

SPECIALIST MATHEMATICS

UNITS 1 & 2

CAMBRIDGE SENIOR MATHEMATICS FOR QUEENSLAND

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- 21B** Antidifferentiation of polynomial functions
- 21C** Applications of antidifferentiation to kinematics
- 21D** Constant acceleration
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22 Sampling and sampling distributions

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- Review of Chapter 22

Online Appendices: Guides to using technology

These online guides are accessed through the Interactive Textbook or PDF Textbook

- A1A** Online guide to the TI-Nspire CX Non-CAS graphics calculator
- A2A** Online guide to the TI-84 Plus CE graphics calculator
- A3A** Online guide to the Casio fxCG20AU and Casio fxCG50AU graphics calculators
- A4A** Online guide to the Desmos graphing calculator

Introduction and overview

Cambridge Senior Mathematics for Queensland Specialist Mathematics Units 1 & 2 provides complete and close coverage of the QCAA syllabus to be implemented in Year 11 from 2019. It also provides a thorough preparation for *Specialist Mathematics Units 3 & 4*. Its four components — the print book, downloadable PDF textbook, online Interactive Textbook (ITB) and Online Teaching Resource (OTS), both powered by the HOTmaths platform — contain a huge range of resources, including worked solutions, available to schools in a single package at one convenient price (the OTS is included with class adoptions, conditions apply). There are no extra subscriptions or per-student charges to pay.

Additional material: Material that is additional to the syllabus is provided in Chapters 1 to 5, 20 to 22 and several other sections of the book to prepare the students for this course, giving sufficient background, and to provide the opportunity for students to have a thorough preparation for Specialist Mathematics Units 3&4.

- Chapter 1 provides an opportunity to review algebra from Year 9 and 10.
- Chapter 2 is a preparation for Chapter 8: Number and proof. It also contains a review of surd arithmetic which is useful in many topics. Expressing rational numbers as terminating or eventually recurring decimals and vice versa is covered in this chapter.
- Chapter 3 includes a review of the sine rule and cosine rule which are utilised in Chapter 6: Vectors in the plane and will also be required in units 3 and 4.
- Chapter 4 provides opportunities to strengthen the student's algebra and provides a broader background for Units 3 and 4. It also includes the study of equating coefficients and partial fractions which is required in Unit 4.
- Proof by mathematical induction is included in Chapter 8: Number and proof.
- Chapter 9 contains a review of geometry from earlier years and the *Specialist Mathematics Units 1 & 2* topic Geometric proofs using vectors.
- Chapter 20 contains several topics which can be used for preparation and deeper understanding of *Specialist Mathematics Units 3 & 4* topics. In particular it is very useful preparation for aspects of both vectors and complex numbers.
- There are also two online chapters that provide valuable preparation for *Specialist Mathematics Units 3 & 4*.

Degree of difficulty classification of questions: in the exercises, questions are classified as **simple familiar** **SF**, **complex familiar** **CF**, and **complex unfamiliar** **CU** questions. The revision chapters described below also contain model questions for each of these categories, and tests are also provided in the teacher resources, made up of such categorised model questions.

Four revision chapters of material covered in the course: These four revision chapters contain sections on *Technology-free questions*, *Multiple-choice questions*, *Extended-response questions*, *Problem-solving and modelling questions*, *Problem-solving and modelling investigations* and *Degree-of-difficulty classified questions*.

Problem-solving and modelling: The problem-solving and modelling questions are multi-part questions where the students undertake modelling or problem solving with an indication of the path to be taken. These can be used in introducing the techniques of problem-solving and modelling. The Investigation section is less structured and the student will have to make decisions about the pathway they will follow. Answers for them are not given in the textbook, but model solutions are in the Online Teaching Suite. See the QCAA Specialist Mathematics syllabus for further guidelines for such tasks.

Additional problem-solving and modelling investigations are provided in the ITB, linked via an icon from the review section of each chapter.

Calculator guidance: Throughout the book there is guidance for the use of the TI-Nspire CX non-CAS and the Casio fxCG20AU and fxCG50AU graphics calculators for the solution of problems. Guidance on the TI-84Plus CE is included in the interactive textbook, accessed via icons next to the TI-Nspire boxes. There are also online guides for the general use of each of these calculators.

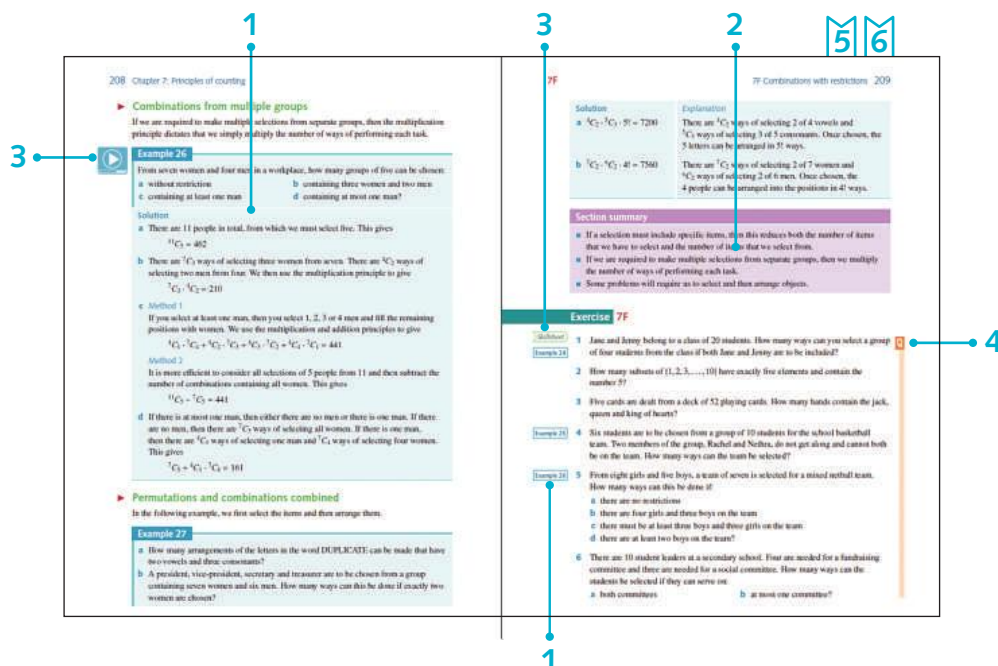
The online graphing calculator from Desmos.com is also embedded in the interactive textbook, as blank screens that students and teachers can use for their own calculations, or as widgets which have been set up for a variety of activities. The new Desmos geometry tool is also embedded in the interactive textbook, and activities and widgets using the tool will be added as they are developed.

Assessment: examination practice questions and assessment tasks are provided in the revision chapters and the Online Teaching Suite. Check the updates there for developments as more guidance is published closer to implementation of the new assessment.

► Overview of the print books in the series (shown opposite)

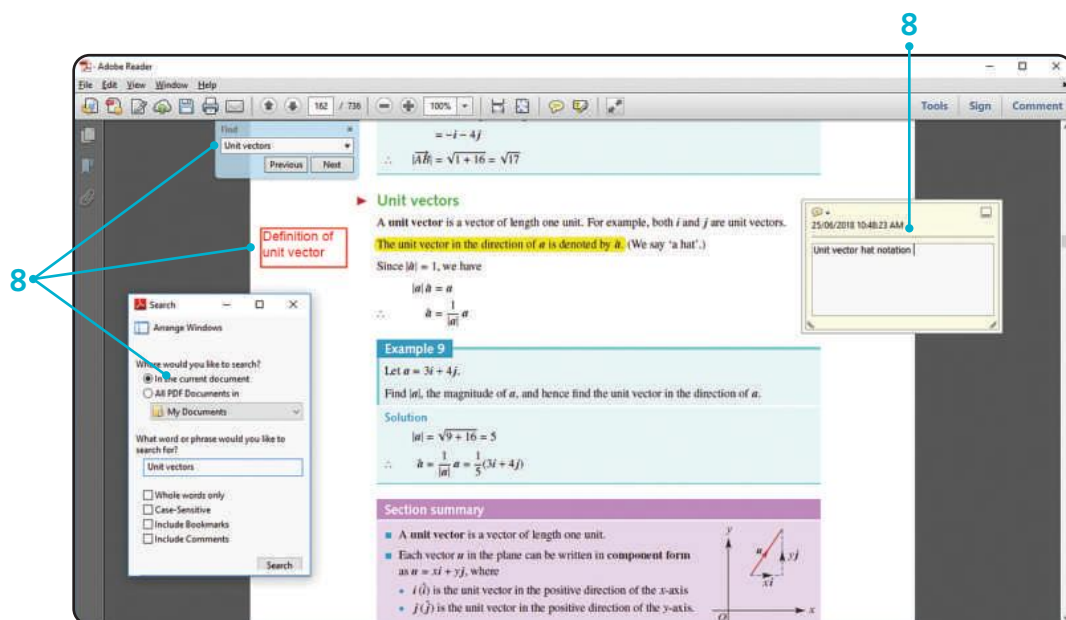
- 1 Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercises.
- 2 Section summaries provide important concepts in boxes for easy reference.
- 3 Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.
- 4 Degree of difficulty categories are indicated in exercises (similar familiar, complex familiar and complex unfamiliar).
- 5 Chapter reviews contain a chapter summary and technology-free, multiple-choice, and extended-response questions. The latter are classified by degree of difficulty.
- 6 The glossary includes page numbers of the main explanation of each term.

Numbers refer to descriptions in the text.



► Overview of the downloadable PDF textbooks in the series

- 7 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.
- 8 PDF annotation and search features are enabled.



► Overview of the Interactive Textbooks in the series (shown on the page opposite)

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

- 9 The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.
- 10 The new **Workspaces** enable students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing.
- 11 The new **self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.
- 12 Selected examples have **video versions** to encourage independent learning.
- 13 **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.
- 14 Interactive **Desmos widgets** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.
- 15 The **Desmos graphics calculator, scientific calculator, and geometry tool** are also embedded for students to use for their own calculations and exploration.
- 16 **Quick quizzes** containing automarked multiple choice questions enable students to check their understanding.
- 17 **Definitions** pop up for key terms in the text, and are also provided in a **dictionary**.
- 18 Messages from the teacher assign tasks and tests.
- 19 **Practice exam-style questions** are provided in downloadable PDF and Word files.
- 20 Additional problem-solving and modelling questions and investigations are provided as downloadable PDFs and editable Word documents.

INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

Note: Screenshots show content from Mathematical Methods Units 1 & 2.

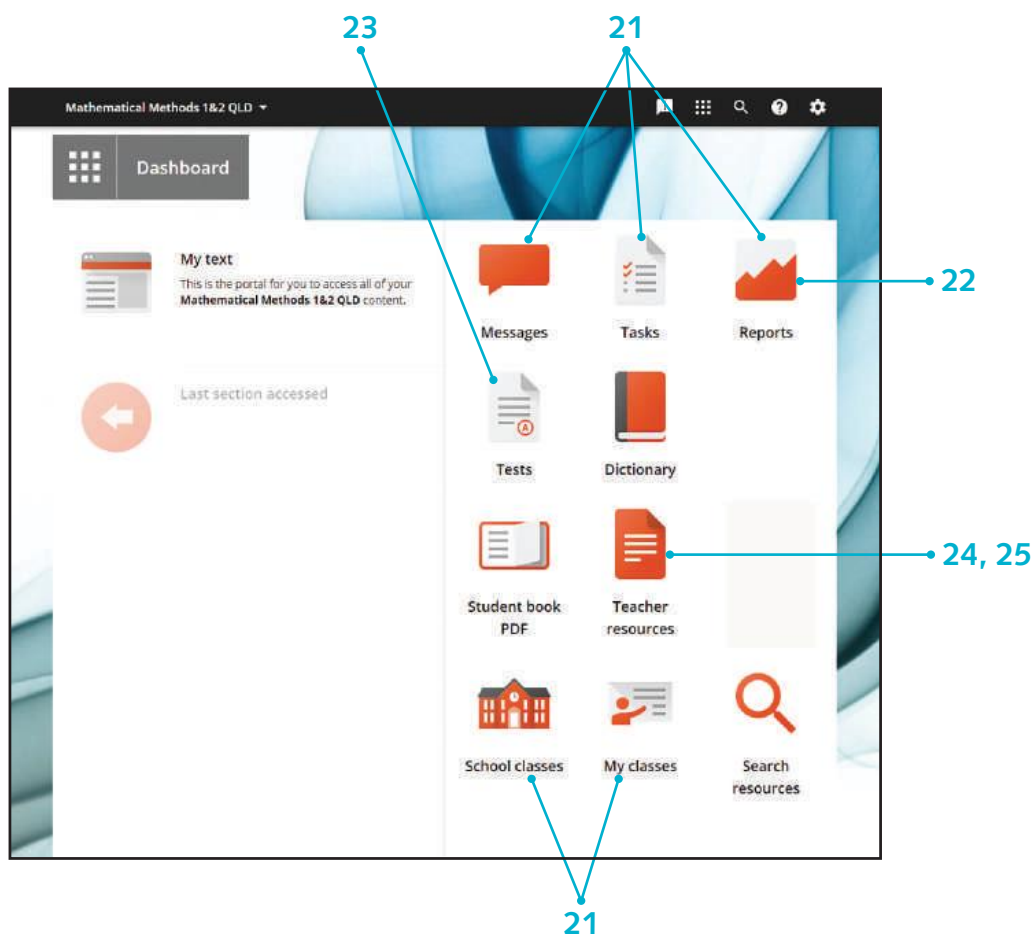
A selection of features is shown. Numbers refer to the descriptions on the opposite page.
HOTmaths platform features are updated regularly

WORKSPACES AND SELF-ASSESSMENT

► Overview of the Online Teaching Suite powered by the HOTmaths platform (shown below)

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

- 21** The HOTmaths learning management system with class and student analytics and reports, and communication tools.
- 22** Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had difficulty.
- 23** A HOTmaths-style test generator.
- 24** Chapter test worksheets and additional assessment items.
- 25** Editable curriculum grids and teaching programs.



About the lead author and consultants

About the lead author

Michael Evans was a consultant to ACARA on the writing of the Australian Curriculum on which the new Queensland syllabus is based. He is a consultant with the Australian Mathematical Sciences Institute, and is coordinating author of the ICE-EM 7–10 series also published by Cambridge.

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1

Algebra

Objectives

- ▶ To solve **linear equations**.
- ▶ To solve problems with linear equations and **simultaneous linear equations**.
- ▶ To use **substitution** and **transposition** with formulas.
- ▶ To add and multiply algebraic fractions.
- ▶ To solve **literal equations**.
- ▶ To solve **simultaneous literal equations**.

This chapter is provided for review of previous years.

Algebra is the language of mathematics. Algebra helps us to state ideas more simply. It also enables us to make general statements about mathematics, and to solve problems that would be difficult to solve otherwise.

We know by basic arithmetic that $9 \times 7 + 2 \times 7 = 11 \times 7$. We could replace the number 7 in this statement by any other number we like, and so we could write down infinitely many such statements. These can all be captured by the algebraic statement $9x + 2x = 11x$, for any number x . Thus algebra enables us to write down general statements.

Formulas enable mathematical ideas to be stated clearly and concisely. An example is the well-known formula for compound interest. Suppose that an initial amount P is invested at an interest rate R , with interest compounded annually. Then the amount, A_n , that the investment is worth after n years is given by $A_n = P(1 + R)^n$.

In this chapter we review some of the techniques which you have met in previous years. Algebra plays a central role in Specialist Mathematics at Years 11 and 12. It is important that you become fluent with the techniques introduced in this chapter and in Chapter 4.

1A Indices

This section revises algebra involving indices.

Review of index laws

For all non-zero real numbers a and b and all integers m and n :

$$\begin{array}{llll} \blacksquare a^m \times a^n = a^{m+n} & \blacksquare a^m \div a^n = a^{m-n} & \blacksquare (a^m)^n = a^{mn} & \blacksquare (ab)^n = a^n b^n \\ \blacksquare \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} & \blacksquare a^{-n} = \frac{1}{a^n} & \blacksquare \frac{1}{a^{-n}} = a^n & \blacksquare a^0 = 1 \end{array}$$

Rational indices

If a is a positive real number and n is a natural number, then $a^{\frac{1}{n}}$ is defined to be the n th root of a . That is, $a^{\frac{1}{n}}$ is the positive number whose n th power is a . For example: $9^{\frac{1}{2}} = 3$.

If n is odd, then we can define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose n th power is a . For example: $(-8)^{\frac{1}{3}} = -2$.

In both cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

In general, the expression a^x can be defined for rational indices, i.e. when $x = \frac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

To employ this definition, we will always first write the fractional power in simplest form.

Note: The index laws hold for rational indices m and n whenever both sides of the equation are defined (for example, if a and b are positive real numbers).

Example 1

Simplify each of the following:

a $x^2 \times x^3$

b $\frac{x^4}{x^2}$

c $x^{\frac{1}{2}} \div x^{\frac{4}{5}}$

d $(x^3)^{\frac{1}{2}}$

Solution

a $x^2 \times x^3 = x^{2+3} = x^5$

b $\frac{x^4}{x^2} = x^{4-2} = x^2$

c $x^{\frac{1}{2}} \div x^{\frac{4}{5}} = x^{\frac{1}{2}-\frac{4}{5}} = x^{-\frac{3}{10}}$

d $(x^3)^{\frac{1}{2}} = x^{\frac{3}{2}}$

Explanation

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

Example 2

Evaluate:

a $125^{\frac{2}{3}}$ **b** $\left(\frac{1000}{27}\right)^{\frac{2}{3}}$

Solution

a $125^{\frac{2}{3}} = \left(125^{\frac{1}{3}}\right)^2 = 5^2 = 25$

b $\left(\frac{1000}{27}\right)^{\frac{2}{3}} = \left(\left(\frac{1000}{27}\right)^{\frac{1}{3}}\right)^2 = \left(\frac{10}{3}\right)^2 = \frac{100}{9}$

Explanation

$$125^{\frac{1}{3}} = \sqrt[3]{125} = 5$$

$$\left(\frac{1000}{27}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{1000}{27}} = \frac{10}{3}$$

Example 3Simplify $\frac{\sqrt[4]{x^2y^3}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$.**Solution**

$$\begin{aligned} \frac{\sqrt[4]{x^2y^3}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} &= \frac{(x^2y^3)^{\frac{1}{4}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} = \frac{x^{\frac{2}{4}}y^{\frac{3}{4}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} \\ &= x^{\frac{2}{4}-\frac{1}{2}}y^{\frac{3}{4}-\frac{2}{3}} \\ &= x^0y^{\frac{1}{12}} \\ &= y^{\frac{1}{12}} \end{aligned}$$

Explanation

$$(ab)^n = a^n b^n$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$a^0 = 1$$

Section summary■ **Index laws**

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{-n} = \frac{1}{a^n}$
- $\frac{1}{a^{-n}} = a^n$
- $a^0 = 1$

■ **Rational indices**

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Exercise 1A**Example 1****1** Simplify each of the following using the appropriate index laws:

a $x^3 \times x^4$

b $a^5 \times a^{-3}$

c $x^2 \times x^{-1} \times x^2$

d $\frac{y^3}{y^7}$

e $\frac{x^8}{x^{-4}}$

f $\frac{p^{-5}}{p^2}$

g $a^{\frac{1}{2}} \div a^{\frac{2}{3}}$

h $(a^{-2})^4$

i $(y^{-2})^{-7}$	j $(x^5)^3$	k $(a^{-20})^{\frac{3}{5}}$	l $(x^{-\frac{1}{2}})^{-4}$
m $(n^{10})^{\frac{1}{5}}$	n $2x^{\frac{1}{2}} \times 4x^3$	o $(a^2)^{\frac{5}{2}} \times a^{-4}$	p $\frac{1}{x^{-4}}$
q $(2n^{-\frac{2}{5}})^5 \div (4^3 n^4)$	r $x^3 \times 2x^{\frac{1}{2}} \times -4x^{-\frac{3}{2}}$	t $(2^2 p^{-3} \times 4^3 p^5 \div (6p^{-3}))^0$	
s $(ab^3)^2 \times a^{-2} b^{-4} \times \frac{1}{a^2 b^{-3}}$			

Example 2 2 Evaluate each of the following:

a $25^{\frac{1}{2}}$	b $64^{\frac{1}{3}}$	c $\left(\frac{16}{9}\right)^{\frac{1}{2}}$	d $16^{-\frac{1}{2}}$
e $\left(\frac{49}{36}\right)^{-\frac{1}{2}}$	f $27^{\frac{1}{3}}$	g $144^{\frac{1}{2}}$	h $64^{\frac{2}{3}}$
i $9^{\frac{3}{2}}$	j $\left(\frac{81}{16}\right)^{\frac{1}{4}}$	k $\left(\frac{23}{5}\right)^0$	l $128^{\frac{3}{7}}$

3 Use your calculator to evaluate each of the following, correct to two decimal places:

a 4.35^2	b 2.4^5	c $\sqrt{34.6921}$	d $(0.02)^{-3}$	e $\sqrt[3]{0.729}$
f $\sqrt[4]{2.3045}$	g $(345.64)^{-\frac{1}{3}}$	h $(4.568)^{\frac{2}{5}}$	i $\frac{1}{(0.064)^{-\frac{1}{3}}}$	

4 Simplify each of the following, giving your answer with positive index:

a $\frac{a^2 b^3}{a^{-2} b^{-4}}$	b $\frac{2a^2(2b)^3}{(2a)^{-2} b^{-4}}$	c $\frac{a^{-2} b^{-3}}{a^{-2} b^{-4}}$
d $\frac{a^2 b^3}{a^{-2} b^{-4}} \times \frac{ab}{a^{-1} b^{-1}}$	e $\frac{(2a)^2 \times 8b^3}{16a^{-2} b^{-4}}$	f $\frac{2a^2 b^3}{8a^{-2} b^{-4}} \div \frac{16ab}{(2a)^{-1} b^{-1}}$

5 Write $\frac{2^n \times 8^n}{2^{2n} \times 16}$ in the form 2^{an+b} .

6 Write $2^{-x} \times 3^{-x} \times 6^{2x} \times 3^{2x} \times 2^{2x}$ as a power of 6.

7 Simplify each of the following:

a $2^{\frac{1}{3}} \times 2^{\frac{1}{6}} \times 2^{-\frac{2}{3}}$	b $a^{\frac{1}{4}} \times a^{\frac{2}{5}} \times a^{-\frac{1}{10}}$	c $2^{\frac{2}{3}} \times 2^{\frac{5}{6}} \times 2^{-\frac{2}{3}}$
d $(2^{\frac{1}{3}})^2 \times (2^{\frac{1}{2}})^5$	e $(2^{\frac{1}{3}})^2 \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}}$	

Example 3 8 Simplify each of the following:

a $\sqrt[3]{a^3 b^2} \div \sqrt[3]{a^2 b^{-1}}$	b $\sqrt{a^3 b^2} \times \sqrt{a^2 b^{-1}}$	c $\sqrt[5]{a^3 b^2} \times \sqrt[5]{a^2 b^{-1}}$
d $\sqrt{a^{-4} b^2} \times \sqrt{a^3 b^{-1}}$	e $\sqrt[3]{a^3 b^2 c^{-3}} \times \sqrt{a^2 b^{-1} c^{-5}}$	f $\sqrt[5]{a^3 b^2} \div \sqrt[5]{a^2 b^{-1}}$
g $\frac{\sqrt{a^3 b^2}}{a^2 b^{-1} c^{-5}} \times \frac{\sqrt{a^{-4} b^2}}{a^3 b^{-1}} \times \sqrt{a^3 b^{-1}}$		

1B Standard form

Often when dealing with real-world problems, the numbers involved may be very small or very large. For example:

- The distance from Earth to the Sun is approximately 150 000 000 kilometres.
- The mass of an oxygen atom is approximately 0.000 000 000 000 000 000 026 grams.

To help deal with such numbers, we can use a more convenient way to express them. This involves expressing the number as a product of a number between 1 and 10 and a power of 10 and is called **standard form** or **scientific notation**.

These examples written in standard form are:

- 1.5×10^8 kilometres
- 2.6×10^{-23} grams

Multiplication and division with very small or very large numbers can often be simplified by first converting the numbers into standard form. When simplifying algebraic expressions or manipulating numbers in standard form, a sound knowledge of the index laws is essential.

Example 4

Write each of the following in standard form:

a 3 453 000

b 0.00675

Solution

a $3\,453\,000 = 3.453 \times 10^6$

b $0.00675 = 6.75 \times 10^{-3}$

Example 5

Find the value of $\frac{32\,000\,000 \times 0.000\,004}{16\,000}$.

Solution

$$\begin{aligned} \frac{32\,000\,000 \times 0.000\,004}{16\,000} &= \frac{3.2 \times 10^7 \times 4 \times 10^{-6}}{1.6 \times 10^4} \\ &= \frac{12.8 \times 10^1}{1.6 \times 10^4} \\ &= 8 \times 10^{-3} \\ &= 0.008 \end{aligned}$$

► Significant figures

When measurements are made, the result is recorded to a certain number of significant figures. For example, if we say that the length of a piece of ribbon is 156 cm to the nearest centimetre, this means that the length is between 155.5 cm and 156.5 cm. The number 156 is said to be correct to three significant figures. Similarly, we may record π as being 3.1416, correct to five significant figures.

When rounding off to a given number of significant figures, first identify the last significant digit and then:

- if the next digit is 0, 1, 2, 3 or 4, round down
- if the next digit is 5, 6, 7, 8 or 9, round up.

It can help with rounding off if the original number is first written in scientific notation.

So $\pi = 3.141\,592\,653\dots$ is rounded off to

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, etc.

depending on the number of significant figures required.

Writing a number in scientific notation makes it clear how many significant figures have been recorded. For example, it is unclear whether 600 is recorded to one, two or three significant figures. However, when written in scientific notation as

$$6.00 \times 10^2, \quad 6.0 \times 10^2 \quad \text{or} \quad 6 \times 10^2$$

it is clear how many significant figures are recorded.

Example 6

Evaluate $\frac{\sqrt[5]{a}}{b^2}$ if $a = 1.34 \times 10^{-10}$ and $b = 2.7 \times 10^{-8}$.

Solution

$$\begin{aligned} \frac{\sqrt[5]{a}}{b^2} &= \frac{\sqrt[5]{1.34 \times 10^{-10}}}{(2.7 \times 10^{-8})^2} \\ &= \frac{(1.34 \times 10^{-10})^{\frac{1}{5}}}{2.7^2 \times (10^{-8})^2} \\ &= 1.45443\dots \times 10^{13} \\ &= 1.45 \times 10^{13} \quad \text{to three significant figures} \end{aligned}$$

Note: Many calculators can display numbers in scientific notation. The format will vary from calculator to calculator. For example, the number $3\,245\,000 = 3.245 \times 10^6$ may appear as 3.245E6 or 3.245^{06} .

Section summary

- A number is said to be in **scientific notation** (or **standard form**) when it is written as a product of a number between 1 and 10 and an integer power of 10.
For example: $6547 = 6.547 \times 10^3$ and $0.789 = 7.89 \times 10^{-1}$
- Writing a number in scientific notation makes it clear how many **significant figures** have been recorded.
- When rounding off to a given number of significant figures, first identify the last significant digit and then:
 - if the next digit is 0, 1, 2, 3 or 4, round down
 - if the next digit is 5, 6, 7, 8 or 9, round up.

Exercise 1B

Example 4

- 1 Express each of the following numbers in standard form:

a 47.8	b 6728	c 79.23	d 43 580
e 0.0023	f 0.000 000 56	g 12.000 34	h 50 million
i 23 000 000 000	j 0.000 000 0013	k 165 thousand	l 0.000 014 567

- 2 Express each of the following in scientific notation:

- a The mass of a water molecule is 0.000 000 000 000 000 000 0299 g.
- b There are 31 536 000 seconds in one year.
- c There are 3 057 000 000 000 000 000 000 atoms in one gram of gold.

- 3 Express each of the following as an ordinary number:

- a The diameter of the Sun is 1.39×10^9 m.
- b The diameter of a red blood cell is 7.5×10^{-6} m.
- c The diameter of an electron is 5.6×10^{-15} m.

- 4 Write each of the following in scientific notation, correct to the number of significant figures indicated in the brackets:

a 456.89 (4)	b 34567.23 (2)	c 5679.087 (5)
d 0.04536 (2)	e 0.09045 (2)	f 4568.234 (5)

Example 5

- 5 Without using a calculator, find the value of:

a $\frac{5\,000\,000 \times 0.08}{400\,000}$	b $\frac{320\,000 \times 0.0006}{4000}$	c $\frac{5\,400\,000 \times 0.8}{0.000\,02}$
d $\frac{1200 \times 0.000\,03}{14.4}$	e $\frac{\sqrt{3.6 \times 10^7}}{3 \times 10^{-2}}$	f $\frac{2.2 \times 10^9}{\sqrt{1.21 \times 10^4}}$

Example 6

- 6 Evaluate the following correct to three significant figures:

- a $\frac{\sqrt[3]{a}}{b^4}$ if $a = 2 \times 10^9$ and $b = 3.215$
- b $\frac{\sqrt[4]{a}}{4b^4}$ if $a = 2 \times 10^{12}$ and $b = 0.05$

SF

1C Solving linear equations and simultaneous linear equations

Many problems may be solved by first translating them into mathematical equations and then solving the equations using algebraic techniques. An equation is solved by finding the value or values of the variables that would make the statement true.

Linear equations are simple equations that can be written in the form $ax + b = 0$. There are a number of standard techniques that can be used for solving linear equations.

Example 7

a Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

b Solve $\frac{x-3}{2} - \frac{2x-4}{3} = 5$.

Solution

- a** Multiply both sides of the equation by the lowest common multiple of 3 and 5:

$$\frac{x}{5} - 2 = \frac{x}{3}$$

$$\frac{x}{5} \times 15 - 2 \times 15 = \frac{x}{3} \times 15$$

$$3x - 30 = 5x$$

$$3x - 5x = 30$$

$$-2x = 30$$

$$x = \frac{30}{-2}$$

$$\therefore x = -15$$

- b** Multiply both sides of the equation by the lowest common multiple of 2 and 3:

$$\frac{x-3}{2} - \frac{2x-4}{3} = 5$$

$$\frac{x-3}{2} \times 6 - \frac{2x-4}{3} \times 6 = 5 \times 6$$

$$3(x-3) - 2(2x-4) = 30$$

$$3x - 9 - 4x + 8 = 30$$

$$3x - 4x = 30 + 9 - 8$$

$$-x = 31$$

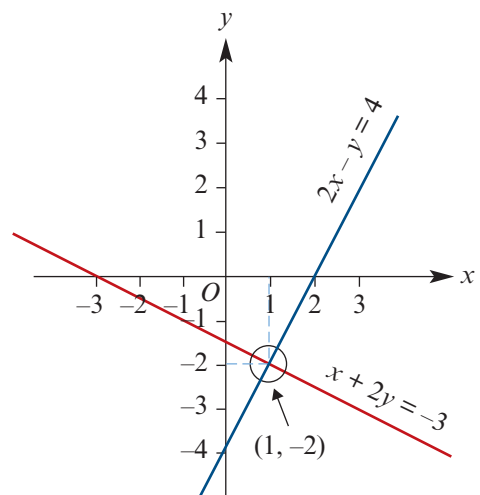
$$x = \frac{31}{-1}$$

$$\therefore x = -31$$

► Simultaneous linear equations

The intersection point of two straight lines can be found graphically; however, the accuracy of the solution will depend on the accuracy of the graphs.

Alternatively, the intersection point may be found algebraically by solving the pair of simultaneous equations. We shall consider two techniques for solving simultaneous equations.



Example 8

Solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution**Method 1: Substitution**

$$2x - y = 4 \quad (1)$$

$$x + 2y = -3 \quad (2)$$

From equation (2), we get $x = -3 - 2y$.

Substitute in equation (1):

$$2(-3 - 2y) - y = 4$$

$$-6 - 4y - y = 4$$

$$-5y = 10$$

$$y = -2$$

Substitute the value of y into (2):

$$x + 2(-2) = -3$$

$$x = 1$$

Check in (1): LHS = $2(1) - (-2) = 4$

$$\text{RHS} = 4$$

Method 2: Elimination

$$2x - y = 4 \quad (1)$$

$$x + 2y = -3 \quad (2)$$

To eliminate x , multiply equation (2) by 2 and subtract the result from equation (1).

When we multiply equation (2) by 2, the pair of equations becomes:

$$2x - y = 4 \quad (1)$$

$$2x + 4y = -6 \quad (2')$$

Subtract (2') from (1):

$$-5y = 10$$

$$y = -2$$

Now substitute for y in equation (2) to find x , and check as in the substitution method.

Explanation

Using one of the two equations, express one variable in terms of the other variable.

Then substitute this expression into the other equation (reducing it to an equation in one variable, y). Solve the equation for y .

Substitute this value for y in one of the equations to find the other variable, x .

A check can be carried out with the other equation.

If one of the variables has the same coefficient in the two equations, we can eliminate that variable by subtracting one equation from the other.

It may be necessary to multiply one of the equations by a constant to make the coefficients of x or y the same in the two equations.

Note: This example shows that the point $(1, -2)$ is the point of intersection of the graphs of the two linear relations.

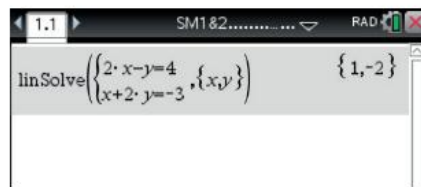


Using the TI-Nspire CX non-CAS

Method 1: Using a Calculator application

Simultaneous linear equations can be solved in a **Calculator** application.

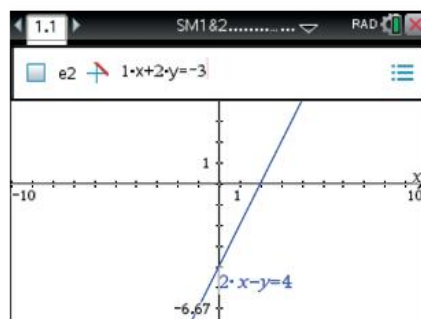
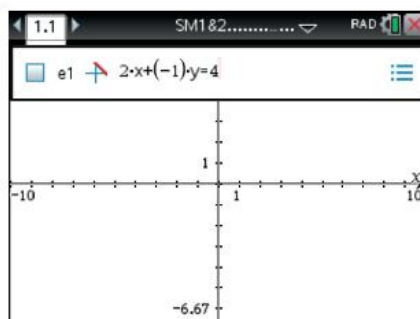
- Use **menu** > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen.
- Enter the equations as shown to give the solution to the simultaneous equations $2x - y = 4$ and $x + 2y = -3$.
- Hence the solution is $x = 1$ and $y = -2$.



Method 2: Using a Graphs application

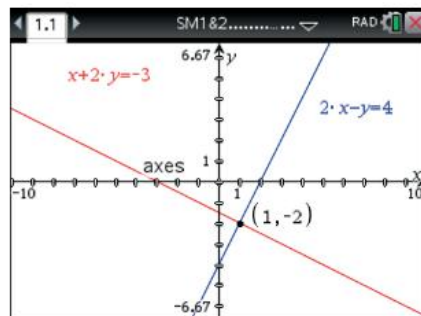
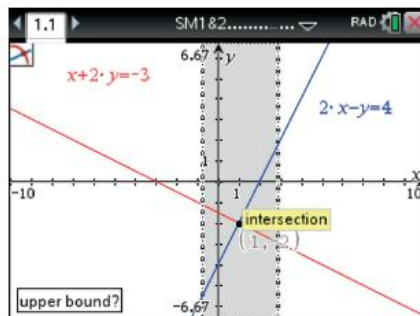
The simultaneous equations can also be solved graphically in a **Graphs** application.

- Equations of the form $a \cdot x + b \cdot y = c$ can be entered directly using **menu** > **Graph Entry/Edit** > **Equation Templates** > **Line** > $a * x + b * y = c$.
- Alternatively, rearrange each equation to make y the subject, and enter as a standard function (e.g. $f1(x) = 2x - 4$).



Note: If the entry line is not visible, press **tab**. Pressing **enter** will hide the entry line. If you want to add more equations, use **▼** to add the next equation.

- The intersection point can be found using **menu** > **Analyze Graph** > **Intersection**.
- Move the cursor to the left of the intersection point (lower bound), click, move to the right of the intersection point (upper bound) and click again. The intersection point's coordinates will appear on the screen.

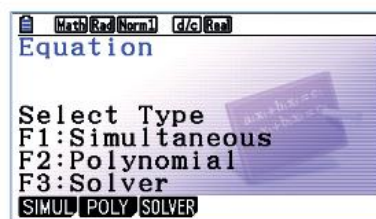


Using the Casio

Simultaneous linear equations can be solved in **Equation** mode or in **Graph** mode.

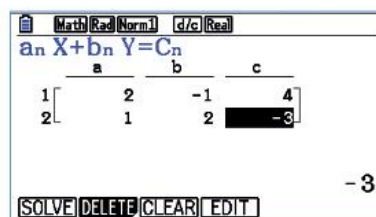
Method 1: Using Equation mode

- Press **MENU** and then select **Equation** mode by pressing **(ALPHA) (X,θ,T)**.
- Select **Simultaneous** **(F1)**.

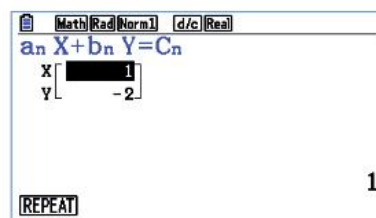


- Enter the coefficients of the two equations $2x - y = 4$ and $x + 2y = -3$ as shown:

(2) (EXE) (-) (1) (EXE) (4) (EXE)
(1) (EXE) (2) (EXE) (-) (3) (EXE)



- Select **Solve** **(F1)**.
- Hence the solution is $x = 1$ and $y = -2$.



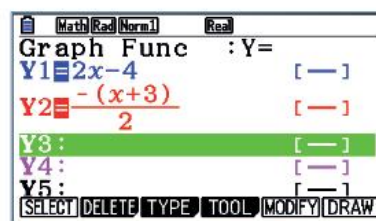
Method 2: Using Graph mode

- Press **MENU** **(5)** to select **Graph** mode.
- Transpose $2x - y = 4$ to make y the subject, and enter the equation in $Y1$:

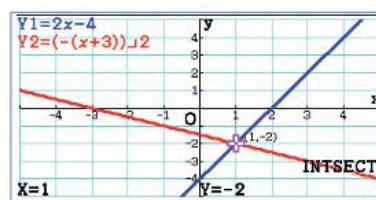
(2) (X,θ,T) (-) (4) (EXE)

- Transpose $x + 2y = -3$ to make y the subject, and enter the equation in $Y2$:

(a b/c) (-) (((X,θ,T) + (3))) (▼) (2) (EXE)



- Select **Draw** **(F6)**.
- Go to the **G-Solve** menu **(SHIFT) (F5)** and select **Intersection** **(F5)**.



Section summary

- An equation is solved by finding the value or values of the variables that would make the statement true.
- A linear equation is one in which the ‘unknown’ is to the first power.
- There are often several different ways to solve a linear equation. The following steps provide some suggestions:
 - 1 Expand brackets and, if the equation involves fractions, multiply through by the lowest common denominator of the terms.
 - 2 Group all of the terms containing a variable on one side of the equation and the terms without the variable on the other side.
- Methods for solving simultaneous linear equations in two variables by hand:

Substitution

- Make one of the variables the subject in one of the equations.
- Substitute for that variable in the other equation.

Elimination

- Choose one of the two variables to eliminate.
- Obtain the same or opposite coefficients for this variable in the two equations. To do this, multiply both sides of one or both equations by a number.
- Add or subtract the two equations to eliminate the chosen variable.

Exercise 1C

Example 7a

- 1 Solve the following linear equations:

a $3x + 7 = 15$

b $8 - \frac{x}{2} = -16$

c $42 + 3x = 22$

d $\frac{2x}{3} - 15 = 27$

e $5(2x + 4) = 13$

f $-3(4 - 5x) = 24$

g $3x + 5 = 8 - 7x$

h $2 + 3(x - 4) = 4(2x + 5)$

i $\frac{2x}{5} - \frac{3}{4} = 5x$

j $6x + 4 = \frac{x}{3} - 3$

Example 7b

- 2 Solve the following linear equations:

a $\frac{x}{2} + \frac{2x}{5} = 16$

b $\frac{3x}{4} - \frac{x}{3} = 8$

c $\frac{3x - 2}{2} + \frac{x}{4} = -8$

d $\frac{5x}{4} - \frac{4}{3} = \frac{2x}{5}$

e $\frac{x - 4}{2} + \frac{2x + 5}{4} = 6$

f $\frac{3 - 3x}{10} - \frac{2(x + 5)}{6} = \frac{1}{20}$

g $\frac{3 - x}{4} - \frac{2(x + 1)}{5} = -24$

h $\frac{-2(5 - x)}{8} + \frac{6}{7} = \frac{4(x - 2)}{3}$

Example 8 3 Solve each of the following pairs of simultaneous equations:

a $3x + 2y = 2$

$2x - 3y = 6$

b $5x + 2y = 4$

$3x - y = 6$

c $2x - y = 7$

$3x - 2y = 2$

d $x + 2y = 12$

$x - 3y = 2$

e $7x - 3y = -6$

$x + 5y = 10$

f $15x + 2y = 27$

$3x + 7y = 45$

1D Solving problems with linear equations

Many problems can be solved by translating them into mathematical language and using an appropriate mathematical technique to find the solution. By representing the unknown quantity in a problem with a symbol (called a pronumeral or a variable) and constructing an equation from the information, the value of the unknown can be found by solving the equation.

Before constructing the equation, each variable and what it stands for (including the units) should be stated. All the elements of the equation must be in units of the same system.



Example 9

For each of the following, form the relevant linear equation and solve it for x :

a The length of the side of a square is $(x - 6)$ cm. Its perimeter is 52 cm.

b The perimeter of a square is $(2x + 8)$ cm. Its area is 100 cm^2 .

Solution

a Perimeter = $4 \times$ side length

Therefore

$$4(x - 6) = 52$$

$$x - 6 = 13$$

and so $x = 19$

b The perimeter of the square is $2x + 8$.

The length of one side is $\frac{2x + 8}{4} = \frac{x + 4}{2}$.

Thus the area is

$$\left(\frac{x + 4}{2}\right)^2 = 100$$

As the side length must be positive, this gives the linear equation

$$\frac{x + 4}{2} = 10$$

Therefore $x = 16$.

Example 10

An athlete trains for an event by gradually increasing the distance she runs each week over a five-week period. If she runs an extra 5 km each successive week and over the five weeks runs a total of 175 km, how far did she run in the first week?

Solution

Let the distance run in the first week be x km.

Then the distance run in the second week is $x + 5$ km, and the distance run in the third week is $x + 10$ km, and so on.

The total distance run is $x + (x + 5) + (x + 10) + (x + 15) + (x + 20)$ km.

$$\therefore 5x + 50 = 175$$

$$5x = 125$$

$$x = 25$$

The distance she ran in the first week was 25 km.

Example 11

A man bought 14 CDs at a sale. Some cost \$15 each and the remainder cost \$12.50 each. In total he spent \$190. How many \$15 CDs and how many \$12.50 CDs did he buy?

Solution

Let n be the number of CDs costing \$15.

Then $14 - n$ is the number of CDs costing \$12.50.

$$\therefore 15n + 12.5(14 - n) = 190$$

$$15n + 175 - 12.5n = 190$$

$$2.5n + 175 = 190$$

$$2.5n = 15$$

$$n = 6$$

He bought 6 CDs costing \$15 and 8 CDs costing \$12.50.

Section summary**Steps for solving a word problem with a linear equation**

- Read the question carefully and write down the known information clearly.
- Identify the unknown quantity that is to be found.
- Assign a variable to this quantity.
- Form an expression in terms of x (or the variable being used) and use the other relevant information to form the equation.
- Solve the equation.
- Write a sentence answering the initial question.

Exercise 1D

Skillsheet

- 1 For each of the cases below, write down a relevant equation involving the variables defined, and solve the equation for parts **a**, **b** and **c**.

Example 9

- a** The length of the side of a square is $(x - 2)$ cm. Its perimeter is 60 cm.
- b** The perimeter of a square is $(2x + 7)$ cm. Its area is 49 cm^2 .
- c** The length of a rectangle is $(x - 5)$ cm. Its width is $(12 - x)$ cm. The rectangle is twice as long as it is wide.
- d** The length of a rectangle is $(2x + 1)$ cm. Its width is $(x - 3)$ cm. The perimeter of the rectangle is y cm.
- e** n people each have a meal costing $\$p$. The total cost of the meal is $\$Q$.
- f** S people each have a meal costing $\$p$. A 10% service charge is added to the cost. The total cost of the meal is $\$R$.
- g** A machine working at a constant rate produces n bolts in 5 minutes. It produces 2400 bolts in 1 hour.
- h** The radius of a circle is $(x + 3)$ cm. A sector subtending an angle of 60° at the centre is cut off. The arc length of the minor sector is a cm.

Example 10

- 2 Bronwyn and Noel have a women's clothing shop in Summerland. Bronwyn manages the shop and her sales are going up steadily over a particular period of time. They are going up by \$500 per week. If over a five-week period her sales total \$17 500, how much did she earn in the first week?

Example 11

- 3 Bronwyn and Noel have a women's clothing shop in Summerland. Sally, Adam and baby Lana came into the shop and Sally bought dresses and handbags. The dresses cost \$65 each and the handbags cost \$26 each. Sally bought 11 items and in total she spent \$598. How many dresses and how many handbags did she buy?
- 4 A rectangular courtyard is three times as long as it is wide. If the perimeter of the courtyard is 67 m, find the dimensions of the courtyard.
- 5 A wine merchant buys 50 cases of wine. He pays full price for half of them, but gets a 40% discount on the remainder. If he paid a total of \$2260, how much was the full price of a single case?
- 6 A real-estate agent sells 22 houses in six months. He makes a commission of \$11 500 per house on some and \$13 000 per house on the remainder. If his total commission over the six months was \$272 500, on how many houses did he make a commission of \$11 500?
- 7 Three boys compare their marble collections. The first boy has 14 fewer than the second boy, who has twice as many as the third. If between them they have 71 marbles, how many does each boy have?

- 8** Three girls are playing Scrabble. At the end of the game, their three scores add up to 504. Annie scored 10% more than Belinda, while Cassie scored 60% of the combined scores of the other two. What did each player score?
- 9** A biathlon event involves running and cycling. Kim can cycle 30 km/h faster than she can run. If Kim spends 48 minutes running and a third as much time again cycling in an event that covers a total distance of 60 km, how fast can she run?
- 10** The mass of a molecule of a certain chemical compound is 2.45×10^{-22} g. If each molecule is made up of two carbon atoms and six oxygen atoms and the mass of an oxygen atom is one-third that of a carbon atom, find the mass of an oxygen atom.
- 11** Mother's pearl necklace fell to the floor. One-sixth of the pearls rolled under the fridge, one-third rolled under the couch, one-fifth of them behind the bookshelf, and nine were found at her feet. How many pearls are there?
- 12** Parents say they don't have favourites, but everyone knows that's a lie. A father distributes \$96 to his three children according to the following instructions: The middle child receives \$12 less than the oldest, and the youngest receives one-third as much as the middle child. How much does each receive?
- 13** Kavindi has achieved an average mark of 88% on her first four maths tests. What mark would she need on her fifth test to increase her average to 90%?
- 14** In a particular class, 72% of the students have black hair. Five black-haired students leave the class, so that now 65% of the students have black hair. How many students were originally in the class?
- 15** Two tanks are being emptied. Tank A contains 100 litres of water and tank B contains 120 litres of water. Water runs from Tank A at 2 litres per minute, and water runs from tank B at 3 litres per minute. After how many minutes will the amount of water in the two tanks be the same?
- 16** Suppose that candle A is initially 10 cm tall and burns out after 2 hours. Candle B is initially 8 cm tall and burns out after 4 hours. Both candles are lit at the same time. Assuming 'constant burning rates':
- a** When is the height of candle A the same as the height of candle B?
 - b** When is the height of candle A half the height of candle B?
 - c** When is candle A 1 cm taller than candle B?
- 17** A motorist drove 320 km in $\frac{10}{3}$ hours. He drove part of the way at an average speed of 100 km/h and the rest of the way at an average speed of 90 km/h. What is the distance he travelled at 100 km/h?

- 18** Jarmila travels regularly between two cities. She takes $\frac{14}{3}$ hours if she travels at her usual speed. If she increase her speed by 3 km/h, she can reduce her time taken by 20 minutes. What is her usual speed?

1E Solving problems with simultaneous linear equations

When the relationship between two quantities is linear, we can find the constants which determine this linear relationship if we are given two sets of information satisfying the relationship. Simultaneous linear equations enable this to be done.

Another situation in which simultaneous linear equations may be used is where it is required to find the point of the Cartesian plane which satisfies two linear relations.



Example 12

There are two possible methods for paying gas bills:

Method A A fixed charge of \$25 per quarter + 50c per unit of gas used

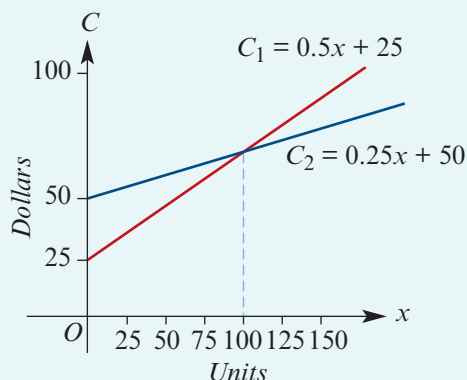
Method B A fixed charge of \$50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

Let C_1 = charge (\$) using method A
 C_2 = charge (\$) using method B
 x = number of units of gas used

Then $C_1 = 25 + 0.5x$
 $C_2 = 50 + 0.25x$



From the graph we see that method B is cheaper if the number of units exceeds 100.

The solution can be obtained by solving simultaneous linear equations:

$$\begin{aligned} C_1 &= C_2 \\ 25 + 0.5x &= 50 + 0.25x \\ 0.25x &= 25 \\ x &= 100 \end{aligned}$$

Example 13

If 3 kg of jam and 2 kg of butter cost \$29, and 6 kg of jam and 3 kg of butter cost \$54, find the cost per kilogram of jam and butter.

Solution

Let the cost of 1 kg of jam be x dollars and the cost of 1 kg of butter be y dollars.

Then $3x + 2y = 29$ (1)

and $6x + 3y = 54$ (2)

Multiply (1) by 2: $6x + 4y = 58$ (1')

Subtract (1') from (2): $-y = -4$

$$y = 4$$

Substitute in (2): $6x + 3(4) = 54$

$$6x = 42$$

$$x = 7$$

Jam costs \$7 per kilogram and butter costs \$4 per kilogram.

Section summary**Steps for solving a word problem with simultaneous linear equations**

- Read the question carefully and write down the known information clearly.
- Identify the two unknown quantities that are to be found.
- Assign variables to these two quantities.
- Form expressions in terms of x and y (or other suitable variables) and use the other relevant information to form the two equations.
- Solve the system of equations.
- Write a sentence answering the initial question.

Exercise 1E**Example 12**

- 1 A car hire firm offers the option of paying \$108 per day with unlimited kilometres, or \$63 per day plus 32 cents per kilometre travelled. How many kilometres would you have to travel in a given day to make the unlimited-kilometres option more attractive?
- 2 Company A will cater for your party at a cost of \$450 plus \$40 per guest. Company B offers the same service for \$300 plus \$43 per guest. How many guests are needed before Company A's charge is less than Company B's?

Example 13

- 3 A basketball final is held in a stadium which can seat 15 000 people. All the tickets have been sold, some to adults at \$45 and the rest for children at \$15. If the revenue from the tickets was \$525 000, find the number of adults who bought tickets.

- 4 A contractor employed eight men and three boys for one day and paid them a total of \$2240. Another day he employed six men and eighteen boys for \$4200. What was the daily rate he paid each man and each boy?
- 5 The sum of two numbers is 212 and their difference is 42. Find the two numbers.
- 6 Each adult paid \$30 to attend a concert and each student paid \$20. A total of 1600 people attended. The total paid was \$37 000. How many adults and how many students attended the concert?
- 7 A chemical manufacturer wishes to obtain 700 litres of a 24% acid solution by mixing a 40% solution with a 15% solution. How many litres of each solution should be used?
- 8 Two children had 220 marbles between them. After one child had lost half her marbles and the other had lost 40 marbles, they had an equal number of marbles. How many did each child start with and how many did each child finish with?
- 9 An investor received \$31 000 interest per annum from a sum of money, with part of it invested at 10% and the remainder at 7% simple interest. She found that if she interchanged the amounts she had invested she could increase her return by \$1000 per annum. Calculate the total amount she had invested.

1F Substitution and transposition of formulas

An equation that states a relationship between two or more quantities is called a **formula**; e.g. the area of a circle is given by $A = \pi r^2$. The value of A , the subject of the formula, may be found by substituting a given value of r and the value of π .

Example 14

Using the formula $A = \pi r^2$, find the value of A correct to two decimal places if $r = 2.3$ and $\pi = 3.142$ (correct to three decimal places).

Solution

$$\begin{aligned} A &= \pi r^2 \\ &= 3.142(2.3)^2 \\ &= 16.62118 \end{aligned}$$

$\therefore A = 16.62$ correct to two decimal places

The formula $A = \pi r^2$ can also be transposed to make r the subject.

When transposing a formula, follow a similar procedure to solving a linear equation.

Whatever has been done to the variable required is 'undone'.

Example 15

- a** Transpose the formula $A = \pi r^2$ to make r the subject.
b Hence find the value of r correct to three decimal places if $A = 24.58$ and $\pi = 3.142$ (correct to three decimal places).

Solution

a $A = \pi r^2$

$$\frac{A}{\pi} = r^2$$

$$r = \sqrt{\frac{A}{\pi}}$$

b $r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{24.58}{3.142}}$
 $= 2.79697 \dots$

$r = 2.797$ correct to three decimal places

Section summary

- A formula relates different quantities: for example, the formula $A = \pi r^2$ relates the radius r with the area A of the circle.
- The variable on the left is called the subject of the formula: for example, in the formula $A = \pi r^2$, the subject is A .
- To calculate the value of a variable which is not the subject of a formula:
Method 1 Substitute the values for the known variables, then solve the resulting equation for the unknown variable.
Method 2 Rearrange to make the required variable the subject, then substitute values.

Exercise 1F**Example 14**

- 1** Substitute the specified values to evaluate each of the following, giving the answers correct to two decimal places:

a v if $v = u + at$ and $u = 15, a = 2, t = 5$

b I if $I = \frac{PrT}{100}$ and $P = 600, r = 5.5, T = 10$

c V if $V = \pi r^2 h$ and $r = 4.25, h = 6$

d S if $S = 2\pi r(r + h)$ and $r = 10.2, h = 15.6$

e V if $V = \frac{4}{3}\pi r^2 h$ and $r = 3.58, h = 11.4$

f s if $s = ut + \frac{1}{2}at^2$ and $u = 25.6, t = 3.3, a = -1.2$

g T if $T = 2\pi\sqrt{\frac{\ell}{g}}$ and $\ell = 1.45, g = 9.8$

h f if $\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$ and $v = 3, u = 7$

i c if $c^2 = a^2 + b^2$ and $a = 8.8, b = 3.4$

j v if $v^2 = u^2 + 2as$ and $u = 4.8, a = 2.5, s = 13.6$

Example 15

- 2 Transpose each of the following to make the symbol in brackets the subject:

a $v = u + at$ (a)

b $S = \frac{n}{2}(a + \ell)$ (ℓ)

c $A = \frac{1}{2}bh$ (b)

d $P = I^2R$ (I)

e $s = ut + \frac{1}{2}at^2$ (a)

f $E = \frac{1}{2}mv^2$ (v)

g $Q = \sqrt{2gh}$ (h)

h $-xy - z = xy + z$ (x)

i $\frac{ax + by}{c} = x - b$ (x)

j $\frac{mx + b}{x - b} = c$ (x)

- 3 The formula $F = \frac{9C}{5} + 32$ is used to convert temperatures given in degrees Celsius (C) to degrees Fahrenheit (F).

- a** Convert 28 degrees Celsius to degrees Fahrenheit.
b Transpose the formula to make C the subject and find C if $F = 135^\circ$.

- 4 The sum, S, of the interior angles of a polygon with n sides is given by the formula $S = 180(n - 2)$.

- a** Find the sum of the interior angles of an octagon.
b Transpose the formula to make n the subject and hence determine the number of sides of a polygon whose interior angles add up to 1260° .

- 5 The volume, V, of a right cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone.

- a** Find the volume of a cone with radius 3.5 cm and height 9 cm.
b Transpose the formula to make h the subject and hence find the height of a cone with base radius 4 cm and volume 210 cm^3 .
c Transpose the formula to make r the subject and hence find the radius of a cone with height 10 cm and volume 262 cm^3 .

- 6 For a particular type of sequence of numbers, the sum (S) of the terms in the sequence is given by the formula

$$S = \frac{n}{2}(a + \ell)$$

where n is the number of terms in the sequence, a is the first term and ℓ is the last term.

- a** Find the sum of such a sequence of seven numbers whose first term is -3 and whose last term is 22.
b What is the first term of such a sequence of 13 numbers whose last term is 156 and whose sum is 1040?
c How many terms are there in the sequence $25 + 22 + 19 + \cdots + (-5) = 110$?

1G Algebraic fractions

The principles involved in addition, subtraction, multiplication and division of algebraic fractions are the same as for simple numerical fractions.

► Addition and subtraction

To add or subtract, all fractions must be written with a common denominator.



Example 16

Simplify:

a $\frac{x}{3} + \frac{x}{4}$

b $\frac{2}{x} + \frac{3a}{4}$

c $\frac{5}{x+2} - \frac{4}{x-1}$

d $\frac{4}{x+2} - \frac{7}{(x+2)^2}$

Solution

$$\begin{aligned} \text{a } \frac{x}{3} + \frac{x}{4} &= \frac{4x + 3x}{12} \\ &= \frac{7x}{12} \end{aligned}$$

$$\text{b } \frac{2}{x} + \frac{3a}{4} = \frac{8 + 3ax}{4x}$$

$$\begin{aligned} \text{c } \frac{5}{x+2} - \frac{4}{x-1} &= \frac{5(x-1) - 4(x+2)}{(x+2)(x-1)} \\ &= \frac{5x - 5 - 4x - 8}{(x+2)(x-1)} \\ &= \frac{x - 13}{(x+2)(x-1)} \end{aligned}$$

$$\begin{aligned} \text{d } \frac{4}{x+2} - \frac{7}{(x+2)^2} &= \frac{4(x+2) - 7}{(x+2)^2} \\ &= \frac{4x + 1}{(x+2)^2} \end{aligned}$$

► Multiplication and division

Before multiplying and dividing algebraic fractions, it is best to factorise numerators and denominators where possible so that common factors can be readily identified.

Example 17

Simplify:

a $\frac{3x^2}{10y^2} \times \frac{5y}{12x}$

b $\frac{2x-4}{x-1} \times \frac{x^2-1}{x-2}$

c $\frac{x^2-1}{2x-2} \times \frac{4x}{x^2+4x+3}$

d $\frac{x^2+3x-10}{x^2-x-2} \div \frac{x^2+6x+5}{3x+3}$

Solution

$$\text{a } \frac{3x^2}{10y^2} \times \frac{5y}{12x} = \frac{x}{8y}$$

$$\begin{aligned} \text{b } \frac{2x-4}{x-1} \times \frac{x^2-1}{x-2} &= \frac{2(x-2)}{x-1} \times \frac{(x-1)(x+1)}{x-2} \\ &= 2(x+1) \end{aligned}$$

$$\begin{aligned} \text{c } \frac{x^2-1}{2x-2} \times \frac{4x}{x^2+4x+3} &= \frac{(x-1)(x+1)}{2(x-1)} \times \frac{4x}{(x+1)(x+3)} \\ &= \frac{2x}{x+3} \end{aligned}$$

$$\begin{aligned} \text{d } \frac{x^2+3x-10}{x^2-x-2} \div \frac{x^2+6x+5}{3x+3} &= \frac{(x+5)(x-2)}{(x-2)(x+1)} \times \frac{3(x+1)}{(x+1)(x+5)} \\ &= \frac{3}{x+1} \end{aligned}$$

► More examples

The following two examples involve algebraic fractions and rational indices.

Example 18

Express $\frac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x}$ as a single fraction.

Solution

$$\begin{aligned} \frac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x} &= \frac{3x^3 + 3x^2\sqrt{4-x}\sqrt{4-x}}{\sqrt{4-x}} \\ &= \frac{3x^3 + 3x^2(4-x)}{\sqrt{4-x}} \\ &= \frac{12x^2}{\sqrt{4-x}} \end{aligned}$$



Example 19

Express $(x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}}$ as a single fraction.

Solution

$$\begin{aligned} (x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}} &= (x-4)^{\frac{1}{5}} - \frac{1}{(x-4)^{\frac{4}{5}}} \\ &= \frac{(x-4)^{\frac{1}{5}}(x-4)^{\frac{4}{5}} - 1}{(x-4)^{\frac{4}{5}}} \\ &= \frac{x-5}{(x-4)^{\frac{4}{5}}} \end{aligned}$$

Section summary

■ Simplifying algebraic fractions

- First factorise the numerator and denominator.
- Then cancel any factors common to the numerator and denominator.

■ Adding and subtracting algebraic fractions

- First obtain a common denominator and then add or subtract.

■ Multiplying and dividing algebraic fractions

- First factorise each numerator and denominator completely.
- Then complete the calculation by cancelling common factors.

Exercise 1G

Skillsheet

1 Simplify each of the following:

Example 16

a $\frac{2x}{3} + \frac{3x}{2}$

b $\frac{3a}{2} - \frac{a}{4}$

c $\frac{3h}{4} + \frac{5h}{8} - \frac{3h}{2}$

d $\frac{3x}{4} - \frac{y}{6} - \frac{x}{3}$

e $\frac{3}{x} + \frac{2}{y}$

f $\frac{5}{x-1} + \frac{2}{x}$

g $\frac{3}{x-2} + \frac{2}{x+1}$

h $\frac{2x}{x+3} - \frac{4x}{x-3} - \frac{3}{2}$

i $\frac{4}{x+1} + \frac{3}{(x+1)^2}$

j $\frac{a-2}{a} + \frac{a}{4} + \frac{3a}{8}$

k $2x - \frac{6x^2-4}{5x}$

l $\frac{2}{x+4} - \frac{3}{x^2+8x+16}$

m $\frac{3}{x-1} + \frac{2}{(x-1)(x+4)}$

n $\frac{3}{x-2} - \frac{2}{x+2} + \frac{4}{x^2-4}$

o $\frac{5}{x-2} + \frac{3}{x^2+5x+6} + \frac{2}{x+3}$

p $x - y - \frac{1}{x-y}$

q $\frac{3}{x-1} - \frac{4x}{1-x}$

r $\frac{3}{x-2} + \frac{2x}{2-x}$

Example 17

2 Simplify each of the following:

a $\frac{x^2}{2y} \times \frac{4y^3}{x}$

b $\frac{3x^2}{4y} \times \frac{y^2}{6x}$

c $\frac{4x^3}{3} \times \frac{12}{8x^4}$

d $\frac{x^2}{2y} \div \frac{3xy}{6}$

e $\frac{4-x}{3a} \times \frac{a^2}{4-x}$

f $\frac{2x+5}{4x^2+10x}$

g $\frac{(x-1)^2}{x^2+3x-4}$

h $\frac{x^2-x-6}{x-3}$

i $\frac{x^2-5x+4}{x^2-4x}$

j $\frac{5a^2}{12b^2} \div \frac{10a}{6b}$

k $\frac{x-2}{x} \div \frac{x^2-4}{2x^2}$

l $\frac{x+2}{x(x-3)} \div \frac{4x+8}{x^2-4x+3}$

m $\frac{2x}{x-1} \div \frac{4x^2}{x^2-1}$

n $\frac{x^2-9}{x+2} \times \frac{3x+6}{x-3} \div \frac{9}{x}$

o $\frac{3x}{9x-6} \div \frac{6x^2}{x-2} \times \frac{2}{x+5}$

3 Express each of the following as a single fraction:

a $\frac{1}{x-3} + \frac{2}{x-3}$

b $\frac{2}{x-4} + \frac{2}{x-3}$

c $\frac{3}{x+4} + \frac{2}{x-3}$

d $\frac{2x}{x-3} + \frac{2}{x+4}$

e $\frac{1}{(x-5)^2} + \frac{2}{x-5}$

f $\frac{3x}{(x-4)^2} + \frac{2}{x-4}$

g $\frac{1}{x-3} - \frac{2}{x-3}$

h $\frac{2}{x-3} - \frac{5}{x+4}$

i $\frac{2x}{x-3} + \frac{3x}{x+3}$

j $\frac{1}{(x-5)^2} - \frac{2}{x-5}$

k $\frac{2x}{(x-6)^3} - \frac{2}{(x-6)^2}$

l $\frac{2x+3}{x-4} - \frac{2x-4}{x-3}$

Example 18

4 Express each of the following as a single fraction:

a $\sqrt{1-x} + \frac{2}{\sqrt{1-x}}$

b $\frac{2}{\sqrt{x-4}} + \frac{2}{3}$

c $\frac{3}{\sqrt{x+4}} + \frac{2}{\sqrt{x+4}}$

d $\frac{3}{\sqrt{x+4}} + \sqrt{x+4}$

e $\frac{3x^3}{\sqrt{x+4}} - 3x^2\sqrt{x+4}$

f $\frac{3x^3}{2\sqrt{x+3}} + 3x^2\sqrt{x+3}$

Example 19

5 Simplify each of the following:

a $(6x-3)^{\frac{1}{3}} - (6x-3)^{-\frac{2}{3}}$

b $(2x+3)^{\frac{1}{3}} - 2x(2x+3)^{-\frac{2}{3}}$

c $(3-x)^{\frac{1}{3}} - 2x(3-x)^{-\frac{2}{3}}$

1H Literal equations

A literal equation in x is an equation whose solution will be expressed in terms of pronumerals rather than numbers.

For the equation $2x + 5 = 7$, the solution is the number 1.

For the literal equation $ax + b = c$, the solution is $x = \frac{c-b}{a}$.

Literal equations are solved in the same way as numerical equations. Essentially, the literal equation is transposed to make x the subject.

Example 20

Solve the following for x :

a $px - q = r$

b $ax + b = cx + d$

c $\frac{a}{x} = \frac{b}{2x} + c$

Solution

a $px - q = r$

$$px = r + q$$

$$\therefore x = \frac{r+q}{p}$$

b $ax + b = cx + d$

$$ax - cx = d - b$$

$$x(a - c) = d - b$$

$$\therefore x = \frac{d-b}{a-c}$$

c Multiply both sides by $2x$:

$$\frac{a}{x} = \frac{b}{2x} + c$$

$$2a = b + 2xc$$

$$2a - b = 2xc$$

$$\therefore x = \frac{2a-b}{2c}$$

► Simultaneous literal equations

Simultaneous literal equations are solved by the same methods that are used for solving simultaneous equations, i.e. substitution and elimination.

Example 21

Solve each of the following pairs of simultaneous equations for x and y :

a $y = ax + c$
 $y = bx + d$

b $ax - y = c$
 $x + by = d$

Solution

a Equate the two expressions for y :

$$ax + c = bx + d$$

$$ax - bx = d - c$$

$$x(a - b) = d - c$$

Thus $x = \frac{d - c}{a - b}$

and $y = a\left(\frac{d - c}{a - b}\right) + c$
 $= \frac{ad - ac + ac - bc}{a - b}$
 $= \frac{ad - bc}{a - b}$

b We will use the method of elimination, and eliminate y .

First number the two equations:

$$ax - y = c \quad (1)$$

$$x + by = d \quad (2)$$

Multiply (1) by b :

$$abx - by = bc \quad (1')$$

Add (1') and (2):

$$abx + x = bc + d$$

$$x(ab + 1) = bc + d$$

$$\therefore x = \frac{bc + d}{ab + 1}$$

Substitute in (1):

$$y = ax - c$$

$$= a\left(\frac{bc + d}{ab + 1}\right) - c$$

$$= \frac{ad - c}{ab + 1}$$

Section summary

- An equation for the variable x in which all the coefficients of x , including the constants, are pronumerals is known as a **literal equation**.
- The methods for solving linear literal equations or simultaneous linear literal equations are exactly the same as when the coefficients are given numbers.

Exercise 1H

Example 20

1 Solve each of the following for x :

a $ax + n = m$

c $\frac{ax}{b} + c = 0$

e $mx + n = nx - m$

g $\frac{b}{x-a} = \frac{2b}{x+a}$

i $-b(ax+b) = a(bx-a)$

k $\frac{x}{a} - 1 = \frac{x}{b} + 2$

m $\frac{p-qx}{t} + p = \frac{qx-t}{p}$

b $ax + b = bx$

d $px = qx + 5$

f $\frac{1}{x+a} = \frac{b}{x}$

h $\frac{x}{m} + n = \frac{x}{n} + m$

j $p^2(1-x) - 2pqx = q^2(1+x)$

l $\frac{x}{a-b} + \frac{2x}{a+b} = \frac{1}{a^2-b^2}$

n $\frac{1}{x+a} + \frac{1}{x+2a} = \frac{2}{x+3a}$

2 For the simultaneous equations $ax + by = p$ and $bx - ay = q$, show that $x = \frac{ap + bq}{a^2 + b^2}$ and $y = \frac{bp - aq}{a^2 + b^2}$.3 For the simultaneous equations $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, show that $x = y = \frac{ab}{a+b}$.

Example 21

4 Solve each of the following pairs of simultaneous equations for x and y :

a $ax + y = c$

$x + by = d$

c $ax + by = t$

$ax - by = s$

e $(a+b)x + cy = bc$

$(b+c)y + ax = -ab$

b $ax - by = a^2$

$bx - ay = b^2$

d $ax + by = a^2 + 2ab - b^2$

$bx + ay = a^2 + b^2$

f $3(x-a) - 2(y+a) = 5 - 4a$

$2(x+a) + 3(y-a) = 4a - 1$

5 Write s in terms of a only in the following pairs of equations:

a $s = ah$

$h = 2a + 1$

b $s = ah$

$h = a(2+h)$

c $as = a + h$

$h + ah = 1$

d $as = s + h$

$ah = a + h$

e $s = h^2 + ah$

$h = 3a^2$

f $as = a + 2h$

$h = a - s$

g $s = 2 + ah + h^2$

$h = a - \frac{1}{a}$

h $3s - ah = a^2$

$as + 2h = 3a$



Chapter summary



■ Indices

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{-n} = \frac{1}{a^n}$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$

- A number is expressed in **standard form** or **scientific notation** when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 1.5×10^8

■ Linear equations

First identify the steps done to construct an equation; the equation is then solved by ‘undoing’ these steps. This is achieved by doing ‘the opposite’ in ‘reverse order’.

e.g.: Solve $3x + 4 = 16$ for x .

Note that x has been multiplied by 3 and then 4 has been added.

Subtract 4 from both sides: $3x = 12$

Divide both sides by 3: $x = 4$

- An equation that states a relationship between two or more quantities is called a **formula**; e.g. the area of a circle is given by $A = \pi r^2$. The value of A , the subject of the formula, may be found by substituting a given value of r and the value of π .

A formula can be transposed to make a different variable the subject by using a similar procedure to solving linear equations, i.e. whatever has been done to the variable required is ‘undone’.

- A **literal equation** is solved using the same techniques as for a numerical equation: transpose the literal equation to make the required variable the subject.

Technology-free questions

- 1 Simplify the following:

a $(x^3)^4$ **b** $(y^{-12})^{\frac{3}{4}}$ **c** $3x^{\frac{3}{2}} \times -5x^4$ **d** $(x^3)^{\frac{4}{3}} \times x^{-5}$

- 2 Express the product $23 \times 10^{-6} \times 14 \times 10^{15}$ in standard form.

- 3 Simplify the following:

a $\frac{3x}{5} + \frac{y}{10} - \frac{2x}{5}$ **b** $\frac{4}{x} - \frac{7}{y}$ **c** $\frac{5}{x+2} + \frac{2}{x-1}$

d $\frac{3}{x+2} + \frac{4}{x+4}$ **e** $\frac{5x}{x+4} + \frac{4x}{x-2} - \frac{5}{2}$ **f** $\frac{3}{x-2} - \frac{6}{(x-2)^2}$

- 4 Simplify the following:

a $\frac{x+5}{2x-6} \div \frac{x^2+5x}{4x-12}$ **b** $\frac{3x}{x+4} \div \frac{12x^2}{x^2-16}$

c $\frac{x^2-4}{x-3} \times \frac{3x-9}{x+2} \div \frac{9}{x+2}$ **d** $\frac{4x+20}{9x-6} \times \frac{6x^2}{x+5} \div \frac{2}{3x-2}$

- 5** Phoebe has bought a 3 terabyte external hard drive (3×10^{12} bytes).
- How many photos could be stored on this hard drive if the average size of a photo is 1.5 megabytes (1.5×10^6 bytes)?
 - How long would it take to copy 3 terabytes of data onto the hard drive if the data transfer rate is 120 Mbps (120×10^6 bits per second, where 1 byte equals 8 bits)?
- 6** Swifts Creek Soccer Team has played 54 matches over the past three seasons. They have drawn one-third of their games and won twice as many games as they have lost. How many games have they lost?
- 7** A music store specialises in three types of CDs: classical, blues and heavy metal. In one week they sold a total of 420 CDs. They sold 10% more classical than blues, while sales of heavy metal CDs constituted 50% more than the combined sales of classical and blues CDs. How many of each type of CD did they sell?
- 8** The volume, V , of a cylinder is given by the formula $V = \pi r^2 h$, where r is the radius of the base and h is the height of the cylinder.
- Find the volume of a cylinder with base radius 5 cm and height 12 cm.
 - Transpose the formula to make h the subject and hence find the height of a cylinder with a base radius of 5 cm and a volume of 585 cm^3 .
 - Transpose the formula to make r the subject and hence find the radius of a cylinder with a height of 6 cm and a volume of 768 cm^3 .
- 9** Solve for x :
- $xy + ax = b$
 - $\frac{a}{x} + \frac{b}{x} = c$
 - $\frac{x}{a} = \frac{x}{b} + 2$
 - $\frac{a - dx}{d} + b = \frac{ax + d}{b}$
- 10** Simplify:
- $\frac{p}{p+q} + \frac{q}{p-q}$
 - $\frac{1}{x} - \frac{2y}{xy - y^2}$
 - $\frac{x^2 + x - 6}{x+1} \times \frac{2x^2 + x - 1}{x+3}$
 - $\frac{2a}{2a+b} \times \frac{2ab + b^2}{ba^2}$
- 11** A is three times as old as B. In three years' time, B will be three times as old as C. In fifteen years' time, A will be three times as old as C. What are their present ages?
- 12** **a** Solve the following simultaneous equations for a and b :
- $$a - 5 = \frac{1}{7}(b + 3) \quad b - 12 = \frac{1}{5}(4a - 2)$$
- b** Solve the following simultaneous equations for x and y :
- $$(p - q)x + (p + q)y = (p + q)^2$$
- $$qx - py = q^2 - pq$$

- 13** A man has to travel 50 km in 4 hours. He does it by walking the first 7 km at x km/h, cycling the next 7 km at $4x$ km/h and motoring the remainder at $(6x + 3)$ km/h. Find x .
- 14** Simplify each of the following:
- a** $2n^2 \times 6nk^2 \div (3n)$ **b** $\frac{8c^2x^3y}{6a^2b^3c^3} \div \frac{\frac{1}{2}xy}{15abc^2}$
- 15** Solve the equation $\frac{x+5}{15} - \frac{x-5}{10} = 1 + \frac{2x}{15}$.

CF

SF

Multiple-choice questions



- 1** For non-zero values of x and y , if $5x + 2y = 0$, then the ratio $\frac{y}{x}$ is equal to
A $-\frac{5}{2}$ **B** $-\frac{2}{5}$ **C** $\frac{2}{5}$ **D** 1 **E** $\frac{5}{4}$
- 2** The solution of the simultaneous equations $3x + 2y = 36$ and $3x - y = 12$ is
A $x = \frac{20}{3}$, $y = 8$ **B** $x = 2$, $y = 0$ **C** $x = 1$, $y = -3$
D $x = \frac{20}{3}$, $y = 6$ **E** $x = \frac{3}{2}$, $y = -\frac{3}{2}$
- 3** The solution of the equation $t - 9 = 3t - 17$ is
A $t = -4$ **B** $t = \frac{11}{2}$ **C** $t = 4$ **D** $t = 2$ **E** $t = -2$
- 4** If $m = \frac{n-p}{n+p}$, then $p =$
A $\frac{n(1-m)}{1+m}$ **B** $\frac{n(m-1)}{1+m}$ **C** $\frac{n(1+m)}{1-m}$ **D** $\frac{n(1+m)}{m-1}$ **E** $\frac{m(n-1)}{m+1}$
- 5** $\frac{3}{x-3} - \frac{2}{x+3} =$
A 1 **B** $\frac{x+15}{x^2-9}$ **C** $\frac{15}{x-9}$ **D** $\frac{x+3}{x^2-9}$ **E** $-\frac{1}{9}$
- 6** $9x^2y^3 \div (15(xy)^3)$ is equal to
A $\frac{9x}{15}$ **B** $\frac{18xy}{5}$ **C** $\frac{3y}{5x}$ **D** $\frac{3x}{5}$ **E** $\frac{3}{5x}$
- 7** Transposing the formula $V = \frac{1}{3}h(\ell + w)$ gives $\ell =$
A $\frac{hw}{3V}$ **B** $\frac{3V}{h} - w$ **C** $\frac{3V-2w}{h}$ **D** $\frac{3Vh}{2} - w$ **E** $\frac{1}{3}h(V+w)$
- 8** $\frac{(3x^2y^3)^2}{2x^2y} =$
A $\frac{9}{2}x^2y^7$ **B** $\frac{9}{2}x^2y^5$ **C** $\frac{9}{2}x^6y^7$ **D** $\frac{9}{2}x^6y^6$ **E** $\frac{9}{2}x^2y^4$

- 9 If X is 50% greater than Y and Y is 20% less than Z , then
A X is 30% greater than Z **B** X is 20% greater than Z **C** X is 20% less than Z
D X is 10% less than Z **E** X is 10% greater than Z
- 10 The average of two numbers is $5x + 4$. One of the numbers is x . The other number is
A $4x + 4$ **B** $9x + 8$ **C** $9x + 4$ **D** $10x + 8$ **E** $3x + 1$

Extended-response questions

- 1 Jack cycles home from work, a distance of $10x$ km. Benny leaves at the same time and drives the $40x$ km to his home. **CU**
- Write an expression in terms of x for the time taken for Jack to reach home if he cycles at an average speed of 8 km/h.
 - Write an expression in terms of x for the time taken for Benny to reach home if he drives at an average speed of 70 km/h.
 - In terms of x , find the difference in times of the two journeys.
 - If Jack and Benny arrive at their homes 30 minutes apart:
 - find x , correct to three decimal places
 - find the distance from work of each home, correct to the nearest kilometre.
- 2 Sam's plastic dinghy has sprung a leak and water is pouring in the hole at a rate of $27\,000\text{ cm}^3$ per minute. He grabs a cup and frantically starts bailing the water out at a rate of 9000 cm^3 per minute. The dinghy is shaped like a circular prism (cylinder) with a base radius of 40 cm and a height of 30 cm.
- How fast is the dinghy filling with water?
 - Write an equation showing the volume of water, $V\text{ cm}^3$, in the dinghy after t minutes.
 - Find an expression for the depth of water, h cm, in the dinghy after t minutes.
 - If Sam is rescued after 9 minutes, is this before or after the dinghy has completely filled with water?
- 3 Henry and Thomas Wong collect basketball cards. Henry has five-sixths the number of cards that Thomas has. The Wright family also collect cards. George Wright has half as many cards again as Thomas, Sally Wright has 18 fewer than Thomas, and Zeb Wright has one-third the number Thomas has.
- Write an expression for each child's number of cards in terms of the number Thomas has.
 - The Wright family owns six more cards than the Wong family. Write an equation representing this information.
 - Solve the equation from part **b** and use the result to find the number of cards each child has collected.

