Specialist Mathematics Units 3 & 4

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Insight Publications acknowledges the Traditional Custodians of the Country on which we meet and work, the Boonwurrung People of the Kulin Nation. We pay our respects to their Elders past and present, and extend that respect to all Aboriginal and Torres Strait Islander peoples.

Contents

Introduction

Insight's *VCE Revisions Questions: Specialist Mathematics Units 3 & 4* contains questions, worked solutions, mark allocations and tips to help you develop skills for your assessment tasks. The questions cover all areas of study in Units 3 and 4 of VCE Specialist Mathematics. A good habit to implement is to test yourself by working through this resource. The process of actively recalling information assists with deeper learning, and you will be able to identify any errors or omissions in your working by comparing your answers with the provided solutions.

Questions are grouped by areas of study, and then under the headings 'Exam 1' and 'Exam 2' to clearly signal the questions for which a CAS calculator can be used. Calculator screenshots are included in the worked solutions for Exam 2 questions. For reasons of space, these are all from a TI-Nspire CAS calculator.

By using this resource as part of your study regime throughout the year, you will be prepared for questions you may encounter in your end-of-year VCE exam.

We wish you well with your studies.

The Insight Team

Questions

Area of Study 1 Discrete mathematics

EXAM 1

Question 1 (4 marks)

Consider the inequality $n! > 2^n$ for $n \geq n_0$, where $n \in \mathbf{N}$.

a. Show that $n_0 = 4$. 1 mark

b. Prove by mathematical induction that $n! > 2^n$ for $n \geq n_0$, where $n \in \mathbf{N}$. 3 marks

Question 2 (3 marks)

Use proof by contradiction to prove that $\sqrt{2} + \sqrt{7} < \sqrt{17}$.

Question 3 (4 marks)

Prove by mathematical induction that the number $11^n - 8^n$ is divisible by 3 for all $n \in \mathbf{N}$.

Question 4 (4 marks)

Prove by mathematical induction that the number $7^n - 1$ is divisible by 3 for all $n \in \mathbf{N}$.

Question 5 (3 marks)

Use proof by contradiction to prove that if n is odd, where $n \in \mathbf{N}$, then $n^2 + 2n + 3$ is even.

EXAM 2

Section A

Question 1

Consider the following statement.

For all integers n, if $3n + 1$ is odd, then n is even.

Which one of the following is the **negation** of this statement?

A. There exists an integer n such that $3n + 1$ is even and n is odd.

B. There exists an integer n such that $3n + 1$ is even and n is even.

C. There exists an integer n such that $3n + 1$ is odd and n is odd.

D. For all integers n, if $3n + 1$ is even, then n is odd.

Question 2

Consider the following statement.

For all integers m and n, if mn is odd, then m and n are both odd.

Which one of the following is the **contrapositive** of this statement?

A. There exist integers m and n such that mn is even and m and n are not both odd.

B. There exist integers m and n such that m or n are both odd and mn is even.

C. For all integers m and n, if m and n are both even, then mn is even.

D. For all integers m and n, if m or n are not both odd, then mn is even.

Use the following information to answer questions 3 and 4.

The algorithm below, described in pseudocode, uses Euler's method to estimate coordinates for a solution to a differential equation $\frac{dy}{dx} = g(x, y)$.

```
Inputs:
  g(x, y), the right-hand side of the DE, as a function on x
  and y
  x_0, the x-coordinate of the starting point
  y_0, the y-coordinate of the starting point
  h, the step size
  n, the number of iterations to perform
define eulers(g(x,y),x_0,y_0,h,n)
  x ← x_0
  y ← y_0
for i from 1 to n
  [                              ]
  x ← x + h
  print (x,y)
end for
return
```

Question 3

Which one of the following options would be the most appropriate to fill the empty box?

A. $y \leftarrow g(x,\ y)$

B. $y \leftarrow y + g(x,\ y)$

C. $y \leftarrow y + h \times g(x,\ y)$

D. $y \leftarrow y + g(x + h,\ y)$

Question 4

Consider the algorithm implemented with the following inputs.

```
eulers(sin(x + y), 1, 4, 0.2, 10)
```

The value of the variable x after **two** iterations of the **for** loop would be closest to

A. 1.2

B. 1.4

C. 1.6

D. 3.6

Question 5

The procedure below has been written in pseudocode.

```
declare integer i
declare integer f
declare integer t
set f to −4
set t to −2
set i to 0
while i < 3
  f ← f + t^2
  t ← f − t
  print f
  i ← i + 1
end while
```

The output of the pseudocode is a list of numbers.

The final number in the list is

A. 6

B. 8

C. 16

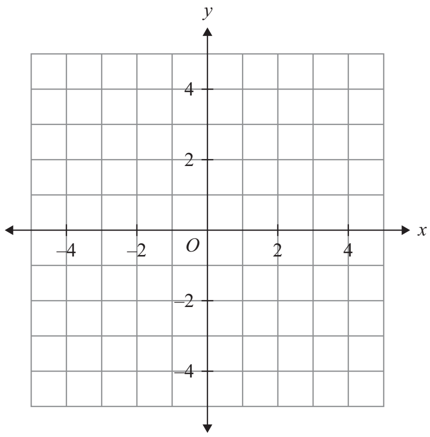
D. 38

Area of Study 2 Functions, relations and graphs

EXAM 1

Question 1 (4 marks)

Sketch the graph of $y = \frac{x^2 - 1}{x^2 - 4}$ on the axes below. Label any axis intercepts and turning points with their coordinates, and any asymptotes with their equation.



Question 2 (4 marks)

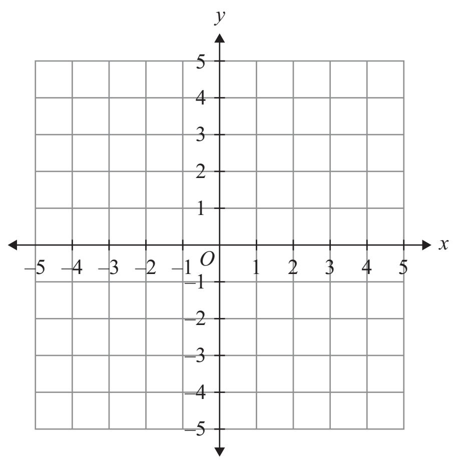
Given that $\sin(x + y) = \frac{1}{2}$ and $\frac{\tan(x)}{\tan(y)} = 3$, find $\sin(x - y)$.

Question 3 (3 marks)

Find $\tan(t)$, given that $t = \arcsin\left(\frac{4}{5}\right) - \arccos\left(\frac{5}{13}\right)$.

Question 4 (3 marks)

Sketch the graph of $y = \frac{x^3 + 2}{x}$ on the axes below. Label any axis intercepts and turning points with their coordinates, and any asymptotes with their equation.



Question 5 (6 marks)

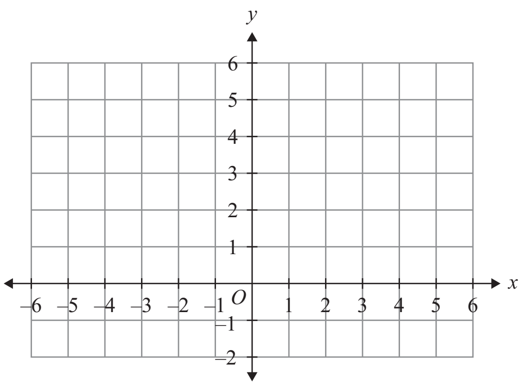
Let $f(x) = \dfrac{(x+1)^2}{(x-2)^2}$.

The rule $f(x)$ can be written in the form $f(x) = A + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$, where $A, B, C \in R$.

a. Show that $A = 1$, $B = 6$ and $C = 9$. 1 mark

b. Find the coordinates of the turning point on the graph of $y = f(x)$. 2 marks

c. Sketch the graph of $y = f(x)$ on the axes below. Label any axis intercepts and turning points with their coordinates, and any asymptotes with their equation. 3 marks



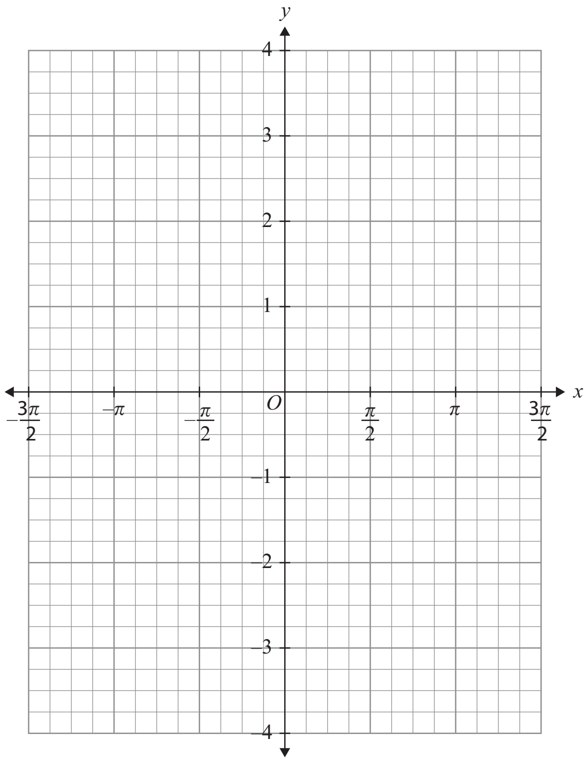
Question 6 (3 marks)

Find $\tan(t)$, given that $t = \arcsin\left(\frac{8}{10}\right) + \arccos\left(\frac{12}{13}\right)$.

Question 7 (3 marks)

Sketch the graphs of $f(x) = \sin(|x|)$ and $g(x) = \frac{2}{\pi}|x|$ on the axes below for $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$.

Label any intercepts and points of intersection with their coordinates.



Question 8 (5 marks)

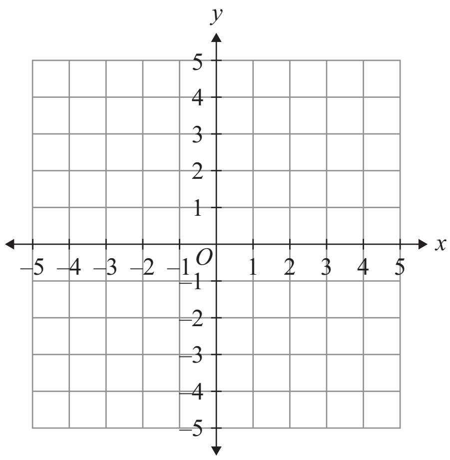
a. Consider the function f with the rule $f(x) = \frac{2\sqrt{3}x^3}{3(x^2 - 1)}$.

Express $f(x)$ in the form $Ax + \frac{B}{x-1} + \frac{C}{x+1}$, where A, B and C are real constants. 3 marks

b. Hence or otherwise, state the equations of the asymptotes for the graph of $y = f(x)$. 2 marks

Question 9 (5 marks)

a. Sketch the graphs of $f(x) = |x + 1|$ and $g(x) = -|2x - 1| + 4$ on the axes below.
Label all axial intercepts with their coordinates and label any point(s) of intersection $f(x)$ and $g(x)$ with their coordinates. 3 marks



b. Determine the area bound by $f(x)$ and $g(x)$. 2 marks

EXAM 2

Section A

Question 1

For the graph $f(x) = \frac{x^4 + 5}{x^2}$, which of the following statements is **not** true?

A. There is a vertical asymptote at $x = 0$.

B. The graph has a stationary point of inflection.

C. The graph has two stationary points.

D. The graph has no axis intercepts.

Question 2

Consider the graph of $y = \frac{x^2 + 1}{x^2 - 1}$.

Which one of the following statements is **true**?

A. The graph has a point of inflection.

B. The graph has an asymptote of $y = x^2 - 1$.

C. The graph is concave down at $x = 0$.

D. The graph has only two asymptotes.

Question 3

An asymptote of the graph of $f(x) = \frac{ax^2 + b}{x - 2}$ has equation $y = ax + c$.
The value of c is

A. 2

B. a

C. b

D. $2a$

Question 4

If a function $f(x)$ has $f''(0) = 0$, which one of the following statements is **true**?

A. The graph of $f(x)$ has a point of inflection at $x = 0$.

B. $f'(0) = 0$

C. The graph of $f(x)$ has a local stationary point at $x = 0$.

D. $f(x)$ is differentiable at $x = 0$.

Question 5

For the graph of $f(x) = \frac{xe^x}{(x - 1)^2}$, which one of the following statements is **true**?

A. $f(x)$ has a vertical asymptote at $x = 0$.

B. $f(x)$ has two local minimum turning points.

C. $f(x)$ has a horizontal asymptote at $x = 0$.

D. $f(x)$ has an oblique asymptote of $y = x$.

Question 6

The graph of $f(x) = \dfrac{x^2 + 3x - 4}{x^2 - 9}$ has which **one** of the following set of asymptotes?

A. two horizontal only

B. two vertical only

C. two horizontal and two vertical

D. one horizontal and two vertical

Question 7

The maximal domain of $f(x) = \dfrac{\sqrt{2 - x}}{\log_e(x)}$ is

A. $[0, 2]$

B. $(0, 2]\backslash\{0\}$

C. $(0, 2)\backslash\{1\}$

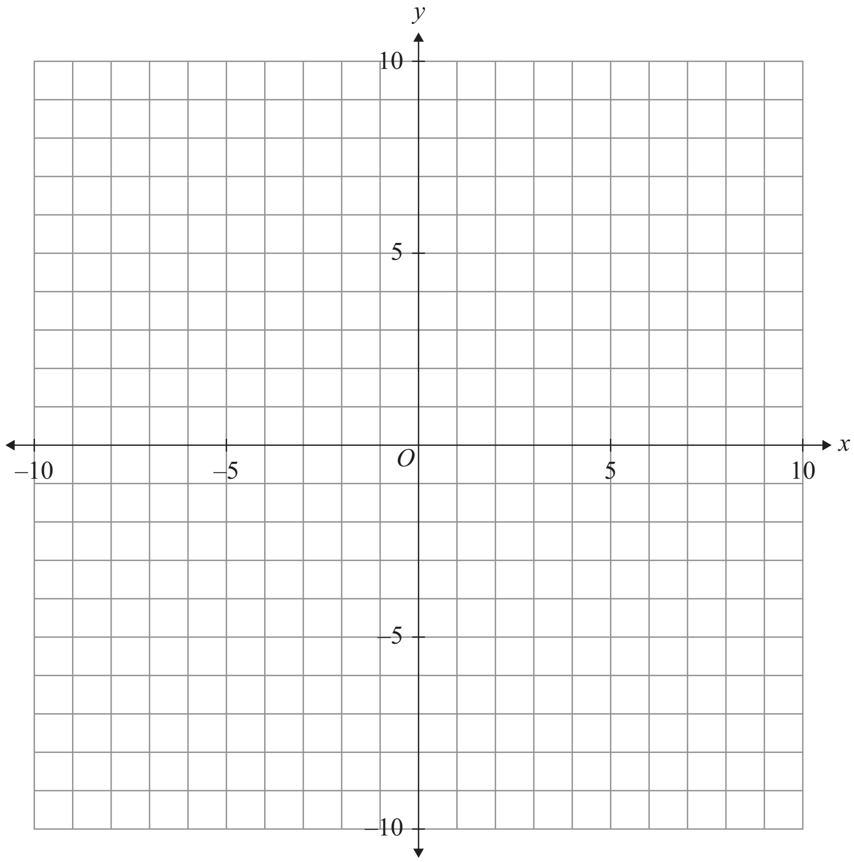
D. $(0, 2]\backslash\{1\}$

Section B

Question 1 (3 marks)

Consider the function $f(x) = \dfrac{x^2 - 2x + 5}{x - 1}$, $x \in R\backslash\{1\}$. The turning points of $f(x)$ are (–1, –4) and (3, 4).

Sketch the graph $y = f(x)$ on the axes below. Label any turning points with their coordinates and any asymptotes with their equation.



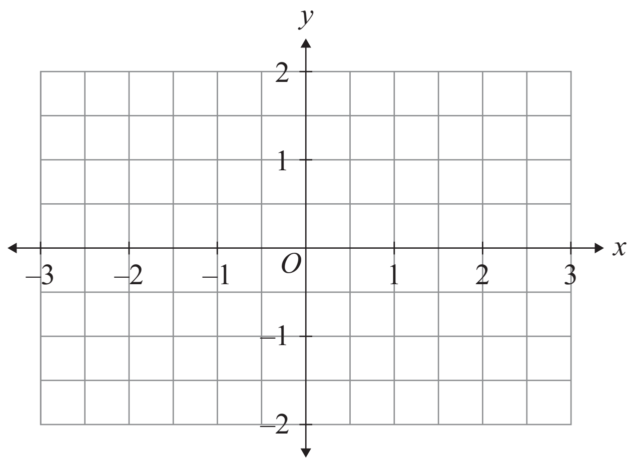
Question 2 (4 marks)

a. Let $f: D \to R, f(x) = \frac{2x}{(x-1)^2} - 1$, where D is the maximal domain of f.

State the equations of any asymptotes of the graph of f. 1 mark

b. Sketch the graph of $f(x) = \frac{2x}{(x-1)^2} - 1$ from $x = -3$ to $x = 3$ on the axes provided below.

Label all stationary points, end points, points of inflection and intercepts with axes, labelling them with their coordinates. Show any asymptotes and label them with their equations. 3 marks



Question 3 (6 marks)

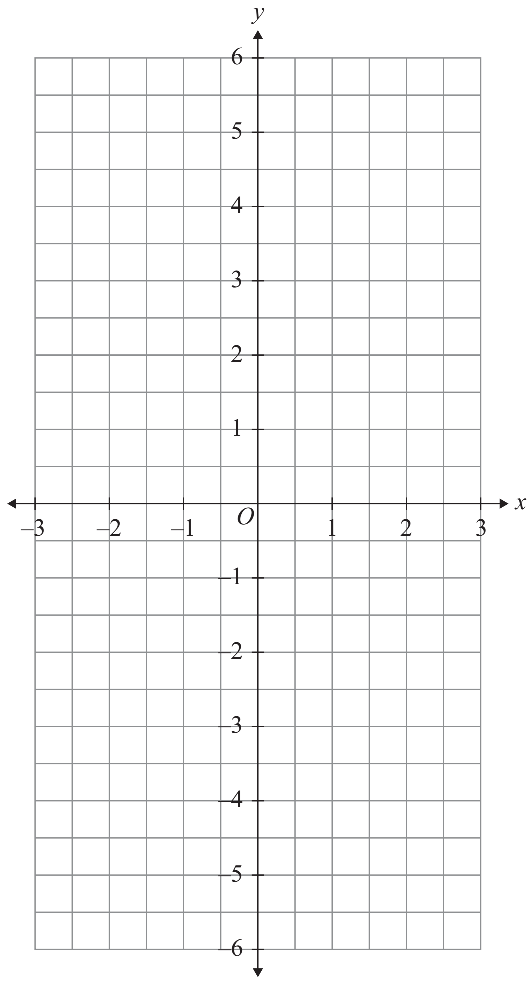
Consider the function $f: D \to R$, where $f(x) = \frac{2x^3 - 3x^2 - 9x + 10}{x^2 - x - 2}$.

a. State the maximal domain D of f. 1 mark

b. Find the equations of any asymptotes of the graph of f. 2 marks

c. $f(x)$ has a point of inflection at $(0.81, -0.84)$.

Sketch the graph of $y = f(x)$ on the axes below, labelling any asymptotes with their equations, and any intercepts and points of inflection with their coordinates. 3 marks



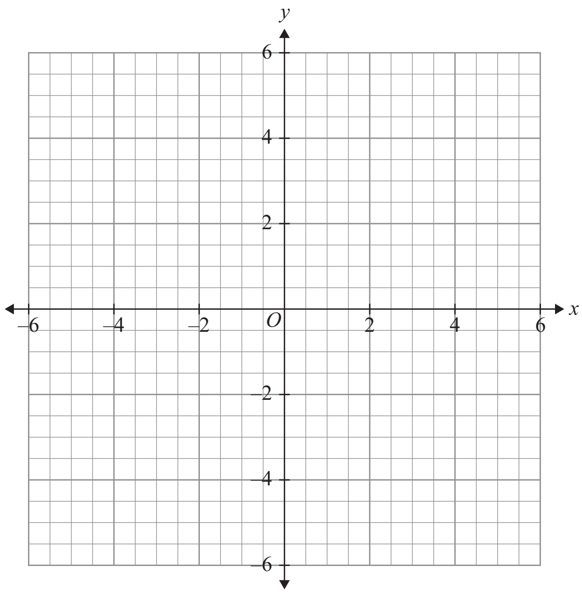
Question 4 (12 marks)

Consider $f(x) = \dfrac{x^2 - x + 1}{x - 1}$, $x \in R\backslash\{1\}$.

a. Find the coordinates of the stationary point(s) of the graph of $f(x)$. 1 mark

b. Find $f''(x)$ and, hence, show that the graph of $f(x)$ does not have any points of inflection. 2 marks

c. Sketch the graph of $y = f(x)$ on the axes below. Label stationary points with their coordinates and show any asymptotes, labelling them with their equation. 2 marks



d. Let $g(x) = \arctan(x)$.

Find the implied domain and range of $g(f(x))$. Give your answers correct to four decimal places. 2 marks

e. Let M be the region bounded by $f(x) = \dfrac{x^2 - x + 1}{x - 1}$ and $h(x) = \dfrac{41}{4} - x$.

Write an integral expression for the area of M and evaluate it, correct to three decimal places. 2 marks

f. Find the volume of the solid of revolution generated by rotating M around the line $y = 1$.

Give your answer correct to three decimal places. 3 marks

Question 5 (4 marks)

Consider the function $f(x) = \dfrac{x^2 - 2x + 5}{x - 1}, x \in R\backslash\{1\}$.

a. Find the coordinates of the stationary points of $f(x)$. 2 marks

b. Show that the stationary point with a positive x-coordinate is a minimum. 2 marks

Question 6 (6 marks)

Let $f: D \to R, f(x) = \dfrac{2x}{(x-1)^2} - 1$, where D is the maximal domain of f.

a. Find the coordinates of the stationary point of the graph of f. 2 marks

b. Use the second derivative to show that the stationary point is a minimum turning point. 2 marks

c. Find the coordinates of any points of inflection of the graph of f. 2 marks

Question 7 (2 marks)

Consider the function $f: D \to R$, where $f(x) = \dfrac{2x^3 - 3x^2 - 9x + 10}{x^2 - x - 2}$.

Find the coordinates of the point for which $f''(x) = 0$ and verify that this is a point of inflection. Give your answer correct to two decimal places.

Question 8 (11 marks)

Let $f(x) = 2e^{\frac{x}{2}}\sqrt{5x^2 - 1}$.

a. State the maximal domain and range of $f(x)$. 2 marks

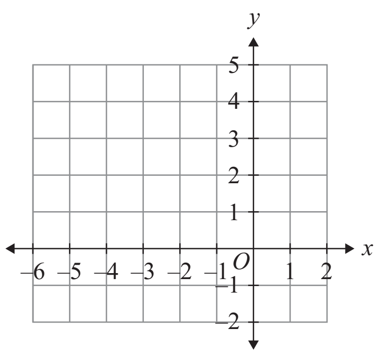
b. State the equation of any asymptotes of $f(x)$. 1 mark

c. Find an expression for $f(x)$ and state the x-coordinate(s) of any stationary points of $f(x)$. 2 marks

d. i. Find the values of x for which $f''(x) = 0$. Give your answer correct to two decimal places. 1 mark

ii. Show that the solutions from **part d.i.** are x-coordinates for points of inflection. 2 marks

e. Sketch the graph of $y = f(x)$ on the axes provided below. Label the local maximum stationary point and all points of inflection with their coordinates, correct to two decimal places, and label any asymptotes with their equations. 3 marks



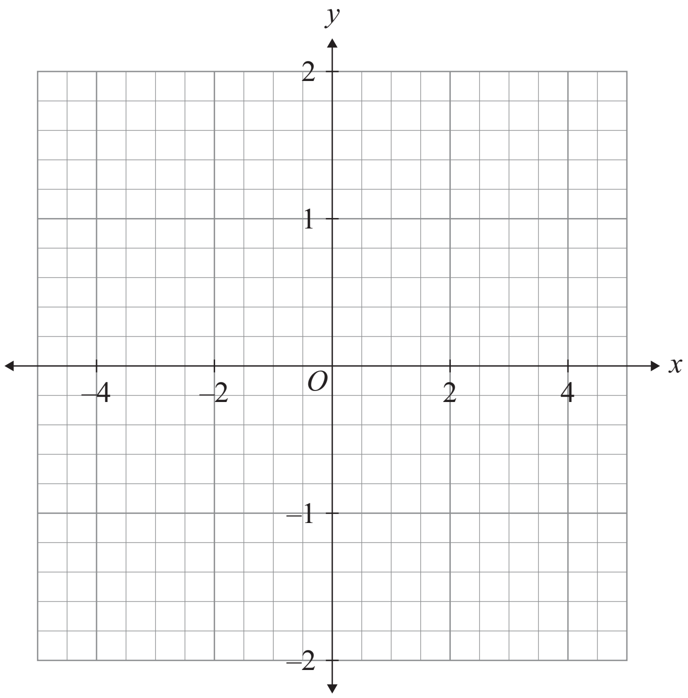
Question 9 (4 marks)

A curve is defined parametrically by the equations $x = 2\cos(t) - 2$ and $y = \sin(t)$, where $t \in [0, \pi]$.

a. State the domain and range of the curve. 2 marks

b. Sketch the curve $y = \sqrt{1 - \frac{(x+2)^2}{4}}$ on the axes below, labelling end points with their coordinates.

2 marks



Area of Study 3 Algebra, number and structure

EXAM 1

Question 1 (3 marks)

Solve $z^4 - (1 + i)z^3 = z^2 - z - iz$, where $z \in C$.

Question 2 (4 marks)

Consider the complex numbers $z = 2 + i$ and $v = 1 + 3i$.

a. Show that $\frac{v}{z} = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$. 2 marks

b. Find $\text{Arg}\left(\frac{\overline{v}}{\overline{z}}\right)$ and $\text{Arg}\left(\frac{v}{zi}\right)$. 2 marks

Question 3 (4 marks)

Let $P(z) = z^3 - (a - 1)z^2 + (b^2 + 4)z - 8$, $z \in C$, where a and b are real constants.

Given that $P(1 - \sqrt{3}i) = 0$, determine all the solutions to $P(z) = 0$ and find the values of a and b.

Question 4 (3 marks)

Express $\dfrac{\sqrt{2} + \sqrt{6}i}{2\sqrt{3} - 2i}$ in polar form.

Question 5 (3 marks)

Let $z^3 - 6z^2 + az - a + 3 = 0$, $z \in R$, where a is a real constant.

Given that $z = 2 - i$ is a solution to the equation, find the value of a.

Question 6 (4 marks)

Solve $z^6 + z^3 + 1 = 0$ for z, where $z \in C$. Express your answers in polar form, using principal values of the argument.

Question 7 (3 marks)

Find the fourth roots of $8 - 8\sqrt{3}i$. Express your answers in polar form, using principal values of the argument.

Question 8 (4 marks)

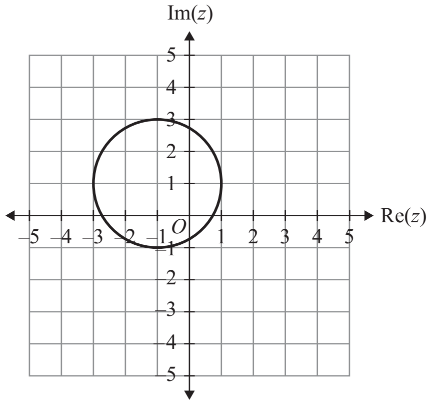
a. Evaluate $(1 + \sqrt{3}i)^5 + (1 - \sqrt{3}i)^5$. 2 marks

b. Find the values of k for which $(1 + \sqrt{3}i)^k + (1 - \sqrt{3}i)^k = 128$, where k is a positive integer. 2 marks

EXAM 2

Section A

Question 1



Given that $z \in C$, the graph shown above can be represented by

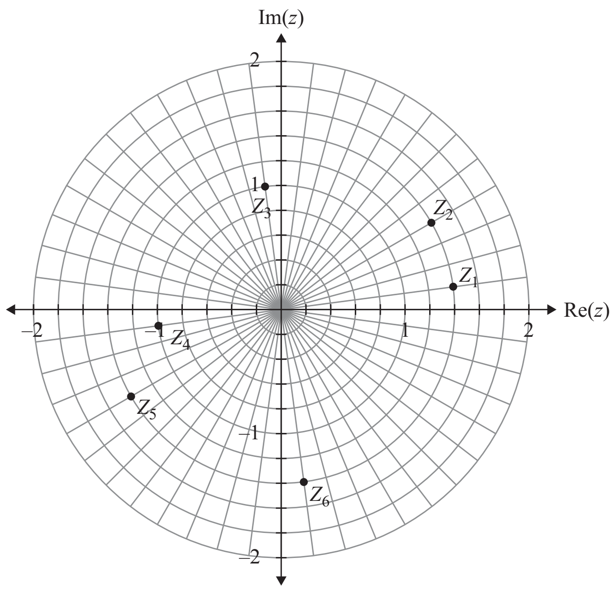
A. $\text{Arg}(z + 1 + i) = 2\pi$

B. $\{z : |z + 1 - i| = 4\}$

C. $\{z : |z - (i - 1)| = 2\}$

D. $\text{Arg}(z + 1 - i) = \pi$

Question 2



Which of the points shown on the complex plane above are solutions to $z^4 = 2\sqrt{3} + 2i$?

A. Z_5 only

B. Z_3 and Z_4 only

C. Z_1, Z_3, Z_4 and Z_6 only

D. Z_1 and Z_6 only

Question 3

For the polynomial $P(z) = z^3 + 2iz^2 + 9z + 18i$, which of the following statements is **true**?

A. $P(z) = 0$ has only one real root.

B. $P(z) = 0$ has two complex roots.

C. $P(-3) = 0$

D. $P(z) = 0$ has no real roots.

Question 4

If $z = r\text{cis}(\theta)$, then $\frac{z^3}{\overline{z}}$ is equivalent to

A. $r^2\,\text{cis}(4\theta)$

B. $3\,\text{cis}(4\theta)$

C. $r^3\text{cis}(4\theta)$

D. $r^2\text{cis}(2\theta)$

Question 5

The relation $|z - 2 + 3i| = 2$ has the same Cartesian equation as

A. $|z - 2 + 3i| = |z - 2 - 3i|$

B. $(z - 2 + 3i)(\overline{z} - 2 - 3i) = 2$

C. $|z - 2| = |z + 3i|$

D. $(z - 2 + 3i)(\overline{z} - 2 - 3i) = 4$

Question 6

The polynomial equation $P(z) = 0$ has real coefficients and roots that include $z = 1, z = i, z = 2 + i$.

The minimum possible degree of $P(z)$ is

A. 3

B. 4

C. 5

D. 6

Question 7

Which one of the following, where A, B, C and D are non-zero real numbers, is the partial fraction form for the expression $\frac{3x^2 + 8x + 4}{(3x + 2)^2(x^2 - 4)}$?

A. $\frac{A}{x - 2} + \frac{B}{3x + 2}$

B. $\frac{A}{x - 2} + \frac{B}{3x + 2} + \frac{C}{(3x + 2)^2}$

C. $\frac{A}{3x + 2} + \frac{B}{3x + 2} + \frac{Cx + D}{x^2 - 4}$

D. $\frac{A}{x - 2} + \frac{B}{x + 2}$

Question 8

A quartic polynomial with all real coefficients has $z = 3ai$ and $z = 2a - 2ai$ as two of its solutions, where a is a non-zero real number.

The polynomial could be

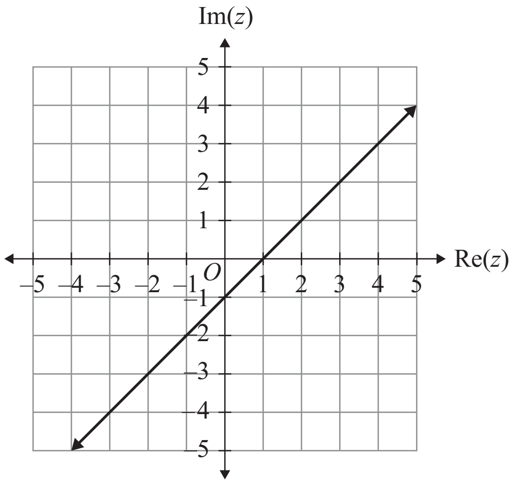
A. $z^4 - 4az^3 + 17a^2z^2 - 36a^3z + 72a^4$

B. $z^2 - (2a + ai)z + 6a^2 + 6a^2i$

C. $z^4 + (13a^2)z^2 - 36a^2$

D. $z^4 + (17a^2 - 4a)z^2 - 36a^3z + 72a^4$

Question 9



Given that $z \in C$, the equation that best represents the graph shown above is

A. $\text{Arg}(z + i) = -\frac{3\pi}{4}$

B. $\{z : |z - 2 + 2i| = |z + 1 - i|\}$

C. $\{z : |z - 2 - i| = |z + 1 + 2i|\}$

D. $\{z : (z + 2 + 3i)(\bar{z} + 2 - 3i) = 4\}$

Question 10

Let z_1 be a complex number with $0 < \text{Arg}(z_1) < \frac{\pi}{2}$. Let z_2 also be a complex number such that z_1 and $z_1 z_2$ are symmetric in the imaginary axis on an Argand diagram.

Which one of the following statements is **true**?

A. $|z_1| = |z_2|$

B. $\text{Arg}(z_2)$

C. $\text{Arg}(z_2) = \pi - 2\text{Arg}(z_1)$

D. $z_2 = 1$

Question 11

Let $z = x + yi$, where $x, y \in R$. The ray $\text{Arg}(z + 2 + i) = \theta$ is normal to the curve defined by $|z - 3 + 4i| = 2$ at the points of intersection of the ray and the path.

The value of $\tan(\theta)$ is

A. $\frac{5}{3}$

B. $\frac{3}{5}$

C. $-\frac{5}{3}$

D. $-\frac{3}{5}$

Question 12

Consider the equation $z^4 = a + bi$, where $z \in C$ and $a, b \in R$.

The sum of all solutions of the equation is

A. $4a + 4bi$

B. $a + bi$

C. 0

D. 1

Question 13

Given the complex number $z = a + bi$, where $a \in R\backslash\{0\}$ and $b \in R$, $\frac{(z\bar{z}i)^2}{z + \bar{z}}$ is equivalent to

A. $\text{Re}(z)[\text{Re}(z) - \text{Im}(z)^2]^2$

B. $\frac{-1}{2\text{Re}(z)}[\text{Re}(z)^2 + \text{Im}(z)^2]^2$

C. $2\text{Re}(z)[\text{Re}(z) + \text{Im}(z)]^2$

D. $\frac{-\text{Re}(z)^2}{2} + \text{Re}(z)\text{Im}(z)^2 - \text{Im}(z)^2$

Question 14

Let $z = x + yi$, where $x, y \in R$. The rays $\text{Arg}(z + 4 - 5i) = -\frac{\pi}{3}$ and $\text{Arg}(z + 2 + i) = \frac{\pi}{4}$, where $z \in C$, intersect on the complex plane at point (a, b).

The coordinates of point (a, b) are

A. $(4\sqrt{3} - 8, 4\sqrt{3} - 9)$

B. $(-4\sqrt{3} + 8, 4\sqrt{3} + 7)$

C. $(-4\sqrt{3} + 8, -4\sqrt{3} + 7)$

D. $(4\sqrt{3} - 8, 4\sqrt{3} - 7)$

Question 15

The equation $az^3 + bz^2 + cz + d = 0$, where $a, b, c, d \in R$, could have solutions

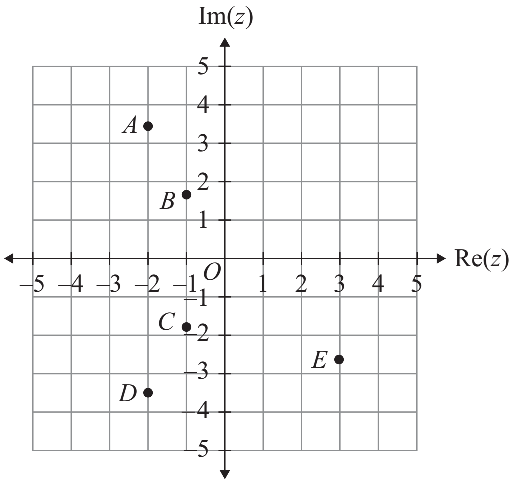
A. $z = 2 + 3i, \quad z = -2 + 3i, \quad z = 3$

B. $z = 2 + 3i, \quad z = -3, \quad z = 3$

C. $z = 2 + 3i, \quad z = 2 - 3i, \quad z = 3$

D. $z = 2 + 3i, \quad z = -2 + 3i, \quad z = 3 + 2i$

Question 16



If $z = 2\,\text{cis}\left(-\frac{4\pi}{3}\right)$, then on the Argand diagram shown above, point z^2 is represented by

A. A

B. B

C. C

D. D

Question 17

The modulus and argument of the solutions to the equation $z^n = \sqrt{2} + \sqrt{2}i$ are, respectively,

A. $\sqrt[n]{2}$ and $\frac{1}{n}\left(\frac{\pi}{4} + 2\pi k\right),\ k \in \mathbf{Z}$

B. $\sqrt[n]{2}$ and $n\left(\frac{\pi}{4} + 2\pi k\right),\ k \in \mathbf{Z}$

C. $\sqrt{2}$ and $\frac{1}{n}\left(\frac{\pi}{4} + 2\pi k\right),\ k \in \mathbf{Z}$

D. 2^n and $\frac{1}{n}\left(\frac{\pi}{4} + 2\pi k\right),\ k \in \mathbf{Z}$

Question 18

Let $z = x + yi$, where $x, y \in R$. The rays $\text{Arg}(z - 2i) = \frac{\pi}{4}$, $\text{Arg}(z - 8) = \frac{\pi}{2}$ and $\text{Arg}(z - (10 + 2i)) = \pi$, where $z \in C$, intersect to form a triangle.

The area of this triangle, in units^2, is

A. 16

B. 20

C. 24

D. 32

Section B

Question 1 (12 marks)

a. Let the complex number $u = 2 + 2i$. Express u in polar form. 2 marks

b. Find the Cartesian equation of the line given by $v = \{z : |\overline{z} + z| = 4\}$, where $z \in C$ and $x > 0$. 2 marks

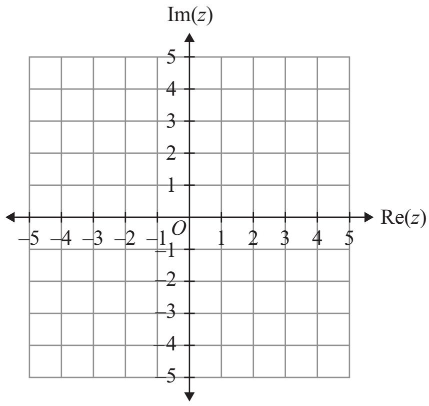
c. Find the Cartesian equation of the circle $w = \{z : (z - 2 - 2i)(\overline{z} - 2 + 2i) - 4 = 0\}$, where $z \in C$. 3 marks

d. On the Argand diagram below, sketch the:

i. line segment from $z = 0$ to $z = u$ from **part a.**

ii. line v from **part b.**

iii. circle w from **part c.** 3 marks



e. The line from (0, 0) to u and the line v form a sector of the circle.

Calculate the area of the sector formed in **part d.** 2 marks

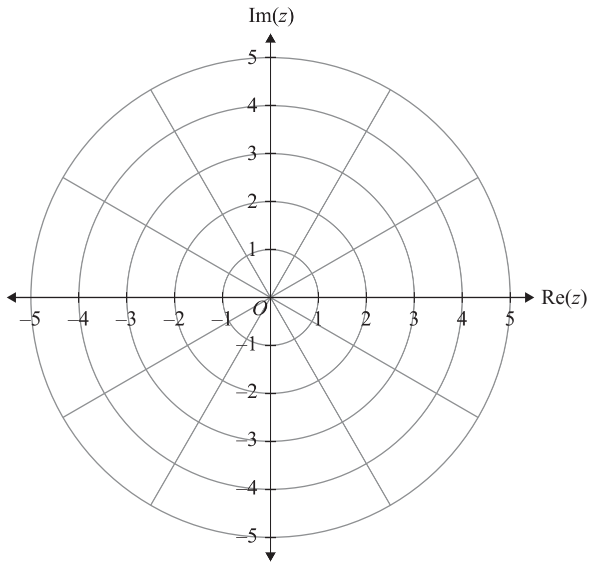
Question 2 (11 marks)

a. Consider the complex number $z_1 = 2\sqrt{3} + 2i$.

i. Express $z_1 = 2\sqrt{3} + 2i$ in polar form. 1 mark

ii. Given that $z_1 = 2\sqrt{3} + 2i$ is a root of the equation $z^3 + 4(1 - \sqrt{3})z^2 + 16(1 - \sqrt{3})z + 64 = 0$, find the other two roots. 2 marks

b. On the Argand diagram below, plot and label the roots of $z^3 + 4(1 - \sqrt{3})z^2 + 16(1 - \sqrt{3})z + 64$. 3 marks



c. The relation $|z - 4\sqrt{3}| = |z|$ represents the Cartesian equation of a line.

Determine the Cartesian equation of this line and sketch it on the Argand diagram in **part b.** 3 marks

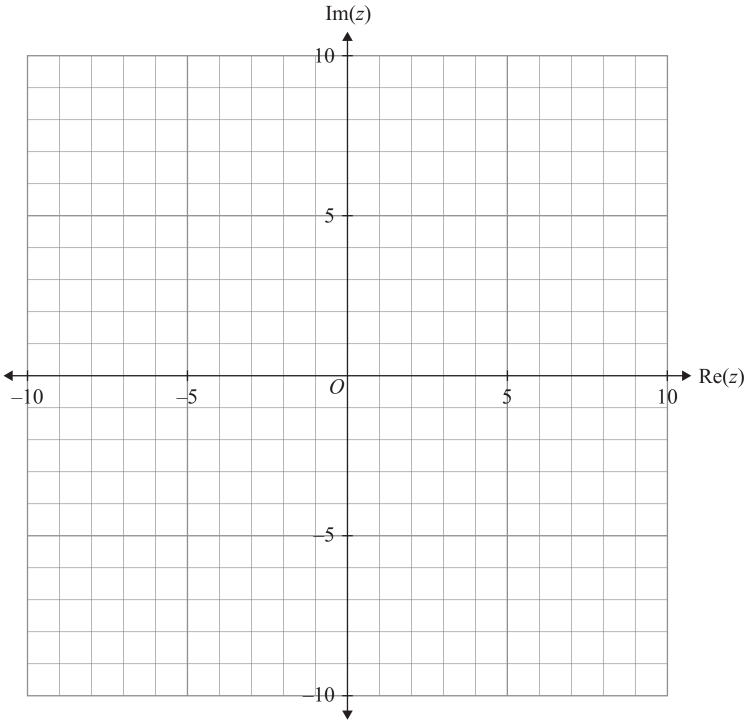
d. The root $z_1 = 2\sqrt{3} + 2i$ and its conjugate lie on the boundary of the circle given by $|z| = 4$.

Determine the area of the minor segment bounded by the line passing through $z_1 = 2\sqrt{3} + 2i$ and its conjugate and the minor arc of the circle given by $|z| = 4$. 2 marks

Question 3 (9 marks)

a. Find the Cartesian equation of the relation given by $|z + 1 + i| = \sqrt{2}|z - 1 - i|$, where $z \in C$. 3 marks

b. Sketch the graph of $|z + 1 + i| = \sqrt{2}|z - 1 - i|$ on the axes below. 2 marks



c. The solutions to the equation $z^2 - (10 + 2i)z + 24 + 2i = 0$, where $z \in C$, lie on the relation sketched in **part b.**

Solve the equation and plot the solutions on the axes provided in **part b.**, labelling the solutions as A and B, where A is the solution located in the first quadrant. 2 marks

d. Let z_1 be the complex number located at the point A.

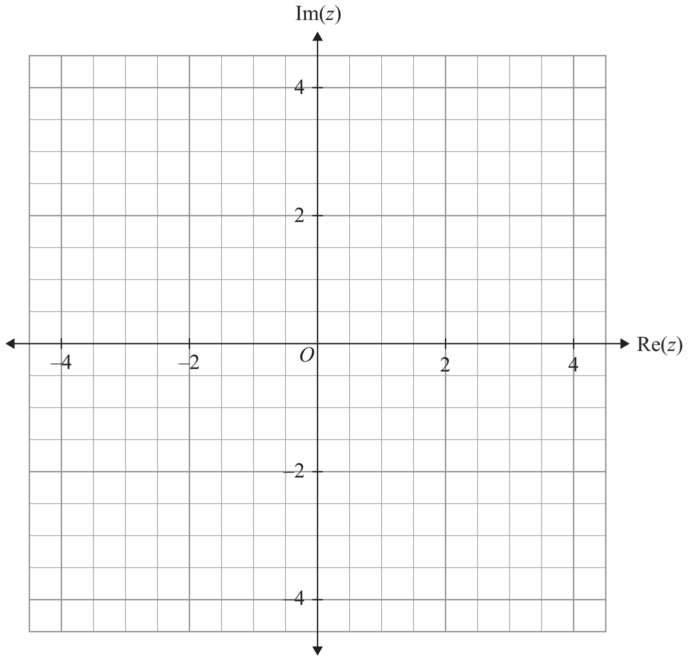
The complex number $z_2 = \overline{z}_1 i$ also lies on the relation $|z + 1 + i| = \sqrt{2}|z - 1 - i|$.

Find and plot z_2 on the axes provided in **part b.** 2 marks

Question 4 (8 marks)

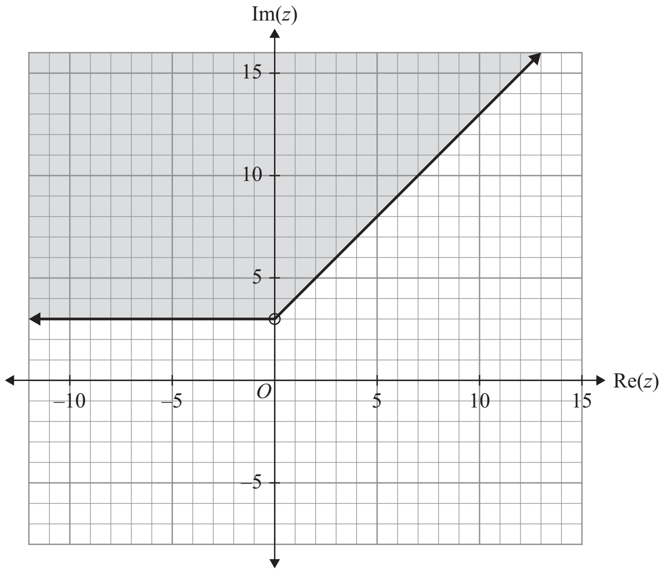
a. Given $z - 3i$ is a factor of $z^3 - 3iz^2 + 4z - 12i$, show that the other two factors are $z + 2i$ and $z - 2i$, where $z \in C$. 2 marks

b. Sketch $|z + 2i| = 2$ on the axes below. 1 mark



c. Determine the area bounded by the graph of $|z + 2i| = 2$ that satisfies $\text{Im}(z) > \text{Re}(z)$. 2 marks

d. Consider the shaded region below.



i. State the Cartesian equation of the boundary of the region defined by $\text{Arg}(z - 3i) = \frac{\pi}{4}$. 1 mark

ii. The line $|z + 2i| = |z + k + 5i|$, where $k \in R$, passes through the shaded region in **part d.**

Find the set of all possible values of k. 2 marks

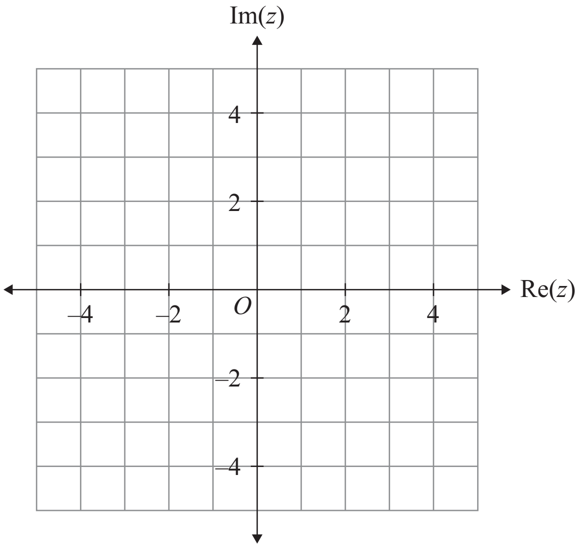
Question 5 (11 marks)

Let $P(z) = z^3 - 2\sqrt{3}z^2 + 12z$.

a. **i.** Show that $w = \sqrt{3} + 3i$ is a solution to $P(z) = 0$. 1 mark

ii. Hence, find the other solutions to $P(z) = 0$. 2 marks

b. Plot and label the solutions on the Argand diagram below. 2 marks



c. i. Find $\overline{w}i^2$. 1 mark

ii. Plot and label $\overline{w}i^2$ on the Argand diagram given in **part b.** 1 mark

iii. Explain how point $\overline{w}i^2$ relates geometrically to point w. 1 mark

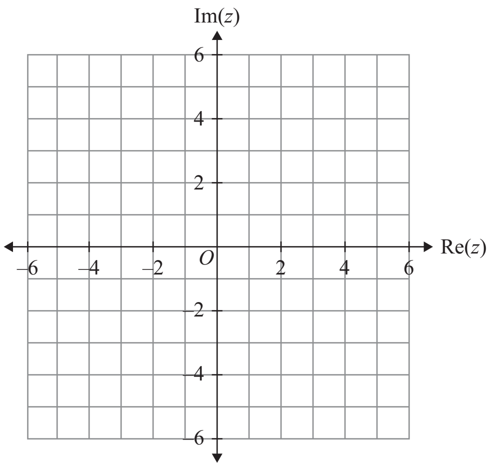
d. The points representing w, $\overline{w}i^2$ and $4i$ lie on the boundary of a circle given by $|z - c| = r$, where c is the centre of the circle and r is the radius.

Find c in the form $a + bi$, where $a, b \in R$, and find the radius r. 3 marks

Question 6 (12 marks)

Two complex numbers, u and v, are defined as $u = 1 + 5i$ and $v = 4 + i$.

a. Sketch u and v on the Argand diagram below. 1 mark



b. State $|u - v|$, the distance between u and v. 1 mark

c. **i.** On the Argand diagram shown in **part a.**, sketch the relation given by $|z - v| = 1$. 2 marks

ii. Evaluate $(-i)^3$. 1 mark

iii. On the Argand diagram given in **part a.**, plot three points that represent the solutions to the equation $(z - v)^3 = i$. 2 marks

d. On the Argand diagram shown in **part a.**, sketch the ray given by $\text{Arg}(z) = \frac{2\pi}{3}$. 2 marks

e. Find the area of the triangle formed by the imaginary axis and the rays $\text{Arg}(z) = \frac{2\pi}{3}$ and $\text{Arg}(z - u) = \frac{-3\pi}{4}$. 3 marks

Area of Study 4 Calculus

EXAM 1

Question 1 (4 marks)

The velocity, v m s^{-1}, of an object moving in a straight line in terms of its position x metres is given by the relation $(x+1)e^{v} - x^2 - 2x - 3 = 0$.

Find the acceleration, in m s^{-2}, when $x = 1$.

Question 2 (3 marks)

Evaluate $\int_0^{\frac{\pi}{3}} \tan^2(x) + \tan^4(x)\,dx$.

Question 3 (3 marks)

Solve the differential equation $\frac{dy}{dx} = 3y\sqrt{x}$, given that $y = e$ when $x = 1$. Express y as a function of x.

Question 4 (2 marks)

A curve is defined by the parametric equations:

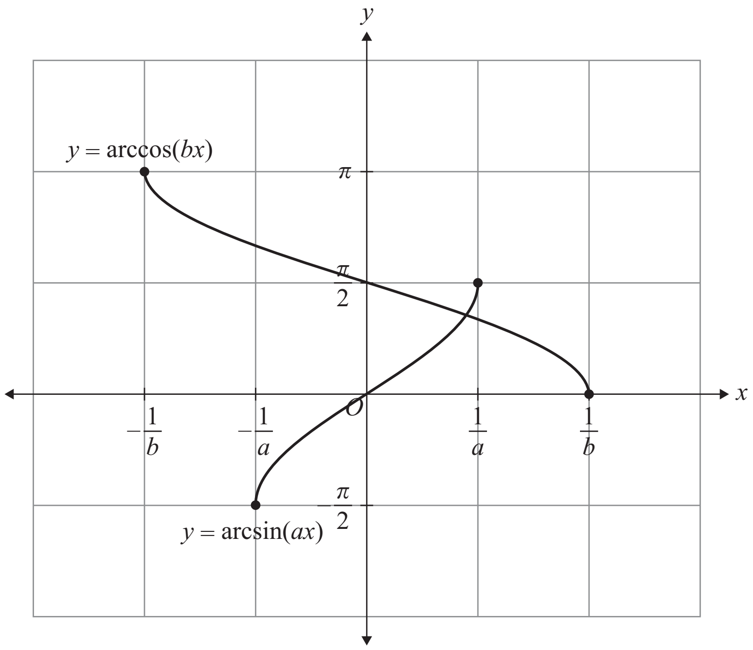
$x(t) = 2 + \sin(2t)$

$y(t) = 2\sin^2(t)$ where $t \geq 0$.

Find the length of the curve from $t = 0$ to $t = \frac{\pi}{3}$.

Question 5 (4 marks)

The graphs of $y = \arcsin(ax)$ and $y = \arccos(bx)$, where $a, b \in R$ and $a > b > 0$, are shown below.



The derivative of $x\arcsin(ax) + \frac{\sqrt{1 - a^2x^2}}{a}$ is equal to $\arcsin(ax)$.

a. Show that $\frac{d}{dx}\left(x\arccos(bx) - \frac{\sqrt{1 - b^2x^2}}{b}\right) = \arccos(bx)$. 1 mark

b. The solution of the equation $\arcsin(ax) = \arccos(bx)$, in terms of a and b, is $x = \frac{1}{\sqrt{a^2 + b^2}}$.

Hence, find the area bounded by $y = \arcsin(2x)$, $y = \arccos(x)$ and the y-axis. 3 marks

Question 6 (4 marks)

Consider the relation $\log_e\left(\frac{x}{y}\right) + x^2 - y = 0, \quad x, y > 0$

a. Show that (1, 1) is a point on the curve defined by this relation. 1 mark

b. Find the equation of the tangent to the curve at point (1, 1) in the form $ax + by + c = 0$. 3 marks

Question 7 (3 marks)

Evaluate $\int_0^{\sqrt{e^2-1}} \frac{4x\log_e(x^2+1)}{x^2+1}\,dx$.

Question 8 (3 marks)

Solve the differential equation $\frac{dy}{dx} = \frac{e^{0.5x}}{3\sqrt{y}}$ for y, given that $y = 1$ when $x = 0$.

Question 9 (4 marks)

A lift is travelling downwards at a constant speed of v m s^{-1}. A person standing in the lift drops a cup of coffee from a height of h metres above the floor of the lift. The coffee cup hits the floor of the lift after 0.5 seconds.

Find the value of h.

Question 10 (3 marks)

Find the gradient of the curve with equation $2xy^2 + \arccos\left(\frac{x}{4}\right) - \frac{\pi xy}{3} = 1$ at point $\left(2, \frac{1}{2}\right)$.

Give your answer in the form $\frac{\pi - a + \sqrt{a}}{b - c\pi}$, where a, b and c are integers.

Question 11 (4 marks)

A fish tank initially holds 200 L of water in which 10 kg of salt has been dissolved. Pure water then flows into the tank at a rate of 5 L per minute. The mixture is stirred continuously and flows out at a rate of 10 L per minute.

a. Show that the differential equation for x, the number of kilograms of salt in the tank after t minutes, is given by $\frac{dx}{dt} = -\frac{2x}{40 - t}$. 1 mark

b. Solve the differential equation given in **part a.** to find x as a function of t. 3 marks

Question 12 (4 marks)

The acceleration, a m s^{-2}, of a motorcycle moving with a velocity of v m s^{-2} in a straight line relative to the origin, is given by $a = v^3 + 16v$.

a. Given that $v = 0$ when $x = \frac{\pi}{16}$, where x is the displacement in metres, find an equation for the velocity of the motorcycle in terms of x. 3 marks

b. Hence, calculate the velocity, in m s^{-1}, of the motorcycle when $x = \frac{\pi}{8}$. 1 mark

Question 13 (4 marks)

Consider the function $f(x) = \arctan(2x - \pi)$.

The function $f(x)$ is translated $\frac{\pi}{2}$ units to the left to form the function $y = g(x)$. The function $y = g(x)$ is then rotated about the y-axis.

Determine the volume of the solid generated when the region bounded by the graph of $y = g(x)$, the y-axis and the line $y = \frac{\pi}{3}$ is rotated about the y-axis.

Give your answer in the form $\frac{\pi}{a}(b\sqrt{c} - \pi)$, where a, b and c are integers.

Question 14 (5 marks)

The position vector of a particle moving along a curve at time t seconds is given by $\underset{\sim}{r}(t) = (2t^6 + 3)\underset{\sim}{i} + \left(\frac{6}{7}t^7 + 1\right)\underset{\sim}{j}$, $t \geq 0$, where distances are measured in metres.

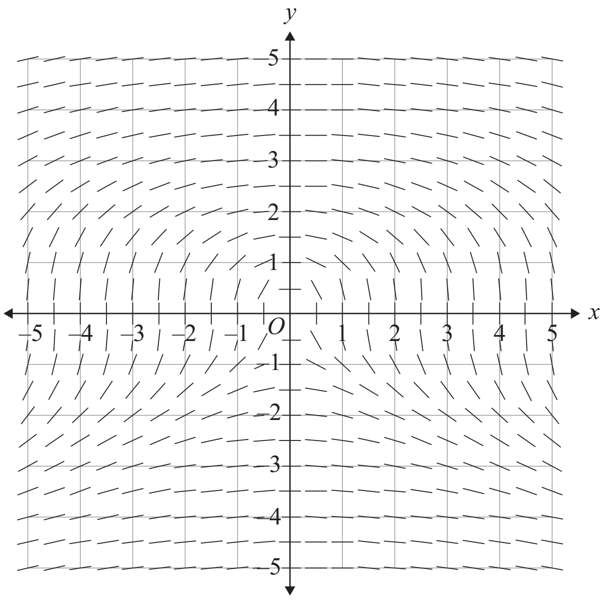
a. Show that the distance, d, that the particle travels along the curve in the first 2 seconds is given by $d = \int_0^2 6t^5\sqrt{t^2 + 4}\, dt$. 1 mark

b. Using a suitable substitution, show that the distance travelled by the particle along the curve in the first 2 seconds can be expressed as $d = 3\left[au^{\frac{7}{2}} - bu^{\frac{5}{2}} + cu^{\frac{3}{2}}\right]_4^8$, where u is the substituted variable and a, b and c are real numbers. State the values of a, b and c. 4 marks

Question 15 (5 marks)

a. Using Euler's method with increments of 0.5, find an estimate (x_2, y_2) for the differential equation $\frac{dy}{dx} = -\frac{x}{y^2}$, given that $y_0(0) = 3$. 2 marks

b. On the slope field below, sketch the solution curve of the differential equation $\frac{dy}{dx} = -\frac{x}{y^2}$ that corresponds to the condition $y(0) = 3$. 1 mark



c. Solve the differential equation $\frac{dy}{dx} = -\frac{x}{y^2}$, given that $y(0) = 3$. 2 marks

Question 16 (3 marks)

Evaluate $\int_0^2 \frac{3}{2x^2 + 3x + 1}\, dx$. Give your answer in the form $a\log_e\left(\frac{b}{a}\right)$, where $a, b \in R$.

Question 17 (5 marks)

Find the equation of the line that is normal to the curve with equation $x\sin(xy) + xy^2 = \pi$ at point $\left(\frac{\pi}{2}, 1\right)$.

Question 18 (4 marks)

Evaluate $\int_0^{2\pi} 2x\cos\left(\frac{x}{4}\right)dx$.

Question 19 (4 marks)

Find the volume, V, of the solid of revolution formed when the graph of $y = \sqrt{\frac{2x+8}{x^2+2}}$ is rotated about the x-axis over the interval $[0, \sqrt{2}]$.

Give your answer in the form $V = \pi\left(\log_e(a) + \sqrt{a}\pi\right)$, $a \in R$.

Question 20 (3 marks)

Evaluate $\int_1^4 \sqrt{x}\log_e(2x)\,dx$. Give your answer in the form $a\log_e(2) + b$, where $a, b \in R$.

Question 21 (4 marks)

The curve given by $y = x^2$ is rotated about the y-axis to form a solid of revolution. Find the surface area of the part of this solid of revolution, where $x \in [\sqrt{2}, \sqrt{6}]$.

Question 22 (3 marks)

Find the surface area of revolution formed when the curve defined by the parametric equations $x = 4 + 2t^2$ and $y = 4t$, where $0 \leq t \leq \sqrt{3}$, is rotated about the x-axis.

Question 23 (5 marks)

Evaluate $\int_1^2 e^x \cos(\pi x)\, dx$.

Question 24 (5 marks)

The population of rats, $P(t)$, in a swamp satisfies the logistic differential equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{200}\right)$, where t is measured in years and the initial population of rats is 5.

a. Find the maximum number of rats predicted by this model. 1 mark

b. Solve the differential equation to find P as a function of t. 4 marks

Question 25 (3 marks)

Consider the curve defined parametrically by $x = 2\cos(t) + \cos(2t)$, $y = 2\sin(t) - \sin(2t)$, where $0 \le t \le 2\pi$.

Find the arc length of the curve from $t = 0$ to $t = \frac{\pi}{3}$.

EXAM 2

Section A

Question 1

Using a suitable substitution, $\int_{e^a}^{e^{5b}} \frac{[\log_e(x)]^4}{5x}\,dx$, where a and b are real constants, can be written as

A. $\int_a^{5b} 5u^4\,du$

B. $\int_{\log_e(a)}^{\log_e(5b)} \frac{u^4}{5}\,du$

C. $\int_a^{5b} \frac{u^4}{5}\,du$

D. $\int_{e^a}^{e^{5b}} \frac{u^4}{5}\,du$

Question 2

Given that $y_0 = y(2) = 0$ for $\frac{dy}{dx} = x^3 - 2x$, using Euler's method, with increments of 0.1, an approximation for the value of y_3 is closest to

A. 1.5309

B. 0.9686

C. 0.12303

D. 1.8876

Question 3

A particle travelling in a straight line with position x and velocity v at time t is given by $v = \sin(x)$. The acceleration of this particle can be given by

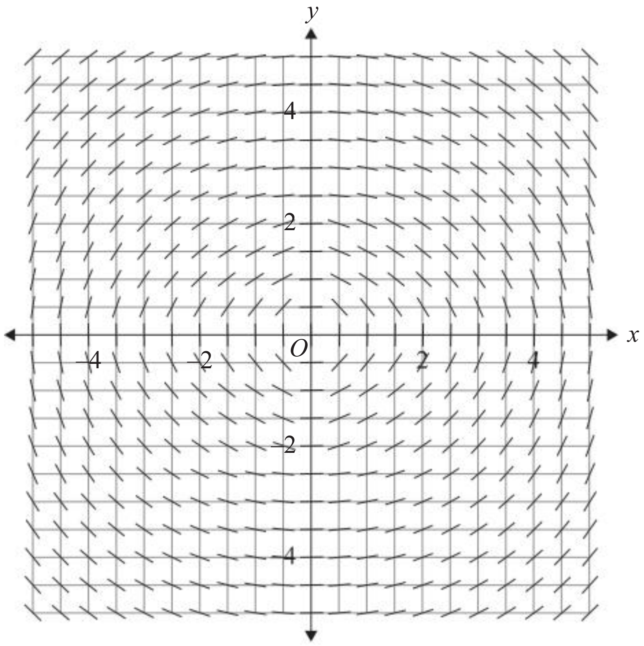
A. $2\sin(x)$

B. $\sin(2x)$

C. $\frac{1}{2}\sin(x)\cos(x)$

D. $\frac{1}{2}\sin(2x)$

Question 4



The direction field shown above best represents the differential equation

A. $\frac{dy}{dx} = -\frac{y}{x}$

B. $\frac{dy}{dx} = -\frac{x}{y}$

C. $\frac{dy}{dx} = x^2$

D. $\frac{dy}{dx} = \frac{1}{y}$

Question 5

An equivalent expression for $\int_{-5}^{-1} \frac{3-2x}{\sqrt{2-x}}\, dx$ is

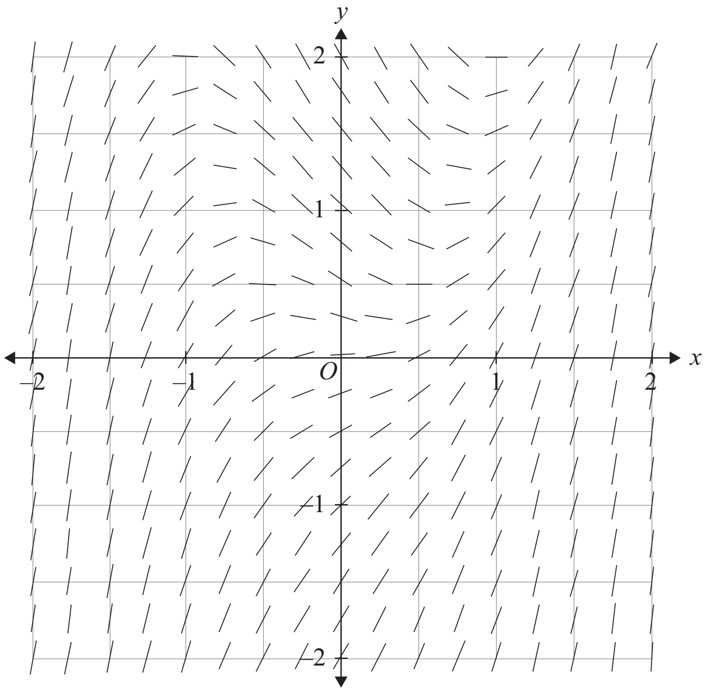
A. $\int_{7}^{3} \left(2u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

B. $\int_{-5}^{-1} \left(2u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

C. $\int_{3}^{7} \left(2u - u^{\frac{1}{2}}\right) du$

D. $\int_{3}^{7} \left(2u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

Question 6



The direction field shown above best represents the solution to the differential equation

A. $\frac{dy}{dx} = 2x - y$

B. $\frac{dy}{dx} = 2x^2 + y$

C. $\frac{dy}{dx} = y - 2x$

D. $\frac{dy}{dx} = 2x^2 - y$

Question 7

Given that $x\cos(y) + y\sin(x) = 2$, the value of $\frac{dy}{dx}$ at point $\left(\frac{\pi}{2}, \frac{\pi}{3}\right)$ is

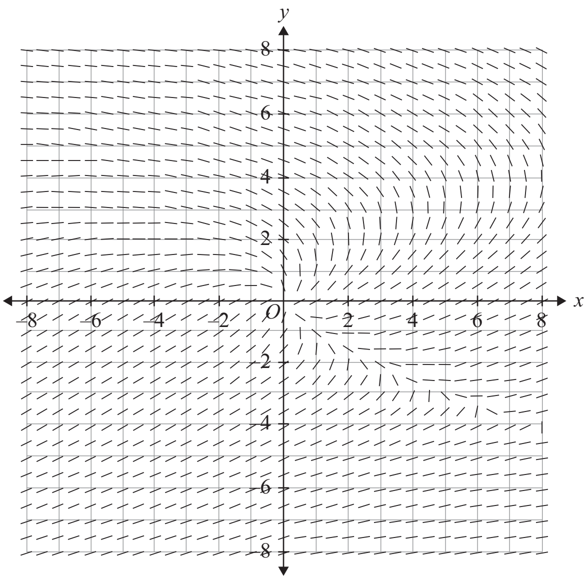
A. $\frac{2}{\pi\sqrt{3} + 4}$

B. $\frac{-3\pi}{4\pi + 6\sqrt{3}}$

C. $\frac{2}{\pi\sqrt{3} - 4}$

D. $\frac{3\pi}{4\pi - 6\sqrt{3}}$

Question 8



The direction field shown above best represents the differential equation

A. $\frac{dy}{dx} = \frac{x + y}{y - x}$

B. $\frac{dy}{dx} = \frac{x + 2y}{2x - y^2}$

C. $\frac{dy}{dx} = \frac{y - 2x}{2x - y}$

D. $\frac{dy}{dx} = \frac{y - 2x}{x - 2y}$

Question 9

With a suitable substitution, $\int_{-1}^{2} (x^2 + 1)\sqrt{x + 2}\, dx$ can be expressed as

A. $\int_{-1}^{4} \left(u^{\frac{5}{2}} - 4u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\right) du$

B. $\int_{1}^{4} \left(u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\right) du$

C. $\int_{1}^{4} \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\right) du$

D. $\int_{1}^{4} \left(u^{\frac{5}{2}} - 4u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\right) du$

Question 10

Given that $\frac{dy}{dx} = \arcsin(x)e^{2y}$, the value of $\frac{d^2y}{dx^2}$ at point $\left(\frac{1}{2}, 0\right)$ is

A. $\frac{\pi^2}{18} + \frac{2\sqrt{3}}{3}$

B. $\frac{\pi^2}{18} + \sqrt{2}$

C. $\frac{\pi}{3} + \frac{2\sqrt{3}}{3}$

D. $2\pi^2 e^2 + e$

Question 11

The length of the curve defined by the parametric equations $x = -4\sin(2t)$ and $y = 2\cos(2t)$ for $0 \le t \le \frac{\pi}{4}$ is given by

A. $\int_0^{\frac{\pi}{4}} \sqrt{8\cos^2(2t) + 4\sin^2(2t)}\, dt$

B. $2\int_0^{\frac{\pi}{4}} \sqrt{3\cos^2(2t) + 1}\, dt$

C. $\int_0^{\frac{\pi}{4}} \sqrt{-64\cos^2(2t) + 16\sin^2(2t)}\, dt$

D. $4\int_0^{\frac{\pi}{4}} \sqrt{3\cos^2(2t) + 1}\, dt$

Question 12

A curve is defined by the pair of parametric equations $x(\theta) = \theta - \sin(\theta)$ and $y(\theta) = 1 - \cos(\theta)$ for $0 \le \theta \le 2\pi$.

