = 0$

Question 20 Determine the number of solutions to the following equations by drawing appropriate sketches.

(a) $x^3 - x + 2 = 0$ (b) $x^3 - x^2 - 1 = 0$ (c) $x^4 - x^3 + x - 2 = 0$

Investigation Task

Investigating asymptotes

Curves like $f(x) = \dfrac{x}{x^2-1}$ or $f(x) = \dfrac{1-x^2}{1+x^2}$ are called *rational functions* in general. Previously we have studied ones with horizontal asymptotes. But not all asymptotes are as simple as $y = k$ for various values of k.

Question 1 Consider the set of functions in the form $f(x) = \dfrac{A(x)}{B(x)}$, where $A(x)$ and $B(x)$ are polynomials.

(a) Describe why they are called 'rational' functions.

(b) Show that when $\deg A = \deg B$, the asymptotes will be in the form $y = k$ for some $k \neq 0$.

(c) Bob claims that just by looking at the rational function where $\deg A \leq \deg B$, he can state the equation of the horizontal asymptote without writing anything down. Explain how he can do this. Your answer should contain two cases.

Question 2 Consider the set of functions in the form $f(x) = \dfrac{A(x)}{B(x)}$, where $A(x)$ and $B(x)$ are polynomials and $\deg A = 1 + \deg B$. Since the $\deg A > \deg B$, we can use polynomial long division and hence the division transformation to express $f(x)$ in a different form

$$f(x) = D(x)Q(x) + R(x),$$

where $D(x)$ is some divisor and $R(x)$ is some remainder.

(a) Explain why $f(x)$ does not have a horizontal asymptote.

(b) Explain what happens to $R(x)$ as x gets large in either direction.

(c) Explain the geometric significance of $D(x)$.

(d) How can $R(x)$ be used to help draw the sketch of the curve?

Question 3 Use graphing software to produce the graphs of two rational functions with linear but not horizontal asymptotes. For each graph, write down the equation of the asymptote and show how to obtain the equation of the asymptote algebraically. The two curves you submit should look very different.

Question 4 Use graphing software to produce the graphs of two rational functions with non-linear asymptotes. For each graph, write down the equation of the asymptote and show how to obtain the equation of the asymptote algebraically. The two curves you submit should look very different.

Investigation Task

Investigating roots of polynomials

The general cubic polynomial is $y = ax^3+bx^2+cx+d$ and similarly the general quartic polynomial is $y = ax^4 + bx^3 + cx^2 + dx + e$. Changing the coefficients will obviously change the shape of the graph in various ways and consequently, the number of roots. This investigation task aims to allow the student to explore further the effects of the coefficients on the number of roots.

Question 1 Consider $y = x^3 + bx^2 + cx + d$.

(a) Suppose $b = 0$. Use graphing software to sketch curves with various types of shapes and numbers of roots for various values of c and d.

(b) For each of the cases you found above, use the techniques from this chapter to verify the number of roots by using the graphs of 'smaller' curves instead.

(c) Discuss the relationship between the roots as c and d changes for each scenario. For example, do the roots move further away or closer together if d is fixed but c is increased?

(d) Repeat the above parts exercise but for $c = 0$.

Question 2 Consider $y = ax^4+bx^3+cx^2+dx+e$. Repeat the previous question, but for $b = 0$, $c = 0$ and $d = 0$ separately.

Investigation Task

Curves in the form $y = f(ax + b)$

Pretend that you are mentoring a new Year 12 student who is learning how to sketch $y = f(ax+b)$ for various values of a and b, given the sketch of $y = f(x)$.

Write a technical document that is sufficiently detailed so that a new Year 12 student can sketch $y = f(ax + b)$ from $y = f(x)$, given ANY variations of a and b.

Your document should include

- plenty of examples covering polynomials, trigonometric functions, exponential functions and logarithmic functions.
- underlying theory so that the student not only knows how to produce the sketches, but why they need to do particular steps.
- a section explaining the importance of the order of transformations and examples demonstrating this.
- a description of when the student should be wary of order, and when they do not need to worry as much.

2

TRIGONOMETRY

- Trigonometric graphs
- Trigonometric equations
- Applications of trigonometric equations

CHAPTER 2

Exercise 2A

Trigonometric graphs

Fundamentals

Fundamentals 1

(a) The a________ of a sine or cosine curve is the distance between the m_____ position and the m________ or m________ values of the function.

(b) The p_____ is the distance required for a function to complete one full cycle.

Fundamentals 2

Describe the effect on the graph of changing each of the following features on the sine curve.

(a) Amplitude (b) Period (c) Phase shift (d) Vertical shift

Fundamentals 3

The general cosine function is

$$f(x) = k\cos\left(a(x+b)\right) + c$$

Write down which parameter affects each of the following features below.

(a) Amplitude (b) Period (c) Phase shift (d) Vertical shift

Fundamentals 4

State the period and range of the following standard trigonometric functions.

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

Question 1 Use graphing software, or otherwise, to draw accurately on the number plane the following trigonometric functions for the values of $a = \pm 1, \pm 2, \pm 5$. State the amplitude and range in terms of a.

(a) $y = a\sin x$ (b) $y = a\cos x$

Question 2 Use graphing software, or otherwise, to draw accurately on the number plane the following trigonometric functions for the values of $b = 1, 2, 3$. State the period for each value of b.

(a) $y = \sin bx$ (b) $y = \cos bx$ (c) $y = \tan bx$

Question 3 Use graphing software, or otherwise, to draw accurately on the number plane the following trigonometric functions for the values of $b = \frac{1}{2}, \frac{1}{3}$. State the period for each value of b.

(a) $y = \sin bx$ (b) $y = \cos bx$ (c) $y = \tan bx$

Question 4 Use graphing software, or otherwise, to draw accurately on the number plane the following trigonometric functions for the values of $c = \pm 1, \pm 2$. Describe your findings in relation to the vertical shift of each curve

(a) $y = \sin x + c$ (b) $y = \cos x + c$ (c) $y = \tan x + c$

Question 5 Use graphing software, or otherwise, to draw accurately on the number plane the following trigonometric functions for the values of $b = \pm\frac{\pi}{2}, \pm\frac{\pi}{4}$. Describe your findings in relation to the phase shift of each curve

(a) $y = \sin(x + b)$ (b) $y = \cos(x + b)$ (c) $y = \tan(x + b)$

Question 6 Use graphing software, or otherwise, to draw accurately on the number plane the following trigonometric functions for the values of $k = \pm 2$, $c = \pm 1$. Describe your findings in relation to the mean value and range of the function

(a) $y = k \sin x + c$ (b) $y = k \cos x + c$ (c) $y = k \tan x + c$

Question 7 Use graphing software, or otherwise, to draw the graphs of the following and describe the transformation on the standard trigonometric graphs. Describe what do you notice, and explain your findings mathematically.

(a) $y = \sin(x + 2\pi)$ (b) $y = \sin(x - 2\pi)$ (c) $y = \cos(x + 2\pi)$

(d) $y = \cos(x - 2\pi)$ (e) $y = \tan(x + \pi)$ (f) $y = \tan(x - \pi)$

Question 8 Use graphing software, or otherwise, to draw the graphs of the following. Describe what do you notice, and explain your findings mathematically.

(a) $y = \cos x$ and $y = \sin\left(x + \frac{\pi}{2}\right)$. (b) $y = \sin x$ and $y = \cos\left(x - \frac{\pi}{2}\right)$.

Question 9 Find the range and period of each function below.

(a) $y = 3\sin x$ (b) $y = 5\cos 2x$ (c) $y = -5\cos\left(\frac{x}{2}\right)$

(d) $y = -2\sin 4x$ (e) $y = \tan 2x$ (f) $y = -6\tan\left(\frac{x}{6}\right)$

Question 10 Find the period of the following.

(a) $y = \sin \pi x$ (b) $y = \cos\left(\frac{\pi}{2}x\right)$ (c) $y = \tan \pi x$

Question 11 Find the range of each function.

(a) $y = -3\sin(x + \pi)$ (b) $y = 2\cos\left(x - \frac{\pi}{3}\right)$ (c) $y = -3\sin\left(2x - \frac{\pi}{3}\right)$

(d) $y = 2\sin(x) + 1$ (e) $y = 2\cos(x) - 5$ (f) $y = -5\sin(x) - 4$

(g) $y = 2 - \cos\left(\frac{2x}{3}\right)$ (h) $y = \frac{1}{2} - \frac{3}{2}\sin\left(2x + \frac{\pi}{3}\right)$ (i) $y = 1 + 2\tan 2x$

Question 12 If $f(x) = a\sin\left(\frac{x}{b}\right)$ find the equation of the function if

(a) amplitude $= 2$ and period $= \pi$.

(b) amplitude $= 5$ and period $= 1$.

Question 13 State the range and the period of the function $f(x) = 3 + 4\cos\frac{\pi x}{2}$

Question 14 Write down the new equation for each function after the given translation has been applied, and sketch the graph of the new curve.

(a) $y = \cos x$

Right $\frac{\pi}{2}$, down 3 units

(b) $y = 2\sin x$

Left $\frac{\pi}{6}$, up 2 units

(c) $y = -\tan x$

Right $\frac{\pi}{4}$, up 1 unit

Question 15 Consider the general sine function

$$y = k\sin\left(a(x+b)\right) + c$$

Write down the equation of the trigonometric function satisfying the parameters below, and hence find the period and range of the curve.

(a) $k = 3,\ a = 2,\ b = \frac{\pi}{6},\ c = -1$

(b) $k = -2,\ a = 3,\ b = -\frac{\pi}{6},\ c = 3$

Question 16 Write down the new curve if the graphs of the following were transformed in the following ways.

(a) $f(x) = \sin x$, squash horizontally by a factor of 2, then translate right by $\frac{\pi}{6}$ units.

(b) $f(x) = \sin x$, translate right by $\frac{\pi}{6}$ units, then squash horizontally by a factor of 2.

(c) $f(x) = \tan x$, squash horizontally by a factor of 3, then translate left by $\frac{\pi}{2}$ units.

(d) $f(x) = \tan x$, translate left by $\frac{\pi}{2}$ units, then squash horizontally by a factor of 3.

(e) $f(x) = \tan x$, stretch horizontally by a factor of 2 and translate left by $\frac{\pi}{2}$ units.

(f) $f(x) = \tan x$, translate left by $\frac{\pi}{2}$ units, then stretch horizontally by a factor of 2.

Question 17 Describe the transformations that were made to the graph of $f(x) = \sin(x)$ to obtain the following.

(a) $y = 2\sin\left(\frac{x}{3}\right)$

(b) $y = \sin(2x) + 1$

(c) $y = -\sin\left(\frac{x}{3}\right)$

(d) $y = \frac{1}{2}\sin\left(x + \frac{\pi}{2}\right)$

Question 18 Describe the transformations that were made to the graph of $f(x) = \tan(x)$ to obtain the following.

(a) $y = \tan\left(2x - \frac{\pi}{2}\right)$

(b) $y = \tan\left(\frac{\pi}{3} - x\right)$

(c) $y = \tan\left(\frac{2\pi}{3} - 2x\right)$

(d) $y = \tan\left(-\frac{\pi}{2} - \frac{x}{3}\right)$

Question 19 Sketch the following graphs.

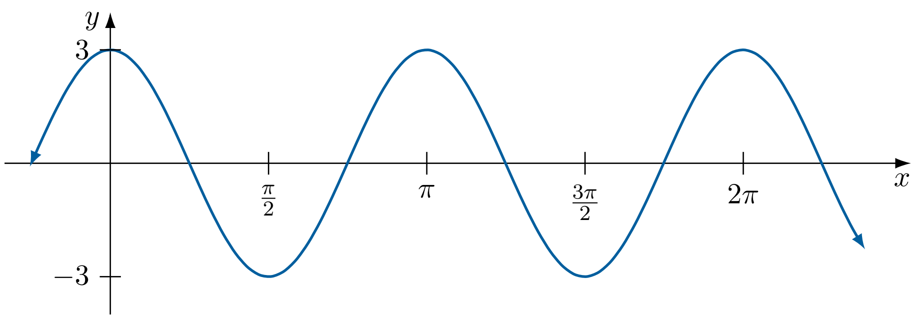
(a) $y = 2\cos\left(x - \frac{\pi}{2}\right)$

(b) $y = 3 - 2\cos(x)$

(c) $y = -\sin(2x) + 3$

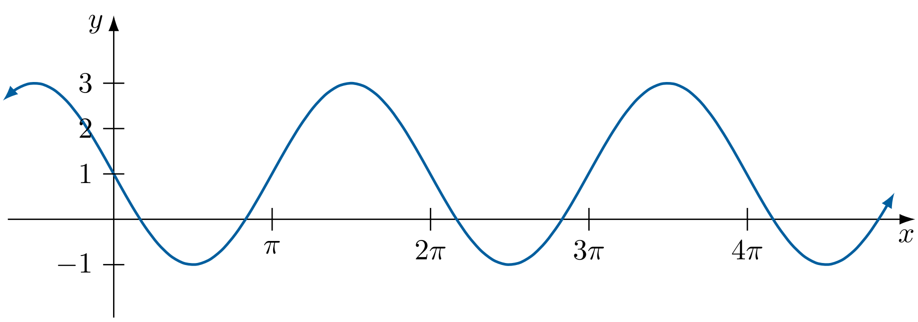
(d) $y = \sin\left(2x + \frac{\pi}{2}\right)$

Question 20 Part of the graph of $y = k\cos(ax)$ is drawn below.



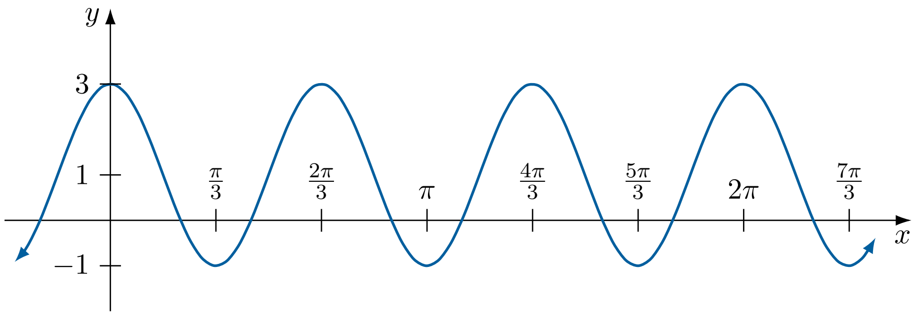
Work out the period and amplitude, and hence find the value of k and a.

Question 21 Part of the graph of $y = k\sin(x) + c$ is drawn below.



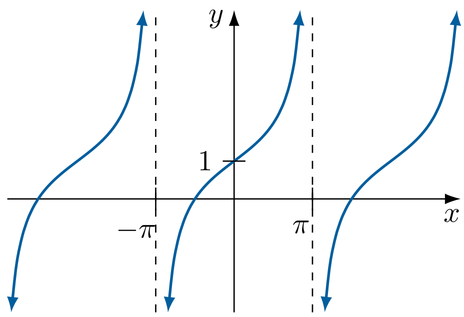
Work out the period and amplitude, and hence find the value of k and c

Question 22 Part of the graph of $y = k\cos(ax) + c$ is drawn below.



Work out the period and amplitude, and hence find the value of a, k and c.

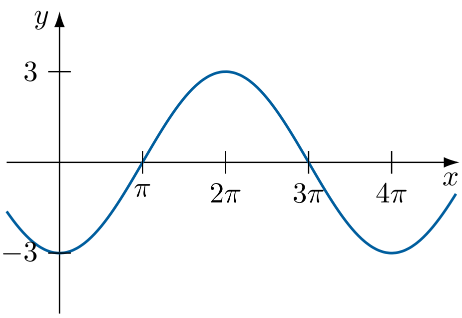
Question 23 Part of the graph of $y = \tan(ax) + c$ is drawn below.



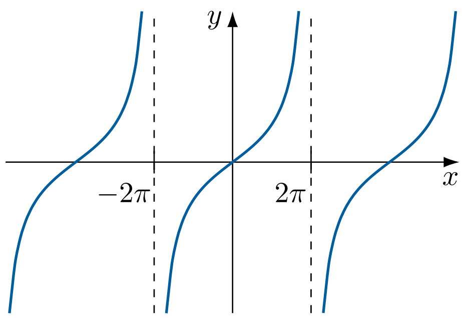
Work out the period and amplitude, and hence find the value of a, k and c.

Question 24 Find a possible equation of the following curves.

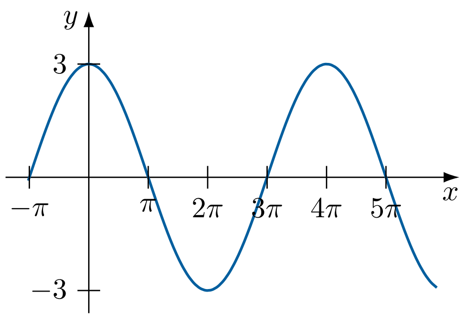
(a)



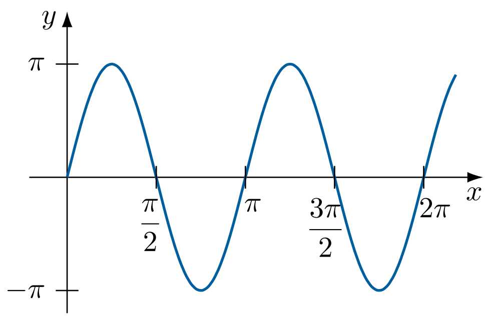
(b)



(c)



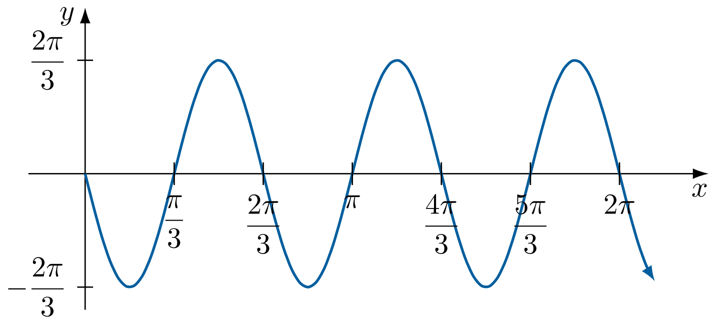
(d)



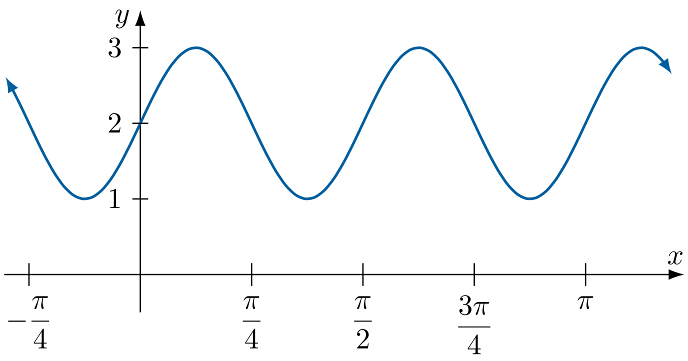
Challenge Problems

Problem 1 Find two possible equations for the following graphs.

(a)



(b)



Problem 2 Use graphing software to draw the curve $y = 3 - 2\cos^2(2x)$. By examining this curve carefully, express it in the form $y = c - k\cos(4x)$. Find the values of c and k and hence find the amplitude, range and period of this function.

Problem 3 Describe an appropriate sequence of transformation that will transform the graph of $f(x) = \cos(x)$ into the graph of $y = -3\cos\left(2x - \frac{\pi}{2}\right) + 1$. In each step of the sequence, write down the equation of the curve to demonstrate the progression of transformations.

Exercise 2B

Trigonometric equations

Fundamentals

Fundamentals 1

Consider the equation $\sin(x - b) = k$ in the domain $-2\pi \leq x \leq 2\pi$. The following below shows the set of steps required to solve the equation.

(a) We need to first modify the d_____ to match the inside of the brackets.

(b) This means that $\_\_ \leq x - b \leq \_\_$.

(c) Solve for _____ and exhaust the domain.

(d) Finally, make __ the subject to obtain the final set of solutions.

Fundamentals 2

Consider the equation $\sin(ax) = k$ in the domain $-2\pi \leq x \leq 2\pi$. The following below shows the set of steps required to solve the equation.

(a) Modify the domain so $\_\_ \leq ax \leq \_\_$.

(b) Solve for __ and exhaust the domain.

(c) Finally, make __ the subject to obtain the final set of solutions.

Fundamentals 3

Describe roughly the set of steps needed to solve $\sin(ax + b) = k$ in the domain $-2\pi \leq x \leq 2\pi$.

Question 1 [Revision with special exact values]

Solve the following equations in the domain $0 \leq x \leq 2\pi$

(a) $\sin x = 0$
(b) $\sin x = 1$
(c) $\sin x = -1$
(d) $\cos x = 0$
(e) $\cos x = 1$
(f) $\cos x = -1$

Question 2 [Revision with standard exact values]

Solve the following equations in the domain $0 \leq x \leq 2\pi$

(a) $\sin x = \frac{1}{2}$
(b) $\sin x = -\frac{1}{\sqrt{2}}$
(c) $\sin x = \frac{\sqrt{3}}{2}$
(d) $\cos x = -\frac{1}{2}$
(e) $\cos x = \frac{1}{\sqrt{2}}$
(f) $\cos x = -\frac{\sqrt{3}}{2}$
(g) $\tan x = \frac{1}{\sqrt{3}}$
(h) $\tan x = -1$
(i) $\tan x = \sqrt{3}$

Question 3 [Breaking down the steps]

Consider the equation $\sin\left(2x+\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$ in the domain $0 \le x \le 2\pi$.

(a) Complete the domain

$$\_\_ \le 2x \le \_\_$$

(b) Complete the domain

$$\_\_ \le 2x+\frac{\pi}{3} \le \_\_$$

(c) Solving, we have

$$2x+\frac{\pi}{3}=\frac{\pi}{3},\ \_\_,\ \_\_,\ \_\_,\ \_\_$$

(d) Hence, the solutions of the equation are

$$x = \_\_,\ \_\_,\ \_\_,\ \_\_,\ \_\_$$

Question 4 Solve the following equations in the domain $0 \le x \le 2\pi$

(a) $\sin(2x)=-\frac{1}{2}$ (b) $\sin\left(\frac{x}{2}\right)=\frac{\sqrt{3}}{2}$ (c) $\sin\left(\frac{2x}{3}\right)=-\frac{1}{\sqrt{2}}$

(d) $\cos(3x)=\frac{1}{\sqrt{2}}$ (e) $\cos\left(\frac{x}{3}\right)=-\frac{\sqrt{3}}{2}$ (f) $\cos\left(\frac{3x}{2}\right)=\frac{1}{2}$

(g) $\tan(3x)=-1$ (h) $\tan\left(\frac{x}{2}\right)=\frac{1}{\sqrt{3}}$ (i) $\tan\left(\frac{3x}{2}\right)=-\sqrt{3}$

Question 5 Solve the following equations in the domain $-\pi \le x \le \pi$

(a) $\sin\left(x+\frac{\pi}{3}\right)=-\frac{1}{2}$ (b) $\cos\left(x-\frac{2\pi}{3}\right)=-\frac{1}{\sqrt{2}}$ (c) $\tan\left(x+\frac{5\pi}{6}\right)=\frac{1}{\sqrt{3}}$

Question 6 Solve the following equations in the domain $-\pi \le x \le \pi$

(a) $\sin\left(2x+\frac{\pi}{3}\right)=1$ (b) $\cos\left(\frac{x}{3}-\frac{\pi}{2}\right)=-1$ (c) $\tan\left(2x+\frac{\pi}{3}\right)=0$

(d) $\sin\left(3x-\frac{3\pi}{2}\right)=\frac{1}{2}$ (e) $\cos\left(2x+\frac{2\pi}{3}\right)=-\frac{1}{2}$ (f) $\tan\left(\frac{x}{2}-\frac{\pi}{6}\right)=\sqrt{3}$

Question 7 Solve the following equations in the domain $-2\pi \leq x \leq 2\pi$

(a) $\sin(2x) = 0$ (b) $\cos(3x) = 0$ (c) $\tan(2x) = 0$

(d) $\sin\left(\frac{x}{2}\right) = -1$ (e) $\cos\left(\frac{x}{3}\right) = 1$ (f) $\tan\left(\frac{2x}{3}\right) = 0$

(g) $\sin\left(2x - \frac{\pi}{3}\right) = 1$ (h) $\cos\left(2x - \frac{2\pi}{3}\right) = -1$ (i) $\tan\left(\frac{x}{2} + \frac{3\pi}{4}\right) = 0$

Question 8 Use graphing software to sketch each straight line and trigonometric equation in the domain $-\pi \leq x \leq \pi$, and hence read off their points of intersection.

Note: The answers to the following questions are exact values, so you will need to choose the scale on the x-axis carefully so that the solutions can be read.

(a) $y = \frac{1}{2}$, $y = \cos(2x)$ (b) $y = 1$, $y = \tan x$

(c) $y = 1$, $y = \tan\left(x + \frac{\pi}{4}\right)$ (d) $y = -1$, $y = \tan(2x)$

(e) $y = -1$, $y = \tan\left(2x - \frac{\pi}{2}\right)$ (f) $y = -1$, $y = \sin\left(2x - \frac{\pi}{6}\right)$

Challenge Problems

Problem 1 Solve the following equations for $0 \leq x \leq 2\pi$.

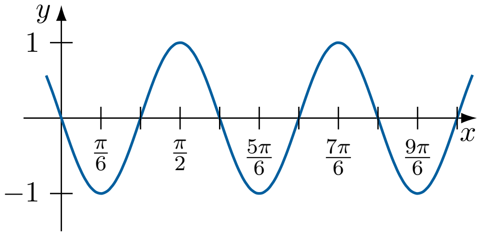
(a) $\sec\left(x + \frac{\pi}{4}\right) = \sqrt{2}$ (b) $\operatorname{cosec}\left(x - \frac{3\pi}{2}\right) = 2$ (c) $\cot\left(x + \frac{2\pi}{3}\right) = \sqrt{3}$

Problem 2 Solve the following equations for $0 \leq x \leq 2\pi$.

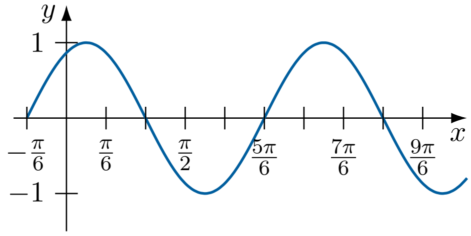
(a) $\tan^2(2x) = 1$ (b) $4\cos^3\left(\frac{x}{2}\right) = 3\cos\left(\frac{x}{2}\right)$ (c) $\sin(2x) = 2\operatorname{cosec}(2x) - 1$

Problem 3 Each graph below is part of a curve with equation $y = \sin(ax - b)$ where $a > 0$, $0 < b \leq \pi$. Use the information on the diagram to find the values of a and b, and hence the equation of the curve

(a)



(b)



Exercise 2C

Applications of trigonometric equations

Question 1 The displacement of a particle x metres from the origin after t seconds is given by $x = 3\cos(\pi t)$.

(a) What is the initial position of this particle?

(b) After how many seconds does the particle return to this position?

(c) Between which two points is the particle oscillating?

(d) Find the centre of motion of the particle

(e) What is the period of this motion

Question 2 The displacement of a particle x metres from the origin after t seconds is in the form of $x = k\sin at$. Its period is 24 seconds and its amplitude is 80 metres. Initially it is at the origin.

(a) Find the equation of this motion

(b) Find the first time it is 40 metres to the right of the origin and when 40 metres to the left of the origin.

(c) Where is the particle 10 seconds after motion has started?

Question 3 The displacement of a particle x metres from the origin after t seconds is $x = 6\cos\left(2t - \frac{\pi}{3}\right)$

(a) State the amplitude and period of this motion

(b) How is this motion different from $x = 6\cos 2t$?

(c) Using a graphing program otherwise sketch the graph of $x = 6\cos 2t$ and $x = 6\cos\left(2t - \frac{\pi}{3}\right)$

(d) What is the initial position of each motion?

(e) What main features are the same?

(f) How many times is each particle at the origin during the first 2π seconds?

(g) How would you read off from your graph when the particle is at $x = -3$ m on both graphs. State the times in seconds for each.

(h) The displacement of a third particle is given by $x = 6\cos 2\left(t - \frac{\pi}{6}\right) + 6$. Describe the similarities and differences between this motion and the original one.

Question 4 Mary was riding her bicycle at a steady speed along a straight line and Bob noticed that the height h (cm) of the pedal in terms of the time t (seconds) is given by the equation

$$h = 28 + 14\sin \pi t$$

(a) Sketch the graph of the height vs. the time

(b) How long will the pedal take to complete a revolution?

(c) If Mary cycled faster, how would the above equation change?

Question 5 It was stated that the average monthly temperatures taken over a year, starting in January for a small town in NSW model the function $T = -10\cos\left(\frac{\pi}{6}t\right) + 20$, where t is time starting in January and T is temperature recorded

(a) Use a graphing program to draw this function

(b) What was the highest temperature recorded and in what month?

(c) What was the lowest temperature recorded and in what month?

Challenge Problems

Problem 1 A particle moves in a straight line and its position at t is given by $x = k\cos(at+b)$. The particle is at the origin at $t = \frac{\pi}{6}, \frac{5\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \ldots$ and oscillates between -2 and 2. Find the equation of motion.

Problem 2 The equation of motion of a particle is in the form $x = x_0 - k\cos nt$.

(a) State the amplitude, period and range of this motion

(b) Use graphing software to sketch the curve $y = 2\sin^2 t$.

(c) Mary tells Bob that the equation $y = 2\sin^2 t$ will in fact be the same equation as the above for certain values of x_0, k and n. She tells him to sketch $y = -\cos(2t)$ to convince him. Draw $y = -\cos(2t)$ on the same diagram as the above and hence find the values of x_0, k and n.

(d) Write down the equation of motion in the original form and now state the centre, amplitude, range and period of the motion.

Chapter 2 Review

Trigonometry

Review

Question 1 Find the period and range of each trigonometric function

(a) $y = -2\cos(2x)$

(b) $y = 2\sin\left(\frac{x}{3}\right)$

(c) $y = \tan\left(\frac{x}{2}\right)$

(d) $y = 2\cos\left(x + \frac{\pi}{3}\right)$

(e) $y = 2\sin 3\left(x - \frac{\pi}{2}\right)$

(f) $y = 4 - \sin(2x)$

(g) $y = 5 - \cos\left(\frac{x}{2}\right)$

(h) $y = -3\sin\left(2x - \frac{\pi}{2}\right)$

(i) $y = 2 - \cos\left(\frac{x}{3}\right)$

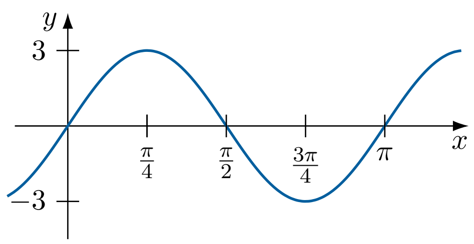
Question 2 What trigonometric identity do you find when you shift $y = \cos x$

(a) to the left by $\frac{\pi}{2}$?

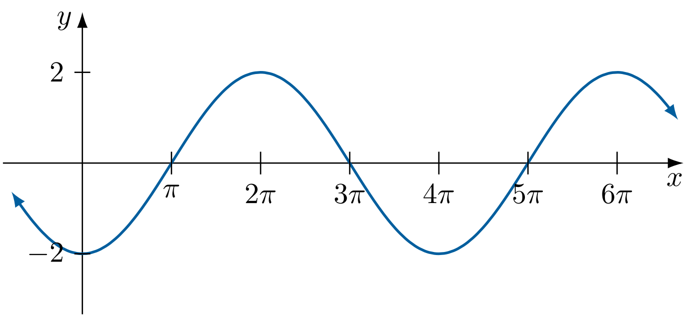
(b) to the right by π?

(c) to the right by 2π?

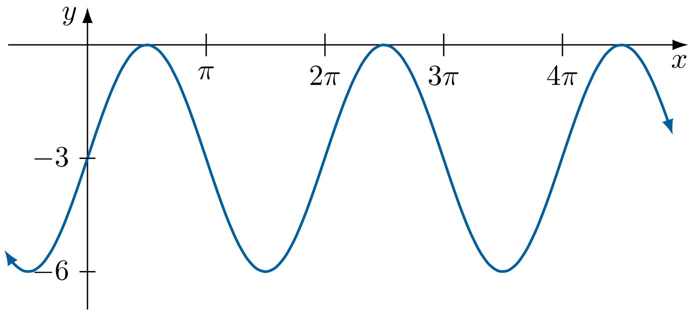
Question 3 Part of the graph of $y = k\sin ax$ is drawn below in the diagram. Work out the period and range of the function hence the values of k and a.



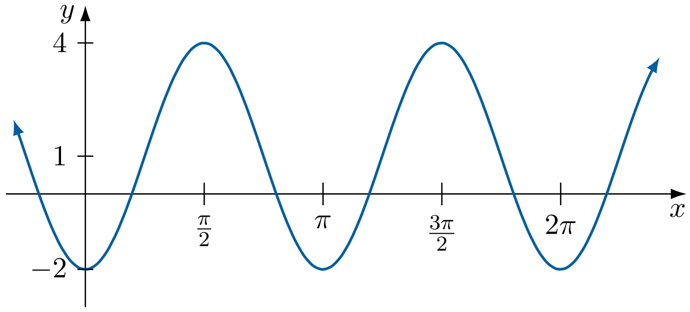
Question 4 Part of the graph of $y = k\cos ax$ is drawn below. Work out the period and amplitude, and hence find the value of k and a.



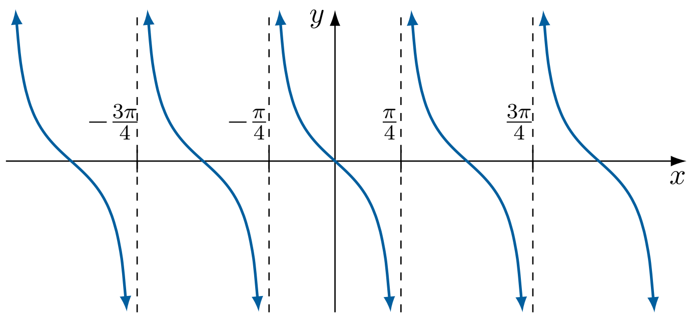
Question 5 Part of the graph of $y = k\sin x + c$ is drawn below. Work out the period and amplitude, and hence find the value of k and c



Question 6 Part of the graph of $y = k\cos(ax) + c$ is drawn below. Work out the period and amplitude, and hence find the value of a, k and c.



Question 7 Part of the graph of $y = \pm\tan ax + c$ is drawn below. Work out the period and amplitude, and hence find the value of a, k and c.



Question 8 Describe an appropriate sequence of transformations that you could use to obtain the following graphs from the standard trigonometric functions $f(x) = \sin x$, $f(x) = \cos x$ and $f(x) = \tan x$.

(a) $y = \sin\left(2x - \frac{\pi}{4}\right)$ (b) $y = \cos\left(\frac{x}{3} + \frac{\pi}{6}\right)$ (c) $y = \tan\left(\frac{\pi}{4} - x\right)$

Question 9 Describe a sequence of transformations that turns $f(x) = \cos(x)$ into

(a) $y = 2\cos(x) - 3$

(b) $y = -\cos(x) + 2$

(c) $y = \cos\left(2x - \frac{\pi}{2}\right)$

(d) $y = \cos\left(\frac{x}{2} + \frac{\pi}{3}\right)$

Question 10 Sketch the following.

(a) $y = -2\cos\left(x - \frac{\pi}{2}\right)$

(b) $y = \sin\left(2x + \frac{\pi}{2}\right) - 2$

Question 11 Draw the following graphs for the domain $0 \le x \le 2\pi$

(a) $y = \sin(2x)$

(b) $y = \sin(3x)$

(c) $y = -3\cos(2x)$

(d) $y = 3\sin\left(\frac{x}{2}\right)$

(e) $y = \sin\left(x + \frac{\pi}{6}\right)$

(f) $y = 1 - \sin(\pi x)$

(g) $y = 2\cos\left(x - \frac{\pi}{4}\right)$

(h) $y = 2\cos(x) + 1$

Question 12 Using graphing software to sketch each straight line and trigonometric equation in the domain $-\pi \le x \le \pi$ and read off their points of intersection.

(a) $y = 0,\ y = \cos\left(2x - \frac{\pi}{6}\right)$

(b) $y = 1,\ y = 2\cos 3x + 1$

(c) $y = 3,\ y = 2\cos 3x + 1$

(d) $y = -1,\ y = 2\cos 3x + 1$

Question 13 Solve the following equations for the domain $0 \le x \le 2\pi$

(a) $\sin\left(\frac{x}{3}\right) = -\frac{1}{2}$

(b) $\sin 2x = -\frac{1}{2}$

(c) $\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$

(d) $\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Question 14 Solve the following equations in the domain $-\pi \le x \le \pi$

(a) $\cos(2x) = \frac{1}{2}$

(b) $\tan(2x) = -1$

(c) $\tan\left(\frac{x}{3}\right) = 1$

(d) $\tan\left(x + \frac{\pi}{4}\right) = -1$

(e) $\tan(2x) = -\sqrt{3}$

(f) $\sin(3x) = \frac{1}{2}$

(g) $\cos(2x) = -\frac{1}{\sqrt{2}}$

(h) $\cos\left(\frac{x}{3}\right) = -\frac{1}{\sqrt{2}}$

(i) $\cos\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}$

(j) $\cos 3x = \frac{1}{\sqrt{2}}$

(k) $\cos\left(3x - \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}$

(l) $\sin\left(3x - \frac{\pi}{6}\right) = \frac{1}{2}$

Question 15 The displacement of a particle is given by $x = 8\cos \pi t$ where t is time and x is in metres

(a) Find the maximum displacement of the object

(b) Find the period of this motion

(c) Sketch the graph

Question 16 A mass is placed at the end of an elastic string and its height h (metres) above the ground at time t (seconds) is given by $h = 1.6 + 0.4\cos 2\pi t$

(a) Between what height is the mass bouncing between?

(b) What is the time period for the mass to beat the highest point in consecutive bounces?

Question 17 Assume the formula $H = -4\cos(0.0172t) + 12$ gives the number of hours of daylight in Sunnytown when you put in any day of the year. Letting t be the day of the year (from 1 to 365), you can figure the number of hours of sunlight, H, if you enter a value for t in the equation above.

(a) Write down the amplitude of this curve

(b) Write down the period and explain your answer

(c) What is the highest number of daylight hours and when does it occur?

Question 18 Tides are measured continuously at a particular location and it is noticed that it goes through two complete cycles in a 24-hour span. It is found that $y = -8\cos\left(\frac{\pi t}{6}\right) + 12$ can be used to model the motion of the tide.

(a) Write down the height of high tide and low tide.

(b) Use graphing software to draw a sketch of the graph.

(c) If low tide occurred at 4 a.m. label on your diagram the times for all high and low tides.

Investigation Task

Modelling tides

At Sydney Harbour on a certain day, a low tide of 4 metres occurs at 4am and a high tide of 16 metres occurs at 10am. Assume that the rise and fall of the tide can be modelled by a trigonometric equation. A ship can safely enter and leave the harbour if the depth is at least 12 metres.

Question 1

(a) Does it make more sense to use $y = A\sin(nt) + x_0$ or $y = A\cos(nt) + x_0$ to model the above scenario? Explain your answer.

(b) Find the period and hence n.

(c) Find A and x_0 and hence, write down the equation of the trigonometric function that models the tide.

(d) Draw the graph of the trigonometric function, and shade the section of the curve where it is safe for a ship to enter or leave the harbour.

(e) Hence, find between which times when the ship can safely enter or leave the harbour.

Question 2 If you go on any website that logs the tides at various bays and ports in Sydney, and you will see that the tides do not really follow a normal trigonometric graph in the form $y = A\cos(nt + \alpha) + x_0$.

(a) What is the behaviour of the data instead that prevents us from using the trigonometric function above?

(b) Use graphing software to investigate graphs in the form

$$y = A\sin(mt + \alpha) + B\sin(nt + \alpha)$$

Explain briefly why these graphs may better model the data that you see.

(c) Pick a location of your choice and come up with an equation in the form

$$y = A\sin(mt + \alpha) + B\sin(nt + \alpha)$$

that models the data more closely than the standard trigonometric function.

3

SEQUENCES AND SERIES

- Arithmetic progression
- Geometric progression
- Simple and compound interest
- Annuities, present and future values
- Regular instalments
- Loan repayments

Exercise 3A

Arithmetic progression

Fundamentals

Fundamentals 1

(a) A number s________ is an ordered list of numbers defined by a rule.

(b) A s_____ is the sum of the terms of a sequence.

Fundamentals 2

(a) An arithmetic progression is a sequence of numbers where any two consecutive terms differ by a c________.

(b) The recursion for an arithmetic progression is $T_n =$ ________, where $T_1 =$ ___.

(c) For example, if the first term T_1 of an arithmetic progression is a and the difference is d, then the next two terms are _____ and _____.

(d) In general, the n^{th} term of an arithmetic progression is $T_n =$ __________.

Fundamentals 3
The test for a sequence being an arithmetic sequence is that

$$T_2 - T_1 = T_3 - \_\_\_\_\_$$

or more generally

$$T_n - T_{n-1} = T_{n-1} - \_\_\_\_\_$$

Fundamentals 4
Let $S_n = T_1 + T_2 + T_3 + \cdots + T_n$.

(a) An arithmetic sum is the sum of the terms in an arithmetic p________.

(b) If you know the last term l of an arithmetic sum, use the formula $S_n =$ __________

(c) If you know only the number of terms of an arithmetic sum, use the formula $S_n =$ __________

(d) Complete the formula

$$T_n = S_n - \_\_\_$$

Fundamentals 5
State the arithmetic mean of two numbers a and b.

Question 1 Write down the first four terms of the sequence defined by the recursive relationship given

(a) $T_n = T_{n-1} + 3$, $T_1 = 4$

(b) $T_n = T_{n-1} + 4$, $T_1 = -3$

Question 2 Write a recursive rule for T_n in terms of T_{n-1} which defines the sequence below and an initial condition for T_1

(a) Each term is obtained by increasing the previous term by 8. The first term is -10.

(b) Each term is obtained by decreasing the previous term by 5. The first term is 9.

(c) The first three term of the arithmetic progression 5, 11, 17, ...

(d) The first three term of the arithmetic progression 5, -2, -9, ...

(e) The n^{th} term in an arithmetic progression is given by $T_n = -6n + 20$

Question 3 The first term of an arithmetic sequence is 4 and the fifth term is 32

(a) Find d the common difference of the sequence

(b) Write a recursive rule for T_n in terms of T_{n-1} which defines this sequence

Question 4 Construct the first 4 terms of each arithmetic sequence in the following questions using the given values of T_1, and d and the recursive formula $T_n = T_{n-1} + d$

(a) $T_1 = -2$ and $d = 11$

(b) $T_1 = 50$ and $d = -8$

Question 5 Find the 10^{th} term of the arithmetic sequence

(a) $-1, 3, 7, 11, \ldots$

(b) $\sqrt{5}, \sqrt{5} + \sqrt{2}, \sqrt{5} + 2\sqrt{2}, \ldots$

Question 6 Find a so the numbers form an arithmetic progression

(a) $a - 7, a + 1, 3a + 1, \ldots$

(b) $a + 4, 12 - a, 3a + 2$

Question 7 The n^{th} term for an arithmetic progression is given by $T_n = 4n + 8$

(a) Find a the first term in the arithmetic progression

(b) Determine d, the common difference

(c) Find T_8, the 8^{th} term in the sequence

Question 8

(a) Find the general term of the sequence $-108, -100, -92, -84, \ldots$

(b) What is the first positive term of this sequence

Question 9 If the general term of an arithmetic progression is $T_n = 5n - 134$

(a) What term is 26?

(b) What is the first positive term of this sequence?

Question 10

(a) Find the general term and then the first negative term in the sequence 28, 25, 22, ...

(b) Find the general term of the sequence −1, 3, 7, 11, ... and then find which term of the sequence is 71?

(c) In an arithmetic sequence

(i) $T_4 = 27$ and $T_7 = 12$ find T_9

(ii) $T_6 = 5$ and $T_{13} = -16$ find T_9

Question 11 Find the number of terms in the series

(a) $\frac{3}{4} + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^5 + \cdots + \left(\frac{3}{4}\right)^{37}$

(b) $\frac{3}{4} + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^5 + \cdots + \left(\frac{3}{4}\right)^{2k+1}$

Question 12 Find the arithmetic mean of

(a) $\frac{1}{4}$ and $\frac{1}{8}$

(b) $\sqrt{5} + 1$ and $\sqrt{5} - 1$

Question 13 Insert 5 arithmetic means between 25 and 7.
Hint: Which term of the sequence is 7?

Question 14 Insert 7 arithmetic means between −3 and 45

Question 15 [Proof of the formula for the arithmetic sum]

We know that

$$S_n = T_1 + T_2 + T_3 + \cdots + T_{n-1} + T_n$$
$$S_n = T_n + T_{n-1} + T_{n-2} + \cdots + T_3 + T_2 + T_1$$

Using the above prove the formula for S_n

Question 16

(a) How many terms of the series $5 + 7 + 9 + 11 + \cdots$ must be added to give a sum of 525

(b) How many terms of the series $21 + 18 + 15 + \cdots$ must be added to give a sum of 0.

Question 17 For the sequence 70, 64, 58, 52, ...

(a) Find the smallest value of n for which $T_n < 0$

(b) Find the smallest value of n for which $S_n < 0$

Question 18 What is the value of the sum $4 + 9 + 14 + \cdots + 84$

Question 19 In a particular arithmetic sequence, there are 14 terms between the first term -11 and the last term 34.

(a) Find the first three terms of the series

(b) Find the sum of all the terms of this series

Question 20 Given the sequence $\log 4, \log 8, \log 16, \log 32, \ldots$

(a) Prove it is arithmetic

(b) Find the 7^{th} term

(c) Find the sum of the first 7 terms

Question 21

(a) Given $T_n = 3n - 2$, find S_n.

(b) Given $T_n = 15n - 21$, find T_4 and S_{10}.

Question 22 Simplify the following.

(a) $S_6 - S_5$

(b) $S_6 + T_7$

(c) $S_6 - T_6$

(d) $S_n - S_{n-1}$

(e) $S_n + T_{n+1}$

(f) $S_n - T_n$

Question 23 An arithmetic series has a sum given by $S_n = 3n^2 - 4n$.

(a) Write down an expression for T_9 in terms of S_n.

(b) Find T_9.

(c) Find the general formula for T_n.

Question 24 The positive multiples of 9 are 9, 18, 27, ...

(a) What is the largest multiple of 9 that is less than 600?

(b) What is the sum of the positive multiples of 9 that are less than 600?

Question 25 In a particular arithmetic sequence, the tenth term is -3. Also, the ninth term is equal to the sum of the first seventeen terms. Find the first term and common difference of the sequence.

Question 26

(a) How many terms of the series $75 + 69 + 63 + \cdots$ will give a sum of 495?

(b) Explain briefly why there are two solutions.

Question 27 In the first week of the snow season 3 cm of snow falls. In each following weeks the snowfalls increase by 2 cm, so in the second week there is 5 cm, in the third week there is 7 cm. How much snow falls in the 12^{th} week?

Question 28 Mary has started a business selling cupcakes. The first week she sold 30 cupcakes.

(a) Set up a recurrence relation if the number of cupcakes Mary sells increases by 8 each week.

(b) Write down an expression for T_n in terms of n, for the recurrence relation found in (a).

(c) Find the number of cupcakes Mary is expected to sell in the fifth week.

Question 29 Cans of fruit in a supermarket are stacked so that there are 3 cans in the top row, 5 in the next row, 7 in the next row and so on. If there are 8 rows in the display, find

(a) the number of cans in the bottom row.

(b) the total number of cans in the display.

Question 30 Boxes are stacked in layers, where each layer contains three boxes fewer than the layer below. There are 8 boxes in the top layer, 11 boxes in the next layer, and so on. There are n layers altogether.

(a) Write an expression for the number of boxes in the bottom layer?

(b) Show that there are $\frac{n}{2}(3n + 13)$ boxes in the stack

Question 31 The cost of building a 12 storey apartment block varies for each floor. The cost of the first floor is \$300 000, the second is \$305 000 and the third \$310 000 and so on until a maximum cost of \$335 000 is reached after which the cost remains the same for each subsequent floor. Find:

(a) cost of the 9^{th} floor.

(b) the total cost of the building.

Question 32 Find the arithmetic mean of of $\ln 4$, $\ln 9$

Question 33 Find x if $3x$ is the arithmetic mean of $5x - 3$ and 15

Challenge Problems

Problem 1

(a) Find the sum of the first 8 terms of $\log_3\left(\frac{1}{x}\right)+\log_3\left(\frac{1}{x^2}\right)+\log_3\left(\frac{1}{x^3}\right)+\cdots$ where $x>0$.

(b) Find the value of x if the sum is -144

Problem 2 x, $2x^2$ and $7x$ are three consecutive terms of an arithmetic series.

(a) Show that $x^2-2x=0$

(b) Find the first 4 terms of this series.

Problem 3 The sum of the first n terms of a certain arithmetic series is given by

$$S_n=\frac{n(5n-1)}{2}$$

Find an expression for the n-th term and the first 3 terms of the sequence

Problem 4 In an arithmetic progression we are given that $2a=-9d$

(a) Show that $S_{10}=0$

(b) Express $T_{11}+T_{12}$ in terms of d

(c) Simplify S_6+S_{12}

Problem 5 Find the sum of all the numbers between 200 and 800 (non-inclusive), which are *not* divisible by 7.

Problem 6 Show that the sum of the first 10 terms of any arithmetic sequence is

(a) five times the sum of the fourth and seventh term

(b) five times the sum of the fifth and sixth term

(c) five times the sum of T_m and T_n as long as $m+n=11$.

Exercise 3B

Geometric progression

Fundamentals

Fundamentals 1

(a) A geometric progression is a sequence of numbers each successive term can be obtained by multiplying the previous term by a fixed value called a c_____ r___.

(b) For example, if the first term of a geometric progression is a and the ratio is r, then the next two terms are _____ and _____.

(c) The recursion for a geometric progression is $T_n =$ _______, where $T_1 =$ ___.

(d) In general, the n^{th} term of a geometric progression is $T_n =$ __________.

(e) To test if a sequence forms a geometric sequence, show that

$$\frac{T_3}{T_2} = \frac{T_2}{}$$

or more generally

$$\frac{T_n}{T_{n-1}} = \frac{T_{n-1}}{\phantom{T_{n-2}}}$$

Fundamentals 2

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$.

(a) A geometric sum is the sum of the terms in a geometric p_______.

(b) If $r > 1$, then use the formula $S_n =$ __________

(c) If $r < 1$, then use the formula $S_n =$ __________

Fundamentals 3

(a) The l_____ sum of a geometric sum is the value that the sum approaches as we add i_______ many terms.

(b) This sum, sometimes denoted by S_∞, only exists if $|r| <$ ___ or otherwise sometimes expressed as ___ $< r <$ ___.

(c) The formula for S_∞ is

$$S_\infty = a + ar + ar^2 + ar^3 + \cdots = \_\_\_\_\_\_\_$$

Fundamentals 4

The geometric mean of two numbers a and b is _____.

Question 1 Find the recurrence relation for the sequence:

(a) $-9, 3, -1, \frac{1}{3}, \ldots$ (b) $\frac{1}{2}, 1, 2, 4, \ldots$

Question 2 Find a if the numbers $a-6$, $a+3$, $6a-6$, ... are in geometric progression

Question 3 Use the recursive formula $T_n = rT_{n-1}$ to find the first 4 terms of the geometric progression given

(a) $T_1 = -\frac{1}{8}$ and $T_n = 2T_{n-1}$ (b) $T_1 = 12$ and $T_n = -\frac{1}{4}T_{n-1}$ (c) $T_1 = \sqrt{2}$, and $T_n = 2\sqrt{2}T_{n-1}$

Question 4 Find the 10^{th} term of the geometric progression 4, 12, 36, ...

Question 5 Consider the sequence 2, 6, 18, 54, ...

(a) Find the 10^{th} term

(b) Is 72 a member of the sequence?

(c) How many terms of the sequence are less than 300?

(d) Find the first term of the sequence which exceeds 5000.

Question 6 Which term of the geometric progression 48, −24, 12, −6, ... is $\frac{3}{16}$

Question 7 In each geometric sequence below defined by two terms, find T_6

(a) $T_1 = 81$ and $T_4 = \frac{1}{9}$. (b) $T_2 = 3$ and $T_5 = \frac{81}{8}$. (c) $T_3 = \frac{5}{2}$ and $T_7 = 40$.

Question 8 In a particular geometric series there are 7 terms between the first term 8 and the last term $\frac{1}{32}$. Find the first three terms of this series.

Question 9 A certain number is added to each term of $\frac{1}{5}, \frac{1}{2}, \frac{9}{10}$ and a geometric sequence is the result. Find the number

Question 10 Given each recurrence model for the number of students in a university after n years, describe a possible scenario for the rule given.

(a) $P_0 = 1800,\ P_{n+1} = 0.95P_n$ (b) $P_0 = 20000,\ P_{n+1} = 1.2P_n$

Question 11 Complete the following to solve $0.85^n > 0.3$.

$$\begin{aligned} \log 0.85^n &> \log 0.3 \\ n \log 0.85 &\ \_\ \_\_\_\_\_ \\ n &\ \_\ \frac{\log 0.3}{\log 0.85} \end{aligned}$$

Question 12 Solve the following by taking the logarithm of both sides.

(a) $1.05^n < 5.3$ (b) $0.95^n < 0.3$

Question 13 Sales of board games for a manufacturer X have been declining by 6% a year since the availability of computer games. In 2000 manufacturer X sold 35000 board games.

(a) What is the predicted number X will sell in 2020?

(b) Once sales fall below 5000, X will not find it viable to continue and stop production completely. What year is this likely to occur?

Question 14 Find the geometric mean of

(a) $\frac{1}{4}$ and $\frac{1}{8}$ (b) $\sqrt{5}+1$ and $\sqrt{5}-1$

Question 15 Insert 3 geometric means between 6 and 96

Question 16 We know that

$$\begin{aligned} S_n &= T_1 + T_2 + T_3 + \cdots + T_{n-1} + T_n \\ &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} \\ rS_n &= ar + \cdots \end{aligned}$$

Continue the above and prove the formula for S_n.

Question 17 Find the sum of 2, 6, 18, 54, ... to 15 terms

Question 18 Find the sum of $0.4 + 0.04 + 0.004 + 0.0004 + \cdots$ to n terms

Question 19

(a) How many terms of the series $2 + 4 + 8 + \cdots$ must be added to give a sum of 254?

(b) How many terms of the series $1 + 3 + 9 + \cdots$ must be added to give a sum of 3280?

Question 20 When a certain number is added to 1, 3, 6 the three term will be in a geometric progression. Find this number.

Question 21 Find the sum to infinity for the series 96, 72, 54, ...

Question 22 Given the series $0.3 + 0.33 + 0.333 + 0.3333 + \cdots$

(a) Simplify the eighth term.

(b) Write down the n^{th} term in simplified form.

(c) Show that the sum of the first n terms of the series in simplified form is $\frac{1}{3}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$

Question 23

(a) Express $0.3\dot{5}$ as an infinite series and hence as a simplified fraction

(b) Express $0.5\dot{2}\dot{3}$ as an infinite series and hence as a rational number

Question 24 An infinite geometric series has a first term of 8 and a limiting sum of 16. What is the common ratio?

Question 25 For a certain geometric progression, the sum of the first 4 terms is $21\frac{2}{3}$, while the sum to infinity is 27. Find the first three terms.

Question 26 Find x if $(x+1)+3(x+1)^2+9(x+1)^3\ldots$ if the limiting sum exists for this geometric progression.

Question 27 Consider the series $1-\cos^2 x+\cos^4 x-\cos^6 x+\cdots$

(a) For what values of x does this series have a sum to infinity?

(b) For these restricted values of x find the limiting sum.

Question 28 Consider the series $1+\tan^2 x+\tan^4 x+\tan^6 x+\cdots$ for $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

(a) For what values of x does this series have a sum to infinity?

(b) For these restricted values of x find the limiting sum.

Question 29 A rubber ball is dropped from a height of 16 m rebounds each time it strikes the ground to half the height from which it falls.

(a) How high will it rise on the 5^{th} rebound?

(b) How far has it travelled at the moment of the 5^{th} rebound?

(c) Find the total distance the ball bounces before coming to rest

Question 30 In a walking marathon, Bob walked at 4 km/h for the first hour then for each succeeding hour walked at 2/3 of his speed the previous hour. If Bob walked for 4 hours what was the total distance he covered?

Challenge Problems

Problem 1 In a geometric sequence the difference between the 4^{th} term and the 2^{nd} term is 48 and the difference between the 6^{th} term and the 4^{th} term is 432. Find the first term and the common ratio

Problem 2 The terms T_4, T_7 and T_{16} of an arithmetic sequence form three consecutive terms of a geometric sequence. Find a possible sequence.

Problem 3 If $\dfrac{1}{p+q}, \dfrac{1}{q+m}, \dfrac{1}{m+p}$ form an arithmetic sequence, prove that q^2, p^2, m^2 form an arithmetic sequence.

Problem 4 For what values of x does the following geometric series have sum to infinity

$$\left(\frac{3}{x+1}\right)+\left(\frac{3}{x+1}\right)^2+\left(\frac{3}{x+1}\right)^3+\cdots$$

Problem 5 For the geometric progression

$$5^n+5^{n-1}+5^{n-2}+\cdots+5^{1-n}+5^{-n}$$

(a) Find how many terms in the sequence.

(b) Find the sum of the series.

Problem 6 For the geometric series $36+12+4+\cdots$ find the smallest value of n so that $S_\infty - S_n \leq 10^{-4}$

Problem 7 Given that $0<\theta<\dfrac{\pi}{2}$, show that the limiting sum of the series

$$\sin\theta+\sin\theta\cos^2\theta+\sin\theta\cos^4\theta+\sin\theta\cos^6\theta+\cdots$$

is equal to $\operatorname{cosec}\theta$. Explain why you can find the limiting sum.

Problem 8 Simplify the following sums.

(a) $2^n+2^{n-1}+2^{n-2}+\cdots+2^{-2}$

(b) $2^n+2^{n-1}+2^{n-2}+\cdots+2^{-n+1}+2^{-n}$

Problem 9 Given the series $2^x+2^{-x}+2^{-3x}+2^{-5x}+\cdots+2^{-(2p+1)x}$

(a) What sequence do the powers form

(b) How many terms are there in this series?

(c) Find the sum of this series and simplify your answer

Problem 10 A geometric series has a common ratio of $\frac{1}{p}$ and limiting sum $\frac{1}{1-p}$. Find the series.

Problem 11 For a particular series

$$T_n = A\left(\frac{1}{2}\right)^{n-1} + B\left(\frac{1}{3}\right)^{n-1}$$

it is given that $T_1 = 35$ and $T_2 = 13$. Find the values of A and B and hence, find S_∞.

Problem 12 Can there be an infinite geometric series with a limiting sum of $\frac{2}{3}$ and a first term 2? Explain your reasons carefully.

Problem 13 [The error term vanishes as n gets large]

(a) Find an expression for the difference between the limiting sum and sum to n terms of the general geometric sequence

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + \cdots$$

(b) Show that if $-1 < r < 1$, then the term in (a) goes to zero as n gets large.

Problem 14 Jacob threw a very bouncy ball from his balcony. After bouncing on the pavement below, it reached a height of 4 metres. On the next bounce it reached 3 metres, then 2.25 metres and so on.

(a) To what height will the ball reach next?

(b) What was the height of the balcony that Jacob threw his ball?

(c) What was the height of the 5^{th} bounce?

(d) What was the height of the n^{th} bounce?

(e) What is the total limiting distance travelled by the ball?
Hint: Don't forget to include the original height from which it was dropped!

Exercise 3C

Simple and compound interest

Fundamentals

Fundamentals 1

(a) Simple interest is when the amount of interest earned at regular intervals is a f_____ percentage of the amount.

(b) Compound interest is when the principle increases by a fixed p_______ at regular intervals. It is called compound interest because the interest earned is re-invested into the p_____.

Fundamentals 2

Let P_0 be the principle of an investment and r be the simple interest rate. After n periods, the investment is valued at $P_0 +$ _______________.

Fundamentals 3

Let P_0 be the principle of an investment and r be the compound interest rate. After n periods, the investment is valued at _______________.

Fundamentals 4

If the annual interest rate is compounding

(a) bi-annually, we h_____ the annual rate and multiply the frequency of the periods by ___.

(b) quarterly, we q_____ the annual rate and multiply the frequency of the periods by ___.

(c) monthly, we divide the annual rate by ___, and multiply the frequency of the periods by ___.

Question 1 The following recurrence relation was used to model a simple interest investment.

$$P_{n+1} = P_n + \$1250$$

where $P_0 = \$50000$.

(a) How much interest was paid every year?

(b) How much was the annual interest rate of this investment?

(c) How many years will it take for the value of the investment to reach $75 000?

Question 2 Zach invests \$3 000 in an investment account that pays 4% per annum simple interest

(a) Write down the value of the investment after 2 years

(b) Write down a recurrence relation to model Zach's investment

(c) Use the model to determine Zach's investment after 4 years

(d) When will Zach's investment first exceed \$4 000?

Question 3 Amanda borrows \$18 000 from a bank to buy a car

(a) She is charged an annual simple interest rate of 4.8%.

(i) Work out how much interest is charged each year.

(ii) Find the recurrence relation to model the value of her loan P_n from year to year.

(iii) Find the rule for the value of the loan P_n after n years

(iv) How much will Amanda owe the bank after 5 years

(b) She is charged compound interest of 4.8% per annum, compounding quarterly.

(i) Write a recurrence relation to model the value of Amanda's loan from quarter to quarter.

(ii) Complete the equation below to reflect how much she owes the bank after 1 year

$$P_4 = (\_\_\_\_)^{\_\_} \times 18\,000$$

(iii) Write a general rule for how much Amanda owes the bank after 2 years

(iv) Write a general rule for how much Amanda owes the bank after n years

Question 4 A new car purchased for \$28 000 depreciates by 15% of its purchase price each year.

(a) Write down the recurrence relation to model the flat rate depreciation of the car.

(b) What is the value of the car after 3 years?

(c) When will the value of the car be less than half of the purchase price?

Question 5 Bob and Mary both start a new job. Bob's starting salary is \$45 000 per annum, with annual increases of \$2 800. Mary's starting salary is \$40 000 per annum with annual increases of 10%.

(a) Draw up a table showing Bob's and Mary's annual wage for the first 5 years.

(b) Find when Mary's salary first exceeds Bob's

(c) Bob had hoped to double his salary after 16 years. What annual increase was he hoping for?

(d) After how many years will Mary double her salary?

Question 6 The average price of a 3 bedroom house in town W was $\$1,200\,000$ in 2005.

(a) Each year the price rises by 5% due to inflation. According to these figures

(i) What will be the average price of a 3 bedroom house in 2020? Round you answer to the nearest $\$10,000$.

(ii) In what year will the average price of a 3 bedroom house exceed $\$2,000\,000$?

(b) Each year the price rises by $r\%$ due to inflation. After 6 years the average price of a 3 bedroom house is $\$1,475\,000$. Find the percentage increase correct to 3 decimal places

Question 7 Michelle bought 4 new tyres for her car costing her \$600. The value of the tyres depreciates by 12 cents for every hour it is in use.

(a) Write down the value of the tyres after n hours of use

(b) What is the value of the tyres after 100 hours of use.

(c) On average Michelle drives 15 hours each week. How many weeks will it take the value of the tyres to halve in value?

Question 8 Mr & Mrs Jones, a retired couple invest $\$100,000$ at 4.8% per annum and at the end of each year are paid out the interest. Work out the interest they will be earning if interest is compounded

(a) yearly (b) quarterly (c) monthly

Question 9 Paul invested $\$20\,000$ at 6% per annum in a fund for his children's education fees. If the investment has doubled find the time period of the investment if

(a) compounding was yearly.

(b) compounding was quarterly.

(c) compounding was monthly.

Question 10 In the following questions write down the statement you would use to work out the answer. There is no need to find the solution, but simplify where possible.

(a) An amount of \$500 is invested for 20 years at 6% p.a compounding quarterly.

(b) An amount of \$500 is invested for 20 years at 6% p.a compounding monthly.

(c) How much interest will be earned from the investment in (b)?

(d) An item bought for \$500 depreciates at 6% p.a for 10 years.

(e) After how many years will the item in (d) be worth half it's original value?

Question 11 Mr & Mrs Jones visit a different bank and are given the option of investing the $\$100,000$ either at 4.7% p.a compounding weekly or 4.75% p.a compounding fortnightly. Which option should they choose?

Question 12 Consider an investment of \$4 000 that pays 6% interest per annum.

(a) Suppose interest is compounding yearly,

(i) Explain why the recurrence formula that we use is $P_{n+1} = 1.06P_n$ where $P_0 = \$4\,000$

(ii) Find the value of the investment after 2 years.

(iii) Find the amount of interest earned after 2 years.

(iv) Find the amount of interest earned in the second year.

(v) Determine when the value of the investment will first exceed \$5 000.

(b) Suppose interest is compounded half yearly.

(i) Write down the recurrence formula.

(ii) Write down the formula to calculate the value of this investment after 2 years.

(iii) Calculate the value of the investment after 2 years.

(c) If interest is compounded monthly calculate the value of the investment after 2 years.

Question 13 Zach invests \$3 000 in an investment account that pays 3% per annum compounding monthly.

(a) Write a recurrence relation to model Zach's investment.

(b) How much is Zach's investment worth after 3 months?

Question 14 A new car purchased for \$28 000 depreciates by 15% of its present value each year.

(a) Write down the value of the car after one year.

(b) Write down the value of the car after two years.

(c) Write down the recurrence relation that we can use to model the value of the car after n years.

(d) Write down the value of the car after six years.

Challenge Problems

Problem 1 A truck has a value after 6 years of \$21 000. If it depreciated at a reducing-balance rate of 15% per annum, what was the initial value of the truck?

Problem 2 How much money must you deposit at a fixed rate of 5.2% per annum if you require \$10 000 in 3 years time

(a) if compounding was yearly?

(b) if compounding was quarterly?

(c) if compounding was monthly?

(d) if compounding was fortnightly?

Problem 3 Elly invests \$12 000 in an account and needs it to have a value of \$18 000 in 5 years. What interest rate is she hoping to achieve if the money invested

(a) is compounding yearly?

(b) is compounding quarterly?

(c) is compounding monthly?

Problem 4 What reducing balance depreciation rate would cause the value of a car to drop from \$36 000 to \$12 000 in 4 years?

Problem 5 When Noah started university he bought a second-hand car for \$9 000. One dealer told him it would have a flat rate depreciation of only 8% per annum and another said it would have a reducing-balance depreciation of 10% per annum.

(a) Write down a recurrence model to work out the value of Noah's car after 3 years of purchase for each option

(b) Use the above model to work out the value of Noah's car after 3 years.

Exercise 3D

Annuities, present and future values

Fundamentals

Fundamentals 1

(a) The **present value** of a single investment is how much it is worth t________.

(b) The **f**________ of a single investment is how much it is worth after a set amount of time.

(c) The present value PV and future value FV is linked by the formula

$$FV = PV\,(1+i)^n$$

where i is the i________ rate per time period and n is the n________ of time periods.

Fundamentals 2

(a) When an amount $\$P$ is invested or paid regularly to an account at $r\%$ for n periods, the series of equal payments is called an a________.

(b) The total amount saved up at the end of the annuity is called the f________ value of the annuity, denoted by FVA.

(c) Annuity tables can be used to predict the FVA of the annuity by showing how much will be in the account if $1 is invested regularly for various p________ and interest rates.

(d) To find out the FVA if $\$x$ is invested regularly, just multiply the corresponding value in the table by ___.

(e) When given the FVA, the rate and the number of periods, the annuity tables can be used in r________ to find the amount of the regular instalments.

Fundamentals 3

(a) Annuities that pay $\$A$ per period over n periods for a total payout of $\$nA$ can be purchased. However, you should not pay $\$nA$ today for that option because once each instalment is paid to you, you can earn interest on that instalment. Hence, you should really pay more/less (circle one) than $\$nA$. This amount is called the p________ value of an annuity, or PVA.

(b) The PVA is the s___ of the PV of each annuity payment. Similarly to the FVA, the PVA can be calculated using a table of values.

Fundamentals 4

If annual interest is compounded

(a) bi-anually, then we h________ the interest rate, and double the number of periods.

(b) q_______, then we divide the annual rate by 4 then multiply the number of periods by 4.

(c) monthly, then we divide the annual rate by 12 then multiply the number of periods by ___.

Question 1 The table below shows the compounded values of $1 at different interest rates over a number of different periods.

Periods	1%	2%	3%	4%	5%
2	1.0201	1.0404	1.0609	1.0816	1.1025
4	1.0406	1.0824	1.1255	1.1699	1.2155
6	1.0615	1.1262	1.1941	1.2653	1.3401
8	1.0829	1.1717	1.2668	1.3686	1.4775
10	1.1046	1.2190	1.3439	1.4802	1.6289
12	1.1268	1.2682	1.4258	1.6010	1.7959

Sam invests a lump-sum of $5000 for 2 years without making any further deposits. Calculate the future value of his investment if the account pays an interest rate of 4% per annum, compounded

(a) yearly. (b) bi-annually. (c) quarterly.

Question 2 The table below shows the future values of an annuity for different interest rates and compounding periods, where $1 is made at the *end* of each compounding period.

n	1%	2%	3%	4%	5%	6%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600
3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836
4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746
5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371
6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753

Use the table of values to answer the following problems.

(a) Stephen contributes $1500 at the end of every 6-month period over the span of 3 years into an account that pays an interest rate of 4% p.a. compounded bi-annually. Calculate the final amount in his account after the last contribution is made.

(b) Bob contributes $2000 at the end of each quarter into an account that pays an interest rate of 8% p.a. compounded quarterly. How much interest did Bob earn after one year?

(c) Helen opens an annuity account and contributes $3000 at the end of each year for 4 years. For the first 3 years, the interest rate is 5% p.a compounded yearly. In the fourth year, the interest rate increases to 6% p.a. Calculate the amount in the account immediately after the fourth contribution is made.

(d) Mary deposits $\$B$ into an annuity account every 6 months for a period of three years. The account pays an interest rate of 2% p.a. compounded bi-annually. At the end of the three-year period, the account has an amount of $\$36,912$. What was the amount of Mary's bi-annual deposits?

Question 3 The table below shows the present value of a $1 annuity for varying interest rates and loan periods, in years

Present value of $1				
n	**3%**	**4%**	**5%**	**6%**
5	4.5797	4.4518	4.3295	4.2124
6	5.4172	5.2421	5.0757	4.9173
7	6.2303	6.0021	5.7864	5.5824
8	7.0197	6.7327	6.4632	6.2098

What is the present value of an annuity where $1000 is contributed

(a) each year for 7 years at 4% p.a. interest compounded yearly?

(b) every 6 months for 3 years at 6% p.a. interest compounded bi-annually?

Question 4 The table below shows the present value of a $1 annuity for varying interest rates and loan periods, in months.

	Interest rate per period					
n	**0.45%**	**0.50%**	**0.75%**	**1.00%**	**1.25%**	**1.50%**
60	52.4796	51.7256	48.1734	44.9550	42.0346	39.3803
120	92.5656	90.0735	78.9417	69.7005	61.9828	55.4985
180	123.1851	118.5035	98.5934	83.3217	71.4496	62.0956
240	146.5735	139.5808	111.1450	90.8194	75.9423	64.7957
300	164.4385	155.2069	119.1616	94.9466	78.0743	65.9009
360	178.0846	166.7916	124.2819	97.2183	79.0861	66.3532

Julia borrows $18,000 to buy a car and makes monthly repayments over 5 years. She is charged an interest rate of 9% p.a. compounding monthly.

(a) Find Julia's monthly repayments.

(b) Calculate the total amount of interest Julia will pay over the life of the loan.

Question 5 The table below shows the present value of a $1 annuity for varying interest rates and loan periods, in months.

	Interest rate per period					
n	**0.45%**	**0.50%**	**0.75%**	**1.00%**	**1.25%**	**1.50%**
60	52.4796	51.7256	48.1734	44.9550	42.0346	39.3803
120	92.5656	90.0735	78.9417	69.7005	61.9828	55.4985
180	123.1851	118.5035	98.5934	83.3217	71.4496	62.0956
240	146.5735	139.5808	111.1450	90.8194	75.9423	64.7957
300	164.4385	155.2069	119.1616	94.9466	78.0743	65.9009
360	178.0846	166.7916	124.2819	97.2183	79.0861	66.3532

Anthony has two choices for a loan of $250,000.

- Loan A can be taken over a period of 15 years at 6% p.a. compounding monthly.
- Loan B can be taken over a period of 20 years at 5.4% p.a. compounding monthly.

(a) Find the monthly instalment for each option.

(b) Find the interest paid on each of the loans.

(c) Which loan has Anthony paying less interest overall? Discuss why Anthony may opt for the loan where he pays *more* interest.

Question 6 The table below shows the present value of a $1 annuity for varying interest rates and loan periods, in months.

	Interest rate per period					
n	**0.60%**	**0.65%**	**0.70%**	**0.75%**	**0.80%**	**0.85%**
45	39.33406	38.90738	38.48712	38.07318	37.66545	37.26383
46	40.09350	39.64965	39.21263	38.78231	38.35859	37.94133
47	40.84841	40.38714	39.93310	39.48617	39.04622	38.61311
48	41.59882	41.11986	40.64856	40.18478	39.72839	39.27924
49	42.34475	41.84785	41.35905	40.87820	40.40515	39.93975
50	43.08623	42.57113	42.06459	41.56645	41.07653	40.59470

(a) Calculate the monthly loan repayment for a car loan of $12,000 at 9.6% p.a. compounding monthly over 4 years.

(b) What would be the present value of a $100 per month annuity at an interest rate of 0.65% per month over the span of 50 months?

(c) Calculate the present value of an annuity of $250 per month for 46 months at an annual interest rate of 7.8% p.a. interest per month.

Challenge Problems

Problem 1 The table below shows the future values of an annuity for different interest rates and compounding periods, where \$1 is made at the end of each compounding period.

n	1%	2%	3%	4%	5%	6%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600
3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836
4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746
5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371
6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753

Find the period of the investment if an annuity of \$1500 is invested every 6 months at 6% p.a. compounding bi-annually, yielding a total of \$6275.40.

Problem 2 The table below shows the present value of a \$1 annuity for varying interest rates and loan periods, in months.

Present value of \$1					
n	0.60%	0.65%	0.70%	0.75%	0.80%
46	40.09350	39.64965	39.21263	38.78231	38.35859
47	40.84841	40.38714	39.93310	39.48617	39.04622
48	41.59882	41.11986	40.64856	40.18478	39.72839
49	42.34475	41.84785	41.35905	40.87820	40.40515

Use the table to answer the following problems.