- 4 The gravitational force between two objects, F N, is given by the formula

$$F = \frac{6.67 \times 10^{-11} m_1 m_2}{r^2}$$

where m_1 and m_2 are the masses (in kilograms) of the two objects and r is the distance (in metres) between them.

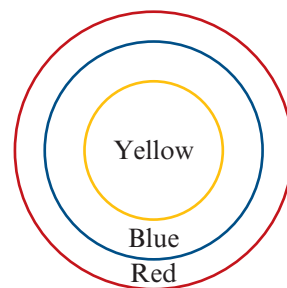
- a What is the gravitational force between two objects each weighing 200 kg if they are 12 m apart? Express the answer in standard form (to two significant figures).
 - b Transpose the above formula to make m_1 the subject.
 - c The gravitational force between a planet and an object 6.4×10^6 m away from the centre of the planet is found to be 2.4×10^4 N. If the object has a mass of 1500 kg, calculate the approximate mass of the planet, giving the answer in standard form (to two significant figures).
- 5 A water storage reservoir is 3 km wide, 6 km long and 30 m deep. (The water storage reservoir is assumed to be a cuboid.)

- a Write an equation to show the volume of water, V m³, in the reservoir when it is d metres full.
 - b Calculate the volume of water, V_F m³, in the reservoir when it is completely filled.
- The water flows from the reservoir down a long pipe to a hydro-electric power station in a valley below. The amount of energy, E J, that can be obtained from a full reservoir is given by the formula

$$E = kV_F h$$

where k is a constant and h m is the length of the pipe.

- c Find k , given that $E = 1.06 \times 10^{15}$ when $h = 200$, expressing the answer in standard form correct to three significant figures.
 - d How much energy could be obtained from a full reservoir if the pipe was 250 m long?
 - e If the rate of water falling through the pipe is 5.2 m³/s, how many days without rain could the station operate before emptying an initially full reservoir?
- 6 A new advertising symbol is to consist of three concentric circles as shown, with the outer circle having a radius of 10 cm. It is desired that the three coloured regions cover the same area. Find the radius of the innermost circle in the figure shown.



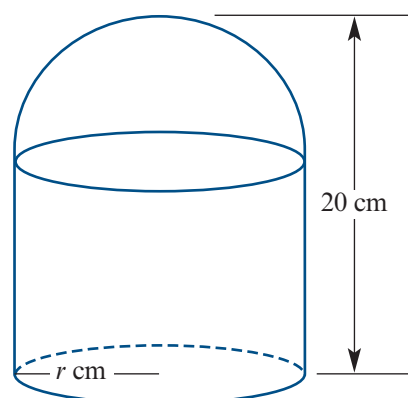
- 7 Temperatures in Fahrenheit (F) can be converted to Celsius (C) by the formula

$$C = \frac{5}{9}(F - 32)$$

Find the temperature which has the same numerical value in both scales.

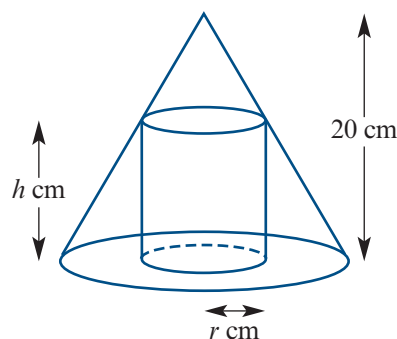
- 8** A cyclist goes up a long slope at a constant speed of 15 km/h. He turns around and comes down the slope at a constant speed of 40 km/h. Find his average speed over a full circuit.

- 9** A container has a cylindrical base and a hemispherical top, as shown in the figure. The height of the container is 20 cm and its capacity is to be exactly 2 litres. Let r cm be the radius of the base.

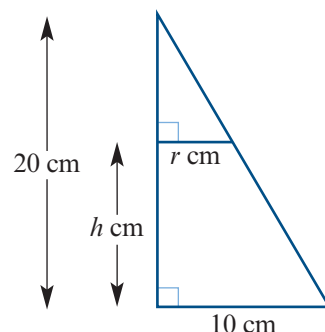


- a** Express the height of the cylinder, h cm, in terms of r .
- b i** Express the volume of the container in terms of r .
- ii** Find r and h if the volume is 2 litres.
- 10 a** Two bottles contain mixtures of wine and water. In bottle A there is two times as much wine as water. In bottle B there is five times as much water as wine. Bottle A and bottle B are used to fill a third bottle, which has a capacity of 1 litre. How much liquid must be taken from each of bottle A and bottle B if the third bottle is to contain equal amounts of wine and water?
- b** Repeat for the situation where the ratio of wine to water in bottle A is 1 : 2 and the ratio of wine to water in bottle B is 3 : 1.
- c** Generalise the result for the ratio $m : n$ in bottle A and $p : q$ in bottle B .

- 11** A cylinder is placed so as to fit into a cone as shown in the diagram. The cone has a height of 20 cm and a base radius of 10 cm. The cylinder has a height of h cm and a base radius of r cm.



- a** Use similar triangles to find h in terms of r .
- b** The volume of the cylinder is given by the formula $V = \pi r^2 h$. Find the volume of the cylinder in terms of r .
- c** Use a calculator to find the values of r and h for which the volume of the cylinder is 500 cm^3 .



2

Number systems and sets

Objectives

- ▶ To understand and use **set notation**, including the symbols $\in$, $\subseteq$, $\cup$, $\cap$, $\emptyset$ and ξ .
- ▶ To be able to identify sets of numbers, including the natural numbers, integers, rational numbers, irrational numbers and real numbers.
- ▶ To know and apply the rules for working with **surds**, including:
 - ▷ simplification of surds
 - ▷ rationalisation of surds.
- ▶ To know and apply the definitions of **factor**, **prime**, **highest common factor** and **lowest common multiple**.
- ▶ To be able to solve problems with sets.

This chapter is provided as preparation for Chapter 8 Number and proof.

This chapter introduces set notation and discusses sets of numbers and their properties. Set notation is used widely in mathematics and in this book it is employed where appropriate. In this chapter we discuss natural numbers, integers and rational numbers, and then continue on to consider irrational numbers.

Irrational numbers such as $\sqrt{2}$ naturally arise when applying Pythagoras' theorem. When solving a quadratic equation, using the method of completing the square or the quadratic formula, we obtain answers such as $x = \frac{1}{2}(1 \pm \sqrt{5})$. These numbers involve surds.

Since these numbers are irrational, we cannot express them in exact form using decimals or fractions. Sometimes we may wish to approximate them using decimals, but mostly we prefer to leave them in exact form. Thus we need to be able to manipulate these types of numbers and to simplify combinations of them which arise when solving a problem.

2A Set notation

A **set** is a general name for any collection of things or numbers. There must be a way of deciding whether any particular object is a member of the set or not. This may be done by referring to a list of the members of the set or a statement describing them.

For example: $A = \{-3, 3\} = \{x : x^2 = 9\}$

Note: $\{x : \dots\}$ is read as ‘the set of all x such that $\dots$ ’.

- The symbol $\in$ means ‘is a member of’ or ‘is an element of’.

For example: $3 \in \{\text{prime numbers}\}$ is read ‘3 is a member of the set of prime numbers’.

- The symbol $\notin$ means ‘is not a member of’ or ‘is not an element of’.

For example: $4 \notin \{\text{prime numbers}\}$ is read ‘4 is not a member of the set of prime numbers’.

- Two sets are **equal** if they contain exactly the same elements, not necessarily in the same order. For example: if $A = \{\text{prime numbers less than 10}\}$ and $B = \{2, 3, 5, 7\}$, then $A = B$.

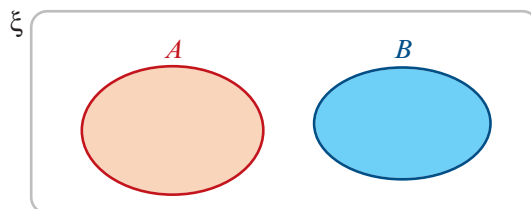
- The set with no elements is called the **empty set** and is denoted by $\emptyset$.

- The **universal set** will be denoted by ξ . The universal set is the set of all elements which are being considered.

- If all the elements of a set B are also elements of a set A , then the set B is called a **subset** of A . This is written $B \subseteq A$. For example: $\{a, b, c\} \subseteq \{a, b, c, d, e, f, g\}$ and $\{3, 9, 27\} \subseteq \{\text{multiples of 3}\}$. We note also that $A \subseteq A$ and $\emptyset \subseteq A$.

Venn diagrams are used to illustrate sets.

For example, the diagram on the right shows two subsets A and B of a universal set ξ such that A and B have no elements in common. Two such sets are said to be **disjoint**.



► The union of two sets

The set of all the elements that are members of set A or set B (or both) is called the **union** of A and B . The union of A and B is written $A \cup B$.

Example 1

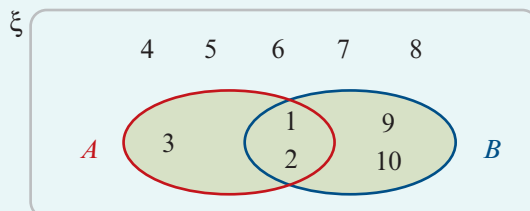
Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3\}$ and $B = \{1, 2, 9, 10\}$.

Find $A \cup B$ and illustrate on a Venn diagram.

Solution

$$A \cup B = \{1, 2, 3, 9, 10\}$$

The shaded area illustrates $A \cup B$.



► The intersection of two sets

The set of all the elements that are members of both set A and set B is called the **intersection** of A and B . The intersection of A and B is written $A \cap B$.

Example 2

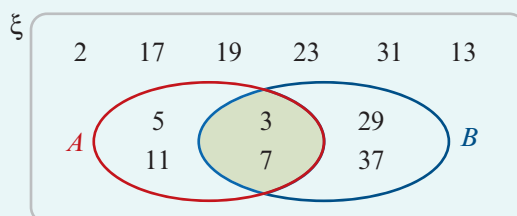
Let $\xi = \{\text{prime numbers less than 40}\}$, $A = \{3, 5, 7, 11\}$ and $B = \{3, 7, 29, 37\}$.

Find $A \cap B$ and illustrate on a Venn diagram.

Solution

$$A \cap B = \{3, 7\}$$

The shaded area illustrates $A \cap B$.



► The complement of a set

The **complement** of a set A is the set of all elements of ξ that are not members of A . The complement of A is denoted by A' .

If $\xi = \{\text{students at Highland Secondary College}\}$ and $A = \{\text{students with blue eyes}\}$, then A' is the set of all students at Highland Secondary College who do not have blue eyes.

Similarly, if $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$, then $A' = \{2, 4, 6, 8, 10\}$.

Example 3

Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{\text{odd numbers}\} = \{1, 3, 5, 7, 9\}$

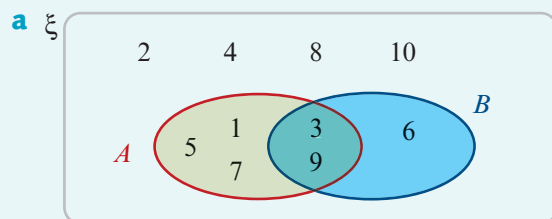
$B = \{\text{multiples of 3}\} = \{3, 6, 9\}$

a Show these sets on a Venn diagram.

b Use the diagram to list the following sets:

- i** A' **ii** B' **iii** $A \cup B$ **iv** the complement of $A \cup B$, i.e. $(A \cup B)'$ **v** $A' \cap B'$

Solution



b From the diagram:

- i** $A' = \{2, 4, 6, 8, 10\}$
ii $B' = \{1, 2, 4, 5, 7, 8, 10\}$
iii $A \cup B = \{1, 3, 5, 6, 7, 9\}$
iv $(A \cup B)' = \{2, 4, 8, 10\}$
v $A' \cap B' = \{2, 4, 8, 10\}$

► Finite and infinite sets

When all the elements of a set may be counted, the set is called a **finite** set. For example, the set $A = \{\text{months of the year}\}$ is finite. The number of elements of a set A will be denoted $|A|$. In this example, $|A| = 12$. If $C = \{\text{letters of the alphabet}\}$, then $|C| = 26$.

Sets which are not finite are called **infinite** sets. For example, the set of real numbers, $\mathbb{R}$, and the set of integers, $\mathbb{Z}$, are infinite sets.

Section summary

- If x is an element of a set A , we write $x \in A$.
- If x is not an element of a set A , we write $x \notin A$.
- The **empty set** is denoted by $\emptyset$ and the **universal set** by ξ .
- If every element of B is an element of A , we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cup B$ is the **union** of A and B , where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
- The set $A \cap B$ is the **intersection** of A and B , where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The **complement** of A , denoted by A' , is the set of all elements of ξ that are not in A .
- If two sets A and B have no elements in common, we say that they are **disjoint** and write $A \cap B = \emptyset$.

Exercise 2A

Example 1–3

- 1 Let $\xi = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3, 5\}$ and $B = \{2, 4\}$.
Show these sets on a Venn diagram and use the diagram to find:
 - a A'
 - b B'
 - c $A \cup B$
 - d $(A \cup B)'$
 - e $A' \cap B'$
- 2 Let $\xi = \{\text{natural numbers less than 17}\}$, $P = \{\text{multiples of 3}\}$ and $Q = \{\text{even numbers}\}$.
Show these sets on a Venn diagram and use it to find:
 - a P'
 - b Q'
 - c $P \cup Q$
 - d $(P \cup Q)'$
 - e $P' \cap Q'$
- 3 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{\text{multiples of 4}\}$ and $B = \{\text{even numbers}\}$.
Show these sets on a Venn diagram and use this diagram to list the sets:
 - a A'
 - b B'
 - c $A \cup B$
 - d $(A \cup B)'$
 - e $A' \cap B'$
- 4 Let $\xi = \{\text{natural numbers from 10 to 25}\}$, $P = \{\text{multiples of 4}\}$ and $Q = \{\text{multiples of 5}\}$.
Show these sets on a Venn diagram and use this diagram to list the sets:
 - a P'
 - b Q'
 - c $P \cup Q$
 - d $(P \cup Q)'$
 - e $P' \cap Q'$
- 5 Let $\xi = \{p, q, r, s, t, u, v, w\}$, $X = \{r, s, t, w\}$ and $Y = \{q, s, t, u, v\}$.
Show ξ , X and Y on a Venn diagram, entering all members. Hence list the sets:
 - a X'
 - b Y'
 - c $X' \cap Y'$
 - d $X' \cup Y'$
 - e $X \cup Y$
 - f $(X \cup Y)'$

Which two sets are equal?

SF

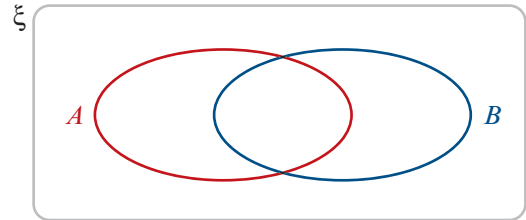
- 6 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $X = \{\text{factors of } 12\}$ and $Y = \{\text{even numbers}\}$. Show ξ , X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' b Y' c $X' \cup Y'$ d $(X \cap Y)'$ e $X \cup Y$ f $(X \cup Y)'$

Which two sets are equal?

- 7 Draw this diagram six times. Use shading to illustrate each of the following sets:

a A' b B' c $A' \cap B'$
d $A' \cup B'$ e $A \cup B$ f $(A \cup B)'$



- 8 Let $\xi = \{\text{different letters in the word } GENERAL\}$,
 $A = \{\text{different letters in the word } ANGEL\}$,
 $B = \{\text{different letters in the word } LEAN\}$

Show these sets on a Venn diagram and use this diagram to list the sets:

a A' b B' c $A \cap B$ d $A \cup B$ e $(A \cup B)'$ f $A' \cup B'$

- 9 Let $\xi = \{\text{different letters in the word } MATHEMATICS\}$
 $A = \{\text{different letters in the word } ATTIC\}$
 $B = \{\text{different letters in the word } TASTE\}$

Show ξ , A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' b B' c $A \cap B$ d $(A \cup B)'$ e $A' \cup B'$ f $A' \cap B'$

2B Sets of numbers

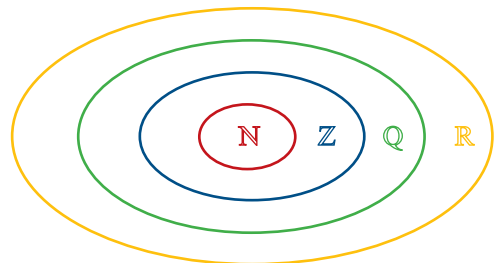
Recall that the elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational**. Some examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, π , $\pi + 2$ and $\sqrt{6} + \sqrt{7}$. These numbers cannot be written in the form $\frac{p}{q}$, for integers p, q ; the decimal representations of these numbers do not terminate or repeat.

- The set of real numbers is denoted by $\mathbb{R}$.
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of integers is denoted by $\mathbb{Z}$.
- The set of natural numbers is denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



We can use set notation to describe subsets of the real numbers.

For example:

- $\{x : 0 < x < 1\}$ is the set of all real numbers strictly between 0 and 1
- $\{x : x \geq 3\}$ is the set of all real numbers greater than or equal to 3
- $\{2n : n \in \mathbb{Z}\}$ is the set of all even integers.

The set of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$. That is,

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

This set is known as the **Cartesian product** of $\mathbb{R}$ with itself.

► Rational numbers

Every rational number can be expressed as a terminating or recurring decimal.

To find the decimal representation of a rational number $\frac{m}{n}$, perform the division $m \div n$.

For example, to find the decimal representation of $\frac{3}{7}$, divide 3.000000... by 7.

$$\begin{array}{r} 0.4285714\dots \\ 7 \overline{) 3.000000\dots} \\ \underline{21} \\ 90 \\ \underline{84} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \end{array}$$

Therefore $\frac{3}{7} = 0.428571\dot{}$.

Theorem

Every rational number can be written as a terminating or recurring decimal.

Proof Consider any two natural numbers m and n . At each step in the division of m by n , there is a remainder. If the remainder is 0, then the division algorithm stops and the decimal is a terminating decimal.

If the remainder is never 0, then it must be one of the numbers $1, 2, 3, \dots, n-1$.

(In the above example, $n = 7$ and the remainders can only be 1, 2, 3, 4, 5 and 6.)

Hence the remainder must repeat after at most $n-1$ steps.

Further examples:

$$\frac{1}{2} = 0.5, \quad \frac{1}{5} = 0.2, \quad \frac{1}{10} = 0.1, \quad \frac{1}{3} = 0.\dot{3}, \quad \frac{1}{7} = 0.142857\dot{}$$

Theorem

A real number has a terminating decimal representation if and only if it can be written as

$$\frac{m}{2^\alpha \times 5^\beta}$$

for some $m \in \mathbb{Z}$ and some $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

Proof Assume that $x = \frac{m}{2^\alpha \times 5^\beta}$ with $\alpha \geq \beta$. Multiply the numerator and denominator by $5^{\alpha-\beta}$. Then

$$x = \frac{m \times 5^{\alpha-\beta}}{2^\alpha \times 5^\alpha} = \frac{m \times 5^{\alpha-\beta}}{10^\alpha}$$

and so x can be written as a terminating decimal. The case $\alpha < \beta$ is similar.

Conversely, if x can be written as a terminating decimal, then there is $m \in \mathbb{Z}$ and $\alpha \in \mathbb{N} \cup \{0\}$ such that $x = \frac{m}{10^\alpha} = \frac{m}{2^\alpha \times 5^\alpha}$.

The method for finding a rational number $\frac{m}{n}$ from its decimal representation is demonstrated in the following example.

Example 4

Write each of the following in the form $\frac{m}{n}$, where m and n are integers:

a 0.05

b $0.\dot{4}2857\dot{1}$

Solution

a $0.05 = \frac{5}{100} = \frac{1}{20}$

b We can write

$$0.\dot{4}2857\dot{1} = 0.428571428571 \dots \quad (1)$$

Multiply both sides by 10^6 :

$$0.\dot{4}2857\dot{1} \times 10^6 = 428571.428571428571 \dots \quad (2)$$

Subtract (1) from (2):

$$0.\dot{4}2857\dot{1} \times (10^6 - 1) = 428571$$

$$\therefore 0.\dot{4}2857\dot{1} = \frac{428571}{10^6 - 1}$$

$$= \frac{3}{7}$$

► Real numbers

The set of real numbers is made up of two important subsets: the **algebraic numbers** and the **transcendental numbers**.

An algebraic number is a solution to a polynomial equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0, \quad \text{where } a_0, a_1, \dots, a_n \text{ are integers}$$

Every rational number is algebraic. The irrational number $\sqrt{2}$ is algebraic, as it is a solution of the equation

$$x^2 - 2 = 0$$

It can be shown that π is not an algebraic number; it is a transcendental number. The proof is too difficult to be given here.

The proof that $\sqrt{2}$ is irrational is presented in Chapter 8.

► Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$\begin{aligned}(a, b) &= \{x : a < x < b\} & [a, b] &= \{x : a \leq x \leq b\} \\(a, b] &= \{x : a < x \leq b\} & [a, b) &= \{x : a \leq x < b\} \\(a, \infty) &= \{x : a < x\} & [a, \infty) &= \{x : a \leq x\} \\(-\infty, b) &= \{x : x < b\} & (-\infty, b] &= \{x : x \leq b\}\end{aligned}$$

Intervals may be represented by diagrams as shown in Example 5.

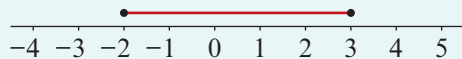
Example 5

Illustrate each of the following intervals of real numbers:

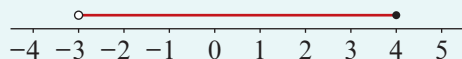
- a** $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 2]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

Solution

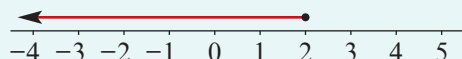
a $[-2, 3]$



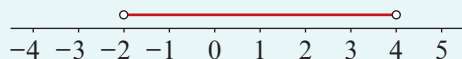
b $(-3, 4]$



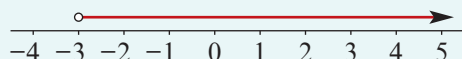
c $(-\infty, 2]$



d $(-2, 4)$



e $(-3, \infty)$



Explanation

The square brackets indicate that the endpoints are included; this is shown with closed circles.

The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as 'negative infinity'. The right endpoint is included.

Both brackets are round; the endpoints are not included.

The symbol ∞ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as 'infinity'. The left endpoint is not included.

Note: The 'closed' circle (●) indicates that the number is included.

The 'open' circle (○) indicates that the number is not included.

The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{x : x > 0\}$
- Negative real numbers: $\mathbb{R}^- = \{x : x < 0\}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

Section summary

■ Sets of numbers

- Real numbers: $\mathbb{R}$
- Rational numbers: $\mathbb{Q}$
- Integers: $\mathbb{Z}$
- Natural numbers: $\mathbb{N}$

- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$\begin{aligned} (a, b) &= \{x : a < x < b\} & [a, b] &= \{x : a \leq x \leq b\} \\ (a, b] &= \{x : a < x \leq b\} & [a, b) &= \{x : a \leq x < b\} \\ (a, \infty) &= \{x : a < x\} & [a, \infty) &= \{x : a \leq x\} \\ (-\infty, b) &= \{x : x < b\} & (-\infty, b] &= \{x : x \leq b\} \end{aligned}$$

Exercise 2B

- 1 **a** Is the sum of two rational numbers also rational?
b Is the product of two rational numbers also rational?
c Is the quotient of two rational numbers also rational (if defined)?
- 2 **a** Is the sum of two irrational numbers always irrational?
b Is the product of two irrational numbers always irrational?
c Is the quotient of two irrational numbers always irrational?

Example 4

- 3 Write each of the following in the form $\frac{m}{n}$, where m and n are integers:

- a** 0.45 **b** $0.\dot{2}\dot{7}$ **c** 0.12
d $0.\dot{2}8571\dot{4}$ **e** $0.\dot{3}\dot{6}$ **f** $0.\dot{2}$

- 4 Give the decimal representation of each of the following rational numbers:

- a** $\frac{2}{7}$ **b** $\frac{5}{11}$ **c** $\frac{7}{20}$ **d** $\frac{4}{13}$ **e** $\frac{1}{17}$

Example 5

- 5 Illustrate each of the following intervals of real numbers:

- a** $[-1, 4]$ **b** $(-2, 2]$ **c** $(-\infty, 3]$ **d** $(-1, 5)$ **e** $(-2, \infty)$

- 6 Write each of the following sets using interval notation:

- a** $\{x : x < 3\}$ **b** $\{x : x \geq -3\}$ **c** $\{x : x \leq -3\}$
d $\{x : x > 5\}$ **e** $\{x : -2 \leq x < 3\}$ **f** $\{x : -2 \leq x \leq 3\}$
g $\{x : -2 < x \leq 3\}$ **h** $\{x : -5 < x < 3\}$

2C Surds

A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.

Note: $\sqrt{a}$ is taken to mean the positive square root.

In general, a **surd of order n** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect n th power.

For example:

- $\sqrt{7}$, $\sqrt{24}$, $\sqrt{\frac{9}{7}}$, $\sqrt{\frac{1}{2}}$ are quadratic surds
- $\sqrt{9}$, $\sqrt{16}$, $\sqrt{\frac{9}{4}}$ are *not* surds
- $\sqrt[3]{7}$, $\sqrt[3]{15}$ are surds of order 3
- $\sqrt[4]{100}$, $\sqrt[4]{26}$ are surds of order 4

Quadratic surds hold a prominent position in school mathematics. For example, the solutions of quadratic equations often involve surds:

$$x = \frac{1 + \sqrt{5}}{2} \text{ is a solution of the quadratic equation } x^2 - x - 1 = 0.$$

Some well-known values of trigonometric functions involve surds. For example:

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Exact solutions are often required in Mathematical Methods Units 3 & 4 and Specialist Mathematics Units 3 & 4.

► Properties of square roots

The following properties of square roots are often used.

For positive numbers a and b :

- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ e.g. $\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ e.g. $\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{\sqrt{9}} = \frac{\sqrt{7}}{3}$

► Properties of surds

Simplest form

If possible, a factor which is the square of a rational number is ‘taken out’ of a square root. When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.

Example 6

Write each of the following in simplest form:

a $\sqrt{72}$

b $\sqrt{28}$

c $\sqrt{\frac{700}{117}}$

d $\sqrt{\frac{99}{64}}$

Solution

a $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$

b $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$

$$\begin{aligned} \text{c } \sqrt{\frac{700}{117}} &= \frac{\sqrt{700}}{\sqrt{117}} = \frac{\sqrt{7 \times 100}}{\sqrt{9 \times 13}} \\ &= \frac{10}{3} \sqrt{\frac{7}{13}} \end{aligned}$$

$$\begin{aligned} \text{d } \sqrt{\frac{99}{64}} &= \frac{\sqrt{99}}{\sqrt{64}} = \frac{\sqrt{9 \times 11}}{8} \\ &= \frac{3\sqrt{11}}{8} \end{aligned}$$

Like surdsSurds which have the same 'irrational factor' are called **like surds**.For example: $3\sqrt{7}$, $2\sqrt{7}$ and $\sqrt{7}$ are like surds.

The sum or difference of two like surds can be simplified:

$$\blacksquare m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p}$$

$$\blacksquare m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$$

Example 7

Express each of the following as a single surd in simplest form:

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

c $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

Solution

$$\begin{aligned} \text{a } \sqrt{147} + \sqrt{108} - \sqrt{363} &= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3} - \sqrt{11^2 \times 3} \\ &= 7\sqrt{3} + 6\sqrt{3} - 11\sqrt{3} \\ &= 2\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{b } \sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48} &= \sqrt{3} + \sqrt{5} + 2\sqrt{5} + 3\sqrt{3} - 3\sqrt{5} - 4\sqrt{3} \\ &= 0\sqrt{3} + 0\sqrt{5} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{c } \sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8} &= 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2} \\ &= 8\sqrt{2} - 6\sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

► Rationalising the denominator

In the past, a labour-saving procedure with surds was to **rationalise** any surds in the denominator of an expression. This is still considered to be a neat way of expressing final answers.

For $\sqrt{5}$, a rationalising factor is $\sqrt{5}$, as $\sqrt{5} \times \sqrt{5} = 5$.

For $1 + \sqrt{2}$, a rationalising factor is $1 - \sqrt{2}$, as $(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1$.

For $\sqrt{3} + \sqrt{6}$, a rationalising factor is $\sqrt{3} - \sqrt{6}$, as $(\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6}) = 3 - 6 = -3$.



Example 8

Rationalise the denominator of each of the following:

a $\frac{1}{2\sqrt{7}}$

b $\frac{1}{2 - \sqrt{3}}$

c $\frac{1}{\sqrt{3} - \sqrt{6}}$

d $\frac{3 + \sqrt{8}}{3 - \sqrt{8}}$

Solution

a $\frac{1}{2\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}}{14}$

b $\frac{1}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{2 + \sqrt{3}}{4 - 3}$
 $= 2 + \sqrt{3}$

c $\frac{1}{\sqrt{3} - \sqrt{6}} \times \frac{\sqrt{3} + \sqrt{6}}{\sqrt{3} + \sqrt{6}} = \frac{\sqrt{3} + \sqrt{6}}{3 - 6}$
 $= \frac{-1}{3}(\sqrt{3} + \sqrt{6})$

d $\frac{3 + \sqrt{8}}{3 - \sqrt{8}} = \frac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \times \frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}}$
 $= \frac{9 + 12\sqrt{2} + 8}{9 - 8}$
 $= 17 + 12\sqrt{2}$

Example 9

Expand the brackets in each of the following and collect like terms, expressing surds in simplest form:

a $(3 - \sqrt{2})^2$

b $(3 - \sqrt{2})(1 + \sqrt{2})$

Solution

a $(3 - \sqrt{2})^2$
 $= (3 - \sqrt{2})(3 - \sqrt{2})$
 $= 3(3 - \sqrt{2}) - \sqrt{2}(3 - \sqrt{2})$
 $= 9 - 3\sqrt{2} - 3\sqrt{2} + 2$
 $= 11 - 6\sqrt{2}$

b $(3 - \sqrt{2})(1 + \sqrt{2})$
 $= 3(1 + \sqrt{2}) - \sqrt{2}(1 + \sqrt{2})$
 $= 3 + 3\sqrt{2} - \sqrt{2} - 2$
 $= 1 + 2\sqrt{2}$

Exercise 2C

Skillsheet

- 1 Express each of the following in terms of the simplest possible surds:

Example 6

a $\sqrt{8}$	b $\sqrt{12}$	c $\sqrt{27}$	d $\sqrt{50}$
e $\sqrt{45}$	f $\sqrt{1210}$	g $\sqrt{98}$	h $\sqrt{108}$
i $\sqrt{25}$	j $\sqrt{75}$	k $\sqrt{512}$	

Example 7

- 2 Simplify each of the following:

a $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$	b $\sqrt{75} + 2\sqrt{12} - \sqrt{27}$
c $\sqrt{28} + \sqrt{175} - \sqrt{63}$	d $\sqrt{1000} - \sqrt{40} - \sqrt{90}$
e $\sqrt{512} + \sqrt{128} + \sqrt{32}$	f $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

- 3 Simplify each of the following:

a $\sqrt{75} + \sqrt{108} + \sqrt{14}$	b $\sqrt{847} - \sqrt{567} + \sqrt{63}$
c $\sqrt{720} - \sqrt{245} - \sqrt{125}$	d $\sqrt{338} - \sqrt{288} + \sqrt{363} - \sqrt{300}$
e $\sqrt{12} + \sqrt{8} + \sqrt{18} + \sqrt{27} + \sqrt{300}$	f $2\sqrt{18} + 3\sqrt{5} - \sqrt{50} + \sqrt{20} - \sqrt{80}$

Example 8

- 4 Express each of the following with rational denominators:

a $\frac{1}{\sqrt{5}}$	b $\frac{1}{\sqrt{7}}$	c $-\frac{1}{\sqrt{2}}$	d $\frac{2}{\sqrt{3}}$	e $\frac{3}{\sqrt{6}}$
f $\frac{1}{2\sqrt{2}}$	g $\frac{1}{\sqrt{2}+1}$	h $\frac{1}{2-\sqrt{3}}$	i $\frac{1}{4-\sqrt{10}}$	j $\frac{2}{\sqrt{6}+2}$
k $\frac{1}{\sqrt{5}-\sqrt{3}}$	l $\frac{3}{\sqrt{6}-\sqrt{5}}$	m $\frac{1}{3-2\sqrt{2}}$		

Example 9

- 5 Express each of the following in the form
- $a + b\sqrt{c}$
- :

a $\frac{2}{3-2\sqrt{2}}$	b $(\sqrt{5}+2)^2$	c $(1+\sqrt{2})(3-2\sqrt{2})$	d $(\sqrt{3}-1)^2$
e $\sqrt{\frac{1}{3}} - \frac{1}{\sqrt{27}}$	f $\frac{\sqrt{3}+2}{2\sqrt{3}-1}$	g $\frac{\sqrt{5}+1}{\sqrt{5}-1}$	h $\frac{\sqrt{8}+3}{\sqrt{18}+2}$

- 6 Expand and simplify each of the following:

a $(2\sqrt{a}-1)^2$	b $(\sqrt{x+1} + \sqrt{x+2})^2$

- 7 For real numbers
- a
- and
- b
- , we have
- $a > b$
- if and only if
- $a - b > 0$
- . Use this to state the larger of:

a $5 - 3\sqrt{2}$ and $6\sqrt{2} - 8$	b $2\sqrt{6} - 3$ and $7 - 2\sqrt{6}$

- 8 For positive real numbers
- a
- and
- b
- , we have
- $a > b$
- if and only if
- $a^2 - b^2 > 0$
- . Use this to state the larger of:

a $\frac{2}{\sqrt{3}}$ and $\frac{3}{\sqrt{2}}$	b $\frac{\sqrt{7}}{3}$ and $\frac{\sqrt{5}}{2}$	c $\frac{\sqrt{3}}{7}$ and $\frac{\sqrt{5}}{5}$	d $\frac{\sqrt{10}}{2}$ and $\frac{8}{\sqrt{3}}$

- 9 For each of the following, find the values of b and c for which the quadratic equation $x^2 + bx + c = 0$ has the two given solutions:

a $\sqrt{3}, -\sqrt{3}$

b $2\sqrt{3}, -2\sqrt{3}$

c $1 - \sqrt{2}, 1 + \sqrt{2}$

d $2 - \sqrt{3}, 2 + \sqrt{3}$

e $3 - 2\sqrt{2}, 3 + 2\sqrt{2}$

f $4 - 7\sqrt{5}, 3 + 2\sqrt{5}$

- 10 Express $\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

- 11 **a** Show that $a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$.

- b** Express $\frac{1}{1 - 2^{\frac{1}{3}}}$ with a rational denominator.

2D Natural numbers

► Factors and composites

The factors of 8 are 1, 2, 4 and 8.

The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

The factors of 5 are 1 and 5.

A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.

A natural number greater than 1 is called a **prime number** if its only factors are itself and 1.

The prime numbers less than 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

A natural number m is called a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m .

► Prime decomposition

Expressing a composite number as a product of powers of prime numbers is called **prime decomposition**. For example:

$$3000 = 3 \times 2^3 \times 5^3$$

$$2294 = 2 \times 31 \times 37$$

This is useful for finding the factors of a number. For example, the prime decomposition of 12 is given by $12 = 2^2 \times 3$. The factors of 12 are

$$1, \quad 2, \quad 2^2 = 4, \quad 3, \quad 2 \times 3 = 6 \quad \text{and} \quad 2^2 \times 3 = 12$$

This property of natural numbers is described formally by the following theorem.

Fundamental theorem of arithmetic

Every natural number greater than 1 either is a prime number or can be represented as a product of prime numbers. Furthermore, this representation is unique apart from rearrangement of the order of the prime factors.



Example 10

Give the prime decomposition of 17 248 and hence list the factors of this number.

Solution

The prime decomposition can be found by repeated division, as shown on the right.

The prime decomposition of 17 248 is

$$17\,248 = 2^5 \times 7^2 \times 11$$

Therefore each factor must be of the form

$$2^\alpha \times 7^\beta \times 11^\gamma$$

where $\alpha = 0, 1, 2, 3, 4, 5$, $\beta = 0, 1, 2$ and $\gamma = 0, 1$.

2	17 248
2	8624
2	4312
2	2156
2	1078
7	539
7	77
11	11
	1

The factors of 17 248 can be systematically listed as follows:

1	2	2^2	2^3	2^4	2^5
7	2×7	$2^2 \times 7$	$2^3 \times 7$	$2^4 \times 7$	$2^5 \times 7$
7^2	2×7^2	$2^2 \times 7^2$	$2^3 \times 7^2$	$2^4 \times 7^2$	$2^5 \times 7^2$
11	2×11	$2^2 \times 11$	$2^3 \times 11$	$2^4 \times 11$	$2^5 \times 11$
7×11	$2 \times 7 \times 11$	$2^2 \times 7 \times 11$	$2^3 \times 7 \times 11$	$2^4 \times 7 \times 11$	$2^5 \times 7 \times 11$
$7^2 \times 11$	$2 \times 7^2 \times 11$	$2^2 \times 7^2 \times 11$	$2^3 \times 7^2 \times 11$	$2^4 \times 7^2 \times 11$	$2^5 \times 7^2 \times 11$

► Highest common factor

The **highest common factor** of two natural numbers a and b is the largest natural number that is a factor of both a and b . It is denoted by $\text{HCF}(a, b)$.

For example, the highest common factor of 15 and 24 is 3. We write $\text{HCF}(15, 24) = 3$.

Note: The highest common factor is also called the **greatest common divisor**.

Using prime decomposition to find HCF

Prime decomposition can be used to find the highest common factor of two numbers.

For example, consider the numbers 140 and 110. Their prime factorisations are

$$140 = 2^2 \times 5 \times 7 \quad \text{and} \quad 110 = 2 \times 5 \times 11$$

A number which is a factor of both 140 and 110 must have prime factors which occur in both these factorisations. The highest common factor of 140 and 110 is $2 \times 5 = 10$.

Next consider the numbers

$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the highest common factor, we take the lower power of each prime factor:

$$\text{HCF}(396\,000, 1\,960\,200) = 2^3 \cdot 3^2 \cdot 5^2 \cdot 11$$

Example 11

- Find the highest common factor of 528 and 3168.
- Find the highest common factor of 3696 and 3744.

Solution

$$\text{a} \quad 528 = 2^4 \times 3 \times 11$$

$$3168 = 2^5 \times 3^2 \times 11$$

$$\therefore \text{HCF}(528, 3168) = 2^4 \times 3 \times 11 \\ = 528$$

$$\text{b} \quad 3696 = 2^4 \times 3 \times 7 \times 11$$

$$3744 = 2^5 \times 3^2 \times 13$$

$$\therefore \text{HCF}(3696, 3744) = 2^4 \times 3 \\ = 48$$



Using the TI-Nspire CX non-CAS

- The prime decomposition of a natural number can be obtained using **menu** > **Number** > **Factor** as shown.

Command	Result
factor(24)	$2^3 \cdot 3$
factor(-24)	$-1 \cdot 2^3 \cdot 3$
factor(1024)	2^{10}
factor(1001)	$7 \cdot 11 \cdot 13$

- The highest common factor of two numbers (also called their *greatest common divisor*) can be found by using the command **gcd()** from **menu** > **Number** > **Greatest Common Divisor**, or by just typing it in, as shown.

Command	Result
gcd(250,800)	50
gcd(gcd(50,745),585)	5

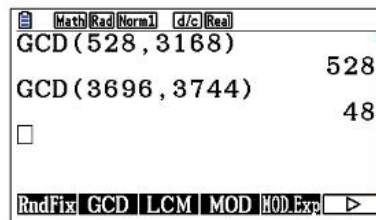
Note: Nested **gcd()** commands may be used to find the greatest common divisor of several numbers.

Using the Casio

To find the highest common factor of two natural numbers:

- Press **MENU** **1** to select **Run-Matrix** mode.
- Go to the **Numeric** menu **OPTN** **F6** **F4** and select **GCD** **F6** **F2**.
- Enter the two numbers as shown and press **EXE**.

Note: The highest common factor is also called the *greatest common divisor* (GCD).



► Lowest common multiple

A natural number a is a **multiple** of a natural number b if there exists a natural number k such that $a = kb$.

The **lowest common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b . It is denoted by $\text{LCM}(a, b)$.

For example:

- $\text{LCM}(24, 36) = 72$
- $\text{LCM}(256, 100) = 6400$

Using prime decomposition to find LCM

Consider again the numbers

$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the lowest common multiple, we take the higher power of each prime factor:

$$\begin{aligned} \text{LCM}(396\,000, 1\,960\,200) &= 2^5 \cdot 3^4 \cdot 5^3 \cdot 11^2 \\ &= 39\,204\,000 \end{aligned}$$



Using the TI-Nspire CX non-CAS

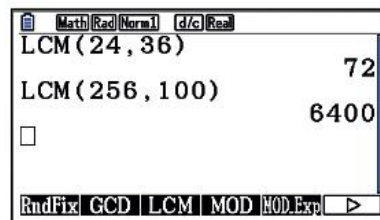
The lowest common multiple of two numbers (also called their *least common multiple*) can be found by using the command **lcm()** from **menu** **> Number > Least Common Multiple**, or by just typing it in, as shown.



Using the Casio

To find the lowest common multiple of two natural numbers:

- Press **MENU** **1** to select **Run-Matrix** mode.
- Go to the **Numeric** menu **OPTN** **F6** **F4** and select **LCM** **F6** **F3**.
- Enter the two numbers as shown and press **EXE**.



Section summary

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m .
- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b , denoted by $\text{HCF}(a, b)$, is the largest natural number that is a factor of both a and b .
- The **lowest common multiple** of two natural numbers a and b , denoted by $\text{LCM}(a, b)$, is the smallest natural number that is a multiple of both a and b .

Exercise 2D

Example 10

- 1 Give the prime decomposition of each of the following numbers:

- a** 60 **b** 676 **c** 228 **d** 900 **e** 252
f 6300 **g** 68 640 **h** 96 096 **i** 32 032 **j** 544 544

Example 11

- 2 Find the highest common factor of each of the following pairs of numbers:

- a** 4361, 9281 **b** 999, 2160 **c** 5255, 716 845 **d** 1271, 3875 **e** 804, 2358

- 3 **a** List all the factors of 18 and all the factors of 36.
b Why does 18 have an even number of factors and 36 an odd number of factors?
c Find the smallest number greater than 100 with exactly three factors.
- 4 A woman has three children and two of them are teenagers, aged between 13 and 19. The product of their three ages is 1050. How old is each child?
- 5 By using prime decomposition, find a natural number n such that $22^2 \times 55^2 = 10^2 \times n^2$.

- 6** Find the smallest natural number n such that $60n$ is a square number.

Hint: First find the prime decomposition of 60.

- 7** The natural number n has exactly eight different factors. Two of these factors are 15 and 21. What is the value of n ?
- 8** Let n be the smallest of three natural numbers whose product is 720. What is the largest possible value of n ?
- 9** When all eight factors of 30 are multiplied together, the product is 30^k . What is the value of k ?
- 10** A bell rings every 36 minutes and a buzzer rings every 42 minutes. If they sound together at 9 a.m., when will they next sound together?
- 11** The LCM of two numbers is $2^5 \times 3^3 \times 5^3$ and the HCF is $2^3 \times 3 \times 5^2$. Find all the possible numbers.

2E Problems involving sets

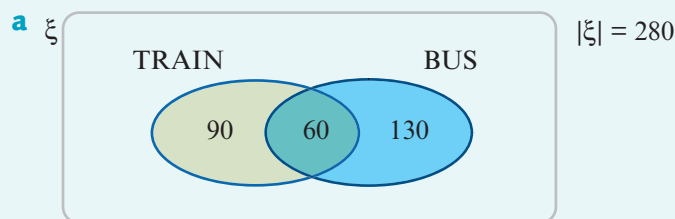
Sets can be used to help sort information, as each of the following examples demonstrates. Recall that, if A is a finite set, then the number of elements in A is denoted by $|A|$.

Example 12

Two hundred and eighty students each travel to school by either train or bus or both. Of these students, 150 travel by train, and 60 travel by both train and bus.

- a** Show this information on a Venn diagram.
- b** Hence find the number of students who travel by:
- bus
 - train but not bus
 - just one of these modes of transport.

Solution



- b**
- $|BUS| = 130 + 60 = 190$
 - $|TRAIN \cap (BUS)'| = 90$
 - $|TRAIN \cap (BUS)'| + |(TRAIN)' \cap BUS| = 90 + 130 = 220$



Example 13

An athletics team has 18 members. Each member competes in at least one of three events: sprints (S), jumps (J) and hurdles (H). Every hurdler also jumps or sprints. The following additional information is available:

$$|S| = 11, \quad |J| = 10, \quad |J \cap H' \cap S'| = 5, \quad |J' \cap H' \cap S| = 5 \quad \text{and} \quad |J \cap H'| = 7$$

a Draw a Venn diagram.

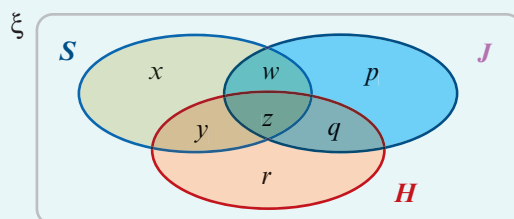
b Find:

- i** $|H|$ **ii** $|S \cap H \cap J|$ **iii** $|S \cup J|$ **iv** $|S \cap J \cap H'|$

Solution

a Assign a variable to the number of members in each region of the Venn diagram.

The information in the question can be summarised in terms of these variables:



$$x + y + z + w = 11 \quad \text{as } |S| = 11 \quad (1)$$

$$p + q + z + w = 10 \quad \text{as } |J| = 10 \quad (2)$$

$$x + y + z + w + p + q + r = 18 \quad \text{as all members compete} \quad (3)$$

$$p = 5 \quad \text{as } |J \cap H' \cap S'| = 5 \quad (4)$$

$$x = 5 \quad \text{as } |J' \cap H' \cap S| = 5 \quad (5)$$

$$r = 0 \quad \text{as every hurdler also jumps or sprints} \quad (6)$$

$$w + p = 7 \quad \text{as } |J \cap H'| = 7 \quad (7)$$

From (4) and (7): $w = 2$.

Equation (3) now becomes

$$5 + y + z + 2 + 5 + q = 18$$

$$\therefore y + z + q = 6 \quad (8)$$

Equation (1) becomes

$$y + z = 4$$

Therefore from (8): $q = 2$.

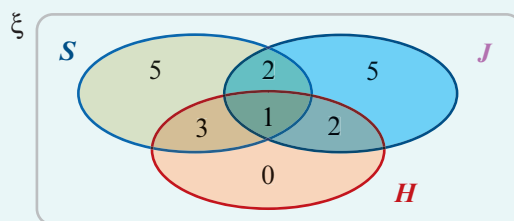
Equation (2) becomes

$$5 + 2 + z + 2 = 10$$

$$\therefore z = 1$$

$$\therefore y = 3$$

The Venn diagram can now be completed as shown.



- b i** $|H| = 6$ **ii** $|S \cap H \cap J| = 1$ **iii** $|S \cup J| = 18$ **iv** $|S \cap J \cap H'| = 2$

Exercise 2E

Skillsheet

Example 12

- 1 There are 28 students in a class, all of whom take either History or Economics or both. Of the 14 students who take History, five also take Economics.

- a Show this information on a Venn diagram.
 b Hence find the number of students who take:
 i Economics ii History but not Economics iii just one of these subjects.

- 2 a Draw a Venn diagram to show three sets A , B and C in a universal set ξ . Enter numbers in the correct parts of the diagram using the following information:

$$|A \cap B \cap C| = 2, \quad |A \cap B| = 7, \quad |B \cap C| = 6, \\ |A \cap C| = 8, \quad |A| = 16, \quad |B| = 20, \quad |C| = 19, \quad |\xi| = 50$$

- b Use the diagram to find:
 i $|A' \cap C'|$ ii $|A \cup B|$ iii $|A' \cap B \cap C|$

- 3 In a border town in the Balkans, 60% of people speak Bulgarian, 40% speak Greek and 20% speak neither. What percentage of the town speak both Bulgarian and Greek?
- 4 At an international conference there were 105 delegates. Seventy spoke English, 50 spoke French and 50 spoke Japanese. Twenty-five spoke English and French, 15 spoke French and Japanese and 30 spoke Japanese and English.
- a How many delegates spoke all three languages?
 b How many spoke Japanese only?

- 5 A restaurant serves lunch to 350 people. It offers three desserts: profiteroles, gelati and fruit. Forty people have all three desserts, 70 have gelati only, 50 have profiteroles only and 60 have fruit only. Forty-five people have fruit and gelati only, 30 people have gelati and profiteroles only and 10 people have fruit and profiteroles only. How many people do not have a dessert?

Example 13

- 6 Forty travellers were questioned about the various methods of transport they had used the previous day. Every traveller used at least one of the following methods: car (C), bus (B), train (T). Of these travellers:

- eight had used all three methods of transport
- four had travelled by bus and car only
- two had travelled by car and train only
- the number (x) who had travelled by train only was equal to the number who had travelled by bus and train only.

If 20 travellers had used a train and 33 had used a bus, find:

- a the value of x
 b the number who travelled by bus only
 c the number who travelled by car only.

- 7 Let ξ be the set of all integers and let

$$X = \{x : 21 < x < 37\}, \quad Y = \{3y : 0 < y \leq 13\}, \quad Z = \{z^2 : 0 < z < 8\}$$

a Draw a Venn diagram representing these sets.

b i Find $X \cap Y \cap Z$. ii Find $|X \cap Y|$.

- 8 A number of students bought red, green and black pens. Three bought one of each colour. Of the students who bought two colours, three did not buy red, five not green and two not black. The same number of students bought red only as bought red with other colours. The same number bought black only as bought green only. More students bought red and black but not green than bought black only. More bought only green than bought green and black but not red. How many students were there and how many pens of each colour were sold?

- 9 For three subsets B , M and F of a universal set ξ ,

$$|B \cap M| = 12, \quad |M \cap F \cap B| = |F'|, \quad |F \cap B| > |M \cap F|,$$

$$|B \cap F' \cap M'| = 5, \quad |M \cap B' \cap F'| = 5, \quad |F \cap M' \cap B'| = 5, \quad |\xi| = 28$$

Find $|M \cap F|$.

- 10 A group of 80 students were interviewed about which sports they play. It was found that 23 do athletics, 22 swim and 18 play football. If 10 students do athletics and swim only, 11 students do athletics and play football only, six students swim and play football only and 46 students do none of these activities on a regular basis, how many students do all three?
- 11 At a certain secondary college, students have to be proficient in at least one of the languages Italian, French and German. In a particular group of 33 students, two are proficient in all three languages, three in Italian and French only, four in French and German only and five in German and Italian only. The number of students proficient in Italian only is x , in French only is x and in German only is $x + 1$. Find x and then find the total number of students proficient in Italian.
- 12 At a certain school, 201 students study one or more of Mathematics, Physics and Chemistry. Of these students: 35 take Chemistry only, 50% more students study Mathematics only than study Physics only, four study all three subjects, 25 study both Mathematics and Physics but not Chemistry, seven study both Mathematics and Chemistry but not Physics, and 20 study both Physics and Chemistry but not Mathematics. Find the number of students studying Mathematics.



Chapter summary



Sets

■ Set notation

$x \in A$ x is an element of A

$x \notin A$ x is not an element of A

ξ the universal set

$\emptyset$ the empty set

$A \subseteq B$ A is a subset of B

$A \cup B$ the union of A and B consists of all elements that are in either A or B or both

$A \cap B$ the intersection of A and B consists of all elements that are in both A and B

A' the complement of A consists of all elements of ξ that are not in A

■ Sets of numbers

$\mathbb{N}$ Natural numbers $\mathbb{Z}$ Integers

$\mathbb{Q}$ Rational numbers $\mathbb{R}$ Real numbers

Surd

- A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.
- A **surd of order n** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect n th power.
- When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.
- Surds which have the same 'irrational factor' are called **like surds**. The sum or difference of two like surds can be simplified:

$$m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p} \quad \text{and} \quad m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$$

Natural numbers

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m .
- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b , denoted by $\text{HCF}(a, b)$, is the largest natural number that is a factor of both a and b .
- The **lowest common multiple** of two natural numbers a and b , denoted by $\text{LCM}(a, b)$, is the smallest natural number that is a multiple of both a and b .

Technology-free questions

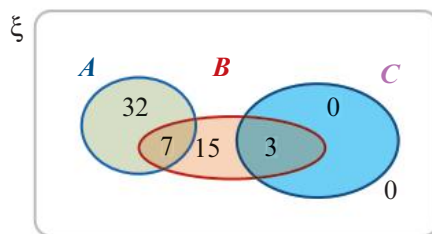
- 1 Express the following as fractions in their simplest form:
a 0.07 **b** 0.45 **c** 0.005 **d** 0.405 **e** 0.26 **f** 0.1714285
- 2 Express 504 as a product of powers of prime numbers.
- 3 Express each of the following with a rational denominator:
a $\frac{2\sqrt{3}-1}{\sqrt{2}}$ **b** $\frac{\sqrt{5}+2}{\sqrt{5}-2}$ **c** $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$
- 4 Express $\frac{3+2\sqrt{75}}{3-\sqrt{12}}$ in the form $a+b\sqrt{3}$, where $a, b \in \mathbb{Q} \setminus \{0\}$.
- 5 Express each of the following with a rational denominator:
a $\frac{6\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$ **b** $\frac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$
- 6 In a class of 100 students, 55 are girls, 45 have blue eyes, 40 are blond, 25 are blond girls, 15 are blue-eyed blonds, 20 are blue-eyed girls, and five are blue-eyed blond girls. Find:
a the number of blond boys
b the number of boys who are neither blond nor blue-eyed.
- 7 A group of 30 students received prizes in at least one of the subjects of English, Mathematics and French. Two students received prizes in all three subjects. Fourteen received prizes in English and Mathematics but not French. Two received prizes in English alone, two in French alone and five in Mathematics alone. Four received prizes in English and French but not Mathematics.
a How many received prizes in Mathematics and French but not English?
b How many received prizes in Mathematics?
c How many received prizes in English?
- 8 Fifty people are interviewed. Twenty-three people like Brand X, 25 like Brand Y and 19 like Brand Z. Eleven like X and Z. Eight like Y and Z. Five like X and Y. Two like all three. How many like none of them?
- 9 Three rectangles A, B and C overlap (intersect). Their areas are 20 cm^2 , 10 cm^2 and 16 cm^2 respectively. The area common to A and B is 3 cm^2 , that common to A and C is 6 cm^2 and that common to B and C is 4 cm^2 . How much of the area is common to all three if the total area covered is 35 cm^2 ?
- 10 Express $\sqrt{112} - \sqrt{63} - \frac{224}{\sqrt{28}}$ in simplest form.
- 11 If $\frac{\sqrt{7}-\sqrt{3}}{x} = \frac{x}{\sqrt{7}+\sqrt{3}}$, find the values of x .

12 Express $\frac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} + \frac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}}$ in the form $a\sqrt{5} + b\sqrt{6}$.

13 Simplify $\sqrt{27} - \sqrt{12} + 2\sqrt{75} - \sqrt{\frac{48}{25}}$.

- 14** A , B and C are three sets and $\xi = A \cup B \cup C$. The number of elements in the regions of the Venn diagram are as shown. Find:

- a** the number of elements in $A \cup B$
b the number of elements in C
c the number of elements in $B' \cap A$.



- 15** Using the result that $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$, determine the square root of $17 + 6\sqrt{8}$.
- 16** Prove that the product of two odd integers is odd. (You may assume that the sum and product of any two integers is an integer.)
- 17** The sum of the ages of Tom and Fred is 63. Tom is twice as old as Fred was when Tom was as old as Fred is now. What are the ages of Tom and Fred?

Multiple-choice questions



1 $\frac{4}{3 + 2\sqrt{2}}$ expressed in the form $a + b\sqrt{2}$ is

A $12 - 8\sqrt{2}$

B $3 + 2\sqrt{2}$

C $\frac{3}{17} - \frac{8}{17}\sqrt{2}$

D $\frac{3}{17} + \frac{8}{17}\sqrt{2}$

E $12 + 8\sqrt{2}$

- 2** The prime decomposition of 86 400 is

A $2^5 \times 3^2 \times 5$

B $2^6 \times 3^3 \times 5^2$

C $2^7 \times 3^3 \times 5$

D $2^7 \times 3^3 \times 5^2$

E $2^6 \times 3^3 \times 5^3$

- 3** $(\sqrt{6} + 3)(\sqrt{6} - 3)$ is equal to

A $3 - 12\sqrt{6}$

B $-3 - 6\sqrt{6}$

C $-3 + 6\sqrt{6}$

D -3

E 3

- 4** For the Venn diagram shown, ξ is the set of natural numbers less than 20, A is the set of natural numbers less than 10, and B is the set of natural numbers less than 20 that are divisible by 3. The set $B' \cap A$ is

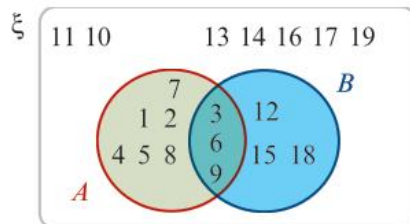
A $\{3, 6, 9\}$

B $\{12, 15, 18\}$

C $\{10, 11, 13, 14, 16, 17, 19\}$

D $\{1, 2, 4, 5, 7, 8\}$

E $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 18\}$



- 5 $(3, \infty) \cap (-\infty, 5] =$
A $(-\infty, 3)$ **B** $(-\infty, 5]$ **C** $(3, 5]$ **D** $\mathbb{R}$ **E** $[3, 5]$
- 6 A bell is rung every 6 minutes and a gong is sounded every 14 minutes. If these occur together at a particular time, then the smallest number of minutes until the bell and the gong are again heard simultaneously is
A 10 **B** 20 **C** 72 **D** 42 **E** 84
- 7 If X is the set of multiples of 2, Y the set of multiples of 7, and Z the set of multiples of 5, then $X \cap Y \cap Z$ can be described as
A the set of multiples of 2 **B** the set of multiples of 70
C the set of multiples of 35 **D** the set of multiples of 14
E the set of multiples of 10
- 8 In a class of students, 50% play football, 40% play tennis and 30% play neither. The percentage that plays both is
A 10 **B** 20 **C** 30 **D** 50 **E** 40
- 9 $\frac{\sqrt{7} - \sqrt{6}}{\sqrt{7} + \sqrt{6}} =$
A $5 + 2\sqrt{7}$ **B** $13 + 2\sqrt{6}$ **C** $13 - 2\sqrt{42}$ **D** $1 + 2\sqrt{42}$ **E** $13 - 2\sqrt{13}$
- 10 There are 40 students in a class, all of whom take either Literature or Economics or both. Twenty take Literature and five of these also take Economics. The number of students who take only Economics is
A 20 **B** 5 **C** 10 **D** 15 **E** 25
- 11 The number of factors that the integer $2^p 3^q 5^r$ has is
A $\frac{(p+q+r)!}{p!q!r!}$ **B** pqr **C** $p+q+r$
D $(p+1)(q+1)(r+1)$ **E** $p+q+r+1$
- 12 The number of pairs of integers (m, n) which satisfy the equation $m+n=mn$ is
A 1 **B** 2 **C** 3 **D** 4 **E** more than 4

Extended-response questions

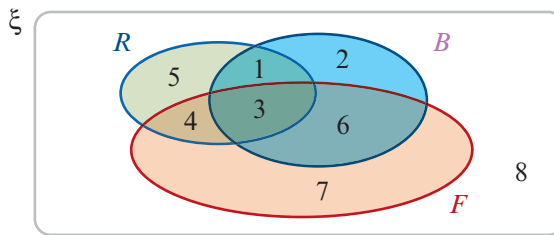
- 1 **a** Show that $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$.
b Substitute $x = 3$ and $y = 5$ in the identity from part **a** to show that

$$\sqrt{3} + \sqrt{5} = \sqrt{8 + 2\sqrt{15}}$$

c Use this technique to find the square root of:
i $14 + 2\sqrt{33}$ (Hint: Use $x = 11$ and $y = 3$.) **ii** $15 - 2\sqrt{56}$ **iii** $51 - 36\sqrt{2}$

- 2** In this question, we consider the set $\{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$. In Chapter 15, the set $\mathbb{C}$ of complex numbers is introduced, where $\mathbb{C} = \{a + b\sqrt{-1} : a, b \in \mathbb{R}\}$.
- a** If $(2 + 3\sqrt{3}) + (4 + 2\sqrt{3}) = a + b\sqrt{3}$, find a and b .
 - b** If $(2 + 3\sqrt{3})(4 + 2\sqrt{3}) = p + q\sqrt{3}$, find p and q .
 - c** If $\frac{1}{3 + 2\sqrt{3}} = a + b\sqrt{3}$, find a and b .
 - d** Solve each of the following equations for x :
 - i** $(2 + 5\sqrt{3})x = 2 - \sqrt{3}$
 - ii** $(x - 3)^2 - 3 = 0$
 - iii** $(2x - 1)^2 - 3 = 0$
 - e** Explain why every rational number is a member of $\{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$.
- 3**
- a** Show that $\frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$.
 - b** Use the substitution $t = (\sqrt{2 + \sqrt{3}})^x$ and part **a** to show that the equation $(\sqrt{2 + \sqrt{3}})^x + (\sqrt{2 - \sqrt{3}})^x = 4$ can be written as $t + \frac{1}{t} = 4$.
 - c** Show that the solutions of the equation are $t = 2 - \sqrt{3}$ and $t = 2 + \sqrt{3}$.
 - d** Use this result to solve the equation $(\sqrt{2 + \sqrt{3}})^x + (\sqrt{2 - \sqrt{3}})^x = 4$.
- 4** Use Venn diagrams to illustrate:
- a** $|A \cup B| = |A| + |B| - |A \cap B|$
 - b** $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$
- 5** A quadratic equation with integer coefficients $x^2 + bx + c = 0$ has a solution $x = 2 - \sqrt{3}$.
- a** Find the values of b and c .
Hint: Use the result that, for m, n rational, if $m + n\sqrt{3} = 0$, then $m = 0$ and $n = 0$.
 - b** Find the other solution to this quadratic equation.
 - c** Now consider a quadratic equation with integer coefficients $x^2 + bx + c = 0$ that has a solution $x = m - n\sqrt{q}$, where q is not a perfect square. Show that:
 - i** $b = -2m$
 - ii** $c = m^2 - n^2q$
- Hence show that:
- iii** $x^2 + bx + c = (x - (m - n\sqrt{q}))(x - (m + n\sqrt{q}))$
- 6** A **Pythagorean triple** (x, y, z) consists of three natural numbers x, y, z such that $x^2 + y^2 = z^2$. For example: $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples. A Pythagorean triple is in simplest form if x, y, z have no common factor. Up to swapping x and y , all Pythagorean triples in simplest form may be generated by:
- $$x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2 \quad \text{where } m, n \in \mathbb{N}$$
- For example, if $m = 2$ and $n = 1$, then $x = 4, y = 3$ and $z = 5$.
- a** Find the Pythagorean triple for $m = 5$ and $n = 2$.
 - b** Verify that, if $x = 2mn, y = m^2 - n^2$ and $z = m^2 + n^2$, where $m, n \in \mathbb{N}$, then $x^2 + y^2 = z^2$.

- 7 a** The Venn diagram shows the set ξ of all students enrolled at Argos Secondary College. Set R is the set of all students with red hair. Set B is the set of all students with blue eyes. Set F is the set of all female students.



The numbers on the diagram are to label the eight different regions.

- i** Identify the region in the Venn diagram which represents male students who have neither red hair nor blue eyes.
 - ii** Describe the gender, hair colour and eye colour of students represented in region 1 of the diagram.
 - iii** Describe the gender, hair colour and eye colour of students represented in region 2 of the diagram.
- b** It is known that, at Argos Secondary College, 250 students study French (F), Greek (G) or Japanese (J). Forty-one students do not study French. Twelve students study French and Japanese but not Greek. Thirteen students study Japanese and Greek but not French. Thirteen students study only Greek. Twice as many students study French and Greek but not Japanese as study all three. The number studying only Japanese is the same as the number studying both French and Greek.
- i** How many students study all three languages?
 - ii** How many students study only French?
- 8** In a certain city, three Sunday newspapers (A , B and C) are available. In a sample of 500 people from this city, it was found that:
- nobody regularly reads both A and C
 - a total of 100 people regularly read A
 - 205 people regularly read only B
 - of those who regularly read C , exactly half of them also regularly read B
 - 35 people regularly read A and B but not C
 - 35 people don't read any of the papers at all.
- a** Draw a Venn diagram showing the number of regular readers for each possible combination of A , B and C .
 - b** How many people in the sample were regular readers of C ?
 - c** How many people in the sample regularly read A only?
 - d** How many people are regular readers of A , B and C ?

3

Trigonometric ratios and applications

Objectives

- ▶ To review the trigonometric ratios **sine**, **cosine** and **tangent**.
- ▶ To solve practical problems using the trigonometric ratios.
- ▶ To find standard **exact values** of the trigonometric ratios.
- ▶ To use the **sine rule** and the **cosine rule** to solve problems.
- ▶ To solve problems involving **angles of elevation** and **angles of depression**.
- ▶ To solve problems involving **compass bearings**.
- ▶ To find the sine and cosine of angles between -360° and 360° .

This chapter is provided to review previous years, in preparation for Chapter 6 Vectors in the plane, and as a foundation for Units 3 and 4.

Trigonometry deals with the side lengths and angles of a triangle: the word *trigonometry* comes from the Greek words for triangle and measurement.

In this chapter, we review and extend your study of trigonometry from previous years. An understanding of trigonometry will be important for our study of vectors in Chapter 6.

We start this chapter by revising the four standard congruence tests for triangles. If you have the information about a triangle given in one of the congruence tests, then the triangle is uniquely determined (up to congruence). You can find the unknown side lengths and angles of the triangle using the **sine rule** or the **cosine rule**.

In this chapter, we establish these two rules, and apply them in practical problems involving angles of elevation, angles of depression and compass bearings. We will explore similar problems using vectors in Chapter 6.

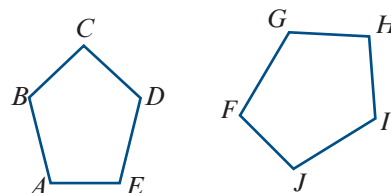
3A Reviewing trigonometry

► Congruent triangles

Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.

Congruent figures have exactly the same shape and size. For example, the two figures shown are congruent. We can write:

$$\text{pentagon } ABCDE \equiv \text{pentagon } FGHIJ$$



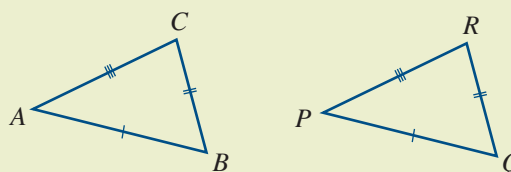
When two figures are congruent, we can find a transformation that pairs up every part of one figure with the corresponding part of the other, so that:

- paired angles have the same size
- paired line segments have the same length
- paired regions have the same area.

There are four standard tests for two triangles to be congruent.

■ The SSS congruence test

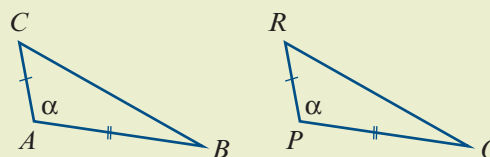
If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.



$$\triangle ABC \equiv \triangle PQR$$

■ The SAS congruence test

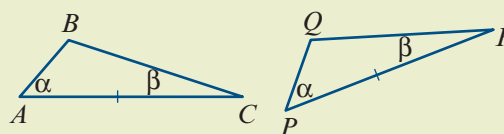
If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.



$$\triangle ABC \equiv \triangle PQR$$

■ The AAS congruence test

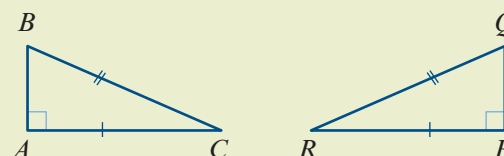
If two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, then the two triangles are congruent.



$$\triangle ABC \equiv \triangle PQR$$

■ The RHS congruence test

If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent.

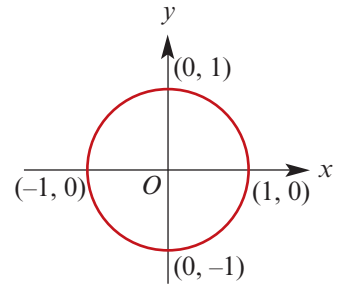


$$\triangle ABC \equiv \triangle PQR$$

► Defining sine and cosine

In this section we review sine, cosine and tangent for angles between 0° and 180° .

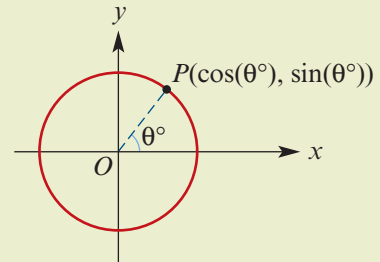
We can define the sine and cosine of any angle by using the **unit circle**. The unit circle is a circle of radius 1 with centre at the origin.



Unit-circle definition of sine and cosine

For each angle θ° , there is a point P on the unit circle as shown. The angle is measured anticlockwise from the positive direction of the x -axis.

- $\cos(\theta^\circ)$ is defined as the x -coordinate of the point P
- $\sin(\theta^\circ)$ is defined as the y -coordinate of the point P



► The trigonometric ratios

For acute angles, the unit-circle definition of sine and cosine given above is equivalent to the ratio definition.

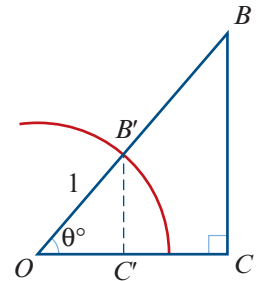
For a right-angled triangle OBC , we can construct a similar triangle $OB'C'$ that lies in the unit circle. From the diagram:

$$B'C' = \sin(\theta^\circ) \quad \text{and} \quad OC' = \cos(\theta^\circ)$$

As triangles OBC and $OB'C'$ are similar, we have

$$\frac{BC}{OB} = \frac{B'C'}{1} \quad \text{and} \quad \frac{OC}{OB} = \frac{OC'}{1}$$

$$\therefore \frac{BC}{OB} = \sin(\theta^\circ) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta^\circ)$$

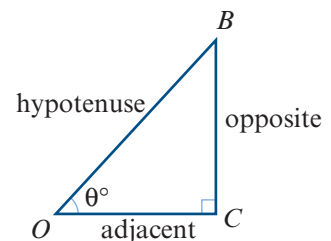


This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.

$$\sin(\theta^\circ) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta^\circ) = \frac{\text{opposite}}{\text{adjacent}}$$



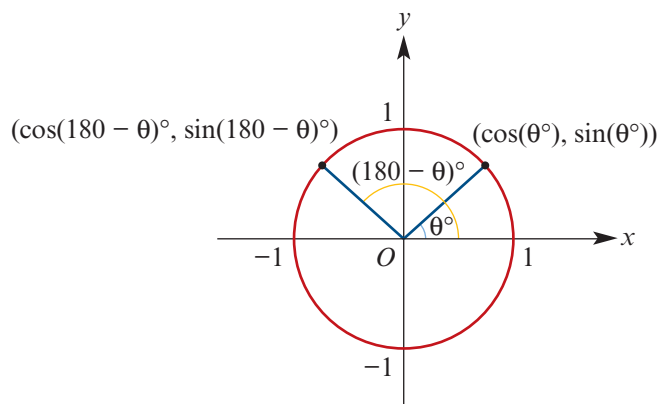
► Obtuse angles

From the unit circle, we see that

$$\begin{aligned}\sin(180 - \theta)^\circ &= \sin(\theta^\circ) \\ \cos(180 - \theta)^\circ &= -\cos(\theta^\circ)\end{aligned}$$

For example:

$$\begin{aligned}\sin 135^\circ &= \sin 45^\circ \\ \cos 135^\circ &= -\cos 45^\circ\end{aligned}$$



In this chapter, we will generally use the ratio definition of tangent for acute angles. But we can also find the tangent of an obtuse angle by defining

$$\tan(\theta^\circ) = \frac{\sin(\theta^\circ)}{\cos(\theta^\circ)}$$

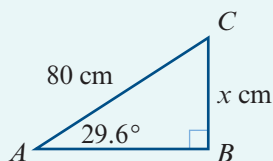
For working with triangles, we only need to consider angles between 0° and 180° . We will consider angles less than 0° and greater than 180° in Section 3F.

► Solving right-angled triangles

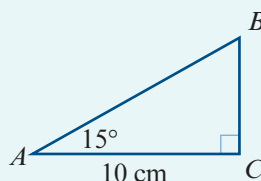
Here we provide some examples of using the trigonometric ratios.

Example 1

- a** Find the value of x correct to two decimal places.



- b** Find the length of the hypotenuse correct to two decimal places.



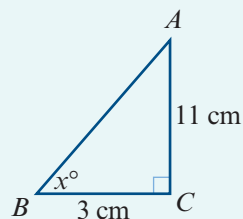
Solution

$$\begin{aligned}\mathbf{a} \quad \frac{x}{80} &= \sin 29.6^\circ \\ \therefore x &= 80 \sin 29.6^\circ \\ &= 39.5153 \dots\end{aligned}$$

Hence $x = 39.52$, correct to two decimal places.

$$\begin{aligned}\mathbf{b} \quad \frac{10}{AB} &= \cos 15^\circ \\ 10 &= AB \cos 15^\circ \\ \therefore AB &= \frac{10}{\cos 15^\circ} \\ &= 10.3527 \dots\end{aligned}$$

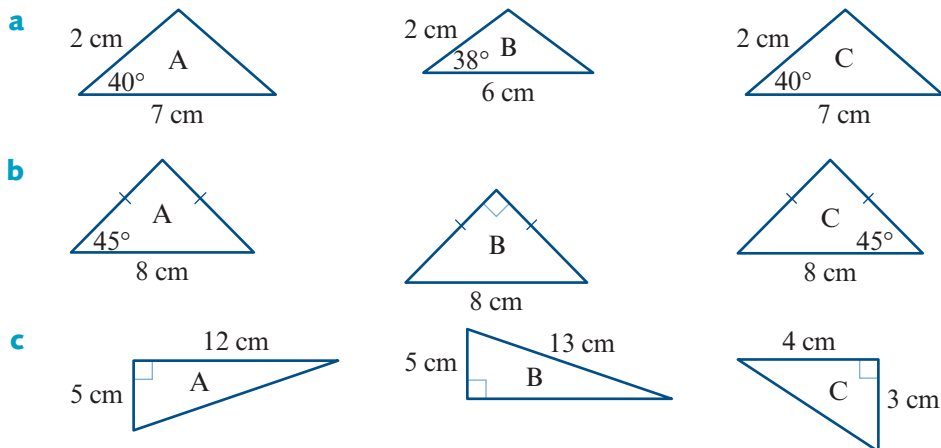
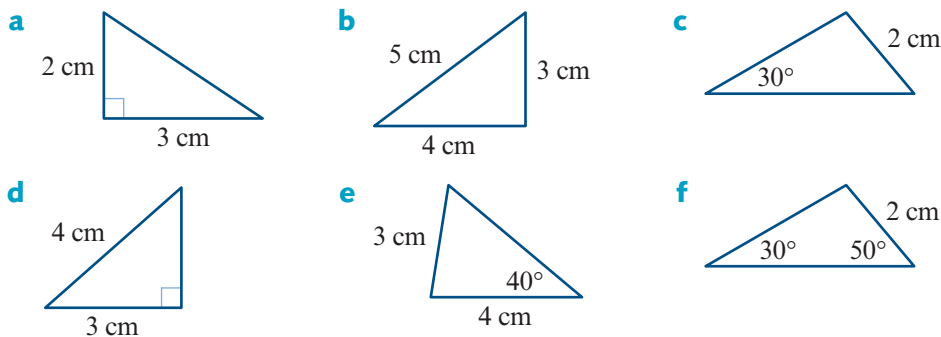
The length of the hypotenuse is 10.35 cm, correct to two decimal places.

Example 2Find the magnitude of $\angle ABC$.**Solution**

$$\tan x = \frac{11}{3}$$

$$\therefore x = \tan^{-1}\left(\frac{11}{3}\right)$$

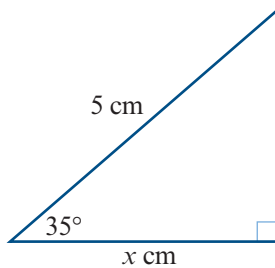
$$= (74.7448\dots)^\circ$$

Hence $x = 74.74^\circ$, correct to two decimal places.**Exercise 3A****1** In each part, find pairs of congruent triangles. State the congruence tests used.**2** For each of the following, state whether or not the given labelling uniquely determines a triangle (up to congruence):

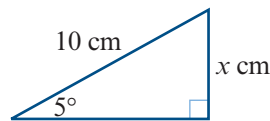
Example 1, 2

3 Find the value of x in each of the following:

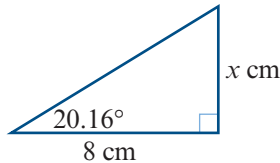
a



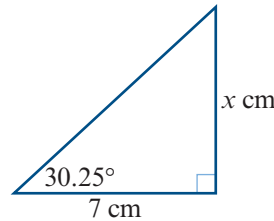
b



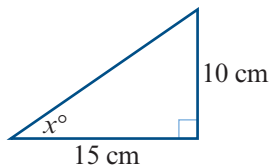
c



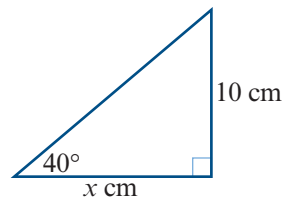
d



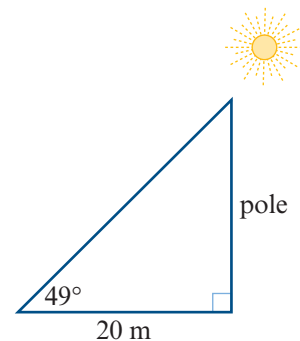
e



f

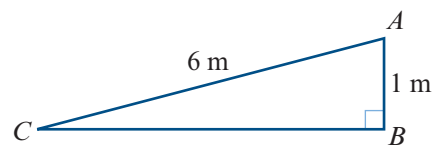


- 4 A pole casts a shadow 20 m long when the altitude of the sun is 49° . Calculate the height of the pole.



- 5 This figure represents a ramp.

- a Find the magnitude of angle ACB .
b Find the distance BC .



- 6 A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate:

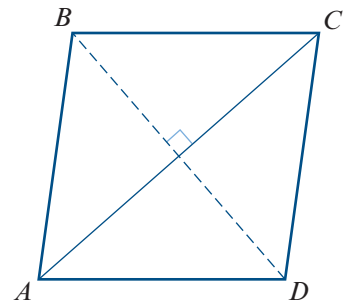
- a the length of the ladder
b the height it reaches above the ground.