The length of the curve is closest to

A. 6.28

B. 11.72

C. 21.71

D. 8.00

Question 13

A solid is formed when the graph of $y = 4\sin(2x)$, where $0 \le x \le \frac{\pi}{2}$, is rotated about the x-axis. Which one of the following integrals is **not** equal to the volume of the solid?

A. $\int_0^{\frac{\pi}{2}} (4\sqrt{\pi}\sin(2x))^2\, dx$

B. $16\pi\int_0^{\frac{\pi}{2}} (1 - \sin(x))\, dx$

C. $64\pi\int_0^{\frac{\pi}{2}} (\sin^2(x)\cos^2(x))\, dx$

D. $8\int_0^{\pi} (x)\, dx$

Question 14

A waste fuel tank initially contains 130 litres of waste fuel, of which 15 litres is diesel. Further diesel is added at a rate of 25 litres per minute, while simultaneously the mixed waste fuel is drawn off at a rate of 20 litres per minute. The tank is continuously stirred and the quantity of diesel in the tank at any time is Q.

Which one of the following equations describes the change in Q?

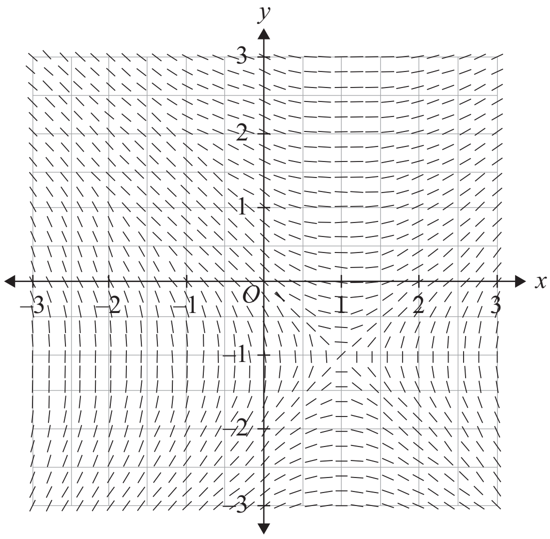
A. $\frac{dQ}{dt} = 25 - 20Q$

B. $\frac{dQ}{dt} = 25 - \frac{20Q}{130}$

C. $\frac{dQ}{dt} = \frac{25 - 20Q}{130 + 5t}$

D. $\frac{dQ}{dt} = 25 - \frac{20Q}{130 + 5t}$

Question 15



The direction field shown above best represents the differential equation

A. $\frac{dy}{dx} = \frac{x-1}{y+1}$

B. $\frac{dy}{dx} = \frac{x+1}{y+1}$

C. $\frac{dy}{dx} = \frac{x}{y+1} - 1$

D. $\frac{dy}{dx} = \frac{x-1}{y-1}$

Question 16

Given that $y(3) = 0$ and $y' = (x-2)(y+1)$, Euler's method, with increments of 0.1, gives $y(4)$ as

A. 5.000

B. 3.482

C. 2.861

D. 3.495

Section B

Question 1 (1 mark)

Consider the function $f(x) = \frac{x^2 - 2x + 5}{x-1}, x \in R\backslash\{1\}$.

Find the length of the curve over the interval $x \in [3, 6]$, correct to two decimal places.

Question 2 (12 marks)

A glass company is investigating the shape of one of the wine glasses it manufactures.

The company uses an inverse sine curve to model the shape of the glasses.

a. **i.** Differentiate $\sqrt{1-x^2} + x\sin^{-1}(x)$ and state the domain for which the derivative exists. 2 marks

ii. Hence, find an antiderivative of $\int 2\sin^{-1}\left(\frac{x}{2}\right)$. 2 marks

The company has decided to model the shape of the wine glass using the equation $y = \sin^{-1}\left(\frac{x}{2} - 2\right) + \frac{\pi}{2}$, where x and y are the respective radius and height of the glass, in centimetres.

b. A solid can be generated by rotating the area bounded by the section of the curve from $x = 2$ to $x = 6$ and the y-axis about the y-axis.

i. Write an integral that represents the volume of the glass. 1 mark

ii. Hence, find the volume of the glass, correct to two decimal places. 1 mark

Water is added to the empty glass such that the height of the water in the glass changes at a rate of $3t$, where $t \geq 0$.

c. Write a differential equation relating the radius of water in the glass, x cm, to the length of time, t seconds, that water is added to the glass. Express your answer in terms of x and t. 3 marks

d. Solve the differential equation found in **part c.** if the radius of water in the glass is initially 2 cm. 3 marks

Question 3 (8 marks)

The side profile of a plastic coffee cup can be modelled by the equation $f(x) = 3\sin^{-1}\left(\frac{x-7}{2}\right) + \frac{3\pi}{2}$, $x \in [5, 9]$, where x is measured in centimetres.

a. Show that the length of the side profile of the coffee cup is given by the definite integral $\int_5^9 \sqrt{1 - \frac{a}{(x+b)(x+c)}}\, dx$, where $a, b, c \in R$. State the values of a, b and c. 2 marks

b. Find the length of the side profile of the coffee cup. Give the answer in centimetres, correct to three decimal places. 1 mark

c. To determine the volume of the coffee cup, the side profile is rotated around the y-axis.

i. Write a definite integral that gives the volume of the coffee cup. 1 mark

ii. Hence, find the volume of the coffee cup, in cubic centimetres, correct to two decimal places. 1 mark

To test the insulation properties of the coffee cup, coffee at a temperature of 90°C is poured into the cup. The coffee then takes 15 minutes to cool to a temperature of 60°C. The coffee cools according to the differential equation $\frac{dT}{dt} = -k(T - T_s)$, where T is the temperature of the coffee after t minutes, T_s is the temperature of the room and $k \geq 0$.

d. The temperature of the room that the cup is in is 20°C.

Solve the differential equation. Give your answer in the form $T = a + be^{-kt}$, where $a, b, k, t \geq 0$. Write k correct to two decimal places. 3 marks

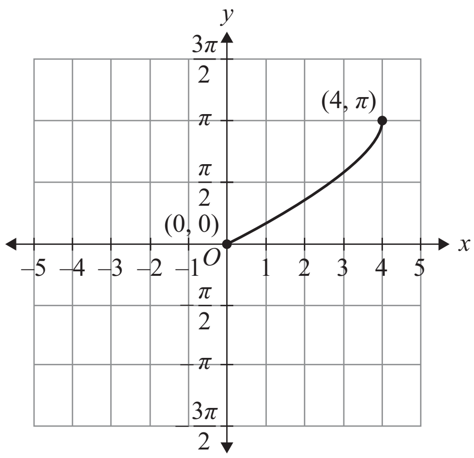
Question 4 (2 marks)

Consider the function $f : D \to R$, where $f(x) = \frac{2x^3 - 3x^2 - 9x + 10}{x^2 - x - 2}$.

Write a definite integral that gives the length, L, of the curve $f(x)$ over the interval $x \in [0.5, 1.5]$, and then state the value of the length of the curve, correct to two decimal places.

Question 5 (8 marks)

Consider the function $f : [0, a] \rightarrow R, f(x) = 2\arcsin\left(\frac{x}{4}\right)$, where a is the largest value for which f is defined. The graph of $f(x)$ is given below.



The curve is rotated about the y-axis to form a volume of revolution that is to model the shape of a birdbath, where the length units are in centimetres.

a. Show that the volume, V cubic centimetres, of water in the birdbath when it is filled to a depth of h centimetres is given by $V = 8\pi\,[h - \sin(h)]$. 2 marks

b. The birdbath is initially empty. Water is poured into the birdbath at a rate of $16\pi^2\sqrt{h}$ cm^3 s^{-1}, where h is the depth, in centimetres, at time t seconds.

i. Show that the expression for the rate at which the height of the water is changing is given by $\frac{dh}{dt} = \frac{2\pi\sqrt{h}}{1-\cos(h)}$. 2 marks

ii. Find the rate, in centimetres per second, at which the height of the water is changing when the birdbath is filled to one-quarter of its maximum depth. Give your answer correct to two decimal places. 1 mark

c. Due to a hot summer, the water in the birdbath evaporates so that the depth of water in it is $\frac{\pi}{4}$ cm before being refilled.

i. Using Euler's method, with a step size of $\frac{\pi}{4}$ cm, find an estimate of the volume of water in the birdbath when $h = \frac{\pi}{2}$ cm. Give your answer in cubic centimetres, correct to two decimal places. 2 marks

ii. Determine if the estimate in **part c.i.** is an overestimate or an underestimate of the volume of water in the birdbath for a depth of $\frac{\pi}{2}$ cm. 1 mark

Question 6 (8 marks)

A water treatment plant has discovered that one of its 12 000 litre water tanks has a fluoride concentration of 3 milligrams per litre, which is above the prescribed limit of 1.5 milligrams per litre. To rectify the situation, pure water is pumped into the tank at a rate of 50 litres per minute and the tank's water is pumped out at a rate of 100 litres per minute.

Let x be the total amount of fluoride in the tank at any time t, where x is measured in milligrams and t is measured in minutes.

a. Show that $\frac{dx}{dt} = \frac{-100x}{12\,000 - 50t}$. 1 mark

b. Show that $x = \frac{5t^2}{8} - 300t + 36\,000$. 3 marks

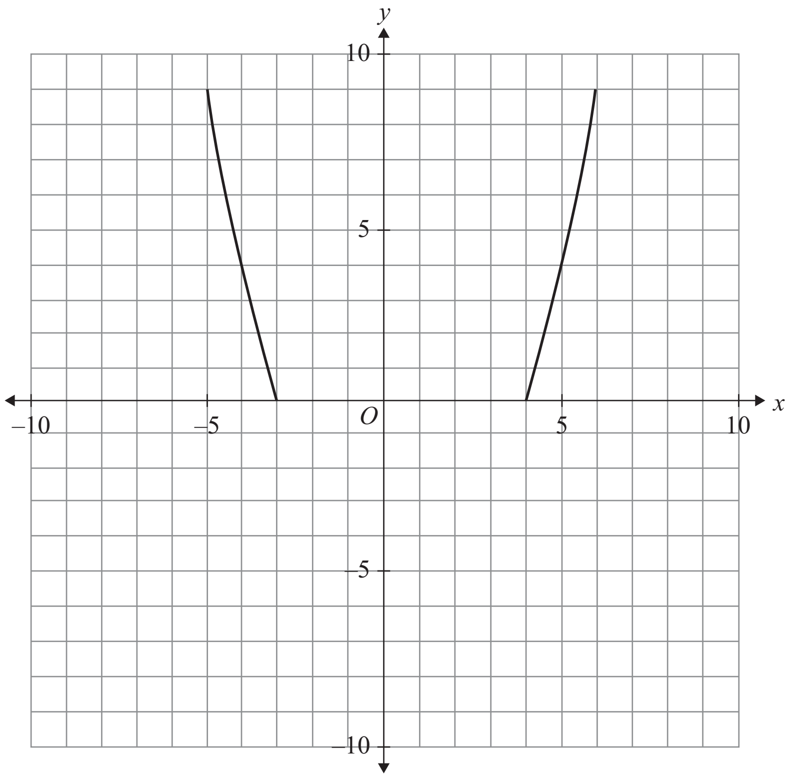
c. i. When the concentration in the tank reaches 2 milligrams per litre, the water being pumped into the tank is switched to fluoride-treated water that has a concentration of 1 milligram per litre, and the rate of inflow is increased to 100 litres per minute.

Find the volume of water in the tank when the fluoride concentration returns to 2 milligrams per litre, and show that the new differential equation representing the rate of change of fluoride with respect to time is given by $\frac{dx}{dt} = 100 - \frac{x}{80}$. 2 marks

ii. Find how long it takes, following the switch to pumping in fluoride-treated water, for the fluoride concentration in the tank to drop to the prescribed limit. Give your answer correct to the nearest minute. 2 marks

Question 7 (11 marks)

The vertical cross-section of a water tank is shown below, where each unit on the x- and y-axes represents 1 metre. The curved sides of the cross-section shown are parts of the parabola with the rule $y = \frac{5x^2}{9} - 5$. The height of the water tank is 9 metres.



The tank is being filled with water and the depth of water in the tank at any given time is h metres.

The volume of water in the tank can be found by rotating the curve around the y-axis.

a. Show that the volume, V cubic metres, of water in the tank when it is filled to a depth of h metres is given by $V = \pi\left[\frac{9h^2}{10} + 9h\right]$. 2 marks

The water tank is initially empty and is being filled at a rate of $\sqrt{3}h^2$ m^3 s^{-1} when the depth is h metres. The tank has a leak in it and, at the same time, water flows out of the tank at a rate of 0.2 m^3 s^{-1}.

b. Show that $\frac{dh}{dt} = \frac{5\sqrt{3}\,h^2 - 1}{9\pi(h+5)}$. 2 marks

c. After 15 seconds the depth of the water in the tank is 3.44 metres.

Using Euler's method, with a step size of 2 seconds, find an estimate of the depth 17 seconds after the water tank begins to fill. Give your answer in metres, correct to two decimal places. 2 marks

The tank is filled to a volume of 200 m^3. The tank has 20 kilograms of salt dissolved into it. Pure water then flows into the tank at a rate of 15 m^3 min^{-1}. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 10 m^3 min^{-1}.

The amount of salt in the tank after t minutes is x kilograms.

d. **i.** Show that the differential equation relating x and t is $\frac{dx}{dt} + \frac{2x}{40+t} = 0$. 2 marks

ii. Solve this differential equation to find x in terms of t. 3 marks

Question 8 (6 marks)

A curve is defined parametrically by the equations $x = 2\cos(t) - 2$ and $y = \sin(t)$, where $t \in [0, \pi]$.

a. Find $\frac{dy}{dx}$ in terms of $\cot(t)$. 2 marks

b. Find the coordinates of the point on the curve where the gradient $\frac{dy}{dx} = -\frac{\sqrt{3}}{6}$. 2 marks

c. The curve $y = \sqrt{1 - \frac{(x+2)^2}{4}}$ is rotated around the x-axis to form a solid of revolution.

Write down an integral that gives the volume of the solid formed. Hence, calculate the volume of the solid formed. 2 marks

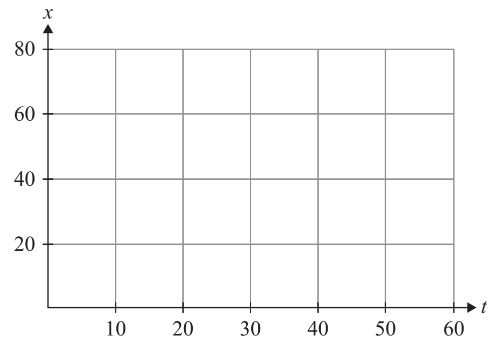
Question 9 (10 marks)

A tank contains 100 litres of well-mixed paint that is made up of red and blue paint in the ratio 3:1, respectively. The paint manufacturer produces paint in various shades of purple and achieves this by pouring the mixture out of the tank at 5 litres per minute while simultaneously adding red and blue paint into the tank at rates of 2 and 3 litres per minute, respectively. The paint mixture is constantly mixed.

a. If the paint mixture in the tank contains x litres of red paint at time t, measured in minutes, show that $\frac{dx}{dt} = \frac{40 - x}{20}$. 2 marks

b. Solve this differential equation to express x as a function of t. 4 marks

c. Sketch a graph on the axes below showing how the volume of red paint in the tank varies with time. Label all axial intercepts with coordinates and any asymptotes with their equation. 2 marks



d. Paint is drawn off at 8 litres per minute but no other changes are made.

Write down, but do not solve, a differential equation relating x and t for this new situation. 2 marks

Question 10 (9 marks)

A coconut falls, from a tree that is 44.1 metres above the water, into a deep tank. After the coconut hits the water, it is subject to an acceleration given by $a = -0.4(v + 0.6)^2$ m s^{-2}, where v is the velocity in the water t seconds after impact.

a. Assuming a constant acceleration of 9.8 m s^{-2} before the coconut hits the water, show that the velocity of the coconut is 29.4 m s^{-1} upon impact with the water. 2 marks

b. Show that the velocity of the coconut in the water, as a function of time, is given by $v = \dfrac{30}{12t + 1} - 0.6$. 3 marks

c. How long after falling from the tree does the coconut float back to the surface of the water? Give your answer in seconds, correct to one decimal place. 4 marks

Question 11 (9 marks)

The population, P, of a sample of bacteria t minutes after the bacteria were introduced to a petri dish can be modelled by the logistic differential equation $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$.

a. 3000 bacteria were initially introduced to the petri dish and the initial rate of growth of the population was 8850 bacteria per minute. When the population of bacteria had doubled, the rate of growth had increased by a factor of $\frac{29}{5}$. Find the values of r and K. 2 marks

b. The particular solution of the differential equation can be written in the form $P(t) = \frac{ae^{bt}}{e^{bt} + c}$, $a, b, c \in R$. Find the values of a, b, and c. 3 marks

c. After how many minutes is the population growing at the maximum rate and what is the population at that time? 2 marks

d. State the values of t for which both the population and rate of growth are increasing. Give your answer in interval notation. 1 mark

e. How many new bacteria were produced in the third minute after the bacteria were introduced? Give your answer to the nearest integer. 1 mark

Area of Study 5 Space and measurement

EXAM 1

Question 1 (2 marks)

A curve is defined by the parametric equations

$x(t) = 2 + \sin(2t)$

$y(t) = 2\sin^2(t)$ where $t \geq 0$.

Determine the Cartesian equation of the curve.

Question 2 (5 marks)

Consider the three vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k}$ and $\underset{\sim}{c} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}$.

$\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the axes x, y and z, respectively where $n \in R$.

a. Find the value of n if vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are perpendicular. 1 mark

b. Find the values of n if vector $\underset{\sim}{b}$ makes an angle of θ with the x–z plane, where $\cos(\theta) = \frac{3}{\sqrt{10}}$. 2 marks

c. Find the values of n if vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly independent. 2 marks

Question 3 (3 marks)

A parallelogram, $OABC$, is defined by the vectors $\overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\overrightarrow{OC} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$.

N is the point on OA that is closest to the point C.

a. Find the scalar resolute of $\overrightarrow{OC}$ in the direction of $\overrightarrow{OA}$. 1 mark

b. Hence, find the length of CN. 2 marks

Question 4 (4 marks)

The acceleration of a particle at time t seconds is given by $\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 4\underset{\sim}{j}$ m s^{-2}.

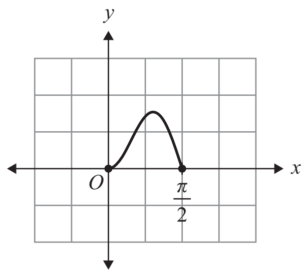
Find the distance of the particle from its initial position when $t = 2$, given that $\dot{\underset{\sim}{r}}(0) = -2\underset{\sim}{i} - 3\underset{\sim}{k}$ m s^{-1} and $\underset{\sim}{r}(0) = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ m.

Question 5 (1 mark)

The curve shown below can be defined parametrically by

$x = \arcsin(t)$

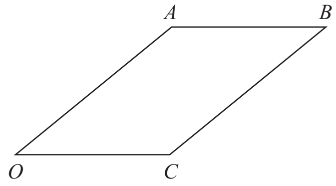
$y = 2t^2\sqrt{1 - t^2}$, where $0 \le t \le 1$.



Find the Cartesian equation of the curve in the form $y = f(x)$.

Question 6 (4 marks)

Consider the parallelogram $OABC$ shown below, where $\overrightarrow{OA} = m\underset{\sim}{i} + \sqrt{39}\underset{\sim}{k}$ and $\overrightarrow{OC} = -6\underset{\sim}{i} + 8\underset{\sim}{j}$.



a. Find the possible values of m if $|\overrightarrow{OA}| = 2|\overrightarrow{OC}|$. 2 marks

b. Find the possible values of m if the angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\cos^{-1}\left(\frac{1}{5}\right)$.
Give your answer in the form $\frac{\sqrt{a}}{b}$, where a and b are integers. 2 marks

Question 7 (3 marks)

Find the value of d for which the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{c} = 5\underset{\sim}{i} + \underset{\sim}{j} + d\underset{\sim}{k}$ are linearly independent.

Question 8 (5 marks)

A right pyramid, $OABCD$, with vertex located at D, has a square base with opposite corners located at the points (0, 0, 0) and (15, 15, 0). The position vector of the vertex of the pyramid relative to the origin is given by $\overrightarrow{OD} = \frac{15}{2}\underset{\sim}{i} + \frac{15}{2}\underset{\sim}{j} + 4\sqrt{2}\,\underset{\sim}{k}$.

a. Find the position vector $\overrightarrow{OM}$ of the centre of the base relative to the origin. 1 mark

b. Find the cosine of the acute angle, θ, formed between vectors $\overrightarrow{OM}$ and $\overrightarrow{OD}$. 3 marks

c. Find the volume of the pyramid. Give your answer in the form $a\sqrt{b}$, where $a, b \in R$. 1 mark

Question 9 (4 marks)

Let $\underset{\sim}{a} = 6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = m\underset{\sim}{i} - 2\underset{\sim}{j} + n\underset{\sim}{k}$, where m and n are positive integers. The scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{20}{3}$ and the magnitude of $\underset{\sim}{b}$ is 3.

Find the values of m and n.

Question 10 (6 marks)

Consider the points $P(3, 1, 5)$, $Q(-1, 3, 3)$ and $R(-2, -4, 7)$.

a. Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. 2 marks

b. Find a vector normal to the plane containing P, Q and R. 2 marks

c. Find a Cartesian equation for the plane containing P, Q and R. 2 marks

Question 11 (6 marks)

The line L is given by $\underset{\sim}{r}(t) = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k} + t(3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$, $t \in R$, and the plane Π is given by $x + 3y - 5z = 9$.

a. Find the coordinates of the point of intersection of L and Π. 3 marks

b. Find the value of $\sin(\theta)$, where θ is the acute angle between L and Π.

Give your answer in the form $\frac{\sqrt{a}}{b}$, where $a, b \in \mathbf{N}$. 3 marks

Question 12 (8 marks)

A triangle has vertices at $A(2, 3, 1)$, $B(3, -4, 5)$ and $C(1, -2, 4)$.

a. Find a vector equation of the line passing through A and B. 2 marks

b. Find the area of the triangle ABC. 3 marks

c. Find the three possible sets of coordinates for a point, D, such that A, B, C and D are the vertices of a parallelogram $ABCD$. 3 marks

Question 13 (8 marks)

Point A has coordinates $(3, -2, -2)$ and line L_1 is given by $\underset{\sim}{r}_1(t) = \underset{\sim}{i} + 4\underset{\sim}{j} + t(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$.

a. Find the point on L_1 that is closest to A. 3 marks

b. Find the minimum distance from A to L_1. 1 mark

c. Line L_2 passes through A and is parallel to the vector $-8\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$. Find the point of intersection of L_1 and L_2. 4 marks

Question 14 (6 marks)

Consider the Cartesian equation of two planes:

$\Pi_1: x + 2y - z = 15$

$\Pi_2: x + y + pz = 5$

a. Given that Π_1 and Π_2 are perpendicular, show that $p = 3$. 1 mark

b. Find the Cartesian equation of the plane that is perpendicular to both Π_1 and Π_2, and also passes through the point (1, 2, 1). 2 marks

The line L, given by the equation $\underset{\sim}{r}(t) = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + t(\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k})$, $t \in R$, intersects Π_1 at point A and Π_2 at point B.

c. Find the distance AB. 3 marks

EXAM 2

Section A

Question 1

Let $\underset{\sim}{a} = 3\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, where m is a real constant.

If the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is 5, then the value of m is

A. 1

B. 6

C. 15

D. 4

Question 2

The position vectors of two moving particles are given by $\underset{\sim}{r} = (2t + 1)\underset{\sim}{i} + (6 - 4t)\underset{\sim}{j}$ and $\underset{\sim}{s} = (3t^2 + 2)\underset{\sim}{i} + (3t + 2)\underset{\sim}{j}$, where $t > 0$.

At what time, t, will the velocity vectors of the two particles be perpendicular?

A. 12

B. 1

C. 0

D. 6

Question 3

The vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{c} = 3\underset{\sim}{i} + \underset{\sim}{j} + \lambda\underset{\sim}{k}$, where λ is a real constant, are linearly independent if

A. $\lambda = 2$

B. $\lambda \in R\backslash\{-2\}$

C. $-2 < \lambda \leq 2$

D. $\lambda \in R\backslash\{2\}$

Question 4

The vector resolute of $\underset{\sim}{a} = \underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$ perpendicular to $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ is

A. $\frac{8}{9}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$

B. $\frac{1}{9}(-7\underset{\sim}{i} + 20\underset{\sim}{j} + 26\underset{\sim}{k})$

C. $\frac{1}{21}(34\underset{\sim}{i} + 10\underset{\sim}{j} - 5\underset{\sim}{k})$

D. $\frac{1}{21}(8\underset{\sim}{i} + 32\underset{\sim}{j} + 16\underset{\sim}{k})$

Question 5

Let $\underset{\sim}{a} = -4\underset{\sim}{i} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + c\underset{\sim}{k}$, where c is a real constant.

If the scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$ is 8, then the value of c is

A. 16

B. $\frac{32}{3}$

C. 5

D. 8

Question 6

The vectors $\underset{\sim}{a} = 4\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = -\frac{3}{2}\underset{\sim}{i} + 6\underset{\sim}{j} - 12\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} + \gamma\underset{\sim}{k}$, where γ is a real constant, are linearly dependent if

A. $\gamma \in R$

B. $\gamma = 7$

C. $\gamma \in R\backslash\{-7\}$

D. $\gamma = -7$

Question 7

The angle that the vector $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ makes with the y-axis, correct to the nearest degree, is

A. 109°

B. 42°

C. 48°

D. 61°

Question 8

If M, P and Q are distinct points, are collinear and $\overrightarrow{MP} = 2\underset{\sim}{i} + 7\underset{\sim}{j} - 3\underset{\sim}{k}$, $\overrightarrow{OP} = -5\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$ and $|\overrightarrow{MP}| = |\overrightarrow{QP}|$, then $\overrightarrow{OQ}$ is

A. $3\underset{\sim}{i} + 9\underset{\sim}{j} + 7\underset{\sim}{k}$

B. $-3\underset{\sim}{i} - 9\underset{\sim}{j} - 7\underset{\sim}{k}$

C. $3\underset{\sim}{i} + 9\underset{\sim}{j} - 7\underset{\sim}{k}$

D. $-3\underset{\sim}{i} + 9\underset{\sim}{j} - 7\underset{\sim}{k}$

Question 9

The angle between vectors $2\underset{\sim}{i} + 5\underset{\sim}{k}$ and $m\underset{\sim}{i} - 3\underset{\sim}{j}$ is acute.

What is the range of values that m can take?

A. $m \geq 0$

B. $m > 0$

C. $m \geq 3$

D. $m > 3$

Question 10

The points A, B and C relative to a fixed origin have positions of $-\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j}$, $\underset{\sim}{i} - \sqrt{3}\underset{\sim}{j}$ and $\sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}$, respectively. They lie on the boundary of a circle and form the triangle ABC.

For the triangle ABC, which one of the following statements is **true**?

A. ABC is an equilateral triangle.

B. ABC is a right-angled triangle.

C. ABC has an obtuse angle.

D. ABC has an area of 2 square units.

Question 11

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j} + 3\underset{\sim}{k}$, where the acute angle between these vectors is θ.

The value of $\cos(2\theta)$ is

A. $\dfrac{4 - \sqrt{2}}{18}$

B. $-\dfrac{16\sqrt{2}}{81} + \dfrac{16}{27}$

C. $-\dfrac{4\sqrt{2}}{81} - \dfrac{8}{9}$

D. $\dfrac{16 - \sqrt{2}}{9}$

Question 12

The vectors $\overrightarrow{PQ} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\overrightarrow{PS} = -6\underset{\sim}{i} + 8\underset{\sim}{k}$ form two sides of the parallelogram $PQRS$.

The area of the parallelogram, in square units, is

A. $5\sqrt{5}$

B. 20

C. 30

D. $10\sqrt{5}$

Question 13

The vector $\underset{\sim}{v} = a\underset{\sim}{i} - 6\underset{\sim}{j} + b\underset{\sim}{k}$ has magnitude $2\sqrt{14}$ and is perpendicular to $\underset{\sim}{u} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$.

The possible values of a and b are

A. $a = 2, b = 4$

B. $a = 4, b = 2$

C. $a = 2, b = 4$ and $a = \frac{278}{101}, b = -\frac{356}{101}$

D. $a = 4, b = 2$ and $a = \frac{278}{101}, b = -\frac{356}{101}$

Question 14

A particle moves with velocity vector $\dot{\underset{\sim}{r}}(t) = 3t\underset{\sim}{i} - e^t\underset{\sim}{j}$. Its initial position is given by $\underset{\sim}{r}(0) = 7\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} - \underset{\sim}{k}$.

The particle's average velocity for the interval $t \in [2, 6]$ is

A. $12\underset{\sim}{i} + \left(\frac{e^2 - e^6}{4}\right)\underset{\sim}{j}$

B. $12\underset{\sim}{i} + (e^2 - e^6)\underset{\sim}{j}$

C. $18\underset{\sim}{i} - e^6\underset{\sim}{j}$

D. $3\underset{\sim}{i} + \left(\frac{e^2 - e^6}{4}\right)\underset{\sim}{j}$

Question 15

Consider two points, A and B, with coordinates (1, –3, 7) and (4, –6, –2), respectively.

Which one of the following is an equation of the straight line passing through point A that is perpendicular to line segment $\overrightarrow{AB}$?

A. $\underset{\sim}{r}(t) = \underset{\sim}{i} - 3\underset{\sim}{j} + 7\underset{\sim}{k} + t(3\underset{\sim}{i} - 3\underset{\sim}{j} - 9\underset{\sim}{k})$

B. $\underset{\sim}{r}(t) = \underset{\sim}{i} - 3\underset{\sim}{j} + 7\underset{\sim}{k} - t(\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k})$

C. $\underset{\sim}{r}(t) = \underset{\sim}{i} - 3\underset{\sim}{j} + 7\underset{\sim}{k} + t(4\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

D. $\underset{\sim}{r}(t) = 4\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k} + t(4\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

Question 16

Consider two parallel planes, given by the equations Π_1: $2x + 3y + 6z = 14$ and Π_2: $4x + 6y + 12z = -42$.

The equation of a third plane, Π_3, parallel to both Π_1 and Π_2, and at the same distance from Π_1 as Π_2, is

A. $2x + 3y + 6z = -56$

B. $4x + 6y + 12z = -14$

C. $4x + 6y + 12z = -5$

D. $6x + 9y + 18z = 147$

Question 17

The acute angle between the line $\underset{\sim}{r}(t) = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + t(2\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k})$ and the plane $2x - 3y - 4z = 12$ is closest to

A. 24.5°

B. 71.0°

C. 65.5°

D. 12.3°

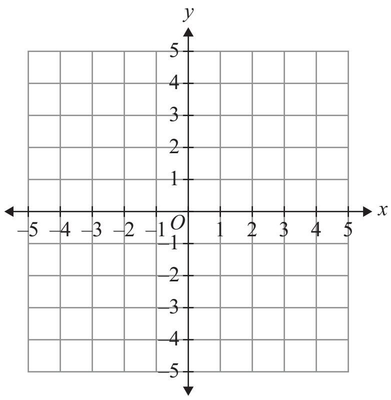
Section B

Question 1 (11 marks)

A walker, W, is observed from an origin, O, walking laps around a park. The position vector, in metres, of the walker relative to the origin at any time t seconds after the walk began is given by $\underset{\sim}{r}_W = (2 - 3\cos(t))\underset{\sim}{i} + (1 - 2\sin(t))\underset{\sim}{j}$, $t \geq 0$.

a. Determine the Cartesian equation of the walker's path. 2 marks

b. Sketch the path of the walker on the axes provided below. Indicate the walker's direction of motion with an arrow. 2 marks



At the same time as the walker starts walking, a dog, D, escapes control of its owner and runs with a position vector, relative to the same origin as the walker, of

$\underset{\sim}{r}_D = (2 - 3\cos(t))\underset{\sim}{i} + \left(\frac{24t}{5\pi} - 4\right)\underset{\sim}{j}$.

c. Verify that the dog will collide with the walker when $t = \frac{5\pi}{6}$ seconds. 2 marks

d. Find the angle, in degrees, between the velocity vectors of the walker and the dog at the time of the collision. Give your answer correct to one decimal place. 3 marks

e. Determine the speed of the dog, correct to two decimal places, at the time of the collision. 2 marks

Question 2 (11 marks)

Geoff is riding his bicycle in a park and his position can be described by the vector $\underset{\sim}{r}_G(t) = (1 - 2\cos(2t))\underset{\sim}{i} + (2\sin(2t) + 3)\underset{\sim}{j}$ at any time for $t \geq 0$ minutes. All distances are measured in kilometres.

a. After riding for $\frac{3\pi}{4}$ minutes, Geoff passes a drinking fountain.

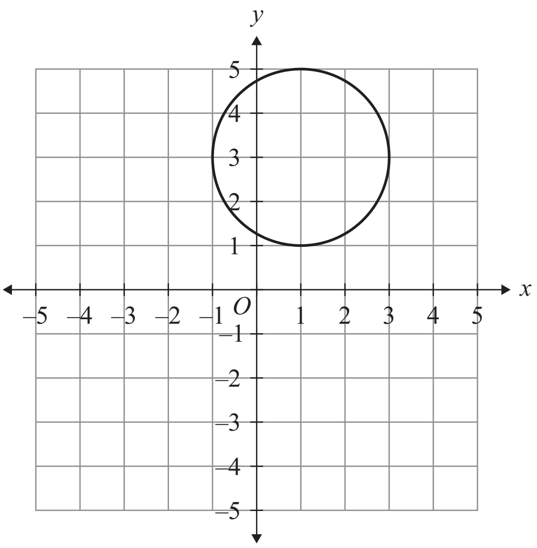
i. Find the distance, in kilometres, from his initial position to the drinking fountain. 1 mark

ii. Find the speed, in kilometres per minute, at which Geoff passes the drinking fountain. 2 marks

A second person, Daneesha, is also riding her bicycle through the park. Daneesha's position is described by the vector $\underset{\sim}{r}_D = (a + \sin(t))\underset{\sim}{i} + (3 - \cos(t))\underset{\sim}{j}$, where a is a real constant. After some time, where $t < \pi$, Geoff and Daneesha will collide.

b. Find the value of a for which Geoff and Daneesha collide and state the coordinates of the collision. 3 marks

c. The graph below shows the path of Geoff on his bicycle. On the same axes, sketch the path of Daneesha on her bicycle, given the value of a found in **part b.** Use arrows to indicate the direction of Daneesha's movement. 2 marks



d. Determine the angle between Geoff and Daneesha at the time of collision. 3 marks

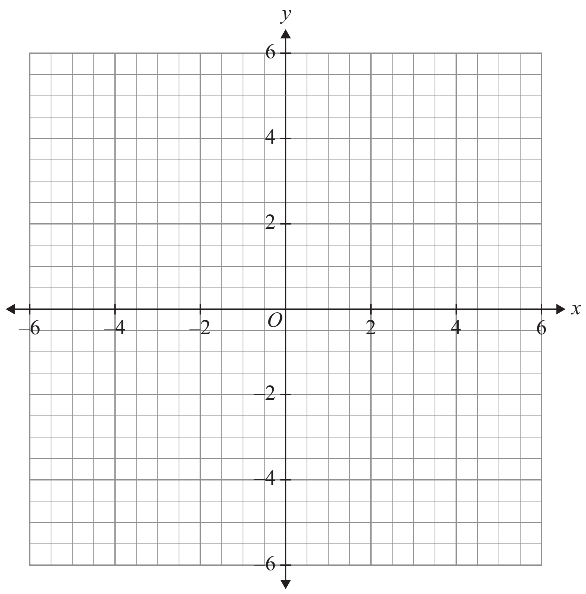
Question 3 (12 marks)

A surveillance drone, S, and a delivery drone, D, are observed flying at the same altitude above a suburb. The position vectors of each drone t minutes after they are initially observed moving relative to some origin, O, are

$\underset{\sim}{r}_S(t) = \left(4 - \cos\left(\frac{t}{2}\right)\right)\underset{\sim}{i} + \left(2 - 2\sin\left(\frac{t}{2}\right)\right)\underset{\sim}{j}$ and $\underset{\sim}{r}_D(t) = \frac{t^2}{2}\underset{\sim}{i} + \left(4 - \frac{t^2}{3}\right)\underset{\sim}{j},\ t \geq 0.$

All distances are measured in kilometres. Both drones are initially observed at the same time.

a. Sketch the path of each drone on the axes below, labelling each curve, indicating the direction of movement and labelling any intercepts with their coordinates. 3 marks



b. When $t = \sqrt{6}$ minutes, find the angle between the direction of motion of drone S and drone D. Give your answer in degrees, correct to one decimal place. 2 marks

c. Show that the paths of drone S and drone D intersect but that the drones do not collide. 3 marks

d. Determine the maximum speed of drone S, in kilometres per minute. 2 marks

e. Find the minimum distance between the two drones and state when this occurs, in minutes since the drones were initially observed. Give your answers correct to one decimal place. 2 marks

Question 4 (9 marks)

The position of an ant relative to its nest and to a beetle passing by is given by $\underset{\sim}{r}_A(t) = e^{-t}\underset{\sim}{i} + e^{-3t+1}\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (e^{-2t+1} - 1)\underset{\sim}{i} + e^{-t}\underset{\sim}{j}$, respectively, for $t \geq 0$, with distances measured in metres and time in seconds.

a. Find the initial distance between the ant and the beetle. 2 marks

b. Write down, but do not solve, an equation that could be solved to show that the ant and the beetle are at the same x-coordinate when $t = \log_e\left(-\frac{1}{2} + \sqrt{e + \frac{1}{4}}\right)$. 1 mark

c. Show that the ant and the beetle never collide. 2 marks

d. Find the point of intersection between the path of the ant and the path of the beetle, correct to four decimal places. 2 marks

e. Find the angle between the paths of the ant and the beetle at the point you found in **part d.**

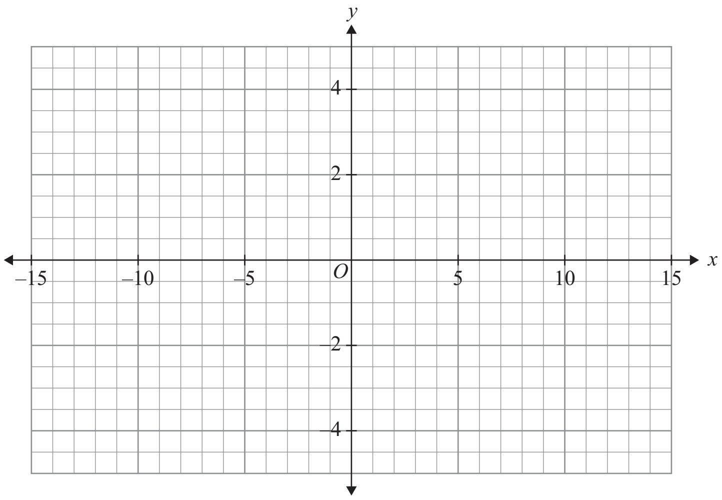
Give your answer in degrees, correct to two decimal places. 2 marks

Question 5 (10 marks)

A baseball is hit and its position at time t relative to a fixed origin, O, is given by $\underset{\sim}{r}_B(t) = (4\sec(t) - 3)\underset{\sim}{i} + (\tan(t) + 1)\underset{\sim}{j}$, where $\underset{\sim}{i}$ is a unit vector in a horizontal direction and $\underset{\sim}{j}$ is a unit vector in a vertical direction. Displacement components of the baseball are measured in metres, and time t is measured in seconds, where $0 \le t < \frac{5\pi}{12}$.

a. i. Determine the Cartesian equation of $\underset{\sim}{r}_B(t)$. 2 marks

ii. Sketch the path of the baseball on the axes provided below. Show any asymptotes with their equations, label end points with their coordinates (correct to two decimal places) and show the direction of motion of the baseball with an arrow. 3 marks



At the same time that the baseball is hit, a pigeon takes flight, with its position at time t relative to the fixed origin, O, given by $\underset{\sim}{r}_P(t) = 2\cos(2t)\underset{\sim}{i} + 3\sin(2t)\underset{\sim}{j}$, where $\underset{\sim}{i}$ is a unit vector in a horizontal direction and $\underset{\sim}{j}$ is a unit vector in a vertical direction. Displacement components of the pigeon are measured in metres, and time t is measured in seconds, where $0 \le t \le \frac{\pi}{4}$.

b. Show that the baseball and the pigeon cross paths but do not collide. 3 marks

c. Find the minimum distance between the baseball and the pigeon and state the time at which it occurs. Give your answer correct to two decimal places. 2 marks

Question 6 (2 marks)

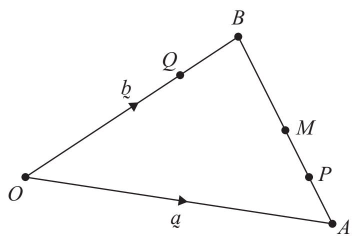
A curve is defined parametrically by the equations $x = 2\cos(t) - 2$ and $y = \sin(t)$, where $t \in [0, \pi]$.

Show that the curve can be defined by the Cartesian equation $y = \sqrt{1 - \frac{(x+2)^2}{4}}$.

Question 7 (8 marks)

Consider the triangle OAB shown below, where $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$.

Point Q lies on OB such that $OQ = 3QB$. Point P lies on AB such that $3AP = PB$. M is the mid-point of line AB.



a. Show that $\overrightarrow{OP} = \frac{3}{4}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}$ and that $\overrightarrow{QM} = \frac{1}{2}\underset{\sim}{a} - \frac{1}{4}\underset{\sim}{b}$. 2 marks

Let X be a point outside the triangle such that $\overrightarrow{OX} = \lambda\overrightarrow{OP}$, where $\lambda > 1$.

b. i. State $\overrightarrow{OX}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and λ. 1 mark

ii. For what value of λ is $\overrightarrow{AX}$ parallel to $\overrightarrow{OB}$? 2 marks

Let point X lie on the line QM, such that $\overrightarrow{QX} = \mu\overrightarrow{QM}$, where $\mu > 1$.

c. i. Give an expression for $\overrightarrow{OX}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and μ if $\overrightarrow{QX} = \mu\overrightarrow{QM}$. 1 mark

ii. Find the values of λ and μ. 2 marks

Question 8 (12 marks)

A plane, Π_1, is described by the parametric equations

$x = 2 + s + 3t$

$y = 5 + s + 2t$ where $s, t \in R$.

$z = 8 - 2s - t$

a. Find a vector equation of the plane Π_1 in the form $\underset{\sim}{r} = \underset{\sim}{a} + s\underset{\sim}{b} + t\underset{\sim}{c}$. 1 mark

b. Hence, find a Cartesian equation of the plane Π_1. 2 marks

A second plane, Π_2, is perpendicular to Π_1 and intersects it along the line L, which is given by $\underset{\sim}{r}_2 = 3\underset{\sim}{i} + \underset{\sim}{j} + 13\underset{\sim}{k} + t(-5\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k})$.

c. Find the Cartesian equation of the plane Π_2. 2 marks

d. Find the reflection of point $P(5, 4, 14)$ across plane Π_2. 3 marks

e. A third plane, Π_3, intersects Π_1 and Π_2 at L and has an equation of the form $13x + 29y + cz = d$.

i. Find the values of c and d. 2 marks

ii. Find the acute angle, in degrees, between Π_1 and Π_3, correct to one decimal place. 2 marks

Question 9 (7 marks)

A plane, Π, intersects the x-axis at $P(1, 0, 0)$, intersects the y-axis at $Q(0, 2, 0)$ and intersects the z-axis at $R(0, 0, r)$, where $r \in R$.

a. Find the values of r such that the area of the triangle PQR is less than 2, and state the minimum area for these values of r. 3 marks

b. Find the normal vector to the plane Π in terms of r, and hence find the Cartesian equation for the plane. 2 marks

c. Find the distance, d, of the plane Π from the origin, in terms of r. 1 mark

d. State the possible values for d. 1 mark

Question 10 (6 marks)

Consider the planes with the following equations.

$\Pi_1: x - y + 3z = 8$

$\Pi_2: -2x + ay + bz = 10$ where $a, b \in R$.

a. Find the values of a and b such that the planes are parallel, and find the distance between them. 3 marks

b. Find the values of a and b such that the planes are perpendicular and Π_2 contains the point (3, 1, 4). 3 marks

Area of Study 6 Data analysis, probability and statistics

EXAM 1

Question 1 (5 marks)

The mass of the oranges grown in an orchard is normally distributed with a mean of μ grams and a standard deviation of 10 grams.

A sample of 100 oranges has a mean mass of 160 grams and a standard deviation of 10 grams.

a. Find an approximate 90% confidence interval for the population mean, given that $\Pr(z > 1.645) \approx 0.05$, where z has the standard normal distribution.

Give your answer correct to one decimal place. 2 marks

Another orchard produces oranges with a mean mass of 157 grams and a standard deviation of 6 grams.

The owner of the orchard introduces a new fertiliser, which is expected to increase the mass of the oranges.

After using the new fertiliser, a sample of 36 oranges has a mean mass of 159 grams and the standard deviation remains at 6 grams.

b. i. Write down the appropriate null and alternative hypotheses to test whether the mean mass of the oranges has increased as a result of the new fertiliser. 1 mark

ii. Find an approximate *p*-value for this test, correct to three decimal places. 1 mark

iii. Explain why the null hypothesis should be rejected at the 5% level of significance. 1 mark

Question 2 (6 marks)

The waiting time, in minutes, for visitors entering the Royal Melbourne Show is normally distributed with mean μ.

In 2023 a random sample of 16 visitors entering the Royal Melbourne Show gave an approximate 95% confidence interval for μ as $5.02 < \mu < 6.98$.

a. Find the sample mean, $\bar{x}$, and hence show that the sample standard deviation is 2. 2 marks

In 2024 the number of inspectors at entry points to the Royal Melbourne Show was increased and the waiting time, in minutes, for visitors was claimed to be normally distributed with a mean waiting time of 4.5 minutes and a standard deviation of 1.5 minutes.

A random sample of 36 visitors entering the Royal Melbourne Show had a mean waiting time of 4 minutes. Assume that the population standard deviation remains at 1.5 minutes.

b. Test the hypothesis that the mean waiting time is less than that claimed by finding an approximate *p*-value for this test. Hence, explain whether or not the null hypothesis should be rejected at the 5% level of significance. Assume that $\Pr(z > 2) \approx 0.025$. 4 marks

Question 3 (4 marks)

Two types of chocolate blocks are selected and their masses are measured. Block X's masses are normally distributed with a mean of 175 grams and a standard deviation of 3 grams. Block Y's masses are normally distributed with a mean of 165 grams and a standard deviation of 4 grams. The distributions are independent of each other.

a. Find the approximate probability that the mass of a randomly selected block X chocolate is greater than the mass of a randomly selected block Y chocolate. 2 marks

b. A quality control officer suspects that the mean difference in mass of the two chocolate blocks is not 10 grams. The officer collects 100 samples of block X chocolate and 100 samples of block Y chocolate.

Calculate the 95% confidence interval for the mean difference in the mass of block X chocolate and block Y chocolate. In your calculations, use an integer multiple of the standard deviation and assume that $\Pr(z > 2) = 0.025$. 2 marks

Question 4 (3 marks)

X and Y are independent random variables. X has a mean of m and a standard deviation of 1, and Y has a mean of 5 and a standard deviation of $\sqrt{n}$, where m and n are real numbers.

If $2X + 4Y$ has a mean of 26 and a standard deviation of 6, find the values of m and n.

Question 5 (3 marks)

The individual points scored per game by two basketball players, Angela and Banhi, can be represented by random normal variables A and B, respectively. A has a mean of 12 and a standard deviation of 3. B has a mean of 12 and a standard deviation of 4.

Use $\Pr(-1.96 \le Z \le 1.96) = 0.95$ to answer the following question.

Find the probability that in any given game the difference between the number of points scored by Angela and Banhi will be greater than 9.8.

Question 6 (3 marks)

A farmer grows two types of apples, green and red, which are sold in bags at a local market. Each bag contains 6 green apples and 7 red apples.

The mass, in grams, of green apples is a random variable, G, with a mean of 75 grams and a standard deviation of 1 gram. The mass, in grams, of red apples is a random variable, R, with a mean of 70 grams and a standard deviation of 2 grams. The masses of red apples and green apples are independent.

The mass of a bag of apples, in grams, is a random variable, B.

Find the mean and standard deviation of B, in grams.

Question 7 (3 marks)

An ice cube tray with 6 cuboid compartments is filled with water and placed in a freezer to make 6 ice cubes. Each compartment has a 3 cm by 3 cm square base.

The height of each ice cube, H, is normally distributed with a mean of 2 cm and a standard deviation of $\frac{1}{6}$ cm. Assume that the height of each ice cube is independent.

a. Find the expected total volume of ice in the ice cube tray, in cm^3. 1 mark

b. Find the standard deviation of the total volume of ice in the ice cube tray, in cm^3. Give your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are positive integers. 2 marks

EXAM 2

Section A

Question 1

X and Y are both independent random variables. X has a mean of 9 and a standard deviation of 5, and Y has a mean of 6 and a standard deviation of 2.

If the random variable Z is defined as $Z = 2X - 3Y$, then the mean, μ, and standard deviation, σ, of Z is given by

A. $\mu = 0, \sigma = 4$

B. $\mu = 0, \sigma = 136$

C. $\mu = 0, \sigma = 2\sqrt{34}$

D. $\mu = 0, \sigma = 8$

Question 2

A fitness company claims that after 6 weeks of completing their fitness program their clients have a mean weight loss of 4 kilograms. A prospective client wishes to test this claim at a significance level of 0.05, using the null and alternative hypothesis of H_0: $\mu = 4$ and H_1: $\mu \neq 4$.

If the *p*-value of the prospective client's test is 0.002, which of the following statements is **true**?

A. A type I error would occur if the null hypothesis is true and not accepted.

B. The *p*-value is less than the significance level, so the null hypothesis is accepted.

C. The *p*-value is greater than the significance level, so the null hypothesis is not rejected.

D. A type II error would occur if the null hypothesis is false and not rejected.

Question 3

A random sample of the height of 150 students has a mean height of 168 cm and a standard deviation of 12 cm.

An approximate 90% confidence interval for the mean height of the students is closest to

A. (166.080, 169.920)

B. (166.388, 169.612)

C. (165.472, 170.528)

D. (167.794, 168.206)

Question 4

The weight of a certain brand of chocolate bar is normally distributed with a mean of 65 grams and a standard deviation of 2.5 grams. A random sample of 10 chocolate bars is selected.

The probability that the mean weight of this sample of chocolate bars is less than 63.5 grams is approximately

A. 0.0289

B. 0.9711

C. 0.2743

D. 0.7257

Question 5

Let X be a random variable with a mean of 9 and a standard deviation of 4, and let Y be a random variable with a mean of 7 and a standard deviation of 2.

If X and Y are independent and $W = 2X - 3Y$, then the mean, μ, and standard deviation, σ, of W will be

A. $\mu = -3, \sigma = 28$

B. $\mu = -3, \sigma = 2\sqrt{7}$

C. $\mu = -13, \sigma = 10$

D. $\mu = -3, \sigma = 10$

Question 6

The weight of a block of chocolate is normally distributed with a mean of 200 grams and a standard deviation of 5 grams.

The probability that the mean weight of 16 chocolate blocks does not exceed 202 grams is closest to

A. 0.345

B. 0.945

C. 0.055

D. 0.655

Question 7

Researchers test the hypothesis that crops given a special fertiliser will have a higher growth rate than average.

Which one of the following represents a type I error?

A. Concluding that the growth rate is the same when in fact it is higher than average when using the special fertiliser.

B. Concluding that the growth rate is higher than average when using the special fertiliser when in fact it is not.

C. Concluding that the growth rate is lower than average when using the special fertiliser when in fact it is not.

D. Concluding that the growth rate is the same as the average when using the special fertiliser when there is no noticeable difference in the growth rate.

Question 8

A mathematics teacher wishes to estimate the average exam score for the state. The teacher uses a sample size of 60 to create a 95% confidence interval for the population mean, μ.

If a confidence interval is (65.18, 71.32), then the sample mean, $\bar{x}$, and sample standard deviation, s, respectively, are closest to

A. 68.52 and 12.1

B. 68.25 and 93.98

C. 68.25 and 12.15

D. 68.25 and 14.5

Use the following information to answer questions 9 and 10.

The times taken to recharge two different brands of electric car, brand X and brand Y, expressed in minutes, are both normally distributed. The mean recharge time and standard deviation of brand X are 125 and 8, respectively, and the mean recharge time and standard deviation of brand Y are 138 and 15, respectively. Assume that the recharge times for each brand are independent of each other.

Question 9

Given that $\frac{m+n}{2} = 125$, where $m < n$ and $\Pr(m < X < n) = 0.95$ for a randomly selected brand X electric car, the values of m and n, correct to one decimal place, are

A. 111.8 and 138.2

B. 114.7 and 135.3

C. 109.3 and 140.7

D. 104.4 and 145.6

Question 10

The probability, correct to four decimal places, that a randomly chosen brand Y electric car has a recharge time within 10 minutes of a randomly chosen brand X car is closest to

A. 0.0014

B. 0.6581

C. 0.9120

D. 0.3419

Question 11

A popular fish and chip shop sells hot chips for \$4 and flake for \$5.50. The mean and standard deviation of hot chips sold in a given week is 220 and 23, respectively, and the mean and standard deviation of flake sold in a given week is 170 and 15, respectively. Assume that the number of hot chips sold and the number of flake sold are independent of each other.

The probability, correct to four decimal places, that in a given week more money is spent on hot chips than on flake is

A. 0.3281

B. 0.3763

C. 0.4986

D. 0.6237

Question 12

Victorian students were asked how many hours they sleep per night. A random sample of n students had a mean of 8.2 hours. The standard deviation of hours of sleep per night is known to be 0.7 hours.

If a 90% confidence interval for the mean hours of sleep of all Victorian students has a width of 0.05, the value of n is closest to

A. 25

B. 46

C. 2109

D. 2121

Question 13

A common illness is known to have a mean duration of 11 days when treated with drug A. A new treatment that is potentially more effective, drug B, is tested on a random sample of 1000 patients and it is found that the mean duration of the illness in these patients is 9 days.

A researcher who wishes to test whether drug B leads to improved patient outcomes should perform a hypothesis test with

A. $H_0: \mu = 11$ and $H_1: \mu = 9$

B. $H_0: \mu = 11$ and $H_1: \mu \neq 11$

C. $H_0: \mu = 11$ and $H_1: \mu > 11$

D. $H_0: \mu = 11$ and $H_1: \mu < 11$

Question 14

The mass of a large block of chocolate is normally distributed with a mean of μ grams and a standard deviation of $\sigma = 5$ grams. The sample mean of n randomly selected large blocks of chocolate is found to be 222 grams.

If a 97% confidence interval for the mass of the large block of chocolate is (220.644, 223.356), the number of large blocks of chocolate, n, in the randomly selected sample is

A. 37

B. 48

C. 57

D. 64

Question 15

The total mass of lemons produced by lemon trees is normally distributed with a mean of 48 kilograms and a variance of 16.

The probability that a sample of 6 randomly selected lemon trees produces an average total mass of lemons less than 50 kilograms is closest to

A. 0.1103

B. 0.3085

C. 0.6915

D. 0.8897

Question 16

ElectroMotion Motors claims that their latest model of electric car has a range of 650 kilometres per full charge. A rival electric car company is doubtful of this claim and believes that it is an overestimate.

If μ represents the average range per full charge for ElectroMotion Motors' latest model, the null and alternative hypotheses that should be used to test ElectroMotion Motors' claim is

A. $H_0: \mu > 650$ and $H_1: \mu = 650$

B. $H_0: \mu = 650$ and $H_1: \mu < 650$

C. $H_0: \mu < 650$ and $H_1: \mu = 650$

D. $H_0: \mu = 650$ and $H_1: \mu > 650$

Question 17

In a game, the number of points scored, S, is modelled by a normal distribution with a mean of 20 and a variance of 5; that is, $S \sim N(20, 5)$. Ten independent games are played and the total score is represented by T.

How is T distributed?

A. $T \sim N(20, 2000)$

B. $T \sim N(20, 5)$

C. $T \sim N(20, 50)$

D. $T \sim N(200, 50)$

Question 18

A mango producer wishes to find a value of μ, the mean weight of mangoes harvested. The producer generates a 90% confidence interval of (248.2, 388.4) for the weight of mangoes, in grams. This confidence interval is generated from a sample of n mangoes. It is known that the standard deviation of the weight of mangoes is 210 grams.

The value of n is closest to

A. 60

B. 97

C. 5

D. 24

Section B

Question 1 (6 marks)

A particular shoe manufacturer knows that the mean shoe size of the adult population is 9.5, with a standard deviation of 1.5. A sports shoe store believes that the mean shoe size of the adult population is not 9.5, so it selects 60 customers at random to test this claim.

a. Let the random variable $\overline{X}$ represent the mean shoe size from the random sample. Determine the standard deviation of $\overline{X}$. 1 mark

From the sample of 60 customers that are randomly selected, the store found that the mean adult shoe size of its customers is 9.

b. Give suitable hypotheses, H_0 and H_1, to test the shoe store's belief about the mean adult shoe size. 2 marks

c. **i.** Find the *p*-value for the sports shoe store's test, correct to four decimal places. 2 marks

ii. Determine whether the sample selected supports the sports shoe store's belief that the mean shoe size of the adult population is not 9.5 for a 5% level of significance. 1 mark

Question 2 (8 marks)

International tennis regulations require that tennis balls used in tournaments must have a minimum diameter of 66.5 mm. To ensure that this requirement is met, a particular company produces tennis balls that have a diameter that is normally distributed with a mean of 66.9 mm and a standard deviation of 3 mm.

For quality control testing, samples of 100 balls are selected and their diameters measured.

a. Show that the standard deviation of the sample mean of the diameter of the tennis balls that undergo quality control testing is 0.3 mm. 1 mark

b. The 90% confidence interval for the mean diameter of the tennis balls has the form $(66.9 - k_1, 66.9 + k_2)$. Find the values of k_1 and k_2, correct to one decimal place. 2 marks

The company's quality control tester claims that tennis balls are being produced with a mean diameter that exceeds its production standards. To test this claim, a sample of 100 tennis balls is selected at random and the mean diameter of the tennis balls is found to be 67 mm.

c. Write down the two hypotheses that would be used in a one-sided statistical test. 1 mark

d. Write down an expression for the *p*-value and evaluate it, correct to four decimal places. 2 marks

e. Explain whether or not the quality control tester's claim should be rejected at the 5% level of significance. 1 mark

f. For this test, determine the minimum value of the sample mean that would lead the quality control tester's claim being rejected at the 1% level of significance. Give your answer correct to one decimal place. 1 mark

Question 3 (8 marks)

The height, X, of mature sunflowers in eastern Victoria is normally distributed with a mean of 270 cm and a standard deviation of 15 cm.

A fertiliser company claims it has the best fertiliser on the market and advertises that its fertiliser can increase the height of mature sunflowers.

Flora uses this fertiliser and decides to test this claim. She measures the heights of 50 mature sunflowers and finds the sample mean to be 274 cm.

Flora then performs a statistical test.

Let $\overline{X}$ denote the mean height of the random sample of 50 mature sunflowers.

a. State suitable hypotheses, H_0 and H_1, for the statistical test. 1 mark

b. Assume that the sample size is large enough that reasonable approximations for the standard deviation of the population and shape of the distribution of the sample means can be made.

Calculate the *p*-value of the statistical test, correct to four decimal places. Hence, state with a reason whether H_0 should be rejected at the 5% level of significance. 2 marks

c. Determine the largest sample size for which H_0 would **not** be rejected at the 5% level of significance. 2 marks

d. For the statistical test conducted by Flora, describe the conclusion that would be made about the fertiliser company's claim if a type I error occurred. 1 mark

e. Given that $\Pr(\overline{X} > 274.935 | \mu = 270) = 0.01$, calculate the probability of committing a type II error at the 1% level of significance if the true mean height of all mature sunflowers in eastern Victoria is 274 cm. Give your answer correct to four decimal places. 2 marks

Question 4 (11 marks)

Two salespeople, Jafan and Liv, are comparing their sales records. Jafan's weekly sales are normally distributed with a mean of 21.5 sales. Liv's weekly sales are also normally distributed, with a mean of 15.8 sales. Both distributions have a standard deviation of 3.8 sales per week.

a. Find the probability that in a given week Jafan will make at least 20 sales. Give your answer correct to four decimal places. 1 mark

b. Jafan's commission per sale is \$110 and Liv's is \$125.

Find the probability that Liv earns more money in a given week than Jafan. Give your answer correct to four decimal places. 2 marks

c. What would Liv's commission need to be so that she and Jafan have an equal probability of earning the most money in a week? Give your answer correct to the nearest cent. 2 marks

d. In pursuit of higher sales, Liv decides to purchase and wear an expensive new suit to all her sales calls. In the 8 weeks that follow, her mean sales per week rises to 17.5. Liv wishes to carry out a statistical test to determine whether the new suit has increased her weekly sales. Assume that the standard deviation remains the same.

State suitable hypotheses, H_0 and H_1, for Liv's statistical test. 1 mark

e. Write down an expression for the *p*-value of the statistical test and evaluate the answer, correct to four decimal places. 2 marks

f. Based on your answer to **part e.** above, decide whether Liv has enough evidence to support the conclusion that the new suit has improved sales at the 5% level of significance. 1 mark

g. Liv, wishing to justify her investment in her expensive suit, decides that she must have made a type II error. In order to reduce the probability of making type II errors in future tests, Liv decides to conduct all future tests at the 10% significance level.

State whether Liv's decision to switch to a 10% significance level will reduce the probability of making a type II error. Justify your answer and discuss a potential drawback of this approach. 2 marks

Question 5 (9 marks)

A company manufactures batteries for electric cars. It is known from past experience that the charging time for the batteries is normally distributed with a mean of 335 minutes and a standard deviation of 12 minutes.