(a) How much can Bob borrow to start a business and pay off the loan in monthly repayments over 4 years if he is charged an interest rate of 8.4% p.a. compounded monthly, and he can afford to repay \$615 a month?

(b) Mary borrowed \$100, 000 to renovate her house, and repays it over the span of four years. The interest rate is 9.6% p.a. compounded monthly, and she budgets to make monthly repayments of \$2, 500. Is this sufficient? If not, how much extra does Mary need to repay per month, to the nearest dollar?

Exercise 3E

Regular instalments

Fundamentals

Fundamentals 1

Suppose M is invested at the beginning of each month into an account that pays an interest rate of $R\%$ p.a, compounding monthly. Let A_n represent the amount of money in the account at the end of the n^{th} month, and let $r = 1 + \dfrac{R}{12}$.

(a) Consider the very first deposit of M. How much will it amount to at the end of n months?

(b) Consider the second deposit of M. How much will it amount to at the end of n months?

(c) Consider the final i.e. n^{th} deposit of M. How much will it amount to at the end?

(d) Hence, how much overall is in the account at the end?

Fundamentals 2

Suppose M is invested at the beginning of each month into an account that pays an interest rate of $R\%$ p.a, compounding monthly. Let A_n represent the amount of money in the account at the end of the n^{th} month, and let $r = 1 + \dfrac{R}{12}$.

(a) Write down an expression for A_1 in terms of M and r.

(b) Explain why $A_2 = (A_1 + M)r$

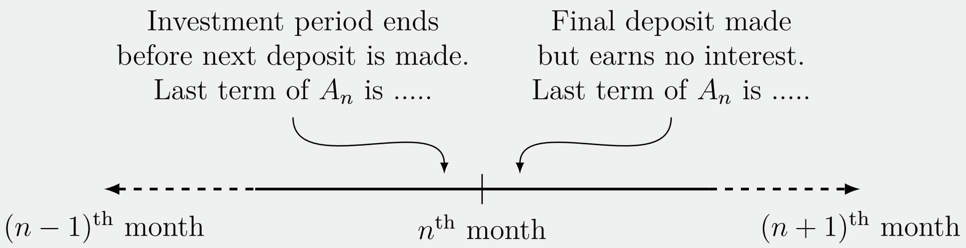
(c) Write down an expression for A_2.

(d) Repeat the process and write down an expression for A_3.

(e) Hence, write down an expression for A_n.

Fundamentals 3

The final term of A_n can be slightly different depending on the scenario. Some scenarios like birthday gifts may include a final deposit that earns no interest. In these cases, the last term is M / Mr (circle one). In most cases though, the final deposit *does* earn interest, so the final term is M / Mr (circle one).



Note to students and teachers: Your answers may vary slightly from the values at the end of the book. Since many answers rely on other (possibly) rounded values, the answer will vary slightly depending on whether you chose to use the rounded version or exact calculator version. However, the difference should be relatively negligible.

Question 1 [One method of solving questions with regular instalments]

When Leon was born, an amount of $300 is placed in an account that earns 6% interest p.a. compounding annually. Every birthday, another $300 is deposited into Leon's account until and including his 21st birthday when he is given the full amount in the account.

(a) Consider the initial $300 that was deposited. How much did this amount grow to become when Leon turns 21?

(b) Consider the second $300 that was deposited i.e. the deposit on Leon's first birthday. How much did this amount grow to become when Leon turns 21?

(c) How much was in the account when Leon turned 21?

Question 2 [Recurrence-based method of solving the same type of question]

When Leon is born, an amount of $300 is placed in an account that earns 6% interest p.a. compounding annually. Every birthday, another $300 is deposited into Leon's account until and including his 21st birthday when he is given the full amount in the account. Let A_n be the amount in the account on Leon's n^{th} birthday, so $A_0 = 300$.

(a) Show that $A_1 = 300(1 + 1.06)$.

(b) Show that $A_2 = 300(1 + 1.06 + 1.06^2)$.

(c) Hypothesise a formula for A_{21}.

(d) How much was in the account when Leon turns 21?

Question 3 [Drill question to calculate interest rate]

Below are annual interest rates, compounded to a given compounding frequency. Find the equivalent interest rate corresponding to the given frequency.

(a) 6% per annum, compounded monthly

(b) 6% per annum, compounded fortnightly

(c) 3% per annum, compounded monthly

(d) 3% per annum, compounded fortnightly

(e) 4.8% per annum, compounded monthly

(f) 4.8% per annum, compounded quarterly

Question 4 Bill invests $500 at the beginning of the year and continues to do annually for 30 years. The account earns an annual interest rate of 6%, compounded annually.

(a) Find the amount to which his 1^{st} investment grew 30 years later.

(b) Find the total amount in the account at the end of the 30 years.

(c) How much interest had Bill earned?

Question 5 Grandpa Doug puts \$20 a month into Toby's bank account starting on his 7th birthday. The account earns an annual interest rate of 4.8%, compounding monthly.

(a) How much is in the account on Toby's 8th birthday just after the corresponding deposit?

(b) How long will it take for the account to have at least \$2 000?

(c) How much money will there be in the account when Grandpa Doug gives Toby this investment on his 18th birthday including the last deposit on his birthday?

(d) Calculate the total interest earned on this investment.

(e) Suppose that the interest rate was instead 6% per annum. By calculating (c) again but using the new interest rate, find how much more money Toby would have received on his 18th birthday, compared to the original interest rate.

Question 6 At the beginning of every month, Mary deposits money into a bank account earning 6% p.a interest, compounded monthly. For the first five years (60 deposits), she deposits \$300 per month. For the next five years (next 60 deposits), she instead deposits \$400 per month.

(a) Calculate how much money is in the account at the end of the first five years. Remember to include interest for the final deposit too.

(b) Calculate how much money is in the account after another five years.

Question 7 When Nathan was born his grandfather deposited \$1 000 and invested \$1 000 on each of his birthdays till he turned 21 years old earning 6% p.a. His grandfather made the last deposit on his 21st birthday and he presented the total investment to Nathan.

(a) How much did Nathan receive on his 21st birthday?

(b) On Nathan's 16th birthday he convinced his grandfather to give him the \$1 000 rather than deposit it into the investment, so he could go on band camp with the school. In this case, how much did Nathan receive on his 21st birthday?

Question 8 Lauren contributes \$500 every quarter from the beginning of the year, into a superannuation fund earning 6% p.a. compounded quarterly. Contributions are made for 5 years. Then Lauren stops her contributions for 3 years whilst she is not working looking after her children. She returns to work, and contributions resume under the previous conditions. Lauren works for a further 10 years. Write an expression to calculate

(a) the amount that the initial 4 contributions are worth when she finally stops working

(b) the amount that her last contribution (before she stopped working to look after her children) is worth when she finally stops working.

(c) the total amount in the superannuation fund when Lauren stops working after working a further 10 years.

Question 9 Leah would like to have $\$10,000$ saved at the end of 3 years for a trip she is planning. She makes a monthly contribution at the beginning of each month.

(a) What should her monthly contribution be if she is investing with a fund giving 6% p.a, compounding monthly?

(b) How much does Leah have in her account after two years?

(c) At the end of two years, she realises that she in fact needs $\$12,000$ for her trip. By how much should she *increase* her monthly contribution?

Question 10 Gabriella pays an annual premium into an account at the beginning of each year. The fund pays compound interest at 12% p.a. If on retirement, after 25 years her payout from the fund was $\$600\,000$. How much was she contributing each year?

Question 11 Mary makes a monthly deposit into an account at the beginning of each month. The fund pays interest at a rate of 8% p.a, compounding monthly. After 25 years, she saves up a total of $\$600,000$. How much was her monthly contribution?

Question 12 Zoe makes a monthly deposit into an account at the beginning of each month, which pays interest at a rate of 3% per annum, compounded monthly.

(a) If Zoe deposits $1500 per month for 20 years, how much will she have at the end of 20 years?

(b) If Zoe wishes to have $\$800,000$ after 20 years, how much should her monthly deposit be?

Question 13 Jeff makes a *quarterly* deposit of $4000 into an account at the beginning of each quarter. The account pays an interest rate of 6% per annum, compounding monthly.

(a) Show that Jeff will have $\$16,612.16$ in the account at the end of the first year.

(b) Show that at the end of the second year, he will have $\$34,248.93$ in his account.

(c) How much will he have at the end of 10 years?

(d) If Jeff wishes to have $\$250,000$ after 10 years, how much should his quarterly deposit be?

Question 14 Elle deposits $500 at the beginning of every month into an account that earns an annual interest rate of 6%, compounded monthly. She wishes to save up $\$20,000$ for a new car.

(a) Show that after n months, immediately after the further deposit, the amount in the account is

$$A_n = 500(1 + r + r^2 + \cdots + r^n),$$

where $r = 1.005$.

(b) Hence, show that

$$A_n = 100000(r^{n+1} - 1)$$

(c) Let $A_n = 20000$ and solve for n.

(d) Hence, after how many months will she have enough for her car?

Question 15 Bob joins a superannuation fund and invests \$240 at the beginning of each month. The compound interest of 4% p.a. is calculated monthly. Let A_n be the amount in the account at the end of the n^{th} month.
For how many months will he have to invest in order to save at least \$24 000?

Challenge Problems

Problem 1 Matthew deposits \$2000 into a savings account every month. The bank pays an interest rate of 3% per annum, compounded monthly. For the first ten years, Matthew is consistent with his deposits. After his 120^{th} deposit (after ten years) he stops his deposits for five years and lets the account grow. After five years has elapsed, he then continues again with his regular \$2000 per month deposits. How much is in Matthew's account

(a) at the end of 10 years?

(b) just before he resumes making deposits again?

(c) overall at the end of the entire 20 years?

Problem 2 Michael invests \$1000 every 3 months into a superannuation scheme. He continues to invest this amount for 10 years. For the first 5 years the interest rate is 6% p.a. compounded quarterly. However for the last 5 years, he manages to get 7.6% p.a. compounded quarterly on his investment. Find the total amount he has saved over the 10 years.

Problem 3 Cody deposits \$1000 into a savings account every month. The bank pays an interest rate of 3% per annum, compounded monthly. After ten years (120th deposit), her interest rate doubles to be 6% per annum. She continues with her monthly deposits at this new interest rate for another ten years. How much did Cody have at the end of the entire 20 years?

Problem 4 Mrs Wise sets up a fund for her grand-daughter Amanda. She deposits \$500 into the fund when her grand-daughter is born, and increases her annual contribution by 5% each year. The investment earns 5% compound interest per annum, compounding annually. Mrs Wise contributes to the fund on each of Amanda's birthdays until she turns 18, which includes her 18^{th} birthday.

(a) What is the first contribution worth on Amanda's 18^{th} birthday?

(b) What is the second contribution worth on Amanda's 18^{th} birthday?

(c) What is the third contribution worth on Amanda's 18^{th} birthday?

(d) Find the total of all the contributions on Amanda's 18^{th} birthday.

Problem 5 $\$P$ is invested at the beginning of the year in a superannuation scheme at a compound interest rate of r% per annum each year, compounding annually.

(a) Show that after 2 years the investment is worth $P(1+r)(2+r)$.

(b) Show that after n years the investment is worth $P\left(1+\frac{1}{r}\right)\left((1+r)^n-1\right)$

Exercise 3F

Loan repayments

Fundamentals

Fundamentals 1
Suppose an amount B is borrowed and the account is charged an interest rate of $R\%$ p.a, compounding monthly. Monthly repayments of M are made. Let A_n represent the amount of money owing after n months, so $A_0 = B$, and let $r = 1 + \dfrac{R}{12}$.

(a) Write down an expression for A_1 in terms of B, M and r.

(b) Write down an expression for A_2.

(c) Write down an expression for A_3.

(d) Hence, write down an expression for A_n.

Question 1 Richard borrows \$2000 at an reducible annual interest rate of 6% p.a, compounded monthly. He wishes to repay the loan over the span of four years with monthly repayments of \$$M$. Let A_n be the amount owing after n repayments, so that $A_0 = 2000$.

(a) Let $r = 1.005$. Explain why $A_1 = 2000r - M$.

(b) Show that $A_2 = 2000r^2 - rM - M$.

(c) Show that $A_3 = 2000r^3 - M(1 + r + r^2)$.

(d) Hypothesise a similar formula for A_n.

(e) Hence, show that

$$A_{48} = 2000r^{48} - M\left(\frac{r^{48} - 1}{r - 1}\right).$$

(f) Explain why conceptually, A_{48} is just zero.

(g) Hence, find the value of M.

(h) How much interest did Richard pay over the course of four years?

Question 2 Lucy borrows \$500 000 at an annual reducible rate of 9% p.a, compounded monthly, and repaid in quarterly instalments of \$$Q$ for 15 years. Let A_n be the amount owing after n repayments, so that $A_0 = 500,000$.

(a) Let r be the interest rate factor that is added to the amount owing each quarter. Show that $r = 1.0075^3$.

(b) Show that after two instalments, the amount owing is

$$A_2 = 500000r^2 - Q(1 + r)$$

(c) Write down an expression for the amount owing after n instalments.

(d) How many instalments were made in total?

(e) How much is each instalment?
Hint: Remember that the loan is repaid after 15 years.

(f) Find the total interest which will be paid over the 15 years.

(g) Calculate the equivalent flat rate of interest over the 15 years, correct to three decimal places.

Question 3 Noah borrows \$100 000 to set up a small business. The interest is calculated monthly at a rate of 12% p.a. Noah intends to repay the loan in 2 equal annual instalments of $\$A$ at the end of the first and second years.

(a) How much does Noah owe at the end of the first year, immediately after making his annual repayment?

(b) Find the amount of Noah's annual repayment.

(c) Calculate the total interest Noah paid over the 2 year period

(d) What is the equivalent flat rate of interest, per annum?

Question 4 A bank offers a loan of \$10 000 charging interest at 1% per month and as a special offer it will not charge interest for the first 2 months. Each monthly repayment on this loan is $\$M$ and the first repayment is at the end of the first month. How much is owing at the end of the 2^{nd} and 3^{rd} month?

Question 5 Sam borrows \$15 000 to buy a car when he starts university. He can afford to pay \$200 a month. The interest is calculated monthly at a rate of 9% p.a. Find after how many months it will take for Sam to repay entirely his loan.

Question 6 Mary borrows \$50 000 at a reducible rate of 1.5% per month and repays \$1000 at the end of each month.

(a) How long will it take Mary to pay off this loan?

(b) Calculate the total interest Mary ended up paying when she cleared her mortgage.

Question 7 [The compounding frequency is not the same as the repayment frequency.]

Anthony borrows \$600 000 from the bank at an annual interest rate of 3% p.a, compounding monthly. How long will it take for Anthony to repay the loan if he makes

(a) monthly repayments of \$2000.

(b) quarterly repayments of \$6000.

(c) bi-annual repayments of \$12, 000.

Question 8 [What happens when you miscalculate repayments!]

Scott borrows \$800 000 and is charged an annual interest rate of 3% p.a, compounding monthly. He makes monthly repayments of \$1500.

(a) Show that after n months, Scott owes

$$800000r^n - 1500\left(\frac{r^n - 1}{r - 1}\right),$$

where $r = 1.0025$.

(b) Let the amount in (a) be zero, and solve for n. What do you notice?

(c) Explain your result.

Question 9 A family borrows \$500 000 to buy a house. Interest is charged at an annual rate of 6% p.a, compounding monthly. The loan is to be repaid in 10 years with equal monthly repayments.

(a) Calculate the monthly repayments.

(b) Calculate how much is still owing after 5 years.

(c) Halfway through the loan, after 60 monthly repayments, the bank increases the annual interest rate to 6.6%, but still compounding monthly. How many more monthly repayments will the family need to make in order to repay the loan, as a result of this increase?

Question 10 Jacob borrows \$32 000 to buy a car at 8% p.a. reducible interest, compounded quarterly. Payment is made by n equal quarterly instalments of \$3 000. Let $r = 1.02$ and let Q_n be the amount owing after n quarters, so that $Q_0 = 32\,000$.

(a) Show that $Q_4 = 32000r^4 - 150000(r^4 - 1)$.

(b) Calculate how many instalments Jacob must make before his loan is paid off fully.

(c) After the first year (starting with 5^{th} instalment), Jacob can only afford to repay \$2 000 each quarter instead of his usual \$3 000 each quarter. He continues with this new quarterly repayment for the remaining life of the loan.

(i) How much does Jacob owe at the end of the first year?

(ii) Show that after n quarters, the expression for the amount Jacob owes is

$$22273(1.02)^n - 100000(1.02^n - 1)$$

(iii) Hence, find how much *longer* will it take for Jacob to fully pay off his loan?

Question 11 [Combining superannuation and withdrawals]

Monique deposits \$3000 at the beginning of every month for 30 years into a superannuation account, which pays an interest rate of 3% per annum, compounding monthly.

(a) Show that when Monique retires, she has $A = \$1,752,581.18$ in her account.

(b) When Monique retires, she ceases all deposits and now withdraws \$6000 at the end of every month from the same account, which continues to earn the same amount of interest. Show that after n withdrawals, the account has

$$M_n = 2400000 - (2400000 - A)r^n$$

where $r = 1.0025$.

(c) For how many months will Monique be able to make the full withdrawal amount of \$6000?

Question 12 Ruthvik wishes to borrow from a bank for a home loan. The bank offers him an annual interest rate of 3% per annum, compounding monthly. He knows that he can afford to repay \$2000 per month, and that he wishes for the entire loan to be repaid in exactly 25 years. How much can he borrow at most from the bank?

Question 13 Mary wins \$75,000 and invests the winnings into an account earning interest at a rate of 0.04% per month. Each month, immediately after the interest has been paid, Mary withdraws \$1000. The amount in the account immediately after the n^{th} withdrawal can be determined using the recurrence relation

$$A_n = A_{n-1}(1.004) - 1000$$

where $n = 1, 2, 3, \ldots$ and $A_0 = 75000$.

(a) Use the recurrence relation to find the amount in the account immediately after the second withdrawal.

(b) Calculate the amount of interest earned in the first 2 months.

(c) Show that $A_n = 250000 - 175000(1.004)^n$.

(d) Calculate for how many months Mary can make full withdrawals of \$1000 from this account.

Challenge Problems

Problem 1 Jennifer borrows \$600 000 to buy a house. Interest is charged at an annual rate of 4% p.a, compounding monthly. The loan is to be repaid in 20 years with equal monthly repayments.

(a) Find the amount of the monthly repayments.

(b) Calculate how much is still owing after 10 years.

(c) Halfway through the loan, after 120 monthly repayments, Jennifer receives a lump sum gift of \$50 000. She puts all of it into the loan and continues with her monthly repayments as usual until the loan is fully repaid. How many more months from here will it take for Jennifer to fully repay the loan?

(d) Hence, find how much less time it took for Jennifer to repay the loan as a result of the lump sum payment.

Problem 2 Bill decides to make monthly deposits for 30 years so that he can have monthly withdrawals of \$1000 for 10 years upon retirement. The bank pays an annual interest rate of 3% p.a, compounded monthly.

(a) How much must be in Richard's account upon retirement for this to happen?

(b) Hence, find how much Bill should deposit every month into his bank account until his retirement.

Problem 3 A loan of $\$P$ at a reducible interest rate of $r\%$ per month is to be repaid in n equal monthly instalments of $\$M$.

(a) Show that after 2 instalments amount owing is $P(1+r)^2 - M(2+r)$

(b) Show that $M = \dfrac{Pr(1+r)^n}{(1+r)^n - 1}$

Problem 4 An amount $\$P$ is borrowed at an annual reducible interest rate of $r\%$ per annum, compounding monthly. The loan is repaid in equal monthly instalments of $\$M$. Let $R = 1 + \dfrac{r}{1200}$ and A_n be the amount owing after n monthly repayments have been made. Show that

$$A_n = PR^n - M\left(\frac{R^n - 1}{R - 1}\right)$$

Chapter 3 Review

Sequences and Series

Review

Question 1 Establish if the following sequence is an arithmetic progression or geometric progression, then find the 5$^{\text{th}}$ term of each

(a) $-\frac{1}{3}, \frac{1}{2}, -\frac{3}{4}, \ldots$

(b) $\sqrt{3}, \sqrt{12}, \sqrt{27}, \ldots$

(c) $\frac{6}{\sqrt{3}}, 3\sqrt{3}, \frac{12}{\sqrt{3}}, \ldots$

(d) $\frac{6}{\sqrt{3}}, 2\sqrt{6}, \frac{12}{\sqrt{3}}, \ldots$

(e) $a-3b, 2a-5b, 3a-7b, \ldots$

(f) $p-3q, -q, -p+q, \ldots$

Question 2 Find the recurrence relation for the sequence:

(a) $9, 7, 5, 3, \ldots$

(b) $5, 8, 11, 14, \ldots$

Question 3 In each of the following, a recursive definition for a sequence is given. Find the first four terms.

(a) $T_n = T_{n-1} - 3, T_1 = 2$

(b) $T_n = -3T_{n-1}, T_1 = 2$

Question 4

(a) In an arithmetic sequence $T_6 = 5$ and $T_{11} = 40$ find T_8

(b) The first term of an Arithmetic Sequence is 3 and the seventh term is four times the second term. Find the common difference.

(c) Which term of the sequence $3, 16, 29, \ldots$ is 224?

(d) Find the sum of the sequence $1 + 5 + 9 + \cdots + 301$

(e) In an arithmetic sequence $T_5 = 5$, and $S_7 = 0$. Find the first three terms of the sequence.

Question 5 An arithmetic sequence is given by $-92, -88, -84, \ldots, 212$

(a) Find an expression for the n^{th} term of the sequence

(b) How many terms are in this sequence?

(c) Find the sum of the terms in this sequence

(d) Find the sum of the first n terms of this sequence, i.e. S_n

(e) Find the value of n such that $S_n = 0$ and work out T_n for this value of n

(f) If this sequence started with first term -40 what would be the value of T_n so that $S_n = 0$.

Question 6 In a park trees have been planted along one path. The first one is 20 m from the gate, the second is 32 m from the gate, the third 44 m and so on.

(a) How far from the gate is the sixth tree?

(b) How far from the sixth tree to the tenth tree?

Question 7 An amphitheatre has 30 seats in row A, 34 seats in row B, 38 seats in row C and so on till row T being the last row.

(a) How many seats are there in row M?

(b) How many seats are there in the last row T?

(c) Which row has 90 seats?

(d) How many seats are there in this amphitheatre?

Question 8 Given $S_n = 24n - n^2$

(a) Find T_n

(b) Find the first 4 term of this sequence

(c) Find the first value of n for which the sequence is negative

(d) Find the values of n for which the sum of the sequence is positive

Question 9 If p^2, q^2, r^2 form an arithmetic progression, prove that $\dfrac{1}{p+q}$, $\dfrac{1}{p+r}$, $\dfrac{1}{q+r}$ also form an arithmetic progresion.

Question 10 x, $2x$ and x^2 are three consecutive terms of an arithmetic series, find x.

Question 11 Greg starts on a salary of \$80 000 his salary increases by a fixed amount every year. If the total amount earned after 10 years is \$818 000, then find

(a) the yearly increase in Greg's salary. (b) Greg's final salary.

Question 12

(a) Find $0.1\dot{2}\dot{5}$ as a rational number by first expressing it as a geometric series.

(b) Express $0.4\dot{6}$ as a fraction.

Question 13

(a) Find the first term of the sequence $3, 3\sqrt{2}, 6, 6\sqrt{2}, \ldots$ which exceeds 800

(b) In a geometric progression $T_3 = 54$ and $T_6 = 2$. Find the common ratio and T_5

(c) Insert 5 geometric means between 6 and 48

(d) In a geometric progression $S_4 = 30$, while the sum to infinity is 32. Find the first three terms.

Question 14 Test if the following series are an arithmetic progression, or geometric progression, or neither

(a) $\log 5$, $\log 15$, $\log 45$,... (b) $\log 3$, $\log 9$, $\log 27$, ... (c) $\log 3$, $\log 9$, $\log 81$, ...

Question 15

(a) Find the sum of $\log 5 + \log 15 + \log 45 + \cdots$ to 6 terms

(b) Find the sum of $\log 3 + \log 9 + \log 27 + \cdots$ to 6 terms

(c) Find the sum of $\log 3 + \log 9 + \log 81 + \cdots$ to 6 terms

Question 16 For the geometric progression a, ar, ar^2, ar^3, ... show that $\log a$, $\log ar$, $\log ar^2$, $\log a^3$, ... is an arithmetic progression.

Question 17 Find two unequal numbers m and n so that 2, m, n are in an arithmetic sequence and 2, n, m are in geometric sequence.

Question 18 Daniel is sending his son Bob to the local school where parents have to pay \$300 for books and resources. Each year these fees are increased by 8%. If Bob starts in year 7 and leaves school at the end of year 12, how much did Daniel pay in total for school fees to the school?

Question 19 How much did Bob invest at 4.8% p.a. simple interest if after 5 years

(a) he now has \$892.80, including his original principle?

(b) he receives a total of \$600 in interest?

Question 20 Marc is investing \$10 000 for 5 years. He has been offered interest rates of either 6% p.a compounded annually, or 5.5% p.a. compounded half yearly, or 5.4% p.a. compounded monthly. Work out each offer and decide which he should chose.

Question 21 After 6 years a compound interest investment of \$4 500 earned a total of \$1 280 in interest. Find the annual interest rate of this investment.

Question 22 How long would it take for \$1 000 to exceed \$2 000 if it was invested at

(a) 4% per annum compounded annually (b) 4% per annum compounded monthly

Question 23

(a) In how many years will a sum of money double at 6% p.a. compound interest?

(b) A \$30 000 new car depreciates at a rate of 15% p.a. Find its value after 6 years.

(c) A car depreciates at 10% p.a. When will it be worth half its original value?

(d) Work out the principle that must be invested to obtain an investment of \$30 000 after 15 years at a rate of 12% p.a compounding quarterly.

Question 24 The table below shows the compounded values of \$1 at different interest rates over a number of different periods.

Periods	**1%**	**2%**	**3%**	**4%**	**5%**
2	1.0201	1.0404	1.0609	1.0816	1.1025
4	1.0406	1.0824	1.1255	1.1699	1.2155
6	1.0615	1.1262	1.1941	1.2653	1.3401
8	1.0829	1.1717	1.2668	1.3686	1.4775
10	1.1046	1.2190	1.3439	1.4802	1.6289
12	1.1268	1.2682	1.4258	1.6010	1.7959

(a) Bob invests a lump-sum for 2 years without making any further deposits into an account that pays an annual interest rate of 4% p.a., compounded bi-annually. Calculate his initial investment if the value of his investment at the end of the 2-year period is \$9200.40.

(b) Mary opens an account today which pays an interest rate of 4% p.a. compounded bi-annually. Calculate the single sum of money Mary needs to invest today if she hopes to have $\$22,000$ in 4-years' time to buy a car.

Question 25 The table below shows the future values of an annuity for different interest rates and compounding periods, where \$1 is made at the end of each compounding period.

n	**1%**	**2%**	**3%**	**4%**	**5%**	**6%**
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600
3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836
4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746
5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371
6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753

(a) Find the future value of a \$8000 per half-year investment at 8% p.a. compounding bi-annually over a period of 3 years.

(b) An annuity of \$2500 is invested every 4 months at 6% p.a. for 2 years. What is the future value of the annuity?

(c) An amount of \$400 is deposited at the end of each month for 6 months at 12% p.a. compounding monthly. The annuity has a future value of F. Calculate the equivalent lump-sum amount P that has to be invested at the beginning of the annuity in order to also have a future value of F at the end of the 6 month period.

Question 26 The table below shows the present value of a $1 annuity for varying interest rates and loan periods, in months.

Present value of $1					
n	**0.60%**	**0.65%**	**0.70%**	**0.75%**	**0.80%**
46	40.09350	39.64965	39.21263	38.78231	38.35859
47	40.84841	40.38714	39.93310	39.48617	39.04622
48	41.59882	41.11986	40.64856	40.18478	39.72839
49	42.34475	41.84785	41.35905	40.87820	40.40515

Bob borrows $65,000 to start a business and pays off the loan with monthly repayments over 4 years. He is charged an interest rate of 9.6% p.a.

(a) Calculate Bob's monthly repayments.

(b) Calculate the total amount of interest that Bob will pay over the life of the loan.

Question 27 On the birth of his son Bob invests a sum of money and would like the sum to be $10 000 on his son's 18$^{\text{th}}$ birthday. How much did Bob invest if the bank was giving a fixed rate of 6% per annum compounded quarterly?

Question 28 Mary bought a car for $48 000. After 10 years is worth $16 000. Calculate the annual rate of depreciation.

Question 29 Mary pays an annual premium into an account at the beginning of each year. The fund pays compound interest at 12% p.a. If on retirement, after 25 years her payout from the fund was $600 000. How much was she contributing each year?

Question 30 Sam borrows $15 000 to buy a car when he starts university. He has a part-time job so can afford to pay $200 a month. The interest is calculated monthly at a rate of 9% p.a. Calculate the period of his loan in months.

Question 31 Mary invests $200 at the beginning of every month for 6 years into an account that pays an annual interest rate of 9% p.a, compounding monthly.

(a) How much will Mary have at the end of the 6 years?

(b) After 2 years, the interest rate drops to 6% p.a, but still compounding monthly. How much less money does Mary have at the end of the 6 years compared to the original rate?

Question 32 Dhang would like to save up for a holiday in 2 years' time. She plans to invest a fixed amount at the beginning of each month, and will receive an interest rate of 6% p.a, compounded monthly.

(a) How much will she have after 2 years if she deposits $240 at the beginning of every month?

(b) How much should she save every month if she wishes to have $5000 after 2 years?

Question 33 Chloe invests \$2000 at the beginning of every quarter for 10 years, into an account that pays an annual interest rate of 3% p.a, compounding monthly.

(a) How much will Chloe have at the end of the 10 years?

(b) After 6 years, Chloe stops her investments for the remainder of the investment period. How much will be in the account at the end of the 10 years?

(c) Calculate how much interest Chloe earned in both part (a) and (b).

Question 34 Bob opens an account for his daughter Jo when she is born. He makes fixed monthly deposits into this account starting from the day Jo is born, until and including her 18^{th} birthday. Bob wants the monthly deposits to be such that Jo receives \$20 000 on her 18^{th} birthday. The account earns an interest rate of 4% p.a, compounding monthly.

(a) How much should Bob deposit into the account every month?

(b) How much is in the account after three years?

(c) After three years, Bob negotiates with the bank and they agree to pay the account an interest rate of 6% p.a, compounding monthly. How much should the monthly repayments now be, if Bob still wishes for the account to have \$20 000 at the end of 18 years?

Question 35 Harry borrows \$40 000 at a rate of 3% per annum, compounded monthly. The loan is to be repaid in 60 equal monthly instalments.

(a) Find the amount of the monthly instalment.

(b) With the twelfth repayment, Harry pays an additional \$5000 but then continues with his normal monthly instalments as usual. How many more repayments are needed?

Question 36 A local library borrows \$50 000 to purchase new computers. The interest is calculated monthly at a rate of 0.5% per month, compounded monthly. The library intends on repaying the loan in two years with four equal bi-annual instalments of $\$B$.

(a) Find the amount of the bi-annual repayments.

(b) How much interest did the library pay over the period of the loan?

Question 37 Paul borrows \$32 000 at 9% p.a. reducible interest, calculated monthly. He pays off the loan over 72 equal monthly instalments.

(a) Show that the monthly repayment is \$576.82.

(b) With the 24^{th} repayment, Paul pays an additional \$8 000. His regular repayments continue after this month. How many more repayments will be needed?

(c) How much interest did Paul save by paying this extra \$8 000, rounded to the nearest hundred dollars?

Investigation Task

The Rule of 72

Any student studying financial mathematics will have at some point heard the 'Rule of 72'. The mathematics behind it is definitely accessible by a student studying Mathematics Advanced, and this investigation task will allow the student to explore this interesting rule.

Question 1 Write a one page document explaining what the rule is, what it is used for and the derivation of it. Your document should contain worked examples comparing values that the rule gives and actual values. Also, discuss any limitations of the rule and cases where the rule does not work very well.

Question 2 When researching the above question, you will find that there are *other* rules as well that work similarly. List out as many as you can find, and explain the mathematics behind each of them and where they are best used. Your answer should also contain an explanation of why the Rule of 72 is known more commonly, despite there being better rules out there.

Investigation Task

Continuous Compounding

In this course, we study monthly compounding, quarterly compounding and annual compounding. Realistically, compounding is done continuously rather than every month.

Question 1 When you deposit money, the bank pays you interest. When you borrow money, you pay the bank interest. Intuitively speaking, for each scenario, is it 'better' for you to have continuous compounding, or monthly compounding?

Question 2 Write down the formula for continuous compounding, and define all the parameters in the formula.

Question 3 Compare continuous compounding with annual compounding, quarterly compounding and monthly compounding for an account with principle $\$100,000$ that pays an interest rate of 3% p.a. How much more do we get after one year using continuous compounding compared to the other compounding frequencies?

Question 4 In the proof of the continuous compounding formula, there is a *crucial* well-known limit that is used. State that limit.

Question 5 Assuming the limit from the previous question, derive fully the formula for continuous compounding.

Investigation Task

Formulas in Financial Mathematics

When studying finance, there are a large numbers of formulae used. A large number of them can be found at https://en.wikipedia.org/wiki/Time_value_of_money. This page also contains many familiar-looking formulae and

Question 1 Define the *closed form* of a series.

Question 2 Identify the formulae that are the closed forms of the sums that you learn to develop when doing finance problems.

Question 3 Pick any three that are *not* just ones covered in the Mathematics Advanced course, and prove them.

4

FURTHER DIFFERENTIATION

- Differentiation of exponential functions
- Differentiation of logarithmic functions
- Differentiation of trigonometric functions
- Mixed differentiation problems

CHAPTER 4

Exercise 4A

Differentiation of exponential functions

Fundamentals

Fundamentals 1

Write down the derivative of the following.

(a) e^x (b) e^{ax+b} (c) $e^{f(x)}$

Fundamentals 2

Remember that the index laws can be used to simplify an expression first before differentiating. For example, instead of differentiating $\sqrt{e^x}$ using the chain rule, we can instead differentiate ______, which is the same thing.

Question 1 [Revision of Preliminary content]

Differentiate the following with respect to x.

(a) e^{5x} (b) $8e^{-3x}$ (c) $4e^{-\frac{x}{2}}$

(d) $e^{3x} - e^{-2x}$ (e) $\dfrac{e^{4x+1}}{2}$ (f) $2e^{\frac{3x-1}{2}}$

(g) $\dfrac{4}{e^{4x}} + \dfrac{e^{4x}}{4}$ (h) $\dfrac{e^{px}}{p} - \dfrac{e^{-qx}}{q}$ (i) $\sqrt{e^{5x}}$

(j) $\sqrt[3]{e^{5x}}$ (k) $\dfrac{e^{2x}+1}{e^{2x}}$ (l) $\dfrac{e^{2x}+e^{-2x}}{e^{2x}}$

Question 2 Use the fact that $\dfrac{d}{dx}\left[e^{f(x)}\right] = f'(x)e^{f(x)}$ to differentiate the following.

(a) e^{x^2} (b) e^{x^3} (c) e^{x^2+x}

(d) $e^{\sqrt{x}}$ (e) $e^{\frac{1}{x}}$ (f) $e^{\frac{1}{x^2}}$

Question 3 Differentiate the following using the chain rule.

(a) $(3-e^{2x})^4$ (b) $\left(x - e^{-x^2}\right)^3$ (c) $\dfrac{1}{e^x+1}$

(d) $\dfrac{1}{1-e^{x^2}}$ (e) $\sqrt{e^{2x}-2x}$ (f) $\dfrac{1}{\sqrt{e^{x^2}-1}}$

Question 4 Differentiate the following using the product rule.

(a) x^2e^x (b) xe^{-2x} (c) xe^{x^2}

(d) $(x^2 - 2x)e^{-x}$ (e) $(3x-4)^2e^{-x}$ (f) $\sqrt{x}e^{\sqrt{x}}$

Question 5 Differentiate the following using the quotient rule.

(a) $\dfrac{e^x}{x}$ (b) $\dfrac{x^2}{e^x}$ (c) $\dfrac{e^{3x}}{1+e^{3x}}$

(d) $\dfrac{e^{2x}-1}{e^{2x}+1}$ (e) $\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$ (f) $\dfrac{x}{e^{x^2}}$

Question 6 Prove the following results.

(a) If $y = xe^{3x}$, then $xy'' - 3xy' - 3y = 0$.

(b) If $y = 2e^{3x} + 3e^{2x}$, then $y'' - 5y' + 6y = 0$.

(c) If $y = e^{2x} + e^{8x}$, then $y'' - 10y' + 16y = 0$.

(d) If $y = xe^{-x^2}$, then $y'' = 2y(2x^2 - 3)$

(e) If $y = xe^{-x}$, then $xy' = (1-x)y$

(f) If $y = xe^{-\frac{x^2}{2}}$, then $xy' = (1-x^2)y$

Question 7 Find the value of k such that

(a) $y = e^{kx}$ satisfies $2y'' - y' - 3y = 0$.

(b) $f(x) = e^{kx} - x$ satisfies $f'(0) = 4$.

Question 8 Show that $y = (Ax+B)e^{3x}$ is a solution of $y'' - 6y' + 9y = 0$ for any real A and B.

Question 9

(a) Differentiate $y = \dfrac{e^x}{x}$.

(b) Differentiate $y = \dfrac{e^x}{x^2}$.

(c) Differentiate $y = \dfrac{e^x}{x^3}$.

(d) Hence, write down the derivative of $y = \dfrac{e^x}{x^n}$

Challenge Problems

Problem 1 [General solution of a second-order ordinary differential equation]

Let α and β be distinct real roots of the quadratic equation $y = ax^2 + bx + c$.

Show that

$$y = Ae^{\alpha x} + Be^{\beta x},$$

where A and B are any real numbers, is a solution of the differential equation

$$ay'' + by' + cy = 0.$$

Exercise 4B

Differentiation of logarithmic functions

Fundamentals

Fundamentals 1

Write down the derivative of the following.

(a) $\ln x$ (b) $\ln(ax+b)$ (c) $\ln\left(f(x)\right)$

Fundamentals 2

Complete the following.

(a) If $y = \ln\left(x^n\right)$, then $y =$ ________ and hence $y' =$ ________.

(b) If $y = \ln\left(f(x) \times g(x)\right)$, then $y =$ ____________ and hence $y' =$ ____________.

(c) If $y = \ln\left(\dfrac{f(x)}{g(x)}\right)$, then $y =$ ____________ and hence $y' =$ ____________.

(d) If $y = \ln\left(\dfrac{1}{f(x)}\right)$, then $y =$ ________ and hence $y' =$ ________.

Fundamentals 3

(a) The formula $\dfrac{d}{dx}\left(\log x\right) = \dfrac{1}{x}$ only works if the base is ___. If the base is anything else, then we must first c________ the base to ___, and then differentiate as usual.

(b) For example, to differentiate $y = \log_a x$, where $a \neq e$, first re-express y as $y =$ ________, and then differentiate to get $y' =$ ________.

Fundamentals 4

(a) If $y = e^{\ln x} =$ ___, then $y' =$ ___. (b) If $y = e^{2\ln x} =$ ___, then $y' =$ ___.

Fundamentals 5

Make sure to always s______ any logarithms before differentiating.

Question 1 Use the fact that $\frac{d}{dx}\left[\ln\left(f(x)\right)\right] = \frac{f'(x)}{f(x)}$ to differentiate the following.

(a) $\ln(4x-3)$

(b) $\ln(5-2x)$

(c) $\ln(x^2+1)$

(d) $\frac{1}{2}\ln(x^2-2x)$

(e) $\ln(\ln x)$

(f) $\ln(x^2-\ln x)$

Question 2 Consider $f(x) = \ln\left((2x-3)(3x+1)\right)$.

(a) Use the fact that $\frac{d}{dx}\left[\ln\left(f(x)\right)\right] = \frac{f'(x)}{f(x)}$ to find $f'(x)$.

(b) Use an appropriate logarithmic law to simplify $f(x)$.

(c) Hence, find another way of differentiating $f(x)$ and show that it yields the same result as (a).

Question 3 Use the logarithmic laws to first simplify, and then differentiate the following.

(a) $\ln\left((x+1)^4\right)$

(b) $\ln\left((3x-2)^5\right)$

(c) $\ln\left((x-1)(x-2)\right)$

(d) $\ln\left((5-2x)(4x-3)\right)$

(e) $\ln(5x^2)$

(f) $\ln\left(\frac{5}{4x-3}\right)$

(g) $\ln\left(\frac{5-2x}{4x-3}\right)$

(h) $\ln\sqrt{5x}$

(i) $\ln\left(\sqrt{2x+1}\right)$

(j) $\ln\left(\frac{1}{\sqrt{2x-1}}\right)$

Question 4 Differentiate the following using the chain rule.

(a) $(\ln x)^2$

(b) $(1-\ln x)^3$

(c) $\frac{1}{\ln x}$

(d) $\sqrt{1+\ln x}$

Question 5 Differentiate the following using the product rule.

(a) $x\ln x$

(b) $x^2\ln x$

(c) $(2x+3)\ln(2x+3)$

(d) $x\ln(x^2+1)$

Question 6 Differentiate the following using the quotient rule.

(a) $\frac{\ln x}{x}$

(b) $\frac{x^2}{\ln x}$

(c) $\frac{\ln(2x+1)}{2x+1}$

(d) $\frac{\ln x}{\sqrt{x}}$

Question 7 Differentiate the following.

(a) $\ln\left((2x-1)^3(3x+2)^4\right)$

(b) $\ln\sqrt{x^2-1}$

(c) $\ln\left(\dfrac{(3x-1)^2}{(2x+1)^3}\right)$

(d) $\ln\left(\sqrt{\dfrac{x+2}{x-2}}\right)$

Question 8 [Differentiating with different bases]

Consider the function $f(x) = \log_5 x$.

(a) Express $f(x)$ using base e.

(b) Hence, differentiate $f(x)$.

(c) Use a similar technique to differentiate $\log_a x$, where $a > 1$.

Question 9 Differentiate the following.

(a) $\log_2(x)$

(b) $\log_3(2x+1)$

(c) $\log_4(x^2+1)$

(d) $\left(\log_5(x)\right)^2$

Question 10 Differentiate the following by first simplifying.

(a) $y = e^{\ln x}$

(b) $y = e^{3\ln x}$

(c) $y = e^{-2\ln x}$

(d) $y = \ln\left(e^{2x}\right)$

(e) $y = e^{\ln(3x)}$

(f) $y = e^{2\ln(3x)}$

Question 11 [Logarithmic differentiation]

Let $f(x) = 2^x$.

(a) By taking the natural logarithm of both sides, show that $\ln\left(f(x)\right) = x\ln 2$.

(b) By differenting both sides with respect to x, show that $f'(x) = 2^x \ln 2$.
Hint: You will need to use a familiar formula here.

Question 12 Complete the following steps to differentiate $y = a^x$.

(a) First re-express y as

$$\begin{aligned} y &= (e^{\ln \_\_})^x \\ &= e^{x\ln \_\_} \end{aligned}$$

(b) Differentiating this, we get

$$\begin{aligned} y' &= \_\_\_ \times e^{x\ln \_\_} \\ &= \_\_\_ \times a^x \end{aligned}$$

Question 13 Use the method outlined in the above question to differentiate the following.

(a) 3^x

(b) 4^{-x}

Question 14 [Basic proof of the derivative of $\ln x$]

Consider $y = \ln x$.

(a) By first making x the subject, show that $\dfrac{dx}{dy} = e^y$.

(b) Use the fact that $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ to show that $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Question 15 Show that if $y = \dfrac{x}{\ln x}$, then

$$y^2 - xy + x^2y' = 0.$$

Challenge Problems

Problem 1 Differentiate the following with respect to x.

(a) $\ln\left(\dfrac{x^2-1}{x^2+1}\right)$

(b) $\ln\left(\dfrac{x}{\sqrt{x^2-1}}\right)$

Problem 2 Show that

$$\frac{d^2}{dx^2}\Big(\ln f(x)\Big) = \frac{f''(x)}{f(x)} - \left(\frac{f'(x)}{f(x)}\right)^2.$$

Exercise 4C

Differentiation of trigonometric functions

Fundamentals

Fundamentals 1
Write down the derivative of the following.

(a) $\sin x$ (b) $\cos x$ (c) $\tan x$

(d) $\sin(ax+b)$ (e) $\cos(ax+b)$ (f) $\tan(ax+b)$

(g) $\sin(f(x))$ (h) $\cos(f(x))$ (i) $\tan(f(x))$

Fundamentals 2

(a) When differentiating a trigonometric function, the 'angle' must be in r________.

(b) If the 'angle' is in degrees instead, then it must first be converted to r________ using the formula $x° =$ ________

Question 1 Differentiate the following.

(a) $\sin(2x)$ (b) $\cos(3x)$ (c) $\tan(4x)$

(d) $4\sin\left(\frac{x}{2}\right)$ (e) $-6\cos\left(\frac{x}{3}\right)$ (f) $8\tan\left(\frac{x}{4}\right)$

(g) $\sin\left(\frac{\pi}{2}-x\right)$ (h) $\cos\left(3x-\frac{\pi}{6}\right)$ (i) $\tan\left(\frac{2x+\pi}{3}\right)$

Question 2 [Differentiation with degrees]

By first expressing the following in terms of radians, find the derivative.

(a) $\sin(x°)$ (b) $\tan(2x°)$ (c) $\cos(90x°)$

Question 3 Prove the following results.
Hint: First express in terms of $\sin x$ and/or $\cos x$, and then differentiate.

(a) $\frac{d}{dx}(\tan x) = \sec^2 x$ (b) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(c) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ (d) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Question 4 Differentiate the following using the chain rule.

(a) $y = \sin(x^2)$ (b) $y = \cos(3x^2)$ (c) $y = \tan(2 - 5x^2)$

(d) $y = \sin\left(\frac{1}{x}\right)$ (e) $y = \cos(\sqrt{x})$ (f) $y = \tan\left(\sqrt{x^2+1}\right)$

Question 5 Differentiate the following using the chain rule.

(a) $y = \sin^2 x$ (b) $y = \cos^3(2x)$ (c) $y = \tan^2(3x)$

(d) $y = \frac{1}{\sin x}$ (e) $y = \sqrt{\cos x}$ (f) $y = \sqrt{\tan(2x)}$

Question 6 Differentiate the following using the product rule.

(a) $x \sin x$ (b) $x^2 \tan\left(\frac{x}{2}\right)$ (c) $\sin^2 x \cos x$

(d) $\sin(2x)\cos(3x)$ (e) $\sqrt{x}\sin x$ (f) $x^2 \sin^2 x$

Question 7 Differentiate the following using the quotient rule.

(a) $\frac{x}{\sin x}$ (b) $\frac{\cos x}{x^2}$ (c) $\frac{\sin^2 x}{\cos x}$

(d) $\frac{\sin x}{\sqrt{x}}$ (e) $\frac{\sin x}{\sin x + \cos x}$ (f) $\frac{\tan x - \sec x}{\tan x + \sec x}$

Question 8

(a) Show that $y = \sin(2x)$ satisfies the differential equation $y'' = -4y$.

(b) Show that $x = 6\sin\left(2t + \frac{\pi}{2}\right)$ satisfies the differential equation $\ddot{x} = -4x$.

(c) Show that $x = 12 - 2\cos(3t)$ satisfies the differential equation $\ddot{x} = -9(x - 12)$.

(d) Show that $y = x\sin x$ satisfies the differential equation $y'' + y = 2\cos x$.

(e) Show that $y = \frac{A}{x^2} + \frac{\sin x}{x^2} - \frac{\cos x}{x}$ satisfies the differential equation $xy' + 2y = \sin x$.

(f) Find the value of A and B so that $y = A\cos(2x) + B\sin(2x)$ satisfies the differential equation $y'' - 4y' - 12y = \sin(2x)$ for any real values of A and B.

Question 9 [Differential equation of simple harmonic motion]

Let A, n, α and x_0 be constants. Show that

$$x = A\cos(nt + \alpha) + x_0$$

satisfies the differential equation

$$\ddot{x} = -n^2(x - x_0).$$

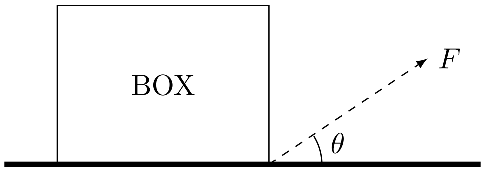
Question 10

(a) Let $f(x) = A\sin x + B\cos x$, where A and B are constants. Find the value of A and B if $f\left(\frac{\pi}{4}\right) = 3$ and $f'\left(\frac{\pi}{4}\right) = 1$.

(b) Let $f(x) = A\tan x + B\sec x$, where A and B are constants. Find the value of A and B if $f\left(\frac{\pi}{4}\right) = 2$ and $f'\left(\frac{\pi}{4}\right) = 0$.

Question 11 [Application to kinetic friction]

A box of mass M on the ground has a rope tied to the base. The rope is pulled at an angle of θ from the ground with force F.



The amount of force required to overcome friction and move the box at a constant velocity is

$$F = \frac{\mu M}{\cos\theta + \mu\sin\theta},$$

where μ is the *coefficient of dynamic friction* between the box and the ground. Suppose the box has mass 50 kg and $\mu = 0.1$. Find the value of $\frac{dF}{d\theta}$, correct to two decimal place, when $\theta = \frac{\pi}{6}$.

Challenge Problems

Problem 1 [More difficult differentiation problems]

Differentiate the following.

(a) $x\sin\left(\frac{1}{x}\right)$ (b) $\frac{1}{x}\tan\left(\frac{1}{x}\right)$ (c) $x^2\sin\left(\frac{1}{x^2}\right)$

Problem 2 [Equations for simple harmonic motion]

Show that the following displacement-time equations all satisfy the differential equation, for any values of A and B.

$$\ddot{x} = -n^2 x$$

(a) $x = A\cos(nt)$ (b) $x = B\sin(nt)$ (c) $x = A\cos(nt) + B\sin(nt)$

Problem 3 The equation $x = A\cos^2(mt)$ satisfies the differential equation

$$\ddot{x} = -n^2(x - x_0),$$

for some value of x_0 and n.

(a) Show that

$$\ddot{x} = -2m^2 A\Big(\cos^2(mt) - \sin^2(mt)\Big)$$

(b) Show that $x_0 = \dfrac{A}{2}$ and $n = 2m$.

Exercise 4D

Mixed differentiation problems

Fundamentals

Fundamentals 1

Differentiate the following.

(a) $\dfrac{1}{f(x)}$ (b) $\sqrt{f(x)}$

(c) $(f(x))^n$ (d) $e^{f(x)}$

(e) $\sin(f(x))$ (f) $\cos(f(x))$

(g) $\tan(f(x))$ (h) $\ln(f(x))$

Question 1 Differentiate the following.

(a) $\sin(e^x)$ (b) $\tan(e^{2x})$ (c) $\cos\left(e^{-x^2}\right)$

(d) $\sin(\ln x)$ (e) $\cos(\ln x)$ (f) $\tan^2(\ln x)$

Question 2 Differentiate the following.

(a) $e^x \sin x$ (b) $e^{\sin x}$ (c) $e^{\tan x}$

(d) $e^{-x}\sin(2x)$ (e) $e^{\sin^2 x}$ (f) $e^{x\tan x}$

Question 3 Differentiate the following.

(a) $\ln(e^x)$ (b) $\ln(e^{2x})$ (c) $\ln(e^{-x})$

(d) $\ln(xe^{2x})$ (e) $\ln\left((1+e^{2x})^3\right)$ (f) $\ln\left(\dfrac{2e}{x^2}\right)$

(g) $\ln(\sin x)$ (h) $\ln\sqrt{\tan x}$ (i) $\ln(1+\tan x)$

(j) $\ln(1-e^{-2x})$ (k) $\ln(\cos^2 x - \sin^2 x)$ (l) $\ln(e^x \sin x)$

(m) $\ln\left(\dfrac{e^x-1}{e^x+1}\right)$ (n) $\ln\left(\dfrac{\sin x+\cos x}{\sin x-\cos x}\right)$ (o) $\ln\sqrt{1-\tan^2 x}$

Question 4 [Anti-derivative of $\sec x$]

Let $y = \ln(\sec x + \tan x)$.

(a) Show that $y = \ln\left(\dfrac{1+\sin x}{\cos x}\right)$.

(b) Hence, show that $\dfrac{dy}{dx} = \sec x$.

Question 5 Show that $y = e^{ax}\sin(bx)$ satisfies the differential equation

$$y'' - 2ay' + (a^2 + b^2)y = 0$$

for any real values of a and b.

Challenge Problems

Problem 1 [Under-damped oscillator]

Show that $y = Ae^{-t}\cos t + Be^{-t}\sin t$ satisfies the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0$$

for any real values of A and B.

Problem 2 [Differential equation of the catenary]

The hyperbolic cosine function $\cosh x$ has can be expressed as the sum of exponential functions

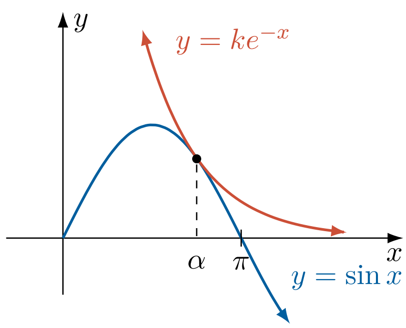
$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

When a cable is hung from two poles of equal height, the curve that the cable forms has equation

$$y = a\cosh\left(\frac{x}{a}\right),$$

where a is a constant. Show that $\dfrac{d^2y}{dx^2} = \dfrac{1}{a}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$.

Problem 3 The diagram below shows the graph of $y = \sin x$ and $y = ke^{-x}$ for some value of k such that the two curves are tangential.



(a) Show that $\sin\alpha = ke^{-\alpha}$ and $\cos\alpha = -ke^{-\alpha}$.

(b) Hence, find the value of k such that the two curves are tangential, and the coordinates of the point of tangency.

Chapter 4 Review

Further differentiation

Review

Question 1 Calculate the derivatives of the following functions

(a) $y = e^{3x^2}$ (b) $y = e^{\frac{1}{x}}$ (c) $y = x^3 e^x$

(d) $y = xe^{3x^2}$ (e) $y = \dfrac{e^{3x}}{x}$ (f) $y = e^{2\ln(x+4)}$

Question 2 Show that $y = xe^{-x}$ is a solution of $y'' + 2y' + y = 0$

Question 3 Find the value of k if

(a) $f(x) = e^{kx}$ is a solution of $f''(x) - 5f'(x) + 4f(x) = 0$.

(b) $f(x) = 4e^{kx}$ is a solution of $f''(x) - 4f'(x) - 5f(x) = 0$.

Question 4 Differentiate the following

(a) $y = \ln(3x - 5)$ (b) $y = \ln\left(3x^{-4}\right)$ (c) $y = \log_2(x)$

(d) $y = \log_x(2)$ (e) $y = \ln\left(\dfrac{e^{2x}+1}{e^{2x}-1}\right)$ (f) $y = \ln\left(\sin 3x\right)$

(g) $y = \ln\left(e^2 x^2\right)$ (h) $y = \ln\left(x\sqrt{x^2 - 1}\right)$ (i) $y = \ln\left(\dfrac{1+\sin x}{1-\sin x}\right)$

Question 5 Differentiate the following.

(a) $y = 6^x$ (b) $y = 3^{2x}$

(c) $y = \sqrt{9^x}$ (d) $y = x^2 4^x$

Question 6 Differentiate the following

(a) $y = \sin(2x - 3)$ (b) $y = \cos\left(\dfrac{\pi - 3x}{2}\right)$ (c) $y = \tan\left(\dfrac{1}{x}\right)$

(d) $y = \tan(1 - x^2)$ (e) $y = \sin(\sqrt{x})$ (f) $y = \sqrt{\sin(x)}$

Question 7 Differentiate the following

(a) $y = 2\cos^2(\sqrt{x})$	(b) $y = \operatorname{cosec} x$	(c) $y = \sec x$	(d) $y = \sec^2 4x$
(e) $y = e^{\tan x}$	(f) $y = \cot 3x$	(g) $y = e^{2x}\cos 3x$	(h) $y = \ln \tan x$
(i) $y = e^{2x} + \ln(2x)$	(j) $y = e^{\sqrt{x}} + \ln\sqrt{x}$	(k) $y = e^{\cos x}$	(l) $y = \cos(e^x)$
(m) $y = \cos(\tan x)$	(n) $y = e^{2x}\sin 4x$	(o) $y = \sin x \cos x$	(p) $y = \tan x \cos x$
(q) $y = 4x^2 \ln\left(\frac{x}{2}\right)$	(r) $y = e^{2x}\ln(2x)$	(s) $y = e^{2x}\ln\left(e^{2x}\right)$	(t) $y = \frac{\sin 2x}{x}$
(u) $y = \frac{e^x}{\cos x}$	(v) $y = \frac{\sin x + \cos x}{e^x}$	(w) $y = \frac{\ln 2x}{e^{2x}}$	(x) $y = \frac{\ln x}{x^2}$

Question 8 If $x = 6\cos\left(3t + \frac{3\pi}{4}\right)$ show that $\ddot{x} = -9x$.

Question 9 If $y = \ln(\tan 3x)$, show that $\frac{dy}{dx} = 3\sec 3x \operatorname{cosec} 3x$.

Question 10 Given $y = e^{3x}$

(a) Find $\frac{dy}{dx}$.

(b) Express x in terms of y.

(c) Find $\frac{dx}{dy}$.

(d) Show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.

Investigation Task

Logarithmic Differentiation

Currently, we have techniques such as the Chain Rule, Product Rule and Quotient Rule to differentiate large expressions like

$$y = x\sqrt{1-x^2}$$

But consider an *even larger* expression like

$$y = (2x+1)^3(3x+1)^4(4x+1)^5$$

Doing this using using conventional techniques would be a nightmare, but *logarithmic differentiation* makes quick work of such problems.

Question 1 Research what logarithmic differentiation is, and explain why it is so effective with differentiating large expressions.

Question 2 Use logarithmic differentiation to differentiate the following.

(a) $y = (2x+1)^5(3x-2)^4$

(b) $y = x\sqrt{1-x^2}$

Question 3 Use logarithmic differentiation to differentiate the following.

(a) x^x

(b) x^{x^2}

(c) $x^{\sqrt{x}}$

Question 4 Write a short one-page document that is sufficiently detailed so that if a Year 12 student follows it, they will be able to understand fully what logarithmic differentiation is, and how to use it to differentiate large expressions.

5

APPLICATIONS OF DIFFERENTIATION

- Tangents and normals
- The first derivative
- The second derivative
- Introduction to curve sketching and global maxima/minima
- Harder curve sketching
- Optimisation

Exercise 5A

Tangents and normals

Fundamentals

Fundamentals 1

Let $f(x)$ be a differentiable function at (x_1, y_1).

(a) The gradient of the tangent is ________.

(b) The equation of the tangent is $y - y_1 =$ ______________.

(c) The normal is the line p________ to the tangent, but passing through the same point from which the tangent was subtended.

(d) The equation of the normal is therefore $y - y_1 =$ ______________.

Fundamentals 2

(a) If $f'(a) > 0$, then $f(x)$ is i________ at $x = a$.

(b) If $f'(a) < 0$, then $f(x)$ is d________ at $x = a$.

(c) If $f'(a) = 0$, then $f(x)$ is s________ at $x = a$.

(d) A function is stationary whenever $f'(x)$ ___ 0.

Fundamentals 3

To find where

(a) $f(x)$ is increasing, let $f'(x)$ _____

(b) $f(x)$ is decreasing, let $f'(x)$ _____

(c) $f(x)$ is stationary, let $f'(x)$ _____

Fundamentals 4

If a line is inclined at an angle of θ from the positive horizontal axis, then the gradient of the line is $m =$ _____.

Question 1

(a) Find the gradient of the tangent to $y = x^2 - x + 5$ when $x = 2$.

(b) Find the equation of the tangent when $x = 2$.

(c) Find the equation of the normal when $x = 2$.

Question 2

(a) Find the point on $y = x^2 - 5x$ where the tangent has gradient 1.

(b) Hence, find the equation of the tangent at this point.

Question 3

Find the value of x on $y = x^2 - 2x + 3$ where the tangent is

(a) inclined at 45° from the positive x-axis.

(b) inclined at 135° from the positive x-axis.

(c) parallel to $y = -2x - 3$.

(d) perpendicular to $x + 4y - 1 = 0$.

Question 4

(a) The tangent to $y = ax^2 + bx - 3$ at $(-1, 1)$ has a gradient of -6. Find the values of a and b.

(b) The line $y = ax + b$ is a tangent to $y = x^3 - x + 4$ at $(1, 5)$. Find the values of a and b.

Question 5

The curve $y = ax^2 + bx + c$ has a gradient of -1 when $x = 3$. The tangent to the curve at $(2, -3)$ is inclined at an angle of 45° from the positive x-axis. Find the values of a, b and c.

Question 6

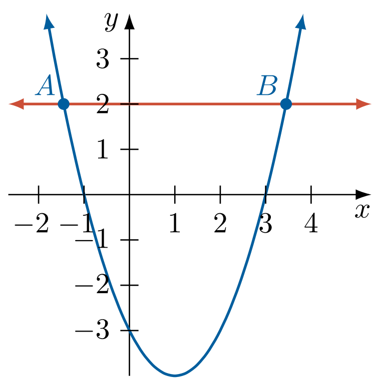
(a) Find the gradient of the tangents to $y = 6 + x - x^2$ when $x = 0$ and $x = 1$. What do you notice?

(b) Repeat part (a), but for $x = -2$ and $x = 3$. What do you notice?

(c) Explain your observations.

Hint: Consider the axis of symmetry $x = \dfrac{1}{2}$.

(d) The diagram below shows the graph of a general parabola and a horizontal line that cuts it at two points A and B. Let the gradient of the tangents from A and B be m and n respectively.



Write down a relationship between m and n.

Question 7

(a) Find the coordinates of the points on the curve $y = 2x^3 - 2x^2$ where the tangent is parallel to the line $2x - y - 5 = 0$.

(b) Find the equation of the normal to $y = 2x^2 - 3$ that is parallel to $x - y + 4 = 0$.

(c) Find the point(s) on $y = \cos(3x) - 1$ over the domain $x \in [0, \pi]$, where the tangent is parallel to $3x + y = 0$.

Question 8

(a) Show that the parabola $y = x^2$ and the line $y = x + 2$ intersect at $A(-1, 1)$ and $B(2, 4)$.

(b) Find the equation of the tangent to the curve at A and B.

(c) Find the coordinates of the point P where the tangent is parallel to AB.

Question 9

(a) Show that the curves $y = \sqrt{3x + 1}$ and $y = \sqrt{5x - x^2}$ intersect when $x = 1$.

(b) Find the equation of the tangent to both curves at this point. What do you notice?

(c) Explain your findings.

(d) Use graphing software to verify your findings.

Question 10 Find the equation of the tangent to each of the following curves at the point indicated.

(a) $y = e^x$ when $x = 0$.

(b) $y = \ln x$ when $x = e$.

(c) $y = 2\sin(3x)$ when $x = \frac{\pi}{9}$.

(d) $y = xe^{-x}$ when $x = 1$.

(e) $y = \tan^2(x)$ when $x = \frac{\pi}{4}$.

(f) $y = \frac{\ln x}{x}$ when $x = e$.

Question 11 Find the equation of the normal to each of the following curves at the point indicated. Express your answers in the form $ax + by + c = 0$.

(a) $y = e^{\frac{1}{x}}$ when $x = -1$.

(b) $y = \ln(x^2 + 1)$ when $x = 1$.

(c) $y = 2\tan(x)$ when $x = \frac{\pi}{4}$.

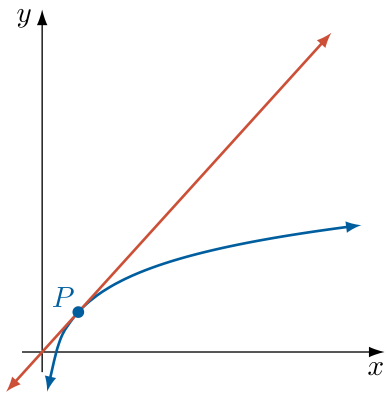
Question 12 [Further simplification with exponentials and logarithms]

For the following questions, the result $e^{\ln x} = x$ for $x > 0$ is required. Express your answers in the form $y = ax + by + c$.

(a) Find the equation of the tangent to $y = e^{2x}$ when $x = \ln 2$.

(b) Find the equation of the normal to $y = e^{-x}$ when $x = 2\ln 2$.

Question 13 The diagram below shows the point $P\,(k, \ln(3k))$ on the curve $y = \ln(3x)$, chosen so that the tangent from P passes through the origin.

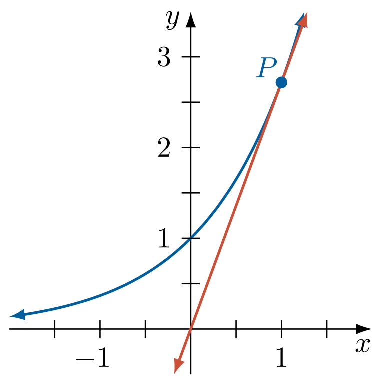


(a) Find the equation of the tangent at P, in terms of k.

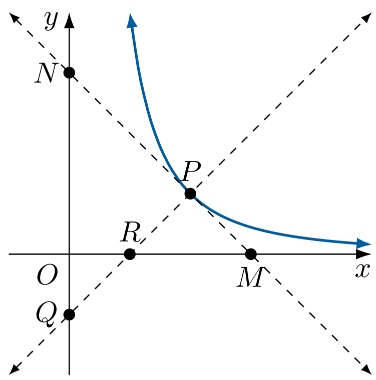
(b) Show that $k = \dfrac{e}{3}$.

(c) Write down the equation of the tangent that passes through the origin.

Question 14 Find the point on $y = e^x$ so that the tangent passes through the origin.



Question 15 The diagram below shows a sketch of $y = \dfrac{4}{x^2}$ for $x > 0$.



A tangent is drawn from P and cuts the x and y-axes at M and N respectively. The point P is chosen such that $\angle PNO = \angle PMO$. A normal is drawn from P, and cuts the x and y-axes respectively at R and Q.

(a) Explain why the gradient of MN is -1.

(b) Show that the coordinates of P are $(2, 1)$.

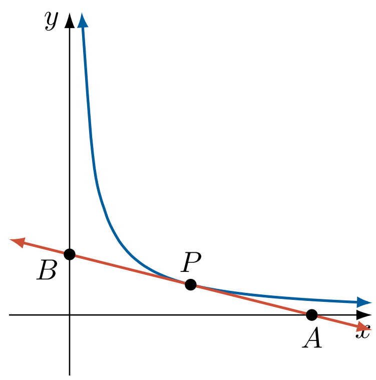
(c) Find the equation of MN and hence find the coordinates of M and N.

(d) Find the equation of the normal PQ, and hence find the coordinates of R and Q.

(e) Hence, find the area of $\triangle RPM$ and $\triangle NPQ$.

Challenge Problems

Problem 1 The diagram below shows the tangent from a general point $P(x_0, y_0)$ on the curve $y = \dfrac{1}{x}$.



A tangent is drawn from P and intersects the x and y-axes at A and B respectively.

(a) Show that the equation of the tangent from P is

$$y - y_0 = -\frac{1}{x_0^2}(x - x_0)$$

(b) Find the coordinates of A and B.

(c) Hence, show that the area of $\triangle OAB$ is independent of the position of P.

Problem 2 [Proving an inequality]

(a) Find the equation of the tangent to $y = e^x$ when $x = 0$.

(b) Deduce that $e^x \geq 1 + x$ for $x \geq 0$.

Exercise 5B

The first derivative

Fundamentals

Fundamentals 1

(a) A stationary point occurs when $f'(x) =$ ____.

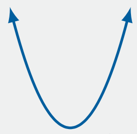
(b) This means that at the stationary point, the tangent is h________.

(c) A t________ point is a special type of stationary point where the curve changes d________. Not all stationary points are t________ points.

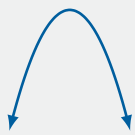
Fundamentals 2

The diagrams below show three different types of stationary points. Classify them accordingly.

(a)



(b)



(c)

Fundamentals 3

(a) One of the methods of classifying stationary points is to draw a t______ of values.

x	$\alpha - \epsilon$	α	$\alpha + \epsilon$
$f'(x)$			

(b) This tests the g________ of the tangent on either side of the stationary point.

Fundamentals 4

(a) A function is increasing over a domain if $f'(x)$ ___ 0 for all x in that domain.

(b) A function is d________ over a domain if $f'(x) < 0$ for all x in that domain.

Fundamentals 5

(a) The following table indicates that α is a local maximum/minimum (circle one).

x	$\alpha - \epsilon$	α	$\alpha + \epsilon$
$f'(x)$	$+$	0	$-$

(b) The following table indicates that α is a local maximum/minimum (circle one).

x	$\alpha - \epsilon$	α	$\alpha + \epsilon$
$f'(x)$	$-$	0	$+$

Question 1 Consider the curve $y = x^3 + x$.

(a) Write down y'.

(b) Explain briefly why this is always a positive number, regardless of the value of x.

(c) Hence, this is an i________ function.

Question 2 Show that the following functions are increasing for all x in the function's domain.

(a) $y = 3x - 5$ (b) $y = \sqrt{x}$ (c) $y = -\dfrac{2}{x}$

(d) $y = \dfrac{3x}{4 - x}$ (e) $y = e^x - e^{-x}$ (f) $y = \dfrac{x^2 - 1}{x}$

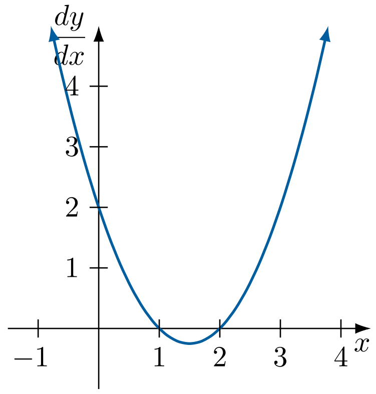
Question 3 Show that the following functions are decreasing for all x in the function's domain.

(a) $y = -2x + 6$ (b) $y = -x^3$ (c) $y = \dfrac{4x}{x - 2}$

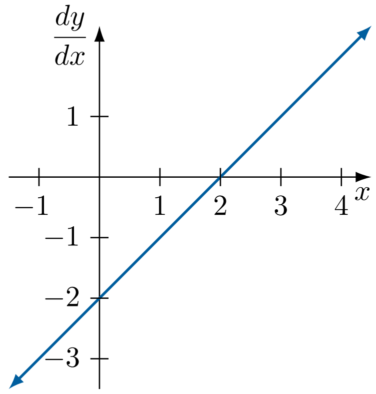
(d) $y = \dfrac{1}{2x - 5}$ (e) $y = \ln(4 - 2x)$ (f) $y = -\tan x$

Question 4 The diagram below shows the sketch of a gradient function $\dfrac{dy}{dx}$.
Write down the value(s) of x for which $y = f(x)$ is (i) increasing (ii) decreasing (iii) stationary.

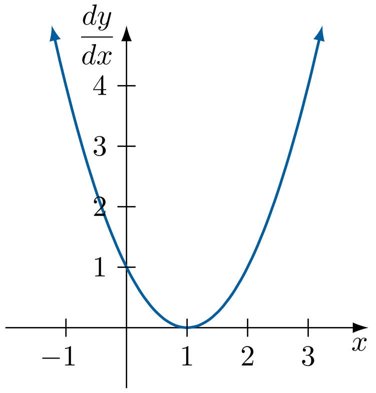
(a)



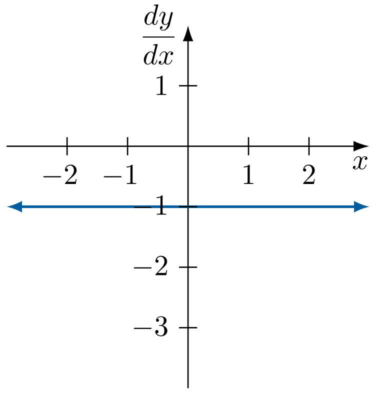
(b)



(c)



(d)



Question 5 For each of the following functions, state the values of x for which $f(x)$ is increasing.

(a) $y = x^2 - 4x$ (b) $y = 6 + 4x - x^2$

(c) $y = \dfrac{2}{x^2}$ (d) $y = x^3 - 3x^2 - 9x + 27$

Question 6 For each of the following functions, state the values of x for which $f(x)$ is decreasing.

(a) $y = x^2 - 6x + 5$ (b) $y = 10 - 4x - 2x^2$

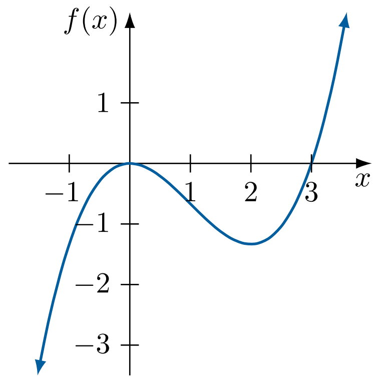
(c) $y = \dfrac{x}{x-4}$ (d) $y = x^3 - 3x^2 - 24x$

Question 7 For each of the following functions, state the value(s) of x for which $f(x)$ is stationary.

(a) $y = 8x - x^2$ (b) $y = 2x^3 + 3x^2 - 36x$

(c) $y = x - \sqrt{x}$ (d) $y = \dfrac{x}{1-x}$

Question 8 The diagram below shows the graph of $y = f(x)$.



Write down the values of x for which

(a) $f'(x) > 0$ (b) $f'(x) < 0$ (c) $f'(x) = 0$

Question 9 Consider the curve

$$f(x) = x^3 - 6x^2 + 9x + 1.$$

(a) Find the x-coordinates of the stationary points.

(b) Complete the following table.

x	0	1	2	3	4
$f'(x)$		0		0	
Direction					

(c) Hence, determine the nature of the stationary points.

Question 10 Find the x-coordinates of the point(s) where $f'(x) = 0$ for each of the following curves.

(a) $y = x^2 - 8x + 16$ (b) $y = 6x^2 - x^3$ (c) $y = 2x^3 + 3x^2 - 36x + 4$

(d) $y = e^{-x^2}$ (e) $y = xe^x$ (f) $y = x^2e^{-x}$

(g) $y = x\ln x$ (h) $y = x^2 \ln x$ (i) $y = \dfrac{\ln x}{x}$

Question 11 For each of the parts in Question 10, determine the nature of the stationary point(s).

Question 12 Show that $y = x^3 - 6x^2 + 18x + 1$ is strictly increasing.
Hint: At some point, you may need to complete the square.

Question 13 Determine the nature of the stationary point of the curves satisfying the following first derivatives.

(a) $y' = (x+1)(x-2)(x-4)$ (b) $y' = x(x+3)(x-5)^2$

(c) $y' = (x+2)^2(x-3)^3$ (d) $y' = (x+1)^2(x-2)^2$

Question 14 Show that any function in the form

$$f(x) = \ln(ax+b),$$

where a and b are constants, will never have any stationary points.

Challenge Problems

Problem 1 Define the function

$$f(x) = \frac{ax+b}{cx+d},$$

where a, b, c and d are constants.