- 7 An engineer is designing a straight concrete entry ramp, 60 m long, for a car park that is 13 m above street level. Calculate the angle of the ramp to the horizontal.

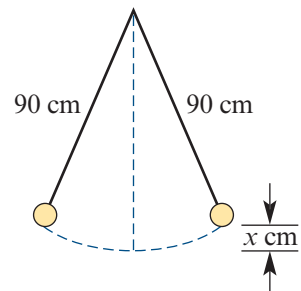
- 8** A vertical mast is secured from its top by straight cables 200 m long fixed at the ground. The cables make angles of 66° with the ground. What is the height of the mast?
- 9** A mountain railway rises 400 m at a uniform slope of 16° with the horizontal. What is the distance travelled by a train for this rise?
- 10** The base of an isosceles triangle is 12 cm long and the equal sides are 15 cm long. Find the magnitude of each of the three angles of the triangle.

- 11** The diagonals of a rhombus bisect each other at right angles. If $BD = AC = 10$ cm, find:

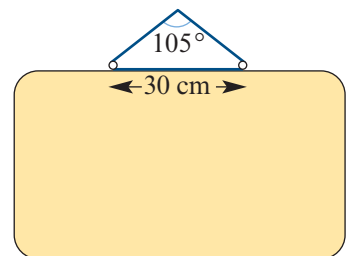
- a** the length of the sides of the rhombus
b the magnitude of angle ABC .



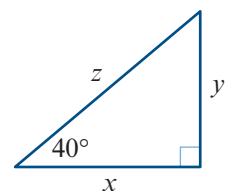
- 12** A pendulum swings from the vertical through an angle of 15° on each side of the vertical. If the pendulum is 90 cm long, what is the distance, x cm, between its highest and lowest points?



- 13** A picture is hung symmetrically by means of a string passing over a nail, with the ends of the string attached to two rings on the upper edge of the picture. The distance between the rings is 30 cm, and the string makes an angle of 105° at the nail. Find the length of the string.



- 14** A ladder 4.7 m long is placed against a wall. The foot of the ladder must not be placed in a flower bed, which extends a distance of 1.7 m out from the base of the wall. How high up the wall can the ladder reach?
- 15** The area of the triangle shown is 7 square units. Find the lengths of the three sides of the triangle, correct to two decimal places.



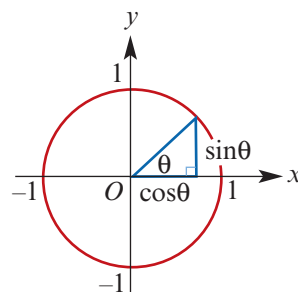
3B Exact values of sine, cosine and tangent

A calculator can be used to find the values of sine, cosine and tangent for different angles. For many angles, the calculator gives an approximation. We now consider some angles for which the exact values can be found.

► Exact values for 0° and 90°

From the unit circle:

$$\begin{aligned}\sin 0^\circ &= 0 & \sin 90^\circ &= 1 \\ \cos 0^\circ &= 1 & \cos 90^\circ &= 0 \\ \tan 0^\circ &= 0 & \tan 90^\circ &\text{ is undefined}\end{aligned}$$

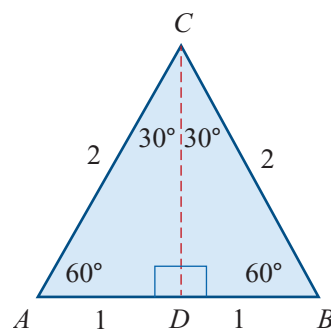


► Exact values for 30° and 60°

Consider an equilateral triangle ABC of side length 2 units.

In $\triangle ACD$, by Pythagoras' theorem, $CD = \sqrt{AC^2 - AD^2} = \sqrt{3}$.

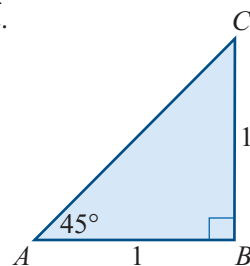
$$\begin{aligned}\sin 30^\circ &= \frac{AD}{AC} = \frac{1}{2} & \sin 60^\circ &= \frac{CD}{AC} = \frac{\sqrt{3}}{2} \\ \cos 30^\circ &= \frac{CD}{AC} = \frac{\sqrt{3}}{2} & \cos 60^\circ &= \frac{AD}{AC} = \frac{1}{2} \\ \tan 30^\circ &= \frac{AD}{CD} = \frac{1}{\sqrt{3}} & \tan 60^\circ &= \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}\end{aligned}$$



► Exact values for 45°

For the triangle ABC shown on the right, we have $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.

$$\begin{aligned}\sin 45^\circ &= \frac{BC}{AC} = \frac{1}{\sqrt{2}} \\ \cos 45^\circ &= \frac{AB}{AC} = \frac{1}{\sqrt{2}} \\ \tan 45^\circ &= \frac{BC}{AB} = 1\end{aligned}$$



Example 3

Evaluate:

a $\cos 150^\circ$

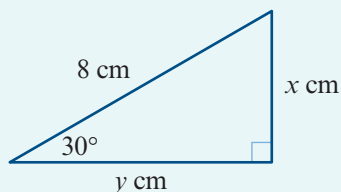
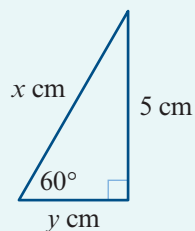
b $\sin 120^\circ$

Solution

a $\cos 150^\circ = \cos(180 - 30)^\circ$
 $= -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

b $\sin 120^\circ = \sin(180 - 60)^\circ$
 $= \sin 60^\circ = \frac{\sqrt{3}}{2}$

Note: Remember that $\sin(180 - \theta)^\circ = \sin(\theta)^\circ$ and $\cos(180 - \theta)^\circ = -\cos(\theta)^\circ$.

Example 4Find the exact values of x and y in each of the following:**a****b****Solution**

a $x = 8 \sin 30^\circ$

$$= 8 \times \frac{1}{2}$$

$$= 4$$

$$y = 8 \cos 30^\circ$$

$$= 8 \times \frac{\sqrt{3}}{2}$$

$$= 4\sqrt{3}$$

b $\frac{5}{x} = \sin 60^\circ$

$$\therefore x = \frac{5}{\sin 60^\circ} = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$$

$$\frac{5}{y} = \tan 60^\circ$$

$$\therefore y = \frac{5}{\tan 60^\circ} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$

Section summary

As an aid to memory, the exact values of the trigonometric functions can be tabulated.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

Exercise 3B

SF

Example 3

1 Find the exact value of each of the following:

a $\cos 135^\circ$

b $\sin 135^\circ$

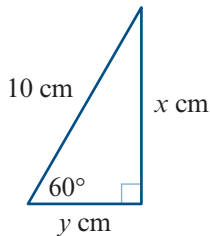
c $\cos 120^\circ$

d $\sin 150^\circ$

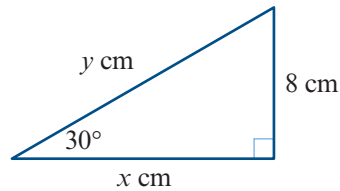
Example 4

2 Find the exact values of x and y in each of the following:

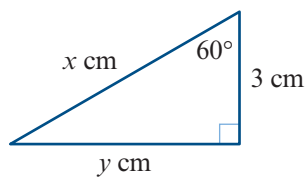
a



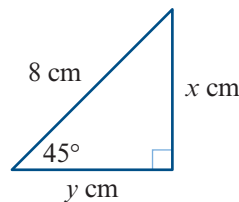
b



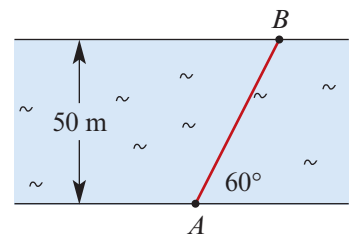
c



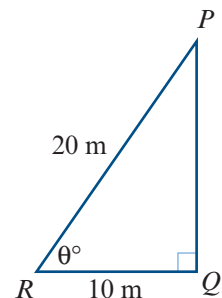
d



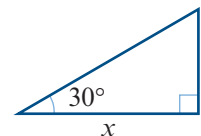
- 3 A river is known to be 50 m wide. A swimmer sets off from A to cross the river, and the path of the swimmer AB is as shown. How far does the person swim?



- 4 An equilateral triangle has altitudes of length 20 cm. Find the length of one side.
- 5 A rope is tied to the top of a flagpole. When it hangs straight down, it is 2 m longer than the pole. When the rope is pulled tight with the lower end on the ground, it makes an angle of 60° to the horizontal. How tall is the flagpole?
- 6 This figure shows a vertical mast PQ , which stands on horizontal ground. A straight wire 20 m long runs from P at the top of the mast to a point R on the ground, which is 10 m from the foot of the mast.
- a Find the angle of inclination, θ° , of the wire to the ground.
- b Find the height of the mast.



- 7 The triangle shown has perimeter 10. Find the value of x .



3C The sine rule

In the previous two sections, we focused on right-angled triangles. In this section and the next, we consider non-right-angled triangles.

The **sine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

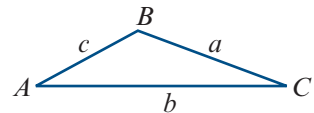
- 1 one side and two angles are given
- 2 two sides and a non-included angle are given (that is, the given angle is not 'between' the two given sides).

In the first case, the triangle is uniquely defined up to congruence. In the second case, there may be two triangles.

Labelling triangles

The following convention is used in the remainder of this chapter:

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

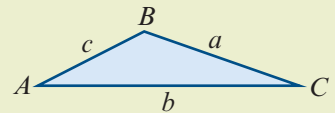


For example, the magnitude of angle BAC is denoted by A , and the length of side BC is denoted by a .

Sine rule

For triangle ABC :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle ACD :

$$\sin A = \frac{h}{b}$$

$$\therefore h = b \sin A$$

In triangle BCD :

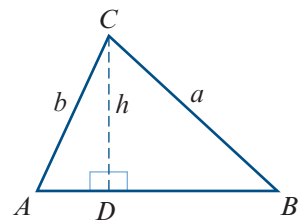
$$\sin B = \frac{h}{a}$$

$$\therefore a \sin B = b \sin A$$

$$\text{i.e. } \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, starting with a perpendicular from A to BC would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

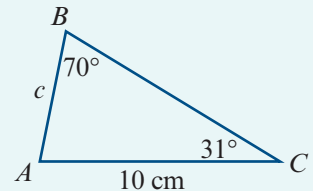


► One side and two angles

When one side and two angles are given, this corresponds to the AAS congruence test. The triangle is uniquely defined up to congruence.

Example 5

Use the sine rule to find the length of AB .



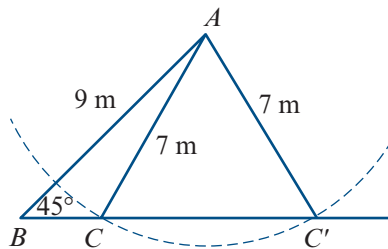
Solution

$$\begin{aligned}\frac{c}{\sin 31^\circ} &= \frac{10}{\sin 70^\circ} \\ \therefore c &= \frac{10 \sin 31^\circ}{\sin 70^\circ} \\ &= 5.4809 \dots\end{aligned}$$

The length of AB is 5.48 cm, correct to two decimal places.

► Two sides and a non-included angle

Suppose that we are given the two side lengths 7 m and 9 m and a non-included angle of 45° . There are two triangles that satisfy these conditions, as shown in the diagram.



Warning

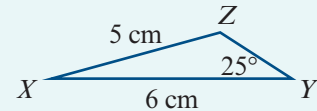
- When you are given two sides and a non-included angle, you must consider the possibility that there are two such triangles.
- An angle found using the sine rule is possible if the sum of the given angle and the found angle is less than 180° .

Note: If the given angle is obtuse or a right angle, then there is only one such triangle.

The following example illustrates the case where there are two possible triangles.

Example 6

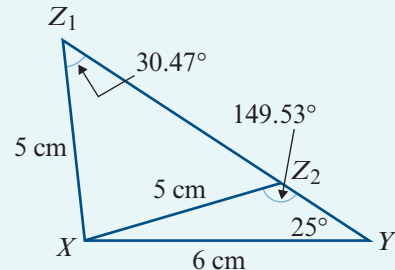
Use the sine rule to find the magnitude of angle XZY in the triangle, given that $Y = 25^\circ$, $y = 5$ cm and $z = 6$ cm.

**Solution**

$$\begin{aligned}\frac{5}{\sin 25^\circ} &= \frac{6}{\sin Z} \\ \frac{\sin Z}{6} &= \frac{\sin 25^\circ}{5} \\ \sin Z &= \frac{6 \sin 25^\circ}{5} \\ &= 0.5071 \dots\end{aligned}$$

$$\therefore Z = (30.473 \dots)^\circ \quad \text{or} \quad Z = (180 - 30.473 \dots)^\circ$$

Hence $Z = 30.47^\circ$ or $Z = 149.53^\circ$, correct to two decimal places.



Note: Remember that $\sin(180 - \theta)^\circ = \sin(\theta)^\circ$.

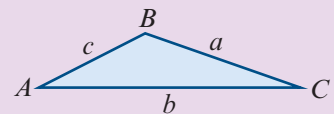
Section summary

■ **Sine rule** For triangle ABC :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

■ When to use the sine rule:

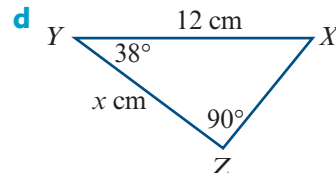
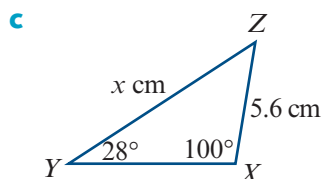
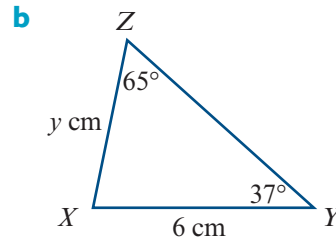
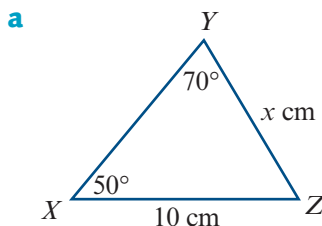
- one side and two angles are given (AAS)
- two sides and a non-included angle are given.

**Exercise 3C**

Skillsheet

1 Find the value of the pronumeral for each of the following triangles:

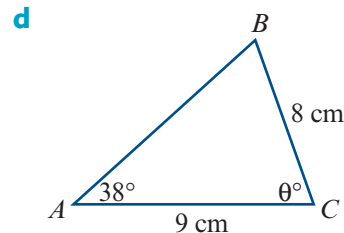
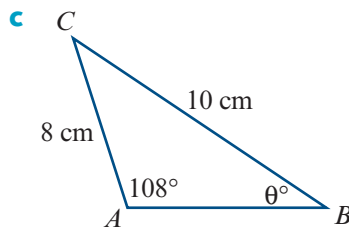
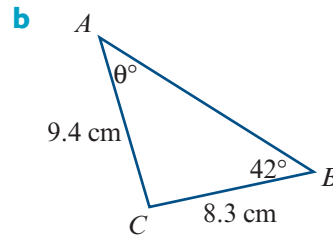
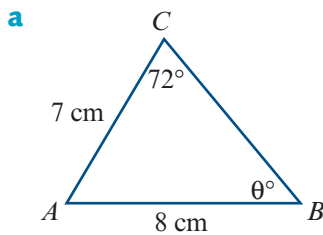
Example 5



SF

Example 6

2 Find the value of θ for each of the following triangles:



3 Solve the following triangles (i.e. find all sides and angles):

a $a = 12$, $B = 59^\circ$, $C = 73^\circ$

b $A = 75.3^\circ$, $b = 5.6$, $B = 48.25^\circ$

c $A = 123.2^\circ$, $a = 11.5$, $C = 37^\circ$

d $A = 23^\circ$, $a = 15$, $B = 40^\circ$

e $B = 140^\circ$, $b = 20$, $A = 10^\circ$

4 Solve the following triangles (i.e. find all sides and angles):

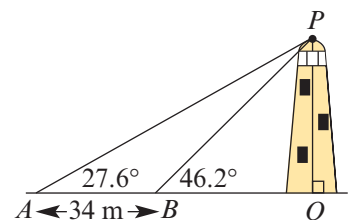
a $b = 17.6$, $C = 48.25^\circ$, $c = 15.3$

b $B = 129^\circ$, $b = 7.89$, $c = 4.56$

c $A = 28.35^\circ$, $a = 8.5$, $b = 14.8$

5 A landmark A is observed from two points B and C , which are 400 m apart. The magnitude of angle ABC is measured as 68° and the magnitude of angle ACB as 70° . Find the distance of A from C .

6 P is a point at the top of a lighthouse. Measurements of the length AB and angles PBO and PAO are as shown in the diagram. Find the height of the lighthouse.

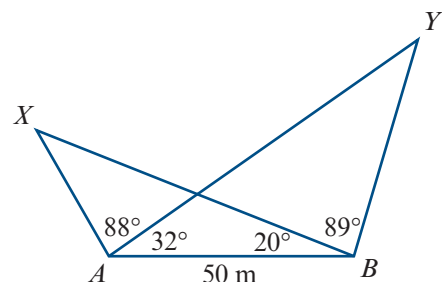


7 A and B are two points on a coastline, and C is a point at sea. The points A and B are 1070 m apart. The angles CAB and CBA have magnitudes of 74° and 69° respectively. Find the distance of C from A .

8 Find:

a AX

b AY



3D The cosine rule

The **cosine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

- 1 two sides and the included angle are given
- 2 three sides are given.

In each case, the triangle is uniquely defined up to congruence.

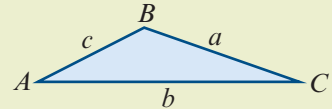
Cosine rule

For triangle ABC :

$$a^2 = b^2 + c^2 - 2bc \cos A$$

or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle ACD :

$$\cos A = \frac{x}{b}$$

$$\therefore x = b \cos A$$

Using Pythagoras' theorem in triangles ACD and BCD :

$$b^2 = x^2 + h^2$$

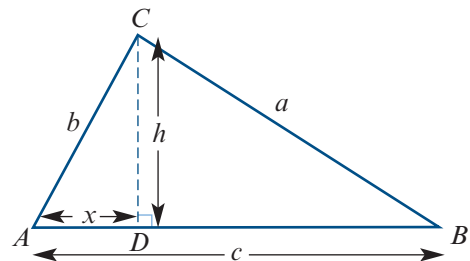
$$a^2 = (c - x)^2 + h^2$$

Expanding gives

$$a^2 = c^2 - 2cx + x^2 + h^2$$

$$= c^2 - 2cx + b^2 \quad (\text{as } b^2 = x^2 + h^2)$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A \quad (\text{as } x = b \cos A)$$



Note: By symmetry, the following results also hold:

$$b^2 = a^2 + c^2 - 2ac \cos B$$

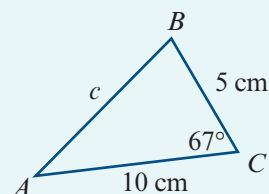
$$c^2 = a^2 + b^2 - 2ab \cos C$$

► Two sides and the included angle

When two sides and the included angle are given, this corresponds to the SAS congruence test. The triangle is uniquely defined up to congruence.

Example 7

For triangle ABC , find the length of AB in centimetres correct to two decimal places.

**Solution**

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$

$$= 85.9268 \dots$$

$$\therefore c = 9.2696 \dots$$

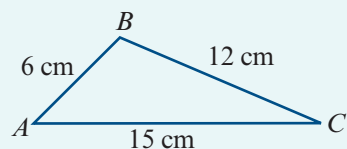
The length of AB is 9.27 cm, correct to two decimal places.

► **Three sides**

When three sides are given, this corresponds to the SSS congruence test. The triangle is uniquely defined up to congruence.

Example 8

Find the magnitude of angle ABC .

**Solution**

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$

$$= -0.3125$$

$$\therefore B = (108.2099 \dots)^\circ$$

The magnitude of angle ABC is 108.21° , correct to two decimal places.

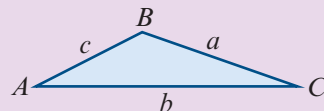
Section summary

- **Cosine rule** For triangle ABC :

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- When to use the cosine rule:

- two sides and the included angle are given (SAS)
- three sides are given (SSS).

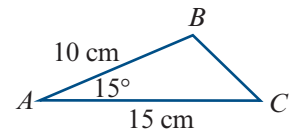


Exercise 3D

Skillsheet

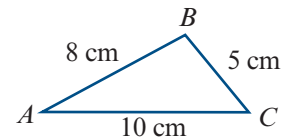
- 1 Find the length of
- BC
- .

Example 7



Example 8

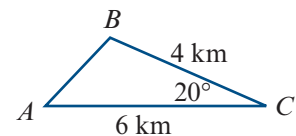
- 2 Find the magnitudes of angles
- ABC
- and
- ACB
- .



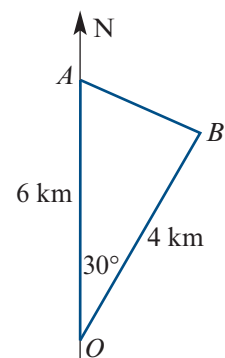
- 3 For triangle
- ABC
- with:

- a** $A = 60^\circ$ $b = 16$ $c = 30$, find a
b $a = 14$ $B = 53^\circ$ $c = 12$, find b
c $a = 27$ $b = 35$ $c = 46$, find the magnitude of angle ABC
d $a = 17$ $B = 120^\circ$ $c = 63$, find b
e $a = 31$ $b = 42$ $C = 140^\circ$, find c
f $a = 10$ $b = 12$ $c = 9$, find the magnitude of angle BCA
g $a = 11$ $b = 9$ $C = 43.2^\circ$, find c
h $a = 8$ $b = 10$ $c = 15$, find the magnitude of angle CBA .

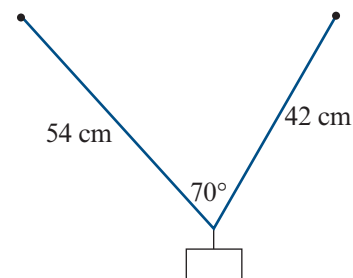
- 4 A section of an orienteering course is as shown. Find the length of leg
- AB
- .



- 5 Two ships sail in different directions from a point
- O
- . At a particular time, their positions
- A
- and
- B
- are as shown. Find the distance between the ships at this time.

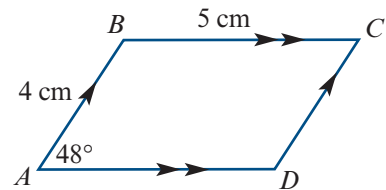


- 6 A weight is hung from two hooks in a ceiling by strings of length
- 54
- cm and
- 42
- cm, which are inclined at
- 70°
- to each other. Find the distance between the hooks.



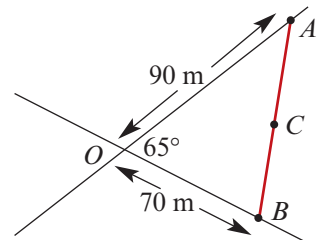
- 7 $ABCD$ is a parallelogram. Find the lengths of the diagonals:

- a AC
b BD



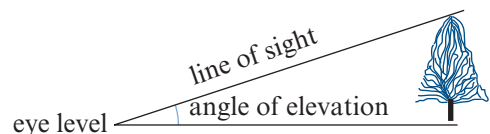
- 8 Two straight roads intersect at an angle of 65° . A point A on one road is 90 m from the intersection and a point B on the other road is 70 m from the intersection, as shown.

- a Find the distance of A from B .
b If C is the midpoint of AB , find the distance of C from the intersection.

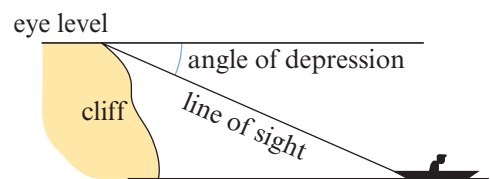


3E Angles of elevation, angles of depression and bearings

The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.



The **angle of depression** is the angle between the horizontal and a direction below the horizontal.



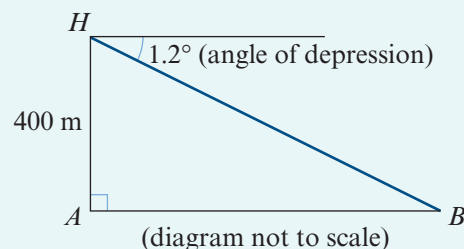
Example 9

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2° . Calculate the horizontal distance of the boat to the helicopter.

Solution

Note that $\angle ABH = 1.2^\circ$, using alternate angles.
Thus

$$\begin{aligned}\frac{400}{AB} &= \tan 1.2^\circ \\ \therefore AB &= \frac{400}{\tan 1.2^\circ} \\ &= 19\,095.800\dots\end{aligned}$$



The horizontal distance is 19 100 m, correct to the nearest 10 m.

**Example 10**

From the point A , a man observes that the angle of elevation of the summit of a hill is 10° . He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of 14° . Find the height of the hill above the level of A .

Solution

Magnitude of $\angle HBA = (180 - 14)^\circ = 166^\circ$

Magnitude of $\angle AHB = (180 - (166 + 10))^\circ = 4^\circ$

Using the sine rule in triangle ABH :

$$\frac{500}{\sin 4^\circ} = \frac{HB}{\sin 10^\circ}$$

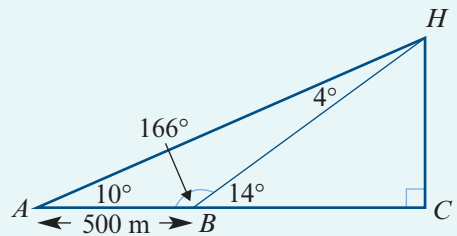
$$\begin{aligned}\therefore HB &= \frac{500 \sin 10^\circ}{\sin 4^\circ} \\ &= 1244.67 \dots\end{aligned}$$

In triangle BCH :

$$\frac{HC}{HB} = \sin 14^\circ$$

$$\begin{aligned}\therefore HC &= HB \sin 14^\circ \\ &= 301.11 \dots\end{aligned}$$

The height of the hill is 301 m, correct to the nearest metre.

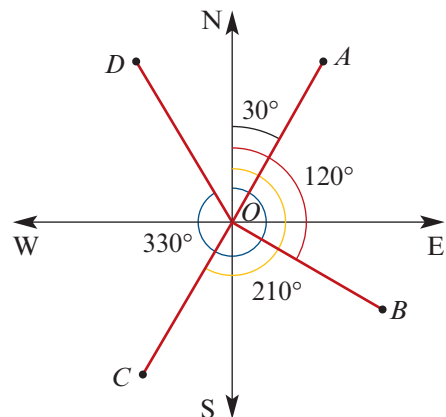


► Bearings

The **bearing** (or compass bearing) is the direction measured from north clockwise.

For example:

- The bearing of A from O is 030° .
- The bearing of B from O is 120° .
- The bearing of C from O is 210° .
- The bearing of D from O is 330° .



Example 11

The road from town A runs due west for 14 km to town B . A television mast is located due south of B at a distance of 23 km. Calculate the distance and bearing of the mast from the centre of town A .

Solution

$$\tan \theta = \frac{23}{14}$$

$$\therefore \theta = 58.67^\circ \quad (\text{to two decimal places})$$

Thus the bearing is

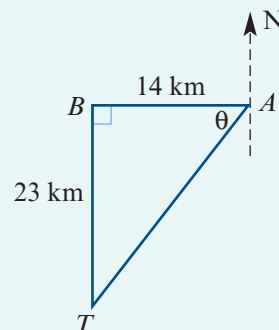
$$180^\circ + (90 - 58.67)^\circ = 211.33^\circ$$

To find the distance, use Pythagoras' theorem:

$$\begin{aligned} AT^2 &= AB^2 + BT^2 \\ &= 14^2 + 23^2 \\ &= 725 \end{aligned}$$

$$\therefore AT = 26.925 \dots$$

The mast is 27 km from the centre of town A (to the nearest kilometre) and on a bearing of 211.33° .

**Example 12**

A yacht starts from a point A and sails on a bearing of 038° for 3000 m. It then alters its course to a bearing of 318° and after sailing for a further 3300 m reaches a point B . Find:

- the distance AB
- the bearing of B from A .

Solution

- The magnitude of angle ACB needs to be found so that the cosine rule can be applied in triangle ABC :

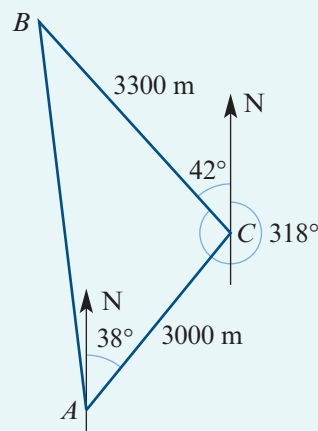
$$\angle ACB = (180 - (38 + 42))^\circ = 100^\circ$$

In triangle ABC :

$$\begin{aligned} AB^2 &= 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \cos 100^\circ \\ &= 23\,328\,233.917 \dots \end{aligned}$$

$$\therefore AB = 4829.931 \dots$$

The distance of B from A is 4830 m (to the nearest metre).

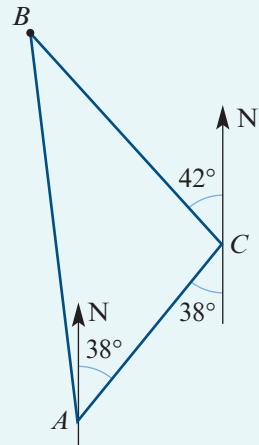


- b** To find the bearing of B from A , the magnitude of angle BAC must first be found. Using the sine rule:

$$\begin{aligned}\frac{3300}{\sin A} &= \frac{AB}{\sin 100^\circ} \\ \therefore \sin A &= \frac{3300 \sin 100^\circ}{AB} \\ &= 0.6728 \dots \\ \therefore A &= (42.288 \dots)^\circ\end{aligned}$$

$$\begin{aligned}\text{The bearing of } B \text{ from } A &= 360^\circ - (42.29^\circ - 38^\circ) \\ &= 355.71^\circ\end{aligned}$$

The bearing of B from A is 356° to the nearest degree.



Exercise 3E

Example 9

- From the top of a vertical cliff 130 m high, the angle of depression of a buoy at sea is 18° . What is the distance of the buoy from the foot of the cliff?
- The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41° . Find the height of the chimney.
- A hiker standing on top of a mountain observes that the angle of depression to the base of a building is 41° . If the height of the hiker above the base of the building is 500 m, find the horizontal distance from the hiker to the building.

Example 10

- A person standing on top of a cliff 50 m high is in line with two buoys whose angles of depression are 18° and 20° . Calculate the distance between the buoys.

Example 11

- A ship sails 10 km north and then sails 15 km east. What is its bearing from the starting point?
- A ship leaves port A and travels 15 km due east. It then turns and travels 22 km due north.

- What is the bearing of the ship from port A ?
- What is the bearing of port A from the ship?

Example 12

- A yacht sails from point A on a bearing of 035° for 2000 m. It then alters course to a direction with a bearing of 320° and after sailing for 2500 m it reaches point B .

- Find the distance AB .
- Find the bearing of B from A .

- The bearing of a point A from a point B is 207° . What is the bearing of B from A ?
- The bearing of a ship S from a lighthouse A is 055° . A second lighthouse B is due east of A . The bearing of S from B is 302° . Find the magnitude of angle ASB .

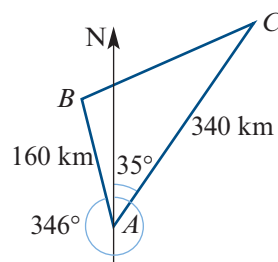
SF

CF

SF

CF

- 10** A yacht starts from L and sails 12 km due east to M . It then sails 9 km on a bearing of 142° to K . Find the magnitude of angle MLK .
- 11** The bearing of C from A is 035° . The bearing of B from A is 346° . The distance of C from A is 340 km. The distance of B from A is 160 km.
- a** Find the magnitude of angle BAC .
- b** Use the cosine rule to find the distance from B to C .
- 12** From a ship S , two other ships P and Q are on bearings 320° and 075° respectively. The distance PS is 7.5 km and the distance QS is 5 km. Find the distance PQ .



3F Angles in the four quadrants

The unit-circle definition of sine and cosine given in Section 3A can be applied to any angle. In this section, we consider the sine and cosine of angles less than 0° and greater than 180° .

► Negative angles

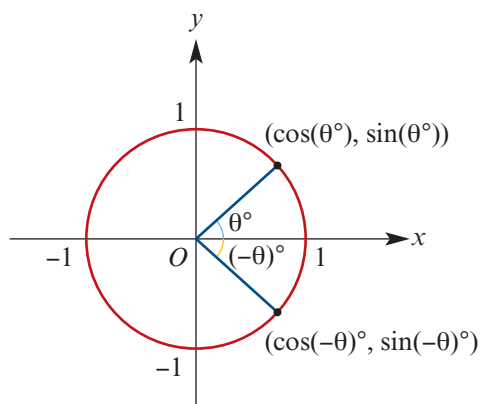
Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

From the diagram shown, we see that

$$\begin{aligned}\sin(-\theta)^\circ &= -\sin(\theta)^\circ \\ \cos(-\theta)^\circ &= \cos(\theta)^\circ\end{aligned}$$

For example:

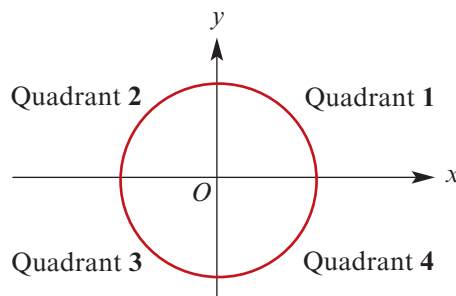
$$\begin{aligned}\sin(-60)^\circ &= -\sin 60^\circ \\ \cos(-60)^\circ &= \cos 60^\circ\end{aligned}$$



► The four quadrants

The coordinate axes divide the unit circle into four quadrants, which are numbered anticlockwise as shown.

You are already familiar with angles in the first and second quadrants. We now consider angles in the third and fourth quadrants.



Third quadrant: angles between 180° and 270°

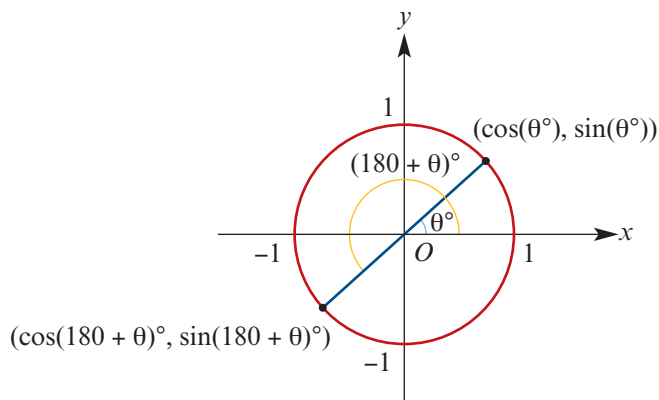
From the unit circle, we see that

$$\begin{aligned}\sin(180 + \theta)^\circ &= -\sin(\theta^\circ) \\ \cos(180 + \theta)^\circ &= -\cos(\theta^\circ)\end{aligned}$$

For example:

$$\begin{aligned}\blacksquare \quad \sin 240^\circ &= \sin(180 + 60)^\circ \\ &= -\sin 60^\circ\end{aligned}$$

$$\begin{aligned}\blacksquare \quad \cos 240^\circ &= \cos(180 + 60)^\circ \\ &= -\cos 60^\circ\end{aligned}$$

**Fourth quadrant: angles between 270° and 360°**

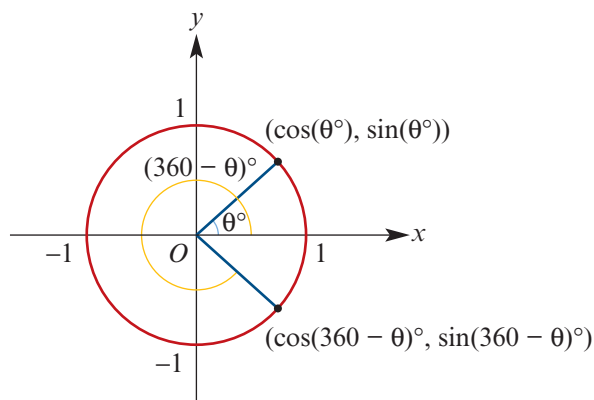
From the unit circle, we see that

$$\begin{aligned}\sin(360 - \theta)^\circ &= -\sin(\theta^\circ) \\ \cos(360 - \theta)^\circ &= \cos(\theta^\circ)\end{aligned}$$

For example:

$$\begin{aligned}\blacksquare \quad \sin 300^\circ &= \sin(360 - 60)^\circ \\ &= -\sin 60^\circ\end{aligned}$$

$$\begin{aligned}\blacksquare \quad \cos 300^\circ &= \cos(360 - 60)^\circ \\ &= \cos 60^\circ\end{aligned}$$

**Example 13**

Find the exact value of each of the following:

a $\sin(-45)^\circ$

b $\cos(-150)^\circ$

Solution

$$\begin{aligned}\mathbf{a} \quad \sin(-45)^\circ &= -\sin 45^\circ \\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \cos(-150)^\circ &= \cos 150^\circ \\ &= \cos(180 - 30)^\circ \\ &= -\cos 30^\circ \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Note: Remember that $\sin(180 - \theta)^\circ = \sin(\theta^\circ)$ and $\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$.

Example 14

If $\cos x^\circ = 0.8$, find the value of:

- a** $\cos(180 - x)^\circ$ **b** $\cos(180 + x)^\circ$ **c** $\cos(360 - x)^\circ$ **d** $\cos(-x)^\circ$

Solution

a $\cos(180 - x)^\circ$	b $\cos(180 + x)^\circ$	c $\cos(360 - x)^\circ$	d $\cos(-x)^\circ$
$= -\cos x^\circ$	$= -\cos x^\circ$	$= \cos x^\circ$	$= \cos x^\circ$
$= -0.8$	$= -0.8$	$= 0.8$	$= 0.8$

Section summary■ **Negative angles**

$$\sin(-\theta)^\circ = -\sin(\theta)^\circ$$

$$\cos(-\theta)^\circ = \cos(\theta)^\circ$$

■ **Second quadrant**

$$\sin(180 - \theta)^\circ = \sin(\theta)^\circ$$

$$\cos(180 - \theta)^\circ = -\cos(\theta)^\circ$$

■ **Third quadrant**

$$\sin(180 + \theta)^\circ = -\sin(\theta)^\circ$$

$$\cos(180 + \theta)^\circ = -\cos(\theta)^\circ$$

■ **Fourth quadrant**

$$\sin(360 - \theta)^\circ = -\sin(\theta)^\circ$$

$$\cos(360 - \theta)^\circ = \cos(\theta)^\circ$$

Exercise 3F**Example 13**

- 1** Without using a calculator, evaluate the sine and cosine of each of the following angles:

a 120°	b 135°	c 180°	d 210°
e 315°	f 330°	g 225°	h 150°
i 360°	j -30°	k -45°	l -120°
m -330°	n -225°	o -315°	p -360°

Example 14

- 2** If $\sin x^\circ = 0.4$ and $\cos \theta^\circ = 0.6$, write down the values of:

a $\sin(180 + x)^\circ$	b $\cos(180 + \theta)^\circ$	c $\sin(360 - x)^\circ$	d $\cos(180 - \theta)^\circ$
e $\sin(-x)^\circ$	f $\cos(360 - \theta)^\circ$	g $\cos(-\theta)^\circ$	h $\sin(-x - 180)^\circ$

- 3**
- a** If $\sin x^\circ = \sin 60^\circ$ and $90 < x < 180$, find the value of x .
- b** If $\sin x^\circ = -\sin 60^\circ$ and $180 < x < 270$, find the value of x .
- c** If $\sin x^\circ = -\sin 60^\circ$ and $-90 < x < 0$, find the value of x .
- d** If $\cos x^\circ = -\cos 60^\circ$ and $90 < x < 180$, find the value of x .
- e** If $\cos x^\circ = -\cos 60^\circ$ and $180 < x < 270$, find the value of x .
- f** If $\cos x^\circ = \cos 60^\circ$ and $270 < x < 360$, find the value of x .



Chapter summary

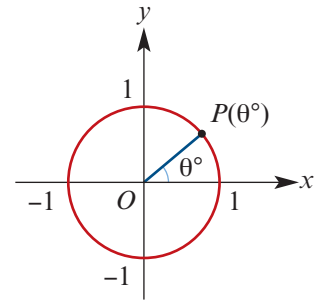


Trigonometric functions

■ Sine and cosine

For each angle θ° , there is a point $P(\theta^\circ)$ on the unit circle as shown. The angle is measured *anticlockwise* from the positive direction of the x -axis.

- $\cos(\theta^\circ)$ is defined as the x -coordinate of the point $P(\theta^\circ)$
- $\sin(\theta^\circ)$ is defined as the y -coordinate of the point $P(\theta^\circ)$



■ Tangent

- $\tan(\theta^\circ) = \frac{\sin(\theta^\circ)}{\cos(\theta^\circ)}$ for $\cos(\theta^\circ) \neq 0$

■ Exact values

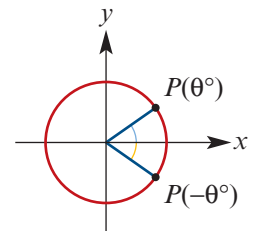
θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

■ Negative angles

Angles measured *clockwise* from the positive direction of the x -axis are defined as *negative*.

$$\sin(-\theta)^\circ = -\sin(\theta)^\circ$$

$$\cos(-\theta)^\circ = \cos(\theta)^\circ$$



■ Angles in the four quadrants

- Second quadrant:

$$\sin(180 - \theta)^\circ = \sin(\theta)^\circ$$

$$\cos(180 - \theta)^\circ = -\cos(\theta)^\circ$$

- Third quadrant:

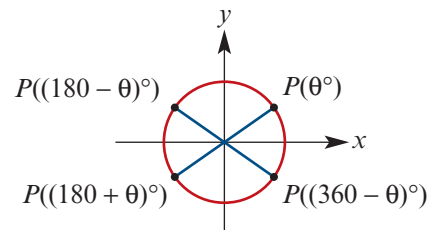
$$\sin(180 + \theta)^\circ = -\sin(\theta)^\circ$$

$$\cos(180 + \theta)^\circ = -\cos(\theta)^\circ$$

- Fourth quadrant:

$$\sin(360 - \theta)^\circ = -\sin(\theta)^\circ$$

$$\cos(360 - \theta)^\circ = \cos(\theta)^\circ$$

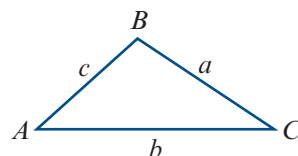


Triangles

■ Labelling triangles

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

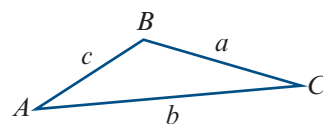
For example, the magnitude of angle BAC is denoted by A , and the length of side BC by a .



■ Sine rule

For triangle ABC :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle in the following cases:

- one side and two angles are given
- two sides and a non-included angle are given.

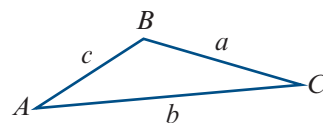
In the first case, the triangle is uniquely defined. But in the second case, there may be two triangles.

■ Cosine rule

For triangle ABC :

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



The symmetrical results also hold:

$$b^2 = a^2 + c^2 - 2ac \cos B$$

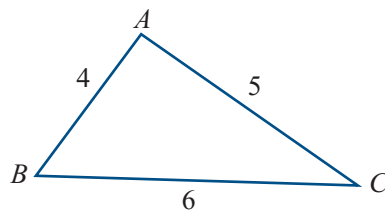
$$c^2 = a^2 + b^2 - 2ab \cos C$$

The cosine rule is used to find unknown quantities in a triangle in the following cases:

- two sides and the included angle are given
- three sides are given.

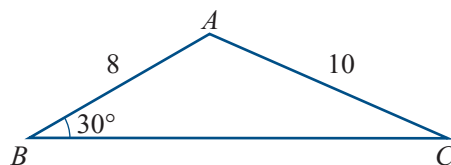
Technology-free questions

- 1 The triangle shown has sides $AB = 4$, $BC = 6$ and $CA = 5$. Find $\cos(\angle BAC)$.



SF

- 2 In triangle ABC , $AB = 8$, $AC = 10$ and $\angle ABC = 30^\circ$. Find $\sin(\angle ACB)$.



- 3 Triangle ABC has $AB = BC = 10$ cm and $\angle ABC = 120^\circ$. Find AC .
- 4 If $\sin x = \sin 37^\circ$ and x is obtuse, find x .
- 5 A point T is 10 km due north of a point S . A point R , which is east of the straight line joining T and S , is 8 km from T and 7 km from S . Calculate the cosine of the bearing of R from S .
- 6 In $\triangle ABC$, $AB = 5$ cm, $\angle BAC = 60^\circ$ and $AC = 6$ cm. Calculate the sine of $\angle ABC$.
- 7 A boat sails 11 km from a harbour on a bearing of 220° . It then sails 15 km on a bearing of 340° . How far is the boat from the harbour?
- 8 A helicopter leaves a heliport A and flies 2.4 km on a bearing of 150° to a checkpoint B . It then flies due east to its base C . If the bearing of C from A is 120° , find the distances AC and BC .

Multiple-choice questions



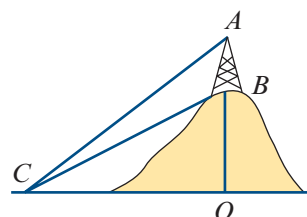
- 1 In a triangle XYZ , $x = 21$ cm, $y = 18$ cm and $\angle YXZ = 62^\circ$. The magnitude of $\angle XYZ$, correct to one decimal place, is
A 0.4° **B** 0.8° **C** 1.0° **D** 49.2° **E** 53.1°
- 2 In a triangle ABC , $a = 30$, $b = 21$ and $\cos C = \frac{51}{53}$. The value of c , to the nearest whole number, is
A 9 **B** 10 **C** 11 **D** 81 **E** 129
- 3 In a triangle ABC , $a = 5.2$ cm, $b = 6.8$ cm and $c = 7.3$ cm. The magnitude of $\angle ACB$, correct to the nearest degree, is
A 43° **B** 63° **C** 74° **D** 82° **E** 98°
- 4 From a point on a cliff 500 m above sea level, the angle of depression to a boat is 20° . The distance from the foot of the cliff to the boat, to the nearest metre, is
A 182 m **B** 193 m **C** 210 m **D** 1374 m **E** 1834 m
- 5 A tower 80 m high is 1.3 km away from a point on the ground. The angle of elevation to the top of the tower from this point, correct to the nearest degree, is
A 1° **B** 4° **C** 53° **D** 86° **E** 89°

- 6** A man walks 5 km due east followed by 7 km due south. The bearing he must take to return to the start is
A 036° **B** 306° **C** 324° **D** 332° **E** 348°
- 7** A boat sails at a bearing of 215° from A to B . The bearing it must take from B to return to A is
A 035° **B** 055° **C** 090° **D** 215° **E** 250°
- 8** If $\sin(\theta^\circ) = -\frac{\sqrt{3}}{2}$, where $270^\circ < \theta^\circ < 360^\circ$, then the angle θ° is
A 285° **B** 300° **C** 315° **D** 330° **E** 345°

Extended-response questions

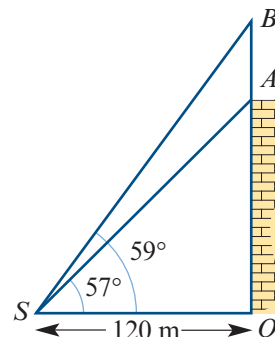
- 1** AB is a tower 60 m high on top of a hill. The magnitude of $\angle ACO$ is 49° and the magnitude of $\angle BCO$ is 37° .

- a** Find the magnitudes of $\angle ACB$, $\angle CBO$ and $\angle CBA$.
b Find the length of BC .
c Find the height of the hill, i.e. the length of OB .



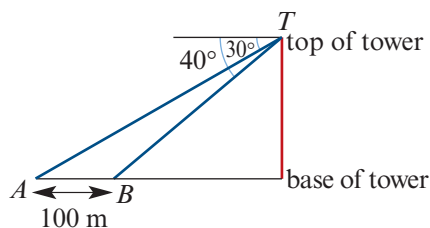
- 2** Point S is a distance of 120 m from the base of a building. On the building is an aerial, AB . The angle of elevation from S to A is 57° . The angle of elevation from S to B is 59° . Find:

- a** the distance OA
b the distance OB
c the distance AB .



- 3** From the top of a communications tower, T , the angles of depression of two points A and B on a horizontal line through the base of the tower are 30° and 40° . The distance between the points is 100 m. Find:

- a** the distance AT
b the distance BT
c the height of the tower.



4

Further algebra

Objectives

- ▶ To understand equality of **polynomials**.
- ▶ To use **equating coefficients** to solve problems.
- ▶ To solve **quadratic equations** by various methods.
- ▶ To use quadratic equations to solve problems involving **rates**.
- ▶ To resolve rational algebraic expressions into **partial fractions**.
- ▶ To find the coordinates of the points of intersection of straight lines with parabolas, circles and rectangular hyperbolas.

This chapter is provided to strengthen the student's algebra and prepare for Unit 4.

In this chapter we first consider equating coefficients of polynomial functions, and then apply this technique to establish partial fractions.

In Chapter 1 we added and subtracted algebraic fractions such as

$$\frac{2}{x+3} + \frac{4}{x-3} = \frac{6(x+1)}{x^2-9}$$

In this chapter we learn how to go from right to left in similar equations. This process is sometimes called **partial fraction decomposition**. Another example is

$$\frac{4x^2 + 2x + 6}{(x^2 + 3)(x - 3)} = \frac{2}{x^2 + 3} + \frac{4}{x - 3}$$

This is a useful tool in integral calculus, and partial fractions are applied this way in Specialist Mathematics Units 3 & 4.

This chapter also includes further study of quadratic functions: solving quadratic equations, using the discriminant, applying quadratic functions to problems involving rates and using quadratic equations to find the intersection of straight lines with parabolas, circles and rectangular hyperbolas.

4A Polynomial identities

Polynomials are introduced in Mathematical Methods Units 1 & 2.

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a natural number or zero, and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.

- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x .)

Two polynomials are **equal** if they give the same value for all x . It can be proved that, if two polynomials are equal, then they have the same degree and corresponding coefficients are equal. For example:

- If $ax + b = cx^2 + dx + e$, then $c = 0$, $d = a$ and $e = b$.
- If $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

This process is called **equating coefficients**.

Example 1

If the expressions $(a + 2b)x^2 - (a - b)x + 8$ and $3x^2 - 6x + 8$ are equal for all x , find the values of a and b .

Solution

Assume that

$$(a + 2b)x^2 - (a - b)x + 8 = 3x^2 - 6x + 8 \quad \text{for all } x$$

Then by equating coefficients:

$$a + 2b = 3 \quad (1)$$

$$-(a - b) = -6 \quad (2)$$

Solve as simultaneous equations.

Add (1) and (2):

$$3b = -3$$

$$\therefore b = -1$$

Substitute into (1):

$$a - 2 = 3$$

$$\therefore a = 5$$

Example 2

Express x^2 in the form $c(x-3)^2 + a(x-3) + d$.

Solution

$$\begin{aligned}\text{Let } x^2 &= c(x-3)^2 + a(x-3) + d \\ &= c(x^2 - 6x + 9) + a(x-3) + d \\ &= cx^2 + (a-6c)x + 9c - 3a + d\end{aligned}$$

This implies that

$$c = 1 \quad (1)$$

$$a - 6c = 0 \quad (2)$$

$$9c - 3a + d = 0 \quad (3)$$

From (2): $a = 6$

From (3): $9 - 18 + d = 0$

i.e. $d = 9$

Hence $x^2 = (x-3)^2 + 6(x-3) + 9$.

**Example 3**

Find the values of a , b , c and d such that

$$x^3 = a(x+2)^3 + b(x+1)^2 + cx + d \quad \text{for all } x$$

Solution

Expand the right-hand side and collect like terms:

$$\begin{aligned}x^3 &= a(x^3 + 6x^2 + 12x + 8) + b(x^2 + 2x + 1) + cx + d \\ &= ax^3 + (6a + b)x^2 + (12a + 2b + c)x + (8a + b + d)\end{aligned}$$

Equate coefficients:

$$a = 1 \quad (1)$$

$$6a + b = 0 \quad (2)$$

$$12a + 2b + c = 0 \quad (3)$$

$$8a + b + d = 0 \quad (4)$$

Substituting $a = 1$ into (2) gives

$$6 + b = 0$$

$$\therefore b = -6$$

Substituting $a = 1$ and $b = -6$ into (3) gives

$$12 - 12 + c = 0$$

$$\therefore c = 0$$

Substituting $a = 1$ and $b = -6$ into (4) gives

$$8 - 6 + d = 0$$

$$\therefore d = -2$$

$$\text{Hence } x^3 = (x + 2)^3 - 6(x + 1)^2 - 2.$$

Example 4

Show that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x + b)^3 + c$.

Solution

Suppose that

$$2x^3 - 5x^2 + 4x + 1 = a(x + b)^3 + c$$

for some constants a , b and c .

Then expanding the right-hand side gives

$$\begin{aligned} 2x^3 - 5x^2 + 4x + 1 &= a(x^3 + 3bx^2 + 3b^2x + b^3) + c \\ &= ax^3 + 3abx^2 + 3ab^2x + ab^3 + c \end{aligned}$$

Equating coefficients:

$$a = 2 \quad (1)$$

$$3ab = -5 \quad (2)$$

$$3ab^2 = 4 \quad (3)$$

$$ab^3 + c = 1 \quad (4)$$

From (2), we have $b = -\frac{5}{6}$. But from (3), we have $b = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$.

This is a contradiction, and therefore we have shown that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x + b)^3 + c$.

Section summary

- A **polynomial function** can be written in the form

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where n is a natural number or zero, and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$. The **leading term** is a_nx^n (the term of highest index) and the **constant term** is a_0 (the term not involving x).

- The **degree** of a polynomial is the index n of the leading term.

Equating coefficients

Two polynomials are equal if they give the same value for all x . If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.

For example: if $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

Exercise 4A

- 1 If $ax^2 + bx + c = 10x^2 - 7$, find the values of a , b and c .

SF

- Example 1** 2 If $(2a - b)x^2 + (a + 2b)x + 8 = 4x^2 - 3x + 8$, find the values of a and b .

- 3 If $(2a - 3b)x^2 + (3a + b)x + c = 7x^2 + 5x + 7$, find the values of a , b and c .

- 4 If $2x^2 + 4x + 5 = a(x + b)^2 + c$, find the values of a , b and c .

- Example 2** 5 Express x^2 in the form $c(x + 2)^2 + a(x + 2) + d$.

- 6 Express x^3 in the form $(x + 1)^3 + a(x + 1)^2 + b(x + 1) + c$.

- Example 3** 7 Find the values of a , b and c such that $x^2 = a(x + 1)^2 + bx + c$.

- Example 4** 8 a Show that $3x^3 - 9x^2 + 8x + 2$ cannot be expressed in the form $a(x + b)^3 + c$.

CF

- b If $3x^3 - 9x^2 + 9x + 2$ can be expressed in the form $a(x + b)^3 + c$, then find the values of a , b and c .

- 9 Show that constants a , b , c and d can be found such that

$$n^3 = a(n + 1)(n + 2)(n + 3) + b(n + 1)(n + 2) + c(n + 1) + d$$

- 10 a Show that no constants a and b can be found such that

$$n^2 = a(n + 1)(n + 2) + b(n + 2)(n + 3)$$

- b Express n^2 in the form $a(n + 1)(n + 2) + b(n + 1) + c$.

- 11 a Express $a(x + b)^2 + c$ in expanded form.

SF

- b Express $ax^2 + bx + c$ in completed-square form.

- 12 Prove that, if $ax^3 + bx^2 + cx + d = (x - 1)^2(px + q)$, then $b = d - 2a$ and $c = a - 2d$.

CF

- 13 If $3x^2 + 10x + 3 = c(x - a)(x - b)$ for all values of x , find the values of a , b and c .

SF

- 14 For any number n , show that n^2 can be expressed as $a(n - 1)^2 + b(n - 2)^2 + c(n - 3)^2$, and find the values of a , b and c .

CF

- 15 If $x^3 + 3x^2 - 9x + c$ can be expressed in the form $(x - a)^2(x - b)$, show that either $c = 5$ or $c = -27$, and find a and b for each of these cases.

- 16 A polynomial P is said to be **even** if $P(-x) = P(x)$ for all x . A polynomial P is said to be **odd** if $P(-x) = -P(x)$ for all x .

- a Show that, if $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, then $b = d = 0$.

- b Show that, if $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, then $b = d = f = 0$.

4B Quadratic equations

A polynomial function of degree 2 is called a **quadratic function**. The general quadratic function can be written as $P(x) = ax^2 + bx + c$, where $a \neq 0$.

Quadratic functions are studied extensively in Mathematical Methods Units 1 & 2. In this section we provide further practice exercises.

A quadratic equation $ax^2 + bx + c = 0$ may be solved by factorising, by completing the square or by using the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The following example demonstrates each method.

Example 5

Solve the following quadratic equations for x :

a $2x^2 + 5x = 12$ **b** $3x^2 + 4x = 2$ **c** $9x^2 + 6x + 1 = 0$

Solution

a $2x^2 + 5x - 12 = 0$
 $(2x - 3)(x + 4) = 0$
 $2x - 3 = 0$ or $x + 4 = 0$
 Therefore $x = \frac{3}{2}$ or $x = -4$.

b $3x^2 + 4x - 2 = 0$
 $x^2 + \frac{4}{3}x - \frac{2}{3} = 0$
 $x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 - \frac{2}{3} = 0$
 $\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{2}{3} = 0$
 $\left(x + \frac{2}{3}\right)^2 = \frac{10}{9}$
 $x + \frac{2}{3} = \pm \frac{\sqrt{10}}{3}$
 $x = -\frac{2}{3} \pm \frac{\sqrt{10}}{3}$
 Therefore $x = \frac{-2 + \sqrt{10}}{3}$ or $x = \frac{-2 - \sqrt{10}}{3}$.

Explanation

Rearrange the quadratic equation.
 Factorise.
 Use the null factor theorem.

Rearrange the quadratic equation.
 Divide both sides by 3.

Add and subtract $\left(\frac{b}{2}\right)^2$ to 'complete the square'.

c If $9x^2 + 6x + 1 = 0$, then

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4 \times 9 \times 1}}{2 \times 9} \\ &= \frac{-6 \pm \sqrt{0}}{18} \\ &= -\frac{1}{3} \end{aligned}$$

Use the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Alternatively, the equation can be solved by noting that $9x^2 + 6x + 1 = (3x + 1)^2$.

The discriminant: real solutions

The number of solutions to a quadratic equation $ax^2 + bx + c = 0$ can be determined by the **discriminant** Δ , where $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the equation has two real solutions.
- If $\Delta = 0$, the equation has one real solution.
- If $\Delta < 0$, the equation has no real solutions.

Note: In parts **a** and **b** of Example 5, we have $\Delta > 0$ and so there are two real solutions. In part **c**, we have $\Delta = 6^2 - 4 \times 9 \times 1 = 0$ and so there is only one real solution.

The discriminant: rational solutions

For a quadratic equation $ax^2 + bx + c = 0$ such that a , b and c are rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Note: In part **a** of Example 5, we have $\Delta = 121$, which is a perfect square.

Example 6

Consider the quadratic equation $x^2 - 4x = t$. Make x the subject and give the values of t for which real solution(s) to the equation can be found.

Solution

$$\begin{aligned} x^2 - 4x &= t \\ x^2 - 4x + 4 &= t + 4 && \text{(completing the square)} \\ (x - 2)^2 &= t + 4 \\ x - 2 &= \pm \sqrt{t + 4} \\ x &= 2 \pm \sqrt{t + 4} \end{aligned}$$

For real solutions to exist, we must have $t + 4 \geq 0$, i.e. $t \geq -4$.

Note: In this case the discriminant is $\Delta = 16 + 4t$. There are real solutions when $\Delta \geq 0$.



Example 7

- Find the discriminant of the quadratic $x^2 + px - \frac{25}{4}$ in terms of p .
- Solve the quadratic equation $x^2 + px - \frac{25}{4} = 0$ in terms of p .
- Prove that there are two solutions for all values of p .
- Find the values of p , where p is a non-negative integer, for which the quadratic equation has rational solutions.

Solution

Here we have $a = 1$, $b = p$ and $c = -\frac{25}{4}$.

- $\Delta = b^2 - 4ac = p^2 + 25$
- The quadratic formula gives $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-p \pm \sqrt{p^2 + 25}}{2}$.
- We have $\Delta = p^2 + 25 > 0$, for all values of p . Thus there are always two solutions.
- If there are rational solutions, then $\Delta = p^2 + 25$ is a perfect square. Since p is an integer, we can write $p^2 + 25 = k^2$, where k is an integer with $k \geq 0$.

Rearranging, we have

$$k^2 - p^2 = 25$$

$$\therefore (k - p)(k + p) = 25$$

We can factorise 25 as 5×5 or 1×25 .

Note: We do not need to consider negative factors of 25, as p and k are non-negative, and so $k + p \geq 0$. Since p is non-negative, we also know that $k - p \leq k + p$.

The table on the right shows the values of k and p in each of the two cases.

Hence $p = 0$ and $p = 12$ are the only values for which the solutions are rational.

$k - p$	$k + p$	k	p
5	5	5	0
1	25	13	12

Example 8

Solve the equation $x - 4\sqrt{x} - 12 = 0$ for x .

Solution

For $\sqrt{x}$ to be defined, we must have $x \geq 0$.

Let $x = a^2$, where $a \geq 0$. The equation becomes

$$a^2 - 4\sqrt{a^2} - 12 = 0$$

$$a^2 - 4a - 12 = 0$$

$$(a - 6)(a + 2) = 0$$

$$\therefore a = 6 \text{ or } a = -2$$

But $a \geq 0$. Hence $a = 6$ and so $x = 36$.

Example 9

A rectangle has an area of 288 cm^2 . If the width is decreased by 1 cm and the length increased by 1 cm, the area would be decreased by 3 cm^2 . Find the original dimensions of the rectangle.

Solution

Let w and ℓ be the width and length, in centimetres, of the original rectangle.

Then $w\ell = 288$ (1)

The dimensions of the new rectangle are $w - 1$ and $\ell + 1$, and the area is 285 cm^2 .

Thus $(w - 1)(\ell + 1) = 285$ (2)

Rearranging (1) to make ℓ the subject and substituting in (2) gives

$$(w - 1)\left(\frac{288}{w} + 1\right) = 285$$

$$288 - \frac{288}{w} + w - 1 = 285$$

$$w - \frac{288}{w} + 2 = 0$$

$$w^2 + 2w - 288 = 0$$

Using the general quadratic formula gives

$$\begin{aligned} w &= \frac{-2 \pm \sqrt{2^2 - 4 \times (-288)}}{2} \\ &= \frac{-2 \pm 34}{2} \\ &= -18 \text{ or } 16 \end{aligned}$$

But $w > 0$, and so $w = 16$. The original dimensions of the rectangle are 16 cm by 18 cm.

Section summary

- Quadratic equations can be solved by completing the square. This method allows us to deal with all quadratic equations, even though some have no solutions.
- To complete the square of $x^2 + bx + c$:
 - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
 - First take out a as a factor and then complete the square inside the bracket.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

For the equation $ax^2 + bx + c = 0$:

- If $\Delta > 0$, there are two solutions.
 - If $\Delta = 0$, there is one solution.
 - If $\Delta < 0$, there are no real solutions.
- For the equation $ax^2 + bx + c = 0$, where a , b and c are rational numbers:
- If Δ is a perfect square and $\Delta \neq 0$, there are two rational solutions.
 - If $\Delta = 0$, there is one rational solution.
 - If Δ is not a perfect square and $\Delta > 0$, there are two irrational solutions.

Exercise 4B

Skillsheet

Example 5

- 1 Solve the following quadratic equations for x :

a $x^2 + 2x = -1$

b $x^2 - 6x + 9 = 0$

c $5x^2 - 10x = 1$

d $-2x^2 + 4x = 1$

e $2x^2 + 4x = 7$

f $6x^2 + 13x + 1 = 0$

- 2 The following equations have the number of solutions shown in brackets. Find the possible values of m .

a $x^2 + 3x + m = 0$ (0)

b $x^2 - 5x + m = 0$ (2)

c $mx^2 + 5x - 8 = 0$ (1)

d $x^2 + mx + 9 = 0$ (2)

e $x^2 - mx + 4 = 0$ (0)

f $4x^2 - mx - m = 0$ (1)

Example 6

- 3 Make x the subject in each of the following and give the values of t for which real solution(s) to the equation can be found:

a $2x^2 - 4t = x$

b $4x^2 + 4x - 4 = t - 2$

c $5x^2 + 4x + 10 = t$

d $tx^2 + 4tx + 10 = t$

Example 7

- 4 **a** Solve the quadratic equation $x^2 + px - 16 = 0$ in terms of p .
b Find the values of p , where p is an integer with $0 \leq p \leq 10$, for which the quadratic equation in **a** has rational solutions.
- 5 **a** Show that the solutions of the equation $2x^2 - 3px + (3p - 2) = 0$ are rational for all integer values of p .
b Find the value of p for which there is only one solution.
c Solve the equation when:
i $p = 1$ **ii** $p = 2$ **iii** $p = -1$
- 6 **a** Show that the solutions of the equation $4(4p - 3)x^2 - 8px + 3 = 0$ are rational for all integer values of p .
b Find the value of p for which there is only one solution.
c Solve the equation when:
i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

Example 7 Solve each of the following equations for x :

a $x - 8\sqrt{x} + 12 = 0$

b $x - 8 = 2\sqrt{x}$

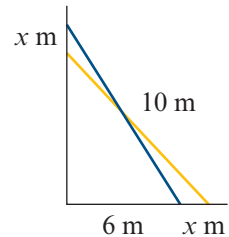
c $x - 5\sqrt{x} - 14 = 0$

d $\sqrt[3]{x^2} - 9\sqrt[3]{x} + 8 = 0$

e $\sqrt[3]{x^2} - \sqrt[3]{x} - 6 = 0$

f $x - 29\sqrt{x} + 100 = 0$

Example 8 A pole 10 m long leans against a wall. The bottom of the pole is 6 m from the wall. If the bottom of the pole is pulled away x m so that the top slides down by the same amount, find x .



- 9** A wire of length 200 cm is cut into two parts and each part is bent to form a square. If the area of the larger square is 9 times the area of the smaller square, find the length of the sides of the larger square.
- 10** Find constants a , b and c such that $3x^2 - 5x + 1 = a(x + b)^2 + c$ for all values of x . Hence find the minimum value of $3x^2 - 5x + 1$.
- 11** Show that the graphs of $y = 2 - 4x - x^2$ and $y = 24 + 8x + x^2$ do not intersect.
- 12** Solve the quadratic equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ for x .
- 13** The two solutions of the equation $2x^2 - 6x - m = 0$ differ by 5. Find the value of m .
- 14** For the equation $(b^2 - 2ac)x^2 + 4(a + c)x = 8$:
- a** Prove that there are always (real) solutions of the equation.
- b** Find the conditions for which there is only one solution.
- 15** The equation $\frac{1}{2} + \frac{1}{x+k} = \frac{1}{x}$ has no solutions. Find the possible values of k .
- 16** Find the smallest positive integer p for which the equation $3x^2 + px + 7 = 0$ has solutions.

4C Applying quadratic equations to rate problems

A **rate** describes how a certain quantity changes with respect to the change in another quantity (often time). An example of a rate is 'speed'. A speed of 60 km/h gives us a measure of how fast an object is travelling. A further example is 'flow', where a rate of 20 L/min is going to fill an empty swimming pool faster than a rate of 6 L/min.

Many problems are solved using rates, which can be expressed as fractions. For example, a speed of 60 km/h can be expressed in fraction form as

$$\frac{\text{distance (km)}}{\text{time taken (h)}} = \frac{60}{1}$$

When solving rate problems, it is often necessary to add two or more fractions with different denominators, as shown in the following examples.

Example 10

- a** Express $\frac{6}{x} + \frac{6}{x+8}$ as a single fraction. **b** Solve the equation $\frac{6}{x} + \frac{6}{x+8} = 2$ for x .

Solution

$$\begin{aligned} \mathbf{a} \quad \frac{6}{x} + \frac{6}{x+8} &= \frac{6(x+8)}{x(x+8)} + \frac{6x}{x(x+8)} \\ &= \frac{6x+48+6x}{x(x+8)} \\ &= \frac{12(x+4)}{x(x+8)} \end{aligned}$$

$$\mathbf{b} \quad \text{Since } \frac{6}{x} + \frac{6}{x+8} = \frac{12(x+4)}{x(x+8)}, \text{ we have}$$

$$\frac{12(x+4)}{x(x+8)} = 2$$

$$12(x+4) = 2x(x+8)$$

$$6(x+4) = x(x+8)$$

$$6x+24 = x^2+8x$$

$$x^2+2x-24 = 0$$

$$(x+6)(x-4) = 0$$

$$\text{Therefore } x = -6 \text{ or } x = 4.$$

**Example 11**

A car travels 500 km at a constant speed. If it had travelled at a speed of 10 km/h less, it would have taken 1 hour more to travel the distance. Find the speed of the car.

Solution

Let x km/h be the speed of the car.

It takes $\frac{500}{x}$ hours for the journey.

If the speed is 10 km/h less, then the new speed is $(x-10)$ km/h.

The time taken is $\frac{500}{x-10} + 1$ hours.

We can now write:

$$500 = (x-10) \times \left(\frac{500}{x-10} + 1 \right)$$

$$500(x-10) = (x-10)(500+x)$$

$$500x - 5000 = 500x - 5000 + x^2 - 10x$$

Thus

$$x^2 - 10x - 5000 = 0$$

and so

$$x = \frac{10 \pm \sqrt{100 + 4 \times 5000}}{2}$$

$$= 5(1 \pm \sqrt{201})$$

The speed is $5(1 + \sqrt{201}) \approx 75.887$ km/h.

Explanation

For an object travelling at a constant speed in one direction:

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

and so

$$\text{time taken} = \frac{\text{distance travelled}}{\text{speed}}$$

and

$$\text{distance travelled} = \text{speed} \times \text{time taken}$$

Example 12

A tank is filled by two pipes. The smaller pipe alone will take 24 minutes longer than the larger pipe alone, and 32 minutes longer than when both pipes are used. How long will each pipe take to fill the tank alone? How long will it take for both pipes used together to fill the tank?

Solution

Let C cubic units be the capacity of the tank, and let x minutes be the time it takes for the larger pipe alone to fill the tank.

Then the average rate of flow for the larger pipe is $\frac{C}{x}$ cubic units per minute.

Since the smaller pipe alone takes $(x + 24)$ minutes to fill the tank, the average rate of flow for the smaller pipe is $\frac{C}{x + 24}$ cubic units per minute.

The average rate of flow when both pipes are used together is the sum of these two rates:

$$\frac{C}{x} + \frac{C}{x + 24} \text{ cubic units per minute}$$

Expressed as a single fraction:

$$\begin{aligned} \frac{C}{x} + \frac{C}{x + 24} &= \frac{C(x + 24) + Cx}{x(x + 24)} \\ &= \frac{2C(x + 12)}{x(x + 24)} \end{aligned}$$

The time taken to fill the tank using both pipes is

$$\begin{aligned} C \div \frac{2C(x + 12)}{x(x + 24)} &= C \times \frac{x(x + 24)}{2C(x + 12)} \\ &= \frac{x(x + 24)}{2(x + 12)} \end{aligned}$$

Therefore the time taken for the smaller pipe alone to fill the tank can be also be expressed as $\frac{x(x + 24)}{2(x + 12)} + 32$ minutes.

$$\text{Thus } \frac{x(x + 24)}{2(x + 12)} + 32 = x + 24$$

$$\frac{x(x + 24)}{2(x + 12)} = x - 8$$

$$x(x + 24) = 2(x + 12)(x - 8)$$

$$x^2 + 24x = 2x^2 + 8x - 192$$

$$x^2 - 16x - 192 = 0$$

$$(x - 24)(x + 8) = 0$$

But $x > 0$, and hence $x = 24$.

It takes 24 minutes for the larger pipe alone to fill the tank, 48 minutes for the smaller pipe alone to fill the tank, and 16 minutes for both pipes together to fill the tank.

Exercise 4C

Skillsheet

- 1 a Express $\frac{6}{x} - \frac{6}{x+3}$ as a single fraction.

Example 10

- b Solve the equation $\frac{6}{x} - \frac{6}{x+3} = 1$ for x .

- 2 Solve the equation $\frac{300}{x+5} = \frac{300}{x} - 2$ for x .

- 3 The sum of the reciprocals of two consecutive odd numbers is $\frac{36}{323}$. Form a quadratic equation and hence determine the two numbers.

Example 11

- 4 A cyclist travels 40 km at a speed of x km/h.

- a Find the time taken in terms of x .
 b Find the time taken when his speed is reduced by 2 km/h.
 c If the difference between the times is 1 hour, find his original speed.

- 5 A car travels from town A to town B, a distance of 600 km, in x hours. A plane, travelling 220 km/h faster than the car, takes $5\frac{1}{2}$ hours less to cover the same distance.

- a Express, in terms of x , the average speed of the car and the average speed of the plane.
 b Find the actual average speed of each of them.

- 6 A car covers a distance of 200 km at a speed of x km/h. A train covers the same distance at a speed of $(x+5)$ km/h. If the time taken by the car is 2 hours more than that taken by the train, find x .

- 7 A man travels 108 km, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled at an average speed 2 km/h faster. What was the man's average speed when he made the trip?

- 8 A bus is due to reach its destination 75 km away at a certain time. The bus usually travels with an average speed of x km/h. Its start is delayed by 18 minutes but, by increasing its average speed by 12.5 km/h, the driver arrives on time.

- a Find x . b How long did the journey actually take?

- 9 Ten minutes after the departure of an express train, a slow train starts, travelling at an average speed of 20 km/h less. The slow train reaches a station 250 km away 3.5 hours after the arrival of the express. Find the average speed of each of the trains.

- 10 When the average speed of a car is increased by 10 km/h, the time taken for the car to make a journey of 105 km is reduced by 15 minutes. Find the original average speed.

- 11 A tank can be filled with water by two pipes running together in $11\frac{1}{9}$ minutes. If the larger pipe alone takes 5 minutes less to fill the tank than the smaller pipe, find the time that each pipe will take to fill the tank.

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CF

- Example 12** **12** At first two different pipes running together will fill a tank in $\frac{20}{3}$ minutes. The rate that water runs through each of the pipes is then adjusted. If one pipe, running alone, takes 1 minute less to fill the tank at its new rate, and the other pipe, running alone, takes 2 minutes more to fill the tank at its new rate, then the two running together will fill the tank in 7 minutes. Find in what time the tank will be filled by each pipe running alone at the new rates.
- 13** The journey between two towns by one route consists of 233 km by rail followed by 126 km by sea. By a second route the journey consists of 405 km by rail followed by 39 km by sea. If the time taken for the first route is 50 minutes longer than for the second route, and travelling by rail is 25 km/h faster than travelling by sea, find the average speed by rail and the average speed by sea.
- 14** A sea freighter travelling due north at 12 km/h sights a cruiser straight ahead at an unknown distance and travelling due east at unknown speed. After 15 minutes the vessels are 10 km apart and then, 15 minutes later, they are 13 km apart. (Assume that both travel at constant speeds.) How far apart are the vessels when the cruiser is due east of the freighter?
- 15** Cask A, which has a capacity of 20 litres, is filled with wine. A certain quantity of wine from cask A is poured into cask B, which also has a capacity of 20 litres. Cask B is then filled with water. After this, cask A is filled with some of the mixture from cask B. A further $\frac{20}{3}$ litres of the mixture now in A is poured back into B, and the two casks now have the same amount of wine. How much wine was first taken out of cask A?
- 16** Two trains travel between two stations 80 km apart. If train A travels at an average speed of 5 km/h faster than train B and completes the journey 20 minutes faster, find the average speeds of the two trains, giving your answers correct to two decimal places.
- 17** A tank is filled by two pipes. The smaller pipe running alone will take 24 minutes longer than the larger pipe alone, and a minutes longer than when both pipes are running together.
- a** Find, in terms of a , how long each pipe takes to fill the tank.
 - b** Find how long each pipe takes to fill the tank when:
 - i** $a = 49$ **ii** $a = 32$ **iii** $a = 27$ **iv** $a = 25$
- 18** Train A leaves Armadale and travels at constant speed to Bundong, which is a town 300 km from Armadale. At the same time, train B leaves Bundong and travels at constant speed to Armadale. They meet at a town Yunga, which is between the two towns. Nine hours after leaving Yunga, train A reaches Bundong, and four hours after leaving Yunga, train B reaches Armadale.
- a** Find the distance of Yunga from Armadale.
 - b** Find the speed of each of the trains.

4D Partial fractions

A **rational function** is the quotient of two polynomials. If $P(x)$ and $Q(x)$ are polynomials, then $f(x) = \frac{P(x)}{Q(x)}$ is a rational function; e.g. $f(x) = \frac{4x+2}{x^2-1}$.

- If the degree of $P(x)$ is less than the degree of $Q(x)$, then $f(x)$ is a **proper fraction**.
- If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then $f(x)$ is an **improper fraction**.

By convention, we consider a rational function for its maximal domain. For example, the function $f(x) = \frac{4x+2}{x^2-1}$ is only defined for $x \in \mathbb{R} \setminus \{-1, 1\}$.

A rational function may be expressed as a sum of simpler functions by resolving it into what are called **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

This technique can help when sketching the graphs of rational functions or when performing other mathematical procedures such as integration.

► Proper fractions

For proper fractions, the technique used for obtaining partial fractions depends on the type of factors in the denominator of the original algebraic fraction. We only consider examples where the denominators have factors that are either degree 1 (linear) or degree 2 (quadratic).

- For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\frac{A}{ax+b}$.
- For every repeated linear factor $(cx + d)^2$ in the denominator, there will be partial fractions of the form $\frac{B}{cx+d}$ and $\frac{C}{(cx+d)^2}$.
- For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\frac{Dx+E}{ax^2+bx+c}$.

Note: A quadratic expression is said to be **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, both $x^2 + 1$ and $x^2 + 4x + 10$ are irreducible.

To resolve an algebraic fraction into its partial fractions:

- Step 1** Write a statement of identity between the original fraction and a sum of the appropriate number of partial fractions.
- Step 2** Express the sum of the partial fractions as a single fraction, and note that the numerators of both sides are equivalent.
- Step 3** Find the values of the introduced constants $A, B, C, \dots$ by substituting appropriate values for x or by equating coefficients.

**Example 13**

Resolve $\frac{3x+5}{(x-1)(x+3)}$ into partial fractions.

Solution**Method 1**

Let

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \quad (1)$$

for all $x \in \mathbb{R} \setminus \{1, -3\}$.

Express the right-hand side as a single fraction:

$$\begin{aligned} \frac{3x+5}{(x-1)(x+3)} &= \frac{A(x+3) + B(x-1)}{(x-1)(x+3)} \\ \therefore \frac{3x+5}{(x-1)(x+3)} &= \frac{(A+B)x + 3A - B}{(x-1)(x+3)} \end{aligned}$$

$$\therefore 3x + 5 = (A+B)x + 3A - B$$

Equate coefficients:

$$A + B = 3$$

$$3A - B = 5$$

Solving these equations simultaneously gives

$$4A = 8$$

and so $A = 2$ and $B = 1$.