For quality control purposes, two independent random samples of 100 batteries are selected and the mean charging time for each sample is calculated.

a. Find the probability that neither random sample has a mean charging time of between 334 minutes and 336 minutes. Give your answer correct to three decimal places. 2 marks

b. Find the probability that the means of the two random samples differ by more than 5 minutes. Give your answer correct to three decimal places. 2 marks

To test whether batteries with a mean charging time of 335 minutes are being produced, the two random samples are combined and the mean charging time of the sample of 200 batteries is found to be 337 minutes. Assuming that the standard deviation is still 12 minutes, a one-tailed hypothesis test is to be conducted at the 5% level of significance.

c. Write down suitable hypotheses, H_0 and H_1, for this test. 1 mark

d. i. Find the *p*-value for this test. Give your answer correct to three decimal places. 1 mark

ii. State with a reason whether H_0 should be rejected at the 5% level of significance. 1 mark

e. What is the largest value of the sample mean charging time that could be observed for H_0 **not** to be rejected? Give your answer in minutes, correct to four decimal places. 1 mark

f. If the true mean charging time of the batteries is 337 minutes, what is the probability of a type II error? Give your answer correct to three decimal places. 1 mark

Question 6 (9 marks)

The distance athletes can jump, X cm, is distributed as follows, with mean μ and a standard deviation of 20.

$X \sim N(\mu, 400)$

A sample of jump distances, in centimetres, by 10 athletes is recorded in the table below.

751	721	778	761	740
710	767	749	797	739

a. Find the mean, $\bar{x}$, of this sample. Give your answer correct to one decimal place. 1 mark

b. Find a 95% confidence interval for μ. Give your answer correct to one decimal place. 1 mark

c. State the margin of error in this calculation of μ. Give your answer correct to one decimal place. 1 mark

d. If further 95% confidence intervals were created from each of 200 different samples, how many of these confidence intervals would be expected to contain μ? 1 mark

e. What value of k, correct to two decimal places, should be used in the following interval to create a 75% confidence interval for μ?

$\left(\bar{x} - k\frac{20}{\sqrt{10}}, \bar{x} + k\frac{20}{\sqrt{10}}\right)$ 2 marks

f. To be really sure of the value of μ, the athletes would like to calculate a 99% confidence interval with a margin of error of 3 cm.

What size sample will be needed for this calculation? 2 marks

Worked solutions

Area of Study 1 Discrete mathematics

EXAM 1

Question 1a.

Worked solution

A table can be constructed to show that $n_0 = 4$.

n	1	2	3	4	5
$n!$	1	2	6	24	120
2^n	2	4	8	16	32

Mark allocation: 1 mark

- 1 mark for correct evaluations of $n!$ and 2^n for $n = \{1, 2, 3, 4\}$ or otherwise showing that $n_0 = 4$

Question 1b.

Worked solution

Assume that the statement is true for $n = k$, that is, $k! > 2^k$.

Prove the statement true for $n = k + 1$, that is, $(k + 1)! > 2^{k+1}$.

$\text{LHS} = (k + 1)!$

$= (k + 1)k!$

$(k + 1)k! > (k + 1) \cdot 2^k$ (from initial assumption)

$(k + 1) \cdot 2^k > 2 \cdot 2^k$ (as $n \geq 4$)

$(k + 1) \cdot 2^k > 2^{k+1}$ (RHS)

Therefore it follows that $(k + 1)! > 2^{k+1}$.

So, by mathematical induction, the statement is true for all $n \geq 4$, where $n \in \mathbf{N}$.

Mark allocation: 3 marks

- 1 mark for the correct statement of the assumption
- 1 mark for the correct statement for $k + 1$
- 1 mark for the correct logical conclusion

Question 2

Worked solution

Assume the opposite of the statement to be true: $\sqrt{2} + \sqrt{7} < \sqrt{17}$.

As both the LHS and RHS of the above statement are positive, it is also true that

$$(\sqrt{2} + \sqrt{7})^2 \geq (\sqrt{17})^2$$

$$2 + 7 + 2\sqrt{14} \geq 17$$

$$2\sqrt{14} \geq 8$$

$$\sqrt{14} \geq 4$$

$$14 \geq 16$$

The last line above is a contradiction.

Therefore the statement $\sqrt{2} + \sqrt{7} \geq \sqrt{17}$ is false.

Therefore the statement $\sqrt{2} + \sqrt{7} < \sqrt{17}$ has been proven to be true by contradiction.

Mark allocation: 3 marks

- 1 mark for the correct statement of the opposite assumption
- 1 mark for squaring both sides of the opposite assumption
- 1 mark for the correct logical conclusion

Question 3

Worked solution

Start by proving the statement for $n = 1$, that is, $11^1 - 8^1$ is divisible by 3.

LHS $= 11 - 8 = 3$, which is divisible by 3.

So the statement is true for $n = 1$.

Assume that the statement is true for $n = k$.

$11^k - 8^k = 3a$, where $a \in \mathbf{N}$.

Now prove that the statement is true for $n = k + 1$, that is, $11^{k+1} - 8^{k+1} = 3b$, where $b \in \mathbf{N}$.

$$\begin{aligned} \text{LHS} &= 11^{k+1} - 8^{k+1} \\ &= 11 \times 11^k - 8 \times 8^k \\ &= 3 \times 11^k + 8 \times 11^k - 8 \times 8^k \\ &= 3 \times 11^k + 8(11^k - 8^k) \\ &= 3 \times 11^k + 8(3a) \\ &= 3(11^k + 8a) \\ &= 3b \\ &= \text{RHS} \end{aligned}$$

Note: As both 11^k and $8a \in \mathbf{N}$, $11^k + 8a$ can be taken to be some $b \in \mathbf{N}$.

Therefore, by mathematical induction, the statement is true for all $n \geq 1$, $n \in \mathbf{N}$.

Mark allocation: 4 marks

- 1 mark for proving the statement true for $n = 1$
- 1 mark for stating the correct assumption: $n = k$
- 1 mark for the correct statement for $n = k + 1$
- 1 mark for the correct logical proof

Question 4

Worked solution

Start by proving the statement for $n = 1$, that is, $7^1 - 1$ is divisible by 3.

LHS $= 7^1 - 1 = 6$, which is divisible by 3.

Therefore the statement is true for $n = 1$.

Assume that the statement is true for $n = k$.

$7^k - 1 = 3a$, $a \in \mathbf{N}$

Now prove that the statement is true for $n = k + 1$, that is, $7^{k+1} - 1 = 3b,\ b \in \mathbf{N}$.

$$\begin{aligned}\text{LHS} &= 7^{k+1} - 1\\ &= 7 \times 7^k - 1\\ &= 6 \times 7^k + 7^k - 1\\ &= 6 \times 7^k + 3a\\ &= 3(2 \times 7k + a)\\ &= 3b,\ b \in \mathbf{N}\\ &= \text{RHS}\end{aligned}$$

Therefore, by mathematical induction, the statement is true for all $n \geq 1,\ n \in \mathbf{N}$.

Mark allocation: 4 marks

- 1 mark for proving the statement true for $n = 1$
- 1 mark for stating the correct assumption: $n = k$
- 1 mark for the correct statement for $n = k + 1$
- 1 mark for the correct logical proof

Question 5

Worked solution

Assume the opposite of the statement to be true:

If n is odd, then $n^2 + 2n + 3$ is odd, or mathematically:

$n = 2p + 1,\ p \in \mathbf{N}$ and $n^2 + 2n + 3 = 2k + 1,\ k \in \mathbf{N}$

From this definition of odd integers, it follows that

$$\begin{aligned}n^2 + 2n + 3 &= (2p + 1)^2 + 2(2p + 1) + 3\\ &= 4p^2 + 4p + 1 + 4p + 2 + 3\\ &= 4p^2 + 8p + 6\\ &= 2(2p^2 + 4p + 3)\end{aligned}$$

This is a contradiction because $n^2 + 2n + 3$ is assumed to be odd and yet $2(2p^2 + 4p + 3)$ is an even number, as it is a multiple of 2.

Therefore the statement that ‘If n is odd, then $n^2 + 2n + 3$ is odd’ is false.

Therefore the statement that ‘If n is odd, where $n \in \mathbf{N}$, then $n^2 + 2n + 3$ is even’ has been proven to be true, using proof by contradiction.

Mark allocation: 3 marks

- 1 mark for the correct statement of the opposite assumption using the mathematical definition of odd numbers
- 1 mark for the correct substitution of the mathematical definition of an odd number into $n^2 + 2n + 3$
- 1 mark for the correct logical conclusion

EXAM 2

Section A

Question 1

Answer: ***C***

Worked solution

The **negation** of a statement, S, is the opposite statement, that is, it is equivalent to S being false. The statement given is a 'for all' statement. Its general form is 'For all x, if A is the case, then B is the case'. The negation of a 'for all' statement is 'There exists an x such that A is the case but B is not the case'. This matches only option C.

Question 2

Answer: ***D***

Worked solution

A **contrapositive** is formed by negating both sides of the implication and changing its direction; that is, by making the consequent (the 'then' clause) the antecedent (the 'if' clause) and vice versa. So the contrapositive of 'If A is the case, then B is the case' is 'If B is not the case, then A is not the case'. Additionally, the negation of 'P and Q' is 'not P or not Q'. This matches only option D.

Question 3

Answer: ***C***

Worked solution

The function $g(x, y)$ is used to determine the gradient and calculate an approximation for the new y value by moving along that gradient with a horizontal distance of h. Option C gives the correct formula to determine the next successive y-coordinate, using Euler's method.

Question 4

Answer: ***B***

Worked solution

The x value starts at 1 and increases by 0.2 for each loop, so its value after two loops will be 1.4.

Question 5

Answer: ***B***

Worked solution

A table can be constructed to help you determine the values of f and t after the three loops.

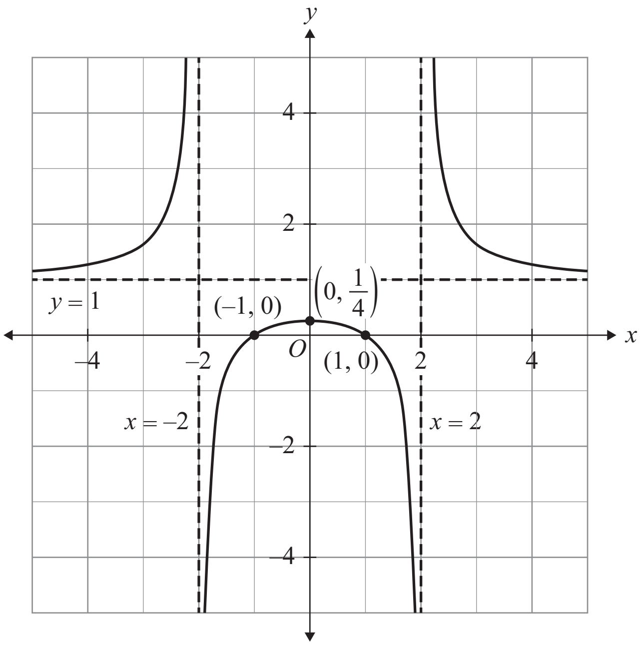
	f	t
Initial values	–4	–2
First loop $(i = 0)$	0	2
Second loop $(i = 1)$	4	2
Third loop $(i = 2)$	8	6

At the end of the third loop, i is set to 3 and will not satisfy the condition of $i < 3$ to continue. So the final value printed is the last value of f, which is 8.

EXAM 1

Question 1

Worked solution



Since the degree of the denominator is equal to the degree of the numerator, long division of polynomials can be used.

Therefore, $y = \frac{x^2 - 1}{x^2 - 4} = 1 + \frac{3}{x^2 - 4}$.

It can be seen that there are vertical asymptotes at $x = 2$ and $x = -2$.

Considering the long-term behaviour as $x \to \pm\infty$, it can be seen that there is a horizontal asymptote at $y = 1$.

To find the x-intercepts, consider $\frac{x^2 - 1}{x^2 - 4} = 0$, which results in $x = \pm 1$.

To find the y-intercept, evaluate the function at $x = 0$, which results in $y = \frac{1}{4}$.

The graph of $y = \frac{x^2 - 1}{x^2 - 4} = 1 + \frac{3}{x^2 - 4}$ can also be viewed as the reciprocal graph $y = \frac{3}{x^2 - 4}$ that has been translated in the positive y direction by 1 unit.

Mark allocation: 4 marks

- 1 mark for vertical asymptotes correctly drawn and labelled
- 1 mark for the horizontal asymptotes correctly drawn and labelled
- 1 mark for all intercepts correctly labelled
- 1 mark for correct shape (i.e. graph converges towards asymptotes. Graph doesn't cross or 'feather' away from asymptotes)

TIPS

- Vertical asymptotes, if they exist, can be obtained by setting the denominator of a rational function equal to zero.
- To obtain any horizontal or oblique asymptotes, consider resolving the rational function into partial fractions. If partial fractions cannot be used, consider using long division to resolve the rational function and to identify the horizontal or oblique asymptote.
- Consider the long-term behaviour of the graph, by considering $x \to \pm\infty$, to identify any horizontal or oblique asymptotes.

Question 2

Worked solution

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) = \frac{1}{2}$

Let $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y) = A$

$\Rightarrow \sin(x+y) + \sin(x-y) = 2\sin(x)\cos(y) = \frac{1}{2} + A$

$\Rightarrow \sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y) = \frac{1}{2} - A$

$$\Rightarrow \frac{2\sin(x)\cos(y)}{2\sin(y)\cos(x)} = \tan(x)\cot(y) = \frac{\frac{1}{2} + A}{\frac{1}{2} - A}$$

But $\tan(x)\cot(y) = \dfrac{\tan(x)}{\tan(y)} = \dfrac{\frac{1}{2} + A}{\frac{1}{2} - A} = 3.$

$$\begin{aligned} \frac{1}{2} + A &= \frac{3}{2} - 3A \\ 4A &= 1 \\ A &= \frac{1}{4} \end{aligned}$$

$\therefore A = \sin(x-y) = \frac{1}{4}$

Alternatively:

$$\frac{\tan(x)}{\tan(y)} = \frac{\frac{\sin(x)}{\cos(x)}}{\frac{\sin(y)}{\cos(y)}} = 3$$

$$\Rightarrow \frac{\sin(x)\cos(y)}{\cos(x)\sin(y)} = 3$$

$\Rightarrow \sin(x)\cos(y) = 3\cos(x)\sin(y)$

Since

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) = \frac{1}{2}$

$$\begin{aligned} \sin(x+y) &= 3\cos(x)\sin(y) + \cos(x)\sin(y) \\ &= 4\cos(x)\sin(y) = \frac{1}{2} \end{aligned}$$

So $\cos(x)\sin(y) = \frac{1}{8}$, therefore:

$$\begin{aligned}\sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y)\\ &= 3\cos(x)\sin(y) - \cos(x)\sin(y)\\ &= 2\cos(x)\sin(y)\\ &= \frac{1}{4}\end{aligned}$$

Mark allocation: 4 marks

- 1 mark for correctly using the compound angle formula to correctly set up equations for $\sin(x+y)$ and $\sin(x-y)$
- 1 mark for correctly using the compound angle formula to add and subtract the first two equations
- 1 mark for obtaining the correct expression for $\frac{\tan(x)}{\tan(y)}$
- 1 mark for the correct answer

OR

- 1 mark for the correct equation $\sin(x)\cos(y) = 3\cos(x)\sin(y)$
- 1 mark for the correct equation $\cos(x)\sin(y) = \frac{1}{8}$
- 1 mark for correctly obtaining $\sin(x-y) = 2\cos(x)\sin(y)$
- 1 mark for the correct answer

Question 3

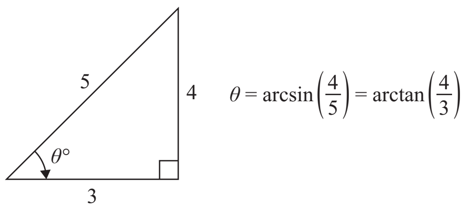
Worked solution

$\tan(t) = \tan\left(\arcsin\left(\frac{4}{5}\right) - \arccos\left(\frac{5}{13}\right)\right)$

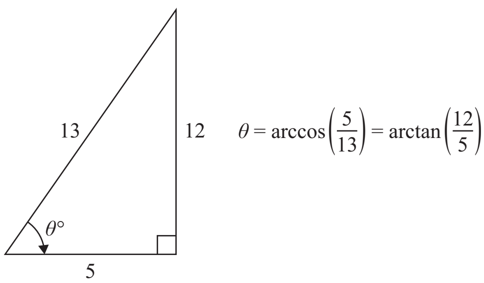
A compound angle formula can be used, which leads to

$$\tan\left(\arcsin\left(\frac{4}{5}\right) - \arccos\left(\frac{5}{13}\right)\right) = \frac{\tan\left(\arcsin\left(\frac{4}{5}\right)\right) - \tan\left(\arccos\left(\frac{5}{13}\right)\right)}{1 + \tan\left(\arcsin\left(\frac{4}{5}\right)\right)\tan\left(\arccos\left(\frac{5}{13}\right)\right)}$$

Since the angle $\arcsin\left(\frac{4}{5}\right)$ is from a 3-4-5 triangle, the corresponding arctan expression is $\arctan\left(\frac{4}{3}\right)$.



Since the angle $\arccos\left(\frac{5}{13}\right)$ is from a 5-12-13 triangle, the corresponding arctan expression is $\arctan\left(\frac{12}{5}\right)$.



Substituting these new expressions into the compound angle formula gives

$$\frac{\tan\left(\arctan\left(\frac{4}{3}\right)\right)-\tan\left(\arctan\left(\frac{12}{5}\right)\right)}{1+\tan\left(\arctan\left(\frac{4}{3}\right)\right)\tan\left(\arctan\left(\frac{12}{5}\right)\right)}=\frac{\frac{4}{3}-\frac{12}{5}}{1+\frac{4}{3}\times\frac{12}{5}}=\frac{-\frac{16}{15}}{\frac{63}{15}}=-\frac{16}{63}$$

Therefore $\tan(t)=-\frac{16}{63}$.

Mark allocation: 3 marks

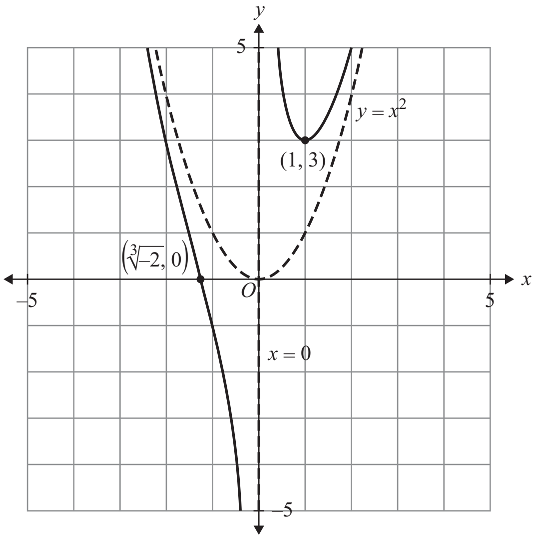
- 1 mark for using the appropriate compound angle formula
- 1 mark for finding $\arctan\left(\frac{4}{3}\right)$ and $\arctan\left(\frac{12}{5}\right)$
- 1 mark for the correct answer

TIP

» Become familiar with recognising when to apply compound angle formulas.

Question 4

Worked solution



To find the x-intercept, consider $\frac{x^3+2}{x}=0$. This results in $x^3+2=0$. Therefore $x=\sqrt[3]{-2}$.

To identify the asymptotes, the expression $\frac{x^3+2}{x}$ can be re-expressed as $\frac{x^3+2}{x}=\frac{x^3}{x}+\frac{2}{x}=x^2+\frac{2}{x}$. It can be seen that $x=0$ will be an asymptote. Considering the long-term behaviour as $x\to\pm\infty$, it can be seen that there is also an asymptote at $y=x^2$.

To determine the coordinates of any turning points, the derivative $\frac{dy}{dx}$ should be considered.

$\frac{dy}{dx}=2x-\frac{2}{x^2}$. For turning points, it must be the case that $\frac{dy}{dx}=0$. Hence

$$2x-\frac{2}{x^2}=0$$
$$x^3=1$$
$$x=1$$
$$y=3$$

Therefore there is a turning point at $(1, 3)$.

Mark allocation: 3 marks

- 1 mark for sketching the graph accurately and showing its asymptotic behaviour
- 1 mark for labelling the intercepts and asymptotes accurately
- 1 mark for labelling the turning point correctly

Question 5a.

Worked solution

$$\frac{(x+1)^2}{(x-2)^2} = A + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\frac{(x+1)^2}{(x-2)^2} = \frac{A(x-2)^2 + B(x-2) + C}{(x-2)^2}$$

$$(x+1)^2 = A(x-2)^2 + B(x-2) + C$$

The equation $(x+1)^2 = A(x-2)^2 + B(x-2) + C$ is true for all values of x. Therefore to determine the values of $A, B, C,$ three different values for x can be substituted to solve for $A, B, C,$ or both sides of the equation can be expanded and coefficients can be equated to determine the values for A, B, C.

$A = 1$, $B = 6$ and $C = 9$

Mark allocation: 1 mark

- 1 mark for correctly using partial fractions and using a method to show how to obtain the given values of A, B, C

Question 5b.

Worked solution

$$y = 1 + \frac{6}{x-2} + \frac{9}{(x-2)^2}$$

$$\frac{dy}{dx} = -\frac{6}{(x-2)^2} - \frac{18}{(x-2)^3} = 0$$

$$x = -1$$

$$y = -1$$

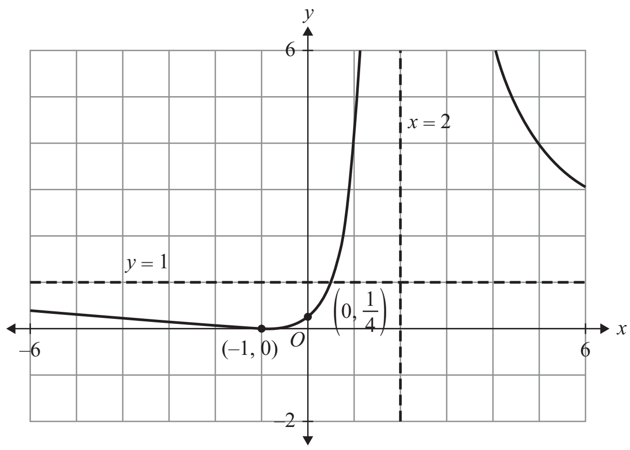
So $(-1, 0)$.

Mark allocation: 2 marks

- 1 mark for correctly finding the derivative $\frac{dy}{dx} = -\frac{6}{(x-2)^2} - \frac{18}{(x-2)^3}$
- 1 mark for the correct answer $(-1, 0)$

Question 5c.

Worked solution



Mark allocation: 3 marks

- 1 mark for correctly drawing and labelling asymptotes
- 1 mark for correctly labelling axial intercepts and turning point
- 1 mark for sketching the graph accurately and showing its asymptotic behaviour

Question 6

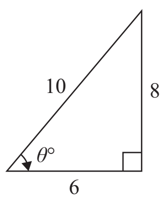
Worked solution

$\tan(t) = \tan\left(\arcsin\left(\frac{8}{10}\right) + \text{arcsec}\left(\frac{13}{12}\right)\right)$

Recognising that a compound angle formula can be used leads to

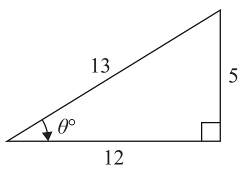
$$\tan\left(\arcsin\left(\frac{8}{10}\right) + \arccos\left(\frac{12}{13}\right)\right) = \frac{\tan\left(\arcsin\left(\frac{8}{10}\right)\right) + \tan\left(\arccos\left(\frac{12}{13}\right)\right)}{1 - \tan\left(\arcsin\left(\frac{8}{10}\right)\right)\tan\left(\arccos\left(\frac{12}{13}\right)\right)}$$

The angle $\arcsin\left(\frac{8}{10}\right)$ is the angle θ shown below in a 6-8-10 right-angled triangle.



Using the ratios in the triangle gives $\tan\left(\arcsin\left(\frac{8}{10}\right)\right) = \tan(\theta) = \frac{8}{6} = \frac{4}{3}$.

The angle $\arccos\left(\frac{12}{13}\right)$ is the angle θ shown below in a 5-12-13 right-angled triangle.



Using the ratios in the triangle gives

$\tan\left(\arccos\left(\frac{12}{13}\right)\right) = \tan(\theta) = \frac{5}{12}$

Substituting these new expressions into the appropriate compound angle formula gives

$$\tan\left(\arcsin\left(\frac{8}{10}\right)+\arccos\left(\frac{12}{13}\right)\right)=\frac{\tan\left(\arcsin\left(\frac{8}{10}\right)\right)+\tan\left(\arccos\left(\frac{12}{13}\right)\right)}{1-\tan\left(\arcsin\left(\frac{8}{10}\right)\right)\tan\left(\arccos\left(\frac{12}{13}\right)\right)}$$

$$=\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\cdot\frac{5}{12}}=\frac{\frac{7}{4}}{\frac{4}{9}}=\frac{63}{16}$$

Therefore $\tan(t)=\frac{63}{16}$.

Mark allocation: 3 marks

- 1 mark for using the appropriate compound angle formula
- 1 mark for finding the correct values of $\tan\left(\arccos\left(\frac{12}{13}\right)\right)$ and $\tan\left(\arcsin\left(\frac{8}{10}\right)\right)$
- 1 mark for deriving $\tan(t)=\frac{63}{16}$

TIPS

» Recognise that $\arccos\left(\frac{12}{13}\right)$ and $\arcsin\left(\frac{8}{10}\right)$ represent angles in right-angled triangles, and that by drawing a diagram of an equivalent right-angled triangle, the trigonometric ratios for those angles can be easily seen.

» Practise recognising the Pythagorean triples that are commonly used in exams, such as 3-4-5, 6-8-10 and 5-12-13.

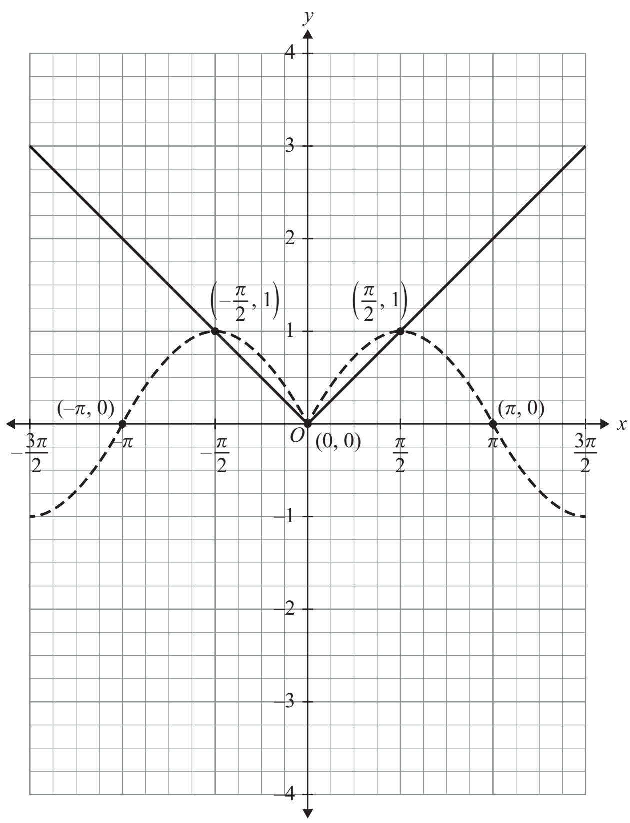
Question 7

Worked solution

The graph of $f(x)=\sin(|x|)$ will be symmetrical about the y-axis. The graph of $f(x)$ for $x \geq 0$ will be the graph of $y=\sin(x)$, and the graph of $f(x)$ for $x<0$ can be obtained by reflecting $y=\sin(x)$, $x>0$ in the y-axis. The graph of $g(x)=\frac{2}{\pi}|x|$ can be obtained by graphing the piecewise function:

$$y=\begin{cases}\frac{2}{\pi}x, & x \geq 0\\ -\frac{2}{\pi}x, & x<0\end{cases}$$

The points of intersection of $f(x)$ with $g(x)$ are $(0, 0)$, $\left(-\frac{\pi}{2}, 1\right)$, $\left(\frac{\pi}{2}, 1\right)$.



Mark allocation: 3 marks

- 1 mark for accurately graphing $f(x)$ with labelled x-intercepts
- 1 mark for accurately graphing $g(x)$ with labelled x-intercepts
- 1 mark for correctly labelling the three points of intersection with their correct coordinates

Question 8a.

Worked solution

It can be seen that $\frac{2\sqrt{3}x^3}{3(x^2-1)} = \frac{2\sqrt{3}}{3}\left(\frac{x^3}{x^2-1}\right)$. Since the degree of the numerator is larger than the degree of the denominator, long division must be used to divide the two polynomial expressions.

Therefore $\frac{2\sqrt{3}}{3}\left(\frac{x^3}{x^2-1}\right) = \frac{2\sqrt{3}}{3}\left(x + \frac{x}{x^2-1}\right) = \frac{2\sqrt{3}}{3}\left(x + \frac{x}{(x-1)(x+1)}\right)$.

The term $\frac{x}{x^2-1}$ can expressed as partial fractions. That is, $\frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x-1}$.

Combining the right-hand side and equating the numerators gives $x = A(x+1) + B(x-1)$.

By evaluating at $x = \pm 1$, or equating coefficients, it can be found that $A = \frac{1}{2}, B = \frac{1}{2}$.

Therefore $\frac{2\sqrt{3}}{3}\left(\frac{x^3}{x^2-1}\right) = \frac{2\sqrt{3}}{3}\left(x + \frac{1}{2(x-1)} + \frac{1}{2(x-1)}\right) = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3(x-1)} + \frac{\sqrt{3}}{3(x-1)}$.

Mark allocation: 3 marks

- 1 mark for correctly applying long division of polynomials, or equivalent to obtain $\frac{2\sqrt{3}}{3}\left(x + \frac{x}{x^2-1}\right)$
- 1 mark for correctly applying partial fractions
- 1 mark for the correct answer $\frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3(x-1)} + \frac{\sqrt{3}}{3(x-1)}$

Question 8b.

Worked solution

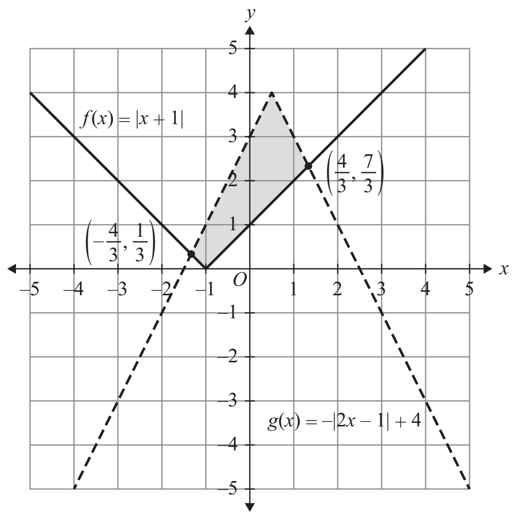
The asymptotes of the graph of $y = f(x)$ are $y = \frac{2\sqrt{3}}{3}x, x = -1, x = 1$.

Mark allocation: 2 marks

- 1 mark for both $x = -1, x = 1$
- 1 mark for $y = \frac{2\sqrt{3}}{3}x$

Question 9a.

Worked solution



Mark allocation: 3 marks

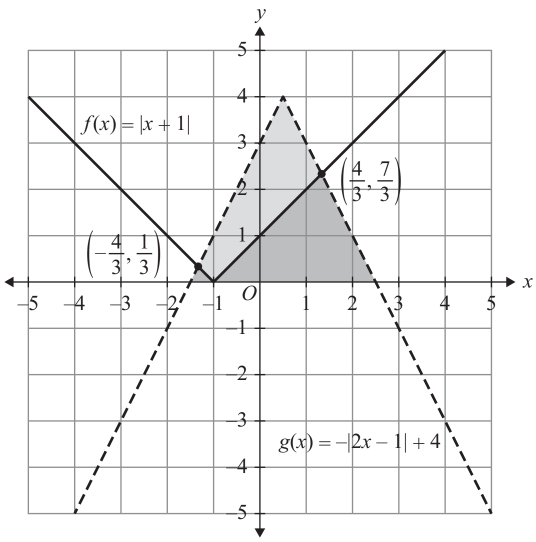
- 1 mark for each of the graphs of $f(x)$ and $g(x)$ (up to 2 marks)

Note: Graphs must have the correct shape and intercepts.

- 1 mark for the correct coordinates of the two points of intersection of $\left(-\frac{4}{3}, \frac{1}{3}\right)$ and $\left(\frac{4}{3}, \frac{7}{3}\right)$

Question 9b.

Worked solution



Method 1: Geometry

The area of the required quadrilateral (shaded in light grey in the diagram above) can be determined by subtracting the areas of the two smaller triangles (shaded in dark grey) from the larger triangle (light and dark grey).

$$A_1 = \frac{1}{2} \times 4 \times 4 = 8$$

$$A_2 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$$

$$A_3 = \frac{1}{2} \times \frac{7}{2} \times \frac{7}{3} = \frac{49}{12}$$

$$A_{\text{required}} = A_1 - A_2 - A_3 = 8 - \frac{1}{12} - \frac{49}{12} = \frac{23}{6}$$

Method 2: Integration

While not recommended, the required area can be determined using three definite integrals.

$$A_{\text{required}} = \int_{-\frac{4}{3}}^{-1} (2x - 1) + 4 - (-x - 1)\,dx + \int_{-1}^{\frac{1}{2}} (2x - 1) + 4 - (x + 1)\,dx + \int_{\frac{1}{2}}^{\frac{4}{3}} -(2x - 1) + 4 - (x + 1)\,dx$$

$$= \int_{-\frac{4}{3}}^{-1} 3x + 4\,dx + \int_{-1}^{\frac{1}{2}} x + 2\,dx + \int_{\frac{1}{2}}^{\frac{4}{3}} -3x + 4\,dx$$

$$= \left[\frac{3}{2}x^2 + 4x\right]_{-\frac{4}{3}}^{-1} + \left[\frac{1}{2}x^2 + 2x\right]_{-1}^{\frac{1}{2}} + \left[-\frac{3}{2}x^2 + 4x\right]_{\frac{1}{2}}^{\frac{4}{3}}$$

$$= \frac{23}{6} \text{ square units}$$

Mark allocation: 2 marks

- 1 mark for setting up a correct geometric or integral expression for the required bound area
- 1 mark for the correct answer of $\frac{23}{6}$ square units

TIP

» **Always look to use symmetry and the geometry of known shapes to find lengths, areas and volumes before considering a calculus approach.**

EXAM 2

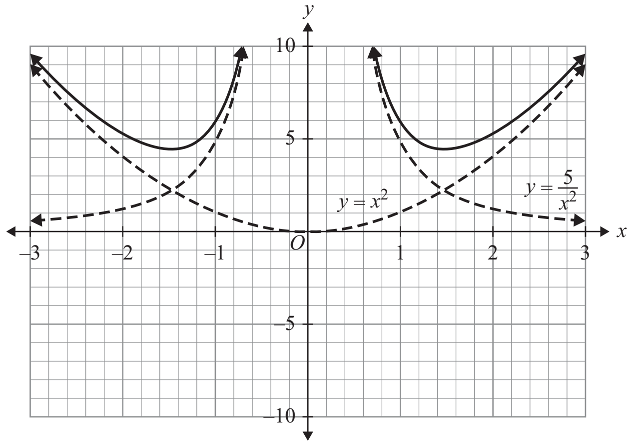
Section A

Question 1

Answer: ***B***

Worked solution

The graph of $f(x) = \dfrac{x^4 + 5}{x^2}$ is shown below.



The graph has asymptotes of $x = 0$, $y = x^2$ and $y = \dfrac{5}{x^2}$, making options A and D true.

The second derivative of $f(x)$ is $f''(x) = 2 + \dfrac{30}{x^4}$ so $f''(x) > 0$ for all x, which means $f(x)$ does not have any stationary points of inflection, making option B false.

TIP

» When given an equation and asked a question relating to the graph of the equation, it is helpful to graph the equation on a CAS to visualise the shape of the graph. This will help eliminate incorrect multiple-choice options.

Question 2

Answer: ***C***

Worked solution

Option A is false because $\dfrac{d^2y}{dx^2} \neq 0$.

Option B is false because the graph has only vertical and horizontal asymptotes.

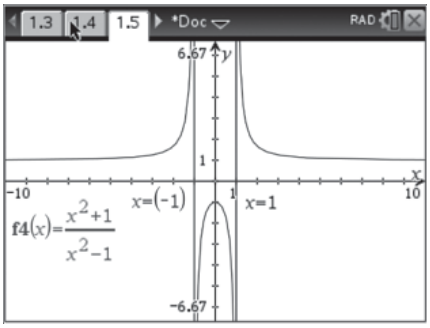
Option C is true since $\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = -4 < 0$ and $\left.\dfrac{dy}{dx}\right|_{x=0} = 0$, making it concave down.

Also, the point is a local maximum.

Option D is false because there are three asymptotes:
$x = -1,\ x = 1,\ y = 1.$

This can be seen using a CAS.

Therefore, option C is the only correct option.



Question 3

Answer: ***D***

Worked solution

$$f(x) = \frac{ax^2 + b}{x - 2} = \frac{a(x^2 - 2x) + 2ax + b}{x - 2}$$

$$= \frac{ax(x - 2) + 2a(x - 2) + 4a + b}{x - 2}$$

$$= ax + 2a + \frac{4a + b}{x - 2}$$

With $x \to \infty$ and $y \to \infty$, $\frac{4a + b}{x - 2} \to 0$.

Therefore a non-vertical asymptote has an equation $y = ax + 2a$. Since the asymptote has the equation $y = ax + c$, it follows that $c = 2a$.

Question 4

Answer: ***D***

Worked solution

Option A is incorrect because $f''(0) = 0$ does not guarantee a point of inflection at $x = 0$. A simple counterexample is $f(x) = x^4$ at $x = 0$.

Option B is incorrect because $f'''(0) = 0$ indicates that $f''(x)$ is stationary at $x = 0$. Being stationary, $f''(x)$ can be either constant or have a stationary point at $x = 0$. Either way, $f''(0)$ may not be zero.

Option C is incorrect because $f'(0) = 0$ indicates that $x = 0$ is a stationary point. Not enough information is given to determine the nature of this stationary point. A simple counterexample is $f(x) = x^3 + x$.

Option D is correct because $f''(0) = 0$ indicates that $f'(x)$ is differentiable at $x = 0$. If $f'(x)$ is differentiable, then $f(x)$ must be differentiable at $x = 0$.

TIP

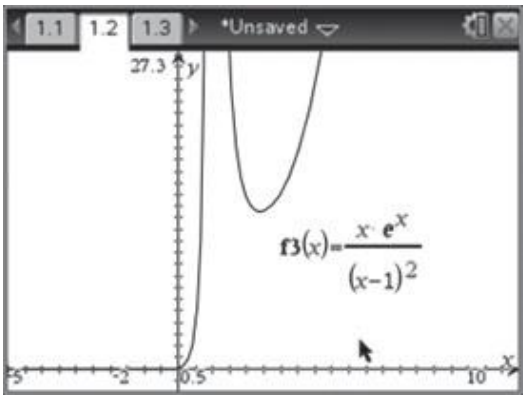
» Thinking of possible counterexamples for questions that ask you to evaluate the truth of five statements helps to quickly eliminate incorrect options.

Question 5

Answer: ***B***

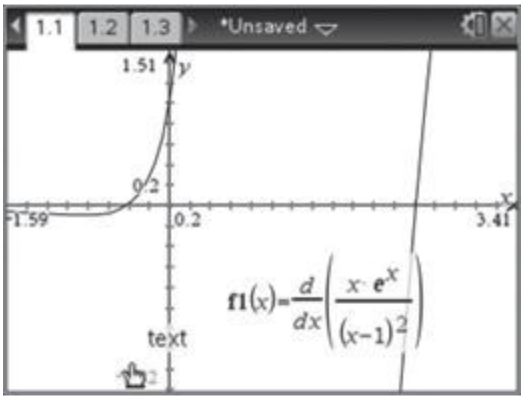
Worked solution

A graph of $f(x) = \frac{xe^x}{(x - 1)^2}$ is shown below.



Looking at this graph and the equation, it can be seen that there is a horizontal asymptote at $y = 0$ and a vertical asymptote at $x = 1$, and that there is no oblique asymptote of $y = x$. Therefore options A, C and D are all false.

A graph of $f'(x)$ is shown below.



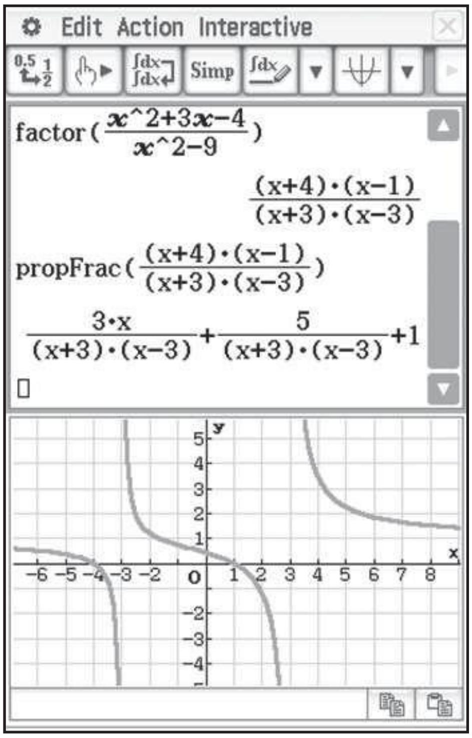
The graph of $f'(x)$ has two points at which $f'(x) = 0$. On the left of each point the gradient is negative; on the right of each point the gradient is positive. Therefore these are minimum turning points. We can also see that these are minimum turning points by looking at the graph of $f(x)$ shown at the left. Therefore option B is true.

Question 6

Answer: ***D***

Worked solution

Use a CAS. First, factorise the denominator to reveal two vertical asymptotes at $x = -3$, $x = 3$. Then use the proper fraction command 'propfrac' to find the horizontal asymptote at $y = 1$. Alternatively (but less reliably), sketch the graph on a CAS and identify the vertical and horizontal asymptotes.

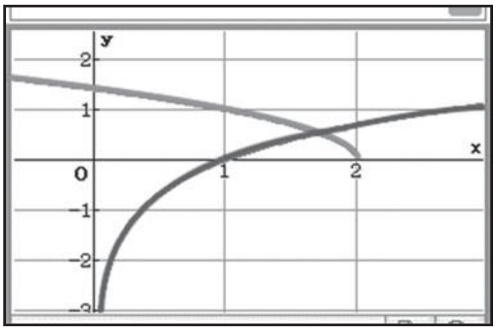


Question 7

Answer: ***D***

Worked solution

Use a CAS to sketch the graph of the numerator and denominator, and note that the domain from (0, 2] is included in both.

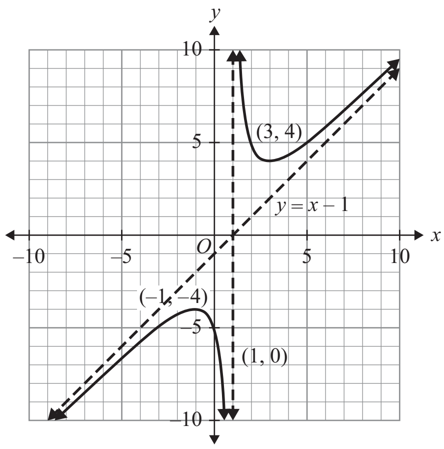


Secondly, note that the denominator in the question is zero when $x = 1$. Therefore the value of 1 cannot be part of the domain. Hence the correct answer is option D.

Section B

Question 1

Worked solution



Mark allocation: 3 marks

- 1 mark for an accurately shaped graph that passes through the point (0, –5)
- 1 mark for accurately showing and labelling the asymptotes
- 1 mark for accurately labelling the turning points

Question 2a.

Worked solution

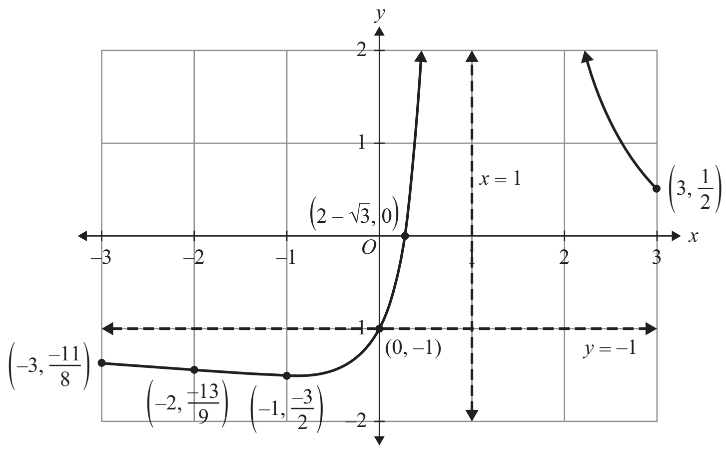
$x = 1$ and $y = -1$.

Mark allocation: 1 mark

- 1 mark for writing the equation of each asymptote

Question 2b.

Worked solution



Mark allocation: 3 marks

- 1 mark for a sketch with an accurate shape that shows the asymptotes with their equations and labelled end points
- 1 mark for accurately showing and labelling all stationary points and points of inflection
- 1 mark for accurately showing and labelling all intercepts

TIP

» Sketch the graph using your calculator. Make the scale and size of the axes on the calculator the same as those in the question to improve the accuracy of the sketch.

Question 3a.

Worked solution

The denominator of f factorises to $(x-2)(x+1)$.

Hence two vertical asymptotes exist.

Therefore the maximal domain is $x \in R\backslash\{-1, 2\}$.

Mark allocation: 1 mark

- 1 mark for stating $x \in R\backslash\{-1, 2\}$

TIPS

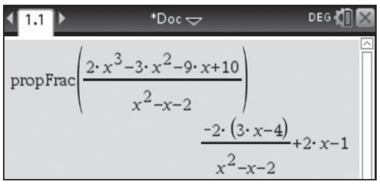
» For questions that ask for the domain of a function, solve for the denominators equal to zero for rational functions. This will show the values of x for which the function is undefined.

» When a curve is going to be used repeatedly throughout a question, it is useful to define its equation on a CAS at the start to avoid having to retype it multiple times.

Question 3b.

Worked solution

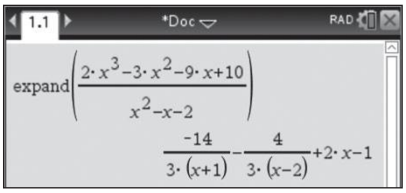
The function f can be expressed as $f(x) = \dfrac{-2(3x-4)}{x^2-x-2} + 2x - 1$.



$$f(x) = \frac{ax^2+b}{x-2} = \frac{a(x^2-2x)+2ax+b}{x-2}$$

Therefore, using the result from **part a.**, there are two vertical asymptotes, $x = -1$ and $x = 2$, and one oblique asymptote, $y = 2x - 1$.

Alternatively, using the 'expand' command on a CAS will yield an expression where all three asymptotes, $x = -1$, $x = 2$ and $y = 2x - 1$, can be seen.



Mark allocation: 2 marks

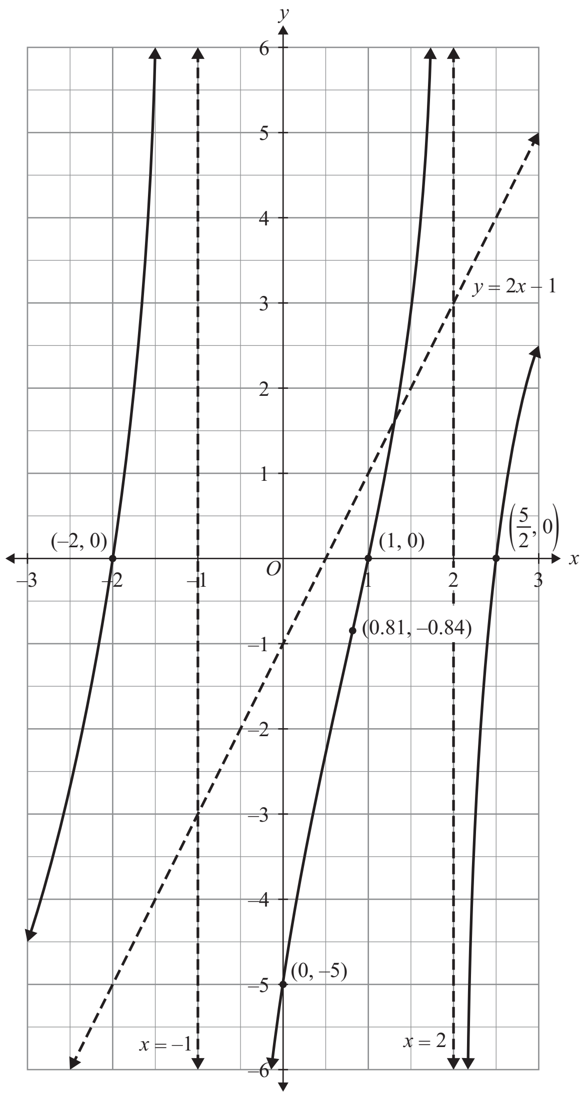
- 1 mark for stating $x = -1,\ x = 2$
- 1 mark for stating $y = 2x - 1$

TIP

» Rational functions, where the degree of the numerator minus the degree of the denominator equals 1, will have an oblique asymptote of the form $y = mx + c$.

Question 3c.

Worked solution



Mark allocation: 3 marks

- 1 mark for the accurate shape of the graph, showing behaviours towards the asymptotes
- 1 mark for accurately labelling intercepts and the point of inflection (or marking the point of inflection within its vicinity)
- 1 mark for showing the equation of each asymptote

Question 4a.

Worked solution

Stationary points occur where $f'(x) = 0$.

$$f'(x) = \frac{x^2 - 2x}{(x-1)^2} = 0$$

$$x^2 - 2x = 0$$

So $x = 0$ or $x = 2$.

So $f(0) = -1$ and $f(2) = 3$.

$\therefore$ Stationary points are $(0, -1)$ and $(2, 3)$.

Mark allocation: 1 mark

- 1 mark for finding both stationary points

TIP

» **As this question is worth just 1 mark, simply graphing $f(x)$ and using your calculator to find the stationary points is sufficient.**

Question 4b.

Worked solution

To find points of inflection we first solve $f''(x) = 0$.

$$f'(x) = \frac{x^2 - 2x}{(x-1)^2}$$

$$f''(x) = \frac{(2x-2)(x-1)^2 - (x^2-2x)(2(x-1))}{(x-1)^4} = 0$$

$$(2x-2)(x^2-2x+1) - (x^2-2x)(2x-2) = 0$$

Take out $(2x - 2)$ as a common factor:

$$(2x-2)(x^2-2x+1-x^2+2x) = 0$$

$$(2x-2) \times 1 = 0$$

$$2x = 2$$

$$x = 1$$

However, at $x = 1$, $f(x)$ is undefined since $x = 1$ is outside the domain. Therefore, $f(x)$ has no points of inflection.

Mark allocation: 2 marks

- 1 mark for using differentiation to find $x = 1$
- 1 mark for stating that $f(x)$ is undefined at $x = 1$

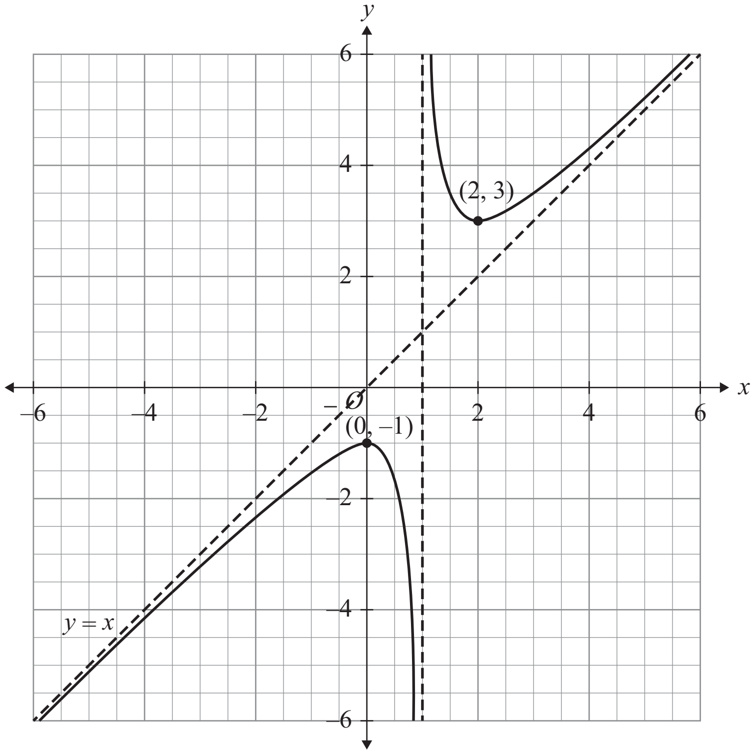
OR

- 1 mark for finding the $\frac{2}{(x-1)^3}$ form using a CAS calculator
- 1 mark for stating that there are no solutions to $f''(x) = 0$ and no points of inflection

Question 4c.

Worked solution

$x = 1$



Mark allocation 2 marks

- 1 mark for the correct shape of the graph with labelled asymptotes
- 1 mark for labelling stationary points

Question 4d.

Worked solution

$f(x) = \dfrac{x^2 - x + 1}{x - 1}$, $g(x) = \tan^{-1}(x)$

	$f(x)$	$g(x)$
Domain	$R\backslash\{1\}$	R
Range	$(-\infty, -1] \cup [3, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\text{ran } f(x) \subseteq \text{dom } g(x)$, therefore the composite function exists.

$\therefore \text{dom } g(f(x)) = \text{dom } f(x) = R\backslash\{1\}$

Notice that $\text{ran } f(x) = (-\infty, -1] \cup [3, \infty)$ is the pre-image of $g(x)$, therefore the range of $g(x)$ is limited to $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right] \cup \left[\tan^{-1}(3), \frac{\pi}{2}\right)$.

$\therefore \text{ran } g(f(x)) = (-1.5708, -0.7854) \cup (1.2490, 1.5708)$

Mark allocation: 2 marks

- 1 mark for the implied domain
- 1 mark for the range

Question 4e.

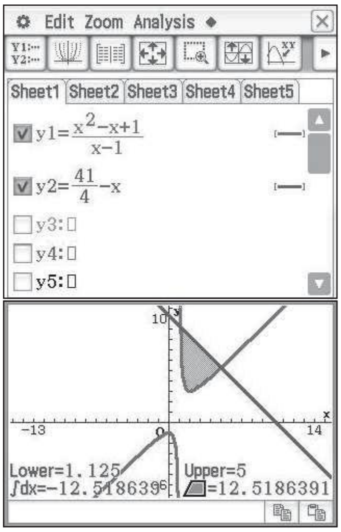
Worked solution

Find the points of intersection of $f(x)$ and $g(x)$:

$f(x) = h(x)$

So $x = 1.125$ or $x = 5$.

Area of $M = \int_{1.125}^{5} h(x) - f(x)\, dx = 12.519$



Mark allocation: 2 marks

- 1 mark for the integral with correct limits
- 1 mark for the correct answer

TIP

» **Always sketch the graph to see the area required.**

Question 4f.

Worked solution

The volume of the solid of revolution generated by rotating M around the line $y = 1$ can be found by translating both $f(x)$ and $h(x)$ 1 unit in the negative direction of the y-axis, and rotating the bounded area around the line $y = 0$ (i.e. the x-axis).

Volume $= \pi\int_{1.125}^{5} (h(x) - 1)^2 - (f(x) - 1)^2\, dx = 365.591$

Mark allocation: 3 marks

- 1 mark for using the volume of the solid of revolution formula to find the area bounded by the two functions
- 1 mark for the translation of graphs or any other equivalent methods
- 1 mark for the correct answer

Question 5a.

Worked solution

$\frac{dy}{dx} = 1 - \frac{4}{(x-1)^2}$

Turning points occur when $\frac{dy}{dx} = 0$.

$1 - \frac{4}{(x-1)^2} = 0$

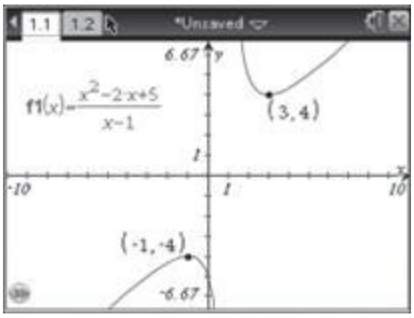
$\Rightarrow x = -1$ and $x = 3$

$f(-1) = -4$

$f(3) = 4$

So the coordinates of the turning points are (–1, –4) and (3, 4).

Using a CAS:



Mark allocation: 2 marks

- 1 mark for finding the turning point (3, 4)
- 1 mark for finding the turning point (–1, –4)

TIP

» Using a CAS to sketch the graph and find the turning points is a good way of checking the accuracy of the solution, as well as any graphs sketched in subsequent questions.

Question 5b.

Worked solution

$\frac{d^2y}{dx^2} = \frac{8}{(x-1)^3}$

When $x = 3$:

$\frac{d^2y}{dx^2} = \frac{8}{(3-1)^3}$

$= 1$

As $\frac{d^2y}{dx^2} > 0$, when $x = 3$, (3, 4) is a local minimum.

Mark allocation: 2 marks

- 1 mark for using the second derivative at $x = 3$ to determine the type of turning point
- 1 mark for concluding that (3, 4) is a local minimum

TIP

» In 'show that' questions, even though the result may be known or obvious, some form of working must be given that proves or justifies the result; for example, showing how the second derivative has been used to determine the type of turning point.

Question 6a.

Worked solution

Stationary points occur when the derivative is equal to zero.

$f'(x) = -\dfrac{2(x+1)}{(x-1)^3} = 0 \;\Rightarrow\; x = -1$

So $f(-1) = \dfrac{2(-1)}{(-1-1)^2} - 1 = -\dfrac{3}{2}$.

So the coordinates of the stationary point are $\left(-1, -\frac{3}{2}\right)$.

Mark allocation: 2 marks

- 1 mark for equating the first derivative to zero
- 1 mark for the correct coordinate

Question 6b.

Worked solution

A function is concave up when $f''(a) > 0$.

$f''(x) = \dfrac{4(x+2)}{(x-1)^4}$

$f''(-1) = \dfrac{4(-1+2)}{(-1-1)^4} = \dfrac{1}{4}$

Since $f''(-1) = \frac{1}{4} > 0$, the stationary point $\left(-1, -\frac{3}{2}\right)$ is concave up. Since it is concave up, the point must be a minimum turning point.

Mark allocation: 2 marks

- 1 mark for finding the second derivative, substituting $x = -1$ into it and evaluating
- 1 mark for concluding that the stationary point is concave up – and hence a minimum turning point – based on the evaluation of the second derivative

Question 6c.

Worked solution

A point of inflection occurs when $f''(a) = 0$.

$f''(x) = \dfrac{4(x+2)}{(x-1)^4} = 0$

$\Rightarrow\; x = -2$

So $f(-2) = \dfrac{2(-2)}{(-2-1)^2} - 1 = -\dfrac{13}{9}$.

So there is a point of inflection at $\left(-2, -\frac{13}{9}\right)$.

Explanatory note

The result $f''(a) = 0$. doesn't always result in an inflection point at $x = a$. An inflection point is when the concavity of a curve changes at that point; that is, the sign of the second derivative changes so further investigation is sometimes necessary.

For instance, the second derivative of $f(x) = x^4$ is $f''(x) = 12x^2$. Substituting $x = 0$ into the second derivative would give $f''(0) = 0$. However when you check the sign of the second derivative either side of $x = 0$, $f''(x)$ will always be greater than zero. Therefore there is no sign change at $x = 0$ and the concavity of the curve has not changed. Thus $x = 0$ is not a point of inflection.

For $f''(x) = \dfrac{4(x+2)}{(x-1)^4}$, $f''(x) < 0$ for $x < -2$ and $f''(x) > 0$ for $x > -2$. Therefore there is a sign change at $x = -2$, so a point of inflection exists at $x = -2$.

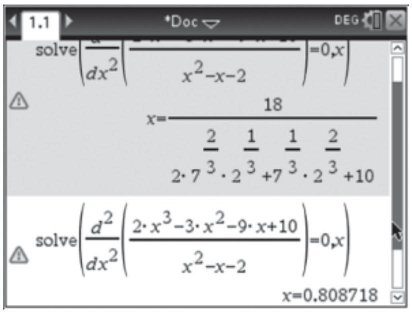
Mark allocation: 2 marks

- 1 mark for equating the second derivative to zero and finding $x = -2$
- 1 mark for the correct coordinate of the point of inflection

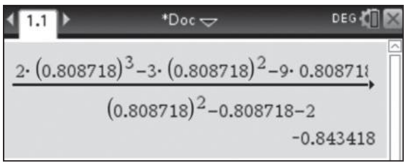
Question 7

Worked solution

Solving for the second derivative equal to zero gives $x = 0.808718$. So the x coordinate of the point of inflection is 0.81, to two decimal places.

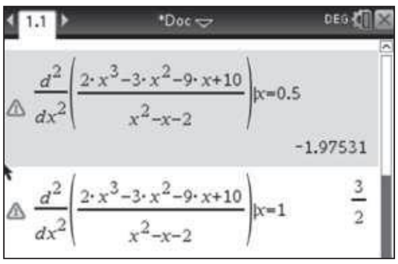


Substituting $x = 0.808718$ into f gives $f(0.808718) = -0.84$, to two decimal places.



Taking a random value on either side of $x = 0.808718$ and calculating the second derivative will show if the point is a point of inflection:

$f''(0.5) = -1.97531$ and $f''(1) = 1.5$



As there is a change of sign either side of $x = 0.808718$, the point $(0.81, -0.84)$, to two decimal places, is a point of inflection.

Mark allocation: 2 marks

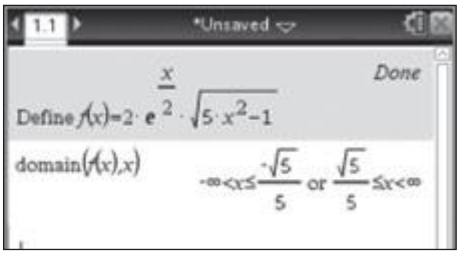
- 1 mark for finding $(0.81, -0.84)$
- 1 mark for showing that the second derivative changes sign either side of $x = 0.808718$

Question 8a.

Worked solution

$2e^{\frac{x}{2}}$ is defined for all values of x, however $\sqrt{5x^2 - 1}$ is undefined for values of x that give a negative value under the square root sign, which are $x \in \left(\frac{-\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$.

This can also be found using a CAS.



Therefore the domain of the function is $x \in R\backslash\left(\frac{-\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$ or $x \in \left(-\infty, \frac{-\sqrt{5}}{5}\right] \cup \left[\frac{\sqrt{5}}{5}, \infty\right)$.

Finding the range, at $x = \frac{-\sqrt{5}}{5}$ or $x = \frac{\sqrt{5}}{5}$, $f(x) = 0$.

As $x \to \infty$, $2e^{\frac{x}{2}} \to \infty$. Therefore $f(x) \to \infty$ as $x \to \infty$.

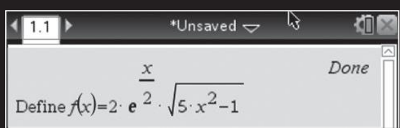
So the range of $f(x)$ is $[0, \infty)$.

Mark allocation: 2 marks

- 1 mark for a domain of $x \in R\backslash\left(\frac{-\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$, or similar
- 1 mark for a range of $[0, \infty)$

TIP

» **For a question where $f(x)$ is used extensively throughout, defining $f(x)$ on a CAS at the start of the question allows for increased efficiency in working through the question.**



Question 8b.

Worked solution

As $x \to -\infty$, $f(x) \to 0$.

Therefore there is an asymptote at $y = 0$.

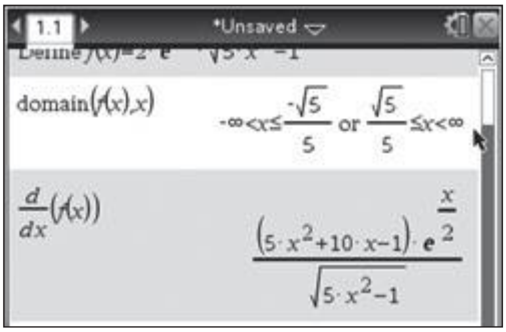
Mark allocation: 1 mark

- 1 mark for giving the equation of the asymptote: $y = 0$.

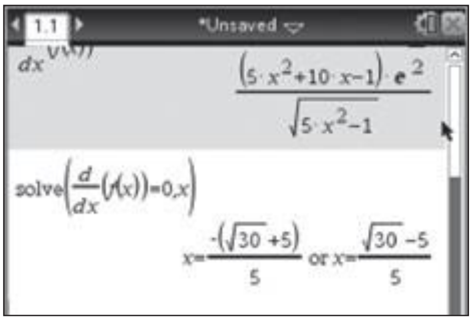
Question 8c.

Worked solution

Calculate the derivative by hand using the product rule, or use the derivative command on a CAS to find the first derivative of $f'(x) = \dfrac{(5x^2 + 10x - 1)e^{\frac{x}{2}}}{\sqrt{5x^2 - 1}}$.



Solving $f'(x) = 0$ for x gives $x = \dfrac{-(\sqrt{30} + 5)}{5}$ or $x = \dfrac{\sqrt{30} - 5}{5}$.



As $x = \dfrac{\sqrt{30} - 5}{5}$ is not part of the domain of $f(x)$ (see **part a.**), we can discard it.

So the x-coordinate of the stationary point is $x = \dfrac{-(\sqrt{30} + 5)}{5}$.

Mark allocation: 2 marks

- 1 mark for calculating $f'(x) = \dfrac{(5x^2 + 10x - 1)e^{\frac{x}{2}}}{\sqrt{5x^2 - 1}}$ or equivalent
- 1 mark for calculating the x-coordinate of the stationary point: $x = \dfrac{-(\sqrt{30} + 5)}{5}$

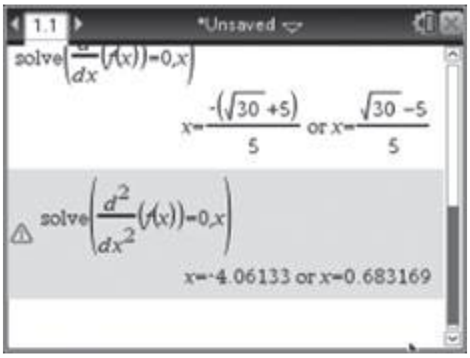
TIP

» When finding solutions, such as x-coordinates, ensure that you check that they are located within the domain of the relation.