(a) Show that $f(x)$ cannot have any stationary points.

(b) Show that if $f(x)$ is increasing, then $ad - bc > 0$.

(c) State what happens to $f(x)$ when $ad - bc < 0$.

Problem 2 Show that $y = \dfrac{\sin x}{1 + \cos x}$ is always increasing in the domain $-\pi < x < \pi$.

Problem 3 Consider the curve $y = x\sqrt{x+1}$.

(a) Find any stationary points and determine their nature.

(b) Find any points where $y'' = 0$.

Problem 4 Consider the curve $y = (x+1)\sqrt{x}$.

(a) Show that $y' = \dfrac{3x+1}{2\sqrt{x}}$

(b) Bob claims that there is therefore a stationary point at $x = -\dfrac{1}{3}$. Mary claims that there is no stationary point. Who is correct, and why?

Problem 5 Consider the polynomial $y = x^3 + ax^2 + bx + 1$.

(a) Show that if $a^2 > 3b$, then the curve has two distinct stationary points.

(b) Describe what happens to the curve if $a^2 = 3b$.

(c) Describe what happens to the curve if $a^2 < 3b$.

Exercise 5C

The second derivative

Fundamentals

Fundamentals 1

(a) A function $f(x)$ is concave up where $y'' > 0/y'' < 0$ (circle one).

(b) A function $f(x)$ is concave down where $y'' > 0/y'' < 0$ (circle one).

Fundamentals 2

(a) To find the possible locations for points of i________, we can let $y'' =$ ___.

(b) However, the solutions of $y'' =$ ___ may NOT be points of inflection necessarily.

(c) To verify if the point is indeed a point of inflection, a t_____ of values with x and $f''(x)$ needs to be constructed.

x	$\alpha - \epsilon$	α	$\alpha + \epsilon$
$f''(x)$			

(d) It is a point of inflection only if $f''(x)$ has changes s____ about the point being tested.

(e) Does the following table indicate that there is a point of inflection at $x = \alpha$?

x	$\alpha - \epsilon$	α	$\alpha + \epsilon$
$f''(x)$	$+$	0	$+$

(f) Does the following table indicate that there is a point of inflection at $x = \alpha$?

x	$\alpha - \epsilon$	α	$\alpha + \epsilon$
$f''(x)$	$-$	0	$+$

Fundamentals 3

(a) Another method of determining the nature of a stationary point at $x = \alpha$ is to use the s________ derivative test.

(b) If $f''(\alpha) > 0$, then there is a maximum/minimum (circle one) turning point at $x = \alpha$.

(c) If $f''(\alpha) < 0$, then there is a maximum/minimum (circle one) turning point at $x = \alpha$.

(d) If $f''(\alpha) = 0$, then there MAY be a h________ point of inflection at $x = \alpha$.

Question 1 Consider the curve

$$f(x) = x^3 - 6x^2 + 9x + 1.$$

(a) Find the x-coordinates of the points when $f''(x) = 0$.

(b) Complete the following table.

x	1	2	3
$f''(x)$			
Sign			

(c) Hence, determine whether or not $x = 2$ is a point of inflection.

Question 2 Find the x-coordinates of the point(s) where $f''(x) = 0$ for each of the following curves.

(a) $f(x) = x^4 + 2x^3 - 12x^2 + 36x - 12$

(b) $f(x) = x^4 + 8x^3 + 24x^2 + 32x + 8$

(c) $f(x) = e^{-x^2}$

(d) $f(x) = xe^{-x}$

(e) $f(x) = x^2 \ln x$

(f) $f(x) = (e^x - 1)^4$

Question 3 For each of the parts in Question 2, use a table of values to determine which of the solutions of $f''(x) = 0$ are points of inflection.

Question 4 Find the values of x for which the following functions are concave up.

(a) $y = x^3 - 3x^2$

(b) $y = xe^x$

(c) $y = x^4 + 2x^3 - 36x^2 + 12x - 5$

Question 5 Find the values of x for which the following functions are concave down.

(a) $y = xe^{-x}$

(b) $y = x - \dfrac{1}{x}$

(c) $y = \ln(x^2 + 1)$

Question 6 Prove that the following functions are always concave up.

(a) $y = e^x + e^{-x}$

(b) $y = x \ln x$

(c) $y = x^2 + \dfrac{1}{x^2}$

Question 7 [$y'' = 0$ is not sufficient for an inflection point]

Consider the equation $y = x^4 - 8x^3 + 24x^2 - 32x + 16$.

(a) Find y' and y''.

(b) Show that $y' = y'' = 0$ when $x = 2$.

(c) Bob claims that there is a horizontal point of inflection at $x = 2$ since both $y' = 0$ and $y'' = 0$. Use graphing software to sketch the graph of the polynomial, and describe your findings. Is Bob correct?

(d) What should Bob have done to verify that indeed there was a point of inflection at $x = 2$?

Question 8 Consider the curve $y = 20x^3 - 12x^5$.

(a) Find all stationary points and determine their nature.

(b) Find all points of inflection, and verify that they are indeed inflection points.

Question 9 How many points of inflections does a curve have, if the second derivative satisfies the following?

(a) $f''(x) = (x-1)(x-2)(x-3)$

(b) $f''(x) = (x-2)^2(x+2)$

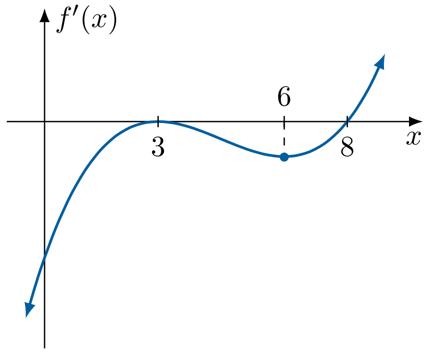
(c) $f''(x) = (x+1)^2(x-2)^3$

(d) $f''(x) = x^4 - 2x^2 + 1$

Challenge Problems

Problem 1 [Useful for motion problems]

The diagram below shows a sketch of $y = f'(x)$.



Write down the value(s) of x for which the original function $y = f(x)$ is

(a) stationary.

(b) increasing.

(c) concave down.

(d) has a point of inflection.

Problem 2 Find the x-coordinates of the inflection points of $y = \dfrac{x}{x^2+1}$ and verify that they are inflection points.

Problem 3 Consider the curve $y = \ln\left(\dfrac{x}{x-1}\right)$

(a) Show that $y'' = \dfrac{1}{(x-1)^2} - \dfrac{1}{x^2}$

(b) Solve $y'' > 0$ and show that the solution is $x > \dfrac{1}{2}$.

(c) Bob claims that the curve is therefore concave up for $x > \frac{1}{2}$. Mary claims instead that the curve is concave up for $x > 1$. Who is correct, and why?

Exercise 5D

Introduction to curve sketching and global maxima/minima

Fundamentals

Fundamentals 1

The results from the following questions also apply for the minimum.

(a) A local maximum can be found by letting $y' =$ ___ .

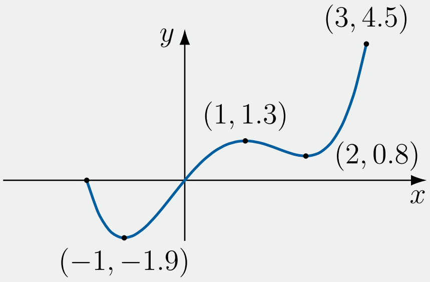
(b) However, the local maximum may not necessarily be the overall maximum value of the curve. The overall maximum value of the curve is called the g_____ maximum.

(c) The global maximum can/cannot (circle one) be the same as the local maximum.

(d) To find the global maximum, we need to find the local maximum and compare it with the endpoints of the given d_____, if provided.

Fundamentals 2

For the diagram below, state the local maximum/minimum, and the global maximum/minimum.



Fundamentals 3

(a) The vertical asymptotes of a rational function $f(x) = \dfrac{A(x)}{B(x)}$, where $A(x)$ and $B(x)$ are polynomials, occur when $B(x) =$ ___ .

(b) When a vertical asymptote $x = \alpha$ is found, it is often useful to find $\lim\limits_{x \to \alpha^+} f(x)$ and _____ .

Fundamentals 4

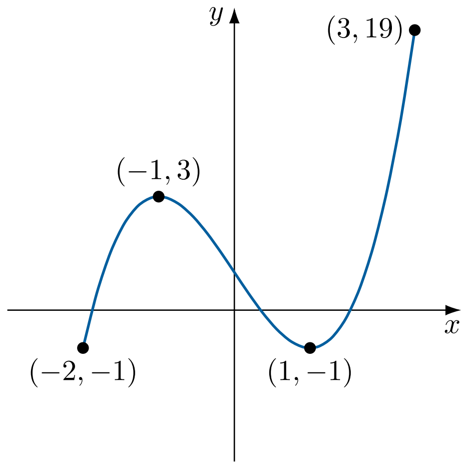
(a) To find the horizontal asymptotes, find the limit as $x \to \pm$ ___ .

(b) The horizontal asymptote of $y = \dfrac{ax + b}{Ax^2 + Bx + C}$ is $y =$ ___ .

(c) The horizontal asymptote of $y = \dfrac{ax^2 + bx + c}{Ax^2 + Bx + C}$ is $y =$ ___ .

(d) When a horizontal asymptote is found, it is useful to determine whether the curve approaches it from a_____ or b_____. This is done by substituting large values of x in the p_______ and n_______ directions.

Question 1 For the diagram below, state the local maximum and minimum, and the global maximum and minimum values.



Question 2 In the following question, there are multiple possible correct answers. Give an example of a function where the

(a) local maximum is the global maximum.

(b) local maximum isn't the global maximum.

(c) local minimum is the global minimum.

(d) local minimum isn't the global minimum.

Question 3 Consider the curve $y = x^3 - 6x^2 + 9x$.

(a) Find the x-intercepts.

(b) Find any stationary points and determine their nature.

(c) Find any points of inflection.

(d) Sketch the curve.

(e) Find the global maximum and minimum of the curve over the domain $x \in [-1, 4]$.

Question 4 Consider the curve $y = 3x^2 - x^3$.

(a) Find the x-intercepts.

(b) Find any stationary points and determine their nature.

(c) Find the point of inflection.

(d) Hence, sketch the curve showing the above features.

(e) Find the global maximum and minimum of the curve over the domain $x \in [-2, 3]$.

Question 5 Consider the curve $y = x^3 - 9x^2 + 24x - 15$

(a) Find any stationary points and determine their nature.

(b) Find the point of inflection.

(c) Hence, sketch the curve showing the above features.

(d) Find the global maximum and minimum of the curve over the domain $x \in [1, 4]$.

Question 6 Consider the curve $y = x^3 + x + 1$.

(a) Show that the curve has no stationary points.

(b) Find the point of inflection.

(c) Hence, sketch the curve showing the above features.

(d) Find the global maximum and minimum of the curve over the domain $x \in [-1, 2]$.

Question 7 Consider the curve $y = x^3 - 6x^2 + 12x - 5$.

(a) Find the coordinates of the y-intercept.

(b) Find any stationary points and determine their nature.

(c) Find the point of inflection.

(d) Hence, sketch the curve showing the above features.

(e) Find the global maximum and minimum of the curve over the domain $x \in [1, 4]$.

Question 8 Consider the curve $y = \dfrac{x-1}{x^2-4}$.

(a) Find the x and y-intercepts.

(b) Find any stationary points, and determine their nature.

(c) State the domain of the function.

(d) State the equations of the vertical asymptotes.

(e) Fill in the following table to determine the behaviour of the curve near the vertical asymptote $x = 2$.

x	**1.99**	**2**	**2.01**
y		Undefined	

(f) Draw and fill in a similar table to determine the behaviour of the curve near the vertical asymptote $x = -2$.

(g) Does the curve have a horizontal asymptote? If so, what is it?

(h) Find the behaviour of the curve as $x \to \pm\infty$

(i) Hence, sketch the curve.

Question 9 Consider the curve $y = \dfrac{1}{1+x^2}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 10 Consider the curve $y = \dfrac{1}{x^2}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Show that the curve has no stationary points.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 11 Consider the curve $y = \dfrac{2x}{x^2+1}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 12 Consider the curve $y = \dfrac{x^2}{x^2+1}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 13 Consider the curve $y = \dfrac{x^2}{x^2-4}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 14 Consider the curve $y = \dfrac{x^2}{(x-1)^2}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 15 Consider the curve $y = \dfrac{8x}{(x-1)^2}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Challenge Problems

Problem 1 [The horizontal asymptotes of rational functions]

Let $y = \dfrac{A(x)}{B(x)}$, where $A(x)$ and $B(x)$ are polynomials with degree m and n respectively.

(a) Prove that if $m < n$, then there will be a horizontal asymptote $y = 0$.

(b) Prove that if $m = n$, then there will be a horizontal asymptote $y = \dfrac{a_m}{b_n}$, where a_m and b_n are the leading coefficients of $A(x)$ and $B(x)$ respectively.

Problem 2 Sketch the graph of the following curves.

(a) $y = (1 - \sqrt{x})^2$

(b) $y = \sqrt{x^2 - 1}$

(c) $y = x - \sqrt{x}$

(d) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

Problem 3 Sketch the graph of the following curves.

(a) $y = \dfrac{x^3}{x^3 - 1}$

(b) $y = \dfrac{x^2}{x^3 + 1}$

Exercise 5E

Harder curve sketching

Fundamentals

Fundamentals 1

Use your calculator to help find the following limits. You are not required to memorise them, but it is worthwhile practising and understanding these concepts.

(a) $\lim_{x\to\infty} xe^{-x} = \_\_$

(b) $\lim_{x\to-\infty} xe^{-x} = \_\_$

(c) $\lim_{x\to 0} x\ln x = \_\_$

(d) $\lim_{x\to\infty} x\ln x = \_\_$

(e) $\lim_{x\to 0} \dfrac{x}{\ln x} = \_\_$

(f) $\lim_{x\to\infty} \dfrac{x}{\ln x} = \_\_$

(g) $\lim_{x\to 0} \dfrac{\ln x}{x} = \_\_$

(h) $\lim_{x\to\infty} \dfrac{\ln x}{x} = \_\_$

Fundamentals 2

The following are useful features to find when producing a sketch of a curve. In general not all of the following are needed to produce the sketch but the more features you have, the more certain you will be about the shape of the curve.

(a) __ and __-intercepts

(b) The d____ of the curve

(c) Behaviour of the curve as $x \to \pm$__.

(d) Any vertical or horizontal a______

(e) Behaviour of the curve on either end of the vertical a______

(f) If there are horizontal asymptotes, whether it approaches it from a____ or b____.

(g) If the domain is restricted, then the the coordinates of any e__-points.

(h) Any possible symmetry i.e. is the curve e____ or o____?

(i) S______ points, and their ____

(j) If easy to find, then also the r____ of the curve.

(k) If needed, any points of i______

Question 1 Consider the curve $y = xe^{-x}$.

(a) Find any x or y-intercepts.

(b) Find any stationary points, and determine their nature.

(c) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(d) Hence, sketch the curve.

Question 2 Consider the curve $y = e^{-x^2}$.

(a) Find any x or y-intercepts.

(b) Find any stationary points, and determine their nature.

(c) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(d) Hence, sketch the curve.

Question 3 Consider the curve $y = (1 + x)e^x$.

(a) Find any x or y-intercepts.

(b) Find any stationary points, and determine their nature.

(c) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(d) Hence, sketch the curve.

Question 4 Consider the curve $y = x^2e^x$.

(a) Find any x or y-intercepts.

(b) Find any stationary points, and determine their nature.

(c) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(d) Hence, sketch the curve.

Question 5 Consider the curve $y = \dfrac{e^x}{x}$.

(a) Find any x or y-intercepts.

(b) Find any stationary points, and determine their nature.

(c) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(d) Hence, sketch the curve.

Question 6 Consider the curve $y = \dfrac{e^x}{e^x + 1}$.

(a) Find any x or y-intercepts.

(b) Find any stationary points, and determine their nature.

(c) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(d) Hence, sketch the curve.

Question 7 Consider the curve $y = 2e^{-x} - e^{-2x}$.

(a) Find any x or y-intercepts.

(b) Find any stationary points, and determine their nature.

(c) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(d) Hence, sketch the curve.

Question 8 Consider the curve $y = x \ln x$.

(a) State the domain of the curve.

(b) Find any x or y-intercepts.

(c) Find any stationary points, and determine their nature.

(d) Find any points of inflection.

(e) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to 0$ and as $x \to \infty$.

(f) Hence, sketch the curve.

Question 9 Consider the curve $y = x - \ln x$.

(a) State the domain of the curve.

(b) Find any stationary points, and determine their nature.

(c) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to 0$ and as $x \to \infty$.

(d) Hence, sketch the curve.

Question 10 Consider the curve $y = \ln(x^2 + 1)$.

(a) Find any stationary points, and determine their nature.

(b) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(c) Hence, sketch the curve.

Question 11 Consider the curve $y = \dfrac{\ln x}{x}$.

(a) Find any stationary points, and determine their nature.

(b) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \infty$ and as $x \to 0$.

(c) Hence, sketch the curve.

Question 12 Consider the curve $y = (\ln x)^2$.

(a) Find any stationary points, and determine their nature.

(b) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \infty$ and $x \to 0^+$.

(c) Hence, sketch the curve.

Question 13 Consider the curve $y = x + \sin x$ in the domain $-2\pi \leq x \leq 2\pi$.

(a) Find the y-intercept.

(b) Find any stationary points in the given domain, and determine their nature.

(c) Find any points of inflection.

(d) Show that the curve is increasing or stationary in the given domain.

(e) Show that the curve represents an odd function.

(f) Hence, sketch the curve.

Question 14 Consider the curve $y = \sin^2 x$ in the domain $-\pi \leq x \leq \pi$.

(a) Find any stationary points, and determine their nature.

(b) Find the coordinates of the endpoints.

(c) Hence, sketch the curve.

Question 15 Consider the curve $y = \sin x + \cos x$ in the domain $-\pi \leq x \leq \pi$.

(a) Show that the x-intercepts occur when $x = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

(b) Find any stationary points, and determine their nature.

(c) Find the coordinates of the endpoints.

(d) Hence, sketch the curve.

Question 16 Consider the curve $y = \dfrac{\sin x}{1 + \cos x}$ in the domain $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$.

(a) Find any stationary points, and determine their nature.

(b) Find the coordinates of the endpoints.

(c) Hence, sketch the curve.

Challenge Problems

Problem 1 [Harder curves involving trigonometric functions]

Sketch the following curves over the domain $-\pi \le x \le \pi$.

(a) $y = \dfrac{1}{1 + \sin x}$

(b) $y = \ln(1 + \sin x)$

(c) $y = e^{\sin x}$

(d) $y = \sqrt{\sin x}$

Problem 2 [Damped oscillators]

Sketch the graph of $y = e^{-x} \sin x$ for $-\pi \le x \le \pi$, labelling stationary points and axes intercepts. You do not need to find points of inflection.

Exercise 5F

Optimisation

Fundamentals

Fundamentals 1

(a) In optimisation problems, we usually seek to m________ or m________ a function of one variable. The function may represent anything such as volume, area, length, time, and money.

(b) Optimisation problems always have some kind of restriction on the resources. For example, a fixed volume may be given, or a fixed cost may be provided. This restriction is sometimes called the c________.

(c) The goal of optimisation problems is to optimise the desired q________, subject to a certain constraint.

Fundamentals 2

Complete the following general set of steps used to solve standard optimisation problems.

(a) Write down an expression for the quantity that needs to be o________. This expression often has more than one variable in it.

(b) Use the c________ to obtain a relationship between the variables, and use the relationship to make the expression for the quantity s_____-variable.

(c) D________ the function to be optimised with respect to the appropriate variable, and find any local maxima or minima.

(d) Verify the nature of the local maxima or minima, using either by constructing a t_____ of values, or using the s________-derivative method.

(e) Check the extremities of the variables to ensure that the l_____ maxima/minima are also the g_____ maxima/minima.

(f) Ensure that the actual question has actually been answered. Check if you are finding the value of x that m________ or m________ a function, or if you need to find the minimum or maximum value of the function.

Question 1 The sum of two numbers x and y is 12.

(a) Write down the value of the product P, in terms of x and y.

(b) Show that $P = 12x - x^2$.

(c) Show that P attains a maximum when $x = 6$.

(d) Hence, find the value of the two numbers such that the sum is 12 and the product is maximised.

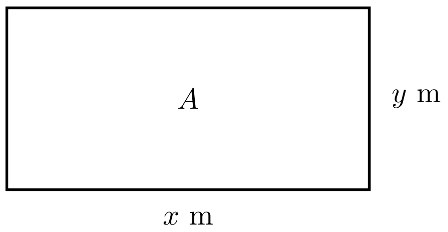
Question 2 The sum of two numbers x and y is 12. Let the sum of their squares be $S = x^2 + y^2$.

(a) Show that $S = 2(x^2 - 12x + 72)$.

(b) Hence, find the value of x that minimises S, and verify this fact using the second derivative test.

(c) Write down the two numbers x and y that sum to 12, such that the sum of their squares is least.

Question 3 The diagram below shows a rectangle with length x m and width y m, with perimeter 36 m.



(a) Show that the area $A = 18x - x^2$.

(b) Hence, find the value of x that maximises the area of the rectangle.

(c) What is special about the rectangle at this value of x?

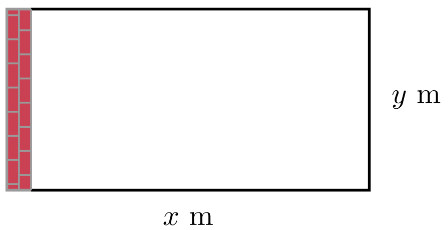
Question 4 The height h and radius r of a cylinder have a sum of 24 cm.

(a) Show that the volume of the cylinder is

$$V = 24\pi r^2 - \pi r^3$$

(b) Show that the maximum volume of the cylinder is 2048π cm^3

Question 5 The diagram below shows a rectangular enclosure with length x m and width y m. A farmer has 300 m of fencing to enclose a rectangular area, using a wall as one of the sides of the enclosure.



(a) Show that the area of the enclosure is $A = 300x - 2x^2$.

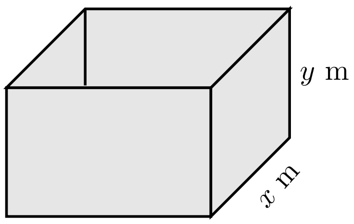
(b) Hence, find the maximum area that can be enclosed.

Question 6 A cylindrical can with height h and radius r has volume 400 cm^3.

(a) Show that the surface area $A = 2\pi r^2 + \dfrac{800}{r}$.

(b) Find the radius, in exact form, that minimises the surface area.

Question 7 The diagram below shows an open rectangular box with a square base of side-length x m and height y m. The box is to be made so that it has constant volume 4 m^3.



(a) Show that the area of cardboard needed to produce the box is $A = x^2 + \dfrac{16}{x}$.

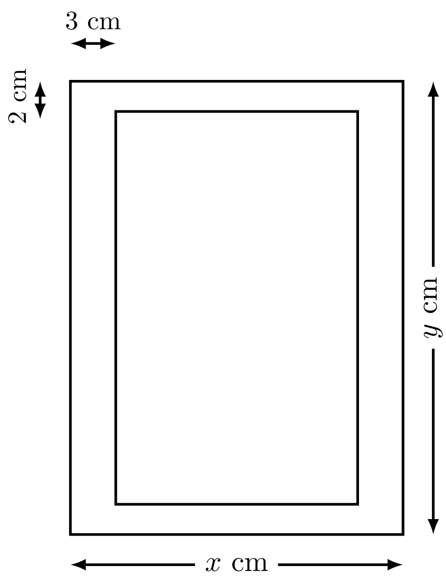
(b) Find the dimensions of the box so that the area of cardboard used is minimised.

(c) The material used to make the sides of the box costs \$8 per square metre, whereas the base must be stronger and costs \$10 per square metre.

(i) Write down an expression for the cost of the box, in terms of x.

(ii) Hence, find the value of x that minimises the cost of making the box.

Question 8 The area of a rectangular sheet of paper with dimensions $x \times y$ cm is 600 cm^2. The printed area has a margin of 2 cm allowed from the top and bottom of the sheet, and a margin of 3 cm allowed from the left and right of the sheet, as shown in the diagram below.



Find the dimensions of the sheet of paper to maximise the printing area.

Question 9 The slant height of a right circular cone is 15 cm. The height of the cone is h cm and the radius is r cm.

(a) Show that

$$V = \frac{\pi}{3}(225h - h^3).$$

(b) Show that the volume of the cone is maximised when $h = 5\sqrt{3}$ cm.

(c) Hence, find the maximum volume of the cone.

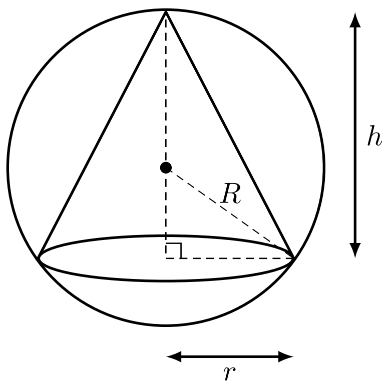
Question 10 A sector with radius r and angle α is formed from a piece of wire of length 40 cm.

(a) Show that the area of the sector is

$$A = 20r - r^2.$$

(b) Hence, show that the area of the sector is maximised when $r = 10$ cm.

Question 11 A cone with radius r and height h is inscribed within a sphere of radius R, as shown in the diagram below.



(a) Show that $r^2 = 2hR - h^2$.

(b) Show that the volume of the cone is

$$V = \frac{\pi}{3}\left(2h^2R - h^3\right).$$

(c) Hence, find the value of h, in terms of R, that maximises the volume of the cone.

Question 12 A box has base length three times as long as the base width. The material used to construct the top and bottom faces is \$10 per square centimetre, a/nd the material used to build the sides costs \$6 per square centimetre. If the box is to have a fixed volume of 100 cm^3, find the dimensions of the box that will minimise the cost, and hence find the minimum cost.

Question 13 A company manufactures objects at a cost of \$1 per unit, and sells them for $\$x$ per unit. The company sells $\dfrac{12000}{x^2}$ units per week. Find the value of x for which the company should sell the objects so that the profit is maximised, and state the maximum profit.

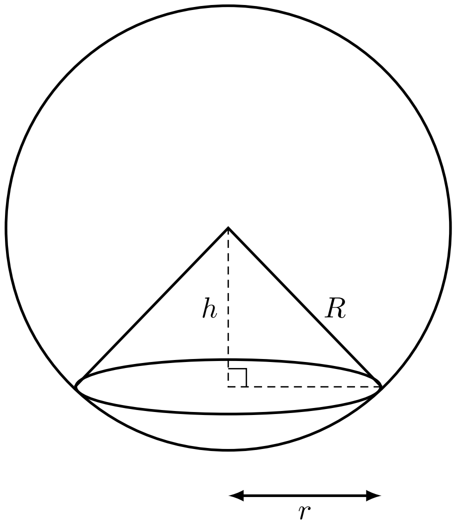
Question 14 A company arranges excursions to Canberra for the day at a cost of \$300 a person for groups of 16 or more. The price is reduced by \$10 for each additional person over 16 travellers.

(a) Show that the revenue for the company in a day is $R = 460p - 10p^2$, where p is the number of people in the group.

(b) What number of people will produce the greatest revenue for the company?

(c) How many extra travellers will give the company the same revenue as 16 people?

Question 15 The diagram below shows a cone inscribed within a sphere so that the vertex of the cone lies at the centre of the sphere. Let the radius of the cone be r and let the radius of the sphere be R. Let the height of the cone be h.

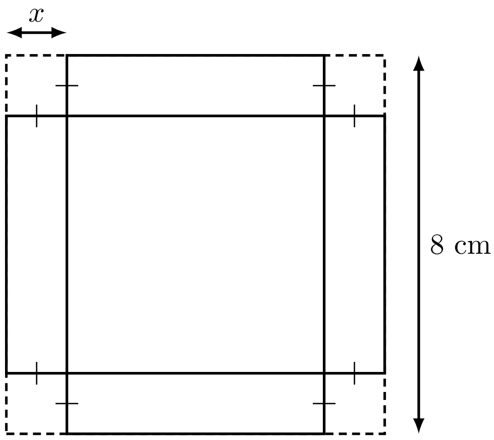


(a) Show that the volume of the cone is

$$V = \frac{\pi}{3}\left(R^2h - h^3\right).$$

(b) Hence, show that the volume is maximised when $h = \dfrac{R}{\sqrt{3}}$.

Question 16 An open box is formed by obtaining a square 8 cm × 8 cm cardboard sheet, cutting out squares of side length x cm from each corner, then folding up the flaps.



(a) Show that the volume of the box is

$$V = 4x^3 - 32x^2 + 64x.$$

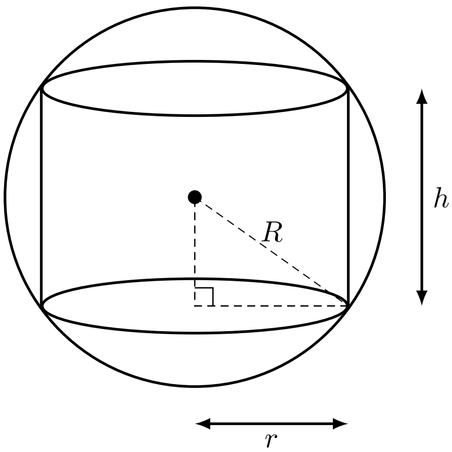
(b) Find the length of x that maximises V.

(c) Find the length of x that minimises V.

(d) Explain how the answer from (c) could have been found intuitively without the use of calculus.

(e) Find the maximum volume of the box.

Question 17 A cylinder of height h and radius r is inscribed in a sphere of radius 20 cm, as shown in the diagram below.



(a) Show that $r^2 = 400 - \dfrac{h^2}{4}$.

(b) Show that the volume of the cylinder is given by

$$V = \frac{\pi}{4}\left(1600h - h^3\right)$$

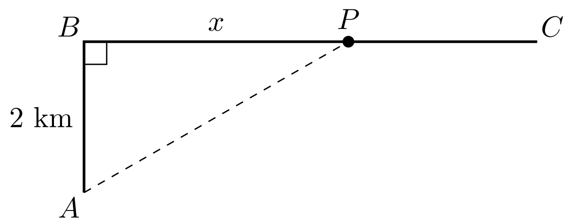
(c) Find $\dfrac{dV}{dh}$.

(d) Show that the volume of the cylinder is maximised when $h = \dfrac{40}{\sqrt{3}}$.

(e) Find the exact value of the maximum possible volume.

Question 18 [Check extremities!]

Bob runs in a cross-country event where the route is to go from A to B which spans 2 kilometres, then from B to C which spans 6 kilometres as shown in the diagram below. However, Bob wishes to cheat and decides that it might be faster to instead cut from A to another point P along BC, and then continue the rest of the route normally.



When running the normal route, he runs at a pace of 10 kilometres per hour. However, when taking the 'shortcut', he runs at a slower pace of 6 kilometres per hour, since the 'shortcut' runs through bushland. Let x be the distance from B to P.

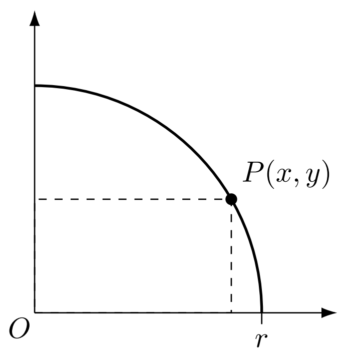
(a) Show that the time taken to go from A to P, then P to C is

$$T = \frac{\sqrt{x^2+4}}{6} + \frac{6-x}{10}$$

(b) Solve $\dfrac{dT}{dx} = 0$ for x.

(c) Hence, find the value of x that minimises the time spent overall.

Question 19 The diagram below shows a point $P(x, y)$ on the circumference of a quarter-circle with radius r units. A rectangle is produced using the origin O as one vertex and P as the opposite vertex.



(a) Show that the area A of the rectangle is $A = x\sqrt{r^2 - x^2}$.

(b) Show that the maximum area of the rectangle is $A = \dfrac{r^2}{2}$.

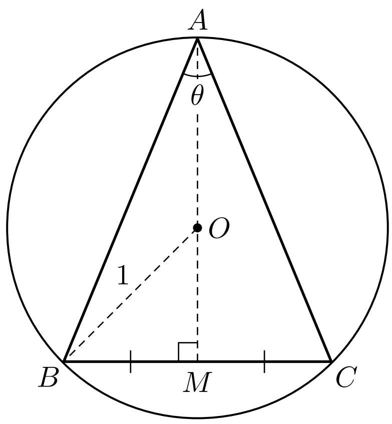
Question 20 A wire of length L is bent to form a sector with radius r and angle θ. Show that the area A of the sector is maximised when

(a) $r = \dfrac{L}{4}$ (b) $\theta = 2$ (c) $A = \dfrac{L^2}{16}$

Challenge Problems

Problem 1 A cylindrical can of height h and radius r is to be made with a fixed volume V. Show that the area of material used is minimised when $h = 2r$.

Problem 2 An isosceles triangle $\triangle ABC$ is inscribed within a unit circle centred at O, as shown in the diagram below. Let M be the mid-point of BC.



(a) Show that $\angle BOM = \theta$.

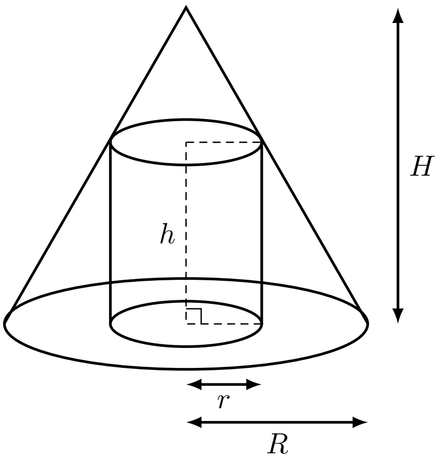
(b) Show that the area of $\triangle ABC$ is

$$T = \sin\theta(1 + \cos\theta).$$

(c) Show that $T' = 2\cos^2\theta + \cos\theta - 1$.

(d) Hence, prove that the area of the isosceles triangle is maximised when it is equilateral.

Problem 3 A cylinder of radius r and height h is inscribed within a cone with radius R and height H, as shown in the diagram below.



(a) Show that $h = \dfrac{H(R-r)}{R}$.

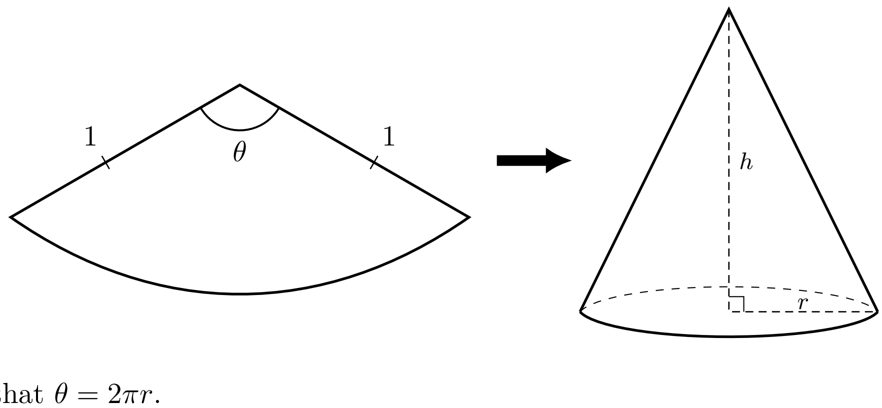
Hint: Use similar triangles.

(b) Show that the volume of the cylinder is

$$\frac{\pi H}{R}\left(r^2 R - r^3\right).$$

(c) Hence, show that the volume of the cylinder is maximised when $r = \dfrac{2R}{3}$.

Problem 4 A sector of unit radius and angle θ is folded to form a cone with height h and radius r.



(a) Show that $\theta = 2\pi r$.

(b) Show that

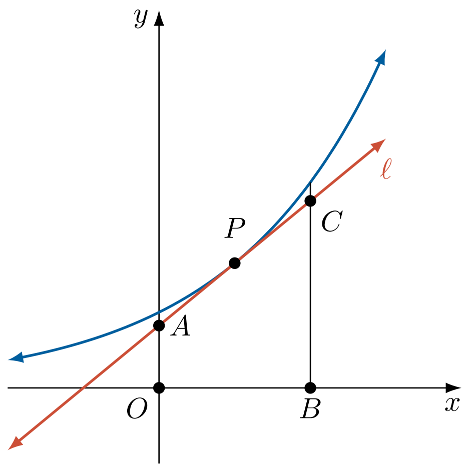
$$V = \frac{\theta^2}{24\pi^2}\sqrt{4\pi^2 - \theta^2}.$$

(c) Show that the value of θ that maximises the volume of the cone is

$$\theta = \frac{2\pi}{3}\sqrt{6}$$

For this question, you may assume that solving $\dfrac{dV}{d\theta} = 0$ yields a maximum.

Problem 5 The diagram below shows the graph of $y = e^x$. A tangent ℓ is drawn from $P(p, q)$ in the domain $x \in [0, 1]$.



The tangent ℓ intersects $x = 0$ and $x = 1$ at A and C respectively, forming a trapezium $AOBC$.

(a) Explain why P has coordinates $P(p, e^p)$.

(b) Show that the tangent from P has equation $y = e^p x + e^p(1 - p)$.

(c) Show that the trapezium has area

$$A = \frac{e^p}{2}(3 - 2p).$$

(d) Hence, find the value of p that maximises the area of the trapezium.

(e) Find the value of p that minimises the area of the trapezium.

Chapter 5 Review

Applications of differentiation

Review

Question 1 The curve $y = ax^2 + bx + 4$ has a stationary point at $(3, -5)$. Find a and b.

Question 2 Find the values of x for which

(a) the curve $y = x^3 - 6x^2 + 9x - 8$ is decreasing.

(b) the curve $y = 2x^3 - x^2$ is concave up.

Question 3

(a) Explain why $y = x^3 + 3$ is an increasing function except at $x = 0$.

(b) Explain why $y = \dfrac{1}{x}$ is a decreasing function except at $x = 0$.

Question 4 Find the equation of the tangent to the curve

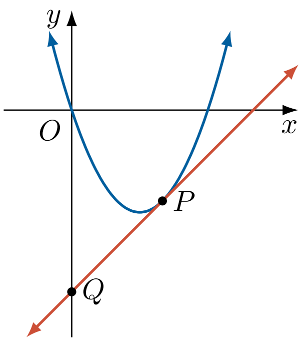
(a) $y = \ln x$ at $x = e^2$.

(b) $y = xe^{-x}$ at $x = 1$.

(c) $y = x \sin x$ at $x = \dfrac{\pi}{2}$.

(d) $y = x \ln x$ at $x = e$.

Question 5 The diagram shows the parabola $y = x^2 - 3x$ and a tangent drawn at P. This tangent has gradient 1 and it cuts the y-axis at Q.



(a) Find the coordinates of P.

(b) Find the equation of the tangent at P.

(c) Find the coordinate of Q.

(d) A normal is drawn from P and cuts the y-axis at R. Find the equation of the normal, and the coordinates of R.

(e) Find the area of triangle RPQ.

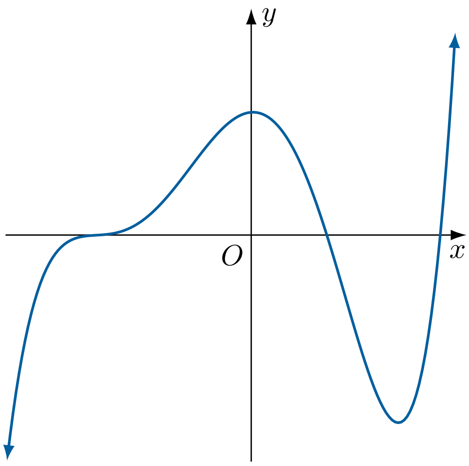
Question 6

(a) Sketch the curves $y = x^2$ and $y = -\frac{1}{2}\ln x$ and show that they meet in one point only $x = x_1$

(b) Find the gradient of the tangent to both curves at $x = x_1$

(c) Prove that at this point the tangents to the two curves meet at right angles

Question 7 On the graph below mark the points satisfying the given condition



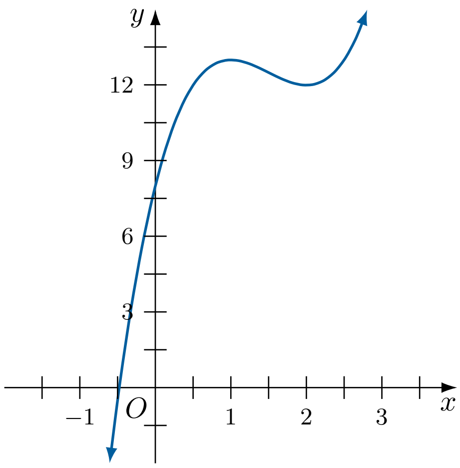
(a) P such that $y < 0$, $y' = 0$

(b) Q such that $y = 0$, $y' = 0$, $y'' = 0$

(c) S such that $y' < 0$, $y'' = 0$

(d) R such that $y > 0$, $y' = 0$, $y'' < 0$

Question 8 For the given curve $y = f(x)$ below, maximum and minimum turning points occur at $x = 1$ and $x = 2$ respectively and the point of inflexion is at $x = \frac{3}{2}$.



Given the information find the range of values of x for which

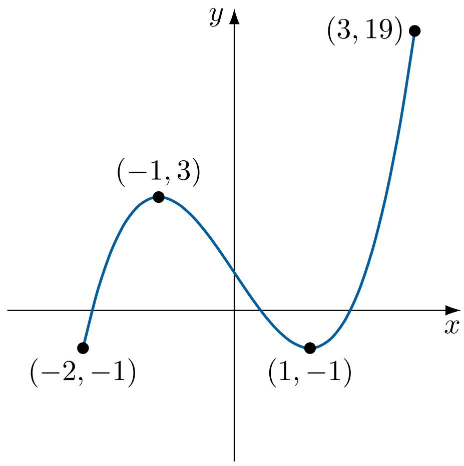
(a) $f(x)$ is increasing

(b) $f(x)$ is concave up

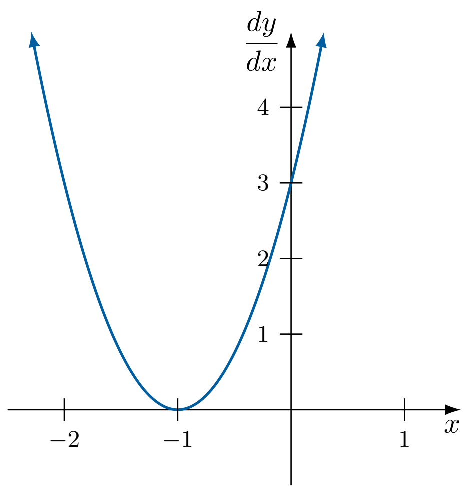
(c) $f(x)$ is decreasing and concave down.

(d) $f(x)$ is increasing and concave down.

Question 9 The curve $y = 1 - 3x + x^3$ is drawn below in the domain $[-2, 3]$. Use the graph provided to write down the values of x for which $f'(x) > 0$, $f'(x) < 0$ and $f'(x) = 0$.



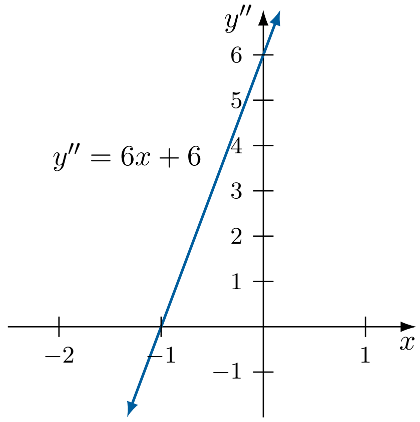
Question 10 The gradient function $\frac{dy}{dx} = 3(x+1)^2$ of a curve is illustrated by the graph below.



(a) A stationary point is located at $x = -1$. Justify this statement by reference to the graph.

(b) Comment on the sign of $\frac{dy}{dx}$ for all real x, except $x = -1$, and state what this implies.

(c) The graph of $\dfrac{d^2y}{dx^2}$ is given below.

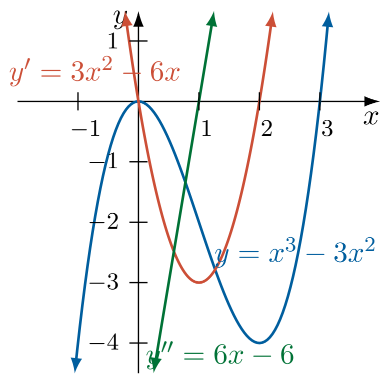


Copy and complete this table:

x	-2	-1	0
sign of $\dfrac{d^2y}{dx^2}$			

(d) What is the nature of the stationary point at $x = -1$?

Question 11 The functions $y = x^3 - 3x^2$, $\dfrac{dy}{dx} = 3x^2 - 6x$ and $y'' = 6x - 6$ are drawn below.



(a) Write down the domain where the function is increasing.

(b) Write down the domain where the function is concave down.

(c) What is the significance of the red graph being below the x-axis, with regards to the blue graph?

(d) What is the significance of the green graph being above the x-axis, with regards to the blue graph?

Question 12 Let $f(x) = \ln(\cos x)$.

(a) State the domain within $x \in [0, 2\pi]$ for which $f(x)$ is defined.

(b) Find the maximum value of $f(x)$, and state when it occurs.

Question 13

(a) Find the equation of the tangent and normal to the curve $y = \ln x$ at $x = \frac{1}{e}$.

(b) Find the area formed by above equations and the x axis.

Question 14 Consider the function $y = 2x^3 + 3x^2 - 12x$ over the domain $-3 \leq x \leq 3$

(a) Determine the locations of the local and global maximum and minimum values in the specified domain.

(b) Sketch the curve in the specified domain.

Question 15 Consider the curve $y = \frac{1}{4}x^4 - 2x^2$

(a) Find where the curve crosses the x-axis.

(b) Find any stationary points and determine their nature.

(c) Find any points of inflection.

(d) Sketch the curve.

(e) For what values of x is the curve concave down?

Question 16 Consider the curve $y = \frac{x}{x^2 - 9}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 17 Consider the curve $y = \frac{x^2 - 4}{x^2 + 4}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 18 Consider the curve $y = \dfrac{x^2+1}{x^2-1}$.

(a) Find any horizontal and/or vertical asymptotes.

(b) Find the coordinates of any stationary points, and determine their nature.

(c) Determine whether the function is odd, even or neither.

(d) Hence, sketch the curve.

Question 19 Consider the curve $y = (1+x)e^{-2x}$

(a) Find the x and y intercepts.

(b) Find any stationary points, and determine their nature.

(c) Find the point of inflexion.

(d) Discuss the behaviour of the curve as $x \to \pm\infty$.

(e) Sketch the curve showing the above features.

Question 20 Consider the curve $y = x - 5\ln x$ for $1 \leq x \leq 18$.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Find the coordinates of the turning point and determine its nature

(c) Discuss the concavity of the curve

(d) Sketch the curve for the given domain

(e) Find the global maximum and minimum of the function

Question 21 Define the function $f(x) = \dfrac{e^x-1}{e^x+1}$.

(a) Prove that $f(x)$ is odd.

(b) Show that $f(x)$ has no stationary points.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Hence, sketch the graph of the curve.

Question 22 Consider the curve $y = x\ln(x^2+1)$.

(a) Show that $(0,0)$ is a horizontal point of inflection.

(b) Use your calculator, or otherwise, to determine the behaviour of the curve as $x \to \pm\infty$.

(c) Hence, sketch the curve.

Question 23 Consider the function $f(x) = 4xe^{-\frac{x}{2}}$.

(a) Find the coordinates of the stationary point, and determine the nature.

(b) Describe the behaviour of the curve as $x \to \pm\infty$.

(c) Hence, sketch the curve in the given domain.

Question 24 Consider the function $f(x) = e^x + 9e^{-x}$.

(a) Find the coordinates of the stationary point, and determine the nature.

(b) Describe the behaviour of the curve as $x \to \pm\infty$.

(c) Show that the curve is always concave up.

(d) Hence, sketch the curve in the given domain.

Question 25 Consider the function $f(x) = e^{x-1} + e^{1-x}$ in the domain $x \in [0, 3]$.

(a) Find the coordinates of the stationary point, and determine the nature.

(b) Find the global maximum and minimum values.

(c) Hence, sketch the curve in the given domain.

Question 26 Consider the curve $y = \tan^2 x$ in the domain $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

(a) Find any stationary points, and determine their nature.

(b) Show that the function is always concave up in the given domain.

(c) Hence, sketch the curve.

Question 27 Tickets to a school dance cost \$10 and the projected attendance is 240 people. For every \$1 increase in the ticket price, the dance attendance committee projects that attendance will decrease by 4. Let x be the number of price increases.

(a) Show that the expected revenue function is $R(x) = 2400 + 200x - 4x^2$.

(b) Determine the ticket price that will produce the greatest revenue.

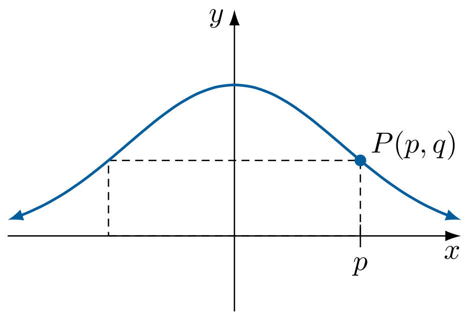
(c) Find the greatest possible revenue.

Question 28 A rectangular room has an area of 64 m^2.

(a) Show that the perimeter P metres of this room is given by $P = 2\left(x + \dfrac{64}{x}\right)$ where the length is x metres

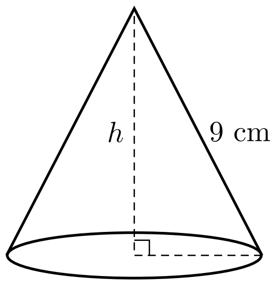
(b) Hence show that the perimeter is a minimum when the rectangle is square.

Question 29 The diagram below shows a rectangle formed from the point $P(p, q)$ on the graph of $y = e^{-x^2}$.



Find the x-coordinate of P so that the rectangle has maximum area.

Question 30 The slant edge of a right circular cone of height h cm is 9 cm.

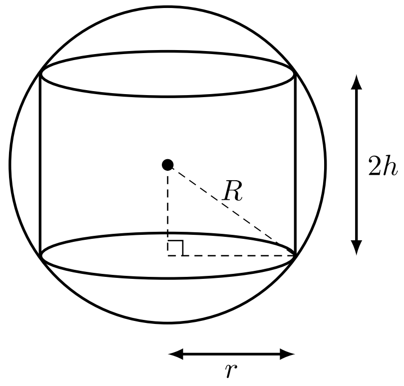


(a) Show that the volume of the cone is given by

$$V = \frac{\pi}{3}(81h - h^3)$$

(b) Find the vertical height of the cone when the volume is a maximum.

Question 31 The diagram below shows a cylinder of height $2h$ and radius r inscribed within a sphere of radius R.



Let the volume of the cylinder be V

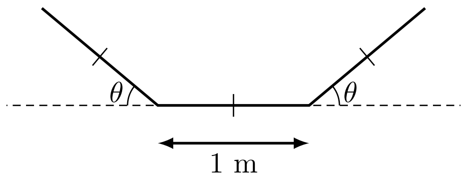
(a) Show that $V = 2\pi\left(R^2h - h^3\right)$.

(b) Show that V is maximised when the height of the cylinder is $\dfrac{2R}{\sqrt{3}}$.

(c) Find the ratio of the radius of the cylinder to the radius of the sphere when the volume is maximised.

(d) Find the ratio of the maximised volume of the cylinder to the volume of the sphere.

Question 32 A flat sheet of metal is bent at an angle of θ from the horizontal to form a cross section of a drain, as shown in the diagram below. The resulting shape is an isosceles trapezium with three side lengths of 1 metre.



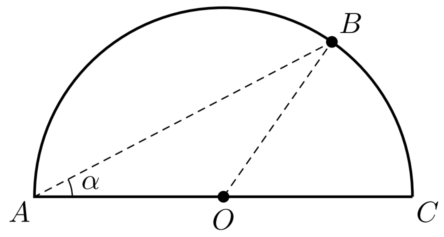
(a) Show that the cross sectional area is given by

$$A = \sin\theta(1+\cos\theta).$$

(b) Find the value of θ that maximises the cross sectional area, and find the maximum area.

Question 33 [Demonstrating why it is necessary to check boundaries]

A circular lake with radius 1 kilometre is surrounded by a jogging path. Robin wishes to go from one end of the lake to another in the shortest possible time. He decides to swim from A to B at a pace of 3 kilometres per hour, and then run from B to C at a pace of 8 kilometres per hour.



Let $\angle BAC = \alpha$.

(a) Show that $\angle BOC = 2\alpha$, and hence show that Robin runs 2α kilometres.

(b) Show that the time taken for the total journey is

$$T = \frac{2}{3}\cos\alpha + \frac{\alpha}{4}.$$

(c) Find the value for α for which $\frac{dT}{d\alpha} = 0$, and determine whether this gives the maximum or minimum value of T. Round your answer to two decimal places.

(d) Find how long it takes Robin to complete his journey if he proceeds with the above path. Round to the nearest minute.

(e) Describe the two extremities that Robin can take to go from A to C, and find the time taken to complete the journey for each case, rounded to the nearest minute.

(f) Hence, determine what Robin should do to go from A to C in the least amount of time, and state the value of α then.

Investigation Task

Creating optimisation scenarios

Optimisation problems arise mostly in scenarios where a quantity has a 'sweet spot', and the goal is generally to find that sweet spot. For example, suppose we wish to maximise the volume of a cylinder inscribed within a sphere. If the cylinder is too narrow, then the cylinder will be like a thin vertical wire and the volume is close to zero. If the cylinder is too wide, the cylinder is like a disc and again the volume is close to zero. So, there is some 'in between' ratio that will yield maximal volume.

Question 1 Research what the *Intermediate Value Theorem* is, and explain it's significance in the context of optimisation problems.

Question 2 Come up with three optimisation scenarios and structure them like textbook questions. One should be involving money, one should be involving the volume of a solid and one should be involving the area of a region. Each of these problems should have very clear extremities that imply the existence of a sweet spot in the middle. Then, solve your own scenarios as if answering exam questions.

Question 3 Sometimes, we are seeking a maximum but calculus gives us a minimum, and vice versa. Such an example is the last question in Chapter Review 6. In these scenarios, the desired maximum or minimum is actually just one of the end-points. Create a scenario where the calculus gives the opposite of what we want, thus forcing the desired maximum/minimum to lie at one of the end-points instead.

Investigation Task

Interesting family of curves

The family of curves

$$y = e^{ax} + ke^{bx}$$

can take on many different shapes and forms depending on the values of a, b, and k. For this task, assume that none of the constants are zero.

Question 1 Use graphing software to investigate all the different types of shapes that can be formed, and the values of a, b and k that result in these types. Also, show screenshots of the resultant shape when $a > b$, $a < b$, $k > 0$, $k < 0$, a positive or negative, b positive or negative, and different combinations of a and b being positive and/or negative.

Question 2 Use calculus to find the stationary points of the general curve $y = e^{ax} + ke^{bx}$. Use your findings to explain why certain values of a, b and k result in certain shapes of the curve. You should explore all possible scenarios of a, b, k, and their corresponding shapes.

Question 3 One of the applications of this family of curves is in pharmaceuticals, where a body absorbs a drug quickly at first and then the concentration in the blood decreases as the body metabolises the drug over time. If the curve is used to model the concentration of the drug in the blood over time, what values of a, b and k will result in a curve that is appropriate in modelling such a scenario?

Investigation Task

Newton's Method

Some equations can be solved and an exact solution found. For example, $x^2 - 5x + 6 = 0$, $\ln(x-2) = 0$, $e^{-x} - e^{-2x} = 0$ can be explicitly solved to give nice exact solutions. However, most equations in general do not have exact solutions e.g. $x\sin x - 1 = 0$. Newton's Method is an algorithmic process that allows us to get better approximations of roots that we may not know how to find explicitly.

Question 1 Research and find what a *closed-form* solution is, and why it is relevant to the discussion of solving equations.

Question 2

(a) Write down the key formula for Newton's Method, and derive the formula.

(b) Explain how it works and how repeated iterations yields better approximations of the roots of equations.

(c) Use Newton's Method with three iterations to find an approximate root of $e^x - x^2 = 0$.

(d) Use Newton's Method repeatedly increasingly accurate approximations of $\sqrt{2}$, and determine how many iterations were needed to be correct to six decimal places.

(e) Explain how Newton's Method can be used to find approximate values for $\sqrt{n}$, where n is not a perfect square.

Question 3 Produce two curves of your choice that intersect at some point, and use Newton's Method to find an approximation for the point where they intersect.

Question 4 Newton's Method is said to have *quadratic convergence*. Research what this term means and explains why having quadratic convergence is a good thing when it comes to approximating the roots of an equation.

Question 5 Newton's Method is a powerful technique that yields accurate approximations with relatively few iterations. However, it does not always work. Find and categorise all the scenarios where it may not work. Your answer should include a discussion of why they did not work, and what could have been done (if anything) instead to ensure that the algorithm works.

6

INTEGRATION

- The anti-derivative
- Integration with exponential functions
- Integration involving the logarithmic function
- Integration involving trigonometric functions
- Reverse chain rule

Exercise 6A

The anti-derivative

Fundamentals

Fundamentals 1

(a) Anti-differentiation, also known as integration, is the reverse of d________.

(b) A function $F(x)$, that differentiates to become $f(x)$, is called a p________ function of $f(x)$.

(c) The p________ function is/is not (circle one) unique because $F(x)+C$ for any constant C produces the same derivative as just $F(x)$. So, in other words

$$\frac{d}{dx}(F(x)+C) = \_\_\_\_\_$$

(d) $F(x)+C$ is also called the indefinite i_____ of $f(x)$. Using integration notation, this is represented by $\int f(x)\,dx = \_\_\_\_\_\_\_\_\_\_$.

Fundamentals 2

Write down the formula for the following, for $n \neq -1$.

(a) $\int x^n\,dx$ (b) $\int (ax+b)^n\,dx$ (c) $\int f'(x)(f(x))^n\,dx$

Fundamentals 3

Complete the following rules of integration.

(a) $\int f(x)+g(x)\,dx$ (b) $\int f(x)-g(x)\,dx$

(c) $\int k\,f(x)\,dx$ (d) $\int k\,dx$

Fundamentals 4

(a) The definite integral is similar to the indefinite integral, except it has two l_____.

(b) Let $F(x)$ be a primitive function of $f(x)$. Write down the formula for $\int_a^b f(x)\,dx$.

Note: At this point the definite integral is introduced from a purely calculation-based perspective. The geometric interpretation of the definite integral is covered in the next chapter.

Question 1 Find the following.

(a) $\int x^2\,dx$ (b) $\int 4\,dx$ (c) $\int 6x^3\,dx$

(d) $\int -7x^6\,dx$ (e) $\int \frac{1}{2}x^{-2}\,dx$ (f) $\int 6x^{\frac{1}{2}}\,dx$

Question 2 Find the following.

(a) $\int 2x + 3x^2\,dx$ (b) $\int \frac{2x+1}{3}\,dx$ (c) $\int \frac{x^2}{3} - \frac{4x^3}{3}\,dx$

(d) $\int x^2(3-2x)\,dx$ (e) $\int (x-1)(x+2)\,dx$ (f) $\int \left(2x^2-3\right)^2\,dx$

Question 3 Complete the following.

$$\begin{aligned}\int \frac{1}{x^2}\,dx &= \int x^{\_}\,dx\\ &= \_\_\_\_ + C\\ &= -\frac{1}{x} + C\end{aligned}$$

Question 4 By first changing the following to the form $k\,x^n$, find

(a) $\int \frac{1}{x^3}\,dx$ (b) $\int \sqrt{x}\,dx$ (c) $\int \frac{1}{\sqrt{x}}\,dx$

(d) $\int \frac{1}{x\sqrt{x}}\,dx$ (e) $\int \frac{1}{x^2\sqrt{x}}\,dx$ (f) $\int \frac{x}{\sqrt{x}}\,dx$

(g) $\int \frac{\sqrt{x}}{x^2}\,dx$ (h) $\int \frac{1+x}{\sqrt{x}}\,dx$ (i) $\int \frac{x^2-2x+1}{\sqrt{x}}\,dx$

Question 5 By first changing the following to the form $k\,(ax+b)^n$, find

(a) $\int (2x+3)^4\,dx$ (b) $\int (5-3x)^7\,dx$ (c) $\int \frac{1}{(3x-2)^2}\,dx$

(d) $\int \sqrt{6x-1}\,dx$ (e) $\int (4-3x)\sqrt{4-3x}\,dx$ (f) $\int \frac{1}{\sqrt{5-6x}}\,dx$

Question 6 [Now practising with limits of integration]

(a) $\int_1^2 x^2\,dx$ (b) $\int_2^4 x\,dx$ (c) $\int_1^4 \sqrt{x}\,dx$

(d) $\int_{-1}^2 x^2 + x^3\,dx$ (e) $\int_{-1}^1 (2x+1)^3\,dx$ (f) $\int_1^2 \frac{1}{(2x-1)^2}\,dx$

Question 7 By first changing the following to the form $k\,(ax+b)^n$, find

(a) $\displaystyle\int_0^1 \frac{1}{(3x+1)^2}\,dx$ (b) $\displaystyle\int_0^2 \frac{4}{(5-2x)^3}\,dx$ (c) $\displaystyle\int_0^1 \frac{3}{2(4-3x)^3}\,dx$

(d) $\displaystyle\int_1^2 \sqrt{3x-2}\,dx$ (e) $\displaystyle\int_0^{\frac{2}{5}} (2-5x)\sqrt{2-5x}\,dx$ (f) $\displaystyle\int_0^3 \frac{1}{\sqrt{x+1}}\,dx$

(g) $\displaystyle\int_0^1 \frac{1}{\sqrt{3-2x}}\,dx$ (h) $\displaystyle\int_0^3 \frac{1}{(4-x)\sqrt{4-x}}\,dx$ (i) $\displaystyle\int_{-\frac{b}{a}}^0 \frac{1}{\sqrt{ax+b}}\,dx$

Question 8

(a) Write down the family of curves satisfying $f'(x) = 2x$.

(b) Find the equation of the particular curve that satisfies the above and passes through $(3, 10)$.

(c) If instead the curve passed through $(2, 6)$, what is the equation of the particular curve?

Question 9 Find the equation of the curve if the curve satisfies the following derivatives, and passes through the following points.

(a) $\dfrac{dy}{dx} = 4x+1$, $(0,3)$ (b) $\dfrac{dy}{dx} = 3-4x$, $(1,5)$

(c) $\dfrac{dy}{dx} = 1-x^2$, $(-3,1)$ (d) $\dfrac{dy}{dx} = x^2-3x$, $(1,0)$

(e) $\dfrac{dy}{dx} = 3x^2-2x+1$, $(2,-2)$ (f) $\dfrac{dy}{dx} = \sqrt{x}$, $(9,2)$

(g) $\dfrac{dy}{dx} = 3(1-x)^2$, $(2,2)$ (h) $\dfrac{dy}{dx} = (3-2x)^3$, $(1,0)$

Question 10 Find the equation of the curve if the curve satisfies the following.

(a) $f'(x) = 3x^2-2x+a,\quad f'(1) = 0,\quad f(1) = 5$

(b) $f''(x) = 4-6x,\quad f'(1) = 1,\quad f(1) = 0$

(c) $f''(x) = -10,\quad f'(0) = 5,\quad f(0) = 20$

(d) $f''(x) = 6x-8,\quad f'(1) = 1,\quad f(1) = 0$

Challenge Problems

Problem 1

(a) Simplify

$$\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}$$

(b) Hence, find

$$\int \frac{x^2+1}{\left(x^2-1\right)^2}\,dx$$

Problem 2

(a) Show that

$$\frac{2x}{(2x+1)^3}=\frac{1}{(2x+1)^2}-\frac{1}{(2x+1)^3}$$

Hint: $2x = 2x+1-1$

(b) Hence, find

$$\int \frac{x}{(2x+1)^3}\,dx$$

Problem 3 Find the equation of $f(x)$ if $f'(x) = px + q$ and $f(x)$ has a minimum stationary point at $(1, -9)$, and passes through $(0, -8)$.

Problem 4 Find the following integrals.

(a) $\displaystyle\int \frac{\left(x^2-1\right)^2}{\sqrt{x}}\,dx$ (b) $\displaystyle\int \frac{1}{4x^2-12x+9}\,dx$ (c) $\displaystyle\int \frac{x-1}{x^3-3x^2+3x-1}\,dx$

Exercise 6B

Integration with exponential functions

Fundamentals

Fundamentals 1
Write down the following integrals.

(a) $\displaystyle\int e^x\,dx$ (b) $\displaystyle\int e^{ax+b}\,dx$ (c) $\displaystyle\int a^x\,dx$

Fundamentals 2
Complete the following set of steps to derive the integral of a^x.

(a) Using the fact that $e^{\ln x} = x$ for $x > 0$, we have

$$\begin{aligned} a^x &= \left(e^{\_\_\_}\right)^x \\ &= e^{\_\_\_\_\_} \end{aligned}$$

(b) Therefore

$$\begin{aligned} \int a^x\,dx &= \int e^{\_\_\_\_\_}\,dx \\ &= \frac{1}{\_\_\_\_}e^{\_\_\_\_\_} + C \\ &= \frac{1}{\_\_\_\_}a^x + C \end{aligned}$$

Question 1 Find the following integrals.

(a) $\displaystyle\int e^{2x}\,dx$ (b) $\displaystyle\int e^{-3x}\,dx$ (c) $\displaystyle\int e^{\frac{x}{2}}\,dx$

(d) $\displaystyle\int e^{2x-1}\,dx$ (e) $\displaystyle\int e^{2-3x}\,dx$ (f) $\displaystyle\int e^{\frac{3x+1}{2}}\,dx$

(g) $\displaystyle\int 3e^{2x} - 6e^{-3x}\,dx$ (h) $\displaystyle\int 6e^{-2x}\,dx$ (i) $\displaystyle\int pe^{-px+q}\,dx$

Question 2 By first converting to the form e^{kx} for an appropriate value of k, integrate the following.

(a) $\int (e^x)^2\,dx$ (b) $\int \frac{1}{e^x}\,dx$ (c) $\int \sqrt{e^x}\,dx$

(d) $\int \sqrt{e^{3x}}\,dx$ (e) $\int \sqrt[3]{e^x}\,dx$ (f) $\int \frac{1}{\sqrt{e^x}}\,dx$

Question 3 Integrate the following.

(a) $\int (1+e^x)^2\,dx$ (b) $\int (e^x+e^{-x})^2\,dx$ (c) $\int \left(e^x+\frac{1}{e^x}\right)\left(e^x-\frac{1}{e^x}\right)dx$

(d) $\int \frac{1+e^x}{e^x}\,dx$ (e) $\int \frac{e^{2x}+e^x}{e^{4x}}\,dx$ (f) $\int \frac{e^x+1}{\sqrt{e^x}}\,dx$

Question 4 [Drills needed for the next question]

Simplify the following, using the fact that $e^{\ln x}=x$ for $x>0$.

(a) $e^{\ln 2}$ (b) $e^{2\ln 2}$ (c) $e^{-\ln 2}$ (d) $e^{\frac{1}{2}\ln 2}$

Question 5 Calculate the following.

(a) $\int_0^1 e^{2x}\,dx$ (b) $\int_{-1}^1 \frac{1}{e^{2x}}\,dx$ (c) $\int_0^1 e^{\frac{2-x}{2}}\,dx$

(d) $\int_0^{\ln 2} e^{3x}\,dx$ (e) $\int_{\ln 4}^{\ln 9} \sqrt{e^x}\,dx$ (f) $\int_{\ln 2}^{2\ln 3} (1-e^x)^2\,dx$

Question 6 Integrate the following.

(a) $\int 2^x\,dx$ (b) $\int 3^x\,dx$ (c) $\int 5^{2x}\,dx$

Question 7 Find the equation of the curve if

(a) $\frac{dy}{dx}=e^{x-1}$ and the curve passes through $(1,1)$.

(b) $\frac{dy}{dx}=e^{2x}$ and the curve passes through $(\ln 2, 3)$.

Question 8 ['Differentiate and hence integrate' problems]

(a) Differentiate e^{x^2} and hence find $\int xe^{x^2}\,dx$.

(b) Differentiate e^{x^3} and hence find $\int x^2e^{x^3}\,dx$.

(c) Differentiate e^{2x^2} and hence find $\int xe^{2x^2}\,dx$.

(d) Differentiate xe^x and hence find $\int xe^x\,dx$.

Challenge Problems

Problem 1

(a) Expand $(u-1)(u^2+u+1)$.

(b) Hence, find $\int \frac{e^{3x}-1}{e^x-1}\,dx$.

Problem 2 Find the following integrals by first simplifying.

(a) $\int \frac{e^{2x}-1}{e^x+1}\,dx$

(b) $\int \frac{e^x-1}{e^{\frac{x}{2}}+1}\,dx$

(c) $\int \frac{e^{2x}-1}{e^x-e^{-x}}\,dx$

(d) $\int \frac{e^x-e^{-x}}{1-e^{-2x}}\,dx$

Problem 3 [Reduction formula]

Define the integral

$$I_n = \int x^n e^x\,dx$$

(a) By differentiating x^ne^x, show that

$$I_n = x^ne^x - nI_{n-1}$$

(b) Hence, find $\int x^3e^x\,dx$.

Exercise 6C

Integration involving the logarithmic function

Fundamentals

Fundamentals 1
Write down the following integrals.

(a) $\int \frac{1}{x}\,dx$ (b) $\int \frac{1}{ax+b}\,dx$ (c) $\int \frac{f'(x)}{f(x)}\,dx$

Question 1 Find the following integrals.

(a) $\int \frac{2}{x}\,dx$ (b) $\int \frac{1}{2x}\,dx$ (c) $\int \frac{1}{2x+1}\,dx$

(d) $\int \frac{3}{3x-4}\,dx$ (e) $\int \frac{1}{3-2x}\,dx$ (f) $\int \frac{4}{2x-5}\,dx$

Question 2 By first simplifying the integrand, find the following.

(a) $\int \frac{x+1}{x}\,dx$ (b) $\int \frac{4-x}{2x}\,dx$ (c) $\int \frac{x^2-2}{3x}\,dx$

(d) $\int \frac{x-2}{x^2-4}\,dx$ (e) $\int \frac{3x+2}{9x^2-4}\,dx$ (f) $\int \frac{5x-4}{16-25x^2}\,dx$

Question 3 Use the formula $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$ to find the following.

(a) $\int \frac{2x}{x^2+4}\,dx$ (b) $\int \frac{3x^2}{x^3-4}\,dx$ (c) $\int \frac{2x+1}{x^2+x+1}\,dx$

(d) $\int \frac{x}{x^2-1}\,dx$ (e) $\int \frac{x}{1-2x^2}\,dx$ (f) $\int \frac{x^2}{8-x^3}\,dx$

(g) $\int \frac{x+1}{x^2+2x+2}\,dx$ (h) $\int \frac{x-3}{x^2-6x+5}\,dx$ (i) $\int \frac{4x-8}{x^2-4x+3}\,dx$

Question 4 Calculate the following.

(a) $\int_1^2 \frac{1}{x}\,dx$ (b) $\int_2^3 \frac{1}{2x-1}\,dx$ (c) $\int_{-1}^0 \frac{1}{2-3x}\,dx$

(d) $\int_1^2 \frac{x}{x^2+1}\,dx$ (e) $\int_{-1}^2 \frac{x}{2+3x^2}\,dx$ (f) $\int_0^1 \frac{x-1}{x^2-2x+4}\,dx$

Question 5 Use the formula $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$ to find the following.

(a) $\displaystyle\int \frac{e^x}{1+e^x}\,dx$ (b) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx$ (c) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

Question 6 Calculate the following.

(a) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^x}{1+e^x}\,dx$ (b) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^{2x}}{3-e^{2x}}\,dx$ (c) $\displaystyle\int_0^1 \frac{x+e^{2x}}{x^2+e^{2x}}\,dx$

Question 7 Find the equation of the curve if

(a) $\dfrac{dy}{dx} = \dfrac{1}{2x}$ and the curve passes through $(e^2, 1)$.

(b) $\dfrac{dy}{dx} = \dfrac{1}{1+x}$ and the curve passes through $(0, 1)$.

(c) $\dfrac{dy}{dx} = \dfrac{1}{4-x}$ and the curve passes through $(3, 1)$.

Question 8 Find the equation of the curve if

$$\frac{d^2y}{dx^2} = -\frac{4}{(2x-1)^2},$$

the curve passes through $(5, 2\ln 3)$, and the gradient of the tangent when $x = 1$ is 2.

Question 9

(a) Simplify $\dfrac{\sec^2 x}{\tan x}$. (b) Hence, find $\displaystyle\int \frac{1}{\sin x \cos x}\,dx$.

Question 10

(a) Show that $\dfrac{1+e^x}{1-e^x} = 1 + \dfrac{2e^x}{1-e^x}$. (b) Hence, find $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$.

Challenge Problems

Problem 1 By first manipulating the integrand to obtain the form $\dfrac{f'(x)}{f(x)}$, find the following.

(a) $\displaystyle\int \frac{1}{x\ln x}\,dx$

(b) $\displaystyle\int \frac{1}{1+e^x}\,dx$

(c) $\displaystyle\int \frac{1}{x+x^4}\,dx$

(d) $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$

Problem 2 [Useful identity]

(a) Show that

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

(b) Hence, show that

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Exercise 6D

Integration involving trigonometric functions

Fundamentals

Fundamentals 1

Write down the following.

(a) $\int \sin x \, dx$ (b) $\int \cos x \, dx$ (c) $\int \sin(ax+b) \, dx$

(d) $\int \cos(ax+b) \, dx$ (e) $\int \sec^2 x \, dx$ (f) $\int \sec^2(ax+b) \, dx$

Question 1 Find the following.

(a) $\int \sin(2x) \, dx$ (b) $\int \sin\left(\frac{x}{3}\right) dx$ (c) $\int \sin\left(\frac{\pi}{2} - 2x\right) dx$

Question 2 Find the following.

(a) $\int \cos(3x) \, dx$ (b) $\int \cos\left(\frac{x}{2}\right) dx$ (c) $\int \cos\left(\frac{\pi - 2x}{3}\right) dx$

Question 3 Find the following.

(a) $\int \sec^2(3x) \, dx$ (b) $\int \sec^2\left(\frac{x}{2}\right) dx$ (c) $\int \sec^2\left(\frac{\pi - 2x}{3}\right) dx$

Question 4 Find the following by first expressing in terms of sines and cosines.

(a) $\int \tan(2x) \, dx$ (b) $\int \cot(3x) \, dx$

Question 5 Calculate the following.

(a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(2x - \frac{\pi}{3}\right) dx$ (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\left(\frac{\pi}{6} - 3x\right) dx$ (c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan\left(2x + \frac{\pi}{3}\right) dx$

Question 6

(a) Complete the identity $1 + \tan^2 x =$ ______. (b) Hence, find $\int \tan^2 x \, dx$.