Therefore

$$\frac{3x+5}{(x-1)(x+3)} = \frac{2}{x-1} + \frac{1}{x+3}$$

Method 2

From equation (1) we can write:

$$3x + 5 = A(x+3) + B(x-1) \quad (2)$$

Substitute $x = 1$ in equation (2):

$$8 = 4A$$

$$\therefore A = 2$$

Substitute $x = -3$ in equation (2):

$$-4 = -4B$$

$$\therefore B = 1$$

Explanation

Since the denominator has two linear factors, there will be two partial fractions of the form $\frac{A}{x-1}$ and $\frac{B}{x+3}$.

We know that equation (2) is true for all $x \in \mathbb{R} \setminus \{1, -3\}$.

But if this is the case, then it also has to be true for $x = 1$ and $x = -3$.

Note: You could substitute any values of x to find A and B in this way, but these values simplify the calculations.

Example 14

Resolve $\frac{2x+10}{(x+1)(x-1)^2}$ into partial fractions.

Solution

Since the denominator has a repeated linear factor and a single linear factor, there are three partial fractions:

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\therefore \frac{2x+10}{(x+1)(x-1)^2} = \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

This gives the equation

$$2x+10 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

We will use a combination of methods to find A , B and C .

$$\text{Let } x = 1: \quad 2(1) + 10 = C(1+1)$$

$$12 = 2C$$

$$\therefore C = 6$$

$$\text{Let } x = -1: \quad 2(-1) + 10 = A(-1-1)^2$$

$$8 = 4A$$

$$\therefore A = 2$$

Now substitute these values for A and C :

$$\begin{aligned} 2x+10 &= 2(x-1)^2 + B(x+1)(x-1) + 6(x+1) \quad (1) \\ &= 2(x^2 - 2x + 1) + B(x^2 - 1) + 6(x+1) \\ &= (2+B)x^2 + 2x + 8 - B \end{aligned}$$

Equate coefficients:

$$2 + B = 0$$

$$8 - B = 10$$

Therefore $B = -2$ and hence

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{2}{x+1} - \frac{2}{x-1} + \frac{6}{(x-1)^2}$$

Alternatively, the value of B could be found by substituting $x = 0$ into equation (1).

Note: In Exercise 4D, you will show that it is impossible to find A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

Example 15

Resolve $\frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)}$ into partial fractions.

Solution

Since the denominator has a single linear factor and an irreducible quadratic factor (i.e. cannot be reduced to linear factors), there are two partial fractions:

$$\frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + x + 1}$$

$$\therefore \frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)} = \frac{A(x^2 + x + 1) + (Bx + C)(x - 2)}{(x - 2)(x^2 + x + 1)}$$

This gives the equation

$$x^2 + 6x + 5 = A(x^2 + x + 1) + (Bx + C)(x - 2) \quad (1)$$

Substituting $x = 2$:

$$2^2 + 6(2) + 5 = A(2^2 + 2 + 1)$$

$$21 = 7A$$

$$\therefore A = 3$$

We can rewrite equation (1) as

$$\begin{aligned} x^2 + 6x + 5 &= A(x^2 + x + 1) + (Bx + C)(x - 2) \\ &= A(x^2 + x + 1) + Bx^2 - 2Bx + Cx - 2C \\ &= (A + B)x^2 + (A - 2B + C)x + A - 2C \end{aligned}$$

Since $A = 3$, this gives

$$x^2 + 6x + 5 = (3 + B)x^2 + (3 - 2B + C)x + 3 - 2C$$

Equate coefficients:

$$3 + B = 1 \quad \text{and} \quad 3 - 2C = 5$$

$$\therefore B = -2 \quad \therefore C = -1$$

Check: $3 - 2B + C = 3 - 2(-2) + (-1) = 6$

Therefore

$$\begin{aligned} \frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)} &= \frac{3}{x - 2} + \frac{-2x - 1}{x^2 + x + 1} \\ &= \frac{3}{x - 2} - \frac{2x + 1}{x^2 + x + 1} \end{aligned}$$

Note: The values of B and C could also be found by substituting $x = 0$ and $x = 1$ in equation (1).

► Improper fractions

Improper algebraic fractions can be expressed as a sum of partial fractions by first dividing the denominator into the numerator to produce a quotient and a proper fraction. This proper fraction can then be resolved into its partial fractions using the techniques just introduced.

Example 16

Express $\frac{x^5 + 2}{x^2 - 1}$ as partial fractions.

Solution

Dividing through:

$$\begin{array}{r}
 x^3 + x \\
 x^2 - 1 \overline{) x^5 + 2} \\
 \underline{x^5 - x^3} \\
 x^3 + 2 \\
 \underline{x^3 - x} \\
 x + 2
 \end{array}$$

Therefore

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x + \frac{x + 2}{x^2 - 1}$$

By expressing $\frac{x + 2}{x^2 - 1} = \frac{x + 2}{(x - 1)(x + 1)}$ as partial fractions, we obtain

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x - \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}$$

Section summary

- A rational function may be expressed as a sum of simpler functions by resolving it into **partial fractions**.

- Examples of resolving a proper fraction into partial fractions:

Single linear factors

$$\frac{3x - 4}{(2x - 3)(x + 5)} = \frac{A}{2x - 3} + \frac{B}{x + 5}$$

Repeated linear factor

$$\frac{3x - 4}{(2x - 3)(x + 5)^2} = \frac{A}{2x - 3} + \frac{B}{x + 5} + \frac{C}{(x + 5)^2}$$

Irreducible quadratic factor

$$\frac{3x - 4}{(2x - 3)(x^2 + 5)} = \frac{A}{2x - 3} + \frac{Bx + C}{x^2 + 5}$$

- A quadratic polynomial is **irreducible** if it cannot be factorised over $\mathbb{R}$.
For example, the quadratics $x^2 + 5$ and $x^2 + 4x + 10$ are irreducible.
- If $f(x) = \frac{P(x)}{Q(x)}$ is an improper fraction, i.e. if the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then the division must be performed first.

Exercise 4D

SF

Example 13

1 Resolve the following rational expressions into partial fractions:

a $\frac{5x+1}{(x-1)(x+2)}$

b $\frac{-1}{(x+1)(2x+1)}$

c $\frac{3x-2}{x^2-4}$

d $\frac{4x+7}{x^2+x-6}$

e $\frac{7-x}{(x-4)(x+1)}$

Example 14

2 Resolve the following rational expressions into partial fractions:

a $\frac{2x+3}{(x-3)^2}$

b $\frac{9}{(1+2x)(1-x)^2}$

c $\frac{2x-2}{(x+1)(x-2)^2}$

Example 15

3 Resolve the following rational expressions into partial fractions:

a $\frac{3x+1}{(x+1)(x^2+x+1)}$

b $\frac{3x^2+2x+5}{(x^2+2)(x+1)}$

c $\frac{x^2+2x-13}{2x^3+6x^2+2x+6}$

Example 16

4 Resolve $\frac{3x^2-4x-2}{(x-1)(x-2)}$ into partial fractions.

5 Show that it is not possible to find values of A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

6 Express each of the following as partial fractions:

a $\frac{1}{(x-1)(x+1)}$

b $\frac{x}{(x-2)(x+3)}$

c $\frac{3x+1}{(x-2)(x+5)}$

d $\frac{1}{(2x-1)(x+2)}$

e $\frac{3x+5}{(3x-2)(2x+1)}$

f $\frac{2}{x^2-x}$

g $\frac{3x+1}{x^3+x}$

h $\frac{3x^2+8}{x(x^2+4)}$

i $\frac{1}{x^2-4x}$

j $\frac{x+3}{x^2-4x}$

k $\frac{x^3-x^2-1}{x^2-x}$

l $\frac{x^3-x^2-6}{2x-x^2}$

m $\frac{x^2-x}{(x+1)(x^2+2)}$

n $\frac{x^2+2}{x^3-3x-2}$

o $\frac{2x^2+x+8}{x(x^2+4)}$

p $\frac{1-2x}{2x^2+7x+6}$

q $\frac{3x^2-6x+2}{(x-1)^2(x+2)}$

r $\frac{4}{(x-1)^2(2x+1)}$

s $\frac{x^3-2x^2-3x+9}{x^2-4}$

t $\frac{x^3+3}{(x+1)(x-1)}$

u $\frac{2x-1}{(x+1)(3x+2)}$

4E Simultaneous equations

In this section, we look at methods for finding the coordinates of the points of intersection of a linear graph with different non-linear graphs: parabolas, circles and rectangular hyperbolas. We also consider the intersections of two parabolas. These types of graphs are studied further in Mathematical Methods Units 1 & 2.

Example 17

Find the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ and the straight line with equation $y = x + 4$.

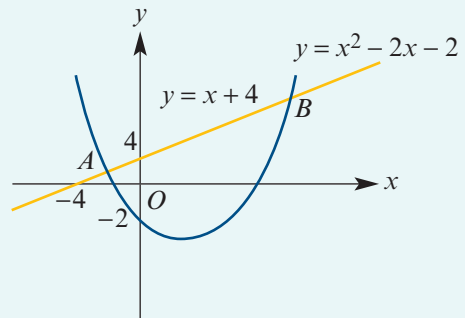
Solution

Equate the two expressions for y :

$$x^2 - 2x - 2 = x + 4$$

$$x^2 - 3x - 6 = 0$$

$$\begin{aligned}\therefore x &= \frac{3 \pm \sqrt{9 - 4 \times (-6)}}{2} \\ &= \frac{3 \pm \sqrt{33}}{2}\end{aligned}$$



The points of intersection are $A\left(\frac{3 - \sqrt{33}}{2}, \frac{11 - \sqrt{33}}{2}\right)$ and $B\left(\frac{3 + \sqrt{33}}{2}, \frac{11 + \sqrt{33}}{2}\right)$.

Example 18

Find the points of intersection of the circle with equation $(x - 4)^2 + y^2 = 16$ and the line with equation $x - y = 0$.

Solution

Rearrange $x - y = 0$ to make y the subject.

Substitute $y = x$ into the equation of the circle:

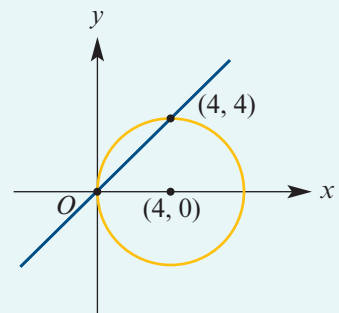
$$(x - 4)^2 + x^2 = 16$$

$$x^2 - 8x + 16 + x^2 = 16$$

$$2x^2 - 8x = 0$$

$$2x(x - 4) = 0$$

$$\therefore x = 0 \text{ or } x = 4$$



The points of intersection are $(0, 0)$ and $(4, 4)$.

Example 19

Find the point of contact of the straight line with equation $\frac{1}{9}x + y = \frac{2}{3}$ and the curve with equation $xy = 1$.

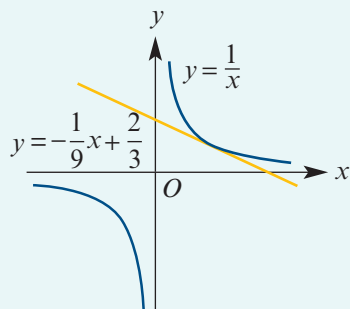
Solution

Rewrite the equations as $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = \frac{1}{x}$.

Equate the expressions for y :

$$\begin{aligned} -\frac{1}{9}x + \frac{2}{3} &= \frac{1}{x} \\ -x^2 + 6x &= 9 \\ x^2 - 6x + 9 &= 0 \\ (x - 3)^2 &= 0 \\ \therefore x &= 3 \end{aligned}$$

The point of intersection is $\left(3, \frac{1}{3}\right)$.

**Example 20**

Find the coordinates of the points of intersection of the graphs of $y = -3x^2 - 4x + 1$ and $y = 2x^2 - x - 1$.

Solution

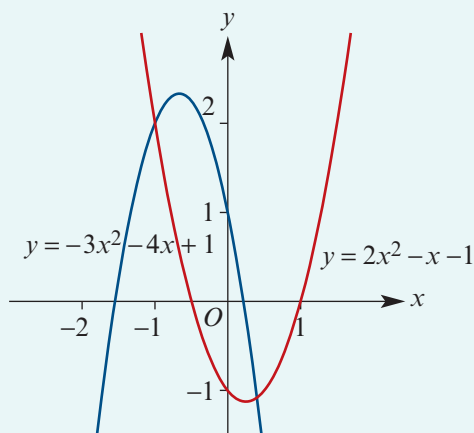
$$\begin{aligned} -3x^2 - 4x + 1 &= 2x^2 - x - 1 \\ -5x^2 - 3x + 2 &= 0 \\ 5x^2 + 3x - 2 &= 0 \\ (5x - 2)(x + 1) &= 0 \\ \therefore x &= \frac{2}{5} \text{ or } x = -1 \end{aligned}$$

Substitute in $y = 2x^2 - x - 1$:

When $x = -1$, $y = 2$.

When $x = \frac{2}{5}$, $y = 2 \times \frac{4}{25} - \frac{2}{5} - 1 = -\frac{27}{25}$.

The points of intersection are $(-1, 2)$ and $\left(\frac{2}{5}, -\frac{27}{25}\right)$.



Exercise 4E

Skillsheet

Example 17

- 1 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$

b $y - 2x^2 = 0$

c $y = x^2 - x$

$y = x$

$y - x = 0$

$y = 2x + 1$

Example 18

- 2 Find the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$

b $x^2 + y^2 = 125$

c $x^2 + y^2 = 185$

$x + y = 16$

$x + y = 15$

$x - y = 3$

d $x^2 + y^2 = 97$

e $x^2 + y^2 = 106$

$x + y = 13$

$x - y = 4$

Example 19

- 3 Find the coordinates of the points of intersection for each of the following:

a $x + y = 28$

b $x + y = 51$

c $x - y = 5$

$xy = 187$

$xy = 518$

$xy = 126$

- 4 Find the coordinates of the points of intersection of the straight line with equation
- $y = 2x$
- and the circle with equation
- $(x - 5)^2 + y^2 = 25$
- .

- 5 Find the coordinates of the points of intersection of the curves with equations
- $y = \frac{1}{x-2} + 3$
- and
- $y = x$
- .

- 6 Find the coordinates of the points A and B where the line with equation
- $x - 3y = 0$
- meets the circle with equation
- $x^2 + y^2 - 10x - 5y + 25 = 0$
- .

- 7 Find the coordinates of the points of intersection of the line with equation
- $\frac{y}{4} - \frac{x}{5} = 1$
- and the circle with equation
- $x^2 + 4x + y^2 = 12$
- .

- 8 Find the coordinates of the points of intersection of the curve with equation
- $y = \frac{1}{x+2} - 3$
- and the line with equation
- $y = -x$
- .

- 9 Find the point where the line
- $4y = 9x + 4$
- touches the parabola
- $y^2 = 9x$
- .

- 10 Find the coordinates of the point where the line with equation
- $y = 2x + 3\sqrt{5}$
- touches the circle with equation
- $x^2 + y^2 = 9$
- .

- 11 Find the coordinates of the point where the straight line with equation
- $y = \frac{1}{4}x + 1$
- touches the curve with equation
- $y = -\frac{1}{x}$
- .

- 12 Find points of intersection of the curve
- $y = \frac{2}{x-2}$
- and the line
- $y = x - 1$
- .

Example 20

- 13 Find the coordinates of the points of intersection of the graphs of the following pairs of quadratic functions:

a $y = 2x^2 - 4x + 1$, $y = 2x^2 - x - 1$

b $y = -2x^2 + x + 1$, $y = 2x^2 - x - 1$

c $y = x^2 + x + 1$, $y = x^2 - x - 2$

d $y = 3x^2 + x + 2$, $y = x^2 - x + 2$

SF

CF

SF



Chapter summary



Polynomials

- A **polynomial function** can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.

- The **degree** of a polynomial is the index n of the leading term (the term of highest index among those terms with a non-zero coefficient).

■ Equating coefficients

Two polynomials are equal if they give the same value for all x . If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.

For example: if $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

Quadratics

- A quadratic function can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$.
- A quadratic equation $ax^2 + bx + c = 0$ may be solved by:
 - Factorising
 - Completing the square
 - Using the **general quadratic formula** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The number of solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found from the **discriminant** $\Delta = b^2 - 4ac$:
 - If $\Delta > 0$, the quadratic equation has two real solutions.
 - If $\Delta = 0$, the quadratic equation has one real solution.
 - If $\Delta < 0$, the quadratic equation has no real solutions.

Partial fractions

- A **rational function** has the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

For example: $f(x) = \frac{2x + 10}{x^3 - x^2 - x + 1}$

- Some rational functions may be expressed as a sum of **partial fractions**:
 - For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\frac{A}{ax + b}$.
 - For every repeated linear factor $(cx + d)^2$ in the denominator, there will be partial fractions of the form $\frac{B}{cx + d}$ and $\frac{C}{(cx + d)^2}$.
 - For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\frac{Dx + E}{ax^2 + bx + c}$.

For example: $\frac{2x + 10}{(x + 1)(x - 1)^2} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}$, where $A = 2$, $B = -2$ and $C = 6$

Technology-free questions

- 1** If $(3a + b)x^2 + (a - 2b)x + b + 2c = 11x^2 - x + 4$, find the values of a , b and c .
- 2** Express x^3 in the form $(x - 1)^3 + a(x - 1)^2 + b(x - 1) + c$.
- 3** Prove that, if $ax^3 + bx^2 + cx + d = (x + 1)^2(px + q)$, then $b = 2a + d$ and $c = a + 2d$.
- 4** Prove that, if $ax^3 + bx^2 + cx + d = (x - 2)^2(px + q)$, then $b = -4a + \frac{1}{4}d$ and $c = 4a - d$.
- 5** Solve the following quadratic equations for x :
- a** $x^2 + x = 12$ **b** $x^2 - 2 = x$ **c** $-x^2 + 3x + 11 = 1$
d $2x^2 - 4x + 1 = 0$ **e** $3x^2 - 2x + 5 = t$ **f** $tx^2 + 4 = tx$
- 6** Solve the equation $\frac{2}{x-1} - \frac{3}{x+2} = \frac{1}{2}$ for x .
- 7** Express each of the following as partial fractions:
- a** $\frac{-3x+4}{(x-3)(x+2)}$ **b** $\frac{7x+2}{x^2-4}$ **c** $\frac{7-x}{x^2+2x-15}$
d $\frac{3x-9}{x^2-4x-5}$ **e** $\frac{3x-4}{(x+3)(x+2)^2}$ **f** $\frac{6x^2-5x-16}{(x-1)^2(x+4)}$
g $\frac{x^2-6x-4}{(x^2+2)(x+1)}$ **h** $\frac{-x+4}{(x-1)(x^2+x+1)}$ **i** $\frac{-4x+5}{(x+4)(x-3)}$
j $\frac{-2x+8}{(x+4)(x-3)}$
- 8** Express each of the following as partial fractions:
- a** $\frac{14(x-2)}{(x-3)(x^2+x+2)}$ **b** $\frac{1}{(x+1)(x^2-x+2)}$ **c** $\frac{3x^3}{x^2-5x+4}$
- 9** Find the coordinates of the points of intersection for each of the following:
- a** $y = x^2$ **b** $x^2 + y^2 = 16$ **c** $x + y = 5$
 $y = -x$ $x + y = 4$ $xy = 4$
- 10** Find the coordinates of the points of intersection of the line with equation $3y - x = 1$ and the circle with equation $x^2 + 2x + y^2 = 9$.
- 11** A motorist makes a journey of 135 km at an average speed of x km/h.
- a** Write an expression for the number of hours taken for the journey.
b Owing to road works, on a certain day his average speed for the journey is reduced by 15 km/h. Write an expression for the number of hours taken on that day.
c If the second journey takes 45 minutes longer than the first, form an equation in x and solve it.
d Find his average speed for each journey.

Multiple-choice questions



- 1** If x^2 is written in the form $(x+1)^2 + b(x+1) + c$, then the values of b and c are
A $b = 0, c = 0$ **B** $b = -2, c = 0$ **C** $b = -2, c = 1$
D $b = 1, c = 2$ **E** $b = 1, c = -2$
- 2** If $x^3 = a(x+2)^3 + b(x+2)^2 + c(x+2) + d$, then the values of a, b, c and d are
A $a = 0, b = -8, c = 10, d = -6$ **B** $a = 0, b = -6, c = 10, d = -8$
C $a = 1, b = -8, c = 10, d = -6$ **D** $a = 1, b = -6, c = 12, d = -8$
E $a = 1, b = -8, c = 12, d = -6$
- 3** The quadratic equation $3x^2 - 6x + 3 = 0$ has
A two real solutions, $x = \pm 1$ **B** one real solution, $x = -1$
C no real solutions **D** one real solution, $x = 1$
E two real solutions, $x = 1$ and $x = 2$
- 4** The quadratic equation whose solutions are 4 and -6 is
A $(x+4)(x-6) = 0$ **B** $x^2 - 2x - 24 = 0$ **C** $2x^2 + 4x = 48$
D $-x^2 + 2x - 24 = 0$ **E** $x^2 + 2x + 24 = 0$
- 5** $\frac{3}{x+4} - \frac{5}{x-2}$ is equal to
A $\frac{-2}{(x+4)(x-2)}$ **B** $\frac{2(x+1)}{(x+4)(x-2)}$ **C** $\frac{-2(x-7)}{(x+4)(x-2)}$
D $\frac{2(4x+13)}{(x+4)(x-2)}$ **E** $\frac{-2(x+13)}{(x+4)(x-2)}$
- 6** $\frac{4}{(x+3)^2} + \frac{2x}{x+1}$ is equal to
A $\frac{8x}{(x+3)^2(x+1)}$ **B** $\frac{2(3x^2+x+18)}{(x+3)^2(x+1)}$ **C** $\frac{3x^2+13x+18}{(x+3)^2(x+1)}$
D $\frac{2(3x^2+13x+18)}{(x+3)^2(x+1)}$ **E** $\frac{2(x^3+6x^2+11x+2)}{(x+3)^2(x+1)}$
- 7** If $\frac{7x^2+13}{(x-1)(x^2+x+2)}$ is expressed in the form $\frac{a}{x-1} + \frac{bx+c}{x^2+x+2}$, then
A $a = 5, b = 0, c = -13$ **B** $a = 5, b = 0, c = -10$ **C** $a = 5, b = 2, c = -3$
D $a = 7, b = 2, c = 3$ **E** $a = 7, b = 3, c = 13$
- 8** $\frac{4x-3}{(x-3)^2}$ is equal to
A $\frac{3}{x-3} + \frac{1}{x-3}$ **B** $\frac{4x}{x-3} - \frac{3}{x-3}$ **C** $\frac{9}{x-3} + \frac{4}{(x-3)^2}$
D $\frac{4}{x-3} + \frac{9}{(x-3)^2}$ **E** $\frac{4}{x-3} - \frac{15}{(x-3)^2}$

9 $\frac{8x+7}{2x^2+5x+2}$ is equal to

A $\frac{2}{2x+1} - \frac{3}{x+2}$

B $\frac{2}{2x+1} + \frac{3}{x+2}$

C $\frac{-4}{2x+2} - \frac{1}{x+1}$

D $\frac{-4}{2x+2} + \frac{1}{x+1}$

E $\frac{4}{2x+2} - \frac{1}{x+1}$

10 $\frac{-3x^2+2x-1}{(x^2+1)(x+1)}$ is equal to

A $\frac{2}{x^2+1} + \frac{3}{x+1}$

B $\frac{2}{x^2+1} - \frac{3}{x+1}$

C $\frac{5}{x^2+1} + \frac{2}{x+1}$

D $\frac{3}{x^2+1} - \frac{2}{x+1}$

E $\frac{3}{x^2+1} + \frac{2}{x+1}$

Extended-response questions

1 A train completes a journey of 240 km at a constant speed.

- a** If the train had travelled 4 km/h slower, it would have taken two hours more for the journey. Find the actual speed of the train.
- b** If the train had travelled a km/h slower and still taken two hours more for the journey of 240 km, what would have been the actual speed? (Answer in terms of a .) Discuss the practical possible values of a and also the possible values for the speed of the train.
- c** If the train had travelled a km/h slower and taken a hours more for the journey of 240 km, and if a is an integer and the speed is an integer, find the possible values for a and the speed of the train.

2 Two trains are travelling at constant speeds. The slower train takes a hours longer to cover b km. It travels 1 km less than the faster train in c hours.

- a** What is the speed of the faster train, in terms of a , b and c ?
- b** If a , b , c and the speeds of the trains are all rational numbers, find five sets of values for a , b and c . Choose and discuss two sensible sets of values.

3 In each of the following, use the discriminant of the resulting quadratic equation:

- a** Find the possible values of k for which the straight line $y = k(1 - 2x)$ touches but does not cross the parabola $y = x^2 + 2$.
- b** Find the possible values of c for which the line $y = 2x + c$ intersects the circle $x^2 + y^2 = 20$ in two distinct points.
- c** Find the value of p for which the line $y = 6$ meets the parabola $y = x^2 + (1 - p)x + 2p$ at only one point.

5

Revision of Chapters 1–4

5A Technology-free questions

- 1 Rewrite each fraction with an integer denominator:

a $\frac{1}{\sqrt{2}-3}$

b $\frac{3}{\sqrt{5}-1}$

c $\frac{2}{2\sqrt{2}-1}$

d $\frac{3}{\sqrt{5}-\sqrt{3}}$

e $\frac{1}{\sqrt{7}-\sqrt{2}}$

f $\frac{1}{2\sqrt{5}-\sqrt{3}}$

- 2 **a** If the equations $x^2 + x - 1 = 0$ and $x^2 + bx + 1 = 0$ have a common solution, show that $b = \pm\sqrt{5}$.

- b** Find the common solution when:

i $b = \sqrt{5}$

ii $b = -\sqrt{5}$

- 3 Find constants a , b and c such that $(n+1)(n-7) = a + bn + cn(n-1)$ for all n .

- 4 Prove that, if $n = \text{HCF}(a, b)$, then n divides $a - b$.

- 5 Write down the prime factorisation of each of the following numbers, and hence determine the square root of each number:

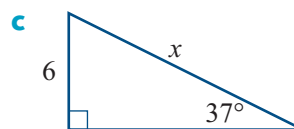
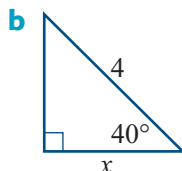
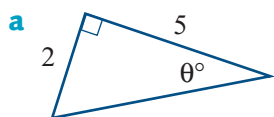
a 576

b 1225

c 1936

d 1296

- 6 Find the value of the pronumeral:

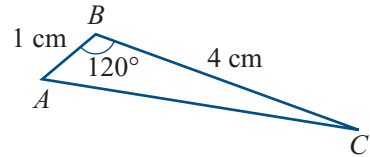


7 Solve the equation $\frac{x+b}{x-c} = 1 - \frac{x}{x-c}$ for x .

8 Solve the equation $\frac{1}{x-a} + \frac{1}{x-b} = \frac{2}{x}$ for x .

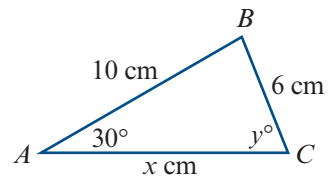
- 9 The triangle ABC has side lengths $AB = 1$ cm and $BC = 4$ cm. The magnitude of angle ABC is 120° .

- a** Find the length of AC .
b Find the sine of angle BAC .



- 10 A triangle ABC satisfies $\sin A : \sin B = 1 : 2$. If $BC = 6$ cm, find AC .

- 11 For the triangle shown on the right:
a Find the two possible values for x .
b Find the two possible values for y .



- 12 Find two sets of values of λ, a, b such that, for all values of x ,

$$x^2 - 4x - 8 + \lambda(x^2 - 2x - 5) = a(x - b)^2$$

- 13 Transpose each of the following to make x the subject:

a $y = 3 + \sqrt{2x - 1}$

b $y = \frac{2}{\sqrt{3x + 1}} - 2$

- 14 Two bushwalkers are 20 km apart on a track and travelling towards each other. One walks at 3 km/h and the other at 5 km/h. After how many minutes will they meet?

- 15 Solve each pair of simultaneous equations:

a $\frac{x}{3} + \frac{y}{4} = 1$
 $3x - 4y = 1$

b $\frac{x}{a} + \frac{y}{b} = 1$
 $ax - by = 1$

- 16 A group of 50 students were interviewed about the types of movies that they watch: 25 of the students like action movies, 26 like comedy, 17 like drama, 11 like action and comedy, 5 like action and drama, 8 like comedy and drama, and 3 like all three. How many of these students:

- a** like none of these types of movies
b like action movies only
c like action and comedy but not drama?

- 17** Consider the quadratic polynomial $P(x) = ax^2 - 2ax + 1$, where $a \neq 0$.
- a** Find the discriminant of $P(x)$.
 - b** Find the values of a for which the graph of $y = P(x)$:
 - i** touches the x -axis once only
 - ii** crosses the x -axis twice
 - iii** does not intersect the x -axis.

5B Multiple-choice questions

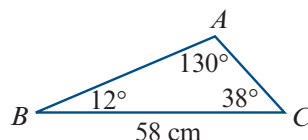
- 1** In algebraic form, five is seven less than three times one more than x can be written as
A $5 = 7 - 3(x + 1)$ **B** $3x + 1 = 5 - 7$ **C** $(x + 1) - 7 = 5$
D $5 = 7 - 3x + 1$ **E** $5 = 3x - 4$
- 2** $\frac{3}{x-3} - \frac{2}{x+3}$ is equal to
A 1 **B** $\frac{x+15}{x^2-9}$ **C** $\frac{15}{x-9}$ **D** $\frac{x-3}{x^2-9}$ **E** $-\frac{1}{6}$
- 3** If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5, 6\}$ and $C = \{3, 4, 5, 6, 7\}$, then $A \cap (B \cup C)$ is equal to
A $\{1, 2, 3, 4, 5, 6, 7\}$ **B** $\{1, 2, 3, 4, 5, 6\}$ **C** $\{2, 3, 4\}$
D $\{3, 4\}$ **E** $\{2, 3, 4, 5, 6, 7\}$
- 4** The recurring decimal $0.\dot{7}2$ is equal to
A $\frac{72}{101}$ **B** $\frac{72}{100}$ **C** $\frac{72}{99}$ **D** $\frac{72}{90}$ **E** $\frac{73}{90}$
- 5** $\frac{-4}{x-1} - \frac{3}{1-x} + \frac{x}{x-1}$ is equal to
A 1 **B** -1 **C** $\frac{7x}{x-1}$ **D** $\frac{1}{1-x}$ **E** none of these
- 6** $\frac{x+2}{3} - \frac{5}{6}$ is equal to
A $\frac{x-3}{6}$ **B** $\frac{2x+4}{6}$ **C** $\frac{2x-1}{6}$ **D** $\frac{2x-5}{6}$ **E** $\frac{x-3}{3}$
- 7** If $a = 1 + \frac{1}{1+b}$, then b equals
A $1 - \frac{1}{a-1}$ **B** $1 + \frac{1}{a-1}$ **C** $\frac{1}{a-1} - 1$ **D** $\frac{1}{a+1} + 1$ **E** $\frac{1}{a+1} - 1$
- 8** When the repeating decimal $0.\dot{3}\dot{6}$ is written in simplest fractional form, the sum of the numerator and denominator is
A 15 **B** 45 **C** 114 **D** 135 **E** 150

- 9 If $\frac{2x-y}{2x+y} = \frac{3}{4}$, then $\frac{x}{y}$ equals
A $\frac{2}{7}$ **B** $\frac{7}{2}$ **C** $\frac{3}{4}$ **D** $\frac{4}{3}$ **E** cannot be determined
- 10 The coordinates of the point where the lines with equations $3x + y = -7$ and $2x + 5y = 4$ intersect are
A $(3, -16)$ **B** $(-3, 2)$ **C** $(3, -2)$ **D** $(-2, 3)$ **E** no solution
- 11 If $\frac{m+2}{4} - \frac{2-m}{4} = \frac{1}{2}$, then m is equal to
A 1 **B** -1 **C** $\frac{1}{2}$ **D** 0 **E** $-\frac{1}{2}$
- 12 The number 46 200 can be written as
A $2 \times 3 \times 5 \times 7 \times 11$ **B** $2^2 \times 3^2 \times 5^2 \times 7 \times 11$ **C** $2 \times 3^2 \times 5 \times 7^2 \times 11$
D $2^3 \times 3 \times 5^2 \times 7 \times 11$ **E** $2^2 \times 3 \times 5^3 \times 7 \times 11$
- 13 If the positive integers $n+1, n-1, n-6, n-5, n+4$ are arranged in increasing order of magnitude, then the middle number is
A $n+1$ **B** $n-1$ **C** $n-6$ **D** $n-5$ **E** $n+4$
- 14 The expression $\frac{4}{n+1} + \frac{3}{n-1}$ is equal to
A $\frac{7n-1}{1-n^2}$ **B** $\frac{1-7n}{1-n^2}$ **C** $\frac{7n-1}{n^2+1}$ **D** $\frac{7}{n^2-1}$ **E** $\frac{7}{n}$
- 15 The second number is twice the first number; the third number is half the first number; the three numbers sum to 28. These numbers are
A $(8, 16, 4)$ **B** $(2, 3, 12)$ **C** $(7, 9, 11)$ **D** $(6, 8, 16)$ **E** $(12, 14, 2)$
- 16 $(\sqrt{7}+3)(\sqrt{7}-3)$ is equal to
A -2 **B** 10 **C** $\sqrt{14}-19$ **D** $2\sqrt{7}-9$ **E** 45
- 17 If $\frac{13x-10}{2x^2-9x+4} = \frac{P}{x-4} + \frac{Q}{2x-1}$, then the values of P and Q are
A $P = 1$ and $Q = 1$ **B** $P = -1$ and $Q = 1$ **C** $P = 6$ and $Q = 1$
D $P = -6$ and $Q = 1$ **E** $P = 1$ and $Q = -6$
- 18 If $\frac{5x}{(x+2)(x-3)} = \frac{P}{x+2} + \frac{Q}{x-3}$, then
A $P = 2$ and $Q = 3$ **B** $P = 2$ and $Q = -3$ **C** $P = -2$ and $Q = 3$
D $P = -2$ and $Q = -3$ **E** $P = 1$ and $Q = 1$

- 19** $\sin(180 - \theta)^\circ + \cos(180 - \theta)^\circ$ is equal to
A $\sin(\theta^\circ) + \cos(\theta^\circ)$ **B** $-\sin(\theta^\circ) + \cos(\theta^\circ)$ **C** $\sin(\theta^\circ) - \cos(\theta^\circ)$
D $-\sin(\theta^\circ) - \cos(\theta^\circ)$ **E** $-\sin(\theta^\circ)\cos(\theta^\circ)$

- 20** Which one of the following gives the correct value for c ?

- A** $\frac{58 \cos 38^\circ}{\cos 130^\circ}$ **B** $\frac{58 \sin 38^\circ}{\sin 130^\circ}$ **C** $58 \sin 38^\circ$
D $\frac{58 \cos 130^\circ}{\cos 38^\circ}$ **E** $\frac{58 \sin 130^\circ}{\sin 38^\circ}$



- 21** A building of height 60 m is 1.1 km away from a point on the ground. The angle of elevation to the top of the building from this point, correct to the nearest degree, is
A 1° **B** 3° **C** 50° **D** 87° **E** 89°

- 22** If the natural number n is a perfect square, then the next perfect square is
A $n + 1$ **B** $n^2 + 1$ **C** $n^2 + 2n + 1$ **D** $n^2 + n$ **E** $n + 2\sqrt{n} + 1$

- 23** Which of the following is *not* a rational number?
A 0.4 **B** $\frac{3}{8}$ **C** $\sqrt{5}$ **D** $\sqrt{16}$ **E** 4.125

- 24** If $\frac{1}{x} = \frac{a}{b}$ and $\frac{1}{y} = a - b$, then $x + y$ equals
A $\frac{2}{a}$ **B** $\frac{a^2 - b^2}{a}$ **C** $\frac{ba - b^2 + a}{a(a - b)}$ **D** $\frac{2a}{a^2 - b^2}$ **E** $\frac{-2b}{a^2 - b^2}$

- 25** $9x^2 - 4mx + 4$ is a perfect square when m equals
A 5 **B** ± 12 **C** 2 **D** ± 1 **E** ± 3

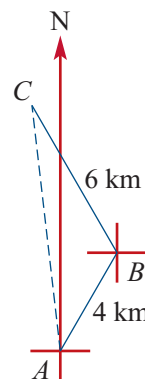
- 26** If $x = (n + 1)(n + 2)(n + 3)$, for some positive integer n , then x is not always divisible by
A 1 **B** 2 **C** 3 **D** 5 **E** 6

- 27** If both n and p are odd numbers, which one of the following numbers must be even?
A $n + p$ **B** np **C** $np + 2$ **D** $n + p + 1$ **E** $2n + p$

- 28** $4a^2b^4 \times 3(ab^3)^{-2}$ is equal to
A $12b^{-2}$ **B** $12ab^{-2}$ **C** $12a^{-4}b^{-2}$ **D** $12a^3b^5$ **E** $12ab^5$

- 29** $\frac{3 \times 10^8}{\sqrt{0.144 \times 10^5}}$ is equal to
A 3.6×10^6 **B** 8×10^3 **C** 8×10^{-3} **D** 2.5×10^6 **E** 2.5×10^5

- 30** If $\text{LCM}(12, n) = 60$ and $\text{HCF}(12, n) = 6$, then $n =$
A 10 **B** 15 **C** 20 **D** 30 **E** 60
- 31** If x^2 is written in the form $(x - 2)^2 + b(x - 2) + c$, then the values of b and c are
A $b = 2, c = 0$ **B** $b = -4, c = -4$ **C** $b = 4, c = 4$
D $b = 2, c = 2$ **E** $b = 0, c = 2$
- 32** A car covers a distance of 50 km at an average speed of x km/h. Over the same period of time, a train covers a distance of 70 km at an average speed of $(x + 25)$ km/h. An equation that can be used to find x is
A $50x = 70(x + 25)$ **B** $70x = 50(x + 25)$ **C** $50x = 70x + 25$
D $70x = 50x + 25$ **E** $70 = 50 + 25x$
- 33** A hiker walks 4 km from point A on a bearing of 030° to point B , and then walks 6 km on a bearing of 330° to point C . The distance AC , in kilometres, is
A $\frac{4}{\sin 30^\circ}$
B $\sqrt{6^2 + 4^2 - 48 \cos 120^\circ}$
C $\sqrt{6^2 + 4^2 + 48 \cos 120^\circ}$
D $6 \sin 60^\circ$
E $\sqrt{52}$
- 34** From a port P , a ship Q is 20 km away on a bearing of 112° , and a ship R is 12 km away on a bearing of 052° . The distance between the two ships is
A 304 km **B** 28 km **C** $4\sqrt{19}$ km **D** 784 km **E** $2\sqrt{106}$ km

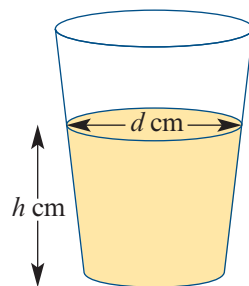


5C Extended-response questions

- 1** The diagram represents a glass containing milk. When the height of the milk in the glass is h cm, the diameter, d cm, of the surface of the milk is given by the formula

$$d = \frac{h}{5} + 6$$

- a** Find d when $h = 10$.
b Find d when $h = 8.5$.
c What is the diameter of the bottom of the glass?
d The diameter of the top of the glass is 9 cm. What is the height of the glass?

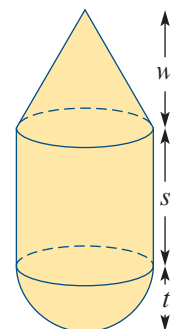


- 2** The cost, \$ C , of manufacturing each jacket of a particular type is given by the formula

$$C = an + b \quad \text{for } 0 < n \leq 300$$

where a and b are constants and n is the size of the production run of this type of jacket. For making 100 jackets, the cost is \$108 each. For 120 jackets, the cost is \$100 each.

- a** Find the values of a and b .
 - b** Sketch the graph of C against n for $0 < n \leq 300$.
 - c** Find the cost of manufacturing each jacket if 200 jackets are made.
 - d** If the cost of making each jacket is \$48.80, find the size of the production run.
- 3** The formula $A = 180 - \frac{360}{n}$ gives the size of each interior angle, A° , of a regular polygon with n sides.
- a** Find the value of A when n equals:
 - i** 180 **ii** 360 **iii** 720 **iv** 7200
 - b** As n becomes very large:
 - i** What value does A approach? **ii** What shape does the polygon approach?
 - c** Find the value of n when $A = 162$.
 - d** Make n the subject of the formula.
 - e** Three regular polygons, two of which are octagons, meet at a point so that they fit together without any gaps. Describe the third polygon.
- 4** The figure shows a solid consisting of three parts – a cone, a cylinder and a hemisphere – all of the same base radius.



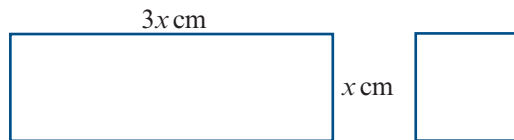
- a** Find, in terms of w , s , t and π , the volume of each part.
 - b**
 - i** If the volume of each of the three parts is the same, find the ratio $w : s : t$.
 - ii** If also $w + s + t = 11$, find the total volume in terms of π .
- 5** At the beginning of 2007, Andrew and John bought a small catering business. The profit, \$ P , in a particular year is given by the rule $P = an + b$, where n is the number of years of operation and a and b are constants.

- a** Given the table, find the values of a and b .

Year	2007	2011
Number of years of operation (n)	1	5
Profit (P)	−9000	15 000

- b** Find the profit when $n = 12$.
- c** In which year was the profit \$45 000?

- 6** A piece of wire 28 cm long is cut into two parts: one to make a rectangle three times as long as it is wide, and the other to make a square.



- What is the perimeter of the rectangle in terms of x ?
 - What is the perimeter of the square in terms of x ?
 - What is the length of each side of the square in terms of x ?
- Let A be the sum of the areas of the two figures.
- Show that $A = 7(x^2 - 4x + 7)$.
 - Use a calculator to help sketch the graph of $A = 7(x^2 - 4x + 7)$ for $0 < x < 3.5$.
 - Find the minimum value that A can take and the corresponding value of x .
- 7** A particular plastic plate manufactured at a factory sells at \$1.50. The cost of production consists of an initial cost of \$3500 and then \$0.50 per plate. Let x be the number of plates produced.
- Let $\$C$ be the cost of production of x plates. Write an expression for C in terms of x .
 - Let $\$I$ be the income from selling x plates. Write an expression for I in terms of x .
 - On the one set of axes, sketch the graphs of I against x and C against x .
 - How many plates must be sold for the income to equal the cost of production?
 - How many plates must be sold for a profit of \$2000 to be made?
 - Let $P = I - C$. Sketch the graph of P against x . What does P represent?

- 8 a i** For the equation $\sqrt{7x - 5} - \sqrt{2x} = \sqrt{15 - 7x}$, square both sides to show that this equation implies

$$8x - 10 = \sqrt{14x^2 - 10x}$$

- ii** Square both sides of this new equation and simplify to form the equation

$$x^2 - 3x + 2 = 0 \quad (1)$$

- iii** The solutions to equation (1) are $x = 1$ and $x = 2$. Test these solutions for the equation

$$\sqrt{7x - 5} - \sqrt{2x} = \sqrt{15 - 7x}$$

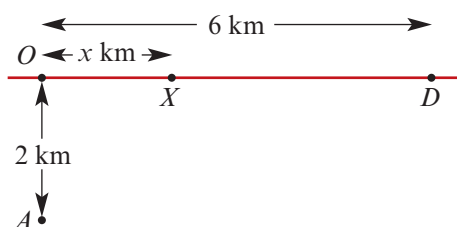
and hence show that $x = 2$ is the only solution to the original equation.

- b** Use the techniques of part **a** to solve the equations:

i $\sqrt{x+2} - 2\sqrt{x} = \sqrt{x+1}$

ii $2\sqrt{x+1} + \sqrt{x-1} = 3\sqrt{x}$

- 9** Let n be a natural number less than 50 such that $n + 25$ is a perfect square.
- Show that there exists an integer a such that $n = a(a + 10)$.
 - Any natural number less than 100 can be written in the form $10p + q$, where p and q are digits. For this representation of n , show that $q = p^2$.
 - Give all possible values of n .
- 10** If an object of mass m kg is at a height of h m above the ground, then its potential energy (P joules) is given by the formula $P = mgh$, where g is a constant.
- Find the value of the constant g given that $P = 980$ when $h = 20$ and $m = 5$.
 - Sketch the graph of P against h for an object of mass 5 kg.
 - Find m if $P = 2058$ and $h = 30$.
 - What is the effect on the potential energy if the height (h m) is doubled and the mass remains constant?
 - What is the effect on the potential energy if the object has one-quarter of the original height (h m) and double the original mass (m kg)?
 - If an object is dropped from a height (h m) above ground level, its speed (V m/s) when it reaches the ground is given by $V = \sqrt{19.6h}$.
 - Find V when $h = 10$.
 - Find V when $h = 90$.
 - In order to double the speed that a given object has when it hits the ground, by what factor must the height from which it is dropped be increased?
- 11** The diagram shows a straight road OD , where $OD = 6$ km.



A hiker is at A , which is 2 km from O . The hiker walks directly to X and then walks along the road to D . The hiker can walk at 3 km/h off-road, but at 8 km/h along the road.

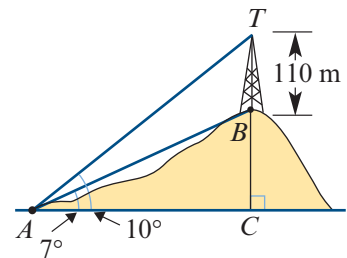
- If $OX = 3$ km, calculate the total time taken for the hiker to walk from A to D via X in hours and minutes, correct to the nearest minute.
- If the total time taken was $1\frac{1}{2}$ hours, calculate the distance OX in kilometres, correct to one decimal place.

- 12** Seventy-six photographers submitted work for a photographic exhibition in which they were permitted to enter not more than one photograph in each of three categories: black and white (B), colour prints (C), transparencies (T). Eighteen entrants had all their work rejected, while 30 B , 30 T and 20 C were accepted.

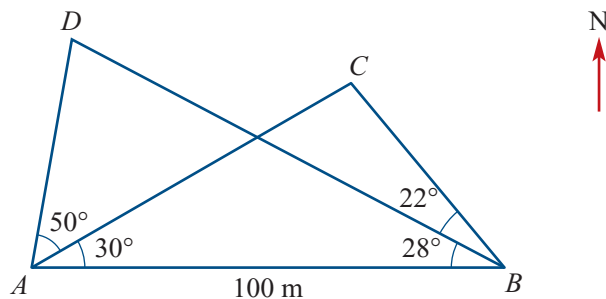
- From the exhibitors, as many showed T only as showed T and C .
 - There were three times as many exhibitors showing B only as showing C only.
 - Four exhibitors showed B and T but not C .
- a** Write the last three sentences in symbolic form.
- b** Draw a Venn diagram representing the information.
- c** **i** Find $|B \cap C \cap T|$.
- ii** Find $|B \cap C \cap T'|$.

- 13** A tower 110 m high stands on the top of a hill. From a point A at the foot of the hill, the angle of elevation of the bottom of the tower is 7° and that of the top is 10° .

- a** Find the magnitudes of angles TAB , ABT and ATB .
- b** Use the sine rule to find the length of AB .
- c** Find CB , the height of the hill.



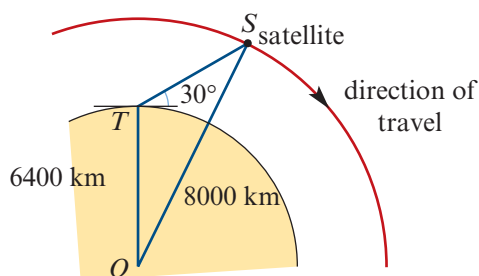
- 14** A triangular survey method was used to calculate the distance from a point A , on land, to two platforms C and D , out to sea. Sightings were made from two accessible points A and B , which are 100 m apart.



- a** Find the distance from A to each of the platforms (to the nearest metre).
- b** Find the distance between the two platforms (to the nearest metre).
- c** Find the bearing of the platform D from the platform C (to the nearest degree).

5D Problem-solving and modelling questions

- 1** A satellite travelling in a circular orbit 1600 km above the Earth is due to pass directly over a tracking station at 12 p.m. Assume that the satellite takes two hours to make an orbit and that the radius of the Earth is 6400 km.



- a** If the tracking station antenna is aimed at 30° above the horizon, at what time will the satellite pass through the beam of the antenna?
- b** Find the distance between the satellite and the tracking station at 12:06 p.m.
- c** At what angle above the horizon should the antenna be aimed so that its beam will intercept the satellite at 12:06 p.m.?

- 2** The factors of 12 are 1, 2, 3, 4, 6, 12.

- a** How many factors does each of the following numbers have?

i 2^3 **ii** 3^7

- b** How many factors does 2^n have?

- c** How many factors does each of the following numbers have?

i $2^3 \cdot 3^7$ **ii** $2^n \cdot 3^m$

- d** Every natural number greater than 1 may be expressed as a product of powers of primes; this is called prime decomposition. For example: $1080 = 2^3 \times 3^3 \times 5$.

Let x be a natural number greater than 1 and let

$$x = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_n^{\alpha_n}$$

be its prime decomposition, where each $\alpha_i \in \mathbb{N}$ and each p_i is a prime number.

How many factors does x have? (Answer to be given in terms of α_i .)

- e** Find the smallest number which has eight factors.

- 3 a** Give the prime decompositions of 1080 and 25 200.

- b** Use your answer to part **a** to find the lowest common multiple of 1080 and 25 200.

- c** Carefully explain why, if m and n are natural numbers, then

$$mn = \text{LCM}(m, n) \times \text{HCF}(m, n)$$

- d i** Find four consecutive even numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.
- ii** Find four consecutive natural numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

- 4 a** Prove that $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

Note: This is known as Sophie Germain's identity.

- b** Use this identity to prove that, if n is an odd number greater than 1, then $n^4 + 4^n$ is not prime.
- c** Hence show that the number $4^{545} + 545^4$ is not prime.

5E Problem-solving and modelling investigations

For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

1 Reciprocals of primes

- a** Find all ordered pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$$

- b** Find all ordered pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{11}$$

- c** Find all ordered pairs of positive integers (x, y) such that

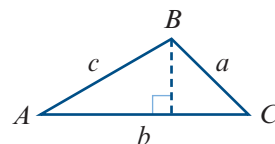
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$$

where p is a prime number. Give your answers with x and y in terms of p .

2 Area of a triangle

- a** Consider a triangle ABC with side lengths a , b and c .
Prove that the area of the triangle is given by

$$\text{Area} = \frac{1}{2}bc \sin A$$



- b** Now let $s = \frac{a+b+c}{2}$. Using part **a**, prove that the area of the triangle is given by

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

This is known as Heron's formula.

Hint: From the unit circle, we have $(\cos A)^2 + (\sin A)^2 = 1$. Use this result and the cosine rule in triangle ABC to show that

$$(\sin A)^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{(2bc)^2}$$

- c** Investigate **Heronian triangles**, which are triangles such that the side lengths and the area are all integers. For example: Explain why there are no equilateral Heronian triangles. Find an isosceles Heronian triangle. Explain why every right-angled triangle with integer side lengths is Heronian.

- 3 Filling a tank** A tank can be filled using two pipes. The smaller pipe alone will take a minutes longer than the larger pipe alone to fill the tank. Also, the smaller pipe will take b minutes longer to fill the tank than when both pipes are used.
- a** In terms of a and b , how long will each of the pipes take to fill the tank?
 - b** If $a = 24$ and $b = 32$, how long will each of the pipes take to fill the tank?
 - c** If a and b are consecutive positive integers, find five pairs of values of a and b such that $b^2 - ab$ is a perfect square. Interpret these results in the context of this problem.

4 Applications of partial fractions

- a i** Show that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

- ii** Hence show that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{99 \times 100} = \frac{99}{100}$$

- iii** Hence evaluate

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+99}$$

- b i** Find A and B such that

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

- ii** Hence evaluate

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \cdots + \frac{1}{99 \times 101}$$

- iii** By using partial fractions in a similar way, evaluate

$$\frac{1}{1 \times 6} + \frac{1}{6 \times 11} + \frac{1}{11 \times 16} + \cdots + \frac{1}{96 \times 101}$$

- c i** Show that

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{2} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

- ii** Hence evaluate

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \cdots + \frac{1}{98 \times 99 \times 100}$$

- d i** By rationalising the denominator of the left-hand side, show that

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} = \sqrt{n+1} - \sqrt{n}$$

- ii** Hence show that

$$\frac{1}{\sqrt{2} + \sqrt{1}} + \frac{1}{\sqrt{3} + \sqrt{2}} + \frac{1}{\sqrt{4} + \sqrt{3}} + \cdots + \frac{1}{\sqrt{100} + \sqrt{99}} = 9$$

- iii** By using a similar approach, evaluate

$$\frac{1}{\sqrt{3} + \sqrt{1}} + \frac{1}{\sqrt{5} + \sqrt{3}} + \frac{1}{\sqrt{7} + \sqrt{5}} + \cdots + \frac{1}{\sqrt{121} + \sqrt{119}}$$

5F Degree-of-difficulty classified questions

► Simple familiar questions

- 1 Write each of the following in the form $\frac{m}{n}$, where m and n are integers:

a 0.5 **b** 0.54 **c** 0.605 **d** 2.3 **e** 4.35673

- 2 Express each of the following as a product of powers of prime numbers:

a 2002 **b** 555 **c** 7007 **d** 10 000

- 3 Simplify $\frac{5m - 2p}{4m^2 + mp - 3p^2} - \frac{1}{4m - 3p}$.

- 4 Expand each of the following and collect like terms:

a $(\sqrt{3} + \sqrt{2})(\sqrt{3} - 1)$ **b** $(5\sqrt{3} - \sqrt{6})(2\sqrt{6} + 3\sqrt{3})$
c $(2\sqrt{x} - 3)^2$ **d** $(\sqrt{x-2} - 3)^2$

- 5 For each of the following, find the values of A , B and C such that the equation holds for all real numbers x :

a $(Ax + B)(x + 1) + C(x^2 + 3) = x - 3$ **b** $16 + 4x - x^2 = A - (B - Cx)^2$

- 6 Find the points of intersection of the circle with equation $(x - 9)^2 + y^2 = 25$ and the line with equation $x - 3y = 0$.

- 7 Find the points of intersection of the graphs of $y = 2x^2 - 4x - 2$ and $y = -2x^2 - 4x + 2$.

- 8 Solve the following equation for x :

$$\frac{4}{x^2 - x - 2} + \frac{3}{x^2 - 4} = \frac{2}{x^2 + 3x + 2}$$

- 9 A train travels at a constant speed of 55 km/h for 2 hours and then at a constant speed of 70 km/h for 3 hours. Find the train's average speed over the 5-hour journey.

- 10 Simplify $\frac{\sqrt{2}}{\sqrt{3} + \sqrt{2}}$.

- 11 Resolve each of the following into partial fractions:

a $\frac{2x}{3(x-2)(x+2)}$ **b** $\frac{2x+5}{(x+2)(x+3)}$ **c** $\frac{5x^2+4x+4}{(x+2)(x^2+4)}$
d $\frac{2(x^2-2x-1)}{(x+1)(x-1)^2}$ **e** $\frac{2x^2-3x+1}{x^3-3x^2+x-3}$ **f** $\frac{3x^2-x+6}{(x^2+4)(x-2)}$

- 12 A triangle ABC has side lengths $AB = 3.6$ cm, $BC = 5.2$ cm and $CA = 4.3$ cm. Determine the magnitudes of the angles of this triangle. (Give answers correct to one decimal place.)

- 13 In triangle ABC , $AC = 4$ cm, $BC = 5$ cm and $\angle CAB = 42^\circ$. Find the magnitudes of $\angle ABC$ and $\angle ACB$.

► **Complex familiar questions**

- 1 Given that $x = \frac{\sqrt{7} + \sqrt{5}}{\sqrt{7} - \sqrt{5}}$, find each of the following:

a $x + \frac{1}{x}$

b $x^2 + \frac{1}{x^2}$

- 2 Solve for x :

$$\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = 3$$

- 3 Find the values of m such that the line $y = mx + 6$ touches (but does not cross) the circle $x^2 + y^2 - 6x - 4y + 8 = 0$.
- 4 Solve the equation $(2x - 1)^2 + (x - 3)^2 = 36$ for x .
- 5 A group of 200 Year 11 students each study one or more of the subjects Mathematical Methods, Physics and Chemistry. Suppose that in this group:
- 40 students study Chemistry only
 - 50% more students study Mathematical Methods than study Physics
 - 5 students study all three subjects
 - 25 students study both Mathematical Methods and Physics
 - 8 students study both Mathematical Methods and Chemistry
 - 20 students study both Physics and Chemistry.

Determine the number of students in the group studying each of the subjects.

- 6 Lighthouse B is 7 km due east of lighthouse A . From A , the bearings of two ships are 031° and 073° respectively. From B , the bearings are 275° and 343° respectively. Find the distance between the two ships.
- 7 Evaluate $(\sqrt{8 + 2\sqrt{7}} - \sqrt{8 - 2\sqrt{7}})^2$.
- 8 Find rational numbers a and b such that $(2 + 2\sqrt{3})^2 = a + \sqrt{b}$.
- 9 **a** Fully factorise $n^5 - 5n^3 + 4n$.
b Explain why:
 - i** the product of any five consecutive integers is divisible by 5
 - ii** the product of any three consecutive integers is divisible by 3
 - iii** the product of any four consecutive integers is divisible by 8.**c** Explain why $n^5 - 5n^3 + 4n$ is divisible by 120, for each integer n .
- 10 In order to measure the width of a river with parallel banks, an object C on one bank is sighted from points A and B , which are 60 m apart, on the opposite bank. It is found that $\angle CAB = 40^\circ$ and $\angle CBA = 139^\circ$. Calculate the width of the river.

- 11** A car travels due north for 30 km and then travels on a bearing of 120° for 50 km. Find the car's distance and bearing from the starting point.
- 12** A group of 30 children were surveyed to find out which of three sports they play: cricket, basketball or volleyball. It was found that:
- 2 children play all three sports ■ 3 children do not play any of these sports
 - 16 children play basketball ■ 6 children play basketball and volleyball only
 - 12 children play volleyball ■ 3 children play cricket and volleyball only.
- Using a Venn diagram, determine how many of the children play cricket only.

► Complex unfamiliar questions

- 1** Find all ordered pairs of positive integers (x, y) such that $\frac{xy}{x+y} = 3$.
- 2** There are two ways to express the number 10 as a product of three positive integers:
- $$10 = 1 \times 1 \times 10 \quad \text{and} \quad 10 = 1 \times 2 \times 5$$
- (Note that the order of the factors does not matter.) How many ways are there to express the number 100 as a product of three positive integers?
- 3** **a** Find the remainder when $3^{1969} + 11$ is divided by 9.
b Find all possible remainders when $3^n + 11$ is divided by 9, for $n \in \mathbb{N}$.
- 4** Express $2^{25} \times 3^{40}$ in the form a^b , where a and b are natural numbers with $b > 1$.
- 5** A car leaves town A at 10 a.m. and arrives in town B at 11:15 a.m. During the first hour of the journey, the car travels at a constant speed of 80 km/h. The average speed of the car between 10:15 a.m. and 11:15 a.m. is 2 km/h less than the average speed for the whole journey.
- a** Find the distance travelled by the car from town A to town B .
b Find the average speed of the car between 10:15 a.m. and 11:15 a.m.
- 6** **a** There is a two-digit number $\square\square$ such that, if you add the digit 1 at the front and the back to obtain a four-digit number $1\square\square 1$, then the new four-digit number is 21 times larger than the original two-digit number. Find this two-digit number.
b Find a five-digit number $\square\square\square\square\square$ such that the six-digit number obtained by adding a 1 at the back, $\square\square\square\square\square 1$, is three times larger than the six-digit number obtained by adding a 1 at the front, $1\square\square\square\square\square$.
- 7** Find the possible integer values of k if:
- a** the quadratic equation $x^2 + kx - 16 = 0$ has integer solutions
b the quadratic equation $x^2 + kx + 20 = 0$ has integer solutions
c the quadratic equation $x^2 + 12x + k = 0$ has integer solutions (k positive).

8 a Show that $x^2 + (1 - x)^2 = 2 \left[\left(x - \frac{1}{2} \right)^2 + \frac{1}{4} \right]$.

b Hence show that, if $0 \leq x \leq 1$, then

$$\frac{1}{2} \leq x^2 + (1 - x)^2 \leq 1$$

c A quadrilateral has one vertex on each side of a unit square (that is, a square of side length 1). Show that the side lengths a , b , c and d of the quadrilateral satisfy

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$$

9 Consider the quadratic expression $x^2 + bx + c$, where b and c are real numbers.

a Given that the equation $x^2 + bx + c + 1 = 0$ has only one solution, find c in terms of b .

b Given that the expression $x^2 + bx + c - 3$ can be factorised as $(x - k)(x - 2k)$, for some non-zero real number k , find c in terms of b .

c If the conditions of both parts **a** and **b** are satisfied, find the possible values of b and c .

10 a Find positive integers m and n such that $\sqrt{9 - 4\sqrt{5}} = \sqrt{m} - n$.

b Given that $x = \sqrt{9 - 4\sqrt{5}}$ is a solution of the quadratic equation

$$x^2 + bx + c = 0 \quad \text{where } b \text{ and } c \text{ are integers}$$

find the values of b and c .

11 Prove that the sum of the squares of five consecutive integers

$$(n - 2)^2 + (n - 1)^2 + n^2 + (n + 1)^2 + (n + 2)^2$$

is divisible by 5.

12 a Write down the equation of the line that passes through the point $P(a, b)$ and has gradient m .

b Given that this line touches (but does not cross) the circle with equation $x^2 + y^2 = r^2$, show that

$$(a^2 - r^2)m^2 - 2abm + (b^2 - r^2) = 0$$

c Using part **b**, find the equations of the two tangents from the point $P(-3, 2)$ to the circle with equation $x^2 + y^2 = 4$.

6

Vectors in the plane

Objectives

- ▶ To understand the concept of a **vector** and to apply the basic operations on vectors.
- ▶ To recognise when two vectors are **parallel**.
- ▶ To use the unit vectors i and j to represent vectors in two dimensions.
- ▶ To express vectors in **polar form** and **component form**, and to be able to convert from one form to the other.
- ▶ To find the **scalar product** of two vectors.
- ▶ To use the scalar product to find the magnitude of the angle between two vectors.
- ▶ To use the scalar product to recognise when two vectors are **perpendicular**.
- ▶ To resolve a vector into **rectangular components**, where one component is parallel to a given vector and the other component is perpendicular.
- ▶ To apply vectors to displacement, velocity, relative velocity and equilibrium.

In scientific experiments, some of the things that are measured are completely determined by their magnitude. Mass, length and time are determined by a number and an appropriate unit of measurement.

length 30 cm is the length of the page of a particular book

time 10 s is the time for one athlete to run 100 m

More is required to describe displacement, velocity or force. The direction must be recorded as well as the magnitude.

displacement 30 km in the direction north

velocity 60 km/h in the direction south-east

A quantity that has both a magnitude and a direction is called a **vector**.

6A Introduction to vectors

Suppose that you are asked: ‘Where is your school in relation to your house?’

It is not enough to give an answer such as ‘four kilometres’. You need to specify a direction as well as a distance. You could give the answer ‘four kilometres north-east’.

Position is an example of a vector quantity.

► Directed line segments

A quantity that has a direction as well as a magnitude can be represented by an arrow:

- the arrow points in the direction of the action
- the length of the arrow gives the magnitude of the quantity in terms of a suitably chosen unit.

Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

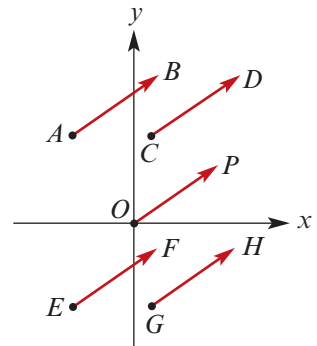
The five directed line segments shown all have the same length and direction, and so they are equivalent.

A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.

For simplicity of language, this is also called vector $\overrightarrow{AB}$.

That is, the set of equivalent segments can be named through one member of the set.

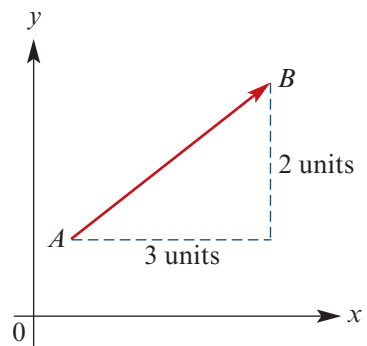
Note: The five directed line segments in the diagram all name the same vector: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$.



► Column vectors

We can also represent a vector using a column of numbers. The column of numbers corresponds to a set of equivalent directed line segments.

For example, the column $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the directed line segments which go 3 across and 2 up.



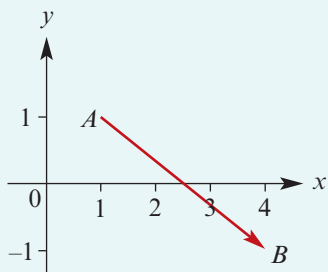
► Vector notation

A vector is often denoted by a single bold lowercase letter. The vector from A to B can be denoted by $\overrightarrow{AB}$ or by a single letter $\mathbf{v}$. That is, $\mathbf{v} = \overrightarrow{AB}$.

When a vector is handwritten, the notation is $\underline{v}$.

Example 1

Draw a directed line segment corresponding to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Solution**Explanation**

The vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is '3 to the right and 2 down'.

Note: Here the segment starts at (1, 1) and goes to (4, -1). It can start at any point.

Example 2

The vector $\mathbf{u}$ is defined by the directed line segment from (2, 6) to (3, 1).

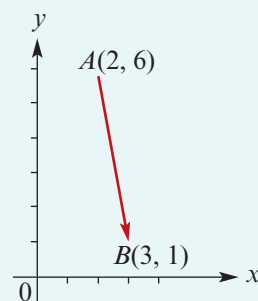
If $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b .

Solution

The vector is

$$\mathbf{u} = \begin{bmatrix} 3 - 2 \\ 1 - 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Hence $a = 1$ and $b = -5$.

Explanation**► Addition of vectors****Adding vectors geometrically**

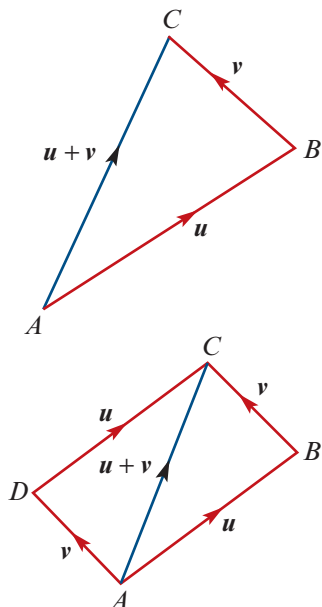
Two vectors $\mathbf{u}$ and $\mathbf{v}$ can be added geometrically by drawing a line segment representing $\mathbf{u}$ from A to B and then a line segment representing $\mathbf{v}$ from B to C .

The sum $\mathbf{u} + \mathbf{v}$ is the vector from A to C . That is,

$$\mathbf{u} + \mathbf{v} = \overrightarrow{AC}$$

The same result is achieved if the order is reversed. This is represented in the diagram on the right:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \overrightarrow{AC} \\ &= \mathbf{v} + \mathbf{u} \end{aligned}$$

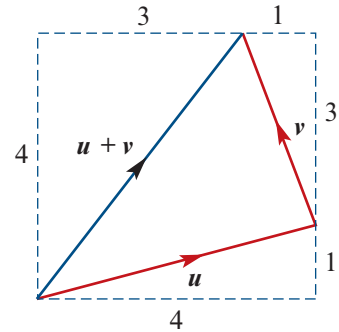


Adding column vectors

Two vectors can be added using column-vector notation.

For example, if $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then

$$u + v = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



► Scalar multiplication

Multiplication by a real number (scalar) changes the length of the vector. For example:

- $2u$ is twice the length of u
- $\frac{1}{2}u$ is half the length of u

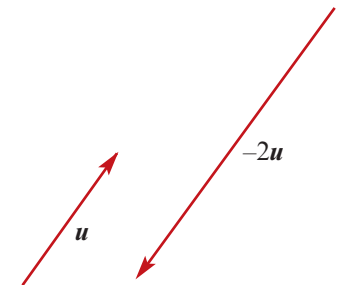
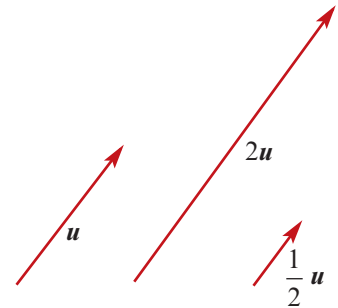
We have $2u = u + u$ and $\frac{1}{2}u + \frac{1}{2}u = u$.

In general, for $k \in \mathbb{R}^+$, the vector ku has the same direction as u , but its length is multiplied by a factor of k .

When a vector is multiplied by -2 , the vector's direction is reversed and the length is doubled.

When a vector is multiplied by -1 , the vector's direction is reversed and the length remains the same.

If $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $-u = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $2u = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2u = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.



If $u = \overrightarrow{AB}$, then

$$-u = -\overrightarrow{AB} = \overrightarrow{BA}$$

The directed line segment $-\overrightarrow{AB}$ goes from B to A .

► Zero vector

The **zero vector** is denoted by $\mathbf{0}$ and represents a line segment of zero length. The zero vector has no direction.

► Subtraction of vectors

To find $u - v$, we add $-v$ to u .



Example 3

For the vectors $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, find $2\mathbf{u} + 3\mathbf{v}$.

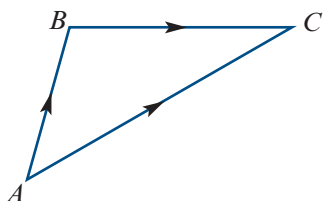
Solution

$$\begin{aligned} 2\mathbf{u} + 3\mathbf{v} &= 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{aligned}$$

► Polygons of vectors

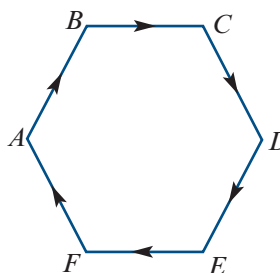
■ For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



■ For a polygon $ABCDEF$, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$$

**► Parallel vectors**

Two parallel vectors have the same direction or opposite directions.

Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\mathbf{u} = k\mathbf{v}$.

For example, if $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, then the vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel as $\mathbf{v} = 3\mathbf{u}$.

► Position vectors

We can use a point O , the origin, as a starting point for a vector to indicate the position of a point A in space relative to O .

In this chapter, we study vectors in two dimensions and the point O is the origin of the Cartesian plane. (Vectors in three dimensions are studied in Specialist Mathematics Units 3 & 4.)

For a point A , the **position vector** is $\overrightarrow{OA}$.

► Finding the midpoint of a line segment

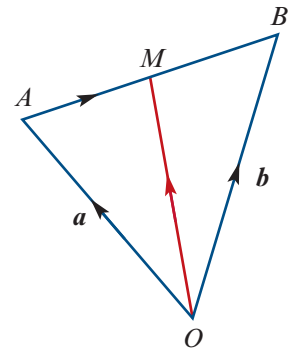
Let M be the midpoint of a line segment AB , where the points A and B have position vectors $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$ respectively.

Then

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

The position vector of the midpoint M is

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} \\ &= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) \\ &= \frac{1}{2}(\mathbf{a} + \mathbf{b})\end{aligned}$$



If M is the midpoint of line segment AB , then

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

► Linear combinations of non-parallel vectors

If two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are not parallel, then

$$m\mathbf{a} + n\mathbf{b} = p\mathbf{a} + q\mathbf{b} \quad \text{implies} \quad m = p \text{ and } n = q$$

Proof Assume that $m\mathbf{a} + n\mathbf{b} = p\mathbf{a} + q\mathbf{b}$. Then

$$m\mathbf{a} - p\mathbf{a} = q\mathbf{b} - n\mathbf{b}$$

$$\therefore (m - p)\mathbf{a} = (q - n)\mathbf{b}$$

If $m \neq p$ or $n \neq q$, we could therefore write

$$\mathbf{a} = \frac{q - n}{m - p} \mathbf{b} \quad \text{or} \quad \mathbf{b} = \frac{m - p}{q - n} \mathbf{a}$$

But this is not possible, as $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors that are not parallel.

Therefore $m = p$ and $n = q$.

Parallelograms

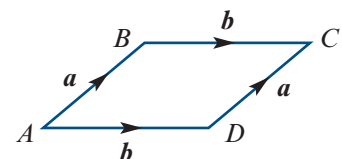
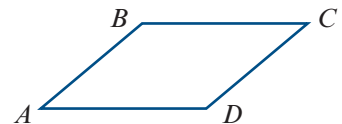
A parallelogram $ABCD$ is a quadrilateral whose opposite sides are parallel. Therefore $\overrightarrow{DC} = k\overrightarrow{AB}$ and $\overrightarrow{BC} = \ell\overrightarrow{AD}$, for some $k, \ell \in \mathbb{R} \setminus \{0\}$. But we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AD} + \overrightarrow{DC}$$

$$\therefore \overrightarrow{AB} + \ell\overrightarrow{AD} = \overrightarrow{AD} + k\overrightarrow{AB}$$

So the previous result gives $k = 1$ and $\ell = 1$.

We have proved that every parallelogram is spanned by two vectors $\mathbf{a}$ and $\mathbf{b}$ as shown on the right.



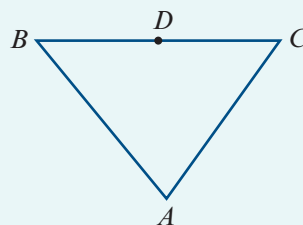
Example 4

Let A , B and C be the vertices of a triangle, and let D be the midpoint of BC .

Let $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$.

Find each of the following in terms of $\mathbf{a}$ and $\mathbf{b}$:

- a** $\overrightarrow{BD}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{AC}$
d $\overrightarrow{AD}$ **e** $\overrightarrow{CA}$

**Solution**

$$\mathbf{a} \quad \overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\mathbf{b}$$

$$\mathbf{b} \quad \overrightarrow{DC} = \overrightarrow{BD} = \frac{1}{2}\mathbf{b}$$

$$\mathbf{c} \quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{a} + \mathbf{b}$$

$$\mathbf{d} \quad \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \mathbf{a} + \frac{1}{2}\mathbf{b}$$

$$\mathbf{e} \quad \overrightarrow{CA} = -\overrightarrow{AC} = -(\mathbf{a} + \mathbf{b})$$

Explanation

same direction and half the length

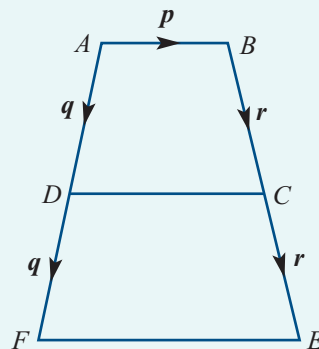
equivalent vectors

since $\overrightarrow{CA} + \overrightarrow{AC} = \mathbf{0}$

**Example 5**

In the figure, $\overrightarrow{DC} = k\mathbf{p}$ where $k \in \mathbb{R} \setminus \{0\}$.

- a** Express $\mathbf{p}$ in terms of k , $\mathbf{q}$ and $\mathbf{r}$.
b Express $\overrightarrow{FE}$ in terms of k and $\mathbf{p}$ to show that FE is parallel to DC .
c If $\overrightarrow{FE} = 4\overrightarrow{AB}$, find the value of k .

**Solution**

$$\begin{aligned} \mathbf{a} \quad \mathbf{p} &= \overrightarrow{AB} \\ &= \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB} \\ &= \mathbf{q} + k\mathbf{p} - \mathbf{r} \end{aligned}$$

Therefore

$$(1 - k)\mathbf{p} = \mathbf{q} - \mathbf{r}$$

$$\begin{aligned} \mathbf{b} \quad \overrightarrow{FE} &= -2\mathbf{q} + \mathbf{p} + 2\mathbf{r} \\ &= 2(\mathbf{r} - \mathbf{q}) + \mathbf{p} \end{aligned}$$

From part **a**, we have

$$\begin{aligned} \mathbf{r} - \mathbf{q} &= k\mathbf{p} - \mathbf{p} \\ &= (k - 1)\mathbf{p} \end{aligned}$$

Therefore

$$\begin{aligned} \overrightarrow{FE} &= 2(k - 1)\mathbf{p} + \mathbf{p} \\ &= 2k\mathbf{p} - 2\mathbf{p} + \mathbf{p} \\ &= (2k - 1)\mathbf{p} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \overrightarrow{FE} &= 4\overrightarrow{AB} \\ (2k - 1)\mathbf{p} &= 4\mathbf{p} \\ 2k - 1 &= 4 \\ \therefore k &= \frac{5}{2} \end{aligned}$$

Section summary

■ A **vector** is a set of equivalent **directed line segments**.

■ **Addition of vectors**

If $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$, then $\mathbf{u} + \mathbf{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

■ **Scalar multiplication**

- For $k \in \mathbb{R}^+$, the vector $k\mathbf{u}$ has the same direction as $\mathbf{u}$, but its length is multiplied by a factor of k .
- If $\mathbf{u} = \overrightarrow{AB}$, then $-\mathbf{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.

■ **Zero vector**

The **zero vector**, denoted by $\mathbf{0}$, has zero length and has no direction.

■ **Subtraction of vectors**

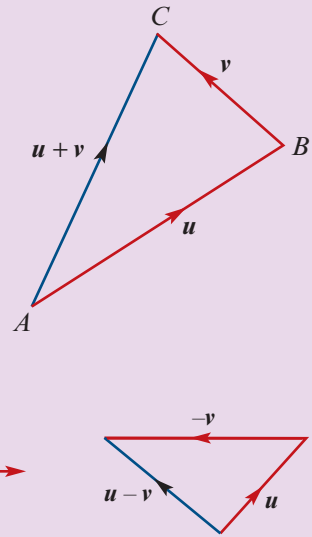
$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

■ **Parallel vectors**

Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\mathbf{u} = k\mathbf{v}$.

■ **Midpoint of a line segment**

If M is the midpoint of line segment AB , then $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.



Exercise 6A

Skillsheet

1 On the same graph, draw arrows which represent the following vectors:

Example 1

a $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

b $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

c $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$

d $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Example 2

2 The vector $\mathbf{u}$ is defined by the directed line segment from $(1, 5)$ to $(6, 6)$.

If $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b .

3 The vector $\mathbf{v}$ is defined by the directed line segment from $(-1, 5)$ to $(2, -10)$.

If $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b .

4 Let $A = (1, -2)$, $B = (3, 0)$ and $C = (2, -3)$ and let O be the origin.

Express each of the following vectors in the form $\begin{bmatrix} a \\ b \end{bmatrix}$:

a $\overrightarrow{OA}$

b $\overrightarrow{AB}$

c $\overrightarrow{BC}$

d $\overrightarrow{CO}$

e $\overrightarrow{CB}$

Example 3

5 Let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

a Find:

i $\mathbf{a} + \mathbf{b}$ **ii** $2\mathbf{c} - \mathbf{a}$ **iii** $\mathbf{a} + \mathbf{b} - \mathbf{c}$

b Show that $\mathbf{a} + \mathbf{b}$ is parallel to $\mathbf{c}$.