Question 8d.i.

Worked solution

Making use of the 'solve' function on a CAS to evaluate $f''(x) = 0$ gives



$x = -4.06133...,\ x = 0.683169...$

Rounded to two decimal places gives $x = -4.06,\ x = 0.68$.

Mark allocation: 1 mark

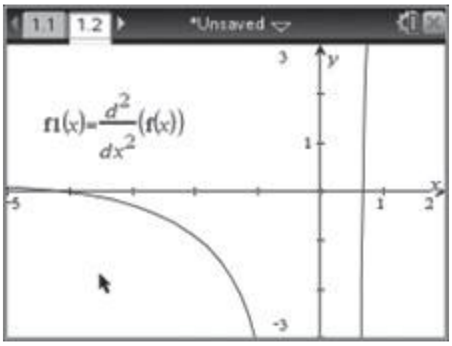
- 1 mark for solving $f''(x) = 0$ and getting $x = -4.06,\ x = 0.68$

Question 8d.ii.

Worked solution

To show that the found x values are points of inflection, we need to show that there is a change of concavity around those points. This can be done by substituting points on either side of those x values into $f''(x)$ and seeing if there is a change in sign on either side of that point.

Alternatively, you can sketch a graph of $f''(x)$ on a CAS and look for a change in concavity around the x values.



First, consider x values in the vicinity of $x = -4.06$.

It can be seen that $f''(x) > 0$ for $x < -4.06$ and $f''(x) < 0$ for $x > -4.06$. As $f''(x) = 0$ at $x = -4.06$ and changes sign on either side of $x = -4.06$, there is a change of concavity around $x = -4.06$. Therefore $x = -4.06$ is a point of inflection.

Now consider x values in the vicinity of $x = 0.68$.

It can be seen that $f''(x) < 0$ for $x < 0.68$ and $f''(x) > 0$ for $x > 0.68$. As there is a change in sign on $f''(x)$, there is a change of concavity around $x = 0.68$. Therefore, $x = 0.68$ is a point of inflection.

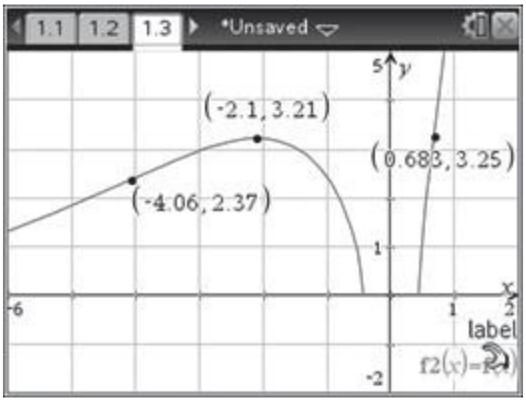
Mark allocation: 2 marks

- 1 mark for showing that $x = -4.06$ is a point of inflection
- 1 mark for showing that $x = 0.68$ is a point of inflection

Question 8e.

Worked solution

Use the 'analyse' function on a CAS graphing window to help find the coordinates of the points of interest.



Mark allocation: 3 marks

- 1 mark for the accurate shape of the graph
- 1 mark for accurately labelled points of inflection and stationary points, correct to two decimal places
- 1 mark for showing the horizontal asymptote of $y = 0$ with a label

TIP

» To increase the accuracy of your sketch, make sure the scale on your calculator graphing window matches that of the axes provided in the question and add grid lines.

Question 9a.

Worked solution

$t \in [0, \pi]$

$\therefore \cos(t) \in [-1, 1]$

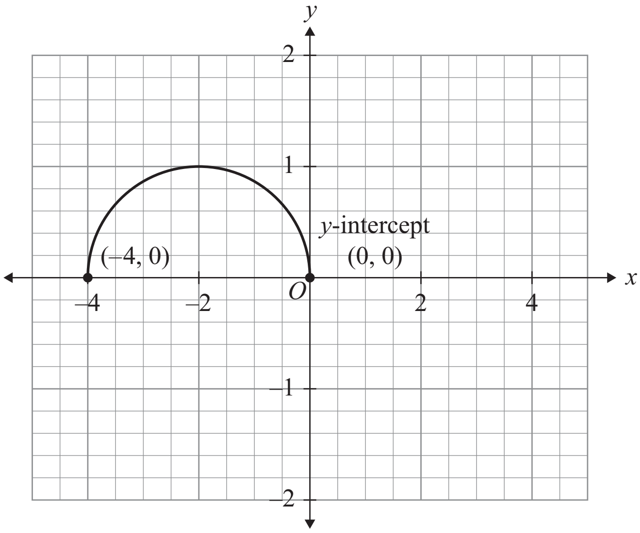
Hence the domain is $2\cos(t) - 2 \in [-4, 0]$ and the range is $\sin(t) \in [0, 1]$.

Mark allocation: 2 marks

- 1 mark for the correct domain
- 1 mark for the correct range

Question 9b.

Worked solution



Mark allocation: 2 marks

- 1 mark for the correct elliptical shape
- 1 mark for both end points labelled with their coordinates

TIP

» **Ensure your curves are drawn with a single smooth line.**

Area of Study 3 Algebra, number and structure

EXAM 1

Question 1

Worked solution

$$z^4 - (1+i)z^3 = z^2 - z - iz$$
$$z^3(z - (1+i)) = z(z - 1 - i)$$
$$z^3(z - (1+i)) - z(z - (1+i)) = 0$$
$$z(z - (1+i))(z^2 - 1) = 0$$
$$\therefore z = 0,\ 1+i,\ \pm 1$$

Mark allocation: 3 marks

- 1 mark for correctly factorising each side of the equation
- 1 mark for setting the equation equal to zero and factorising
- 1 mark for the four correct solutions

Question 2a.

Worked solution

$$\frac{v}{z} = \frac{1+3i}{2+i} \times \frac{2-i}{2-i}$$
$$= \frac{5+5i}{5}$$
$$= 1+i$$
$$\left|\frac{v}{z}\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\text{Arg}\left(\frac{v}{z}\right) = \tan^{-1}\left(\frac{1}{1}\right)$$
$$\Rightarrow \quad \frac{v}{z} = \sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$$

Mark allocation: 2 marks

- 1 mark for correctly simplifying $\frac{v}{z}$
- 1 mark for correctly showing $\left|\frac{v}{z}\right| = \sqrt{2}$ and $\text{Arg}\left(\frac{v}{z}\right) = \frac{\pi}{4}$

TIPS

» Do not attempt to convert z and v into polar form. You must simplify $\frac{v}{z}$ first as $\frac{v\bar{z}}{z\bar{z}}$ and then convert to polar form.

» Minimise errors by recognising that $1 + i = \sqrt{2}\ \text{cis}\frac{\pi}{4}$, without having to calculate the modulus and argument.

Question 2b.

Worked solution

$$\operatorname{Arg}\left(\frac{\bar{v}}{\bar{z}}\right) = \operatorname{Arg}\overline{\left(\frac{v}{z}\right)}$$
$$= -\frac{\pi}{4}$$

$$\operatorname{Arg}\left(\frac{v}{zi}\right) = \operatorname{Arg}\left(\frac{v}{z} \div i\right)$$
$$= \operatorname{Arg}\left(\frac{v}{z}\right) - \operatorname{Arg}(i)$$
$$= \frac{\pi}{4} - \frac{\pi}{2}$$
$$= -\frac{\pi}{4}$$

Alternatively, solving for $\operatorname{Arg}\left(\frac{v}{zi}\right)$:

$$\frac{v}{zi} = \frac{1+3i}{-1+2i} \times \frac{-1-2i}{-1-2i}$$
$$= \frac{5-5i}{5}$$
$$= 1 - i$$

$$\operatorname{Arg}\left(\frac{v}{zi}\right) = \tan^{-1}\left(-\frac{1}{1}\right)$$
$$= -\tan^{-1}(1) = -\frac{\pi}{4}$$

Mark allocation: 2 marks

- 1 mark for the correct answer for $\operatorname{Arg}\left(\frac{\bar{v}}{\bar{z}}\right)$
- 1 mark for the correct answer for $\operatorname{Arg}\left(\frac{v}{zi}\right)$

Question 3

Worked solution

Since $P(1-\sqrt{3}i) = 0 \Rightarrow z = 1 - \sqrt{3}i$ is a solution.

As all the coefficients are real, $z = 1 + \sqrt{3}i$ is also a solution (by the complex conjugate root theorem).

Therefore

$$P(z) = (z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i)(z - c)$$
$$= \left((z-1)^2 + 3\right)(z - c)$$
$$= (z^2 - 2z + 4)(z - c), \text{ where } c \in R.$$

Since $(z^2 - 2z + 4)(z - c) = z^3 - (a-1)z^2 + (b^2 + 4)z - 8$, it follows that $c = 2$. So $z = 2$ is a solution.

Therefore the solutions of $P(z)$ are $z = 1 - \sqrt{3}i,\ 1 + \sqrt{3}i,\ 2$.

Hence

$$(z^2 - 2z + 4)(z - 2) = z^3 - 4z^2 + 8z - 8$$
$$= z^3 - (a-1)z^2 + (b^2 + 4)z - 8$$

Equating the coefficients and solving for a and b gives

$a - 1 = 4 \Rightarrow a = 5$

$b^2 + 4 = 8 \Rightarrow b = \pm 2$

Mark allocation: 4 marks

- 1 mark for using the complex conjugate root theorem to find the solutions $z = 1 \pm \sqrt{3}i$
- 1 mark for finding the third solution $(z = 2)$
- 1 mark for correctly calculating a
- 1 mark for correctly calculating b

TIPS

» **When finding solutions to a complex polynomial in which all coefficients are real, make use of the complex conjugate root theorem to find additional solutions.**

» **When expanding $(z - 1 - \sqrt{3}i)(z - 1 + \sqrt{3}i)$, use the difference of two squares formula: $(z-1)^2 - \left(\sqrt{(3)}i\right)^2 = (z-1)^2 + 3$.**

Question 4

Worked solution

Method 1: Complex division

Multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{\sqrt{2} + \sqrt{6}i}{2\sqrt{3} - 2i} \times \frac{2\sqrt{3} + 2i}{2\sqrt{3} + 2i} = \frac{2\sqrt{6} + 2\sqrt{2}i + 2\sqrt{18}i - 2\sqrt{6}}{\left(2\sqrt{3}\right)^2 + (2)^2}$$

$$= \frac{2\sqrt{2}i + 6\sqrt{2}i}{16} = \frac{8\sqrt{2}}{16}i$$

$$= \frac{\sqrt{2}}{2}i$$

The modulus of the number is

$$\left|\frac{\sqrt{2}}{2}\right| = \frac{\sqrt{2}}{2}$$

As $\frac{\sqrt{2}}{2}i$ lies on the positive imaginary axis of the Argand plane, in polar form it will have a principal argument of $\frac{\pi}{2}$.

Therefore the polar form of $\frac{\sqrt{2} + \sqrt{6}i}{2\sqrt{3} - 2i}$ is $\frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{2}\right)$.

Method 2: Polar form

Let the numerator be $z_1 = \sqrt{2} + \sqrt{6}i$ and the denominator be $z_2 = 2\sqrt{3} - 2i$.

The modulus of z_1 is $\left|\sqrt{2} + \sqrt{6}i\right| = 2\sqrt{2}$.

The argument of z_1 is $\tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{2}}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$.

Therefore the polar form of z_1 is $2\sqrt{2}\text{cis}\left(\frac{\pi}{3}\right)$.

The modulus of z_2 is $\left|2\sqrt{3} - 2i\right| = 4$.

The argument of z_2 is $\tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = -\frac{\pi}{6}$.

Therefore, in polar form, z_2 is $4\text{cis}\left(-\frac{\pi}{6}\right)$.

It follows that $\frac{\sqrt{2}+\sqrt{6}i}{2\sqrt{3}-2i} = \frac{2\sqrt{2}\text{cis}\left(\frac{\pi}{3}\right)}{4\text{cis}\left(-\frac{\pi}{6}\right)}$.

Using complex division in polar form gives

$$\begin{aligned}\frac{2\sqrt{2}\text{cis}\left(\frac{\pi}{3}\right)}{4\text{cis}\left(-\frac{\pi}{6}\right)} &= \frac{2\sqrt{2}}{4}\text{cis}\left(\frac{\pi}{3} - -\frac{\pi}{6}\right) \\ &= \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{2}\right)\end{aligned}$$

Therefore the polar form of $\frac{\sqrt{2}+\sqrt{6}i}{2\sqrt{3}-2i}$ is $\frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{2}\right)$.

Mark allocation: 3 marks

- 1 mark for using an appropriate technique to derive $\frac{\sqrt{2}}{2}i$
- 1 mark for determining the modulus and argument of $\frac{\sqrt{2}}{2}i$
- 1 mark for deriving the correct final answer of $\frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{2}\right)$

OR

- 1 mark for finding the correct polar forms of the numerator and denominator of $\frac{\sqrt{2}+\sqrt{6}i}{2\sqrt{3}-2i}$
- 1 mark for using division of complex numbers in polar form
- 1 mark for deriving the correct final answer of $\frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{2}\right)$

TIP

» **Become familiar with how to perform multiplication and division of complex numbers in polar form, as sometimes it is a more efficient way to work through a problem than using Cartesian form.**

Question 5

Worked solution

Since all the coefficients of z are real, conjugate root theorem applies. Thus $z = 2 + i$ is another solution of the equation.

Since $z = 2 - i$ and $z = 2 + i$ are solutions of the equation, $(z - 2 + i)$ and $(z - 2 - i)$ are both linear factors of $z^3 - 6z^2 + az - a + 3$.

Expanding $(z - 2 + i)(z - 2 - i)$ gives $z^2 - 4z + 5$, which is a quadratic factor of $z^3 - 6z^2 + az - a + 3$.

Method 1: Equating coefficients

Let $z - b$ be a linear factor of $z^3 - 6z^2 + az - a + 3$.

Expand the linear and quadratic factors:

$(z - b)(z^2 - 4z + 5) = z^3 + (-b - 4)z^2 + (4b + 5)z - 5b$

Now compare the coffecients of $z^3 + (-b - 4)z^2 + (4b + 5)z - 5b$ to $z^3 - 6z^2 + az - a + 3$.

For coefficients of z^2:

$-b - 4 = -6$

$b = 2$

For coefficients of z:

$4b + 5 = a$

$a = 13$

Method 2: Polynomial long division

To find the value of a, use polynomial long division to divide $z^3 - 6z^2 + az - a + 3$ by its quadratic factor, $z^2 - 4z + 5$:

$$\begin{array}{r} z - 2 \\ z^2 - 4z + 5 \overline{\smash{)}\, z^3 - 6z^2 + az - a + 3} \end{array}$$

This gives a third linear factor of $z - 2$ and, since this quotient should have a remainder of zero, gives two equations for the value of a.

(1) $a - 5 = 8$

(2) $-a + 3 = -10$

$\therefore a = 13$

$a = 13$ satisfies both equations and gives the required remainder of zero.

Method 3: Substituting the original solution

Substituting the original solution into the equation gives

$(2 - i)^3 - 6(2 - i)^2 + a(2 - i) - a + 3 = 0$

$-13 + a + 13i - ai = 0$

Equating the coefficients of either the real part or the imaginary part of the equation gives a solution of $a = 13$.

Mark allocation: 3 marks

- 1 mark for determining the second solution using the conjugate root theorem
- 1 mark for setting up the correct quotient for polynomial long division, or correctly expanding and equating coefficients
- 1 mark for deriving the correct value of $a = 13$

OR

- 1 mark for correctly substituting the solution of $z = 2 - i$ into the equation
- 1 mark for equating the coefficients of either the real or the imaginary part of the equation
- 1 mark for deriving the correct value of $a = 13$

TIP

» **A complex polynomial of degree n, where n is an odd integer and all coefficients are real, will always have $n - 1$ complex solutions that appear as conjugate pairs and one real solution.**

Question 6

Worked solution

Let $w = z^3$.

$$w^2 + w + 1 = 0$$

$$w^2 + w + \frac{1}{4} + -\frac{1}{4} + 1 = 0$$

$$\left(w + \frac{1}{2}\right)^2 = -\frac{3}{4}$$

$$w = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$z^3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $z^3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

In polar form:

$z^3 = \text{cis}\left(\frac{2\pi}{3}\right)$ or $z^3 = \text{cis}\left(-\frac{\pi}{3}\right)$

Letting $z = r\text{cis}(\theta)$, de Moivre's theorem can be used to solve for z.

First three solutions:

$$r^3\text{cis}(3\theta) = \text{cis}\left(\frac{2\pi}{3}\right)$$

$$r = 1$$

$$3\theta = \frac{2\pi}{3} + 2k\pi,\ k \in \mathbf{Z}$$

$$\theta = \frac{(2 + 6k)\pi}{9}$$

$$\theta = \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{14\pi}{9}$$

As the principal values of the argument have the domain $\theta \in (-\pi, \pi)$, $\theta = -\frac{4\pi}{9}, \frac{2\pi}{9}, \frac{8\pi}{9}$.

Therefore the first three solutions are $z = \text{cis}\left(-\frac{4\pi}{9}\right),\ \text{cis}\left(\frac{2\pi}{9}\right),\ \text{cis}\left(\frac{8\pi}{9}\right)$.

Other three solutions:

Since the coefficients of $z^6 + z^3 + 1$ are all real, the solutions of $z^6 + z^3 + 1 = 0$ will obey the conjugate root theorem.

Therefore the other three solutions are $z = \text{cis}\left(\frac{4\pi}{9}\right),\ \text{cis}\left(-\frac{2\pi}{9}\right),\ \text{cis}\left(-\frac{8\pi}{9}\right)$.

Mark allocation: 4 marks

- 1 mark for rewriting the equation as a quadratic and finding the two correct solutions for z^3 of $-\frac{1}{2} + \frac{3}{2}i$ and $-\frac{1}{2} - \frac{3}{2}i$
- 1 mark for correctly writing the solutions for z^3 in polar form
- 1 mark for using de Moivre's theorem to find the first three solutions in polar form, using the correct principal arguments
- 1 mark for using the conjugate root theorem (or other method) to determine the other three solutions in polar form, using the correct principal arguments

TIP

» When using de Moivre's theorem to find the roots of a complex polynomial in polar form $r\text{cis}(\theta)$, always check the argument of each root, θ, at the end of your calculations to ensure that $\theta \in (-\pi, \pi]$, as required.

Question 7

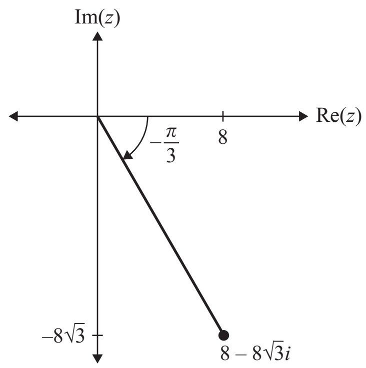
Worked solution

First, convert $8 - 8\sqrt{3}i$ to polar form by finding the modulus and argument:

$$r = \sqrt{8^2 + \left(8\sqrt{3}\right)^2}$$
$$= \sqrt{64 + 64 \times 3}$$
$$= \sqrt{4 \times 64} = 2 \times 8 = 16$$

$$\theta = \tan^{-1}\left(-\frac{8\sqrt{3}}{8}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

Alternatively, the argument, θ, can be determined by sketching a diagram on the Argand plane and using trigonometry:



$$\therefore\ 8 - 8\sqrt{3}i = 16\text{cis}\left(-\frac{\pi}{3}\right)$$

To find the fourth roots we must solve the equation $z^4 = 8 - 8\sqrt{3}i,\ z \in C$.

Letting $z = r\text{cis}(\theta)$ and applying de Moivre's theorem gives

$$r^4\text{cis}(4\theta) = 16\text{cis}\left(-\frac{\pi}{3}\right)$$

$$r = \sqrt[4]{16} = 2$$

$$4\theta = -\frac{\pi}{3} + 2k\pi,\ \ k \in \mathbf{Z}$$

$$\theta = \frac{-\pi + 6k\pi}{12},\ \ k \in \mathbf{Z}$$

$$\theta_1 = -\frac{\pi}{12},\ \theta_2 = \frac{5\pi}{12},\ \theta_3 = \frac{11\pi}{12},\ \theta_4 = \frac{17\pi}{12} = -\frac{7\pi}{12}$$

Therefore, in polar form using the principal argument, the fourth roots of $8 - 8\sqrt{3}i$ are

$$z_1 = 2\text{cis}\left(\frac{-7\pi}{12}\right),\ z_2 = 2\text{cis}\left(\frac{-\pi}{12}\right),\ z_3 = 2\text{cis}\left(\frac{5\pi}{12}\right),\ z_4 = 2\text{cis}\left(\frac{11\pi}{12}\right).$$

Alternatively, we could use the periodicity of the solutions $\left(\frac{2\pi}{n}=\frac{2\pi}{4}=\frac{\pi}{2}\right)$ to write

$z^4 = 16\text{cis}\left(-\frac{\pi}{3}\right),\ 16\text{cis}\left(\frac{5\pi}{3}\right),\ 16\text{cis}\left(\frac{11\pi}{3}\right),\ 16\text{cis}\left(\frac{17\pi}{3}\right).$

$\therefore\ z = 2\text{cis}\left(-\frac{\pi}{12}\right),\ 2\text{cis}\left(\frac{5\pi}{12}\right),\ 2\text{cis}\left(\frac{11\pi}{12}\right),\ 2\text{cis}\left(\frac{17\pi}{12}\right) = 2\text{cis}\left(-\frac{7\pi}{12}\right)$

Mark allocation: 3 marks

- 1 mark for correctly converting $8 - 8\sqrt{3}i$ to the polar form $16\text{cis}\left(-\frac{\pi}{3}\right)$
- 1 mark for using de Moivre's theorem to set up the equation $r^4\text{cis}(4\theta) = 16\text{cis}\left(-\frac{\pi}{3}\right)$
- 1 mark for finding the four roots in polar form with the correct modulus and principal arguments:

  $z_1 = 2\text{cis}\left(\frac{-7\pi}{12}\right),\ z_2 = 2\text{cis}\left(\frac{-\pi}{12}\right),\ z_3 = 2\text{cis}\left(\frac{5\pi}{12}\right),\ z_4 = 2\text{cis}\left(\frac{11\pi}{12}\right)$

TIPS

» You should be able to recall the exact values for all trigonometric ratios in right-angled triangles involving angles of $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ radians.

» Remember that an equation of the form $z^n = a,\ a \in C$ will have n solutions that lie on the circumference of a circle in the Argand diagram at an angular separation of $\frac{2\pi}{n}$ radians. Once you find one solution in polar form, you can add $\frac{2\pi}{n}$ to the argument of that solution to find another solution.

Question 8a.

Worked solution

$$\begin{aligned}(1+\sqrt{3}i)^5 + (1-\sqrt{3}i)^5 &= \left(2\text{cis}\left(\frac{\pi}{3}\right)\right)^5 + \left(2\text{cis}\left(-\frac{\pi}{3}\right)\right)^5\\ &= 2^5\text{cis}\left(\frac{5\pi}{3}\right) + 2^5\text{cis}\left(-\frac{5\pi}{3}\right)\\ &= 32\cos\left(\frac{5\pi}{3}\right) + 32\sin\left(\frac{5\pi}{3}\right)i + 32\cos\left(-\frac{5\pi}{3}\right) + 32\sin\left(-\frac{5\pi}{3}\right)i\\ &= 2\times 32\cos\left(\frac{5\pi}{3}\right)\\ &= 64\times\frac{1}{2}\\ &= 32\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for converting both $1 + \sqrt{3}i$ and $1 - \sqrt{3}i$ into polar form
- 1 mark for correct answer of 32

Question 8b.

Worked solution

$2^k \operatorname{cis}\left(\frac{k\pi}{3}\right) + 2^k \operatorname{cis}\left(-\frac{k\pi}{3}\right) = 128$

$2^k \cos\left(\frac{k\pi}{3}\right) + 2^k \sin\left(\frac{k\pi}{3}\right)i + 2^k \cos\left(-\frac{k\pi}{3}\right) + 2^k \sin\left(-\frac{k\pi}{3}\right)i = 128$

$2^k \cos\left(\frac{k\pi}{3}\right) + 2^k \sin\left(\frac{k\pi}{3}\right)i + 2^k \cos\left(\frac{k\pi}{3}\right) - 2^k \sin\left(\frac{k\pi}{3}\right)i = 128$

$2^{k+1} \cos\left(\frac{k\pi}{3}\right) = 128 = 2^7$

Case 1:

$\cos\left(\frac{k\pi}{3}\right) = 1$

Therefore:

$\frac{k\pi}{3} = 2\pi$

$k = 6$

Case 2: Given that $\cos\left(\frac{k\pi}{3}\right) = \frac{1}{2}$:

$2^{k+1} \times \frac{1}{2} = 2^7$

$2^k = 2^7$

Alternatively, since $\cos\left(\frac{k\pi}{3}\right) = 1$, then $2^{k+1} = 2^7$

$k = 6$

Mark allocation: 2 marks

- 1 mark for correct answer of $k = 6$
- 1 mark for correct answer of $k = 7$

EXAM 2

Section A

Question 1

Answer: ***C***

Worked solution

The graph is a circle centred at (–1, 1) with a radius of 2.

Options A and D represent rays, so can be eliminated.

Option B represents a circle centred at (–1, 1) with a radius of 4, so can be eliminated.

Option C represents a circle centred at (–1, 1) with a radius of 2, so is therefore the correct response.

So the circle can be represented by $\{z: |z - (i - 1)| = 2\}$.

Question 2

Answer: ***D***

Worked solution

Let $z = r\operatorname{cis}\theta$ and $2\sqrt{3} + 2i = \sqrt{(2\sqrt{3})^2 + 2^2}\ \operatorname{cis}\left(\frac{\pi}{6}\right) = 4\operatorname{cis}\left(\frac{\pi}{6}\right)$.

Hence $z^4 = 4\operatorname{cis}\left(\frac{\pi}{6}\right)$.

From de Moivre's theorem, $z^n = r^n \operatorname{cis}(n\theta)$, it follows that

$$z^4 = 4\operatorname{cis}\left(\frac{\pi}{6}\right)$$

$$z = \sqrt[4]{4}\operatorname{cis}\left(\frac{1}{4}\left(\frac{\pi}{6} + 2k\pi\right)\right), \text{ where } k \in \mathbf{Z}$$

$$z = \sqrt{2}\operatorname{cis}\left(\frac{1}{4}\left(\frac{\pi}{6} + 2k\pi\right)\right), \text{ where } k \in \mathbf{Z}$$

So $z = \sqrt{2}\operatorname{cis}\left(-\frac{23\pi}{24}\right)$, $\sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{24}\right)$, $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{24}\right)$ and $\sqrt{2}\operatorname{cis}\left(\frac{13\pi}{24}\right)$.

On the diagram given in the question, only points z_1 and z_6 match solutions to the equation. Therefore the answer is option D.

Question 3

Answer: ***D***

Worked solution

The polynomial $P(z)$ factorises as follows:

$$\begin{aligned} z^3 + 2iz^2 + 9z + 18i &= z^2(z + 2i) + 9(z + 2i) \\ &= (z^2 + 9)(z + 2i) \\ &= (z - 3i)(z + 3i)(z + 2i) \end{aligned}$$

So the solutions to $P(z) = 0$ are $z = -2i, -3i, 3i$.

Since all roots of $P(z)$ are complex, $P(z)$ has no real roots and the correct response is option D.

Question 4

Answer: ***A***

Worked solution

Using de Moivre's theorem:

$z^3 = r^3\operatorname{cis}(3\theta)$ and $\bar{z} = r\operatorname{cis}(-\theta)$.

Then, using division in polar form:

$$\frac{z^3}{\bar{z}} = \frac{r^3\operatorname{cis}(3\theta)}{r\operatorname{cis}(-\theta)} = r^2\operatorname{cis}(3\theta - -\theta) = r^2\operatorname{cis}(4\theta)$$

Question 5

Answer: ***D***

Worked solution

The relation $|z - 2 + 3i| = 2$ represents the circle $(x - 2)^2 + (y + 3)^2 = 4$. This can be shown as follows:

$$\begin{aligned} (z - 2 + 3i)(\bar{z} - 2 - 3i) &= 4 \\ (x + yi - 2 + 3i)(x - iy - 2 - 3i) &= 4 \\ [(x - 2) + i(y + 3)][(x - 2) - i(y + 3)] &= 4 \\ (x - 2)^2 + (y + 3)^2 &= 4 \end{aligned}$$

Option A represents the perpendicular bisector between two points, which is a straight line.

Option B represents the equation of a circle with radius $\sqrt{2}$.

Option C represents the perpendicular bisector between the two points, which is a straight line.

Option D represents the equation of a circle with radius 2.

TIP

» **Remembering the different representations of the subsets of a plane (such as rays, lines and circles) is an efficient way to eliminate multiple-choice options without resorting to algebra.**

Question 6

Answer: ***C***

Worked solution

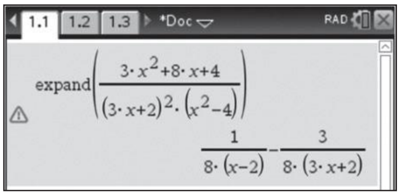
Since all the coefficients of $P(z) = 0$ are real, the complex conjugates $z = -i$ and $z = 2 - i$ are also roots. Since three roots have already been stated, the addition of the two complex conjugate roots means that the minimum degree of $P(z)$ is 5.

Question 7

Answer: ***A***

Worked solution

Use the 'expand' command on a CAS.



Therefore a partial fraction form of the expression is $\frac{A}{x-2} + \frac{B}{3x+2}$, where $A = \frac{1}{8}$ and $B = -\frac{3}{8}$.

Question 8

Answer: ***A***

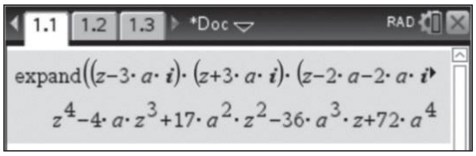
Worked solution

From the conjugate root theorem, the other solutions are $z = -3ai$ and $z = 2a - 2ai$.

So the quartic polynomial can be expressed as

$$\begin{aligned}(z-3ai)(z+3ai)(z-2a-2ai)(z-2a+2ai) &= (z^2-(3ai)^2)\left((z-2a)^2-(2ai)^2\right) \\ &= (z^2+9a^2)(z^2-4az+8a^2) \\ &= z^4-4az^3+17a^2z^2-36a^3z+72a^4\end{aligned}$$

Alternatively, use the 'expand' function on a CAS to get the correct answer.



TIP

» **When all coefficients are real, make use of the conjugate root theorem to find additional solutions to complex polynomials.**

Question 9

Answer: ***B***

Worked solution

The Cartesian equation of the line is $y = x - 1$.

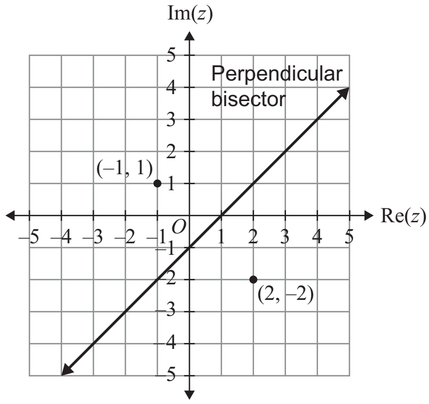
Option A is incorrect, as it represents rays.

Option D is incorrect, as it represents the equation of a circle centred at (–2, –3) with a radius of 4.

Options B and C represent equations of perpendicular bisectors.

Furthermore, option C cannot be correct. Substituting $z = 2 + i$, a point on the line, into $|z - 2 - i| = |z + 1 + 2i|$ gives $|0| \neq |3 + 3i|$. Therefore the line shown is not the perpendicular bisector described.

Option B is correct because the perpendicular bisector of the points (2, –2) and (–1, 1) is the line $x = -1, x = 1, y = 1$, which matches the line shown in the graph.



Alternatively, if you define z as $x + yi$ on a CAS, you can generate the Cartesian relation of the line, as shown below.

$x+y\cdot\boldsymbol{i}\rightarrow z$	$x+y\cdot\boldsymbol{i}$
$\lvert z-2+2\cdot\boldsymbol{i}\rvert=\lvert z+1-\boldsymbol{i}\rvert$	$\sqrt{x^2-4\cdot x+y^2+4\cdot y+8}=\sqrt{x^2+2\cdot x+y^2-2\cdot y+2}$
$\text{solve}\left(\sqrt{x^2-4\cdot x+y^2+4\cdot y+8}=\sqrt{x^2+2\cdot x+y^2-2\cdot y+2},y\right)$	$y=x-1$

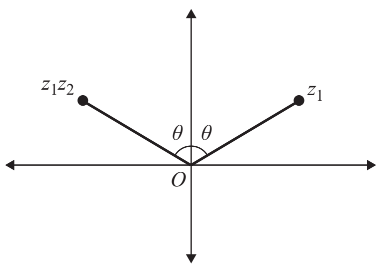
Question 10

Answer: ***C***

Worked solution

Given that $0 < \text{Arg}(z_1) < \frac{\pi}{2}$, z_1 is in the first quadrant.

Given that z_1 and z_1z_2 are symmetric in the imaginary axis, z_1z_2 is in the second quadrant, as illustrated by the following diagram.



Let $\theta = \frac{\pi}{2} - \text{Arg}(z_1)$.

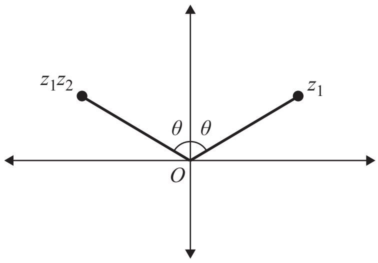
$\text{Arg}(z_1z_2) = \frac{\pi}{2} + \theta = \frac{\pi}{2} + \frac{\pi}{2} - \text{Arg}(z_1) = \pi - \text{Arg}(z_1)$

Also, $\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$.

$\therefore\ \text{Arg}(z_1) + \text{Arg}(z_2) = \pi - \text{Arg}(z_1)$

$\therefore\ \text{Arg}(z_2) = \pi - 2\text{Arg}(z_1)$

Another approach, using the same diagram:



Let $\text{Arg}(z_1) = \theta_1$ and $\text{Arg}(z_2) = \theta_2$, then:

$$\text{Arg}(z_1z_2) = \theta_1 + \theta_2$$
$$\theta_1 + \theta_2 = \pi - \theta_1$$
$$\theta_2 = \pi - 2\theta_1$$

Therefore option C is true.

Option A is incorrect because the question gives information only about angles, not magnitudes. Not enough information is given to determine whether option A is true.

Option B is incorrect because $\text{Arg}(z_1)$ may not be greater than $\text{Arg}(z_2)$. That depends on the size of $\text{Arg}(z_1)$.

Option D is incorrect because if $z_2 = 1$, then $\text{Arg}(z_2) = 0$. In this case, $\text{Arg}(z_1) = \text{Arg}(z_1z_2)$ and therefore z_1 and z_1z_2 are in the same quadrant and not symmetric in the imaginary axis.

TIP

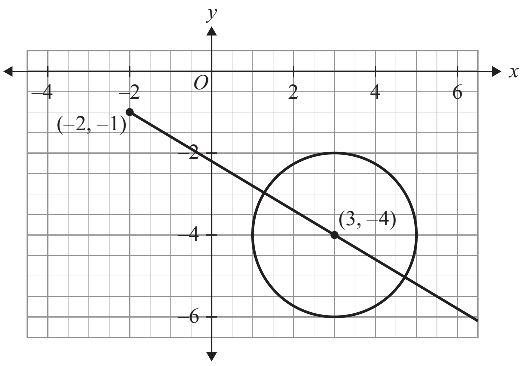
» Questions that ask 'which one of the following is (or is not) true?' will usually require a process-of-elimination approach. Working through the four multiple-choice options systematically, and not rushing to select the first one that seems plausible, will reduce the likelihood of making a careless error.

Question 11

Answer: ***D***

Worked solution

By circle geometry, the radius of a circle is always perpendicular to its tangent at the point where the radius intersects the circumference. Therefore the ray $\text{Arg}(z + 2 + i) = \theta$ is perpendicular to the path $|z - 3 + 4i| = 2$ when it goes through centre of the circle at (3, −4).



Forming a right-angled triangle using the points (–2, –1) and (3, –4) allows us to deduce that $\tan(\theta) = -\frac{3}{5}$.

TIP

» Visualising your thinking is an essential tool in Specialist Maths. An algebraic approach to this question would prove very laborious, but a rough sketch often shows the solution very quickly.

Question 12

Answer: ***C***

Worked solution

$z^4 = a + bi$ has four solutions, equally spaced around the circle with radius $|z|$. So each solution can be found by rotating a previous solution by $360° \div 4 = 90°$.

Let one solution be $x + yi$. The next solution must then be $i(x + yi)$.

Sum of all solutions $= (x + yi) + i(x + yi) + i^2(x + yi) + i^3(x + yi)$

$$= x + yi + xi - y - x - yi - xi + y$$

$$= 0$$

TIP

» Roughly sketching four equally spaced points on an Argand diagram produces the same result without the need for algebra.

Question 13

Answer: ***B***

Worked solution

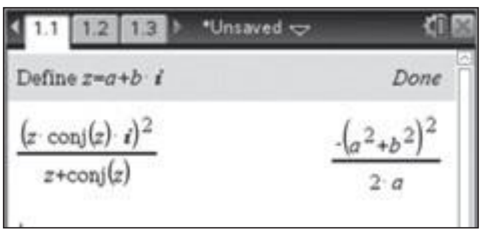
If $z = a + bi$, then

$$\frac{(z\bar{z}i)^2}{z + \bar{z}} = \frac{[(a + bi)(a - bi)i]^2}{a + bi + a - bi}$$

$$= \frac{i^2(a^2 + b^2)^2}{2a}$$

$$= \frac{-(a^2 + b^2)^2}{2a}$$

$$= \frac{-1}{2\text{Re}(z)}[\text{Re}(z) + \text{Im}(z)]^2$$



TIP

» Defining $z = a + bi$ on a CAS and using it to perform the algebraic operations is an efficient way to solve a problem such as this.

Question 14

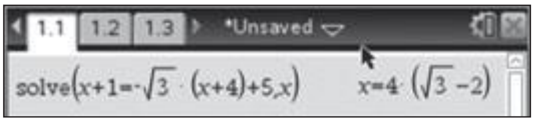
Answer: ***D***

Worked solution

$\text{Arg}(z + 4 - 5i) = -\frac{\pi}{3}$ can be expressed as the Cartesian equation $y = -\sqrt{3}(x + 4) + 5, x > -4$.

$\text{Arg}(z + 2 + i) = \frac{\pi}{4}$ can be expressed as the Cartesian equation $y = x + 1, x > -2$.

Equating these two equations and solving for x gives $x = 4\sqrt{3} - 8$.



Substituting this x value into either Cartesian equation gives $y = 4\sqrt{3} - 7$.

Therefore the point of intersection is $(4\sqrt{3} - 8, 4\sqrt{3} - 7)$.

TIP

» Sketching the rays is a good way to visualise the situation and gain a sense of where the point of intersection is.

Question 15

Answer: ***C***

Worked solution

If $a, b, c, d \in R$, then we know that all solutions must be either real or occur in conjugate pairs. This comes from the conjugate root theorem. Option C is the only option that satisfies this condition.

TIP

» You must remember that conjugate pairs are reflections in the real axis; that is, they have the same real component but the sign of the imaginary component is reversed.

Question 16

Answer: ***D***

Worked solution

$z = 2\text{cis}\left(-\frac{4\pi}{3}\right) = 2\text{cis}\left(\frac{2\pi}{3}\right)$

Then $z^2 = 2^2\text{cis}\left(2 \times \frac{2\pi}{3}\right) = 4\text{cis}\left(\frac{4\pi}{3}\right) = 4\text{cis}\left(-\frac{2\pi}{3}\right)$

$\Rightarrow 4\text{cis}\left(-\frac{2\pi}{3}\right) = 4\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= -2 - 2\sqrt{3}i$

Question 17

Answer: ***A***

Worked solution

First, rewrite the question as $z^n = 2\text{cis}\left(\frac{\pi}{4}\right)$.

Next, using de Moivre's theorem, note that the modulus of the solutions must be $\sqrt[n]{2}$ and that the argument of the first solution must be $\frac{\pi}{4n}$. It follows that the answer must be option A.

Question 18

Answer: ***D***

Worked solution

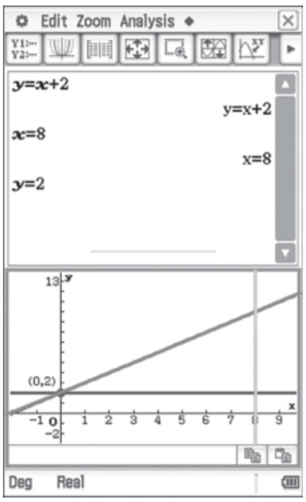
Using $m = \tan(\theta)$ and remembering that a line with equation $\text{Arg}(z - (a + bi)) = \theta$ passes through the point $a + bi$, we find the Cartesian equations of each line to be

$\text{Arg}(z - 2i) = \frac{\pi}{4} \quad \Rightarrow \quad \text{Im}(z) = \text{Re}(z) + 2$

$\text{Arg}(z - 8) = \frac{\pi}{2} \quad \Rightarrow \quad \text{Re}(z) = 8$

$\text{Arg}(z - (10 + 2i)) = \pi \quad \Rightarrow \quad \text{Im}(z) = 2$

Now sketch each line and use the formula for the area of a triangle to find that the bounded area is 32 units^2. Use a CAS for convenience.



Section B

Question 1a.

Worked solution

The polar form of a complex number has the general form $u = r\operatorname{cis}(\theta)$.

So $r = \sqrt{2^2 + 2^2}$

$r = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{2}{2}\right)$

$\theta = \frac{\pi}{4}$

So $u = 2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$.

Mark allocation: 2 marks

- 1 mark for showing calculations to find $r = 2\sqrt{2}$ and $\theta = \frac{\pi}{4}$
- 1 mark for stating $u = 2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$

Question 1b.

Worked solution

If $z = x + iy$, then

$$|\bar{z} + z| = 4$$
$$|x - iy + x + iy| = 4$$
$$|2x| = 4$$
$$2x = 4$$
$$x = 2$$

Mark allocation: 2 marks

- 1 mark for showing calculations that lead to $|2x| = 4$
- 1 mark for the line equation $x = 2$

Question 1c.

Worked solution

If $z = x + iy$, then

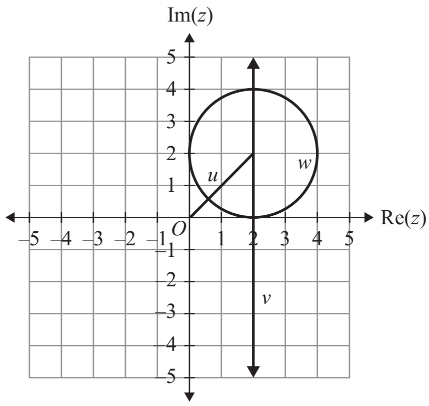
$$(z - 2 - 2i)(\bar{z} - 2 + 2i) - 4 = 0$$
$$(x + iy - 2 - 2i)(x - iy - 2 + 2i) - 4 = 0$$
$$[(x - 2) + i(y - 2)]\,[(x - 2) - i(y - 2)] - 4 = 0$$
$$(x - 2)^2 - i(x - 2)(y - 2) + i(x - 2)(y - 2) - i^2(y - 2)^2 - 4 = 0$$
$$(x - 2)^2 + (y - 2)^2 - 4 = 0$$
$$\Rightarrow (x - 2)^2 + (y - 2)^2 = 4$$

Mark allocation: 3 marks

- 1 mark for showing that $[(x - 2) + i(y - 2)]\,[(x - 2) - i(y - 2)] - 4 = 0$
- 1 mark for expanding the expression using difference of two squares
- 1 mark for the Cartesian equation $(x - 2)^2 + (y - 2)^2 = 4$

Question 1d.

Worked solution



Mark allocation: 3 marks

- 1 mark for accurately sketching the line from (0, 0) to u
- 1 mark for accurately sketching $v = \{z : |\bar{z} + z| = 4\}$
- 1 mark for accurately sketching $w = \{z : (z - 2 - 2i)(\bar{z} - 2 + 2i) - 4 = 0\}$

Question 1e.

Worked solution

The area of a sector is given by $A = \frac{1}{2}r^2\theta$.

Given that $\theta = \frac{\pi}{4}$, then the area of the sector is $A = \frac{1}{2} \times 2^2 \times \frac{\pi}{4} = \frac{\pi}{2}$ units^2.

Mark allocation: 2 marks

- 1 mark for using the formula $A = \frac{1}{2}r^2\theta$
- 1 mark for the answer $\frac{\pi}{2}$ units^2

Question 2a.i.

Worked solution

Polar form is given by $r\text{cis}(\theta)$.

$r = \sqrt{(2\sqrt{3})^2 + 2^2} = 4$

$\theta = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6}$

So, in polar form, $z_1 = 4\text{cis}\left(\frac{\pi}{6}\right)$.

Alternatively, using technology:

Mark allocation: 1 mark

- 1 mark for the correct polar form of $4\text{cis}\left(\frac{\pi}{6}\right)$

Question 2a.ii.

Worked solution

Since all coefficients of the equation are real, one of the other solutions is the complex conjugate of z_1.

So $z_2 = 2\sqrt{3} - 2i$.

Then

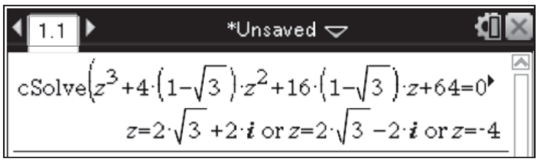
$$(z - 2\sqrt{3} - 2i)(z - 2\sqrt{3} + 2i)(z + a) = z^3 + 4(1 - \sqrt{3})z^2 + 16(\sqrt{3} + 1)z + 64$$

$$(z^2 - 4\sqrt{3}z + 16)(z + a) = z^3 + 4(1 - \sqrt{3})z^2 + 16(1 - \sqrt{3})z + 64$$

By equating coefficients, it can be seen that $a = 4$.

So $z_3 = -4$.

Alternatively, using technology:

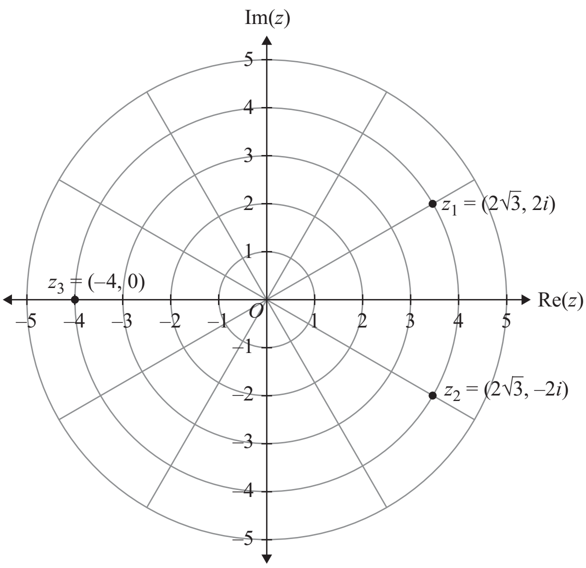


Mark allocation: 2 marks

- 1 mark for the correct root of $z_2 = 2\sqrt{3} - 2i$
- 1 mark for the correct root of $z_3 = -4$

Question 2b.

Worked solution



Mark allocation: 3 marks

- 1 mark for the correctly plotted and labelled root $z_1 = 2\sqrt{3} + 2i$
- 1 mark for the correctly plotted and labelled root $z_1 = 2\sqrt{3} - 2i$
- 1 mark for the correctly plotted and labelled root $z_3 = -4$

Question 2c.

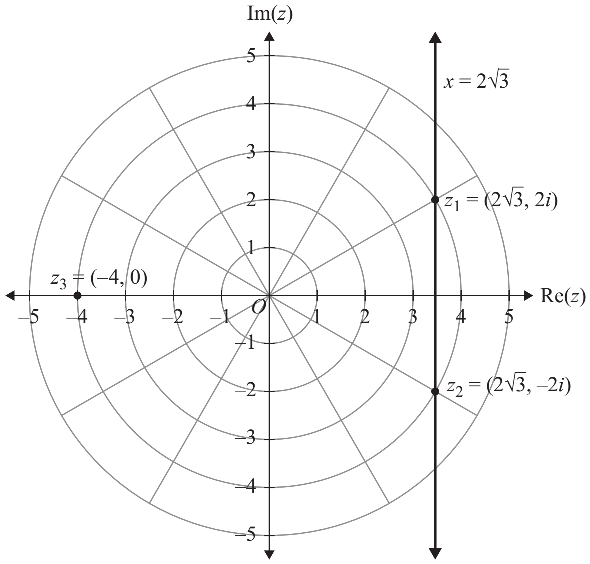
Worked solution

$|z - 4\sqrt{3}| = |z|$ is equivalent to

$$|x + iy - 4\sqrt{3}| = |x + iy|$$
$$\sqrt{(x - 4\sqrt{3})^2 + y^2} = \sqrt{x^2 + y^2}$$
$$(x - 4\sqrt{3})^2 + y^2 = x^2 + y^2$$

Expanding and collecting like terms leads to

$$-8\sqrt{3}x + 48 = 0$$
$$\Rightarrow x = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$



Mark allocation: 3 marks

- 1 mark for showing that $\sqrt{(x - 4\sqrt{3})^2 + y^2} = \sqrt{x^2 + y^2}$
- 1 mark for getting the Cartesian equation $x = 2\sqrt{3}$
- 1 mark for correctly sketching the line $|z - 4\sqrt{3}| = |z|$ going through the points $z_1 = 2\sqrt{3} + 2i$ and $z_2 = 2\sqrt{3} - 2i$

Question 2d.

Worked solution

The segment described is the area of a sector (formed by z_1, z_2 and the origin) less than the area of a triangle (formed with vertices at z_1, z_2 and the origin).

$$\text{Area} = \frac{\frac{\pi}{3}}{2\pi} \times \pi \times 4^2 - \frac{1}{2} \times 4 \times 2\sqrt{3}$$
$$= \frac{8\pi}{3} - 4\sqrt{3}$$
$$= \frac{8\pi - 12\sqrt{3}}{3}$$

Alternatively, use the area of a segment formula: $A = \frac{1}{2}r^2(\theta - \sin(\theta))$, with $r = 4$ and $\theta = \frac{\pi}{3}$.

So the area is

$$A = \frac{1}{2}4^2\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right)$$
$$= 8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$
$$= \frac{8\pi - 12\sqrt{3}}{3}$$

Mark allocation: 2 marks

- 1 mark for subtracting the area of a triangle from the area of a sector, or using the area of a segment formula
- 1 mark for the correct answer of $\frac{8\pi - 12\sqrt{3}}{3}$

Question 3a.

Worked solution

Let $z = x + iy$, then $|z + 1 + i| = \sqrt{2}|z - 1 - i|$.

$$\Rightarrow \quad |x + iy + 1 + i| = \sqrt{2}|x + iy - 1 - i|$$
$$\Rightarrow \sqrt{(x+1)^2 + (y+1)^2} = \sqrt{2}\sqrt{(x-1)^2 + (y-1)^2}$$
$$(x+1)^2 + (y+1)^2 = 2(x-1)^2 + 2(y-1)^2$$
$$x^2 + 2x + 1 + y^2 + 2y + 1 = 2x^2 - 4x + 2 + 2y^2 - 4y + 2$$
$$0 = x^2 - 6x + 1 + y^2 - 6y + 1$$

Completing the square:

$$0 = (x-3)^2 - 9 + 1 + (y-3)^2 - 9 + 1$$
$$\Rightarrow (x-3)^2 + (y-3)^2 = 16$$

Alternatively, squaring both sides gives $|z + 1 + i|^2 = 2|z - 1 - i|^2$.

Using the fact that $z\bar{z} = |z|^2$, gives

$$(z + (1+i))(\bar{z} + (1-i)) = 2(z - (1+i))(\bar{z} - (1-i))$$
$$z\bar{z} + (1-i)z + (1+i)\bar{z} + (1+i)(1-i) = 2\left[z\bar{z} - (1-i)z - (1+i)\bar{z} + (1+i)(1-i)\right]$$
$$(z - 3(1+i))(\bar{z} - 3(1-i)) - 18 + 2 = 0$$
$$(z - 3(1+i))(\bar{z} - 3(1-i)) = 16$$

Once again, using the fact that $z\bar{z} = |z|^2$, the equation can be rewritten as

$$|z - 3(1+i)|^2 = 16$$
$$|z - (3+3i)| = 4$$

From here, the Cartesian equation of the circle can be read as $(x-3)^2 + (y-3)^2 = 16$.

Alternatively, if you define z as $x + iy$ on a CAS, you can generate the Cartesian relation of the curve, $(x-3)^2 + (y-3)^2 = 16$, as shown below.

$x+y\cdot\boldsymbol{i}\to z$	$x+y\cdot\boldsymbol{i}$
$\lvert z+1+\boldsymbol{i}\rvert=\sqrt{2}\cdot\lvert z-1-\boldsymbol{i}\rvert$	$\sqrt{x^2+2\cdot x+y^2+2\cdot y+2}=\sqrt{2\cdot(x^2-2\cdot x+y^2-2\cdot y+2)}$
$\text{completeSquare}\left(x^2+2\cdot x+y^2+2\cdot y+2=2\cdot(x^2-2\cdot x+y^2-2\cdot y+2),x,y\right)$	$(x-3)^2-(y-3)^2=16$

Mark allocation: 3 marks

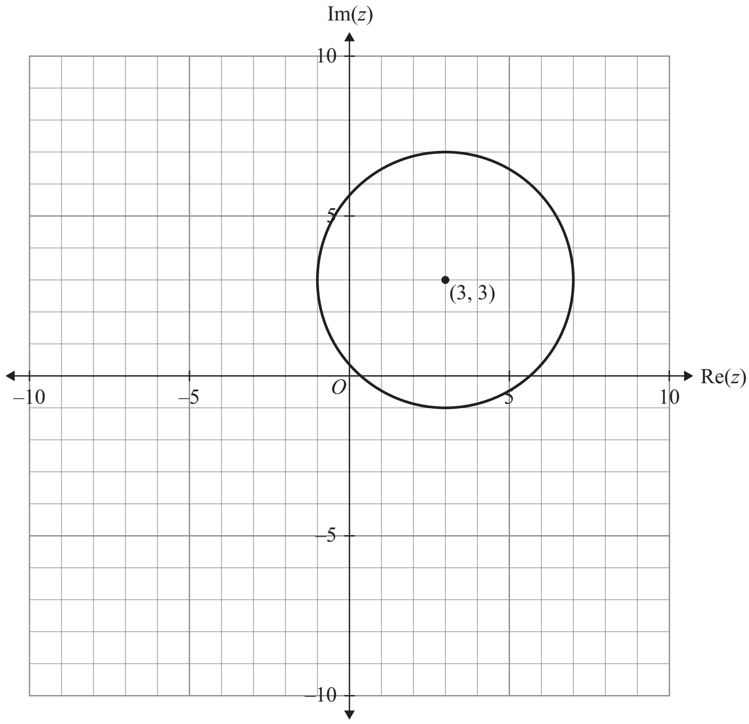
- 1 mark for showing that $\sqrt{(x+1)^2+(y+1)^2} = \sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$
- 1 mark for showing that $0 = x^2 - 6x + 1 + y^2 - 6y + 1$ or similar
- 1 mark for showing that $(x-3)^2 + (y-3)^2 = 16$

OR

- 1 mark for expressing $|z+1+i|^2 = 2|z-1-i|^2$ as $(z+(1+i))(\bar{z}+(1-i)) = 2(z-(1+i))(\bar{z}-(1-i))$
- 1 mark for showing that $|z-(3+3i)| = 4$
- 1 mark for showing that $(x-3)^2 + (y-3)^2 = 16$

Question 3b.

Worked solution



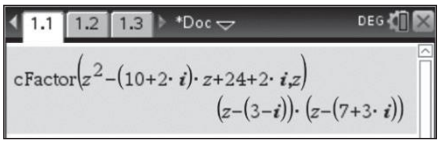
Mark allocation: 2 marks

- 1 mark for sketching a circle centred at (3, 3)
- 1 mark for giving the circle the correct radius

Question 3c.

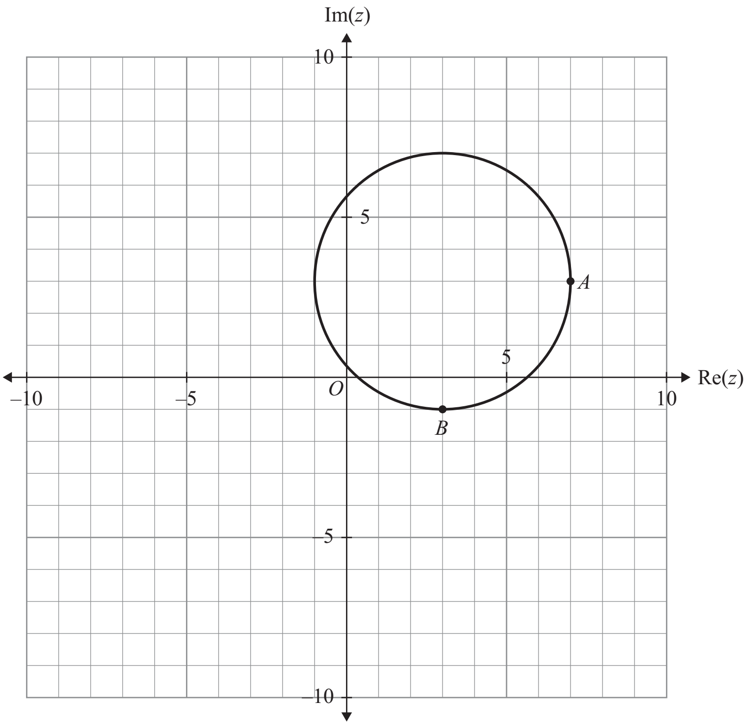
Worked solution

Factorising the equation using a CAS gives $(z - 3 + i)(z - 7 - 3i) = 0$.



So the solutions are $A = 7 + 3i$ and $B = 3 - i$.

The plotted points are



Mark allocation: 2 marks

- 1 mark for $A = 7 + 3i$ and $B = 3 - i$
- 1 mark for plotting and labelling points correctly

TIP

» When labelling points with coordinates on the complex plane, an 'i' should not be included on the vertical coordinate, as the labelling of the $\text{Im}(z)$ axes already implies that the coordinate is imaginary.

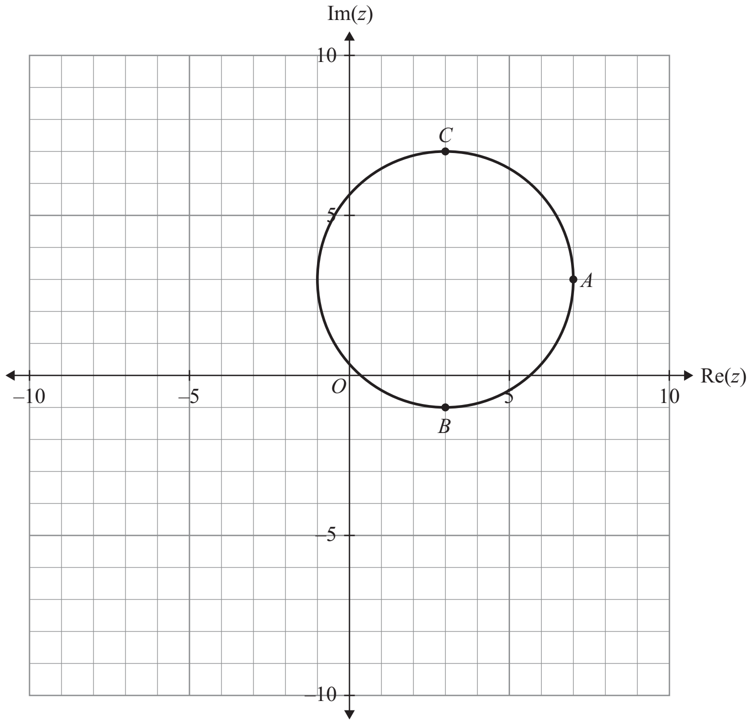
Question 3d.

Worked solution

$z_2 = \bar{z}_1 i$

$= (7 - 3i)i$

$= 3 + 7i$



Mark allocation: 2 marks

- 1 mark for showing that $z_2 = 3 + 7i$
- 1 mark for plotting and labelling the point correctly

Question 4a.

Worked solution

Divide $z^3 - 3iz^2 + 4z - 12i$ by $z - 3i$:

$$\begin{array}{r|l} & z^2 \quad\quad +4 \\ z - 3i & z^3 - 3iz^2 + 4z - 12i \\ & \underline{z^3 - 3iz^2} \\ & \quad\quad\quad 4z - 12i \\ & \quad\quad\quad \underline{4z - 12i} \end{array}$$

$\therefore z^3 - 3iz^2 + 4z - 12i = (z - 3i)(z^2 + 4)$

$= (z - 3i)(z^2 - (2i)^2) = (z - 3i)(z + 2i)(z - 2i)$

Therefore the other two factors are $z + 2i$ and $z - 2i$.

Mark allocation: 2 marks

- 1 mark for using long division or an equivalent method to derive $z^2 + 4$
- 1 mark for factorising $z^2 + 4$ to find the other two factors

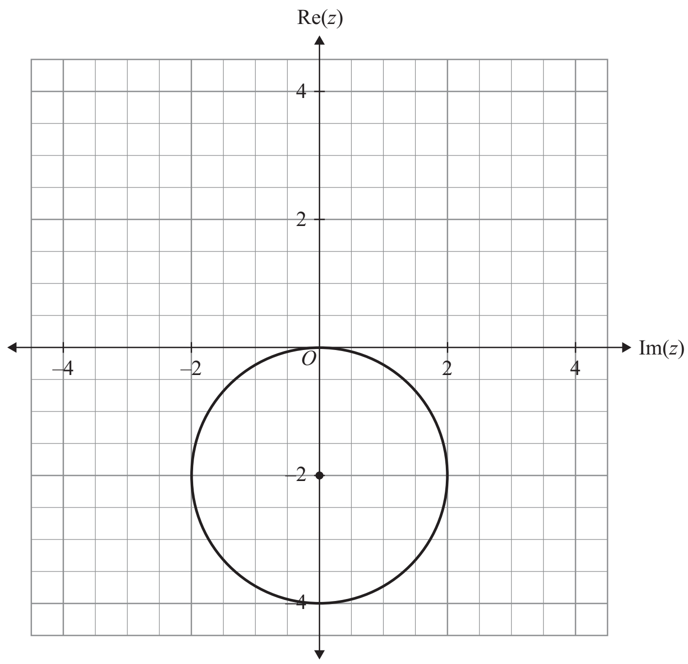
TIP

» It is very important that you know the difference between the terms 'show' and 'verify'. Because this is a 'show that' question, simply multiplying the three factors together and getting $z^3 - 3iz^2 + 4z - 12i$ will not suffice. Were it a 'verify that' question instead, the multiplication of the three factors would be enough.

Question 4b.

Worked solution

$|z + 2i| = 2$ represents a circle translated 2 units down with a radius of 2.



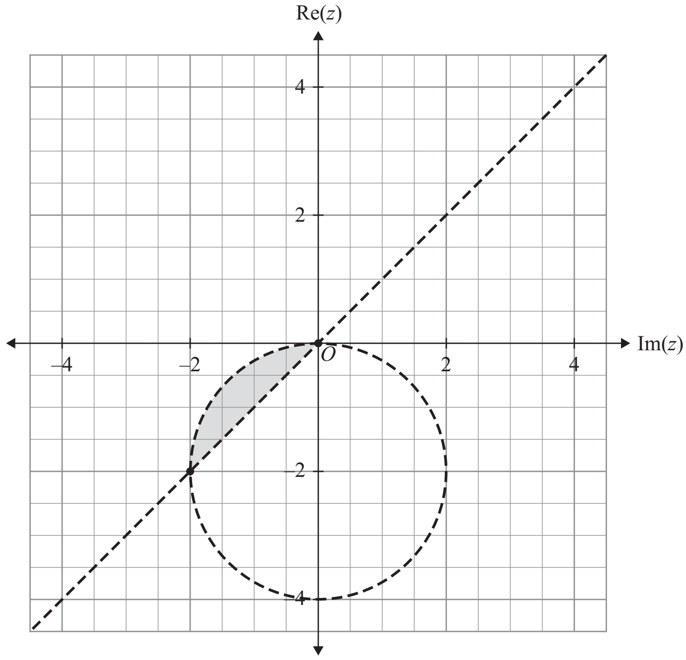
Mark allocation: 1 mark

- 1 mark for sketching the correct circle

Question 4c.

Worked solution

$\text{Im}(z) > \text{Re}(z)$ represents the region $y > x$.



Points (0, 0), (–2, –2) and the centre of the circle (0, –2) form a right-angled triangle. The area of the shaded region is a quarter of the area of the circle less than the area of the right-angled triangle.

$\text{Area} = \frac{1}{4} \times \pi(2^2) - \frac{1}{2}(2 \times 2) = \pi - 2$

Mark allocation: 2 marks

- 1 mark for showing the shaded area or describing the required area as the subtraction of the area of a triangle from the area of a quarter of a circle
- 1 mark for the correct area

Question 4d.i.

Worked solution

The boundary defined by $\text{Arg}(z - 3i) = \frac{\pi}{4}$ has a y-intercept at (0, 3) and a gradient of 1, therefore $y = x + 3$.