Question 7 Find the following.

(a) $\int_0^{\frac{\pi}{3}} \frac{\sec^2 x}{1 + \tan x} \, dx$ (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x} \, dx$ (c) $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{\sin x + \cos x} \, dx$

Question 8 By using complementary identities, show that $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$.

Question 9 [Using degrees]

Find the following.

(a) $\int \cos(180°x)\,dx$ (b) $\int \sin(x°)\,dx$ (c) $\int \tan(x°)\,dx$

Question 10 [Easier 'Differentiate and hence integrate' problems]

(a) Differentiate $y = \sec x$ and hence find $\int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$.

(b) Differentiate $y = \sin^2 x$ and hence find $\int_0^{\frac{\pi}{2}} \sin x \cos x\,dx$.

(c) Differentiate $y = \sin^3 x$ and hence find $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x\,dx$.

(d) Differentiate $y = \sin(x^2)$ and hence find $\int_0^{\frac{\sqrt{\pi}}{2}} x\cos(x^2)\,dx$.

(e) Differentiate $y = e^{\tan x}$ and hence find $\int_0^{\frac{\pi}{4}} e^{\tan x}\sec^2 x\,dx$.

Question 11 Find $\int_0^{\frac{\pi}{2}} \sin^2 x + \cos^2 x\,dx$.

Question 12

(a) Show that $\dfrac{1}{\cos^2 x + \sin x \cos x} = \dfrac{\sec^2 x}{1 + \tan x}$.

(b) Hence, find $\int_0^{\frac{\pi}{4}} \dfrac{1}{\cos^2 x + \sin x \cos x}\,dx$.

Challenge Problems

Problem 1 [Application of double-angle identities]

It can be shown that $\cos(2\theta) = 2\cos^2\theta - 1$. Use this identity to find the following.

(a) $\int \cos^2 x \, dx$

(b) $\int \sin^2 x \, dx$

(c) $\int \cos^2(2x) \, dx$

(d) $\int \sin^2\left(\frac{x}{2}\right) dx$

Problem 2 [Harder 'Differentiate and hence integrate' problems]

(a) Differentiate $y = x\sin x$ and hence find $\int_0^{\frac{\pi}{2}} x\cos x \, dx$.

(b) Differentiate $y = x\tan x$ and hence find $\int_0^{\frac{\pi}{3}} x\sec^2 x \, dx$.

(c) Differentiate $y = \ln\left(\frac{1+\sin x}{\cos x}\right)$ and hence find $\int_0^{\frac{\pi}{4}} \sec x \, dx$.

Problem 3 Differentiate $y = \frac{x}{\tan x}$ and hence show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x} \, dx = \frac{1}{2}\ln 3 + \frac{\pi\sqrt{3}}{18}.$$

Problem 4 [Integral of $\sec x$]

(a) Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$

(b) Hence, find $\int \sec x \, dx$.

Hint: Multiply the integrand by $\frac{\sec x + \tan x}{\sec x + \tan x}$.

Problem 5 Define the integrals

$$I = \int \frac{\cos x}{\sin x + \cos x} \, dx$$

$$J = \int \frac{\sin x}{\sin x + \cos x} \, dx$$

(a) Find $I + J$.

(b) Find $I - J$.

(c) Hence, find I and J.

Exercise 6E

Reverse chain rule

Fundamentals

Fundamentals 1

(a) If $y = \dfrac{1}{n+1}(f(x))^{n+1}$, then $\dfrac{dy}{dx} =$ ________.

(b) Hence we have

$$\int f'(x)(f(x))^n \, dx = \_\_\_\_\_\_$$

Question 1 Find the following.

(a) $\displaystyle\int (3x-2)^4 \, dx$

(b) $\displaystyle\int (3-2x)^6 \, dx$

(c) $\displaystyle\int \sqrt{4x-1} \, dx$

(d) $\displaystyle\int \sqrt{3-2x}$

(e) $\displaystyle\int \frac{1}{(4x-3)^5} \, dx$

(f) $\displaystyle\int \frac{1}{\sqrt{5-2x}} \, dx$

Question 2 Consider the integral $\int x(x^2-1)^5 \, dx$. Complete the following set of steps.

$$\begin{aligned}\int x(x^2-1)^5 \, dx &= \_\_ \int \_\_ x(x^2-1)^5 \, dx \\ &= \_\_ \times \frac{1}{6}(x^2-1)^6 + C, \quad \text{using} \int f'(x)[f(x)]^n \, dx \\ &= \_\_ (x^2-1)^6 + C\end{aligned}$$

Question 3 Use a similar set of steps to Question 2 to find the following.

(a) $\displaystyle\int x(x^2+1)^3 \, dx$

(b) $\displaystyle\int x^2(x^3-1)^5 \, dx$

(c) $\displaystyle\int x^2(1-4x^3)^5 \, dx$

(d) $\displaystyle\int \frac{x}{(x^2-1)^3} \, dx$

(e) $\displaystyle\int x\sqrt{x^2+1} \, dx$

(f) $\displaystyle\int \frac{x}{\sqrt{x^2-1}} \, dx$

Question 4 Find the following.

(a) $\int e^x(1-e^x)^4\,dx$

(b) $\int e^{2x}(e^{2x}+1)^3\,dx$

(c) $\int \frac{e^x}{(3e^x-2)^2}\,dx$

(d) $\int e^{2x}\sqrt{e^{2x}-1}\,dx$

(e) $\int \frac{e^x}{\sqrt{e^x+1}}\,dx$

(f) $\int \frac{e^{3x}}{(1-2e^{3x})^4}\,dx$

Question 5 Find the following.

(a) $\int \sin^2 x\cos x\,dx$

(b) $\int \cos^3(2x)\sin(2x)\,dx$

(c) $\int \tan^2(3x)\sec^2(3x)\,dx$

(d) $\int \frac{\cos x}{\sin^3 x}\,dx$

(e) $\int \frac{\sin(2x)}{\cos^2(2x)}\,dx$

(f) $\int \cos x\sqrt{1+\sin x}\,dx$

(g) $\int \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$

(h) $\int \frac{\sec^2 x}{(1-\tan x)^2}\,dx$

Question 6 [Importance of the constant of integration]

(a) Find $\int \sin x\cos x\,dx$ by letting $f(x)=\sin x$.

(b) Find $\int \cos x\sin x\,dx$ by letting $f(x)=\cos x$.

(c) Explain why the two answers are different.

Question 7 [Partial Fractions]

(a) Show that

$$\frac{6x^2-5x-2}{(x+1)(2x-1)^2}=\frac{1}{2x-1}-\frac{2}{(2x-1)^2}+\frac{1}{x+1}$$

Hint: You are not obliged to begin the proof from the left-hand-side.

(b) Hence, show that

$$\int_1^2 \frac{6x^2-5x-2}{(x+1)(2x-1)^2}\,dx=\ln\left(\frac{3\sqrt{3}}{2}\right)-\frac{2}{3}$$

Challenge Problems

Problem 1 Define $I = \int_0^{\frac{\pi}{2}} \cos^3 x \, dx$

(a) Show that $\cos^3 x = \cos x - \sin^2 x \cos x$.

(b) Hence, find the value of I.

Problem 2 Recall that $\frac{d}{dx}(\sec x) = \sec x \tan x$. Use this fact to find $\int \sec^4 x \tan x \, dx$.

Problem 3 Use a similar technique to above to find $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx$.

Problem 4 Let $n \geq 0$ and define

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$$

(a) Show that

$$I_n = \frac{1}{n-1} - I_{n-2}$$

for $n \geq 2$.
Hint: $\tan^n x = \tan^{n-2} x \times \tan^2 x$

(b) Use the above formula to express I_5 in terms of I_1.

(c) Hence, calculate $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$.

(d) Similarly, calculate $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$.

Problem 5 Find the following.

(a) $\int \frac{1}{x \ln x} \, dx$

(b) $\int \frac{\ln x}{x} \, dx$

(c) $\int \frac{e^x}{1 - 2e^x + e^{2x}} \, dx$

(d) $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} \, dx$

Chapter 6 Review

Integration

Review

Question 1 Find the following.

(a) $\displaystyle\int \left(\frac{3x^2}{2} - \sqrt{x}\right) dx$ (b) $\displaystyle\int x\sqrt{x}\, dx$ (c) $\displaystyle\int x^2(x-6)\, dx$

(d) $\displaystyle\int (x^2+2)^2\, dx$ (e) $\displaystyle\int (5x-3)^2\, dx$ (f) $\displaystyle\int \sqrt{5x-3}\, dx$

(g) $\displaystyle\int \frac{1}{\sqrt{5x-3}}\, dx$ (h) $\displaystyle\int \frac{3}{\sqrt{1+2x}}\, dx$ (i) $\displaystyle\int \frac{1}{(3x-1)^2}\, dx$

Question 2 Find the following.

(a) $\displaystyle\int e^{3x}\, dx$ (b) $\displaystyle\int e^{2-3x}\, dx$ (c) $\displaystyle\int e^{-\frac{x}{2}}\, dx$

(d) $\displaystyle\int \left(\frac{x}{3} + \frac{3}{x}\right) dx$ (e) $\displaystyle\int \left(\frac{x}{e} + \frac{e}{x}\right) dx$ (f) $\displaystyle\int \frac{5}{e^{3x}}\, dx$

(g) $\displaystyle\int e^{x}\left(e^{-3x} + e^{3x}\right) dx$ (h) $\displaystyle\int (1+e^{2x})^2\, dx$

Question 3 Calculate the following.

(a) $\displaystyle\int_1^2 \frac{1}{2}\, dx$ (b) $\displaystyle\int_1^2 \left(3x^2 - 5x\right) dx$ (c) $\displaystyle\int_1^2 \frac{2}{x^2}\, dx$

(d) $\displaystyle\int_1^4 \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) dx$ (e) $\displaystyle\int_0^1 \sqrt{5x+4}\, dx$ (f) $\displaystyle\int_0^2 e^{2x}\, dx$

Question 4 Find the following.

(a) $\displaystyle\int \frac{3}{4x}\, dx$ (b) $\displaystyle\int \frac{6}{1+6x}\, dx$ (c) $\displaystyle\int \frac{-2}{1-2x}\, dx$

(d) $\displaystyle\int \frac{2}{5-2x}\, dx$ (e) $\displaystyle\int \frac{3}{1+2x}\, dx$ (f) $\displaystyle\int \frac{x}{1-x^2}\, dx$

(g) $\displaystyle\int \frac{x-2}{x^2-4x}\, dx$ (h) $\displaystyle\int \frac{4x+2}{x^2+x}\, dx$ (i) $\displaystyle\int \frac{x^2-2x}{x^3-3x^2}\, dx$

Question 5 Calculate the following.

(a) $\displaystyle\int_1^2 \frac{1}{1+x}\, dx$ (b) $\displaystyle\int_{-1}^2 \frac{4}{3+2x}\, dx$ (c) $\displaystyle\int_{-2}^2 \frac{4x}{x^2+1}\, dx$

Question 6 Find the primitive of.

(a) 2^x (b) 4^{2x} (c) 3^{-x}

Question 7 Find the following.

(a) $\int (3\sin 2x - \cos 2x)\, dx$ (b) $\int \cos\left(\frac{3x}{2}\right) dx$ (c) $\int 2\sec^2(3x)\, dx$

(d) $\int \sin\left(\frac{2x}{3}\right) dx$ (e) $\int -\cos\left(\frac{3x}{4}\right) dx$ (f) $\int \sec^2\left(\frac{\pi}{4}x\right) dx$

(g) $\int \sin\left(\frac{\pi}{2} - 3x\right) dx$ (h) $\int \cos\left(3x + \frac{\pi}{3}\right) dx$ (i) $\int \sec^2\left(\frac{\pi}{6} - 8x\right) dx$

Question 8 Calculate the following.

(a) $\int_0^{\pi} \sin x\, dx$ (b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx$ (c) $\int_0^2 \sin\left(\frac{\pi x}{2}\right) dx$

(d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin 2x - 3\cos x)\, dx$ (e) $\int_0^{\frac{\pi}{6}} \sec^2\left(\frac{3x}{2}\right) dx$ (f) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\, dx$

Question 9 Find the following.

(a) $\int \frac{4x^2}{1+x^3}\, dx$ (b) $\int \frac{e^{3x}}{e^{3x}-2}\, dx$

(c) $\int \frac{3\cos x}{\sin x + 2}\, dx$ (d) $\int \frac{\sec^2 3x}{\tan 3x - 2}\, dx$

(e) $\int \frac{x^2+2}{x}\, dx$ (f) $\int \left(\frac{1}{3x+2} + \frac{1}{(3x+2)^2}\right) dx$

Question 10

(a) Find $\frac{d}{dx}(x^2+3)^6$ and hence find $\int 12x(x^2+3)^5\, dx$

(b) Find $\frac{d}{dx}\left(\ln\left(\frac{2+x}{2-x}\right)\right)$ and hence find $\int \frac{1}{4-x^2}\, dx$

(c) Find $\frac{d}{dx}(x\ln x)$ and hence find $\int \ln x\, dx$

Question 11

(a) Differentiate $y = e^{3x^2}$ and hence find $\int 6xe^{3x^2}\, dx$

(b) Differentiate $y = e^{\sin x}$ and hence find $\int \cos x e^{\sin x}\, dx$

(c) Differentiate xe^{3x} and hence $\int xe^{3x}\, dx$

Question 12 Calculate the following.

(a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x\, dx$ (b) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x\, dx$ (c) $\displaystyle\int_{\frac{\pi}{3}}^{\pi} \cos\frac{x}{2}\, dx$

(d) $\displaystyle\int_{0}^{\frac{\pi}{3}} (\sin 3x + \cos 3x)\, dx$ (e) $\displaystyle\int_{0}^{\frac{\pi}{2}} \left(2\cos 2x + \sin\frac{x}{2}\right) dx$ (f) $\displaystyle\int_{0}^{\frac{5\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) dx$

Question 13 Find the following.

(a) $\displaystyle\int x(x^2-2)^5\, dx$ (b) $\displaystyle\int x(3x^2-1)^6\, dx$

(c) $\displaystyle\int e^x(3e^x+2)^2\, dx$ (d) $\displaystyle\int \frac{1}{x}(\ln x)^3\, dx$

Question 14 Evaluate

(a) $\displaystyle\int_{1}^{2} \left(x - \frac{1}{x}\right)^2 dx$ (b) $\displaystyle\int_{-1}^{1} \left(e^x - \frac{1}{e^x}\right)^2 dx$

Question 15 Find the value of k such that $\displaystyle\int_{k}^{1} \frac{3}{x^2}\, dx = 6$.

Question 16

(a) Differentiate $y = x\sin(2x)$ and hence find $\displaystyle\int_{0}^{\frac{\pi}{2}} x\cos(2x)\, dx$

(b) Differentiate $y = xe^x$ and hence find $\displaystyle\int_{0}^{1} xe^x\, dx$

(c) Differentiate $y = xe^{-3x}$ and hence find $\displaystyle\int_{0}^{1} xe^{-3x}\, dx$

(d) Differentiate $y = x^2\ln x$ and hence find $\displaystyle\int_{1}^{e^2} x\ln x\, dx$

(e) Differentiate $y = \ln(\cos x)$ and hence find $\displaystyle\int_{0}^{\frac{\pi}{3}} \tan x\, dx$

(f) Differentiate $y = (\ln x)^2$ and hence find $\displaystyle\int_{1}^{e} \frac{\ln x}{x}\, dx$

(g) Differentiate $y = x\cos x$ and hence find $\displaystyle\int_{0}^{\frac{\pi}{2}} x\sin x\, dx$

Question 17 Find the following.

(a) $\displaystyle\int \frac{\cos x}{1+\sin x}\, dx$ (b) $\displaystyle\int \frac{\sec^2 x}{1-\tan x}\, dx$ (c) $\displaystyle\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$

(d) $\displaystyle\int \cot x\, dx$ (e) $\displaystyle\int \tan x\, dx$ (f) $\displaystyle\int \frac{\sin x\cos x}{1+\sin^2 x}\, dx$

Investigation Task

Integrating powers of trigonometric functions

Using the reverse chain rule, a student can find integrals such as $\int \sin^2 x \cos x \, dx$. The study of integrating expressions involving powers of trigonometric functions can be stretched a bit further to make a wider range of integrals accessible to the student. This investigation task will guide the student towards this goal.

Note: Teachers reading this may be concerned about integrals such as $\int \sin^4 x \, dx$, which require double-angle formulae in Mathematics Extension 1, but these will not be treated in this task. However, this task can easily be extended to include such examples for Extension 1 classes

Question 1 Consider the integral $\int \sin^3 x \, dx$.

(a) Show that this is equivalent to $\int \sin x - \cos^2 x \sin x \, dx$.

(b) Hence, find $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$.

(c) Use a similar technique to find $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$.

(d) Explain why it is desirable to have an *odd* power in these cases.

Question 2 Repeat a similar process to find the following integrals.

(a) $\int \cos^5 x \, dx$

(b) $\int \tan^5 x \, dx$

(c) $\int \sec^4 x \, dx$

(d) $\int \tan^4 x \, dx$

Question 3 Find the following integrals.

(a) $\int \cos^3 \sin^2 x \, dx$

(b) $\int \sin^3 x \cos^2 \, dx$

(c) $\int \cos^4 \sin^3 x \, dx$

(d) $\int \frac{\sec^4 x}{\tan x} \, dx$

(e) $\int \tan^4 x \sec^4 x \, dx$

(f) $\int \frac{\sin^3 x}{\cos x} \, dx$

7

APPLICATIONS OF INTEGRATION

- Area and the integral
- Area under a curve
- Area involving two curves
- Trapezoidal rule
- Applications involving integration

Exercise 7A

Area and the integral

Fundamentals

Fundamentals 1

(a) Regions above the x-axis have a p_____ signed area.

(b) Regions below the x-axis have a n_____ signed area.

(c) Suppose $f(x) \geq 0$ in the domain $x \in [a, b]$. Then $\int_a^b f(x)\,dx$ yields the a_____ between the curve and the ___-axis.

(d) Calculating $\int_a^b f(x)\,dx$ yields the s_____ area enclosed by the graph of $y = f(x)$ and the x-axis in the domain $x \in [\_\_\_, \_\_\_]$.

(e) If $f(x)$ is below the x-axis for $x \in [a, b]$, then $\int_a^b f(x)\,dx$ will be n_______.

Question 1 Calculate the following by sketching the integrand and shading the appropriate region.

(a) $\int_0^4 3\,dx$

(b) $\int_0^4 2x\,dx$

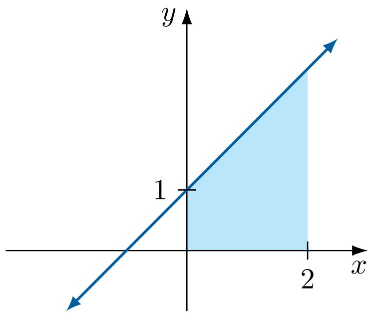
(c) $\int_0^4 4 - x\,dx$

(d) $\int_0^4 8 - 2x\,dx$

(e) $\int_{-5}^0 5 + x\,dx$

(f) $\int_{-2}^4 x + 2\,dx$

Question 2 The diagram below shows the region that is represented by $\int_0^2 x + 1\,dx$.



(a) Find the area of the region using the formula for the area of a trapezium.

(b) Find the area of the region by calculating the definite integral directly.

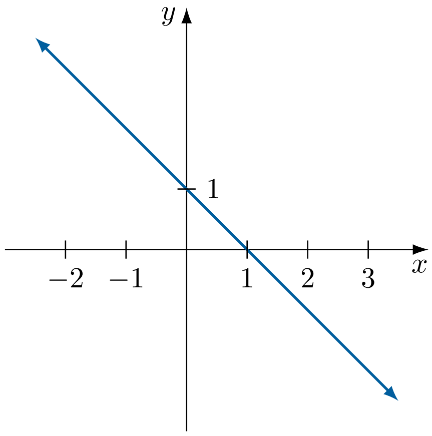
Question 3 Calculate the following by sketching the integrand and shading the appropriate region.

(a) $\int_0^2 x + 2\,dx$ (b) $\int_0^2 4 - x\,dx$ (c) $\int_1^3 3x - 2\,dx$

(d) $\int_0^4 2x + 3\,dx$ (e) $\int_{-1}^4 2x + 4\,dx$ (f) $\int_{-1}^2 5 - 2x\,dx$

(g) $\int_2^5 2x + 1\,dx$ (h) $\int_{-6}^{-2} x + 8\,dx$

Question 4 The diagram below shows a sketch of $y = f(x)$. Determine whether the following integrals should yield a positive, negative, or zero result.



(a) $\int_0^1 f(x)\,dx$ (b) $\int_1^2 f(x)\,dx$ (c) $\int_0^2 f(x)\,dx$

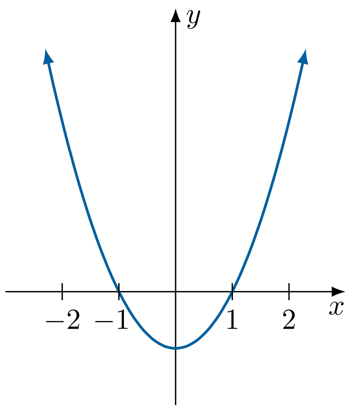
(d) $\int_{-2}^2 f(x)\,dx$ (e) $\int_0^3 f(x)\,dx$ (f) $\int_{-1}^3 f(x)\,dx$

Question 5 Calculate the following by sketching the integrand and shading the appropriate region. Remember that regions below the x-axis yield negative 'area' whereas regions above the x-axis yield positive 'area'.

(a) $\int_0^4 -2\,dx$ (b) $\int_0^4 -2x\,dx$

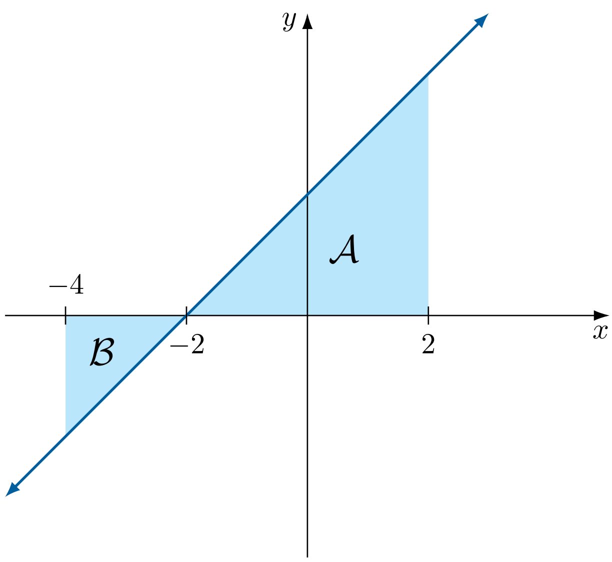
(c) $\int_0^4 x - 4\,dx$ (d) $\int_0^4 -1 - x\,dx$

Question 6 The diagram below shows the graph of $y = x^2 - 1$. Copy the diagram into your book, and shade the region that corresponds to the following.



(a) $\int_{1}^{2} x^2 - 1\,dx$ (b) $\int_{0}^{1} x^2 - 1\,dx$ (c) $\int_{-1}^{1} x^2 - 1\,dx$ (d) $\int_{-2}^{0} x^2 - 1\,dx$

Question 7 Consider the shaded region below, bounded by the curve $y = x + 2$ and the x-axis in the domain $x \in [-4, 2]$.



(a) Calculate the area of region $\mathcal{A}$.

(b) Explain briefly why $\int_{-4}^{-2} x + 2\,dx$ does *not* give the area of region $\mathcal{B}$, and state what it instead provides.

(c) Calculate the area of region $\mathcal{B}$.

(d) Hence, calculate the total area of the region bounded by $y = x + 2$ and the x-axis in the domain $x \in [-4, 2]$.

(e) Bob tries to calculate the total area by instead calculating $\int_{-4}^{2} x+2\,dx$. Explain why his answer will be incorrect, and describe what it instead provides.

(f) Mary claims that Bob's answer is the same as calculating $\int_{0}^{2} x+2\,dx$. Without explicitly calculating Mary's answer, explain why her answer is correct.

Question 8 Calculate the area of the region bounded by the graph of $y = 6 - x$ and the x-axis in the following domains.

(a) $x \in [0,\ 6]$ (b) $x \in [4,\ 6]$ (c) $x \in [0,\ 4]$ (d) $x \in [0,\ 8]$

(e) $x \in [0,\ 12]$ (f) $x \in [-4,\ 0]$ (g) $x \in [-4,\ 4]$ (h) $x \in [-4,\ 10]$

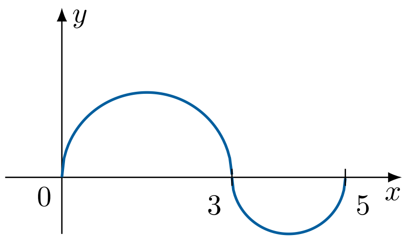
Question 9 Calculate the following by sketching and shading an appropriate region, then using familiar area formulae.

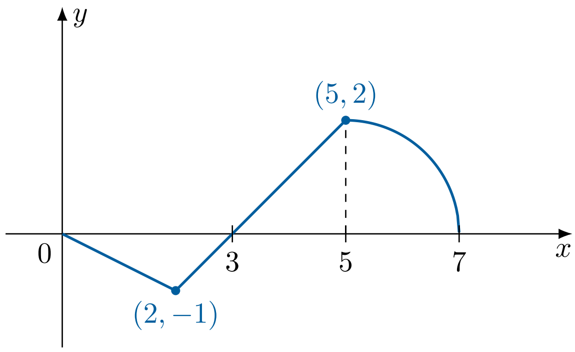
(a) $\int_{0}^{2} \sqrt{4-x^2}\,dx$ (b) $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

(c) $\int_{-1}^{1} -\sqrt{1-x^2}\,dx$ (d) $\int_{0}^{-5} \sqrt{25-x^2}\,dx$

Question 10 The diagrams below show sketches of $y = f(x)$. Calculate the following.

(a) $\int_{0}^{5} f(x)\,dx$ (b) $\int_{0}^{7} f(x)\,dx$





Question 11 By drawing a sketch, write down the result of the following without explicitly evaluating the integral.

(a) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \tan x\,dx$ (b) $\int_{0}^{\pi} \sin 2x\,dx$ (c) $\int_{0}^{\pi} \cos x\,dx$

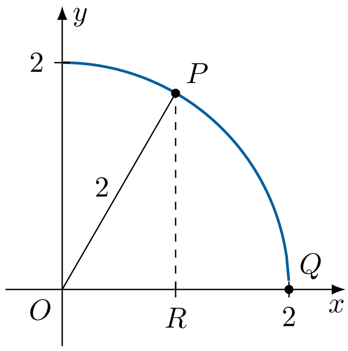
Challenge Problems

Problem 1 Find $\int_2^8 \sqrt{-x^2+10x-16}\,dx$.

Hint: Sketch the integrand first.

Problem 2 [Using sectors]

The diagram below shows a point P on a part of $y = \sqrt{4-x^2}$. The point P is vertically above R, and Q has coordinates $(2, 0)$. The point R has coordinates $(1, 0)$.



(a) Find $\angle POR$.

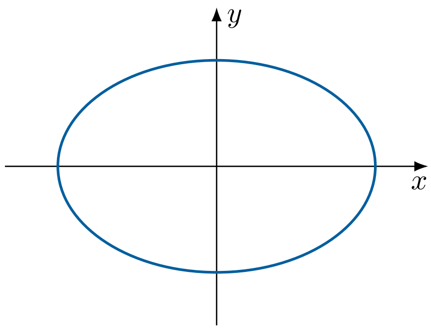
(b) Find the area of sector OPQ.

(c) Find the area of $\triangle OPR$.

(d) Hence, find $\int_0^1 \sqrt{4-x^2}\,dx$.

Problem 3 [Area of an ellipse]

The diagram below shows the equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



(a) Find the x and y-intercepts of the ellipse.

(b) Find a definite integral that represents the area of the ellipse in the first quadrant.

(c) Hence, show that the area of the ellipse is πab.

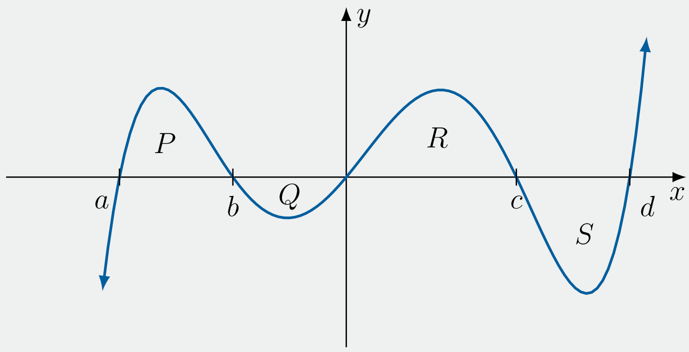
Exercise 7B

Area under a curve

Fundamentals

Fundamentals 1

The diagram below shows the graph of $y = f(x)$. Let P, Q, R and S represent the *positive* areas of their corresponding regions.



(a) Name the regions with positive signed area, and the regions with negative signed area.

(b) $\int_a^d f(x)\,dx$ will calculate $P - Q + R$___.

(c) The area of the region enclosed by $f(x)$ and the x-axis where $x \in [c, d]$ is ______.

(d) The total area between the $f(x)$ and the x-axis is ________.

Fundamentals 2

(a) When finding the area of a region, you should produce a s_____ of the region first. This is so then you can see if there are any 'areas' b_____ the x-axis that you have to make positive.

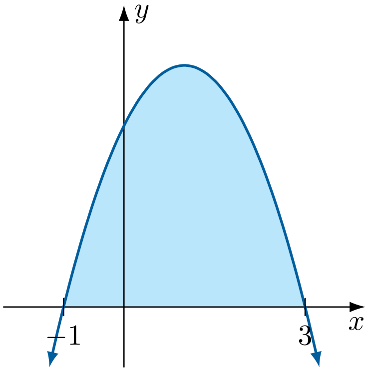
(b) However, if the function is clearly p________ in the domain of integration, then a s_____ is not necessary.

Fundamentals 3

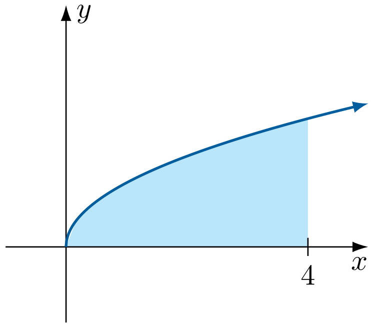
If the function is o___ or e_____, use symmetry whenever possible to make it easier to find the area.

Question 1 Calculate the area of the shaded regions below, given the equation of the curve.

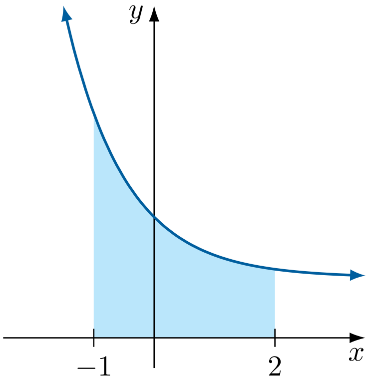
(a) $y = (x+1)(3-x)$



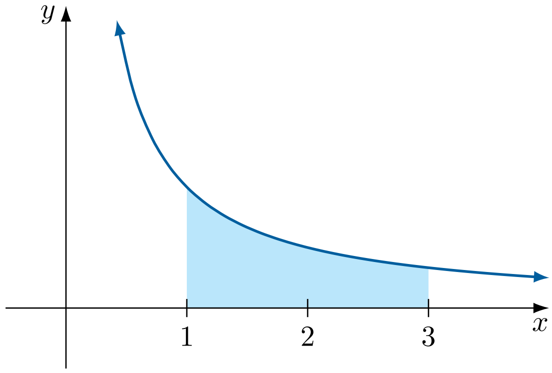
(b) $y = \sqrt{x}$



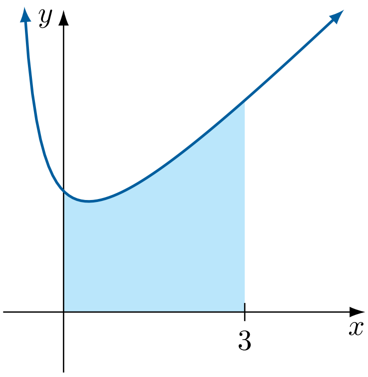
(c) $y = \dfrac{e^x + 1}{e^x}$



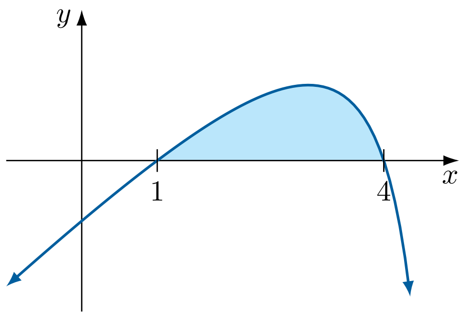
(d) $y = \dfrac{1}{x}$



(e) $y = x + \dfrac{2}{x+1}$



(f) $y = x + \dfrac{4}{x-5}$



Question 2 By first sketching the following graphs, find the area of the region bounded by the curve and the x-axis.

(a) $y = 4x - x^2$

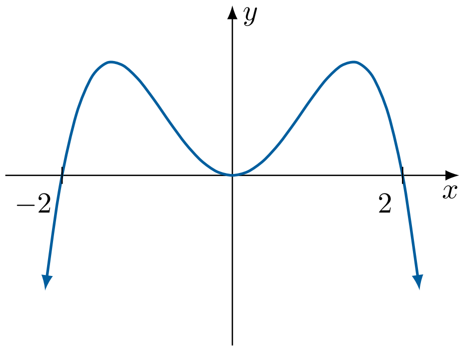
(b) $y = 2 - x - x^2$

Question 3 [A sketch is not always necessary under certain circumstances]

A teacher asks Bob and Mary to find the area under the curve $y = \dfrac{1}{x}$ in the domain $x \in [1, 5]$. Bob draws a sketch of the curve, shades the region, and then finds the area. Mary claims that the answer is simply $\displaystyle\int_1^5 \frac{1}{x}\,dx$ and that a sketch is not necessary. Explain why her claim is true for this particular equation.

Question 4 Find the area of the region bounded by the curve $y = \dfrac{x}{x^2+1}$ in the domain $x \in [0, 2]$.

Question 5 The diagram below shows a sketch of $f(x) = 4x^2 - x^4$.

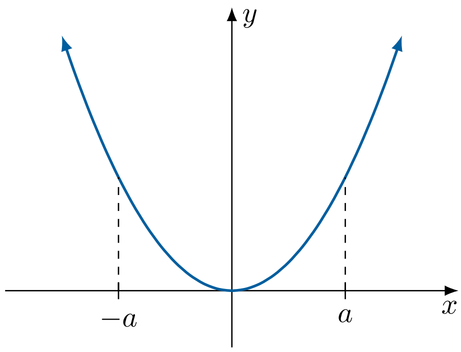


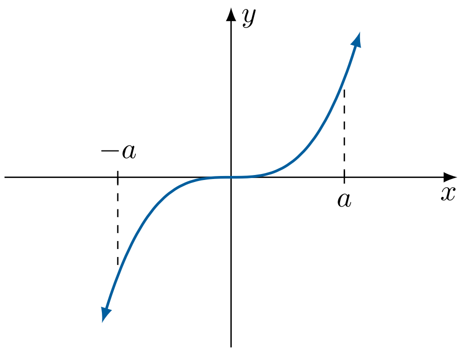
(a) Explain why $\displaystyle\int_{-2}^{2} f(x)\,dx = 2\int_0^2 f(x)\,dx$ for this curve.

(b) Hence, find the area of the region enclosed by the curve and the x-axis.

Question 6 [Odd and even functions]

Explain, with the aid of the diagrams below, how the following identities work.

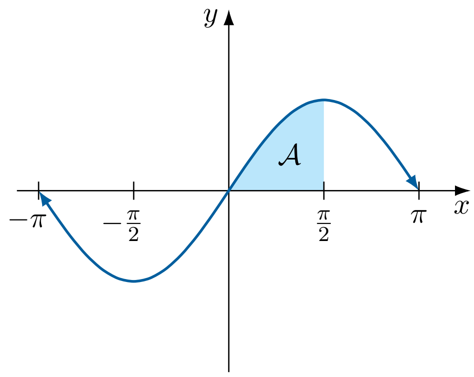




(a) If $f(x)$ is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If $g(x)$ is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Question 7 The diagram below shows a sketch of $y = \sin x$ for $x \in [-\pi, \pi]$.

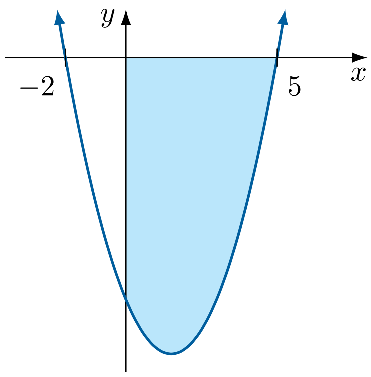


(a) Find the area of $\mathcal{A}$.

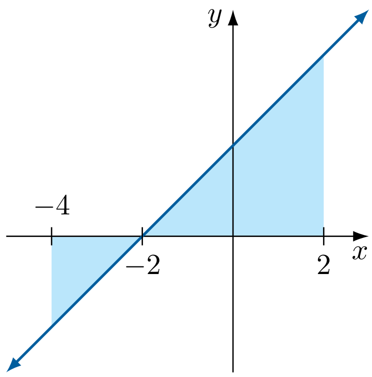
(b) Hence, without explicitly evaluating the integral, find $\int_{-\frac{\pi}{2}}^{\pi} \sin x \, dx$

Question 8 Calculate the area of the shaded regions below, given the equation of the curve.

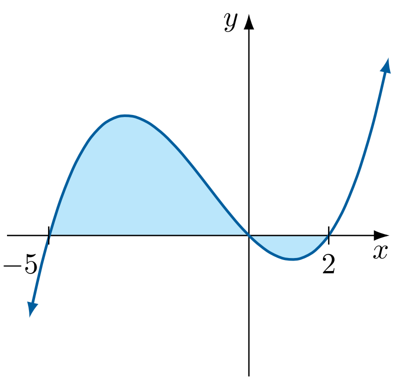
(a) $y = (x + 2)(x - 5)$



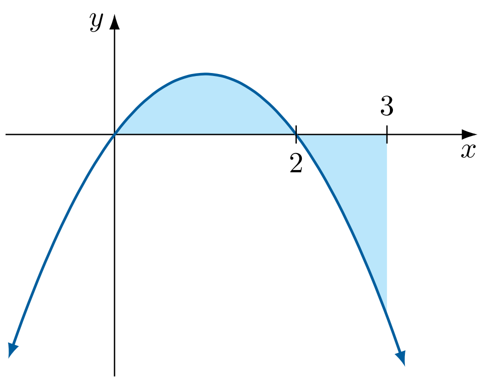
(b) $y = x + 2$



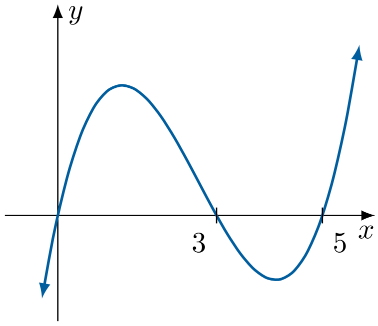
(c) $y = x^3 + 3x^2 - 10x$



(d) $y = 2x - x^2$



Question 9 The diagram below shows the sketch of $f(x) = x^3 - 8x^2 + 15x$.



(a) Calculate $\int_0^5 f(x)\,dx$.

(b) Find the area of the region bounded by $f(x)$ and the x-axis.

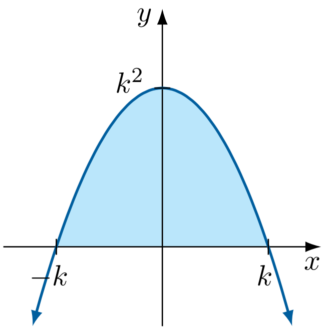
(c) Explain briefly how your answer in (a) implies that the region where $x \in [0, 3]$ is larger than the region where $x \in [3, 5]$.

Question 10 Let $f(x) = x^2 - 6x$.

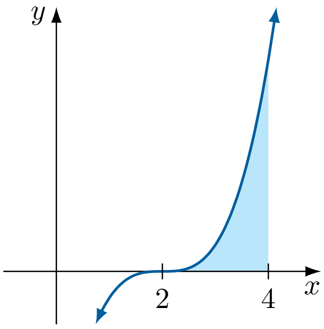
(a) Sketch $y = f(x)$. You do not need to find the exact coordinates of the vertex.

(b) Find the area of the region bounded by $f(x)$ and the x-axis in the domain $x \in [2, 8]$.

Question 11 The diagram shows a region bounded by $y = k^2 - x^2$ and the x-axis. If the area of this region is $\dfrac{500}{3}$ square units, find the values of k.

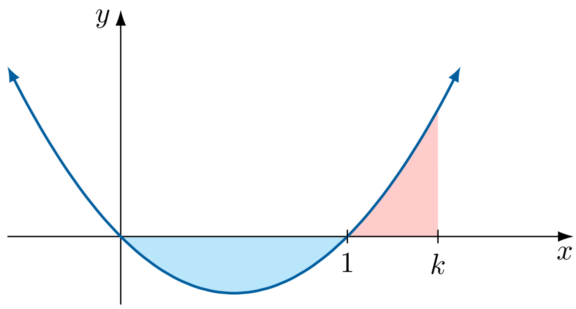


Question 12 The graph with equation $y = k(x - 2)^3$ is shown below.



The area of the shaded region is 32 square units. Find the value of k.

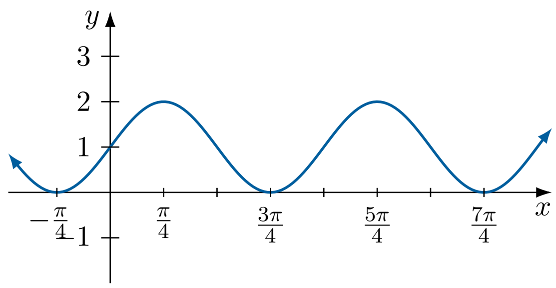
Question 13 The diagram below shows the graph of $y = x^2 - x$.



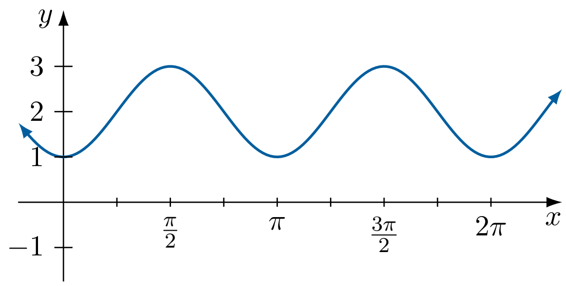
Find the value of k such that the red and blue regions have the same area.

Question 14 Use the diagrams below to find the appropriate values of A and B so that the following equalities work.

(a) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (1 + \sin 2x)\, dx = \int_{\frac{5\pi}{4}}^{B} (1 + \sin 2x)\, dx$



(b) $\displaystyle\int_{0}^{\pi} (2 - \cos 2x)\, dx = \int_{A}^{B} (2 - \cos 2x)\, dx$



Question 15

(a) Find $\displaystyle\int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right) dx$

(b) Sketch the graph of $y = \left|\cos\left(\frac{x}{2}\right)\right|$ for $-4\pi \le x \le 4\pi$.

(c) Hence, evaluate $\displaystyle\int_{-4\pi}^{4\pi} \left|\cos\left(\frac{x}{2}\right)\right| dx$.

Question 16 Find $\int_0^{\pi} \sin\left(2x+\frac{\pi}{4}\right)\,dx$ and explain your answer by considering the graph and period of the function.

Question 17 [True or false questions]

(a) If $\int_{-a}^{a} f(x)\,dx = 0$, then $f(x)$ is an odd function.

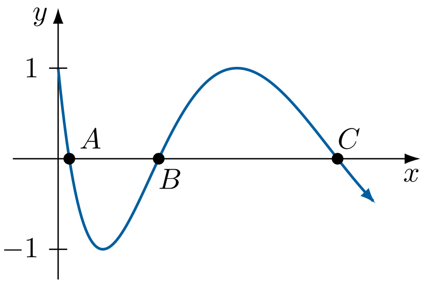
(b) If $\int_{-a}^{a} f(x)\,dx = 2 \times \int_0^{a} f(x)\,dx$, then $f(x)$ is an even function.

Challenge Problems

Problem 1 Consider $y = 2\cos\sqrt{x} + 2\sqrt{x}\sin\sqrt{x}$

(a) Find $\frac{dy}{dx}$ and hence, find $\int \cos\sqrt{x}\,dx$.

(b) The graph of $y = \cos\sqrt{x}$ is drawn below. Write down the coordinates of A, B and C the points where the curve cuts the x-axis



(c) Show that the area from B to C is double the area from A to B.

Problem 2

(a) Find $\int_{-a}^{a} \frac{e^x}{e^x+1} - \frac{1}{2}\,dx$.

(b) Use graphing software to sketch $y = \frac{e^x}{e^x+1} - \frac{1}{2}$.

(c) Prove that $y = \frac{e^x}{e^x+1} - \frac{1}{2}$ is an odd function and hence justify your answers above.

Problem 3

(a) Find the equation of the parabola that passes through $(-\pi, 0)$, $(\pi, 0)$, and $(0, k)$.

(b) If the area between the parabola and the x-axis is 4π unit^2, find the value of k.

(c) Is your answer the only value of k satisfying this condition?

Exercise 7C

Area involving two curves

Fundamentals

Fundamentals 1

If $f(x)$ and $g(x)$ are two continuous functions such that $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area of the region bounded by the two curves in the domain $x \in [a, b]$ is given by

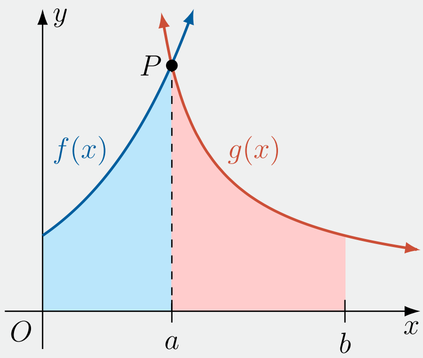
$$A = \int_a^b \cdots \, dx$$

Fundamentals 2

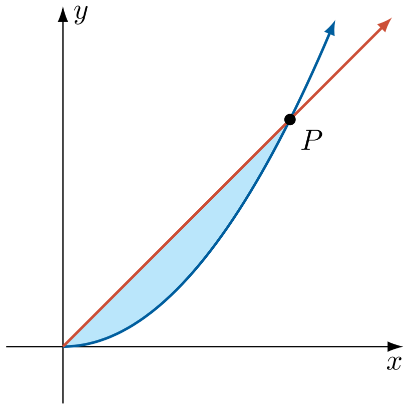
If you are asked to find the area enclosed by two curves, without being given any bounds, it means that you are required to first find the points of i__________, and those will be your l_____ of integration.

Fundamentals 3

For the diagram below, write down an integral that represents the total area of the shaded regions.



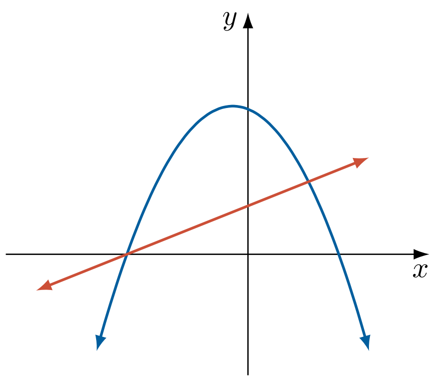
Question 1 The diagram below shows a sketch of $y = \frac{1}{2}x^2$, $y = x$, and the region enclosed between them. Let P be the point where the two curves intersect.



(a) Find the coordinates of P.

(b) Hence, find the area of the shaded region.

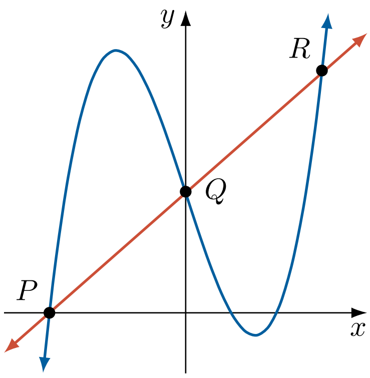
Question 2 The diagram below shows a sketch of $y = x + 4$ and $y = 12 - x - x^2$.



(a) Find the coordinates of the points where the two curves intersect.

(b) Hence, find the area of the region bounded by the two curves.

Question 3 The diagram below shows a sketch of $y = x^3 - 7x + 6$ and $y = 2x + 6$.

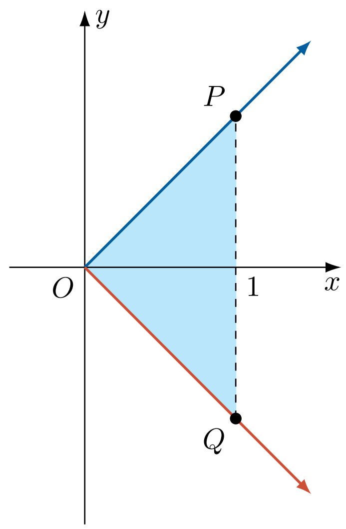


(a) Find the coordinates of their points of intersection P, Q and R.

(b) Hence, find the area of the region bounded by $y = x^3 - 7x + 6$ and $y = 2x + 6$.

Question 4 [Verifying that being below the x-axis does not change the formula]

The diagram below shows the graph of $y = x$ and $y = -x$.

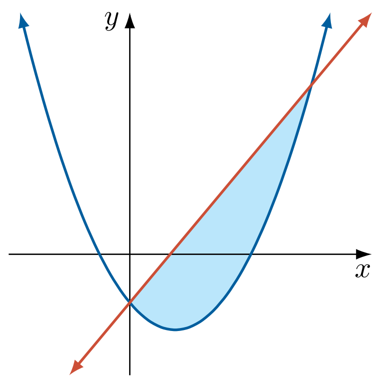


Let P and Q the points on the lines where $x = 1$.

(a) Find the area of the triangle using $A = \frac{1}{2}bh$.

(b) Find the area of the triangle using $A = \int_a^b f(x) - g(x)\, dx$ and verify that it is the same as your answer from (a).

Question 5 The diagram below shows a sketch of $y = 3x - 4$ and $y = x^2 - 3x - 4$.



(a) Find the coordinates of the points where the two curves intersect.

(b) Hence, find the area of the shaded region.

Question 6

(a) Draw a sketch of $y = x - 2$ and $y = -x^2 - x + 6$.

(b) Find the coordinates of the points where the two curves intersect.

(c) Hence, find the area of the region between the two curves.

Question 7

(a) Draw a sketch of $y = x^3$ and $y = x$.

(b) Find the coordinates of the points where the two curves intersect.

(c) Hence, find the area of the region bounded by the two curves.

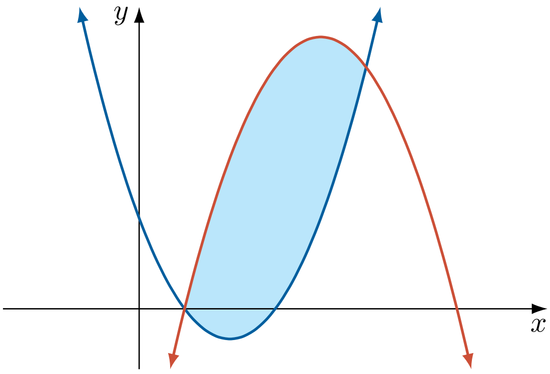
Question 8

(a) Draw a sketch of $y = 2\sqrt{x}$ and $y = \dfrac{x}{4}$.

(b) Find the coordinates of the points where the two curves intersect.

(c) Hence, find the area of the region bounded by the two curves.

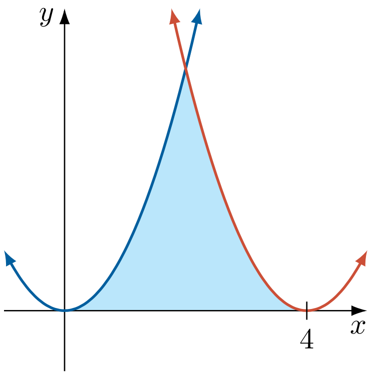
Question 9 The diagram below shows a sketch of $y = 8x - 7 - x^2$ and $y = x^2 - 4x + 3$.



(a) Find their point of intersection.

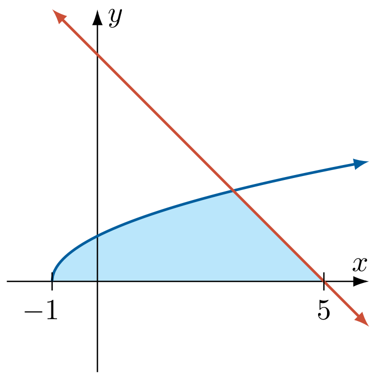
(b) Hence, find the area of the region between the two curves.

Question 10 The diagram below shows a sketch of $y = x^2$ and $y = (x - 4)^2$.



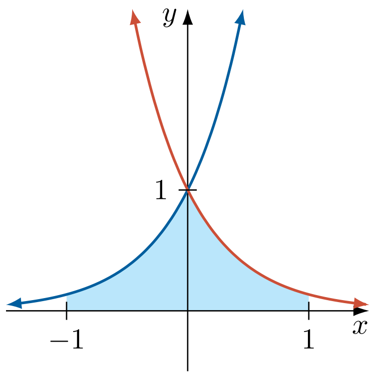
By first finding the point of intersection, find the area of the shaded region.

Question 11 The diagram below shows a sketch of $y = \sqrt{x+1}$ and $y = 5 - x$.

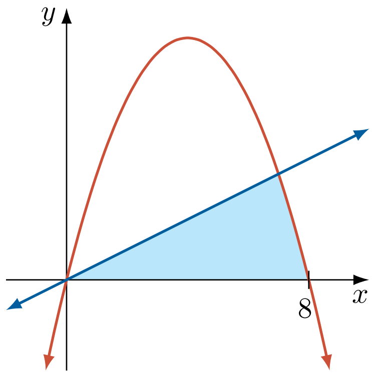


By first finding the point of intersection, find the area of the shaded region.

Question 12 Calculate the area bounded by $y = e^{2x}$, $y = e^{-2x}$, the x-axis and the lines $x = 1$ and $x = -1$

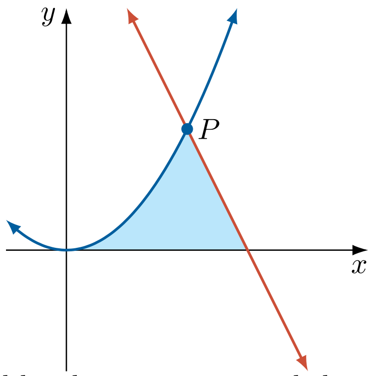


Question 13 The diagram below shows a sketch of $y = x$ and $y = 8x - x^2$.



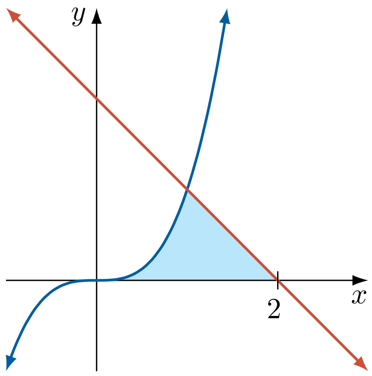
By first finding the point of intersection, find the area of the shaded region.

Question 14 The diagram shows the parabola $y = x^2$ and the line $y = 3 - 2x$ intersecting at the point P, in the first quadrant.



Find the area of the region bounded by the two curves and the x-axis.

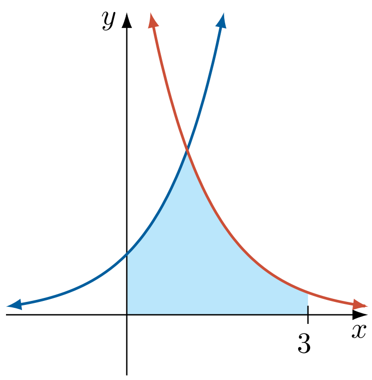
Question 15 The diagram below shows a sketch of $y = x^3$ and $y = 2 - x$.



(a) Show that the two curves intersect at $x = 1$.

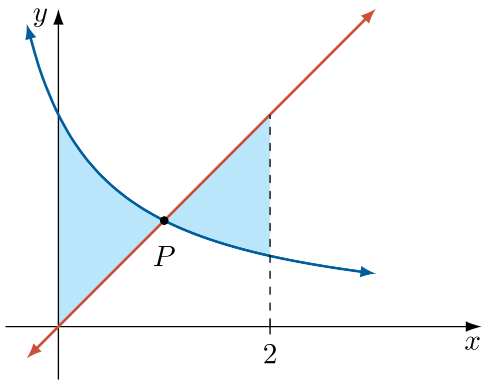
(b) Hence, find the area of the shaded region.

Question 16 The diagram below shows a sketch of $y = e^x$ and $y = e^{2-x}$.



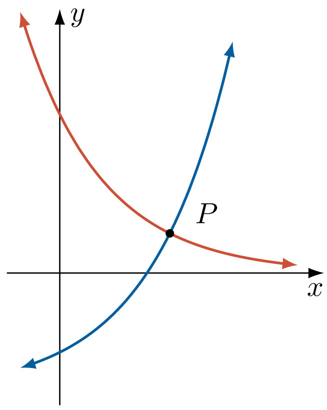
By first finding the point of intersection, find the area of the shaded region.

Question 17 The diagram below shows the graphs of $y = \dfrac{2}{x+1}$ and $y = x$.



Find the area of the shaded region.

Question 18 The diagram below shows the graphs of $y = 4e^{-x}$ and $y = e^{x} - 3$.

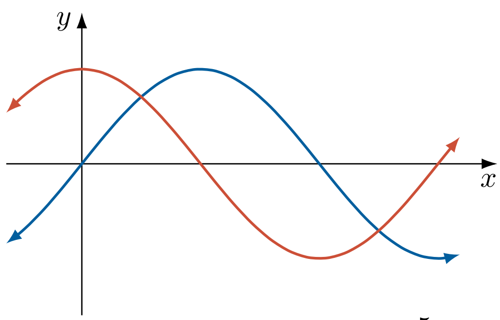


(a) Show that the curves intersect when $e^{2x} - 3e^{x} - 4 = 0$.

(b) Hence show that the x-coordinate P is $x = \ln 4$.

(c) Find the exact area bounded by the curves and the y-axis.

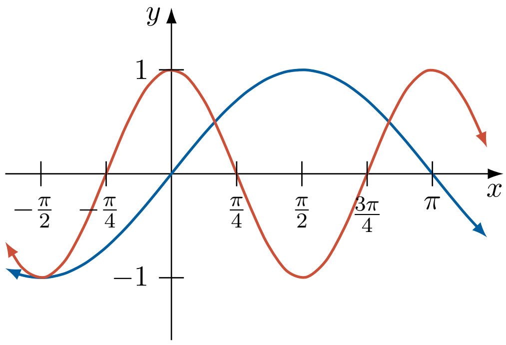
Question 19 The diagram below shows a sketch of $y = \sin x$ and $y = \cos x$.



(a) Show that the two curves intersect when $x = \dfrac{\pi}{4}$ and $x = \dfrac{5\pi}{4}$.

(b) Hence, find the area of the region bounded between the two curves.

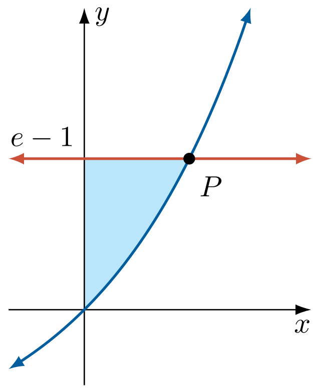
Question 20 The diagram shows the graphs of $y = \sin x$ and $y = \cos 2x$. The curves intersect at $x = -\dfrac{\pi}{2}, \dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$ in this domain.



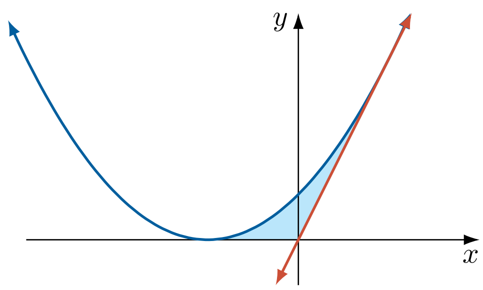
(a) Calculate the area between the curves in the domain $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{6}\right]$.

(b) Calculate the area between the curves in the domain $x \in \left[\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right]$.

Question 21 The diagram below shows a sketch of $y = e^x - 1$. Find the area of the shaded region.
Hint: First, find the area of a rectangle.

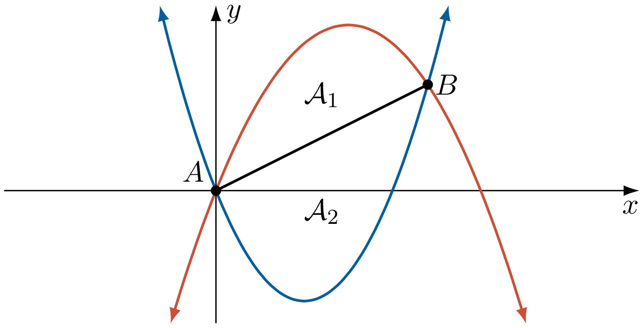


Question 22 Find the area of the region bounded by the x-axis, the curve $f(x) = (2x + 1)^2$ and the tangent at $x = \dfrac{1}{2}$.



Challenge Problems

Problem 1 The diagram below shows a sketch of $y = 6x^2 - 5x$ and $y = 5x - 4x^2$. Let $\mathcal{A}_1$ represent the area of the region bounded by $y = 5x - 4x^2$ and the chord AB. Similarly, let $\mathcal{A}_2$ represent the area of the region between $y = 6x^2 - 5x$ and AB.

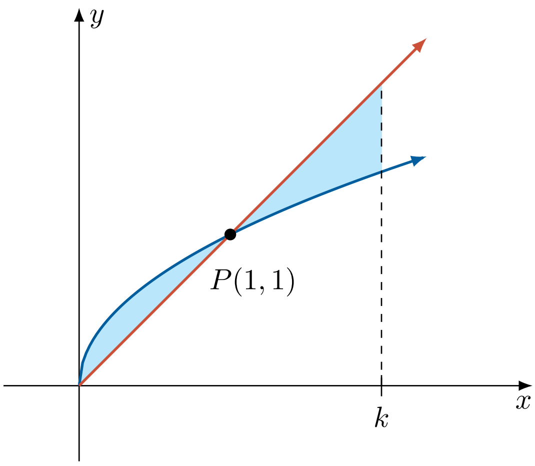


(a) Find the coordinates of B.

(b) Hence, find the equation of the chord AB.

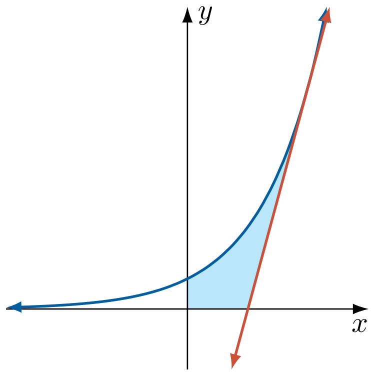
(c) Hence, show that that the ratio of the areas $\mathcal{A}_1 : \mathcal{A}_2$ is $2 : 3$.

Problem 2 The diagram below shows the graph of $y = \sqrt{x}$ and $y = x$. Consider the region bounded by the two curves and $x = k$, where $k > 1$.



Find the value of k such that the area to the left of P is equal to the area to the right of P.

Problem 3 The diagram shows the curve $y = e^x$ and the tangent drawn at $x = 2$.



(a) Find the equation of the tangent to the curve $y = e^x$ at $x = 2$.

(b) Find the area of the shaded region.

Exercise 7D

Trapezoidal rule

Fundamentals

Fundamentals 1

Suppose the domain $x \in [a, b]$ were being divided as described below. The length of the sub-intervals when the domain is split with

(a) 2 function values i.e. ___ sub-interval, is $h =$ _____.

(b) 3 function values i.e. ___ sub-intervals, is $h =$ _____.

(c) 4 function values i.e. ___ sub-intervals, is $h =$ _____.

(d) n function values i.e. ___ sub-intervals, is $h =$ _____.

Fundamentals 2

In the formula for the trapezoidal rule, we have $h =$ _____, where n is the number of function values/sub-intervals (circle one).

Fundamentals 3

Write down the formula for the trapezoidal rule with the followng number of sub-intervals. Let the function values be y_0, y_1, y_2, y_3, $\ldots$, y_n, and let the width be h.

(a) two sub-intervals.

(b) three sub-intervals.

(c) four sub-intervals.

(d) five sub-intervals.

(e) n sub-intervals.

Question 1 [Drill exercise to find the sub-interval width]

Write down the width of each sub-interval in the domain $x \in [1, 5]$, if the number of sub-intervals is

(a) two.

(b) four.

(c) five.

(d) six.

Question 2 [Drill exercise to construct the table of values]

Draw a table that splits the given domain into the given number of sub-intervals.

(a) $x \in [0, 4]$, 4 sub-intervals.

(b) $x \in [1, 3]$, 4 sub-intervals.

(c) $x \in [1, 3]$, 5 sub-intervals.

(d) $x \in [-1, 3]$, 6 sub-intervals.

Question 3 A surveyor measures the width of a garden area at regular increments, and records the data in a table. Use the trapezoidal rule to approximate the surface area of the garden.

(a)

x_k	0	10	20	30
y_k	120	80	90	100

(b)

x_k	0	5	10	15	20
y_k	26	32	48	30	36

(c)

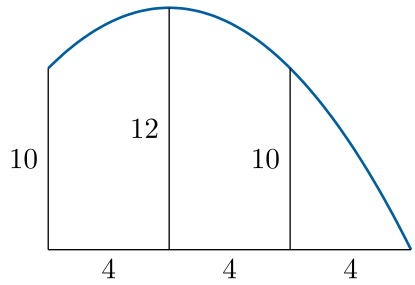
x_k	10	20	30	40	50	60
y_k	15	35	70	45	45	0

(d)

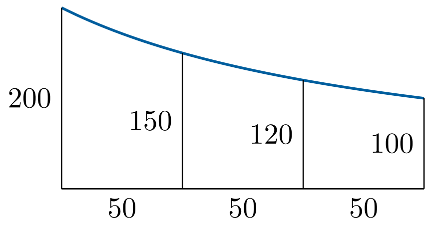
x_k	0	30	60	90	120	150
y_k	10	40	30	20	50	30

Question 4 The following diagrams represent the findings from a surveyor measuring irregularly-shaped area of land. Use the trapezoidal rule to approximate the area of the land areas below.

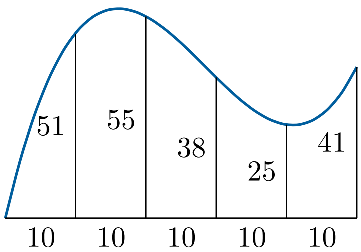
(a)



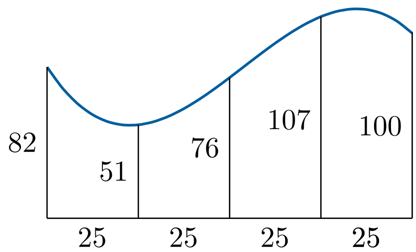
(b)



(c)



(d)



Question 5 Use the trapezoidal rule with three sub-intervals to approximate the following.

(a) $\displaystyle\int_0^3 2^x\,dx$

(b) $\displaystyle\int_1^4 \ln x\,dx$

(c) $\displaystyle\int_1^3 \frac{1}{x^2+1}\,dx$

(d) $\displaystyle\int_0^1 xe^{-x}\,dx$

Question 6 Use the trapezoidal rule with five function values to approximate the following.

(a) $\displaystyle\int_0^1 2^{-x}\,dx$

(b) $\displaystyle\int_1^3 \frac{1}{x}\,dx$

(c) $\displaystyle\int_0^\pi \sin^2 x\,dx$

(d) $\displaystyle\int_0^1 e^{-x^2}\,dx$

Question 7 [Under-estimation and over-estimation]

For each of the following (i) sketch the region. (ii) use the trapezoidal rule with five function values. (iii) determine whether the approximation will over-estimate or under-estimate the integral.

(a) $\displaystyle\int_0^2 \sqrt{x}\,dx$

(b) $\displaystyle\int_1^5 \frac{1}{x}\,dx$

(c) $\displaystyle\int_1^2 \ln x\,dx$

(d) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx$

Question 8 [Increasing the number of sub-intervals refines the approximation]

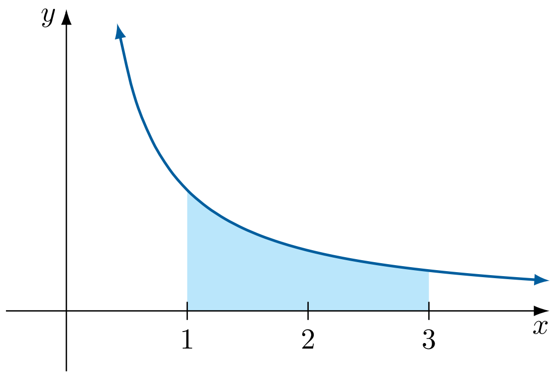
Consider the integral $\int_0^1 x^2\,dx$.

(a) Use the trapezoidal with three sub-intervals to approximate the integral.

(b) Use the trapezoidal with five sub-intervals to approximate the integral.

(c) For each approximation above, calculate the percentage error, correct to two decimal places.

Question 9 The diagram below shows the graph of $y = \dfrac{1}{x}$ in the first quadrant.



(a) Find the exact area of the shaded region.

(b) Use the trapezoidal rule with three function values to show that $\ln 3 < \dfrac{7}{6}$.

(c) Use the trapezoidal rule with five function values to find a more accurate value of p such that $\ln 3 < p$.

Question 10 [True or false questions]

Determine whether the following statements are true or false for a continuous function $f(x)$ in the domain $x \in [a, b]$.

(a) If $f(x)$ is concave up, then the trapezoidal rule will under-estimate $\int_a^b f(x)\,dx$.

(b) As we increase the number of sub-intervals, the trapezoidal rule will yield better approximations of $\int_a^b f(x)\,dx$.

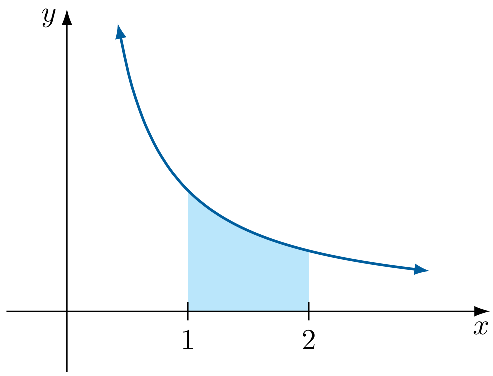
(c) If $f(x) = mx + c$, then the trapezoidal rule will give the exact value of $\int_a^b f(x)\,dx$.

(d) We know that $\int_{-a}^a x^3\,dx = 0$ since $y = x^3$ is an odd function. Using the trapezoidal rule with any number of sub-intervals will also yield zero.

Challenge Problems

Problem 1 [Approximating e]

Consider the graph of $f(x) = \dfrac{1}{x}$ in the domain $x \in [1, 2]$.



(a) Approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$ using six function values.

(b) Find the exact value of $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

(c) Hence, find an approximation for e, correct to two decimal places.

(d) Without using a calculator, determine whether the answer from (a) under-estimates or over-estimates the exact value from (b).

(e) Does this mean that your approximation of e under-estimates or over-estimates the exact value of e?

Problem 2 [Approximating π]

Consider $f(x) = \sqrt{1 - x^2}$.

(a) Approximate $\displaystyle\int_0^1 f(x)\,dx$ using six function values.

(b) Find the exact value of $\displaystyle\int_0^1 f(x)\,dx$.

(c) Hence, find an approximation for π correct to three decimal places.

(d) Use your calculator to find the percentage error from the actual value of π, correct to one decimal place.

(e) Show, without the use of a calculator, how we could have found that this result is an under-estimation of π.

Exercise 7E

Applications involving integration

Fundamentals

Fundamentals 1

Suppose we are given a function that models the rate of a quantity.

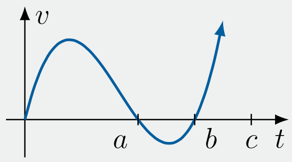
(a) The anti-derivative of that function will give us the q_____ itself.

(b) However, there will always be a c_______ of integration.

(c) To find it, s_______ the given information.

Fundamentals 2

Let $v(t)$ be the velocity-time function of a particle that changes direction at $t = a$ and $t = b$.



The particle is initially at rest at the origin. We wish to find the total distance travelled over the interval $t \in [0, c]$, where $c > b$.

(a) First, find the displacement-time equation by i_______ the $v(t)$ function. Remember to include the c_____ of integration in your answer!

(b) Find the c_____ by substituting the fact that when $t =$ ___, $x =$ ___.

(c) First find the distance travelled in the interval $t \in [0, a]$ by finding $x(a) - x($___$)$.

(d) Then, find the distance travelled in the interval $t \in$ _____ by finding _____.

(e) Then, find the distance travelled in the interval $t \in$ _____ by finding _____. When you have all three distances, a___ them all to find the total distance travelled.

Fundamentals 3

Using a definite integral can be a convenient way of calculating the total change in quantity.

(a) Suppose the rate of a quantity is given by $Q'(t)$. Then the definite integral

$$\int_a^b Q'(t)\,dt$$

gives the n___ change in q_______ over the time interval $t \in$ _____.

(b) When using this method you need/do not need (circle one) the initial quantity to find the total change in quantity.

Question 1 A particle moves in a straight line with velocity given by

$$v = 2t - 4$$

Initially, the particle is at $x = 1$.

(a) Find the displacement-time equation of the particle.

(b) Find the position of the particle when $t = 3$.

(c) Find the times when the particle is at $x = -2$.

Question 2 A particle moves in a straight line with acceleration given by

$$a = -40e^{-2t}$$

The particle is initially at the origin with velocity 20m/s.

(a) Find the velocity-time equation of the particle.

(b) Find the displacement-time equation of the particle.

(c) Find the limiting position of the particle.

(d) Find how long it takes for the particle to reach half-way between it's starting position and the limiting position.

(e) Find the particle's velocity at this moment in time.

Question 3 A particle undergoes straight line motion with velocity

$$v = \frac{6}{\sqrt{3t+4}},$$

where t is time in seconds and distance is in metres.

(a) Find the particle's position x at time t, if initially the particle is at the origin.

(b) Find the position 7 seconds later.

Question 4 The velocity of a body travelling in a straight line is $v = 24 - 16e^{-0.2t}$ m/s

(a) What is the initial velocity?

(b) Find the acceleration of this particle and show that it is always positive.

(c) Find the formula for the displacement, if the initial position is 8m in the positive direction.

Question 5 The acceleration of a particle moving along a straight line is given by

$$a = 12t - 4,$$

where t is time in seconds. Initially the particle is at rest 8 units to the right of the origin. Find when the particle returns to its original position.