SF

- 6** If $A = (2, -3)$, $B = (4, 0)$, $C = (1, -4)$ and O is the origin, sketch the following vectors:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

- 7** On graph paper, sketch the vectors joining the following pairs of points in the direction indicated:

a $(0, 0) \rightarrow (2, 1)$ **b** $(3, 4) \rightarrow (0, 0)$ **c** $(1, 3) \rightarrow (3, 4)$
d $(2, 4) \rightarrow (4, 3)$ **e** $(-2, 2) \rightarrow (5, -1)$ **f** $(-1, -3) \rightarrow (3, 0)$

- 8** Identify vectors from Question 7 which are parallel to each other.

- 9 a** Plot the points $A(-1, 0)$, $B(1, 4)$, $C(4, 3)$ and $D(2, -1)$ on a set of coordinate axes.

b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$.

c Show that:

i $\overrightarrow{AB} = \overrightarrow{DC}$ **ii** $\overrightarrow{BC} = \overrightarrow{AD}$

d Describe the shape of the quadrilateral $ABCD$.

- 10** Find the values of m and n such that $m \begin{bmatrix} 3 \\ -3 \end{bmatrix} + n \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -19 \\ 61 \end{bmatrix}$

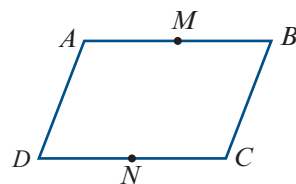
Example 4

- 11** Points A, B, C, D are the vertices of a parallelogram, and M and N are the midpoints of AB and DC respectively. Let $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AD}$.

a Express the following in terms of $\mathbf{a}$ and $\mathbf{b}$:

i $\overrightarrow{MD}$ **ii** $\overrightarrow{MN}$

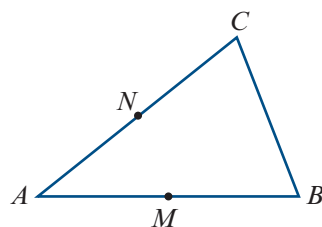
b Find the relationship between $\overrightarrow{MN}$ and $\overrightarrow{AD}$.



- 12** The figure represents the triangle ABC , where M and N are the midpoints of AB and AC respectively. Let $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AC}$.

a Express $\overrightarrow{CB}$ and $\overrightarrow{MN}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Hence describe the relation between the two vectors (or directed line segments).

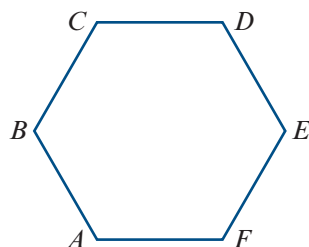


Example 5

- 13** The figure shows a regular hexagon $ABCDEF$. Let $\mathbf{a} = \overrightarrow{AF}$ and $\mathbf{b} = \overrightarrow{AB}$.

Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$:

a $\overrightarrow{CD}$ **b** $\overrightarrow{ED}$ **c** $\overrightarrow{BE}$ **d** $\overrightarrow{FC}$
e $\overrightarrow{FA}$ **f** $\overrightarrow{FB}$ **g** $\overrightarrow{FE}$



- 14** In parallelogram $ABCD$, let $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$. Express each of the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$:

a $\overrightarrow{DC}$ **b** $\overrightarrow{DA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{CA}$ **e** $\overrightarrow{BD}$

- 15** In triangle OAB , let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$. The point P on AB is such that $\overrightarrow{AP} = 2\overrightarrow{PB}$ and the point Q is such that $\overrightarrow{OP} = 3\overrightarrow{PQ}$. Express each of the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$:

a $\overrightarrow{BA}$ **b** $\overrightarrow{PB}$ **c** $\overrightarrow{OP}$ **d** $\overrightarrow{PQ}$ **e** $\overrightarrow{BQ}$

- 16** $PQRS$ is a quadrilateral in which $\overrightarrow{PQ} = \mathbf{u}$, $\overrightarrow{QR} = \mathbf{v}$ and $\overrightarrow{RS} = \mathbf{w}$. Express each of the following vectors in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$:

a $\overrightarrow{PR}$ **b** $\overrightarrow{QS}$ **c** $\overrightarrow{PS}$

- 17** $OABC$ is a parallelogram. Let $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OC}$. Let M be the midpoint of AB .

- a** Express $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.
b Express $\overrightarrow{CM}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.
c If P is a point on CM and $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$, express $\overrightarrow{CP}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.
d Find $\overrightarrow{OP}$ and hence show that P lies on the line segment OB .
e Find the ratio $OP : PB$.

6B Components of vectors

The vector $\overrightarrow{AB}$ in the diagram is described by the column vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

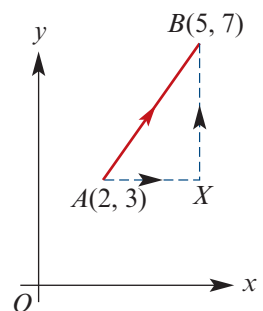
From the diagram, we see that the vector $\overrightarrow{AB}$ can also be expressed as the sum

$$\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$$

Using column-vector notation:

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

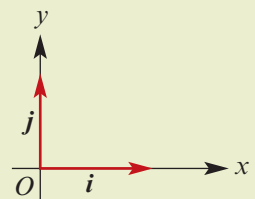
This suggests the introduction of two important vectors.



Standard unit vectors in two dimensions

- Let $\mathbf{i}$ be the vector of unit length in the positive direction of the x -axis.
- Let $\mathbf{j}$ be the vector of unit length in the positive direction of the y -axis.

Using column-vector notation, we have $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Sometimes you write $\mathbf{i}$ and $\mathbf{j}$ as $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

For the example at the start of this section, we have $\overrightarrow{AX} = 3\mathbf{i}$ and $\overrightarrow{XB} = 4\mathbf{j}$. Therefore

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AX} + \overrightarrow{XB} \\ &= 3\mathbf{i} + 4\mathbf{j}\end{aligned}$$

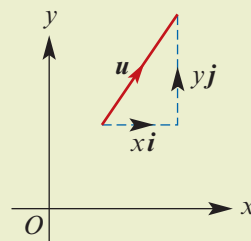
It is possible to describe any two-dimensional vector in this way.

Component form

- We can write the vector $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ as $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$.

We say that $\mathbf{u}$ is the sum of the two **components** $x\mathbf{i}$ and $y\mathbf{j}$.

- The **magnitude** of vector $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$ is denoted by $|\mathbf{u}|$ and is given by $|\mathbf{u}| = \sqrt{x^2 + y^2}$.



Operations with vectors now look more like basic algebra:

- $(x\mathbf{i} + y\mathbf{j}) + (m\mathbf{i} + n\mathbf{j}) = (x + m)\mathbf{i} + (y + n)\mathbf{j}$
- $k(x\mathbf{i} + y\mathbf{j}) = kx\mathbf{i} + ky\mathbf{j}$

Two vectors are equal if and only if their components are equal:

$$x\mathbf{i} + y\mathbf{j} = m\mathbf{i} + n\mathbf{j} \quad \text{if and only if} \quad x = m \text{ and } y = n$$

Example 6

Let A and B be points in the Cartesian plane such that $\overrightarrow{OA} = 3\mathbf{i}$ and $\overrightarrow{OB} = 2\mathbf{i} - \mathbf{j}$.

a Find $\overrightarrow{AB}$.

b Find $\overrightarrow{OM}$, where M is the midpoint of AB .

Solution

$$\begin{aligned}\mathbf{a} \quad \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -3\mathbf{i} + (2\mathbf{i} - \mathbf{j}) \\ &= -\mathbf{i} - \mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \overrightarrow{OM} &= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) \\ &= \frac{1}{2}(3\mathbf{i} + 2\mathbf{i} - \mathbf{j}) \\ &= \frac{5}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\end{aligned}$$

Example 7

Find the magnitude of the vector $2\mathbf{i} - 3\mathbf{j}$.

Solution

$$\begin{aligned}|2\mathbf{i} - 3\mathbf{j}| &= \sqrt{2^2 + (-3)^2} \\ &= \sqrt{4 + 9} \\ &= \sqrt{13}\end{aligned}$$

Example 8

Let A and B be points in the Cartesian plane such that $\overrightarrow{OA} = 2\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OB} = \mathbf{i} - 3\mathbf{j}$.
Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

Solution

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ \therefore \overrightarrow{AB} &= -(2\mathbf{i} + \mathbf{j}) + \mathbf{i} - 3\mathbf{j} \\ &= -\mathbf{i} - 4\mathbf{j} \\ \therefore |\overrightarrow{AB}| &= \sqrt{1 + 16} = \sqrt{17}\end{aligned}$$

► Unit vectors

A **unit vector** is a vector of length one unit. For example, both $\mathbf{i}$ and $\mathbf{j}$ are unit vectors.

The unit vector in the direction of $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$. (We say ‘a hat’.)

Since $|\hat{\mathbf{a}}| = 1$, we have

$$\begin{aligned}|\mathbf{a}| \hat{\mathbf{a}} &= \mathbf{a} \\ \therefore \hat{\mathbf{a}} &= \frac{1}{|\mathbf{a}|} \mathbf{a}\end{aligned}$$

Example 9

Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$.

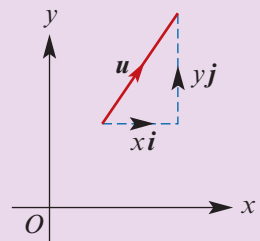
Find $|\mathbf{a}|$, the magnitude of $\mathbf{a}$, and hence find the unit vector in the direction of $\mathbf{a}$.

Solution

$$\begin{aligned}|\mathbf{a}| &= \sqrt{9 + 16} = 5 \\ \therefore \hat{\mathbf{a}} &= \frac{1}{|\mathbf{a}|} \mathbf{a} = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j})\end{aligned}$$

Section summary

- A **unit vector** is a vector of length one unit.
- Each vector $\mathbf{u}$ in the plane can be written in **component form** as $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$, where
 - $\mathbf{i}$ ($\hat{\mathbf{i}}$) is the unit vector in the positive direction of the x -axis
 - $\mathbf{j}$ ($\hat{\mathbf{j}}$) is the unit vector in the positive direction of the y -axis.
- The **magnitude** of vector $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$ is given by $|\mathbf{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\mathbf{a}$ is given by $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \mathbf{a}$.



Exercise 6B

Skillsheet

Example 6a

1 If A and B are points in the plane such that $\vec{OA} = \mathbf{i} + 2\mathbf{j}$ and $\vec{OB} = 3\mathbf{i} - 5\mathbf{j}$, find $\vec{AB}$.

2 $OAPB$ is a rectangle with $\vec{OA} = 5\mathbf{i}$ and $\vec{OB} = 6\mathbf{j}$. Express each of the following vectors in terms of $\mathbf{i}$ and $\mathbf{j}$:

a $\vec{OP}$

b $\vec{AB}$

c $\vec{BA}$

Example 6b

3 Points A and B have position vectors $\vec{OA} = 10\mathbf{i}$ and $\vec{OB} = 4\mathbf{i} + 5\mathbf{j}$. If M is the midpoint of AB , find $\vec{OM}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

Example 7

4 Determine the magnitude of each of the following vectors:

a $5\mathbf{i}$

b $-2\mathbf{j}$

c $3\mathbf{i} + 4\mathbf{j}$

d $-5\mathbf{i} + 12\mathbf{j}$

5 The vectors $\mathbf{u}$ and $\mathbf{v}$ are given by $\mathbf{u} = 7\mathbf{i} + 8\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$.

a Find $|\mathbf{u} - \mathbf{v}|$.b Find constants x and y such that $x\mathbf{u} + y\mathbf{v} = 44\mathbf{j}$.

6 $OPAQ$ is a rectangle with $\vec{OP} = 2\mathbf{i}$ and $\vec{OQ} = \mathbf{j}$. Let M be the point on OP such that $OM = \frac{1}{5}OP$ and let N be the point on MQ such that $MN = \frac{1}{6}MQ$.

a Find each of the following vectors in terms of $\mathbf{i}$ and $\mathbf{j}$:

i $\vec{OM}$

ii $\vec{MQ}$

iii $\vec{MN}$

iv $\vec{ON}$

v $\vec{OA}$

b i Hence show that N is on the diagonal OA .ii State the ratio of the lengths $ON : NA$.

7 The position vectors of A and B are given by $\vec{OA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{OB} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Find the distance between A and B .

8 Find the pronumerals in the following equations:

a $\mathbf{i} + 3\mathbf{j} = 2(\ell\mathbf{i} + k\mathbf{j})$

b $(x - 1)\mathbf{i} + y\mathbf{j} = 5\mathbf{i} + (x - 4)\mathbf{j}$

c $(x + y)\mathbf{i} + (x - y)\mathbf{j} = 6\mathbf{i}$

d $k(\mathbf{i} + \mathbf{j}) = 3\mathbf{i} - 2\mathbf{j} + \ell(2\mathbf{i} - \mathbf{j})$

Example 8

9 Let $A = (2, 3)$ and $B = (5, 1)$. Find $\vec{AB}$ and $|\vec{AB}|$.

10 Let $\vec{OA} = 3\mathbf{i}$, $\vec{OB} = \mathbf{i} + 4\mathbf{j}$ and $\vec{OC} = -3\mathbf{i} + \mathbf{j}$. Find:

a $\vec{AB}$

b $\vec{AC}$

c $|\vec{BC}|$

11 Let $A = (5, 1)$, $B = (0, 4)$ and $C = (-1, 0)$. Find:

a D such that $\vec{AB} = \vec{CD}$

b F such that $\vec{AF} = \vec{BC}$

c G such that $\vec{AB} = 2\vec{GC}$

SF

- 12** Let $\mathbf{a} = \mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} + 2\mathbf{j}$. Points A , B and C are such that $\overrightarrow{AO} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{BC} = 2\mathbf{a}$, where O is the origin. Find the coordinates of A , B and C .

SF

- 13** A , B , C and D are the vertices of a parallelogram and O is the origin.
 $A = (2, -1)$, $B = (-5, 4)$ and $C = (1, 7)$.

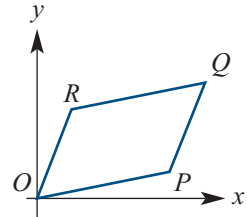
CF

a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{BC}$ **v** $\overrightarrow{AD}$

b Hence find the coordinates of D .

- 14** The diagram shows a parallelogram $OPQR$.
 The points P and Q have coordinates $(12, 5)$ and $(18, 13)$ respectively.



a Find $\overrightarrow{OP}$ and $\overrightarrow{PQ}$.

b Find $|\overrightarrow{RQ}|$ and $|\overrightarrow{OR}|$.

- 15** $A(1, 6)$, $B(3, 1)$ and $C(13, 5)$ are the vertices of a triangle ABC .

a Find:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

b Hence show that ABC is a right-angled triangle.

- 16** $A(4, 4)$, $B(3, 1)$ and $C(7, 3)$ are the vertices of a triangle ABC .

a Find the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

b Find:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

c Hence show that triangle ABC is a right-angled isosceles triangle.

- 17** Consider points $A(-3, 2)$ and $B(0, 7)$ in the Cartesian plane. Let O be the origin and let M be the point on the line segment AB such that $AM = \frac{1}{3}AB$.

a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{BA}$ **iv** $\overrightarrow{BM}$

b Hence find the coordinates of M . (Hint: $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$.)

Example 9

- 18** Find the unit vector in the direction of each of the following vectors:

a $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$

b $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$

c $\mathbf{c} = -\mathbf{i} + \mathbf{j}$

d $\mathbf{d} = \mathbf{i} - \mathbf{j}$

e $\mathbf{e} = \frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j}$

f $\mathbf{f} = 6\mathbf{i} - 4\mathbf{j}$

SF

6C Polar form of a vector

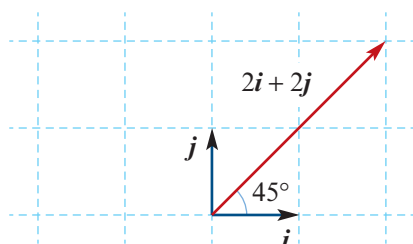
So far in this chapter, we have represented vectors using directed line segments, column vectors and component form. In this section, we introduce another way to describe vectors.

Each vector $\mathbf{v}$ in the plane can be written in **polar form** as $\mathbf{v} = [r, \theta]$, where

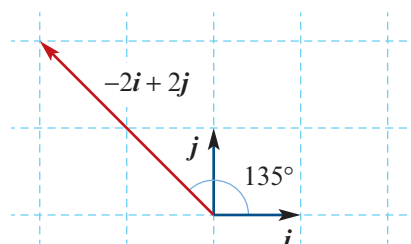
- r is the magnitude of $\mathbf{v}$
- θ describes the angle that $\mathbf{v}$ makes with the i -direction, measured anticlockwise.

For example:

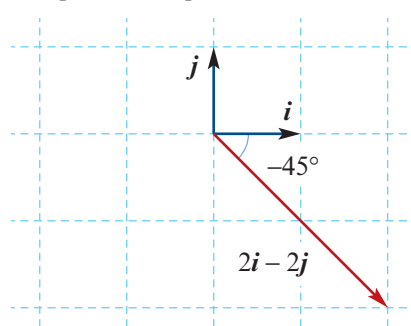
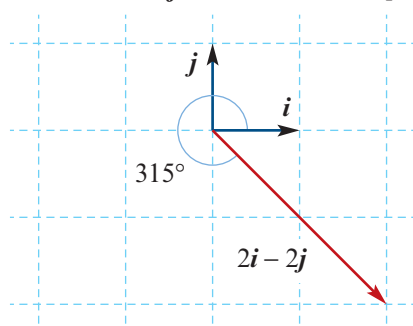
The vector $2\mathbf{i} + 2\mathbf{j}$ can be written as $[2\sqrt{2}, 45^\circ]$.



The vector $-2\mathbf{i} + 2\mathbf{j}$ can be written as $[2\sqrt{2}, 135^\circ]$.



The vector $2\mathbf{i} - 2\mathbf{j}$ can be written as $[2\sqrt{2}, 315^\circ]$ or as $[2\sqrt{2}, -45^\circ]$.



Note: From the example $2\mathbf{i} - 2\mathbf{j}$ above, we see that the same vector can have different representations in polar form. Generally, we will prefer to choose $-180^\circ < \theta \leq 180^\circ$, so that the representation is unique.

We can convert between polar form and component form using basic trigonometry.

■ Converting from polar form to component form

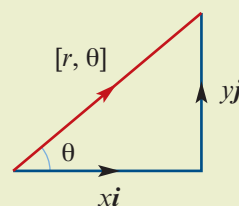
Given $\mathbf{v} = [r, \theta]$, we can write $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, where

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

■ Converting from component form to polar form

Given $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, we can write $\mathbf{v} = [r, \theta]$, where

$$r = \sqrt{x^2 + y^2}, \quad \cos \theta = \frac{x}{r} \quad \text{and} \quad \sin \theta = \frac{y}{r}$$



Example 10

Write each of the following vectors in component form:

a $[3, 30^\circ]$

b $[4, -60^\circ]$

c $[2, 50^\circ]$

Solution

a $x = 3 \cos 30^\circ$

$y = 3 \sin 30^\circ$

$$\therefore [3, 30^\circ] = \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$$

b $x = 4 \cos(-60^\circ)$

$y = 4 \sin(-60^\circ)$

$$\therefore [4, -60^\circ] = 2\mathbf{i} - 2\sqrt{3}\mathbf{j}$$

c $x = 2 \cos 50^\circ$

$y = 2 \sin 50^\circ$

$$\therefore [2, 50^\circ] \approx 1.29\mathbf{i} + 1.53\mathbf{j}$$

Example 11

Write each of the following vectors in polar form:

a $-4\sqrt{3}\mathbf{i} + 4\mathbf{j}$

b $-4\mathbf{i} - 4\mathbf{j}$

Solution**a** We have $x = -4\sqrt{3}$ and $y = 4$, giving

$$\begin{aligned}
 r &= \sqrt{x^2 + y^2} \\
 &= \sqrt{48 + 16} \\
 &= 8
 \end{aligned}$$

The vector $-4\sqrt{3}\mathbf{i} + 4\mathbf{j}$ points to the 2nd quadrant, and so $90^\circ < \theta < 180^\circ$.

We know that

$$\cos \theta = \frac{x}{r} = -\frac{\sqrt{3}}{2}$$

$$\text{and } \sin \theta = \frac{y}{r} = \frac{1}{2}$$

Hence $\theta = 150^\circ$ and therefore

$$-4\sqrt{3}\mathbf{i} + 4\mathbf{j} = [8, 150^\circ]$$

b We have $x = -4$ and $y = -4$, giving

$$\begin{aligned}
 r &= \sqrt{x^2 + y^2} \\
 &= \sqrt{16 + 16} \\
 &= 4\sqrt{2}
 \end{aligned}$$

The vector $-4\mathbf{i} - 4\mathbf{j}$ points to the 3rd quadrant, and so $-180^\circ < \theta < -90^\circ$.

We know that

$$\cos \theta = \frac{x}{r} = -\frac{1}{\sqrt{2}}$$

$$\text{and } \sin \theta = \frac{y}{r} = -\frac{1}{\sqrt{2}}$$

Hence $\theta = -135^\circ$ and therefore

$$-4\mathbf{i} - 4\mathbf{j} = [4\sqrt{2}, -135^\circ]$$

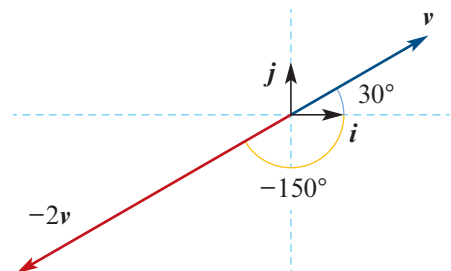
► Basic operations on vectors in polar form

We can easily perform scalar multiplication for vectors written in polar form.

For example, if $\mathbf{v} = [3, 30^\circ]$, then

$$\begin{aligned}
 -2\mathbf{v} &= [2 \times 3, (30 - 180)^\circ] \\
 &= [6, -150^\circ]
 \end{aligned}$$

Here the magnitude is doubled and the direction is reversed.



The next example demonstrates two different methods for finding the sum of two vectors given in polar form.

Example 12

Find the sum of the vectors $\mathbf{u} = [10, 30^\circ]$ and $\mathbf{v} = [12, 90^\circ]$.

Solution

Method 1: Using trigonometry

In the diagram, we have $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{AB}$.

So $\mathbf{u} + \mathbf{v} = \overrightarrow{OB}$.

Use the cosine rule to find OB :

$$\begin{aligned} OB^2 &= 10^2 + 12^2 - 2 \times 10 \times 12 \cos 120^\circ \\ &= 100 + 144 - 240 \times \left(-\frac{1}{2}\right) \\ &= 364 \end{aligned}$$

$$\begin{aligned} \therefore OB &= \sqrt{364} \\ &= 2\sqrt{91} \end{aligned}$$

Now use the sine rule to find $\angle BOA$:

$$\begin{aligned} \frac{12}{\sin(\angle BOA)} &= \frac{2\sqrt{91}}{\sin 120^\circ} \\ \sin(\angle BOA) &= \frac{12 \sin 120^\circ}{2\sqrt{91}} \end{aligned}$$

$$\therefore \angle BOA \approx 33.004^\circ$$

The angle that $\overrightarrow{OB}$ makes with the i -direction is approximately 63.004° .

Hence $\mathbf{u} + \mathbf{v} = \overrightarrow{OB} \approx [2\sqrt{91}, 63.004^\circ]$.

Method 2: Using components

$$\begin{aligned} \text{We have } \mathbf{u} &= 10 \cos 30^\circ \mathbf{i} + 10 \sin 30^\circ \mathbf{j} \\ &= 5\sqrt{3}\mathbf{i} + 5\mathbf{j} \end{aligned}$$

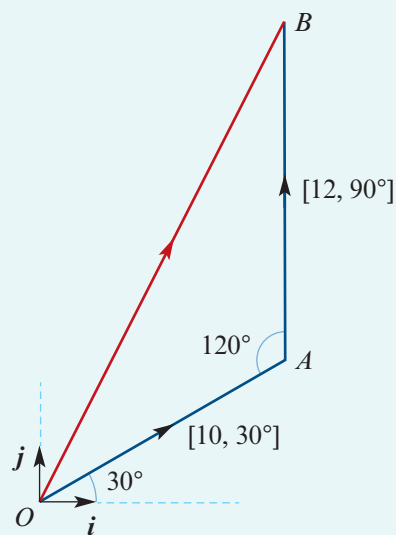
$$\text{and } \mathbf{v} = 12\mathbf{j}$$

$$\text{Hence } \mathbf{u} + \mathbf{v} = 5\sqrt{3}\mathbf{i} + 17\mathbf{j}$$

Note: If required, we can convert $\mathbf{u} + \mathbf{v}$ into polar form $[r, \theta]$ using

$$r = \sqrt{(5\sqrt{3})^2 + 17^2} = 2\sqrt{91} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{17}{5\sqrt{3}}\right) \approx 63.004^\circ$$

However, the answer is neater in component form.

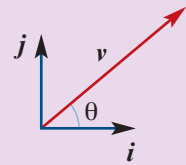


Section summary

- Each vector $\mathbf{v}$ in the plane can be written in **polar form** as $\mathbf{v} = [r, \theta]$, where
 - r is the magnitude of $\mathbf{v}$
 - θ describes the angle that $\mathbf{v}$ makes with the i -direction, measured anticlockwise.
- **Converting from polar form to component form**
If $\mathbf{v} = [r, \theta]$, then $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, where

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$
- **Converting from component form to polar form**
If $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, then $\mathbf{v} = [r, \theta]$, where

$$r = \sqrt{x^2 + y^2}, \quad \cos \theta = \frac{x}{r} \quad \text{and} \quad \sin \theta = \frac{y}{r}$$



Exercise 6C

- 1 For each of the following, draw a sketch to illustrate the vector and write the vector in component form:

a $[2, 30^\circ]$ **b** $[4, -30^\circ]$ **c** $[2, 135^\circ]$ **d** $[4, -120^\circ]$ **e** $[4, 150^\circ]$

Example 10

- 2 Write each of the following vectors in component form:

a $[8, 60^\circ]$ **b** $[10, 30^\circ]$ **c** $[6, 150^\circ]$ **d** $[8, -45^\circ]$ **e** $[12, -150^\circ]$

Example 11

- 3 Write each of the following vectors in polar form:

a $4\mathbf{i} - 4\mathbf{j}$ **b** $-3\mathbf{i} + 3\mathbf{j}$ **c** $-2\sqrt{3}\mathbf{i} + 2\mathbf{j}$ **d** $3\mathbf{i} + 4\mathbf{j}$ **e** $-5\mathbf{i} - 12\mathbf{j}$

- 4 Write each of the following vectors in polar form:

a $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ **d** $\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$ **e** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

- 5 Write each of the following vectors in component form, giving values correct to two decimal places:

a $[7, 40^\circ]$ **b** $[10, 35^\circ]$ **c** $[11, 155^\circ]$ **d** $[9, -35^\circ]$ **e** $[12, -125^\circ]$

- 6 Express each of the following vectors in polar form:

a $2\mathbf{u}$, where $\mathbf{u} = [10, 60^\circ]$ **b** $-\frac{1}{2}\mathbf{v}$, where $\mathbf{v} = [2, 15^\circ]$ **c** $-3\mathbf{w}$, where $\mathbf{w} = [6, -20^\circ]$

Example 12

- 7 Find each vector sum in polar form:

a $[10, 60^\circ] + [15, 90^\circ]$ **b** $[10, 60^\circ] + [10, 45^\circ]$ **c** $[8, -60^\circ] + [8, 50^\circ]$

Example 12

- 8 Find each vector sum in component form:

a $[8, 30^\circ] + [12, 60^\circ]$ **b** $[10, 45^\circ] + [10, -90^\circ]$ **c** $[4, 20^\circ] + [6, 80^\circ]$

6D Scalar product of vectors

The scalar product is an operation that takes two vectors and gives a real number.

Definition of the scalar product

We define the **scalar product** of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$$

For example:

$$\begin{aligned}(2\mathbf{i} + 3\mathbf{j}) \cdot (\mathbf{i} - 4\mathbf{j}) &= 2 \times 1 + 3 \times (-4) \\ &= -10\end{aligned}$$

The scalar product is often called the **dot product**.

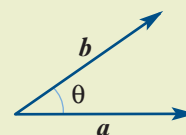
Note: If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} \cdot \mathbf{b} = 0$.

Geometric description of the scalar product

For vectors $\mathbf{a}$ and $\mathbf{b}$, we have

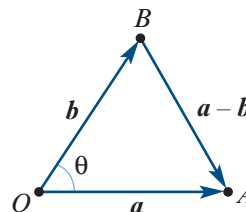
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



Proof Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$. Then using the cosine rule in $\triangle OAB$ gives

$$\begin{aligned}|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}| |\mathbf{b}| \cos \theta &= |\mathbf{a} - \mathbf{b}|^2 \\ (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2|\mathbf{a}| |\mathbf{b}| \cos \theta &= (a_1 - b_1)^2 + (a_2 - b_2)^2 \\ 2(a_1b_1 + a_2b_2) &= 2|\mathbf{a}| |\mathbf{b}| \cos \theta \\ a_1b_1 + a_2b_2 &= |\mathbf{a}| |\mathbf{b}| \cos \theta \\ \therefore \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos \theta\end{aligned}$$

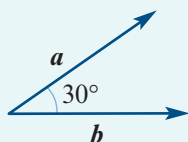


Example 13

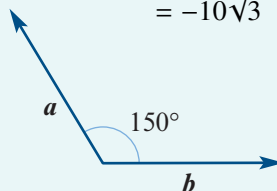
- a** If $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 30° , find $\mathbf{a} \cdot \mathbf{b}$.
b If $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 150° , find $\mathbf{a} \cdot \mathbf{b}$.

Solution

a $\mathbf{a} \cdot \mathbf{b} = 4 \times 5 \times \cos 30^\circ$
 $= 20 \times \frac{\sqrt{3}}{2}$
 $= 10\sqrt{3}$



b $\mathbf{a} \cdot \mathbf{b} = 4 \times 5 \times \cos 150^\circ$
 $= 20 \times \left(-\frac{\sqrt{3}}{2}\right)$
 $= -10\sqrt{3}$



► Properties of the scalar product

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ ■ $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$ ■ $\mathbf{a} \cdot \mathbf{0} = 0$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ ■ $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- If the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a} \cdot \mathbf{b} = 0$.
- If $\mathbf{a} \cdot \mathbf{b} = 0$ for non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.
- For parallel vectors $\mathbf{a}$ and $\mathbf{b}$, we have

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} |\mathbf{a}||\mathbf{b}| & \text{if } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel and in the same direction} \\ -|\mathbf{a}||\mathbf{b}| & \text{if } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel and in opposite directions} \end{cases}$$

► Finding the magnitude of the angle between two vectors

To find the angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$, we can use the two different forms of the scalar product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \quad \text{and} \quad \mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$$

Therefore

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_1b_1 + a_2b_2}{|\mathbf{a}||\mathbf{b}|}$$



Example 14

A , B and C are points defined by the position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j}, \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} \quad \text{and} \quad \mathbf{c} = \mathbf{i} - 2\mathbf{j}$$

Find the magnitude of $\angle ABC$.

Solution

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = -\mathbf{i} + 2\mathbf{j}$$

$$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = -\mathbf{i} - 3\mathbf{j}$$

We will apply the scalar product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos(\angle ABC)$$

We have

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-\mathbf{i} + 2\mathbf{j}) \cdot (-\mathbf{i} - 3\mathbf{j}) = 1 - 6 = -5$$

$$|\overrightarrow{BA}| = \sqrt{1 + 4} = \sqrt{5}$$

$$|\overrightarrow{BC}| = \sqrt{1 + 9} = \sqrt{10}$$

Therefore

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{-5}{\sqrt{5}\sqrt{10}} = \frac{-1}{\sqrt{2}}$$

Hence $\angle ABC = 135^\circ$.

► Scalar product of vectors in polar form

If $\mathbf{a}_1 = [r_1, \theta_1]$ and $\mathbf{a}_2 = [r_2, \theta_2]$, then the geometric description of the scalar product gives

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = r_1 r_2 \cos(\theta_1 - \theta_2)$$

Note: In this chapter, we restrict to $\theta_1, \theta_2 \in (-180^\circ, 180^\circ]$ to ensure $\theta_1 - \theta_2 \in (-360^\circ, 360^\circ)$.

We will consider the cosine of angles outside this range in Chapter 13.

Example 15

Find the scalar product of each of the following pairs of vectors:

- a** $[4, 60^\circ]$ and $[5, 30^\circ]$ **b** $[7, -30^\circ]$ and $[6, 120^\circ]$ **c** $[11, 60^\circ]$ and $[8, -20^\circ]$

Solution

a $[4, 60^\circ] \cdot [5, 30^\circ]$	b $[7, -30^\circ] \cdot [6, 120^\circ]$	c $[11, 60^\circ] \cdot [8, -20^\circ]$
$= 4 \times 5 \cos(60 - 30)^\circ$	$= 7 \times 6 \cos(-30 - 120)^\circ$	$= 11 \times 8 \cos(60 + 20)^\circ$
$= 20 \cos 30^\circ$	$= 42 \cos(-150)^\circ$	$= 88 \cos 80^\circ$
$= 10\sqrt{3}$	$= 42 \cos 150^\circ$	≈ 15.28
	$= -21\sqrt{3}$	

Section summary

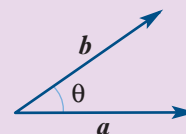
- The **scalar product** of vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

- The scalar product can be described geometrically by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



- Therefore $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.
- Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
- For vectors in polar form $\mathbf{a}_1 = [r_1, \theta_1]$ and $\mathbf{a}_2 = [r_2, \theta_2]$, the scalar product is given by

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = r_1 r_2 \cos(\theta_1 - \theta_2)$$

Exercise 6D

Skillsheet

- 1** Let $\mathbf{a} = \mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c} = -2\mathbf{i} - 2\mathbf{j}$. Find:

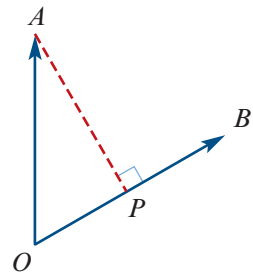
a $\mathbf{a} \cdot \mathbf{a}$	b $\mathbf{b} \cdot \mathbf{b}$	c $\mathbf{c} \cdot \mathbf{c}$	d $\mathbf{a} \cdot \mathbf{b}$
e $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$	f $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{c})$	g $(\mathbf{a} + 2\mathbf{b}) \cdot (3\mathbf{c} - \mathbf{b})$	

- 2** Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{c} = -\mathbf{i} + 3\mathbf{j}$. Find:

a $\mathbf{a} \cdot \mathbf{a}$	b $\mathbf{b} \cdot \mathbf{b}$	c $\mathbf{a} \cdot \mathbf{b}$
d $\mathbf{a} \cdot \mathbf{c}$	e $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b})$	

Example 13

- 3** **a** If $|a| = 5$, $|b| = 6$ and the angle between a and b is 45° , find $a \cdot b$.
b If $|a| = 5$, $|b| = 6$ and the angle between a and b is 135° , find $a \cdot b$.
- 4** Expand and simplify:
a $(a + 2b) \cdot (a + 2b)$ **b** $|a + b|^2 - |a - b|^2$
c $a \cdot (a + b) - b \cdot (a + b)$ **d** $\frac{a \cdot (a + b) - a \cdot b}{|a|}$
- 5** If A and B are points defined by the position vectors $a = 2i + 2j$ and $b = -i + 3j$ respectively, find:
a $\overrightarrow{AB}$ **b** $|\overrightarrow{AB}|$ **c** the magnitude of the angle between vectors $\overrightarrow{AB}$ and a .
- 6** Let C and D be points with position vectors c and d respectively. If $|c| = 5$, $|d| = 7$ and $c \cdot d = 4$, find $|\overrightarrow{CD}|$.
- 7** Solve each of the following equations:
a $(i + 2j) \cdot (5i + xj) = -6$ **b** $(xi + 7j) \cdot (-4i + xj) = 10$
c $(xi + j) \cdot (-2i - 3j) = x$ **d** $x(2i + 3j) \cdot (i + xj) = 6$
- 8** Points A and B are defined by the position vectors $a = i + 4j$ and $b = 2i + 5j$. Let P be the point on OB such that AP is perpendicular to OB . Then $\overrightarrow{OP} = qb$, for a constant q .
a Express $\overrightarrow{AP}$ in terms of q , a and b .
b Use the fact that $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$ to find the value of q .
c Find the coordinates of the point P .



Example 14

- 9** Find the angle, in degrees, between each of the following pairs of vectors, correct to two decimal places:
a $i + 2j$ and $i - 4j$ **b** $-2i + j$ and $-2i - 2j$
c $2i - j$ and $4i$ **d** $7i + j$ and $-i + j$
- 10** Let a and b be non-zero vectors such that $a \cdot b = 0$. Use the geometric description of the scalar product to show that a and b are perpendicular vectors.

Example 15

- 11** Find the scalar product of each of the following pairs of vectors:
a $[10, 45^\circ]$ and $[5, 15^\circ]$ **b** $[8, 70^\circ]$ and $[5, 10^\circ]$ **c** $[12, -60^\circ]$ and $[10, -30^\circ]$
d $[4, 120^\circ]$ and $[9, -60^\circ]$ **e** $[7, 40^\circ]$ and $[2, 70^\circ]$ **f** $[10, -25^\circ]$ and $[4, 35^\circ]$

For Questions 12–13, find the angles in degrees correct to two decimal places.

- 12** Let A and B be the points defined by the position vectors $a = i + j$ and $b = 2i - j$ respectively. Let M be the midpoint of AB . Find:
a $\overrightarrow{OM}$ **b** $\angle AOM$ **c** $\angle BMO$

- 13** Let A , B and C be the points defined by the position vectors $3\mathbf{i}$, $4\mathbf{j}$ and $-2\mathbf{i} + 6\mathbf{j}$ respectively. Let M and N be the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively. Find:

a i $\overrightarrow{OM}$ ii $\overrightarrow{ON}$

b $\angle MON$

c $\angle MOC$

6E Vector projections

It is often useful to decompose a vector $\mathbf{a}$ into a sum of two vectors, one parallel to a given vector $\mathbf{b}$ and the other perpendicular to $\mathbf{b}$.

From the diagram, it can be seen that

$$\mathbf{a} = \mathbf{u} + \mathbf{w}$$

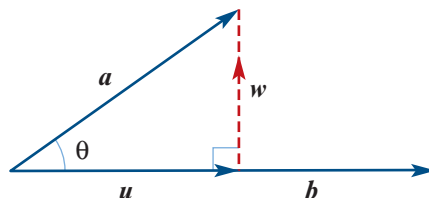
where $\mathbf{u} = k\mathbf{b}$ and so $\mathbf{w} = \mathbf{a} - \mathbf{u} = \mathbf{a} - k\mathbf{b}$.

For $\mathbf{w}$ to be perpendicular to $\mathbf{b}$, we must have

$$\mathbf{w} \cdot \mathbf{b} = 0$$

$$(\mathbf{a} - k\mathbf{b}) \cdot \mathbf{b} = 0$$

$$\mathbf{a} \cdot \mathbf{b} - k(\mathbf{b} \cdot \mathbf{b}) = 0$$



Hence $k = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}$ and therefore $\mathbf{u} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}$.

This vector $\mathbf{u}$ is called the **vector projection** (or **vector resolute**) of $\mathbf{a}$ in the direction of $\mathbf{b}$.

Vector resolute

The **vector resolute** of $\mathbf{a}$ in the direction of $\mathbf{b}$ can be expressed in any one of the following equivalent forms:

$$\mathbf{u} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b} = \left(\mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} \right) \left(\frac{\mathbf{b}}{|\mathbf{b}|} \right) = (\mathbf{a} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}$$

Note: The quantity $\mathbf{a} \cdot \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$ is the ‘signed length’ of the vector resolute $\mathbf{u}$ and is called the **scalar resolute** of $\mathbf{a}$ in the direction of $\mathbf{b}$.

Note that, from our previous calculation, we have

$$\mathbf{w} = \mathbf{a} - \mathbf{u} = \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}$$

Expressing $\mathbf{a}$ as the sum of the two components, the first parallel to $\mathbf{b}$ and the second perpendicular to $\mathbf{b}$, gives

$$\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} + \left(\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} \right)$$

This is sometimes described as resolving the vector $\mathbf{a}$ into **rectangular components**, one parallel to $\mathbf{b}$ and the other perpendicular to $\mathbf{b}$.

**Example 16**

Let $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$. Find the vector resolute of:

a $\mathbf{a}$ in the direction of $\mathbf{b}$

b $\mathbf{b}$ in the direction of $\mathbf{a}$.

Solution

a $\mathbf{a} \cdot \mathbf{b} = 1 - 3 = -2$

$$\mathbf{b} \cdot \mathbf{b} = 1 + 1 = 2$$

The vector resolute of $\mathbf{a}$ in the direction of $\mathbf{b}$ is

$$\begin{aligned} \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} &= \frac{-2}{2}(\mathbf{i} - \mathbf{j}) \\ &= -1(\mathbf{i} - \mathbf{j}) \\ &= -\mathbf{i} + \mathbf{j} \end{aligned}$$

b $\mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} = -2$

$$\mathbf{a} \cdot \mathbf{a} = 1 + 9 = 10$$

The vector resolute of $\mathbf{b}$ in the direction of $\mathbf{a}$ is

$$\begin{aligned} \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} &= \frac{-2}{10}(\mathbf{i} + 3\mathbf{j}) \\ &= -\frac{1}{5}(\mathbf{i} + 3\mathbf{j}) \end{aligned}$$

**Example 17**

Resolve $\mathbf{i} + 3\mathbf{j}$ into rectangular components, one of which is parallel to $2\mathbf{i} - 2\mathbf{j}$.

Solution

Let $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j}$.

The vector resolute of $\mathbf{a}$ in the direction of $\mathbf{b}$ is given by $\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}$.

We have

$$\mathbf{a} \cdot \mathbf{b} = 2 - 6 = -4$$

$$\mathbf{b} \cdot \mathbf{b} = 4 + 4 = 8$$

Therefore the vector resolute is

$$\begin{aligned} \frac{-4}{8}(2\mathbf{i} - 2\mathbf{j}) &= -\frac{1}{2}(2\mathbf{i} - 2\mathbf{j}) \\ &= -\mathbf{i} + \mathbf{j} \end{aligned}$$

The perpendicular component is

$$\begin{aligned} \mathbf{a} - (-\mathbf{i} + \mathbf{j}) &= (\mathbf{i} + 3\mathbf{j}) - (-\mathbf{i} + \mathbf{j}) \\ &= 2\mathbf{i} + 2\mathbf{j} \end{aligned}$$

Hence we can write

$$\mathbf{i} + 3\mathbf{j} = (-\mathbf{i} + \mathbf{j}) + (2\mathbf{i} + 2\mathbf{j})$$

Check: We can check our calculation by verifying that the second component is indeed perpendicular to $\mathbf{b}$. We have $(2\mathbf{i} + 2\mathbf{j}) \cdot (2\mathbf{i} - 2\mathbf{j}) = 4 - 4 = 0$, as expected.

Example 18

Find the scalar resolute of $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j}$ in the direction of $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$.

Solution

$$\mathbf{a} \cdot \mathbf{b} = -2 + 6 = 4$$

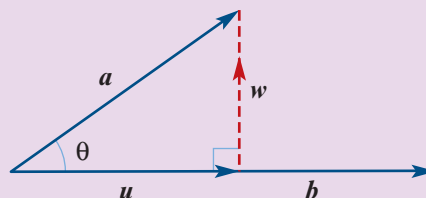
$$|\mathbf{b}| = \sqrt{1 + 9} = \sqrt{10}$$

The scalar resolute of $\mathbf{a}$ in the direction of $\mathbf{b}$ is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{4}{\sqrt{10}} = \frac{2\sqrt{10}}{5}$$

Section summary

- Resolving a vector $\mathbf{a}$ into rectangular components is expressing the vector $\mathbf{a}$ as a sum of two vectors, one parallel to a given vector $\mathbf{b}$ and the other perpendicular to $\mathbf{b}$.
- The **vector resolute** of $\mathbf{a}$ in the direction of $\mathbf{b}$ is given by $\mathbf{u} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}$.
- The **scalar resolute** of $\mathbf{a}$ in the direction of $\mathbf{b}$ is the 'signed length' of the vector resolute $\mathbf{u}$ and is given by $\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$.

**Exercise 6E**

- 1 Points A and B are defined by the position vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$.
 - a Find $\hat{\mathbf{a}}$.
 - b Find $\hat{\mathbf{b}}$.
 - c Find $\hat{\mathbf{c}}$, where $\mathbf{c} = \overrightarrow{AB}$.
- 2 Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$.
 - a Find $\hat{\mathbf{a}}$ and $|\mathbf{b}|$.
 - b Find the vector with the same magnitude as $\mathbf{b}$ and with the same direction as $\mathbf{a}$.
- 3 Points A and B are defined by the position vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 5\mathbf{i} + 12\mathbf{j}$.
 - a Find $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.
 - b Find the unit vector which bisects $\angle AOB$.
- 4 For each pair of vectors, find the vector resolute of $\mathbf{a}$ in the direction of $\mathbf{b}$:
 - a $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j}$
 - b $\mathbf{a} = \mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j}$
 - c $\mathbf{a} = 4\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = 4\mathbf{i}$
- 5 For each of the following pairs of vectors, find the resolution of the vector $\mathbf{a}$ into rectangular components, one of which is parallel to $\mathbf{b}$:
 - a $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 5\mathbf{i}$
 - b $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$
 - c $\mathbf{a} = -\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$

Example 16**Example 17**

Example 18

6 For each of the following pairs of vectors, find the scalar resolute of the first vector in the direction of the second vector:

a $a = 2i + j$ and $b = i$

b $a = 3i + j$ and $c = i - 2j$

c $b = 2j$ and $a = 2i + \sqrt{3}j$

d $b = i - \sqrt{5}j$ and $c = -i + 4j$

7 Let A and B be the points defined by the position vectors $a = i + 3j$ and $b = i + j$ respectively. Find:

a the vector resolute of a in the direction of b

b a unit vector perpendicular to OB

8 Let A and B be the points defined by the position vectors $a = 4i + j$ and $b = i - j$ respectively. Find:

a the vector resolute of a in the direction of b

b the vector component of a perpendicular to b

c the shortest distance from A to line OB

9 Points A , B and C have position vectors $a = i + 2j$, $b = 2i + j$ and $c = 2i - 3j$. Find:

a i $\overrightarrow{AB}$ ii $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to line AC

d the area of triangle ABC

6F Applications of vectors: displacement and velocity

For the remainder of this chapter, we will be working with vector and scalar quantities:

- A **vector quantity** has both magnitude and direction. We will introduce the vector quantities displacement, velocity and force.
- A **scalar quantity** has only magnitude. We will use the scalar quantities distance, time, speed and mass.

► Displacement

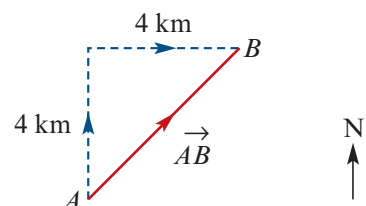
We have been describing points in the plane using position vectors. Points A and B have position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.

If an object moves from point A to point B , then the **displacement** of the object is the change in position of the object; it is described by the vector $\overrightarrow{AB}$.

For example, suppose that a person walks 4 km north and then 4 km east.

The person's displacement is $4\sqrt{2}$ km north-east.

Note: The total distance that the person has walked is 8 km, which is not equal to the magnitude of the displacement vector.



Example 19

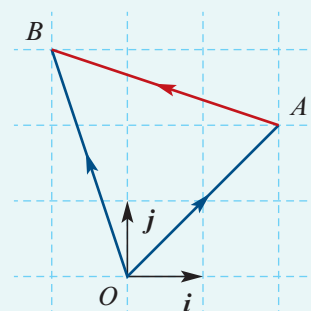
A particle moves from point $A(2, 2)$ to point $B(-1, 3)$. Express the displacement vector of the particle in component form.

Solution

We have $\vec{OA} = 2\mathbf{i} + 2\mathbf{j}$ and $\vec{OB} = -\mathbf{i} + 3\mathbf{j}$.

The displacement vector is

$$\begin{aligned}\vec{AB} &= \vec{AO} + \vec{OB} \\ &= -(2\mathbf{i} + 2\mathbf{j}) + (-\mathbf{i} + 3\mathbf{j}) \\ &= -3\mathbf{i} + \mathbf{j}\end{aligned}$$



Displacement problems involving bearings can be solved using the trigonometric techniques demonstrated in Example 12 and in Section 3E.

► Velocity

Velocity is the rate of change of position with respect to time.

Velocity is a vector quantity; it has magnitude and direction. The units of velocity which will be used in this chapter are metres per second (m/s) and kilometres per hour (km/h).

Some examples of velocity vectors are:

- 80 km/h in the direction north
- 10 km/h on a bearing of 080°
- $3\mathbf{i} + 4\mathbf{j}$ m/s

The first two vectors have magnitudes 80 km/h and 10 km/h respectively. The third vector has magnitude $|3\mathbf{i} + 4\mathbf{j}| = \sqrt{3^2 + 4^2} = 5$ m/s. The magnitude of velocity is called **speed**.

Motion with constant velocity

In this chapter, we only deal with constant velocity (that is, the velocity does not change over a particular time interval). Consider the following two examples:

- If a car travels for 2 hours with a constant velocity of 80 km/h north, then its displacement is $2 \times 80 = 160$ km north.
- If a particle starts at the origin and moves with a velocity of $3\mathbf{i} + 4\mathbf{j}$ m/s for 2 seconds, then its position is $2(3\mathbf{i} + 4\mathbf{j}) = 6\mathbf{i} + 8\mathbf{j}$ m.

If an object moves with a constant velocity of $\mathbf{v}$ m/s for t seconds, then its displacement vector, $\mathbf{s}$ m, is given by

$$\mathbf{s} = t\mathbf{v}$$

Note: Here $\mathbf{s}$ and $\mathbf{v}$ are vector quantities and t is a scalar quantity. So this is an example of scalar multiplication.

Example 20

A particle starts at the point A with position vector $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j}$, where the unit is metres. The particle begins moving with a constant velocity of $2\mathbf{i} + 4\mathbf{j}$ m/s. Find the position vector of the particle after:

a 5 seconds

b t seconds.

Solution

a Let P be the point that the particle reaches after 5 seconds. Then

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + 5(2\mathbf{i} + 4\mathbf{j}) \\ &= \mathbf{i} + 3\mathbf{j} + 10\mathbf{i} + 20\mathbf{j} \\ &= 11\mathbf{i} + 23\mathbf{j}\end{aligned}$$

b Let Q be the point that the particle reaches after t seconds. Then

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OA} + t(2\mathbf{i} + 4\mathbf{j}) \\ &= \mathbf{i} + 3\mathbf{j} + 2t\mathbf{i} + 4t\mathbf{j} \\ &= (1 + 2t)\mathbf{i} + (3 + 4t)\mathbf{j}\end{aligned}$$

► Direction of motion

The velocity vector is in the direction of motion. We often use the unit vector of the velocity vector to describe the direction of motion.

For example, if $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$, then the unit vector $\hat{\mathbf{v}} = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$ is in the direction of motion.

Example 21

Particle A starts moving from point O with a constant velocity of $\mathbf{v}_A = 3\mathbf{i} + 4\mathbf{j}$ m/s. Three seconds later, particle B starts from O and moves in the same direction as A with a constant speed of 7 m/s. When and where will B catch up to A ?

Solution

At time t seconds, particle A is at the point with position vector

$$\overrightarrow{OP}_A = t(3\mathbf{i} + 4\mathbf{j})$$

At time t seconds, for $t \geq 3$, particle B has been moving for $t - 3$ seconds and is at the point with position vector

$$\overrightarrow{OP}_B = \frac{7(t-3)}{5}(3\mathbf{i} + 4\mathbf{j})$$

The two particles are at the same point when

$$\frac{7(t-3)}{5} = t$$

$$7(t-3) = 5t$$

$$2t = 21$$

$$\therefore t = \frac{21}{2}$$

Particle B catches up to particle A at time $t = 10.5$ seconds.

At this time, both particles have position vector $31.5\mathbf{i} + 42\mathbf{j}$.

Example 22

A particle starts from O with a constant velocity of $\mathbf{v}_1 = 3\mathbf{i} + 4\mathbf{j}$ m/s. At the same time, a second particle starts moving with constant velocity from point B , where $\overrightarrow{OB} = 25\mathbf{j}$.

Given that the two particles meet and their paths are at right angles, find:

- a** the position vector of the point where they meet
- b** the velocity of the second particle.

Solution

- a** Assume that the particles meet at the point P at time t seconds. Since their paths are at right angles, we have

$$\overrightarrow{OP} \cdot \overrightarrow{BP} = 0$$

At time t seconds, the position vector of the first particle is

$$\overrightarrow{OP} = t(3\mathbf{i} + 4\mathbf{j}) = 3t\mathbf{i} + 4t\mathbf{j}$$

Therefore

$$\begin{aligned}\overrightarrow{BP} &= \overrightarrow{BO} + \overrightarrow{OP} \\ &= -25\mathbf{j} + (3t\mathbf{i} + 4t\mathbf{j}) \\ &= 3t\mathbf{i} + (4t - 25)\mathbf{j}\end{aligned}$$

Since $\overrightarrow{OP} \cdot \overrightarrow{BP} = 0$, we obtain

$$\begin{aligned}(3t\mathbf{i} + 4t\mathbf{j}) \cdot (3t\mathbf{i} + (4t - 25)\mathbf{j}) &= 0 \\ 9t^2 + 4t(4t - 25) &= 0 \\ 25t^2 - 100t &= 0\end{aligned}$$

$$\therefore t(t - 4) = 0$$

The particles do not meet at time 0 s, so they meet at time $t = 4$ s.

The position vector of the point where they meet is

$$\overrightarrow{OP} = 4(3\mathbf{i} + 4\mathbf{j}) = 12\mathbf{i} + 16\mathbf{j}$$

- b** Let $\mathbf{v}$ m/s be the velocity of the second particle. We use the formula $s = t\mathbf{v}$.

At time $t = 4$, the displacement of the second particle is $\overrightarrow{BP}$. Therefore

$$\begin{aligned}\overrightarrow{BP} &= 4\mathbf{v} \\ \overrightarrow{BO} + \overrightarrow{OP} &= 4\mathbf{v} \\ -25\mathbf{j} + (12\mathbf{i} + 16\mathbf{j}) &= 4\mathbf{v} \\ 12\mathbf{i} - 9\mathbf{j} &= 4\mathbf{v}\end{aligned}$$

Hence the velocity of the second particle is $\mathbf{v} = 3\mathbf{i} - \frac{9}{4}\mathbf{j}$ m/s.

Section summary

- The **displacement** of a particle is the change in its position. If a particle moves from point A to point B , then its displacement is $\overrightarrow{AB}$.
- The **velocity** of a particle is the rate of change of its position with respect to time.
- Displacement and velocity are vector quantities. The magnitude of velocity is **speed**.
- **Motion with constant velocity** If a particle moves with a constant velocity of $\mathbf{v}$ m/s for t seconds, then its displacement vector, $\mathbf{s}$ m, is given by $\mathbf{s} = t\mathbf{v}$.

Exercise 6F

Example 19

- 1 For each of the following, find the displacement vector in component form for a particle that moves from point A to point B :

- a** $A(3, 7), B(2, -4)$ **b** $A(-2, 4), B(3, -2)$ **c** $A(3, 1), B(4, 6)$
d $A(3, 7), B(3, -4)$ **e** $A(-2, -7), B(2, -7)$ **f** $A(5, -6), B(11, 5)$

- 2 From the point O , a hiker walks 5 km north and then 8 km on a bearing of 330° , finishing at a point A . Describe the displacement vector $\overrightarrow{OA}$ by giving a distance and a bearing.
- 3 From the point O , a yacht sails 3 km east and then 5 km on a bearing of 060° , finishing at a point A . Describe the displacement vector $\overrightarrow{OA}$ by giving a distance and a bearing.
- 4 Give the corresponding speed for each of the following velocity vectors:
- a** $5\mathbf{i} + 4\mathbf{j}$ m/s **b** $3\mathbf{i} - 4\mathbf{j}$ m/s **c** $-\mathbf{i} + 4\mathbf{j}$ m/s
d $-2\mathbf{i} - 6\mathbf{j}$ m/s **e** $5\mathbf{i} - 12\mathbf{j}$ m/s **f** $-7\mathbf{i} + 11\mathbf{j}$ m/s

In each of the following questions, the unit of distance is metres.

Example 20

- 5 A particle starts from the point A with position vector $\overrightarrow{OA} = -\mathbf{i} + 2\mathbf{j}$ and moves with a constant velocity of $5\mathbf{i} + 12\mathbf{j}$ m/s. Find the position vector of the particle after:
- a** 5 seconds **b** t seconds.
- 6 An object takes 5 seconds to move with constant velocity from point A to point B , where $\overrightarrow{OA} = 5\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{OB} = -15\mathbf{i} + 24\mathbf{j}$. Find the velocity of the object.
- 7 A particle starts from the point B with position vector $\overrightarrow{OB} = -2\mathbf{i} + 3\mathbf{j}$ and moves with a constant velocity of $7\mathbf{i} + 24\mathbf{j}$ m/s.
- a** Find the position vector of the particle after:
i 4 seconds **ii** t seconds.
b Find the particle's distance from the origin after:
i 4 seconds **ii** t seconds.

- 8** Let O be the origin and let A and B be the points with $\vec{OA} = 5\mathbf{i} + 2\mathbf{j}$ and $\vec{OB} = -5\mathbf{i} - 3\mathbf{j}$. A particle moves with constant velocity from A to B in 10 seconds. Find:

- a** the velocity of the particle **b** the speed of the particle.

Example 21

- 9** Particle A starts moving from point O with a constant velocity of $\mathbf{v}_A = \mathbf{i} + 2\mathbf{j}$ m/s. Two seconds later, particle B starts from O and moves in the same direction as A with a constant speed of 6 m/s. When and where will B catch up to A ?

Example 22

- 10** A particle starts from O with a constant velocity of $\mathbf{v}_1 = 2\mathbf{i} + \mathbf{j}$ m/s. At the same time, a second particle starts moving with constant velocity from point B , where $\vec{OB} = 20\mathbf{j}$. Given that the two particles meet and their paths are at right angles, find:
- a** the position vector of the point where they meet
b the velocity of the second particle.
- 11** Points A and B have position vectors $\vec{OA} = 10\mathbf{j}$ and $\vec{OB} = 20\mathbf{i}$. A particle starts moving from point A with a constant velocity of $\mathbf{v}_1 = 2\mathbf{i}$ m/s. At the same time, a second particle starts moving from point B with constant velocity. Given that the two particles meet and their paths are at right angles, find:
- a** the position vector of the point where they meet
b the velocity of the second particle.

6G Applications of vectors: relative velocity

► Resultant velocity

If two or more velocity vectors are added, then the sum is called a **resultant velocity**.

Example 23

A river is flowing north at 5 km/h. Mila can swim at 2 km/h in still water. She dives in from the west bank of the river and swims towards the opposite bank.

- a** In which direction does she travel? **b** What is her actual speed?

Solution

The swimmer's actual velocity, $\mathbf{v}$, is the vector sum of her velocity relative to the water (2 km/h east) and the water's velocity (5 km/h north).

- a** From the diagram, we have

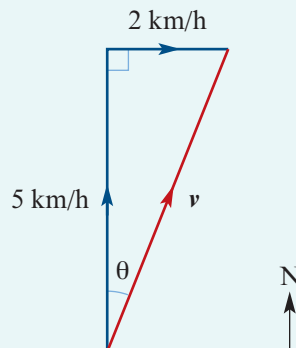
$$\tan \theta = \frac{2}{5}$$

$$\therefore \theta \approx 21.8^\circ$$

She is travelling on a bearing of 022° .

- b** Her actual speed is

$$|\mathbf{v}| = \sqrt{2^2 + 5^2} \approx 5.39 \text{ km/h}$$



► Relative velocity

In the previous example, the velocity of the water is given *relative to the bank* and the velocity of the swimmer is given *relative to the water*. The velocity of the swimmer *relative to the bank* is found by taking the vector sum. That is:

$$\boxed{\text{Velocity of swimmer relative to bank}} = \boxed{\text{Velocity of swimmer relative to water}} + \boxed{\text{Velocity of water relative to bank}}$$

The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B .

Consider another example: A train is travelling north at 60 km/h, and a passenger walks at 3 km/h along the corridor towards the back of the train.

$$\boxed{\text{Velocity of passenger relative to Earth}} = \boxed{\text{Velocity of passenger relative to train}} + \boxed{\text{Velocity of train relative to Earth}}$$

The passenger is moving with a velocity of 57 km/h north relative to Earth.

In general, if an object A is in motion relative to another object B , then we can find the velocity of A using a vector sum:

$$\boxed{\text{Velocity of } A \text{ relative to Earth}} = \boxed{\text{Velocity of } A \text{ relative to } B} + \boxed{\text{Velocity of } B \text{ relative to Earth}}$$

Velocities measured relative to Earth are often called **true velocities** or **actual velocities**.

Example 24

A train is moving with a constant velocity of 80 km/h north. A passenger walks straight across a carriage from the west side to the east side at 3 km/h. What is the true velocity of the passenger?

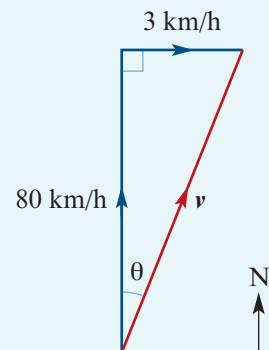
Solution

The passenger's true velocity, v , is the vector sum of his velocity relative to the train (3 km/h east) and the train's velocity (80 km/h north).

Speed: $|v| = \sqrt{80^2 + 3^2}$
 $= \sqrt{6409}$
 $\approx 80.06 \text{ km/h}$

Direction: $\tan \theta = \frac{3}{80}$
 $\therefore \theta \approx 2.15^\circ$

The passenger's true velocity is 80.06 km/h on a bearing of 002° .



Example 25

Car A is moving with a velocity of 50 km/h due north, while car B is moving with a velocity of 120 km/h due west. What is the velocity of car A relative to car B?

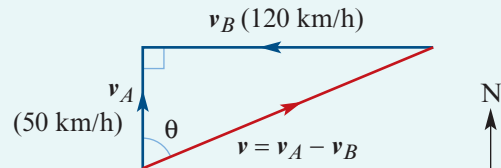
Solution

Let v_A be the velocity of car A, and let v_B be the velocity of car B.

The velocity of car A relative to car B is given by $v = v_A - v_B$.

Speed: $|v| = \sqrt{50^2 + 120^2}$
 $= 130 \text{ km/h}$

Direction: $\theta = \tan^{-1}\left(\frac{12}{5}\right)$
 $\approx 67.38^\circ$



The velocity of car A relative to car B is 130 km/h on a bearing of 067° .

► **Wind effect on flight paths**

The **airspeed** of an aircraft is its speed relative to air. In the next example, we see how the wind affects the actual velocity of an aircraft.

Example 26

A light aircraft has an airspeed of 250 km/h. The pilot sets a course due north. If the wind is blowing from the north-west at 80 km/h, what is the true speed and direction of the aircraft?

Solution

We can use the cosine rule to find the true speed:

$$|v| = \sqrt{250^2 + 80^2 - 2 \times 250 \times 80 \cos 45^\circ}$$

$$= 201.5334 \dots \text{ km/h}$$

We can now use the sine rule to find the angle θ :

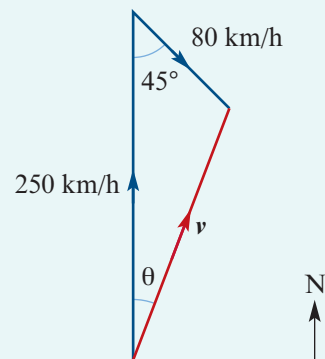
$$\frac{80}{\sin \theta} = \frac{|v|}{\sin 45^\circ}$$

$$\sin \theta = \frac{80 \sin 45^\circ}{|v|}$$

$$= 0.2806 \dots$$

$$\therefore \theta \approx 16.30^\circ$$

The aircraft is flying at 201.53 km/h on a bearing of 016° .



To fly an aircraft in a given direction, the pilot must compensate for the effect of the wind.

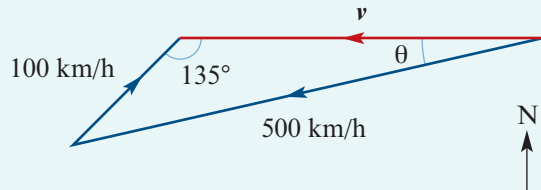
Example 27

An aeroplane is scheduled to travel from a point P to a point Q , which is 1000 km due west of P . The aeroplane's airspeed is 500 km/h and the wind is blowing from the south-west at 100 km/h.

- a** In which direction should the pilot set the course?
b How long will the flight take?

Solution

We want to ensure that the plane's true velocity, $\mathbf{v}$, is due west.



- a** Use the sine rule to find θ :

$$\begin{aligned}\frac{500}{\sin 135^\circ} &= \frac{100}{\sin \theta} \\ \sin \theta &= \frac{100 \sin 135^\circ}{500} \\ &= 0.1414 \dots \\ \therefore \theta &= (8.130 \dots)^\circ\end{aligned}$$

The pilot should head on a bearing of 262° .

- b** Use the sine rule to find $|\mathbf{v}|$:

$$\begin{aligned}\frac{500}{\sin 135^\circ} &= \frac{|\mathbf{v}|}{\sin(36.869 \dots)^\circ} \\ \therefore |\mathbf{v}| &= \frac{500 \sin(36.869 \dots)^\circ}{\sin 135^\circ} \\ &= 424.264 \dots \\ &\approx 424.26 \text{ km/h}\end{aligned}$$

The plane's speed relative to the ground is approximately 424 km/h. The flight will take approximately 2.4 hours.

Section summary

- If two or more velocity vectors are added, then the sum is called a **resultant velocity**.
- The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B .
- If an object A is in motion relative to another object B , we can find the velocity of A using a vector sum:

Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

Exercise 6G

Example 23

- 1** A river is flowing south at 4 km/h. Max can swim at 3 km/h in still water. He dives in from the west bank of the river and swims towards the opposite bank.

- a** In which direction does he travel? **b** What is his actual speed?



Example 24

- 2 A train is moving due north at 100 km/h. A passenger walks straight across a carriage from the east side to the west side at 4 km/h. What is the true velocity of the passenger?
- 3 Cars *A* and *B* are driving along a straight level road that runs east–west.
 - a If car *A* has a velocity of 100 km/h west and car *B* has a velocity of 80 km/h west, what is the velocity of car *A* relative to car *B*?
 - b If car *A* has a velocity of 100 km/h west and car *B* has a velocity of 80 km/h east, what is the velocity of car *A* relative to car *B*?
- 4 A cricketer is on a moving walkway which runs from south to north at 2 m/s. He bowls his fastest delivery, which is 45 m/s, again in a direction north. What is the velocity of the ball (relative to Earth)?
- 5 A ship is moving in a straight line at 15 m/s. A bird flies horizontally from the front of the ship towards the back of the ship at a speed of 5 m/s relative to the ship. What is the speed of the bird relative to the sea?
- 6 Car *A* is travelling north at 60 km/h along a straight level road. Car *B* is on the same road travelling north at 40 km/h. Find:
 - a the velocity of car *A* relative to car *B*
 - b the velocity of car *B* relative to car *A*.
- 7 A plane is heading due north, its airspeed is 240 km/h and there is an 80 km/h wind blowing from west to east. What is the velocity of the plane relative to Earth?

Example 25

- 8 Car *A* is moving with a velocity of 60 km/h due north, while car *B* is moving with a velocity of 80 km/h due west. What is the velocity of car *A* relative to car *B*?
- 9 A glider *P* is travelling due north at 60 km/h, and another glider *Q* is travelling north-west at 40 km/h. Find the velocity of *P* relative to *Q*.
- 10 Two particles, *A* and *B*, are moving with constant velocities of $\mathbf{v}_A = 4\mathbf{i} - 3\mathbf{j}$ m/s and $\mathbf{v}_B = 5\mathbf{i} - 7\mathbf{j}$ m/s respectively.
 - a Find the velocity of *B* relative to *A*.
 - b Find the magnitude of this relative velocity.
- 11 A ship is moving in a straight line at 15 m/s. A bird flies at an angle of 18° to the horizontal from the front of the ship towards the back of the ship at a speed of 5 m/s relative to the ship. What is the speed of the bird relative to the sea?

Example 26

- 12 A light aircraft has an airspeed of 240 km/h. The pilot sets a course due north. The wind is blowing from the north-east at 70 km/h. What is the true speed and direction of the aircraft?

Example 27

- 13 An aeroplane with an airspeed of 200 km/h is flying to an airport south-west of its present position. There is a wind blowing at 70 km/h from the east.
 - a Find the course that the pilot must set.
 - b Find the speed of the aeroplane relative to the ground.

- 14** A canoeist can paddle at 2 m/s in still water. He wishes to go straight across a river so that his path is at right angles to the banks of the river. The river is flowing at 1.5 m/s.
- Find the direction in which he must paddle.
 - If the river is 60 m wide, how long will it take him to cross the river?

CF

6H Applications of vectors: forces and equilibrium

A **force** is a measure of the strength of a *push* or *pull*. Forces can start motion, stop motion, make objects move faster or slower, and change the direction of motion.

Force can be defined as the physical quantity that causes a change in motion.

We will focus on situations where the forces ‘cancel each other out’. For example, if an inflatable raft is floating in a swimming pool, then the water is exerting an upwards force on the raft (called the buoyant force) that cancels out the downwards force of gravity.

► Introduction to forces

A force has both magnitude and direction – it may be represented by a vector.

When considering the forces that act on an object, it is convenient to treat the forces as acting on a single particle. The single particle may be thought of as a point at which the entire mass of the object is concentrated.

Weight and units of force

Every object near the surface of the Earth is subject to the force of gravity. We refer to this force as the **weight** of the object. Weight is a force that acts vertically downwards on an object (actually towards the centre of the Earth).

The unit of force used in this section is the **kilogram weight** (kg wt). If an object has a **mass** of 1 kg, then the force due to gravity acting on the object is 1 kg wt.

This unit is convenient for objects near the Earth’s surface. An object with a mass of 1 kg would have a different weight on the moon.

Note: The standard unit of force is the newton (N). At the Earth’s surface, a mass of m kg has a force of m kg wt = mg N acting on it, where g is the acceleration due to gravity ($g \text{ m/s}^2 \approx 9.8 \text{ m/s}^2$).

Resultant force and equilibrium

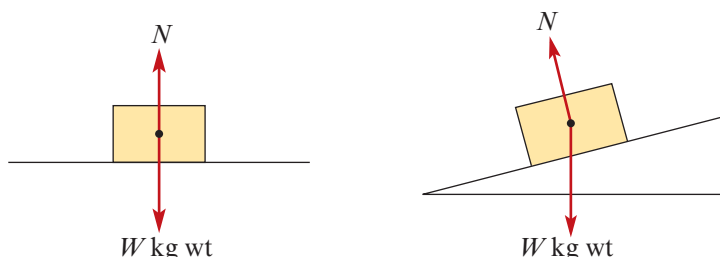
When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**. The resultant force is the vector sum of the forces acting on the particle.

If the resultant force acting on an object is zero, the object will remain at rest or continue moving with constant velocity. The object is said to be in **equilibrium**.

Note: Planet Earth is moving and our galaxy is moving, but we use Earth as our frame of reference and so our observation of an object being at rest is determined in this way.

Normal force

Any mass placed on a surface, either horizontal or inclined, experiences a force perpendicular to the surface. This force is referred to as a **normal force**.

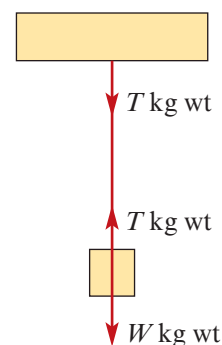


For example, a book sitting on a table is obviously being subjected to a force due to gravity. But the fact that it does not fall to the ground indicates that there must be a second force on the book. The table is exerting a force on the book equal in magnitude to gravity, but in the opposite direction. Hence the book remains at rest; it is in **equilibrium**.

Tension force

The diagram shows a string attached to the ceiling supporting a mass, which is at rest. The force of gravity, W kg wt, acts downwards on the mass and the string exerts an equal force, T kg wt, upwards on the mass. The force exerted by the string is called the **tension force**.

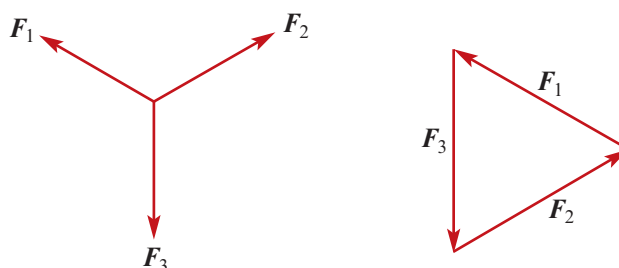
Note that there is a force, equal in magnitude but opposite in direction, acting on the ceiling at the point of contact.



► Triangle of forces

If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.

Suppose that three forces F_1 , F_2 and F_3 are acting on a particle in equilibrium, as shown in the diagram on the left. Since the particle is in equilibrium, we must have $F_1 + F_2 + F_3 = \mathbf{0}$. Therefore the three forces can be rearranged into a triangle as shown on the right.



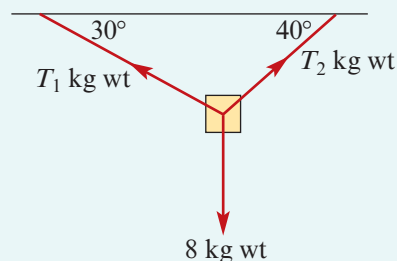
The magnitudes of the forces and the angles between the forces can now be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.

In the following examples and exercise, strings and ropes are considered to have negligible mass. A smooth light pulley is considered to have negligible mass and the friction between a rope and pulley is considered to be negligible.



Example 28

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 40° to the horizontal, find the tension in each string.



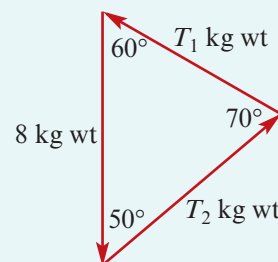
Solution

Represent the forces in a triangle. The sine rule gives

$$\frac{T_1}{\sin 50^\circ} = \frac{T_2}{\sin 60^\circ} = \frac{8}{\sin 70^\circ}$$

$$T_1 = \frac{8 \sin 50^\circ}{\sin 70^\circ} \approx 6.52 \text{ kg wt}$$

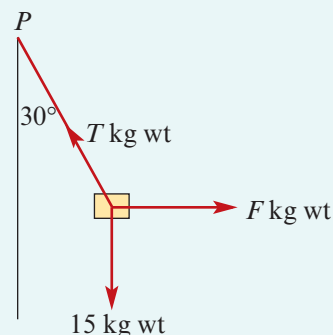
$$T_2 = \frac{8 \sin 60^\circ}{\sin 70^\circ} \approx 7.37 \text{ kg wt}$$



Example 29

A particle of mass 15 kg is suspended vertically from a point P by a string. The particle is pulled horizontally by a force of F kg wt so that the string makes an angle of 30° with the vertical.

Find the value of F and the tension in the string.



Solution

Representing the forces in a triangle gives

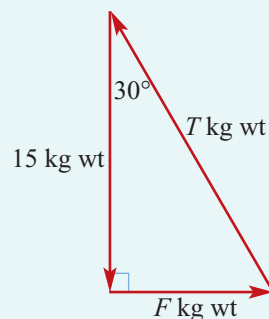
$$\frac{F}{15} = \tan 30^\circ$$

$$F = 15 \tan 30^\circ = 5\sqrt{3}$$

and $\frac{15}{T} = \cos 30^\circ$

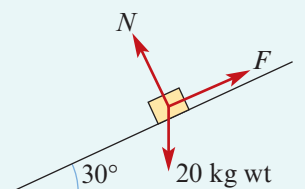
$$T = \frac{15}{\cos 30^\circ} = 10\sqrt{3}$$

The tension in the string is $10\sqrt{3}$ kg wt.



Example 30

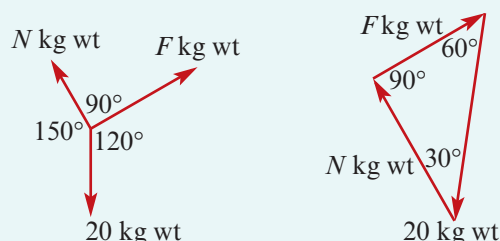
A body of mass 20 kg is placed on a smooth plane inclined at 30° to the horizontal. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.

**Solution**

The three forces form a triangle (as the body is in equilibrium). Therefore

$$F = 20 \sin 30^\circ = 10 \text{ kg wt}$$

$$N = 20 \cos 30^\circ = 10\sqrt{3} \text{ kg wt}$$



Note: Force is a vector quantity, but it is often useful to employ only the magnitude of a force in calculations, and the direction is evident from the context. In this section, and in particular in diagrams, we often denote the magnitude of a force (for example, F) by the same unbolded letter (in this case, F).

► Resolution of forces

Obviously there are many situations where more than three forces (or in fact only two forces) will be acting on a body. An alternative method is required to solve such problems.

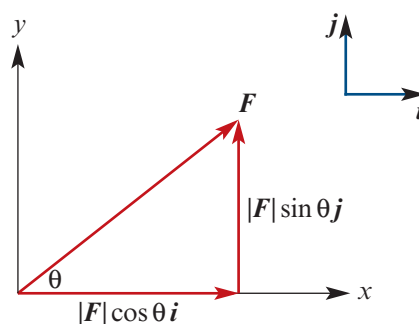
If all forces under consideration are acting in the same plane, then these forces and the resultant force can each be expressed as a sum of its i - and j -components.

If a force F acts at an angle of θ to the x -axis, then F can be written as the sum of two forces, one 'horizontal' and the other 'vertical':

$$F = |F| \cos \theta \mathbf{i} + |F| \sin \theta \mathbf{j}$$

The force F is **resolved** into two components:

- the i -component is parallel to the x -axis
- the j -component is parallel to the y -axis.

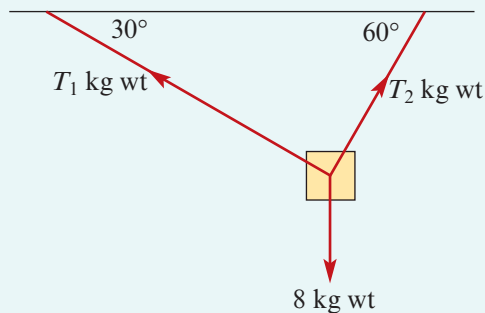
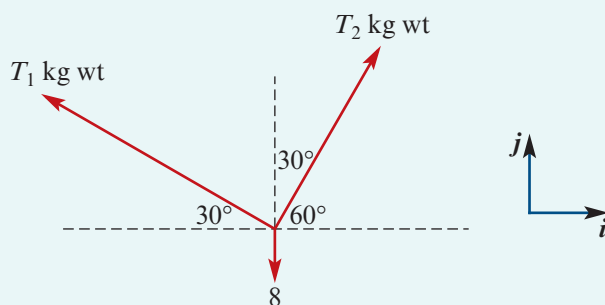


For a particle that is in equilibrium, if all the forces acting on the particle are resolved into their i - and j -components, then:

- the sum of all the i -components is zero
- the sum of all the j -components is zero.