Mark allocation: 1 mark

- 1 mark for the correct equation

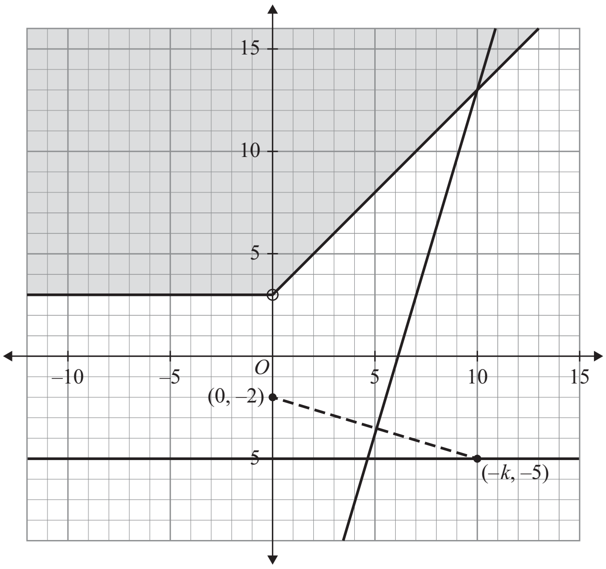
TIP

» **The principal argument of z is restricted: $\text{Arg}(z) \in (-\pi, \pi)$.**

Question 4d.ii.

Worked solution

$|z + 2i| = |z + k + 5i|$ represents the perpendicular bisector of the line between the points (0, –2) and (–k, –5).



This bisector intersects the shaded region when it has a gradient less than zero or greater than 1.

The gradient between points (0, –2) and (–k, –5) is $\frac{-5-(-2)}{-k-0} = \frac{3}{k}$.

For perpendicular lines, the product of the gradients of the two lines will be equal to –1.

The gradient perpendicular to the line between the points (0, –2) and (–k, –5) is $-\frac{k}{3}$.

This perpendicular gradient needs to be less than zero or greater than 1.

$$-\frac{k}{3} < 0 \cup -\frac{k}{3} > 1$$

$$k > 0 \cup k < -3$$

$$\therefore k < -3 \cup k > 0$$

Mark allocation: 2 marks

- 1 mark for calculating the required gradients of the line or alternative correct working
- 1 mark for the correct values of k

Question 5a.i.

Worked solution

Substitute $w = \sqrt{3} + 3i$ into $P(z)$:

$$P(w) = (\sqrt{3} + 3i)^3 - 2\sqrt{3}(\sqrt{3} + 3i)^2 + 12(\sqrt{3} + 3i)$$

$$= -24\sqrt{3} + 12\sqrt{3} - 36i + 12\sqrt{3} + 36i$$

$$= 0$$

Therefore $w = \sqrt{3} + 3i$ is a solution to $P(z) = 0$.

Alternatively, assume that $w = \sqrt{3} + 3i$ is a solution of $P(z) = 0$. It follows that $z - \sqrt{3} - 3i$ must also be a factor. Now use polynomial division with the new factor as the divisor:

$$\begin{array}{r l}
 & z^2 + (-\sqrt{3} + 3i)z \\
z - \sqrt{3} - 3i \,\big) & z^3 - 2\sqrt{3}z^2 + 12z \\
 & \underline{z^3 + (-\sqrt{3} - 3i)z^2} \\
 & 0z^3 + (-\sqrt{3} + 3i)z^2 + 12z \\
 & \underline{(-\sqrt{3} + 3i)z^2 + 12z} \\
 & 0z^2 + 0z
\end{array}$$

As the remainder is zero, it has been shown that $z - \sqrt{3} - 3i$ is a factor, and therefore that $w = \sqrt{3} + 3i$ is a solution of $P(z) = 0$.

Alternatively, taking out a common factor of z from $P(z) = z^3 - 2\sqrt{3}z^2 + 12z$ gives

$z(z^2 - 2\sqrt{3}z + 12)$.

Completing the square on the bracketed term:

$\Rightarrow z\left((z - \sqrt{3})^2 - 3 + 12\right)$

$= z\left((z - \sqrt{3})^2 + 9\right)$

$= z\left((z - \sqrt{3})^2 - 9i^2\right)$

Using the difference of two squares gives

$z(z - \sqrt{3} + 3i)(z - \sqrt{3} - 3i)$

From this it can be seen that $z - \sqrt{3} - 3i$ is a factor, which means $z = \sqrt{3} + 3i$ is a solution. Therefore it has been shown that $w = \sqrt{3} + 3i$ is a solution of $P(z) = 0$.

Mark allocation: 1 mark

- 1 mark for showing that $w = \sqrt{3} + 3i$ is a solution to $P(z) = 0$, using substitution, polynomial division or factorisation techniques

Question 5a.ii.

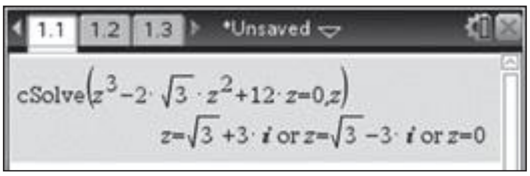
Worked solution

$P(z) = z^3 - 2\sqrt{3}z^2 + 12z = 0$ can also be expressed as $P(z) = z(z^2 - 2\sqrt{3}z + 12) = 0$.

Therefore $z = 0$ is a solution.

Using the conjugate root theorem, $z = \sqrt{3} - 3i$ is the other solution.

Solving on a CAS could also be used for this question.

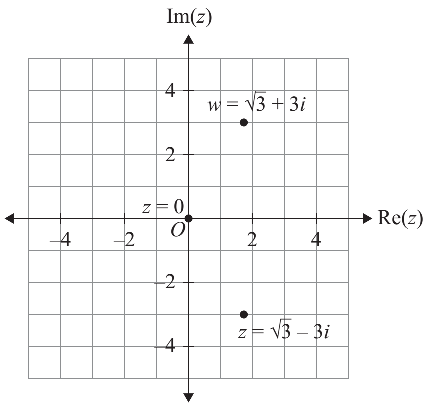


Mark allocation: 2 marks

- 1 mark for calculating $z = 0$
- 1 mark for calculating $z = \sqrt{3} - 3i$

Question 5b.

Worked solution



Mark allocation: 2 marks

- 1 mark for correctly plotting and labelling any two solutions
- 1 mark for correctly plotting and labelling the third solution correctly

Question 5c.i.

Worked solution

$$\overline{w}i^2 = \overline{(\sqrt{3} + 3i)} \times i^2$$

$$= (\sqrt{3} - 3i) \times -1$$

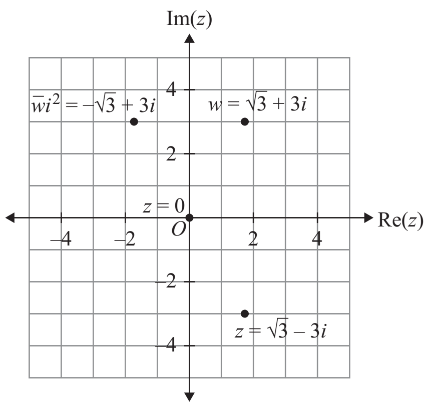
$$= -\sqrt{3} + 3i$$

Mark allocation: 1 mark

- 1 mark for calculating $\overline{w}i^2 = -\sqrt{3} + 3i$

Question 5c.ii.

Worked solution



Mark allocation: 1 mark

- 1 mark for correctly plotting and labelling $\overline{w}i^2$ on the Argand diagram

Question 5c.iii.

Worked solution

Looking at the graph, it can be seen that $\overline{w}i^2$ is a reflection of w in the imaginary axis.

OR

It can be calculated that point w makes an angle of $\frac{\pi}{3}$ with the real axis, hence $\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) = \frac{\pi}{3}$.

It can be calculated that $\overline{w}i^2$ makes an angle of $\frac{2\pi}{3}$ with the real axis, hence $\tan^{-1}\left(\frac{3}{-\sqrt{3}}\right) = \frac{2\pi}{3}$.
Therefore $\overline{w}i^2$ is a rotation of $\frac{\pi}{3}$ anti-clockwise about the origin from w.

Mark allocation: 1 mark

- 1 mark for explaining that $\overline{w}i^2$ is a reflection of w in the imaginary axis

OR

- 1 mark for explaining that $\overline{w}i^2$ is a rotation of $\frac{\pi}{3}$ anti-clockwise about the origin from w

Question 5d.

Worked solution

The Cartesian equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$, which can be stated as $(x - a)^2 + (y - b)^2 = r^2$ in the context of this question.

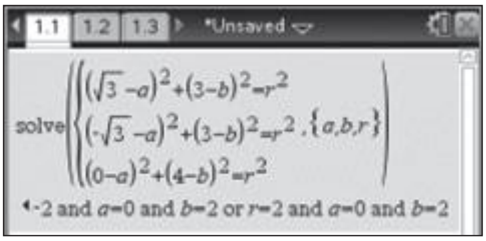
Given that $w = \sqrt{3} + 3i$, $\overline{w}i^2 = -\sqrt{3} + 3i$ and $4i$ are on the boundary of the circle, we can set up the following simultaneous equations:

$(\sqrt{3} - a)^2 + (3 - b)^2 = r^2$

$(-\sqrt{3} - a)^2 + (3 - b)^2 = r^2$

$(0 - a)^2 + (4 - b)^2 = r^2$

Solving these equations gives $a = 0$, $b = 2$, $r = 2$.



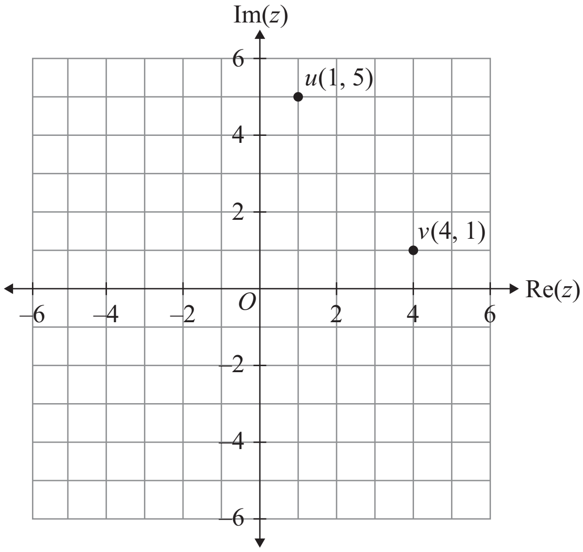
Therefore $c = 2i$ and $r = 2$.

Mark allocation: 3 marks

- 1 mark for setting up the simultaneous equations
- 1 mark for calculating $c = 2i$
- 1 mark for calculating $r = 2$

Question 6a.

Worked solution



Mark allocation: 1 mark

- 1 mark for both points labelled and in the correct locations, as shown

> **TIP**
>
> » Consider the real component to be the x value and the imaginary component to be the y value.

Question 6b.

Worked solution

$|u - v|$ represents the distance between the two points, which is 5 units. Use the Pythagorean distance formula or recognise that these points lie on a 3-4-5 triangle. Hence:

$\sqrt{(5-1)^2 + (1-4)^2}$

$= \sqrt{4^2 + 3^2}$

$= 5$ units

Mark allocation: 1 mark

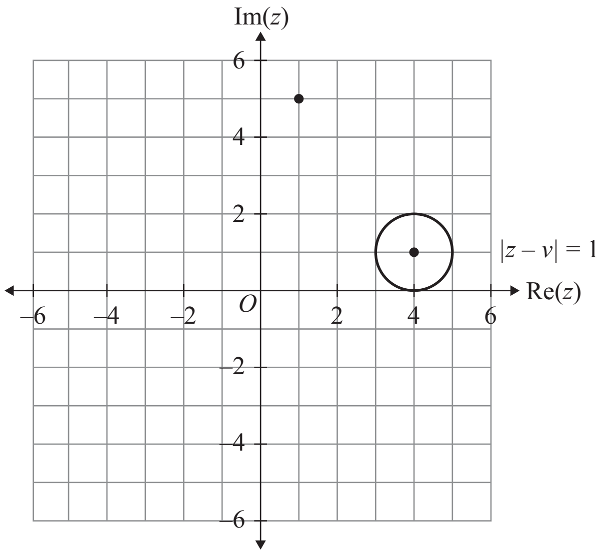
- 1 mark for the answer of 5 units

> **TIP**
>
> » When you see questions with the vertical bars of the modulus operator, you should be thinking about the distance between the two points on either side of the negative sign.

Question 6c.i.

Worked solution

The equation represents all points on the complex plane (Argand diagram) that are 1 unit away from point v. These points are represented by the circle below.



Mark allocation: 2 marks

- 1 mark for any circle drawn
- 1 mark for drawing a circle with its centre at (4, 1) and with a radius of 1

TIP

» Remember that z represents any complex number on the plane that satisfies the condition given by the equation. Compare this with v, which represents a defined, fixed point.

Question 6c.ii.

Worked solution

$(-i)^3 = (-i) \times (-i) \times (-i) = (-1) \times (-i) = i$

Mark allocation: 1 mark

- 1 mark for the answer of i

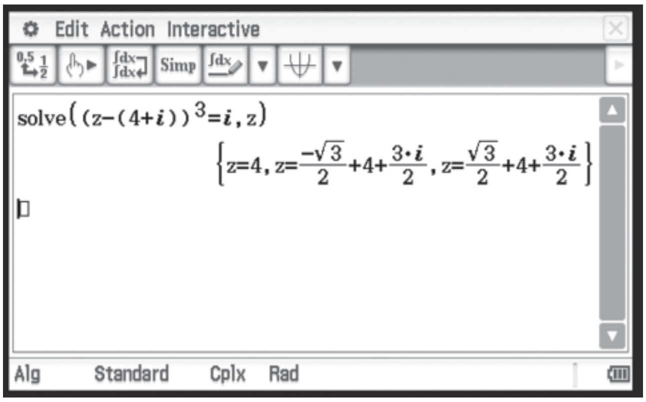
TIP

» If using a CAS for this calculation, ensure that you have set it to complex mode.

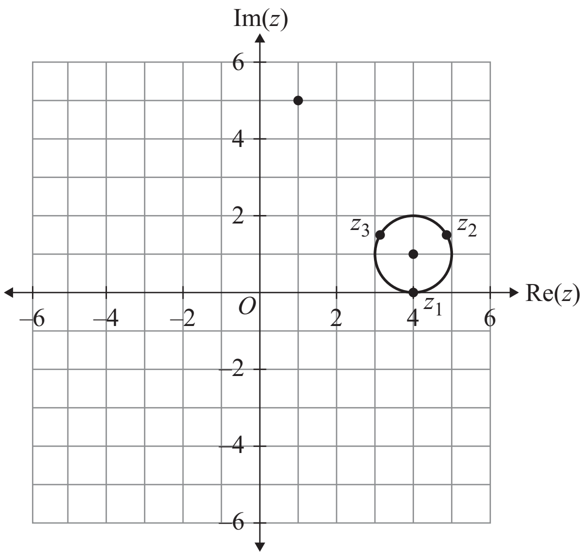
Question 6c.iii.

Worked solution

Method 1: Use a CAS to solve for z, then plot each solution as a point on the Argand diagram.



Method 2: Use the answer from **part c.i.** to identify that one solution will lie at $v - i = 4$.
Then use de Moivre's theorem to identify that the remaining solutions will lie on the circle with radius 1 and be separated by an angle of 120°.



Mark allocation: 2 marks

- 1 mark for the solution at (4, 0)
- 1 mark for other two solutions in the correct locations

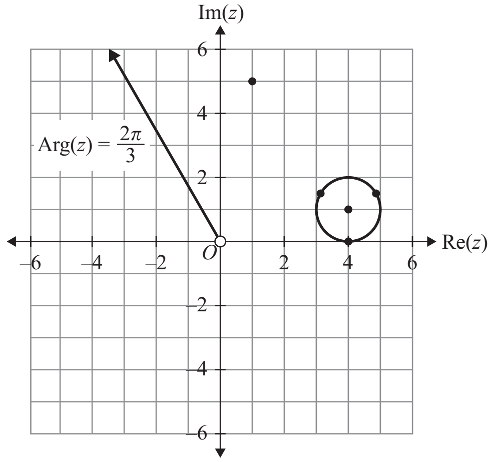
TIP

» Be on the look out for questions like these where you can use the answer to a previous part to help find the solution.

Question 6d.

Worked solution

Rays are straight lines that emanate from a given point. Their orientation is measured anti-clockwise from the real axis. Hence the answer is a straight line with an open end point at the origin that makes an angle of 120° with the real axis.



Mark allocation: 2 marks

- 1 mark for any line making the correct angle with the real axis
- 1 mark for an open end point at the origin and the line in the correct position

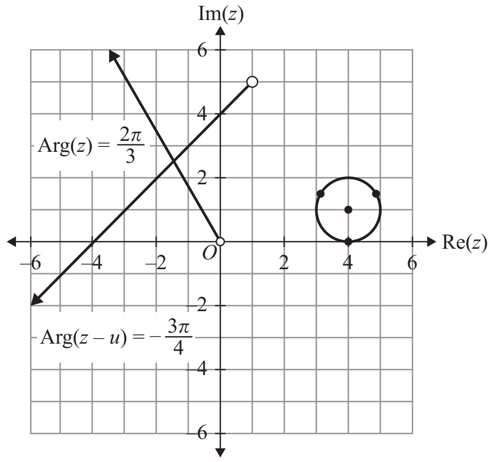
TIP

» Think of $\text{Arg}(z)$ as being the same as $\text{Arg}(z - 0)$ to help you to remember that this ray emanates from the origin.

Question 6e.

Worked solution

First, sketch the rays on the diagram. Then identify the triangle of interest. Finally, calculate its area.



Find the equations of the rays in Cartesian form and then the point of intersection:

$\text{Arg}(z) = \frac{2\pi}{3} \rightarrow y = -\sqrt{3}x$

$\text{Arg}(z - u) = \frac{-3\pi}{4} \rightarrow y = x + 4$

$-\sqrt{3}x = x + 4$

$x = 2 - 2\sqrt{3}$

$\therefore \text{Area} = \frac{1}{2} \times 4 \times \left(-2 + 2\sqrt{3}\right)$

$\text{Area} = 4\sqrt{3} - 4$

Mark allocation: 3 marks

- 1 mark for sketching the ray from μ or for finding its Cartesian equations
- 1 mark for finding the point of intersection of the rays
- 1 mark for the correct answer

TIP

» **When finding Cartesian equations of rays, it can be useful to remember that the gradient is $m = \tan(\theta)$.**

Area of Study 4 Calculus

EXAM 1

Question 1

Worked solution

$(x+1)e^{v} - x^2 - 2x - 3 = 0$

When $x = 1$:

$$\begin{aligned} 2e^{v} - 6 &= 0 \\ e^{v} &= 3 \\ v &= \log_e 3 \end{aligned}$$

Differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}\left((x+1)e^{v} - x^2 - 2x - 3\right) = \frac{d}{dx}(0)$$

$$e^{v} + (x+1)e^{v}\frac{dv}{dx} - 2x - 2 = 0$$

$$(x+1)e^{v}\frac{dv}{dx} = 2x + 2 - e^{v}$$

$$\frac{dv}{dx} = \frac{2x + 2 - e^{v}}{(x+1)e^{v}}$$

Substituting $x = 1$ and $v = \log_e 3$ gives

$\Rightarrow \frac{dv}{dx} = \frac{2+2-3}{6} = \frac{1}{6}$

Acceleration $= v\frac{dv}{dx}$

$\therefore$ The acceleration when $x = 1$ is $\frac{1}{6}\log_e 3$ m s^{-2}.

Alternative method for finding $\frac{dv}{dx}$:

$(x+1)e^{v} - x^2 - 2x - 3 = 0$

$$e^{v} = \frac{x^2 + 2x + 3}{x + 1}$$

$$v = \log_e\left(\frac{x^2 + 2x + 3}{x + 1}\right)$$

Let $u = \frac{x^2 + 2x + 3}{x + 1}$.

$$\begin{aligned} \frac{du}{dx} &= \frac{(2x+2)(x+1) - (x^2 + 2x + 3)}{(x+1)^2} \\ &= \frac{x^2 + 2x - 1}{(x+1)^2} \end{aligned}$$

$v = \log_e u$

$\Rightarrow \frac{dv}{du} = \frac{1}{u} = \frac{x+1}{x^2 + 2x + 3}$

$$\begin{aligned} \therefore \frac{dv}{dx} &= \frac{x^2 + 2x - 1}{(x+1)^2} \times \frac{x+1}{x^2 + 2x + 3} \\ &= \frac{x^2 + 2x - 1}{(x+1)(x^2 + 2x + 3)} \end{aligned}$$

When $x = 1$:

$\frac{dv}{dx} = \frac{2}{2 \times 6} = \frac{1}{6}$ and $v = \log_e 3$

Acceleration $= v\frac{dv}{dx}$

$\therefore$ The acceleration when $x = 1$ is $\frac{1}{6}\log_e 3$ m s^{-2}.

Mark allocation: 4 marks

- 1 mark for the correct value of v when $x = 1$
- 1 mark for correctly differentiating both sides of the equation with respect to x
- 1 mark for the correct evaluation of $\frac{dv}{dx}$
- 1 mark for the correct answer

TIP

» It is easier to obtain $v\frac{dv}{dx}$ using implicit differentiation rather than explicitly expressing v as a function of x and then differentiating.

Question 2

Worked solution

$$\int_0^{\frac{\pi}{3}} (\tan^2 x + \tan^4 x)\,dx = \int_0^{\frac{\pi}{3}} \tan^2 x(1 + \tan^2 x)\,dx$$

$$= \int_0^{\frac{\pi}{3}} \tan^2 x(\sec^2 x)\,dx$$

Let $u = \tan(x)$:

$\Rightarrow \frac{du}{dx} = \sec^2 x$

$x = \frac{\pi}{3} \Rightarrow u = \sqrt{3}$

$x = 0 \Rightarrow u = 0$

$$\therefore \int_0^{\frac{\pi}{3}} (\tan^2 x + \tan^4 x)\,dx = \int_0^{\sqrt{3}} u^2\,du$$

$$= \left[\frac{1}{3}u^3\right]_0^{\sqrt{3}}$$

$$= \frac{1}{3} \times \sqrt{3}^3 - 0$$

$$= \sqrt{3}$$

Mark allocation: 3 marks

- 1 mark for correctly setting up the integrand for a substitution of $u = \tan(x)$
- 1 mark for obtaining a correct integrand with respect to u
- 1 mark for the correct answer

TIP

» Use the substitution $u = \tan(x)$, as its derivative $\sec^2 x$ is a factor of the integrand.

Question 3

Worked solution

$\frac{dy}{dx} = 3y\sqrt{x}$

Use the separation of variables technique to antidifferentiate.

$\Rightarrow \int \frac{1}{y}\,dy = \int 3\sqrt{x}\,dx$

$\log_e|y| = 2x^{\frac{3}{2}} + c$

$y = e$ when $x = 1$, so

$\Rightarrow \log_e e = 1 = 2 + c$

$\Rightarrow c = -1$

$$\begin{aligned}\log_e|y| &= 2x^{\frac{3}{2}} - 1\\ |y| &= e^{2x^{\frac{3}{2}}-1}\\ y &= \pm e^{2x\sqrt{x}-1}\end{aligned}$$

$\therefore\ y = e^{2x\sqrt{x}-1}$, as $y > 0$.

Mark allocation: 3 marks

- 1 mark for correctly integrating using the separation of variables
- 1 mark for correctly evaluating the constant of antidifferentiation
- 1 mark for the correct answer

Question 4

Worked solution

$\frac{dx}{dt} = 2\cos(2t)$

$\frac{dy}{dt} = 2\sin(2t)$

$$\begin{aligned}\text{Length} &= \int_0^{\frac{\pi}{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\\ &= \int_0^{\frac{\pi}{3}} \sqrt{(2\cos(2t))^2 + (2\sin(2t))^2}\,dt\\ &= \int_0^{\frac{\pi}{3}} \sqrt{4\cos^2(2t) + 4\sin^2(2t)}\,dt\\ &= \int_0^{\frac{\pi}{3}} \sqrt{4}\,dt = \int_0^{\frac{\pi}{3}} 2\,dt\\ &= [2t]_0^{\frac{\pi}{3}}\\ &= \frac{2\pi}{3} - 0\\ &= \frac{2\pi}{3}\text{ units}\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for correctly setting up the integrand representing the curve length
- 1 mark for the correct answer

Question 5a.

Worked solution

$\frac{d}{dx}\left(x \cdot \arccos(bx) - \frac{\sqrt{1-b^2x^2}}{b}\right)$

$= \arccos(bx) - x \cdot \frac{b}{\sqrt{1-b^2x^2}} - \frac{1}{2} \cdot \frac{1}{b} \cdot (-2b^2x)(1-b^2x^2)^{-\frac{1}{2}}$

$= \arccos(bx) - \frac{bx}{\sqrt{1-b^2x^2}} + \frac{bx}{\sqrt{1-b^2x^2}}$

$= \arccos(bx)$

Mark allocation: 1 mark

- 1 mark for correctly differentiating using both the product rule and the chain rule

Question 5b.

Worked solution

$y = \arcsin(2x)$ and $y = \arccos(x)$ intersect at $x = \frac{1}{\sqrt{a^2+b^2}}$, where $a = 2$ and $b = 1$, giving $x = \frac{1}{\sqrt{2^2+1^2}} = \frac{1}{\sqrt{5}}$.

$\Rightarrow$ Area required $= \int_0^{\frac{1}{\sqrt{5}}} [\arccos(x) - \arcsin(2x)]dx$

Area $= \left[x \cdot \arccos(x) - \sqrt{1-x^2} - \left(x \cdot \arcsin(2x) + \frac{\sqrt{1-4x^2}}{2}\right)\right]_0^{\frac{1}{\sqrt{5}}}$

Area $= \left[x \cdot \arccos(x) - \sqrt{1-x^2} - x \cdot \arcsin(2x) - \frac{\sqrt{1-4x^2}}{2}\right]_0^{\frac{1}{\sqrt{5}}}$

Area $= \left(\frac{1}{\sqrt{5}}\arccos\left(\frac{1}{\sqrt{5}}\right) - \sqrt{\frac{4}{5}} - \frac{1}{\sqrt{5}}\arcsin\left(\frac{2}{\sqrt{5}}\right) - \frac{1}{2} \times \sqrt{\frac{1}{5}}\right) - \left(-1 - \frac{1}{2}\right)$

But $\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{5}{5} = 1$

$\therefore \arccos\left(\frac{1}{\sqrt{5}}\right) = \arcsin\left(\frac{2}{\sqrt{5}}\right)$

$\Rightarrow$ Area $= \left(\frac{1}{\sqrt{5}}\arccos\left(\frac{1}{\sqrt{5}}\right) - \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}\arccos\left(\frac{1}{\sqrt{5}}\right) - \frac{1}{2\sqrt{5}}\right) + \frac{3}{2}$

$\therefore$ Area $= \frac{3}{2} - \frac{5}{2\sqrt{5}} = \frac{3-\sqrt{5}}{2}$ square units

Mark allocation: 3 marks

- 1 mark for setting up the correct integrand that represents the required area
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

Question 6a.

Worked solution

LHS = $\log_e(1) + 1^2 - 1 = 0 + 1 - 1 = 0$ = RHS

Mark allocation: 1 mark

- 1 mark for the correct evaluation

Question 6b.

Worked solution

Using log laws, the relation can be written as $\log_e x - \log_e y + x^2 - y = 0$.

Now use implicit differentiation:

$$\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} + 2x - \frac{dy}{dx} = 0$$

At point (1, 1), we have

$$1 - \frac{dy}{dx} + 2 - \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}$$

Since the tangent passes through (1, 1), the equation of the tangent (using the formula $y - y_1 = m(x - x_1)$) is

$$y - 1 = \frac{3}{2}(x - 1)$$
$$2y - 2 = 3x - 3$$

and so $3x - 2y - 1 = 0$.

Mark allocation: 3 marks

- 1 mark for correctly differentiating both sides of the equation with respect to x
- 1 mark for the correct evaluation of $\frac{dy}{dx}$
- 1 mark for the correct equation

Question 7

Worked solution

$$\int_0^{\sqrt{e^2-1}} \frac{4x\log_e(x^2+1)}{x^2+1}\,dx$$

$$= 2\int_0^{\sqrt{e^2-1}} \frac{2x}{x^2+1}\log_e(x^2+1)\,dx$$

Let $u = \log_e(x^2 + 1)$, giving

$$\frac{du}{dx} = \frac{2x}{x^2+1}$$

$x = \sqrt{e^2 - 1} \Rightarrow u = \log_e(e^2) = 2$

$x = 0 \Rightarrow u = \log_e(1) = 0$

$$= 2\int_0^{\sqrt{e^2-1}} \frac{2x}{x^2+1}\log_e(x^2+1)\,dx$$

$= 2\int_0^2 u \frac{du}{dx}\, dx = \int_0^2 2u\, du$

$= [u^2]_0^2$

$= 4 - 0 = 4$

Mark allocation: 3 marks

- 1 mark for correctly setting up the integrand for a substitution of $u = \log_e(x^2 + 1)$
- 1 mark for obtaining a correct integrand with respect to u
- 1 mark for the correct answer

TIP

» **Use the substitution $u = \log_e(x^2 + 1)$ since its derivative, $\frac{2x}{x^2+1}$, is a factor of the integrand.**

Question 8

Worked solution

$\frac{dy}{dx} = \frac{e^{0.5x}}{3\sqrt{y}}$

$\Rightarrow \int 3y^{\frac{1}{2}}\, dy = \int e^{0.5x}\, dx$

$2y^{\frac{3}{2}} = 2e^{0.5x} + c$

If $y = 1$ when $x = 0$, then

$\Rightarrow 2 = 2 + c$

$\Rightarrow c = 0$

$2y^{\frac{3}{2}} = 2e^{0.5x}$

$y^{\frac{3}{2}} = e^{0.5x}$

$y = e^{\frac{x}{3}}$

Mark allocation: 3 marks

- 1 mark for correctly integrating using the separation of variables
- 1 mark for correctly evaluating the constant of antidifferentiation
- 1 mark for the correct answer

TIP

» **Use the separation of variables technique to antidifferentiate.**

Question 9

Worked solution

Let $s_L(t)$ = the downward displacement of the lift floor t seconds after the cup is dropped.

Let $s_C(t)$ = the downward displacement of the cup t seconds after the cup is dropped.

$$\Rightarrow h = s_C\left(\frac{1}{2}\right) - s_L\left(\frac{1}{2}\right)$$

Using $s = ut + \frac{1}{2}at^2$:

lift floor: $u = v,\ a = 0$

$$\Rightarrow \quad s_L(t) = vt$$

$$\Rightarrow s_L\left(\frac{1}{2}\right) = \frac{v}{2}$$

coffee cup: $u = v,\ a = g$

$$\Rightarrow \quad s_C(t) = vt + \frac{gt^2}{2}$$

$$\Rightarrow s_C\left(\frac{1}{2}\right) = \frac{v}{2} + \frac{g}{8}$$

Solve for h:

$$h = s_C\left(\frac{1}{2}\right) - s_L\left(\frac{1}{2}\right) = \frac{g}{8}$$

or $h = \dfrac{9.8}{8} = 1.225$ metres

Mark allocation: 4 marks

- 1 mark for correctly expressing h as $s_C\left(\frac{1}{2}\right) - s_L\left(\frac{1}{2}\right)$
- 1 mark for the correct evaluation of $s_L\left(\frac{1}{2}\right)$
- 1 mark for the correct evaluation of $s_C\left(\frac{1}{2}\right)$
- 1 mark for the correct answer

TIP

» There are several ways to answer this question. The simplest method is to realise that since the initial speed downwards of the coffee cup equals the (constant) speed of the lift, we can model this situation as an object dropped from rest at a height h metres above the ground.

Question 10

Worked solution

Using implicit differentiation gives

$$2y^2 + 4xy\frac{dy}{dx} - \frac{1}{\sqrt{16-x^2}} - \frac{\pi y}{3} - \frac{\pi x}{3}\frac{dy}{dx} = 0$$

Substitute the coordinates of point $\left(2, \frac{1}{2}\right)$ into the differentiated equation to find the gradient.

$$\frac{1}{2} + 4\frac{dy}{dx} - \frac{1}{2\sqrt{3}} - \frac{\pi}{6} - \frac{2\pi}{3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(4 - \frac{2\pi}{3}\right) = \frac{1}{2\sqrt{3}} + \frac{\pi}{6} - \frac{1}{2}$$

$$\frac{dy}{dx}\left(\frac{12-2\pi}{3}\right) = \frac{\sqrt{3}+\pi-3}{6}$$

$$\frac{dy}{dx} = \frac{\pi - 3 + \sqrt{3}}{2(12-2\pi)} = \frac{\pi - 3 + \sqrt{3}}{24 - 4\pi}$$

Mark allocation: 3 marks

- 1 mark for correctly differentiating the equation on both sides, using implicit differentiation
- 1 mark for substituting the coordinates of the point into the equation
- 1 mark for giving the gradient in the correct form

Question 11a.

Worked solution

The inflow of salt, in kg s^{-1}, is $0 \times 5 = 0$.

The outflow of salt, in kg s^{-1}, is $\frac{x}{200 + 5t - 10t} \times 10 = \frac{10x}{200 - 5t} = \frac{2x}{40 - t}$.

$$\frac{dx}{dt} = \text{inflow} - \text{outflow} = 0 - \frac{2x}{40-t}$$

$$\frac{dx}{dt} = -\frac{2x}{40-t}$$

Mark allocation: 1 mark

- 1 mark for showing appropriate working

Question 11b.

Worked solution

From the result given in **part a.**, $\frac{dx}{dt} + \frac{2x}{40-t} = 0$.

Separate the variables:

$$\frac{dx}{x} = \frac{2\,dt}{t-40}$$

Integrate both sides of the equation:

$$\int\frac{dx}{x} = \int\frac{2\,dt}{t-40}$$

$$\ln|x| + c = 2\ln|t-40|$$

$$\ln(x) + c = \ln(t-40)^2$$

where c is a real constant and $x > 0$.

Apply the initial conditions to find c:

$$\ln(10) + c = \ln(-40)^2$$
$$c = \ln(1600) - \ln(10) = \ln(160)$$

Solve for x:

$$\ln(x) + \ln(160) = \ln(t - 40)^2$$
$$\ln(x) = \ln(t - 40)^2 - \ln(160)$$
$$\ln(x) = \ln\left(\frac{(t-40)^2}{160}\right)$$
$$x = \frac{(t-40)^2}{160}$$

Mark allocation: 3 marks

- 1 mark for separating the variables and correctly integrating both sides of the differential equation
- 1 mark for applying the initial conditions to find c
- 1 mark for providing the correct answer of $x = \frac{(t-40)^2}{160}$

Question 12a.

Worked solution

Express the acceleration as $v\frac{dv}{dx}$:

$$v\frac{dv}{dx} = v^3 + 16v$$

Separate the variables and integrate both sides:

$$\int \frac{v \cdot dv}{v^3 + 16v} = \int dx$$
$$\int \frac{dv}{v^2 + 16} = \int dx$$

$\Rightarrow \frac{1}{4}\arctan\left(\frac{v}{4}\right) = x + c$, where c is a real constant.

Apply the initial conditions to find c:

$$\frac{1}{4}\arctan\left(\frac{0}{4}\right) = \frac{\pi}{16} + c$$
$$\Rightarrow c = -\frac{\pi}{16}$$
$$\Rightarrow \frac{1}{4}\arctan\left(\frac{v}{4}\right) = x - \frac{\pi}{16}$$

Rearrange to get the velocity:

$$v = 4\tan\left(4x - \frac{\pi}{4}\right)$$

Mark allocation: 3 marks

- 1 mark for selecting an appropriate acceleration formula, such as $a = v\frac{dv}{dx}$, to equate the acceleration to, and for appropriate integration techniques
- 1 mark for using the initial conditions to find c
- 1 mark for finding the equation of the velocity in terms of x

TIP

» When given acceleration as a function of velocity and the initial conditions are given in terms of displacement and velocity, express the acceleration in the form $a = v\frac{dv}{dx}$.

Question 12b.

Worked solution

Substitute $x = \frac{\pi}{8}$ into this equation:

$$\begin{aligned} v &= 4\tan\left(4 \times \frac{\pi}{8} - \frac{\pi}{4}\right) \\ &= 4\tan\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\ &= 4 \text{ m s}^{-1} \end{aligned}$$

Mark allocation: 1 mark

- 1 mark for correctly calculating the velocity

Question 13

Worked solution

The translated function is $g(x) = f\left(x + \frac{\pi}{2}\right) = \arctan\left(2\left(x + \frac{\pi}{2}\right) - \pi\right) = \arctan(2x)$.

The volume when rotated about the y-axis is given by $V = \pi \int_{y_1}^{y_2} x^2 \, dy$.

Rearranging the function to make x the subject gives $x = \frac{1}{2}\tan(y)$.

Therefore the volume is

$$\begin{aligned} V &= \int_0^{\frac{\pi}{3}} \frac{1}{4}\tan^2(y) \, dy \\ &= \frac{\pi}{4}\int_0^{\frac{\pi}{3}} \tan^2(y) \, dy = \int_0^{\frac{\pi}{3}} \sec^2(y) - 1 \, dy \\ &= \frac{\pi}{4}\left[\tan(y) - y\right]_0^{\frac{\pi}{3}} \\ &= \frac{\pi}{4}\left[\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3} - \tan(0) + 0\right] \\ &= \frac{\pi}{4}\left(\sqrt{3} - \frac{\pi}{3}\right) \\ &= \frac{\pi}{12}(3\sqrt{3} - \pi) \end{aligned}$$

Mark allocation: 4 marks

- 1 mark for defining $g(x)$
- 1 mark for deriving the equation $V = \pi \int_0^{\frac{\pi}{3}} \frac{1}{4}\tan^2(y) \, dy$
- 1 mark for correctly integrating the integrand
- 1 mark for deriving the equation $V = \frac{\pi}{12}(3\sqrt{3} - \pi)$

TIP

» Remember to use the identity $1 + \tan^2(\theta) = \sec^2(\theta)$ to integrate the volume.

Question 14a.

Worked solution

Use the formula $\int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$ for the length of a curve.

The parametric equations for the position vector are

$x(t) = 2t^6 + 3$ and $y(t) = \frac{6}{7}t^7 + 1$

$\Rightarrow x'(t) = 12t^5$ and $y'(t) = 6t^6$

Substituting these values into the formula for the length of a curve and simplifying gives the required result:

$$\int_0^2 \sqrt{(12t^5)^2 + (6t^6)^2}\,dt = \int_0^2 \sqrt{144t^{10} + 36t^{12}}\,dt$$

$$= \int_0^2 6t^5\sqrt{4 + t^2}\,dt$$

$$= \int_0^2 6t^5\sqrt{t^2 + 4}\,dt$$

Mark allocation: 1 mark

- 1 mark for showing the appropriate working to achieve the required result

Question 14b.

Worked solution

Begin with $\int_0^2 6t^5\sqrt{t^2+4}\,dt$.

Now let $u = t^2 + 4$.

When $t = 0$ then $u = 4$, and when $t = 2$, $u = 8$.

Therefore $\frac{du}{dt} = 2t \Rightarrow dt = \frac{du}{2t}$.

So $\int_0^2 6t^5\sqrt{t^2+4}\,dt = \int_4^8 6t^5\sqrt{u}\ \frac{du}{2t} = \int_4^8 3t^4\sqrt{u}\,du$.

Since $u = t^2 + 4 \Rightarrow t^2 = u - 4$.

$$\int_4^8 3t^4\sqrt{u}\ du = 3\int_4^8 (u-4)^2 \times u^{\frac{1}{2}}du$$

$$= 3\int_4^8 (u^2 - 8u + 16) \times u^{\frac{1}{2}}du$$

$$= 3\int_4^8 u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}du$$

$$= 3\left[\frac{2}{7}u^{\frac{7}{2}} - \frac{16}{5}u^{\frac{5}{2}} + \frac{32}{3}u^{\frac{3}{2}}\right]_4^8$$

This is as required.

So it follows that $a = \frac{2}{7}$, $b = \frac{16}{5}$, $c = \frac{32}{3}$.

Alternatively:

To evaluate $\int_0^2 6t^5\sqrt{t^2+4}\,dt$, let $u = t^2 + 4$ so that $\frac{du}{dt} = 2t \Rightarrow \frac{du}{2} = t\,dt$.

Note also that $t^2 = u - 4 \Rightarrow t^4 = (u-4)^2$.

Thus when $t = 0,\ u = 4$ and when $t = 2,\ u = 8$.

The integral becomes

$$\begin{aligned}\frac{6}{2}\int_4^8 (u-4)^2\sqrt{u}\,du &= 3\int_4^8 (u^2 - 8u + 16)\,u^{\frac{1}{2}}\,du \\ &= 3\int_4^8 \left(u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}\right)du \\ &= 3\left[\frac{2}{7}u^{\frac{7}{2}} - 8\times\frac{2}{5}u^{\frac{5}{2}} + 16\times\frac{2}{3}u^{\frac{3}{2}}\right]_4^8 \\ &= 3\left[\frac{2}{7}u^{\frac{7}{2}} - \frac{16}{5}u^{\frac{5}{2}} + \frac{32}{3}u^{\frac{3}{2}}\right]_4^8\end{aligned}$$

$\therefore a = \frac{2}{7},\ b = \frac{16}{5},\ c = \frac{32}{3}$

Mark allocation: 4 marks

- 1 mark for demonstrating the appropriate substitution $u = t^2 + 4$ and adjusting the terminals
- 1 mark for correctly stating $3\int_4^8 (u-4)^2 \times u^{\frac{1}{2}}\,du$
- 1 mark for correctly deriving $d = 3\left[au^{\frac{7}{2}} - bu^{\frac{5}{2}} + cu^{\frac{3}{2}}\right]_4^8$
- 1 mark for correctly determining the values of a, b and c

TIPS

» Remember to adjust the integrand terminals when performing integration by substitution.

» Be aware that once a variable has been substituted, a linear substitution may be possible to simplify the integral.

Question 15a.

Worked solution

Apply Euler's method to find x_1 and y_1:

$$\begin{aligned}x_1 &= x_0 + h \\ &= 0 + 0.5 \\ &= 0.5\end{aligned}$$

$$\begin{aligned}y_1 &= y_0 + h\left.\frac{dy}{dx}\right|_{x_0,\,y_0} \\ &= 3 + 0.5 \times -\frac{0}{3^2} \\ &= 3\end{aligned}$$

Apply Euler's method again to find x_2 and y_2:

$$\begin{aligned} x_2 &= x_1 + h \\ &= 0.5 + 0.5 \\ &= 1 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h\left.\frac{dy}{dx}\right|_{x_1,\, y_1} \\ &= 3 + 0.5 \times -\frac{0.5}{3^2} \\ &= \frac{107}{36} \end{aligned}$$

Thus an estimate of (x_2, y_2) is $\left(1, \frac{107}{36}\right)$.

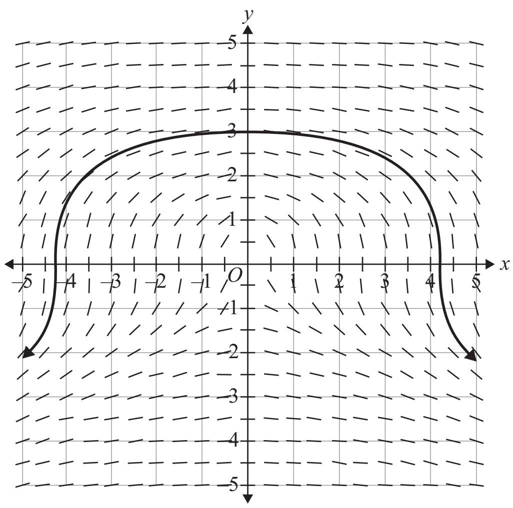
Mark allocation: 2 marks

- 1 mark for correctly applying Euler's method
- 1 mark for deriving the correct estimate of $\left(1, \frac{107}{36}\right)$

Question 15b.

Worked solution

The curve needs to go through the point (0, 3).



Mark allocation: 1 mark

- 1 mark for accurately sketching the solution curve

TIPS

» Work ahead and use the solution from **part c.** to produce an accurate sketch.

» The solution curve should always be drawn between slope lines or right on top of a slope line. It should not cross a slope line at a different angle to that line.

Question 15c.

Worked solution

Separate the variables and integrate both sides:

$$\frac{dy}{dx} = -\frac{x}{y^2}$$

$$\int y^2\,dy = \int -x\,dx$$

$$\frac{y^3}{3} = -\frac{x^2}{2} + c$$

where c is a constant of integration.

Apply the condition $y(0) = 3$ to find c:

$$\frac{(3)^3}{3} = -\frac{0^2}{2} + c$$

$$c = 9$$

$$\frac{y^3}{3} = -\frac{x^2}{2} + 9$$

Making y the subject gives

$$y = \sqrt[3]{-\frac{3x^2}{2} + 27}$$

Mark allocation: 2 marks

- 1 mark for using the appropriate integration techniques
- 1 mark for deriving the solution $y = \sqrt[3]{-\frac{3x^2}{2} + 27}$ or $\frac{y^3}{3} = -\frac{x^2}{2} + 9$

Question 16

Worked solution

$$\int_0^2 \frac{3}{2x^2 + 3x + 1}\,dx = 3\int_0^2 \frac{1}{(2x+1)(x+1)}\,dx$$

Using decomposition by partial fractions on the integrand gives

$$\frac{1}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(2x+1)$$

Letting $x = -\frac{1}{2}$ gives $A = 2$ and letting $x = -1$ gives $B = -1$.

Hence $\frac{1}{(2x+1)(x+1)} = \frac{2}{2x+1} - \frac{1}{x+1}$ and the required integral becomes $\int_0^2 \frac{3}{2x^2 + 3x + 1}\,dx$.

$$= 3\int_0^2 \frac{1}{(2x+1)(x+1)}\,dx$$

$$= 3\int_0^2 \frac{2}{(2x+1)} - \frac{1}{(x+1)}\,dx$$

$$= 3\left[\log_e|2x+1| - \log_e|x+1|\right]_0^2$$

$$= 3\left[\log_e\left|\frac{2x+1}{x+1}\right|\right]_0^2$$

$$= 3\left(\log_e\left(\frac{5}{3}\right) - \log_e(1)\right)$$

$$= 3\log_e\left(\frac{5}{3}\right)$$

Mark allocation: 3 marks

- 1 mark for using decomposition by partial fractions
- 1 mark for the correct antiderivative of the integrand
- 1 mark for the correct answer in the specified form: $3\log_e\left(\frac{5}{3}\right)$

TIP

» Remember that for any linear function $ax + b$, $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + c$.

Question 17

Worked solution

$$\sin(xy) + x\cos(xy) \times \left(x\frac{dy}{dx} + y\right) + y^2 + 2xy\frac{dy}{dx} = 0$$

$$\sin(xy) + x^2\cos(xy)\frac{dy}{dx} + xy\cos(xy) + y^2 + 2xy\frac{dy}{dx} = 0$$

At point $\left(\frac{\pi}{2}, 1\right)$:

$$\sin\left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right)^2\cos\left(\frac{\pi}{2}\right)\frac{dy}{dx} + \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) + 1^2 + 2\left(\frac{\pi}{2}\right)(1)\frac{dy}{dx} = 0$$

$$1 + 1 + \pi\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2}{\pi}$$

The gradient of the normal is given by $-\frac{dx}{dy} = \frac{\pi}{2}$.

The equation of the normal line at point $\left(\frac{\pi}{2}, 1\right)$ can be determined using point slope form:

$$y - 1 = \frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$$

$$y = \frac{\pi}{2}x + 1 - \frac{\pi^2}{4}$$

Mark allocation: 5 marks

- 2 marks for correctly performing implicit differentiation on each term of the relation to obtain $\sin(xy) + x^2\cos(xy)\frac{dy}{dx} + xy\cos(xy) + y^2 + 2xy\frac{dy}{dx} = 0$ or an equivalent expression

Note: Award only 1 mark if at least two terms have the correct implicit differentiation.

- 1 mark for correctly evaluating $\frac{dy}{dx}$ at $\left(\frac{\pi}{2}, 1\right)$
- 1 mark for the correct gradient of the normal line: $\frac{\pi}{2}$
- 1 mark for the correct equation of the normal line: $y = \frac{\pi}{2}x + 1 - \frac{\pi^2}{2}$ or equivalent

TIP

» **Both the product and chain rule must be used to implicitly differentiate $\sin(xy)$.**

$$\frac{d}{dx}(\sin(xy))$$
$$= \cos(xy) \times \frac{d}{dx}(xy)$$
$$= \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

Question 18

Worked solution

The integral can be evaluated using integration by parts.

Let $u = 2x$:

$$\frac{du}{dx} = 2$$

Let $\frac{dv}{dx} = \cos\left(\frac{x}{4}\right)$:

$$v = \int dv = \int \cos\left(\frac{x}{4}\right)dx = 4\sin\left(\frac{x}{4}\right)$$

Using the formula for integration by parts, $\int u \cdot \frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$, it follows that

$$\int_0^{2\pi} 2x\cos\left(\frac{x}{4}\right)dx = \left[2x \cdot 4\sin\left(\frac{x}{4}\right)\right]_0^{2\pi} - \int_0^{2\pi} 4\sin\left(\frac{x}{4}\right) \cdot 2dx$$
$$= \left[8x\sin\left(\frac{x}{4}\right)\right]_0^{2\pi} - 8\int_0^{2\pi} \sin\left(\frac{x}{4}\right) dx$$
$$= \left[8 \cdot 2\pi \cdot \sin\left(\frac{2\pi}{4}\right) - 0\right] - 8\left[-4\cos\left(\frac{x}{4}\right)\right]_0^{2\pi}$$
$$= 16\pi + 32\left(\cos\left(\frac{2\pi}{4}\right) - \cos(0)\right)$$
$$= 16\pi - 32$$

Mark allocation: 4 marks

- 1 mark for the correct expressions for u, v and their derivatives
- 1 mark for the correct integral derived using the formula for integration by parts
- 1 mark for the correct antiderivative
- 1 mark for the correct answer of $16\pi - 32$

Question 19

Worked solution

$$V = \pi \int_0^{\sqrt{2}} \left(\sqrt{\frac{2x+8}{x^2+2}} \right)^2 dx$$

$$= \pi \int_0^{\sqrt{2}} \left(\frac{2x+8}{x^2+2} \right) dx$$

$$= \pi \left[\int_0^{\sqrt{2}} \frac{2x}{x^2+2} dx + \int_0^{\sqrt{2}} \frac{8}{x^2+2} dx \right]$$

$$= \pi [\log_e(x^2+2)]_0^{\sqrt{2}} + \pi \left[\frac{8}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}} \right) \right]_0^{\sqrt{2}}$$

$$= \pi(\log_e(4) - \log_e(2)) + \pi\left(\frac{8}{\sqrt{2}} \arctan(1) - \frac{8}{\sqrt{2}} \arctan(0) \right)$$

$$= \pi\left(\log_e\left(\frac{4}{2} \right) + \frac{8}{\sqrt{2}} \times \frac{\pi}{4} \right)$$

$$= \pi(\log_e(2) + \sqrt{2}\,\pi)$$

Mark allocation: 4 marks

- 1 mark for writing the correct integral for the volume of rotation
- 2 marks for the correct antiderivatives
- 1 mark for the correct value for the volume of rotation in the specified form

Question 20

Worked solution

To perform integration by parts, we must identify which terms we will assign u and v', respectively.

Let $u = \log_e(2x)$, then $u' = \frac{1}{x}$.

Let $v' = x^{\frac{1}{2}}$, then, $v = \frac{2}{3}x^{\frac{3}{2}}$.

Therefore

$$\int_1^4 \sqrt{x} \log_e(2x)\, dx = \left[\frac{2}{3} x^{\frac{3}{2}} \log_e(2x) \right]_1^4 - \frac{2}{3} \int_1^4 x^{\frac{1}{2}} dx$$

$$= \frac{2}{3}(8)\log_e(8) - \frac{2}{3}\log_e(2) - \frac{2}{3}\left[\frac{2}{3} x^{\frac{3}{2}} \right]_1^4$$

$$= 16\log_e(2) - \frac{2}{3}\log_e(2) - \frac{4}{9}(8-1)$$

$$= \frac{46}{3}\log_e(2) - \frac{28}{9}$$

Mark allocation: 3 marks

- 1 mark for evidence of performing integration by parts (i.e. correctly selecting u and v)
- 1 mark for the correct answer of $\left[\frac{2}{3} x^{\frac{3}{2}} \log_e(2x) \right]_1^4 - \frac{2}{3} \int_1^4 x^{\frac{1}{2}} dx$ (or equivalent)
- 1 mark for correct answer of $\frac{46}{3}\log_e(2) - \frac{28}{9}$

TIP

» If you get stuck when performing integration by parts, look for a term that you know how to differentiate and consider setting that term equal to u.

Question 21

Worked solution

The surface area of the solid of revolution can be found by evaluating the general formula

$S = \int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$

In this specific case:

$$y_1 = (\sqrt{2})^2 = 2$$

$$y_2 = (\sqrt{6})^2 = 6$$

$$x = \sqrt{y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

Therefore the integral to evaluate the surface area is

$$\begin{aligned}
S &= \int_2^6 2\pi\sqrt{y}\cdot\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\, dy \\
&= 2\pi\int_2^6 \sqrt{y}\cdot\sqrt{1+\frac{1}{4y}}\, dy \\
&= 2\pi\int_2^6 \sqrt{y+\frac{1}{4}}\, dy \\
&= 2\pi\cdot\frac{2}{3}\left[\left(y+\frac{1}{4}\right)^{\frac{3}{2}}\right]_2^6 \\
&= \frac{4}{3}\pi\left[\left(6+\frac{1}{4}\right)^{\frac{3}{2}} - \left(2+\frac{1}{4}\right)^{\frac{3}{2}}\right] \\
&= \frac{4}{3}\pi\left[\left(\frac{25}{4}\right)^{\frac{3}{2}} - \left(\frac{9}{4}\right)^{\frac{3}{2}}\right] \\
&= \frac{4}{3}\pi\left[\left(\frac{5}{2}\right)^{3} - \left(\frac{3}{2}\right)^{3}\right] \\
&= \frac{\pi}{6}(125-27) \\
&= \frac{49\pi}{3}
\end{aligned}$$

Mark allocation: 4 marks

- 1 mark for the correct expression for $\frac{dx}{dy}$
- 1 mark for the correct integral for the surface area
- 1 mark for the correct antiderivative
- 1 mark for the correct answer of $\frac{49\pi}{3}$

Question 22

Worked solution

The required surface area can be found by evaluating the integral

$$S = \int_{t_2}^{t_1} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

For this specific parametric curve:

$$\frac{dx}{dt} = 4t$$

$$\frac{dy}{dt} = 4$$

Therefore the integral that gives the surface area becomes

$$\begin{aligned} S &= \int_0^{\sqrt{3}} 2\pi \cdot 4t\sqrt{(4t)^2 + (4)^2}\ dt \\ &= 8\pi \int_0^{\sqrt{3}} t\sqrt{16t^2 + 16}\ dt \\ &= 32\pi \int_0^{\sqrt{3}} t\sqrt{t^2 + 1}\ dt \\ &= 32\pi \left[\frac{1}{3}(t^2 + 1)^{\frac{3}{2}}\right]_0^{\sqrt{3}} \\ &= \frac{32\pi}{3}\left[\left(\sqrt{3}^2 + 1\right)^{\frac{3}{2}} - (0^2 + 1)^{\frac{3}{2}}\right] \\ &= \frac{32\pi}{3}(8 - 1) \\ &= \frac{224\pi}{3} \end{aligned}$$

Mark allocation: 3 marks

- 1 mark for the correct integral for the surface area
- 1 mark for the correct antiderivative
- 1 mark for the correct answer of $\frac{224\pi}{3}$

Question 23

Worked solution

The required integral can be evaluated using two iterations of integration by parts, then using integration by recognition. To find an antiderivative of $e^x\cos(\pi x)$, first use integration by parts to find the indefinite integral $\int e^x \cos(\pi x)\, dx$.

For the first iteration:

Let $u = \cos(\pi x)$, hence $\frac{du}{dx} = -\pi\sin(\pi x)$.

Let $\frac{dv}{dx} = e^x$, hence $v = \int dv = \int e^x\, dx = e^x$.

Using the formula for integration by parts, $\int u \cdot \frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$, it follows that

$$\begin{aligned} \int e^x \cos(\pi x)\, dx &= e^x\cos(\pi x) - \int e^x \cdot -\pi\sin(\pi x)\ dx \\ &= e^x\cos(\pi x) + \pi\int e^x \cdot \sin(\pi x)\ dx \end{aligned}$$

Integration by parts can be used a second time to evaluate $\int e^x \cdot \sin(\pi x)\ dx$:

Let $u = \sin(\pi x)$, hence $\frac{du}{dx} = \pi \cos(\pi x)$.

Let $\frac{dv}{dx} = e^x$, hence $v = \int dv = \int e^x dx = e^x$. So:

$$\begin{aligned}\int e^x \cdot \sin(\pi x)\ dx &= e^x \sin(\pi x) - \int e^x \cdot \pi\cos(\pi x)\, dx \\ &= e^x \sin(\pi x) - \pi \int e^x \cos(\pi x)\, dx\end{aligned}$$

Combining the results from the first and second iteration of integration by parts gives

$$\begin{aligned}\int e^x \cos(\pi x)\, dx &= e^x \cos(\pi x) + \pi \int e^x \cdot \sin(\pi x)\ dx \\ &= e^x \cos(\pi x) + \pi\left(e^x \sin(\pi x) - \pi \int e^x \cos(\pi x)\ dx\right) \\ &= e^x \cos(\pi x) + \pi e^x \sin(\pi x) - \pi^2 \int e^x \cos(\pi x)\ dx\end{aligned}$$

This equation can be rearranged to obtain an expression for the required antiderivative.

$$\begin{aligned}\int e^x \cos(\pi x)\, dx + \pi^2 \int e^x \cos(\pi x)\, dx &= e^x \cos(\pi x) + \pi e^x \sin(\pi x) \\ (\pi^2 + 1) \int e^x \cos(\pi x)\, dx &= e^x(\cos(\pi x) + \pi \sin(\pi x)) \\ \int e^x \cos(\pi x)\ dx &= \frac{e^x(\cos(\pi x) + \pi \sin(\pi x))}{(\pi^2 + 1)}\end{aligned}$$

This result can be used to evaluate the initial definite integral.

$$\begin{aligned}\int_1^2 e^x \cos(\pi x)\ dx &= \left[\frac{e^x(\cos(\pi x) + \pi \sin(\pi x))}{(\pi^2 + 1)}\right]_1^2 \\ &= \frac{1}{\pi^2 + 1}(e^2 + e) \\ &= \frac{e^2 + e}{\pi^2 + 1}\end{aligned}$$

Mark allocation: 5 marks

- 1 mark for the correct first iteration of integration by parts
- 1 mark for the correct second iteration of integration by parts
- 1 mark for the correct expression for antiderivative of $e^x \cos(\pi x)$
- 1 mark for using the antiderivative of $e^x \cos(\pi x)$, with correct terminals to write an expression for the required definite integral
- 1 mark for correct answer of $\frac{e^2 + e}{\pi^2 + 1}$

Question 24a.

Worked solution

The maximum number of rats will occur when the rate of growth is zero.

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{200}\right) = 0$$

$P = 0,\ \ P = 200$

Therefore the maximum number of rats predicted by the model is 200.

Mark allocation: 1 mark

- 1 mark for the correct maximum population of 200 rats

Question 24b.

Worked solution

$$\frac{dP}{dt}=2P\left(1-\frac{P}{200}\right)=2P-\frac{P^2}{100}=\frac{200P-P^2}{100}$$

$$\frac{dt}{dP}=\frac{100}{200P-P^2}$$

$$\frac{dt}{dP}=\frac{100}{P(200-P)}$$

Using decomposition by partial fractions gives

$$\frac{100}{P(200-P)}=\frac{A}{P}+\frac{B}{200-P}$$

$$100=A(200-P)+BP$$

$$P=0\rightarrow A=\frac{1}{2}$$

$$P=200\rightarrow B=\frac{1}{2}$$

$$\frac{100}{P(200-P)}=\frac{1}{2}\left(\frac{1}{P}+\frac{1}{200-P}\right)$$

Using this result:

$$2\frac{dt}{dP}=\left(\frac{1}{P}+\frac{1}{200-P}\right)$$

Integrating both sides with respect to P yields a general solution to the differential equation $2t=\log_e(P)-\log_e(200-P)+C,\ C\in R$.

Note that absolute value signs are not needed on the argument of the natural logarithms, as $P\le 200$.

Substituting the initial conditions to find the value of C gives

$$0=\log_e(5)-\log_e(200-5)+C$$

$$C=\log_e\left(\frac{195}{5}\right)=\log_e(39)$$

Therefore the particular solution to the differential equation is

$$2t=\log_e(P)-\log_e(200-P)+\log_e(39)$$

$$2t=\log_e\left(\frac{39P}{200-P}\right)$$

$$e^{2t}=\frac{39P}{200-P}$$

$$200e^{2t}-Pe^{2t}=39P$$

$$200e^{2t}=39P+Pe^{2t}$$

$$200e^{2t}=P(39+e^{2t})$$

$$P(t)=\frac{200e^{2t}}{39+e^{2t}}$$

Mark allocation: 4 marks

- 1 mark for writing the differential equation in the form $\frac{dt}{dP} = \frac{100}{P(200 - P)}$
- 1 mark for the correct application of decomposition by partial fractions
- 1 mark for the correct general solution to the differential equation of $2t = \log_e(P) - \log_e(200 - P) + C$ or equivalent expression
- 1 mark for the correct particular solution to the differential equation in the form $P(t) = \frac{200e^{2t}}{39 + e^{2t}}$, or equivalent expression with $P(t)$ as the subject

Question 25

Worked solution

As the information provided is in parametric form, the parametric form of arc length must be used.

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Substituting the expression for $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and integrating from $t = 0$ to $t = \frac{\pi}{3}$ gives

$$\int_0^{\frac{\pi}{3}} \sqrt{(-2\sin(t) - 2\sin(2t))^2 + (2\cos(t) - 2\cos(2t))^2}\, dt$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{4\sin^2(t) + 8\sin(t)\sin(2t) + 4\sin^2(2t) + 4\cos^2(t) - 8\cos(t)\cos(2t) + 4\cos^2(2t)}\, dt$$

We have $4\sin^2(t) + 4\cos^2(t) = 4$ and $4\sin^2(2t) + 4\cos^2(2t) = 4$ by Pythagoras' theorem.

$$= \int_0^{\frac{\pi}{3}} \sqrt{8\sin(t)\sin(2t) - 8\cos(t)\cos(2t) + 8}\, dt$$

$8\sin(t)\sin(2t) - 8\cos(t)\cos(2t) = -8(\cos(t)\cos(2t) - \sin(t)\sin(2t)) = -8\cos(3t)$ using compound angle formulas.

$$t = \int_0^{\frac{\pi}{3}} \sqrt{8 - 8\cos(3t)}\, dt$$

$8 - 8\cos(3t)$ can be simplified to $16\sin^2\left(\frac{3t}{2}\right)$, using the double angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$.

$$\cos(3t) = 1 - 2\sin^2\left(\frac{3t}{2}\right)$$

Therefore

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sqrt{8 - 8\cos(3t)}\, dt &= \int_0^{\frac{\pi}{3}} \sqrt{16\sin^2\left(\frac{3t}{2}\right)}\, dt \\ &= \int_0^{\frac{\pi}{3}} 4\sin\left(\frac{3t}{2}\right) dt \\ &= \left[-\frac{8}{3}\cos\left(\frac{3t}{2}\right)\right]_0^{\frac{\pi}{3}} \\ &= \frac{8}{3}\end{aligned}$$

Mark allocation: 3 marks

- 1 mark for constructing the expression for arc length $\int_0^{\frac{\pi}{3}} \sqrt{(-2\sin(t) - 2\sin(2t))^2 + (2\cos(t) - 2\cos(2t))^2}\, dt$
- 1 mark for applying Pythagoras' theorem and compound angle formula to simplify the expression for arc length $\int_0^{\frac{\pi}{3}} \sqrt{8 - 8\cos(3t)}\, dt$
- 1 mark for correct answer of $\frac{8}{3}$

TIPS

» Refer to the formula sheet when working with double angle or compound angle formulae. Whilst you may not remember them, you may be expected to use them within a question.

» When calculating the arclength of a curve, aim to make the expression inside the square root a perfect square either by factorising or using trigonometric identities.

EXAM 2

Section A

Question 1

Answer: ***C***

Worked solution

Let $u = \log_e(x)$.

Then $\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\ du$ and

$x = e^{5b} \Rightarrow u = \log_e(e^{5b}) = 5b$

$x = e^{a} \Rightarrow u = \log_e(e^{a}) = a$

Substituting into $\int_{e^a}^{e^{5b}} \frac{[\log_e(x)]^4}{5x}\, dx$ gives

$\int_a^{5b} \frac{u^4}{5x} x\, du$

$= \int_a^{5b} \frac{u^4}{5} du$

TIP

» Don't forget to change the values of the terminals according to the substitution used.

Question 2

Answer: ***A***

Worked solution

$x_0 = 2,\ y_0 = 0$ and $h = 0.1$

Using Euler's method gives

$x_1 = 2 + 0.1 = 2.1$ and $y_1 = 0 + 0.1(2^3 - 2 \times 2) = 0.4$

$x_2 = 2.1 + 0.1 = 2.2$ and $y_2 = 0.4 + 0.1(2.1^3 - 2 \times 2.1) = 0.9061$

$y_3 = 0.9061 + 0.1(2.2^3 - 2 \times 2.2) = 1.5309$

Question 3

Answer: ***D***

Worked solution

$$\begin{aligned} a &= v\frac{dv}{dx} \\ &= \sin(x)\frac{d(\sin(x))}{dx} \\ &= \sin(x)\cos(x) \end{aligned}$$

Rearranging the double angle formula $\sin(2x) = 2\sin(x)\cos(x)$ gives $a = \frac{1}{2}\sin(2x)$.

Alternatively:

$$\begin{aligned} a &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= \frac{d}{dx}\left(\frac{1}{2}\sin^2(x)\right) \\ &= \sin(x)\cos(x) \\ &= \frac{1}{2}\sin(2x) \end{aligned}$$

Question 4

Answer: ***B***

Worked solution

Option A does not give a gradient of zero when $x = 0$, so can be eliminated.

Option C is the differential equation of a cubic, so can be eliminated.

Option D is the differential equation for a square root function and doesn't always give the correct direction for y in the first and fourth quadrants. Hence option D can be eliminated.

Option B is the differential equation for a circle and also has the correct directions of the gradients, so is the correct answer.