Question 6 A particle moves along a horizontal line so that its velocity is given by $v = \cos \pi t$. Initially the particle is at the origin. When will the particle be at the origin again?

Question 7 Find the displacement function of a particle that falls from rest from a height of 50 metres, given that a falling particle has a constant acceleration of about -10 m s^{-2}

Question 8 A large ice cube with an edge length of 30 cm melts in a fridge so that the volume decreases at a constant rate of 18 cm^3/min and the block remains a cube.

(a) Write down an expression for $\dfrac{dV}{dt}$.

(b) Find the volume after t minutes.

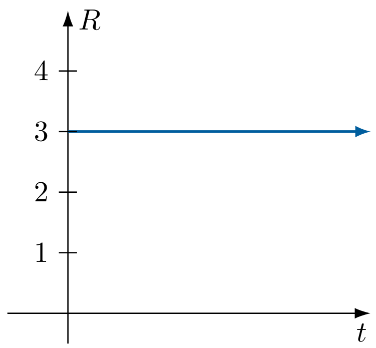
(c) Find the time for the ice cube to melt completely.

Question 9 [Basic example demonstrating the relationship between area and quantity]

Water flows into a container at a constant rate R of 3 litres per second.

(a) How much water has flowed into the container after 15 seconds?

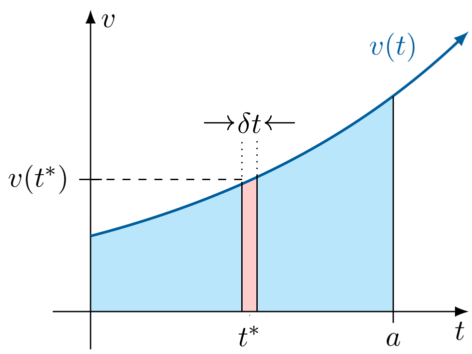
(b) The graph of the rate of flow is shown.



On this graph, illustrate how much water has flowed into the container after 1 minute as an area under the graph.

Question 10 [Developing the theory connecting definite integrals and quantity.]

The diagram below shows the graph of an arbitrary velocity-time function $v(t)$. Consider a small time interval δt, from some time point t^*, as shown in the diagram below.



(a) Explain why the distance travelled in the interval δt is approximately $v(t^*)\delta t$.
Hint: Use the familiar formula Distance = Speed × Time

(b) Write down an approximate area of the shaded region in terms of t^*.
Hint: The shaded region is basically a rectangle.

(c) Hence, what is the connection between the area of the strip and the distance travelled in that short time interval?

(d) Bob claims that therefore if we continue adding similar strips over the domain $t \in [0, a]$, then we will obtain roughly the distance travelled for $t \in [0, a]$. Is he correct?

(e) What do we need to do to δt to make the approximations more and more accurate? What does the process of adding rectangles and making them infinitely small remind you of?

(f) Hence, complete the following statement.

$$\text{“ The distance travelled by the particle for } t \in [0, a] \text{ is } \int_{?}^{?} \_\_\_\_\_\_ \, d? \text{ ”}$$

(g) The above statement is only correct as it currently stands because the $v(t)$ curve is positive in the given domain. What happens if $v(t)$ is below the x-axis?

Question 11 Recall that the integral

$$\int_a^b v(t)\, dt$$

gives the *net change* in quantity in the time interval $t \in [a, b]$. Consider a particle that is initially at $x = 1$.

(a) If $\int_0^2 v(t)\, dt = 3$, then where is the particle after 2 seconds?

(b) If $\int_0^5 v(t)\, dt = 0$, then where is the particle after 5 seconds?

(c) If $\int_0^8 v(t)\, dt = -4$, then where is the particle after 8 seconds?

Question 12 A particle moves along a horizontal line so that its velocity is given by

$$v = 4t - 12,$$

where t is time in seconds and v is velocity in m/s. The particle is initially at the origin.

(a) Find a formula for the displacement of the particle.

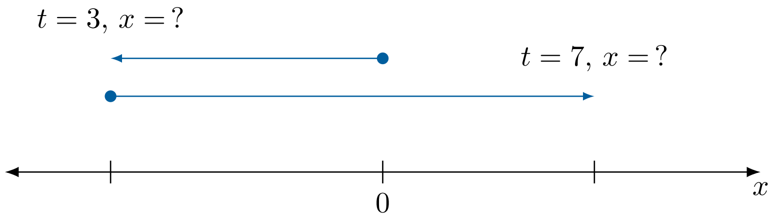
(b) In what direction does the particle move initially?

(c) When and where is the particle at rest?

(d) In what direction does the particle start to move after that?

(e) Where is the particle after 7 seconds?

(f) Complete the following diagram.



(g) Use the diagram to find the total distance travelled by the particle during the first 7 seconds.

(h) Bob calculates $\int_0^3 4t - 12\,dt$ and claims that it represents the distance travelled over $t \in [0, 3]$. Explain why he is not correct, and correct the statement.

(i) Calculate $\int_3^7 4t - 12\,dt$ and give a physical interpretation of your answer. Was this answer expected?

(j) Write down an expression for the distance travelled over $t \in [0, 7]$, using integrals.

Question 13 A particle undergoing straight line motion has velocity $v = 2 - t$, where t is time in seconds and distance is measured in metres.

(a) Find when the particle changes direction.

(b) Hence. find the total distance covered in the first 4 seconds.

Question 14 A particle undergoing straight line motion has velocity $v = t^2 - 4t$, where t is time in seconds and distance is measured in metres. The particle is initially it is at rest.

(a) Find when the particle changes direction.

(b) Hence. find the total distance covered in the first 6 seconds.

Question 15 [Drill to construct the correct definite integral]

Write down an integral that will yield the desired quantities for each of the scenarios below.

(a) A particle has velocity-time equation $v(t)$, which is always positive. Find the distance travelled in the time interval $t \in [0, 4]$.

(b) A particle has velocity-time equation $v(t)$, which is always negative. Find the distance travelled in the time interval $t \in [0, 3]$.

(c) A particle has velocity-time equation $v(t)$, which is always positive. Find the distance travelled in the time interval $t \in [3, 6]$.

(d) A particle has velocity-time equation $v(t)$. The particle moves to the right initially, changes direction at $t = 3$, then moves to the left indefinitely. Find the distance travelled in the time interval $t \in [0, 6]$.

(e) A particle has velocity-time equation $v(t)$. The particle moves to the right initially, changes direction at $t = 3$, then moves to the left indefinitely. Find the net change in displacement in the time interval $t \in [0, 6]$.

(f) A jet engine burns fuel at a rate given by $f(t)$. Find the amount of fuel burned over the time interval $t \in [5, 10]$.

Question 16 A particle undergoing straight line motion has velocity $v = \sqrt{t} - t$ m/s and initially it is at rest.

(a) Find when the particle changes direction.

(b) Find the total distance covered in the first 4 seconds.

Question 17 [Simple harmonic motion]

A particle undergoing straight line motion has acceleration

$$a = -8\cos(2t),$$

and initially it is at rest at $x = 3$. Time is measured in seconds and distance is measured in metres.

(a) Find the velocity-time equation.

(b) Find the displacement-time equation.

(c) Find the first two times the particle changes direction, and the particle's position then.

(d) State the period of motion.

(e) Find the total distance covered in the first π seconds.

Question 18 A particle undergoing straight line motion has velocity

$$v = 1 - t^2,$$

where t is time in seconds and distance is measured in metres.

(a) When does the particle change direction?

(b) Bob calculates $\int_0^2 1 - t^2\, dt$ and Mary calculates $\int_0^1 1 - t^2\, dt - \int_1^2 1 - t^2\, dt$. What is the physical interpretation of Bob and Mary's answers?

Question 19 A particle moves along a horizontal line so that its velocity after t seconds is given by

$$v(t) = (t - 4)(2t - 1),$$

where distance is measured in metres.

(a) When does the particle change direction.

(b) Explain why the distance covered over the first five seconds is given by

$$d = \int_0^{\frac{1}{2}} v(t)\, dt - \int_{\frac{1}{2}}^4 v(t)\, dt + \int_4^5 v(t)\, dt$$

(c) Hence, find the distance covered by the particle during the first 5 seconds.

Question 20 A small pipe pumps out water at a rate given by $\frac{dV}{dt} = 6t - t^2$ for $t \in [0, 6]$, where t is time in minutes and V is in litres.

(a) Write down an expression for the total volume of water that flowed out from the pipe over $t \in [0, 6]$.

(b) Hence, find the volume of water that flowed out from the pipe in total.

Question 21 The rate of flow of water from a tap is given by

$$R(t) = 6\,e^{-0.1t}$$

measured in litres per minute. Find the number of litres, to the nearest litre which flowed out in the first 2 minutes.

Challenge Problems

Problem 1 The valve of a bucket containing honey is opened and 200mL flows out in the first k seconds. It flows out at a rate of

$$\frac{dV}{dt} = 30 - 2t$$

Find the value of k.

Problem 2 The velocity of a particle moving in a straight line after t seconds is

$$v = e^{2t} - 5e^{t},$$

where distance is measured in metres. Find the distance travelled by the particle in the first $\ln(10)$ seconds.

Problem 3 The velocity of a particle moving in a straight line after t seconds is

$$v = t^2 - 6t + 8,$$

where distance is measured in metres. Find the total distance travelled by the particle in the first 6 seconds.

Chapter 7 Review

Applications of integration

Review

Question 1

(a) Sketch the graph of $y = x^2 - 2x$ and shade the region bounded by the curve and the x-axis, in the domain $x \in [0, 3]$.

(b) Explain, with the aid of your sketch, why the shaded area is not just $\int_0^3 x^2 - 2x \, dx$.

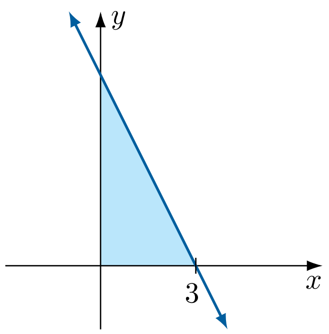
(c) Find the values of a, b and c so that the expression below will yield the correct area.

$$\text{Area} = -\int_a^b x^2 - 2x \, dx + \int_c^3 x^2 - 2x \, dx$$

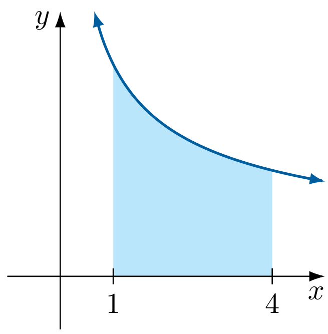
(d) Hence, calculate the area.

Question 2 Calculate the area of the shaded regions, given the equation of the curve.

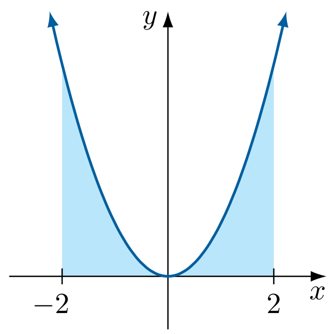
(a) $y = 6 - 2x$



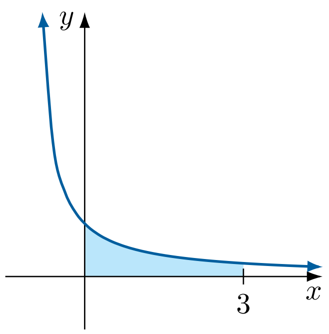
(b) $y = \dfrac{4}{\sqrt{x}}$



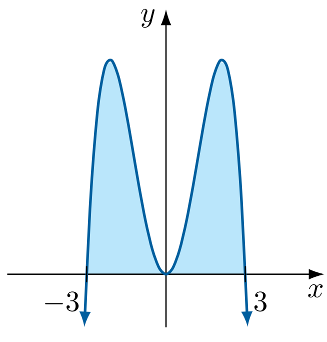
(c) $y = x^2$



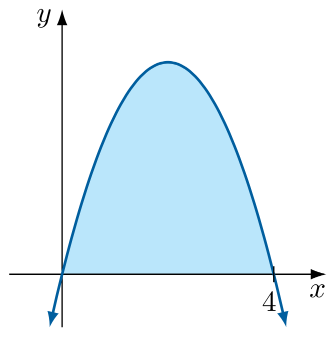
(d) $y = \dfrac{1}{x+1}$



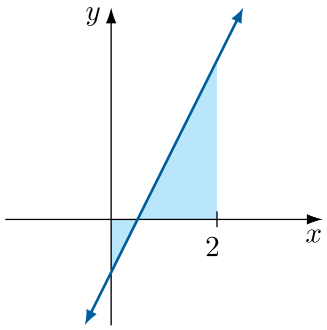
(e) $y = 9x^2 - x^4$



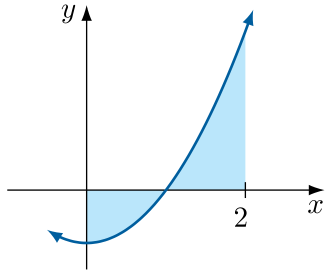
(f) $y = 4x - x^2$



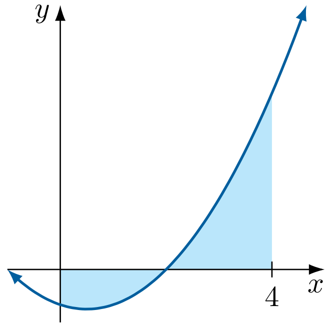
(g) $y = 2x - 1$



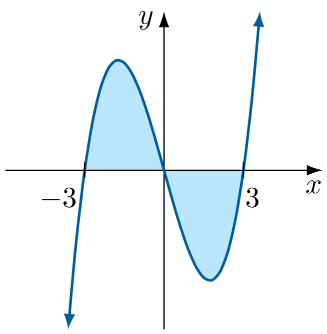
(h) $y = x^2 - 1$



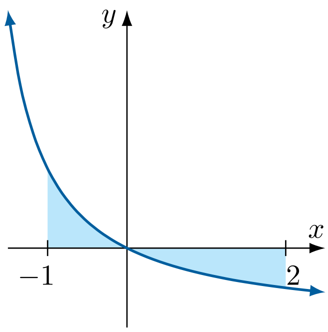
(i) $y = x^2 - x - 2$



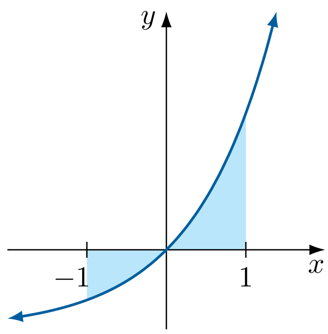
(j) $y = x^3 - 9x$



(k) $y = \dfrac{2}{x+2} - 1$



(l) $y = e^x - 1$



Question 3 By first drawing a sketch, find the area of the region enclosed by the following curves and the x-axis.

(a) $y = 1 - x^2$ (b) $y = x^2 - 9$ (c) $y = -x^2 - 6x - 5$

(d) $y = x^2 - 4x + 3$ (e) $y = x^3 - x$ (f) $y = 4x - x^3$

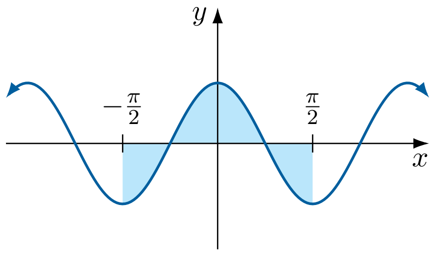
Question 4 By first drawing a sketch, find the area of the region enclosed by the following curves and the coordinate axes.

(a) $y = 4 - 2x$ (b) $y = \sqrt{1 - x}$ (c) $y = 1 - x^3$

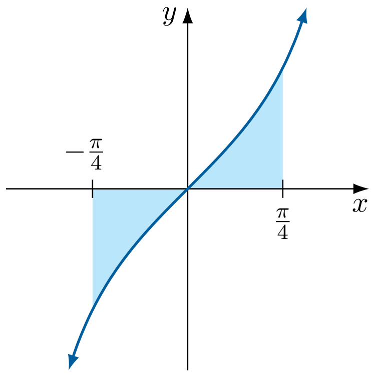
(d) $y = e^x - e$ (e) $y = 1 - \tan x$ (f) $y = 2 - \dfrac{1}{x+1}$

Question 5 Find the area of the shaded region.

(a) $y = \cos(2x)$



(b) $y = \tan(x)$



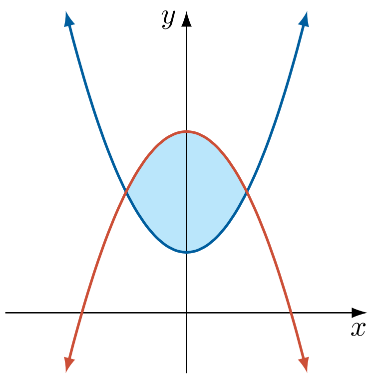
Question 6 Calculate the following.

(a) $\displaystyle\int_0^4 \sqrt{16 - x^2}\,dx$

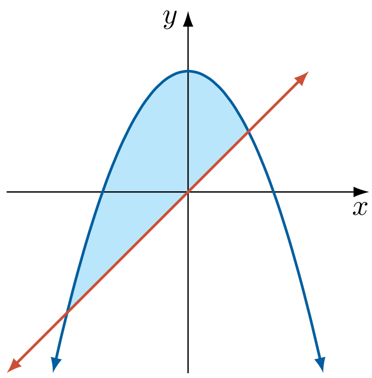
(b) $\displaystyle\int_{-2}^2 \sqrt{4 - x^2}\,dx$

Question 7 Calculate the area of the region enclosed by the following curves.

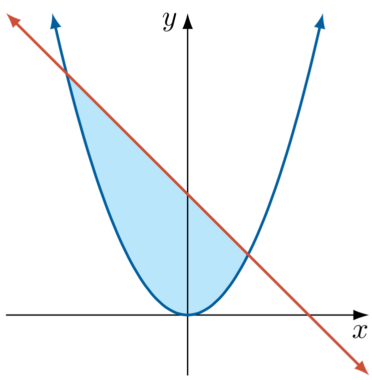
(a) $y = x^2 + 1$ and $y = 3 - x^2$.



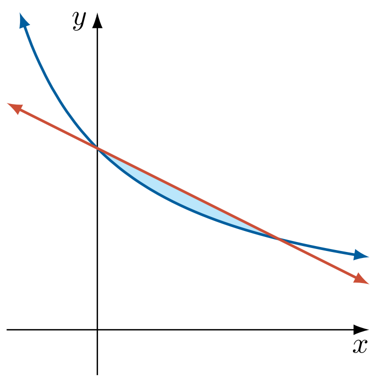
(b) $y = 2 - x^2$ and $y = x$.



(c) $y = x^2$ and $y = 2 - x$.

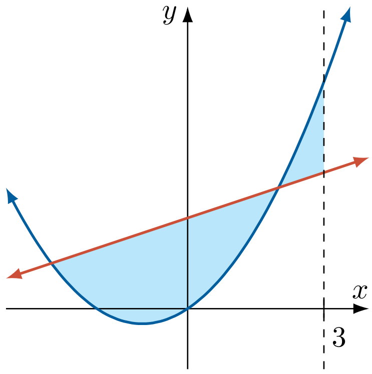


(d) $y = \dfrac{4}{x+2}$ and $y = 2 - \dfrac{x}{2}$.

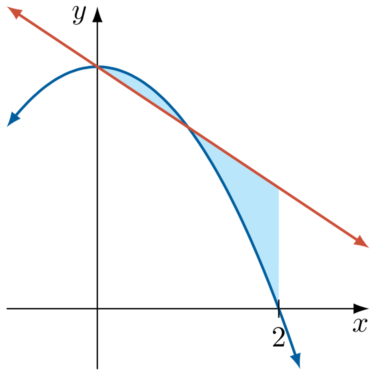


Question 8 Calculate the area of the shaded regions below.

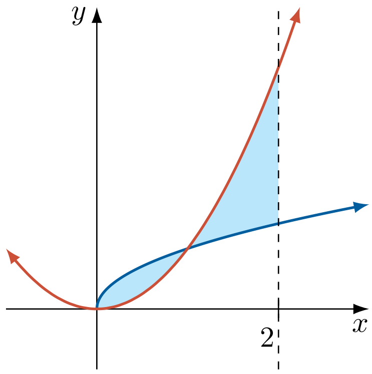
(a) $y = x^2 + 2x$ and $y = x + 6$.



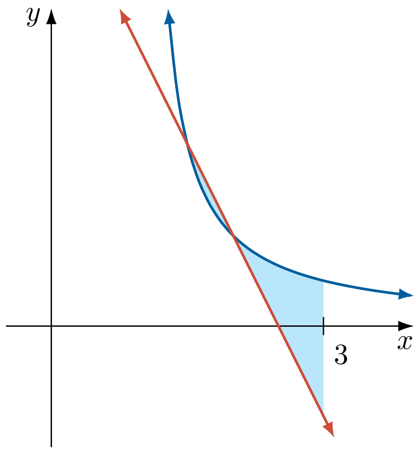
(b) $y = 4 - x^2$ and $y = 4 - x$.



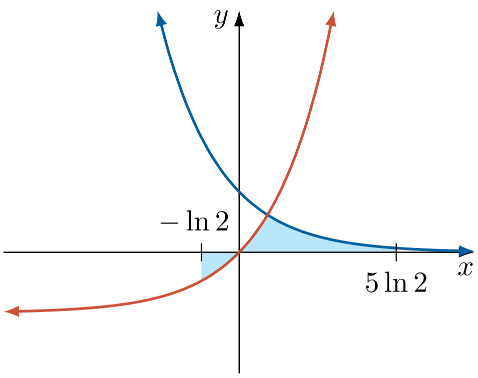
(c) $y = \sqrt{x}$ and $y = x^2$.



(d) $y = \dfrac{1}{x-1}$ and $y = 5 - 2x$.



Question 9 The diagram below shows the graph of $y = 2e^{-x}$ and $y = e^x - 1$.



Find the area of the shaded region.

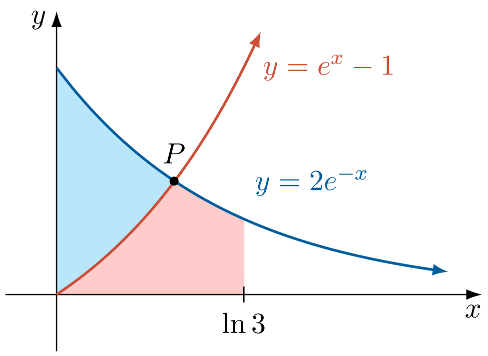
Question 10

(a) Sketch the graphs of $y = 1 - x^2$ and $y = x - x^3$ on the same set of axes.

(b) Find the area of the region enclosed by the two curves.

(c) Verify that the answer above is the same as $2\int_0^1 1 - x^2\,dx$. Explain why this is the case.

Question 11 The diagram below shows the graph of $y = 2e^{-x}$ and $y = e^x - 1$.

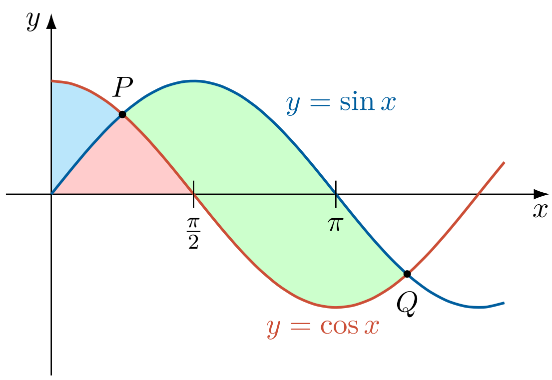


(a) Find the x-coordinate of P.

(b) Find the area of the blue region.

(c) Find the area of the red region.

Question 12 The diagram below shows a sketch of $y = \sin x$ and $y = \cos x$. Let P and Q be the first two intersection points where $x > 0$.



(a) Find the x-coordinates of P and Q.

(b) Find the area of the blue region.

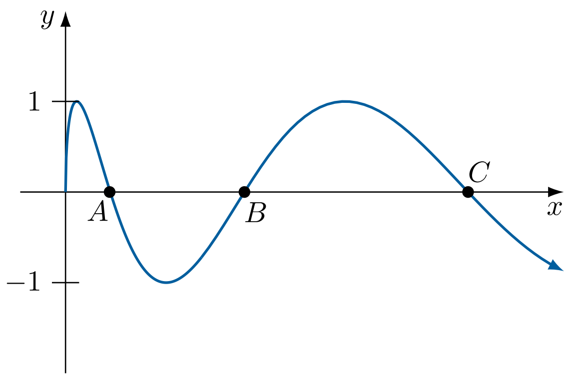
(c) Find the area of the red region.

(d) Find the area of the green region.

Question 13 Consider $y = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x}$

(a) Find $\dfrac{dy}{dx}$, and hence find $\displaystyle\int \sin\sqrt{x}\,dx$.

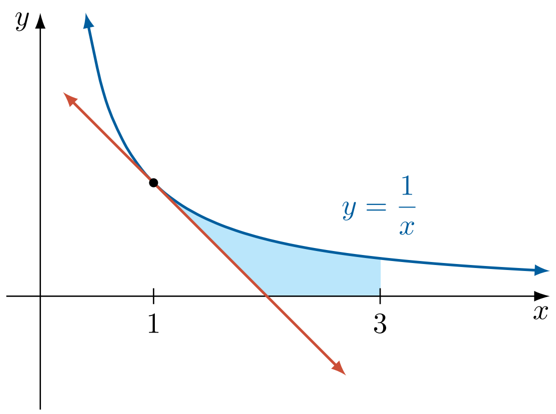
(b) The graph of $y = \sin\sqrt{x}$ is drawn below. Find the x-coordinates of A, B and C.



(c) Find the area of the region bounded by the curve and the x-axis in the three regions indicated.

(d) Find the ratio of the areas of these three regions.

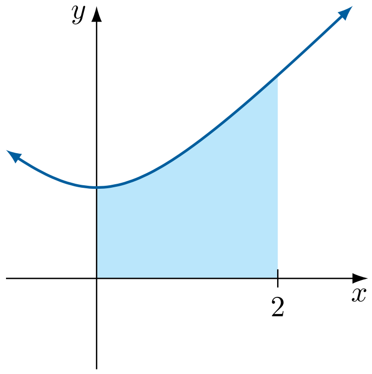
Question 14 The diagram below shows a sketch of $y = \dfrac{1}{x}$ and the tangent when $x = 1$.



(a) Find the equation of the tangent.

(b) Find the area of the shaded region.

Question 15 The diagram below shows the graph of $y = \sqrt{x^2 + 1}$.



(a) Use the trapezoidal rule with four sub-intervals to find an approximation for the shaded area.

(b) Does this approximation over-estimate or under-estimate the actual area?

Question 16 Use the trapezoidal rule with four function values to approximate the following integrals.

(a) $\displaystyle\int_0^2 \frac{1}{x^2+2}\,dx$ (b) $\displaystyle\int_0^\pi \cos^2 x\,dx$ (c) $\displaystyle\int_1^2 \ln x\,dx$

Question 17

(a) Use the trapezoidal rule with three function values to approximate $\displaystyle\int_0^4 \sqrt{16 - x^2}\,dx$.

(b) Use the trapezoidal rule with five function values to approximate $\displaystyle\int_0^4 \sqrt{16 - x^2}\,dx$.

(c) Find the exact value of $\displaystyle\int_0^4 \sqrt{16 - x^2}\,dx$.

(d) Calculate the percentage error using three and five function values.

Question 18 A cube of ice has an edge length of 15 cm. It melts so that the volume decreases at a constant rate of 20 cm^3 per minute. Find the

(a) volume V after t seconds.

(b) time to completely melt the cube, correct to the nearest minute.

Question 19 A particle undergoing straight line motion has velocity $v = \dfrac{4t}{(t^2+1)^2}$ after t seconds, where distance is measured in metres.

(a) What is the initial velocity?

(b) Describe the velocity as $t \to \infty$.

(c) Find the position of the particle when $t = 2$ if initially the particle is at the origin.

(d) Describe what happens to the particle as $t \to \infty$

Question 20 A particle moves along a horizontal line so that its velocity is given by $v = 6 - 2t$, where t is time in seconds. Initially the particle is 5 units to the left of the origin.

(a) In what direction does the particle start to move?

(b) Find the displacement-time equation of the particle.

(c) Where is the particle 3 seconds later?

(d) When is the particle at the origin?

(e) Find the distance travelled in the first 5 seconds

(f) Explain why $\int_0^5 6 - 2t\,dt$ will not give the distance travelled in the first 5 seconds.

Question 21 A car starting initially from rest, moves in a straight line with acceleration given by $a = -\frac{t}{5}$, where distance is measured in metres and time is in seconds. How far does the car move in the first 6 seconds?

Question 22 The acceleration of a particle moving in a straight line is given by

$$\ddot{x} = 4\cos(2t),$$

where the particle is initially at rest 3 units to the right of the origin. Find the velocity-time and displacement-time equation of the particle.

Question 23 A particle undergoing straight line motion has velocity $v = t^2 - 2t$, where t is time in seconds and distance is measured in metres. The particle is initially it is at rest.

(a) Find when the particle changes direction.

(b) Hence. find the total distance covered in the first 4 seconds.

Question 24 A particle undergoing straight line motion has velocity $v = e^{1-t} - 1$, where t is time in seconds and distance is measured in metres. The particle is initially it is at rest.

(a) Find when the particle changes direction.

(b) Hence. find the total distance covered in the first 2 second of motion.

Question 25 A particle undergoing straight line motion has velocity

$$v = \frac{t}{2} - \frac{1}{t+1},$$

where t is time in seconds and distance is measured in metres. The particle is initially at $x = \ln 3$. Where is the particle after 2 seconds?

Question 26 The velocity of a particle over the time interval $t \in [0, 6]$ is given by the following functions, measured in metres per second. Find the distance travelled in the first 3 seconds.

(a) $v = 2t - 12$

(b) $v = 12 \sin \dfrac{\pi t}{6}$

(c) $v = 6(1 - e^{-t})$

Investigation Task

Fundamental Theorem of Calculus

Upon further thought, a student can quickly see that what we are doing currently to find the area underneath a curve seems completely arbitrary. We are doing the 'opposite' of differentiation, then we substitute in two numbers, and somehow that gives the exact area under a curve. But as it currently stands, there doesn't seem to be any reason behind this. The logical leap between area under a curve and anti-differentiation is not at all obvious. This investigation task gives the student an opportunity to explore this further so they can fully understand the relationship between these two seemingly unrelated concepts.

The task is to write an article that answers, in full detail, the following question.

"How does doing the opposite of differentiation and substituting some numbers give the area underneath a curve?"

Your article should

- include plenty of diagrams.
- have a detailed discussion of the Fundamental Theorem of Calculus.
- have any necessary proofs and working.

Investigation Task

Simpson's Rule

In this course, you have learned the Trapezoidal Rule, which is used to approximate integrals using trapezia. However, this is most certainly not the only method of numerical integration. Another well-known method is called *Simpson's Rule*. There are many forms of Simpson's Rule, but the one we will focus on is called 'Simpsons 1/3 Rule'.

Question 1 State the formula for Simpson's Rule.

Question 2 Derive fully the formula for Simpson's Rule using integration.

Question 3 Use Simpson's Rule to approximate the following integrals, using four sub-intervals.

(a) $\displaystyle\int_0^2 x^2\,dx$

(b) $\displaystyle\int_0^2 2^x\,dx$

(c) $\displaystyle\int_1^2 \frac{1}{x}\,dx$

(d) $\displaystyle\int_1^3 \frac{1}{1+x^2}\,dx$

Question 4 What would you expect to happen if you used Simpson's Rule to approximate the area under a quadratic curve?

Question 5 A well-known 'feature' of Simpson's Rule is that it gives the exact answer for cubic functions. In half a page, show why and how this is true.

Question 6 Compare Simpson's Rule with the Trapezoidal Rule. Which one do you think yields better approximations for the same number of sub-intervals? Discuss scenarios where Simpson's Rule may be better, and scenarios where the Trapezoidal Rule may be better.

Investigation Task

Divergence of the Harmonic Series

The *Harmonic Series* is a classic series studied in mathematics that is known for yielding a number of counter-intuitive results. The partial sum is given by the expression.

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

The Harmonic Series is this same series where infinitely many terms are added together i.e. H_∞. One of the natural questions that arises from this is whether the sum converges i.e. tends to a limit, or diverges i.e. goes to infinity. We cannot use the techniques from our study of series in the Mathematics Advanced course, so we need other techniques.
This investigation task will show the student how the techniques from this chapter can be applied to produce some basic results about the Harmonic Series.

Question 1 Research and explain what *upper bound rectangles* and *lower bound rectangles* are. Your answer should contain a number of diagrams the upper and lower rectangles for basic curves like $y = \sqrt{x}$.

Question 2 Draw the curve $y = \frac{1}{x}$ for $x > 0$. In your answer, draw upper and lower-bound rectangles at $x = 1, 2, 3, \ldots, n$.

(a) Show that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} > \ln n$$

(b) Hence, show that $H_n > \ln n + \frac{1}{n}$

(c) Deduce that the Harmonic Series diverges i.e. goes to infinity as n gets large.

(d) Show similarly that $H_n < 1 + \ln n$

(e) Deduce that the value of H_n and $\ln n$ for any value of n never differs by more than 1.

Question 3 Research what the *Euler-Mascheroni Constant* is, and it's relationship to the Harmonic Series and in particular, part (e) of **Question 1**. Your answer should include a diagram that is fundamental to the discussion of the constant.

Investigation Task

Volume of a Solid of Revolution

In this chapter we studied finding the area underneath a curve. Imagine if this region were instead rotated about the x-axis. Of course, this forms a solid. Similarly to how we found areas, we can use integration to find the volume of such solids.

The task is to write an article that will teach a Year 12 Mathematics Advanced student how to find the volume of a solid of revolution when regions are rotated about the x-axis only. Your answer should include

- the main formula
- the derivation of the formula
- a section on how to find the volume when the region rotated is bounded by two curves $f(x)$ and $g(x)$
- a section on how to find the volume involving composite areas
- at least four worked examples

8

CONTINUOUS RANDOM VARIABLES

- Probability density functions
- Cumulative distribution functions
- Normal distributions
- Empirical rules
- The z-score

Exercise 8A

Probability density functions

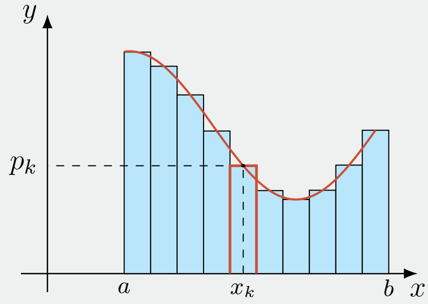
Fundamentals

Fundamentals 1

(a) A random variable X that can take on the value of any real number is called a c________ random variable. Often, it is a quantity that can be m________ like height or time.

(b) With a c________ random variable, we cannot strictly construct a frequency h________ because we do not have discrete categories to place outcomes into. Instead, we create intervals of X and the measured values are placed into the categories.

(c) As we shrink the intervals, the frequency histogram forms a curve.



This curve $f(x)$ is called the p______ d______ function.

(d) The a____ under the curve $f(x)$ in a given domain of X gives the p________ of the corresponding outcomes.

(e) Hence $P(a \leq X \leq b) = \int_{?}^{?} f(x)\,dx$

(f) The mode of a random variable X is the value of X below the h________ point on a probability density function.

Fundamentals 2

A probability density function is a function such that

(a) $f(x)$ ___ 0

(b) $\int_{-\infty}^{\infty} f(x)\,dx =$ ___

Fundamentals 3

If the probability density function $f(x)$ is only non-zero for $x \in [a, b]$, then $\int_{a}^{b} f(x)\,dx =$ ___.

Question 1 [Finite domain]

Show that the following represent valid probability density functions.

(a) $f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{3}{16}x^2, & -2 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{2}{(x+1)^2}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} 2\left(1 - \dfrac{1}{x^2}\right), & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

Question 2 [Infinite domain]

Show that the following represent valid probability density functions.

(a) $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{8}{x^3}, & x \geq 2 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{1}{(x-1)^2}, & x \geq 2 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} \dfrac{1}{10}e^{-0.1x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Question 3 Explain why each of the following are not valid probability density functions.

(a) $f(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & 1 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{1}{(2x-1)^2}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{1}{4}(3x^2 - 2x), & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} \dfrac{3}{16}(x^2 - 2x), & 0 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$

Question 4 Find the value of k so that the following represent valid probability density functions.

(a) $f(x) = \begin{cases} kx, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} k(x - x^2), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{k}{x^2}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} ke^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Question 5

(a) Find $P(1 \leq X \leq 2)$.

$$f(x) = \begin{cases} \frac{2}{27}(x + x^2), & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

(b) Find $P(0 \leq X \leq 2)$.

$$f(x) = \begin{cases} \frac{2}{(2x+1)^2}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) Find $P(X \geq 20)$.

$$f(x) = \begin{cases} \frac{1}{20}e^{-\frac{x}{20}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(d) Find $P(X \leq -2)$.

$$f(x) = \begin{cases} \frac{6}{(2-3x)^2}, & x \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

Question 6

Draw a sketch of the following probability density functions.

(a) $f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} \frac{3}{2}x^2, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} 6x - 6x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Question 7

Find the mode of the following probability density functions.

(a) $f(x) = \begin{cases} \frac{2}{3}(x+1), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} 6x - 6x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} \ln(x), & 1 \leq x \leq e \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} 12x^2 - 12x^3, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Question 8

(a) Calculate $\int_0^{\frac{3\pi}{2}} \sin x \, dx$.

(b) Explain why $f(x) = \sin x$ is not a valid probability density function for $x \in \left[0, \frac{3\pi}{2}\right]$.

Question 9 [Conditional probability]

Consider the PDF given by

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $P(X \geq 3)$.

(b) Find $P(X \geq 2)$.

(c) Write down the formula for $P(A \mid B)$.

(d) Hence, find $P(X \geq 3 \mid X \geq 2)$.

Question 10 Consider the probability density function

$$f(x) = \begin{cases} k(1-x^2), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Find $P(X \le 0.5)$.

(c) Sketch the region from (b).

(d) Find $P(X \ge 0.5)$

(e) Find $P(X \ge 0.5)$, given that $P(X \ge 0.2)$.

(f) Find $P(X \ge 0.5 \,|\, X \ge 0.4)$.

Question 11 Define $f(x) = \dfrac{1}{x}$.

(a) Find the value of k so that $f(x)$ is a valid probability density function for $x \in [1, k]$.

(b) Find $P(a \le X \le 2a)$ for $a \ge 1$.

Question 12 Consider the probability density function $f(x) = kx^4$ for $0 \le x \le p$. Find the value of k and p if $P\left(X \le \dfrac{3}{2}\right) = \dfrac{1}{32}$.

Question 13 [Why we need the zero part of the function]

Consider the probability distribution function

$$f(x) = \begin{cases} k(2x-x^2), & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Bob wishes to find $P(-1 \le X \le 1)$, but Mary tells him that he may as well find $P(0 \le X \le 1)$. Briefly explain why this is the case.

(c) Similarly, Bob wishes to find $P(X \ge 1)$ but Mary tells him that it's just the same as his answer from (b). Briefly explain why.

(d) Write down the values of $P(X \ge 1)$ and $P(X \le 1)$.

Question 14 Consider the probability distribution function

$$f(x) = \begin{cases} k + x, & -1 \le x \le 0 \\ k - x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $k > 0$.

(a) Find the value of k.

(b) Sketch the PDF.

(c) Hence, or otherwise, shade the region corresponding to $P(-0.5 \le X \le 0.5)$, and find it.

Question 15 The reaction speed T, in seconds, of a sample of athletes is modelled by the the probability density function

$$f(t) = \begin{cases} \frac{3}{2}(1-t^2), & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the proportion of athletes who react within one second.

(b) Olympic-level athletes generally have reaction speeds within 0.3 seconds. What proportion of the sample would be considered Olympic-level?

(c) Semi-professional athletes generally have reaction speeds in the interval $t \in [0.3, 0.5]$. What proportion of the sample would be considered semi-professional?

(d) What proportion would be considered neither Olympic-level nor semi-professional?

(e) What is the probability that a randomly chosen athlete is Olympic-level, given that they are at least semi-professional?

Question 16 The continuous piece-wise defined function below forms a PDF.

$$f(x) = \begin{cases} x, & 0 \leq x \leq ? \\ k - 2x, & ? \leq x \leq \frac{k}{2} \\ 0, & \text{otherwise} \end{cases}$$

(a) Fill in the blanks.

(b) Hence, find the value of k.

Challenge Problems

Problem 1 [Mean of a distribution]

The expected value μ of a probability density function is given by

$$\int_a^b x f(x)\, dx$$

Find the expected value of the following distributions.

(a) $f(x) = \begin{cases} \frac{4}{\pi(1+x^2)}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} 2e^{-x^2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Problem 2 Let $f(x) = \frac{2}{x^2}$ be a probability density function. Find the domain of $f(x)$ if the lower bound of X is

(a) 1 (b) 2

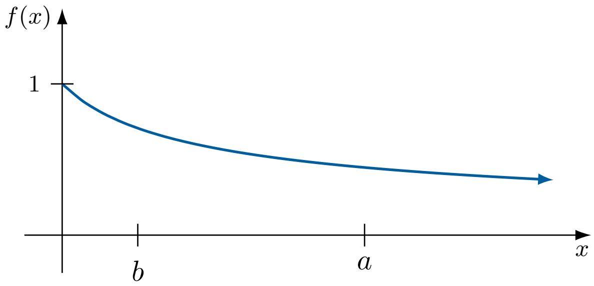
Problem 3 Show that

$$f(x) = \begin{cases} ke^{-kx}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

will always form a PDF regardless of the value of k.

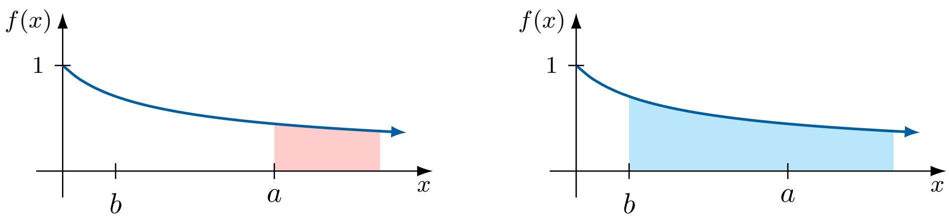
Problem 4 [Proving the general result for conditional probability in continuous PDFs]

Let $f(x)$ be a probability density function in some domain containing $x = a$ and $x = b$, where $a > b$.



(a) State the general formula for $P(A \mid B)$.

(b) Consider the events $P(X \geq a)$ and $P(X \geq b)$.



Use the diagrams above to help you explain briefly why

$$P\left((X \geq a) \cap (X \geq b)\right) = P(X \geq a)$$

(c) Deduce that

$$P(X \geq a \mid X \geq b) = \frac{P(X \geq a)}{P(X \geq b)}$$

Exercise 8B

Cumulative distribution functions

Fundamentals

Fundamentals 1

(a) The cumulative distribution function of a PDF $f(x)$ is a function that 'counts' the cumulative total a____ starting from the lowest possible value of x.

(b) As a result, the CDF is a non-d_____ function.

(c) The y-coordinate $F(t)$ of the CDF gives the probability that the random variable X has a value less than ___.

(d) Hence, if X is unbounded from above, then the CDF $F(x)$ will have a h_____ asymptote.

Fundamentals 2

(a) The formula for the CDF is

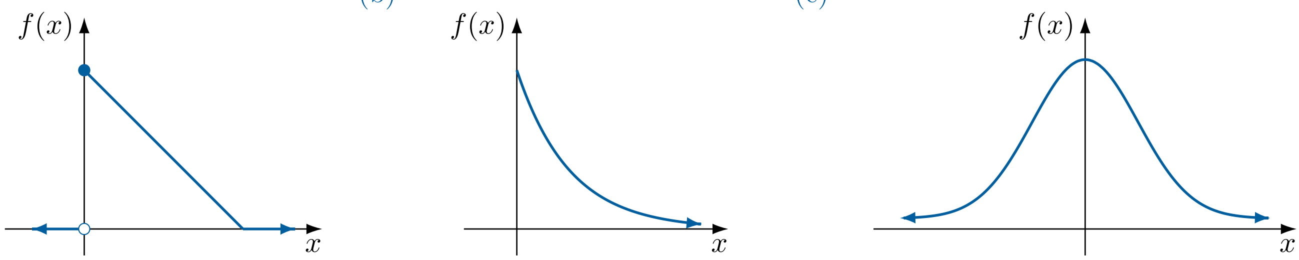
$$F(x) = \int_{?}^{?} f(x)\,dx$$

where $f(x)$ is the equation of a PDF.

(b) To find the median, let $F(x) =$ ___.

Question 1 The following diagrams show the graphs of a PDF. Sketch a rough outline of the corresponding CDF by analysing how the area under the PDF changes.

(a) (b) (c)



Question 2 Consider the probability density function given by

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that the cumulative distribution function is $F(x) = \dfrac{x^2}{4}$ and state the domain of x.

(b) What is the value of the CDF at the median?

(c) Hence, find the median value of X.

Question 3 Consider the probability density function given by

$$f(x) = \begin{cases} \dfrac{k}{x}, & 1 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Find the CDF.

(c) Hence, find the median of X.

Question 4 The lifetime of a light bulb, in years, from a factory is a random variable X with probability density function

$$f(x) = \begin{cases} \dfrac{1}{x^2}, & x \ge 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that the cumulative distribution function is $F(x) = 1 - \dfrac{1}{x}$.

(b) What is the value of the CDF that corresponds to the 80^{th} percentile?

(c) After t years, 80% of light bulbs from the factory are dead. Find the value of t.

(d) Sketch the PDF and shade the corresponding region to your findings in (c).

(e) Find the median lifetime of light bulbs from the factory.

Question 5 Iron ore is mined from a particular region and the proportion of it being usable, called the yield, is given by the random variable with probability density function

$$f(x) = \begin{cases} 6x - 6x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

The ore is considered low quality if the yield is less than 40%.

(a) Find the CDF.

(b) Hence, find the proportion of the iron ore from the region that is considered low quality.

Question 6 A device measures the time period T seconds between detections of particles from a radioactive substance, and models it with the probability density function

$$f(t) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the CDF.

(b) Find the median number of seconds between the detections of particles.

(c) Find $P(1 \leq T \leq 2)$.

Question 7 Find the cumulative distribution function $F(x)$ for the following probability density functions $f(x)$ and hence, find the median m.

(a) $f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} \frac{3}{2}x^2, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} \frac{2}{(x+1)^2}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(e) $f(x) = \begin{cases} \frac{1}{10}e^{-\frac{x}{10}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(f) $f(x) = \begin{cases} \frac{8}{x^3}, & x \geq 2 \\ 0, & \text{otherwise} \end{cases}$

Question 8 Consider the CDF given by

$$F(x) = \begin{cases} 1 - e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch the graph of $y = F(x)$.

(b) Find the median.

(c) Find $P(X \leq 3)$

(d) Find $P(X \leq 2)$

(e) Find $P(2 \leq X \leq 3)$

(f) Find $P(X \geq 4)$

Question 9 [Developing conceptual understanding about the CDF]

Explain the significance of the following about the cumulative distribution function $F(x)$.

(a) The highest point of $F(x)$.

(b) The lowest point of $F(x)$.

(c) The point where $F(x)$ intersects $y = 0.3$.

(d) There is a larger change in y-coordinate for $x \in [a, b]$ compared to $x \in [c, d]$.

Question 10 [True/False questions]

Determine whether the following statements are true or false about the cumulative distribution function.

(a) The CDF must always be non-negative.

(b) The CDF cannot be stationary i.e. it must be strictly increasing.

(c) The CDF must have a horizontal asymptote $y = 1$.

(d) The value of the CDF at $x = t$ corresponds to the area of the region under the PDF up to and including $x = t$.

(e) If the PDF has a horizontal asymptote, then the CDF has horizontal asymptote $y = 1$.

(f) The 10^{th} percentile of a sample space is found by letting $y = 0.1$.

Question 11 [Not your typical median problem]

Consider the probability density function

$$f(x) = \begin{cases} k(2-x), & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

of the random variable X.

(a) Find the value of k.

(b) Find the CDF of $f(x)$.

(c) Find the median of X.

Challenge Problems

Problem 1 [Link to the Fundamental Theorem of Calculus]

Let $F(x)$ be the cumulative distribution function of the probability density function $f(x)$. Explain why $F'(x) = f(x)$.

Problem 2 Let $F(x)$ be the cumulative distribution function of the probability density function $f(x)$. Show that

$$P(a \le x \le b) = F(b) - F(a)$$

Problem 3 [Conditional probability]

Let $F(x)$ be the cumulative distribution function of the probability density function $f(x)$ defined in the domain $x \ge 0$. Explain why if $a > b$, then

$$P(X \ge a \mid X \ge b) = \frac{F(a)}{F(b)}$$

Exercise 8C

Normal distributions

Fundamentals

Fundamentals 1

(a) There are many types of probability distribution functions. One of the most common ones is called the n______ distribution function.

(b) The defining feature of the graph of it is the b____-shaped curve.

(c) The equation of the standard normal distribution curve is $f(x) =$ ____________.

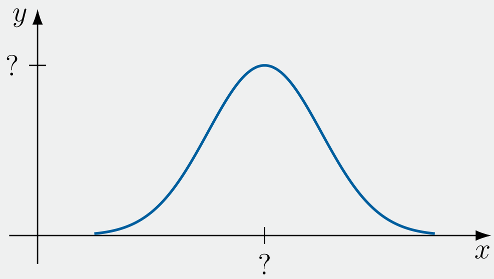
Fundamentals 2

A random variable X that is normally distributed with mean μ and standard deviation σ is represented by writing down

$$X \sim \mathcal{N}(\_\_, \_\_)$$

Fundamentals 3

Complete the following diagram of $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$



Question 1 Write down the equation of the normal distribution function that satisfies the following.

(a) $\mu = 0$, $\sigma = 2$

(b) $\mu = 0$, $\sigma = \dfrac{1}{\sqrt{2}}$

(c) $\mu = 1$, $\sigma = 1$

(d) $\mu = 1$, $\sigma = \sqrt{2\pi}$

(e) $\mu = -2$, $\sigma = \dfrac{1}{2}$

(f) $\mu = -2$, $\sigma = \dfrac{1}{\sqrt{2\pi}}$

Question 2 Draw a sketch of the following normal probability density functions, and state the value of μ and σ.

(a) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}$

(b) $f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{2}\right)^2}$

(c) $f(x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+1}{5}\right)^2}$

(d) $f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(2x)^2}$

Question 3 Let $f(x)$ be the standard normal distribution curve centred at $x = 0$.

(a) Describe the effect of changing μ on the graph of $f(x)$.

(b) Describe the effect of changing σ on the graph of $f(x)$.

Question 4 [CDF of a normally distributed random variable]

The cumulative distribution function of the standard normal distribution is denoted by $\Phi(x)$. Write down the formula for $\Phi(x)$, expressed as an integral.

Question 5 Let $f(x)$ be the PDF of a standard normal distribution function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

(a) Show that $f'(x) = -xf(x)$.

(b) Show that $f''(x) = -f(x) - xf'(x)$.

(c) Deduce that $f''(x) = (x^2 - 1)f(x)$.

(d) Hence, show that the points of inflection are exactly one standard deviation away from the mean.

Question 6 [Significance of the points of inflection]

Show that the points of inflection of the general normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

occur precisely when x is one standard deviation away from the mean μ.

Challenge Problems

Problem 1 [General quadratic exponent]

Consider the probability density function

$$f(x) = e^{ax^2+bx+c}$$

that models a normally distributed random variable X.

(a) Find the value of a, b and c if $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

(b) Find μ and σ^2 of $f(x)$.

Problem 2 [Mean of the normal distribution.]

Recall that expected value μ of a probability density function is given by

$$\int_a^b x f(x)\, dx$$

Show that $\mu = 0$.

Problem 3 [Connection between the CDF and PDF of the normal distribution]

Consider a normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ for any choice of μ. Let $F(x)$ be the cumulative distribution function of $f(x)$.

(a) Draw a rough sketch of $f(x)$ for your choice of μ.

(b) Draw a rough sketch of $F(x)$.

(c) Hypothesise the relationship between the location of the peak of $f(x)$ and the point of inflection of $F(x)$.

(d) Show that $F''(\mu) = 0$.

(e) Hence, describe the relationship between the location of the peak of $f(x)$ and the point of inflection of $F(x)$.

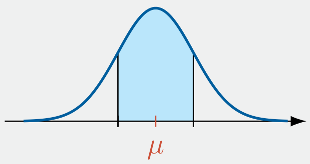
Exercise 8D

Empirical rules

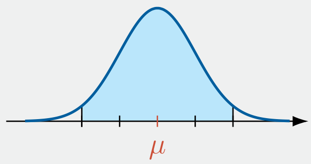
Fundamentals

Fundamentals 1
Consider the graph of the PDF of $X \sim \mathcal{N}(\mu, \sigma^2)$.

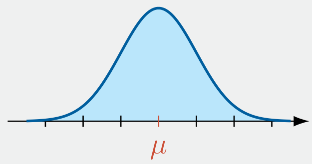
(a) Approximately __% of the values will be within __ standard deviation from μ.



(b) Approximately __% of the values will be within __ standard deviations from μ.



(c) Approximately __% of the values will be within __ standard deviations from μ.



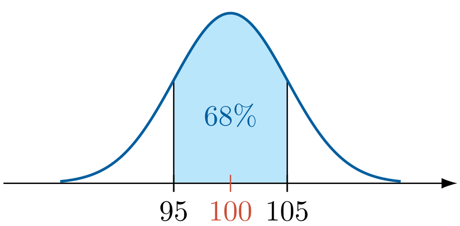
Question 1 A school fitness test has all students running 1 kilometre. The results are normally distributed with mean $\mu = 7$ minutes and $\sigma = 1$. What percentage of students finished the run

(a) between 6 to 8 minutes? (b) between 5 to 9 minutes? (c) between 4 to 10 minutes?

Question 2 The heights of adults in a city are normally distributed with $X \sim \mathcal{N}(160, 225)$. In what range of heights would you expect to find the middle

(a) 68% of heights? (b) 95% of heights? (c) 99.7% of heights?

Question 3 The diagram below shows a normal distribution curve with $\mu = 100$ and $\sigma = 5$.

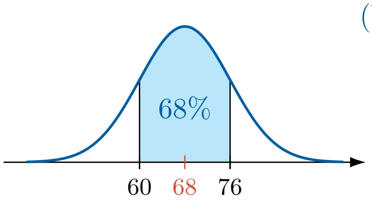


Draw a similar diagram for normally distributed random variables satisfying the following values of μ, σ, and given percentiles.

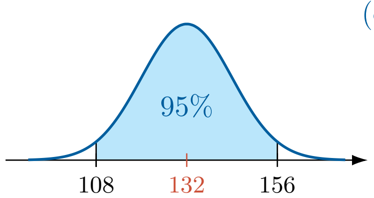
(a) $\mu = 50$, $\sigma = 4$, 68% (b) $\mu = 76$, $\sigma = 6$, 95% (c) $\mu = 250$, $\sigma = 15$, 99.7%

Question 4 The diagrams below show normal distribution curves with shaded percentiles centred at the mean. Find the value of μ and σ.

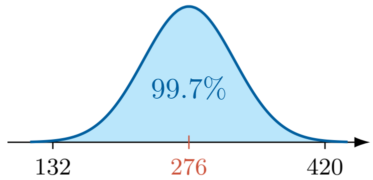
(a)



(b)



(c)



Question 5 [Halving the percentiles]

At the NSW Swimming State championships, the time in seconds X for competitors from all age groups to finish the 50m freestyle is normally distributed with $X \sim \mathcal{N}(27, 2.25)$.

(a) What percentage of competitors swims 50m in the range

(i) $25.5 \leq x \leq 28.5$ (ii) $24 \leq x \leq 30$ (iii) $22.5 \leq x \leq 31.5$

(b) What percentage of competitors swims 50m in *more* than

(i) 28.5 seconds? (ii) 30 seconds? (iii) 31.5 seconds?

(c) What percentage of competitors swims 50m in *less* than

(i) 25.5 seconds? (ii) 24 seconds? (iii) 22.5 seconds?

Question 6 The time in minutes X for a group of runners to finish running 10km is normally distributed with $X \sim \mathcal{N}(75, 225)$. What percentage of runners can run 10km in

(a) more than 75 minutes? (b) more than 60 minutes?

(c) less than 105 minutes? (d) less than 120 minutes?

Question 7 [Using the trapezoidal rule to approximate the empirical rules]

Consider the random variable $X \sim \mathcal{N}(0, 1)$. Use the trapezoidal rule with the given number of sub-intervals below, and show that the values get closer to 68% as we increase the number of sub-intervals.

(a) 4 sub-intervals. (b) 6 sub-intervals. (c) 8 sub-intervals.

Question 8 Let $f(x)$ be the probability density function of the random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Write down the values of the following.

(a) $\displaystyle\int_{\mu}^{\mu+\sigma} f(x)\,dx$ (b) $\displaystyle\int_{\mu}^{\mu+2\sigma} f(x)\,dx$ (c) $\displaystyle\int_{\mu}^{\mu+3\sigma} f(x)\,dx$

(d) $\displaystyle\int_{\mu}^{\infty} f(x)\,dx$ (e) $\displaystyle\int_{\mu+\sigma}^{\infty} f(x)\,dx$ (f) $\displaystyle\int_{-\infty}^{\mu-2\sigma} f(x)\,dx$

Challenge Problems

Problem 1 [More advanced problems using the empirical rules]

Let $f(x)$ be the probability density function of the random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the following.

(a) $\displaystyle\int_{-\infty}^{\mu-\sigma} f(x)\,dx$ (b) $\displaystyle\int_{\mu+\sigma}^{\mu+2\sigma} f(x)\,dx$ (c) $\displaystyle\int_{-\infty}^{\mu+2\sigma} f(x)\,dx$

(d) $\displaystyle\int_{\mu-\sigma}^{\infty} f(x)\,dx$ (e) $\displaystyle\int_{\mu-\sigma}^{\mu+2\sigma} f(x)\,dx$ (f) $\displaystyle\int_{\mu-3\sigma}^{\mu+\sigma} f(x)\,dx$

Problem 2 Let $f(x)$ be the probability density function of the random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Show the following identities with the aid of a diagram.

(a) $\displaystyle\int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx = 2\int_{\mu}^{\mu+\sigma} f(x)\,dx$ (b) $\displaystyle\int_{\mu-3\sigma}^{\infty} f(x)\,dx = 1 - \int_{-\infty}^{\mu-3\sigma} f(x)\,dx$

Exercise 8E

The z-score

Fundamentals

Fundamentals 1

(a) The z-score is the number of s_____ d_______ that a value is from the m____.

(b) Sometimes, the z-score is called the s_____ score.

(c) The formula for the z-score is

$$z = \_\_\_\_$$

(d) If $z > 0$, then $x = t$ is above/below (circle one) the mean μ.

(e) If $z < 0$, then $x = t$ is above/below (circle one) the mean μ.

Fundamentals 2

(a) Without any aids, we can find $P(Z \leq a)$ if $a = \pm 1, \pm 2, \pm 3$ by using the e_______ rules.

(b) If a is not an integer, then a z-score t_____ of values must be used.

(c) The z-score t_____ only gives $F(a) = P(Z \_\_ a)$, where $F(a)$ is the cumulative distribution function evaluated at $a > 0$.

(d) To calculate $P(Z \geq a)$ where $a > 0$, use the fact that

$$P(Z \geq a) = 1 - \_\_\_\_\_\_\_\_$$

(e) To calculate $P(Z \geq -a)$ where $a > 0$, use the fact that

$$P(Z \geq -a) = P(Z \_\_ a)$$

(f) To calculate $P(Z \leq -a)$ where $a > 0$, use the fact that

$$P(Z \leq -a) = P(Z \_\_\_\_ a)$$

(g) To calculate $P(a \leq Z \leq b)$, use the fact that

$$P(a \leq Z \leq b) = F(\_\_) - F(\_\_)$$

Question 1 Consider a data set with $\mu = 50$ and $\sigma = 5$. Find the z-score of the following.

(a) 45 (b) 60 (c) 43 (d) 57

Question 2 Consider the normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Write down the value of the following.

(a) $P(-1 \leq Z \leq 1)$ (b) $P(-2 \leq Z \leq 2)$ (c) $P(-3 \leq Z \leq 3)$

Question 3 Consider the random variable $X \sim \mathcal{N}(4, 0.25)$. Let z be the standard normal random variable so that $Z \sim \mathcal{N}(0, 1)$.

(a) Find a such that $P(X > 5) = P(Z < a)$

(b) Find b such that $P(X > 5.5) = P(Z < b)$

(c) Find c such that $P(X < 3.5) = P(Z > c)$

Question 4 Let $f(x)$ be the probability density function of the random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the following using the empirical rules.

(a) $P(1 \leq Z \leq 2)$ (b) $P(2 \leq Z \leq 3)$ (c) $P(Z \geq 2)$ (d) $P(Z \leq 1)$

Question 5 [Practising using the z-score table]

Use the z-score table to find the following.

(a) $P(z \leq 1.42)$ (b) $P(z \leq -0.69)$ (c) $P(z \leq 2.1)$

(d) $P(z \geq 1.27)$ (e) $P(z \geq -0.98)$ (f) $P(z \geq 2.2)$

(g) $P(1.68 \leq z \leq 2.43)$ (h) $P(-0.93 \leq z \leq -0.28)$ (i) $P(-1.37 \leq z \leq 0.52)$

Question 6 [Verifying the values from the empirical rules]

Use the z-score table to find the following.

(a) $P(-1 \leq z \leq 1)$ (b) $P(-2 \leq z \leq 2)$ (c) $P(-3 \leq z \leq 3)$

Question 7 Let $f(x)$ be the probability density function of the random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Determine whether the following identities are true or false.

(a) $P(-a \leq Z \leq a) = 2P(0 \leq Z \leq a)$ (b) $P(Z \geq a) = 1 - P(Z \leq a)$

(c) $P(Z \leq -a) = P(Z \geq a)$ (d) $P(-a \leq Z \leq a) = 1 - 2P(Z \geq a)$

Question 8 [Common mistakes]

A teacher gives Bob and Mary the normally distributed random variable $X \sim \mathcal{N}(20, 4)$, and asks them to calculate the percentage of scores within two standard deviations from the mean. Bob wishes to use Z and writes down

$$P(-16 \leq Z \leq 24) = 0.95$$

whereas Mary prefers to use X and writes down

$$P(-12 \leq X \leq 28) = 0.95$$

State what is wrong with their answers, and correct them.

Question 9 Bob and Mary both sit a test in Biology and Chemistry respectively. Bob scores 82% in his Biology test but Mary scores 76% in her Chemistry test. The average Biology mark is 74% and the standard deviation is 6. The average Chemistry mark is 68% and the standard deviation is 7%.

(a) Calculate Bob and Mary's z-scores.

(b) Hence, determine who performed better, relative to their peers.

Question 10 The table below shows Bob's scores in his trial HSC examinations, as well as the mean and standard deviations for each subject. He performs above average for every subject, but does not know which one he performed relatively best in.

	Biology	**Chemistry**	**English**	**Mathematics**	**PDHPE**
μ	80	72	73	88	65
σ	3	8	5	4	4
Score	84	83	82	95	74

Determine Bob's z-score for each subject, and hence determine his weakest and strongest subject.

Question 11 [Using the table backwards]

Find the value of $a > 0$ if

(a) $P(Z \leq a) = 0.9306$

(b) $P(Z \leq -a) = 0.2061$

(c) $P(Z \geq a) = 0.0197$

(d) $P(Z \geq -a) = 0.9382$

Question 12 The height of 3-year-old girls is normally distributed with mean 94cm and standard deviation 4cm.

(a) What percentage of girls have heights between 90cm and 102cm?

(b) From a group of 200 girls, how many would you expect to be between 86cm and 102cm?

(c) Find the probability that a randomly chosen girl has height less than 90cm, given that she is less than 98cm?

Question 13 A random variable X is normally distributed with mean 5 and standard deviation 0.6. Find the probability that X is at least 6.2.

Question 14 If the z-scores on an examination are normally distributed and $P(Z < a) = 0.65$ for some value of a, then find the value of $P(-a < Z < a)$.

Question 15 Leah ran 100 metres in 13.6 seconds and 200 metres in 29.3 seconds. The following table is a summary of the mean and standard deviation across all competitors for each event.

Event	μ	σ
100 metre	14.5 seconds	0.8 seconds
200 metre	31.5 seconds	1.4 seconds

In which event did Leah perform better relative to the other competitors?

Question 16 Toby completed two tests this week. His results are shown in the table below with the class mean and standard deviation shown for each subject.

Subject	**Toby's mark**	μ	σ
Mathematics	74	66	4
Chemistry	80	72	10

(a) In which test did Toby perform better, relative to his peers?

(b) Talia is in Toby's classes and sat the same tests. Her z-score for both subjects is 1.5. What mark did Talia receive for each test?

Question 17 A manufacturing company produces pot lids for stove pots of a certain size. In order for the lids to fit, the final product must be near-perfect to the intended dimension. Lids with radius length 14.5 cm to 15.5 cm are accepted and anything else is rejected. Typically, about 5% of all lids produced are rejected. If the radii of the lids are normally distributed, find the mean and standard deviation of the normal distribution.

Question 18 The time taken for a plane to make a trip from A to B is normally distributed with a mean of 200 minutes and a standard deviation of 10 minutes. Find the proportion of times that the flight is

(a) less than 204 minutes.

(b) less than 195 minutes.

(c) greater than 225 minutes.

(d) greater than 192 minutes.

Question 19 In a nursery, the heights of seedlings sold are normally distributed. The mean height of one type of seedling sold is 52mm and the standard deviation is 6.2mm. Seedlings with heights between 45mm and 68mm are sold. What percentage of seedlings are sold, correct to the nearest percentage?

Challenge Problems

Problem 1 A farming company harvests apples to sell. Apples that weigh less than 80 grams are rejected. Typically, about 97.5% of all apples are accepted. Also, apples that are larger than 110 grams are categorised as 'premium' apples. Typically, about 16% of all apples are 'premium' apples. If the weight of the apples are normally distributed, find the mean and standard deviation of the normal distribution.

Problem 2 The average human reaction time is about 215 milliseconds. Ethan takes a test for reaction time, and he beats 99.34% of all other people taking the test. Let μ be the mean reaction time and let t be Ethan's score, in milliseconds.

(a) Show that Ethan's z-score is $z = -2.48$.

(b) If the standard deviation is 40 milliseconds, what did Ethan score?

Problem 3 Zoe speaks at a rate of about 90 words per minute, which places her in the top 2.17% of the population for speaking rate. If the average speaking rate is 80 words per minute, find the proportion of the population that speaks faster than 75 words per minute.

Chapter 8 Review

Continuous Random Variables

Review

Question 1 Consider the probability density function

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{2x+1}}, & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Calculate $P(X \le 1)$

(c) Calculate $P\left(\frac{1}{2} \le X \le 3\right)$

Question 2 Consider the probability density function

$$f(x) = \begin{cases} ke^{-x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Sketch the graph of $f(x)$.

(c) Calculate $P(X \ge 1)$

(d) Calculate $P(X \le 2)$

Question 3 Let X be a random variable with probability density function $f(x)$ with equation

$$f(x) = \begin{cases} \dfrac{k}{(3x-1)^2}, & x \ge 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Find $P(X \ge 2)$

(c) Find $P(2 \le X \le 3)$

(d) Hence, find $P(2 \le X \le 3 \mid X \ge 2)$.

(e) Find the equation of the CDF.

(f) Sketch the PDF.

(g) Sketch the CDF.

(h) Find the median of X.

Question 4 Let X be a random variable with probability density function $f(x)$ with equation

$$f(x) = \begin{cases} k, & -k \le x \le k \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Sketch the PDF.

(c) Find $P\left(X \ge \frac{1}{2}\right)$

(d) Write down the the median m.

Question 5 Let X be a random variable with probability density function $f(x)$ with equation

$$f(x) = \begin{cases} k(12x - x^3), & 0 \leq x \leq 2\sqrt{3} \\ 0, & \text{otherwise} \end{cases}$$

Find the mode of $f(x)$.

Question 6 The lifetime t of a diode, in hours, is given by the probability density function

$$f(t) = \begin{cases} \dfrac{1}{k} e^{-\frac{t}{400}}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) The diode is considered faulty if the lifespan is less than 10 hours. What percentage of manufactured diodes are faulty?

(c) The diode is considered exceptional if the lifespan exceeds 2000 hours. What percentage of manufactured diodes are exceptional?

Question 7 Consider the probability density function for a random variable X given by

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{9 + 2x}}, & 0 \leq x \leq 8 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Find the equation of the CDF.

(c) Hence, find the median value of X.

Question 8 Let X be the random variable that represents the number of minutes it takes for a drug to be absorbed by a patient's body.

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & 1 \leq x \leq k \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Find the probability that a patient took less than 1.5 minutes to absorb the drug, given that we know that it took less than 2 minutes to absorb the drug.

Question 9 When a bus is late, the number of minutes T that it is late by is modelled by the probability density function

$$f(t) = \begin{cases} \frac{1}{10}e^{-\frac{x}{10}}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the CDF.

(b) Find the median length of extra time spent waiting for the bus.

(c) How long should a bus be late by, so that it is later than 80% of other late occasions?

(d) A randomly chosen bus is known to be at least 10 minutes late. What is the probability that it was actually more than half an hour late?

Question 10 Consider the probability density function

$$f(x) = \begin{cases} k(4-x), & 0 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

of the random variable X.

(a) Find the value of k. (b) Find the CDF of $f(x)$. (c) Find the median of X.

Question 11 Consider the probability density function

$$f(x) = \begin{cases} \frac{x+1}{4}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

of the random variable X. Find the value of k such that $P(X \leq k) = \frac{3}{8}$.

Question 12 Consider the probability density function

$$f(x) = \begin{cases} k \sin\left(\frac{\pi x}{2}\right), & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

of the random variable X.

(a) Find the value of k. (b) Find $P\left(X \leq \frac{1}{2}\right)$ given that $P(X \leq 1)$.

Question 13 Consider the probability density function

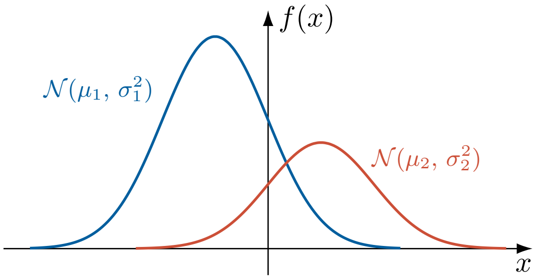
$$f(x) = \begin{cases} |x-1|, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

of the random variable X. Find $P(X < 1.5)$.

Question 14 Write down the equation of the normal distribution function that satisfies the following, and hence draw a sketch of the function.

(a) $\mu = 0,\ \sigma = 1$ (b) $\mu = 1,\ \sigma = 2$ (c) $\mu = -1,\ \sigma = \dfrac{1}{2}$

Question 15 Compare the two graphs of normally distributed random variables drawn below.



Determine which is larger.

(a) μ_1 or μ_2 (b) σ_1 or σ_2

Question 16 An army base has all soldiers doing as many push-ups as possible. The results are normally distributed with mean $\mu = 50$ and $\sigma = 5$. What percentage of soldiers could do

(a) between 45 to 55 push-ups? (b) between 40 to 60 push-ups? (c) between 40 to 50 push-ups?

Question 17 The time T, in seconds, that divers can hold their breath is normally distributed with $T \sim \mathcal{N}(120,\ 400)$. In what range of lengths would you expect to find the middle

(a) 68%? (b) 95%? (c) 99.7%?

Question 18 In an examination, the raw mark X of students is approximately normal with $X \sim \mathcal{N}(324,\ 3136)$. What mark range is needed to land in the

(a) top 2.5% of the cohort? (b) bottom 16% of the cohort? (c) top 0.15% of the cohort?

Question 19 Matthew is running a course at university and wishes to scale the marks in his most recent examination, which was out of 100. The mark X of students is approximately normal with $X \sim \mathcal{N}(72,\ 64)$. Matthew decides the following.

- students who fall below three standard deviations from the mean should not pass the course.
- students who score more than three standard deviations from the mean should receive a scaled mark of 100.

What is the minimum raw mark needed to

(a) pass the exam? (b) receive a scaled mark of 100?

Question 20 The time in minutes X for a large group of swimmers to finish a 1km swim is normally distributed with $X \sim \mathcal{N}(15, 2.25)$. What percentage of swimmers can swim 1km in

(a) less than 15 minutes?

(b) more than 16.5 minutes?

(c) less than 10.5 minutes?

(d) more than 12 minutes?

Question 21 Bob does a Mathematics Advanced test whereas Mary sits a Mathematics Extension 1 test. In the Advanced test, the mean was 84 and the standard deviation was 4. In the Extension 1 test, the mean was 72 and the standard deviation was 6. Bob scored 93 in his test and Mary scored 86 in her test. Who performed better relative to their peers?

Question 22 A farming company harvests oranges to sell to grocery stores. Oranges that weigh less than 150 grams are rejected and used for juicing instead. Typically, about 16% of all oranges are rejected. Also, oranges that are larger than 300 grams are reserved for more up-market grocery stores. Typically, about 2.5% of all oranges are sold to the up-market grocery stores. If the weight of the oranges are normally distributed, find the mean and standard deviation of the normal distribution.

Question 23 The table below shows Jay's scores in his trial HSC examinations, as well as the mean and standard deviations for each subject. He performs below average for every subject, but does not know which one he performed worst in.

	Chemistry	**English**	**Maths Extension 1**	**Maths Extension 2**	**Physics**
μ	78	82	74	62	70
σ	5	4	6	9	4
Score	70	69	64	48	61

Determine Jay's z-score for each subject, and hence determine his weakest and strongest subject relative to his peers.

Question 24 It was found that the heights of children for a ride in an amusement park are normally distributed with a mean of 120cm and a standard deviation of 3.2cm. It is known that 30% of children are not allowed to go on the ride because they do not meet the safe height requirements.

(a) Find the minimum safe height requirement, correct to the nearest centimetre.

(b) Find the proportion of children who are at least 126cm tall.

Question 25 During the past year, data was collected about the time taken to travel from a suburb to the city centre. It was found that the data is normally distributed with the average trip being 40 minutes long, and the standard deviation being 10 minutes. What percentage of trips took between 32 minutes and 38 minutes?

Investigation Task

Mean and variance

In Year 11, you studied the *discrete* random variable. In Year 12 now, you are studying the *continuous* random variable. However, in Year 12, it is not in the syllabus to calculate the mean, variance and standard deviation of a continuous probability density function. However, the construction of these values for the continuous case is actually a very natural extension of the Year 11 content. This investigation task will allow the student to see this, and to find some interesting results using the continuous analogies of the mean and variance.

Question 1

(a) Write down the formula for the expected value of a continuous discrete random variable.

(b) Explain how it is analogous to the formula for the expected value in the discrete scenario.

(c) Show that $E(aX+b) = aE(X)+b$.