**Example 31**

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 60° to the horizontal, find the tension in each string.

**Solution**

Resolution in the j -direction:

$$T_1 \sin 30^\circ + T_2 \sin 60^\circ - 8 = 0$$

$$T_1 \left(\frac{1}{2} \right) + T_2 \left(\frac{\sqrt{3}}{2} \right) - 8 = 0 \quad (1)$$

Resolution in the i -direction:

$$-T_1 \cos 30^\circ + T_2 \cos 60^\circ = 0$$

$$-T_1 \left(\frac{\sqrt{3}}{2} \right) + T_2 \left(\frac{1}{2} \right) = 0 \quad (2)$$

From (2): $\sqrt{3} T_1 = T_2$

Substituting in (1) gives

$$T_1 \left(\frac{1}{2} \right) + \sqrt{3} T_1 \left(\frac{\sqrt{3}}{2} \right) - 8 = 0$$

$$4T_1 = 16$$

$$\therefore T_1 = 4$$

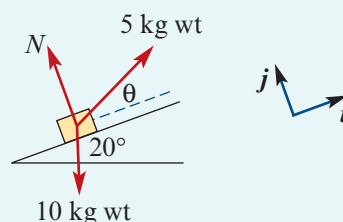
Hence $T_1 = 4$ and $T_2 = 4\sqrt{3}$.

The tensions in the strings are 4 kg wt and $4\sqrt{3}$ kg wt.

Example 32

A body of mass 10 kg is held at rest on a smooth plane inclined at 20° by a string with tension 5 kg wt as shown.

Find the angle between the string and the inclined plane.

**Solution**

We resolve the forces parallel and perpendicular to the plane. Then the normal force N has no parallel component, since it is perpendicular to the plane.

Resolving in the i -direction:

$$5 \cos \theta - 10 \sin 20^\circ = 0$$

$$\cos \theta = \frac{10 \sin 20^\circ}{5}$$

$$= 0.6840 \dots$$

$$\therefore \theta \approx 46.84^\circ$$

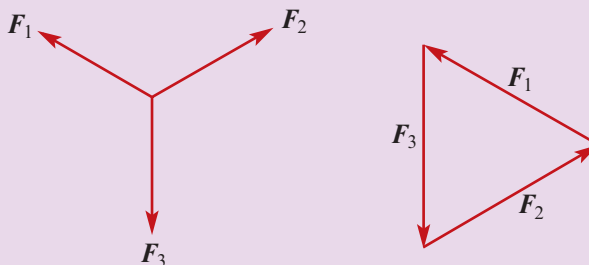
The angle between the string and the inclined plane is 46.84° .

Section summary

- **Force** is a vector quantity.
- The magnitude of a force can be measured using **kilogram weight** (kg wt).
If an object near the surface of the Earth has a mass of 1 kg, then the force due to gravity acting on the object is 1 kg wt.

- **Triangle of forces**

If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.



- **Resolution of forces**

- A force F is **resolved** into components by writing it in the form $F = xi + yj$.
- If forces are acting on a particle that is in equilibrium, then:
 - the sum of the i -components of all the forces is zero
 - the sum of the j -components of all the forces is zero.

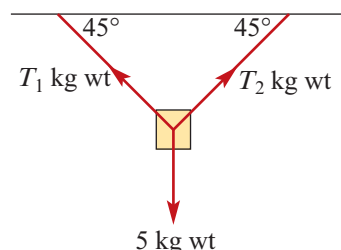
Exercise 6H

Complete Questions 1–10 using triangles of forces.

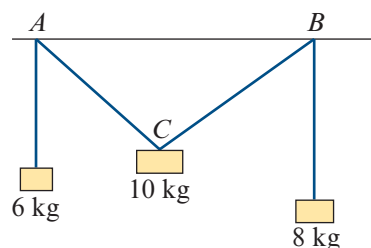
Skillsheet

Example 28

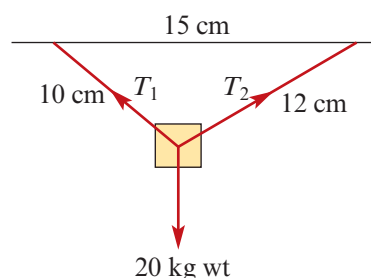
- 1** A particle of mass 5 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 45° with the horizontal, find the tension in each string.



- 2** Using strings and pulleys, three weights of mass 6 kg, 8 kg and 10 kg are suspended in equilibrium as shown. Calculate the magnitude of the angle ACB .



- 3** A mass of 20 kg is suspended from two strings of length 10 cm and 12 cm, the ends of the strings being attached to two points in a horizontal line, 15 cm apart. Find the tension in each string.

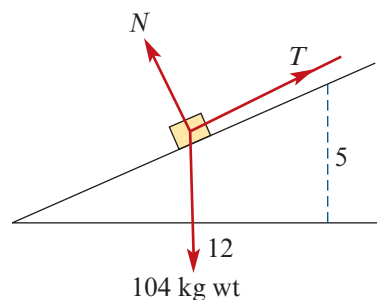


Example 29

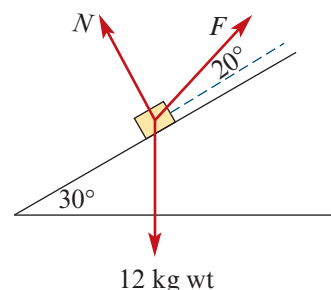
- 4** A boat is being pulled by a force of 40 kg wt towards the east and by a force of 30 kg wt towards the north-west. What third force must be acting on the boat if it remains stationary? Give the magnitude and direction.

Example 30

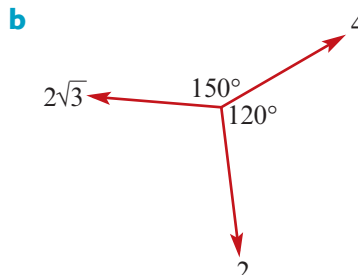
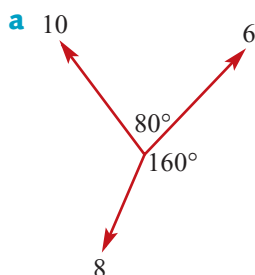
- 5** A body of mass 104 kg is placed on a smooth inclined plane which rises 5 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.



- 6** A body of mass 12 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 20° to the plane. Find the magnitude of the force.



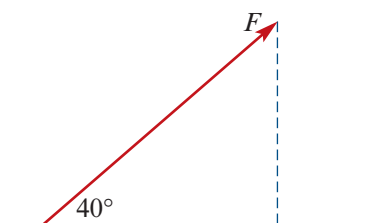
7 In each of the following cases, determine whether the particle is in equilibrium:



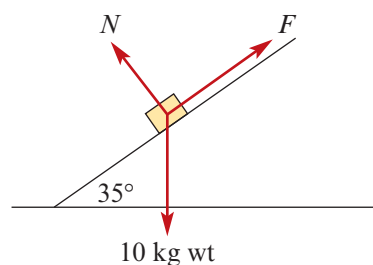
- 8** Three forces of magnitude 4 kg wt, 7 kg wt and 10 kg wt are in equilibrium. Determine the magnitudes of the angles between the forces.
- 9** A mass of 15 kg is maintained at rest on a smooth inclined plane by a string that is parallel to the plane. Determine the tension in the string if:
- a** the plane is at 30° to the horizontal
 - b** the plane is at 40° to the horizontal
 - c** the plane is at 30° to the horizontal, but the string is held at an angle of 10° to the plane.
- 10** A string is connected to two points A and D in a horizontal line and masses of 12 kg and W kg are attached at points B and C with C lower than B . If AB , BC and CD make angles of 40° , 20° and 50° respectively with the horizontal, calculate the tensions in the string and the value of W .

Complete Questions 11–19 using resolution of forces.

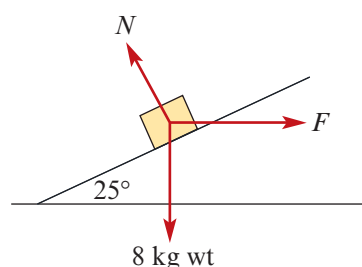
- 11** A force of F kg wt makes an angle of 40° with the horizontal. If its horizontal component is a force of 10 kg wt, find the value of F .



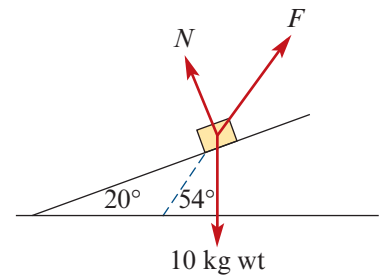
- Example 31** **12** Find the magnitude of the force, acting on a smooth inclined plane of angle 35° , required to support a mass of 10 kg resting on the plane.



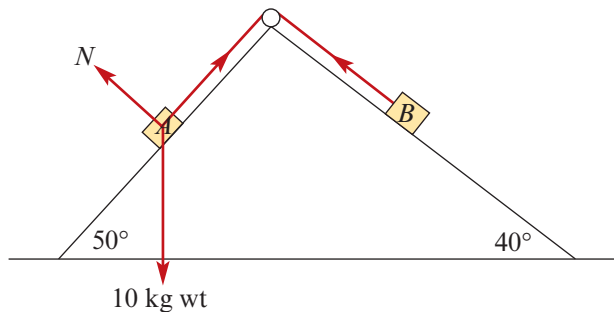
- 13** A body of mass 8 kg rests on a smooth inclined plane of angle 25° under the action of a horizontal force. Find the magnitude of the force and the reaction of the plane on the body.



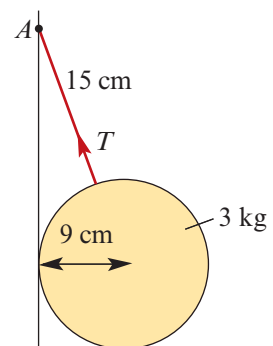
- Example 32** **14** A body of mass 10 kg rests on a smooth inclined plane of angle 20° . Find the force that will keep it in equilibrium when it acts at an angle of 54° with the horizontal.



- 15** If a body of mass 12 kg is suspended by a string, find the horizontal force required to hold it at an angle of 30° from the vertical.
- 16** A force of 20 kg wt acting directly up a smooth plane inclined at an angle of 40° maintains a body in equilibrium on the plane. Calculate the mass of the body and the force it exerts on the plane.
- 17** Two men are supporting a block by ropes. One exerts a force of 20 kg wt, his rope making an angle of 35° with the vertical, and the other exerts a force of 30 kg wt. Determine the mass of the block and the angle of direction of the second rope.
- 18** A body A of mass 10 kg is supported against a smooth plane of angle 50° . Find the force that the body exerts on the plane and the tension in the string, which is parallel to the slope. A body B on a plane of angle 40° is connected to A by a string passing over a smooth pulley on the ridge. If the system is in equilibrium, what is the mass of B?



- 19** A sphere of radius 9 cm is attached to a point A on a vertical wall by a string of length 15 cm. If the mass of the sphere is 3 kg, find the tension in the string.



Chapter summary

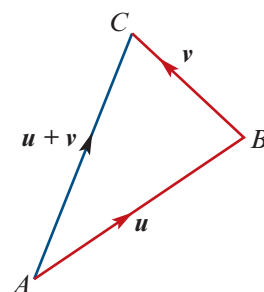


- A **vector** is a set of equivalent **directed line segments**.
- A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.
- The **position vector** of a point A is the vector $\overrightarrow{OA}$, where O is the origin.
- A vector can be written as a column of numbers. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is '2 across and 3 up'.

Basic operations on vectors

Addition

- If $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$.
- The sum $\mathbf{u} + \mathbf{v}$ can also be obtained geometrically as shown.



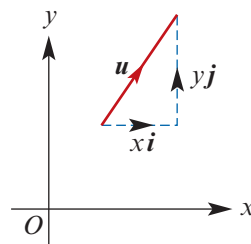
Scalar multiplication

- For $k \in \mathbb{R}^+$, the vector $k\mathbf{u}$ has the same direction as $\mathbf{u}$, but its length is multiplied by a factor of k .
- The vector $-\mathbf{v}$ has the same length as $\mathbf{v}$, but the opposite direction.
- Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there exists $k \in \mathbb{R} \setminus \{0\}$ such that $\mathbf{u} = k\mathbf{v}$.

Subtraction $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

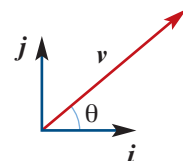
Component form

- In two dimensions, each vector $\mathbf{u}$ can be written in the form $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$, where
 - $\mathbf{i}$ ($\hat{\mathbf{i}}$) is the unit vector in the positive direction of the x -axis
 - $\mathbf{j}$ ($\hat{\mathbf{j}}$) is the unit vector in the positive direction of the y -axis.
- The **magnitude** of vector $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$ is given by $|\mathbf{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\mathbf{a}$ is given by $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \mathbf{a}$.



Polar form

- In two dimensions, each vector $\mathbf{v}$ can be written in the form $\mathbf{v} = [r, \theta]$, where
 - r is the magnitude of $\mathbf{v}$
 - θ describes the angle that $\mathbf{v}$ makes with the $\mathbf{i}$ -direction, measured anticlockwise.

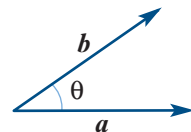


Scalar product

- The **scalar product** of vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$$

- The scalar product is described geometrically by $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- Therefore $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.
- Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.



Vector projections

- Resolving a vector $\mathbf{a}$ into rectangular components is expressing the vector $\mathbf{a}$ as a sum of two vectors, one parallel to a given vector $\mathbf{b}$ and the other perpendicular to $\mathbf{b}$.
- The **vector resolute** of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}$.
- The **scalar resolute** of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$.

Displacement and velocity

- The **displacement** of a particle is the change in its position. If a particle moves from point A to point B , then its displacement is $\overrightarrow{AB}$.
- The **velocity** of a particle is the rate of change of its position with respect to time.
- **Motion with constant velocity** If a particle moves with a constant velocity of $\mathbf{v}$ m/s for t seconds, then its displacement vector, $\mathbf{s}$ m, is given by $\mathbf{s} = t\mathbf{v}$.

Relative velocity

- The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B .
- If an object A is in motion relative to another object B , we can find the velocity of A using a vector sum:

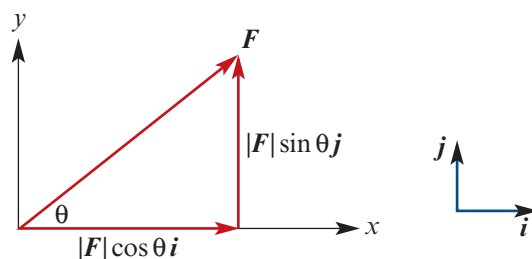
Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

Forces and equilibrium

- **Resultant force** When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**.
- **Equilibrium** If the resultant force acting on an object is zero, then the object is said to be in **equilibrium**; it will remain at rest or continue moving with constant velocity.
- **Triangle of forces** If three forces are acting on a particle in equilibrium, then the vectors representing the forces may be arranged to form a triangle. The magnitudes of the forces and the angles between them can be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.
- **Resolution of forces**

If all forces on a particle are acting in two dimensions, then each force can be expressed in terms of its components in the $\mathbf{i}$ - and $\mathbf{j}$ -directions:

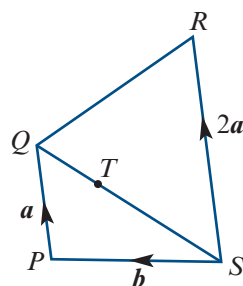
$$\mathbf{F} = |\mathbf{F}| \cos \theta \mathbf{i} + |\mathbf{F}| \sin \theta \mathbf{j}$$



For the particle to be in equilibrium, the sum of all the $\mathbf{i}$ -components must be zero and the sum of all the $\mathbf{j}$ -components must be zero.

Technology-free questions

- 1 Given that $\mathbf{a} = 7\mathbf{i} + 6\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + x\mathbf{j}$, find the values of x for which:
 - a $\mathbf{a}$ is parallel to $\mathbf{b}$
 - b $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude.
- 2 $ABCD$ is a parallelogram where $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j}$, $\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{AD} = -2\mathbf{i} + 5\mathbf{j}$. Find the coordinates of the four vertices of the parallelogram.
- 3 Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j}$. Find the value of k such that $\mathbf{a} + k\mathbf{b}$ is parallel to the x -axis.
- 4 The position vectors of P and Q are $2\mathbf{i} - 2\mathbf{j}$ and $3\mathbf{i} - 7\mathbf{j}$ respectively.
 - a Find $|\overrightarrow{PQ}|$.
 - b Find the unit vector in the direction of $\overrightarrow{PQ}$.
- 5 $\overrightarrow{OA} = 4\mathbf{i} + 3\mathbf{j}$ and C is a point on OA such that $|\overrightarrow{OC}| = \frac{16}{5}$.
 - a Find the unit vector in the direction of $\overrightarrow{OA}$.
 - b Hence find $\overrightarrow{OC}$.
- 6 In the diagram, $ST = 2TQ$, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{SR} = 2\mathbf{a}$ and $\overrightarrow{SP} = \mathbf{b}$.
 - a Find each of the following in terms of $\mathbf{a}$ and $\mathbf{b}$:
 - i $\overrightarrow{SQ}$
 - ii $\overrightarrow{TQ}$
 - iii $\overrightarrow{RQ}$
 - iv $\overrightarrow{PT}$
 - v $\overrightarrow{TR}$
 - b Prove that T lies on the line PR by showing that PT and TR are parallel.
- 7 Let M be the midpoint of the line segment AB , where $\overrightarrow{OA} = 5\mathbf{i} + s\mathbf{j}$ and $\overrightarrow{OB} = t\mathbf{i} + 3\mathbf{j}$. Given that $\overrightarrow{OM} = 7\mathbf{i} - 2\mathbf{j}$, find s and t .
- 8 The vector $\mathbf{p}$ has magnitude 7 units and bearing 050° and the vector $\mathbf{q}$ has magnitude 12 units and bearing 170° . (These are compass bearings on the horizontal plane.) Draw a diagram (not to scale) showing $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{p} + \mathbf{q}$. Calculate the magnitude of $\mathbf{p} + \mathbf{q}$.
- 9 If $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j}$, find:
 - a $\mathbf{a} + 2\mathbf{b}$
 - b $|\mathbf{a}|$
 - c $\hat{\mathbf{a}}$
 - d $\mathbf{a} - \mathbf{b}$
- 10 Let O , A and B be the points $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.
 - a If C is the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$, find the coordinates of C .
 - b If D is the point $(1, 24)$ and $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$, find the values of h and k .
- 11 The points A , B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ relative to an origin O . Write down an equation connecting $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ for each of the following cases:
 - a $OABC$ is a parallelogram
 - b B divides AC in the ratio $3 : 2$. That is, $AB : BC = 3 : 2$.



12 The vector $\mathbf{v}$ is defined by the directed line segment from $(2\sqrt{3}, 1)$ to $(0, 3)$. Express $\mathbf{v}$ in polar form.

13 Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c} = -2\mathbf{i} - 2\mathbf{j}$. Find:

a $\mathbf{a} \cdot \mathbf{a}$

b $\mathbf{b} \cdot \mathbf{b}$

c $\mathbf{c} \cdot \mathbf{c}$

d $\mathbf{a} \cdot \mathbf{b}$

e $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

f $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{c})$

g $(\mathbf{a} + 2\mathbf{b}) \cdot (3\mathbf{c} - \mathbf{b})$

14 Points A , B and C have position vectors $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{c} = -5\mathbf{i} + 3\mathbf{j}$ respectively. Evaluate $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and hence show that $\triangle ABC$ is right-angled at B .

15 Given the vectors $\mathbf{p} = 5\mathbf{i} + 3\mathbf{j}$ and $\mathbf{q} = 2\mathbf{i} + t\mathbf{j}$, find the values of t for which:

a $\mathbf{p} + \mathbf{q}$ is parallel to $\mathbf{p} - \mathbf{q}$

b $\mathbf{p} - 2\mathbf{q}$ is perpendicular to $\mathbf{p} + 2\mathbf{q}$

c $|\mathbf{p} - \mathbf{q}| = |\mathbf{q}|$

16 Points A , B and C have position vectors $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$. Find:

a i $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to the line AC .

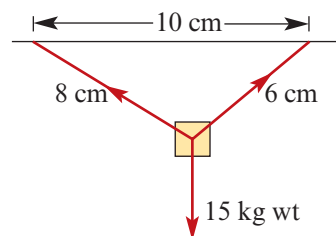
17 Priya can swim at a speed of 1.6 m/s in still water. She swims across a river that is 48 metres wide and flows at 1.2 m/s between parallel banks.

a Find the speed of the swimmer relative to the river bank.

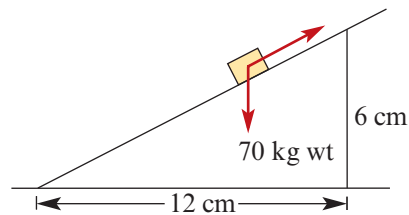
b Find the time that it takes her to cross the river.

c Describe the position at which she arrives on the opposite bank.

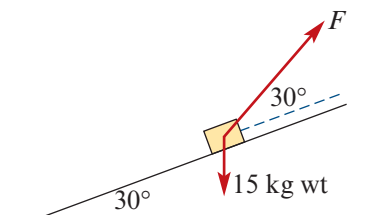
18 A mass of 15 kg is suspended from two strings of length 6 cm and 8 cm, the ends of the strings being attached to two points in a horizontal line, 10 cm apart. Find the tension in each string.



19 A body of mass 70 kg is placed on a smooth inclined plane which rises 6 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.



20 A body of mass 15 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 30° to the plane. Find the magnitude of the force.



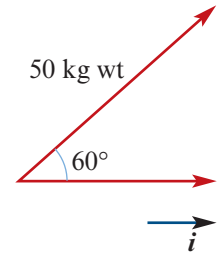
Multiple-choice questions



- The vector $\mathbf{v}$ is defined by the directed line segment from $(1, 1)$ to $(3, 5)$.
If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then
A $a = 3$ and $b = 5$ **B** $a = -2$ and $b = -4$ **C** $a = 2$ and $b = 4$
D $a = 2$ and $b = 3$ **E** $a = 4$ and $b = 2$
- If vector $\overrightarrow{AB} = \mathbf{u}$ and vector $\overrightarrow{AC} = \mathbf{v}$, then vector $\overrightarrow{CB}$ is equal to
A $\mathbf{u} + \mathbf{v}$ **B** $\mathbf{v} - \mathbf{u}$ **C** $\mathbf{u} - \mathbf{v}$ **D** $\mathbf{u} \times \mathbf{v}$ **E** $\mathbf{v} + \mathbf{u}$
- If vector $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and vector $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $2\mathbf{a} - 3\mathbf{b} =$
A $\begin{bmatrix} 9 \\ -13 \end{bmatrix}$ **B** $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ **C** $\begin{bmatrix} 9 \\ -7 \end{bmatrix}$ **D** $\begin{bmatrix} 3 \\ -13 \end{bmatrix}$ **E** $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$
- $PQRS$ is a parallelogram. If $\overrightarrow{PQ} = \mathbf{p}$ and $\overrightarrow{QR} = \mathbf{q}$, then $\overrightarrow{SQ}$ is equal to
A $\mathbf{p} + \mathbf{q}$ **B** $\mathbf{p} - \mathbf{q}$ **C** $\mathbf{q} - \mathbf{p}$ **D** $2\mathbf{q}$ **E** $2\mathbf{p}$
- $|3\mathbf{i} - 5\mathbf{j}| =$
A 2 **B** $\sqrt{34}$ **C** 34 **D** 8 **E** -16
- If $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OB} = \mathbf{i} - 2\mathbf{j}$, then $\overrightarrow{AB}$ equals
A $-\mathbf{i} - 5\mathbf{j}$ **B** $-\mathbf{i} + 5\mathbf{j}$ **C** $-\mathbf{i} - \mathbf{j}$ **D** $-\mathbf{i} + \mathbf{j}$ **E** $\mathbf{i} + \mathbf{j}$
- If $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OB} = \mathbf{i} - 2\mathbf{j}$, then $|\overrightarrow{AB}|$ equals
A 6 **B** 26 **C** $\sqrt{26}$ **D** $\sqrt{24}$ **E** 36
- If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, then the unit vector in the direction of $\mathbf{a}$ is
A $2\mathbf{i} + 3\mathbf{j}$ **B** $\frac{1}{13}(2\mathbf{i} + 3\mathbf{j})$ **C** $\frac{1}{\sqrt{5}}(2\mathbf{i} + 3\mathbf{j})$
D $\frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$ **E** $\sqrt{13}(2\mathbf{i} + 3\mathbf{j})$
- The vector $[\sqrt{6}, -45^\circ]$ corresponds to
A $\begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \end{bmatrix}$ **B** $\begin{bmatrix} -\sqrt{6} \\ \sqrt{6} \end{bmatrix}$ **C** $\begin{bmatrix} -\sqrt{3} \\ \sqrt{3} \end{bmatrix}$ **D** $\begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix}$ **E** $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$
- An aircraft has an airspeed of 100 km/h. The aircraft is heading in the direction $3\mathbf{i} - 4\mathbf{j}$ and the wind is blowing with a velocity of $-5\mathbf{i} + 20\mathbf{j}$ km/h. The velocity of the aircraft relative to the ground (in km/h) is
A $55\mathbf{i} - 60\mathbf{j}$ **B** $65\mathbf{i} - 60\mathbf{j}$ **C** $305\mathbf{i} - 420\mathbf{j}$ **D** $60\mathbf{i} - 40\mathbf{j}$ **E** $295\mathbf{i} - 380\mathbf{j}$
- The velocity of a ship is $20\mathbf{i}$ km/h and the velocity of the wind is $-4\mathbf{i} + 3\mathbf{j}$ km/h. The direction of the smoke trail coming from the ship's funnel is given by
A $16\mathbf{i} + 3\mathbf{j}$ **B** $-24\mathbf{i} + 3\mathbf{j}$ **C** $-4\mathbf{i} + 3\mathbf{j}$ **D** $-16\mathbf{i} - 3\mathbf{j}$ **E** $24\mathbf{i} - 3\mathbf{j}$

- 12** The magnitude of the component of force F in the i -direction is

A 300 kg wt **B** 50 kg wt **C** 40 kg wt
D 20 kg wt **E** 25 kg wt



- 13** Two perpendicular forces have magnitudes 5 kg wt and 4 kg wt. The magnitude of the resultant force is

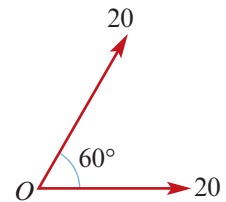
A 3 kg wt **B** $\sqrt{11}$ kg wt **C** $\sqrt{41}$ kg wt **D** 1 kg wt **E** 9 kg wt

- 14** A particle is acted on by two forces: $F_1 = [7, 45^\circ]$ kg wt and $F_2 = [a, -45^\circ]$ kg wt. If the resultant force $F_1 + F_2$ has magnitude 9 kg wt, then the value of a must be

A 2 **B** $4\sqrt{2}$ **C** $\sqrt{130}$ **D** 16 **E** 32

- 15** Two forces of magnitude 20 kg wt act on a particle at O as shown. The magnitude of the resultant force (in kg wt) is

A 40 **B** $20\sqrt{3}$ **C** 0
D 20 **E** 10



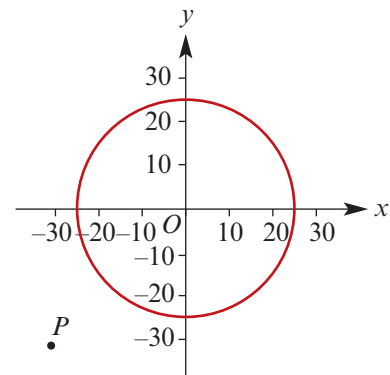
Extended-response questions

- 1** Let $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represent a displacement 1 km due east.

Let $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represent a displacement 1 km due north.

The diagram shows a circle of radius 25 km with centre at $O(0, 0)$. A lighthouse entirely surrounded by sea is located at O . The lighthouse is not visible from points outside the circle.

A ship is initially at point P , which is 31 km west and 32 km south of the lighthouse.



- a** Write down the vector $\overrightarrow{OP}$.

The ship is travelling in the direction of vector $u = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ with speed 20 km/h.

An hour after leaving P , the ship is at point R .

- b** Show that $\overrightarrow{PR} = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$ and hence find the vector $\overrightarrow{OR}$.

- c** Show that the lighthouse first becomes visible when the ship reaches R .

- 2 Given that $\mathbf{p} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{q} = -2\mathbf{i} + 4\mathbf{j}$, find:

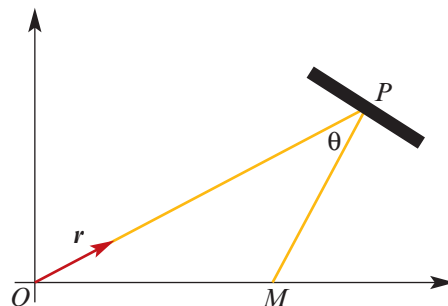
a $|\mathbf{p} - \mathbf{q}|$ **b** $|\mathbf{p}| - |\mathbf{q}|$ **c** $\mathbf{r}$ such that $\mathbf{p} + 2\mathbf{q} + \mathbf{r} = \mathbf{0}$

- 3 The quadrilateral $PQRS$ is a parallelogram. The point P has coordinates $(5, 8)$, the point R has coordinates $(32, 17)$ and the vector $\overrightarrow{PQ}$ is given by $\overrightarrow{PQ} = \begin{bmatrix} 20 \\ -15 \end{bmatrix}$.

- a** Find the coordinates of Q and write down the vector $\overrightarrow{QR}$.
b Write down the vector $\overrightarrow{RS}$ and show that the coordinates of S are $(12, 32)$.

- 4 The diagram shows the path of a light beam from its source at O in the direction of the vector $\mathbf{r} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

At point P , the beam is reflected by an adjustable mirror and meets the x -axis at M . The position of M varies, depending on the adjustment of the mirror at P .



- a** Given that $\overrightarrow{OP} = 4\mathbf{r}$, find the coordinates of P .
b The point M has coordinates $(k, 0)$. Find an expression, in terms of k , for vector $\overrightarrow{PM}$.
c Find the magnitudes of vectors $\overrightarrow{OP}$, $\overrightarrow{OM}$ and $\overrightarrow{PM}$, and hence find the value of k for which θ is equal to 90° .
d Find the value θ for which M has coordinates $(9, 0)$.
- 5 A helicopter can fly at 150 km/h in still air. The wind is blowing at 30 km/h from the east.
- a** How long in total would it take the helicopter to fly directly to a point 180 km due east and back again?
b On what bearing should the helicopter head in order to fly directly to a point 90 km due north? How long would this take?
c On what bearing should the helicopter head in order to fly due south?
- 6 The unit vectors $\mathbf{i}$ and $\mathbf{j}$ represent 1 km east and 1 km north respectively. Two motor boats, A and B , are moving with velocities $\mathbf{v}_A = 12\mathbf{i} + 16\mathbf{j}$ km/h and $\mathbf{v}_B = 8\mathbf{i} + \alpha\mathbf{j}$ km/h, where α is a real number.
- a** Find an expression for the velocity of boat A relative to boat B .
b When the two boats first sight each other, boat A is 10 km due west of boat B .
i Find the value of α for which the two boats would collide if they maintained their current velocities.
ii Find the time between the boats first sighting each other and the collision.

7

Principles of counting

Objectives

- ▶ To solve problems using the **addition** and **multiplication principles**.
- ▶ To solve problems involving **permutations**.
- ▶ To solve problems involving **combinations**.
- ▶ To establish and use identities associated with **Pascal's triangle**.
- ▶ To solve problems using the **pigeonhole principle**.
- ▶ To understand and apply the **inclusion–exclusion principle**.
- ▶ To apply permutations and combinations to **probability** problems.

Take a deck of 52 playing cards. This simple, familiar deck can be arranged in so many ways that if you and every other living human were to shuffle a deck once per second from the beginning of time, then by now only a tiny fraction of all possible arrangements would have been obtained. So, remarkably, every time you shuffle a deck you are likely to be the first person to have created that particular arrangement of cards!

To see this, note that we have 52 choices for the first card, and then 51 choices for the second card, and so on. This gives a total of

$$52 \times 51 \times \cdots \times 2 \times 1 \approx 8.1 \times 10^{67}$$

arrangements. This is quite an impressive number, especially in light of the fact that the universe is estimated to be merely 1.4×10^{10} years old.

Combinatorics is concerned with counting the number of ways of doing something. Our goal is to find thoughtful ways of doing this without explicitly listing all the possibilities. This is particularly important in the study of probability. For instance, we can use combinatorics to explain why certain poker hands are more likely to occur than others without considering all 2 598 960 possible hands.

7A Basic counting methods

► Tree diagrams

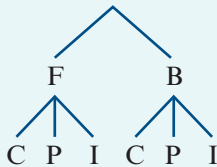
In most combinatorial problems, we are interested in the *number of solutions* to a given problem, rather than the solutions themselves. Nonetheless, for simple counting problems it is sometimes practical to list and then count all the solutions. Tree diagrams provide a systematic way of doing this, especially when the problem involves a small number of steps.

Example 1

A restaurant has a fixed menu, offering a choice of fish or beef for the main meal, and cake, pudding or ice-cream for dessert. How many different meals can be chosen?

Solution

We illustrate the possibilities on a tree diagram:



This gives six different meals, which we can write as

FC, FP, FI, BC, BP, BI

► The multiplication principle

In the above example, for each of the two ways of selecting the main meal, there were three ways of selecting the dessert. This gives a total of $2 \times 3 = 6$ ways of choosing a meal. This is an example of the **multiplication principle**, which will be used extensively throughout this chapter.

Multiplication principle

If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

Example 2

Sandra has three different skirts, four different tops and five different pairs of shoes. How many choices does she have for a complete outfit?

Solution

$$3 \times 4 \times 5 = 60$$

Explanation

Using the multiplication principle, we multiply the number of ways of making each choice.

Example 3

How many paths are there from point P to point R travelling from left to right?

**Solution**

$$4 \times 3 = 12$$

Explanation

For each of the four paths from P to Q , there are three paths from Q to R .

► **The addition principle**

In some instances, we have to count the number of ways of choosing between two alternative tasks. In this case, we use the **addition principle**.

Addition principle

Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

Example 4

To travel from Brisbane to Sydney tomorrow, Kara has a choice between three different flights and two different trains. How many choices does she have?

Solution

$$3 + 2 = 5$$

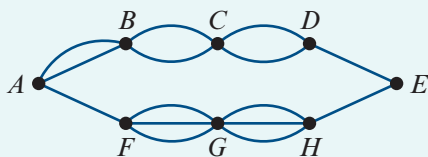
Explanation

The addition principle applies because Kara cannot travel by both plane and train. Therefore, we add the number of ways of making each choice.

Some problems will require use of both the multiplication and the addition principles.

Example 5

How many paths are there from point A to point E travelling from left to right?

**Solution**

We can take *either* an upper path *or* a lower path:

- Going from A to B to C to D to E there are $2 \times 2 \times 2 \times 1 = 8$ paths.
- Going from A to F to G to H to E there are $1 \times 3 \times 3 \times 1 = 9$ paths.

Using the addition principle, there is a total of $8 + 9 = 17$ paths from A to E .

► Harder problems involving tree diagrams

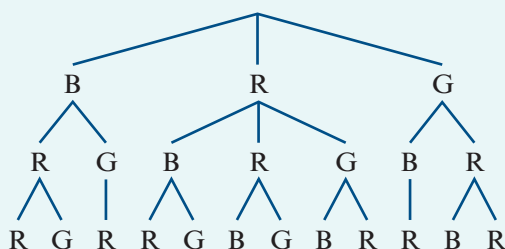
For some problems, a straightforward application of the multiplication and addition principles is not possible.

Example 6

A bag contains one blue token, two red tokens and one green token. Three tokens are removed from the bag and placed in a row. How many arrangements are possible?

Solution

The three tokens are selected without replacement. So once a blue or green token is taken, these cannot appear again. We use a tree diagram to systematically find every arrangement.



The complete set of possible arrangements can be read by tracing out each path from top to bottom of the diagram. This gives 12 different arrangements:

BRR, BRG, BGR, RBR, RBG, RRB, RRG, RGB, RGR, GBR, GRB, GRR

Section summary

Three useful approaches to solving simple counting problems:

■ Tree diagrams

These can be used to systematically list all solutions to a problem.

■ Multiplication principle

If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

■ Addition principle

Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, there are $m + n$ ways to perform one of the tasks.

Some problems require use of both the addition and the multiplication principles.

Exercise 7A

Skillsheet

- Sam has five T-shirts, three pairs of pants and three pairs of shoes. How many different outfits can he assemble using these clothes?

Example 2

SF

2 A restaurant offers five beef dishes and three chicken dishes. How many selections of one main meal does a customer have?

4 Draw a tree diagram showing all the two-digit numbers that can be formed using the digits 7, 8 and 9 if each digit:

b can be repeated.

a as many times as you would like

b at most once?

6 Jack wants to travel from Sydney to Perth via Adelaide. There are four flights and two trains from Sydney to Adelaide. There are two flights and three trains from Adelaide to Perth. How many ways can Jack travel from Sydney to Perth?

7 Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?

a

A graph with three vertices labeled A , a central vertex, and B . There are two curved edges between A and the central vertex, two curved edges between the central vertex and B , and one straight edge between the central vertex and B .

b

A diagram showing a path graph with 4 vertices and 3 edges. The vertices are labeled A and B at the ends. The edges are represented by blue lines connecting the vertices in a sequence.

d

A graph with four vertices labeled A, B, C, and D. There are two edges between A and B, two edges between B and C, and two edges between C and D. There is also a single edge between A and C.

8 A bag contains two blue, one red and two green tokens. Two tokens are removed from the bag and placed in a row. With the help of a tree diagram, list all the different arrangements.

9 How many ways can you make change for 50 cents using 5, 10 and 20 cent pieces?

10 Four teachers decide to swap desks at work. How many ways can this be done if no teacher is to sit at their previous desk?

11 Three runners compete in a race. In how many ways can the runners complete the race assuming:

a there are no tied places

b the runners can tie places?

12 A six-sided die has faces labelled with the numbers 0, 2, 3, 5, 7 and 11. If the die is rolled twice and the two results are multiplied, how many different answers can be obtained?

7B Factorial notation and permutations

► Factorial notation

Factorial notation provides a convenient way of expressing products of consecutive natural numbers. For each natural number n , we define

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$$

where the notation $n!$ is read as ' n factorial'.

We also define $0! = 1$. Although it might seem strange at first, this definition will turn out to be very convenient, as it is compatible with formulas that we will establish shortly.

Another very useful identity is

$$n! = n \cdot (n-1)!$$

Example 7

Evaluate:

a $3!$

b $\frac{50!}{49!}$

c $\frac{10!}{2!8!}$

Solution

a $3! = 3 \cdot 2 \cdot 1$
 $= 6$

b $\frac{50!}{49!} = \frac{50 \cdot 49!}{49!}$
 $= 50$

c $\frac{10!}{2!8!} = \frac{10 \cdot 9 \cdot 8!}{2! \cdot 8!}$
 $= \frac{10 \cdot 9}{2 \cdot 1}$
 $= 45$

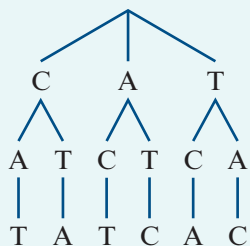
► Permutations of n objects

A **permutation** is an ordered arrangement of a collection of objects.

Example 8

Using a tree diagram, list all the permutations of the letters in the word CAT.

Solution



There are six permutations:

CAT, CTA, ACT, ATC, TCA, TAC

Explanation

There are three choices for the first letter. This leaves only two choices for the second letter, and then one for the third.

Another way to find the number of permutations for the previous example is to draw three boxes, corresponding to the three positions. In each box, we write the number of choices we have for that position.

- We have 3 choices for the first letter (C, A or T).
- We have 2 choices for the second letter (because we have already used one letter).
- We have 1 choice for the third letter (because we have already used two letters).

3	2	1

By the multiplication principle, the total number of arrangements is

$$3 \times 2 \times 1 = 3!$$

So three objects can be arranged in $3!$ ways. More generally:

The number of permutations of n objects is $n!$.

Proof The reason for this is simple:

- The first item can be chosen in n ways.
- The second item can be chosen in $n - 1$ ways, since only $n - 1$ objects remain.
- The third item can be chosen in $n - 2$ ways, since only $n - 2$ objects remain.
- ⋮
- The last item can be chosen in 1 way, since only 1 object remains.

Therefore, by the multiplication principle, there are

$$n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 2 \cdot 1 = n!$$

permutations of n objects.

Example 9

How many ways can six different books be arranged on a shelf?

Solution

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

Explanation

Six books can be arranged in $6!$ ways.

Example 10

Using your calculator, find how many ways 12 students can be lined up in a row.



Using the TI-Nspire CX non-CAS

Evaluate $12!$ as shown.

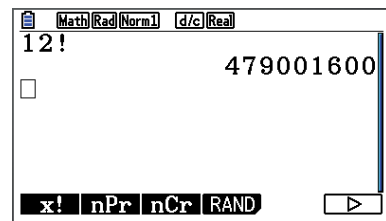
Note: The factorial symbol (!) can be accessed using $\boxed{2\text{nd}} \rightarrow$, the Symbols palette ($\boxed{\text{ctrl}} \rightarrow \boxed{\text{alpha}}$) or $\boxed{\text{menu}} > \text{Probability} > \text{Factorial}$.



Using the Casio

Evaluate $12!$ as follows:

- Press **MENU** **(1)** to select **Run-Matrix** mode.
- Go to the **Probability** menu **(OPTN)** **(F6)** **(F3)**.
- Enter 12, select **x!** **(F1)** and press **(EXE)**.



Example 11

How many four-digit numbers can be formed using the digits 1, 2, 3 and 4 if:

- they cannot be repeated
- they can be repeated?

Solution

- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $4^4 = 4 \times 4 \times 4 \times 4 = 256$

Explanation

Four numbers can be arranged in $4!$ ways.

Using the multiplication principle, there are 4 choices for each of the 4 digits.

► Permutations of n objects taken r at a time

Imagine a very small country with very few cars. Licence plates consist of a sequence of four digits, and repetitions of the digits are not allowed. How many such licence plates are there?

Here, we are asking for the number of permutations of 10 digits taken four at a time. We will denote this number by ${}^{10}P_4$.

To solve this problem, we draw four boxes. In each box, we write the number of choices we have for that position. For the first digit, we have a choice of 10 digits. Once chosen, we have only 9 choices for the second digit, then 8 choices for the third and 7 choices for the fourth.

10	9	8	7

By the multiplication principle, the total number of licence plates is

$$10 \times 9 \times 8 \times 7$$

There is a clever way of writing this product as a fraction involving factorials:

$$\begin{aligned}
 {}^{10}P_4 &= 10 \cdot 9 \cdot 8 \cdot 7 \\
 &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
 &= \frac{10!}{6!} \\
 &= \frac{10!}{(10-4)!}
 \end{aligned}$$

We can easily generalise this procedure to give the following result.

Number of permutations

The number of permutations of n objects taken r at a time is denoted by ${}^nP_r$ and is given by the formula

$${}^nP_r = \frac{n!}{(n-r)!}$$

Proof To establish this formula we note that:

- The 1st item can be chosen in n ways.
- The 2nd item can be chosen in $n - 1$ ways.
- ⋮
- The r th item can be chosen in $n - r + 1$ ways.

Therefore, by the multiplication principle, the number of permutations of n objects taken r at a time is

$$\begin{aligned} {}^nP_r &= n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \\ &= \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \cdot (n-r) \cdot \cdots \cdot 2 \cdot 1}{(n-r) \cdot \cdots \cdot 2 \cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Notes:

- If $r = n$, then we have ${}^nP_n$, which is simply the number of permutations of n objects and so must equal $n!$. The formula still works in this instance, since

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Note that this calculation depends crucially on our decision to define $0! = 1$.

- If $r = 1$, then we obtain ${}^nP_1 = n$. Given n objects, there are n choices of one object, and each of these can be arranged in just one way.

**Example 12**

- a** Using the letters A, B, C, D and E without repetition, how many different two-letter arrangements are there?
- b** Six runners compete in a race. In how many ways can the gold, silver and bronze medals be awarded?

Solution

- a** There are five letters to arrange in two positions:

$$\begin{aligned} {}^5P_2 &= \frac{5!}{(5-2)!} = \frac{5!}{3!} \\ &= \frac{5 \cdot 4 \cdot 3!}{3!} \\ &= 20 \end{aligned}$$

- b** There are six runners to arrange in three positions:

$$\begin{aligned} {}^6P_3 &= \frac{6!}{(6-3)!} = \frac{6!}{3!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} \\ &= 120 \end{aligned}$$

Although the formula developed for ${}^nP_r$ will have an important application later in this chapter, you do not actually have to use it when solving problems. It is often more convenient to simply draw boxes corresponding to the positions, and to write in each box the number of choices for that position.

Example 13

How many ways can seven friends sit along a park bench with space for only four people?

Solution

7	6	5	4

By the multiplication principle, the total number of arrangements is

$$7 \times 6 \times 5 \times 4 = 840$$

Explanation

We draw four boxes, representing the positions to be filled. In each box we write the number of ways we can fill that position.



Using the TI-Nspire CX non-CAS

- To evaluate ${}^7P_4$, use **menu** > **Probability** > **Permutations** as shown.

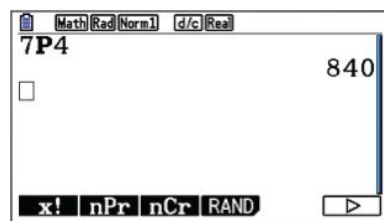


Note: Alternatively, you can simply type $\text{npr}(7,4)$. The command is not case sensitive.

Using the Casio

To evaluate ${}^7P_4$:

- Press **MENU** **(1)** to select **Run-Matrix** mode.
- Go to the **Probability** menu **(OPTN)** **(F6)** **(F3)**.
- Enter 7, select **nPr** **(F2)**, enter 4 and press **(EXE)**.



Section summary

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$ and $0! = 1$
- $n! = n \cdot (n-1)!$
- A **permutation** is an ordered arrangement of objects.
- The number of permutations of n objects is $n!$.
- The number of permutations of n objects taken r at a time is given by

$${}^nP_r = \frac{n!}{(n-r)!}$$

Exercise 7B

- 1 Evaluate $n!$ for $n = 0, 1, 2, \dots, 10$.

Example 7

- 2 Evaluate each of the following:

a $\frac{5!}{4!}$

b $\frac{10!}{8!}$

c $\frac{12!}{10! \cdot 2!}$

d $\frac{100!}{97! \cdot 3!}$

- 3 Simplify the following expressions:

a $\frac{(n+1)!}{n!}$

b $\frac{(n+2)!}{(n+1)!}$

c $\frac{n!}{(n-2)!}$

d $\frac{1}{n!} + \frac{1}{(n+1)!}$

- 4 Evaluate ${}^4P_r$ for $r = 0, 1, 2, 3, 4$.

Example 8

- 5 Use a tree diagram to find all the permutations of the letters in the word DOG.

Example 9

- 6 How many ways can five books on a bookshelf be arranged?

- 7 How many ways can the letters in the word HYPERBOLA be arranged?

Example 12

- 8 Write down all the two-letter permutations of the letters in the word FROG.

Example 13

- 9 How many ways can six students be arranged along a park bench if the bench has:

a six seats

b five seats

c four seats?

- 10 Using the digits 1, 2, 5, 7 and 9 without repetition, how many numbers can you form that have:

a five digits

b four digits

c three digits?

- 11 How many ways can six students be allocated to eight vacant desks?

- 12 How many ways can three letters be posted in five mailboxes if each mailbox can receive:

a more than one letter

b at most one letter?

- 13 Using six differently coloured flags without repetition, how many signals can you make using:

a three flags in a row

b four flags in a row

c five flags in a row?

- 14 You are in possession of four flags, each coloured differently. How many signals can you make using at least two flags arranged in a row?

- 15 Many Australian car licence plates consist of a sequence of three letters followed by a sequence of three digits.

a How many different car licence plates have letters and numbers arranged this way?

b How many of these have no repeated letters or numbers?

- 16 Find all possible values of m and n if $m! \cdot n! = 720$ and $m > n$.

- 17 Show that $n! = (n^2 - n) \cdot (n - 2)!$ for $n \geq 2$.

SF

CF

- 18 a** The three tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



- b** The four tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



- 19** Given six different colours, how many ways can you paint a cube so that all the faces have different colours? Two colourings are considered to be the same when one can be obtained from the other by rotating the cube.

7C Permutations with restrictions

Suppose we want to know how many three-digit numbers have no repeated digits. The answer is *not* simply ${}^{10}P_3$, the number of permutations of 10 digits taken three at a time. This is because the digit 0 cannot be used in the hundreds place.

- There are 9 choices for the first digit (1, 2, 3, ..., 9).
- There are 9 choices for the second digit (0 and the eight remaining non-zero digits).
- This leaves 8 choices for the third digit.

100s	10s	units
9	9	8

By the multiplication principle, there are $9 \times 9 \times 8 = 648$ different three-digit numbers.

When considering permutations with restrictions, we deal with the restrictions first.



Example 14

- a** How many arrangements of the word DARWIN begin and end with a vowel?
b Using the digits 0, 1, 2, 3, 4 and 5 without repetition, how many odd four-digit numbers can you form?

Solution

- a** We draw six boxes. In each box, we write the number of choices we have for that position. We first consider restrictions. There are two choices of vowel (A or I) for the first letter, leaving only one choice for the last letter.

2					1

This leaves four choices for the second letter, three for the next, and so on.

2	4	3	2	1	1

By the multiplication principle, the number of arrangements is

$$2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$$

- b** We draw four boxes. Again, we first consider restrictions. The last digit must be odd (1, 3 or 5), giving three choices. We cannot use 0 in the first position, so this leaves four choices for that position.



Once these two digits have been chosen, this leaves four choices and then three choices for the remaining two positions.



Thus the number of arrangements is

$$4 \times 4 \times 3 \times 3 = 144$$

► Permutations with items grouped together

For some arrangements, we may want certain items to be grouped together. In this case, the trick is to initially treat each group of items as a single object. We then multiply by the numbers of arrangements within each group.

Example 15

- a** How many arrangements of the word EQUALS are there if the vowels are kept together?
- b** How many ways can two chemistry, four physics and five biology books be arranged on a shelf if the books of each subject are kept together?

Solution

a $4! \times 3!$
 $= 144$

b $3! \times 2! \times 4! \times 5!$
 $= 34\,560$

Explanation

We group the three vowels together so that we have four items to arrange: (E, U, A), Q, L, S. They can be arranged in $4!$ ways. Then the three vowels can be arranged among themselves in $3!$ ways. We use the multiplication principle.

There are three groups and so they can be arranged in $3!$ ways. The two chemistry books can be arranged among themselves in $2!$ ways, the four physics books in $4!$ ways and the five biology books in $5!$ ways. We use the multiplication principle.

Section summary

- To count permutations that are subject to restrictions, we draw a series of boxes. In each box, we write the number of choices we have for that position. We always consider the restrictions first.
- When items are to be grouped together, we initially treat each group as a single object. We find the number of arrangements of the groups, and then multiply by the numbers of arrangements within each group.

Exercise 7C

Skillsheet

Example 14

- 1 Using the digits 1, 2, 3, 4 and 5 without repetition, how many five-digit numbers can you form:

- a without restriction
- b that are odd
- c that begin with 5
- d that do not begin with 5?

Example 15

- 2 In how many ways can three girls and two boys be arranged in a row:
- a without restriction
 - b if the two boys sit together
 - c if the two boys do not sit together
 - d if girls and boys alternate?
- 3 How many permutations of the word QUEASY:
- a begin with a vowel
 - b begin and end with a vowel
 - c keep the vowels together
 - d keep the vowels and consonants together?
- 4 How many ways can four boys and four girls be arranged in a row if:
- a boys and girls sit in alternate positions
 - b boys sit together and girls sit together?
- 5 The digits 0, 1, 2, 3, 4 and 5 can be combined without repetition to form new numbers. In how many ways can you form:
- a a six-digit number
 - b a four-digit number divisible by 5
 - c a number less than 6000
 - d an even three-digit number?
- 6 Two parents and four children are seated in a cinema along six consecutive seats. How many ways can this be done:
- a without restriction
 - b if the two parents sit at either end
 - c if the children sit together
 - d if the parents sit together and the children sit together
 - e if the youngest child must sit between and next to both parents?
- 7 12321 is a **palindromic number** because it reads the same backwards as forwards. How many palindromic numbers have:
- a five digits
 - b six digits?
- 8 How many arrangements of the letters in VALUE do not begin and end with a vowel?
- 9 Using each of the digits 1, 2, 3 and 4 at most once, how many even numbers can you form?
- 10 How many ways can six girls be arranged in a row so that two of the girls, A and B:
- a do not sit together
 - b have one person between them?
- 11 How many ways can three girls and three boys be arranged in a row if no two girls sit next to each other?

7D Permutations of like objects

The name for the Sydney suburb of WOOLLOOMOOLOO has the unusual distinction of having 13 letters in total, of which only four are different. Finding the number of permutations of the letters in this word is not as simple as evaluating $13!$. This is because switching like letters does not result in a new permutation.

Our aim is to find an expression for P , where P is the number of permutations of the letters in the word WOOLLOOMOOLOO. First notice that the word has

1 letter W, 1 letter M, 3 letter Ls, 8 letter Os

Replace the three identical Ls with L_1, L_2 and L_3 . These three letters can be arranged in $3!$ different ways. Therefore, by the multiplication principle, there are now

$$P \cdot 3!$$

permutations. Likewise, replace the eight identical Os with $O_1, O_2, \dots, O_8$. These eight letters can be arranged in $8!$ different ways. Therefore there are now

$$P \cdot 3! \cdot 8!$$

permutations.

On the other hand, notice that the 13 letters are now distinct, so there are $13!$ permutations of these letters. Therefore

$$P \cdot 3! \cdot 8! = 13! \quad \text{and so} \quad P = \frac{13!}{3!8!}$$

We can easily generalise this procedure to give the following result.

Permutations of like objects

The number of permutations of n objects of which n_1 are alike, n_2 are alike, $\dots$ and n_r are alike is given by

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Example 16

- Find the number of permutations of the letters in the word RIFFRAFF.
- There are four identical knives, three identical forks and two identical spoons in a drawer. They are taken out of the drawer and lined up in a row. How many ways can this be done?

Solution

$$\text{a} \quad \frac{8!}{4!2!} = 840$$

$$\text{b} \quad \frac{9!}{4!3!2!} = 1260$$

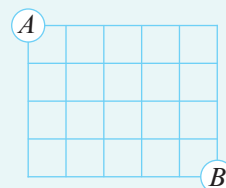
Explanation

There are 8 letters of which 4 are alike and 2 are alike.

There are 9 items of which 4 are alike, 3 are alike and 2 are alike.

Example 17

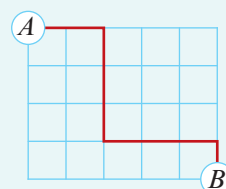
The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?

**Solution**

Each path from A to B can be described by a sequence of four Ds and five Rs in some order. For example, the path shown can be described by the sequence RRDDRRRD.

Since there are 9 letters of which 4 are alike and 5 are alike, the number of permutations of these letters is

$$\frac{9!}{4!5!} = 126$$

**Section summary**

- Switching like objects does not give a new arrangement.
- The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by

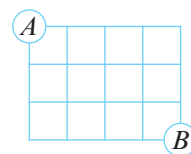
$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Exercise 7D**Skillsheet****Example 16**

- 1 Ying has four identical 20 cent pieces and three identical 10 cent pieces. How many ways can she arrange these coins in a row?
- 2 How many ways can the letters in the word MISSISSIPPI be arranged?
- 3 Find the number of permutations of the letters in the word WARRNAMBOOL.
- 4 Using five 9s and three 7s, how many eight-digit numbers can be made?
- 5 Using three As, four Bs and five Cs, how many sequences of 12 letters can be made?
- 6 How many ways can two red, two black and four blue flags be arranged in a row:
 - a without restriction
 - b if the first flag is red
 - c if the first and last flags are blue
 - d if every alternate flag is blue
 - e if the two red flags are adjacent?

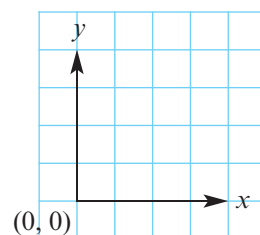
Example 17

- 7 The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



- 8 The grid shown consists of unit squares. By travelling only along the grid lines, how many paths are there:

- a of length 6 from $(0, 0)$ to the point $(2, 4)$
 b of length $m + n$ from $(0, 0)$ to the point (m, n) , where m and n are natural numbers?



- 9 Consider a deck of 52 playing cards.

- a How many ways can the deck be arranged? Express your answer in the form $a!$.
 b If two identical decks are combined, how many ways can the cards be arranged? Express your answer in the form $\frac{a!}{(b!)^c}$.
 c If n identical decks are combined, find an expression for the number of ways that the cards can be arranged.

- 10 An ant starts at position $(0, 0)$ and walks north, east, south or west, one unit at a time. How many different paths of length 8 units finish at $(0, 0)$?
 11 Jessica is about to walk up a flight of 10 stairs. She can take either one or two stairs at a time. How many different ways can she walk up the flight of stairs?

7E Combinations

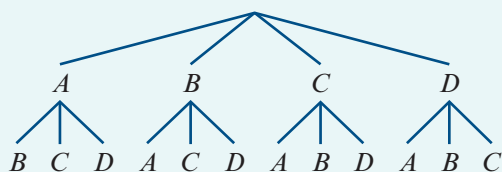
We have seen that a permutation is an ordered arrangement of objects. In contrast, a **combination** is a selection made regardless of order. We use the notation ${}^nP_r$ to denote the number of permutations of n distinct objects taken r at a time. Similarly, we use the notation ${}^nC_r$ to denote the number of combinations of n distinct objects taken r at a time.

Example 18

How many ways can two letters be chosen from the set $\{A, B, C, D\}$?

Solution

The tree diagram below shows the ways that the first and second choices can be made.



This gives 12 arrangements. But there are only six selections, since

$\{A, B\}$ is the same as $\{B, A\}$, $\{A, C\}$ is the same as $\{C, A\}$, $\{A, D\}$ is the same as $\{D, A\}$,
 $\{B, C\}$ is the same as $\{C, B\}$, $\{B, D\}$ is the same as $\{D, B\}$, $\{C, D\}$ is the same as $\{D, C\}$

Suppose we want to count the number of ways that three students can be chosen from a group of seven. Let's label the students with the letters $\{A, B, C, D, E, F, G\}$. One such combination might be BDE . Note that this combination corresponds to $3!$ permutations:

$$BDE, BED, DBE, DEB, EBD, EDB$$

In fact, each combination of three items corresponds to $3!$ permutations, and so there are $3!$ times as many permutations as combinations. Therefore

$${}^7P_3 = 3! \times {}^7C_3$$

and so

$${}^7C_3 = \frac{{}^7P_3}{3!}$$

Since we have already established that ${}^7P_3 = \frac{7!}{(7-3)!}$, we obtain

$${}^7C_3 = \frac{7!}{3!(7-3)!}$$

This argument generalises easily so that we can establish a formula for ${}^nC_r$.

Number of combinations

The number of combinations of n objects taken r at a time is given by the formula

$${}^nC_r = \frac{n!}{r!(n-r)!}$$



Example 19

- a** A pizza can have three toppings chosen from nine options. How many different pizzas can be made?
- b** How many subsets of $\{1, 2, 3, \dots, 20\}$ have exactly two elements?

Solution

- a** Three objects are to be chosen from nine options. This can be done in ${}^9C_3$ ways, and

$$\begin{aligned} {}^9C_3 &= \frac{9!}{3!(9-3)!} \\ &= \frac{9!}{3!6!} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3! \cdot 6!} \\ &= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} \\ &= 84 \end{aligned}$$

- b** Two objects are to be chosen from 20 options. This can be done in ${}^{20}C_2$ ways, and

$$\begin{aligned} {}^{20}C_2 &= \frac{20!}{2!(20-2)!} \\ &= \frac{20!}{2!18!} \\ &= \frac{20 \cdot 19 \cdot 18!}{2! \cdot 18!} \\ &= \frac{20 \cdot 19}{2 \cdot 1} \\ &= 190 \end{aligned}$$

Example 20

Using your calculator, find how many ways 10 students can be selected from a class of 20 students.

**Using the TI-Nspire CX non-CAS**

- To evaluate ${}^{20}C_{10}$, use **menu** > **Probability** > **Combinations** as shown.

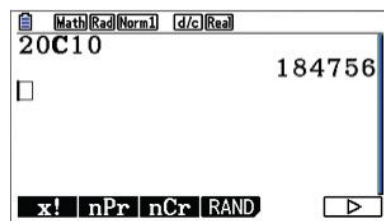


Note: Alternatively, you can simply type $\text{ncr}(20, 10)$. The command is not case sensitive.

Using the Casio

To evaluate ${}^{20}C_{10}$:

- Press **MENU** **(1)** to select **Run-Matrix** mode.
- Go to the **Probability** menu **(OPTN)** **(F6)** **(F3)**.
- Enter 20, select **nCr** **(F3)**, enter 10 and press **(EXE)**.

**Example 21**

Consider a group of six students. In how many ways can a group of:

- a** two students be selected **b** four students be selected?

Solution

$$\begin{aligned} \mathbf{a} \quad {}^6C_2 &= \frac{6!}{2!(6-2)!} \\ &= \frac{6!}{2!4!} \\ &= \frac{6 \cdot 5 \cdot 4!}{2! \cdot 4!} \\ &= \frac{6 \cdot 5}{2 \cdot 1} \\ &= 15 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad {}^6C_4 &= \frac{6!}{4!(6-4)!} \\ &= \frac{6!}{4!2!} \\ &= \frac{6 \cdot 5 \cdot 4!}{4! \cdot 2!} \\ &= \frac{6 \cdot 5}{2 \cdot 1} \\ &= 15 \end{aligned}$$

The fact that parts **a** and **b** of the previous example have the same answer is not a coincidence. Choosing two students out of six is the same as *not choosing* the other four students out of six. Therefore ${}^6C_2 = {}^6C_4$.

More generally:

$${}^nC_r = {}^nC_{n-r}$$

Quick calculations

In some instances, you can avoid unnecessary calculations by noting that:

- ${}^nC_0 = 1$, since there is only one way to select no objects from n objects
- ${}^nC_n = 1$, since there is only one way to select n objects from n objects
- ${}^nC_1 = n$, since there are n ways to select one object from n objects
- ${}^nC_{n-1} = n$, since this corresponds to the number of ways of not selecting one object from n objects.

Example 22

- a** Six points lie on a circle. How many triangles can you make using these points as the vertices?
- b** Each of the 20 people at a party shakes hands with every other person. How many handshakes take place?

Solution

a ${}^6C_3 = 20$

b ${}^{20}C_2 = 190$

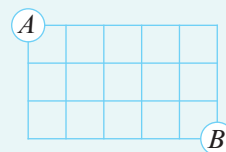
Explanation

This is the same as asking how many ways three vertices can be chosen out of six.

This is the same as asking how many ways two people can be chosen to shake hands out of 20 people.

Example 23

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of three Ds and five Rs in some order. Therefore, the number of paths is equal to the number of ways of selecting three of the eight boxes below to be filled with the three Ds. (The rest will be Rs.) This can be done in ${}^8C_3 = 56$ ways.



Alternative notation

We will consistently use the notation ${}^nC_r$ to denote the number of ways of selecting r objects from n objects, regardless of order. However, it is also common to denote this number by $\binom{n}{r}$.

For example:

$$\binom{6}{4} = \frac{6!}{4!2!} = 15$$

Section summary

- A **combination** is a selection made regardless of order.
- The number of combinations of n objects taken r at a time is given by

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Exercise 7E

- Evaluate ${}^5C_r$ for $r = 0, 1, 2, 3, 4, 5$.
- Evaluate each of the following without the use of your calculator:
a ${}^7C_1$ **b** ${}^6C_5$ **c** ${}^{12}C_{10}$ **d** ${}^8C_5$ **e** ${}^{100}C_{99}$ **f** ${}^{1000}C_{998}$
- Simplify each of the following:
a ${}^nC_1$ **b** ${}^nC_2$ **c** ${}^nC_{n-1}$ **d** ${}^{n+1}C_1$ **e** ${}^{n+2}C_n$ **f** ${}^{n+1}C_{n-1}$
- A playlist contains ten of Nandi's favourite songs. How many ways can he:
a arrange three songs in a list **b** select three songs for a list?
- How many ways can five cards be selected from a deck of 52 playing cards?
- How many subsets of $\{1, 2, 3, \dots, 10\}$ contain exactly:
a 1 element **b** 2 elements **c** 8 elements **d** 9 elements?
- A lottery consists of drawing seven balls out of a barrel of balls numbered from 1 to 45. How many ways can this be done if their order does not matter?
- Eight points lie on a circle. How many triangles can you make using these points as the vertices?
- a** In a hockey tournament, each of the 10 teams plays every other team once. How many games take place?
b In another tournament, each team plays every other team once and 120 games take place. How many teams competed?
- At a party, every person shakes hands with every other person. Altogether there are 105 handshakes. How many people are at the party?
- Prove that ${}^nC_r = {}^nC_{n-r}$.
- Explain why the number of diagonals in a regular polygon with n sides is ${}^nC_2 - n$.
- Ten students are divided into two teams of five. Explain why the number of ways of doing this is $\frac{{}^{10}C_5}{2}$.
- Twelve students are to be divided into two teams of six. In how many ways can this be done? (**Hint:** First complete the previous question.)

SF

CF

- 15** Using the formula for ${}^nC_r$, prove that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, where $1 \leq r < n$.
- 16** Consider the 5×5 grid shown.
- How many ways can three dots be chosen?
 - How many ways can three dots be chosen so that they lie on a straight line?
 - How many ways can three dots be chosen so that they are the vertices of a triangle? (Hint: Use parts **a** and **b**.)



CF

7F Combinations with restrictions

► Combinations including specific items

In some problems, we want to find the number of combinations that include specific items. This reduces both the number of items we have to select and the number of items from which we are selecting.

Example 24

- Grace belongs to a group of eight workers. How many ways can a team of four workers be selected if Grace must be on the team?
- A hand of cards consists of five cards drawn from a deck of 52 playing cards. How many hands contain both the queen and the king of hearts?

Solution

a ${}^7C_3 = 35$

b ${}^{50}C_3 = 19\,600$

Explanation

Grace must be in the selection. Therefore three more workers are to be selected from the remaining seven workers.

The queen and king of hearts must be in the selection. So three more cards are to be selected from the remaining 50 cards.

In some other problems, it can be more efficient to count the selections that we don't want.

Example 25

Four students are to be chosen from a group of eight students for the school tennis team. Two members of the group, Sam and Tess, do not get along and cannot both be on the team. How many ways can the team be selected?

Solution

There are ${}^8C_4$ ways of selecting four students from eight. We then subtract the number of combinations that include both Sam and Tess. If Sam and Tess are on the team, then we can select two more students from the six that remain in ${}^6C_2$ ways. This gives

$${}^8C_4 - {}^6C_2 = 55$$

► Combinations from multiple groups

If we are required to make multiple selections from separate groups, then the multiplication principle dictates that we simply multiply the number of ways of performing each task.



Example 26

From seven women and four men in a workplace, how many groups of five can be chosen:

a without restriction	b containing three women and two men
c containing at least one man	d containing at most one man?

Solution

- a** There are 11 people in total, from which we must select five. This gives

$${}^{11}C_5 = 462$$

- b** There are ${}^7C_3$ ways of selecting three women from seven. There are ${}^4C_2$ ways of selecting two men from four. We then use the multiplication principle to give

$${}^7C_3 \cdot {}^4C_2 = 210$$

c Method 1

If you select at least one man, then you select 1, 2, 3 or 4 men and fill the remaining positions with women. We use the multiplication and addition principles to give

$${}^4C_1 \cdot {}^7C_4 + {}^4C_2 \cdot {}^7C_3 + {}^4C_3 \cdot {}^7C_2 + {}^4C_4 \cdot {}^7C_1 = 441$$

Method 2

It is more efficient to consider all selections of 5 people from 11 and then subtract the number of combinations containing all women. This gives

$${}^{11}C_5 - {}^7C_5 = 441$$

- d** If there is at most one man, then either there are no men or there is one man. If there are no men, then there are ${}^7C_5$ ways of selecting all women. If there is one man, then there are ${}^4C_1$ ways of selecting one man and ${}^7C_4$ ways of selecting four women. This gives

$${}^7C_5 + {}^4C_1 \cdot {}^7C_4 = 161$$

► Permutations and combinations combined

In the following example, we first select the items and then arrange them.

Example 27

- a** How many arrangements of the letters in the word DUPLICATE can be made that have two vowels and three consonants?
- b** A president, vice-president, secretary and treasurer are to be chosen from a group containing seven women and six men. How many ways can this be done if exactly two women are chosen?

Solution

$$\mathbf{a} \quad {}^4C_2 \cdot {}^5C_3 \cdot 5! = 7200$$

$$\mathbf{b} \quad {}^7C_2 \cdot {}^6C_2 \cdot 4! = 7560$$

Explanation

There are ${}^4C_2$ ways of selecting 2 of 4 vowels and ${}^5C_3$ ways of selecting 3 of 5 consonants. Once chosen, the 5 letters can be arranged in $5!$ ways.

There are ${}^7C_2$ ways of selecting 2 of 7 women and ${}^6C_2$ ways of selecting 2 of 6 men. Once chosen, the 4 people can be arranged into the positions in $4!$ ways.

Section summary

- If a selection must include specific items, then this reduces both the number of items that we have to select and the number of items that we select from.
- If we are required to make multiple selections from separate groups, then we multiply the number of ways of performing each task.
- Some problems will require us to select and then arrange objects.

Exercise 7F**Skillsheet****Example 24**

- 1** Jane and Jenny belong to a class of 20 students. How many ways can you select a group of four students from the class if both Jane and Jenny are to be included?

- 2** How many subsets of $\{1, 2, 3, \dots, 10\}$ have exactly five elements and contain the number 5?

- 3** Five cards are dealt from a deck of 52 playing cards. How many hands contain the jack, queen and king of hearts?

Example 25

- 4** Six students are to be chosen from a group of 10 students for the school basketball team. Two members of the group, Rachel and Nethra, do not get along and cannot both be on the team. How many ways can the team be selected?

Example 26

- 5** From eight girls and five boys, a team of seven is selected for a mixed netball team. How many ways can this be done if:

- a** there are no restrictions
- b** there are four girls and three boys on the team
- c** there must be at least three boys and three girls on the team
- d** there are at least two boys on the team?

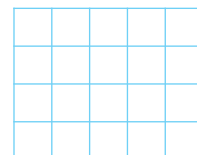
- 6** There are 10 student leaders at a secondary school. Four are needed for a fundraising committee and three are needed for a social committee. How many ways can the students be selected if they can serve on:

- a** both committees
- b** at most one committee?

- 7** There are 18 students in a class. Seven are required for a basketball team and eight are required for a netball team. How many ways can the teams be selected if students can play in:
- a** both teams **b** at most one team?
- 8** From 10 Labor senators and 10 Liberal senators, a committee of five is formed. How many ways can this be done if:
- a** there are no restrictions
b there are at least two senators from each political party
c there is at least one Labor senator?
- 9** Consider the set of numbers $\{1, 2, 3, 4, 5, 6, 7\}$.
- a** How many subsets have exactly five elements?
b How many five-element subsets contain the numbers 2 and 3?
c How many five-element subsets do not contain both 2 and 3?
- 10** Four letters are selected from the English alphabet. How many of these selections will contain exactly two vowels?
- 11** A seven-card hand is dealt from a deck of 52 playing cards. How many distinct hands contain:
- a** four hearts and three spades **b** exactly two hearts and three spades?
- 12** A committee of five people is chosen from four doctors, four dentists and three physiotherapists. How many ways can this be done if the committee contains:
- a** exactly three doctors and one dentist **b** exactly two doctors?

Example 27

- 13** There are four girls and five boys. Two of each are chosen and then arranged on a bench. How many ways can this be done?
- 14** A president, vice-president, secretary and treasurer are to be chosen from a group containing six women and five men. How many ways can this be done if exactly two women must be chosen?
- 15** Using five letters from the word TRAMPOLINE, how many arrangements contain two vowels and three consonants?
- 16** How many rectangles are there in the grid shown on the right?
Hint: Every rectangle is determined by a choice of two vertical and two horizontal lines.
- 17** Five cards are dealt from a deck of 52 playing cards. A full house is a hand that contains 3 cards of one rank and 2 cards of another rank (for example, 3 kings and 2 sevens). How many ways can a full house be dealt?



7G Pascal's triangle

The diagram below consists of the binomial coefficients ${}^nC_r$, for $0 \leq n \leq 5$. They form the first 6 rows of **Pascal's triangle**, named after the seventeenth century French mathematician Blaise Pascal, one of the founders of probability theory.

Interestingly, the triangle was well known to Chinese and Indian mathematicians many centuries earlier.

[illegible]

► Pascal's rule

Pascal's triangle has many remarkable properties. Most importantly:

Each entry in Pascal's triangle is the sum of the two entries immediately above.

Pascal's triangle has this property because of the following identity.

Pascal's rule

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad \text{where } 1 \leq r < n$$

Proof In Question 15 of Exercise 7E, you are asked to prove Pascal's rule using the formula for ${}^nC_r$. However, there is a much nicer argument.

The number of subsets of $\{1, 2, \dots, n\}$ containing exactly r elements is ${}^nC_r$. Each of these subsets can be put into one of two groups:

- 1 those that contain n
- 2 those that do not contain n .

If the subset contains n , then each of the remaining $r - 1$ elements must be chosen from $\{1, 2, \dots, n - 1\}$. Therefore the first group contains ${}^{n-1}C_{r-1}$ subsets.

If the subset does not contain n , then we still have to choose r elements from $\{1, 2, \dots, n-1\}$. Therefore the second group contains ${}^{n-1}C_r$ subsets.

The two groups together contain all ${}^nC_r$ subsets and so

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

which establishes Pascal's rule.

Example 28

Given that ${}^{17}C_2 = 136$ and ${}^{17}C_3 = 680$, evaluate ${}^{18}C_3$.

Solution

$$\begin{aligned} {}^{18}C_3 &= {}^{17}C_2 + {}^{17}C_3 \\ &= 136 + 680 \\ &= 816 \end{aligned}$$

Explanation

We let $n = 18$ and $r = 3$ in Pascal's rule:

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Example 29

Write down the $n = 6$ row of Pascal's triangle and then write down the value of ${}^6C_3$.

Solution

$$\begin{array}{ccccccc} n = 6: & 1 & 6 & 15 & \boxed{20} & 15 & 6 & 1 \\ {}^6C_3 & = & 20 \end{array}$$

Explanation

Each entry in the $n = 6$ row is the sum of the two entries immediately above.

Note that ${}^6C_3$ is the fourth entry in the row, since the first entry corresponds to ${}^6C_0$.

► Subsets of a set

Suppose your friend says to you: 'I have five books that I no longer need, take any that you want.' How many different selections are possible?

We will look at two solutions to this problem.

Solution 1

You could select none of the books (${}^5C_0$ ways), or one out of five (${}^5C_1$ ways), or two out of five (${}^5C_2$ ways), and so on. This gives the answer

$${}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 32$$

Note that this is simply the sum of the entries in the $n = 5$ row of Pascal's triangle.

Solution 2

For each of the five books we have two options: either accept or reject the book. Using the multiplication principle, we obtain the answer

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

There are two important conclusions that we can draw from this example.

1 The sum of the entries in row n of Pascal's triangle is 2^n . That is,

$${}^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

2 A set of size n has 2^n subsets, including the empty set and the set itself.

Example 30

- a** Your friend offers you any of six books that she no longer wants. How many selections are possible assuming that you take at least one book?
- b** How many subsets of $\{1, 2, 3, \dots, 10\}$ have at least two elements?

Solution

a $2^6 - 1 = 63$

b $2^{10} - {}^{10}C_1 - {}^{10}C_0$
 $= 2^{10} - 10 - 1$
 $= 1013$

Explanation

There are 2^6 subsets of a set of size 6. We subtract 1 because we discard the empty set of no books.

There are 2^{10} subsets of a set of size 10. There are ${}^{10}C_1$ subsets containing 1 element and ${}^{10}C_0$ subsets containing 0 elements.

Section summary

- The values of ${}^nC_r$ can be arranged to give Pascal's triangle.
- Each entry in Pascal's triangle is the sum of the two entries immediately above.
- **Pascal's rule:** ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$
- The sum of the entries in row n of Pascal's triangle is 2^n . That is,

$${}^nC_0 + {}^nC_1 + \dots + {}^nC_{n-1} + {}^nC_n = 2^n$$
- A set of size n has 2^n subsets, including the empty set and the set itself.

Exercise 7G**Example 28**

- 1** Evaluate ${}^7C_2$, ${}^6C_2$ and ${}^6C_1$, and verify that the first is the sum of the other two.

Example 29

- 2** Write down the $n = 7$ row of Pascal's triangle. Use your answer to write down the values of ${}^7C_2$ and ${}^7C_4$.
- 3** Write down the $n = 8$ row of Pascal's triangle. Use your answer to write down the values of ${}^8C_4$ and ${}^8C_6$.

Example 30

- 4** Your friend offers you any of six different DVDs that he no longer wants. How many different selections are possible?
- 5** How many subsets does the set $\{A, B, C, D, E\}$ have?
- 6** How many subsets does the set $\{1, 2, 3, \dots, 10\}$ have?
- 7** How many subsets of $\{1, 2, 3, 4, 5, 6\}$ have at least one element?
- 8** How many subsets of $\{1, 2, 3, \dots, 8\}$ have at least two elements?
- 9** How many subsets of $\{1, 2, 3, \dots, 10\}$ contain the numbers 9 and 10?

- 10** You have one 5 cent, one 10 cent, one 20 cent and one 50 cent piece. How many different sums of money can you make assuming that at least one coin is used?
- 11** Let's call a set **selfish** if it contains its size as an element. For example, the set $\{1, 2, 3\}$ is selfish because the set has size 3 and the number 3 belongs to the set.
- a** How many subsets of $\{1, 2, 3, \dots, 8\}$ are selfish?
- b** How many subsets of $\{1, 2, 3, \dots, 8\}$ have the property that both the subset and its complement are selfish?

7H The pigeonhole principle

The pigeonhole principle is an intuitively obvious counting technique which can be used to prove some remarkably counterintuitive results. It gets its name from the following simple observation: If $n + 1$ pigeons are placed into n holes, then some hole contains at least two pigeons. Obviously, in most instances we will not be working with pigeons, so we will recast the principle as follows.

Pigeonhole principle

If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

Proof Suppose that each of the n holes contains at most one object. Then the total number of objects is at most n , which is a contradiction.

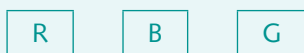
We are now in a position to prove a remarkable fact: There are at least two people in Australia with the same number of hairs on their head. The explanation is simple. No one has more than 1 million hairs on their head, so let's make 1 million holes labelled with the numbers from 1 to 1 million. We now put each of the 24 million Australians into the hole corresponding to the number of hairs on their head. Clearly, some hole contains at least two people, and all the people in that hole will have the same number of hairs on their head.

Example 31

You have thirteen red, ten blue and eight green socks. How many socks need to be selected at random to ensure that you have a matching pair?

Solution

Label three holes with the colours red, blue and green.