Question 5

Answer: ***D***

Worked solution

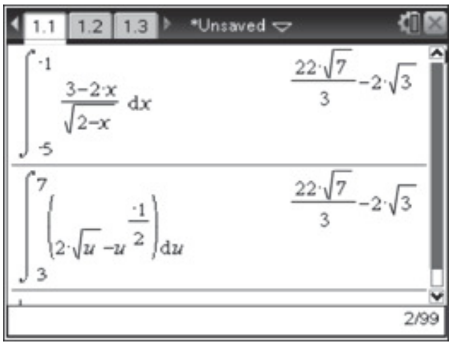
Let $u = 2 - x \Rightarrow x = 2 - u$.

Then $\frac{du}{dx} = -1 \Rightarrow dx = -du$.

When $x = -1 \Rightarrow u = 3$ and when $x = -5 \Rightarrow u = 7$, so

$$\begin{aligned}
\int_{-5}^{-1} \frac{3 - 2x}{\sqrt{2 - x}}\, dx &= \int_{7}^{3} \frac{3 - 2(2 - u)}{\sqrt{u}} \cdot -\, du \\
&= -\int_{7}^{3} \frac{3 - 4 + 2u}{\sqrt{u}}\, du \\
&= -\int_{7}^{3} u^{-\frac{1}{2}}(-1 + 2u)\, du \\
&= -\int_{7}^{3} \left(2u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du \\
&= \int_{3}^{7} \left(2u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du
\end{aligned}$$

Alternatively, a CAS calculator can be used to calculate the value of the given integral and then used to check the value of each option in turn until one option is found to have the same value as the given integral.



TIPS

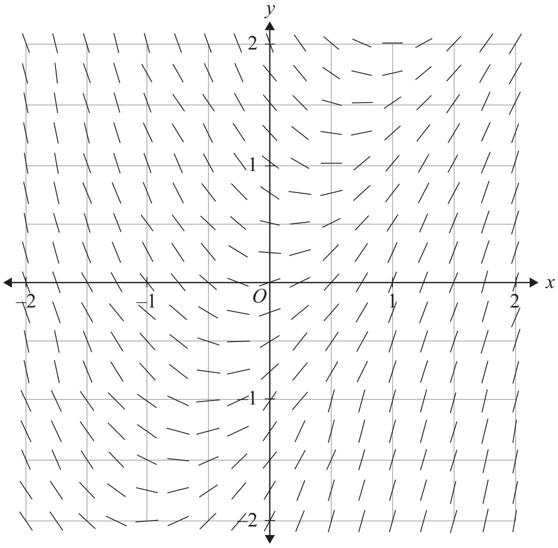
- » **Don't forget to change the values of the terminals to match the substitution used.**
- » **Remember to use the properties of integral terminals, such as $-\int_a^b f(x)dx = \int_b^a f(x)dx$, to get the correct final answer.**

Question 6

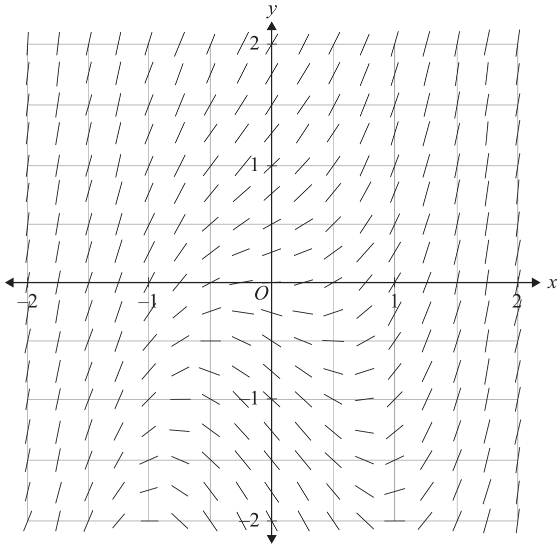
Answer: ***D***

Worked solution

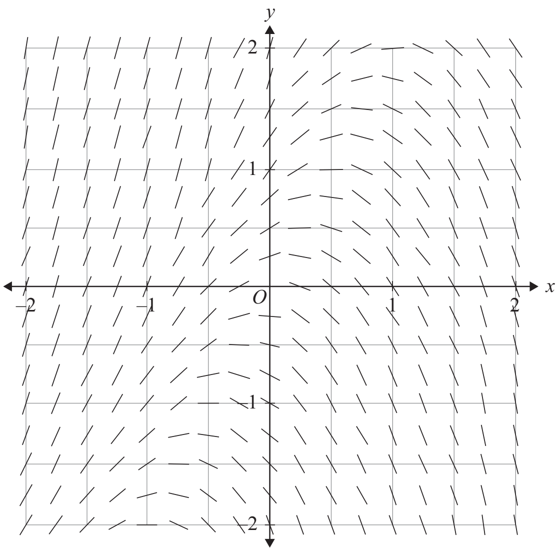
Option A has the direction field:



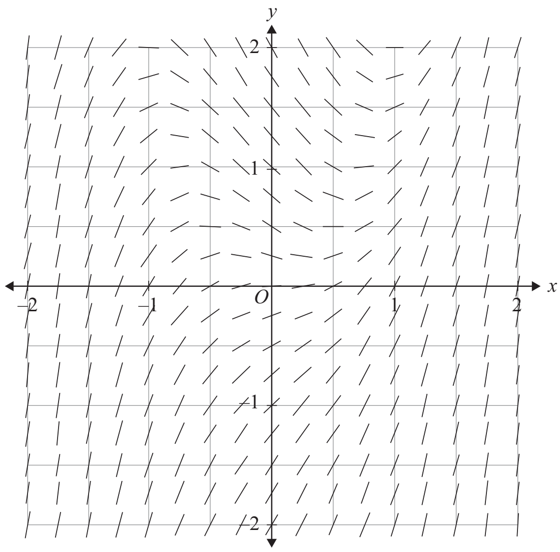
Option C has the direction field:



Option B has the direction field:



Option D has the direction field:



Therefore option D is correct.

Alternatively, if we look at the behaviour of $\frac{dy}{dx}$ for $x = 0$ and $y > 0$, we can see in the direction field given that $\frac{dy}{dx} < 0$. This eliminates options B, C and D because $\frac{dy}{dx} > 0$ for these values of x and y.

If we then look at the behaviour of $\frac{dy}{dx}$ for $x = -1$ and $y > 0$, we can see in the direction field given that $\frac{dy}{dx} > 0$. This then eliminates option A because $\frac{dy}{dx} < 0$ for these values of x and y.

TIP

» When graphing the direction field on a CAS, set the scale and size of the axes to the same as those used in the question. This will make it easier to see and compare the direction field to the one in the question. When doing this, only three options at most need to be checked. Either the correct option is one of the three or, if all three are wrong, the remaining option is the correct response.

Question 7

Answer: ***C***

Worked solution

Implicitly differentiating the function gives $\cos(y) - x\sin(y)\frac{dy}{dx} + \sin(x)\frac{dy}{dx} + y\cos(x) = 0$.

Substituting point $\left(\frac{\pi}{2}, \frac{\pi}{3}\right)$ and then simplifying gives

$$\cos\left(\frac{\pi}{3}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{3}\right)\frac{dy}{dx} + \sin\left(\frac{\pi}{2}\right)\frac{dy}{dx} + \frac{\pi}{3}\cos\left(\frac{\pi}{2}\right) = 0$$

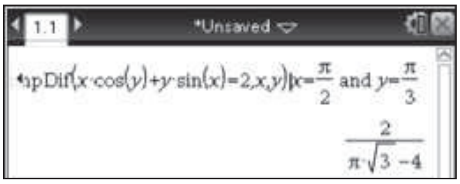
$$\frac{1}{2} - \frac{\sqrt{3}\pi}{4}\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\left(1 - \frac{\sqrt{3}\pi}{4}\right)\frac{dy}{dx} = -\frac{1}{2}$$

$$\left(\frac{4 - \sqrt{3}\pi}{4}\right)\frac{dy}{dx} = -\frac{1}{2}$$

$$\frac{dy}{dx} = \frac{2}{\pi\sqrt{3} - 4}$$

Using a CAS gives



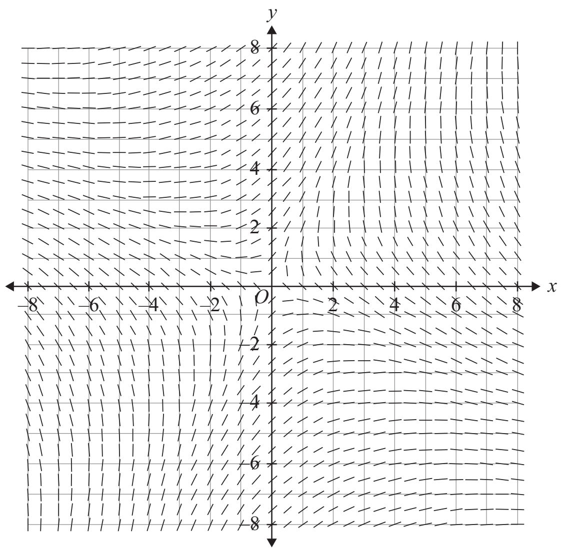
Question 8

Answer: ***B***

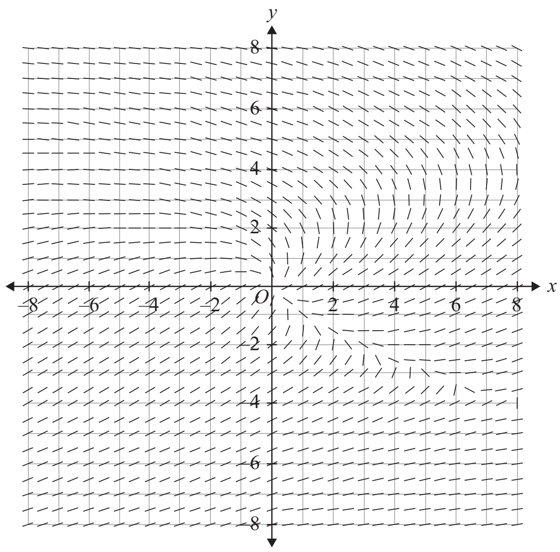
Worked solution

Method 1

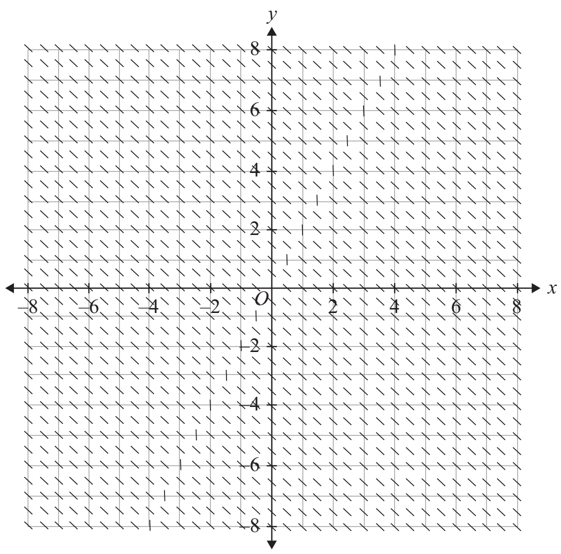
Option A has the direction field:



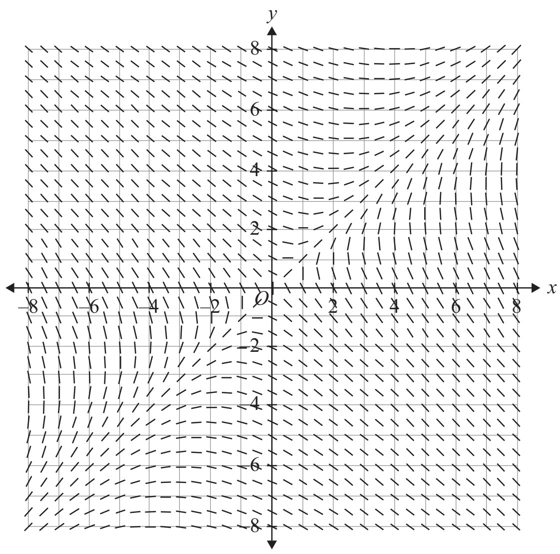
Option C has the direction field:



Option B has the direction field:



Option D has the direction field:



Therefore option B is the correct answer.

Method 2

If we look at the behaviour of $\frac{dy}{dx}$ for $y = 0$ and $x > 0$, we can see in the direction field given that $\frac{dy}{dx} > 0$. This eliminates options A, C and D because $\frac{dy}{dx} < 0$ for these values of x and y.

Question 9

Answer: ***D***

Worked solution

Let $u = x + 2$.

Then $\frac{du}{dx} = 1 \Rightarrow du = dx$ and $x = u - 2$.

If $x = -1,\ 2$, then $u = 1,\ 4$, so

$$\int_{-1}^{2} (x^2 + 1)\sqrt{x+2}\ dx = \int_{1}^{4} ((u-2)^2 + 1)\,u^{\frac{1}{2}}\ du$$

$$\Rightarrow \int_{1}^{4} (u^2 - 4u + 4 + 1)\,u^{\frac{1}{2}}\ du = \int_{1}^{4} (u^2 - 4u + 5)\,u^{\frac{1}{2}}\ du$$

$$\Rightarrow \int_{1}^{4} \left(u^{\frac{5}{2}} - 4u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\right) du$$

TIPS

- If you can't algebraically derive an answer, evaluate the integral given in the question and then evaluate all the options. Whichever option matches the evaluation of the integral in the question will be the correct answer.
- Ensure you change the terminals when making the substitution. Options that have the same terminals as those given in the initial integral will be incorrect and can be eliminated.

Question 10

Answer: ***A***

Worked solution

If $\frac{dy}{dx} = \arcsin(x)e^{2y}$, then

$$\frac{d^2y}{dx^2} = \arcsin(x)2e^{2y}\frac{dy}{dx} + \frac{e^{2y}}{\sqrt{1-x^2}}$$

$$= \arcsin(x)2e^{2y} \times \arcsin(x)e^{2y} + \frac{e^{2y}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\arcsin^2(x)e^{4y} + \frac{e^{2y}}{\sqrt{1-x^2}}$$

Substituting point $\left(\frac{1}{2},\ 0\right)$ gives

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \arcsin^2\left(\frac{1}{2}\right) \times e^{4\times 0} + \frac{e^{2\times 0}}{\sqrt{1-\left(\frac{1}{2}\right)^2}}$$

$$= 2 \times \left(\frac{\pi}{6}\right)^2 \times 1 + \frac{1}{\sqrt{\frac{3}{4}}} = \frac{\pi^2}{18} + \frac{2\sqrt{3}}{3}$$

Question 11

Answer: ***D***

Worked solution

For parametric equations, the length of the curve is given by $\int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Substituting $x'(t) = -8\cos(2t)$ and $y'(t) = -4\sin(2t)$ into the curve length equation, with terminals of $t_1 = 0$ and $t_2 = \frac{\pi}{4}$, gives

$$\int_0^{\frac{\pi}{4}} \sqrt{(-8\cos(2t))^2 + (-4\sin(2t))^2}\, dt$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{64\cos^2(2t) + 16\sin^2(2t)}\, dt$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{4\cos^2(2t) + \sin^2(2t)}\, dt$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{3\cos^2(2t) + (\cos^2(2t) + \sin^2(2t))}\, dt$$

$$= 4\int_0^{\frac{\pi}{4}} \sqrt{3\cos^2(2t) + 1}\, dt$$

Question 12

Answer: ***D***

Worked solution

First, find the derivative of each parametric equation. By hand is easiest. Then use a CAS and the distance formula, as follows:

$x'(\theta) = 1 - \cos(\theta), \quad y'(\theta) = \sin(\theta)$

distance $= \int_0^{2\pi} \sqrt{(1 - \cos(\theta))^2 + (\sin(\theta))^2}\, d\theta$

distance $= 8$

Answering options A, B or C could result from the incorrect use of this distance equation, such as forgetting the square root, forgetting to differentiate or not squaring etc.

Question 13

Answer: ***B***

Worked solution

Remember that the correct volume is given by $\int_0^{\frac{\pi}{2}} (4\sqrt{\pi}\sin(2x))^2\, dx$.

Option A works and simply incorporates π into the integral.

Option B is the correct answer because $1 - \sin(x) \neq \sin^2(2x)$.

Option C works because $2\sin(x)\cos(x) = \sin(2x)$.

Option D works because the integral shown gives the same answer.

Question 14

Answer: ***D***

Worked solution

Consider the flow of diesel in minus the flow of diesel out, as follows:

$$\frac{dQ}{dt}_{\text{IN}} = 25, \quad \frac{dQ}{dt}_{\text{OUT}} = \frac{20Q}{130 + 5t}$$

$$\frac{dQ}{dt} = \frac{dQ}{dt}_{\text{IN}} - \frac{dQ}{dt}_{\text{OUT}}$$

$$\therefore \frac{dQ}{dt} = 25 - \frac{20Q}{130 + 5t}$$

TIP

» **You must remember that the rate of diesel flowing out is equal to the concentration of diesel in the tank multiplied by the rate of flow of mixture out of the tank.**

Question 15

Answer: ***A***

Worked solution

Note that the gradient is zero whenever $x = 1$. This means that the equation must be equal to zero when $x = 1$. Also note that the gradient is undefined when $y = -1$. This means that the denominator must be zero when $y = -1$. Option A is the only answer that satisfies both of these conditions.

TIP

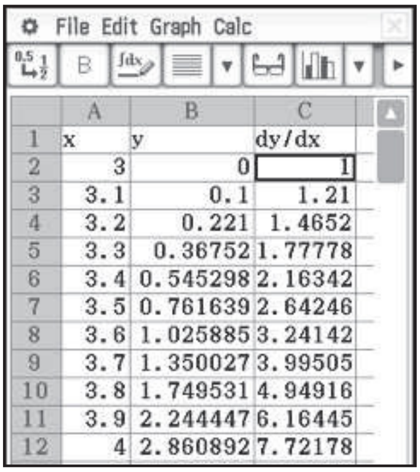
» **Familiarise yourself with your CAS calculator's facility to draw slope fields.**

Question 16

Answer: ***C***

Worked solution

Set up a spreadsheet to calculate y when $x = 4$. The columns are shown below.



	A	B	C
1	x	y	dy/dx
2	3	0	1
3	3.1	0.1	1.21
4	3.2	0.221	1.4652
5	3.3	0.36752	1.77778
6	3.4	0.545298	2.16342
7	3.5	0.761639	2.64246
8	3.6	1.025885	3.24142
9	3.7	1.350027	3.99505
10	3.8	1.749531	4.94916
11	3.9	2.244447	6.16445
12	4	2.860892	7.72178

When using Euler's method, remember to use the formula $y_n = y_{n-1} + hy'(x_{n-1})$, where the increment or step size is represented by h.

Section B

Question 1

Worked solution

This question is asking for the arc length.

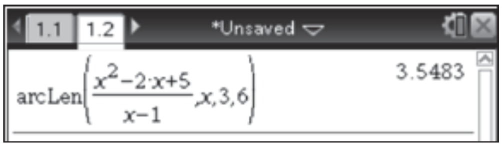
$$\int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_3^6 \sqrt{1 + \left(1 - \frac{4}{(x-1)^2}\right)^2}\, dx$$

$= 3.5483$

$= 3.55$

Or, using a CAS:



Mark allocation: 1 mark

- 1 mark for correct arc length: 3.55

TIP

» Using a CAS to find the arc length of a curve is a much more efficient and accurate method than using the formula and calculating by hand.

Question 2a.i.

Worked solution

$$\frac{d}{dx}\left(\sqrt{1-x^2} + x\sin^{-1}(x)\right) = \frac{d}{dx}\left(\sqrt{1-x^2}\right) + \frac{d}{dx}\left(x\sin^{-1}(x)\right)$$

$$= -\frac{2x}{2\sqrt{1-x^2}} + x\frac{d(\sin^{-1}(x))}{dx} + \frac{d(x)}{dx}\sin^{-1}(x)$$

$$= -\frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x)$$

$$= \sin^{-1}(x)$$

This is an inverse sine function that exists for $x \in [-1, 1]$.

Mark allocation: 2 marks

- 1 mark for the domain $x \in [-1, 1]$
- 1 mark for determining $\sin^{-1}(x)$

Question 2a.ii.

Worked solution

If $\int \sin^{-1}(x)\,dx = \int \sqrt{1-x^2} + x\sin^{-1}(x)$, then $\int \sin^{-1}\left(\frac{x}{2}\right)dx = \sqrt{4-x^2} + x\sin^{-1}\left(\frac{x}{2}\right)$.

Therefore $\int 2\sin^{-1}\left(\frac{x}{2}\right)dx = 2\left(\sqrt{4-x^2} + x\sin^{-1}\left(\frac{x}{2}\right)\right)$.

Mark allocation: 2 marks

- 1 mark for using the result of **part a.i.** to find an antiderivative
- 1 mark for $2\left(\sqrt{4-x^2} + x\sin^{-1}\left(\frac{x}{2}\right)\right)$

Question 2b.i.

Worked solution

When rotated about the y-axis, volume is given by $V = \int_a^b \pi x^2\,dy$.

Rearranging to make x the subject gives $x = 2\sin\left(y - \frac{\pi}{2}\right) + 4$.

When $x = 2 \Rightarrow y = 0$ and when $x = 6 \Rightarrow y = \pi$.

So the volume of the solid generated by rotating the curve about the y-axis is

$V = \int_0^\pi \pi\left(2\sin\left(y - \frac{\pi}{2}\right) + 4\right)^2$.

Alternatively, when solving for x using a CAS, $x = 4 - 2\cos(y)$.

So the volume of the solid generated by rotating the curve about the y-axis is

$V = \int_0^\pi \pi(4 - 2\cos(y))^2\,dy$.

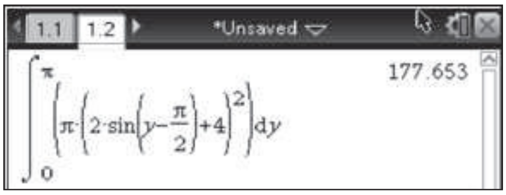
Mark allocation: 1 mark

- 1 mark for showing the integral $V = \int_0^\pi \pi\left(2\sin\left(y - \frac{\pi}{2}\right) + 4\right)^2$ or $V = \int_0^\pi \pi(4 - 2\cos(y))^2\,dy$

Question 2b.ii.

Worked solution

$$V = 177.653$$
$$= 177.65\ \text{cm}^3$$



Mark allocation: 1 mark

- 1 mark for correctly evaluating 177.65 cm^3

Question 2c.

Worked solution

The differential equation describing the change in height is $\frac{dy}{dt} = 3t$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{1 - \left(\frac{x}{2} - 2\right)^2}}$$

So a differential equation relating the change in radius to the change in time is

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{dy} \cdot \frac{dy}{dt} \\ &= 2\sqrt{1 - \left(\frac{x}{2} - 2\right)^2} \cdot 3t \\ &= 6t\sqrt{1 - \left(\frac{x}{2} - 2\right)^2} \\ &= 3t\sqrt{4 - (x-4)^2} \text{ or } 6t\sqrt{1 - \left(\frac{x}{2} - 2\right)^2}\end{aligned}$$

Mark allocation: 3 marks

- 1 mark for recognising that $\frac{dy}{dt} = 3t$
- 1 mark for using substitution to find $\frac{dy}{dx} = \frac{1}{2\sqrt{1 - \left(\frac{x}{2} - 2\right)^2}}$
- 1 mark for $\frac{dx}{dt} = 6t\sqrt{1 - \left(\frac{x}{2} - 2\right)^2}$ or $\frac{dx}{dt} = 3t\sqrt{4 - (x-4)^2}$

TIP

» When a value is given in a question and has a rate of change unit, this value is often a differential equation that is to be applied to solve the problem. For instance, $3t \text{ cm s}^{-1}$ is equivalent to the differential equation $\frac{dy}{dt} = 3t$. Wording such as 'changes with respect to' also indicates a differential equation.

Question 2d.

Worked solution

$\frac{dx}{dt} = 3t\sqrt{4-(x-4)^2}$

$\Rightarrow \int \frac{1}{\sqrt{4-(x-4)^2}} \cdot dx = \int 3t \cdot dt$

$\sin^{-1}\left(\frac{x-4}{2}\right) = \frac{3t^2}{2} + c$

Applying the conditions $x = 2,\ t = 0$ gives

$\sin^{-1}(-1) = c$

$\Rightarrow c = -\frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{x-4}{2}\right) = \frac{3t^2 - \pi}{2}$

$\frac{x-4}{2} = \sin\left(\frac{3t^2-\pi}{2}\right)$

$\Rightarrow x = 4 + 2\sin\left(\frac{3t^2-\pi}{2}\right)$

Alternatively:

From **part c.**

$\frac{dy}{dt} = 3t$

$\Rightarrow y = \frac{3t^2}{2} + c$

Applying initial conditions $y = 0$ when $t = 0$ gives $y = \frac{3t^2}{2}$.

From **part b.** it is known that $x = 2\sin\left(y - \frac{\pi}{2}\right) + 4$.

So substituting $y = \frac{3t^2}{2}$ gives $x = 4 + 2\sin\left(\frac{3t^2-\pi}{2}\right)$.

Mark allocation: 3 marks

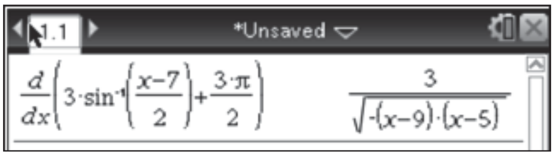
- 1 mark for the integration by separating the variables to get $\sin^{-1}\left(\frac{x-4}{2}\right) = \frac{3t^2}{2} + c$ or for integrating $\frac{dy}{dt} = 3t$ to get $y = \frac{3t^2}{2} + c$
- 1 mark for finding $c = -\frac{\pi}{2}$ or for applying initial conditions to get $y = \frac{3t^2}{2}$
- 1 mark for $x = 4 + 2\sin\left(\frac{3t^2-\pi}{2}\right)$

Question 3a.

Worked solution

Arc length is given by $\int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}\, dx$.

The derivative of the curve $f'(x)$ is $f'(x) = \dfrac{3}{\sqrt{-(x-5)(x-9)}}$.



So the arc length is

$$\int_5^9 \sqrt{1 + \left(\frac{3}{\sqrt{-(x-5)(x-9)}}\right)^2}\, dx$$

$$= \int_5^9 \sqrt{1 + \frac{9}{-(x-5)(x-9)}}\, dx$$

$$= \int_5^9 \sqrt{1 - \frac{9}{(x-5)(x-9)}}\, dx$$

So $a = 9,\ b = -5,\ c = -9$ or $a = 9,\ b = -9,\ c = -5$.

Mark allocation: 2 marks

- 1 mark for finding $f'(x) = \dfrac{3}{\sqrt{-(x-5)(x-9)}}$
- 1 mark for the correct values of a, b and c

Question 3b.

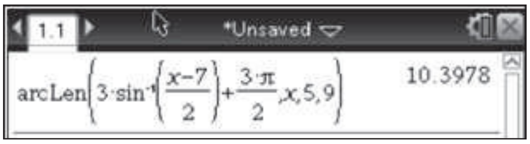
Worked solution

Evaluating the definite integral found in **part a.** gives

$$\int_5^9 \sqrt{1 - \frac{9}{(x+5)(x+9)}}\, dx$$

$= 10.3978...$

$= 10.398$ cm



Mark allocation: 1 mark

- 1 mark for the correct answer of 10.398 cm

Question 3c.i.

Worked solution

The volume generated when a curve is rotated about the y-axis is given by $V = \pi \int_a^b x^2 \, dy$.

Rearranging the equation of the curve in terms of y gives $x = 2\sin\left(\frac{1}{3}\left(y - \frac{3\pi}{2}\right)\right) + 7$.

Therefore the definite integral for the volume is $V = \pi \int_0^{3\pi} \left[2\sin\left(\frac{1}{3}\left(y - \frac{3\pi}{2}\right)\right) + 7\right]^2 dy$.

Alternatively, $V = \pi \int_0^{3\pi} \left[7 - 2\cos\left(\frac{y}{3}\right)\right]^2 dy$.

Mark allocation: 1 mark

- 1 mark for a correct definite integral: $V = \pi \int_0^{3\pi} \left[2\sin\left(\frac{1}{3}\left(y - \frac{3\pi}{2}\right)\right) + 7\right]^2 dy$ or $V = \pi \int_0^{3\pi} \left[7 - 2\cos\left(\frac{y}{3}\right)\right]^2 dy$

Question 3c.ii.

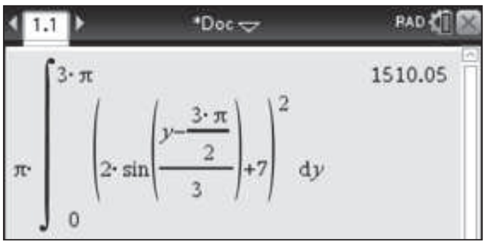
Worked solution

Evaluating the integral found in **part c.i.** gives

$$V = \pi \int_0^{3\pi} \left[2\sin\left(\frac{1}{3}\left(y - \frac{3\pi}{2}\right)\right) + 7\right]^2 dy$$

$$= 1510.05 \text{ cm}^3$$

Using a CAS to evaluate the integral from **part c.i.** gives a volume of 1510.05 cm^3.



Mark allocation: 1 mark

- 1 mark for a correct answer: 1510.05 cm^3

Question 3d.

Worked solution

Newton's law of cooling is $\frac{dT}{dt} = -k(T - T_s)$.

Substituting $T_s = 20$ gives $\frac{dT}{dt} = -k(T - 20)$.

Separating the variables gives $\frac{dT}{(T-20)} = -k \cdot dt$.

Integrating both sides gives

$$\int \frac{dT}{(T-20)} = \int -k \cdot dt$$

$$\Rightarrow \log_e(T - 20) = -kt + c$$

$$\Rightarrow \quad T - 20 = e^{-kt+c}$$

$$\Rightarrow \quad T = 20 + be^{-kt}, \text{ where } b = e^c.$$

So $a = 20$.

Applying the initial conditions gives

$90 = 20 + be^{-k\times 0}$

$\Rightarrow b = 70$

To find k, substitute $T = 60$ and $t = 15$, and solve for k.

$60 = 20 + 70\,e^{-15k}$

$$\Rightarrow k = -\frac{1}{15}\log_e\left(\frac{40}{70}\right)$$

$$= 0.037...$$

$$= 0.04$$

Therefore $a = 20$, $b = 70$ and $k = 0.04$, and the temperature of the coffee is given as $T = 20 + 70\,e^{-0.04t}$.

Alternatively, we have $\frac{dT}{(T-20)} = -k \cdot dt$.

Integrating both sides over the given conditions to find k gives

$$\int_{90}^{60}\frac{dT}{(T-20)} = \int_0^{15} -k \cdot dt$$

$$[\log_e(T-20)]_{90}^{60} = [-kt]_0^{15}$$

$$\Rightarrow \log_e\left(\frac{40}{70}\right) = -15k$$

$$\Rightarrow k = -\frac{1}{15}\log_e\left(\frac{4}{7}\right) = 0.037...$$

$$= 0.04$$

This then gives $\int\frac{dT}{(T-20)} = \int -0.04 \cdot dt$.

Integrating both sides gives

$\log_e(T-20) = -0.04t + c$

$\Rightarrow T - 20 = e^{-0.04t+c}$

$T = 20 + be^{-0.04t}$, where $b = e^c$.

So $a = 20$.

Applying the initial conditions gives

$90 = 20 + be^{-0.04\times 0}$

$\Rightarrow b = 70$

Therefore $a = 20$, $b = 70$, $k = 0.04$ and the temperature of the coffee is given by $T = 20 + 70e^{-0.04t}$.

Mark allocation: 3 marks

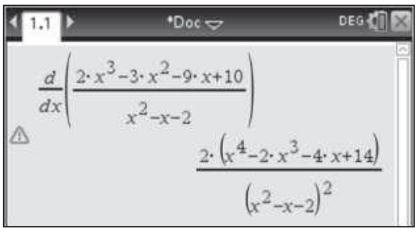
- 1 mark for separating the variables and integrating both sides of the equation
- 1 mark for applying the conditions in the question to get $a = 20$, $b = 70$ and $k = 0.04$, correct to two decimal places
- 1 mark for stating the equation: $T = 20 + 70e^{-0.04t}$

Question 4

Worked solution

The length of the curve is given by $L = \int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}\, dx$.

Use a CAS to find the derivative.



Then the integral is $L = \int_{0.5}^{1.5} \sqrt{1 + \left(\dfrac{2x^4 - 2x^3 - 8x + 28}{(x^2 - x - 2)^2}\right)^2}\, dx$.

Evaluating the integral for L gives 5.12327.

The length, correct to two decimal places, is 5.12.

Mark allocation: 2 marks

- 1 mark for showing $L = \int_{0.5}^{1.5} \sqrt{1 + \left(\dfrac{2x^4 - 2x^3 - 8x + 28}{(x^2 - x - 2)^2}\right)^2}\, dx$
- 1 mark for stating a length of 5.12

Question 5a.

Worked solution

$V = \pi \int_a^b x^2\, dy$

Rearranging the equation to make x the subject gives $x = 4\sin\left(\dfrac{y}{2}\right)$.

$$\Rightarrow V = \pi \int_0^h \left(4\sin\left(\frac{y}{2}\right)\right)^2 dy$$

$$= 16\pi \int_0^h \sin^2\left(\frac{y}{2}\right) dy$$

Using the trigonometric double angle formula $\sin^2(\theta) = \dfrac{1}{2}(1 - \cos(2\theta))$ gives

$$\Rightarrow V = 16\pi \int_0^h \frac{1}{2}(1 - \cos(y))\, dy$$

$$= 8\pi\, [y - \sin(y)]_0^h$$

$$= 8\pi[h - \sin(h) - 0 + \sin(0)]$$

$$= 8\pi[h - \sin(h)]$$

Mark allocation: 2 marks

- 1 mark for $8\pi[y - \sin(y)]_0^h$ or using an appropriate trigonometric identity to get $16\pi \int_0^h \dfrac{1}{2}(1 - \cos(y))\, dy$
- 1 mark for correctly showing the volume is $8\pi[h - \sin(h)]$

Question 5b.i.

Worked solution

$\frac{dV}{dt} = 16\pi^2\sqrt{h}$ and $\frac{dV}{dh} = 8\pi(1 - \cos(h))$.

Then

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$$
$$= \frac{1}{8\pi(1 - \cos(h))} \times 16\pi^2\sqrt{h}$$
$$= \frac{2\pi\sqrt{h}}{1 - \cos(h)}$$

Mark allocation: 2 marks

- 1 mark for $\frac{dV}{dh} = 8\pi(1 - \cos(h))$
- 1 mark for showing $\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \frac{2\pi\sqrt{h}}{1 - \cos(h)}$

Question 5b.ii.

Worked solution

One-quarter of the depth is $\frac{\pi}{4}$ cm.

So $\frac{dh}{dt} = \frac{2\pi\sqrt{\frac{\pi}{4}}}{1 - \cos\left(\frac{\pi}{4}\right)} = 19.01$ cm s^{-1}, correct to two decimal places.

Mark allocation: 1 mark

- 1 mark for $\frac{dh}{dt} = 19.01$ cm s^{-1}

Question 5c.i.

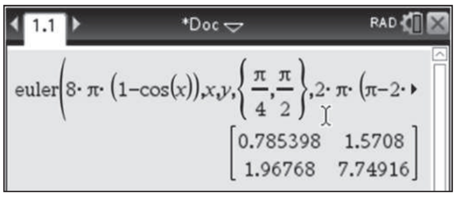
Worked solution

Euler's method, in the context of this question, is $h_{n+1} = h_n + \text{step size}$, $V_{n+1} = V_n + \text{step size} \times \frac{dV}{dh_n}$, where $\text{step size} = \frac{\pi}{4}$, $h_0 = \frac{\pi}{4}$ and $V_0 = 2\pi(\pi - 2\sqrt{2})$, using the result from **part a.**

When $\frac{dV}{dh} = 8\pi(1 - \cos(h))$, then $h_1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$ and

$$V_1 = V_0 + \frac{\pi}{4} \times \frac{dV}{dh_0}$$
$$= 2\pi(\pi - 2\sqrt{2}) + \frac{\pi}{4} \times 8\pi\left(1 - \cos\left(\frac{\pi}{4}\right)\right)$$
$$\approx 7.7492 \ldots$$
$$= 7.75 \text{ cm}^3 \text{ (correct to two decimal places)}$$

Euler's method can also be used with a CAS.



Mark allocation: 2 marks

- 1 mark for evidence of using Euler's method to find the answer
- 1 mark for an estimate of 7.75 cm^3

Question 5c.ii.

Worked solution

From **part a.**, the volume is given by $8\pi[h - \sin(h)]$. So if $h = \frac{\pi}{2}$ cm, then the volume is $8\pi\left[\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\right] \approx 14.346$ cm^3.

Therefore the estimate is an underestimate of the volume.

Mark allocation: 1 mark

- 1 mark for finding the correct volume and concluding that the estimate is an underestimate

Question 6a.

Worked solution

Inflow: pure water with 0 mg L^{-1} pumped into the tank at 50 L min^{-1}.

Outflow: mixture pumped out at 100 L min^{-1}.

$$\frac{dx}{dt} = \text{inflow} - \text{outflow}$$

$$\frac{dx}{dt} = 0 \times 50 - \frac{x}{12\,000 + 50t - 100t} \times 100$$

$$\therefore \frac{dx}{dt} = \frac{-100x}{12\,000 - 50t}$$

Mark allocation: 1 mark

- 1 mark for clearly showing the derivation of inflow and outflow and concluding that $\frac{dx}{dt} = \frac{-100x}{12\,000 - 50t}$

Question 6b.

Worked solution

$$\frac{dx}{dt} = \frac{-100x}{12\,000 - 50t}$$

$$\int \frac{1}{-100x}\,dx = \int \frac{1}{12\,000 - 50t}\,dt$$

$$-\frac{1}{100}\log_e|x| = -\frac{1}{50}\log_e|12\,000 - 50t| + c$$

$$\log_e|x| = 2\log_e|12\,000 - 50t| - 100c$$

$$\log_e|x| = \log_e(|12\,000 - 50t|)^2 + \log_e(e^{-100c})$$

$$\log_e|x| = \log_e(e^{-100c}(|12\,000 - 50t|)^2)$$

Let $A = e^{-100c}$. Then $|x| = A(|12\,000 - 50t|)^2$.

Substituting the initial condition into this equation to find A will determine the sign within the absolute brackets.

$x = A(12\,000 - 50t)^2$

Given that initially there was a volume of 12 000 L, with a concentration of fluoride being 3 mg L^{-1}:

At $t = 0$, $x = 12\,000 \times 3 = 36\,000$, so

$$36\,000 = A(12\,000)^2$$

$$A = \frac{36\,000}{(12\,000)^2} = \frac{1}{4000}$$

$$x = \frac{1}{4000}(12\,000 - 50t)^2$$

$$x = \frac{1}{4000}(12\,000^2 - 1\,200\,000t + 2500t^2)$$

$$x = \frac{5}{8}t^2 - 300t + 36\,000$$

Mark allocation: 3 marks

- 1 mark for establishing the integrals (the second line of the solution above is sufficient)
- 1 mark for correct integration resulting in the expression $x = A(12\,000 - 50t)^2$ or equivalent
- 1 mark for substituting the initial conditions and manipulating algebraically, leading to the answer

TIP

» **A common mistake is to omit the absolute value when integrating logarithms, or to go directly from an expression that includes an absolute value to one that doesn't. High-level responses will use the initial condition to justify the step that takes them from an expression that includes an absolute value to one that doesn't.**

Question 6c.i.

Worked solution

When pure water is pumped into the tank, $V = 12\,000 - 50t$.

When the concentration of fluoride reaches 2 mg L^{-1}, the amount of fluoride is $x = 2V = 2(12\,000 - 50t)$.

Substitute this into the equation for x in terms of t from **part b.** to find the time when the concentration reaches 2 mg L^{-1}:

$$2(12\,000 - 50t) = \frac{5}{8}t^2 - 300t + 36\,000$$

$t = 80$ or $t = 240$

$t = 240$ is not a valid solution, as $V = 0$ at this time.

So at $t = 80$, $V = 8000$, $x = 16\,000$.

New inflow: 1 mg L^{-1} pumped into the tank at 100 L min^{-1}.

Outflow stays the same: mixture pumped out at 100 L min^{-1}.

$$\frac{dx}{dt} = \text{inflow} - \text{outflow}$$

$$\frac{dx}{dt} = 1 \times 100 - \frac{x}{8000 + 100t - 100t} \times 100$$

$$\frac{dx}{dt} = 100 - \frac{x}{80}$$

Mark allocation: 2 marks

- 1 mark for the correct volume
- 1 mark for setting up the differential equation

Question 6c.ii.

Worked solution

With initial conditions $t = 0$ and $x = 16\,000$, using a CAS calculator gives $x = 8000\,e^{-\frac{t}{80}} + 8000$.

When the concentration reaches the prescribed limit of 1.5 mg L^{-1}, $x = 1.5V = 1.5(8000) = 12\,000$.

$$12\,000 = 8000e^{-\frac{t}{80}} + 8000$$
$$t = 80\log_e(2)$$
$$t = 55.45$$

Therefore, to the nearest minute, it will take 56 min for the concentration to drop below the prescribed limit.

Mark allocation: 2 marks

- 1 mark for the correct equation for x in terms of t
- 1 mark for the correct time, to the nearest minute

Question 7a.

Worked solution

The volume of the curve can be found from $y = \pi\int_{y=a}^{y=b} x^2\, dy$.

Rearranging $y = \frac{5x^2}{9} - 5$ to make x^2 the subject gives $x^2 = \frac{9y}{5} + 9$.

So the volume at a depth of h metres is $V = \pi\int_0^h x^2\, dy = \pi\int_0^h \frac{9y}{5} + 9\, dy$.

Integrating gives $V = \pi\left[\frac{9y^2}{10} + 9y\right]_0^h$.

Substituting in the terminals gives a volume of $V = \pi\left[\frac{9h^2}{10} + 9h\right]$.

Mark allocation: 2 marks

- 1 mark for showing $V = \pi\int_0^h \frac{9y}{5} + 9\, dy$
- 1 mark for showing $V = \pi\left[\frac{9y^2}{10} + 9y\right]_0^h = \pi\left[\frac{9h^2}{10} + 9h\right]$

Question 7b.

Worked solution

From **part a.**, $V = \pi\left[\frac{9h^2}{10} + 9h\right]$, so $\frac{dV}{dh} = \frac{9\pi(h+5)}{5}$.

From the information supplied, the change in volume with respect to time can be expressed as $\frac{dV}{dt} = \text{volume in} - \text{volume out} = \sqrt{3}h^2 - 0.2$. So

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \frac{5}{9\pi(h+5)} \times (\sqrt{3}h^2 - 0.2)$$
$$= \frac{5\sqrt{3}h^2 - 1}{9\pi(h+5)}, \text{ as required.}$$

Mark allocation: 2 marks

- 1 mark for finding $\frac{dV}{dh} = \frac{9\pi(h+5)}{5}$ and $\frac{dV}{dt} = \sqrt{3}h^2 - 0.2$
- 1 mark for using $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ and showing calculations that get the required result

Question 7c.

Worked solution

Euler's method in the context of this question is $t_{n+1} = t_n + \text{ step size}, \; h_{n+1} = h_n + \text{step size} \times \left.\frac{dh}{dt}\right|_{h=h_n}$, where the step size is 2.

$t_0 = 17$ and $h_0 = 3.44$.

If $\frac{dh}{dt} = \frac{5\sqrt{3}\,h^2 - 1}{9\pi(h+5)}$, then $t_1 = 15 + 2 = 17$ and

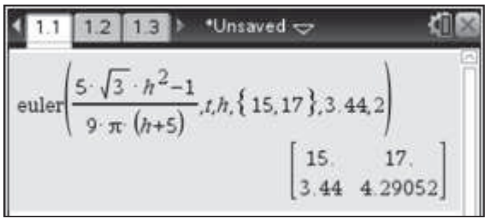
$$h_1 = h_0 + 2 \times \left.\frac{dh}{dt}\right|_{h=h_0}$$

$$= 3.44 + 2 \times \frac{5\sqrt{3}\,(3.44)^2 - 1}{9\pi(3.44+5)}$$

$$\approx 4.29052\ ...$$

$= 4.29$ m, correct to two decimal places.

Using Euler's method on a CAS gives



Mark allocation: 2 marks

- 1 mark for evidence of using Euler's method to find an estimate
- 1 mark for an estimate of 4.29 m

TIP

» In questions involving Euler's method, it is enough just to show the values generated by the formula when more than two steps are required.

Question 7d.i.

Worked solution

The volume of the tank for any given time, t, can be expressed as $200 + 15t - 10t = 200 + 5t$.

From the description:

$\text{inflow} = 0 \times 15 = 0$

$\text{outflow} = \dfrac{x}{200 + 5t} \times 10 = \dfrac{2x}{40 + t}$

$\dfrac{dx}{dt} = \text{inflow} - \text{outflow} = 0 - \dfrac{2x}{40 + t} = -\dfrac{2x}{40 + t}$

Therefore $\dfrac{dx}{dt} + \dfrac{2x}{40 + t} = 0$.

Mark allocation: 2 marks

- 1 mark for setting up the inflow and outflow rates
- 1 mark for correctly showing steps that lead to $\dfrac{dx}{dt} + \dfrac{2x}{40 + t} = 0$

Question 7d.ii.

Worked solution

Starting with $\dfrac{dx}{dt} + \dfrac{2x}{40 + t}$ and then separating the variables gives $\dfrac{dx}{2x} = -\dfrac{dt}{40 + t}$.

Integrating both sides:

$\displaystyle\int \frac{dx}{x} = -\int \frac{2\,dt}{40 + t}$

$\Rightarrow \ln(x) = -2\ln(40 + t) + c, \quad 40 + t > 0$

$\Rightarrow \ln(x) = \ln\left(\dfrac{1}{(40 + t)^2}\right) + c$

Substituting $x = 20$ when $t = 0$ and solving to find c:

$$\ln(20) = \ln\left(\frac{1}{(40 + 0)^2}\right) + c$$

$$c = \ln(20) - \ln\left(\frac{1}{1600}\right) = \ln(20 \times 1600)$$

$$= \ln(32\,000)$$

$$\ln(x) = \ln\left(\frac{1}{(40 + t)^2}\right) + \ln(32\,000) = \ln\left(\frac{32\,000}{(40 + t)^2}\right)$$

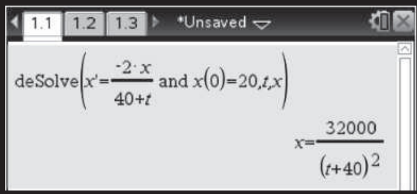
$$x = \frac{32\,000}{(40 + t)^2}$$

Mark allocation: 3 marks

- 1 mark for showing $\displaystyle\int \frac{dx}{x} = -\int \frac{2\,dt}{40 + t}$ or equivalent
- 1 mark for finding $c = \ln(32\,000)$ or an equivalent value
- 1 mark for obtaining the result $x = \dfrac{32\,000}{(40 + t)^2}$

TIP

» Using the deSolve (TI-Nspire) or dSolve (CASIO) function can help you get the final answer if you don't know how to show the steps that lead to it.



Question 8a.

Worked solution

$$\frac{dx}{dt} = -2\sin(t), \ \frac{dy}{dt} = \cos(t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cos(t)}{-2\sin(t)}$$

$$\frac{dy}{dx} = -\frac{\cot(t)}{2}$$

Mark allocation: 2 marks

- 1 mark for each derivative
- 1 mark for using the chain rule

TIP

» Be sure to show that you are using the chain rule by showing how you will multiply the derivatives together.

Question 8b.

Worked solution

$$-\frac{\sqrt{3}}{6} = -\frac{\cot(t)}{2}$$

$$\frac{\sqrt{3}}{3} = \frac{1}{\tan(t)}$$

$$\tan(t) = \sqrt{3}$$

$$t = \frac{\pi}{3}$$

$\therefore x = -1$, giving $\left(-1, \frac{\sqrt{3}}{2}\right)$.

Mark allocation: 2 marks

- 1 mark for the correct t value, found from equating the gradient with the derivative
- 1 mark for the correct coordinates

Question 8c.

Worked solution

$$y = \sqrt{1 - \frac{(x+2)^2}{4}}$$

$$V = \pi \int_{-4}^{0} (y^2)\ dx$$

$$V = \pi \int_{-4}^{0} \left(1 - \frac{(x+2)^2}{4}\right) dx$$

$$V = \frac{8\pi}{3}$$

Mark allocation: 2 marks

- 1 mark for the correct integral expression and terminals
- 1 mark for the final answer

TIP

» You might correctly note that this 3D shape has twice the volume of a unit sphere. If you are using this method, explain your thinking so you are awarded the method mark as well as the answer mark.

Question 9a.

Worked solution

The rate of paint flowing into the tank is equal to the rate flowing out, so we consider the change in the amount of red paint.

$\frac{dx}{dt}_{\text{IN}} = 2\ \text{L min}^{-1}$ and $\frac{dx}{dt}_{\text{OUT}} = \frac{x}{100} \times 5\ \text{L min}^{-1}$

$$\frac{dx}{dt} = \frac{dx}{dt}_{\text{IN}} - \frac{dx}{dt}_{\text{OUT}}$$

$$\frac{dx}{dt} = 2 - \frac{5x}{100} = \frac{200 - 5x}{100}$$

$$\frac{dx}{dt} = \frac{40 - x}{20}$$

Mark allocation: 2 marks

- 1 mark for correctly stating the rate of red paint in and out
- 1 mark for IN – OUT, and for simplifying the fraction

TIP

» You must remember to multiply the concentration of red paint (in litres per litre) by the rate of change of volume (in litres per minute).

Question 9b.

Worked solution

$$\frac{dt}{dx} = \frac{20}{40 - x}$$

$$t = -20\log_e|40 - x| + c$$

$$x = 40 + Ae^{-\frac{t}{20}}, \text{ so } A = \pm e^{\frac{c}{20}}.$$

Because $x = 75,\ t = 0 \Rightarrow A = 35.$

$\therefore x = 40 + 35e^{-\frac{t}{20}}$

Mark allocation: 4 marks

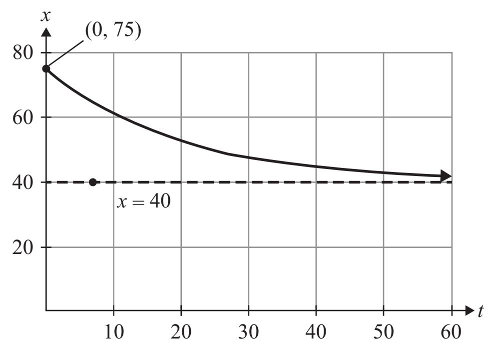
- 1 mark for finding the reciprocal
- 1 mark for finding the antiderivative
- 1 mark for substituting the initial conditions
- 1 mark for the constant of integration and the final answer

TIP

» By changing the constant of integration from $\pm c$ to A, you can remove any confusion caused by the modulus sign.

Question 9c.

Worked solution



Mark allocation: 2 marks

- 1 mark for the end point at (0, 75)
- 1 mark for the shape of the graph, and for the asymptote labelled with its equation

TIP

» You can draw the graph even if you don't correctly answer **part b.** Initially there are 75 litres of red paint, and ultimately it will approach 40 litres. Hence the shape must be as shown.

Question 9d.

Worked solution

This time the volume in the tank is reducing by 3 L min^{-1}.

$$\left.\frac{dx}{dt}\right|_{\text{IN}} = 2\,\text{L min}^{-1} \quad \text{and} \quad \left.\frac{dx}{dt}\right|_{\text{OUT}} = \frac{x}{100-3t} \times 8\,\text{L min}^{-1}$$

$$\frac{dx}{dt} = \left.\frac{dx}{dt}\right|_{\text{IN}} - \left.\frac{dx}{dt}\right|_{\text{OUT}}$$

$$\frac{dx}{dt} = 2 - \frac{8x}{100-3t}$$

Mark allocation: 2 marks

- 1 mark for a correct expression for volume: $(100-3t)$
- 1 mark for the final answer

TIP

» When a differential equation question asks you to 'write down' an equation without asking you to solve it, the answer is likely to involve two variables. In this case, the variables are x and t.

Question 10a.

Worked solution

Using a constant acceleration formula, with $u = 0,\ x = 44.1,\ a = 9.8,\ v = ?$, gives

$$v^2 = u^2 + 2ax$$

$$v^2 = 2(9.8)(44.1)$$

$$v^2 = 864.36$$

$$v = 29.4\ \text{m s}^{-1}$$

Mark allocation: 2 marks

- 1 mark for using the constant acceleration formula
- 1 mark for working leading to 29.4 m s^{-1}

TIP

» State the known values and the required variable to help you to decide which formula to use.

Question 10b.

Worked solution

$a = -0.4(v + 0.6)^2 \text{ m s}^{-2}$

$\dfrac{dv}{dt} = -\dfrac{2(v+0.6)^2}{5}$

$\dfrac{dt}{dv} = -\dfrac{5}{2(v+0.6)^2}$

$t = \displaystyle\int_{29.4}^{v} -\frac{5}{2(p+0.6)^2}\,dp$

$t = \dfrac{5}{2(v+0.6)} - \dfrac{1}{12}$

$v = \dfrac{30}{12t+1} - 0.6$ or $v = -\dfrac{3(12t-49)}{5(12t+1)}$

Mark allocation: 3 marks

- 1 mark for antidifferentiating $\dfrac{dt}{dv}$
- 1 mark for the correct use of initial conditions (terminals) to find the constant of integration
- 1 mark for the final answer in the correct form

Question 10c.

Worked solution

Up until $t = 4.08$ s, the coconut is sinking in the water. After that, it reverses direction and slowly rises to the surface.

There are three parts to the answer.

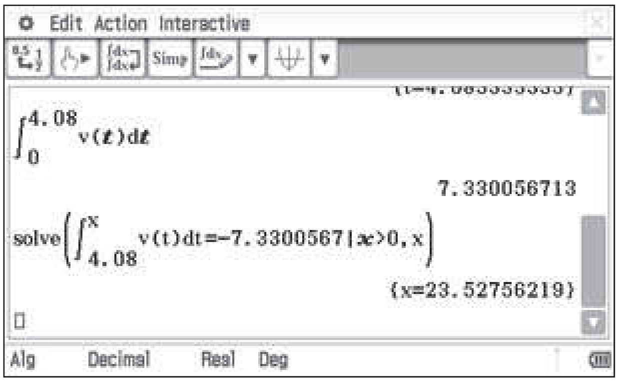
The coconut sinking underwater (a distance of 7.33 m over 4.08 s).

$v(t) = 0$

$t = 4.08$

$\displaystyle\int_0^{4.08} v(t)\,dt = 7.33$

The coconut rising in the water to the surface ($23.53 - 4.08 = 19.45$ s).



The time of 23.528 s is from the time the coconut hits the water and so already incorporates the 4.08 s from above.

The coconut initially falling through the air to get to the water in the first place, so $t = \sqrt{\frac{2 \times 44.1}{9.8}} = 3$ s.

The total of the three components is 26.53 s.

Mark allocation: 4 marks

- 1 mark for finding the time spent in the air
- 1 mark for finding the time spent sinking
- 1 mark for finding the time spent rising in the water
- 1 mark for the final answer, correct to one decimal place

Question 11a.

Worked solutions

Two equations can be written from the information provided:

$$8850 = r \cdot 3000\left(1 - \frac{3000}{K}\right) \qquad [1]$$

$$8850 \cdot \frac{5}{9} = r \cdot 6000\left(1 - \frac{6000}{K}\right) \qquad [2]$$

Solving the simultaneous equations [1] and [2] gives values of $r = 3$ and $K = 18\,000$.

Mark allocation: 2 marks

- 1 mark for the correct value of $r = 3$
- 1 mark for the correct value of $K = 18\,000$.

Question 11b.

Worked solution

Solving the differential equation using a CAS yields

$$P(t) = \frac{180\,000\,e^{3t}}{e^{3t} + 59}$$

$a = 18\,000$

$b = 3$

$c = 59$

Mark allocation: 3 marks

- 1 mark for the correct value of $a = 18\,000$
- 1 mark for the correct value of $b = 3$
- 1 mark for the correct value of $c = 59$

Question 11c.

Worked solution

The maximum rate of growth occurs at a time when

$$\frac{d^2P}{dt^2} = 0$$

$$\frac{-95\,580\,000 \cdot e^{3t}(e^{3t} - 59)}{(e^{3t} + 59)^3} = 0$$

$$t = \frac{\log_e(59)}{3}$$

$$P\left(\frac{\log_e(59)}{3}\right) = 90\,000$$

Mark allocation: 2 marks

- 1 mark for the correct time value of $t = \frac{\log_e(59)}{3}$
- 1 mark for the correct population value of $P = 90\,000$

Question 11d.

Worked solution

The population and rate of growth will both be increasing from $t = 0$ up to the point of inflection of the curve described by $P(t)$, which occurs at $\frac{\log_e(59)}{3}$ (known from **part c.**).

Therefore the correct interval is $t \in \left[0, \frac{\log_e(59)}{3}\right]$.

Mark allocation: 1 mark

- 1 mark for correct time interval value of $t \in \left[0, \frac{\log_e(59)}{3}\right]$

Question 11e.

Worked solution

The third minute is the time interval between $t = 2$ and $t = 3$.

The amount of new bacteria produced during this time interval can be found by evaluating $P(3) - P(2) = 21\,664.6 \approx 21\,665$.

Mark allocation: 1 mark

- 1 mark for the correct integer population value of 21 665 bacteria

Area of Study 5 Space and measurement

EXAM 1

Question 1

Worked solution

$x - 2 = \sin(2t)$

$y = 2\sin^2(t) = 2 \times \dfrac{1 - \cos(2t)}{2} = 1 - \cos(2t)$

$\Rightarrow 1 - y = \cos(2t)$

$\therefore (x - 2)^2 + (1 - y)^2 = \sin^2(2t) + \cos^2(2t) = 1$

or $(x - 2)^2 + (y - 1)^2 = 1$

Mark allocation: 2 marks

- 1 mark for correctly expressing y in terms of $\cos(2t)$
- 1 mark for the correct equation

TIP

» Use the double angle formula to express y in terms of $\cos(2t)$ so that the Cartesian equation is easily found.

Question 2a.

Worked solution

$\underset{\sim}{b} \cdot \underset{\sim}{c} = (2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k}) \cdot (4\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k})$

$\Rightarrow \underset{\sim}{b} \cdot \underset{\sim}{c} = 8 + 3 + 6n = 0$

$\Rightarrow 6n = -11$

$\therefore n = -\dfrac{11}{6}$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 2b.

Worked solution

$$\cos\theta = \frac{(2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k}) \cdot (2\underset{\sim}{i} + n\underset{\sim}{k})}{\sqrt{5+n^2} \times \sqrt{4+n^2}}$$

$$\Rightarrow \cos\theta = \frac{4+n^2}{\sqrt{5+n^2} \times \sqrt{4+n^2}} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \frac{\sqrt{4+n^2}}{\sqrt{5+n^2}} = \frac{3}{\sqrt{10}}$$

$$\sqrt{10(4+n^2)} = 3\sqrt{5+n^2}$$

$$\Rightarrow 40 + 10n^2 = 45 + 9n^2$$

$$\Rightarrow n^2 = 5$$

$$\therefore n = \pm\sqrt{5}$$

Mark allocation: 2 marks

- 1 mark for the correct equation for $\cos(\theta)$ in terms of n
- 1 mark for the correct answer

TIP

» The angle between a vector and a plane is the angle between the vector and the component of the vector in the direction of the plane only.

Question 2c.

Worked solution

Vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent if $\underset{\sim}{b} = m\underset{\sim}{a} + p\underset{\sim}{c}$, where $m, p \in R$ and are not both zero.

Hence $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k} = 3m\underset{\sim}{i} + 2m\underset{\sim}{j} + 2m\underset{\sim}{k} + 4p\underset{\sim}{i} + 3p\underset{\sim}{j} + 6p\underset{\sim}{k}$

$\Rightarrow 3m + 4p = 2$

$2m + 3p = 1$

$2m + 6p = n$

$3m + 4p = 2 \Rightarrow 6m + 8p = 4$

$2m + 3p = 1 \Rightarrow 6m + 9p = 3$

$\Rightarrow p = -1$

$\Rightarrow m = 2$

$\Rightarrow n = -2$

$\therefore$ Vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly independent if $n \in R\backslash\{-2\}$.

Mark allocation: 2 marks

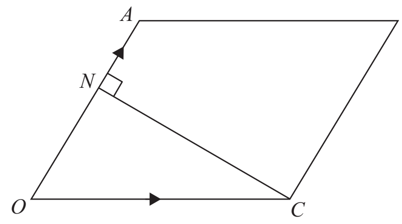
- 1 mark for finding correct values of m and p, where $\underset{\sim}{b} = m\underset{\sim}{a} + p\underset{\sim}{c}$
- 1 mark for the correct answer

TIP

» **First determine the value of n for which the vectors are linearly dependent.**

Question 3a.

Worked solution



For N to be closest to C, $\overrightarrow{CN}$ must be perpendicular to $\overrightarrow{OA}$.

Let $\overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\overrightarrow{OC} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$.

The scalar resolute of $\overrightarrow{OC}$ in the direction of the equation of $\overrightarrow{OA} = |ON|$.

$$ON = \frac{(\text{vec}(c) \cdot \text{vec}(a))}{|\text{vec}(a)|} = \frac{2 - 2 + 2}{\sqrt{1^2 + 2^2 + (-1)^2}} = \frac{2}{\sqrt{6}}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3b.

Worked solution

$$\left|\overrightarrow{OC}\right| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

Using Pythagoras' theorem:

$$\begin{aligned} CN^2 &= OC^2 - ON^2 \\ &= 9 - \frac{4}{6} \\ &= 9 - \frac{2}{3} \\ &= \frac{25}{3} \end{aligned}$$

$$\therefore CN = \frac{5}{\sqrt{3}}$$

Mark allocation: 2 marks

- 1 mark for the correct evaluation of $\left|\overrightarrow{OC}\right|$
- 1 mark for the correct answer

Question 4

Worked solution

$\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 4\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + 4t\underset{\sim}{j} + \underset{\sim}{c}$

$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{c} = -2\underset{\sim}{i} - 3\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + 4t\underset{\sim}{j} - 2\underset{\sim}{i} - 3\underset{\sim}{k}$

$\Rightarrow \dot{\underset{\sim}{r}}(t) = (2t - 2)\underset{\sim}{i} + 4t\underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{r}(t) = (t^2 - 2t)\underset{\sim}{i} + 2t^2\underset{\sim}{j} - 3t\underset{\sim}{k} + \underset{\sim}{d}$

$\underset{\sim}{r}(0) = \underset{\sim}{d} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{r}(t) = (t^2 - 2t)\underset{\sim}{i} + 2t^2\underset{\sim}{j} - 3t\underset{\sim}{k} + \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\Rightarrow \underset{\sim}{r}(t) = (t^2 - 2t + 1)\underset{\sim}{i} + (2t^2 - 1)\underset{\sim}{j} + (2 - 3t)\underset{\sim}{k}$

$\underset{\sim}{r}(2) = \underset{\sim}{i} + 7\underset{\sim}{j} - 4\underset{\sim}{k}$

$\underset{\sim}{r}(2) - \underset{\sim}{r}(0) = 8\underset{\sim}{j} - 6\underset{\sim}{k}$

$|\underset{\sim}{r}(2) - \underset{\sim}{r}(0)| = \sqrt{8^2 + (-6)^2} = 10$

Therefore the particle is 10 metres from its initial position after 2 seconds.

Mark allocation: 4 marks

- 1 mark for the correct velocity vector $\dot{\underset{\sim}{r}}(t)$
- 1 mark for the correct position vector $\underset{\sim}{r}(t)$
- 1 mark for the correct vector $\underset{\sim}{r}(2) - \underset{\sim}{r}(0)$
- 1 mark for the correct answer

Question 5

Worked solution

$x = \arcsin(t) \Rightarrow t = \sin(x)$

$y = 2t^2\sqrt{1 - t^2} \Rightarrow y = 2\sin^2(x)\sqrt{1 - \sin^2(x)}$

$y = 2\sin^2(x)\cos(x)$

Mark allocation: 1 mark

- 1 mark for the correct equation

Question 6a.

Worked solution

$|\overrightarrow{OA}| = 2|\overrightarrow{OC}|$

$\Rightarrow \sqrt{m^2 + (\sqrt{39})^2} = 2|\overrightarrow{OC}| = 20$

$m^2 + 39 = 400$

$m^2 = 361$

$m = \pm\sqrt{361} = \pm 19$

Mark allocation: 2 marks

- 1 mark for showing that $\sqrt{m^2 + (\sqrt{39})^2} = 20$
- 1 mark for the correct answer (expressed as a square root is acceptable)

TIP

» It is important to check the conditions given in the question to ensure that you give the correct number of solutions when more than one solution is possible.

Question 6b.

Worked solution

$$\frac{\overrightarrow{OA} \cdot \overrightarrow{OC}}{|\overrightarrow{OA}||\overrightarrow{OC}|} = \frac{1}{5}$$

$$\Rightarrow \frac{-6 \times m + 0 \times 8 + 0 \times \sqrt{39}}{10\sqrt{m^2 + (\sqrt{39})^2}} = \frac{1}{5}$$

$$\Rightarrow \frac{-3m}{5\sqrt{m^2 + 39}} = \frac{1}{5}$$

$$-3m = \sqrt{m^2 + 39} \quad \therefore m < 0$$

$$9m^2 = m^2 + 39$$

$$8m^2 = 39$$

$$m = \pm\sqrt{\frac{39}{8}} = \pm\frac{\sqrt{78}}{4}$$

But $m < 0, \ \therefore m = -\frac{\sqrt{78}}{4}$.

Mark allocation: 2 marks

- 1 mark for showing that $\frac{-3m}{5\sqrt{m^2 + 39}} = \frac{1}{5}$
- 1 mark for the correct answer

Question 7

Worked solution

Begin by assuming that the vectors are linearly dependent and can be expressed in the form $\underset{\sim}{c} = \gamma\underset{\sim}{a} + \lambda\underset{\sim}{b}$, where $\gamma, \lambda \in R$.

Determine the values of γ and λ by setting up and solving the following simultaneous equations.