Question 2

(a) Write down two formulas for the variance of a continuous discrete random variable.

(b) Explain how it is analogous to the formula for the variance in the discrete scenario.

(c) Derive the formula $\sigma^2 = E(X^2) - \mu^2$ from the other formula you found in (a).

(d) Does every distribution have a mean and variance?

(e) Prove that $\text{Var}(aX+b) = a^2\,\text{Var}(X)$.

Question 3 [Exponential random variable]

(a) Research what the exponential random variable is, and the types of of scenarios where it may be used.

(b) By first differentiating xe^x, find the mean of the exponential random variable.

(c) By first differentiating x^2e^x, find the variance of the exponential random variable.

(d) Use a similar technique to find the variance of the normal random variable.

Question 4 [Cauchy random variable]

(a) Research what the Cauchy random variable is, and the types of of scenarios where it may be used.

(b) Try to find the mean of the Cauchy random variable. What do you notice?

(c) Similarly try to find the variance and comment on your findings.

Investigation Task

Non-integer z-scores

In the chapter, we always have dealt with integer z-scores because we can then make the use of the empirical rules. For example $P(-1 \leq Z \leq 1) = 0.68$ and $P(-2 \leq Z \leq 2) = 0.95$. But it's not difficult to see that generally speaking, z-scores will likely be non-integers, so what do we do then? This investigation task will broaden the scope of the study of the z-score where the empirical rules may no longer be useful.

Question 1 Research and find out how people calculate percentiles of a normally distributed random variable when the z-score is not an integer and instead is any arbitrary number z_0.

Question 2 Consider $X \sim \mathcal{N}(0, 1)$. Use any method you have found above to find z_0 such that such that the following is for

(a) $P(Z < z_0) = 0.7$ (b) $P(Z > z_0) = 0.1$ (c) $P(Z > z_0) = 0.2$

Question 3 Consider $X \sim \mathcal{N}(0, 1)$. Repeat the above questions to try to verify the empirical rules. Are they accurate?

Investigation Task

Buffon's Needle

In the study of probability, a well-known "No way!" statement is this.

"We can approximate π by throwing a stick on a floorboard repeatedly"

This is called *Buffon's Needle Problem.* Your task is to write an article or produce a presentation explaining everything there is to know about the problem. Your answer should include:

- The statement of the main result.
- A proof of the result.
- Any necessary diagrams or graphs needed to clarify any parts of your article.
- How it is related to continuous probability distributions.
- A section where you perform the experiment yourself, log your data and show also how you were able to approximate π by throwing a stick numerous times.

Investigation Task

Hands-on with the normal distribution!

In the chapter, you did all sorts of calculations involving continuous random variables, normal distributions and z-scores. This investigation task puts everything together, so that you can fully understand and appreciate the roles of every component in the course.

Note: For the following investigation task, you may use any measurement you like, but we mention the reaction times because a large data set can be found within a relatively short period of time.

Question 1 Hold out a 30cm ruler vertically in front of a friend who has an open palm ready to grab it. Release the ruler at a random time, and record the number of centimetres that the ruler fell before your friend grabbed it. Repeat this experiment with whoever you may know including teachers, other school friends, family etc until you have at least 50 data points.

(a) Plot your data set into Excel and form a frequency histogram with intervals 0-1cm, 1-2cm, 2-3cm etc until you reach the longest distance i.e. slowest reaction time.

(b) Calculate the mean and standard deviation of your data, and form a normal distribution curve satisfying your data. Superimpose the curve on top of your frequency histogram and comment on your findings.

(c) If the data is normally distributed, what proportion of people should fall within one and two standard deviations of the curve?

(d) Now, sift through the data manually to find the proportion of people who fall within one and two standard deviations of the curve. Compare your findings to the theoretic result.

Question 2 Discuss any potential bias in your experimental methodology, and try to find any possible reason for any discrepancies between your data set and the normal distribution curve you formed. Having some bias is okay for purposes of this investigation task. But it is important that you are able to identify them. Your discussion should mention the idea of *skew*.

1. Transformations

Exercise 1A

Translations and dilations

F1

(a) Up

(b) Down

(c) Right

(d) Left

F2

(a) stretched, vertically

(b) squashed, vertically

(c) squashed, horizontally

(d) stretched, horizontally

F3

$\frac{1}{a}$, squashed, stretched

Q1

(a) Translated 2 units up

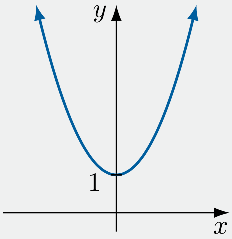
(b) Translated 3 units down

(c) Translated 1 unit to the left

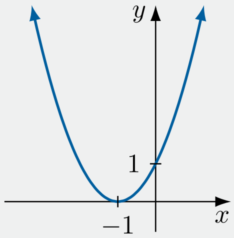
(d) Translated 2 units to the right

Q2

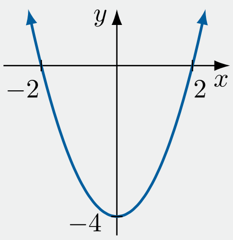
(a)



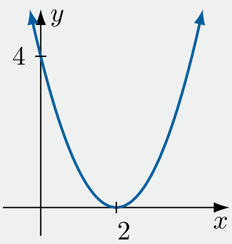
(b)



(c)



(d)



Q3

(a) Translated 1 unit down

(b) Translated 2 units to the right

(c) Translated 1 unit up

(d) Translated 1 unit to the left

Q4

(a) Translated 1 unit to the right

(b) Translated 1 unit up

(c) Translated 3 units to the right

(d) Translated 3 units down

(e) Translated 2 units to the right

(f) Translated 3 units down

(g) Translated 2 units to the right

(h) Translated 2 units down

Q5

(a) $y = 2x + 4$

(b) $y = 2x - 3$

(c) $y = \frac{1}{x+1}$

(d) $y = \frac{1}{x-1} - 3$

(e) $y = 2(x-4)^2$

(f) $y = 2x^2 + 2$

(g) $y = 5 - 2x$

(h) $y = -2x - 3$

Q6

(a) $2x + 3y - 3 = 0$

(b) $(x+4)^2 + y^2 = 4$

(c) $y = \frac{1}{x-3}$

(d) $y = \frac{1}{x+1} - (x+1)$

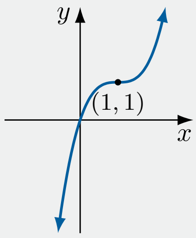
Q7

$P(-2, 1)$ and $Q(-6, -3)$

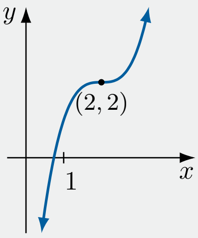
Q8

(a)

(i) $y = (x-1)^3 + 1$

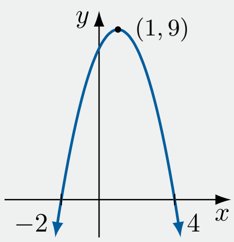


(ii) $y = (x-2)^3 + 2$

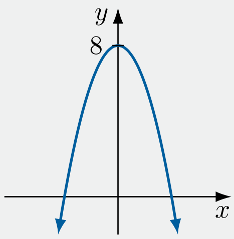


(b)

(i) $y = (4-x)(x+2)$

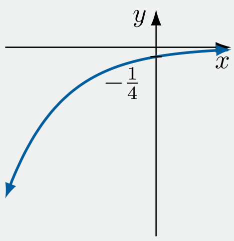


(ii) $y = 8 - x^2$

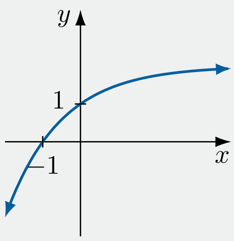


(c)

(i) $y = -2^{-x-2}$

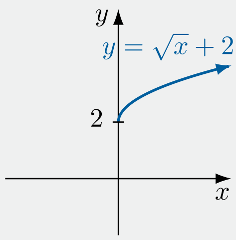


(ii) $y = 2 - 2^{-x}$

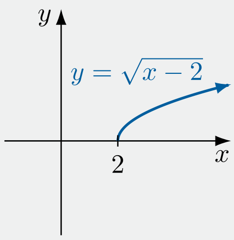


Q9

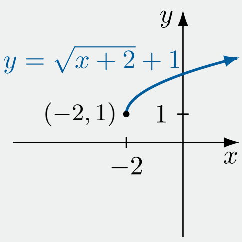
(a)



(b)

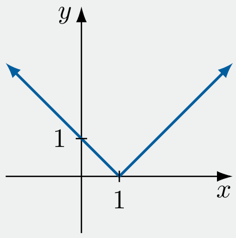


(c)

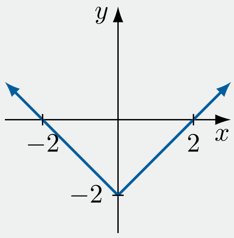


Q10

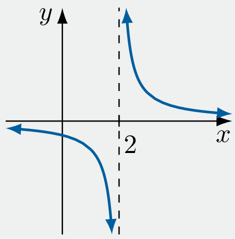
(a)



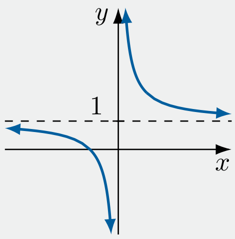
(b)



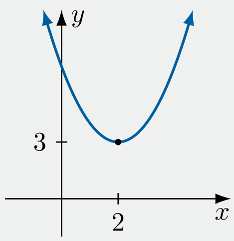
(c)



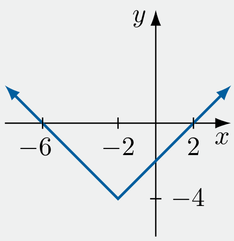
(d)



(e)



(f)



Q11

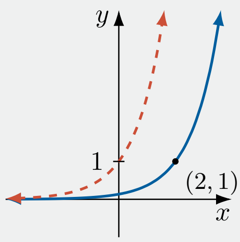
(a) $f(x) = 7 - 2x$

(b) $f(x) = 7 - x^2$

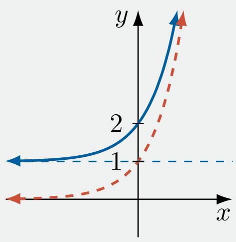
(c) $f(x) = 2^{-x+2}$

Q12

(a)

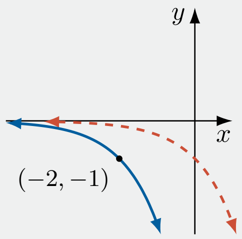


(b)

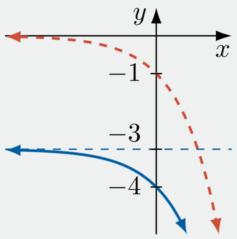


ANSWERS

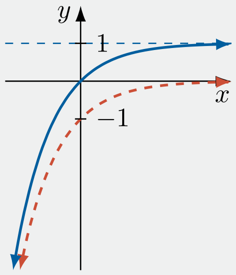
(c)



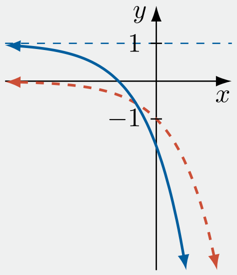
(d)



(e)

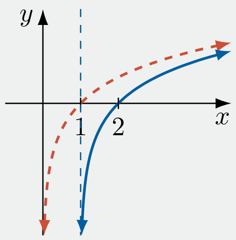


(f)

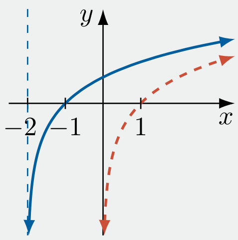


Q13

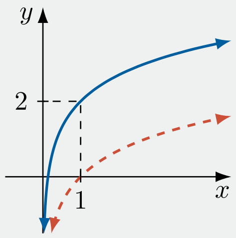
(a)



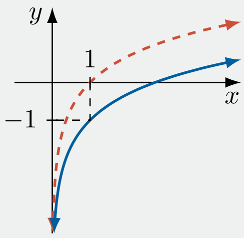
(b)



(c)

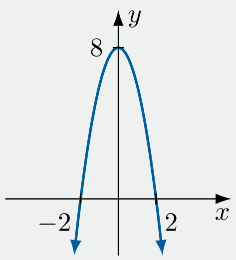


(d)

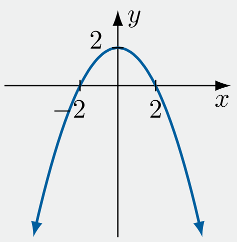


Q14

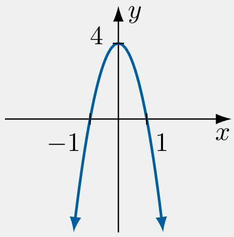
(a) Stretched vertically by a factor of 2



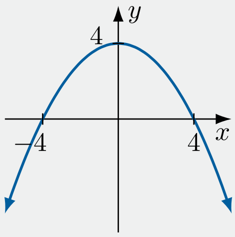
(b) Squashed vertically by a factor of 2



(c) Squashed horizontally by a factor of 2

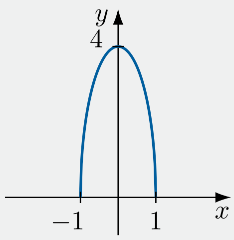


(d) Stretched horizontally by a factor of 2

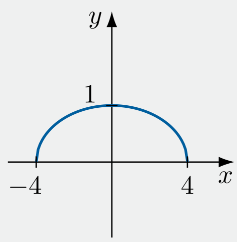


Q15

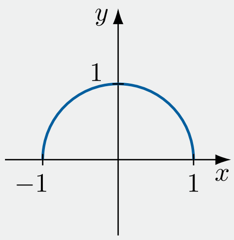
(a) Squashed horizontally by a factor of 2 and stretched vertically by a factor of 2



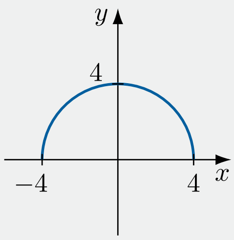
(b) Stretched horizontally by a factor of 2 and squashed vertically by a factor of 2



(c) Squashed horizontally and vertically by a factor of 2

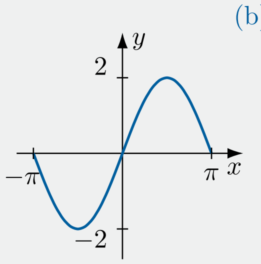


(d) Stretched horizontally and vertically by a factor of 2

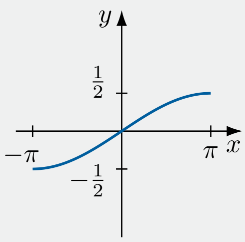


Q16

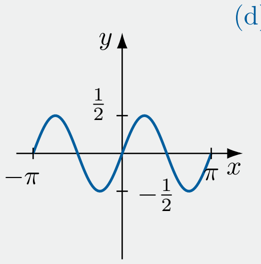
(a)



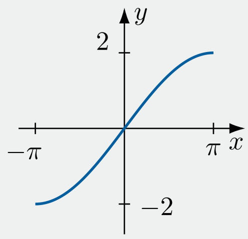
(b)



(c)



(d)



P1

(a) Bob is incorrect because his result is actually

$$f(x-6) = (2(x-6)+1)^2 = (2x-11)^2.$$

(b) $f(x-a) = (2x-2a+1)^2$

(c) 3 units

P2

(a) Horizontal shift of 5 units to the left
(b) Horizontal shift of 2 units to the right
(c) Horizontal shift of 1 unit to the right
(d) Horizontal shift of 2 units to the right

P3

(a) $f(ax) = \ln(ax) = \ln(x) + \ln(a) = f(x) + c$

(b) $f(x+a) = e^{x+a} = e^a \times e^x = k \times f(x)$

(c) $k \times f(x) = kx^2 = \left(x\sqrt{k}\right)^2 = f\left(x\sqrt{k}\right) = f(ax)$

Exercise 1B
Combining transformations

F1

(a) $x-c$, $f(x-c)$
(b) ax, $f(ax-c)$
(c) different

F2

(a) $x+c$, $f(x+c)$
(b) $-x$, $f(-x+c)$
(c) different

F3

(a) ax, $f(ax)$
(b) $-x$, $f(-ax)$
(c) the same

F4

order matters, order does not matter

Q1

(a) 2, $y = (x-2)^2$

(b) $x+1$, $y = \ln(x+1)$

(c) $\frac{x}{2}$, $y = \sin\left(\frac{x}{2}\right)$

(d) 2, $y = \cos(2x)$

Q2

(a) $y = (x-1)^2$

(b) $y = (2x-1)^2$

(c) $x^2 \xrightarrow[\text{translate}]{} (x-1)^2 \xrightarrow[\text{dilate}]{} (2x-1)^2$

(d) $x^2 \xrightarrow[\text{dilate}]{} (2x)^2 \xrightarrow[\text{translate}]{} (2x-2)^2$

(e) Order did matter.

Q3

(a) $y = \ln(x-1)$, $y = \ln(2x-1)$

(b) $y = \ln(2x)$, $y = \ln(2x-2)$

(c) $y = \ln(x+2)$, $y = \ln\left(\frac{x}{3}+2\right)$

(d) $y = \ln\left(\frac{x}{3}\right)$, $y = \ln\left(\frac{x+2}{3}\right)$

Q4

(a) $y = \dfrac{1}{3x+2}$ (b) $y = \dfrac{1}{3x+6}$

(c) $y = \dfrac{2}{x-10}$ (d) $y = \dfrac{2}{x-5}$

ANSWERS

Q5

(a) $y = \sqrt{2x+1}$ (b) $y = \sqrt{2x}$

(c) $a = \frac{1}{2}$ (d) 0.5 units

Q6

(a) $f(3x-2)$ (b) $f\left(\frac{1}{2}x+3\right)$

Q7

(a) $f(2x+2)$ (b) $f\left(\frac{1}{3}x-2\right)$

Q8

(a) Translate left by 4 then squash horizontally by a factor of 2.
Squash horizontally by a factor of 2 then translate left by 2

(b) Translate right by 2 then squash horizontally by a factor of 3.
Squash horizontally by a factor of 3 then translate right by $\frac{2}{3}$.

(c) Translate right by 1 then squash horizontally by a factor of 2.
Squash horizontally by a factor of 2 then translate right by $\frac{1}{2}$.

(d) Translate left by 1 then squash horizontally by a factor of 3.
Squash horizontally by a factor of 3 then translate left by $\frac{1}{3}$.

Q9

(a) $y = e^{-x}$ (b) $y = \ln(-x)$

(c) $y = \sin(-x)$ (d) $y = \sqrt{-x}$

(e) $y = -2x-1$ (f) $y = 3+2x$

(g) $y = \frac{x}{x+1}$ (h) $y = f(-x-1)$

(i) $y = f(-2x+1)$

Q10

(a) $y = \ln(x-1)$

(b) $y = \ln(-x-1)$

(c) $\ln(x) \xrightarrow[\text{translate}]{} \ln(x-1) \xrightarrow[\text{reflect}]{} \ln(-x-1)$

(d) $\ln(x) \xrightarrow[\text{reflect}]{} \ln(-x) \xrightarrow[\text{translate}]{} \ln(1-x)$

(e) Order did matter.

Q11

(a) $y = \sqrt{-x-3}$ (b) $y = \sqrt{3-x}$

(c) $y = \sqrt{-x+2}$ (d) $y = \sqrt{-x-2}$

Q12

(a) $y = f\left(-\frac{1}{2}x-1\right)$

(b) $y = f\left(-\frac{1}{2}x-1\right)$

(c) $y = f\left(-\frac{1}{2}(x-1)\right)$

(d) $y = f\left(-\frac{1}{2}x+1\right)$

(e) $y = f\left(-\frac{1}{2}x-\frac{1}{2}\right)$

(f) $y = f\left(-\frac{1}{2}x+\frac{1}{2}\right)$

Q13

(a) Translate left by 1, then squash by a factor of 2, then reflect across the y-axis.

(b) Reflect across the y-axis, then translate right by 1, then squash by a factor of 2.

(c) Squash by a factor of 2, then reflect across y-axis, then translate right by $\frac{1}{2}$.

Q14

(a) $y = 2e^x$

(b) $y = 2e^x + 1$

(c) $e^x \xrightarrow[\text{dilate}]{} 2e^x \xrightarrow[\text{translate}]{} 2e^x + 1$

(d) $e^x \xrightarrow[\text{translate}]{} e^x + 1 \xrightarrow[\text{dilate}]{} 2(e^x + 1)$

(e) Order did matter.

Q15

Dilating first, and then translating.

ANSWERS

Q16

(a) $y = -e^x$

(b) $y = 1 - e^x$

(c) $e^x \xrightarrow[\text{reflect}]{} -e^x \xrightarrow[\text{translate}]{} -e^x + 1$

(d) $e^x \xrightarrow[\text{translate}]{} e^x + 1 \xrightarrow[\text{reflect}]{} -(e^x + 1)$

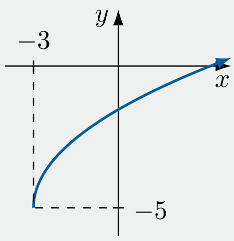
(e) Order did matter.

Q17

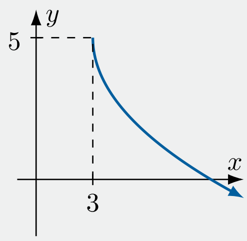
Reflecting, then dilating, then translating

Q18

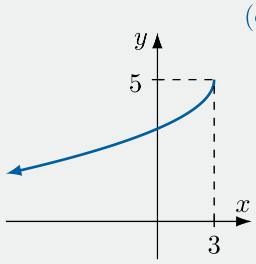
(a)



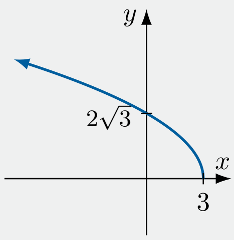
(b)



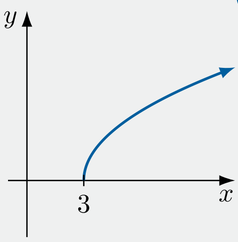
(c)



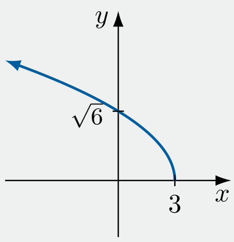
(d)



(e)

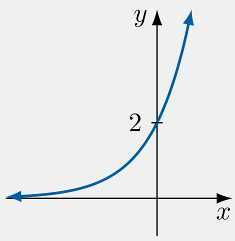


(f)

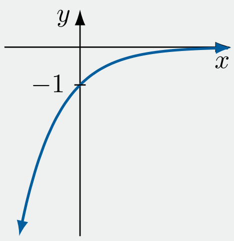


Q19

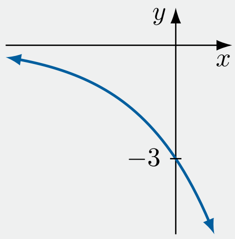
(a)



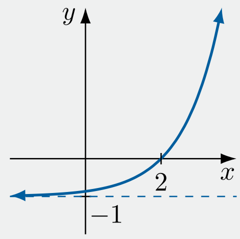
(b)



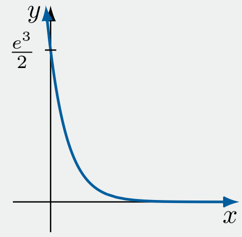
(c)



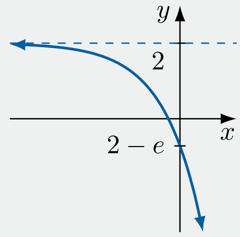
(d)



(e)

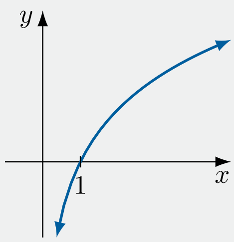


(f)

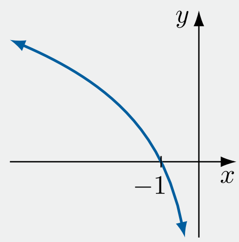


Q20

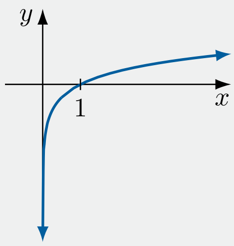
(a)



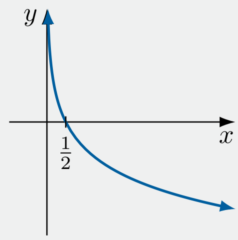
(b)



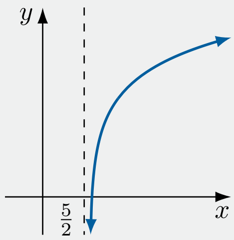
(c)



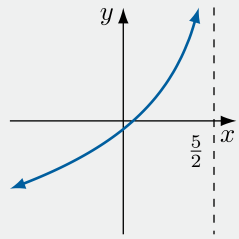
(d)



(e)

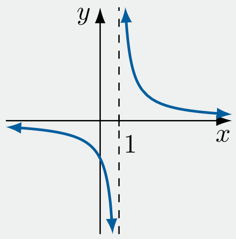


(f)

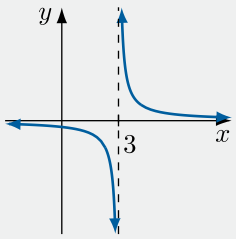


Q21

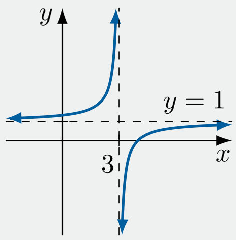
(a)



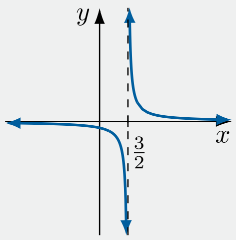
(b)



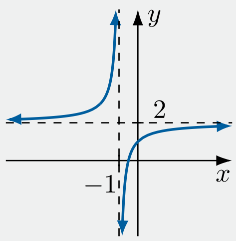
(c)



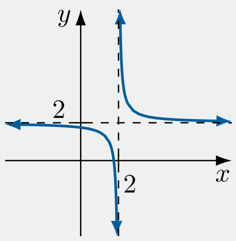
(d)



(e)

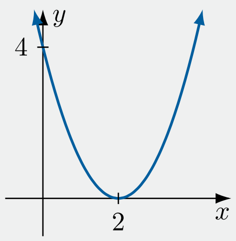


(f)

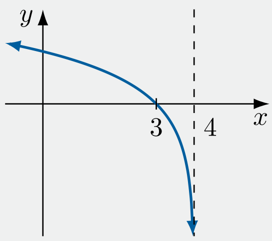


P1

(a)



(b)



(c) See full worked solutions.

P2

Translate down by 1, then reflect across the x-axis, then squash vertically by a factor of 2.
Translate down by 1, then squash vertically by a factor of 2, then reflect across the x-axis.
Reflect across the x-axis, then translate up by 1, then squash vertically by a factor of 2.

P3

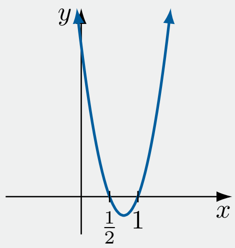
Stretch horizontally by a factor of 3, then translate left by 6.
Translate left by 2, then stretch horizontally by a factor of 3.

P4

Translate left by 1.5, stretch horizontally by a factor of 2, then reflect across y-axis.
Translate left by 1.5, reflect across y-axis, then stretch horizontally by a factor of 2.
Stretch horizontally by a factor of 2, reflect across y-axis, then translate right by 3.

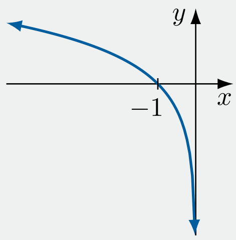
P5

Translate right by 1, then squash horizontally by a factor of 2.



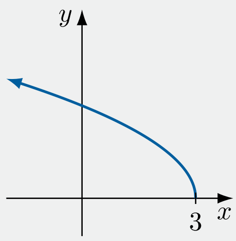
P6

Translate left by 1, then reflect across the y-axis.



P7

Translate left by 4, reflect across y-axis, squash by factor of 2



Exercise 1C

Curve sketching

F1

(a) x, y

(b) domain

(c) ∞

(d) asymptotes

(e) asymptote

(f) above, below

(g) domain

(h) even, odd

(i) range

Q1

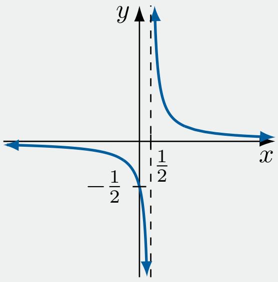
(a) $(0, -2)$

(b) $x = \dfrac{1}{2}$ vertical asymptote $y = 0$, horizontal asymptote

(c) As $x \to \infty$, $y \to 0^+$, $x \to -\infty$, $y \to 0^-$

(d) As $x \to \dfrac{1}{2}^-$, $y \to -\infty$, $x \to \dfrac{1}{2}^+$, $y \to \infty$

(e)



Q2

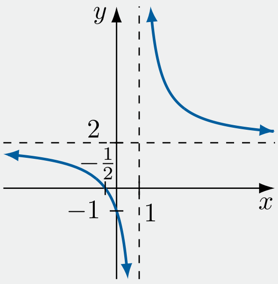
(a) $(0, -1)$, $(-\frac{1}{2}, 0)$

(b) $x = 1$ vertical asymptote $y = 2$, horizontal asymptote

(c) As $x \to \infty$, $y \to 2^+$, $x \to -\infty$, $y \to 2^-$

(d) neither

(e)



Q3

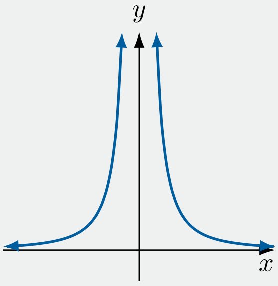
(a) $x = 0$ vertical asymptote $y = 0$, horizontal asymptote

(b) As $x \to \infty$, $y \to 0^+$

(c) See full worked solutions.

(d) $x^2 > 0$ so $\dfrac{1}{x^2} > 0$ i.e. $y > 0$

(e)



Q4

(a) $(0, 1)$

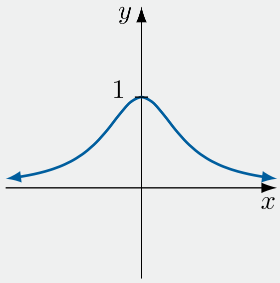
(b) no vertical asymptote $y = 0$, horizontal asymptote

(c) As $x \to \infty$, $y \to 0^+$, and as $x \to -\infty$, $y \to 0^+$

(d) As $x^2 > 0$, $y > 0$

(e) Even.

(f)



Q5

(a) $(0, -\frac{1}{4})$

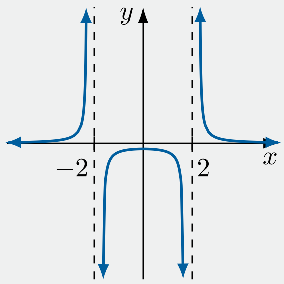
(b) $x = \pm 2$ vertical asymptotes $y = 0$, horizontal asymptote

(c) As $x \to \pm\infty$, $y \to 0^+$

(d) As $x \to 2^-$, $y \to -\infty$. As $x \to 2^+$, $y \to \infty$. As $x \to -2^-$, $y \to -\infty$. As $x \to -2^+$, $y \to \infty$

(e) Even.

(f)



Q6

(a) $(0, 0)$

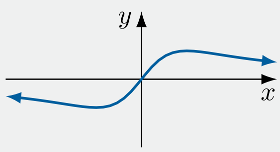
(b) $y = 0$

(c) As $x \to \infty$, $y \to 0^+$
As $x \to -\infty$, $y \to 0^-$

(d) Above for $x > 0$ and below for $x < 0$.

(e) Odd.

(f)



Q7

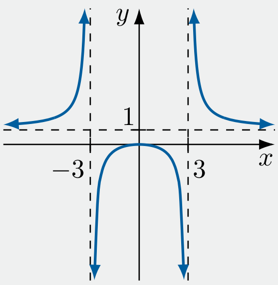
(a) $(0, 0)$

(b) $x = \pm 3$ vertical asymptotes $y = 1$, horizontal asymptote

(c) As $x \to \pm\infty$, $y \to 1^+$

(d) Even.

(e)



Q8

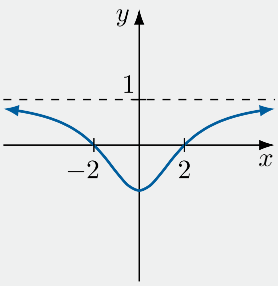
(a) $(-2, 0)$, $(2, 0)$, $(0, -1)$

(b) $y = 1$ horizontal asymptote

(c) As $x \to \pm\infty$, $y \to 1^-$

(d) Even.

(e)



Q9

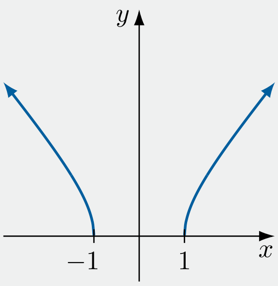
(a) $(-1, 0)$, $(1, 0)$

(b) $x \leq -1$ or $x \geq 1$

(c) As $x \to \pm\infty$, $y \to \infty$

(d) Even.

(e)



Q10

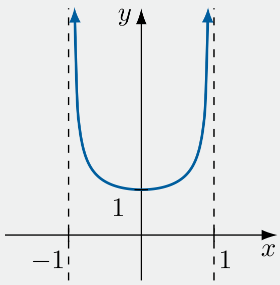
(a) $(0, 1)$

(b) $x = \pm 1$, no horizontal asymptotes

(c) $-1 < x < 1$

(d) Even.

(e)



Q11

(a) $(0, 1)$, $(1, 0)$

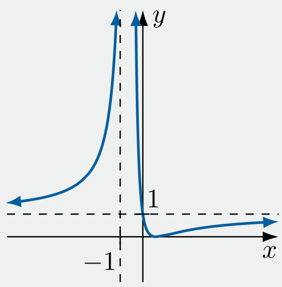
(b) $x = -1$ vertical asymptotes $y = 1$, horizontal asymptote

(c) As $x \to \infty$, $y \to 1^-$, and as $x \to -\infty$, $y \to 1^+$

(d) Other than at $(1, 0)$, the curve is always above the x-axis.

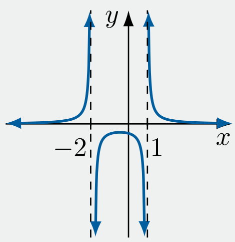
(e) neither

(f)

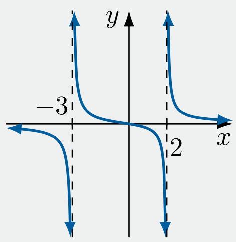


Q12

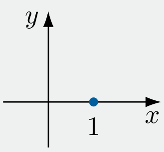
(a)



(b)



Q13



P1

(a) $x < -1$ or $x > 1$

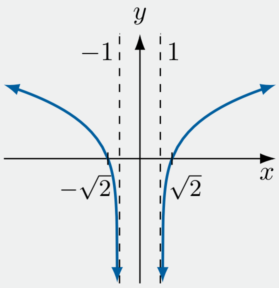
(b) $f(-x) = \ln((-x)^2 - 1)) = \ln(x^2 - 1) = f(x)$

(c) $x = \pm 1$

(d) $y \to -\infty$

(e) $y \to \infty$

(f)



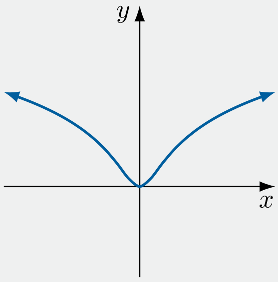
P2

(a) $x \in \mathbb{R}$

(b) $f(-x) = \ln\left((-x)^2 + 1\right) = \ln(x^2 + 1) = f(x)$

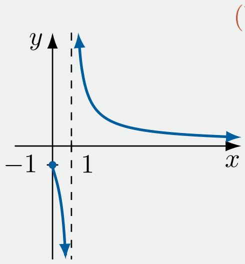
(c) $y \to \infty$

(d)

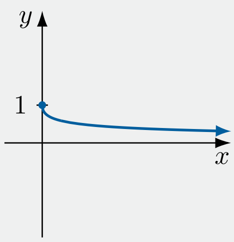


P3

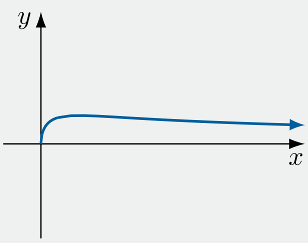
(a)



(b)

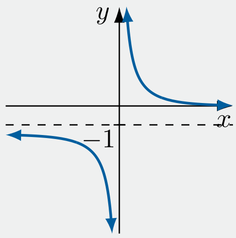


(c)

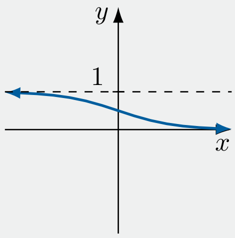


P4

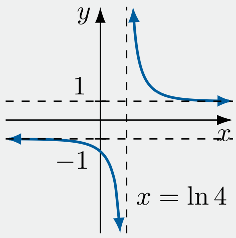
(a)



(b)



(c)



Exercise 1D

Using graphs to solve equations and inequalities

F1

$f(x)$, above, x

F2

(a) $ax^2 + bx + c$

(b) above, below, x

(c) x

F3

(a) $x \leq a$ or $x \geq b$ (b) $a < x < b$

F4

(a) simultaneously

(b) x

(c) $f(x)$, $g(x)$

Q1

(a) $x = -2$ (b) $x < -2$ (c) $x > -2$

Q2

(a) $x \leq 2$ (b) $x \leq \frac{1}{3}$

(c) $x > 4$ (d) $x > -\frac{3}{4}$

Q3
$1 \le x \le 3$

Q4
(a) $k = -1$ (b) $k > -1$ (c) $k < -1$

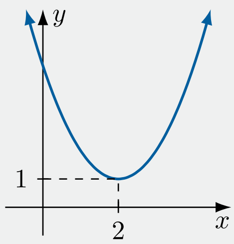
Q5
(a) $k < -2$ or $k > 2$
(b) $k = \pm 2$
(c) $-2 < k < 2$

Q6
(a) $k > -1$
(b) No value of k
(c) $k \le -1$

Q7
(a) $k = 1$
(b) $0 < k < 1$
(c) $k \le 0$ or $k > 1$

Q8
(a) $k = 0, \pm 1$
(b) $-1 < k < 1$
(c) $k < -1$ or $k > 1$

Q9
(a) $k = 0$, or $k = 1$
(b) $0 < k < 1$ or $k > 1$
(c) $k < 0$

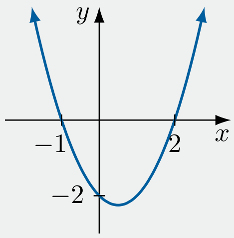
Q10
(a)


(b) $k > 1$
(c) $\Delta = 4k - 4$, which is positive when $k > 1$.

Q11
(a)
(i) $k > 0$ or $k < -4$
(ii) $k = -4$, or $k = 0$
(iii) $-4 < k < 0$

(b)
(i) $k > 0$ or $k < -8$
(ii) $k = -8$, or $k = 0$
(iii) $-8 < k < 0$

(c) These are all the same answers as (a) since the graph is shifted horizontally, which does not affect the number of solutions.
(i) $k > 0$ or $k < -4$
(ii) $k = -4$, or $k = 0$
(iii) $-4 < k < 0$

Q12
(a)


(b)
(i) $x = -1, 2$
(ii) $-1 < x < 2$
(iii) $x < -1$ or $x > 2$

Q13
(a) $-2 < x < 3$
(b) $x \le -4$ or $x \ge 5$
(c) $x \le -1$ or $x \ge 2$
(d) $-1 < x < \frac{1}{2}$
(e) $x = 2$
(f) No solution

Q14
(a) $x \le -1$ or $x \ge 1$
(b) $-2 < x < 2$
(c) $1 \le x \le 4$
(d) $x < -3$ or $x > 2$
(e) $-2\frac{1}{2} < x < -1$
(f) $x \le -\frac{1}{2}$ or $x \ge \frac{1}{3}$

Q15

(a) All real x (b) No solutions

(c) $x = 0$ (d) All real x

(e) No solutions (f) $x = \frac{1}{2}$

Q16

(a) $x = 0$, $x = 4$, $(-1, 5)$, $(5, 5)$, $(-2, 12)$, $(6, 12)$

(b)

- (i) $x < 0$ or $x > 4$
- (ii) $x \leq -2$ or $x \geq 6$
- (iii) $-1 \leq x \leq 5$
- (iv) $-2 < x < -1$ or $5 < x < 6$

Q17

(a) $(-2, 0)$, $(1, 3)$ (b) $-2 < x < 1$

Q18

(a) $(-2, -2)$, $(3, -7)$ (b) $x < -2$ or $x > 3$

Q19

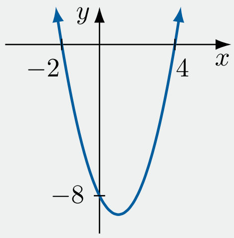
(a) $x < -4$ or $x > 2$ (b) $x \leq 1$ or $x \geq 2$

(c) $-2 < x < 0$ (d) $-2 \leq x \leq 3$

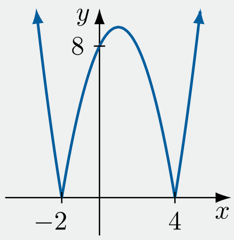
(e) $0 \leq x \leq 12$ (f) $x \leq -4$ or $x \geq 2$

Q20

(a)



(b)



(c) From the diagram, clearly a horizontal line can intersect either two times, three times, or four times. But there cannot be only one intersection point and hence never exactly one solution.

(d)

- (i) $k < 0$
- (ii) $k = 0$ or $k > 9$
- (iii) $k = 9$
- (iv) $0 < k < 9$

Q21

(a) 2 (b) 1 (c) 1 (d) 2

Q22

(a) 2 (b) 2 (c) 1

(d) 2 (e) 1 (f) 0

Q23

(a) $y = x$, $y = \cos x$

(b) $y = x^2$, $y = \ln x$

(c) $y = e^x$, $y = x + 2$

(d) $y = x + 1$, $y = \sqrt{x}$

(e) $y = \ln x$, $y = 1 + x$

(f) $y = 1 - x^2$, $y = \frac{1}{x - 1}$

Q24

(a) 2 (b) 1 (c) 2

Q25

(a) 1 (b) 1 (c) 1

(d) 1 (e) 1 (f) 0

Q26

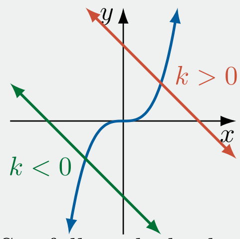
See full worked solutions.

Q27

(a) 2 (b) 2 (c) 2

P1

(a)



(b) See full worked solutions.

P2

Consider $y = x^4 - 1$ and $y = kx$ for varying values of k. No matter what k is, there will always be two intersections with one on either side of the y-axis.

P3

As $k \to \infty$, $\alpha \to \infty$.
As $k \to -\infty$, $\alpha \to 0^+$.

P4

(a) See full worked solutions.

(b) See full worked solutions.

(c) See full worked solutions.

(d) Approaches zero from the positive side.

(e) Approaches zero from the negative side

Chapter Review

R1

(a) Translated 1 unit to the left

(b) Translated 2 units up

(c) Squashed horizontally by a factor of 2

(d) Squashed vertically by a factor of 2

(e) Stretched horizontally by a factor of 2

(f) Translated 3 units down

(g) Translated 4 units to the right

(h) Stretched horizontally by a factor of 2

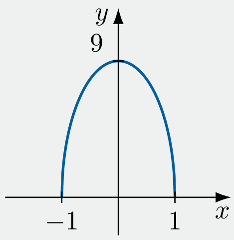
R2

(a) $y = (x+1)^3$ (b) $y = (x-4)^3$

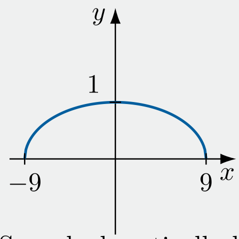
(c) $y = 3x + 11$ (d) $y = 3x - 10$

R3

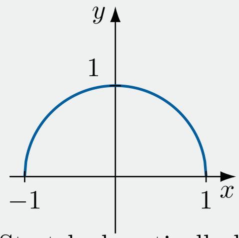
(a) Stretched vertically by a factor of 3 and squashed horizontally by a factor of 3



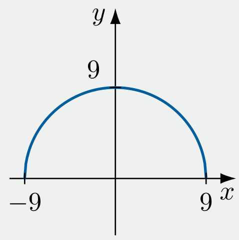
(b) Squashed vertically by a factor of 3 and stretched horizontally by a factor of 3



(c) Squashed vertically by a factor of 3 and squashed horizontally by a factor of 3

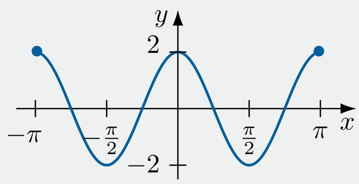


(d) Stretched vertically by a factor of 3 and stretched horizontally by a factor of 3

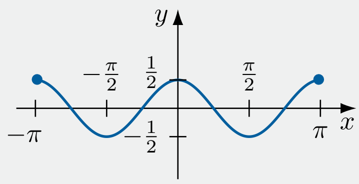


R4

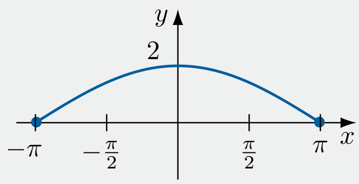
(a)



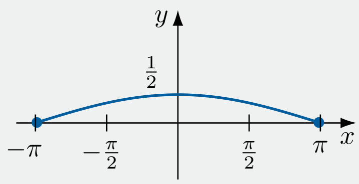
(b)



(c)



(d)



R5

(a) $y = \sqrt{x+2}$, $y = \sqrt{\dfrac{x}{3} + 2}$

(b) $y = \sqrt{\dfrac{x}{3}}$, $y = \sqrt{\dfrac{x+2}{3}}$

(c) $y = \sqrt{x-1}$, $y = \sqrt{2x-1}$

(d) $y = \sqrt{2x}$, $y = \sqrt{2x-2}$

R6

(a) $f(2x+1)$ (b) $f\,(2x+2)$

(c) $f\left(\dfrac{x}{3} - 2\right)$ (d) $f\left(\dfrac{x-2}{3}\right)$

R7

(a) $f(-x-2)$ (b) $f(2-x)$

(c) $f(3-x)$ (d) $f(-x-3)$

R8

(a) Translate left by $\dfrac{\pi}{3}$, then squash horizontally by a factor of 2

(b) Translate right by 6, then squash horizontally by a factor of 3

(c) Translate left by 3, then reflect across the y-axis

(d) Translate left by $\dfrac{\pi}{4}$, then reflect across the y-axis

R9

(a) Translate left by $\dfrac{\pi}{3}$, then squash horizontally by a factor of 2, then reflect across the y-axis

(b) Translate left by 2, then stretch horizontally by a factor of 3, then reflect across the y-axis

R10

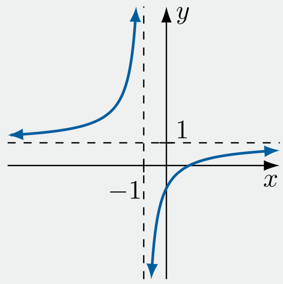
(a) $(0,-1)$, $(1,0)$

(b) $x = -1$ vertical asymptote $y = 1$, horizontal asymptote

(c) As $x \to \infty$, $y \to 1^-$, $x \to -\infty$, $y \to 1^+$

(d) Neither

(e)



R11

(a) $(0,0)$

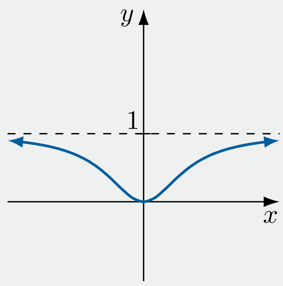
(b) Horizontal asymptote at $y = 1$, no vertical asymptote

(c) As $x \to \pm\infty$, $y \to 1^-$

(d) above the axis

(e) Even.

(f)



R12

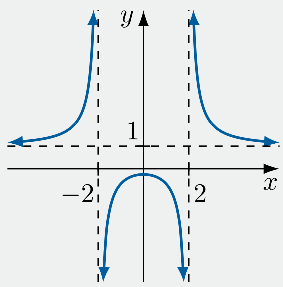
(a) $(0, -\frac{1}{4})$

(b) Horizontal asymptote at $y = 1$, vertical asymptotes at $x = \pm 2$

(c) As $x \to \pm\infty$, $y \to 1^+$

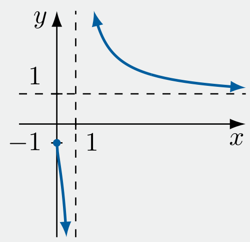
(d) Even.

(e)

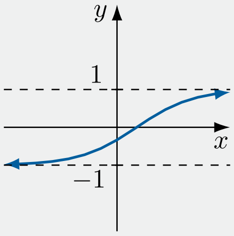


R13

(a)



(b)



R14

(a) $k = -19$

(b) $-19 < k < 8$ or $k > 13$

(c) $k = 8, 13$

(d) $8 < k < 13$

R15

(a) $x \geq 2$ or $x \leq -3$ (b) $-4 < x < \frac{2}{3}$

(c) $-5 \leq x \leq -4$ (d) $-2 \leq x \leq 3$

(e) $x < -\frac{7}{2}$ or $x > \frac{1}{2}$ (f) $-\frac{1}{3} < x < \frac{2}{3}$

R16

(a) All real x.

(b) No solutions.

(c) $x = -\frac{1}{2}$

R17

$-1 \leq x \leq 1$ or $5 \leq x \leq 7$

R18

(a) $(1, 3)$, $(-4, 8)$

(b) $x < -4$ or $x > 1$

R19

(a) 2 (b) 1 (c) 1

(d) 2 (e) 2 (f) 1

R20

(a) 1 (b) 1 (c) 2

2. Trigonometry

Exercise 2A

Trigonometric graphs

F1

(a) amplitude, mean, maximum, minimum

(b) period

F2

(a) Changes how much it oscillates vertically

(b) Changes the frequency of complete oscillations in a fixed interval

(c) Translates the graph horizontally

(d) Translates the graph vertically

F3

(a) k is the amplitude itself

(b) The period is given by $\frac{2\pi}{a}$

(c) b is the phase shift itself

(d) c is the vertical shift itself

F4

(a) Period 2π, range $[-1, 1]$

(b) Period 2π, range $[-1, 1]$

(c) Period π, range all real y

Q1

(a) Amplitude is the positive value of a, range $[-a, a]$

(b) Amplitude is the positive value of a, range $[-a, a]$

Q2

(a) Period is 2π, π, $\frac{2\pi}{3}$ respectively

(b) Period is 2π, π, $\frac{2\pi}{3}$ respectively

(c) Period is π, $\frac{\pi}{2}$, $\frac{\pi}{3}$ respectively

Q3

As b decreases graph is expanding horizontally

(a) Period is 4π, 6π respectively

(b) Period is 4π, 6π respectively

(c) Period is 2π, 3π respectively

Q4

(a) Translate up/down 1 or 2 units

(b) Translate up/down 1 or 2 units

(c) Translate up/down 1 or 2 units

Q5

(a) Translate left/right $\frac{\pi}{2}$ or $\frac{\pi}{4}$ units

(b) Translate left/right $\frac{\pi}{2}$ or $\frac{\pi}{4}$ units

(c) Translate left/right $\frac{\pi}{2}$ or $\frac{\pi}{4}$ units

Q6

(a) The range is $y \in [-2, 2]$ and the vertical shift is ± 1.

(b) The range is $y \in [-2, 2]$ and the vertical shift is ± 1.

(c) The range is still $x \in \mathbb{R}$, and the vertical shift is ± 1.

Q7

In general, the newly obtained graph is either the positive or negative version of the standard graph. This is because of the supplementary identities.

(a) Translate left 2π

(b) Translate right 2π

(c) Translate left 2π

(d) Translate right 2π

(e) Translate left π

(f) Translate right π

Q8

In general, the results below occur because of the complementary identities.

(a) The graph of $y = \sin x$ shifted left $\frac{\pi}{2}$, is $y = \cos x$

(b) The graph of $y = \cos x$ shifted right $\frac{\pi}{2}$, is $y = \sin x$

Q9

(a) Range $[-3, 3]$, period 2π

(b) Range $[-5, 5]$, period π

(c) Range $[-5, 5]$, period 4π

(d) Range $[-2, 2]$, period $\frac{\pi}{2}$

(e) Range all real y, period $\frac{\pi}{2}$

(f) Range all real y, period 6π

Q10

(a) 2 (b) 4 (c) 1

Q11

(a) $[-3, 3]$ (b) $[-2, 2]$ (c) $[-3, 3]$

(d) $[-1, 3]$ (e) $[-3, -7]$ (f) $[-9, 1]$

(g) $[1, 3]$ (h) $[-1, 2]$ (i) All real y

Q12

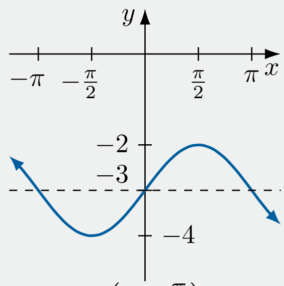
(a) $f(x) = 2\sin(2x)$ (b) $f(x) = 5\sin(2\pi x)$

Q13

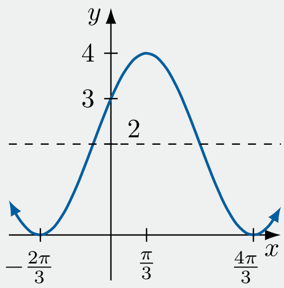
Range $[-1, 7]$, period 4

Q14

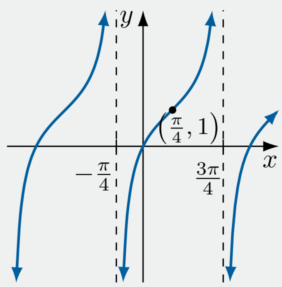
(a) $y = \cos\left(x - \frac{\pi}{2}\right) - 3$



(b) $y = 2\sin\left(x + \frac{\pi}{6}\right) + 2$



(c) $y = \tan\left(x - \frac{\pi}{4}\right) + 1$



Q15

(a) $y = 3\sin\left(2x + \frac{\pi}{3}\right) - 1$
Period π
Range $[-4, 2]$

(b) $y = -2\sin\left(3x - \frac{\pi}{2}\right) + 3$
Period $\frac{2\pi}{3}$
Range $[1, 5]$

Q16

(a) $y = \sin\left(2x - \frac{\pi}{3}\right)$

(b) $y = \sin\left(2x - \frac{\pi}{6}\right)$

(c) $y = \tan\left(3x + \frac{3\pi}{2}\right)$

(d) $y = \tan\left(3x + \frac{\pi}{2}\right)$

(e) $y = \tan\left(\frac{2x + \pi}{4}\right)$

(f) $y = \tan\left(\frac{x}{2} + \frac{\pi}{2}\right)$

Q17

(a) Stretch vertically by a factor of 2, then stretch horizontally by a factor of 3

(b) Translate up by 1, then squash horizontally by a factor of 2

(c) Reflect across x-axis, then stretch horizontally by a factor of 3

(d) Translate left by $\frac{\pi}{2}$, then squash vertically by a factor of 2

Q18

(a) Translate right by $\frac{\pi}{2}$, then squash horizontally by a factor of 2

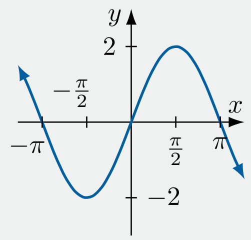
(b) Translate left by $\frac{\pi}{3}$, then reflect across the y-axis

(c) Translate left by $\frac{2\pi}{3}$, then squash horizontally by a factor of 2, then reflect across the y-axis

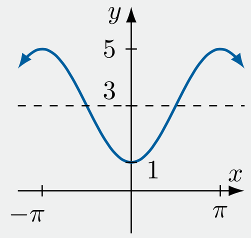
(d) Translate right by $\frac{\pi}{2}$, then stretch horizontally by a factor of 3, then reflect across the y-axis

Q19

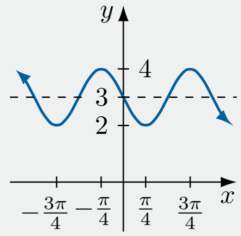
(a)



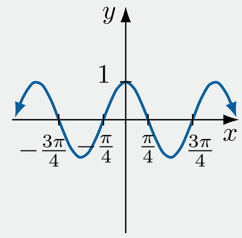
(b)



(c)



(d)



Q20

$y = 3\cos 2x$, period π, range $[-3, 3]$

Q21

$y = -2\sin x + 1$, period 2π, range $[-1, 3]$

Q22

$y = 2\cos 3x + 1$, period $\frac{2\pi}{3}$, range $[-1, 3]$

Q23

$y = \tan\frac{x}{2} + 1$, period 2π

Q24

(a) $y = -3\cos\frac{x}{2}$

(b) $y = \tan\frac{x}{4}$

(c) $y = 3\cos\frac{x}{2}$

(d) $y = \pi\sin 2x$

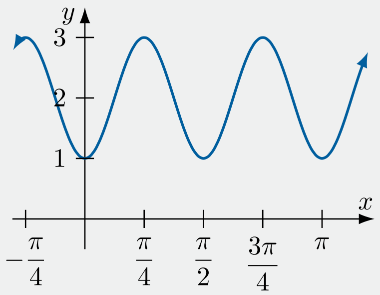
P1

(a) $y=-\frac{2\pi}{3}\sin 3x,\ y=-\frac{2\pi}{3}\cos\left(3x-\frac{\pi}{2}\right)$

(b) $y=\sin 4x+2$ and $y=\cos 4\left(x+\frac{3\pi}{8}\right)+2$

P2

$y=2-\cos(4x)$, amplitude 1, range $[1,3]$, period $\frac{\pi}{2}$



P3

Stretch vertically by a factor of 3
$y=3\cos(x)$
Reflect across the x-axis
$y=-3\cos(x)$
Translate up by 1 unit
$y=-3\cos(x)+1$
Translate right by $\frac{\pi}{4}$ units
$y=-3\cos\left(x-\frac{\pi}{4}\right)+1$
Squash horizontally by a factor of 2
$y=-3\cos\left(2x-\frac{\pi}{2}\right)+1$

Exercise 2B

Trigonometric equations

F1

(a) domain

(b) $-2\pi-b\le x-b\le 2\pi-b$

(c) $x-b$

(d) x

F2

(a) $-2a\pi\le ax\le 2a\pi$

(b) ax

(c) x

F3

Modify the domain so that $-2a\pi+b\le ax+b\le 2a\pi+b$, solve for $ax+b$ until the domain is exhausted, then make x the subject for each solution.

Q1

(a) $0,\ \pi,\ 2\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$

(d) $\frac{\pi}{2},\ \frac{3\pi}{2}$ (e) $0,\ 2\pi$ (f) π

Q2

(a) $\frac{\pi}{6},\ \frac{5\pi}{6}$ (b) $\frac{5\pi}{4},\ \frac{7\pi}{4}$ (c) $\frac{\pi}{3},\ \frac{2\pi}{3}$

(d) $\frac{2\pi}{3},\ \frac{4\pi}{3}$ (e) $\frac{\pi}{4},\ \frac{7\pi}{4}$ (f) $\frac{5\pi}{6},\ \frac{7\pi}{6}$

(g) $\frac{\pi}{6},\ \frac{7\pi}{6}$ (h) $\frac{3\pi}{4},\ \frac{7\pi}{4}$ (i) $\frac{\pi}{3},\ \frac{4\pi}{3}$

Q3

(a) $0\le 2x\le 4\pi$ (b) $\frac{\pi}{3}\le 2x+\frac{\pi}{3}\le\frac{13\pi}{3}$

(c) $\frac{2\pi}{3},\ \frac{7\pi}{3},\ \frac{8\pi}{3},\ \frac{13\pi}{3}$ (d) $0,\ \frac{\pi}{6},\ \pi,\ \frac{7\pi}{6},\ 2\pi$

Q4

(a) $\frac{7\pi}{12},\ \frac{11\pi}{12},\ \frac{19\pi}{12},\ \frac{23\pi}{12}$

(b) $\frac{2\pi}{3},\ \frac{4\pi}{3}$

(c) $\frac{15\pi}{8}$

(d) $\frac{\pi}{12},\ \frac{7\pi}{12},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{17\pi}{12}$

(e) No solution

(f) $\frac{2\pi}{9},\ \frac{10\pi}{9},\ \frac{14\pi}{9}$

(g) $\frac{\pi}{4},\ \frac{7\pi}{12},\ \frac{11\pi}{12},\ \frac{5\pi}{4},\ \frac{19\pi}{12}$

(h) $\frac{\pi}{3}$

(i) $\frac{4\pi}{9},\ \frac{10\pi}{9},\ \frac{16\pi}{9}$

Q5

(a) $-\dfrac{\pi}{2}, \dfrac{5\pi}{6}$

(b) $-\dfrac{\pi}{12}, -\dfrac{7\pi}{12}$

(c) $-\dfrac{2\pi}{3}, \dfrac{\pi}{3}$

Q6

(a) $-\dfrac{11\pi}{12}, \dfrac{\pi}{12}$

(b) No solution

(c) $-\dfrac{2\pi}{3}, -\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{5\pi}{6}$

(d) $-\dfrac{7\pi}{9}, -\dfrac{5\pi}{9}, -\dfrac{\pi}{9}, \dfrac{\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}$

(e) $0, -\pi, -\dfrac{2\pi}{3}, \dfrac{\pi}{3}, \pi$

(f) $-\pi, \pi$

Q7

(a) $\pm 2\pi, \pm\dfrac{3\pi}{2}, \pm\pi, \pm\dfrac{\pi}{2}, 0$

(b) $\pm\dfrac{\pi}{6}, \pm\dfrac{\pi}{2}, \pm\dfrac{5\pi}{6}, \pm\dfrac{7\pi}{6}, \pm\dfrac{3\pi}{2}, \pm\dfrac{11\pi}{6}$

(c) $0, \pm\dfrac{\pi}{2}, \pm\pi, \pm\dfrac{3\pi}{2}, \pm 2\pi$

(d) $-\pi$

(e) 0

(f) $-\dfrac{3\pi}{2}, 0, \dfrac{3\pi}{2}$

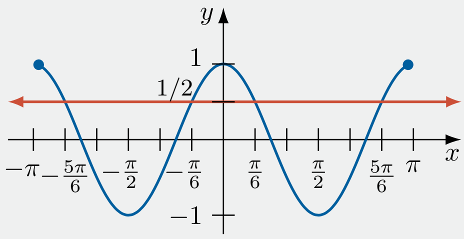
(g) $\dfrac{5\pi}{12}, \dfrac{17\pi}{12}, -\dfrac{7\pi}{12}, -\dfrac{19\pi}{12}$

(h) $-\dfrac{7\pi}{6}, -\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{11\pi}{6}$

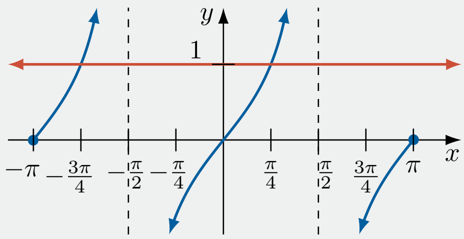
(i) $-\dfrac{3\pi}{2}, \dfrac{\pi}{2}$

Q8

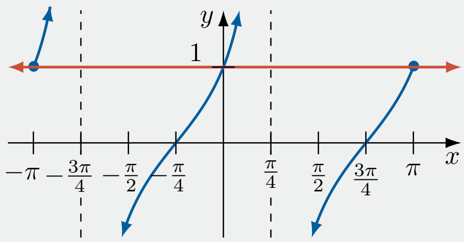
(a) $\cos 2x = \dfrac{1}{2}$, $x = -\dfrac{5\pi}{6}, -\dfrac{\pi}{6}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$



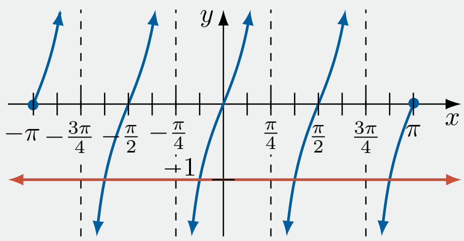
(b) $\tan x = 1$, $x = -\dfrac{3\pi}{4}, \dfrac{\pi}{4}$



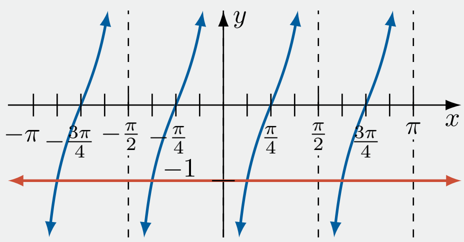
(c) $\tan\left(x + \dfrac{\pi}{4}\right) = 1$, $x = 0, -\pi, \pi$



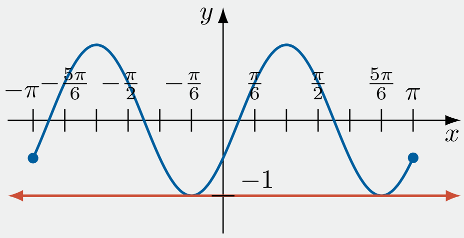
(d) $\tan 2x = -1$, $x = -\dfrac{5\pi}{8}, -\dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{7\pi}{8}$



(e) $\tan\left(2x - \dfrac{\pi}{2}\right) = -1$, $x = -\dfrac{7\pi}{8}, -\dfrac{3\pi}{8}, \dfrac{\pi}{8}, \dfrac{5\pi}{8}$



(f) $\sin\left(2x - \frac{\pi}{6}\right) = -1,\ x = -\frac{\pi}{6}, \frac{5\pi}{6}$



P1

(a) $0, \frac{3\pi}{2}, 2\pi$

(b) $\frac{\pi}{3}, \frac{5\pi}{3}$

(c) $\frac{\pi}{2}, \frac{3\pi}{2}$

P2

(a) $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

(b) $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

(c) $\frac{\pi}{4}, \frac{5\pi}{4}$

P3

(a) $y = \sin(3x - \pi)$

(b) $y = \sin\left(2x + \frac{\pi}{3}\right)$

Exercise 2C

Applications of trigonometric equations

Q1

(a) $x = 3$

(b) 2 seconds

(c) $-3 \le x \le 3$

(d) $x = 0$

(e) 2

Q2

(a) $x = 80 \sin \frac{\pi t}{12}$

(b) $t = 2$ and $t = 14$

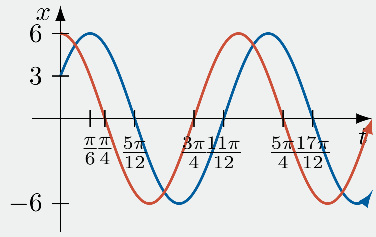
(c) $x = 40$

Q3

(a) $6, \pi$

(b) $x = 6\cos 2t$ is initially at $x = 6$, while $x = 6\cos\left(2t - \frac{\pi}{3}\right)$ starts motion at $x = 3$

(c)



(d) $x = 6,\ x = 3$

(e) Amplitude and period

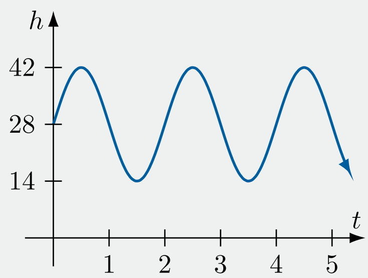
(f) 4 times each

(g) $x = 6\cos 2t = -3$ when
$t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \dots$
$x = 6\cos 2(t - \frac{\pi}{6}) = -3$ when
$t = \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, \dots$

(h) Period the same, but this particle oscillates between 0 and 12

Q4

(a)

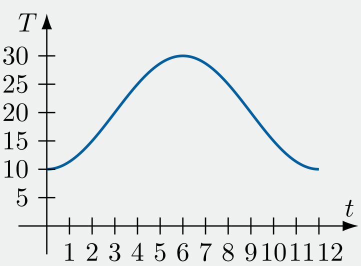


(b) 2 seconds

(c) $h = 28 + 14\sin(at),\ a > \pi$

Q5

(a)

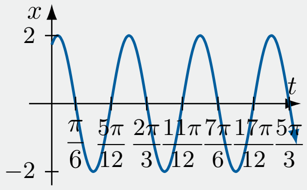


(b) 30°C, July

(c) 10°C, January

P1

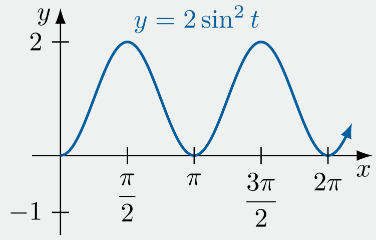
$y = 2\cos\left(4x - \frac{\pi}{6}\right)$



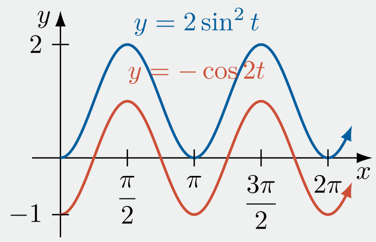
P2

(a) k, $\frac{2\pi}{n}$, $y \in [x_0 - k, x_0 + k]$

(b)



(c)



$x_0 = 1,\ k = 1,\ n = 2$

(d) $y = 1 - \cos(2t)$, 1, 1, $[0, 2]$, π

Chapter Review

R1

(a) π, $[-2, 2]$

(b) 6π, $[-2, 2]$

(c) 2π, all real y

(d) 2π, $[-2, 2]$

(e) $\frac{2\pi}{3}$, $[-2, 2]$

(f) π, $[3, 5]$

(g) 4π, $[4, 6]$

(h) π, $[-3, 3]$

(i) 6π, $[1, 3]$

R2

(a) $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

(b) $\cos(x - \pi) = -\cos x$

(c) $\cos(x - 2\pi) = \cos x$

R3

$y = 3\sin 2x$

R4

$y = -2\cos\frac{x}{2}$, period 4π, range $[-2, 2]$

R5

$y = 3\sin x - 3$, period 2π, range $[-6, 0]$

R6

$y = -3\cos 2x + 1$, period π, range $[-2, 4]$

R7

$y = -\tan 2x$, period $\frac{\pi}{2}$

R8

(a) Translate right by $\frac{\pi}{4}$ then squash horizontally by a factor of 2.

(b) Translate left by $\frac{\pi}{6}$, then stretch horizontally by a factor of 3.

(c) Translate left by $\frac{\pi}{4}$, then reflect across the y-axis.