Selecting just three socks is clearly not sufficient, as you might pick one sock of each colour. Select four socks and place each sock into the hole corresponding to the colour of the sock. As there are four socks and three holes, the pigeonhole principle guarantees that some hole contains at least two socks. This is the required pair.

**Example 32**

- a** Show that for any five points chosen inside a 2×2 square, at least two of them will be no more than $\sqrt{2}$ units apart.
- b** Seven football teams play 22 games of football. Show that some pair of teams play each other at least twice.

Solution

- a** Split the 2×2 square into four unit squares.



Now we have four squares and five points. By the pigeonhole principle, some square contains at least two points. The distance between any two of these points cannot exceed the length of the square's diagonal, $\sqrt{1^2 + 1^2} = \sqrt{2}$.

- b** There are ${}^7C_2 = 21$ ways that two teams can be chosen to compete from seven. There are 22 games of football, and so some pair of teams play each other at least twice.

► **The generalised pigeonhole principle**

Suppose that 13 pigeons are placed into four holes. By the pigeonhole principle, there is some hole with at least two pigeons. In fact, some hole must contain at least four pigeons. The reason is simple: If each of the four holes contained no more than three pigeons, then there would be no more than 12 pigeons.

This observation generalises as follows.

Generalised pigeonhole principle

If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Proof Again, let's suppose that the statement is false. Then each of the n holes contains no more than m objects. However, this means that there are no more than mn objects, which is a contradiction.

Example 33

Sixteen natural numbers are written on a whiteboard. Prove that at least four numbers will leave the same remainder when divided by 5.

Solution

We label five holes with each of the possible remainders on division by 5.



There are 16 numbers to be placed into five holes. Since $16 = 3 \times 5 + 1$, there is some hole with at least four numbers, each of which leaves the same remainder when divided by 5.

► Pigeons in multiple holes

In some instances, objects can be placed into more than one hole.

Example 34

Seven people sit at a round table with 10 chairs. Show that there are three consecutive chairs that are occupied.

Solution

Number the chairs from 1 to 10. There are 10 groups of three consecutive chairs:

$$\begin{array}{cccccc} \{1, 2, 3\}, & \{2, 3, 4\}, & \{3, 4, 5\}, & \{4, 5, 6\}, & \{5, 6, 7\}, \\ \{6, 7, 8\}, & \{7, 8, 9\}, & \{8, 9, 10\}, & \{9, 10, 1\}, & \{10, 1, 2\} \end{array}$$

Each of the seven people will belong to three of these groups, and so 21 people have to be allocated to 10 groups. Since $21 = 2 \times 10 + 1$, the generalised pigeonhole principle guarantees that some group must contain three people.

Section summary

■ Pigeonhole principle

If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

■ Generalised pigeonhole principle

If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Exercise 7H

Example 31

- 1 You have twelve red, eight blue and seven green socks. How many socks need to be selected at random to ensure that you have a matching pair?
- 2 A sentence contains 27 English words. Show that there are at least two words that begin with the same letter.
- 3 Show that in any collection of five natural numbers, at least two will leave the same remainder when divided by 4.
- 4 How many cards need to be dealt from a deck of 52 playing cards to be certain that you will obtain at least two cards of the same:
 - a colour
 - b suit
 - c rank?
- 5 Eleven points on the number line are located somewhere between 0 and 1. Show that there are at least two points no more than 0.1 apart.

SF

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Example 32

- 6** An equilateral triangle has side length 2 units. Choose any five points inside the triangle. Prove that there are at least two points that are no more than 1 unit apart.
- 7** Thirteen points are located inside a rectangle of length 6 and width 8. Show that there are at least two points that are no more than $2\sqrt{2}$ units apart.
- 8** The **digital sum** of a natural number is defined to be the sum of its digits. For example, the digital sum of 123 is $1 + 2 + 3 = 6$.
- a** Nineteen two-digit numbers are selected. Prove that at least two of them have the same digital sum.
- b** Suppose that 82 three-digit numbers are selected. Prove that at least four of them have the same digital sum.

Example 33

- 9** Whenever Eva writes down 13 integers, she notices that at least four of them leave the same remainder when divided by 4. Explain why this is always the case.
- 10** Twenty-nine games of football are played among eight teams. Prove that there is some pair of teams who play each other more than once.
- 11** A teacher instructs each member of her class to write down a different whole number between 1 and 49. She says that there will be at least one pair of students such that the sum of their two numbers is 50. How many students must be in her class?

Example 34

- 12** There are 10 students seated at a round table with 14 chairs. Show that there are three consecutive chairs that are occupied.
- 13** There are four points on a circle. Show that three of these points lie on a half-circle.
Hint: Pick any one of the four points and draw a diameter through that point.
- 14** There are 35 players on a football team and each player has a different number chosen from 1 to 99. Prove that there are at least four pairs of players whose numbers have the same sum.
- 15** Seven boys and five girls sit evenly spaced at a round table. Prove that some pair of boys are sitting opposite each other.
- 16** There are n guests at a party and some of these guests shake hands when they meet. Use the pigeonhole principle to show that there is a pair of guests who shake hands with the same number of people.
Hint: Place the n guests into holes labelled from 0 to $n - 1$, corresponding to the number of hands that they shake. Why must either the first or the last hole be empty?

CF

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71 The inclusion–exclusion principle

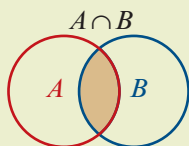
► Basic set theory

A set is any collection of objects where order is not important. The set with no elements is called the **empty set** and is denoted by $\emptyset$. We say that set B is a **subset** of set A if each element of B is also in A . In this case, we can write $B \subseteq A$. Note that $\emptyset \subseteq A$ and $A \subseteq A$.

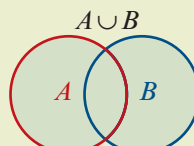
If A is a finite set, then the number of elements in A will be denoted by $|A|$.

Given any two sets A and B we define two important sets:

- 1** The **intersection** of sets A and B is denoted by $A \cap B$ and consists of elements belonging to A and B .



- 2** The **union** of sets A and B is denoted by $A \cup B$ and consists of elements belonging to A or B .



Note: It is important to realise that $A \cup B$ includes elements belonging to A and B .

Example 35

Consider the three sets of numbers $A = \{2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{3, 4, 5\}$.

a Find $B \cap C$.	b Find $A \cup C$.
c Find $A \cap B \cap C$.	d Find $A \cup B \cup C$.
e Find $ A $.	f List all the subsets of C .

Solution

a $B \cap C = \{3, 4\}$	b $A \cup C = \{2, 3, 4, 5\}$
c $A \cap B \cap C = \{3\}$	d $A \cup B \cup C = \{1, 2, 3, 4, 5\}$
e $ A = 2$	f $\emptyset, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}$

Earlier in the chapter we encountered the addition principle. This principle can be concisely expressed using set notation.

Addition principle

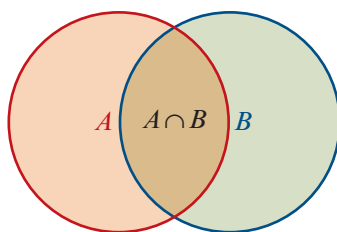
If A and B are two finite sets of objects such that $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B|$$

Our aim is to extend this rule for instances where $A \cap B \neq \emptyset$.

► Two sets

To count the number of elements in the set $A \cup B$, we first add (include) $|A|$ and $|B|$. However, this counts the elements in $A \cap B$ twice, and so we subtract (exclude) $|A \cap B|$.



Inclusion–exclusion principle for two sets

If A and B are two finite sets of objects, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 36

Each of the 25 students in a Year 11 class studies Physics or Chemistry. Of these students, 15 study Physics and 18 study Chemistry. How many students study both subjects?

Solution

$$\begin{aligned} |P \cup C| &= |P| + |C| - |P \cap C| \\ 25 &= 15 + 18 - |P \cap C| \\ 25 &= 33 - |P \cap C| \\ \therefore |P \cap C| &= 8 \end{aligned}$$

Explanation

Let P and C be the sets of students who study Physics and Chemistry respectively.

Since each student studies Physics or Chemistry, we know that $|P \cup C| = 25$.

Example 37

A bag contains 100 balls labelled with the numbers from 1 to 100. How many ways can a ball be chosen that is a multiple of 2 or 5?

Solution

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 50 + 20 - 10 \\ &= 60 \end{aligned}$$

Explanation

Within the set of numbers $\{1, 2, 3, \dots, 100\}$, let A be the set of multiples of 2 and let B be the set of multiples of 5.

Then $A \cap B$ consists of numbers that are multiples of both 2 and 5, that is, multiples of 10.

Therefore $|A| = 50$, $|B| = 20$ and $|A \cap B| = 10$.

We then use the inclusion–exclusion principle.

Example 38

A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly:

- a** two clubs **b** three spades
c two clubs and three spades **d** two clubs or three spades?

Solution

a $^{13}C_2 \cdot ^{39}C_3 = 712\,842$

b ${}^{13}C_3 \cdot {}^{39}C_2 = 211\,926$

c ${}^{13}C_2 \cdot {}^{13}C_3 = 22\,308$

$$\begin{aligned} & |A \cup B| \\ &= |A| + |B| - |A \cap B| \\ &= 712\,842 + 211\,926 - 22\,308 \\ &= 902\,460 \end{aligned}$$

Explanation

There are $^{13}C_2$ ways of choosing 2 clubs from 13 and $^{39}C_3$ ways of choosing 3 more cards from the 39 non-clubs.

There are $^{13}C_3$ ways of choosing 3 spades from 13 and $^{39}C_2$ ways of choosing 2 more cards from the 39 non-spades.

There are $^{13}C_2$ ways of choosing 2 clubs from 13 and $^{13}C_3$ ways of choosing 3 spades from 13.

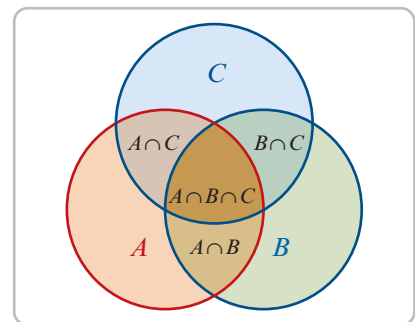
We let A be the set of all hands with 2 clubs and let B be the set of all hands with 3 spades. Then $A \cap B$ is the set of all hands with 2 clubs and 3 spades. We use the inclusion–exclusion principle to find $|A \cup B|$.

► Three sets

For three sets A , B and C , the formula for $|A \cup B \cup C|$ is slightly harder to establish.

We first add $|A|$, $|B|$ and $|C|$. However, we have counted the elements in $A \cap B$, $A \cap C$ and $B \cap C$ twice, and the elements in $A \cap B \cap C$ three times.

Therefore we subtract $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ to compensate. But then the elements in $A \cap B \cap C$ will have been excluded once too often, and so we add $|A \cap B \cap C|$.



Inclusion–exclusion principle for three sets

If A , B and C are three finite sets of objects, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example 39

How many integers from 1 to 140 inclusive are not divisible by 2, 5 or 7?

Solution

Let A , B and C be the sets of all integers from 1 to 140 that are divisible by 2, 5 and 7 respectively. We then have

A	multiples of 2	$ A = 140 \div 2 = 70$
B	multiples of 5	$ B = 140 \div 5 = 28$
C	multiples of 7	$ C = 140 \div 7 = 20$
$A \cap B$	multiples of 10	$ A \cap B = 140 \div 10 = 14$
$A \cap C$	multiples of 14	$ A \cap C = 140 \div 14 = 10$
$B \cap C$	multiples of 35	$ B \cap C = 140 \div 35 = 4$
$A \cap B \cap C$	multiples of 70	$ A \cap B \cap C = 140 \div 70 = 2$

We use the inclusion–exclusion principle to give

$$\begin{aligned}
 |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
 &= 70 + 28 + 20 - 14 - 10 - 4 + 2 \\
 &= 92
 \end{aligned}$$

Therefore the number of integers not divisible by 2, 5 or 7 is $140 - 92 = 48$.

Section summary

- The inclusion–exclusion principle extends the addition principle to instances where the two sets have objects in common.
- The principle works by ensuring that objects belonging to multiple sets are not counted more than once.
- The inclusion–exclusion principles for two sets and three sets:

$$\begin{aligned}
 |A \cup B| &= |A| + |B| - |A \cap B| \\
 |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
 \end{aligned}$$

Exercise 71**Example 35**

- 1 Consider the three sets of numbers $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 3, 4, 6\}$.

a Find $B \cap C$.	b Find $A \cup C$.
c Find $A \cap B \cap C$.	d Find $A \cup B \cup C$.
e Find $ A $.	f List all the subsets of A .

SF

- 2** In an athletics team, each athlete competes in track or field events. There are 25 athletes who compete in track events, 23 who compete in field events and 12 who compete in both track and field events. How many athletes are in the team?

- 3** Fifty patients at a medical clinic are being treated for a disease using two types of medication, A and B . There are 25 patients using medication A and 29 patients using medication B . How many patients are using both types of medication?

- 4** How many integers from 1 to 630 inclusive are multiples of 7 or 9?

- 5** Consider the integers from 1 to 96 inclusive. How many of these are:

a divisible by 2 or 3

b not divisible by 2 or 3?

- 6** How many five-letter arrangements of the word ALIEN:

a begin with a vowel

b end with a vowel

- c** begin and end with a vowel

d begin or end with a vowel?

- 7 a** How many of the integers from 1 to 100 inclusive are perfect squares or perfect cubes?

- b** How many of the integers from 1 to 1000 inclusive are perfect squares or perfect cubes?

- 8** How many of the integers from 1 to 120 inclusive are multiples of 2, 3 or 5?

- 9** How many of the integers from 1 to 220 inclusive are not divisible by 2, 5 or 11?

- 10** There are 90 Year 11 students at a secondary school and each of them must study at least one of Biology, Physics or Chemistry. There are 36 students who study Biology, 42 who study Physics and 40 who study Chemistry. Moreover, 9 study Biology and Physics, 8 study Biology and Chemistry and 7 study Physics and Chemistry. How many students study all three subjects?

- 11** A group of six students is selected from four students in Year 10, five in Year 11 and four in Year 12. How many selections have exactly:

a two Year 10 students

b two Year 11 students

- two Year 10 and two Year 11 students

d two Year 10 or two Year 11 students?

- 12** A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly one heart or exactly two diamonds?

- 13** Find the sum of all the integers from 1 to 100 inclusive that are divisible by 2 or 3.

Hint: Use the formula $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

- 14** There are seventy Year 11 students at a school and each of them must study at least one of three languages. Thirty are studying French, forty-five are studying Chinese, thirty are studying German and fifteen are studying all three languages. How many students are studying exactly two languages?

CF

7J Applications to probability

In this section, we apply our knowledge of permutations and combinations to solving probability problems where there are equally likely outcomes.

Probability is studied in Mathematical Methods Units 1 & 2. For this section, we recall some concepts from your study of probability in previous years.

- The **sample space** for a random experiment is the set of all possible outcomes. For example, the sample space for rolling a die is the set $\{1, 2, 3, 4, 5, 6\}$.
- An **event** is a subset of the sample space. For example, the event of interest when rolling a die may be ‘getting an even number’, which is described by the set $\{2, 4, 6\}$.

When the sample space is finite, the **probability** of an event is equal to the sum of the probabilities of the outcomes in that event.

For example, when rolling a die, there are six possible outcomes that are all equally likely, so we assign a probability of $\frac{1}{6}$ to each outcome. If A is the event ‘getting an even number’, then $A = \{2, 4, 6\}$ and so

$$\begin{aligned}\Pr(A) &= \Pr(2) + \Pr(4) + \Pr(6) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}\end{aligned}$$

Since the outcomes are equally likely, we can calculate this more easily as

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

More generally:

Equally likely outcomes

If the outcomes of a random experiment are all equally likely, then the probability of an event A is given by

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}$$

Establishing the number of outcomes in the event and the total number of outcomes is often achieved by using permutations and combinations.

Example 40

Four-letter ‘words’ are to be made by arranging letters of the word SPECIAL. What is the probability that the ‘word’ will start with a vowel?

Solution

There are 7 letters to be arranged in groups of 4. So the total number of possible arrangements is

$$7 \times 6 \times 5 \times 4 = 840$$

Now consider words which start with a vowel. Since there are three vowels, we have 3 choices for the first letter. Having done this, we have six letters remaining which are to be placed in the three remaining positions, and this can be done in $6 \times 5 \times 4 = 120$ ways.

Thus the number of arrangements which start with a vowel is

$$3 \times 6 \times 5 \times 4 = 360$$

Hence, the probability of the word starting with a vowel is

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}} = \frac{360}{840} = \frac{3}{7}$$

**Example 41**

Three students are to be chosen to represent the class in a debate. If the class consists of six boys and four girls, what is the probability that the team will contain:

- a** exactly one girl
- b** at least two girls?

Solution

Since there is a total of 10 students, the number of possible teams is $^{10}C_3 = 120$.

- a** One girl can be chosen for the team in $^4C_1 = 4$ different ways. Having placed one girl, the other two places must be filled by boys, and this can be done in $^6C_2 = 15$ different ways. Thus the total number of teams containing one girl and two boys is $4 \times 15 = 60$, and the probability that the team contains exactly one girl is $\frac{60}{120} = \frac{1}{2}$.

- b** If the team is to contain at least two girls, then it may contain two *or* three girls. The number of teams containing:

- exactly two girls is $^6C_1 \times ^4C_2 = 36$
- exactly three girls is $^6C_0 \times ^4C_3 = 4$

Thus the total number of teams containing at least two girls is 40, and the probability of this is $\frac{40}{120} = \frac{1}{3}$.

Section summary

Using our knowledge of permutations and combinations, we can calculate probabilities for sample spaces with equally like outcomes:

- First determine the total number of possible outcomes.
- Then determine the number of outcomes in the event of interest.
- The required probability is equal to

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}}$$

Exercise 7J

Skillsheet

- 1 A four-digit number (with no repetitions) is to be formed from the set of digits {1, 2, 3, 4, 5, 6}. Find the probability that the number:

a is even **b** is odd **c** is greater than 4000.

- 2 Eight people, labelled *A, B, C, D, E, F, G* and *H*, are arranged randomly in a line. What is the probability that:

a *A* and *B* are next to each other **b** *A* and *B* are not next to each other?

Example 40

- 3 Three-letter 'words' are to be made by arranging the letters of the word COMPUTER. What is the probability that the 'word' will start with a vowel?

- 4 Three letters are chosen at random from the word HEART and arranged in a row. Find the probability that:

a the letter H is first **b** the letter H is chosen **c** both vowels are chosen.

- 5 A class of 30 children contains 18 girls and 12 boys. Four complimentary theatre tickets are distributed at random to the children in the class. What is the probability that:

a all four tickets go to girls **b** two boys and two girls receive tickets?

- 6 Three men and three women are to be randomly seated in a row. Find the probability that both the end places will be filled by women.

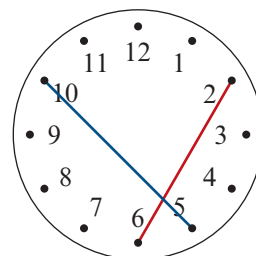
Example 41

- 7 A netball team of seven players is to be chosen randomly from six men and seven women. Find the probability that the selected team contains more men than women.

- 8 Bill is making a sandwich. He may choose any combination of the following: lettuce, tomato, carrot, cheese, cucumber, beetroot, onion, ham. (Bill's sandwich must contain at least one of these ingredients.) Find the probability that:

a the sandwich contains ham
b the sandwich contains three ingredients
c the sandwich contains at least three ingredients.

- 9** A bag contains five white, six red and seven blue balls. If three balls are selected at random, without replacement, find the probability they are:
- a** all red **b** all different colours.
- 10** Susie chooses four pieces of bubble gum from a jar containing five red, two blue and three green pieces of bubble gum. Calculate the probability that Susie selects:
- a** no green bubble gum
b at least one green bubble gum
c at least one green bubble gum and at least one red bubble gum.
- 11** A five-digit number is formed using the digits 1, 2, 3, 4 and 5 without repetition. Find the probability that the number:
- a** is even **b** is odd **c** is greater than 40 000
d does not have two odd digits next to each other
e does not have the two even digits next to each other.
- 12** A four-digit number (with no repetitions) is to be formed from the set of digits $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Find the probability that the number:
- a** is odd **b** is even **c** is less than 4000.
- 13** Three numbers are randomly chosen without replacement from the set $\{1, 2, 3, 4, 5, 6\}$. Find the probability that their sum is greater than 8.
- 14** A team of four people is chosen at random from a group of eight people. Find the probability that the team contains:
- a** the youngest person in the group
b the youngest person in the group and the oldest person in the group.
- 15** A hand of five cards is dealt from a normal pack of 52 cards. Find the probability that the hand will contain:
- a** no aces **b** at least one ace **c** the ace of spades.
- 16** A committee of three is to be chosen from a group of four men and five women. Find the probability that the committee contains:
- a** all women **b** at least one woman.
- 17** Three regular dice are rolled and the three outcomes are recorded. Find the probability that one of the three numbers is the sum of the other two numbers.
- 18** On a clock face, there are 12 points evenly spaced around a circle. Alice randomly chooses two of these points and Bob does the same. Alice then draws a line segment connecting her two points and Bob does the same. Find the probability that their line segments do not meet.



Chapter summary



- The addition and multiplication principles provide efficient methods for counting the number of ways of performing multiple tasks.
- The number of **permutations** (or arrangements) of n objects taken r at a time is given by

$${}^nP_r = \frac{n!}{(n-r)!}$$

- The number of **combinations** (or selections) of n objects taken r at a time is given by

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

- When permutations or combinations involve restrictions, we deal with them first.
- The values of ${}^nC_r$ can be arranged to give **Pascal's triangle**, where each entry is the sum of the two entries immediately above.
- The sum of the entries in row n of Pascal's triangle is 2^n . That is,

$${}^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

- A set of size n has 2^n subsets.
- The **pigeonhole principle** is used to show that some pair or group of objects have the same property.
- The **inclusion–exclusion principle** allows us to count the number of elements in a union of sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- If the outcomes of a random experiment are all equally likely, then the probability of an event A is given by

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}$$

Technology-free questions

- 1 Evaluate:

a ${}^6C_3$

b ${}^{20}C_2$

c ${}^{300}C_1$

d ${}^{100}C_{98}$

- 2 Find the value of n if ${}^nC_2 = 55$.

- 3 How many three-digit numbers can be formed using the digits 1, 2 and 3 if the digits:

a can be repeated

b cannot be repeated?

- 4 How many ways can six students be arranged on a bench seat with space for three?

- 5** How many ways can three students be allocated to five vacant desks? SF
- 6** There are four Year 11 and three Year 12 students in a school debating club. How many ways can a team of four be selected if two are chosen from each year level?
- 7** There are three boys and four girls in a group. How many ways can three children be selected if at least one of them is a boy?
- 8** On a ship's mast are two identical red and three identical black flags that can be arranged to send messages to nearby ships. How many different arrangements using all five flags are possible? CF
- 9** There are 53 English words written on a page. How many are guaranteed to share the same first letter?
- 10** Each of the twenty students in a class plays netball or basketball. Twelve play basketball and four play both sports. How many students play netball?
- 11** Six people are to be seated in a row. Calculate the number of ways this can be done so that two particular people, *A* and *B*, always have exactly one person between them.
- 12** Two parents and their three children are randomly arranged in a row. Find the probability that the parents are at the start and end of the row.
- 13** Three numbers are randomly chosen without replacement from the set $\{1, 2, 3, 4, 5, 6\}$. Find the probability that the selection includes the number 6.

Multiple-choice questions



- 1** Bao plans to study six subjects in Year 12. He has already chosen three subjects and for the remaining three he plans to choose one of four languages, one of three mathematics subjects and one of four science subjects. How many ways can he select his remaining subjects?
A 6 **B** 11 **C** 48 **D** 165 **E** 990
- 2** There are three flights directly from Melbourne to Brisbane. There are also two flights from Melbourne to Sydney and then four choices of connecting flight from Sydney to Brisbane. How many different paths are there from Melbourne to Brisbane?
A 9 **B** 11 **C** 18 **D** 20 **E** 24
- 3** In how many ways can 10 people be arranged in a queue at the bank?
A $10!$ **B** 10^{10} **C** 2^{10} **D** $^{10}C_2$ **E** $^{10}C_1$

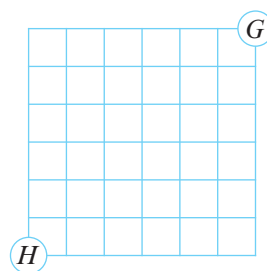
- 4** How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5 and 6 at most once?
A ${}^6C_3$ **B** $3!$ **C** $6!$ **D** $6 \times 5 \times 4$ **E** $6 + 5 + 4$
- 5** How many permutations of the word UTOPIA begin and end with a vowel?
A 90 **B** 288 **C** 384 **D** 720 **E** 4320
- 6** How many ways can four identical red flags and three identical blue flags be arranged in a row?
A 4×3 **B** $\frac{7!}{4! \times 3!}$ **C** $7! \times 3! \times 4!$ **D** $4! \times 3!$ **E** $2 \times 3! \times 4!$
- 7** How many ways can three DVDs be chosen from a collection of nine different DVDs?
A $3!$ **B** $9 \times 8 \times 7$ **C** ${}^9C_3$ **D** $\frac{9!}{3!}$ **E** 3×9
- 8** The number of subsets of $\{A, B, C, D, E, F\}$ with at least one element is
A ${}^6C_2$ **B** ${}^6C_2 - 1$ **C** $2^5 - 1$ **D** $2^6 - 1$ **E** 2^6
- 9** A class consists of nine girls and eight boys. How many ways can a group of two boys and two girls be chosen?
A $\frac{17!}{2!2!}$ **B** ${}^{17}C_4$ **C** ${}^9C_2 \cdot {}^8C_2$ **D** $\frac{17!}{9!8!}$ **E** $9 \times 8 \times 8 \times 7$
- 10** There are six blue balls and five red balls in a bag. How many balls need to be selected at random before you are certain that three will have the same colour?
A 3 **B** 4 **C** 5 **D** 7 **E** 11
- 11** Each of the 30 students in a class studies French, German or Chinese. Of these students, 15 study French, 17 study German and 15 study Chinese. There are 15 students that study more than one subject. How many students study all three subjects?
A 2 **B** 3 **C** 4 **D** 5 **E** 6
- 12** If the letters of the word HEADS are arranged in random order, then the probability that the letters are in alphabetical order is
A $\frac{1}{120}$ **B** $\frac{1}{60}$ **C** $\frac{1}{24}$ **D** $\frac{1}{5}$ **E** $\frac{1}{4}$
- 13** A box contains twelve red balls and four green balls. A ball is selected at random from the box and not replaced and then a second ball is drawn. The probability that the two balls are both green is equal to
A $\frac{1}{4}$ **B** $\frac{1}{16}$ **C** $\frac{3}{64}$ **D** $\frac{1}{8}$ **E** $\frac{1}{20}$

Extended-response questions

- 1** A six-digit number is formed using the digits 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done if:
- a** the first digit is 5
 - b** the first digit is even
 - c** even and odd digits alternate
 - d** the even digits are kept together?
- 2** Three letters from the word AUNTIE are arranged in a row. How many ways can this be done if:
- a** the first letter is E
 - b** the first letter is a vowel
 - c** the letter E is used?
- 3** A student leadership team consists of four boys and six girls. A group of four students is required to organise a social function. How many ways can the group be selected:
- a** without restriction
 - b** if the school captain is included
 - c** if there are two boys
 - d** if there is at least one boy?
- 4** Consider the eight letters N, N, J, J, T, T, T, T. How many ways can all eight letters be arranged if:
- a** there is no restriction
 - b** the first and last letters are both N
 - c** the two Js are adjacent
 - d** no two Ts are adjacent?
- 5** A pizza restaurant offers the following toppings: onion, capsicum, mushroom, olives, ham and pineapple.
- a** How many different kinds of pizza can be ordered with:
 - i** three different toppings
 - ii** three different toppings including ham
 - iii** any number of toppings (between none and all six)?
 - b** Another pizza restaurant boasts that they can make more than 200 varieties of pizza. What is the smallest number of toppings that they could use?
- 6** In how many ways can a group of four people be chosen from five married couples if:
- a** there is no restriction
 - b** any two married couples are chosen
 - c** the selected group cannot contain a married couple?
- 7** The name David Smith has initials DS.
- a** How many different two-letter initials are possible?
 - b** How many different two-letter initials contain at least one vowel?
 - c** Given 50 000 people, how many of them can be guaranteed to share the same two-letter initials?

- 8** Consider the integers from 1 to 96 inclusive. Let sets A and B consist of those integers that are multiples of 6 and 8 respectively.
- What is the lowest common multiple of 6 and 8?
 - How many integers belong to $A \cap B$?
 - How many integers from 1 to 96 are divisible by 6 or 8?
 - An integer from 1 to 96 is chosen at random. What is the probability that it is not divisible by 6 or 8?

- 9** Every morning, Milly walks from her home $H(0, 0)$ to the gym $G(6, 6)$ along city streets that are laid out in a square grid as shown. She always takes a path of shortest distance.



- How many paths are there from H to G ?
- Show that there is some path that she takes at least twice in the course of three years.
- On her way to the gym, she often purchases a coffee at a cafe located at point $C(2, 2)$. How many paths are there from:
 - H to C
 - C to G
 - H to C to G ?
- A new cafe opens up at point $B(4, 4)$. How many paths can Milly take, assuming that she buys coffee at either cafe?

Hint: You will need to use the inclusion–exclusion principle here.

- 10** Danica has 10 dice to distribute into three empty boxes. For each of the 10 dice in turn, she randomly chooses one of the three boxes in which to place it. What is the probability that, after she has finished, there is one box containing four dice and there are two boxes containing three dice?
- 11** An ordinary pack of 52 cards is thoroughly shuffled. The cards are then dealt face up, one at a time, until an ace appears. What is the probability that the first ace appears at:
- the fifth card
 - the k th card
 - the k th card or sooner?
- 12** A box contains 400 balls, each of which is blue, red, green, yellow or orange. The ratio of blue to red to green balls is $1 : 4 : 2$. The ratio of green to yellow to orange balls is $1 : 3 : 6$. What is the smallest number of balls that must be drawn to ensure that at least 50 balls of one colour are selected?

8

Number and proof

Objectives

- ▶ To understand and use various methods of proof, including:
 - ▷ **direct proof**
 - ▷ **proof by contrapositive**
 - ▷ **proof by contradiction.**
 - ▶ To write down the **negation** of a statement.
 - ▶ To write and prove **converse** statements.
 - ▶ To understand when mathematical statements are **equivalent**.
 - ▶ To use the symbols for **implication** ($\Rightarrow$) and **equivalence** ($\Leftrightarrow$).
 - ▶ To understand and use the quantifiers '**for all**' and '**there exists**'.
 - ▶ To disprove statements using **counterexamples**.
 - ▶ To understand and use the **principle of mathematical induction**.
-

A **mathematical proof** is an argument that demonstrates the absolute truth of a statement.

It is certainty that makes mathematics different from other sciences. In science, a theory is never proved true. Instead, one aims to prove that a theory is not true. And if such evidence is hard to come by, then this increases the likelihood that a theory is correct, but never provides a guarantee. The possibility of absolute certainty is reserved for mathematics alone.

When writing a proof you should always aim for three things:

- correctness
- clarity
- simplicity.

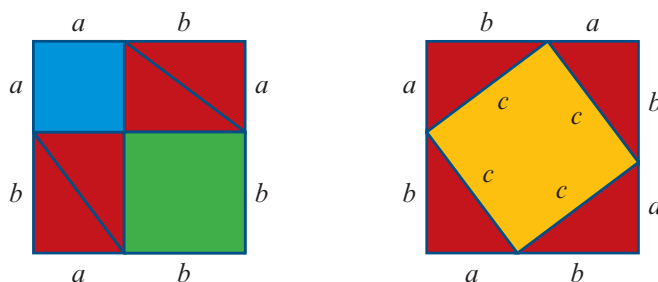
Perhaps the following proof of Pythagoras' theorem exemplifies these three aims.

Pythagoras' theorem

Take any triangle with side lengths a , b and c . If the angle between a and b is 90° , then

$$a^2 + b^2 = c^2$$

Proof Consider the two squares shown below.



The two squares each have the same total area. So subtracting four red triangles from each figure will leave the same area. Therefore $a^2 + b^2 = c^2$.

The ideas introduced in this chapter will be used in proofs throughout the rest of this book.

8A Direct proof

► Conditional statements

Consider the following sentence:

Statement	If	it is raining	then	the grass is wet.

This is called a **conditional statement** and has the form:

Statement	If	P is true	then	Q is true.

This can be abbreviated as

$$P \Rightarrow Q$$

which is read ' P **implies** Q '. We call P the **hypothesis** and Q the **conclusion**.

Not all conditional statements will be true. For example, switching the hypothesis and the conclusion above gives:

Statement	If	the grass is wet	then	it is raining.

Anyone who has seen dewy grass on a cloudless day knows this to be false. In this chapter we will learn how to prove (and disprove) mathematical statements.

► Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.



Example 1

Prove the following statements:

- a** If a is odd and b is even, then $a + b$ is odd.
- b** If a is odd and b is odd, then ab is odd.

Solution

- a** Assume that a is odd and b is even.

Since a is odd, we have $a = 2m + 1$ for some $m \in \mathbb{Z}$. Since b is even, we have $b = 2n$ for some $n \in \mathbb{Z}$. Therefore

$$\begin{aligned} a + b &= (2m + 1) + 2n \\ &= 2m + 2n + 1 \\ &= 2(m + n) + 1 \\ &= 2k + 1 \quad \text{where } k = m + n \in \mathbb{Z} \end{aligned}$$

Hence $a + b$ is odd.

Note: We must use two different pronumerals m and n here, because these two numbers may be different.

- b** Assume that both a and b are odd. Then $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. Therefore

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1 \quad \text{where } k = 2mn + m + n \in \mathbb{Z} \end{aligned}$$

Hence ab is odd.

Example 2

Let $p, q \in \mathbb{Z}$ such that p is divisible by 5 and q is divisible by 3. Prove that pq is divisible by 15.

Solution

Since p is divisible by 5, we have $p = 5m$ for some $m \in \mathbb{Z}$. Since q is divisible by 3, we have $q = 3n$ for some $n \in \mathbb{Z}$. Thus

$$\begin{aligned} pq &= (5m)(3n) \\ &= 15mn \end{aligned}$$

and so pq is divisible by 15.

Example 3

Let x and y be positive real numbers. Prove that if $x > y$, then $x^2 > y^2$.

Solution

Assume that $x > y$. Then $x - y > 0$.

Since x and y are positive, we also know that $x + y > 0$.

Therefore

$$x^2 - y^2 = \overbrace{(x - y)}^{\text{positive}} \overbrace{(x + y)}^{\text{positive}} > 0$$

Hence $x^2 > y^2$.

Explanation

When trying to prove that $x^2 > y^2$, it is easier to first prove that $x^2 - y^2 > 0$.

Also, note that the product of two positive numbers is positive.

**Example 4**

Let x and y be any two positive real numbers. Prove that

$$\frac{x + y}{2} \geq \sqrt{xy}$$

Solution

A **false proof** might begin with the statement that we are trying to prove.

$$\begin{aligned} & \frac{x + y}{2} \geq \sqrt{xy} \\ \Rightarrow & x + y \geq 2\sqrt{xy} \\ \Rightarrow & (x + y)^2 \geq 4xy \quad (\text{using Example 3}) \\ \Rightarrow & x^2 + 2xy + y^2 \geq 4xy \\ \Rightarrow & x^2 - 2xy + y^2 \geq 0 \\ \Rightarrow & (x - y)^2 \geq 0 \end{aligned}$$

Although it is true that $(x - y)^2 \geq 0$, the argument is faulty. We cannot prove that the result is true by assuming that the result is true! However, the above work is not a waste of time.

We can correct the proof by reversing the order of the steps shown above.

Note: In the corrected proof, we need to use the fact that $a > b$ implies $\sqrt{a} > \sqrt{b}$ for all positive numbers a and b . This is shown in Question 8 of Exercise 8B.

► **Breaking a proof into cases**

Sometimes it helps to break a problem up into different cases.

Example 5

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'We are both knaves.' What are Alice and Bob?

Solution

We will prove that Alice is a knave and Bob is a knight.

Case 1

Suppose Alice is a knight.

- $\Rightarrow$ Alice is telling the truth.
- $\Rightarrow$ Alice and Bob are both knaves.
- $\Rightarrow$ Alice is a knave and a knight.

This is impossible.

Case 2

Suppose Alice is a knave.

- $\Rightarrow$ Alice is not telling the truth.
- $\Rightarrow$ Alice and Bob are not both knaves.
- $\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knave and Bob must be a knight.

Section summary

- A **mathematical proof** establishes the truth of a statement.
- A **conditional statement** has the form: If P is true, then Q is true.
This can be abbreviated as $P \Rightarrow Q$, which is read ' P **implies** Q '.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Exercise 8A**Skillsheet**

- 1** Assume that m is even and n is even. Prove that:

- a** $m + n$ is even **b** mn is even.

- 2** Assume that m is odd and n is odd. Prove that $m + n$ is even.

- 3** Assume that m is even and n is odd. Prove that mn is even.

Example 2

- 4** Suppose that m is divisible by 3 and n is divisible by 7. Prove that:

- a** mn is divisible by 21 **b** m^2n is divisible by 63.

- 5** Suppose that m and n are perfect squares. Show that mn is a perfect square.

- 6** Let m and n be integers. Prove that $(m + n)^2 - (m - n)^2$ is divisible by 4.

- 7** Suppose that n is an even integer. Prove that $n^2 - 6n + 5$ is odd.

- 8** Suppose that n is an odd integer. Prove that $n^2 + 8n + 3$ is even.

- 9** Let $n \in \mathbb{Z}$. Prove that $5n^2 + 3n + 7$ is odd.

Hint: Consider the cases when n is odd and n is even.

Example 3

- 10** Let x and y be positive real numbers. Show that if $x > y$, then $x^4 > y^4$.

Example 4

- 11** Let $x, y \in \mathbb{R}$. Show that $x^2 + y^2 \geq 2xy$.

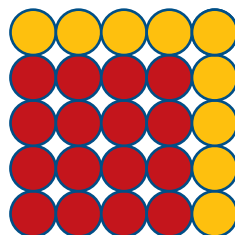
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Example 5 **12** Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Determine whether Alice and Bob are knights or knaves in each of the following separate instances:

- a** Alice says: 'We are both knaves.'
- b** Alice says: 'We are both of the same kind.' Bob says: 'We are of a different kind.'
- c** Alice says: 'Bob is a knave.' Bob says: 'Neither of us is a knave.'

13 The diagram shows that 9 can be written as the difference of two squares: $9 = 5^2 - 4^2$.



- a** Draw another diagram to show that 11 can be written as the difference of two squares.
- b** Prove that every odd number can be written as the difference of two squares.
- c** Hence, express 101 as the difference of two squares.

14 a Consider the numbers $\frac{9}{10}$ and $\frac{10}{11}$. Which is larger?

b Let n be a natural number. Prove that $\frac{n}{n+1} > \frac{n-1}{n}$.

15 a Prove that

$$\frac{1}{10} - \frac{1}{11} < \frac{1}{100}$$

b Let $n > 0$. Prove that

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

16 Let $a, b \in \mathbb{R}$. Prove that $\frac{a^2 + b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$.

17 a Expand $(x-y)(x^2 + xy + y^2)$.

b Prove that $x^2 + yx + y^2 \geq 0$ for all $x, y \in \mathbb{R}$.

Hint: Complete the square by thinking of y as a constant.

c Hence, prove that if $x \geq y$, then $x^3 \geq y^3$.

18 Sally travels from home to work at a speed of 12 km/h and immediately returns home at a speed of 24 km/h.

a Show that her average speed is 16 km/h.

b Now suppose that Sally travels to work at a speed of a km/h and immediately returns home at a speed of b km/h. Show that her average speed is $\frac{2ab}{a+b}$ km/h.

c Let a and b be any two positive real numbers. Prove that

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}$$

Note: This proves that Sally's average speed for the whole journey can be no greater than the average of her speeds for the two individual legs of the journey.

8B Proof by contrapositive

► The negation of a statement

To **negate** a statement P we write its very opposite, which we call '**not P** '. For example, consider the following four statements and their negations.

P	not P
The sky is green. (false)	The sky is not green. (true)
$1 + 1 = 2$ (true)	$1 + 1 \neq 2$ (false)
All prime numbers are odd. (false)	There exists an even prime number. (true)
All triangles have three sides. (true)	Some triangle does not have three sides. (false)

Notice that negation turns a true statement into a false statement, and a false statement into a true statement.



Example 6

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $2 > 1$

b 5 is divisible by 3

c The sum of any two odd numbers is even.

d There are two primes whose product is 12.

Solution

a P : $2 > 1$ (true)
not P : $2 \leq 1$ (false)

b P : 5 is divisible by 3 (false)
not P : 5 is not divisible by 3 (true)

c P : The sum of any two odd numbers is even. (true)
not P : There are two odd numbers whose sum is odd. (false)

d P : There are two primes whose product is 12. (false)
not P : There are no two primes whose product is 12. (true)

► De Morgan's laws

Negating statements that involve 'and' and 'or' requires the use of De Morgan's laws.

De Morgan's laws

not (P and Q) is the same as (not P) or (not Q)

not (P or Q) is the same as (not P) and (not Q)

Example 7

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 6 is divisible by 2 and 3

b 10 is divisible by 2 or 7

Solution

a P : 6 is divisible by 2 and 6 is divisible by 3 (true)

not P : 6 is not divisible by 2 or 6 is not divisible by 3 (false)

b P : 10 is divisible by 2 or 10 is divisible by 7 (true)

not P : 10 is not divisible by 2 and 10 is not divisible by 7 (false)

Example 8

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'I am a knave or Bob is a knight.' What are Alice and Bob?

Solution

We will prove that Alice is a knight and Bob is a knight.

Case 1

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice is a knight AND Bob is a knave.

$\Rightarrow$ Alice is a knight and a knave.

This is impossible.

Case 2

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice is a knave OR Bob is a knight.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knight and Bob must be a knight.

► Proof by contrapositive

Consider this statement:

Statement	If it is the end of term then the students are happy.

By switching the hypothesis and the conclusion and negating both, we obtain the **contrapositive** statement:

Contrapositive	If the students are <i>not</i> happy then it is <i>not</i> the end of term.

Note that the original statement and its contrapositive are logically equivalent:

- If the original statement is true, then the contrapositive is true.
- If the original statement is false, then the contrapositive is false.

This means that to prove a conditional statement, we can instead prove its contrapositive. This is helpful, as it is often easier to prove the contrapositive than the original statement.

- The **contrapositive** of $P \Rightarrow Q$ is the statement $(\text{not } Q) \Rightarrow (\text{not } P)$.
- To prove $P \Rightarrow Q$, we can prove the contrapositive instead.

Example 9

Let $n \in \mathbb{Z}$ and consider this statement: If n^2 is even, then n is even.

- a** Write down the contrapositive. **b** Prove the contrapositive.

Solution

- a** If n is odd, then n^2 is odd.
b Assume that n is odd. Then $n = 2m + 1$ for some $m \in \mathbb{Z}$. Squaring n gives

$$\begin{aligned} n^2 &= (2m + 1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= 2k + 1 \quad \text{where } k = 2m^2 + 2m \in \mathbb{Z} \end{aligned}$$

Therefore n^2 is odd.

Note: Although we proved the contrapositive, remember that we have actually proved that if n^2 is even, then n is even.

**Example 10**

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2 + 4n + 1$ is even, then n is odd.

- a** Write down the contrapositive. **b** Prove the contrapositive.

Solution

- a** If n is even, then $n^2 + 4n + 1$ is odd.
b Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$. Therefore

$$\begin{aligned} n^2 + 4n + 1 &= (2m)^2 + 4(2m) + 1 \\ &= 4m^2 + 8m + 1 \\ &= 2(2m^2 + 4m) + 1 \\ &= 2k + 1 \quad \text{where } k = 2m^2 + 4m \in \mathbb{Z} \end{aligned}$$

Hence $n^2 + 4n + 1$ is odd.

Example 11

Let x and y be positive real numbers and consider this statement: If $x < y$, then $\sqrt{x} < \sqrt{y}$.

- a** Write down the contrapositive. **b** Prove the contrapositive.

Solution

- a** If $\sqrt{x} \geq \sqrt{y}$, then $x \geq y$.
b Assume that $\sqrt{x} \geq \sqrt{y}$. Then $x \geq y$ by Example 3, since $\sqrt{x}$ and $\sqrt{y}$ are positive.

Section summary

- To **negate** a statement we write its opposite.
- For a statement $P \Rightarrow Q$, the **contrapositive** is the statement $(\text{not } Q) \Rightarrow (\text{not } P)$. That is, we switch the hypothesis and the conclusion and negate both.
- A statement and its contrapositive are logically equivalent.
- Proving the contrapositive of a statement may be easier than giving a direct proof.

Exercise 8B

Skillsheet

- 1 Write down each statement and its negation. Which of the statement and its negation is true and which is false?

Example 6

- a** $1 > 0$ **b** 4 is divisible by 8
c Each pair of primes has an even sum.
d Some rectangle has four sides of equal length.

Example 7

- 2 Write down each statement and its negation. Which of the statement and its negation is true and which is false?

- a** 14 is divisible by 7 and 2 **b** 12 is divisible by 3 or 4
c 15 is divisible by 3 and 6 **d** 10 is divisible by 2 or 3

Example 8

- 3 Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'I am a knave and Bob is a knight.' What are Alice and Bob?

Example 9

- 4 Write down the contrapositive version of each of these statements:

- a** If it is raining, then there are clouds in the sky.
b If you are smiling, then you are happy.
c If $x = 1$, then $2x = 2$.
d If $x > y$, then $x^5 > y^5$.
e Let $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.
f Let $m, n \in \mathbb{Z}$. If m and n are odd, then mn is odd.
g Let $m, n \in \mathbb{Z}$. If $m + n$ is even, then m and n are either both even or both odd.

Example 10

- 5 Let $m, n \in \mathbb{Z}$. For each of the following statements, write down and prove the contrapositive statement:

- a** If $3n + 5$ is even, then n is odd. **b** If n^2 is odd, then n is odd.
c If $n^2 - 8n + 3$ is even, then n is odd.
d If n^2 is not divisible by 3, then n is not divisible by 3.
e If $n^3 + 1$ is even, then n is odd.
f If mn is not divisible by 3, then m is not divisible by 3 and n is not divisible by 3.
g If $m + n$ is odd, then $m \neq n$.

- 6** Let $x, y \in \mathbb{R}$. For each of the following statements, write down and prove the contrapositive statement:
- a** If $x^2 + 3x < 0$, then $x < 0$.
 - b** If $x^3 - x > 0$, then $x > -1$.
 - c** If $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.
 - d** If $2x + 3y \geq 12$, then $x \geq 3$ or $y \geq 2$.
- 7** Let $m, n \in \mathbb{Z}$ and consider this statement: If mn and $m + n$ are even, then m and n are even.
- a** Write down the contrapositive.
 - b** Prove the contrapositive. You will have to consider cases.

Example 11

- 8** Let x and y be positive real numbers.

- a** Prove that

$$\sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$

- b** Hence, prove that if $x > y$, then $\sqrt{x} > \sqrt{y}$.
- c** Give a simpler proof by considering the contrapositive.

8C Proof by contradiction

There are various instances when we want to prove mathematically that something cannot be done. To do this, we assume that it can be done, and then show that something goes horribly wrong. Let's first look at a familiar example from geometry.

Example 12

An angle is called **reflex** if it exceeds 180° . Prove that no quadrilateral has more than one reflex angle.

Solution

If there is more than one reflex angle, then the angle sum must exceed $2 \times 180^\circ = 360^\circ$. This contradicts the fact that the angle sum of any quadrilateral is exactly 360° . Therefore there cannot be more than one reflex angle.

The example above is a demonstration of a **proof by contradiction**. The basic outline of a proof by contradiction is:

- 1** Assume that the statement we want to prove is false.
- 2** Show that this assumption leads to mathematical nonsense.
- 3** Conclude that we were wrong to assume that the statement is false.
- 4** Conclude that the statement must be true.

Example 13

A **Pythagorean triple** consists of three natural numbers (a, b, c) satisfying

$$a^2 + b^2 = c^2$$

Show that if (a, b, c) is a Pythagorean triple, then a , b and c cannot all be odd numbers.

Solution

This will be a proof by contradiction.

Let (a, b, c) be a Pythagorean triple. Then $a^2 + b^2 = c^2$.

Suppose that a , b and c are all odd numbers.

$\Rightarrow a^2, b^2$ and c^2 are all odd numbers.

$\Rightarrow a^2 + b^2$ is even and c^2 is odd.

Since $a^2 + b^2 = c^2$, this gives a contradiction.

Therefore a , b and c cannot all be odd numbers.

Possibly the most well-known proof by contradiction is the following.

Theorem

$\sqrt{2}$ is irrational.

Proof This will be a proof by contradiction.

Suppose that $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}$$

We can assume that p and q have no common factors (or else they could be cancelled). Then, squaring both sides and rearranging gives

$$p^2 = 2q^2 \quad (1)$$

$$\Rightarrow p^2 \text{ is divisible by } 2$$

$$\Rightarrow p \text{ is divisible by } 2 \quad (\text{by Example 9})$$

$$\Rightarrow p = 2n \text{ for some } n \in \mathbb{Z}$$

$$\Rightarrow (2n)^2 = 2q^2 \quad (\text{substituting into (1)})$$

$$\Rightarrow q^2 = 2n^2$$

$$\Rightarrow q^2 \text{ is divisible by } 2$$

$$\Rightarrow q \text{ is divisible by } 2 \quad (\text{by Example 9})$$

Therefore both p and q are divisible by 2, which contradicts the fact that they have no common factors.

Hence $\sqrt{2}$ is irrational.

**Example 14**

Suppose x satisfies $5^x = 2$. Show that x is irrational.

Solution

Suppose that x is rational. Since x must be positive, we can write $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$.
Therefore

$$\begin{aligned} 5^x = 2 &\Rightarrow 5^{\frac{m}{n}} = 2 \\ &\Rightarrow \left(5^{\frac{m}{n}}\right)^n = 2^n \quad (\text{raise both sides to the power } n) \\ &\Rightarrow 5^m = 2^n \end{aligned}$$

The left-hand side of this equation is odd and the right-hand side is even. This gives a contradiction, and so x is not rational.

We finish on a remarkable result, which is attributed to Euclid some 2300 years ago.

Theorem

There are infinitely many prime numbers.

Proof This is a proof by contradiction, so we will suppose that there are only finitely many primes. This means that we can create a list that contains *every* prime number:

$$2, 3, 5, 7, \dots, p$$

where p is the largest prime number.

Now for the trick. We create a new number N by multiplying each number in the list and then adding 1:

$$N = 2 \times 3 \times 5 \times 7 \times \dots \times p + 1$$

The number N is not divisible by any of the primes $2, 3, 5, 7, \dots, p$, since it leaves a remainder of 1 when divided by any of these numbers.

However, every natural number greater than 1 is divisible by a prime number. (This is proved in Question 13 of Exercise 8F.) Therefore N is divisible by some prime number q . But this prime number q is not in the list $2, 3, 5, 7, \dots, p$, contradicting the fact that our list contains every prime number.

Hence there are infinitely many prime numbers.

Section summary

- A **proof by contradiction** is used to prove that something cannot be done.
- These proofs always follow the same basic structure:
 - 1 Assume that the statement we want to prove is false.
 - 2 Show that this assumption leads to mathematical nonsense.
 - 3 Conclude that we were wrong to assume that the statement is false.
 - 4 Conclude that the statement must be true.

Exercise 8C

Skillsheet

Example 12

1 Prove that every triangle has some interior angle with a magnitude of at least 60° .

2 Prove that there is no smallest positive rational number.

3 Let p be a prime number. Show that $\sqrt{p}$ is not an integer.

Example 14

4 Suppose that $3^x = 2$. Prove that x is irrational.

5 Prove that $\log_2 5$ is irrational.

6 Suppose that $x > 0$ is irrational. Prove that $\sqrt{x}$ is also irrational.

7 Suppose that a is rational and b is irrational. Prove that $a + b$ is irrational.

8 Suppose that $c^2 - b^2 = 4$. Prove that b and c cannot both be natural numbers.

9 Let a , b and c be real numbers with $a \neq 0$. Prove by contradiction that there is only one solution to the equation $ax + b = c$.

10 a Prove that all primes $p > 2$ are odd.

b Hence, prove that there are no two primes whose sum is 1001.

11 a Prove that there are no integers a and b for which $42a + 7b = 1$.

Hint: The left-hand side is divisible by 7.

b Prove that there are no integers a and b for which $15a + 21b = 2$.

12 a Prove that if n^2 is divisible by 3, then n is divisible by 3.

Hint: Prove the contrapositive by considering two cases.

b Hence, prove that $\sqrt{3}$ is irrational.

13 a Prove that if n^3 is divisible by 2, then n is divisible by 2.

Hint: Prove the contrapositive.

b Hence, prove that $\sqrt[3]{2}$ is irrational.

14 Prove that if $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

15 a Let $a, b, n \in \mathbb{N}$. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

b Hence, show that 97 is a prime number.

16 a Let m be an integer. Prove that m^2 is divisible by 4 or leaves a remainder of 1.

Hint: Suppose that $m = 4n + r$ and consider m^2 for $r = 0, 1, 2, 3$.

b Let $a, b, c \in \mathbb{Z}$. Prove by contradiction: If $a^2 + b^2 = c^2$, then a is even or b is even.

17 a Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$.

b Hence, find $c, d \in \mathbb{Z}$ if $\sqrt{3 + 2\sqrt{2}} = c + d\sqrt{2}$. Hint: Square both sides.

18 Let $a, b, c \in \mathbb{Z}$. Prove that if a , b and c are all odd, then the equation $ax^2 + bx + c = 0$ cannot have a rational solution.

SF

CF

8D Equivalent statements

► The converse of a statement

At the beginning of this chapter, we proved Pythagoras' theorem. Consider any triangle with side lengths a , b and c .

Statement	If the angle between a and b is 90° then $a^2 + b^2 = c^2$.

By switching the hypothesis and the conclusion, we obtain the **converse** statement:

Converse	If $a^2 + b^2 = c^2$ then the angle between a and b is 90° .

In Chapter 9 we will prove that this is also a true statement.

When we switch the hypothesis and the conclusion of a conditional statement, $P \Rightarrow Q$, we obtain the **converse** statement, $Q \Rightarrow P$.

Note: The converse of a true statement may not be true. For example:

Statement	If it is raining, then there are clouds in the sky.	(true)
Converse	If there are clouds in the sky, then it is raining.	(false)

Example 15

Let x and y be positive real numbers. Consider the statement: If $x < y$, then $x^2 < y^2$.

- a** Write down the converse of this statement.
- b** Prove the converse.

Solution

- a** If $x^2 < y^2$, then $x < y$.
- b** Assume that $x^2 < y^2$. Then, since both x and y are positive,

$$\begin{aligned}
 & x^2 - y^2 < 0 && \text{(subtract } y^2) \\
 \Rightarrow & (x - y)(x + y) < 0 && \text{(factorising)} \\
 \Rightarrow & x - y < 0 && \text{(divide both sides by } x + y > 0) \\
 \Rightarrow & x < y
 \end{aligned}$$

as required.

Example 16

Let m and n be integers. Consider the statement: If m and n are even, then $m + n$ is even.

- a** Write down the converse of this statement.
- b** Show that the converse is not true.

Solution

- a** If $m + n$ is even, then m is even and n is even.
b Clearly $1 + 3 = 4$ is even, although 1 and 3 are not.

► **Equivalent statements**

Now consider the following two statements:

P : your heart is beating

Q : you are alive

Notice that both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true statements. In this case, we say that P and Q are **equivalent** statements and we write

$$P \Leftrightarrow Q$$

We will also say that P is true **if and only if** Q is true. So in the above example, we can say

Your heart is beating if and only if you are alive.

To prove that two statements P and Q are equivalent, you have to prove two things:

$$P \Rightarrow Q \quad \text{and} \quad Q \Rightarrow P$$

Example 17

Let $n \in \mathbb{Z}$. Prove that n is even if and only if $n + 1$ is odd.

Solution

($\Rightarrow$) Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$.

Therefore $n + 1 = 2m + 1$, and so $n + 1$ is odd.

($\Leftarrow$) Assume that $n + 1$ is odd. Then $n + 1 = 2m + 1$ for some $m \in \mathbb{Z}$.

Subtracting 1 from both sides gives $n = 2m$. Therefore n is even.

Note: To prove that $P \Leftrightarrow Q$, we have to show that $P \Rightarrow Q$ and $P \Leftarrow Q$. When we are about to prove $P \Rightarrow Q$, we write ($\Rightarrow$). When we are about to prove $P \Leftarrow Q$, we write ($\Leftarrow$).

Section summary

- For a statement $P \Rightarrow Q$, the **converse** is the statement $Q \Rightarrow P$.
That is, we switch the hypothesis and the conclusion.
- If $P \Rightarrow Q$ is true and $Q \Rightarrow P$ is true, then we say that P is **equivalent** to Q , or that P is true **if and only if** Q is true.
- If P and Q are equivalent, we write $P \Leftrightarrow Q$.

Exercise 8D

Skillsheet

Example 15

1 Write down and prove the converse of each of these statements:

- a Let $x \in \mathbb{R}$. If $2x + 3 = 5$, then $x = 1$.
- b Let $n \in \mathbb{Z}$. If n is odd, then $n - 3$ is even.
- c Let $m \in \mathbb{Z}$. If $m^2 + 2m + 1$ is even, then m is odd.
- d Let $n \in \mathbb{Z}$. If n^2 is divisible by 5, then n is divisible by 5.

Example 16

2 Let m and n be integers. Consider the statement: If m and n are even, then mn is a multiple of 4.

- a Write down the converse of this statement.
- b Show that the converse is not true.

3 Which of these pairs of statements are equivalent?

- a P : Vivian is in China.
 Q : Vivian is in Asia.
- b P : $2x = 4$
 Q : $x = 2$
- c P : $x > 0$ and $y > 0$
 Q : $xy > 0$
- d P : m is even or n is even, where $m, n \in \mathbb{Z}$
 Q : mn is even, where $m, n \in \mathbb{Z}$

Example 17

4 Let n be an integer. Prove that $n + 1$ is odd if and only if $n + 2$ is even.5 Let $n \in \mathbb{N}$. Prove that $n^2 - 4$ is a prime number if and only if $n = 3$.6 Let n be an integer. Prove that n^3 is even if and only if n is even.7 Let n be an integer. Prove that n is odd if and only if $n = 4k \pm 1$ for some $k \in \mathbb{Z}$.8 Let $x, y \in \mathbb{R}$. Prove that $(x + y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.9 Let m and n be integers.

- a By expanding the right-hand side, prove that $m^3 - n^3 = (m - n)(m^2 + mn + n^2)$.
- b Hence, prove that $m - n$ is even if and only if $m^3 - n^3$ is even.

10 Prove that an integer is divisible by 4 if and only if the number formed by its last two digits is divisible by 4. **Hint:** 100 is divisible by 4.

SF

CF

8E Disproving statements

► Quantification using ‘for all’ and ‘there exists’

For all

Universal quantification claims that a property holds for *all* members of a given set. For example, consider this statement:

Statement	For all natural numbers n , we have $2n \geq n + 1$.

To prove that this statement is true, we need to give a general argument that applies to every natural number n .

There exists

Existential quantification claims that a property holds for *at least one* member of a given set. For example, consider this statement:

Statement	There exists an integer m such that $m^2 = 25$.

To prove that this statement is true, we just need to give an example: $5 \in \mathbb{Z}$ with $5^2 = 25$.

Example 18

Rewrite each statement using either ‘for all’ or ‘there exists’:

- a** Some real numbers are irrational.
- b** Every integer that is divisible by 4 is also divisible by 2.

Solution

- a** There exists $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$.
- b** For all $m \in \mathbb{Z}$, if m is divisible by 4, then m is divisible by 2.

Negating ‘for all’ and ‘there exists’

To negate a statement involving a quantifier, we interchange ‘for all’ with ‘there exists’ and then negate the rest of the statement.

Example 19

Write down the negation of each of the following statements:

- a** For all natural numbers n , we have $2n \geq n + 1$.
- b** There exists an integer m such that $m^2 = 4$ and $m^3 = -8$.

Solution

- a** There exists a natural number n such that $2n < n + 1$.
- b** For all integers m , we have $m^2 \neq 4$ or $m^3 \neq -8$.

Note: For part **b**, we used one of De Morgan’s laws.

Notation

The words ‘for all’ can be abbreviated using the *turned A* symbol, $\forall$. The words ‘there exists’ can be abbreviated using the *turned E* symbol, $\exists$. For example:

- ‘For all natural numbers n , we have $2n \geq n + 1$ ’ can be written as $(\forall n \in \mathbb{N}) 2n \geq n + 1$.
- ‘There exists an integer m such that $m^2 = 25$ ’ can be written as $(\exists m \in \mathbb{Z}) m^2 = 25$.

Despite the ability of these new symbols to make certain sentences more concise, we do not believe that they make written sentences clearer. Therefore we have avoided using them in this chapter.

► Counterexamples

Consider the quadratic function $f(n) = n^2 - n + 11$. Notice how $f(n)$ is a prime number for small natural numbers n :

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	11	13	17	23	31	41	53	67	83	101

From this, we might be led to believe that the following statement is true:

Statement For all natural numbers n , the number $f(n)$ is prime.

We call this a **universal statement**, because it asserts the truth of a statement without exception. So to disprove a universal statement, we need only show that it is not true in some particular instance. For our example, we need to find $n \in \mathbb{N}$ such that $f(n)$ is not prime. Luckily, we do not have to look very hard.

Example 20

Let $f(n) = n^2 - n + 11$. Disprove this statement: For all $n \in \mathbb{N}$, the number $f(n)$ is prime.

Solution

When $n = 11$, we obtain

$$f(11) = 11^2 - 11 + 11 = 11^2$$

Therefore $f(11)$ is not prime.

To disprove a statement of the form $P \Rightarrow Q$, we simply need to give one example for which P is true and Q is not true. Such an example is called a **counterexample**.

Example 21

Find a counterexample to disprove this statement: For all $x, y \in \mathbb{R}$, if $x > y$, then $x^2 > y^2$.

Solution

Let $x = 1$ and $y = -2$. Clearly $1 > -2$, but $1^2 = 1 \leq 4 = (-2)^2$.

► Disproving existence statements

Consider this statement:

Statement	There exists $n \in \mathbb{N}$ such that $n^2 + 3n + 2$ is a prime number.

We call this an **existence statement**, because it claims the existence of an object possessing a particular property. To show that such a statement is false, we prove that its negation is true:

Negation	For all $n \in \mathbb{N}$, the number $n^2 + 3n + 2$ is not a prime number.

This is easy to prove, as

$$n^2 + 3n + 2 = (n + 1)(n + 2)$$

is clearly a composite number for each $n \in \mathbb{N}$.

Example 22

Disprove this statement: There exists $n \in \mathbb{N}$ such that $n^2 + 13n + 42$ is a prime number.

Solution

We need to prove that, for all $n \in \mathbb{N}$, the number $n^2 + 13n + 42$ is not prime.

This is true, since

$$n^2 + 13n + 42 = (n + 6)(n + 7)$$

is clearly a composite number for each $n \in \mathbb{N}$.

Example 23

Show that this statement is false: There exists some real number x such that $x^2 = -1$.

Solution

We have to prove that the negation is true: For *all* real numbers x , we have $x^2 \neq -1$.

This is easy to prove since, for any real number x , we have $x^2 \geq 0$ and so $x^2 \neq -1$.

Section summary

- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier '**for all**'.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier '**there exists**'.
- A universal statement of the form $P \Rightarrow Q$ can be disproved by giving one example of an instance when P is true but Q is not.
- Such an example is called a **counterexample**.
- To disprove an existence statement, we prove that its negation is true.

Exercise 8E

Skillsheet

- 1 Which of the following are universal statements ('for all') and which are existence statements ('there exists')?

Example 18

- a For each $n \in \mathbb{N}$, the number $5n^2 + 3n + 7$ is odd.
- b There is an even prime number.
- c Every natural number greater than 1 has a prime factorisation.
- d All triangles have three sides.
- e Some natural numbers are primes.
- f At least one real number x satisfies the equation $x^2 - x - 1 = 0$.
- g Any positive real number has a square root.
- h The angle sum of a triangle is 180° .

- 2 Which of the following statements are true and which are false?

- a There exists a real number x such that $x^2 = 2$.
- b There exists a real number x such that $x^2 < 0$.
- c For all natural numbers n , the number $2n - 1$ is odd.
- d There exists $n \in \mathbb{N}$ such that $2n$ is odd.
- e For all $x \in \mathbb{R}$, we have $x^3 \geq 0$.

Example 19

- 3 Write down the negation of each of the following statements:

- a For every natural number n , the number $2n^2 - 4n + 31$ is prime.
- b For all $x \in \mathbb{R}$, we have $x^2 > x$.
- c There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.
- d For all $x, y \in \mathbb{R}$, we have $(x + y)^2 = x^2 + y^2$.
- e There exist $x, y \in \mathbb{R}$ such that $x < y$ and $x^2 > y^2$.

- 4 Prove that each of the following statements is false by finding a counterexample:

Example 20

- a For every natural number n , the number $2n^2 - 4n + 31$ is prime.
- b If $x, y \in \mathbb{R}$, then $(x + y)^2 = x^2 + y^2$.
- c For all $x \in \mathbb{R}$, we have $x^2 > x$.
- d Let $n \in \mathbb{Z}$. If $n^3 - n$ is even, then n is even.
- e If $m, n \in \mathbb{N}$, then $m + n \leq mn$.
- f Let $m, n \in \mathbb{Z}$. If 6 divides mn , then 6 divides m or 6 divides n .

Example 21

- 5 Show that each of the following existence statements is false:

Example 22

- a There exists $n \in \mathbb{N}$ such that $9n^2 - 1$ is a prime number.
- b There exists $n \in \mathbb{N}$ such that $n^2 + 5n + 6$ is a prime number.

Example 23

- c There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.

SF

- 6** Provide a counterexample to disprove each of the following statements.

Hint: $\sqrt{2}$ might come in handy.

- a** If a is irrational and b is irrational, then ab is irrational.
- b** If a is irrational and b is irrational, then $a + b$ is irrational.
- c** If a is irrational and b is irrational, then $\frac{a}{b}$ is irrational.

- 7** Let $a \in \mathbb{Z}$.

- a** Prove that if a is divisible by 4, then a^2 is divisible by 4.
- b** Prove that the converse is not true.

- 8** Let $a, b \in \mathbb{Z}$.

- a** Prove that if $a - b$ is divisible by 3, then $a^2 - b^2$ is divisible by 3.
- b** Prove that the converse is not true.

- 9** Prove that each of the following statements is false:

- a** There exist real numbers a and b such that $a^2 - 2ab + b^2 = -1$.
- b** There exists some real number x such that $x^2 - 4x + 5 = \frac{3}{4}$.

- 10** The numbers $\{1, 2, \dots, 8\}$ can be paired so that the sum of each pair is a square number:

$$1 + 8 = 9, \quad 2 + 7 = 9, \quad 3 + 6 = 9, \quad 4 + 5 = 9$$

- a** Prove that you can also do this with the numbers $\{1, 2, \dots, 16\}$.
- b** Prove that you cannot do this with the numbers $\{1, 2, \dots, 12\}$.

- 11** Let $f(n) = an^2 + bn + c$ be a quadratic function, where a, b, c are natural numbers and $c \geq 2$. Show that there is an $n \in \mathbb{N}$ such that $f(n)$ is not a prime number.

8F Mathematical induction

Consider the sum of the first n odd numbers:

$$1 = 1 = 1^2$$

$$1 + 3 = 4 = 2^2$$

$$1 + 3 + 5 = 9 = 3^2$$

$$1 + 3 + 5 + 7 = 16 = 4^2$$

From this limited number of examples, we could make the following proposition $P(n)$ about the number n : the sum of the first n odd numbers is n^2 . Since the n th odd number is $2n - 1$, we can write this proposition as

$$P(n): 1 + 3 + 5 + \dots + (2n - 1) = n^2$$

However, we have to be careful here: Just because something looks true does not mean that it is true. In this section, we will learn how to prove statements like the one above.

► The principle of mathematical induction

Imagine a row of dominoes extending infinitely to the right. Each of these dominoes can be knocked over provided two conditions are met:

- 1 The first domino is knocked over.
- 2 Each domino is sufficiently close to the next domino.



This scenario provides an accurate physical model of the following proof technique.

Principle of mathematical induction

Let $P(n)$ be some proposition about the natural number n .

We can prove that $P(n)$ is true for every natural number n as follows:

- a Show that $P(1)$ is true.
- b Show that, for every natural number k , if $P(k)$ is true, then $P(k + 1)$ is true.

The idea is simple: Condition **a** tells us that $P(1)$ is true. But then condition **b** means that $P(2)$ will also be true. However, if $P(2)$ is true, then condition **b** also guarantees that $P(3)$ is true, and so on. This process continues indefinitely, and so $P(n)$ is true for all $n \in \mathbb{N}$.

$$P(1) \text{ is true} \Rightarrow P(2) \text{ is true} \Rightarrow P(3) \text{ is true} \Rightarrow \dots$$

Let's see how mathematical induction is used in practice.

Example 24

Prove that

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

for all $n \in \mathbb{N}$.

Solution

For each natural number n , let $P(n)$ be the proposition:

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

Step 1 $P(1)$ is the proposition $1 = 1^2$, that is, $1 = 1$. Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$1 + 3 + 5 + \dots + (2k - 1) = k^2$$

Step 3 We now have to prove that $P(k + 1)$ is true, that is,

$$1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) = (k + 1)^2$$

Notice that we have written the last and the second-last term in the summation. This is so we can easily see how to use our assumption that $P(k)$ is true.

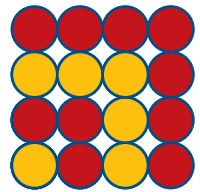
We have

$$\begin{aligned}
 \text{LHS of } P(k+1) &= 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \\
 &= k^2 + (2k+1) && \text{(using } P(k)) \\
 &= (k+1)^2 \\
 &= \text{RHS of } P(k+1)
 \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k .

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number n .

While mathematical induction is good for proving that formulas are true, it rarely indicates why they should be true in the first place. The formula $1 + 3 + 5 + \cdots + (2n-1) = n^2$ can be discovered in the diagram shown on the right.



Example 25

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n , let $P(n)$ be the proposition:

$$7^n - 4 \text{ is divisible by } 3$$

Step 1 $P(1)$ is the proposition $7^1 - 4 = 3$ is divisible by 3. Clearly, $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$7^k - 4 = 3m$$

for some $m \in \mathbb{Z}$.

Step 3 We now have to prove that $P(k+1)$ is true, that is, $7^{k+1} - 4$ is divisible by 3.

We have

$$\begin{aligned}
 7^{k+1} - 4 &= 7 \times 7^k - 4 \\
 &= 7(3m + 4) - 4 && \text{(using } P(k)) \\
 &= 21m + 28 - 4 \\
 &= 21m + 24 \\
 &= 3(7m + 8)
 \end{aligned}$$

Therefore $7^{k+1} - 4$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k .

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

► Proving inequalities

Induction can be used to prove certain inequalities.

For example, consider this table of values:

n	1	2	3	4	5
3^n	3	9	27	81	243
3×2^n	6	12	24	48	96

From the table, it certainly looks as though

$$3^n > 3 \times 2^n \quad \text{for all } n \geq 3$$

We will prove this formally using induction; this time starting with the proposition $P(3)$ instead of $P(1)$.

Example 26

Prove that

$$3^n > 3 \times 2^n$$

for every natural number $n \geq 3$.

Solution

For each natural number $n \geq 3$, let $P(n)$ be the proposition:

$$3^n > 3 \times 2^n$$

Step 1 $P(3)$ is the proposition $3^3 > 3 \times 2^3$, that is, $27 > 24$. Therefore $P(3)$ is true.

Step 2 Let k be a natural number with $k \geq 3$, and assume $P(k)$ is true. That is,

$$3^k > 3 \times 2^k$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$3^{k+1} > 3 \times 2^{k+1}$$

We have

$$\begin{aligned}
 \text{LHS of } P(k+1) &= 3^{k+1} \\
 &= 3 \times 3^k \\
 &> 3 \times 3 \times 2^k && \text{(using } P(k)) \\
 &> 3 \times 2 \times 2^k \\
 &= 3 \times 2^{k+1} \\
 &= \text{RHS of } P(k+1)
 \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number $k \geq 3$.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number $n \geq 3$.

► Applications to sequences

Sequences are studied in Mathematical Methods Units 1 & 2. A **sequence** is a list of numbers, with order being important. An example is the sequence of odd numbers:

$$1, 3, 5, 7, 9, \dots$$

The n th term of a sequence is denoted by t_n . So, for the sequence of odd numbers, we have $t_1 = 1$, $t_2 = 3$ and $t_3 = 5$. In general, we have $t_n = 2n - 1$ for all $n \in \mathbb{N}$.

Some sequences can be defined by a rule that enables each subsequent term to be found from previous terms. We can define the sequence of odd numbers by $t_1 = 1$ and $t_{n+1} = t_n + 2$. This type of rule is called a **recurrence relation**.

Induction proofs are frequently used in the study of sequences. As an example, we will consider the sequence defined by the recurrence relation

$$t_1 = 11 \quad \text{and} \quad t_{n+1} = 10t_n - 9$$

The first five terms of this sequence are listed in the following table.

n	1	2	3	4	5
t_n	11	101	1001	10 001	100 001

Notice that each of these terms is one more than a power of 10. Let's see if we can prove that this is true for *every* term in the sequence.

Example 27

Given $t_1 = 11$ and $t_{n+1} = 10t_n - 9$, prove that $t_n = 10^n + 1$.

Solution

For each natural number n , let $P(n)$ be the proposition: $t_n = 10^n + 1$.

Step 1 Since $t_1 = 11$ and $10^1 + 1 = 11$, it follows that $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$t_k = 10^k + 1$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$t_{k+1} = 10^{k+1} + 1$$

We have

$$\begin{aligned}
 \text{LHS of } P(k+1) &= t_{k+1} \\
 &= 10t_k - 9 \\
 &= 10 \times (10^k + 1) - 9 \quad (\text{using } P(k)) \\
 &= 10^{k+1} + 1 \\
 &= \text{RHS of } P(k+1)
 \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

► Tower of Hanoi

You have three pegs and a collection of n discs of different sizes. Initially, all the discs are stacked in size order on the left-hand peg. Discs can be moved one at a time from one peg to any other peg, provided that a larger disc never rests on a smaller one. The aim of the puzzle is to transfer all the discs to another peg using the smallest possible number of moves.



Example 28

Let a_n be the minimum number of moves needed to solve the Tower of Hanoi with n discs.

- Find a formula for a_{n+1} in terms of a_n .
- Evaluate a_n for $n = 1, 2, 3, 4, 5$. Guess a formula for a_n in terms of n .
- Confirm your formula for a_n using mathematical induction.
- If $n = 20$, how many days are needed to transfer all the discs to another peg, assuming that one disc can be moved per second?

Solution

- Suppose there are $n + 1$ discs on the left-hand peg.

If we want to be able to move the largest disc to the right-hand peg, then first we must transfer the other n discs to the centre peg. This takes a minimum of a_n moves.

It takes 1 move to transfer the largest disc to the right-hand peg. Now we can complete the puzzle by transferring the n discs on the centre peg to the right-hand peg. This takes a minimum of a_n moves.

Hence the minimum number of moves required to transfer all the discs is

$$\begin{aligned} a_{n+1} &= a_n + 1 + a_n \\ &= 2a_n + 1 \end{aligned}$$

- We have $a_1 = 1$, since one disc can be moved in one move. Using the recurrence relation from part **a**, we find that

$$a_2 = 2a_1 + 1 = 2 \times 1 + 1 = 3$$

$$a_3 = 2a_2 + 1 = 2 \times 3 + 1 = 7$$

Continuing in this way, we obtain the following table.

n	1	2	3	4	5
a_n	1	3	7	15	31

It seems as though every term is one less than a power of 2. We guess that

$$a_n = 2^n - 1$$

c For each natural number n , let $P(n)$ be the proposition:

$$a_n = 2^n - 1$$

Step 1 The minimum number of moves required to solve the Tower of Hanoi puzzle with one disc is 1. Since $2^1 - 1 = 1$, it follows that $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$a_k = 2^k - 1$$

Step 3 We now wish to prove that $P(k + 1)$ is true, that is,

$$a_{k+1} = 2^{k+1} - 1$$

We have

$$\begin{aligned} \text{LHS of } P(k + 1) &= a_{k+1} \\ &= 2a_k + 1 && \text{(using part a)} \\ &= 2 \times (2^k - 1) + 1 && \text{(using } P(k)) \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \\ &= \text{RHS of } P(k + 1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k + 1)$ is true, for every natural number k .

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$. Hence we have shown that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

d A puzzle with 20 discs requires a minimum of $2^{20} - 1$ seconds.

Since there are $60 \times 60 \times 24 = 86\,400$ seconds in a day, it will take

$$\frac{2^{20} - 1}{86\,400} \approx 12.14 \text{ days}$$

to complete the puzzle.

Section summary

The basic outline of a proof by mathematical induction is:

- 0** Define the proposition $P(n)$ for $n \in \mathbb{N}$.
- 1** Show that $P(1)$ is true.
- 2** Assume that $P(k)$ is true for some $k \in \mathbb{N}$.
- 3** Show that $P(k + 1)$ is true.
- 4** Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Exercise 8F

Skillsheet

1 Prove each of the following by mathematical induction:

Example 24

$$\mathbf{a} \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\mathbf{b} \quad 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \text{ where } x \neq 1$$

$$\mathbf{c} \quad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\mathbf{d} \quad 1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

$$\mathbf{e} \quad \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

$$\mathbf{f} \quad \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for } n \geq 2$$

Example 25

2 Prove each of the following divisibility statements by mathematical induction:

$$\mathbf{a} \quad 11^n - 1 \text{ is divisible by 10 for all } n \in \mathbb{N}$$

$$\mathbf{b} \quad 3^{2n} + 7 \text{ is divisible by 8 for all } n \in \mathbb{N}$$

$$\mathbf{c} \quad 7^n - 3^n \text{ is divisible by 4 for all } n \in \mathbb{N}$$

$$\mathbf{d} \quad 5^n + 6 \times 7^n + 1 \text{ is divisible by 4 for all } n \in \mathbb{N}$$

Example 26

3 Prove each of the following inequalities by mathematical induction:

$$\mathbf{a} \quad 4^n > 10 \times 2^n \text{ for all integers } n \geq 4$$

$$\mathbf{b} \quad 3^n > 5 \times 2^n \text{ for all integers } n \geq 5$$

$$\mathbf{c} \quad 2^n > 2n \text{ for all integers } n \geq 3$$

$$\mathbf{d} \quad n! > 2^n \text{ for all integers } n \geq 4$$

Example 27

4 Prove each of the following statements by mathematical induction:

$$\mathbf{a} \quad \text{If } a_{n+1} = 2a_n - 1 \text{ and } a_1 = 3, \text{ then } a_n = 2^n + 1.$$

$$\mathbf{b} \quad \text{If } a_{n+1} = 5a_n + 4 \text{ and } a_1 = 4, \text{ then } a_n = 5^n - 1.$$

$$\mathbf{c} \quad \text{If } a_{n+1} = 2a_n - n + 1 \text{ and } a_1 = 3, \text{ then } a_n = 2^n + n.$$

5 Prove that 3^n is odd for every $n \in \mathbb{N}$.6 $\mathbf{a}$ Prove by mathematical induction that $n^2 - n$ is even for all $n \in \mathbb{N}$. $\mathbf{b}$ Find an easier proof by factorising $n^2 - n$.7 $\mathbf{a}$ Prove by mathematical induction that $n^3 - n$ is divisible by 6 for all $n \in \mathbb{N}$. $\mathbf{b}$ Find an easier proof by factorising $n^3 - n$.