$\underset{\sim}{i}$ components: $5 = 3\gamma - 2\lambda$ [1]

$\underset{\sim}{j}$ components: $1 = \gamma - 2\lambda$ [2]

$\underset{\sim}{k}$ components: $d = -2\gamma - 4\lambda$ [3]

[1] – [2] gives

$\Rightarrow 5 - 1 = 3\gamma - \gamma - 2\lambda - -2\lambda$

$\Rightarrow 4 = 2\gamma$

Therefore $\gamma = 2$.

Substituting $\gamma = 2$ into either equation [1] or [2] and solving for λ gives $\lambda = \frac{1}{2}$.

Substituting $\gamma = 2$ and $\lambda = \frac{1}{2}$ into equation [3] gives $d = -6$.

If $d = -6$, then the vectors are linearly dependent.

Therefore for the vectors to be linearly independent, $d \in R \backslash \{-6\}$.

Mark allocation: 3 marks

- 1 mark for expressing the vectors as a linear combination (in the form $\underset{\sim}{c} = \gamma \underset{\sim}{a} + \lambda \underset{\sim}{b}$)
- 1 mark for finding the value of d that makes the vectors linearly dependent
- 1 mark for stating that $d \in R \backslash \{-6\}$

Question 8a.

Worked solution

The centre of the base will be the mid point of the line joining the opposite corners of the square base.

The coordinate of the centre is therefore $\left(\frac{15}{2}, \frac{15}{2}, 0\right)$, with a corresponding position vector of $\overrightarrow{OM} = \frac{15}{2}\underset{\sim}{i} + \frac{15}{2}\underset{\sim}{j} + 0\underset{\sim}{k}$.

Mark allocation: 1 mark

- 1 mark for the correct position vector of $\frac{15}{2}\underset{\sim}{i} + \frac{15}{2}\underset{\sim}{j} + 0\underset{\sim}{k}$ or, more simply, $\frac{15}{2}\underset{\sim}{i} + \frac{15}{2}\underset{\sim}{j}$

Question 8b.

Worked solution

Method 1: Trigonometry

A right-angled triangle is formed by the position vectors $\overrightarrow{OM}$, $\overrightarrow{OD}$ and a straight line from the centre of the pyramid's base to its vertex.

$$\left|\overrightarrow{OM}\right| = \sqrt{\left(\frac{15}{2}\right)^2 + \left(\frac{15}{2}\right)^2} = \frac{15}{\sqrt{2}}$$

$$\left|\overrightarrow{OD}\right| = \sqrt{\left(\frac{15}{2}\right)^2 + \left(\frac{15}{2}\right)^2 + (4\sqrt{2})^2} = \sqrt{\frac{225}{2} + 32} = \frac{17}{\sqrt{2}}$$

$$\cos(\theta) = \frac{\left|\overrightarrow{OM}\right|}{\left|\overrightarrow{OD}\right|} = \frac{\frac{15}{\sqrt{2}}}{\frac{17}{\sqrt{2}}} = \frac{15}{17}$$

Method 2: Dot product

$$\left|\overrightarrow{OM}\right| = \sqrt{\left(\frac{15}{2}\right)^2 + \left(\frac{15}{2}\right)^2} = \frac{15}{\sqrt{2}}$$

$$\left|\overrightarrow{OD}\right| = \sqrt{\left(\frac{15}{2}\right)^2 + \left(\frac{15}{2}\right)^2 + (4\sqrt{2})^2} = \sqrt{\frac{225}{2} + 32} = \frac{17}{\sqrt{2}}$$

$$\overrightarrow{OM} \cdot \overrightarrow{OD} = \left(\frac{15}{2}\right)^2 + \left(\frac{15}{2}\right)^2 = \frac{225}{2}$$

$$\cos(\theta) = \frac{\overrightarrow{OM} \cdot \overrightarrow{OD}}{\left|\overrightarrow{OM}\right|\left|\overrightarrow{OD}\right|} = \frac{\frac{225}{2}}{\frac{15}{\sqrt{2}} \times \frac{17}{\sqrt{2}}} = \frac{15}{17}$$

Mark allocation: 3 marks

- 1 mark for the correct magnitude of $\overrightarrow{OM}$
- 1 mark for the correct magnitude of $\overrightarrow{OD}$
- 1 mark for the correct value for $\cos(\theta)$ of $\frac{15}{17}$

Question 8c.

Worked solution

$a = 15^2 = 225$

$h = 4\sqrt{2}$

$V = \frac{1}{3}ah = \frac{225 \cdot 4\sqrt{2}}{3} = 300\sqrt{2}$

Mark allocation: 1 mark

- 1 mark for the volume of the pyramid in the correct form: $V = 300\sqrt{2}$

TIP

» The VCAA formula sheet contains several useful measurement formulas, including that for the volume of a pyramid.

Question 9

Worked solution

$|\underset{\sim}{a}| = \sqrt{36 + 9 + 4} = \sqrt{49} = 7$

$|\underset{\sim}{b}| = \sqrt{m^2 + n^2 + 4} = 3$

$\Rightarrow m^2 + n^2 = 5$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 6m + 2n + 6$

The scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is given by

$$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$\frac{20}{3} = \frac{6m + 2n + 6}{3}$$

$$14 = 6m + 2n$$

$$7 = 3m + n$$

We now have two equations relating m and n, and they can be solved by substitution:

$n^2 = (7 - 3m)^2 = 9m^2 - 42m + 49$, so

$$\begin{aligned} m^2 + (9m^2 - 42m + 49) &= 5 \\ 10m^2 - 42m + 49 &= 5 \\ 10m^2 - 42m + 44 &= 0 \\ 5m^2 - 21m + 22 &= 0 \\ 5m^2 - 10m - 11m + 22 &= 0 \\ 5m(m - 2) - 11(m - 2) &= 0 \\ (5m - 11)(m - 2) &= 0 \\ m &= 2, \frac{11}{5} \\ m &= 2, \text{ as } m \in Z \\ n &= 7 - 3(2) = 1 \end{aligned}$$

$\therefore\ m = 2,\ n = 1$

Mark allocation: 4 marks

- 1 mark for determining the first equation relating m and n: $m^2 + n^2 = 5$, or equivalent
- 1 mark for determining the second equation relating m and n: $7 = 3m + n$, or equivalent
- 1 mark for the correct value of m: 2
- 1 mark for the correct value of n: 1

Question 10a.

Worked solution

$$\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = -4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$\overrightarrow{PR} = \begin{pmatrix} -2 \\ -4 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ 2 \end{pmatrix} = -5\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}$$

Mark allocation: 2 marks

- 1 mark for the correct expression for $\overrightarrow{PQ}$
- 1 mark for the correct expression for $\overrightarrow{PR}$

Question 10b.

Worked solution

A vector normal to the plane containing P, Q and R can be found by evaluating the vector cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\underset{\sim}{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & 2 & -2 \\ -5 & -5 & 2 \end{vmatrix}$$

$$= (4 - 10)\underset{\sim}{i} - (-8 - 10)\underset{\sim}{j} + (20 + 10)\underset{\sim}{k} = -6\underset{\sim}{i} + 18\underset{\sim}{j} + 30\underset{\sim}{k}$$

Simplifying with a common factor of 6 gives $\underset{\sim}{n} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$.

Mark allocation: 2 marks

- 1 mark for the correct determinant or correct formula to evaluate the cross product
- 1 mark for the correct vector for $\underset{\sim}{n}$.

Note: Any vector parallel to $\underset{\sim}{n} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$ should be taken as a correct answer (i.e. vectors of the form $m(-\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}),\ m \in R$.)

Question 10c.

Worked solution

The vector equation of the plane can be determined using the normal vector $\underset{\sim}{n}$ found previously in **part b.**

$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$, where $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underset{\sim}{a}$ is the position vector of any point on the plane.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$$

Evaluating the above scalar product yields a Cartesian equation for the plane of $x - 3y - 5z = -25$.

Mark allocation: 2 marks

- 1 mark for the correct vector expression for the plane, found using the scalar product or equivalent valid method
- 1 mark for the correct Cartesian equation of the plane of $x - 3y - 5z = -25$ or equivalent

Question 11a.

Worked solution

The coordinates of any point on L will be described by

$$\underset{\sim}{r}(t) = \begin{pmatrix} 2 + 3t \\ 3 - t \\ -4 + 2t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Substituting the coordinates of a point on L into the equation of plane Π, then solving for t, gives

$$\begin{aligned} (2 + 3t) + 3(3 - t) - 5(-4 + 2t) &= 9 \\ -10t + 31 &= 9 \\ -10t &= -22 \\ t &= \frac{11}{5} \end{aligned}$$

It follows that the intersection point will be described by

$$\underset{\sim}{r}(2) = \begin{pmatrix} 2 + 3(2) \\ 3 - (2) \\ -4 + 2(2) \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix}$$

Therefore the point of intersection of L and Π is $\left(\frac{43}{5}, \frac{4}{5}, \frac{2}{5}\right)$.

Mark allocation: 3 marks

- 1 mark for correctly substituting values for x, y, z in terms of t into equation of plane Π
- 1 mark for the correct value of $t = 2$ for the intersection point
- 1 mark for the correct coordinates of the point of intersection: $(8, 1, 0)$

Question 11b.

Worked solution

Let ϕ be the angle between L and the normal vector to plane Π.

By symmetry, we know that $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) = \cos(\phi)$.

Using the scalar product to find $\cos(\phi)$:

$$\cos(\phi) = \frac{\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}}{\left|\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\right|\left|\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}\right|} = \frac{-10}{\sqrt{14} \cdot \sqrt{35}} = -\frac{\sqrt{10}}{7}$$

As ϕ is acute, $\cos(\phi) > 0$.

Therefore $\cos(\phi) = \frac{\sqrt{10}}{7}$.

Therefore $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) = \cos(\phi) = \frac{\sqrt{10}}{7}$.

Mark allocation: 3 marks

- 1 mark for choosing the correct vectors to find the angle between L and the normal vector to plane Π
- 1 mark for setting up the correct scalar product expression or equivalent to determine the cosine of the angle
- 1 mark for the correct value of $\sin(\theta)$, in the specified form of $\frac{\sqrt{a}}{b}$

Question 12a.

Worked solution

Points A, B and C will have position vectors of $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$, respectively.

A vector equation of a line passing through A and B can be found by evaluating

$$\begin{aligned} \underset{\sim}{r}(t) &= \underset{\sim}{a} + t(\underset{\sim}{b} - \underset{\sim}{a}) \\ &= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t\left(\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right) \end{aligned}$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -7 \\ 4 \end{pmatrix}, \quad t \in R$$

Mark allocation: 2 marks

- 1 mark for the correct vector expression for a line using the position vectors of any two points on the line
- 1 mark for any correct vector equation of a line through A and B. **Note:** There are many correct expressions. The expression given by the worked solution is the most common one.

Question 12b.

Worked solution

$$\text{Area} = \frac{1}{2}\left|(\underset{\sim}{b} - \underset{\sim}{a}) \times (\underset{\sim}{c} - \underset{\sim}{a})\right|$$

$$= \frac{1}{2}\left\| \begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -7 & 4 \\ -1 & -5 & 3 \end{matrix} \right\|$$

$$= \frac{1}{2}\left|-\underset{\sim}{i} - 7\underset{\sim}{j} - 12\underset{\sim}{k}\right|$$

$$= \frac{\sqrt{194}}{2}$$

Mark allocation: 3 marks

- 1 mark for the correct expression, found using vector cross product, which could be evaluated to find the required area
- 1 mark for the correct evaluation of vector cross product
- 1 mark for the correct area

Question 12c.

Worked solution

Possible position vectors and coordinates of D are

$$D_1 = \underset{\sim}{a} + (\underset{\sim}{b} - \underset{\sim}{c}) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

D_1 has coordinates $(4, 1, 2)$.

$$D_2 = \underset{\sim}{a} + (\underset{\sim}{c} - \underset{\sim}{b}) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

D_2 has coordinates $(0, 5, 0)$.

$$D_3 = \underset{\sim}{b} + (\underset{\sim}{c} - \underset{\sim}{a}) = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -9 \\ 8 \end{pmatrix}$$

D_3 has coordinates $(2, -9, 8)$.

Mark allocation: 3 marks

- 1 mark for each correct possible coordinate of point D (up to 3 marks)

Question 13a.

Worked solution

From the vector equation L_1 it can be seen that a point on L_1 will have position vector $\begin{pmatrix} 1+2t \\ 4+t \\ t \end{pmatrix}$.

The minimum distance from A to L_1 will occur when the vector from A to L_1 is perpendicular to the direction of L_1.

The vector from A to L_1 is $\begin{pmatrix} 1+2t-3 \\ 4+t+2 \\ t+2 \end{pmatrix} = \begin{pmatrix} 2t-2 \\ t+6 \\ t+2 \end{pmatrix}$.

Now use the dot product to find the value of t such that the vector from A to L_1 is perpendicular to the direction of L_1.

$$\begin{pmatrix} 2t-2 \\ t+6 \\ t+2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$2(2t-2)+(t+6)+(t+2)=0$$

$$6t+4=0$$

$$t=-\frac{2}{3}$$

Therefore the point closest to A has position vector

$\underset{\sim}{r}_1\left(\frac{-2}{3}\right) = \underset{\sim}{i} + 4\underset{\sim}{j} - \frac{2}{3}(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) = -\frac{1}{3}\underset{\sim}{i} + \frac{10}{3}\underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$ and has coordinates $\left(-\frac{1}{3}, \frac{10}{3}, -\frac{2}{3}\right)$.

Mark allocation: 3 marks

- 1 mark for the correct scalar product expression to find the value of t such that the vector from A to L_1 is perpendicular to the direction of L_1
- 1 mark for the correct value of $t = -\frac{2}{3}$
- 1 mark for the correct coordinates of closest point: $\left(-\frac{1}{3}, \frac{10}{3}, -\frac{2}{3}\right)$

Question 13b.

Worked solution

The minimum distance is given by

$$\left|\begin{pmatrix} -\frac{1}{3} \\ \frac{10}{3} \\ -\frac{2}{3} \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}\right| = \left|\begin{pmatrix} -\frac{10}{3} \\ \frac{16}{3} \\ \frac{4}{3} \end{pmatrix}\right| = \frac{2\sqrt{93}}{3}$$

Mark allocation: 1 mark

- 1 mark for the correct minimum distance of $\frac{2\sqrt{93}}{3}$ units

Question 13c.

Worked solution

L_2 will have the vector equation $\underset{\sim}{r}_2(s) = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k} + s(-8\underset{\sim}{i} + 3\underset{\sim}{j} + -\underset{\sim}{k}),\ s \in R$.

L_1 and L_2 intersect when

$$\begin{pmatrix} 1+2t \\ 4+t \\ t \end{pmatrix} = \begin{pmatrix} 3-8s \\ -2+3s \\ -2-s \end{pmatrix}$$

The above vector equation yields three linear equations for s and t:

$1 + 2t = 3 - 8s$ [1]

$4 + t = -2 + 3s$ [2]

$t = -2 - s$ [3]

Substituting equation [3] into [2] gives

$$\begin{aligned} 4 + (-2 - s) &= -2 + 3s \\ s &= 1 \end{aligned}$$

Evaluating equation [3] with $s = 1$ gives $t = -2 - 1 = -3$.

Verifying that solution for s and t also works in equation [1]:

$$\begin{aligned} 1 + 2(-3) &= 3 - 8(1) \\ -5 &= -5 \end{aligned}$$

Therefore L_1 and L_2 intersect when $s = 1$ and $t = -3$.

The point of intersection will have position vector $\underset{\sim}{r}_1(-3) = \underset{\sim}{r}_2(1) = \begin{pmatrix} -5 \\ 1 \\ -3 \end{pmatrix}$, which corresponds to coordinates $(-5, 1, -3)$.

Mark allocation: 4 marks

- 1 mark for the correct vector equation of L_2
- 1 mark for the equation position vectors of points on L_1 and L_2
- 1 mark for the correct values of parameters s and t that correspond to the point of intersection of $s = 1$ and $t = -3$
- 1 mark for the correct coordinates of the point of intersection: $(-5, 1, -3)$

Question 14a.

Worked solution

For two planes to be perpendicular, the scalar (dot) product of the two normal vectors must be zero.

$$\begin{aligned} \left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) \cdot \left(\underset{\sim}{i} + \underset{\sim}{j} + p\underset{\sim}{k}\right) &= 0 \\ 1 - 2 - p &= 0 \\ p &= 3 \end{aligned}$$

Mark allocation: 1 mark

- 1 mark for equating the dot product of the two normal vectors equal to zero and solving for p

Question 14b.

Worked solution

We can find the cross product of the normal vectors of Π_1 and Π_2 to obtain a vector that is perpendicular to both normal vectors Π_1 and Π_2.

$$\underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -1 \\ 1 & 1 & 3 \end{vmatrix} = 7\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k}$$

Therefore the Cartesian equation of a plane is given by $7x - 4y - z = q$, where q is a constant.

Substituting the point $(1, 2, 1)$ to determine the value of q gives $q = -2$.

Therefore the Cartesian equation of the plane is given by $7x - 4y - z = -2$.

Mark allocation: 2 marks

- 1 mark for determining the cross product of the normal vectors of Π_1 and Π_2: $\underset{\sim}{n} = 7\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k}$
- 1 mark for correct answer of $7x - 4y - z = -2$

Question 14c.

Worked solution

To find the distance from A to B, we must first find the coordinates of A and B.

The equation of line L can be expressed in parametric form to determine the coordinates of any point along the line.

$(x, y, z) = (1 + t,\ 1 + 3t,\ 1 - 6t)$

To find point A, the point $(x, y, z) = (1 + t,\ 1 + 3t,\ 1 - 6t)$ can be substituted into the Cartesian equation of Π_1 and then solve for t.

$$\begin{aligned} (1 + t) + 2(1 + 3t) - (1 - 6t) &= 15 \\ 2 + 13t &= 15 \\ t &= 1 \end{aligned}$$

Therefore, the coordinates of A are $(2, 4, -2)$.

To find point A, the point $(x, y, z) = (1 + t,\ 1 + 3t,\ 1 - 6t)$ can be substituted into the Cartesian equation of Π_2 and then solve for t.

$$\begin{aligned} (1 + t) + (1 + 3t) + 3(1 - 6t) &= 5 \\ 5 - 14t &= 5 \\ t &= 0 \end{aligned}$$

Therefore the coordinates of B are $(1, 1, 1)$.

The vector $\overrightarrow{AB}$ is given by $-\underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}$.

Therefore $\left|\overrightarrow{AB}\right| = \sqrt{19}$.

Mark allocation: 3 marks

- 1 mark for determining the coordinate of A
- 1 mark for determining the coordinate of B
- 1 mark for correct answer of $\sqrt{19}$

EXAM 2

Section A

Question 1

Answer: ***B***

Worked solution

The scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is given by $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$, where $\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \frac{1}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$. So

$$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{1}{3}(3\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}) \cdot (2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) = 5$$

$$= \frac{1}{3}(3 + 2m) = 5$$

$\Rightarrow m = 6$

Question 2

Answer: ***B***

Worked solution

Velocity vectors of the two particles are $\dot{\underset{\sim}{r}} = 2\underset{\sim}{i} + -4\underset{\sim}{j}$ and $\dot{\underset{\sim}{s}} = 6t\underset{\sim}{i} + 3\underset{\sim}{j}$.

Two vectors are perpendicular when their dot products are equal to zero. So

$$\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{s}} = 2 \times 6t + -4 \times 3 = 0$$

$$= 12t - 12 = 0$$

$\Rightarrow t = 1$

Question 3

Answer: ***D***

Worked solution

Start off by finding the value of λ that makes the vectors linearly dependent.

Write each component of $\underset{\sim}{c}$ as a linear combination of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$\underset{\sim}{i}$ component: $2m - n = 3$ [1]

$\underset{\sim}{j}$ component: $-m + 3n = 1$ [2]

$\underset{\sim}{k}$ component: $2m - 2n = \lambda$ [3]

Rearranging equation [1] gives $n = 2m - 3$.

Substituting n into equation [2] and solving for m and n gives $m = 2$, $n = 1$.

Substituting m and n into equation [3] gives $\lambda = 2$.

This value of λ makes the vectors linearly dependent, so option A is incorrect because the question is asking for values that make the vectors linearly independent.

Options B and C are all incorrect because they include $\lambda = 2$ in their possible values of λ. Therefore option D is correct because it provides values of λ that make the vectors linearly independent.

> **TIP**
>
> » **When attempting questions about linearly dependent/independent vectors, start off by showing linear dependence. Then take the time to carefully read your answer so that you are giving the correct response for the context of the question; that is, if the question is about linear independence, you give the response for linear independence and not linear dependence.**

Question 4

Answer: ***B***

Worked solution

The vector resolute of $\underset{\sim}{a} = \underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$ perpendicular to $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ is given by

$$\begin{aligned}\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} &= (\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}) - \left((\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}) \cdot \frac{1}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})\right)\frac{1}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) \\ &= \frac{1}{9}(-7\underset{\sim}{i} + 20\underset{\sim}{j} + 26\underset{\sim}{k})\end{aligned}$$

> **TIP**
>
> » **Ensure you are familiar with the language of the vector projections, in particular which formula is associated with the projection. This will help you choose the correct formula when given the context of the question.**

Question 5

Answer: ***A***

Worked solution

The scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$ is given by $\underset{\sim}{b} \cdot \hat{\underset{\sim}{a}}$, where $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{1}{5}(-4\underset{\sim}{i} + 3\underset{\sim}{k})$. So

$$\begin{aligned}\underset{\sim}{b} \cdot \hat{\underset{\sim}{a}} &= \frac{1}{5}(2\underset{\sim}{i} - 3\underset{\sim}{j} + c\underset{\sim}{k}) \cdot (-4\underset{\sim}{i} + 3\underset{\sim}{k}) \\ &= \frac{1}{5}(-8 + 3c) \\ &= 8 \\ \Rightarrow c &= 16\end{aligned}$$

Question 6

Answer: ***D***

Worked solution

If the vectors are linearly dependent, then $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.

Equating the components of the vectors gives

$\underset{\sim}{i}$ component: $1 = 4m - \frac{3}{2}n$

$\underset{\sim}{j}$ component: $1 = -6m + 6n$

$\underset{\sim}{k}$ component: $y = 2m - 12n$

Rearranging the $\underset{\sim}{j}$ component gives $n = \frac{1 + 6m}{6}$.

Substituting n into the $\underset{\sim}{i}$ component and solving for m and n gives $m = \frac{1}{2}$, $n = \frac{2}{3}$.

Substituting m and n into the $\underset{\sim}{k}$ component gives $y = -7$.

Only option D matches this value of y.

Question 7

Answer: ***A***

Worked solution

The angle between the vector $\underset{\sim}{a}$ and the y-axis is given by $\theta = \cos^{-1}\left(\frac{\underset{\sim}{a}_j}{|\underset{\sim}{a}|}\right)$, where $\underset{\sim}{a}_j$ is the vector made of the components of $\underset{\sim}{a}$ in the direction of the y-axis. So

$$\theta = \cos^{-1}\left(\frac{-1}{|2\underset{\sim}{i} - 1\underset{\sim}{j} + 2\underset{\sim}{k}|}\right)$$

$$= \cos^{-1}\left(\frac{-1}{3}\right)$$

$$= 109.471$$

$$\approx 109°$$

Question 8

Answer: ***D***

Worked solution

$\overrightarrow{MP} = 2\underset{\sim}{i} + 7\underset{\sim}{j} - 3\underset{\sim}{k}$, $\overrightarrow{OP} = -5\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$

Since M, P and Q are collinear (i.e. $\overrightarrow{MQ} = k\overrightarrow{PQ}$) and $|\overrightarrow{MP}| = |\overrightarrow{QP}|$, it follows that $\overrightarrow{MP} = \overrightarrow{PQ}$.

$\therefore \ \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OP} + \overrightarrow{MP} = -3\underset{\sim}{i} + 9\underset{\sim}{j} - 7\underset{\sim}{k}$

Question 9

Answer: ***B***

Worked solution

Let the angle be θ.

$$\cos(\theta) = \frac{(2\underset{\sim}{i} + 5\underset{\sim}{k})\cdot(m\underset{\sim}{i} - 3\underset{\sim}{j})}{|2\underset{\sim}{i} + 5\underset{\sim}{k}|\,|m\underset{\sim}{i} - 3\underset{\sim}{j}|} = \frac{2m}{\sqrt{2^2+5^2}\sqrt{m^2+(-3)^2}}$$

Given that θ is less than $\frac{\pi}{2}$, $0 < \cos(\theta) < 1$, then $0 < \frac{2m}{\sqrt{29(m^2+9)}} < 1$.

Consider $\frac{2m}{\sqrt{29(m^2+9)}} < 1$:

$2m < \sqrt{29(m^2+9)}$ is true for $m \in [0, \infty)$.

Now consider $\frac{2m}{\sqrt{29(m^2+9)}} > 0$:

$2m > 0$

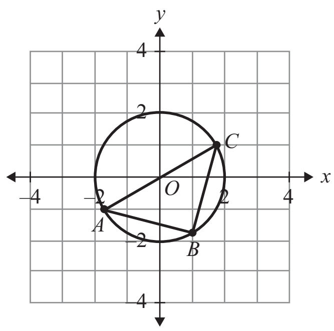
$\therefore\ m > 0$

Question 10

Answer: ***B***

Worked solution

A sketch of the arrangement is shown below.



A circle of origin (0, 0) is shown with a radius of 2.

It can be seen that the line $\overrightarrow{AC}$ passes through the centre of the circle and is the diameter of the circle. From the vector proof, the angle subtended by a diameter in a circle is a right angle. So $\angle ABC$ is a right angle.

Therefore option B is true.

Alternatively, exploring the side lengths:

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(-\sqrt{3}-1\right)^2 + \left(\sqrt{3}-1\right)^2} = 2\sqrt{2}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{\left(2\sqrt{3}\right)^2 + (2)^2} = 4$$

$$\left|\overrightarrow{BC}\right| = \sqrt{\left(\sqrt{3}-1\right)^2 + \left(1+\sqrt{3}\right)^2} = 2\sqrt{2}$$

As $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right|$, we can say that the triangle is isosceles, which eliminates option A.

The diameter of the circle is 4 units, so the base of the triangle formed is 4 and the height is 2. So the area of the triangle is 4 square units, which eliminates option D.

Finding the dot product of $\overrightarrow{BC}$ and $\overrightarrow{BA}$:

$\overrightarrow{BA} = (-\sqrt{3} - 1)\underset{\sim}{i} + (\sqrt{3} - 1)\underset{\sim}{j}$

$\overrightarrow{BC} = (\sqrt{3} - 1)\underset{\sim}{i} + (\sqrt{3} + 1)\underset{\sim}{j}$

$$\begin{aligned}\Rightarrow \overrightarrow{BA} \cdot \overrightarrow{BC} &= (-\sqrt{3} - 1)(\sqrt{3} - 1) + (\sqrt{3} - 1)(\sqrt{3} + 1) \\ &= -2 + 2 \\ &= 0\end{aligned}$$

Therefore there is a right angle at $\angle ABC$, making option B true. This also makes option C false because if one angle in a triangle is a right angle, the remaining two must be acute angles; therefore there are no obtuse angles in the triangle.

Question 11

Answer: ***C***

Worked solution

$$\begin{aligned}\cos(\theta) &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} \\ &= \frac{(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}) \cdot (5\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j} + 3\underset{\sim}{k})}{\sqrt{2^2 + (-1)^2 + (-2)^2}\sqrt{5^2 + (\sqrt{2})^2 + 3^2}} \\ &= \frac{4 - \sqrt{2}}{18}\end{aligned}$$

$$\begin{aligned}\text{Then } \cos(2\theta) &= 2\cos^2(\theta) - 1 \\ &= 2 \times \left(\frac{4 - \sqrt{2}}{18}\right)^2 - 1 \\ &= -\frac{4\sqrt{2}}{81} - \frac{8}{9}\end{aligned}$$

Question 12

Answer: ***D***

Worked solution

To find the area, the vector component of $\overrightarrow{PQ}$ perpendicular to $\overrightarrow{PS}$ is needed to find the height of the parallelogram.

$$\begin{aligned}\overrightarrow{PQ} - \left(\overrightarrow{PQ} \cdot \overrightarrow{PS}\right)\overrightarrow{PS} &= 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} - \left[(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) \cdot \frac{(-6\underset{\sim}{i} + 8\underset{\sim}{k})}{10}\right]\frac{(-6\underset{\sim}{i} + 8\underset{\sim}{k})}{10} \\ &= 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} - [-2]\frac{(-6\underset{\sim}{i} + 8\underset{\sim}{k})}{10} \\ &= 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + \frac{(-6\underset{\sim}{i} + 8\underset{\sim}{k})}{5} \\ &= \frac{4}{5}\underset{\sim}{i} + 2\underset{\sim}{j} + \frac{3}{5}\underset{\sim}{k}\end{aligned}$$

The area of the parallelogram will be

$$\left|\overrightarrow{PQ} - \left(\overrightarrow{PQ} \cdot \overrightarrow{PS}\right)\overrightarrow{PS}\right|\left|\overrightarrow{PS}\right| = \left|\frac{4}{5}\underset{\sim}{i} + 2\underset{\sim}{j} + \frac{3}{5}\underset{\sim}{k}\right|\left|-6\underset{\sim}{i} + 8\underset{\sim}{k}\right|$$

$$= 10\sqrt{5}$$

Alternatively, the area of the parallelogram can be found by calculating $\left|\overrightarrow{PQ} \times \overrightarrow{PS}\right|$.

$$\overrightarrow{PQ} \times \overrightarrow{PS} = 16\underset{\sim}{i} - 10\underset{\sim}{j} + 12\underset{\sim}{k}$$

$$\left|\overrightarrow{PQ} \times \overrightarrow{PS}\right| = 10\sqrt{5}$$

Question 13

Answer: ***C***

Worked solution

First, use the modulus to find a relationship between a and b.

$$\underset{\sim}{v} = a\underset{\sim}{i} - 6\underset{\sim}{j} + b\underset{\sim}{k}$$

$$|\underset{\sim}{v}| = \sqrt{(a)^2 + (-6)^2 + (b)^2} = \sqrt{36 + a^2 + b^2}$$

$$2\sqrt{14} = \sqrt{36 + a^2 + b^2}$$

$$56 = 36 + a^2 + b^2$$

$$20 = a^2 + b^2$$

Now use the scalar (dot) product to find a further relationship.

$$\underset{\sim}{v} \cdot \underset{\sim}{u} = 0$$

$$\left(a\underset{\sim}{i} - 6\underset{\sim}{j} + b\underset{\sim}{k}\right) \cdot \left(5\underset{\sim}{i} + 2\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}\right) = 0$$

$$5a - 12 + \frac{b}{2} = 0$$

$$10a + b = 24$$

$$b = 24 - 10a$$

Finally, solve the two equations simultaneously.

$$20 = a^2 + b^2$$

$$b = 24 - 10a$$

$$20 = a^2 + (24 - 10a)^2$$

$$a = 2,\ b = 4$$

and

$$a = \frac{278}{101},\ b = -\frac{356}{101}$$

Question 14

Answer: ***A***

Worked solution

First, antidifferentiate using the initial position to find the position vector as a function of time.

$$\dot{\underset{\sim}{r}}(t) = 3t\underset{\sim}{i} - e^t\underset{\sim}{j}$$

$$\therefore \underset{\sim}{r}(t) = \frac{3t^2}{2}\underset{\sim}{i} - e^t\underset{\sim}{j} + \underset{\sim}{c}$$

$$\underset{\sim}{r}(0) = 7\underset{\sim}{i} + \sqrt{3}\,\underset{\sim}{j} - \underset{\sim}{k}$$

$$\therefore \underset{\sim}{c} = 7\underset{\sim}{i} + (\sqrt{3} + 1)\underset{\sim}{j} - \underset{\sim}{k}$$

$$\therefore \underset{\sim}{r}(t) = \left(\frac{3t^2}{2} + 7\right)\underset{\sim}{i} + (\sqrt{3} + 1 - e^t)\underset{\sim}{j} - \underset{\sim}{k}$$

Now, find the position at the two end points (i.e. 2 and 6) and divide by 4.

$$\underset{\sim}{r}(t) = \left(\frac{3t^2}{2} + 7\right)\underset{\sim}{i} + (\sqrt{3} + 1 - e^t)\underset{\sim}{j} - \underset{\sim}{k}$$

$$v_{\text{av}} = \frac{\underset{\sim}{r}(6) - \underset{\sim}{r}(2)}{6 - 2}$$

$$v_{\text{av}} = \frac{\left(61\underset{\sim}{i} + (\sqrt{3} + 1 - e^6)\underset{\sim}{j} - \underset{\sim}{k}\right) - \left(13\underset{\sim}{i} + (\sqrt{3} + 1 - e^2)\underset{\sim}{j} - \underset{\sim}{k}\right)}{4}$$

$$v_{\text{av}} = \frac{48\underset{\sim}{i} + (e^2 - e^6)\underset{\sim}{j}}{4}$$

$$v_{\text{av}} = 12\underset{\sim}{i} + \frac{(e^2 - e^6)}{4}\underset{\sim}{j}$$

Question 15

Answer: ***C***

Worked solution

Points A and B have position vectors $\underset{\sim}{a} = \underset{\sim}{i} - 3\underset{\sim}{j} + 7\underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}$, respectively.

The direction of line segment $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = 4\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k} - \left(\underset{\sim}{i} - 3\underset{\sim}{j} + 7\underset{\sim}{k}\right) = 3\underset{\sim}{i} - 3\underset{\sim}{j} - 9\underset{\sim}{k}$.

Only options C and D describe a line that is perpendicular to line segment $\overrightarrow{AB}$ because the scalar product of $3\underset{\sim}{i} - 3\underset{\sim}{j} - 9\underset{\sim}{k}$ and $4\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ is zero.

The line described by option C passes through point A, whereas the line described by option D passes through point B. Therefore option C is correct.

Question 16

Answer: ***D***

Worked solution

The signed distance between Π_1 and the origin is $\frac{14}{\sqrt{2^2 + 3^2 + 6^2}} = 2$.

The signed distance between Π_2 and the origin is $\frac{-21}{\sqrt{2^2 + 3^2 + 6^2}} = -3$.

Therefore the distance between Π_1 and Π_2 is 5.

To be the same distance from Π_1 as Π_2, Π_3 must be 7 units from the origin.

As Π_3 is parallel to Π_1, its equation should be of the form $2x + 3y + 6z = 49$.

If we multiply this equation by a factor of 3, we obtain $6x + 9y + 18z = 147$.

Therefore option D is correct.

Question 17

Answer: ***C***

Worked solution

By inspection, the direction of the line is $\underset{\sim}{d} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$ and a normal vector to the plane is $\underset{\sim}{n} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 4\underset{\sim}{k}$.The angle between $\underset{\sim}{d}$ and $\underset{\sim}{n}$ is given by $\theta = \cos^{-1}\left(\frac{\underset{\sim}{d} \cdot \underset{\sim}{n}}{|\underset{\sim}{d}||\underset{\sim}{n}|}\right) \approx 19.0°$.

The angle between the line and the plane is therefore $90° - 19.0° = 71.0°$, giving option B as the correct answer.

Section B

Question 1a.

Worked solution

Let $x = 2 - 3\cos(t)$ and $y = 1 - 2\sin(t)$.

Hence $\cos(t) = \frac{x-2}{-3}$ and $\sin(t) = \frac{y-1}{-2}$.

Recall that $\cos^2(t) + \sin^2(t) = 1$. It follows that

$$\left(\frac{x-2}{-3}\right)^2 + \left(\frac{y-1}{-2}\right)^2 = 1$$

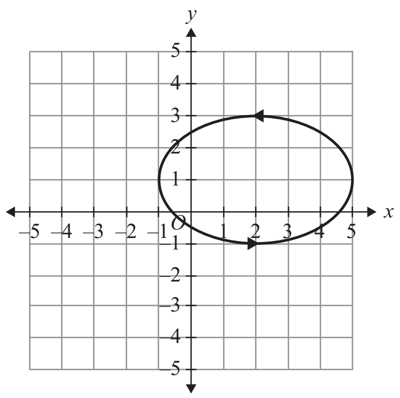
$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$$

Mark allocation: 2 marks

- 1 mark for $\cos(t) = \frac{x-2}{-3}$ and $\sin(t) = \frac{y-1}{-2}$
- 1 mark for $\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$

Question 1b.

Worked solution



When $t = 0$, $x = -1$ and $y = 1$.

When $t = \frac{\pi}{2}$, $x = 2$ and $y = -1$.

When $t = \pi$, $x = 5$ and $y = 1$.

Mark allocation: 2 marks

- 1 mark for accurately drawing the ellipse
- 1 mark for at least one arrow showing an anti-clockwise direction

TIP

» Testing different values of t in parametric equations to find the Cartesian coordinates is a useful way of determining the direction of motion.

Question 1c.

Worked solution

The $\underset{\sim}{i}$ component of the position vector of the walker and the dog is the same for all time, so it needs to be shown that the $\underset{\sim}{j}$ component of the position vector of the walker and the dog is the same at $t = \frac{5\pi}{6}$.

Walker:

When $t = \frac{5\pi}{6}$, the $\underset{\sim}{j}$ component is $1 - 2\sin\left(\frac{5\pi}{6}\right)\underset{\sim}{j} = 1 - 1\underset{\sim}{j} = 0\underset{\sim}{j}$.

Dog:

When $t = \frac{5\pi}{6}$, the $\underset{\sim}{j}$ component is $\frac{24 \times 5\pi}{5\pi \times 6} - 4\underset{\sim}{j} = 4 - 4\underset{\sim}{j} = 0\underset{\sim}{j}$.

Since the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of $\underset{\sim}{r}_W$ and $\underset{\sim}{r}_D$ are the same at $t = \frac{5\pi}{6}$, then the walker and the dog collide at $t = \frac{5\pi}{6}$.

Mark allocation: 2 marks

- 1 mark for equating the $\underset{\sim}{j}$ components of the position vectors at $t = \frac{5\pi}{6}$
- 1 mark for a concluding statement that the walker and the dog collide because they have the same position vector at $t = \frac{5\pi}{6}$

Question 1d.

Worked solution

The velocity vectors can be found by differentiating the position vectors:

$\dot{\underset{\sim}{r}}_W = 3\sin(t)\underset{\sim}{i} - 2\cos(t)\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}_D = 3\sin(t)\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}$

When $t = \frac{5\pi}{6}$:

$\dot{\underset{\sim}{r}}_W = 3\sin\left(\frac{5\pi}{6}\right)\underset{\sim}{i} - 2\cos\left(\frac{5\pi}{6}\right)\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}_D = 3\sin\left(\frac{5\pi}{6}\right)\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}_W = \frac{3}{2}\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}_D = \frac{3}{2}\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}$

The angle between the velocity vectors is given by

$$\theta = \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_W \cdot \dot{\underset{\sim}{r}}_D}{|\dot{\underset{\sim}{r}}_W||\dot{\underset{\sim}{r}}_D|}\right)$$

$$= \cos^{-1}\left(\frac{\left(\frac{3}{2}\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}\right)\cdot\left(\frac{3}{2}\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}\right)}{\sqrt{\left(\frac{3}{2}\right)^2 + (-\sqrt{3})^2}\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{24}{5\pi}\right)^2}}\right)$$

$$= \cos^{-1}\left(\frac{\frac{9}{4} + \frac{24\sqrt{3}}{5\pi}}{\sqrt{\left(\frac{3}{2}\right)^2 + (\sqrt{3})^2}\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{24}{5\pi}\right)^2}}\right)$$

$$= 3.579946343$$

$$= 3.6°$$

Mark allocation: 3 marks

- 1 mark for correctly differentiating the position vectors
- 1 mark for using $\theta = \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_W \cdot \dot{\underset{\sim}{r}}_D}{|\dot{\underset{\sim}{r}}_W||\dot{\underset{\sim}{r}}_D|}\right)$ at $t = \frac{5\pi}{6}$ to find the angle between the velocity vectors
- 1 mark for the correct angle: $\theta = 3.6°$

Question 1e.

Worked solution

Speed of the dog is given by speed = $|\dot{\underset{\sim}{r}}_D|$.

So at $t = \frac{5\pi}{6}$:

speed = $|\dot{\underset{\sim}{r}}_D|$

$$= \left|\frac{3}{2}\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{24}{5\pi}\right)^2} = 2.14 \text{ ms}^{-1}$$

Mark allocation: 2 marks

- 1 mark for showing $\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{24}{5\pi}\right)^2}$
- 1 mark for the correct answer: 2.14 ms^{-1}

Question 2a.i.

Worked solution

The distance can be found using the formula $|\underset{\sim}{r}_G(t_2) - \underset{\sim}{r}_G(t_1)|$.

Substituting $t_1 = 0$ and $t_2 = \frac{3\pi}{4}$ and then evaluating gives

$$\left|\underset{\sim}{r}_G\left(\frac{3\pi}{4}\right) - \underset{\sim}{r}_G(0)\right| = \left|\left(1 - 2\cos\left(2 \times \frac{3\pi}{4}\right) - 1 + 2\cos(2 \times 0)\right)\underset{\sim}{i} + \left(2\sin\left(2 \times \frac{3\pi}{4}\right) + 3 - 2\sin(0) - 3\right)\underset{\sim}{j}\right|$$

$$= \sqrt{2^2 + (-2)^2}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2} \text{ km}$$

Mark allocation: 1 mark

- 1 mark for correct final answer: $2\sqrt{2}$ km

Question 2a.ii.

Worked solution

Speed is given by $\left|\dot{\underset{\sim}{r}}_G(t)\right|$. So $\dot{\underset{\sim}{r}}_G = 4\sin(2t)\underset{\sim}{i} + 4\cos(2t)\underset{\sim}{j}$. Then

$$\left|\dot{\underset{\sim}{r}}_G\left(\frac{3\pi}{4}\right)\right| = \left|4\sin\left(2\times\frac{3\pi}{4}\right)\underset{\sim}{i} + 4\cos\left(2\times\frac{3\pi}{4}\right)\underset{\sim}{j}\right| = \sqrt{\left(4\sin\left(2\times\frac{3\pi}{4}\right)\right)^2 + \left(4\cos\left(2\times\frac{3\pi}{4}\right)\right)^2} = 4 \text{ km min}^{-1}$$

Mark allocation: 2 marks

- 1 mark for finding $\dot{\underset{\sim}{r}}_G = 4\sin(2t)\underset{\sim}{i} + 4\cos(2t)\underset{\sim}{j}$
- 1 mark for the correct final answer: 4 km min^{-1}

Question 2b.

Worked solution

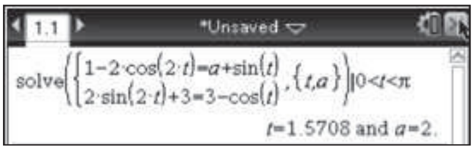
For a collision to occur, Geoff and Daneesha must have the same position at the same time.

Equating the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of the position vectors and solving simultaneously gives

$1 - 2\cos(2t) = a + \sin(t)$ [1]

$2\sin(2t) + 3 = 3 - \cos(t)$ [2]

$\Rightarrow t = \frac{\pi}{2}$ and $a = 2$.



Solving equation [2] to get $t = \frac{\pi}{2}$ can be done as follows.

$$2\sin(2t) = -\cos(t)$$

$\Rightarrow 4\sin(t)\cos(t) + \cos(t) = 0$

$\Rightarrow \quad \cos(t)(4\sin(t) + 1) = 0$

So either $\cos(t) = 0$ or $4\sin(t) + 1 = 0$.

For $\cos(t) = 0$, $t = \frac{\pi}{2}$ is the only solution inside the domain.

For $4\sin(t) + 1 = 0 \Rightarrow \sin(t) = -\frac{1}{4}$, there is no solution inside the domain.

Thus $t = \frac{\pi}{2}$.

Substituting $t = \frac{\pi}{2}$ into either position vector will give the coordinates of the collision, which are

$$\begin{aligned}\underset{\sim}{r}_G\left(\frac{\pi}{2}\right) &= \left(1 - 2\cos\left(2\times\frac{\pi}{2}\right)\right)\underset{\sim}{i} + \left(2\sin\left(2\times\frac{\pi}{2}\right) + 3\right)\underset{\sim}{j} \\ &= (3, 3)\end{aligned}$$

Mark allocation: 3 marks

- 1 mark for equating the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of the position vectors
- 1 mark for correctly finding $a = 2$
- 1 mark for the correct coordinates of the collision: (3, 3)

TIP

» When solving trigonometric equations using a CAS, it is a good idea to restrict the domain to that indicated in the question. This will ignore solutions that are not in the required domain.

Question 2c.

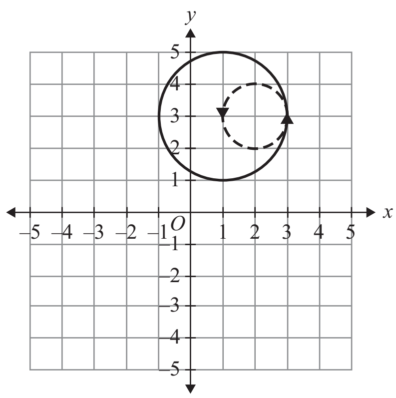
Worked solution

The parametric equations that describe the motion of Daneesha's path are $x - 2 = \sin(t)$ and $-(y - 3) = \cos(t)$.

These give the following Cartesian equation for a circle: $(x - 2)^2 + (y - 3)^2 = 1$.

The initial position of Daneesha is when $t = 0$, which gives the coordinate (2, 2).

From **part b.**, we know that Daneesha and Geoff collide at (3, 3), therefore Daneesha is travelling in an anti-clockwise direction. So Daneesha's path with the direction given is



Mark allocation: 2 marks

- 1 mark for sketching a circle of radius 1 with a centre at (2, 3)
- 1 mark for indicating a path that is anti-clockwise

Question 2d.

Worked solution

The angle between two moving particles is found by finding the angle between the velocity vectors.

From **part a.ii.**, the velocity vector for Geoff is $\dot{\underset{\sim}{r}}_G = 4\sin(2t)\underset{\sim}{i} + 4\cos(2t)\underset{\sim}{j}$.

The velocity vector for Daneesha is $\dot{\underset{\sim}{r}}_D = \cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$.

So the angle between the velocity vectors when $t = \frac{\pi}{2}$ is given by

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_G\left(\frac{\pi}{2}\right) \cdot \dot{\underset{\sim}{r}}_D\left(\frac{\pi}{2}\right)}{\left|\dot{\underset{\sim}{r}}_G\left(\frac{\pi}{2}\right)\right|\left|\dot{\underset{\sim}{r}}_D\left(\frac{\pi}{2}\right)\right|}\right) \\
&= \cos^{-1}\left(\frac{(0\underset{\sim}{i} - 4\underset{\sim}{j}) \cdot (0\underset{\sim}{i} + \underset{\sim}{j})}{\sqrt{(0)^2 + (-4)^2}\sqrt{(0)^2 + (1)^2}}\right) \\
&= \cos^{-1}\left(\frac{-4}{4}\right) \\
&= \pi
\end{aligned}$$

Mark allocation: 3 marks

- 1 mark for finding $\dot{\underset{\sim}{r}}_D = \cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$
- 1 mark for using $\theta = \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_G\left(\frac{\pi}{2}\right) \cdot \dot{\underset{\sim}{r}}_D\left(\frac{\pi}{2}\right)}{|\dot{\underset{\sim}{r}}_G\left(\frac{\pi}{2}\right)||\dot{\underset{\sim}{r}}_D\left(\frac{\pi}{2}\right)|}\right)$ or similar
- 1 mark for the correct angle of π

Question 3a.

Worked solution

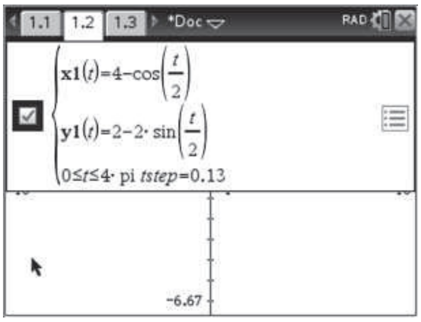
Starting with drone S:

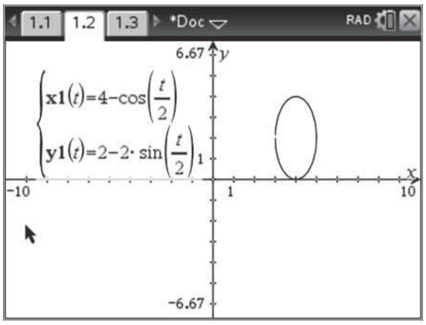
Let $x = 4 - \cos\left(\frac{t}{2}\right)$ and $y = 2 - 2\sin\left(\frac{t}{2}\right)$.

$\Rightarrow -(x-4) = \cos\left(\frac{t}{2}\right)$ and $\frac{y-2}{-2} = \sin\left(\frac{t}{2}\right)$

If $\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) = 1$, then the Cartesian equation of drone S can be modelled by the ellipse $(x-4)^2 + \frac{(y-2)^2}{4} = 1$.

Alternatively, using the parametric graphing feature on a CAS, you could graph the parametric equations to get an ellipse and be able to read the Cartesian equation from the graph.





Test some values of t to determine the direction of movement.

When $t = 0$: $\underset{\sim}{r}_S(0) = (4 - \cos(0))\underset{\sim}{i} + (2 - 2\sin(0))\underset{\sim}{j} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

When $t = \frac{\pi}{2}$:

$$\underset{\sim}{r}_S\left(\frac{\pi}{4}\right) = \left(4 - \cos\left(\frac{\pi}{4}\right)\right)\underset{\sim}{i} + \left(2 - 2\sin\left(\frac{\pi}{4}\right)\right)\underset{\sim}{j}$$

$$= \left(4 - \frac{\sqrt{2}}{2}\right)\underset{\sim}{i} + \left(2 - \frac{\sqrt{2}}{2}\right)\underset{\sim}{j}$$

$$\approx 3.29\underset{\sim}{i} + 1.29\underset{\sim}{j}$$

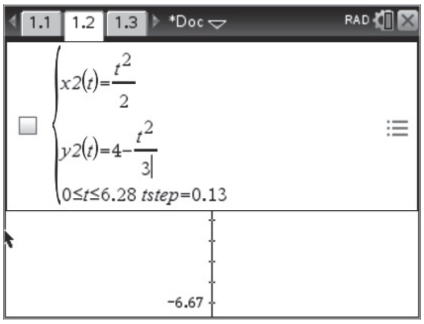
So drone S is moving in an anti-clockwise direction.

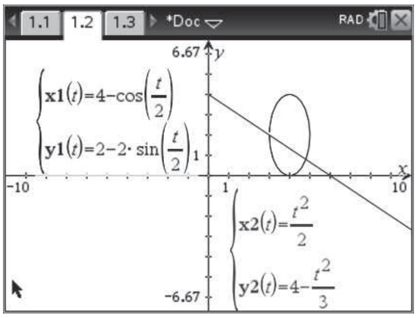
For the delivery drone:

Let $x = \frac{t^2}{2}$ and $y = 4 - \frac{t^2}{3}$, then $2x = t^2 \ \Rightarrow \ y = 4 - \frac{2x}{3}$.

Therefore the Cartesian equation of drone S can be modelled by the straight line $y = 4 - \frac{2x}{3}$.

Alternatively, using the parametric graphing feature on a CAS, you could graph the parametric equations to get a straight line and be able to read the Cartesian equation from the graph.





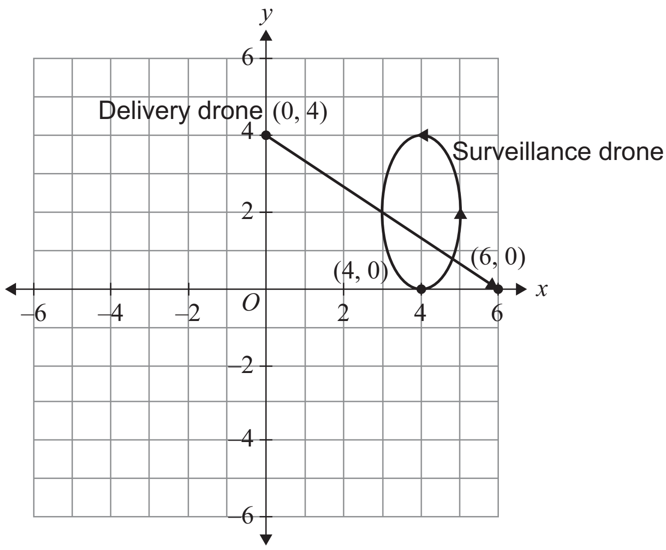
Test some values of t to determine the direction of movement.

When $t = 0$: $\underset{\sim}{r}_D(0) = \left(\frac{0^2}{2}\right)\underset{\sim}{i} + \left(4 - \frac{0^2}{3}\right)\underset{\sim}{j} = 0\underset{\sim}{i} + 4\underset{\sim}{j}$

When $t = \frac{\pi}{2}$: $\underset{\sim}{r}_S(1) = \left(\frac{1^2}{2}\right)\underset{\sim}{i} + \left(4 - \frac{1^2}{3}\right)\underset{\sim}{j} = \frac{1}{2}\underset{\sim}{i} + \frac{11}{3}\underset{\sim}{j}$

Drone D is moving in the positive x direction and negative y direction.

The graph is shown below.



Mark allocation: 3 marks

- 1 mark for showing correct curves with labels
- 1 mark for showing the direction of movement of the drones
- 1 mark for correctly labelling intercepts

Question 3b.

Worked solution

The angle between the drones is given by $\theta = \cos^{-1}\left(\dfrac{\underset{\sim}{v}_S \cdot \underset{\sim}{v}_D}{|\underset{\sim}{v}_S||\underset{\sim}{v}_D|}\right)$ at $t = \sqrt{6}$, so

$$\underset{\sim}{v}_S = \frac{\sin\left(\frac{t}{2}\right)}{2}\underset{\sim}{i} - \cos\left(\frac{t}{2}\right)\underset{\sim}{j}$$

$$\underset{\sim}{v}_D = t\,\underset{\sim}{i} - \frac{2t}{3}\underset{\sim}{j}$$

Therefore

$$\theta = \cos^{-1}\left(\frac{\left(\frac{1}{2}\sin\left(\frac{\sqrt{6}}{2}\right)\underset{\sim}{i} - \cos\left(\frac{\sqrt{6}}{2}\right)\underset{\sim}{j}\right)\cdot\left(\sqrt{6}\underset{\sim}{i} - \frac{2\sqrt{6}}{3}\underset{\sim}{j}\right)}{\left|\frac{1}{2}\sin\left(\frac{\sqrt{6}}{2}\right)\underset{\sim}{i} - \cos\left(\frac{\sqrt{6}}{2}\right)\underset{\sim}{j}\right|\left|\sqrt{6}\underset{\sim}{i} - \frac{2\sqrt{6}}{3}\underset{\sim}{j}\right|}\right)$$

$$= 2.106\ldots$$

$$= 2.1°$$

Mark allocation: 2 marks

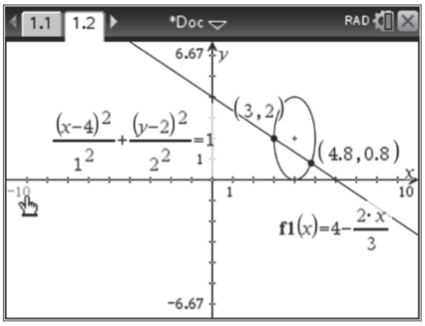
- 1 mark for finding the velocity vectors
- 1 mark for an angle of 2.1°, correct to one decimal place

Question 3c.

Worked solution

Both drones must be shown to go through the same point but at different times.

It can be found that the curves of the drones intersect at the points (3, 2) and (4.8, 0.8).



Take the points and find the value of t for which the delivery drone is at those positions. Then substitute this value of t into the position vector of drone S to show that, although the drones cross paths, they don't collide.

So for point (3, 2):

$$\underset{\sim}{r}_D(t) = \frac{t^2}{2}\underset{\sim}{i} + \left(4 - \frac{t^2}{3}\right)\underset{\sim}{j} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$$

$$\Rightarrow t = \sqrt{6}$$

Then the position of drone S at $t = \sqrt{6}$ is

$$\underset{\sim}{r}_S(\sqrt{6}) = \left(4 - \cos\left(\frac{\sqrt{6}}{2}\right)\right)\underset{\sim}{i} + \left(2 - 2\sin\left(\frac{\sqrt{6}}{2}\right)\right)\underset{\sim}{j}$$

$$\approx 3.66\underset{\sim}{i} + 0.12\underset{\sim}{j}$$

Then for point (4.8, 0.8):

$$\underset{\sim}{r}_D(t) = \frac{t^2}{2}\underset{\sim}{i} + \left(4 - \frac{t^2}{3}\right)\underset{\sim}{j} = 4.8\underset{\sim}{i} + 0.8\underset{\sim}{j}$$

$$\Rightarrow t = \sqrt{9.6}$$

Then the position of drone S at $t = \sqrt{9.6}$ is

$$\underset{\sim}{r}_S(\sqrt{9.6}) = \left(4 - \cos\left(\frac{\sqrt{9.6}}{2}\right)\right)\underset{\sim}{i} + \left(2 - 2\sin\left(\frac{\sqrt{9.6}}{2}\right)\right)\underset{\sim}{j}$$

$$\approx 3.98\underset{\sim}{i} + 0.0004\underset{\sim}{j}$$

Therefore the paths of each drone intersect, as the points (3, 2) and (4.8, 0.8) are common to each path. However, the drones are not at these points at the same time and therefore do not collide.

Mark allocation: 3 marks

- 1 mark for finding the positions that the drones have in common
- 1 mark for showing that the drones are not at (3, 2) at the same time
- 1 mark for showing that the drones are not at (4.8, 0.8) at the same time

Question 3d.

Worked solution

The speed of the surveillance drone is given by

$$|\underset{\sim}{v}_S| = \left|\frac{1}{2}\sin\left(\frac{t}{2}\right)\underset{\sim}{i} - \cos\left(\frac{t}{2}\right)\underset{\sim}{j}\right|$$

$$= \sqrt{\left(\frac{1}{2}\sin\left(\frac{t}{2}\right)\right)^2 + \left(-\cos\left(\frac{t}{2}\right)\right)^2}$$

$$= \sqrt{\frac{1}{4}\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right)} = \sqrt{\frac{1}{4} + \frac{3}{4}\cos^2\left(\frac{t}{2}\right)}$$

As the maximum value of $\cos^2\left(\frac{t}{2}\right) = 1$, it follows that

$$\Rightarrow \underset{\sim}{v}_S = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 \text{ km min}^{-1}$$

Mark allocation: 2 marks

- 1 mark for showing $\sqrt{\frac{1}{4}\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right)}$ or similar
- 1 mark for the maximum speed

Question 3e.

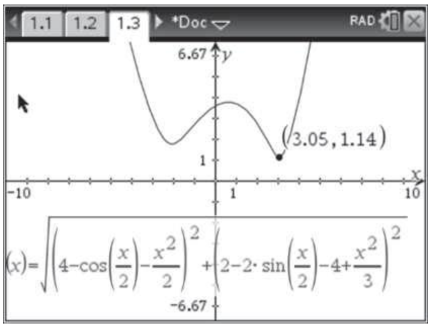
Worked solution

The distance between the two drones is

$$|\underset{\sim}{r}_S - \underset{\sim}{r}_D| = \left|\left(4 - \cos\left(\frac{t}{2}\right) - \frac{t^2}{2}\right)\underset{\sim}{i} + \left(2 - 2\sin\left(\frac{t}{4}\right) - 4 + \frac{t^2}{3}\right)\underset{\sim}{j}\right|$$

$$= \sqrt{\left(4 - \cos\left(\frac{t}{2}\right) - \frac{t^2}{2}\right)^2 + \left(2 - 2\sin\left(\frac{t}{4}\right) - 4 + \frac{t^2}{3}\right)^2}$$

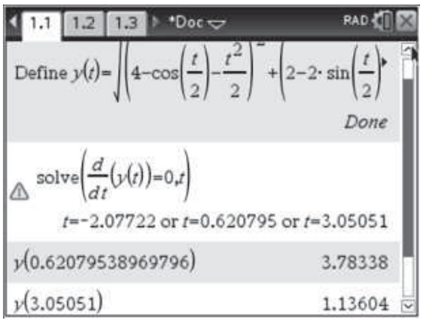
Graphing the distance equation and finding the minimum point yields a minimum distance of 1.1 km at a time of 3.1 min.



Alternatively, using calculus:

Let y be the distance between the drones, then $y = \sqrt{\left(4 - \cos\left(\frac{t}{2}\right) - \frac{t^2}{2}\right)^2 + \left(2 - 2\sin\left(\frac{t}{4}\right) - 4 + \frac{t^2}{3}\right)^2}$.

Now differentiate the distance and set the derivative equal to zero to find the time. Then substitute the time into the distance equation to find the minimum distance.



Therefore the minimum distance is 1.1 km at a time of 3.1 min.

Mark allocation: 2 marks

- 1 mark for setting up the distance relationship
  $$|\underset{\sim}{r}_S - \underset{\sim}{r}_D| = \sqrt{\left(4 - \cos\left(\frac{t}{2}\right) - \frac{t^2}{2}\right)^2 + \left(2 - 2\sin\left(\frac{t}{4}\right) - 4 + \frac{t^2}{3}\right)^2}$$
- 1 mark for a minimum distance of 1.1 km at a time of 3.1 min

TIP

» When finding the maximum/minimum distance between two moving objects, graphing the distance formula is a useful way of visualising where or what is the maximum/minimum point, and thus finding the required value.

Question 4a.

Worked solution

$\underset{\sim}{r}_A(0) = e^{-0}\underset{\sim}{i} + e^{-3\times0+1}\underset{\sim}{j} = \underset{\sim}{i} + e\underset{\sim}{j}$

$\underset{\sim}{r}_B(0) = (e^{-2\times0+1} - 1)\underset{\sim}{i} + e^{-0}\underset{\sim}{j} = (e-1)\underset{\sim}{i} + \underset{\sim}{j}$

initial distance = $\sqrt{((e-1)-1)^2 + (1-e)^2} = \sqrt{2e^2 - 6e + 5}$

Mark allocation: 2 marks

- 1 mark for finding the initial positions
- 1 mark for the correct distance

Question 4b.

Worked solution

Equate the $\underset{\sim}{i}$ components of the ant and beetle:

$e^{-t} = e^{-2t+1} - 1$

Mark allocation: 1 mark

- 1 mark for forming the correct quadratic equation

Question 4c.

Worked solution

Substitute $t = \log_e\left(-\frac{1}{2} + \sqrt{e + \frac{1}{4}}\right)$ into the $\underset{\sim}{j}$ components of the position of the ant and beetle.

For $\underset{\sim}{r}_A(t)$, $e^{-3\log_e\left(-\frac{1}{2}+\sqrt{e+\frac{1}{4}}\right)+1} = 1.4865$. For $\underset{\sim}{r}_B(t)$, $e^{-\log_e\left(-\frac{1}{2}+\sqrt{e+\frac{1}{4}}\right)} = 0.8177$.

Since $t = \log_e\left(-\frac{1}{2} + \sqrt{e + \frac{1}{4}}\right)$ is the only time the ant and the beetle have the same x-coordinate and their y-coordinates are not the same at this time, the ant and beetle never collide.

Alternatively, equating the y-coordinates of the ant and the beetle gives

$$\begin{aligned} e^{-3t+1} &= e^{-t} \\ e^{-3t+1} - e^{-t} &= 0 \\ e^{-t}(e^{-2t+1} - 1) &= 0 \\ e^{-t} &\neq 0 \\ \therefore e^{-2t+1} - 1 &= 0 \\ t &= \frac{1}{2} \end{aligned}$$

This is not the same time at which they have the same x-coordinate, therefore they never collide.

Mark allocation: 2 marks

- 1 mark for finding the y-coordinates of the ant and beetle
- 1 mark for concluding that they never collide, as the two values are not equal

Question 4d.

Worked solution

Equate the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of the ant and the beetle with different variables t and s as they arrive at the same position at different times.

$\underset{\sim}{i}$ components: $e^{-t} = e^{-2s+1} - 1$

$\underset{\sim}{j}$ components: $e^{-3t+1} = e^{-s}$

Solve simultaneously: $t = 0.415531461$ and $s = 0.2465943829$

Substitute into $\underset{\sim}{r}_A(t)$: $\underset{\sim}{r}_A(0.415531461) = 0.6600\underset{\sim}{i} + 0.7815\underset{\sim}{j}$

Therefore the point of intersection of the paths is (0.6600, 0.7815).

Mark allocation: 2 marks

- 1 mark for setting up the two equations for the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components
- 1 mark for the correct x and y values of the intersection

Question 4e.

Worked solution

The direction of travel is given by the gradient of the path.

$\dot{\underset{\sim}{r}}_A(t) = -e^{-t}\underset{\sim}{i} - 3e^{-3t+1}\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}_B(s) = -2e^{-2s+1}\underset{\sim}{i} - e^{-s}\underset{\sim}{j}$

The paths intersect when $t = 0.415531461$ and $s = 0.2465943829$ (from **part 4d.**). Substitute these time values into the gradient functions to find the direction of each path.

$\dot{\underset{\sim}{r}}_A(0.415531461) = -0.6599894288\underset{\sim}{i} - 2.344372805\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}_B(0.2465943829) = -3.319978858\underset{\sim}{i} - 0.7814576018\underset{\sim}{j}$

To find the angle, use dot product.

$$\theta = \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_A(t) \cdot \dot{\underset{\sim}{r}}_B(s)}{|\dot{\underset{\sim}{r}}_A(t)|\,|\dot{\underset{\sim}{r}}_B(s)|}\right)$$

$$= \cos^{-1}\frac{-0.6599894288 \times -3.319978858 + -2.344372805 \times -0.7814576018}{\sqrt{(-0.659)^2 + (-2.344)^2} \cdot \sqrt{(-3.319)^2 + (-0.781)^2}}$$

$$= 61.03°$$

Mark allocation: 2 marks

- 1 mark for differentiation
- 1 mark for finding the angle

TIP

» Use the 'store' function in your CAS calculator to avoid copying and typing lengthy decimal places during calculations.

Question 5a.i.