R9

(a) Stretch vertically by a factor of 2, then translate down by 3

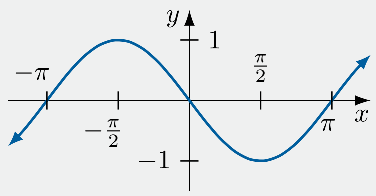
(b) Reflect across the x-axis, then translate up by 2

(c) Translate right by $\frac{\pi}{2}$, then squash horizontally by a factor of 2

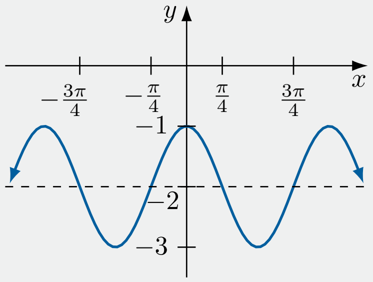
(d) Translate left by $\frac{\pi}{3}$, then stretch horizontally by a factor of 2

R10

(a) Shift to right of $\frac{\pi}{2}$ then reflected in the x axis

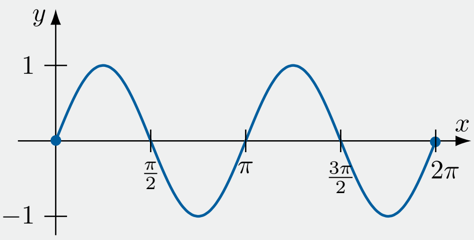


(b) Shift to left of $\frac{\pi}{4}$, then squash horizontally by a factor of 2 and shift down 2

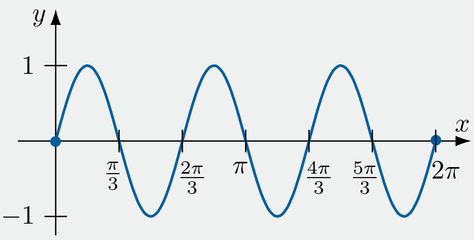


R11

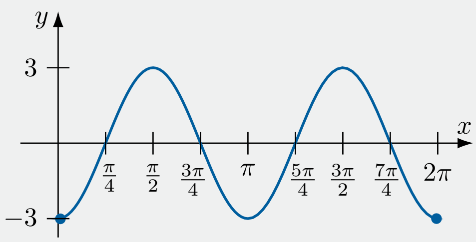
(a)



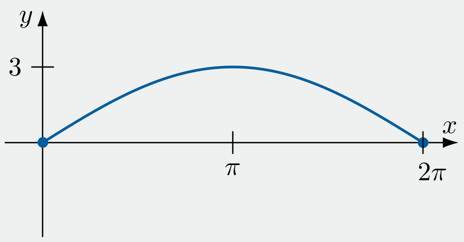
(b)



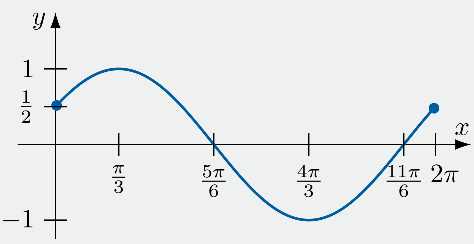
(c)



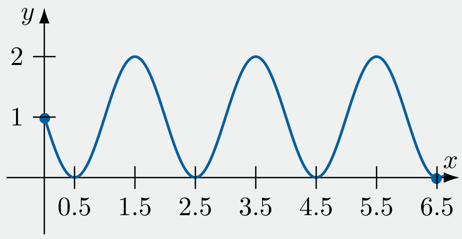
(d)



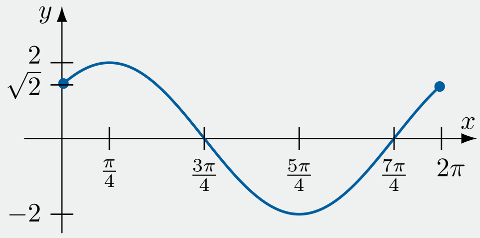
(e)



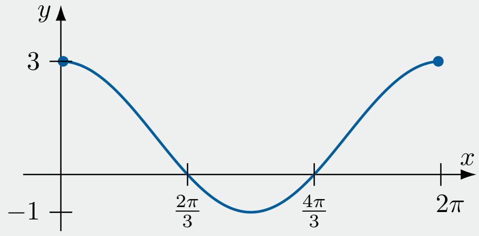
(f)



(g)

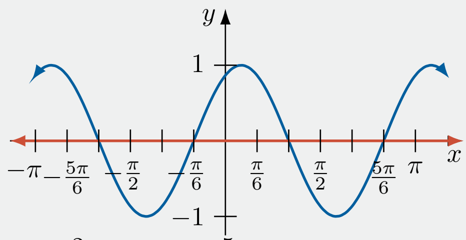


(h)



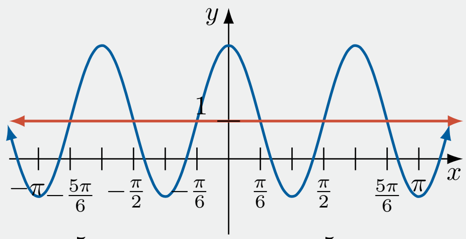
R12

(a)



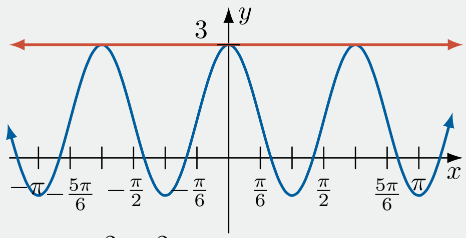
$x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$

(b)



$x = -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

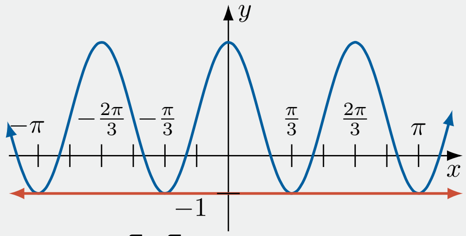
(c)



$x = 0, -\frac{2\pi}{3}, \frac{2\pi}{3}$

ANSWERS

(d)



$x = -\pi, -\frac{\pi}{3}, \frac{\pi}{3}, \pi$

R13

(a) No solution

(b) $x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

(c) $\frac{\pi}{6}, \frac{3\pi}{2}$

(d) $x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$

R14

(a) $x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

(b) $x = -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$

(c) $x = \frac{3\pi}{4}$

(d) $x = -\frac{\pi}{2}, \frac{\pi}{2}$

(e) $x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$

(f) $x = -\frac{11\pi}{18}, -\frac{7\pi}{18}, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$

(g) $x = -\frac{5\pi}{8}, -\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}$

(h) No solution

(i) $x = -\pi, \frac{\pi}{3}, \pi$

(j) $x = -\frac{3\pi}{4}, -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$

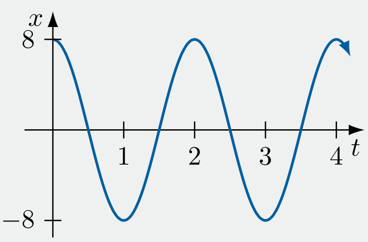
(l) $x = -\pi, -\frac{5\pi}{9}, -\frac{\pi}{3}, \frac{\pi}{9}, \frac{\pi}{3}, \frac{7\pi}{9}, \pi$

R15

(a) $x = \pm 8$

(b) 2

(c)



R16

(a) 1.2 m and 2 m

(b) 1 second

R17

(a) 4

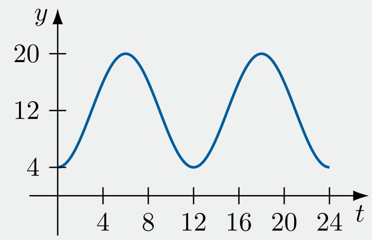
(b) 365 days as formula gives full cycle for the year

(c) 16 hours on 183rd day 2nd July

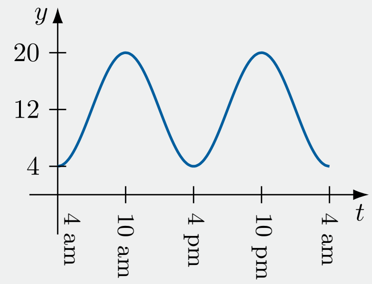
R18

(a) 20 m and 4 m

(b)



(c)



3. Sequences and Series

Exercise 3A

Arithmetic progression

F1

(a) sequence

(b) series

F2

(a) constant

(b) $T_{n-1} + d$, a

(c) $a + d$, $a + 2d$

(d) $a + (n - 1)d$

F3

T_2

T_{n-2}

F4

(a) progression (b) $\frac{n}{2}(a+l)$

(c) $\frac{n}{2}(2a+(n-1)d)$ (d) S_{n-1}

F5

$\frac{a+b}{2}$

Q1

(a) $T_1=4,\ T_2=7,\ T_3=10,\ T_4=13$

(b) $T_1=-3,\ T_2=1,\ T_3=5,\ T_4=9$

Q2

(a) $T_n=T_{n-1}+8,\ T_1=-10$

(b) $T_n=T_{n-1}-5,\ T_1=9$

(c) $T_n=T_{n-1}+6,\ T_1=5$

(d) $T_n=T_{n-1}-7,\ T_1=5$

(e) $T_n=T_{n-1}-6,\ T_1=14$

Q3

(a) $d=7$

(b) $T_n=T_{n-1}+7,\ T_1=4$

Q4

(a) $T_1=-2,\ T_2=9,\ T_3=20,\ T_4=31$

(b) $T_1=50,\ T_2=42,\ T_3=34,\ T_4=26$

Q5

(a) $T_{10}=35$ as $d=4$

(b) $T_{10}=\sqrt{5}+9\sqrt{2}$ as $d=\sqrt{2}$

Q6

(a) $a=4$ (b) $a=3$

Q7

(a) $a=12$

(b) $d=4$

(c) $T_8=40$

Q8

(a) $T_n=8n-116$ (b) $n=15$

Q9

(a) T_{32} (b) T_{27}

Q10

(a) $T_{11}=-2$ (b) $n=19$

(c)

(i) $T_9=2$ (ii) $T_9=-4$

Q11

(a) 19 (b) $k+1$

Q12

(a) $\frac{3}{16}$ (b) $\sqrt{5}$

Q13

22, 19, 16, 13, 10

Q14

3, 9, 15, 21, 27, 33, 39

Q15

See full worked solutions.

Q16

(a) 21 (b) 15

Q17

(a) 13 (b) 25

Q18

748

Q19

(a) $-11,\ -8,\ -5$ (b) 184

Q20

(a) See full worked solutions.

(b) $8\log 2=\log 256$

(c) $35\log 2$

Q21

(a) $S_n=\frac{n}{2}(3n-1)$

(b) $T_4=39$ and $S_{10}=615$

Q22

(a) T_6 (b) S_7 (c) S_5

(d) T_n (e) S_{n+1} (f) S_{n-1}

ANSWERS

Q23

(a) $S_9 - S_8$

(b) 47

(c) $T_n = 6n - 7$

Q24

(a) 594 (b) 19899

Q25

$a = 24$, $d = -3$

Q26

(a) 11 or 15

(b) When $n = 11$, the sequence sums to 495. The next few terms go beyond 495 but then terms start becoming negative, which brings the sum back down to 495.

Q27

25 cm

Q28

(a) $T_n = T_{n-1} + 8$, $T_1 = 30$

(b) $T_n = 8n + 22$

(c) $T_5 = 62$

Q29

(a) 17 (b) 80

Q30

(a) $3n + 5$

(b) See full worked solutions.

Q31

(a) \$335 000 (b) \$3 880 000

Q32

$\ln 6$

Q33

12

P1

(a) $-36 \log_3 x$ (b) 81

P2

(a) See full worked solutions.

(b) 2, 8, 14, 20

P3

$T_1 = 2$, $T_2 = 7$, $T_3 = 12$, $T_n = 5n - 3$

P4

(a) See full worked solutions.

(b) $12d$

(c) 0

P5

There are 86 terms divisible by 7 from 203 to 798, which all sum to 43043. The total sum of integers from 201 to 799 is 299500, so the we can subtract the results to get 256457.

P6

(a) Show that $5(T_4 + T_7) = S_{10}$

(b) Show that $5(T_5 + T_6) = S_{10}$

(c) Show that $5(T_m + T_n) = S_{10}$ and use $m + n = 11$ to help prove it.

Exercise 3B

Geometric progression

F1

(a) common ratio (b) ar, ar^2

(c) rT_{n-1}, a (d) ar^{n-1}

(e) T_1, T_{n-2}

F2

(a) progression

(b) $\dfrac{a(r^n - 1)}{r - 1}$

(c) $\dfrac{a(1 - r^n)}{1 - r}$

F3

(a) limiting, infinitely

(b) 1, −1, 1

(c) $\dfrac{a}{1 - r}$

F4

$\sqrt{ab}$

Q1

(a) $T_n = -\dfrac{1}{3}T_{n-1}$, $T_1 = -9$

(b) $T_n = 2T_{n-1}$, $T_1 = \dfrac{1}{2}$

Q2
$\frac{3}{5}, 9$

Q3
(a) $T_1 = -\frac{1}{8}, T_2 = -\frac{1}{4}, T_3 = -\frac{1}{2}, T_4 = -1$

(b) $T_1 = 12, T_2 = -3, T_3 = \frac{3}{4}, T_4 = -\frac{3}{16}$

(c) $T_1 = \sqrt{2}, T_2 = 4, T_4 = 8\sqrt{2}, T_4 = 32$

Q4
78732

Q5
(a) 39366
(b) No
(c) 5
(d) 13122

Q6
9

Q7
(a) $T_6 = \frac{1}{729}$
(b) $T_6 = \frac{243}{16}$
(c) $T_6 = \pm 20$

Q8
8, ±4, 2

Q9
0.7

Q10
(a) A population of 1800 decreases by 5% every year.
(b) A population of 20000 increases by 20% every year.

See full worked solutions.

Q11
$> \log 0.3, n < 7.408$

Q12
(a) $n < 34.18$
(b) $n > 23.47$

Q13
(a) 10154
(b) 2032

Q14
(a) $\pm\frac{1}{4\sqrt{2}}$
(b) ± 2

Q15
12, 24, 48 or −12, 24, −48

Q16
See full worked solutions.

Q17
$S_{15} = 3^{15} - 1 = 14348906$

Q18
$S_n = \frac{4}{9}\left(1 - \left(\frac{1}{10}\right)^n\right)$

Q19
(a) 7 terms
(b) 8 terms

Q20
3

Q21
384

Q22
(a) $\frac{1}{3}\left(1 - \frac{1}{10^8}\right)$
(b) $\frac{1}{3}\left(1 - \frac{1}{10^n}\right)$
(c) See full worked solutions.

Q23
(a) $\frac{16}{45}$
(b) $\frac{259}{495}$

Q24
$\frac{1}{2}$

Q25
9, 6, 4, ... or 45, −30, 20, ...

Q26
$-1\frac{1}{3} < x < -\frac{2}{3}$

Q27
(a) All values of x except $x = n\pi$, where n an integer
(b) $\frac{1}{1 + \cos^2 x}$

Q28
(a) $-\frac{\pi}{4} < x < \frac{\pi}{4}$
(b) $\frac{1}{1 - \tan^2 x}$

Q29
(a) $\frac{1}{2}$ m
(b) 46m
(c) 48 m

Q30
9.63 km

ANSWERS

P1

$r = -3,\ a = 2$

P2

$\dfrac{T_7}{T_4} = \dfrac{T_{16}}{T_7}$ becomes $2a + 3d = 0$ or in other words $a = -\dfrac{3}{2}d$. So, any AP satisfying this identity will work. Setting $d = 2$ we get the sequence $T_n = 2n - 5$, in which case $T_4 = 3,\ T_7 = 9,\ T_{16} = 27$, which indeed forms a GP as expected.

P3

See full worked solutions.

P4

$x < -4$ or $x > 2$

P5

(a) $2n + 1$ terms

(b) $S_{2n+1} = \dfrac{5^{-n}\left(5^{2n+1} - 1\right)}{4}$

P6

13

P7

$r = \cos^2\theta$, so $S_\infty = \dfrac{\sin\theta}{1 - \cos^2\theta} = \operatorname{cosec}\theta$. The limiting sum exists since $|\cos\theta| < 1$ in the domain provided, so $\left|\cos^2\theta\right| < 1$

P8

(a) $2^{n+1} - 2^{-2}$

(b) $S_{2n+1} = 2^{n+1} - 2^{-n}$

P9

(a) arithmetic progression

(b) $p + 2$

(c) $\dfrac{2^{3x}\left(2^{-2x(p+1)} - 1\right)}{1 - 2^{2x}}$

P10

$-\dfrac{1}{p},\ -\dfrac{1}{p^2},\ -\dfrac{1}{p^3}$

P11

$A = 8,\ B = 27,\ S_\infty = 56.5$

P12

No, as $r = -2$

P13

(a) $\dfrac{ar^n}{1 - r}$

(b) $r^n \to 0$ as n gets large

P14

(a) $1\dfrac{11}{16}$ m

(b) $5\dfrac{1}{3}$ m

(c) $\dfrac{81}{64}$ m

(d) $4\left(\dfrac{3}{4}\right)^n$

(e) $\dfrac{112}{3}$ m

Exercise 3C

Simple and compound interest

F1

(a) fixed

(b) percentage, principle

F2

$n\left(\dfrac{r}{100}\right)P_0$

F3

$P_0\left(1 + \dfrac{r}{100}\right)^n$. Note: Take care to ensure that r is matching the frequency of n.

F4

(a) halve, 2

(b) quarter, 4

(c) 12, 12

Q1

(a) \$1 250

(b) 2.5%

(c) 20 years

Q2

(a) \$3 240

(b) $P_{n+1} = P_n + 120,\ P_0 = \$3\,000$

(c) $P_4 = \$3\,480$

(d) 9 years later

Q3

(a)

(i) 864

(ii) $P_0 = 18\,000,\ P_{n+1} = P_n + 864$

(iii) $P_n = 18\,000 + 864n$

(iv) \$22 320

(b)

(i) $P_{n+1} = (1.012)P_n,\ P_0 = 18\,000$

(ii) $P_4 = (1.012)^4 \times 18\,000$

(iii) $P_8 = (1.012)^8 \times 18\,000$

(iv) $P_{4n} = (1.012)^{4n} \times 18\,000$

Q4

(a) $P_0 = 28\,000,\ P_{n+1} = P_n - 4\,200$

(b) \$15 400

(c) After 4 years

Q5

(a)

	Bob's salary	Mary's salary
1st year	45 000	40 000
2nd year	47 800	44 000
3rd year	50 600	48 400
4th year	53 400	53 240
5th year	56 200	58 564

(b) 5th year

(c) \$3 000

(d) After 8 years of working there

Q6

(a)

(i) \$2 490 000

(ii) 2016

(b) 3.499%

Q7

(a) $P_n = 600 - 0.12n$

(b) 588

(c) 166.67 weeks

Q8

(a) \$4 800

(b) \$4 887.09

(c) \$4 907.02

Q9

(a) 12 years

(b) 46.56 quarters or 11 years 3 quarters

(c) 11 years 7 months

Q10

(a) $A_n = 500(1.015)^{80}$

(b) $A_n = 500(1.005)^{240}$

(c) Interest $= 500(1.005)^{240} - 500$

(d) $A_n = 500(0.94)^{10}$

(e) $500(0.94)^n = 250$

Q11

4.75% fortnightly

Q12

(a)

(i) Each year the investment grows by 6% i.e. 1.06 of its value from the previous year. We use $P_0 = 4000$ because that was the given initial investment.

(ii) \$4 494.4

(iii) \$494.40

(iv) \$254.40

(v) 4 years

(b)

(i) $P_{n+1} = 1.03P_n$ where $P_0 = \$4\,000$

(ii) $P_4 = 1.03^4(4\,000)$

(iii) \$4 502.04

(c) \$4 508.64

Q13

(a) $A_n = 1.0025A_{n-1},\ A_0 = 3000$

(b) \$3022.56

Q14

(a) \$23 800

(b) \$20 230

(c) $V_{n+1} = 0.85V_n$ and $V_0 = 28\,000$

(d) \$10 560

P1

\$55 681

P2

(a) \$8 589.2 (b) \$8 564.2

(c) \$8 558.5 (d) \$8 556.9

P3

(a) 8.45% (b) 8.19% (c) 8.14%

P4

24%

P5

(a) $V_{n+1} = V_n - 720$, $V_0 = \$9\,000$, or $V_{n+1} = 0.9V_n$, $V_0 = \$9\,000$

(b) \$6 840 or \$6 561

Exercise 3D

Annuities, present and future values

F1

(a) today (b) future

(c) interest, number

F2

(a) annuity (b) future

(c) periods (d) x

(e) reverse

F3

(a) less, present (b) sum

F4

(a) halve (b) quarterly (c) 12

Q1

(a) \$5408 (b) \$5412 (c) \$5414.50

Q2

(a) \$9462.15 (b) \$243.20

(c) $\$13,024.95$ (d) \$6000

Q3

(a) \$6002.10 (b) \$5417.20

Q4

(a) \$373.65 (b) \$4419

Q5

(a) Loan A is \$2109.64 whereas Loan B is \$1705.63.

(b) Loan A is $\$129,735.20$ whereas Loan B is $\$159,350.94$

(c) Anthony pays less interest with Loan A. However, he may opt for Loan B if he cannot make the monthly repayments for Loan A, or if he simply prefers lower monthly repayments.

Q6

(a) \$302.05 (b) \$4257.11 (c) \$9912.41

P1

2 years

P2

(a) $\$24,998.86$, so probably about $\$25,000$

(b) At her current rate, she will repay only $\$99,320$ over the first four years. She needs to make repayments of \$2517 per month to repay $\$100,000$ over four years.

Exercise 3E

Regular instalments

F1

(a) Mr^n

(b) Mr^{n-1}

(c) Mr

(d) $A_n = Mr^n + Mr^{n-1} + Mr^{n-2} + \cdots + Mr$

F2

(a) $A_1 = Mr$

(b) At the beginning of the next month, the account now amounts to $A_1 + M$ because of the monthly deposit. Then, this entire amount earns interest at the end of the month, so we multiple the whole thing by r to get $A_2 = (A_1 + M)r$.

(c) $A_2 = Mr^2 + Mr$

(d) $A_3 = Mr^3 + Mr^2 + Mr$

(e) $A_n = Mr^n + Mr^{n-1} + Mr^{n-2} + \cdots + Mr$

F3

M, Mr

Q1

(a) $\$300(1.06)^{21}$ (b) $\$300(1.06)^{20}$ (c) $\$13017.69

Q2

(a) See full worked solutions.

(b) See full worked solutions.

(c) $A_{21} = 300(1 + 1.06 + 1.06^2 + \cdots + 1.06^{21})$

(d) $\$13017.69$

Q3

(a) 0.005% (b) 0.0023% (c) 0.0025%

(d) 0.00115% (e) 0.004% (f) 0.012%

Q4

(a) $\$500(1.06)^{30}$ (b) $41 900.84 (c) $26 900.84

Q5

(a) $266.33 (b) 85 months

(c) $3 502.64 (d) $842.64

(e) $3765.08, $262.44

Q6

(a) $21035.66 (b) $56421.51

Q7

(a) $43 392.29 (b) $42 054.06

Q8

(a) 5 714.08 (b) $500(1.015)^{53}$ (c) $52 993.26

Q9

(a) $252.95 (b) $6465.31 (c) $161.33

Q10

$4 017.84

Q11

$626.72

Q12

(a) $493, 684.13 (b) $2430.70

Q13

(a) See full worked solutions.

(b) See full worked solutions.

(c) $220, 694.49

(d) $4531.15

Q14

(a) See full worked solutions.

(b) See full worked solutions.

(c) $n = 35.56$

(d) 36 months

Q15

87 months

P1

(a) $280, 181.54 (b) $325, 463.58 (c) $507, 680.62

P2

$58, 712.42

P3

$419, 579.43

P4

(a) $\$500(1.05)^{18}$ (b) $\$500(1.05)^{18}$

(c) $\$500(1.05)^{18}$ (d) $22 863

P5

(a) See full worked solutions.

(b) See full worked solutions.

Exercise 3F

Loan repayments

F1

(a) $A_1 = Br - M$

(b) $A_2 = Br^2 - Mr - M$

(c) $A_3 = Br^3 - Mr^2 - Mr - M$

(d) $A_n = Br^n - M(1 + r + r^2 + \cdots + r^{n-1})$

ANSWERS

Q1

(a) First the interest is charged on \$2000, and then the repayment is made.

(b) $A_2 = rA_1 - M$

(c) $A_3 = rA_2 - M$

(d) $A_n = 2000r^n - M\left(1 + r + r^2 + \cdots + r^{n-1}\right)$

(e) See full worked solutions.

(f) The loan is fully paid after four years i.e. 48 monthly repayments.

(g) \$46.97

(h) \$254.56

Q2

(a) Monthly interest rate is 0.0075, so we multiply the principle by 1.0075 each month. But this occurs three times before Lucy makes her regular quarterly repayment.

(b) $A_1 = 500000r - Q,\ A_2 = A_1 r - Q$

(c) $A_n = 500000r^n - Q(1 + r + r^2 + \cdots + r^{n-1})$

(d) 60

(e) \$15 328.39

(f) \$419, 703.40

(g) 5.596%

Q3

(a) $100000(1.01)^{12} - A$ (b) \$59700.95

(c) \$19 401.89 (d) $r = 9.7\%$ p.a.

Q4

$A_1 = 10\,000 - M,\ A_2 = 10\,000 - 2M,$
$A_3 = (10\,000 - 2M)(1.01) - M$

Q5

111 months

Q6

(a) 94 months (b) \$43, 110

Q7

(a) 46 years, 4 months

(b) 46 years, 9 months

(c) 47 years

Q8

(a) See full worked solutions.

(b) No solution for n.

(c) The loan will never be repaid because Scott's repayment is not even enough to cover the interest itself.

Q9

(a) \$5551.03

(b) \$287129.54

(c) It will take 1.03 more months, so technically 2 more monthly repayments, though the second repayment will be very small relative to what it normally is.

Q10

(a) $A_4 = 32000r^4 - 3000\left(\frac{r^4 - 1}{r - 1}\right)$

(b) 13 instalments, which consists of 12 *full* instalments and one final partial instalment.

(c)

- (i) \$22, 273
- (ii) See full worked solutions.
- (iii) With the new repayments, the life of the loan is now 17 quarters, which is 4 quarters more than the original life of the loan.

Q11

(a) See full worked solutions.

(b) See full worked solutions.

(c) Monique can make in total 525 more withdrawals, but only 524 of them are the full amount of \$6000.

Q12

\$421, 752.91

Q13

(a) \$73, 597.20

(b) \$597.20

(c) See full worked solutions.

(d) 89 months

P1

(a) \$3635.88 (b) \$359 116.99

(c) 101 months (d) 19 months less

P2

(a) $\$103,561.75$ (b) $\$177.27$

P3

(a) $A_1 = P(1+r) - M,\ A_2 = M_1(1+r) - M$

(b) See full worked solutions.

P4

See full worked solutions.

Chapter Review

R1

(a) geometric progression, $-\dfrac{27}{16}$

(b) arithmetic progression, $5\sqrt{3}$

(c) arithmetic progression, $6\sqrt{3}$

(d) geometric progression, $\dfrac{24}{\sqrt{3}}$

(e) arithmetic progression, $5a - 11b$

(f) arithmetic progression, $5q - 3p$

R2

(a) $T_n = T_{n-1} - 2,\ T_1 = 9$

(b) $T_n = T_{n-1} + 3,\ T_1 = 5$

R3

(a) $2,\ -1,\ -4,\ -7$ (b) $2,\ -6,\ 18,\ -54$

R4

(a) 19 (b) 4.5

(c) 18 (d) 11 476

(e) $-15,\ -10,\ -5$

R5

(a) $T_n = 4n - 96$ (b) 77

(c) 4620 (d) $n(2n - 94)$

(e) $n = 47,\ T_{47} = 92$ (f) $n = 21,\ T_{21} = 40$

R6

(a) 80 m (b) 48 m

R7

(a) 78 seats (b) 106 seats

(c) Row P (d) 1360

R8

(a) $25 - 2n$ (b) 23, 21, 19, 17

(c) 13 (d) $0 < n < 24$

R9

See full worked solutions.

R10

$x = 3$

R11

(a) \$400 (b) \$83 600

R12

(a) $\dfrac{62}{495}$ (b) $\dfrac{7}{15}$

R13

(a) $T_{18} = 3(\sqrt{2})^{17}$

(b) $r = \dfrac{1}{3},\ T_5 = 6$

(c) $\pm 6\sqrt{2},\ 12,\ \pm 12\sqrt{2},\ 24,\ \pm 24\sqrt{2}$

(d) $r = \pm\dfrac{1}{2}$, 16, 8, 4 or 48, -24, 12

R14

(a) arithmetic progression with $d = \log 3$

(b) arithmetic progression with $d = \log 3$

(c) geometric progression with $r = 2$

R15

(a) $3(2\log 5 + 5\log 3)$

(b) $21\log 3$

(c) $63\log 3$

R16

See full worked solutions.

R17

$m = \dfrac{1}{2},\ n = -1$

R18

\$2 200.78

ANSWERS

R19

(a) \$720 (b) \$2 500

R20

Respectively \$13 382.26 , \$13 116.51, \$13 091.71, and hence Marc should choose 6% p.a. compounded annually.

R21

4.26%

R22

(a) 18 years

(b) 17 years 5 months

R23

(a) 12 years approximately

(b) \$11, 314

(c) 7 years

(d) \$5092

R24

(a) \$8500 (b) \$18, 776.14

R25

(a) \$53, 064

(b) \$15, 770.25

(c) $F = \$2460.80$, so $P = \$2318.18$.

R26

(a) \$1636.11

(b) \$13, 533.28

R27

\$3423.30

R28

10.4%

R29

\$4 500

R30

9 years and 3 months

R31

(a) \$19, 143.92

(b) After two years, Mary earns \$5276.98. This amount continues to earn interest for another 48 months to amount to \$6704.35. The last four years, Mary earns \$10, 873.66, so the total is \$17, 578. Hence, she has \$1565.91 less than the original rate.

R32

(a) \$6134.19 (b) \$195.62

R33

(a) \$93, 627.14

(b) \$59, 503.09

(c) Chloe earned \$13, 627.14 and \$11, 503.09 in (a) and (b) respectively.

R34

(a) \$62.96 (b) \$2411.92 (c) \$48.42

R35

(a) \$718.75

(b) 41 more repayments.

R36

(a) \$13, 463.50 (b) \$3854

R37

(a) See full worked solutions.

(b) 30 more repayments. After the 24th repayment Paul owes \$23, 179.20 so after the lump sum payment he now owes \$15, 179.20.

(c) \$2 400

4. Further differentiation

Exercise 4A

Differentiation of exponential functions

F1

(a) e^x (b) ae^{ax+b} (c) $f'(x)e^{f(x)}$

F2

$e^{\frac{1}{2}x}$

Q1

(a) $5e^{5x}$ (b) $-24e^{-3x}$

(c) $-2e^{-\frac{x}{2}}$ (d) $3e^{3x}+2e^{-2x}$

(e) $2e^{4x+1}$ (f) $3e^{\frac{3x-1}{2}}$

(g) $\frac{-16}{e^{4x}}+e^{4x}$ (h) $e^{px}+e^{qx}$

(i) $\frac{5}{2}\sqrt{e^{5x}}$ (j) $\frac{5}{3}\sqrt[3]{e^{5x}}$

(k) $-2e^{-2x}$ (l) $-4e^{-4x}$

Q2

(a) $2xe^{x^2}$ (b) $3x^2e^{x^3}$

(c) $(2x+1)e^{x^2+x}$ (d) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$

(e) $-\frac{1}{x^2}e^{\frac{1}{x}}$ (f) $-\frac{2}{x^3}e^{\frac{1}{x^2}}$

Q3

(a) $-8e^{2x}(3-e^{2x})^3$

(b) $3\left(x-e^{-x^2}\right)^2\left(1+2xe^{-x^2}\right)$

(c) $\frac{-e^x}{(e^x+1)^2}$

(d) $\frac{2xe^{x^2}}{\left(1-e^{x^2}\right)^2}$

(e) $\frac{e^{2x}-1}{\sqrt{e^{2x}-2x}}$

(f) $\frac{-xe^{x^2}}{(e^{x^2}-1)^{\frac{3}{2}}}$

Q4

(a) $x^2e^x+2xe^x$

(b) $e^{-2x}(1-2x)$

(c) $e^{x^2}(1+2x^2)$

(d) $(-x^2+4x-2)e^{-x}$

(e) $(3x-4)e^{-x}(10-3x)$

(f) $\frac{e^{\sqrt{x}}}{2}\left(1+\frac{1}{\sqrt{x}}\right)$

Q5

(a) $\frac{e^x(x-1)}{x^2}$ (b) $\frac{2x-x^2}{e^x}$

(c) $\frac{3e^{3x}}{(1+e^{3x})^2}$ (d) $\frac{4e^{2x}}{(e^{2x}+1)^2}$

(e) $\frac{-4}{(e^x-e^{-x})^2}$ (f) $\frac{1-2x^2}{e^{x^2}}$

Q6

See full worked solutions.

Q7

(a) $k=-1,\ \frac{3}{2}$ (b) $k=5$

Q8

See full worked solutions.

Q9

(a) $\frac{e^x(x-1)}{x^2}$ (b) $\frac{e^x(x-2)}{x^3}$

(c) $\frac{e^x(x-3)}{x^4}$ (d) $\frac{e^x(x-n)}{x^{n+1}}$

P1

See full worked solutions.

Exercise 4B

Differentiation of logarithmic functions

F1

(a) $\frac{1}{x}$ (b) $\frac{a}{ax+b}$ (c) $\frac{f'(x)}{f(x)}$

F2

(a) $n\ln x,\ \frac{n}{x}$

(b) $\ln\big(f(x)\big)+\ln\big(g(x)\big),\ \frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}$

(c) $\ln\big(f(x)\big)-\ln\big(g(x)\big),\ \frac{f'(x)}{f(x)}-\frac{g'(x)}{g(x)}$

(d) $-\ln\big(f(x)\big),\ -\frac{f'(x)}{f(x)}$

F3

(a) e, change, e (b) $\frac{\ln x}{\ln a},\ \frac{1}{x\ln a}$

F4

(a) x, 1 (b) x^2, $2x$

ANSWERS

F5

simplify

Q1

(a) $\frac{4}{4x-3}$ (b) $\frac{-2}{5-2x}$

(c) $\frac{2x}{x^2+1}$ (d) $\frac{x-1}{x^2-2x}$

(e) $\frac{1}{x\ln x}$ (f) $\frac{2x^2-1}{x\left(x^2-\ln x\right)}$

Q2

(a) $f'(x)=\frac{12x-7}{6x^2-7x-3}$

(b) $f(x)=\ln(2x-3)+\ln(3x+1)$

(c) $\frac{2}{2x-3}+\frac{3}{3x+1}=\frac{12x-7}{6x^2-7x-3}$

Q3

(a) $\frac{4}{x+1}$ (b) $\frac{15}{3x-2}$

(c) $\frac{1}{x-1}+\frac{1}{x-2}$ (d) $\frac{-2}{5-2x}+\frac{4}{4x-3}$

(e) $\frac{2}{x}$ (f) $-\frac{4}{4x-3}$

(g) $\frac{-2}{5-2x}-\frac{4}{4x-3}$ (h) $\frac{1}{2x}$

(i) $\frac{1}{2x+1}$ (j) $\frac{1}{1-2x}$

Q4

(a) $\frac{2\ln x}{x}$ (b) $-\frac{3}{x}(1-\ln x)^2$

(c) $-\frac{1}{x(\ln x)^2}$ (d) $\frac{1}{2x\sqrt{1+\ln x}}$

Q5

(a) $1+\ln x$ (b) $x+2x\ln x$

(c) $2\ln(2x+3)+2$ (d) $\frac{2x^2}{x^2+1}+\ln(x^2+1)$

Q6

(a) $\frac{1-\ln x}{x^2}$ (b) $\frac{2x\ln x-x}{(\ln x)^2}$

(c) $\frac{2-2\ln(2x+1)}{(2x+1)^2}$ (d) $\frac{2-\ln x}{2x\sqrt{x}}$

Q7

(a) $\frac{6}{2x-1}+\frac{12}{3x+2}$ (b) $\frac{x}{x^2-1}$

(c) $\frac{6}{3x-1}-\frac{6}{2x+1}$ (d) $\frac{1}{2}\left(\frac{1}{x+2}-\frac{1}{x-2}\right)$

Q8

(a) $\frac{\ln x}{\ln 5}$ (b) $\frac{1}{x\ln 5}$ (c) $\frac{1}{x\ln a}$

Q9

(a) $\frac{1}{x\ln 2}$ (b) $\frac{2}{(2x+1)\ln 3}$

(c) $\frac{2x}{(x^2+1)\ln 4}$ (d) $\frac{2\log_5(x)}{x\ln 5}$

Q10

(a) 1 (b) $3x^2$ (c) $-\frac{2}{x^3}$

(d) 2 (e) 3 (f) $18x$

Q11

(a) See full worked solutions.

(b) See full worked solutions.

Q12

(a) a, a (b) $\ln a$, a, $\ln a$

Q13

(a) $3^x\ln 3$ (b) $-4^{-x}\ln 4$

Q14

(a) See full worked solutions.

(b) See full worked solutions.

Q15

See full worked solutions.

P1

(a) $\frac{2x}{x^2-1}-\frac{2x}{x^2+1}$ (b) $\frac{1}{x}-\frac{x}{x^2-1}$

P2

See full worked solutions.

Exercise 4C

Differentiation of trigonometric functions

F1

(a) $\cos x$ (b) $-\sin x$
(c) $\sec^2 x$ (d) $a\cos(ax+b)$
(e) $-a\sin(ax+b)$ (f) $a\sec^2(ax+b)$
(g) $f'(x)\cos\left(f(x)\right)$ (h) $-f'(x)\sin\left(f(x)\right)$
(i) $f'(x)\sec^2\left(f(x)\right)$

F2

(a) radians (b) radians, $\dfrac{\pi x}{180}$

Q1

(a) $2\cos(2x)$ (b) $-3\sin(3x)$
(c) $4\sec^2(4x)$ (d) $2\cos\left(\dfrac{x}{2}\right)$
(e) $2\sin\left(\dfrac{x}{3}\right)$ (f) $2\sec^2\left(\dfrac{x}{4}\right)$
(g) $-\cos\left(\dfrac{\pi}{2}-x\right)$ (h) $-3\sin\left(3x-\dfrac{\pi}{6}\right)$
(i) $\dfrac{2}{3}\sec^2\left(\dfrac{2x+\pi}{3}\right)$

Q2

(a) $\dfrac{\pi}{180}\cos\left(\dfrac{x\pi}{180}\right)$ (b) $\dfrac{\pi}{90}\sec^2\left(\dfrac{x\pi}{90}\right)$
(c) $-\dfrac{\pi}{2}\sin\left(\dfrac{\pi x}{2}\right)$

Q3

See full worked solutions.

Q4

(a) $2x\cos\left(x^2\right)$ (b) $-6x\sin\left(3x^2\right)$
(c) $-10x\sec^2\left(2-5x^2\right)$ (d) $-\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)$
(e) $-\dfrac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$ (f) $\dfrac{x}{\sqrt{x^2+1}}\sec^2\left(\sqrt{x^2+1}\right)$

Q5

(a) $2\sin x\cos x$
(b) $-6\cos^2(2x)\sin(2x)$
(c) $6\tan(3x)\sec^2(3x)$
(d) $-\dfrac{\cos x}{\sin^2 x} = -\cot x \operatorname{cosec} x$
(e) $-\dfrac{\sin x}{2\sqrt{\cos x}}$
(f) $\dfrac{\sec^2(2x)}{\sqrt{\tan(2x)}}$

Q6

(a) $x\cos x+\sin x$
(b) $\dfrac{x^2}{2}\sec^2\dfrac{x}{2}+2x\tan\dfrac{x}{2}$
(c) $-\sin^3 x+2\sin x\cos^2 x$
(d) $-3\sin 2x\sin 3x+2\cos 2x\cos 3x$
(e) $\sqrt{x}\cos x+\dfrac{1}{2\sqrt{x}}\sin x$
(f) $2x\sin^2 x+2x^2\sin x\cos x$

Q7

(a) $\dfrac{\sin x - x\cos x}{\sin^2 x}$
(b) $\dfrac{-x\sin x-2\cos x}{x^3}$
(c) $\dfrac{2\cos^2 x\sin x+\sin^3 x}{\cos^2 x}$
(d) $\dfrac{2x\cos x-\sin x}{2x\sqrt{x}}$
(e) $\dfrac{1}{(\sin x+\cos x)^2}$
(f) $\dfrac{2\sec x}{(\tan x+\sec x)^2}$

Q8

(a) See full worked solutions.
(b) See full worked solutions.
(c) See full worked solutions.
(d) See full worked solutions.
(e) See full worked solutions.
(f) $A=\dfrac{1}{40},\ B=-\dfrac{1}{20}$

Q9

See full worked solutions.

Q10

(a) $A=2\sqrt{2},\ B=\sqrt{2}$
(b) $A=-2,\ B=2\sqrt{2}$

Q11

2.46 kilograms per radian

P1

(a) $-\frac{1}{x}\cos\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x}\right)$

(b) $-\frac{1}{x^3}\sec^2\left(\frac{1}{x}\right)-\frac{1}{x^2}\tan\left(\frac{1}{x}\right)$

(c) $-\frac{2}{x}\cos\left(\frac{1}{x^2}\right)+2x\sin\left(\frac{1}{x^2}\right)$

P2

See full worked solutions.

P3

See full worked solutions.

Exercise 4D

Mixed differentiation problems

F1

(a) $-\frac{f'(x)}{\big(f(x)\big)^2}$ (b) $\frac{f'(x)}{2\sqrt{f(x)}}$

(c) $n\big(f(x)\big)^{n-1}$ (d) $f'(x)e^{f(x)}$

(e) $f'(x)\cos\big(f(x)\big)$ (f) $-f'(x)\sin\big(f(x)\big)$

(g) $f'(x)\sec^2\big(f(x)\big)$ (h) $\frac{f'(x)}{f(x)}$

Q1

(a) $e^x\cos e^x$ (b) $2e^{2x}\sec^2 e^{2x}$

(c) $2xe^{-x^2}\sin e^{-x^2}$ (d) $\frac{1}{x}\cos(\ln x)$

(e) $-\frac{1}{x}\sin(\ln x)$ (f) $\frac{2}{x}\tan(\ln x)\sec^2(\ln x)$

Q2

(a) $e^x(\sin x+\cos x)$

(b) $e^{\sin x}\cos x$

(c) $e^{\tan x}\sec^2 x$

(d) $e^{-x}(2\cos 2x-\sin 2x)$

(e) $2\sin x\cos x e^{\sin^2 x}$

(f) $(x\sec^2 x+\tan x)e^{x\tan x}$

Q3

(a) 1 (b) 2

(c) -1 (d) $\frac{1}{x}+2$

(e) $\frac{6e^{2x}}{1+e^{2x}}$ (f) $-\frac{2}{x}$

(g) $\cot x$ (h) $\frac{1}{2\sin x\cos x}$

(i) $\frac{\sec^2 x}{1+\tan x}$ (j) $\frac{2}{e^{2x}-1}$

(k) $\frac{-4\sin x\cos x}{\cos^2 x-\sin^2 x}$ (l) $1+\cot x$

(m) $\frac{e^x}{e^x-1}-\frac{e^x}{e^x+1}$ (n) $\frac{-2}{\sin^2 x-\cos^2 x}$

(o) $\frac{-\tan x\sec^2 x}{1-\tan^2 x}$

Q4

See full worked solutions.

Q5

See full worked solutions.

P1

See full worked solutions.

P2

See full worked solutions.

P3

(a) See full worked solutions.

(b) $k=\frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}$, $\left(\frac{3\pi}{4},\frac{1}{\sqrt{2}}\right)$

Chapter Review

R1

(a) $6xe^{3x^2}$ (b) $-\frac{1}{x^2}e^{\frac{1}{x}}$

(c) $x^2e^x(x+3)$ (d) $e^{3x^2}(6x^2+1)$

(e) $\frac{e^{3x}(3x-1)}{x^2}$ (f) $2(x+4)$

R2

See full worked solutions.

R3

(a) $k=1,\ 4$ (b) $k=-1,\ 5$

R4

(a) $\frac{3}{3x-5}$

(b) $-\frac{4}{x}$

(c) $\frac{1}{x\ln 2}$

(d) $-\frac{\ln 2}{x(\ln x)^2}$

(e) $-\frac{4e^{2x}}{e^{4x}-1}$

(f) $3\cot(3x)$

(g) $\frac{2}{x}$

(h) $\frac{1-2x^2}{x-x^3}$

(i) $2\sec x$

R5

(a) $6^x\ln 6$

(b) $9^x\ln 9$

(c) $3^x\ln 3$

(d) $2x4^x+x^2 4^x\ln 4$

R6

(a) $2\cos(2x-3)$

(b) $\frac{3}{2}\sin\left(\frac{\pi-3x}{2}\right)$

(c) $-\frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)$

(d) $-2x\sec^2\left(1-x^2\right)$

(e) $\frac{\cos(\sqrt{x})}{2\sqrt{x}}$

(f) $\frac{\cos x}{2\sqrt{\sin x}}$

R7

(a) $\frac{-2\cos\sqrt{x}\sin\sqrt{x}}{\sqrt{x}}$

(b) $-\operatorname{cosec} x\cot x$

(c) $\sec x\tan x$

(d) $8\sec^2 4x\tan 4x$

(e) $\sec^2 x\, e^{\tan x}$

(f) $-3\operatorname{cosec}^2 3x$

(g) $2e^{2x}\cos 3x-3e^{2x}\sin 3x$

(h) $\sec x\operatorname{cosec} x$

(i) $2e^{2x}+\frac{1}{x}$

(j) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}+\frac{1}{2x}$

(k) $-e^{\cos x}\sin x$

(l) $-e^x\sin\left(e^x\right)$

(m) $-\sec^2 x\sin(\tan x)$

(n) $2e^{2x}\sin 4x+4e^{2x}\cos 4x$

(o) $\cos^2 x-\sin^2 x$

(p) $\cos x$

(q) $4x\left(2\ln\left(\frac{x}{2}\right)+1\right)$

(r) $\frac{e^{2x}(2x\ln(2x)+1)}{x}$

(s) $2e^{2x}(2x+1)$

(t) $\frac{2x\cos 2x-\sin 2x}{x^2}$

(u) $\frac{e^x(\cos x+\sin x)}{\cos^2 x}$

(v) $-\frac{2\sin x}{e^x}$

(w) $\frac{1-2x\ln(2x)}{xe^{2x}}$

(x) $\frac{1-2\ln x}{x^3}$

R8

See full worked solutions.

R9

See full worked solutions.

R10

(a) $3e^{3x}$

(b) $x=\frac{1}{3}\ln y$

(c) $\frac{1}{3y}$

(d) See full worked solutions.

5. Applications of differentiation

Exercise 5A

Tangents and normals

ANSWERS

F1

(a) $f'(x_1)$ (b) $f'(x_1)(x - x_1)$

(c) perpendicular (d) $-\dfrac{1}{f'(x_1)}(x - x_1)$

F2

(a) increasing (b) decreasing

(c) stationary (d) $=$

F3

(a) > 0 (b) < 0 (c) $= 0$

F4

$\tan\theta$

Q1

(a) 3

(b) $y = 3x + 1$

(c) $x + 3y = 23$

Q2

(a) $(3, -6)$ (b) $y = x - 9$

Q3

(a) $x = \dfrac{3}{2}$ (b) $x = \dfrac{1}{2}$

(c) $x = 0$ (d) $x = 3$

Q4

(a) $a = 2,\ b = -2$ (b) $a = 2,\ b = 3$

Q5

$a = -1,\ b = 5,\ c = -9$

Q6

(a) $m = \pm 1$. They are negatives of each other.

(b) $m = \pm 5$. They are negatives of each other.

(c) See full worked solutions.

(d) $m + n = 0$

Q7

(a) $\left(-\dfrac{1}{3}, -\dfrac{8}{27}\right)$, $(1, 0)$

(b) $8x - 8y - 21 = 0$

(c) $\left(\dfrac{\pi}{6}, -1\right)$, $\left(\dfrac{5\pi}{6}, -1\right)$

Q8

(a) See full worked solutions.

(b) $y = -2x - 1,\ y = 4x - 4$

(c) $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$

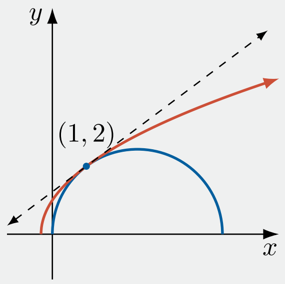
Q9

(a) See full worked solutions.

(b) $3x - 4y + 5 = 0$ for both curves.

(c) It must be a common tangent.

(d)



Q10

(a) $y = x + 1$ (b) $y = \dfrac{1}{e}x$

(c) $y = 3x + \sqrt{3} - \dfrac{\pi}{3}$ (d) $y = \dfrac{1}{e}$

(e) $y = 4x + 1 - \pi$ (f) $y = \dfrac{1}{e}$

Q11

(a) $e^2x - ey + (e^2 + 1) = 0$

(b) $x + y - 1 - \ln 2 = 0$

(c) $4x + 16y - \pi - 32 = 0$

Q12

(a) $8x - y + 4 - 8\ln 2 = 0$

(b) $16x - 4y + 1 - 32\ln 2 = 0$

Q13

(a) $x - ky + k\ln(3k) - k = 0$

(b) See full worked solutions.

(c) $y = \dfrac{3}{e}x$

Q14

$(1, e)$

Q15

(a) $\angle PMO = \dfrac{\pi}{4}$, so the gradient of MN is $\tan\dfrac{3\pi}{4} = -1$.

(b) See full worked solutions.

(c) $y = -x + 3$, $M(3, 0)$, $N(0, 3)$

(d) $y = x - 1$, $R(1, 0)$, $Q(0, -1)$

(e) Area $\triangle NPQ = 4$ and Area $\triangle RPM = 1$.

P1

(a) See full worked solutions.

(b) $A\big(x_0 + x_0^2 y_0, 0\big)$ $B\left(0, y_0 + \dfrac{1}{x_0}\right)$

(c) Use the fact that $x_0 y_0 = 1$ to show that the area is always 2 and hence, independent of P.

P2

(a) $y = x + 1$

(b) See full worked solutions.

Exercise 5B

The first derivative

F1

(a) 0

(b) horizontal

(c) turning, direction, turning

F2

(a) Minimum turning point.

(b) Maximum turning point.

(c) Horizontal point of inflection.

F3

(a) table (b) gradient

F4

(a) $>$ (b) decreasing

F5

(a) Maximum. (b) Minimum.

Q1

(a) $y' = 3x^2 + 1$

(b) $3x^2$ is always either positive or zero, so $y' = 3x^2 + 1$ is guaranteed to be positive.

(c) increasing

Q2

(a) $y' = 3$

(b) $y' = \dfrac{1}{2\sqrt{x}}$ as $x > 0$, $y' > 0$

(c) $y' = \dfrac{2}{x^2}$ as $x^2 > 0$, $y' > 0$

(d) $y' = \dfrac{12}{(4 - x)^2}$ as $(4 - x)^2 > 0$, $y' > 0$

(e) $y' = e^x + e^{-x} > 0$ for all x

(f) $y' = \dfrac{x^2 + 1}{x^2} > 0$ for all x

Q3

(a) $y' = -2$

(b) $y' = -3x^2$ as $x^2 > 0$, $y' < 0$

(c) $y' = \dfrac{-8}{(x - 2)^2}$ as $(x - 2)^2 > 0$, $y' < 0$

(d) $y' = \dfrac{-2}{(2x - 5)^2}$ as $(2x - 5)^2 > 0$, $y' < 0$

(e) $y' = \dfrac{-2}{4 - 2x}$ now domain of function is $4 - 2x > 0$ hence $y' < 0$

(f) $y' = -\sec^2 x < 0$

Q4

(a)

- (i) $x < 1$ and $x > 2$
- (ii) $1 < x < 2$
- (iii) $x = 1, 2$

(b)

- (i) $x > 2$
- (ii) $x < 2$
- (iii) $x = 2$

(c)

(i) All real x except $x = 1$

(ii) No values of x

(iii) $x = 1$

(d)

(i) No values of x

(ii) All values of x

(iii) No values of x

Q5

(a) $x > 2$

(b) $x < 2$

(c) $x < 0$

(d) $x < -1$ or $x > 3$

Q6

(a) $x < 3$

(b) $x > -1$

(c) All except $x = 4$

(d) $-2 < x < 4$

Q7

(a) $x = 4$

(b) $x = 3,\ x = -2$

(c) $x = \frac{1}{4}$

(d) No SP

Q8

(a) $x < 0,\ x > 2$

(b) $0 < x < 2$

(c) $x = 0,\ x = 2$

Q9

(a) $x = 1,\ x = 3$

(b)

x	0	1	2	3	4
$f'(x)$	+	0	−	0	+
Direction	/	–	\	–	/

(c) $x = 1$ maximum, $x = 3$ minimum.

Q10

(a) $x = 4$

(b) $x = 0,\ 4$

(c) $x = -3,\ 2$

(d) $x = 0$

(e) $x = -1$

(f) $x = 0,\ 2$

(g) e^{-1}

(h) $x = e^{-\frac{1}{2}}$

(i) $x = e$

Q11

(a) Minimum

(b) $x = 0$ minimum, $x = 4$ maximum

(c) $x = -3$ maximum, $x = 2$ minimum

(d) Maximum

(e) Minimum

(f) $x = 0$ minimum, $x = 2$ maximum

(g) Minimum

(h) Minimum

(i) Maximum

Q12

$y' = 3(x-2)^2 + 6 > 0$

Q13

(a) $x = -1$ Minimum, $x = 2$ Maximum, $x = 4$ Min

(b) $x = -3$ Maximum, $x = 0$ Minimum, $x = 5$ Horizontal point of inflection

(c) $x = -2$ horizontal point of inflection, $x = 3$ Min

(d) $x = -1$ horizontal point of inflection, $x = 2$ horizontal point of inflection

Q14

$f'(x) = \frac{a}{ax+b} \neq 0$ for any value of x.

P1

(a) $f'(x) = \frac{ad-bc}{(cx+d)^2}$

(b) If $f'(x) > 0$, then the numerator must be positive since the denominator is always positive anyway, so $ad - bc > 0$.

(c) $f(x)$ is a decreasing function.

P2

$f'(x) = \frac{1}{1+\cos x} > 0$ for $-\pi < x < \pi$

P3

(a) $x = -\frac{2}{3}$, minimum

(b) The solution of $y'' = 0$ is $x = -\frac{4}{3}$, but this is not in the domain of the function. Hence, no points where $y'' = 0$.

P4

(a) See full worked solutions.

(b) Mary is correct. Although $x = -\frac{1}{3}$ appears to be a solution of $y' = 0$, the curve is only defined for $x \geq 0$ anyway.

P5

(a) See full worked solutions.

(b) The curve has exactly one stationary point i.e. it has a horizontal point of inflection.

(c) The curve has no stationary points, and is therefore strictly increasing since it is a positive cubic.

Exercise 5C

The second derivative

F1

(a) $y'' > 0$

(b) $y'' < 0$

F2

(a) inflection, 0

(b) 0

(c) table

(d) sign

(e) No.

(f) Yes.

F3

(a) second.

(b) minimum

(c) maximum

(d) horizontal

Q1

(a) $x = 2$

(b)

x	1	2	3
$f''(x)$	-6	0	6
Sign	$-$	0	$+$

(c) Yes $x = 2$ is point of inflection

Q2

(a) $x = -2, x = 1$

(b) $x = -2$

(c) $x = \pm\frac{1}{\sqrt{2}}$

(d) $x = 2$

(e) $x = e^{-\frac{3}{2}}$

(f) $x = -2\ln 2,\ 0$

Q3

(a) Both points of inflection.

(b) Not a point of inflection.

(c) Both points of inflection.

(d) Point of inflection.

(e) Point of inflection.

(f) Only $x = -2\ln 2$ is a point of inflection.

Q4

(a) $x > 1$

(b) $x > -2$

(c) $x > 2$ or $x < -3$

Q5

(a) $x < 2$

(b) $x > 0$

(c) $x > 1$ or $x < -1$

Q6

(a) $y'' = e^x + e^{-x} > 0$

(b) $y'' = \frac{1}{x} > 0$ since the domain is $x > 0$

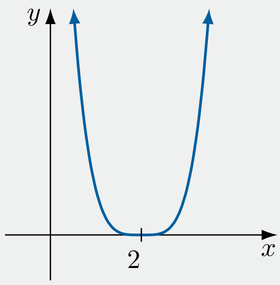
(c) $y'' = 2 + \frac{6}{x^4} > 0$

Q7

(a) $y' = 4x^3 - 24x^2 + 48x - 32$,
$y'' = 12x^2 - 48x + 48$

(b) See full worked solutions.

(c)



No. Bob's claim is incorrect.

(d) Bob should have checked the concavity on either side of $x = 2$.

Q8

(a) $x = -1$ is a minimum turning point.
$x = 1$ is a maximum turning point.
$x = 0$ is a horizontal point of inflection.

(b) $x = 0, \pm\frac{1}{\sqrt{2}}$

ANSWERS

Q9

(a) Three. (b) One.

(c) One. (d) None.

P1

(a) $x = 3, \quad x = 8$ (b) $x > 8$

(c) $x \in (3, 6)$ (d) $x = 3,\ 6$

P2

$x = 0,\ \pm\sqrt{3}$

P3

(a) See full worked solutions.

(b) See full worked solutions.

(c) Mary is correct since the curve is only defined when $x > 1$ or $x < 0$. Bob's answer contains values like $x = \frac{3}{4}$, which are not in the domain of the graph.

Exercise 5D

Introduction to curve sketching and global maxima/minima

F1

(a) 0 (b) global

(c) can (d) domain

F2

Local maximum is 1.3
Local minimum is −1.9
Global maximum is 4.5
Global minimum is −1.9

F3

(a) 0 (b) $\lim_{x \to \alpha^-} f(x)$

F4

(a) ∞

(b) 0

(c) $\frac{a}{A}$

(d) above, below, positive, negative

Q1

Local maximum 3
Local minimum −1
Global maximum 19
Global minimum −1

Q2

(a) $y = 1 - x^2$ (b) $y = x^3 - x$

(c) $y = x^2$ (d) $y = x^3 - x$

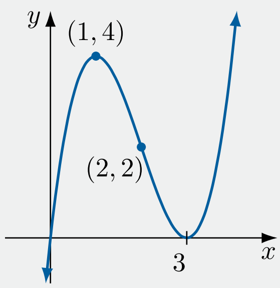
Q3

(a) $x = 0,\ 3$

(b) $(1, 4)$ maximum, $(3, 0)$ minimum

(c) $(2, 2)$

(d)



(e) Global Maximum: $y = 4$
Global Minimum: $y = -16$

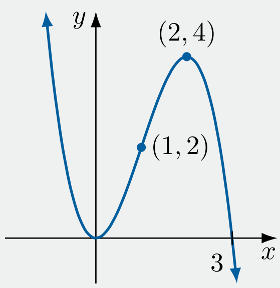
Q4

(a) $x = 0,\ 3$

(b) $(0, 0)$ minimum, $(2, 4)$ maximum

(c) $(1, 2)$

(d)



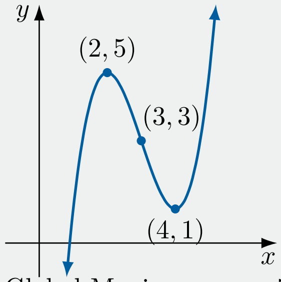
(e) Global Maximum: $y = 20$
Global Minimum: $y = 0$

Q5

(a) $(2, 5)$ Maximum, $(4, 1)$ Minimum

(b) $(3, 3)$

(c)



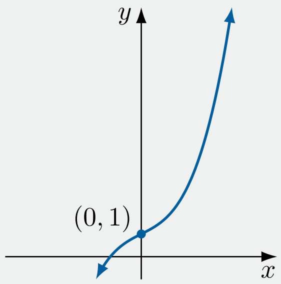
(d) Global Maximum: $y = 5$
Global Minimum: $y = 1$

Q6

(a) $y' = 3x^2 + 1 \neq 0$ for any real x

(b) $(0, 1)$

(c)



(d) Global Maximum: $y = 11$
Global Minimum: $y = -1$

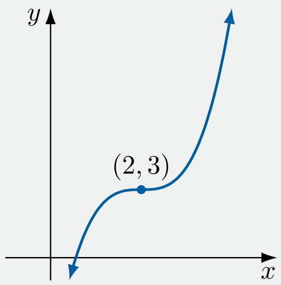
Q7

(a) $(0, -5)$

(b) Stationary point at $(2, 3)$.

(c) $(2, 3)$

(d)



(e) Global Maximum: $y = 11$
Global Minimum: $y = 2$

Q8

(a) $(1, 0)$ and $\left(0, \frac{1}{4}\right)$

(b) No stationary points.

(c) All real x, except $x = \pm 2$.

(d) $x = \pm 2$

(e)

x	1.99	2	2.01
y	-24.8	Undefined	25.2

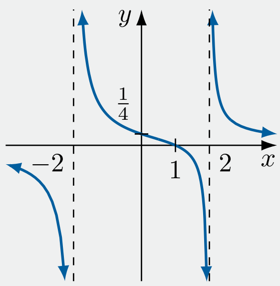
(f)

x	-2.01	-2	-1.99
y	-75.1	Undefined	74.9

(g) $y = 0$ i.e. the x-axis.

(h) As $x \to \infty$, $y \to 0^+$
As $x \to -\infty$, $y \to 0^-$

(i)



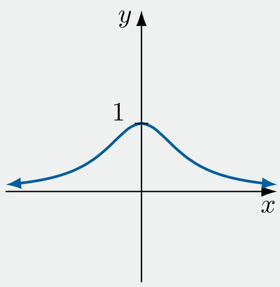
Q9

(a) $y = 0$ horizontal asymptote, no vertical asymptote

(b) $(0, 1)$ Maximum

(c) Even function

(d)



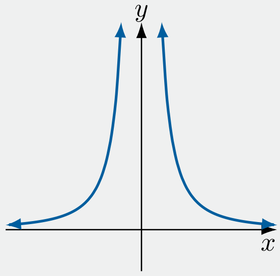
Q10

(a) $y = 0$ horizontal asymptote, $x = 0$ vertical asymptote

(b) See full worked solutions.

(c) Even function

(d)



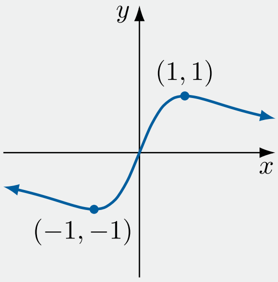
Q11

(a) $y = 0$ horizontal asymptote, no vertical asymptote

(b) $(1, 1)$ Maximum, $(-1, -1)$ Minimum

(c) odd function

(d)



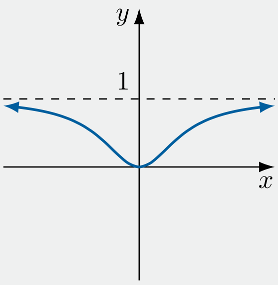
Q12

(a) $y = 1$ horizontal asymptote, no vertical asymptote

(b) $(0, 0)$ Minimum

(c) Even function

(d)



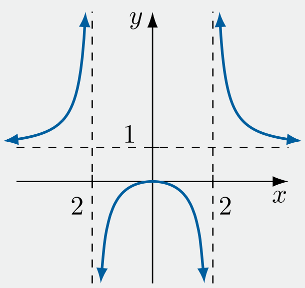
Q13

(a) $y = 1$ horizontal asymptote, $x = \pm 2$ vertical asymptote

(b) $(0, 0)$ maximum

(c) Even function

(d)



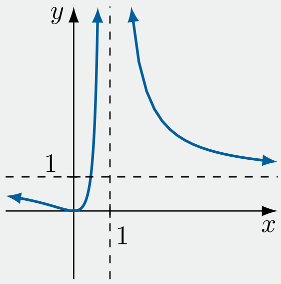
Q14

(a) $y = 1$ horizontal asymptote, $x = 1$ vertical asymptote

(b) $(0, 0)$ minimum

(c) Neither

(d)



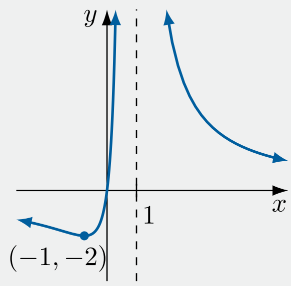
Q15

(a) $y = 0$ horizontal asymptote, $x = 1$ vertical asymptote

(b) $(-1, -2)$ minimum

(c) Neither

(d)

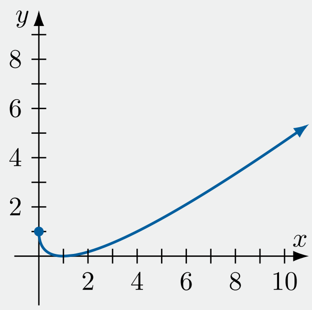


P1

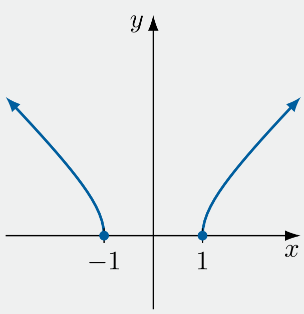
See full worked solutions.

P2

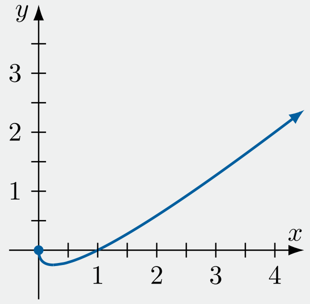
(a)



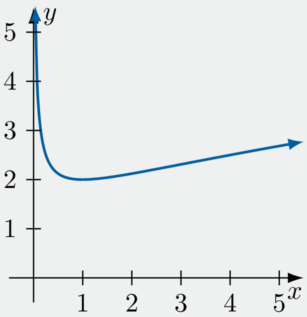
(b)



(c)

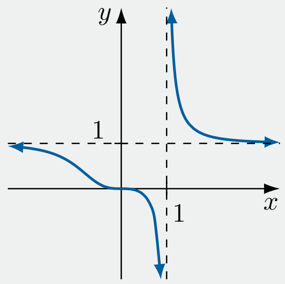


(d)

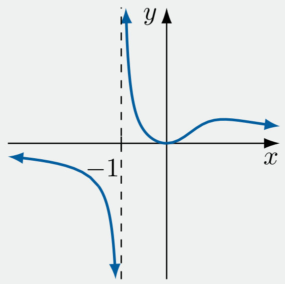


P3

(a)



(b)



Exercise 5E

Harder curve sketching

F1

(a) 0 (b) $-\infty$ (c) 0

(d) ∞ (e) 0 (f) ∞

(g) $-\infty$ (h) 0

F2

(a) x, y (b) domain

(c) ∞ (d) asymptotes

(e) asymptote (f) above, below

(g) end (h) even, odd

(i) stationary, nature (j) range

(k) inflection

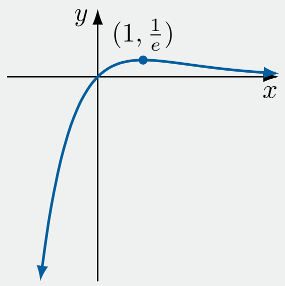
Q1

(a) $(0, 0)$

(b) $\left(1, \frac{1}{e}\right)$, maximum

(c) As $x \to \infty$, $y \to 0^+$
As $x \to -\infty$, $y \to -\infty$

(d)



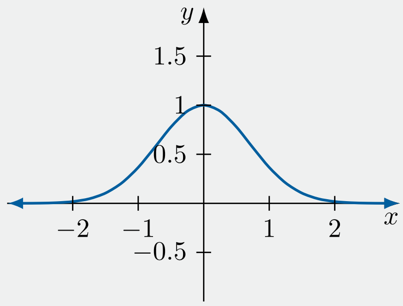
Q2

(a) $(0, 1)$

(b) $(0, 1)$, maximum

(c) As $x \to \infty$, $y \to 0$
As $x \to -\infty$, $y \to 0$

(d)



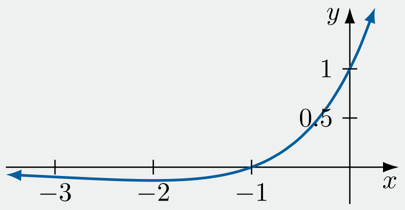
Q3

(a) $(-1, 0)$, $(0, 1)$

(b) $\left(-2, -\frac{1}{e^2}\right)$, minimum

(c) As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to 0^-$

(d)



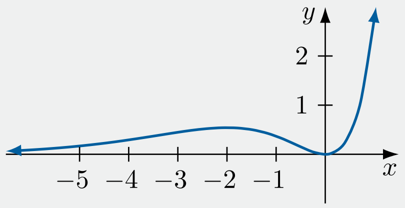
Q4

(a) $(0, 0)$

(b) $(-2, \frac{4}{e^2})$, maximum, and $(0, 0)$, minimum

(c) As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to 0^+$

(d)



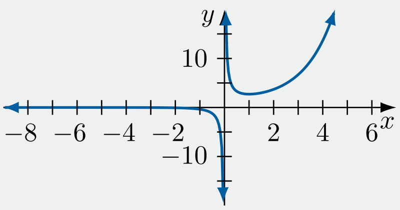
Q5

(a) None.

(b) $(1, e)$, minimum

(c) As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to 0^-$

(d)



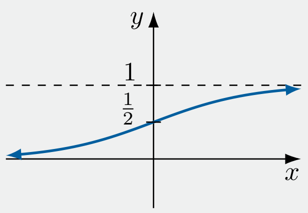
Q6

(a) $(0, \frac{1}{2})$

(b) No stationary points.

(c) As $x \to \infty$, $y \to 1$
As $x \to -\infty$, $y \to 0$

(d)



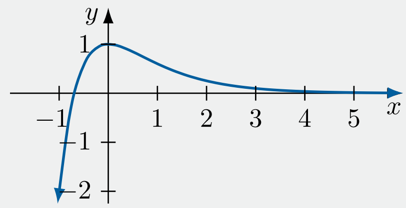
Q7

(a) $(0, 1)$, $(-\ln 2, 0)$

(b) $(0, 1)$, maximum

(c) As $x \to \infty$, $y \to 0$
As $x \to -\infty$, $y \to -\infty$

(d)



Q8

(a) $x > 0$

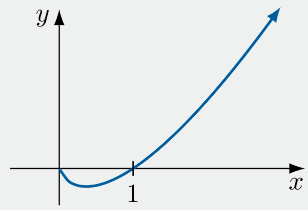
(b) $(1, 0)$

(c) $\left(\dfrac{1}{e}, -\dfrac{1}{e}\right)$, minimum

(d) None.

(e) As $x \to \infty$, $y \to \infty$
As $x \to 0^+$, $y \to 0^-$

(f)



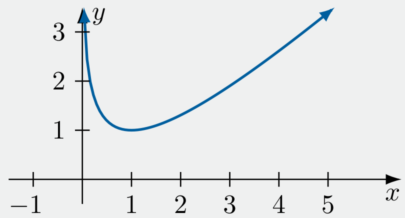
Q9

(a) $x > 0$

(b) $(1, 1)$, minimum

(c) As $x \to \infty$, $y \to \infty$
As $x \to 0$, $y \to \infty$

(d)

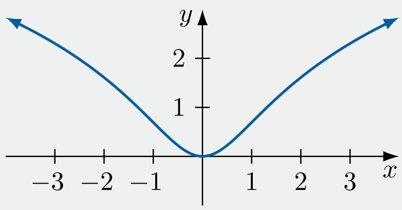


Q10

(a) $(0, 0)$, minimum

(b) As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to \infty$

(c)

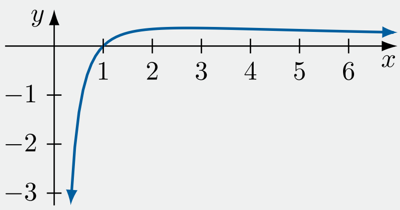


Q11

(a) $\left(e, \frac{1}{e}\right)$, maximum

(b) As $x \to \infty$, $y \to 0^+$
As $x \to 0$, $y \to -\infty$

(c)

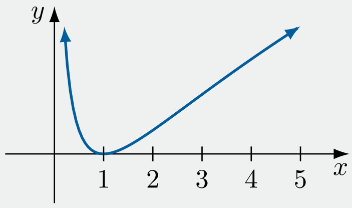


Q12

(a) $(1, 0)$, minimum

(b) As $x \to \infty$, $y \to \infty$
As $x \to 0$, $y \to \infty$

(c)



Q13

(a) $(0, 0)$

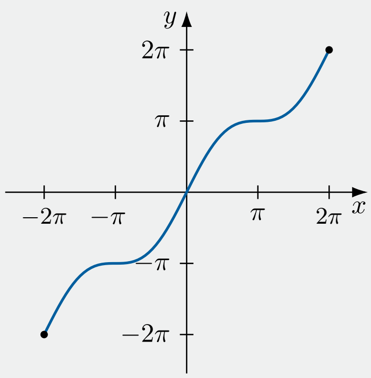
(b) $(-\pi, -\pi), (\pi, \pi)$, horizontal point of inflection

(c) $(-\pi, -\pi), (\pi, \pi)$

(d) $y' = 1 + \cos x \geq 0$ for $-2\pi \leq x \leq 2\pi$.

(e) See full worked solutions.

(f)



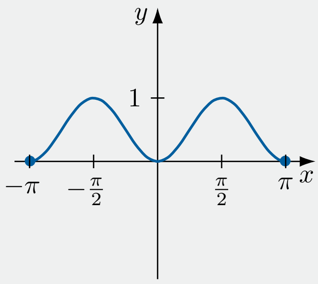
Q14

(a) $(0, 0)$ and $(\pm\pi, 0)$ are minimum stationary points.

$\left(\pm\frac{\pi}{2}, 1\right)$ are maximum stationary points.

(b) $(\pm\pi, 0)$

(c)



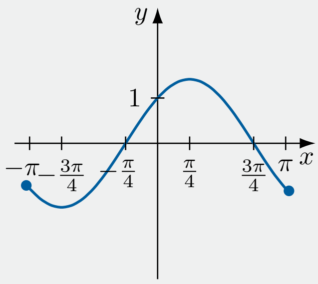
Q15

(a) See full worked solutions.

(b) $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is a maximum, $\left(-\frac{3\pi}{4}, -\sqrt{2}\right)$ is a minimum

(c) $(\pm\pi, -1)$

(d)

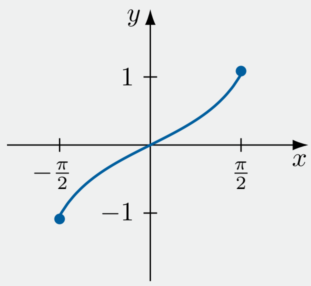


Q16

(a) No stationary points.

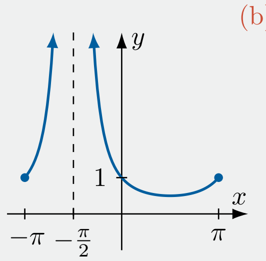
(b) $\left(\pm\frac{\pi}{2}, \pm 1\right)$

(c)

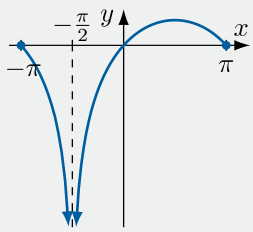


P1

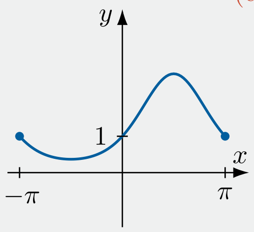
(a)



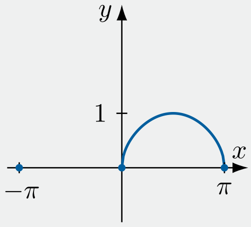
(b)



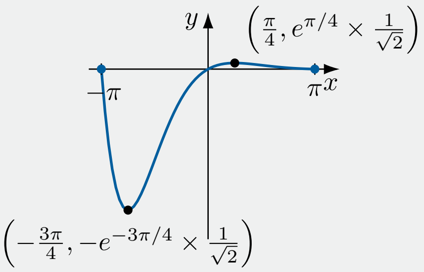
(c)



(d)



P2



Exercise 5F
Optimisation

F1

(a) maximise, minimise

(b) constraint

(c) quantity

F2

(a) optimised

(b) constraint, single

(c) Differentiate

(d) table, second

(e) local, global

(f) minimum, maximum

Q1

(a) $P = xy$

(b) See full worked solutions.

(c) See full worked solutions.

(d) 6, 6

Q2

(a) See full worked solutions.

(b) $x = 6$

(c) $x = y = 6$

Q3

(a) See full worked solutions.

(b) $x = 9$

(c) It is a square.

Q4

See full worked solutions.

Q5

(a) See full worked solutions.

(b) 11250 m^2

Q6

(a) See full worked solutions.

(b) $r = \left(\frac{200}{\pi}\right)^{\frac{1}{3}}$

Q7

(a) See full worked solutions.

(b) $2 \times 2 \times 1$ m

(c)

(i) $C = \frac{128}{x} + 10x^2$

(ii) $x = \frac{4}{\sqrt[3]{10}}$

Q8

30 cm × 20 cm

Q9

(a) See full worked solutions.

(b) See full worked solutions.

(c) $250\pi\sqrt{3}$ centimetres cubed.

Q10

See full worked solutions.

Q11

(a) See full worked solutions.

(b) See full worked solutions.

(c) $h = \frac{4R}{3}$

Q12

Dimensions $l \times w \times h = 7.11 \times 2.37 \times 5.93$
Minimum cost is \$1012.12

Q13

$x = 2$, \$3000

Q14

(a) $R = 300 \times 16 + (p - 16)(300 - 10p) = 460p - 10p^2$

(b) 23

(c) 14

Q15

(a) See full worked solutions.

(b) See full worked solutions.

Q16

(a) See full worked solutions.

(b) $x = \frac{4}{3}$ cm

(c) $x = 4$ cm

(d) If we cut 4 cm from either side of the box, there is no box left to form and hence zero volume.

(e) $37.93\ \text{cm}^3$

Q17

(a) See full worked solutions.

(b) See full worked solutions.

(c) $\frac{\pi}{4}(1600 - 3h^2)$

(d) See full worked solutions.

(e) $\frac{32000\pi}{3\sqrt{3}}\ \text{cm}^3$

Q18

(a) See full worked solutions.

(b) $x = \frac{3}{2}$

(c) $x = \frac{3}{2}$, making sure to verify that this is indeed the minimum.

Q19

(a) See full worked solutions.

(b) See full worked solutions.

Q20

(a) $A = \frac{1}{2}(rL - 2r^2)$

(b) $A = \frac{\theta}{2}\left(\frac{L}{2+\theta}\right)^2$

(c) Substitute $r = \frac{L}{4}$ and $\theta = 2$ into A.

P1

See full worked solutions.

P2

See full worked solutions.

P3

See full worked solutions.

P4

See full worked solutions.

P5

(a) See full worked solutions.

(b) See full worked solutions.

(c) See full worked solutions.