Worked solution

$\underset{\sim}{r}_B(t) = (4\sec(t) - 3)\underset{\sim}{i} + (\tan(t) + 1)\underset{\sim}{j}$ is equivalent to

$x = 4\sec(t) - 3$ and $y = \tan(t) + 1$

$\Rightarrow \frac{x+3}{4} = \sec(t)$ and $y - 1 = \tan(t)$

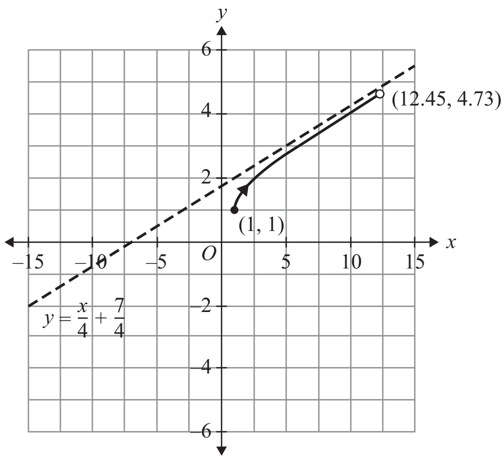
Using $\sec^2(t) - \tan^2(t) = 1$ gives the Cartesian equation $\frac{(x+3)^2}{16} - (y-1)^2 = 1$.

Mark allocation: 2 marks

- 1 mark for $\frac{x+3}{4} = \sec(t)$ and $y - 1 = \tan(t)$, and using the relation $\sec^2(t) - \tan^2(t) = 1$ to get the Cartesian equation
- 1 mark for the Cartesian equation: $\frac{(x+3)^2}{16} - (y-1)^2 = 1$

Question 5a.ii.

Worked solution



When the end points $t = 0$ and $t = \frac{5\pi}{12}$ are substituted into the position vector, it can be seen that the baseball is moving from left to right.

The equations of the asymptotes of a hyperbola are given by $y - k = \pm\frac{b}{a}(x - h)$.

Substituting the relevant values from the Cartesian equation of the hyperbola and rearranging to make y the subject gives

$$y - 1 = \pm\frac{1}{4}(x+3)$$

$$\Rightarrow y = \frac{x}{4} + \frac{3}{4} + 1 \text{ or } y = -\frac{x}{4} - \frac{3}{4} + 1$$

$$\Rightarrow y = \frac{x}{4} + \frac{7}{4} \text{ or } y = -\frac{x}{4} + \frac{1}{4}$$

As there is a restricted domain on the parametric equations, and due to the direction of motion of the baseball, the asymptote with a negative gradient is rejected and the only asymptote to be sketched is $y = \frac{x}{4} + \frac{7}{4}$.

Mark allocation: 3 marks

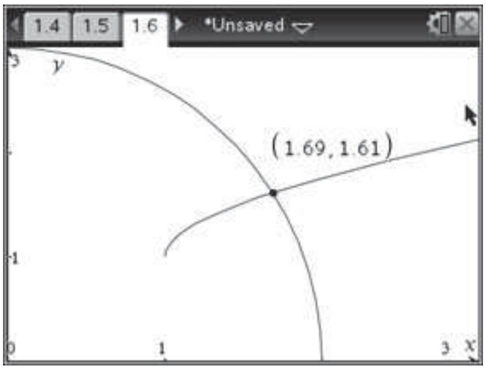
- 1 mark for accurately sketching the hyperbola and a labelled asymptote
- 1 mark for labelling the end points accurately
- 1 mark for correctly indicating the direction of motion of the baseball

Question 5b.

Worked solution

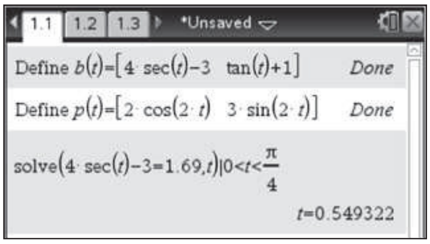
Crossing paths but not colliding means that the baseball and pigeon will occupy the same coordinates but at different times.

Sketching the graphs of the paths of the baseball and the pigeon gives the following, where the intersection is approximately at (1.69, 1.61).



For the baseball, point (1.69, 1.61) is $\underset{\sim}{r}_B(t) = (4\sec(t) - 3)\underset{\sim}{i} + (\tan(t) + 1)\underset{\sim}{j} = 1.69\underset{\sim}{i} + 1.61\underset{\sim}{j}$.

Solving for t over the time interval common to both paths gives $t = 0.549322...$



Substituting $t = 0.549322...$ into $\underset{\sim}{r}_P(t) = 2\cos(2t)\underset{\sim}{i} + 3\sin(2t)\underset{\sim}{j}$ gives

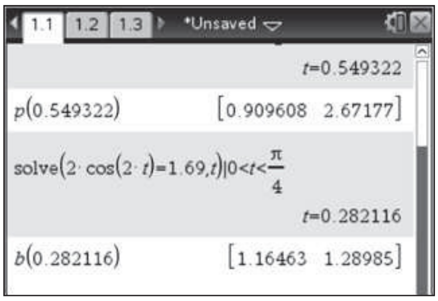
$\underset{\sim}{r}_P(0.549322) = 2\cos(2 \times 0.549322)\underset{\sim}{i} + 3\sin(2 \times 0.549322)\underset{\sim}{j} \approx 0.91\underset{\sim}{i} + 2.67\underset{\sim}{j}$

Similarly, for the pigeon the point (1.69, 1.61) is $\underset{\sim}{r}_P(t) = 2\cos(2t)\underset{\sim}{i} + 3\sin(2t)\underset{\sim}{j} = 1.69\underset{\sim}{i} + 1.61\underset{\sim}{j}$.

Solving for t over the time interval common to both paths gives $t = 0.282116....$

Substituting $t = 0.282116...$ into $\underset{\sim}{r}_B(t) = (4\sec(t) - 3)\underset{\sim}{i} + (\tan(t) + 1)\underset{\sim}{j}$ gives

$\underset{\sim}{r}_B(0.282116) = (4\sec(0.282116) - 3)\underset{\sim}{i} + (\tan(0.282116) + 1)\underset{\sim}{j} \approx 1.16\underset{\sim}{i} + 1.29\underset{\sim}{j}$



Therefore the baseball and the pigeon cross paths at point $1.69\underset{\sim}{i} + 1.61\underset{\sim}{j}$ but at different times. Hence they do not collide.

Alternatively:

Solve the following simultaneous equations to calculate when the baseball and pigeon have the same coordinates, but at different times.

Let t_1 be the time for the baseball and t_2 be the time for the pigeon.

$4\sec(t_1) - 3 = 2\cos(2t_2)$

$\tan(t_1) + 1 = 3\sin(2t_2)$

$t_1 = 0.54835...$ and $t_2 = 0.28340...$

Substituting these values for t into the baseball and pigeon position vectors gives the coordinates (1.6872…, 1.6108…).

Substituting $t = 0.28340...$ into $\underset{\sim}{r}_B(t) = (4\sec(t) - 3)\underset{\sim}{i} + (\tan(t) + 1)\underset{\sim}{j}$ shows that the baseball is at a different position at the time that the pigeon passes through the common point.

$\underset{\sim}{r}_B(0.28340) = (4\sec(0.28340) - 3)\underset{\sim}{i} + (\tan(0.28340) + 1)\underset{\sim}{j} \approx 1.17\underset{\sim}{i} + 1.29\underset{\sim}{j}$

Mark allocation: 3 marks

- 1 mark for finding the position that the baseball and the pigeon have in common
- 1 mark for finding the time at which the baseball and the pigeon pass through the common position
- 1 mark for showing that when either the baseball or pigeon is at position $1.69\underset{\sim}{i} + 1.61\underset{\sim}{j}$, the other object is at a different position

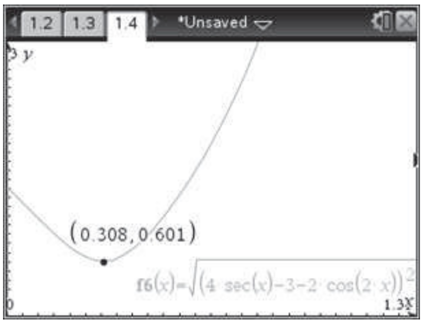
Question 5c.

Worked solution

The distance between the baseball and the pigeon is given by

$$|\underset{\sim}{r}_B - \underset{\sim}{r}_P| = |(4\sec(t) - 3 - 2\cos(2t))\underset{\sim}{i} + (\tan(t) + 1 - 3\sin(2t))\underset{\sim}{j}|$$
$$= \sqrt{(4\sec(t) - 3 - 2\cos(2t))^2 + (\tan(t) + 1 - 3\sin(2t))^2}$$

Graphing the distance equation and finding the minimum point gives a minimum distance of 0.60 m at a time of 0.31 s.

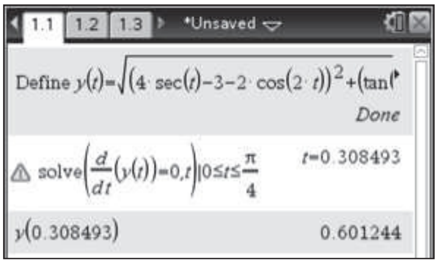


Alternatively, using calculus:

Let $y(t)$ be the distance between the baseball and the pigeon, then

$y(t) = \sqrt{(4\sec(t) - 3 - 2\cos(2t))^2 + (\tan(t) + 1 - 3\sin(2t))^2}$.

Now differentiate the distance and set the derivative equal to zero to find the time. Then substitute the time into the distance equation to find the minimum distance.



Therefore the minimum distance is 0.60 m at a time of 0.31 s.

Mark allocation: 2 marks

- 1 mark for the minimum distance: 0.60 m
- 1 mark for the time: 0.31 s

TIP

» **When finding the minimum/maximum distance between two moving objects, graphing the distance formula is a useful way of visualising where or what is the maximum/minimum point, and thus finding the required value.**

Question 6

Worked solution

$\frac{x+2}{2} = \cos(t)$ and $y = \sin(t)$

$$\therefore \left(\frac{x+2}{2}\right)^2 + y^2 = 1$$

$$\therefore y = \pm\sqrt{1 - \left(\frac{x+2}{2}\right)^2}$$

$$y = \sqrt{1 - \left(\frac{x+2}{2}\right)^2}$$

So $y \geq 0$ because $t \in [0, \pi]$.

Mark allocation: 2 marks

- 1 mark for using Pythagorean identity
- 1 mark for rearranging and explaining why the positive values are chosen

TIP

» **For 'show that' questions, you should show all lines of working as if done tech-free even when you are sitting a tech-active exam.**

Question 7a.

Worked solution

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + \frac{1}{4}\overrightarrow{AB}$

$\overrightarrow{OP} = \underset{\sim}{a} + \frac{1}{4}(\underset{\sim}{b} - \underset{\sim}{a})$

$\overrightarrow{OP} = \frac{3}{4}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}$

$\overrightarrow{QM} = \overrightarrow{QB} + \overrightarrow{BM} = \frac{1}{4}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{BA}$

$\overrightarrow{QM} = \frac{1}{4}\underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$

$\overrightarrow{QM} = \frac{1}{2}\underset{\sim}{a} - \frac{1}{4}\underset{\sim}{b}$

Mark allocation: 2 marks

- 1 mark for deriving the expression for $\overrightarrow{OP}$
- 1 mark for deriving the expression for $\overrightarrow{QM}$

Question 7b.i.

Worked solution

$\overrightarrow{OX} = \lambda\overrightarrow{OP}$

$\overrightarrow{OX} = \lambda\left(\frac{3}{4}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}\right)$

$\overrightarrow{OX} = \frac{3\lambda}{4}\underset{\sim}{a} + \frac{\lambda}{4}\underset{\sim}{b}$

Mark allocation: 1 mark

- 1 mark for the answer, expressed correctly

Question 7b.ii.

Worked solution

First, find vector $\overrightarrow{AX}$.

$\overrightarrow{AX} = -\overrightarrow{OA} + \overrightarrow{OX}$

$\overrightarrow{AX} = -\underset{\sim}{a} + \frac{3\lambda}{4}\underset{\sim}{a} + \frac{\lambda}{4}\underset{\sim}{b}$

$\overrightarrow{AX} = \left(\frac{3\lambda - 4}{4}\right)\underset{\sim}{a} + \frac{\lambda}{4}\underset{\sim}{b}$

If this vector is parallel to $\overrightarrow{OB}$, then there is no vector in the $\underset{\sim}{a}$ direction; that is:

$$\frac{3\lambda - 4}{4} = 0$$

$$\lambda = \frac{4}{3}$$

Mark allocation: 2 marks

- 1 mark for finding vector $\overrightarrow{AX}$
- 1 mark for finding the correct value of λ

Question 7c.i.

Worked solution

$\overrightarrow{OX} = \overrightarrow{OQ} + \mu\overrightarrow{QM}$

$\overrightarrow{OX} = \frac{3}{4}\underset{\sim}{b} + \mu\left(\frac{1}{2}\underset{\sim}{a} - \frac{1}{4}\underset{\sim}{b}\right)$

$\overrightarrow{OX} = \frac{\mu}{2}\underset{\sim}{a} + \frac{3-\mu}{4}\underset{\sim}{b}$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 7c.ii.

Worked solution

We now have two different expressions for $\overrightarrow{OX}$ in terms of λ and μ. We can equate the coefficients of vector $\underset{\sim}{a}$ to give $\frac{3\lambda}{3} = \frac{\mu}{2}$.

Equating the coefficients of vector $\underset{\sim}{b}$ gives $\frac{\lambda}{4} = \frac{3-\mu}{4}$.

Solving these simultaneously gives $\mu = \frac{9}{5}$, $\lambda = \frac{6}{5}$.

Mark allocation: 2 marks

- 1 mark for the two equations
- 1 mark for the correct answer

Question 8a.

Worked solution

By inspection of the parametric equations, $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$.

Mark allocation: 1 mark

- 1 mark for the correct vector equation of Π_1

Question 8b.

Worked solution

A vector normal to Π_1 can be found using the vector cross product of two vectors in the plane.

$\underset{\sim}{n}_1 = \underset{\sim}{b} \times \underset{\sim}{c} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$

A vector equation of Π_1 can now be determined, which can be used to write a Cartesian expression.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \underset{\sim}{n}_1 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \cdot \underset{\sim}{n}_1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$$

$3x - 5y - z = -27$

Mark allocation: 2 marks

- 1 mark for the correct normal vector to Π_1
- 1 mark for the correct Cartesian equation for Π_1: $3x - 5y - z = -27$

Question 8c.

Worked solution

Π_2 contains both $\underset{\sim}{n}_1$ and L. Therefore a vector normal to Π_2 can be found by evaluating the vector cross product of $\underset{\sim}{n}_1$ and a vector in the direction of L.

$$\underset{\sim}{n}_2 = \underset{\sim}{n}_1 \times \underset{\sim}{c} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} \times \begin{pmatrix} -5 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 25 \\ 17 \\ -10 \end{pmatrix}$$

As in the previous part of the question, we can use $\underset{\sim}{n}_2$ to determine a vector, and hence a Cartesian expression for Π_2.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 25 \\ 17 \\ -10 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 25 \\ 17 \\ -10 \end{pmatrix}$$

$$25x + 17y - 10z = -38$$

Mark allocation: 2 marks

- 1 mark for the correct normal vector to Π_2
- 1 mark for the correct Cartesian equation for Π_2: $25x + 17y - 10z = -38$

Question 8d.

Worked solution

First, find a vector from $P(5, 4, 14)$ to any point on Π_2: $\underset{\sim}{u} = \begin{pmatrix} 3 \\ 1 \\ 13 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 14 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$

Next, find the vector projection of $\underset{\sim}{u}$ on $\underset{\sim}{n}_2$, a vector from $P(5, 4, 14)$ to Π_2 that is perpendicular to Π_2.

$$\underset{\sim}{v} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{n}_2}{\underset{\sim}{n}_2 \cdot \underset{\sim}{n}_2}\underset{\sim}{n}_2 = \begin{pmatrix} \frac{-175}{78} \\ \frac{-119}{78} \\ \frac{35}{39} \end{pmatrix}$$

Finally, the position vector of the reflection of $P(5, 4, 14)$ across the plane Π_2 can be found by evaluating $\underset{\sim}{P} + 2\underset{\sim}{v}$, where $\underset{\sim}{P}$ is the position vector of $P(5, 4, 14)$.

$$\begin{pmatrix} 5 \\ 4 \\ 14 \end{pmatrix} + 2\begin{pmatrix} \frac{-175}{78} \\ \frac{-119}{78} \\ \frac{35}{39} \end{pmatrix} = \begin{pmatrix} \frac{20}{39} \\ \frac{37}{39} \\ \frac{616}{39} \end{pmatrix}$$

This corresponds to coordinates $\left(\frac{20}{39}, \frac{37}{39}, \frac{616}{39}\right)$.

Mark allocation: 3 marks

- 1 mark for the correct expression for a vector, $\underset{\sim}{v}$, from $P(5, 4, 14)$ to Π_2 that is perpendicular to Π_2
- 1 mark for the correct expression using $\underset{\sim}{v}$ that can be evaluated to find the position vector of the reflected point
- 1 mark for the correct coordinates of the reflected point: $\left(\frac{20}{39}, \frac{37}{39}, \frac{616}{39}\right)$

Question 8e.i.

Worked solution

Since line L lies in plane Π_3, any point on L will also satisfy the equation for Π_3.

Using the equation for L, $\underset{\sim}{r}_2(t) = 3\underset{\sim}{i} + \underset{\sim}{j} + 13\underset{\sim}{k} + t(-5\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k})$, the position vectors and, hence, coordinates of two points on Π_3 can be determined.

$\underset{\sim}{r}_2(0) = 3\underset{\sim}{i} + \underset{\sim}{j} + 13\underset{\sim}{k}$

$\underset{\sim}{r}_2(1) = -2\underset{\sim}{i} + 6\underset{\sim}{j} + 9\underset{\sim}{k}$

So the coordinates of two points on plane Π_3 are (3, 1, 13) and (–2, 6, 9).

These coordinates can now be substituted into the equation for Π_3 to find the values of c and d.

$13(3) + 29(1) + c(13) = d$ [1]

$13(-2) + 29(6) + c(9) = d$ [2]

Solving the simultaneous equations [1] and [2] gives $c = 20$ and $d = 328$.

Mark allocation: 2 marks

- 1 mark for finding the coordinates of any two points on Π_3
- 1 mark for the correct values of $c = 20$ and $d = 328$

Question 8e.ii.

Worked solution

First, find the angle, θ, between the normal vectors of Π_1 and Π_2, using the dot product.

$$\cos(\theta) = \frac{\begin{pmatrix}13\\29\\20\end{pmatrix}\cdot\begin{pmatrix}3\\-5\\-1\end{pmatrix}}{\left|\begin{pmatrix}13\\29\\20\end{pmatrix}\right|\left|\begin{pmatrix}3\\-5\\-1\end{pmatrix}\right|}.$$

$$\theta = 124.6°$$

The angle between the planes is therefore $180° - 124.6° = 55.4°$.

Mark allocation: 2 marks

- 1 mark for finding the angle between normal vectors of Π_1 and Π_2
- 1 mark for the correct value for the angle between the planes: 55.4°

Question 9a.

Worked solution

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix}-1\\2\\0\end{pmatrix} \times \begin{pmatrix}-1\\0\\r\end{pmatrix}\right|$$

$$= \frac{\sqrt{5r^2+4}}{2}$$

To find the required values of r such that the area of the triangle is less than 2, solve the inequality.

$$\frac{\sqrt{5r^2+4}}{2} < 2$$

$$r \in \left(\frac{-2\sqrt{15}}{5}, \frac{2\sqrt{15}}{5}\right)$$

The minimum area is the minimum value of $\frac{\sqrt{5r^2+4}}{2}$ over the domain $r \in \left(\frac{-2\sqrt{15}}{5}, \frac{2\sqrt{15}}{5}\right)$, which is 1.

Mark allocation: 3 marks

- 1 mark for setting up the correct inequality for the area
- 1 mark for correct range of $r \in \left(\frac{-2\sqrt{15}}{5}, \frac{2\sqrt{15}}{5}\right)$
- 1 mark for the correct minimum area: 1

Question 9b.

Worked solution

$$\underset{\sim}{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix}2r\\r\\2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2r\\r\\2\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}2r\\r\\2\end{pmatrix}$$

$$2rx + ry + 2z = 2r$$

Mark allocation: 2 marks

- 1 mark for the correct normal vector to the plane
- 1 mark for the correct Cartesian equation of the plane

Question 9c.

Worked solution

Distance to the origin is given by

$$d = \frac{|2r|}{|\underset{\sim}{n}|} = \frac{|2r|}{\sqrt{5r^2+4}}$$

Mark allocation: 1 mark

- 1 mark for the correct expression for distance: $d = \frac{|2r|}{\sqrt{5r^2+4}}$

Question 9d.

Worked solution

$d = \dfrac{|2r|}{\sqrt{5r^2 + 4}}$

$d \in \left[0, \dfrac{2\sqrt{5}}{5}\right)$

Mark allocation: 1 mark

- 1 mark for the correct range of values for d: $d \in \left[0, \dfrac{2\sqrt{5}}{5}\right)$

Question 10a.

Worked solution

If planes are parallel, their normal vectors are also parallel. Therefore by inspection:

$a = -2 \times -1 = 2$

$b = -2 \times 3 = -6$

Therefore equation for Π_2 is $-2x + 2y - 6z = 10$ or equivalently $-x + y - 3z = 5$.

Distance between planes will be given by

$\dfrac{8 + 5}{\sqrt{1^2 + (-1)^2 + 3^2}} = \dfrac{13\sqrt{11}}{11}$

Mark allocation: 3 marks

- 1 mark each for correct values of a and b
- 1 mark for correct distance: $\dfrac{13\sqrt{11}}{11}$

Question 10b.

Worked solution

If Π_2 is perpendicular to Π_1, then the normal vector of Π_1 is parallel to Π_2.

Therefore using the point given on Π_2, a second point on Π_2 can be found by adding any scalar multiple of the vector normal to Π_1 to the given point on Π_2.

Given point on Π_2: $(3, 1, 4)$

Second point on Π_2: $(3, 1, 4) + (1, -1, 3) = (4, 0, 7)$

The two points on Π_2 can then be substituted into the general equation for Π_2 to find the values of a and b.

$-2(3) + a(1) + b(4) = 10$ [1]

$-2(4) + a(0) + b(7) = 10$ [2]

Solving the above simultaneous equations yields $a = \dfrac{40}{7}$, $b = \dfrac{18}{7}$.

Mark allocation: 3 marks

- 1 mark for finding two points on Π_2
- 1 mark each for correct values of a and b

Area of Study 6 Data analysis, probability and statistics

EXAM 1

Question 1a.

Worked solution

$\mu = \bar{x} \pm z \times \frac{s}{\sqrt{n}}$

$\mu = 160 \pm 1.645 \times \frac{10}{\sqrt{100}}$

$\mu = 160 \pm 1.645 = 160 \pm 1.6$

$158.4 \le \mu \le 161.6$

Mark allocation: 2 marks

- 1 mark for correctly substituting the values into $\mu = \bar{x} \pm z \times \frac{s}{\sqrt{n}}$
- 1 mark for the correct answer

Question 1b.i.

Worked solution

H_0: $\mu = 157$

H_1: $\mu > 157$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1b.ii.

Worked solution

$p = \Pr\left(z > \frac{159 - 157}{\frac{6}{\sqrt{36}}}\right) = \Pr(z > 2) \approx \Pr(z > 1.96)$

$\Rightarrow p = 0.025$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1b.iii.

Worked solution

∴ Reject the null hypothesis since $p = 0.025 < 0.05$.

Mark allocation: 1 mark

- 1 mark for the correct explanation

Question 2a.

Worked solution

The sample mean, $\bar{x}$, is the middle of the confidence interval, so

$$\bar{x} = \frac{6.98 + 5.02}{2} = 6$$

Therefore

$$6 + 1.96 \times \frac{s}{\sqrt{16}} = 6.98$$

$$\Rightarrow 1.96 \times \frac{s}{\sqrt{16}} = 0.98$$

And so

$$s = \frac{4 \times 0.98}{1.96}$$

$$= \frac{4 \times 0.98}{2 \times 0.98} = 2$$

Mark allocation: 2 marks

- 1 mark for the correct mean
- 1 mark for the correct evaluation of s

Question 2b.

Worked solution

H_0: $\mu = 4.5$

H_1: $\mu < 4.5$

$$p = \Pr\left(z < \frac{4 - 4.5}{\frac{1.5}{\sqrt{36}}}\right) = \Pr\left(\frac{-0.5}{0.25}\right)$$

$$= \Pr(z < -2)$$

$$\Rightarrow p = 0.025$$

∴ Reject the null hypothesis since $p = 0.025 < 0.05$.

Mark allocation: 4 marks

- 1 mark for the correct null and alternative hypotheses
- 1 mark for the correct evaluation of z
- 1 mark for the correct value of p
- 1 mark for the correct reason for rejecting the null hypothesis

Question 3a.

Worked solution

Let W be the normal random variable $X - Y$.

To find $\Pr(X - Y > 0) = \Pr(W > 0)$, the mean and standard deviation of W is needed.

$$\text{E}(W) = \text{E}(X) - \text{E}(Y) = 175 - 165 = 10$$

$$\text{var}(W) = \text{var}(X) + (-1)^2\,\text{var}(Y) = 3^2 + 4^2 = 25$$

$$\Rightarrow \text{sd}(W) = 5$$

So $\Pr(W > 0) = \Pr\left(Z > \frac{0 - 10}{5}\right) = \Pr(Z > -2) \approx 0.975$

Mark allocation: 2 marks

- 1 mark for finding the mean and standard deviation of the normal random variable W
- 1 mark for calculating that the $\Pr(W > 0) \approx 0.975$

TIP

» It is important to remember the approximate probabilities associated with z-values of ±1, ±2 and ±3 in a standard normal distribution.

Question 3b.

Worked solution

Let $\overline{W}$ be the normal random distribution of the mean difference in mass, so $\overline{W} = \overline{X} - \overline{Y}$.

Then $E(\overline{W}) = E(\overline{X}) - E(\overline{Y}) = 175 - 165 = 10$.

For a 95% confidence interval, the value of z is 1.96, so the integer value to use in the calculation is 2.

The confidence interval is given by

$$\left(\overline{w} - z\frac{s}{\sqrt{n}}, \overline{w} + z\frac{s}{\sqrt{n}}\right) = \left(10 - 2\frac{5}{\sqrt{100}}, 10 + 2\frac{5}{\sqrt{100}}\right)$$
$$= (10 - 2 \times 0.5, 10 + 2 \times 0.5)$$
$$= (9, 11)$$

Mark allocation: 2 marks

- 1 mark for defining $\overline{W}$ and finding that $\mathrm{E}(\overline{W}) = 10$
- 1 mark for determining the correct interval

Question 4

Worked solution

First, write an expression for the expected value of $2X + 4Y$ to find the value of m.

$$\mathrm{E}(2X + 4Y) = 2\mathrm{E}(X) + 4\mathrm{E}(Y) = 2(m) + 4(5)$$
$$2m + 20 = 26$$
$$m = 3$$

Find the variances of X, Y and $2X + 4Y$ by squaring the standard deviations.

$\mathrm{Var}(X) = 1^2 = 1$ and $\mathrm{Var}(Y) = (\sqrt{n})^2 = n$.

Next, write an expression for $\mathrm{Var}(2X + 4Y)$ to determine the value of n.

$$\mathrm{Var}(2X + 4Y) = (2^2)\mathrm{Var}(X) + (4^2)\mathrm{Var}(Y)$$
$$36 = 4 + 16n$$
$$n = 2$$

Mark allocation: 3 marks

- 1 mark for writing the correct expression for $E(2X + 4Y)$
- 1 mark for writing the correct expression for $var(2X + 4Y)$
- 1 mark for calculating the correct values of m and n: 3 and 2, respectively

Question 5

Worked solution

We can write a linear combination of random variables A and B to calculate probabilities relating to the difference, D, between A and B. D will also be a random normal variable.

$D = A - B$

$E(D) = E(A) - E(B) = 12 - 12 = 0$

$Var(D) = Var(A) + Var(B) = (3)^2 + (4)^2 = 25$

$\sigma_D = \sqrt{Var(D)} = \sqrt{25} = 5$

$$\begin{aligned} Pr(|D| > 9.8) &= Pr(D < -9.8) + Pr(D > 9.8) = 2 \times Pr(D > 9.8) \\ &= 2 \times Pr\left(\frac{D - E(D)}{\sigma_D} > \frac{9.8 - 0}{5}\right) = 2 \times Pr(Z > 1.96) \\ &= 0.05 \end{aligned}$$

Mark allocation: 3 marks

- 1 mark for finding the correct mean and variance of $A - B$
- 1 mark for writing a correct probability statement for the difference of the scores being greater than 9.8
- 1 mark for the correct value for the required probability of 0.05

TIP

» **The variance of the difference of two random variables is the sum of the variances of those variables: $Var(A - B) = Var(A) + Var(B)$.**

Question 6

Worked solution

G and R are independent random variables.

$E(G) = 75$

$Var(G) = 1^2 = 1$

$E(R) = 70$

$Var(R) = 2^2 = 4$

Let G_i, where $i = 1, ..., 6$, be a random variable with a mean of 75 and standard deviation of 1.

Let R_i, where $i = 1, ..., 7$, be a random variable with a mean of 70 and standard deviation of 2.

A random variable, B, representing the mass of a bag of apples, can be written as

$B = G_1 + G_2 + ... + G_6 + R_1 + R_2 + ... + R_7$.

The mean of B is

$$E(B) = E(G_1) + E(G_2) + ... + E(G_6) + E(R_1) + E(R_2) + ... + E(R_7)$$
$$= 6 \times 75 + 7 \times 70$$
$$= 450 + 490$$
$$= 940$$

The variance of B is

$$Var(B) = Var(G_1 + G_2 + ... + G_6 + R_1 + R_2 + ... + R_7)$$
$$= 1^2 Var(G_1) + 1^2 Var(G_2) + ... + 1^2 Var(G_6) + 1^2 Var(R_1) + 1^2 Var(R_2) + ... + 1^2 Var(R_7)$$
$$= 6 \times 1^2 + 7 \times 2^2$$
$$= 6 + 28$$
$$= 34$$

Hence $sd(B) = \sqrt{34}$.

The mass of a bag of apples that is a random variable, B, has a mean of 940 g and a standard deviation of $\sqrt{34}$ g.

Mark allocation: 3 marks

- 1 mark for writing the correct linear combination of G and R to represent B
- 1 mark for the correct value for the mean of B: 940 g
- 1 mark for the correct value for the standard deviation of B: $\sqrt{34}$ g

TIPS

» Remember that $Var(G_1 + G_2 + ... + G_6 + R_1 + R_2 + ... + R_7) \neq Var(6G + 7R)$

» Writing the incorrect linear combination for the mass of a bag as $B = 6G + 7R$ would suggest that you are taking the mass of one green apple and one red apple and multiplying those masses by 6 and 7, respectively, to obtain the mass for a bag. The correct linear combination for the mass of a bag, B, requires that you obtain a different random value for the mass of each apple in the bag.

Question 7a.

Worked solution

The volume of one ice cube is given by $V = 3 \times 3 \times H = 9H$.

Since the height of each ice cube is independent, the total volume of ice is given by $T = 9H_1 + 9H_2 + ... + 9H_6$.

$$E(T) = 9E(H_1) + 9E(H_2) + ... + 9E(H_6)$$

$$E(T) = 108 \text{ cm}^3$$

Mark allocation: 1 mark

- 1 mark for correct answer of 108

Question 7b.

Worked solution

To determine the standard deviation, we first must calculate the variance.

$\text{Var}(T) = 9^2\text{Var}(H_1) + 9^2\text{Var}(H_2) + ... + 9^2\text{Var}(H_6)$

$\text{Var}(T) = \frac{81}{6}$

$\sigma = \frac{3\sqrt{6}}{2} \text{ cm}^3$

Mark allocation: 2 marks

- 1 mark for calculating the variance of $\frac{81}{6}$.
- 1 mark for correct standard deviation of $\frac{3\sqrt{6}}{2}$.

TIP

» **Ensure that you understand the difference between a sum of independent random variables and a scalar multiple of a random variable. Whilst it may be seen that $\text{E}(X_1 + X_2) = \text{E}(2X)$, where X_1 and X share the same distribution, the variance does not behave in the same way; that is, $\text{Var}(X_1 + X_2) \neq \text{Var}(2X)$.**

EXAM 2

Section A

Question 1

Answer: ***C***

Worked solution

$$\begin{aligned}\mu = \text{E}(Z) &= \text{E}(2X - 3Y)\\ &= 2\text{E}(X) - 3\text{E}(Y)\\ &= 2 \times 9 - 3 \times 6\\ &= 0\end{aligned}$$

$\text{Var}(X) = \sigma_X^2 = 5^2 = 25$

$\text{Var}(Y) = \sigma_Y^2 = 2^2 = 4$

$$\begin{aligned}\text{Var}(Z) &= 2^2\text{Var}(X) + 3^2\text{Var}(Y)\\ &= 2^2 \times 25 + 3^2 \times 4\\ &= 136\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\text{Var}(Z)} = \sqrt{136}\\ &= 2\sqrt{34}\end{aligned}$$

Question 2

Answer: ***D***

Worked solution

A type I error occurs when the null hypothesis is rejected when true, which eliminates option A because the null hypothesis is true. This means that a type I error cannot occur.

The *p*-value is less than the significance level, which eliminates option C.

When the *p*-value is less than the significance level, the null hypothesis is rejected, which means option B is incorrect.

Therefore a type II error would occur if the null hypothesis is false and not rejected, which makes option D the correct response.

Question 3

Answer: ***B***

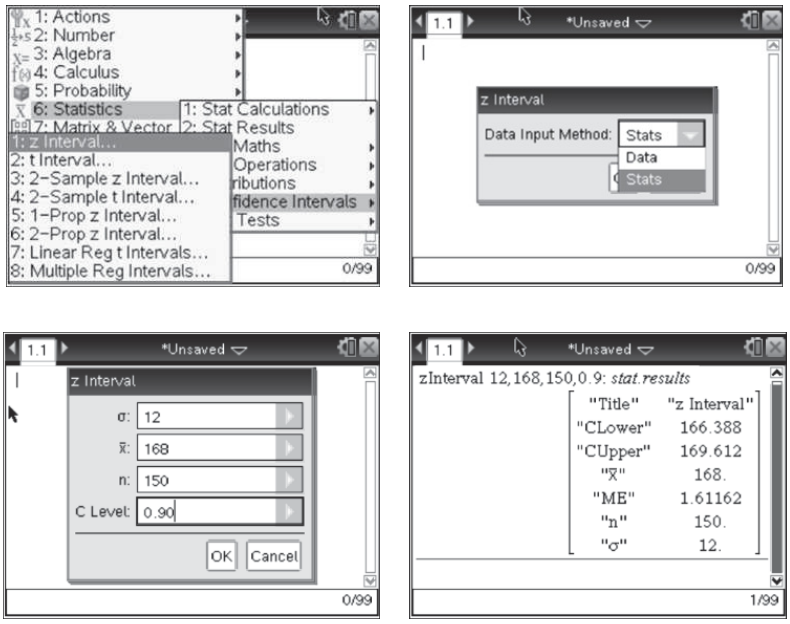
Worked solution

A 90% confidence interval is closest to

$$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right) = \left(168 - 1.645\frac{12}{\sqrt{150}}, 168 + 1.645\frac{12}{\sqrt{150}}\right)$$

$$= (166.388, 169.612)$$

Using a CAS:



TIP

» To avoid rounding errors, always use more accuracy during a calculation than is required in the final answer.

Question 4

Answer: ***A***

Worked solution

$\mu_{\bar{x}} = \mu = 65$

$\text{sd}(\bar{x}) = \frac{\sigma}{\sqrt{n}} = \frac{2.5}{\sqrt{10}}$

$$\Pr(\bar{X} < 63.5) = \Pr\left(Z < \frac{63.5 - 65}{\frac{2.5}{\sqrt{10}}}\right) = 0.02889$$

$$= 0.0289$$

Question 5

Answer: ***D***

Worked solution

The mean is given by

$\mu = \text{E}(W) = \text{E}(2X) + \text{E}(-3Y)$

$= 2\text{E}(X) - 3\text{E}(Y)$

$= 2 \times 9 - 3 \times 7$

$= -3$

The standard deviation is given by

$\sigma = \sqrt{\text{Var}(W)} = \sqrt{\text{Var}(2X) + \text{Var}(-3Y)}$

$= \sqrt{2^2\ \text{Var}(X) + (-3)^2\text{Var}(Y)}$

$= \sqrt{4 \times 4^2 + 9 \times 2^2}$

$= \sqrt{100}$

$= 10$

Question 6

Answer: ***B***

Worked solution

'Does not exceed 202 grams' is the same as 'less than or equal to 202 grams', so

$$\Pr(\bar{X} < 202) = \Pr\left(Z < \frac{202 - 200}{\frac{5}{\sqrt{16}}}\right)$$

$$= 0.945$$

TIP

» In probability questions, pay close attention to the way the question is worded. For instance, in this question, 'does not exceed' is the same as 'less than or equal to'. So be aware of the wording so that you know whether you are trying to calculate the probability for a 'less than' or 'more than' interval.

Question 7

Answer: ***B***

Worked solution

A type I error is when the null hypothesis is rejected when it is true. In this scenario, the null hypothesis is that the growth rate is the same.

Option B is the only option that meets the type I error guidelines. ‘Concluding that the growth rate is higher than average when using the special fertiliser’ represents rejecting the null hypothesis, and ‘when in fact it is not’ represents the null hypothesis is true. Therefore option B is a type I error.

Question 8

Answer: ***C***

Worked solution

The sample mean is the average of a confidence interval: $\bar{x} = \frac{65.18 + 71.32}{2} = 68.25$.

For the lower limit of a confidence interval, we have $65.18 = 68.25 - z\frac{s}{\sqrt{60}}$.

For a 95% confidence interval, use $z = 1.96$.

$\Rightarrow s = \frac{\sqrt{60}(68.25 - 65.18)}{1.96} = 12.13$

These results are closest to option C: 68.25 and 12.15.

Question 9

Answer: ***C***

Worked solution

Since $\Pr(m < X < n) = 0.95$ and $\frac{m+n}{2} = 125$, then $\Pr(X < m) = 0.025$ and $\Pr(X > n) = 0.025$.

Using the inverse normal function, we get $m = 109.3$ and $n = 140.7$, correct to one decimal place.

Note: $\frac{m+n}{2} = 125$ indicates that 125 is the mid point between m and n. This means m and n are the same distance away from 125, making them symmetrical about the mean.

Question 10

Answer: ***D***

Worked solution

For this question we are trying to find $\Pr(-10 < Y - X < 10)$. So

$$\text{E}(Y - X) = \text{E}(Y) - \text{E}(X) = 138 - 125 = 13$$

$$\text{Var}(Y - X) = \text{Var}(Y) + (-1)^2\text{Var}(X) = 15^2 + 8^2 = 289$$

$$\Rightarrow s(Y - X) = \sqrt{289} = 17$$

Using the normal CDF function on a CAS gives $\Pr(-10 < Y - X < 10) = 0.3419$.

Note: The wording ‘within 10 minutes’ means that the recharge time of brand Y could be 10 minutes greater or 10 minutes less than that of brand X, leading to $\Pr(-10 < Y - X < 10)$.

Question 11

Answer: ***A***

Worked solution

Let C be the number of hot chips sold in a randomly selected week.

Let F be the number of flake sold in a randomly selected week.

Consider $\Pr(4C - 5.5F > 0)$:

The mean is given by

$$\begin{aligned}\mu &= \mathrm{E}(4C - 5.5F)\\ &= 4\mathrm{E}(C) - 5.5\mathrm{E}(F)\\ &= 4 \times 220 - 5.5 \times 170 = -55\end{aligned}$$

The standard deviation is given by

$$\begin{aligned}\sigma &= \sqrt{\mathrm{Var}(4C - 5.5F)}\\ &= \sqrt{4^2\mathrm{Var}(C) + (-5.5)^2\,\mathrm{Var}(F)}\\ &= \sqrt{4^2 \times 23^2 + 5.5^2 \times 15^2}\\ &= \sqrt{15\,270.25}\end{aligned}$$

Entering μ and σ into a CAS will show that the probability is $\Pr(4C - 5.5F > 0) = 0.3281$.

Question 12

Answer: ***D***

Worked solution

Let X be the number of hours of sleep per night of a student chosen at random from this population.

The 90% confidence interval is given by $\left(\bar{x} - k\frac{\sigma}{\sqrt{n}},\ \bar{x} + k\frac{\sigma}{\sqrt{n}}\right)$, where $\Pr(-k < Z < k) = 0.90$.

Using a CAS will show that $k = 1.64485$.

The width of this 90% confidence interval is $= 2k\frac{\sigma}{\sqrt{n}} = 0.05$, and given that $\sigma = 0.7$, then

$$n = \left(\frac{2k\sigma}{0.05}\right)^2 = \left(\frac{2 \times 1.64485 \times 0.7}{0.05}\right)^2 = 2121$$

Question 13

Answer: ***D***

Worked solution

To test for improvement, define an alternative hypothesis that postulates fewer days to recover than the null hypothesis, so option D is true.

Question 14

Answer: ***D***

Worked solution

The z-score for a 97% confidence interval is approximately 2.17.

If $\bar{x} + z\frac{\sigma}{\sqrt{n}} = 222 + 2.17 \times \frac{5}{\sqrt{n}} = 223.356$, then solving for n gives $n = 64.0236$.

Therefore the sample size is 64.

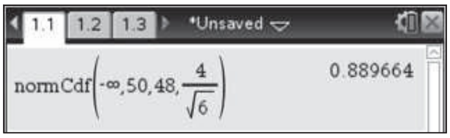
Question 15

Answer: ***D***

Worked solution

For the sample of lemon trees selected $\mu = 48$, $\text{sd}(\bar{X}) = \frac{4}{\sqrt{6}}$.

Therefore $\Pr(\bar{X} < 50) = 0.889664... \approx 0.8897$.



Question 16

Answer: ***B***

Worked solution

The null hypothesis assumes that the mean drawn from the population is the same as that stated and that any difference we observe is due to the variation in the samples. Therefore H_0: $\mu = 650$.

The alternative says that the mean is an overestimate, so the value of 650 is more than the population's true mean. Therefore H_1: $\mu < 650$.

Question 17

Answer: ***D***

Worked solution

The key point here is that the 10 games are independent. Therefore the mean total score is simply the sum of the expected score in each of the 10 games; that is, $10 \times 20 = 200$. Similarly, the variance of the total score is the sum of the variance of each game; that is, $10 \times 5 = 50$. Hence option D is correct.

Contrast this with the situation where the events are not independent. In this situation, we would use the familiar rules $\text{E}(aX + b) = a\text{E}(X) + b$ and $\text{Var}(aX + b) = a^2\text{Var}(X)$. Combinations of these rules provide the incorrect options.

Question 18

Answer: ***D***

Worked solution

An approximate 90% confidence interval for μ is $\left(\bar{x} - 1.644...\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.644...\frac{\sigma}{\sqrt{n}}\right)$.

The width of this interval is $2 \times 1.644...\frac{\sigma}{\sqrt{n}}$. Equating this with the interval given and then solving leads to

$$2 \times 1.6448...\frac{\sigma}{\sqrt{n}} = 388.4 - 248.2$$

$$2 \times 1.6448...\frac{210}{\sqrt{n}} = 140.2$$

$$n = 24.28$$

Therefore the answer is 24, which is option D.

Section B

Question 1a.

Worked solution

$\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{1.5}{\sqrt{60}} = \frac{\sqrt{15}}{20}$

Mark allocation: 1 mark

- 1 mark for $\text{sd}(\bar{X}) = \frac{\sqrt{15}}{20}$

Question 1b.

Worked solution

H_0: $\mu = 9.5$ and H_1: $\mu \neq 9.5$.

Mark allocation: 2 marks

- 1 mark for H_0: $\mu = 9.5$
- 1 mark for H_1: $\mu \neq 9.5$

TIP

» **Looking for words like ‘increase’, ‘decrease’ and ‘not’ in the question’s description will help you to determine the alternative hypothesis and whether a one-tailed test or a two-tailed test must be used.**

Question 1c.i.

Worked solution

Because of the alternative hypothesis, H_1: $\mu \neq 9.5$, a two-tailed test is required. So the *p*-value must be calculated accordingly. So

$$\begin{aligned}
p\text{-value} &= 2 \times \Pr(\overline{X} \leq 9 \mid \mu = 9.5)\\
&= 2 \times \Pr\left(Z \leq \frac{9-9.5}{\frac{\sqrt{15}}{20}}\right)\\
&= 2 \times \Pr(Z \leq -2.58199)\\
&= 2 \times 0.004912\\
&= 0.009824\\
&= 0.0098
\end{aligned}$$

Because a two-tailed test is non-directional, both tails must be considered in the calculation of the *p*-value. So the *p*-value is double the value of a single-tailed test.

Mark allocation: 2 marks

- 1 mark for $2 \times \Pr\left(Z \leq \frac{9-9.5}{\frac{\sqrt{15}}{20}}\right)$
- 1 mark for determining that the *p*-value is 0.0098

Question 1c.ii.

Worked solution

Since the *p*-value found for this sample is less than $\alpha = 0.05$, the null hypothesis, H_0, is rejected and the alternative hypothesis, H_1, is accepted. So the sample selected supports the sports shoe store's belief.

Mark allocation: 1 mark

- 1 mark for stating that H_0 is rejected in favour of H_1, and that the sample supports the sports shoe store's belief

Question 2a.

Worked solution

Let $\overline{X}$ be the distribution of the sample means.

The standard deviation is then

$$\begin{aligned}
\text{sd}(\overline{X}) &= \frac{\sigma}{\sqrt{n}}\\
&= \frac{3}{\sqrt{100}}\\
&= 0.3
\end{aligned}$$

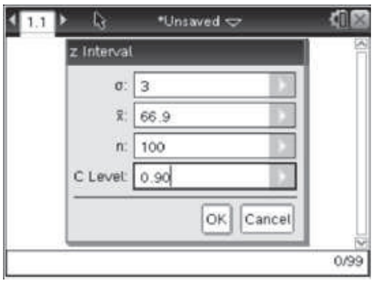
Mark allocation: 1 mark

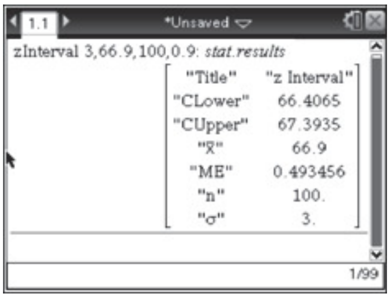
- 1 mark for the correct answer

Question 2b.

Worked solution

Begin by finding the 90% confidence interval by using a CAS.





The confidence interval is $(66.4065, 67.3935)$.

Note that the formula $\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$, where $z = 1.65$, can be used to find the 90% confidence interval.

Equate the confidence interval found with the confidence interval given in the question and solve for k_1 and k_2, correct to one decimal place.

$66.9 - k_1 = 66.4065$ and $66.9 + k_2 = 67.3935$.

$\Rightarrow k_1 = 0.5$ and $k_2 = 0.5$.

Mark allocation: 2 marks

- 1 mark for finding the 90% confidence interval: $(66.4065, 67.3935)$
- 1 mark for the correct answer: $k_1 = 0.5$ and $k_2 = 0.5$

Question 2c.

Worked solution

Since the quality control tester claims the diameter exceeds the production company's mean, this is an uppertail hypothesis test.

The null and alternative hypotheses are then

H_0: $\mu = 66.9$

H_1: $\mu > 66.9$

Mark allocation: 1 mark

- 1 mark for H_0: $\mu = 66.9$ and H_1: $\mu > 66.9$

Question 2d.

Worked solution

An expression for the *p*-value is $\Pr(\bar{X} > 67 \mid \mu = 66.9)$.

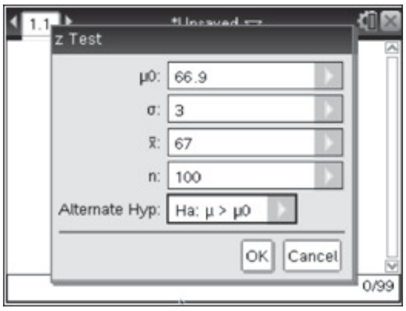
The *p*-value is then

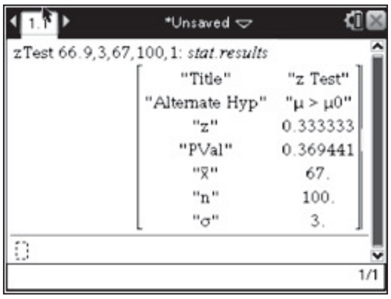
$\Pr(\bar{X} > 67 \mid \mu = 66.9)$

$= \Pr\left(Z > \dfrac{67 - 66.9}{\frac{3}{\sqrt{100}}}\right)$

$= 0.3694$

Alternatively, using a CAS to evaluate the p-value gives





Mark allocation: 2 marks

- 1 mark for the expression $\Pr(\bar{X} > 67 \mid \mu = 66.9)$
- 1 mark for the correct p-value: 0.3694

Question 2e.

Worked solution

Since the p-value (0.3694) is greater than 0.05, we cannot reject the null hypothesis at the 5% level of significance. There is insufficient evidence to support the quality control tester's claim.

Mark allocation: 1 mark

- 1 mark for stating that we cannot reject the null hypothesis and there is insufficient evidence to support the quality control tester's claim since $p > 0.05$

Question 2f.

Worked solution

The critical z-value for the test is 2.326.

So to find the minimum value of the sample mean we solve $2.326 = \dfrac{\bar{x} - 66.9}{\dfrac{3}{\sqrt{100}}}$, which gives the minimum value as $67.5978 \approx 67.6$ mm.

Mark allocation: 1 mark

- 1 mark for the correct answer: 67.6 mm

Question 3a.

Worked solution

$H_0\!: \mu = 270$ cm

$H_1\!: \mu > 270$ cm

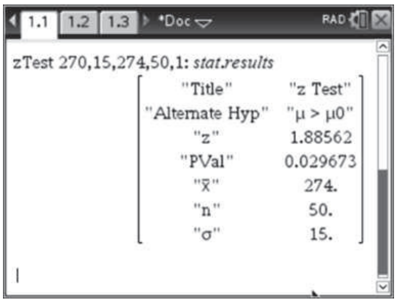
Mark allocation: 1 mark

- 1 mark for stating the correct hypotheses

Question 3b.

Worked solution

p-value $= \Pr(\overline{X} > 274 \mid \mu = 270) = 0.0297$, correct to four decimal places.



As $p < 0.05$, H_0 should be rejected at the 5% significance level.

Mark allocation: 2 marks

- 1 mark for the correct *p*-value: 0.0297
- 1 mark for stating why H_0 should be rejected

Question 3c.

Worked solution

For H_0 not to be rejected at the 5% significance level, the *p*-value must be greater than 0.05.

The critical value for a right-tailed test at the 5% significance level is $z = 1.645$. So

$$1.645 = \frac{274 - 270}{\left(\frac{15}{\sqrt{n}}\right)}$$

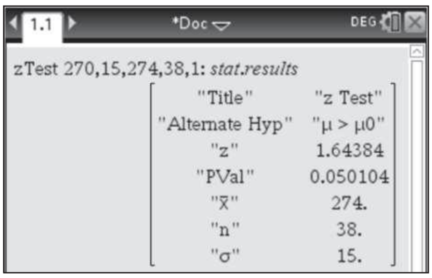
Rearranging for n gives

$$n = \left(\frac{15 \times 1.645}{274 - 270}\right)^2$$

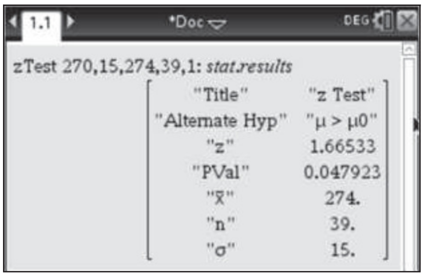
$$n = 38.0535$$

As n must be an integer, sample sizes of 38 and 39 must be checked to determine if their respective *p*-values are greater than 0.05.

For $n = 38$, $p = 0.0501$, which is greater than 0.05.



For $n = 39$, $p = 0.0479$, which is less than 0.05.



So the largest sample size is 38.

Mark allocation: 2 marks

- 1 mark for $1.645 = \dfrac{274 - 270}{\left(\frac{15}{\sqrt{n}}\right)}$
- 1 mark for a sample size of 38

TIP

» Remember that both the rounded-up and rounded-down answers, however unlikely one or the other may appear to be, must be checked to ensure that your answer meets the requirements of the question.

Question 3d.

Worked solution

Conclusion: That the fertiliser increases the height of mature sunflowers when it actually has no effect.

Mark allocation: 1 mark

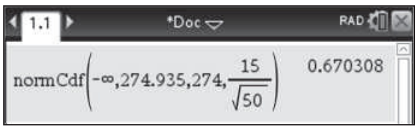
- 1 mark for stating the correct conclusion

Question 3e.

Worked solution

A type II error is failing to reject H_0 when it is false, which would mean selecting a sample mean that is less than 274.935 cm. So

$$\begin{aligned}\Pr(\text{type II error}) &= \Pr(\text{accept } H_0 \text{ when false} \mid H_1 = \mu) \\ &= \Pr(\bar{X} < 274.935 \mid \mu = 274) \\ &= 0.6703\end{aligned}$$



Mark allocation: 2 marks

- 1 mark for $\Pr(\text{type II error}) = \Pr(\bar{X} < 274.935 \mid \mu = 274)$
- 1 mark for a probability of 0.6703

Question 4a.

Worked solution

Let J be the number of sales by Jafan in a randomly selected week.

$\text{E}(J) = 21.5,\ \text{sd}(J) = 3.8$

$\Pr(J \geq 20) = 0.6535$

Mark allocation: 1 mark

- 1 mark for the correct probability

Question 4b.

Worked solution

Let C_{L} be the commission earned by Liv in a randomly selected week.

Let C_{J} be the commission earned by Jafan in a randomly selected week.

Let random variables L and J be the number of weekly sales made by Liv and Jafan respectively.

$\Pr(C_{\text{L}} > C_{\text{J}}) = \Pr(C_{\text{L}} - C_{\text{J}} > 0)$

$$\begin{aligned} \text{E}(C_{\text{L}} - C_{\text{J}}) &= \text{E}(125L - 110J) = 125\text{E}(L) - 110\text{E}(J) \\ &= 125 \times 15.8 - 110 \times 21.5 = -390 \end{aligned}$$

$$\begin{aligned} \text{sd}(C_{\text{L}} - C_{\text{J}}) &= \sqrt{\text{Var}(125L - 110J)} \\ &= \sqrt{125^2\text{Var}(L) + (-110)^2\text{Var}(J)} \\ &= \sqrt{125^2 \times 3.8^2 + 110^2 \times 3.8^2} \end{aligned}$$

$\Pr(C_{\text{L}} - C_{\text{J}} > 0) = 0.2688$

Mark allocation: 2 marks

- 1 mark for forming the expressions for the mean and standard deviation of the difference between Jafan's and Liv's commissions
- 1 mark for the correct probability

TIP

» When random variables are combined, the overall variance generally increases. This is the case even when considering a combination that involves subtraction, such as $C_{\text{L}} - C_{\text{J}}$. If you calculate that the variance of a combination of random variables is much smaller than the variances of the individual random variables, it is likely that you have forgotten to square the negative number in the calculation.

Question 4c.

Worked solution

Let Liv's new commission be x.

$$\mathrm{E}(C_L - C_J) = \mathrm{E}(xL - 110J) = x\mathrm{E}(L) - 110\mathrm{E}(J)$$
$$= x \times 15.8 - 110 \times 21.5 = 15.8x - 2365$$

$$\Pr(C_L - C_J > 0) = 0.5$$

Transform this to the standard normal distribution:

$$\Pr\left(Z > \frac{0 - \mathrm{E}(C_L - C_J)}{\mathrm{sd}(C_L - C_J)}\right) = 0.5$$

$$\frac{0 - \mathrm{E}(C_L - C_J)}{\mathrm{sd}(C_L - C_J)} = 0$$

$$\mathrm{E}(C_L - C_J) = 0$$

$$15.8x - 2365 = 0$$

$$x = 149.68$$

Liv's new commission needs to be \$149.68.

Alternatively:

The expected value of the combination must now be zero (if they each have the same chance of earning more).

$$\mathrm{E}(xL - 110J) = x \times 15.8 - 110 \times 21.5 = 0$$

$$\therefore x = 149.68$$

Mark allocation: 2 marks

- 1 mark for finding an expression of $\mathrm{E}(C_L - C_J)$ for Liv's new commission
- 1 mark for finding Liv's new commission

Question 4d.

Worked solution

H_0: $\mu = 15.8$

H_1: $\mu > 15.8$

Mark allocation: 1 mark

- 1 mark for the correct null hypothesis and the correct alternative hypothesis

TIP

» Look for specific words like 'improved', 'increased', 'decreased', 'bettered' and so on as indications that you should be using a one-tailed test. When no such language is present, a two-tailed test should be used.

Question 4e.

Worked solution

Let X be the number of sales Liv makes when wearing her new suit in a randomly selected week.

$E(\overline{X}) = \mu = 15.8,\ n = 8,\ sd(\overline{X}) = \frac{3.8}{\sqrt{8}}$

p-value $= \Pr(\overline{X} \geq 17.5 \mid \mu = 15.8) = 0.1029$

Mark allocation: 2 marks

- 1 mark for an expression of the *p*-value
- 1 mark for the correct answer

Question 4f.

Worked solution

Since the *p*-value (0.1029) is greater than the significance level (0.05), Liv does not have enough evidence to conclude that the new suit has improved sales.

Mark allocation: 1 mark

- 1 mark for the correct conclusion

Question 4g.

Worked solution

A type II error occurs when the null hypothesis is not rejected when it should be. Increasing the significance level means that the null hypothesis is more likely to be rejected, which would indeed lead to fewer type II errors. However, increasing the significance level increases the probability of type I errors, whereby a null hypothesis is rejected despite being true.

Mark allocation: 2 marks

- 1 mark for explaining why increasing the significance level decreases the chance of making a type II error
- 1 mark for stating that this will increase the chance of making a type I error

Question 5a.

Worked solution

The distribution of the charging time of the sample of batteries can be described as

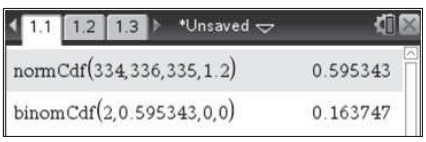
$\overline{X} \sim N\left(335, \left(\frac{12}{10}\right)^2\right)$.

$\Pr(334 < \overline{X} < 336) = 0.595343$.

Let the distribution of the samples be described as $Y \sim Bi(2, 0.595343)$.

Then $\Pr(Y = 0) = 0.1637457...$

So the probability that neither random sample has a charging time between 334 min and 336 min is 0.164.



Alternatively:

$\Pr(\bar{X} < 334 \cup \bar{X} > 336) = 0.40466...$

Hence the probability that both sample means are outside the range of 334 min to 336 min is $0.40466 \times 0.40466 = 0.16375$, which rounded to three decimal places is 0.164.

Mark allocation: 2 marks

- 1 mark for correctly finding $\Pr(334 < \bar{X} < 336) = 0.595343$ and using it as the probability of success in a binomial distribution
- 1 mark for the correctly rounded answer: 0.164

OR

- 1 mark for finding the probability that the charging time is outside the range of 334 min to 336 min is 0.40466
- 1 mark for the correctly rounded answer: 0.164

Question 5b.

Worked solution

For this question we are trying to find $\Pr\left(\left|\bar{X}_1 - \bar{X}_2\right| > 5\right) = 1 - \Pr\left(-5 < \bar{X}_1 - \bar{X}_2 < 5\right)$.

Then let $Y = \bar{X}_1 - \bar{X}_2$, where

$$\mathrm{E}(Y) = \mathrm{E}(\bar{X}_1 - \bar{X}_2) = 0$$

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(\bar{X}_1 - \bar{X}_2) \\ &= \left(\frac{12}{\sqrt{100}}\right)^2 + \left(\frac{12}{\sqrt{100}}\right)^2 \\ &= \frac{72}{25}\end{aligned}$$

$$\Rightarrow \mathrm{sd}(Y) = \frac{6\sqrt{2}}{5}$$

So $Y \sim N\left(0, \frac{72}{25}\right)$.

Therefore

$$\begin{aligned}\Pr\left(\left|\bar{X}_1 - \bar{X}_2\right| > 5\right) &= 1 - \Pr(-5 < \bar{X}_1 - \bar{X}_2 < 5) \\ &= 1 - \Pr(-5 < Y < 5) \\ &= 0.003216...\end{aligned}$$

Correct to three decimals places, the probability is 0.003.

Mark allocation: 2 marks

- 1 mark for using linear combinations and defining the distribution of Y
- 1 mark for the probability of 0.003

Question 5c.

Worked solution

H_0: $\mu = 335$ $\quad\quad$ H_1: $\mu > 335$

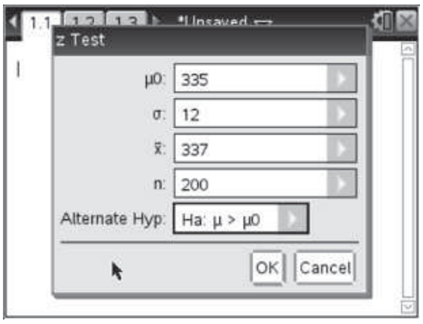
Mark allocation: 1 mark

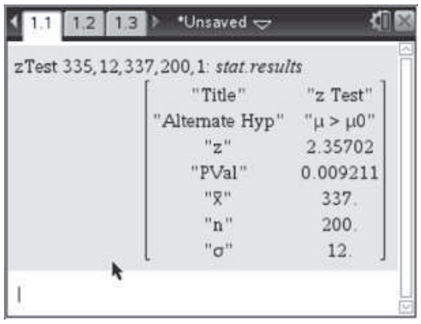
- 1 mark for H_0: $\mu = 335$ and H_1: $\mu > 335$

Question 5d.i.

Worked solution

p-value = $\Pr(\bar{X} > 337 \mid \mu = 335) = 0.009$, correct to three decimal places.





Mark allocation: 1 mark

- 1 mark for a *p*-value of 0.009

Question 5d.ii.

Worked solution

Since $p < 0.05$, H_0 should be rejected at the 5% significance level.

Mark allocation: 1 mark

- 1 mark for stating that since $p < 0.05$, H_0 should be rejected at the 5% significance level

Question 5e.

Worked solution

$\Pr(\bar{X} > \bar{x}_c \mid \mu = 335) = 0.05$

$\Rightarrow \Pr\left(Z > \dfrac{\bar{x}_c - 335}{\frac{12}{\sqrt{200}}} = 1.6449\right) = 0.05$

$\Rightarrow \bar{x}_c = 336.39574...$

$\Rightarrow \bar{x}_c = 336.3957$ min, correct to four decimal places.

Mark allocation: 1 mark

- 1 mark for 336.3957 min

Question 5f.

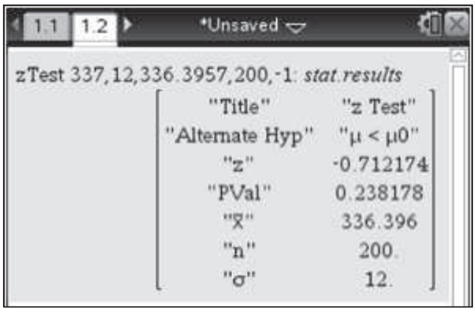
Worked solution

A type II error is failing to reject H_0 when it is false.

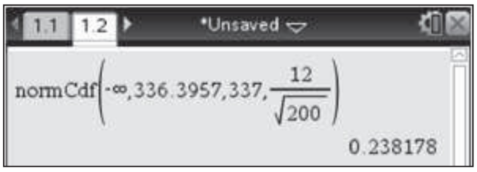
$\Pr(\text{type II error}) = \Pr(\text{do not reject } H_0 \mid \mu = 337)$

$= \Pr(\bar{X} < 336.3957 \mid \mu = 337)$

$= 0.238$, correct to three decimal places



Alternatively:



Mark allocation: 1 mark

- 1 mark for 0.238

Question 6a.

Worked solution

The mean of these 10 values is 751.3.

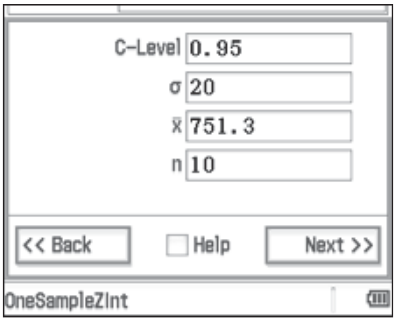
Mark allocation: 1 mark

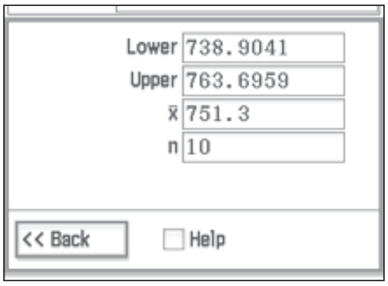
- 1 mark for the correct answer, rounded to one decimal place

Question 6b.

Worked solution

Use a CAS to find a confidence interval, as shown below.





(738.9, 763.7)

Mark allocation: 1 mark

- 1 mark for the correct values, to one decimal place, separated by a comma and in round brackets

Question 6c.

Worked solution

12.4

Mark allocation: 1 mark

- 1 mark for the correct answer, rounded to one decimal place

TIP

» You can think of the margin of error as the distance between the mean and the edge of the confidence interval.

Question 6d.

Worked solution

You need to calculate 95% of 200 to find the answer, 190.

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 6e.

Worked solution

Use the inverse standard normal distribution to find the value of k.

$\Pr(-k < Z < k) = 0.75$

$k = 1.15$

Mark allocation: 2 marks

- 1 mark for using the inverse standard normal distribution
- 1 mark for the correct answer, rounded to two decimal places

TIP

» Use the command 'invnorm' in your CAS.

Question 6f.

Worked solution

Using the formula for margin of error (MOE) gives

$$n = \left(\frac{2.58\sigma}{\text{MOE}}\right)^2 = \left(\frac{2.58 \times 20}{3}\right)^2$$

$$n = 294.88$$

$\therefore n = 295$

Mark allocation: 2 marks

- 1 mark for using the value of 2.58 (as the z-score that corresponds to a 99% confidence interval)
- 1 mark for the correct answer: 295