(d) $p = \frac{1}{2}$

(e) $p = 1$

Chapter Review

R1

$a = 1,\ b = -6$

R2

(a) $1 < x < 3$

(b) $x > \frac{1}{6}$

R3

(a) $y' = 3x^2 > 0$ for all real $x \neq 0$

(b) $y' = -\frac{1}{x^2} < 0$ for all real $x \neq 0$

R4

(a) $x - e^2y + e^2 = 0$

(b) $y = \frac{1}{e}$

(c) $y = x$

(d) $2x - y - e = 0$

R5

(a) $P(2, -2)$

(b) $y = x - 4$

(c) $Q(0, -4)$

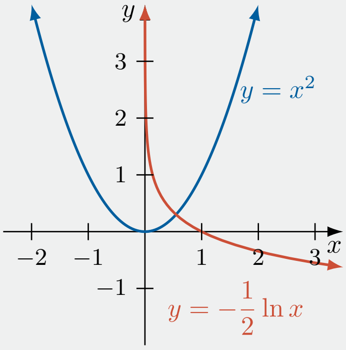
(d) $y = -x,\ R(0, 0)$

(e) 4 units squared.

ANSWERS

R6

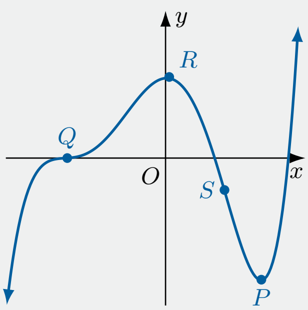
(a)



(b) $2x_1$ and $-\dfrac{1}{2x_1}$

(c) Since $2x_1 \times \dfrac{-1}{2x_1} = -1$, lines are perpendicular

R7



R8

(a) $x < 1,\ x > 2$

(b) $x > \dfrac{3}{2}$

(c) $1 < x < \dfrac{3}{2}$

(d) $x < 1$

R9

$-2 < x < -1$ and $1 < x < 3$, $-1 < x < 1$, $x = -1$ and $x = 1$

R10

(a) See full worked solutions.

(b) The graph is increasing but stationary at $x = -1$. So, there is a horizontal point of inflection at $x = -1$.

(c)

x	-2	-1	0
sign of $\dfrac{d^2y}{dx^2}$	$-$ve	0	$+$ve

(d) Point of inflection as $\dfrac{d^2y}{dx^2} = 0$ and there was a change of sign before and after $x = -1$.

R11

See full worked solutions.

(a) $x > 2$ or $x < 0$

(b) $x < 1$

(c) This shows where the blue graph is decreasing.

(d) This shows where the blue graph is concave up.

R12

(a) $x \in \left[0, \dfrac{\pi}{2}\right)$, $x \in \left(\dfrac{3\pi}{2}, 2\pi\right]$

(b) 0, occurs when $x = 0,\ 2\pi$

R13

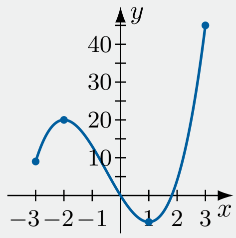
(a) $y = ex - 2$, $x + ey + e - \dfrac{1}{e} = 0$

(b) $\dfrac{1}{2}\left(e + \dfrac{1}{e}\right)$

R14

(a) Global minimum $(1, -7)$ and global maximum $(3, 45)$, local maximum $(-2, 20)$, local minimum $(-3, 9)$

(b)



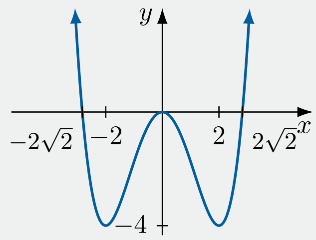
R15

(a) $x = 0,\ x = \pm 2\sqrt{2}$

(b) $(0, 0)$ maximum, $(2, -4)$ and $(-2, -4)$ minimum

(c) $x = \pm\dfrac{2}{\sqrt{3}}$

(d)



(e) $-\dfrac{2}{\sqrt{3}} < x < \dfrac{2}{\sqrt{3}}$

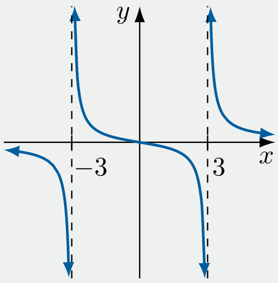
R16

(a) Vertical asymptotes at $x = \pm 3$, horizontal asymptote at $y = 0$

(b) No stationary points.

(c) Odd function

(d)



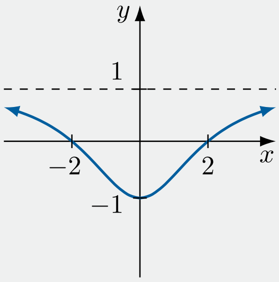
R17

(a) Only horizontal asymptote at $y = 1$

(b) $(0, -1)$ minimum

(c) Even function

(d)



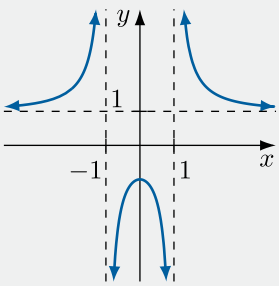
R18

(a) Horizontal asymptote at $y = 1$, vertical asymptotes at $x = \pm 1$

(b) $(0, -1)$ maximum

(c) Even function

(d)



R19

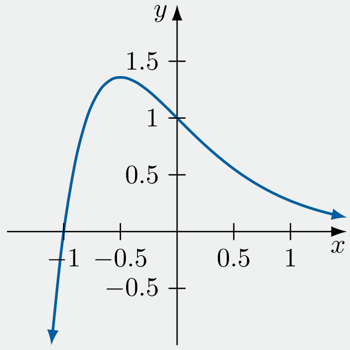
(a) $(-1, 0)$, $(0, 1)$

(b) $(-\frac{1}{2}, \frac{e}{2})$, maximum turning point

(c) $(0, 1)$

(d) As $x \to \infty$, $y \to 0$
$x \to -\infty$, $y \to -\infty$

(e)



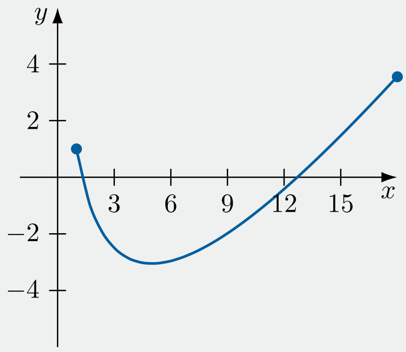
R20

(a) $y' = 1 - \dfrac{5}{x}$, $y'' = \dfrac{5}{x^2}$

(b) $(5, 5 - 5\ln 5)$, minimum turning point

(c) Concave up for all x

(d)



(e) Global max $18 - 5\ln 18$, global min $5 - 5\ln 5$

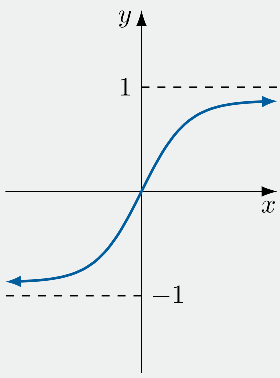
R21

(a) See full worked solutions.

(b) $f'(x) = \dfrac{2e^x}{(e^x + 1)^2} \neq 0$ for all real x

(c) As $x \to \infty$, $y \to 1^-$
As $x \to -\infty$, $y \to -1^+$

(d)

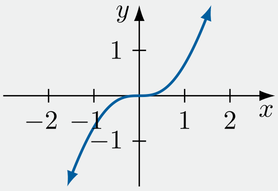


R22

(a) See full worked solutions.

(b) As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to -\infty$

(c)

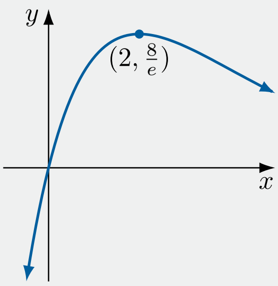


R23

(a) $\left(2, 8e^{-1}\right)$, maximum

(b) As $x \to \infty$, $y \to 0^+$.
As $x \to -\infty$, $y \to -\infty$

(c)



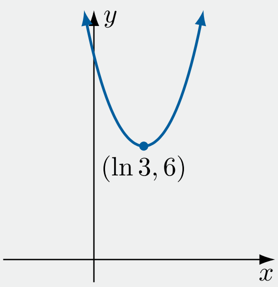
R24

(a) $\left(\ln 3,\ 6\right)$, minimum

(b) As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to \infty$

(c) $f''(x) = e^x + 9e^{-x} > 0$ for all real x

(d)

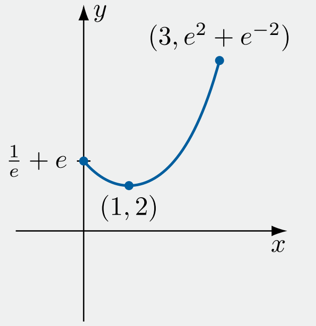


R25

(a) $(1, 2)$, minimum

(b) Global minimum is 2.
Global maximum is $e^2 + e^{-2}$.

(c)

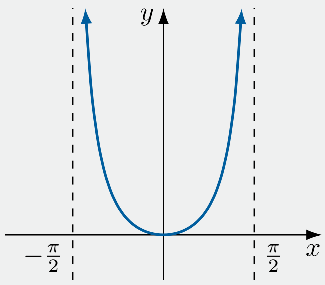


R26

(a) $(0, 0)$ minimum

(b) $f''(x) = 2\sec^2 x\left(2\tan^2 x + \sec^2 x\right) > 0$ for all real x.

(c)



R27

(a) $R(x) = (240-4x)(10+x) = 2400+200x-4x^2$

(b) \$25

(c) \$4900

R28

(a) See full worked solutions.

(b) See full worked solutions.

R29

$x = \dfrac{1}{\sqrt{2}}$

R30

(a) See full worked solutions.

(b) $3\sqrt{3}$

R31

(a) See full worked solutions.

(b) See full worked solutions.

(c) $r : R = \sqrt{2} : \sqrt{3}$

(d) $V_C : V_S = 1 : \sqrt{3}$

R32

(a) See full worked solutions.

(b) $\theta = \frac{\pi}{3}$, $A_{\max} = \frac{3\sqrt{3}}{4}$

R33

(a) See full worked solutions.

(b) See full worked solutions.

(c) $\alpha = 0.38$, maximum

(d) 43 minutes

(e) Swimming the whole journey directly from A to C takes 40 minutes.
Running the whole journey along the circumference from A to C takes 24 minutes.

(f) Robin should run the whole journey. This is equivalent to $\alpha = \frac{\pi}{2}$.

6. Integration

Exercise 6A

The anti-derivative

F1

(a) differentiation (b) primitive

(c) primitive, is not, $f(x)$ (d) integral, $F(x) + C$

F2

(a) $\frac{1}{n+1}x^{n+1} + C$

(b) $\frac{1}{a(n+1)}(ax+b)^{n+1} + C$

(c) $\frac{1}{n+1}\big(f(x)\big)^{n+1} + C$

F3

(a) $\int f(x)\,dx + \int g(x)\,dx$

(b) $\int f(x)\,dx - \int g(x)\,dx$

(c) $k\int f(x)\,dx$

(d) $kx + C$

F4

(a) limits (b) $F(b) - F(a)$

Q1

(a) $\frac{x^3}{3} + C$ (b) $4x + C$ (c) $\frac{3x^4}{2} + C$

(d) $-x^7 + C$ (e) $-\frac{1}{2x} + C$ (f) $4x^{\frac{3}{2}} + C$

Q2

(a) $x^2 + x^3 + C$ (b) $\frac{1}{3}(x^2 + x) + C$

(c) $\frac{x^3}{9} - \frac{x^4}{3} + C$ (d) $x^3 - \frac{x^4}{2} + C$

(e) $\frac{x^3}{3} + \frac{x^2}{2} - 2x + C$ (f) $\frac{4x^5}{5} - 4x^3 + 9x + C$

Q3

-2, $-x^{-1}$

Q4

(a) $-\frac{1}{2x^2} + C$ (b) $\frac{2}{3}x^{\frac{3}{2}} + C$

(c) $2\sqrt{x} + C$ (d) $-\frac{2}{\sqrt{x}} + C$

(e) $-\frac{2}{3x^{\frac{3}{2}}} + C$ (f) $\frac{2}{3}x^{\frac{3}{2}} + C$

(g) $-\frac{2}{\sqrt{x}} + C$ (h) $2\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + C$

(i) $\frac{2}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + 2\sqrt{x} + C$

Q5

(a) $\frac{1}{10}(2x+3)^5 + C$ (b) $-\frac{1}{24}(5-3x)^8 + C$

(c) $-\frac{1}{3(3x-2)} + C$ (d) $\frac{1}{9}(6x-1)^{\frac{3}{2}} + C$

(e) $-\frac{2}{15}(4-3x)^{\frac{5}{2}} + C$ (f) $-\frac{1}{3}\sqrt{5-6x} + C$

Q6

(a) $\frac{7}{3}$ (b) 6 (c) $\frac{14}{3}$

(d) $\frac{27}{4}$ (e) 10 (f) $\frac{1}{3}$

Q7

(a) $\frac{1}{4}$ (b) $\frac{24}{25}$ (c) $\frac{15}{64}$

(d) $\frac{14}{9}$ (e) $\frac{8\sqrt{2}}{25}$ (f) 2

(g) $\sqrt{3} - 1$ (h) 1 (i) $\frac{2\sqrt{b}}{a}$

Q8

(a) $f(x) = x^2 + C$

(b) $f(x) = x^2 + 1$

(c) $f(x) = x^2 + 2$

Q9

(a) $y = 2x^2 + x + 3$ (b) $y = 3x - 2x^2 + 4$

(c) $y = x - \frac{x^3}{3} - 5$ (d) $y = \frac{x^3}{3} - \frac{3x^2}{2} + \frac{7}{6}$

(e) $y = x^3 - x^2 + x - 8$ (f) $y = \frac{2}{3}x^{\frac{3}{2}} - 16$

(g) $y = -(1-x)^3 + 1$ (h) $y = \frac{1}{8} - \frac{(3-2x)^4}{8}$

Q10

(a) $y = x^3 - x^2 - x + 6$

(b) $y = 2x^2 - x^3 - 1$

(c) $y = -5x^2 + 5x + 20$

(d) $y = x^3 - 4x^2 + 6x - 3$

P1

(a) $\frac{2x^2 + 2}{(x^2 - 1)^2}$ (b) $\frac{-x}{x^2 - 1} + C$

P2

(a) See full worked solutions.

(b) $-\frac{4x + 1}{8(2x + 1)^2} + C$

P3

$y = x^2 - 2x - 8$

P4

(a) $\frac{2}{9}x^{\frac{9}{2}} - \frac{4}{5}x^{\frac{3}{2}} + 2\sqrt{x} + C$

(b) $-\frac{1}{2(2x - 3)} + C$

(c) $-\frac{1}{x - 1} + C$

Exercise 6B

Integration with exponential functions

F1

(a) $e^x + C$ (b) $\frac{1}{a}e^{ax+b} + C$

(c) $\frac{1}{\ln a}a^x + C$

F2

(a) $\ln a$, $x \ln a$

(b) $x \ln a$, $\frac{1}{\ln a}e^{x \ln a}$, $\frac{1}{\ln a}$

Q1

(a) $\frac{1}{2}e^{2x} + C$ (b) $-\frac{1}{3}e^{-3x} + C$

(c) $2e^{\frac{x}{2}} + C$ (d) $\frac{1}{2}e^{2x-1} + C$

(e) $-\frac{1}{3}e^{2-3x} + C$ (f) $\frac{2}{3}e^{\frac{3x+1}{2}} + C$

(g) $\frac{3}{2}e^{2x} + 2e^{-3x} + C$ (h) $-3e^{-2x} + C$

(i) $-e^{-px+q} + C$

Q2

(a) $\frac{1}{2}e^{2x} + C$ (b) $-e^{-x} + C$

(c) $2e^{\frac{x}{2}} + C$ (d) $\frac{2}{3}e^{\frac{3x}{2}} + C$

(e) $3e^{\frac{x}{3}} + C$ (f) $-2e^{-\frac{x}{2}} + C$

Q3

(a) $x + 2e^x + \frac{1}{2}e^{2x} + C$

(b) $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

(c) $\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} + C$

(d) $-e^{-x} + x + C$

(e) $-\frac{1}{2}e^{-2x} - \frac{1}{3}e^{-3x} + C$

(f) $2e^{\frac{x}{2}} - 2e^{-\frac{x}{2}} + C$

Q4

(a) 2 (b) 4

(c) $\frac{1}{2}$ (d) $\sqrt{2}$

Q5

(a) $\frac{1}{2}(e^2 - 1)$ (b) $\frac{1}{2}(e^2 - e^{-2})$

(c) $2(e - \sqrt{e})$ (d) $\frac{7}{3}$

(e) 2 (f) $\frac{49}{2} + \ln\left(\frac{9}{2}\right)$

Q6

(a) $\frac{1}{\ln 2}2^x + C$

(b) $\frac{1}{\ln 3}3^x + C$

(c) $\frac{1}{\ln(25)}25^x + C = \frac{1}{2\ln 5}5^{2x} + C$

Q7

(a) $y = e^{x-1}$ (b) $y = \frac{1}{2}e^{2x} + 1$

Q8

(a) $\frac{1}{2}e^{x^2} + C$ (b) $\frac{1}{3}e^{x^3} + C$

(c) $\frac{1}{4}e^{2x^2} + C$ (d) $xe^x - e^x + C$

P1

(a) $u^3 - 1$ (b) $\frac{1}{2}e^{2x} + e^x + x + C$

P2

(a) $e^x - x + C$ (b) $2e^{\frac{x}{2}} - x + C$

(c) $e^x + C$ (d) $e^x + C$

P3

(a) See full worked solutions.

(b) $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

Exercise 6C

Integration involving the logarithmic function

F1

(a) $\ln|x| + C$

(b) $\frac{1}{a}\ln|ax + b| + C$

(c) $\ln|f(x)| + C$ for $f(x) \neq 0$

Q1

(a) $2\ln|x| + C$ (b) $\frac{1}{2}\ln|x| + C$

(c) $\frac{1}{2}\ln|2x + 1| + C$ (d) $\ln|3x - 4| + C$

(e) $-\frac{1}{2}\ln|3 - 2x| + C$ (f) $2\ln|2x - 5| + C$

Q2

(a) $x + \ln|x| + C$ (b) $2\ln|x| - \frac{1}{2}x + C$

(c) $\frac{1}{6}x^2 - \frac{2}{3}\ln|x| + C$ (d) $\ln|x + 2| + C$

(e) $\frac{1}{3}\ln|3x - 2| + C$ (f) $-\frac{1}{5}\ln|4 + 5x| + C$

Q3

(a) $\ln|x^2 + 4| + C$

(b) $\ln|x^3 - 4| + C$

(c) $\ln|x^2 + x + 1| + C$

(d) $\frac{1}{2}\ln|x^2 - 1| + C$

(e) $-\frac{1}{4}\ln|1 - 2x^2| + C$

(f) $-\frac{1}{3}\ln|8 - x^3| + C$

(g) $\frac{1}{2}\ln|x^2 + 2x + 2| + C$

(h) $\frac{1}{2}\ln|x^2 - 6x + 5| + C$

(i) $2\ln|x^2 - 4x + 3| + C$

Q4

(a) $\ln 2$ (b) $\frac{1}{2}\ln\left(\frac{5}{3}\right)$

(c) $\frac{1}{3}\ln\left(\frac{5}{2}\right)$ (d) $\frac{1}{2}\ln\left(\frac{5}{2}\right)$

(e) $\frac{1}{6}\ln\left(\frac{14}{5}\right)$ (f) $\frac{1}{2}\ln\left(\frac{3}{4}\right)$

Q5

(a) $\ln|1 + e^x| + C$ (b) $\frac{1}{2}\ln|1 + e^{2x}| + C$

(c) $\ln|e^x + e^{-x}| + C$

ANSWERS

Q6

(a) $\ln\left(\frac{4}{3}\right)$ (b) $-\frac{1}{2}\ln 6$

(c) $\frac{1}{2}\ln\left(1+e^2\right)$

Q7

(a) $y=\frac{1}{2}\ln|x|$ (b) $y=\ln|1+x|+1$

(c) $y=-\ln|4-x|+1$

Q8

$y=\ln|2x-1|$

Q9

(a) $\frac{1}{\sin x\cos x}$ (b) $\ln|\tan x|+C$

Q10

(a) See full worked solutions.

(b) $x-2\ln|1-e^x|+C$

P1

(a) $\ln\big|\ln|x|\big|+C$ (b) $-\ln|1+e^{-x}|+C$

(c) $-\frac{1}{3}\ln\left(x^{-3}+1\right)+C$ (d) $2\ln|\sqrt{x}+1|+C$

P2

See full worked solutions.

Exercise 6D

Integration involving trigonometric functions

F1

(a) $-\cos x+C$ (b) $\sin x+C$

(c) $-\frac{1}{a}\cos(ax+b)+C$ (d) $\frac{1}{a}\sin(ax+b)+C$

(e) $\tan x+C$ (f) $\frac{1}{a}\tan(ax+b)+C$

Q1

(a) $-\frac{1}{2}\cos 2x+C$ (b) $-3\cos\frac{x}{3}+C$

(c) $\frac{1}{2}\cos\left(\frac{\pi}{2}-2x\right)+C$

Q2

(a) $\frac{1}{3}\sin(3x)+C$ (b) $2\sin\frac{x}{2}+C$

(c) $-\frac{3}{2}\sin\frac{(\pi-2x)}{3}+C$

Q3

(a) $\frac{1}{3}\tan 3x+C$ (b) $2\tan\frac{x}{2}+C$

(c) $-\frac{3}{2}\tan\frac{(\pi-2x)}{3}+C$

Q4

(a) $-\frac{1}{2}\ln|\cos(2x)|+C$

(b) $\frac{1}{3}\ln|\sin(3x)|+C$

Q5

(a) $\frac{1}{4}$ (b) $-\frac{1}{\sqrt{3}}$ (c) 0

Q6

(a) $\sec^2 x$ (b) $\tan x-x+C$

Q7

(a) $\ln(1+\sqrt{3})$ (b) $\ln\left(\frac{4}{3}\right)$

(c) 0

Q8

Use the fact that $\operatorname{cosec}^2 x=\sec^2\left(\frac{\pi}{2}-x\right)$

Q9

(a) $\frac{1}{\pi}\sin 180°x+C$ (b) $-\frac{180}{\pi}\cos x°+C$

(c) $\frac{180}{\pi}\ln|\sec x°|+C$

Q10

(a) $\sec x\tan x,\ \sqrt{2}-1$

(b) $2\sin x\cos x,\ \frac{1}{2}$

(c) $3\sin^2 x\cos x,\ \frac{1}{3}$

(d) $2x\cos(x^2),\ \frac{1}{2\sqrt{2}}$

(e) $e^{\tan x}\sec^2 x,\ e-1$

Q11

$\frac{\pi}{2}$

ANSWERS

Q12

(a) See full worked solutions.

(b) $\ln 2$

P1

(a) $\frac{x}{2}+\frac{1}{4}\sin 2x+C$

(b) $\frac{x}{2}-\frac{1}{4}\sin 2x+C$

(c) $\frac{x}{2}+\frac{1}{8}\sin 4x+C$

(d) $\frac{x}{2}-\frac{1}{2}\sin x+C$

P2

(a) $x\cos x+\sin x,\ \frac{\pi}{2}-1$

(b) $x\sec^2 x+\tan x,\ \frac{\pi}{\sqrt{3}}-\ln 2$

(c) $\frac{\cos x}{1+\sin x}+\tan x,\ \ln(1+\sqrt{2})$

P3

See full worked solutions.

P4

(a) See full worked solutions.

(b) $\ln|\sec x+\tan x|$

P5

(a) $x+C$

(b) $\ln|\sin x+\cos x|+C$

(c) $I=\frac{1}{2}\{x+\ln|\sin x+\cos x|\}+C$ and

$J=\frac{1}{2}\{x-\ln|\sin x+\cos x|\}+C$

Exercise 6E
Reverse chain rule

F1

(a) $f'(x)\{f(x)\}^n$ (b) $\frac{\{f(x)\}^{n+1}}{n+1}+C$

Q1

(a) $\frac{1}{15}(3x-2)^5+C$ (b) $-\frac{1}{14}(3-2x)^7+C$

(c) $\frac{1}{6}(4x-1)^{\frac{3}{2}}+C$ (d) $-\frac{1}{3}(3-2x)^{\frac{3}{2}}+C$

(e) $-\frac{1}{16(4x-3)^4}+C$ (f) $-\sqrt{5-2x}+C$

Q2

$\frac{1}{2},\ 2,\ \frac{1}{2},\ \frac{1}{12}$

Q3

(a) $\frac{1}{8}(x^2+1)^4+C$ (b) $\frac{1}{18}(x^3-1)^6+C$

(c) $-\frac{1}{72}(1-4x^3)^6+C$ (d) $-\frac{1}{4(x^2-1)^2}+C$

(e) $\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C$ (f) $\sqrt{x^2-1}+C$

Q4

(a) $-\frac{1}{5}(1-e^x)^5+C$ (b) $\frac{1}{8}(e^{2x}+1)^4+C$

(c) $\frac{-1}{3(3e^x-2)}+C$ (d) $\frac{1}{3}\left(e^{2x}-1\right)^{\frac{3}{2}}+C$

(e) $2\sqrt{e^x+1}+C$ (f) $\frac{1}{18(1-2e^{3x})^3}+C$

Q5

(a) $\frac{1}{3}\sin^3 x+C$ (b) $-\frac{1}{8}\cos^4 2x+C$

(c) $\frac{1}{9}\tan^3 3x+C$ (d) $-\frac{1}{2\sin^2 x}+C$

(e) $\frac{1}{2\cos(2x)}+C$ (f) $\frac{2}{3}(1+\sin x)^{\frac{3}{2}}+C$

(g) $2\sqrt{1+\tan x}+C$ (h) $\frac{1}{1-\tan x}+C$

Q6

(a) $\frac{1}{2}\sin^2 x+C$

(b) $-\frac{1}{2}\cos^2 x+C$

(c) Use the fact that $\sin^2 x=1-\cos^2 x$

Q7

(a) See full worked solutions.

(b) See full worked solutions.

P1

(a) See full worked solutions.

(b) $\frac{2}{3}$

P2

$\frac{1}{4}\sec^4 x+C$

ANSWERS

P3

$\frac{1}{2}(1-\ln 2)$

P4

(a) See full worked solutions.

(b) $I_5 = I_1 - \frac{1}{4}$

(c) $\frac{1}{2}\ln 2 - \frac{1}{4}$

(d) $\frac{\pi}{4} - \frac{2}{3}$

P5

(a) $\ln\left|\ln x\right| + C$ (b) $\frac{1}{2}(\ln x)^2 + C$

(c) $\frac{1}{1-e^x} + C$ (d) $-\frac{2}{1+\sqrt{x}}$

Chapter Review

R1

(a) $\frac{1}{2}x^3 - \frac{2}{3}x^{\frac{3}{2}} + C$ (b) $\frac{2}{5}x^{\frac{5}{2}} + C$

(c) $\frac{1}{4}x^4 - 2x^3 + C$ (d) $\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x + C$

(e) $\frac{1}{15}(5x-3)^3 + C$ (f) $\frac{2}{15}(5x-3)^{\frac{3}{2}} + C$

(g) $\frac{2}{5}\sqrt{5x-3} + C$ (h) $3\sqrt{1+2x} + C$

(i) $-\frac{1}{3(3x-1)} + C$

R2

(a) $\frac{1}{3}e^{3x} + C$ (b) $-\frac{1}{3}e^{2-3x} + C$

(c) $-2e^{-\frac{x}{2}} + C$ (d) $\frac{x^2}{6} + 3\ln|x| + C$

(e) $\frac{x^2}{2e} + e\ln|x| + C$ (f) $-\frac{5}{3e^{3x}} + C$

(g) $-\frac{1}{2}e^{-2x} + \frac{1}{4}e^{4x} + C$ (h) $x + e^{2x} + \frac{1}{4}e^{4x} + C$

R3

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1

(d) $-\frac{8}{3}$ (e) $\frac{38}{15}$ (f) $\frac{1}{2}(e^4 - 1)$

R4

(a) $\frac{3}{4}\ln|x| + C$ (b) $\ln|1+6x| + C$

(c) $\ln|1-2x| + C$ (d) $-\ln|5-2x| + C$

(e) $\frac{2}{3}\ln|1+2x| + C$ (f) $-\frac{1}{2}\ln|1-x^2| + C$

(g) $\frac{1}{2}\ln|x^2-4x| + C$ (h) $2\ln|x^2+x| + C$

(i) $\frac{1}{3}\ln|x^3-3x^2| + C$

R5

(a) $\ln\left(\frac{3}{2}\right)$ (b) $\ln 49$ (c) 0

R6

(a) $\frac{1}{\ln 2}2^x + C$ (b) $\frac{1}{2\ln 4}4^{2x} + C$

(c) $-\frac{1}{\ln 3}3^{-x} + C$

R7

(a) $-\frac{3}{2}\cos 2x - \frac{1}{2}\sin 2x + C$

(b) $\frac{2}{3}\sin\frac{3x}{2} + C$

(c) $\frac{2}{3}\tan 3x + C$

(d) $-\frac{3}{2}\cos\frac{2x}{3} + C$

(e) $-\frac{4}{3}\sin\frac{3x}{4} + C$

(f) $\frac{4}{\pi}\tan\frac{\pi}{4}x + C$

(g) $\frac{1}{3}\cos\left(\frac{\pi}{2} - 3x\right) + C$

(h) $\frac{1}{3}\sin\left(3x + \frac{\pi}{3}\right) + C$

(i) $-\frac{1}{8}\tan\left(\frac{\pi}{6} - 8x\right) + C$

R8

(a) 2 (b) 2 (c) $\frac{4}{\pi}$

(d) $2 - \frac{3\sqrt{3}}{2}$ (e) $\frac{2}{3}$ (f) 2

ANSWERS

R9

(a) $\frac{4}{3}\ln|1+x^3|+C$

(b) $\frac{1}{3}\ln|e^{3x}-2|+C$

(c) $3\ln|\sin x+2|+C$

(d) $\frac{1}{3}\ln|\tan 3x-2|+C$

(e) $\frac{x^2}{2}+2\ln|x|+C$

(f) $\frac{1}{3}\ln|3x+2|-\frac{1}{3(3x+2)}+C$

R10

(a) $12x(x^2+3)^5,\ (x^2+3)^6+C$

(b) $\frac{1}{2+x}+\frac{1}{2-x},\ \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|+C$

(c) $1+\ln x,\ x\ln|x|-x+C$

R11

(a) $6xe^{3x^2},\ e^{3x^2}+C$

(b) $\cos x e^{\sin x},\ e^{\sin x}+C$

(c) $3xe^{3x}+e^{3x},\ \frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+C$

R12

(a) $\frac{1}{2}$ (b) $\sqrt{3}$ (c) 1

(d) $\frac{2}{3}$ (e) $2-\sqrt{2}$ (f) $1+\frac{1}{\sqrt{2}}$

R13

(a) $\frac{1}{12}(x^2-2)^6+C$ (b) $\frac{1}{42}(3x^2-1)^7+C$

(c) $\frac{1}{9}(3e^x+2)^3+C$ (d) $\frac{1}{4}(\ln x)^4+C$

R14

(a) $\frac{5}{6}$ (b) $e^2-4-\frac{1}{e^2}$

R15

$k=\frac{1}{3}$

R16

(a) $\sin 2x+2x\cos 2x,\ -\frac{1}{2}$

(b) $e^x(x+1),\ 1$

(c) $e^{-3x}(1-3x),\ \frac{1}{9}-\frac{4}{9e^3}$

(d) $2x\ln x+x,\ \frac{1}{4}(1+3e^4)$

(e) $-\tan x,\ \ln 2$

(f) $\frac{2\ln x}{x},\ \frac{1}{2}$

(g) $\cos x-x\sin x,\ 1$

R17

(a) $\ln|1+\sin x|+C$

(b) $-\ln|1-\tan x|+C$

(c) $\ln|\sin x+\cos x|+C$

(d) $\ln|\sin x|+C$

(e) $-\ln|\cos x|+C$

(f) $\frac{1}{2}\ln|1+\sin^2 x|+C$

7. Applications of integration

Exercise 7A

Area and the integral

F1

(a) positive (b) negative

(c) area, x (d) signed, a, b

(e) negative

Q1

(a) 12 (b) 16 (c) 8

(d) 16 (e) 12.5 (f) 18

Q2

(a) 4 (b) 4

Q3

(a) 6 (b) 6 (c) 8

(d) 28 (e) 35 (f) 12

(g) 24 (h) 16

Q4

(a) Positive (b) Negative

(c) Zero (d) Positive

(e) Negative (f) Zero

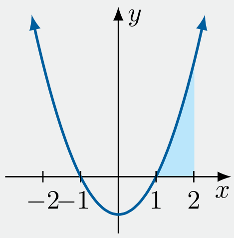
Q5

(a) -8 (b) -16 (c) -8

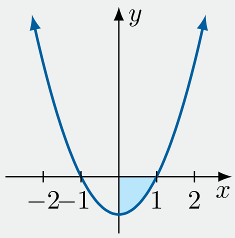
(d) -12

Q6

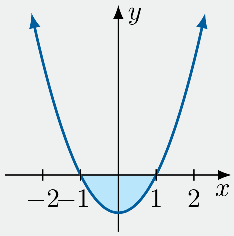
(a)



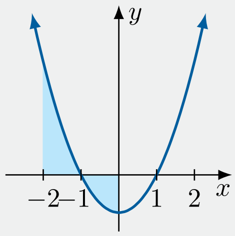
(b)



(c)



(d)



Q7

(a) 8

(b) Region $\mathcal{B}$ is below the x-axis, so the integral will yield 'negative' area. Numerically, it will yield the correct magnitude, but we need to make it positive to give the true area since area cannot be negative.

(c) 2

(d) 10

(e) Bob's answer is the net signed area in the domain $x \in [-4, 2]$. In other words, it is the result when area $\mathcal{B}$ is subtracted from area $\mathcal{A}$.

(f) From above, Bob's answer gives the result when area $\mathcal{B}$ is subtracted from area $\mathcal{A}$. From the diagram, this leaves only part of $\mathcal{A}$ on the right of the x-axis i.e. $\displaystyle\int_0^2 x + 2\,dx$.

Q8

(a) 18 (b) 2 (c) 16

(d) 20 (e) 36 (f) 32

(g) 48 (h) 58

Q9

(a) π (b) $\dfrac{9\pi}{2}$ (c) $-\dfrac{\pi}{2}$

(d) $\dfrac{25\pi}{4}$

Q10

(a) $\dfrac{5\pi}{8}$ (b) $\dfrac{1}{2} + \pi$

Q11

(a) 0 (b) 0 (c) 0

P1

$\dfrac{9\pi}{2}$

P2

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{2\pi}{3}$

(c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$

P3

(a) $(\pm a, 0)$ and $(0, \pm b)$

(b) $\displaystyle\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$

(c) See full worked solutions.

Exercise 7B

Area under a curve

F1

(a) P and R have positive signed areas whereas Q and S have negative signed areas.

(b) $-S$

(c) $\displaystyle -\int_c^d f(x)\,dx$ or $\displaystyle\left|\int_c^d f(x)\,dx\right|$

(d) $P + Q + R + S$

F2

(a) sketch, below (b) positive, sketch

F3

odd, even

Q1

(a) $\frac{32}{3}$ (b) $\frac{16}{3}$

(c) $3 + e - \frac{1}{e^2}$ (d) $\ln 3$

(e) $\frac{9}{2} + 4\ln 2$ (f) $\frac{15}{2} - 8\ln 2$

Q2

(a) $\frac{32}{3}$ (b) $\frac{9}{2}$

Q3

The sketch is to determine if there are any negative areas that need to be flipped. However, this function is positive in the domain of integration so no sketch is necessary.

Q4

$\frac{1}{2}\ln 5$

Q5

(a) The function is even, so the areas on either side are equal.

(b) $\frac{128}{15}$

Q6

(a) The function is even, so the areas on either side are equal.

(b) The function is odd, so the magnitude of the areas on either side are equal. However, since one of the regions is below the x-axis, the signed area from here 'cancels' the signed area from the positive half which results in a net of zero.

Q7

(a) 1 (b) 1

Q8

(a) $\frac{275}{6}$ (b) 10

(c) $\frac{407}{4}$ (d) $\frac{8}{3}$

Q9

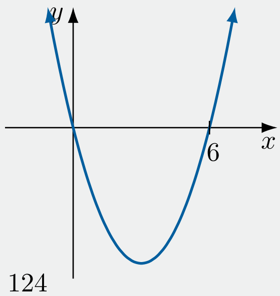
(a) $\frac{125}{12}$

(b) $\frac{253}{12}$

(c) The net signed area $\int_0^5 f(x)\,dx$ is positive, so there must have been more 'positive' area than 'negative' area.

Q10

(a)



(b) $\frac{124}{3}$

Q11

$k = 5$

Q12

$k = 8$

Q13

$k = \frac{3}{2}$

Q14

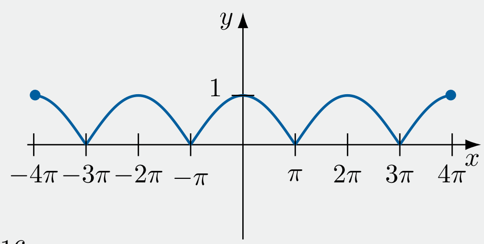
(a) $\frac{9\pi}{4}$

(b) Immediately from the diagram, we can see that as long as they are consecutive integer multiples of π apart, it will work. For example $A = 2\pi$, $B = 3\pi$, or $A = 3\pi$, $B = 4\pi$ etc. However, a less obvious fact is that in actuality *any* interval that is π apart will work, so even something like $A = e + \pi$ and $B = e + 2\pi$ will work!

Q15

(a) 4

(b)



(c) 16

ANSWERS

Q16

0. The curve is symmetric about $x = \frac{\pi}{2}$ in the given domain.

Q17

(a) False. Consider $\int_{-\pi}^{\pi} \cos x \, dx$

(b) True.

P1

(a) $\frac{dy}{dx} = \cos\sqrt{x}$,
$\int \cos\sqrt{x} \, dx = 2\cos\sqrt{x} + 2\sqrt{x}\sin\sqrt{x} + C$

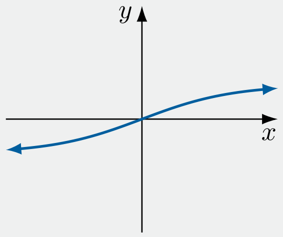
(b) $A\left(\left(\frac{\pi}{2}\right)^2, 0\right)$, $B\left(\left(\frac{3\pi}{2}\right)^2, 0\right)$, $C\left(\left(\frac{5\pi}{2}\right)^2, 0\right)$

(c) Area from B to C is 8π and area from A to B is 4π.

P2

(a) 0

(b)



(c) See full worked solutions.

P3

(a) $y = \frac{k}{\pi^2}(\pi^2 - x^2)$

(b) $k = 3$

(c) $k = -3$ also works because it's just the same region, but flipped across the x-axis.

Exercise 7C
Area involving two curves

F1

$\int_a^b f(x) - g(x) \, dx$

F2

intersection, limits

F3

$\int_0^a f(x) \, dx + \int_a^b g(x) \, dx$

Q1

(a) $(2, 2)$ (b) $\frac{2}{3}$

Q2

(a) $(-4, 0)$ and $(2, 6)$ (b) 36

Q3

(a) $P(-3, 0)$, $Q(0, 6)$, $R(3, 12)$

(b) $\frac{81}{2}$

Q4

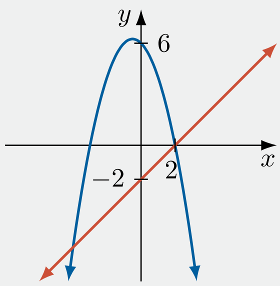
(a) 1 (b) 1

Q5

(a) $(0, -4)$ and $(6, 14)$ (b) 36

Q6

(a)

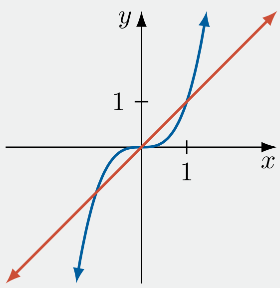


(b) $(-4, -6)$ and $(2, 0)$

(c) 36

Q7

(a)

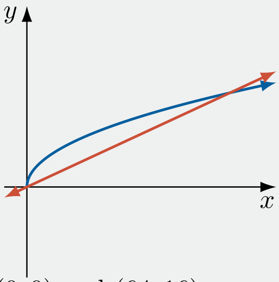


(b) $(\pm 1, \pm 1)$ and $(0, 0)$

(c) $\frac{1}{2}$

Q8

(a)



(b) $(0,0)$ and $(64,16)$

(c) $\frac{512}{3}$

Q9

(a) $(1,0)$ and $(5,8)$ (b) $\frac{64}{3}$

Q10

$(2,4)$, $\frac{16}{3}$

Q11

$(3,2)$, $\frac{22}{3}$

Q12

$1-e^{-2}$

Q13

$(7,7)$, $\frac{169}{6}$

Q14

$\frac{7}{12}$

Q15

(a) Do not solve simultaneously. Instead substitute $x=1$ into both functions.

(b) $\frac{3}{4}$

Q16

$(1,e)$, $2e-1-\frac{1}{e}$

Q17

$P(1,1)$, $1+2\ln\left(\frac{4}{3}\right)$

Q18

(a) See full worked solutions.

(b) See full worked solutions.

(c) $3\ln 4$

Q19

(a) See full worked solutions.

(b) $2\sqrt{2}$

Q20

(a) $\frac{3\sqrt{3}}{4}$ (b) $\frac{3\sqrt{3}}{2}$

Q21

1

Q22

$\frac{1}{3}$

P1

(a) $(1,1)$

(b) $y=x$

(c) Area of A_1 is $\frac{2}{3}$ and area of A_2 is 1.

P2

$k=\frac{16}{9}$

P3

(a) $y=e^2x-e^2$ (b) $\frac{e^2}{2}-1$

Exercise 7D

Trapezoidal rule

F1

(a) $1,\ b-a$ (b) $2,\ \frac{b-a}{2}$

(c) $3,\ \frac{b-a}{3}$ (d) $n-1,\ \frac{b-a}{n}$

F2

$\frac{b-a}{n}$, sub-intervals

F3

(a) $\int_a^b f(x)\,dx \approx \frac{h}{2}\left(y_0+y_2+2y_1\right)$

(b) $\int_a^b f(x)\,dx \approx \frac{h}{2}\left(y_0+y_3+2(y_1+y_2)\right)$

(c) $\int_a^b f(x)\,dx \approx \frac{h}{2}\left(y_0+y_4+2(y_1+y_2+y_3)\right)$

(d) $\int_a^b f(x)\,dx \approx \frac{h}{2}\left(y_0+y_5+2(y_1+y_2+y_3+y_4)\right)$

(e) $\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0+y_n+2(y_1+\cdots+y_{n-1})\right]$

ANSWERS

Q1

(a) 2 (b) 1

(c) $\frac{4}{5}$ (d) $\frac{2}{3}$

Q2

(a)

x	0	1	2	3	4

(b)

x	1	1.5	2	2.5	3

(c)

x	1	1.4	1.8	2.2	2.6	3

(d)

x	-1	$-\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{5}{3}$	$\frac{7}{3}$	3

Q3

(a) 2800 (b) 705

(c) 2025 (d) 4800

Q4

(a) 108 (b) 21000

(c) 1895 (d) 8125

Q5

(a) 10.5 (b) 2.485

(c) 0.48 (d) 0.255

Q6

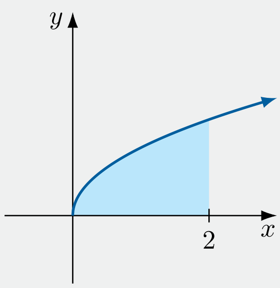
(a) 0.723 (b) 1.117

(c) 1.57 (d) 0.743

Q7

(a)

(i)

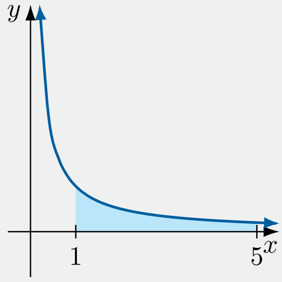


(ii) 1.819

(iii) Under-estimate

(b)

(i)

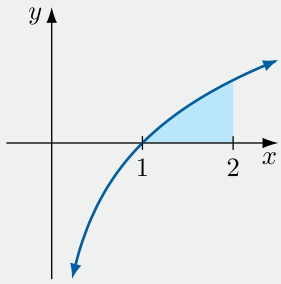


(ii) 1.683

(iii) Over-estimate

(c)

(i)

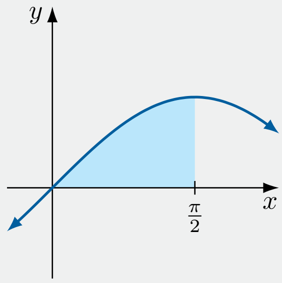


(ii) 0.384

(iii) Under-estimate

(d)

(i)



(ii) 0.987

(iii) Under-estimate

Q8

(a) $\frac{19}{54}$

(b) $\frac{17}{50}$

(c) Three sub-intervals: 5.56%
Five sub-intervals: 2%

Q9

(a) $\ln 3$

(b) See full worked solutions.

(c) $\frac{67}{60}$

Q10

(a) False (b) True

(c) True (d) True

P1

(a) 0.6956

(b) $\ln 2$

(c) 2.71

(d) Over-estimates, because $f(x) = \frac{1}{x}$ is concave up in the domain $x \in [1, 2]$.

(e) Under-estimates.

P2

(a) 0.759

(b) $\frac{\pi}{4}$

(c) 3.037

(d) 3.3%

(e) The curve is concave up, so the trapezia will be below the curve i.e. the areas under-approximate the actual area.

Exercise 7E
Applications involving integration

F1

(a) quantity (b) constant (c) substitute

F2

(a) integrating, constant

(b) 0, 0

(c) 0

(d) $[a, b]$, $|x(b) - x(a)|$

(e) $[b, c]$, $|x(c) - x(b)|$, add

F3

(a) net, quantity, $[a, b]$ (b) do not need

Q1

(a) $x = t^2 - 4t + 1$

(b) $x = -2$

(c) $t = 1, 3$

Q2

(a) $v = 20e^{-2t}$ (b) $x = 10(1 - e^{-2t})$

(c) $x = 10$ (d) $t = \frac{1}{2}\ln 2$

(e) $v = 10\text{m/s}$

Q3

(a) $x = 4\sqrt{3t + 4} - 8$ (b) 12 metres.

Q4

(a) 8 metres per second

(b) $a = 3.2e^{-0.2t} > 0$ for all real t

(c) $x = 24t + 80e^{-0.2t} - 72$

Q5

$t = 1$ second

Q6

$t = 1$

Q7

$x = -5t^2 + 50$

Q8

(a) $\frac{dV}{dt} = -18$

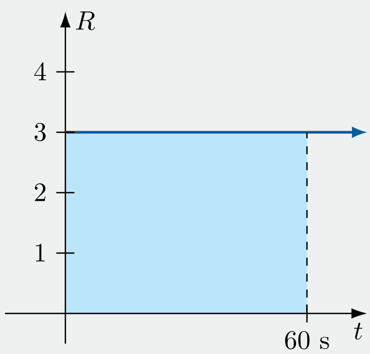
(b) $V = 27000 - 18t$

(c) 25 hours

Q9

(a) 45 litres

(b)



Q10

(a) The speed is $v(t^*)$ and the time is δt, so $D = ST = v(t^*)\delta t$

(b) $v(t^*)\delta t$

(c) The area of the strip represents approximately the distance travelled in that short time interval.

(d) Yes, he is correct.

(e) Take the limit as $\delta t \to 0$. The process reminds us of the construction of the integral, which is based on exactly the same idea of adding an infinitely large number of infinitely small area elements.

(f) The distance travelled by the particle for $t \in [0, a]$ is $\int_0^a v(t)\,dt$

(g) We will still get the distance, but it will be negative since the 'height' of every rectangle is a negative value because the function from which they came is negative.

Q11

(a) $x = 4$ (b) $x = 1$ (c) $x = -3$

Q12

(a) $x = 2t^2 - 12t$

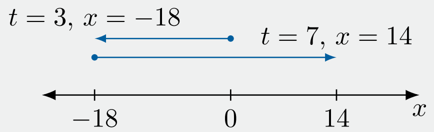
(b) Left

(c) $t = 3$, $x = -18$

(d) Right

(e) $x = 14$

(f)



(g) 50 metres

(h) No, he is not correct. The integral gives the net change in displacement for $t \in [0, 3]$, which is negative because the particle moves to the left. To make it correct he should either have $\left|\int_0^3 4t - 12\,dt\right|$ or $-\int_0^3 4t - 12\,dt$

(i) 32. It represents the distance travelled over $t \in [3, 7]$ because the particle is moving right. This answer is expected because using the diagram, the distance travelled for $t \in [3, 7]$ is $18 + 14 = 32$ metres.

(j) $-\int_0^3 4t - 12\,dt + \int_3^7 4t - 12\,dt$

Q13

(a) $t = 2$ (b) 4 metres

Q14

(a) $t = 4$ (b) $\frac{64}{3}$ metres

Q15

(a) $\int_0^4 v(t)\,dt$

(b) $-\int_0^3 v(t)\,dt$

(c) $\int_3^6 v(t)\,dt$

(d) $\int_0^3 v(t)\,dt - \int_3^6 v(t)\,dt$

(e) $\int_0^6 v(t)\,dt$

(f) $\int_5^{10} f(t)\,dt$

Q16

(a) $t = 1$ (b) 3 metres

Q17

(a) $v = -4\sin(2t)$

(b) $x = 2\cos(2t) + 1$

(c) When $t = \frac{\pi}{2}$, $x = -1$ and when $t = \pi$, $x = 3$

(d) π

(e) 8 metres

Q18

(a) $t = 1$

(b) Bob's answer gives the net change in position after 2 seconds, whereas Mary's answer gives the total distance travelled in the first two seconds.

Q19

(a) $t = \frac{1}{2}, 4$

(b) The particle moves to the right when $t \in [0, 0.5]$, then to the left when $t \in [0.5, 4]$, then to the right again when $t \in [4, 5]$.

(c) $\frac{233}{12}$

Q20

(a) $V_{\text{total}} = \int_0^6 6t - t^2 \, dt$

(b) 36 litres

Q21

$60\left(1 - e^{-0.2}\right)$

P1

$k = 10$

P2

20.5 metres

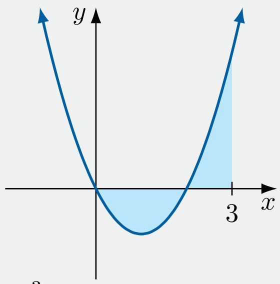
P3

$\frac{44}{3}$ metres

Chapter Review

R1

(a)



(b) $\int_0^3 x^2 - 2x \, dx$ will just give us the net signed area, not area of the shaded region. In other words it will give us the area above the x-axis minus the area below the x-axis.

(c) $a = 0$, $b = 2$, $c = 2$

(d) $\frac{8}{3}$

R2

(a) 9 (b) 8 (c) $\frac{16}{3}$

(d) $2\ln 2$ (e) $\frac{324}{5}$ (f) $\frac{32}{3}$

(g) $\frac{5}{2}$ (h) 2 (i) 12

(j) $\frac{81}{2}$ (k) 1 (l) $e - 2 + \frac{1}{e}$

R3

(a) $\frac{4}{3}$ (b) 36 (c) $\frac{32}{3}$

(d) $\frac{4}{3}$ (e) $\frac{1}{2}$ (f) 8

R4

(a) 4 (b) $\frac{2}{3}$ (c) $\frac{3}{4}$

(d) 1 (e) $\frac{\pi}{4} - \frac{\ln 2}{2}$ (f) $1 - \ln 2$

R5

(a) 2 (b) $\ln 2$

R6

(a) 4π (b) 2π

R7

(a) $\frac{8}{3}$ (b) $\frac{9}{2}$

(c) $\frac{9}{2}$ (d) $3 - 4\ln 2$

R8

(a) $\frac{71}{3}$ (b) 1

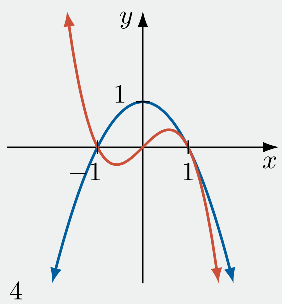
(c) $\frac{1}{3}(10 - 4\sqrt{2})$ (d) $\frac{3}{4}$

R9

$\frac{23}{16}$

R10

(a)



(b) $\frac{4}{3}$

(c) See full worked solutions.

R11

(a) $(\ln 2,\ 1)$ (b) $\ln 2$ (c) $\frac{4}{3} - \ln 2$

R12

(a) $x_P = \frac{\pi}{4}$ and $x_Q = \frac{5\pi}{4}$

(b) $\sqrt{2} - 1$

(c) $2 - \sqrt{2}$

(d) $2\sqrt{2}$

R13

(a) $\sin\sqrt{x}$, $2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C$

(b) π^2, $4\pi^2$, $9\pi^2$

(c) 2π, 6π, 10π

(d) $1 : 3 : 5$

R14

(a) $y = 2 - x$ (b) $\ln 3 - \frac{1}{2}$

R15

(a) 2.9765

(b) Over-approximate, since the curve is concave up.

R16

(a) 0.67 (b) 1.57 (c) 0.38

R17

(a) 10.93

(b) 11.98

(c) 4π

(d) Three function values is 13% and five function values is 4.7%.

R18

(a) $V = 3375 - 20t$ (b) 2 hours 49 minutes

R19

(a) 0 m/s

(b) $v \to 0^+$

(c) $x = 2 - \frac{2}{t^2+1}$, $x = \frac{8}{5}$

(d) $x \to 2^-$ and the particle is slowing down.

R20

(a) Right

(b) $x = -t^2 + 6t - 5$

(c) $x = 4$

(d) $t = 1, 5$

(e) 13 units

(f) See full worked solutions.

R21

7.2 metres

R22

$\dot{x} = 2\sin(2t)$, $x = 4 - \cos(2t)$

R23

(a) $t = 2$ (b) 8 metres

R24

(a) $t = 1$ (b) $e + \frac{1}{e} - 2$ metres

R25

$x = 1$

R26

(a) 27 m (b) $\frac{72}{\pi}$ m (c) $12 + \frac{6}{e^3}$ m

8. Continuous Random Variables

Exercise 8A

Probability density functions

F1

(a) continuous, measured

(b) continuous, histogram

(c) probability, density

(d) area, probability

(e) $\int_a^b f(x)\,dx$

(f) highest

F2

(a) $\geq$ (b) 1

F3

1

Q1

For each part, show that $f(x) \geq 0$ and $\int_a^b f(x)\,dx = 1$ in the domain given.

Q2

For each part, show that $f(x) \geq 0$ and $\int_a^b f(x)\,dx = 1$ in the domain given.

Q3

(a) $\int_1^4 f(x)\,dx \neq 1$

(b) $\int_1^\infty f(x)\,dx \neq 1$

(c) Although $\int_0^4 f(x)\,dx = 1$, $f(x)$ is not always positive in the provided domain.

(d) Although $\int_0^4 f(x)\,dx = 1$, $f(x)$ is not always positive in the provided domain.

Q4

(a) $k = \frac{1}{2}$ (b) $k = 6$

(c) $k = 1$ (d) $k = 2$

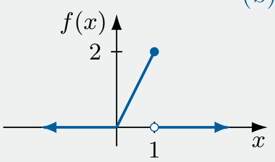
Q5

(a) $\frac{23}{81}$ (b) $\frac{2}{15}$

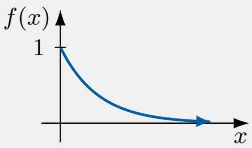
(c) $\frac{1}{e}$ (d) $\frac{3}{4}$

Q6

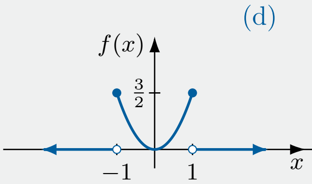
(a)



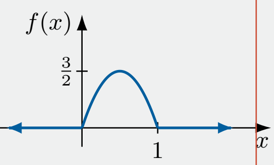
(b)



(c)



(d)



Q7

(a) 1 (b) $\frac{1}{2}$

(c) e (d) $\frac{2}{3}$

Q8

(a) 1

(b) Although the integral is 1, the function is not always positive.

Q9

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

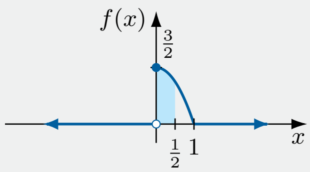
(d) $\frac{2}{3}$

Q10

(a) $\frac{3}{2}$

(b) $\frac{11}{16}$

(c)



(d) $\frac{5}{16}$

(e) $\frac{625}{1408} \approx 0.44$

(f) $\frac{625}{864} \approx 0.72$

Q11

(a) e

(b) $\ln 2$

Q12

$k = \frac{5}{243}$, $p = 3$

Q13

(a) $\frac{3}{4}$

(b) The function is zero for $x \in [-1, 0]$ anyway, so we may as well just integrate over the domain $x \in [0, 1]$.

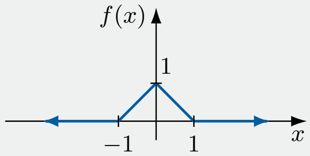
(c) Similarly $P(X \geq 1) = P(0 \leq X \leq 1)$, but by symmetry this is equal to $P(0 \leq X \leq 1)$

(d) Both equal to $\frac{1}{2}$.

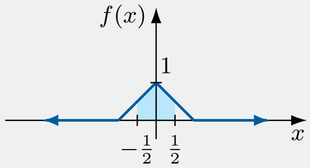
Q14

(a) $k = 1$

(b)



(c) $\frac{3}{4}$



Q15

(a) All of them i.e. 100%

(b) 43.65%

(c) 25.1%

(d) 31.25%

(e) $\frac{873}{1375} \approx 63.5\%$

Q16

(a) $f(x) = \begin{cases} x, & 0 \le x \le \frac{k}{3} \\ k - 2x, & \frac{k}{3} \le x \le \frac{k}{2} \\ 0, & \text{otherwise} \end{cases}$

(b) $k = 2\sqrt{3}$

P1

(a) $\frac{2}{\pi} \ln 2$ (b) 1

P2

(a) $1 \le x \le 2$ (b) $x \ge 2$

P3

See full worked solutions.

P4

(a) $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

(b) The event $P(X \ge b)$ contains the event $P(X \ge a)$. This means that if $X \ge a$, then we necessarily have $X \ge b$ as well. Hence, the probability of $P(X \ge a)$ *and* $P(X \ge b)$ is just $P(X \ge a)$

(c) $P(X \ge a \mid X \ge b)$

$$= \frac{P\left((X \ge a) \cap (X \ge b)\right)}{P(X \ge b)}$$

$$= \frac{P(X \ge a)}{P(X \ge b)}$$

Exercise 8B

Cumulative distribution functions

F1

(a) area (b) decreasing

(c) t (d) horizontal

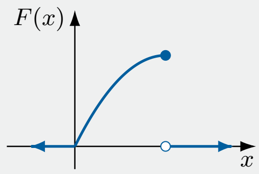
F2

(a) $F(x) = \int_{-\infty}^{x} f(x)\,dx$

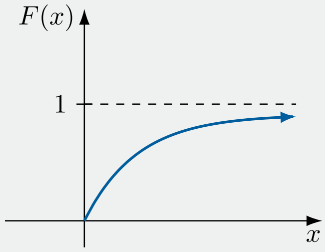
(b) 0.5

Q1

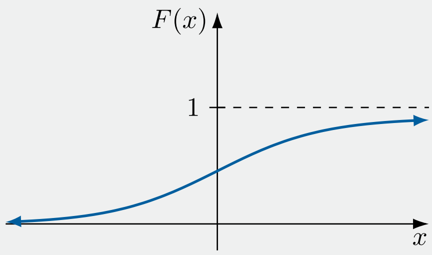
(a)



(b)



(c)



Q2

(a) $0 \le x \le 2$

(b) $\frac{1}{2}$

(c) $x = \sqrt{2}$

Q3

(a) $k = \dfrac{1}{2\ln 2}$

(b) $F(x) = \dfrac{1}{2\ln 2}\ln(x)$

(c) $x = 2$

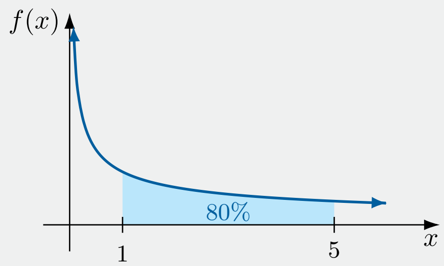
Q4

(a) $F(x) = \int_1^x \dfrac{1}{t^2}\,dt = 1 - \dfrac{1}{x}$

(b) 0.8

(c) $t = 5$

(d)



(e) 2 years

Q5

(a) $F(x) = 3x^2 - 2x^3$ (b) 35.2%

Q6

(a) $F(x) = 1 - e^{-\frac{x}{5}}$

(b) $5\ln 2$

(c) 0.1484

Q7

(a) $F(x) = x^2,\ m = \dfrac{1}{\sqrt{2}}$

(b) $F(x) = \dfrac{1}{2}(1 + x^3),\ m = 0$

(c) $F(x) = 1 - e^{-2x},\ m = \dfrac{1}{2}\ln 2$

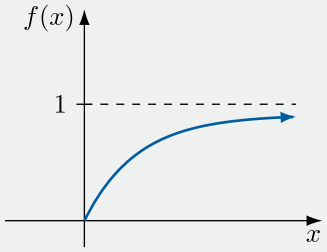
(d) $F(x) = 2 - \dfrac{2}{x+1},\ m = \dfrac{1}{3}$

(e) $F(x) = 1 - e^{-\frac{x}{10}},\ m = 10\ln 2$

(f) $F(x) = 1 - \dfrac{4}{x^2},\ m = 2\sqrt{2}$

Q8

(a)



(b) $x = \ln 2$

(c) $1 - e^{-3}$

(d) $1 - e^{-2}$

(e) $e^{-2} - e^{-3}$

(f) e^{-4}

Q9

(a) The probability threshold that contains all possible values of the random variable. If $F(x)$ has a horizontal asymptote $y = 1$ then the curve has no 'highest point' anyway.

(b) The probability threshold that contains none of the values of the random variable.

(c) The threshold of X, starting from the lowest value, so that 30% of the sample space falls under this category.

(d) $P(a \le X \le B) > P(c \le X \le d)$

Q10

(a) True (b) False (c) False

(d) True (e) True (f) False

Q11

(a) $k = \dfrac{1}{2}$

(b) $F(x) = x - \dfrac{x^2}{4}$

(c) $x = 2 - \sqrt{2}$

P1

$F'(x) = \dfrac{d}{dx}(F(x)) = \dfrac{d}{dx}\left(\int_a^x f(x)\,dx\right) = f(x)$ by the Fundamental Theorem of Calculus.

P2

$P(a \le x \le b) = P(x \le b) - P(x \le a) = F(b) - F(a)$

ANSWERS

P3

$$P(X \geq a \mid X \geq b) = \frac{\int_0^a f(x)\,dx}{\int_0^b f(x)\,dx} = \frac{F(a)}{F(b)}$$

Exercise 8C

Normal distributions

F1

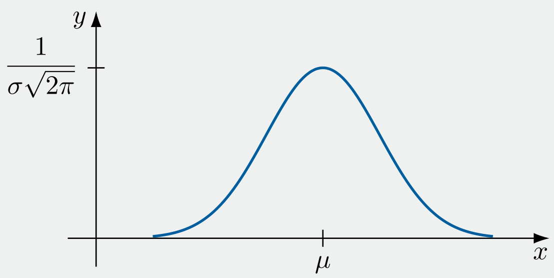
(a) normal (b) bell

(c) $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

F2

$\mathcal{N}(\mu, \sigma^2)$

F3



Q1

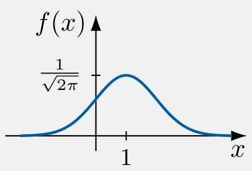
(a) $\dfrac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}$ (b) $\dfrac{1}{\sqrt{\pi}} e^{-x^2}$

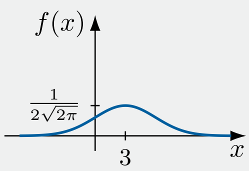
(c) $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}$ (d) $\dfrac{1}{2\pi} e^{-\frac{1}{4\pi}(x-1)^2}$

(e) $\sqrt{\dfrac{2}{\pi}} e^{-2(x+2)^2}$ (f) $e^{-\pi(x+2)^2}$

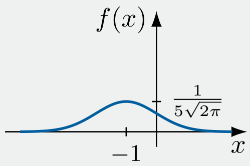
Q2

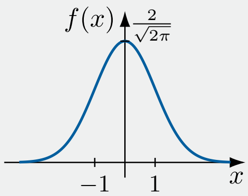
(a) $\mu = 1,\ \sigma = 1$ (b) $\mu = 3,\ \sigma = 2$





(c) $\mu = -1,\ \sigma = 5$ (d) $\mu = 0,\ \sigma = \dfrac{1}{2}$





Q3

(a) Changing μ translates the curve horizontally. If $\mu > 0$ the curve is translated to the right, and if $\mu < 0$ the curve is translated to the left. Note that $x = \mu$ is the centre of the curve.

(b) The larger σ is, the shallower the curve is. The smaller σ is, the steeper the curve is.

Q4

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$$

Q5

See full worked solutions.

Q6

See full worked solutions.

P1

(a) $a = -\dfrac{1}{2},\ b = 0,\ c = -\dfrac{1}{2}\ln(2\pi)$

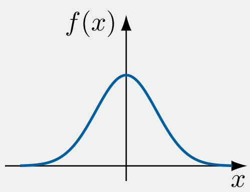
(b) $\mu = -\dfrac{b}{2a},\ \sigma^2 = -\dfrac{1}{2a}$

P2

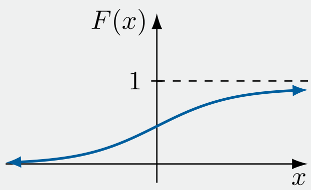
See full worked solutions.

P3

(a)



(b)



(c) The location of the peak of $f(x)$ i.e. $x = \mu$ is the actual location of the inflection point of $F(x)$.

(d) See full worked solutions.

(e) The location of the peak of $f(x)$ i.e. $x = \mu$ is the actual location of the inflection point of $F(x)$.

Exercise 8D
Empirical rules

F1

(a) 68, one

(b) 95, two

(c) 99.7, three

Q1

(a) 68% (b) 95% (c) 99.7%

Q2

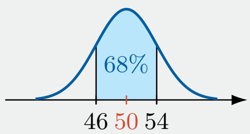
(a) $145 \leq x \leq 175$

(b) $130 \leq x \leq 190$

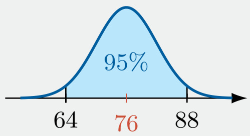
(c) $115 \leq x \leq 205$

Q3

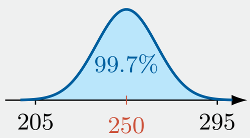
(a)



(b)



(c)



Q4

(a) $\mathcal{N}(68, 64)$ (b) $\mathcal{N}(132, 144)$ (c) $\mathcal{N}(276, 2304)$

Q5

(a)

(i) 68% (ii) 95% (iii) 99.7%

(b)

(i) 16% (ii) 2.5% (iii) 0.15%

(c)

(i) 16% (ii) 2.5% (iii) 0.15%

Q6

(a) 50% (b) 84% (c) 97.5% (d) 99.85%

Q7

(a) 0.6725 (b) 0.6782 (c) 0.6802

Q8

(a) 0.34 (b) 0.475 (c) 0.4985

(d) 0.5 (e) 0.16 (f) 0.025

P1

(a) 0.16 (b) 0.135 (c) 0.975

(d) 0.84 (e) 0.815 (f) 0.8385

P2

See full worked solutions.

Exercise 8E
The z-score

F1

(a) standard, deviations, mean

(b) standard

(c) $\dfrac{x-\mu}{\sigma}$

(d) above

(e) below

F2

(a) empirical (b) table

(c) table, $\leq$ (d) $P(Z \leq a)$

(e) $\leq$ (f) $\geq$

(g) b, a

Q1

(a) -1 (b) 2

(c) $-\dfrac{7}{5}$ (d) $\dfrac{7}{5}$

Q2

(a) 0.68 (b) 0.95 (c) 0.997

Q3

(a) $a = -2$ (b) $b = -3$ (c) $c = 1$

Q4

(a) 0.135 (b) 0.0235
(c) 0.025 (d) 0.84

Q5

(a) 0.9222 (b) 0.2451 (c) 0.9821
(d) 0.102 (e) 0.8365 (f) 0.0139
(g) 0.039 (h) 0.2135 (i) 0.6132

Q6

(a) 0.6826 (b) 0.9544 (c) 0.9974

Q7

(a) True (b) True
(c) True (d) True

Q8

Bob used the X values instead of Z. Instead he should have written down $P(-1 \leq Z \leq 1)$. Mary mis-calculated σ and thought it was 4 when it is actually 2.

Q9

(a) Bob has z-score 1.33 whereas Mary has z-score 1.14.

(b) Bob performed better.

Q10

Biology 1.33, Chemistry 1.38, English 1.80, Maths 1.75, PDHPE 2.25
Bob's strongest subject is PDHPE, and his weakest subject is Biology.

Q11

(a) $a = 1.48$ (b) $a = 0.82$
(c) $a = 2.06$ (d) $a = 1.54$

Q12

(a) 81.5% (b) 190 (c) $\frac{4}{21}$

Q13

$P(X > 6.2) = 0.0228$

Q14

0.3

Q15

z score of the 100 metre event is -1.125
z score of the 200 metre event is -1.571
Leah performed relatively better in the 200 metre event. Note that a lower time is more desirable in the context of racing, so we actually want a more 'negative' z-score.

Q16

(a) Mathematics is $z = 2$.
Chemistry is $z = 0.8$.
Performed better in Mathematics.

(b) Mathematics 72, Chemistry 87.

Q17

$\mu = 15,\ \sigma = \frac{1}{4}$

Q18

(a) 0.6554 (b) 0.3085
(c) 0.0062 (d) 0.7881

Q19

87% are sold

P1

$\mu = 100,\ \sigma = 10$

P2

(a) $P(Z > -a) = P(Z < a) = 0.9934$, then use the table in reverse.

(b) 115.8 milliseconds

P3

0.8438

Chapter Review

R1

(a) $k = \frac{1}{2}$

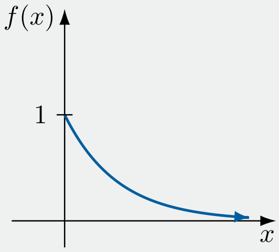
(b) $\frac{\sqrt{3} - 1}{2}$

(c) $\frac{\sqrt{7} - \sqrt{2}}{2}$

R2

(a) $k = 1$

(b)



(c) $\dfrac{1}{e}$

(d) $1 - \dfrac{1}{e^2}$

R3

(a) $k = 6$

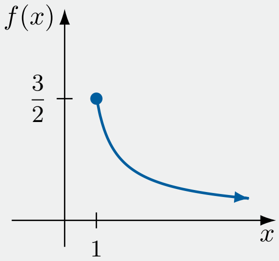
(b) $\dfrac{2}{5}$

(c) $\dfrac{3}{20}$

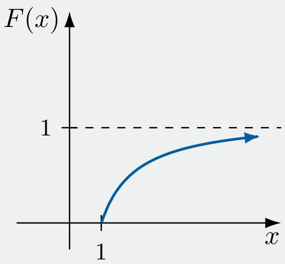
(d) $\dfrac{3}{8}$

(e) $1 + \dfrac{2}{1 - 3x}$

(f)



(g)

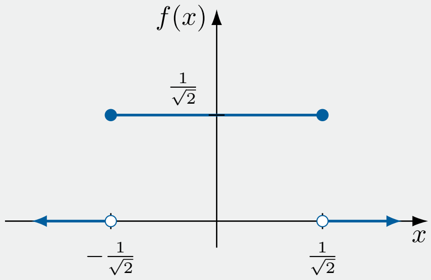


(h) $x = \dfrac{5}{3}$

R4

(a) $k = \dfrac{1}{\sqrt{2}}$

(b)



(c) $\dfrac{2 - \sqrt{2}}{4}$

(d) $x = 0$

R5

$x = 2$

R6

(a) $k = 400$ (b) 2.47% (c) 0.674%

R7

(a) $k = \dfrac{1}{2}$

(b) $F(x) = \dfrac{1}{2}\sqrt{2x + 9} - \dfrac{3}{2}$

(c) $x = \dfrac{7}{2}$

R8

(a) $k = \dfrac{9}{4}$ (b) $\dfrac{\sqrt{6} - 2}{2\sqrt{2} - 2} \approx 0.54$

R9

(a) $F(t) = 1 - e^{-\frac{t}{10}}$

(b) $t = 10 \ln 2 \approx 6.9$ minutes

(c) $t = 10 \ln 5 \approx 16.1$ minutes

(d) $\dfrac{1}{e^2}$

R10

(a) $k = \dfrac{1}{8}$

(b) $F(x) = -\dfrac{1}{16}(x^2 - 8x)$

(c) $x = 4 - 2\sqrt{2}$

R11

$k = 1$

R12

(a) $k = \frac{\pi}{4}$

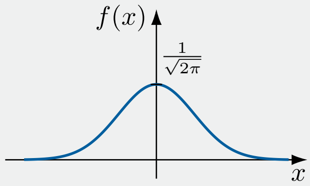
(b) $1 - \frac{1}{\sqrt{2}}$

R13

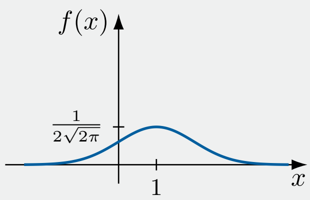
$\frac{5}{8}$

R14

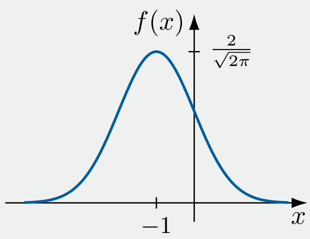
(a) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$



(b) $f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-1)^2}{8}}$



(c) $f(x) = \frac{2}{\sqrt{2\pi}} e^{-2(x+1)^2}$



R15

(a) μ_2 (b) σ_2

R16

(a) 68% (b) 95% (c) 47.5%

R17

(a) $100 \leq x \leq 140$

(b) $80 \leq x \leq 160$

(c) $60 \leq x \leq 180$

R18

(a) $x \geq 436$

(b) $0 \leq x \leq 268$

(c) $x \geq 492$

R19

(a) 48 (b) 96

R20

(a) 50% (b) 16%

(c) 0.15% (d) 97.5%

R21

Bob's z-score was 2.25 whereas Mary's z-score was 2.33, so Mary performed better.

R22

$\mu = 200$, $\sigma = 50$

R23

Chemistry −1.60, English −3.25, Maths Extension 1 −1.670, Maths Extension 2 −1.56, Physics −2.25
Jay's strongest subject is Maths Extension 2 and his weakest subject is English.

R24

(a) 118cm

(b) 3%

R25

$P(32 \leq X \leq 38) = 0.2088$