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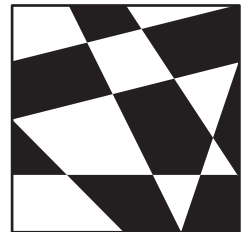
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- 8** Consider the sequence defined by $a_{n+1} = 10a_n + 9$ where $a_1 = 9$.
- Find a_n for $n = 1, 2, 3, 4, 5$.
 - Guess a formula for a_n in terms of n .
 - Confirm that your formula is valid by using mathematical induction.
- 9** The Fibonacci numbers are defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$.
- Find f_n for $n = 1, 2, \dots, 10$.
 - Prove that $f_1 + f_2 + \dots + f_n = f_{n+2} - 1$.
 - Evaluate $f_1 + f_3 + \dots + f_{2n-1}$ for $n = 1, 2, 3, 4$.
 - Try to find a formula for the above expression.
 - Confirm that your formula works using mathematical induction.
 - Using induction, prove that every third Fibonacci number, f_{3n} , is even.
- 10** Prove that $4^n + 5^n$ is divisible by 9 for all odd integers n .
- 11** Prove by induction that, for all $n \in \mathbb{N}$, every set of numbers S with exactly n elements has a largest element.
- 12** Standing around a circle, there are n friends and n thieves. You begin with no money, but as you go around the circle clockwise, each friend will give you \$1 and each thief will steal \$1. Prove that no matter where the friends and thieves are placed, it is possible to walk once around the circle without going into debt, provided you start at the correct point.
- 13** Prove by induction that every natural number $n \geq 2$ is divisible by some prime number.
Hint: Let $P(n)$ be the statement that every integer j such that $2 \leq j \leq n$ is divisible by some prime number.
- 14** If n straight lines are drawn across a sheet of paper, they will divide the paper into regions. Show that it is always possible to colour each region black or white, so that no two adjacent regions have the same colour.

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Chapter summary



- A **conditional statement** has the form: If P is true, then Q is true.
This can be abbreviated as $P \Rightarrow Q$, which is read ' P implies Q '.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.
- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- Statements P and Q are **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$. We write $P \Leftrightarrow Q$.
- The **contrapositive** of $P \Rightarrow Q$ is $(\text{not } Q) \Rightarrow (\text{not } P)$.
- Proving the contrapositive of a statement may be easier than giving a direct proof.
- A **proof by contradiction** begins by assuming the negation of what is to be proved.
- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier '**for all**'.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier '**there exists**'.
- **Counterexamples** can be used to demonstrate that a universal statement is false.
- **Mathematical induction** is used to prove that a statement is true for all natural numbers.

Technology-free questions

- 1 For each of the following statements, if the statement is true, then prove it, and otherwise give a counterexample to show that it is false:
 - a The sum of any three consecutive integers is divisible by 3.
 - b The sum of any four consecutive integers is divisible by 4.
- 2 Assume that n is even. Prove that $n^2 - 3n + 1$ is odd.
- 3 Let $n \in \mathbb{Z}$. Consider the statement: If n^3 is even, then n is even.
 - a Write down the contrapositive of this statement.
 - b Prove the contrapositive.
 - c Hence, prove by contradiction that $\sqrt[3]{6}$ is irrational.
- 4
 - a Show that one of three consecutive integers is always divisible by 3.
 - b Hence, prove that $n^3 + 2n^2 + 2n$ is divisible by 3 for all $n \in \mathbb{Z}$.
- 5
 - a Suppose that both m and n are divisible by d . Prove that $m - n$ is divisible by d .
 - b Hence, prove that the highest common factor of two consecutive integers is 1.
 - c Find the highest common factor of 1002 and 999.
- 6 A student claims that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, for all $x \geq 0$ and $y \geq 0$.
 - a Using a counterexample, prove that the equation is not always true.
 - b Prove that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ if and only if $x = 0$ or $y = 0$.

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- 7** Let $n \in \mathbb{Z}$. Prove that $n^2 + 3n + 4$ is even.
Hint: Consider the cases when n is odd and n is even.

- 8** Suppose that a, b, c and d are positive integers.

- a** Provide a counterexample to disprove the equation

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

- b** Now suppose that $\frac{c}{d} > \frac{a}{b}$. Prove that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

- 9** Prove by mathematical induction that:

- a** $6^n + 4$ is divisible by 10 for all $n \in \mathbb{N}$

- b** $1^2 + 3^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all $n \in \mathbb{N}$

Multiple-choice questions



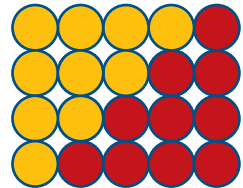
- 1** If m is even and n is odd, then which of these statements is true?
A $m + 2n$ is odd **B** $m + n$ is even **C** $m \times n$ is odd
D $m^2 - n^2$ is even **E** $m - 3n$ is odd
- 2** If m is divisible by 6 and n is divisible by 15, then which of these statements might be false?
A $m \times n$ is divisible by 90 **B** $m \times n$ is divisible by 30 **C** $m \times n$ is divisible by 15
D $m + n$ is divisible by 3 **E** $m + n$ is divisible by 15
- 3** The contrapositive of $P \Rightarrow Q$ is
A $Q \Rightarrow P$ **B** $(\text{not } P) \Rightarrow (\text{not } Q)$ **C** $(\text{not } Q) \Rightarrow (\text{not } P)$
D $Q \Leftrightarrow P$ **E** $(\text{not } P) \Leftrightarrow (\text{not } Q)$
- 4** The converse of $P \Rightarrow Q$ is
A $(\text{not } Q) \Rightarrow (\text{not } P)$ **B** $Q \Rightarrow P$ **C** $Q \Leftrightarrow Q$
D $(\text{not } P) \Leftrightarrow (\text{not } Q)$ **E** $(\text{not } Q) \Leftrightarrow (\text{not } P)$
- 5** The number of pairs of integers (m, n) that satisfy $m + n = mn$ is
A 0 **B** 1 **C** 2 **D** 3 **E** 4
- 6** If a, b and c are any real numbers with $a > b$, the statement that must be true is
A $\frac{1}{a} > \frac{1}{b}$ **B** $\frac{1}{a} < \frac{1}{b}$ **C** $ac > bc$ **D** $a + c > b + c$ **E** $a^2 > b^2$

- 7** If $n = (m-1)(m-2)(m-3)$ where m is an integer, then n will not always be divisible by
- A** 1 **B** 2 **C** 3 **D** 5 **E** 6
- 8** Let $m, n \in \mathbb{Z}$. Which of the following statements is false?
- A** n is even if and only if $n+1$ is odd
- B** $m+n$ is odd if and only if $m-n$ is odd
- C** $m+n$ is even if and only if m and n are even
- D** m and n are odd if and only if mn is odd
- E** mn is even if and only if m is even or n is even

Extended-response questions

- 1 a** Use the diagram on the right to deduce the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad (1)$$



- b** Using equation (1), prove that the sum $1 + 2 + \cdots + 99$ is divisible by 99.
- c** Using equation (1), prove that if n is odd, then the sum of any n consecutive odd natural numbers is divisible by n .
- d** With the help of equation (1) and mathematical induction, prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 \quad \text{for all } n \in \mathbb{N}$$

- 2** Define $n! = n \times (n-1) \times \cdots \times 2 \times 1$.
- a** Prove that $10! + 2, 10! + 3, \dots, 10! + 10$ are nine consecutive composite numbers.
Hint: The first number is divisible by 2.
- b** Find a sequence of ten consecutive composite numbers.
- 3** We call (a, b, c) a Pythagorean triple if a, b, c are natural numbers such that $a^2 + b^2 = c^2$.
- a** Let $n \in \mathbb{N}$. Prove that if (a, b, c) is a Pythagorean triple, then so is (na, nb, nc) .
- b** Prove that there is only one Pythagorean triple (a, b, c) of consecutive natural numbers.
- c** Prove that there is no Pythagorean triple (a, b, c) containing the numbers 1 or 2.
- 4** Let a be an integer that is not divisible by 3. We know that $a = 3k + 1$ or $a = 3k + 2$, for some $k \in \mathbb{Z}$.
- a** Show that a^2 must leave a remainder of 1 when divided by 3.
- b** Hence, prove that if (a, b, c) is any Pythagorean triple, then a or b is divisible by 3.

- 5** **a** Prove by mathematical induction that $n^2 + n$ is even for all $n \in \mathbb{N}$.
b Find an easier proof by factorising $n^2 + n$.
c Hence, prove that if n is odd, then there exists an integer k such that $n^2 = 8k + 1$.
- 6** Let $n \in \mathbb{Z}$ and consider the statement: If n is divisible by 8, then n^2 is divisible by 8.
a Prove the statement.
b Write down the converse of the statement.
c If the converse is true, prove it. Otherwise, give a counterexample.
- 7** Goldbach's conjecture is that every even integer greater than 2 can be expressed as the sum of two primes. To date, no one has been able to prove this, although it has been verified for all integers less than 4×10^{18} .
a Express 100 and 102 as the sum of two prime numbers.
b Prove that 101 cannot be written as the sum of two prime numbers.
c Express 101 as the sum of three prime numbers.
d Assuming that Goldbach's conjecture is true, prove that every odd integer greater than 5 can be written as the sum of three prime numbers.

- 8** **a** Simplify the expression $\frac{1}{n-1} - \frac{1}{n}$.

b Hence, show that

$$\frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \frac{1}{4 \times 3} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$$

- c** Give another proof of the above equation using mathematical induction.
d Using the above equation, prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 \quad \text{for all } n \in \mathbb{N}$$

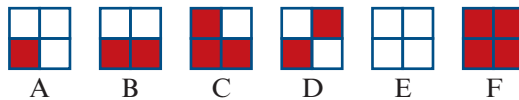
- 9** **a** Let $x \geq 0$ and $y \geq 0$. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

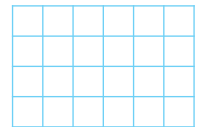
by substituting $x = a^2$ and $y = b^2$ into $\frac{x+y}{2} - \sqrt{xy}$.

- b** Using the above inequality, or otherwise, prove each of the following:
- i** If $a > 0$, then $a + \frac{1}{a} \geq 2$.
 - ii** If a, b and c are positive real numbers, then $(a+b)(b+c)(c+a) \geq 8abc$.
 - iii** If a, b and c are positive real numbers, then $a^2 + b^2 + c^2 \geq ab + bc + ca$.
- c** Take any rectangle of length x and width y . Prove that a square with the same perimeter has an area greater than or equal to that of the rectangle.

- 10** Exactly one of the following three people is lying. Who is the liar?
- Jay says: 'Kaye is lying.'
 - Kaye says: 'Elle is lying.'
 - Elle says: 'I am not lying.'
- 11** There are four sentences written below. Which of them is true?
- Exactly one of these statements is false.
 - Exactly two of these statements are false.
 - Exactly three of these statements are false.
 - Exactly four of these statements are false.
- 12** We will say that a set of numbers can be **split** if it can be divided into two groups so that no two numbers appear in the same group as their sum. For example, the set $\{1, 2, 3, 4, 5, 6\}$ can be split into the two groups $\{1, 2, 4\}$ and $\{3, 5, 6\}$.
- a** Prove that the set $\{1, 2, \dots, 8\}$ can be split.
 - b** Hence, explain why the set $\{1, 2, \dots, n\}$ can be split, where $1 \leq n \leq 8$.
 - c** Prove that it is impossible to split the set $\{1, 2, \dots, 9\}$.
 - d** Hence, prove that it is impossible to split the set $\{1, 2, \dots, n\}$, where $n \geq 9$.
- 13** Consider the set of six 2×2 square tiles shown below.



- a** Tile the 2×12 grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.
- b** Prove that there are only four ways to tile the 4×6 grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.



9

Geometry in the plane and proof

Objectives

- ▶ To consider necessary and sufficient conditions for two lines to be **parallel**.
- ▶ To determine the **angle sum** of a polygon.
- ▶ To define **congruence** of two figures.
- ▶ To determine when two triangles are congruent.
- ▶ To write geometric proofs.
- ▶ To use **Pythagoras' theorem** and its converse.
- ▶ To define **similarity** of two figures.
- ▶ To determine when two triangles are similar.
- ▶ To determine and apply **similarity factors** for areas and volumes.
- ▶ To use **vectors** to prove geometric results.
- ▶ To investigate properties of the **golden ratio**.

There are three main reasons for the study of geometry at school.

The first reason is that the properties of figures in two and three dimensions are helpful in other areas of mathematics. The second reason is that the subject provides a good setting to show how a large body of results may be deduced from a small number of assumptions. The third reason is that it gives you, the student, the opportunity to practise writing coherent, logical mathematical arguments.

In this chapter and the next, we use some of the proof techniques introduced in the previous chapter. Review of geometry from Years 9 and 10 is included, but in such a way that you can see the building of the results.

9A Points, lines and angles

In this section we do not pretend to be fully rigorous, but aim to make you aware that assumptions are being made and that we base the proofs of the results on these assumptions. The assumptions do seem obvious to us, but there are ways of making the study of geometry even more rigorous. However, whatever we do, we will need to accept a set of results as our starting point.

► Points, lines and planes

We begin with a few basic concepts. No formal definitions are given.

Point In geometry, a point is used to indicate position.

Line In the physical world, we may illustrate the idea of a line as a tightly stretched wire or a fold in a piece of paper. A line has no width and is infinite in length.

Plane A plane has no thickness and it extends infinitely in all directions.

We make the following assumptions about points and lines:

- Given a point and a line, the point may or may not lie on the line.
- Two distinct points are contained in exactly one line.
- Two distinct lines do not have more than one point in common.

► Angles

A **ray** is a portion of a line consisting of a point O and all the points on one side of O .

An **angle** is the figure formed by two distinct rays which have a common endpoint O . The common endpoint is called the **vertex** of the angle.

- If the two rays are part of one straight line, the angle is called a **straight angle** and measures 180° .
- A **right angle** is an angle of 90° .
- An **acute angle** is an angle which is less than 90° .
- An **obtuse angle** is an angle which is greater than 90° and less than 180° .
- **Supplementary angles** are two angles whose sum is 180° .
- **Complementary angles** are two angles whose sum is 90° .

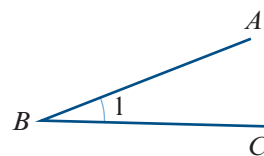
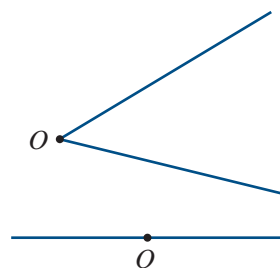
Naming angles

The convention for naming an angle is to fully describe the rays of the angle and the endpoint where the rays meet.

The marked angle is denoted by $\angle ABC$.

When there is no chance of ambiguity, it can be written as $\angle B$.

Sometimes an angle can simply be numbered as shown, and in a proof we refer to the angle as $\angle 1$.



The important thing is that the writing of your argument must be clear and unambiguous. With complicated diagrams, the $\angle ABC$ notation is safest.

Theorem

If two straight lines intersect, then the opposite angles are equal in pairs.

Such angles are said to be **vertically opposite**.

Proof using angle names

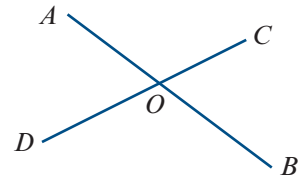
$\angle AOC$ and $\angle COB$ are supplementary.

That is, $\angle AOC + \angle COB = 180^\circ$.

Also, $\angle COB$ and $\angle BOD$ are supplementary.

That is, $\angle COB + \angle BOD = 180^\circ$.

Hence $\angle AOC = \angle BOD$.



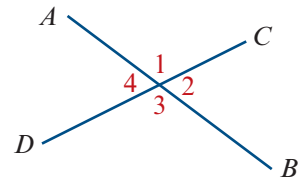
The proof can also be presented with the labelling technique.

Proof using number labels

$\angle 1 + \angle 2 = 180^\circ$ (supplementary angles)

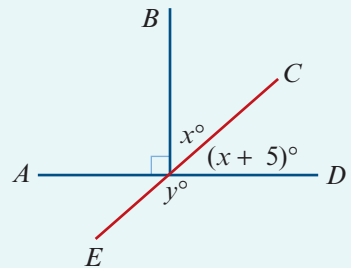
$\angle 2 + \angle 3 = 180^\circ$ (supplementary angles)

$\therefore \angle 1 = \angle 3$



Example 1

Find the values of x and y in the diagram.



Solution

$x + (x + 5) = 90$ (complementary angles)

$$2x = 85$$

$$\therefore x = 42.5$$

$y + (x + 5) = 180$ (supplementary angles)

$$y + 47.5 = 180$$

$$\therefore y = 132.5$$

► Parallel lines

Given two distinct lines ℓ_1 and ℓ_2 in the plane, either the lines intersect in a single point or the lines have no point in common. In the latter case, the lines are said to be **parallel**. We can write this as $\ell_1 \parallel \ell_2$.

Here is another important assumption.

Playfair's axiom

Given any point P not on a line ℓ , there is only one line through P parallel to ℓ .

From this we have the following results for three distinct lines ℓ_1 , ℓ_2 and ℓ_3 in the plane:

- If $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$, then $\ell_1 \parallel \ell_3$.
- If $\ell_1 \parallel \ell_2$ and ℓ_3 intersects ℓ_1 , then ℓ_3 also intersects ℓ_2 .

We prove the first of these and leave the other as an exercise. The proof is by contradiction.

Proof Let ℓ_1 , ℓ_2 and ℓ_3 be three distinct lines in the plane such that $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$.

Now suppose that ℓ_1 is not parallel to ℓ_3 . Then ℓ_1 and ℓ_3 meet at a point P . But by Playfair's axiom, there is only one line parallel to ℓ_2 passing through P . Therefore $\ell_1 = \ell_3$. But this gives a contradiction, as ℓ_1 and ℓ_3 are distinct by assumption.

Corresponding, alternate and co-interior angles

The following types of pairs of angles play an important role in considering parallel lines.

In the diagram, the lines ℓ_1 and ℓ_2 are crossed by a **transversal** ℓ_3 .

Corresponding angles:

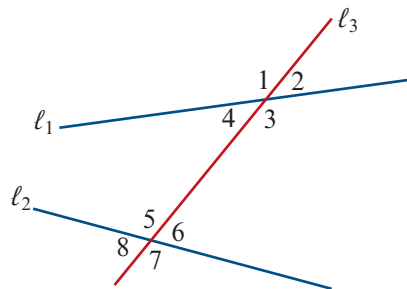
- Angles 1 and 5
- Angles 2 and 6
- Angles 3 and 7
- Angles 4 and 8

Alternate angles:

- Angles 3 and 5
- Angles 4 and 6

Co-interior angles:

- Angles 3 and 6
- Angles 4 and 5



The following result is easy to prove, and you should complete it as an exercise.

Theorem

When two lines are crossed by a transversal, any one of the following three conditions implies the other two:

- a pair of alternate angles are equal
- a pair of corresponding angles are equal
- a pair of co-interior angles are supplementary.

The next result is important as it gives us the ability to establish properties of the angles associated with parallel lines crossed by a transversal, and it also gives us an easily applied method for proving that two lines are parallel.

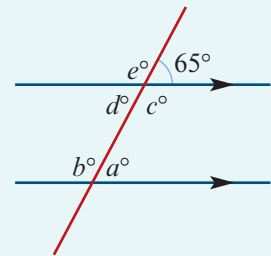
Theorem

- If two parallel lines are crossed by a transversal, then alternate angles are equal.
- Conversely, if two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Example 2

Find the values of the pronumerals.

Note: The arrows indicate that the two lines are parallel.



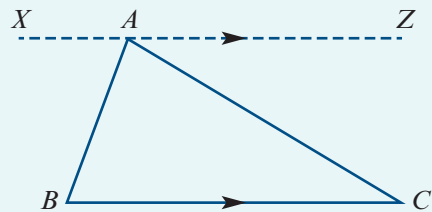
Solution

- $a = 65$ (corresponding)
- $d = 65$ (alternate with a)
- $b = 115$ (co-interior with d)
- $e = 115$ (corresponding with b)
- $c = 115$ (vertically opposite e)

Example 3

For $\triangle ABC$ shown in the diagram, the line XAZ is drawn through vertex A parallel to BC .

Use this construction to prove that the sum of the interior angles of a triangle is a straight angle (180°).



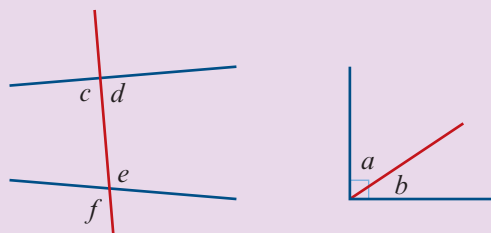
Solution

- $\angle ABC = \angle XAB$ (alternate angles)
- $\angle ACB = \angle ZAC$ (alternate angles)
- $\angle XAB + \angle ZAC + \angle BAC$ is a straight angle.
- Therefore $\angle ABC + \angle ACB + \angle BAC = 180^\circ$.

Section summary

■ Pairs of angles

- complementary (a and b)
- supplementary (c and d)
- vertically opposite (e and f)
- alternate (c and e)
- corresponding (c and f)
- co-interior (d and e)



■ Parallel lines

If two parallel lines are crossed by a transversal, then:

- alternate angles are equal
- corresponding angles are equal
- co-interior angles are supplementary.

If two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Exercise 9A

1 Consider the diagram shown.

a State whether each of the following angles is acute, obtuse, right or straight:

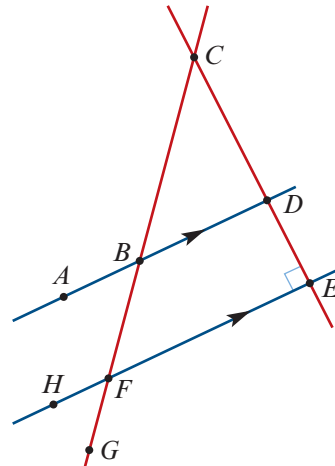
- i $\angle ABC$ ii $\angle HFE$ iii $\angle CBD$ iv $\angle FED$

b State which angle is:

- i corresponding to $\angle ABC$
 ii alternate to $\angle ABF$
 iii vertically opposite $\angle BFE$
 iv co-interior to $\angle DBF$

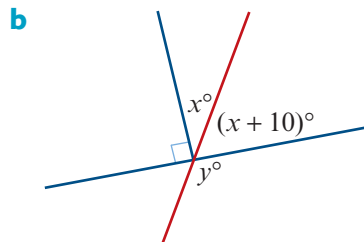
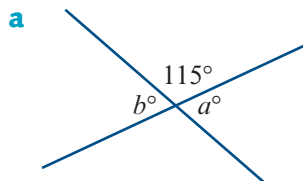
c State which angles are:

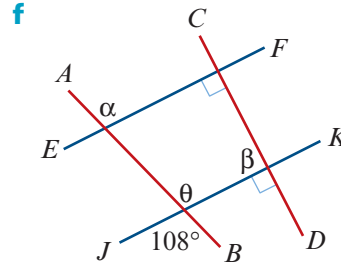
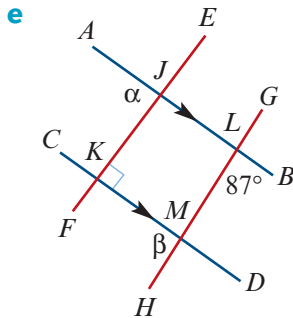
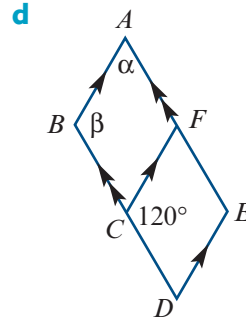
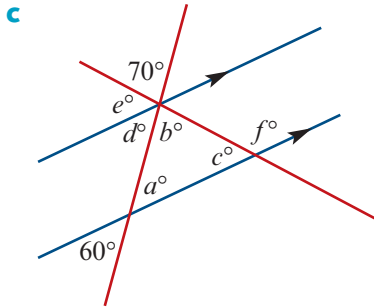
- i complementary to $\angle BCD$
 ii supplementary to $\angle CBD$



Example 1, 2

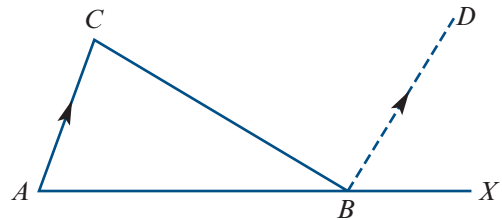
2 Calculate the values of the unknowns for each of the following. Give reasons.



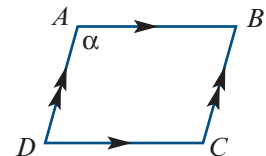


- Example 3** **3** Side AB of $\triangle ABC$ is extended to point X and line BD is drawn parallel to side AC .
Prove that the sum of two interior angles of a triangle is equal to the opposite exterior angle.

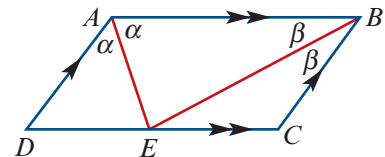
Hint: Using the diagram, this means showing that $\angle CAB + \angle ACB = \angle CBX$.



- 4** Recall that a parallelogram is a quadrilateral whose opposite sides are parallel. A parallelogram $ABCD$ is shown on the right. Let $\angle A = \alpha$.

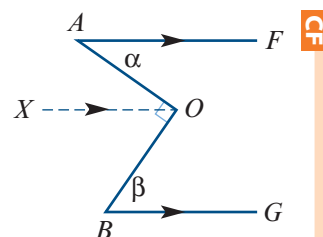


- a** Find the sizes of $\angle B$ and $\angle D$ in terms of α .
b Hence find the size of $\angle C$ in terms of α .
- 5** Prove the converse of the result in Question 4. That is, prove that if the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- 6** Prove that AE is perpendicular to EB .

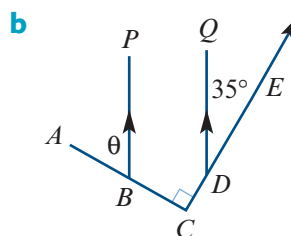
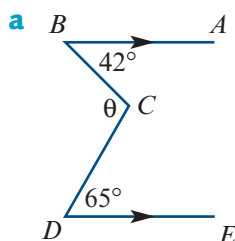


- 7** The lines PQ and RS are parallel. A transversal meets PQ at X and RS at Y . Lines XA and YB are bisectors of the angles PXY and XYR . Prove that XA is parallel to YB .

- 8 For the diagram on the right, show that $\alpha + \beta = 90^\circ$.



- 9 For each of the following, use a construction line to find the angle marked θ :



9B Triangles and polygons

We first define polygons.

A **line segment** AB is a portion of a line consisting of two distinct points A and B and all the points between them.

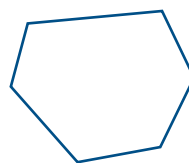
If distinct points $A_1, A_2, \dots, A_n$ in the plane are connected in order by the line segments $A_1A_2, A_2A_3, \dots, A_nA_1$, then the figure formed is a **polygon**. The points $A_1, A_2, \dots, A_n$ are the vertices of the polygon, and the line segments $A_1A_2, A_2A_3, \dots, A_nA_1$ are its sides.

Types of polygons

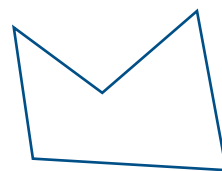
A **simple polygon** is a polygon such that no two sides have a point in common except a vertex.

A **convex polygon** is a polygon that contains each line segment connecting any pair of points on its boundary.

For example, the left-hand figure is convex, while the right-hand figure is not.



A convex polygon



A non-convex polygon

Note: In this chapter we will always assume that the polygons being considered are convex.

A **regular polygon** is a polygon in which all the angles are equal and all the sides are equal.

Names of polygons

■ triangle (3 sides)	■ quadrilateral (4 sides)	■ pentagon (5 sides)
■ hexagon (6 sides)	■ heptagon (7 sides)	■ octagon (8 sides)
■ nonagon (9 sides)	■ decagon (10 sides)	■ dodecagon (12 sides)

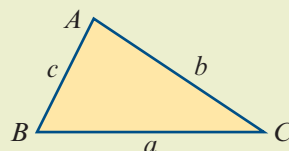
► Triangles

A **triangle** is a figure formed by three line segments determined by a set of three points not on one line. If the three points are A , B and C , then the figure is called triangle ABC and commonly written $\triangle ABC$. The points A , B and C are called the vertices of the triangle.

Triangle inequality

An important property of a triangle is that any side is shorter than the sum of the other two.

In $\triangle ABC$: $a < b + c$, $b < c + a$ and $c < a + b$.



Note: For $\triangle ABC$ labelled as shown, we have $c < b < a$ if and only if $\angle C < \angle B < \angle A$.

The following two results have been proved in Example 3 and in Question 3 of Exercise 9A.

Angles of a triangle

- The sum of the three interior angles of a triangle is 180° .
- The sum of two interior angles of a triangle is equal to the opposite exterior angle.

Classification of triangles

Equilateral triangle a triangle in which all three sides are equal

Isosceles triangle a triangle in which two sides are equal

Scalene triangle a triangle in which all three sides are unequal

Important lines in a triangle

Median A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side.

Altitude An **altitude** of a triangle is a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side.

Example 4

The sides of a triangle are $6 - x$, $4x + 1$ and $2x + 3$. Find the value of x for which the triangle is isosceles, and show that if it is isosceles, then it is equilateral.

Solution

$$\begin{aligned} 6 - x &= 4x + 1 \\ \Rightarrow 5x &= 5 \\ \Rightarrow x &= 1 \end{aligned}$$

When $x = 1$, we have $6 - x = 5$, $4x + 1 = 5$ and $2x + 3 = 5$. Hence the triangle is equilateral with each side of length 5 units.

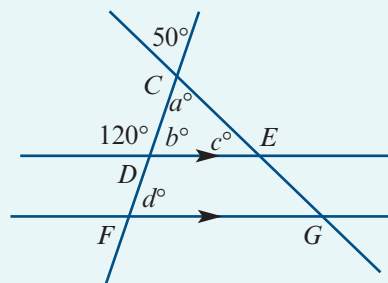
Explanation

We want to show that if any two side lengths are equal, then the third length is the same.

It is enough to show that the three lines $y = 6 - x$, $y = 4x + 1$ and $y = 2x + 3$ intersect in a common point.

Example 5

Find the values of a , b , c and d , giving reasons.

**Solution**

$$a = 50 \quad (\text{vertically opposite angles})$$

$$b = 60 \quad (\text{supplementary angles})$$

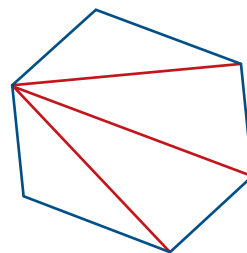
$$c = 180 - (50 + 60) = 70 \quad (\text{angle sum of a triangle})$$

$$d = 60 \quad (\text{corresponding angles } DE \parallel FG)$$

► **Angle sum of a polygon**

If a polygon has n sides, then we can draw $n - 3$ diagonals from a vertex. In this way, we can divide the polygon into $n - 2$ triangles, each with an angle sum of 180° .

We have drawn a hexagon to illustrate this, but we could have used any polygon.

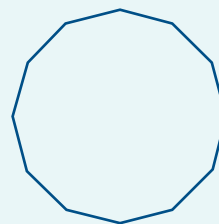
**Angle sum of a polygon**

- The sum of the interior angles of an n -sided polygon is $(n - 2)180^\circ$.
- Each interior angle of a regular n -sided polygon has size $\frac{(n - 2)}{n}180^\circ$.

Example 6

A regular dodecagon is shown to the right.

- a Find the sum of the interior angles of a dodecagon.
- b Find the size of each interior angle of a regular dodecagon.

**Solution**

- a The angle sum of a polygon with n sides is $(n - 2)180^\circ$.
Therefore the angle sum of a dodecagon is 1800° .

- b Each of the interior angles is $\frac{1800}{12} = 150^\circ$.

Section summary

■ Polygons

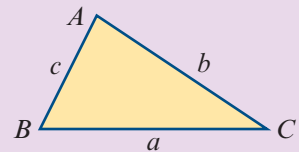
- The sum of the interior angles of an n -sided polygon is $(n - 2)180^\circ$.
- In a **regular polygon**, all the angles are equal and all the sides are equal.
Each interior angle of a regular n -sided polygon has size $\frac{(n - 2)}{n}180^\circ$.

■ Triangles

- An **equilateral triangle** is a triangle in which all three sides are equal.
- An **isosceles triangle** is a triangle in which two sides are equal.
- A **scalene triangle** is a triangle in which all three sides are unequal.
- The sum of the three interior angles of a triangle is 180° .
- The sum of two interior angles of a triangle is equal to the opposite exterior angle.

In $\triangle ABC$:

- $a < b + c$, $b < c + a$ and $c < a + b$
- $c < b < a$ if and only if $\angle C < \angle B < \angle A$



Exercise 9B

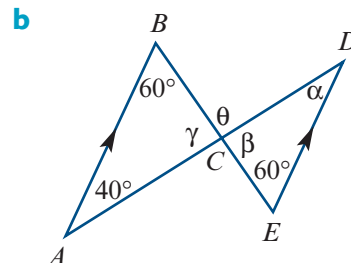
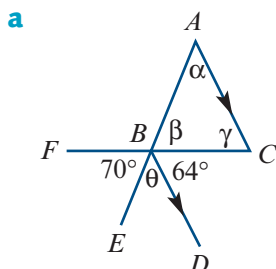
- Is it possible for a triangle to have sides of lengths:
 - 12 cm, 9 cm, 20 cm
 - 24 cm, 24 cm, 40 cm
 - 5 cm, 5 cm, 5 cm
 - 12 cm, 9 cm, 2 cm?
- Describe each of the triangles in Question 1.
- If a triangle has sides 10 cm and 20 cm, what can be said about the third side?
- The sides of a triangle are $2n - 1$, $n + 5$ and $3n - 8$.
 - Find the value(s) of n for which the triangle is isosceles.
 - Is there a value of n which makes the triangle equilateral?

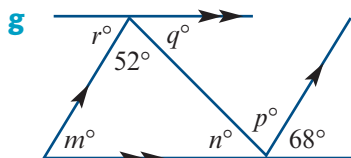
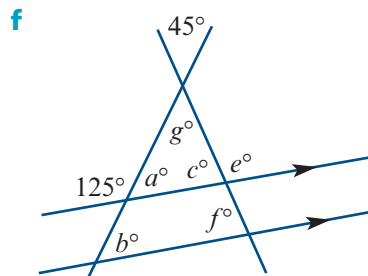
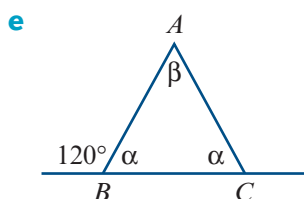
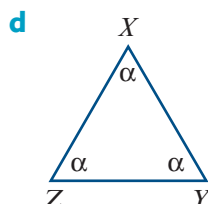
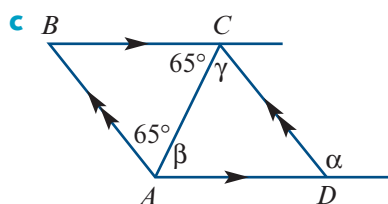
Example 4

- The sides of a triangle are $2n - 1$, $n + 7$ and $3n - 9$. Prove that if the triangle is isosceles, then it is equilateral.

Example 5

- Calculate the value of the unknowns for each of the following. Give reasons.





Example 6 7 Find the interior-angle sum and the size of each angle of a regular polygon with:

a 6 sides

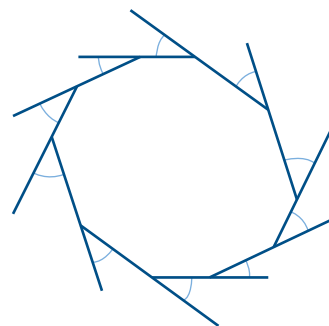
b 12 sides

c 20 sides

8 In the decagon shown on the right, each side has been extended to form an exterior angle.

a Explain why the sum of the interior angles plus the sum of the exterior angles is 1800° .

b Hence find the sum of the decagon's 10 exterior angles.



9 Prove that the sum of the exterior angles of any polygon is 360° .

10 If the sum of the interior angles of a polygon is four times the sum of the exterior angles, how many sides does the polygon have?

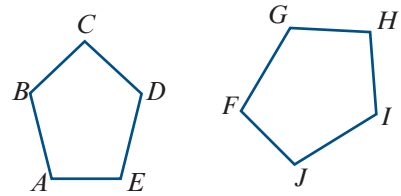
11 Assume that the sum of the interior angles of a polygon is k times the sum of the exterior angles (where $k \in \mathbb{N}$). Prove that the polygon has $2(k + 1)$ sides.

9C Congruence and proofs

Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.

Congruent figures have exactly the same shape and size. For example, the two figures shown are congruent. We can write:

$$\text{pentagon } ABCDE \equiv \text{pentagon } FGHIJ$$



When two figures are congruent, we can find a transformation that pairs up every part of one figure with the corresponding part of the other, so that:

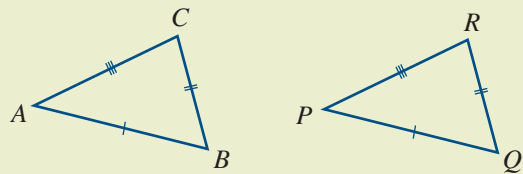
- paired angles have the same size
- paired line segments have the same length
- paired regions have the same area.

► Congruent triangles

We used the four standard congruence tests for triangles in Chapter 3.

■ The SSS congruence test

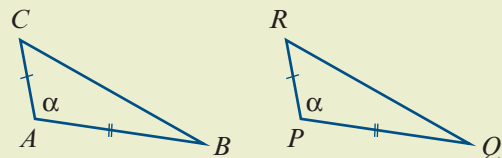
If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.



$$\triangle ABC \equiv \triangle PQR$$

■ The SAS congruence test

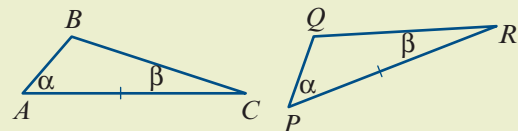
If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.



$$\triangle ABC \equiv \triangle PQR$$

■ The AAS congruence test

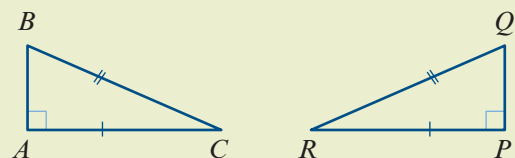
If two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, then the two triangles are congruent.



$$\triangle ABC \equiv \triangle PQR$$

■ The RHS congruence test

If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent.



$$\triangle ABC \equiv \triangle PQR$$

► Classification of quadrilaterals

Trapezium	a quadrilateral with at least one pair of opposite sides parallel
Parallelogram	a quadrilateral with both pairs of opposite sides parallel
Rhombus	a parallelogram with a pair of adjacent sides equal
Rectangle	a quadrilateral in which all angles are right angles
Square	a quadrilateral that is both a rectangle and a rhombus

► Proofs using congruence

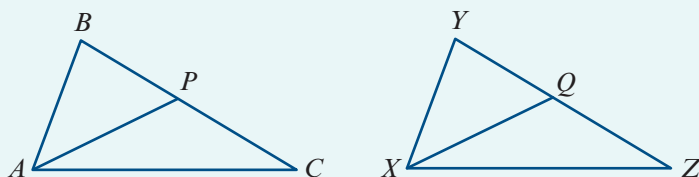


Example 7

Let $\triangle ABC$ and $\triangle XYZ$ be such that $\angle BAC = \angle YXZ$, $AB = XY$ and $AC = XZ$.

If P and Q are the midpoints of BC and YZ respectively, prove that $AP = XQ$.

Solution



From the given conditions, we have $\triangle ABC \equiv \triangle XYZ$ (SAS).

Therefore $\angle ABP = \angle XYQ$ and $BC = YZ$.

Thus $BP = YQ$, as P and Q are the midpoints of BC and YZ respectively.

Hence $\triangle ABP \equiv \triangle XYQ$ (SAS) and so $AP = XQ$.



Example 8

- Prove that, in a parallelogram, the diagonals bisect each other.
- Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Solution

- Note that opposite sides of a parallelogram are equal. (See Question 8 of Exercise 9C.)

In triangles DOC and BOA :

$$\angle ODC = \angle OBA \quad (\text{alternate angles } CD \parallel AB)$$

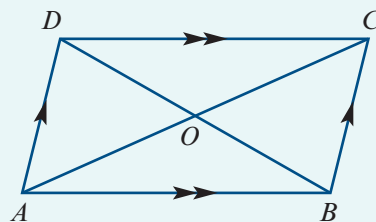
$$\angle OCD = \angle OAB \quad (\text{alternate angles } CD \parallel AB)$$

$$\angle AOB = \angle DOC \quad (\text{vertically opposite})$$

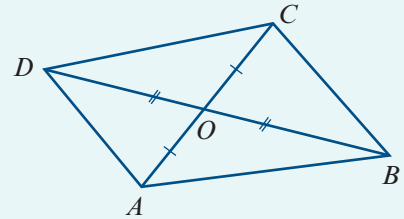
$$AB = CD \quad (\text{opposite sides of parallelogram are equal})$$

$$\triangle DOC \equiv \triangle BOA \quad (\text{AAS})$$

Hence $AO = OC$ and $DO = OB$.



- b** $OD = OB$ (diagonals bisect each other)
 $OA = OC$ (diagonals bisect each other)
 $\angle AOB = \angle DOC$ (vertically opposite)
 $\angle DOA = \angle COB$ (vertically opposite)
 $\triangle DOC \equiv \triangle BOA$ (SAS)
 $\triangle DOA \equiv \triangle BOC$ (SAS)



Therefore $\angle ODC = \angle OBA$ and so $CD \parallel AB$, since alternate angles are equal.
 Similarly, we have $AD \parallel BC$. Hence $ABCD$ is a parallelogram.

Example 9

Prove that the triangle formed by joining the midpoints of the three sides of an isosceles triangle (with the midpoints as the vertices of the new triangle) is also isosceles.

Solution

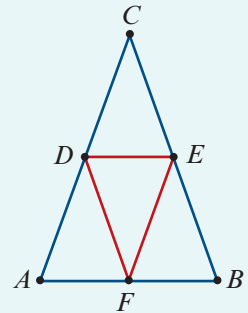
Assume $\triangle ABC$ is isosceles with $CA = CB$ and $\angle CAB = \angle CBA$.
 (See Question 3 of Exercise 9C.)

Then we have $DA = EB$, where D and E are the midpoints of CA and CB respectively.

We also have $AF = BF$, where F is the midpoint of AB .

Therefore $\triangle DAF \equiv \triangle EBF$ (SAS).

Hence $DF = EF$ and so $\triangle DEF$ is isosceles.



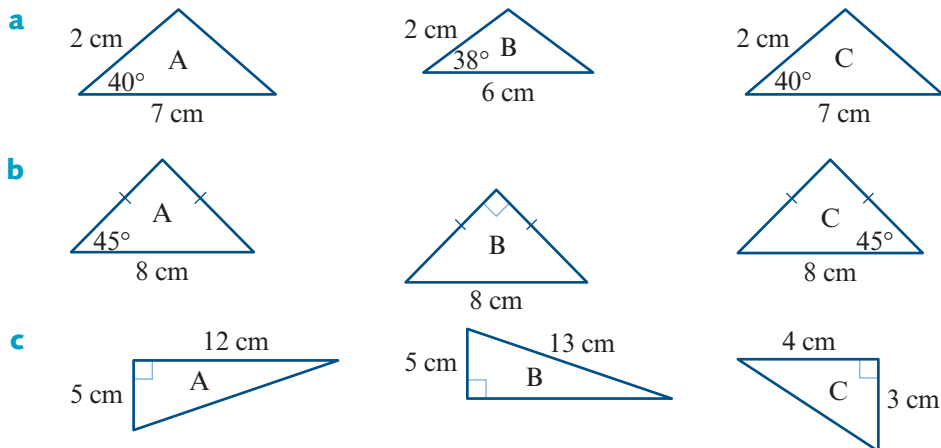
Section summary

- **Congruent figures** have exactly the same shape and size.
- If triangle ABC is congruent to triangle XYZ , this can be written as $\triangle ABC \equiv \triangle XYZ$.
- Two triangles are congruent provided any one of the following four conditions holds:
 - SSS** the three sides of one triangle are equal to the three sides of the other triangle
 - SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
 - AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
 - RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

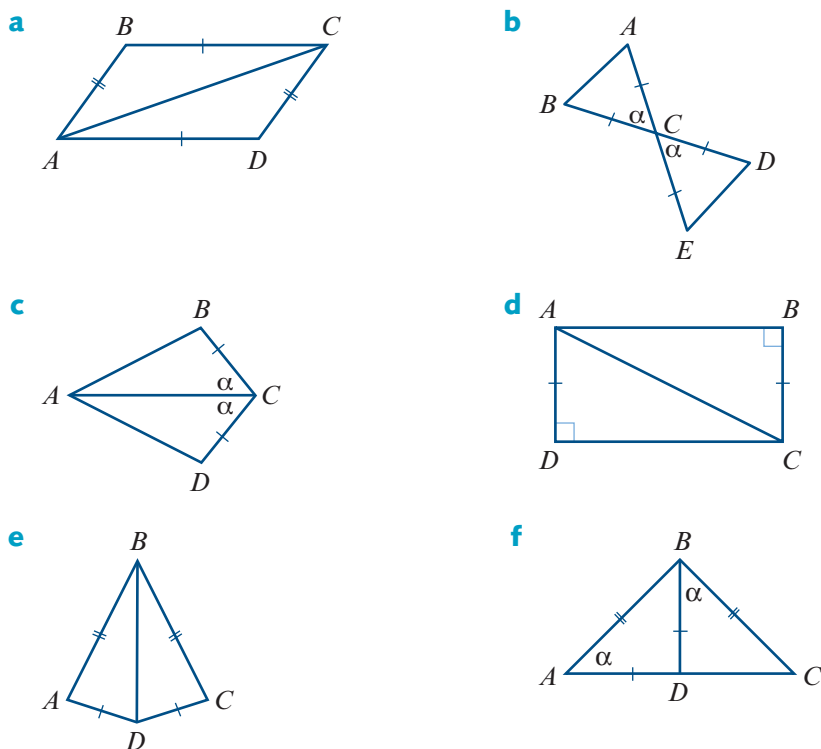
Exercise 9C

SF

- 1 In each part, find pairs of congruent triangles. State the congruence tests used.



- 2 Name the congruent triangles and state the congruence test used:



Example 7

- 3 Prove that if $\triangle ABC$ is isosceles with $AB = AC$, then $\angle ABC = \angle ACB$.

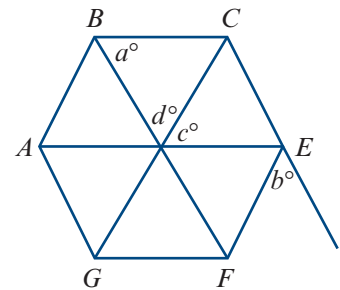
- 4 Prove that if $\triangle ABC$ is such that $\angle ABC = \angle ACB$, then $\triangle ABC$ is isosceles. (This is the converse of Question 3.)

- 5 For the quadrilateral shown, prove that $AB \parallel CD$.



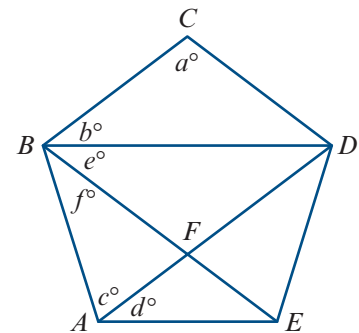
- 6 $ABCEFG$ is a regular hexagon.

- a Find the values of a , b , c and d .
b Prove that $AE \parallel BC$ and $CG \parallel BA$.



- 7 $ABCDE$ is a regular pentagon.

- a Find the values of a , b , c , d , e and f .
b Prove that $AE \parallel BD$ and $BE \parallel CD$.



Example 8

- 8 **Proofs involving parallelograms** Prove each of the following:

- a In a parallelogram, opposite sides are equal and opposite angles are equal.
b If each side of a quadrilateral is equal to the opposite side, then the quadrilateral is a parallelogram.
c If each angle of a quadrilateral is equal to the opposite angle, then the quadrilateral is a parallelogram.
d If one side of a quadrilateral is equal and parallel to the opposite side, then the quadrilateral is a parallelogram.

- 9 Let $ABCD$ be a parallelogram and let P and Q be the midpoints of AB and DC respectively. Prove that $APCQ$ is a parallelogram.
- 10 Let $PQRS$ be a parallelogram whose diagonals meet at O . Let X , Y , Z and W be the midpoints of PO , QO , RO and SO respectively. Prove that $XYZW$ is a parallelogram.
- 11 **Proofs involving rhombuses** Prove each of the following:
- a The diagonals of a rhombus bisect each other at right angles.
b The diagonals of a rhombus bisect the vertex angles through which they pass.
c If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

12 Proofs involving rectangles Prove each of the following:

- a** The diagonals of a rectangle are equal and bisect each other.
- b** A parallelogram with one right angle is a rectangle.
- c** If the diagonals of a quadrilateral are equal and bisect each other, then the quadrilateral is a rectangle.

Example 9 **13** $ABCDE$ is a pentagon in which all the sides are equal and diagonal AC is equal to diagonal AD . Prove that $\angle ABC = \angle AED$.

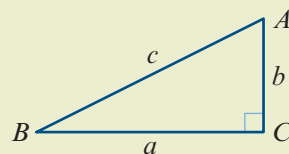
- 14** $\triangle ABC$ is equilateral and its sides are extended to points X, Y and Z so that AY, BZ and CX are all equal in length to the sides of $\triangle ABC$. Prove that $\triangle XYZ$ is also equilateral.
- 15** $ABCD$ is a quadrilateral in which $AB = BC$ and $AD = DC$. The diagonal BD is extended to a point K . Prove that $AK = CK$.
- 16** Prove that if the angle C of a triangle ABC is equal to the sum of the other two angles, then the length of side AB is equal to twice the length of the line segment joining C with the midpoint of AB .
- 17** Prove that if NO is the base of isosceles triangle MNO and if the perpendicular from N to MO meets MO at A , then angle ANO is equal to half of angle NMO .
- 18** If a median of a triangle is drawn, prove that the perpendiculars from the other vertices upon this median are equal. (The median may be extended.)

9D Pythagoras' theorem**Pythagoras' theorem**

Let ABC be a triangle with side lengths a, b and c .

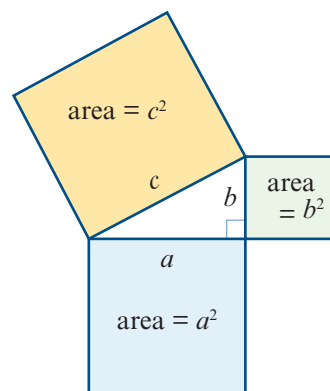
If $\angle C$ is a right angle, then

$$a^2 + b^2 = c^2$$



Pythagoras' theorem can be illustrated by the diagram shown here. The sum of the areas of the two smaller squares is equal to the area of the square on the longest side (hypotenuse).

There are many different proofs of Pythagoras' theorem. One was given at the start of Chapter 8. Here we give another proof, due to James A. Garfield, the 20th President of the United States.



Proof The proof is based on the diagram shown on the right.

$$\text{Area of trapezium } XYZW = \frac{1}{2}(a+b)(a+b)$$

$$\text{Area of } \triangle EWX + \text{area of } \triangle EYZ + \text{area of } \triangle EWZ$$

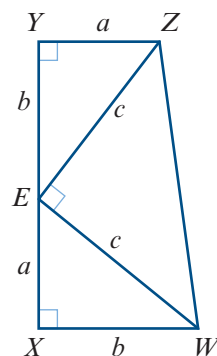
$$= \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$$

$$= ab + \frac{1}{2}c^2$$

$$\text{Thus } \frac{1}{2}(a+b)(a+b) = ab + \frac{1}{2}c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

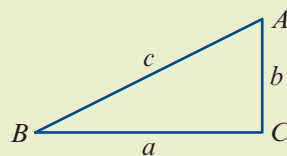
$$\text{Hence } a^2 + b^2 = c^2$$



Converse of Pythagoras' theorem

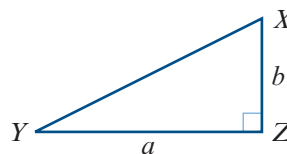
Let ABC be a triangle with side lengths a , b and c .

If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.



Proof Assume $\triangle ABC$ has side lengths $a = BC$, $b = CA$ and $c = AB$ such that $a^2 + b^2 = c^2$.

Construct a second triangle $\triangle XYZ$ with $YZ = a$ and $ZX = b$ such that $\angle XZY$ is a right angle.



By Pythagoras' theorem, the length of the hypotenuse of $\triangle XYZ$ is

$$\sqrt{a^2 + b^2} = \sqrt{c^2}$$

$$= c$$

Therefore $\triangle ABC \equiv \triangle XYZ$ (SSS).

Hence $\angle C$ is a right angle.

Example 10

The diagonal of a soccer field is 130 m and the length of the long side of the field is 100 m. Find the length of the short side, correct to the nearest centimetre.

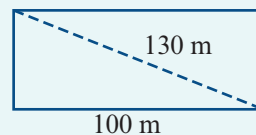
Solution

Let x be the length of the short side. Then

$$x^2 + 100^2 = 130^2$$

$$x^2 = 130^2 - 100^2$$

$$\therefore x = \sqrt{6900}$$



Correct to the nearest centimetre, the length of the short side is 83.07 m.

**Example 11**

Consider $\triangle ABC$ with $AB = 9$ cm, $BC = 11$ cm and $AC = 10$ cm. Find the length of the altitude of $\triangle ABC$ on AC .

Solution

Let BN be the altitude on AC as shown, with $BN = h$ cm.

Let $AN = x$ cm. Then $CN = (10 - x)$ cm.

$$\text{In } \triangle ABN: \quad x^2 + h^2 = 81 \quad (1)$$

$$\text{In } \triangle CBN: \quad (10 - x)^2 + h^2 = 121 \quad (2)$$

Expanding in equation (2) gives

$$100 - 20x + x^2 + h^2 = 121$$

Substituting for $x^2 + h^2$ from (1) gives

$$100 - 20x + 81 = 121$$

$$\therefore x = 3$$

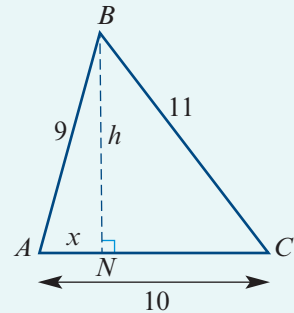
Substituting in (1), we have

$$9 + h^2 = 81$$

$$h^2 = 72$$

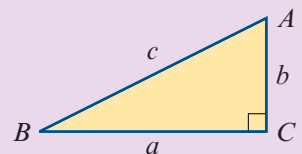
$$\therefore h = 6\sqrt{2}$$

The length of altitude BN is $6\sqrt{2}$ cm.

**Section summary****Pythagoras' theorem and its converse**

Let ABC be a triangle with side lengths a , b and c .

- If $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.
- If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.

**Exercise 9D**

- 1 An 18 m ladder is 7 m away from the bottom of a vertical wall. How far up the wall does it reach?

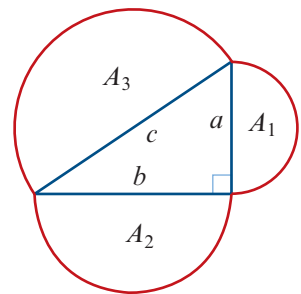
Example 10

- 2 Find the length of the diagonal of a rectangle with dimensions 40 m by 9 m.
- 3 In a circle of centre O , a chord AB is of length 4 cm. The radius of the circle is 14 cm. Find the distance of the chord from O .

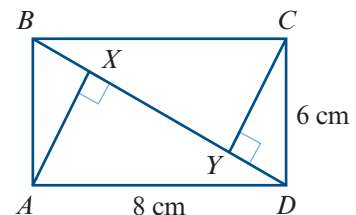
- 4 A square has an area of 169 cm^2 . What is the length of the diagonal?
- 5 Find the area of a square with a diagonal of length:
- 10 cm
 - 8 cm
- 6 $ABCD$ is a square of side length 2 cm. If E is a point on AB extended and $CA = CE$, find the length of DE .
- 7 In a square of side length 2 cm, the midpoints of each side are joined to form a new square. Find the area of the new square.

Example 11

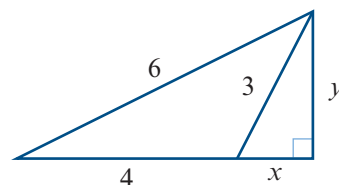
- 8 Consider $\triangle ABC$ with $AB = 7 \text{ cm}$, $BC = 6 \text{ cm}$ and $AC = 5 \text{ cm}$. Find the length of AN , the altitude on BC .
- 9 Which of the following are the three side lengths of a right-angled triangle?
- 5 cm, 6 cm, 7 cm
 - 3.9 cm, 3.6 cm, 1.5 cm
 - 2.4 cm, 2.4 cm, 4 cm
 - 82 cm, 18 cm, 80 cm
- 10 Prove that a triangle with sides lengths $x^2 - 1$, $2x$ and $x^2 + 1$ is a right-angled triangle.
- 11 Consider $\triangle ABC$ such that $AB = 20 \text{ cm}$, $AC = 15 \text{ cm}$ and the altitude AN has length 12 cm. Prove that $\triangle ABC$ is a right-angled triangle.
- 12 Find the length of an altitude in an equilateral triangle with side length 16 cm.
- 13 Three semicircles are drawn on the sides of this right-angled triangle. Let A_1 , A_2 and A_3 be the areas of these semicircles. Prove that $A_3 = A_1 + A_2$.



- 14 Rectangle $ABCD$ has $CD = 6 \text{ cm}$ and $AD = 8 \text{ cm}$. Line segments CY and AX are drawn such that points X and Y lie on BD and $\angle AXD = \angle CYD = 90^\circ$. Find the length of XY .



- 15 Find the values of x and y .



- 16 If P is a point in rectangle $ABCD$ such that $PA = 3$ cm, $PB = 4$ cm and $PC = 5$ cm, find the length of PD .
- 17 Let AQ be an altitude of $\triangle ABC$, where Q lies between B and C . Let P be the midpoint of BC . Prove that $AB^2 + AC^2 = 2PB^2 + 2AP^2$.
- 18 For a parallelogram $ABCD$, prove that $2AB^2 + 2BC^2 = AC^2 + BD^2$.

9E Ratios

This section is revision of work of previous years.

Example 12

Divide 300 in the ratio 3 : 2.

Solution

$$\begin{aligned}\text{one part} &= 300 \div 5 = 60 \\ \therefore \text{two parts} &= 60 \times 2 = 120 \\ \therefore \text{three parts} &= 60 \times 3 = 180\end{aligned}$$

Example 13

Divide 3000 in the ratio 3 : 2 : 1.

Solution

$$\begin{aligned}\text{one part} &= 3000 \div 6 = 500 \\ \therefore \text{two parts} &= 500 \times 2 = 1000 \\ \therefore \text{three parts} &= 500 \times 3 = 1500\end{aligned}$$

Exercise 9E

Skillsheet

Example 12

- 1 Divide 9000 in the ratio 2 : 7.

Example 13

- 2 Divide 15 000 in the ratio 2 : 2 : 1.
- 3 Given that $x : 6 = 9 : 15$, find x .

- 4 The ratio of the numbers of orange flowers to pink flowers in a garden is 6 : 11. There are 144 orange flowers. How many pink flowers are there?
- 5 Given that $15 : 2 = x : 3$, find x .
- 6 The angles of a triangle are in the ratio 6 : 5 : 7. Find the sizes of the three angles.
- 7 Three men X , Y and Z share an amount of money in the ratio 2 : 3 : 7. If Y receives \$2 more than X , how much does Z receive?
- 8 An alloy consists of copper, zinc and tin in the ratio 1 : 3 : 4 (by weight). If there is 10 g of copper in the alloy, find the weights of zinc and tin.
- 9 The ratio of red beads to white beads to green beads in a bag is 7 : 2 : 1. If there are 56 red beads, how many white beads and how many green beads are there?
- 10 On a map, the length of a road is represented by 45 mm. If the scale is 1 : 125 000, find the actual length of the road.
- 11 Five thousand two hundred dollars was divided between a mother and daughter in the ratio 8 : 5. Find the difference between the sums they received.
- 12 Points A , B , C and D are placed in that order on a line so that $AB = 2BC = CD$. Express BD as a fraction of AD .
- 13 If the radius of a circle is increased by two units, find the ratio of the new circumference to the new diameter.
- 14 In a class of 30 students, the ratio of boys to girls is 2 : 3. If six boys join the class, find the new ratio of boys to girls in the class.
- 15 If $a : b = 3 : 4$ and $a : (b + c) = 2 : 5$, find the ratio $a : c$.
- 16 The scale of a map is 1 : 250 000. Find the distance, in kilometres, between two towns which are 3.5 cm apart on the map.
- 17 Prove that if $\frac{a-c}{b-d} = \frac{c}{d}$, then $\frac{a}{b} = \frac{c}{d}$.
- 18 Prove that if $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{2}{3}$, then $\frac{a+b+c}{x+y+z} = \frac{2}{3}$.
- 19 Prove that if $\frac{x}{y} = \frac{m}{n}$, then $\frac{x+y}{x-y} = \frac{m+n}{m-n}$.

9F Introduction to similarity

The two triangles ABC and $A'B'C'$ shown in the diagram are similar.

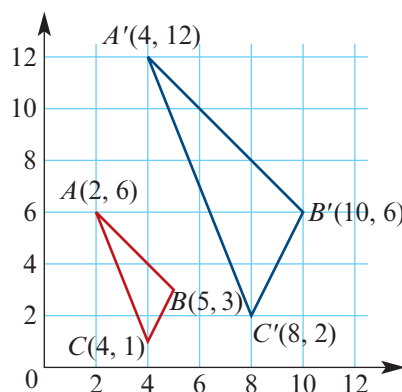
Note: $OA' = 2OA$, $OB' = 2OB$, $OC' = 2OC$

Triangle $A'B'C'$ can be considered as the image of triangle ABC under a mapping of the plane in which the coordinates are multiplied by 2.

This mapping is called an **expansion** from the origin of factor 2. From now on we will call this factor the **similarity factor**.

The rule for this mapping can be written in transformation notation as $(x, y) \rightarrow (2x, 2y)$.

There is also a mapping from $\triangle A'B'C'$ to $\triangle ABC$, which is an expansion from the origin of factor $\frac{1}{2}$. The rule for this is $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$.



Two figures are called **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.

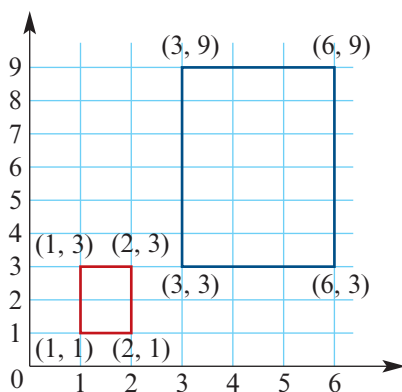
- Matching lengths of similar figures are in the same ratio.
- Matching angles of similar figures are equal.

For example, the rectangle with side lengths 1 and 2 is similar to the rectangle with side lengths 3 and 6.

Here the similarity factor is 3 and the rule for the mapping is $(x, y) \rightarrow (3x, 3y)$.

Notes:

- Any two circles are similar.
- Any two squares are similar.
- Any two equilateral triangles are similar.



► Similar triangles

If triangle ABC is similar to triangle $A'B'C'$, we can write this as

$$\triangle ABC \sim \triangle A'B'C'$$

The triangles are named so that angles of equal magnitude hold the same position. That is, A matches to A' , B matches to B' and C matches to C' . So we have

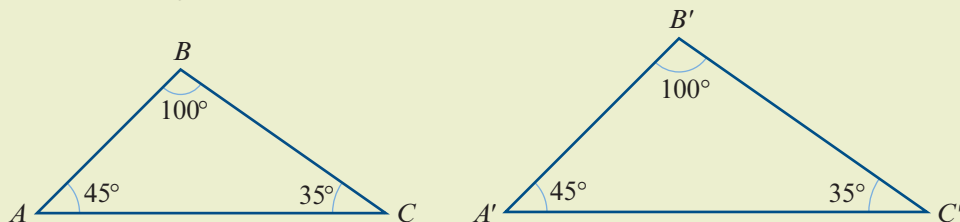
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = k$$

where k is the **similarity factor**.

There are four standard tests for two triangles to be similar.

■ **The AAA similarity test**

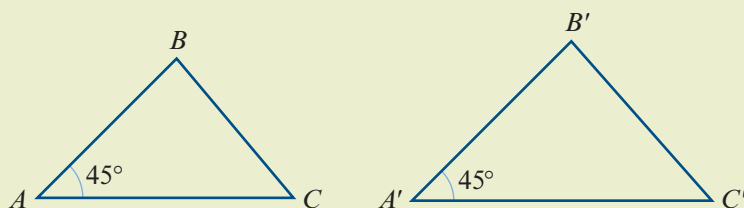
If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.



■ **The SAS similarity test**

If the ratios of two pairs of matching sides are equal and the included angles are equal, then the two triangles are similar.

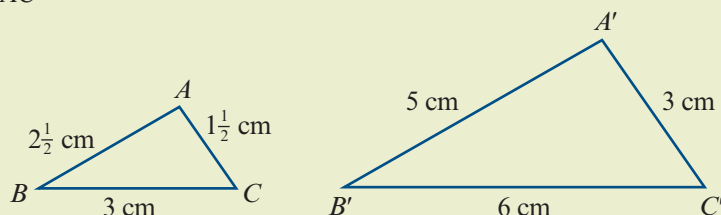
$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$



■ **The SSS similarity test**

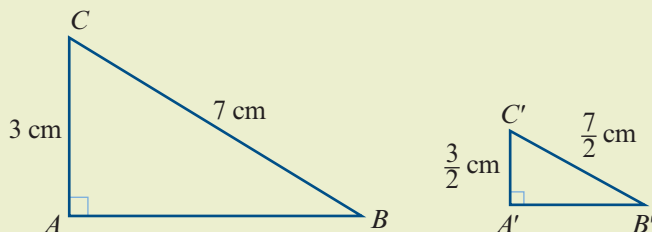
If the sides of one triangle can be matched up with the sides of another triangle so that the ratio of matching lengths is constant, then the two triangles are similar.

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$$



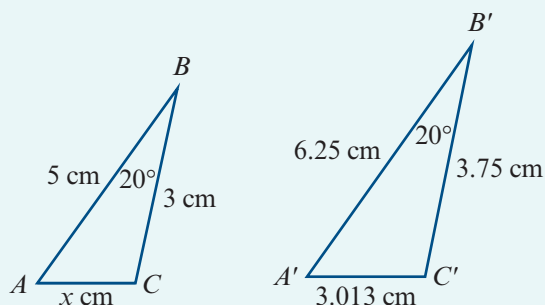
■ **The RHS similarity test**

If the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides, then the two triangles are similar.



Example 14

- a** Give the reason for triangle ABC being similar to triangle $A'B'C'$.
b Find the value of x .

**Solution**

- a** Triangle ABC is similar to triangle $A'B'C'$ by SAS, since

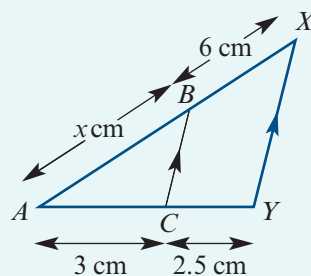
$$\frac{5}{6.25} = 0.8 = \frac{3}{3.75}$$

$$\text{and } \angle ABC = 20^\circ = \angle A'B'C'$$

$$\begin{aligned} \mathbf{b} \quad \frac{x}{3.013} &= \frac{5}{6.25} \\ \therefore x &= \frac{5}{6.25} \times 3.013 \\ &= 2.4104 \end{aligned}$$

**Example 15**

- a** Give the reason for triangle ABC being similar to triangle AXY .
b Find the value of x .

**Solution**

- a** Corresponding angles are of equal magnitude (AAA).

$$\begin{aligned} \mathbf{b} \quad \frac{AB}{AX} &= \frac{AC}{AY} \\ \frac{x}{x+6} &= \frac{3}{5.5} \\ 5.5x &= 3(x+6) \\ 2.5x &= 18 \\ \therefore x &= 7.2 \end{aligned}$$

Section summary

- Two figures are **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.
 - Matching lengths of similar figures are in the same ratio.
 - Matching angles of similar figures are equal.

- Two triangles are similar provided any one of the following four conditions holds:

AAA two angles of one triangle are equal to two angles of the other triangle

SAS the ratios of two pairs of matching sides are equal and the included angles are equal

SSS the sides of one triangle can be matched up with the sides of the other triangle so that the ratio of matching lengths is constant

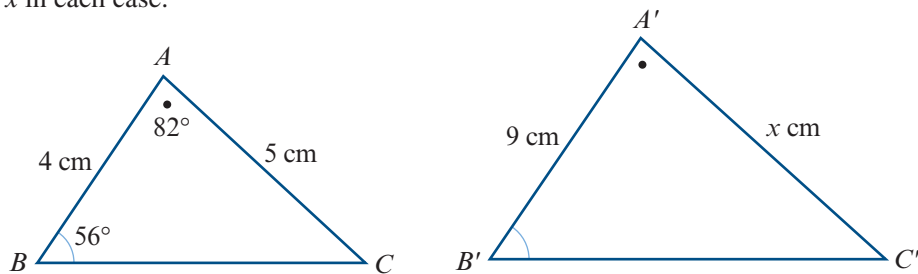
RHS the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

Exercise 9F

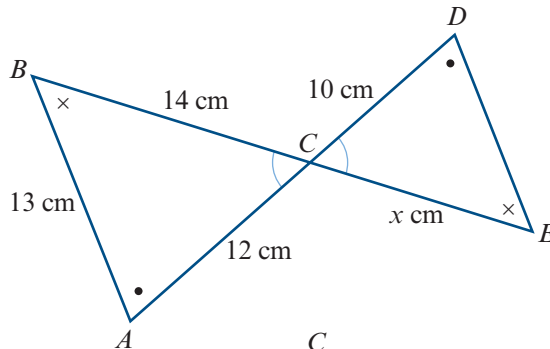
Example 14

- 1 Give reasons why each of the following pairs of triangles are similar and find the value of x in each case:

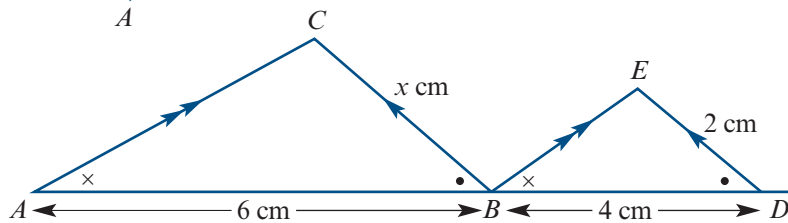
a



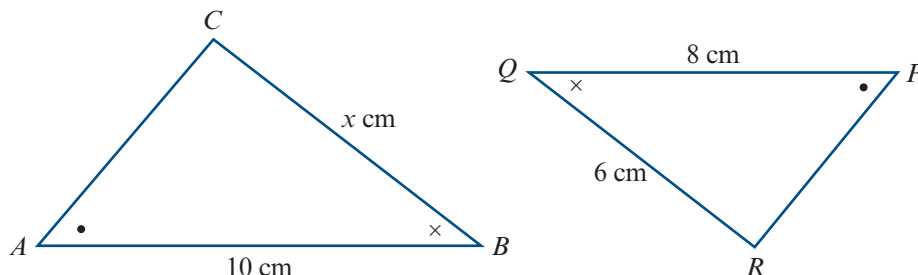
b



c

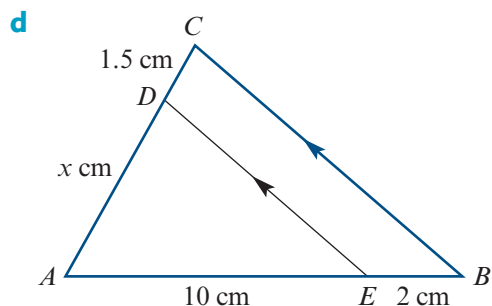
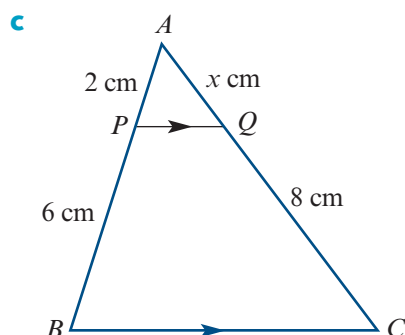
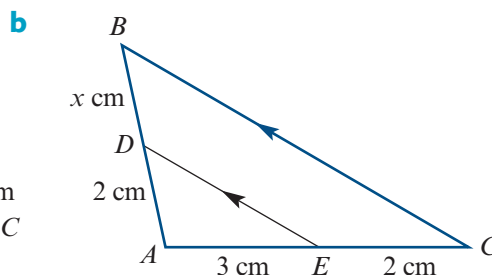
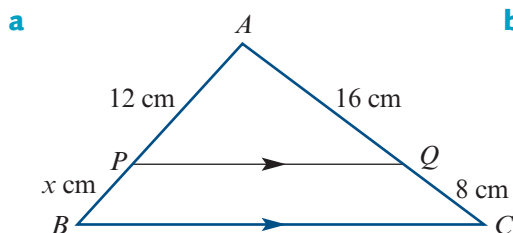


d

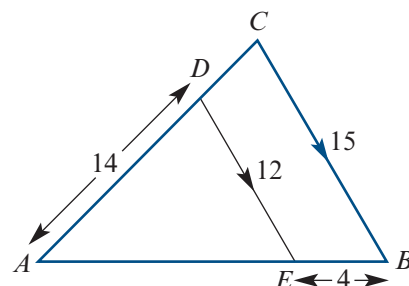


Example 15

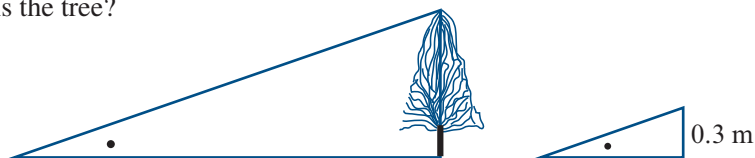
- 2 Give reasons why each of the following pairs of triangles are similar and find the value of x in each case:



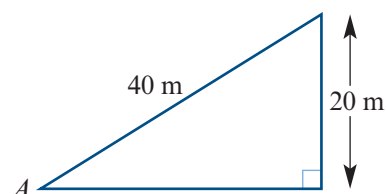
- 3 Given that $AD = 14$, $DE = 12$, $BC = 15$ and $EB = 4$, find AC , AE and AB .



- 4 A tree casts a shadow of 33 m and at the same time a stick 30 cm long casts a shadow of 224 cm. How tall is the tree?

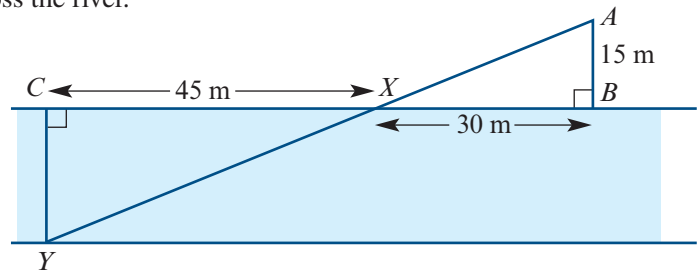


- 5 A 20 m high neon sign is supported by a 40 m steel cable as shown. An ant crawls along the cable starting at A. How high is the ant when it is 15 m from A?



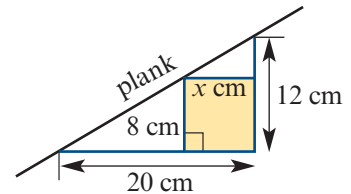
- 6 A hill has a gradient of 1 in 20, i.e. for every 20 m horizontally there is a 1 m increase in height. If you go 300 m horizontally, how high up will you be?

- 7** A man stands at A and looks at point Y across the river. He gets a friend to place a stone at X so that the three points A , X and Y are collinear (that is, they all lie on a single line). He then measures AB , BX and XC to be 15 m, 30 m and 45 m respectively. Find CY , the distance across the river.

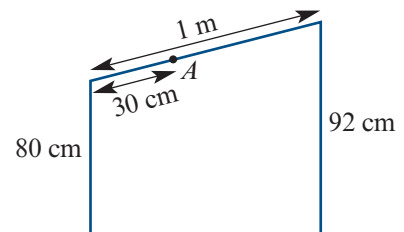


- 8** Find the height, h m, of a tree that casts a shadow 32 m long at the same time that a vertical straight stick 2 m long casts a shadow 6.2 m long.

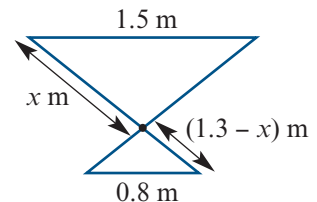
- 9** A plank is placed straight up stairs that are 20 cm wide and 12 cm deep. Find x , where x cm is the width of the widest rectangular box of height 8 cm that can be placed on a stair under the plank.



- 10** The sloping edge of a technical drawing table is 1 m from front to back. Calculate the height above the ground of a point A , which is 30 cm from the front edge.

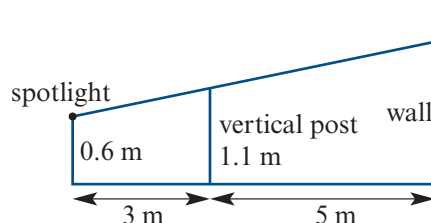


- 11** Two similar rods 1.3 m long have to be hinged together to support a table 1.5 m wide. The rods have been fixed to the floor 0.8 m apart. Find the position of the hinge by finding the value of x .

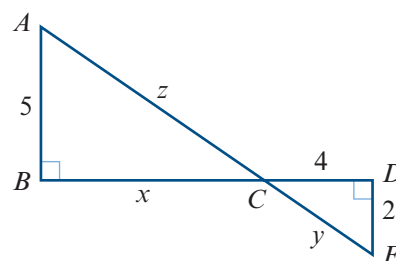


- 12** A man whose eyes are 1.7 m from the ground, when standing 3.5 m in front of a wall 3 m high, can just see the top of a tower that is 100 m away from the wall. Find the height of the tower.
- 13** A man is 8 m up a 10 m ladder, the top of which leans against a vertical wall and touches it at a height of 9 m above the ground. Find the height of the man above the ground.

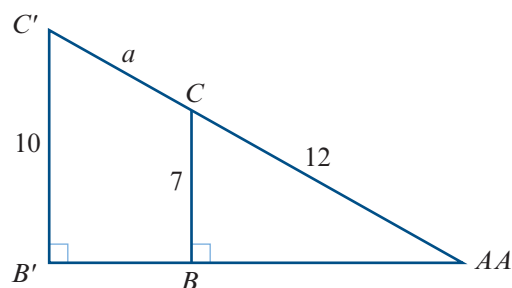
- 14** A spotlight is at a height of 0.6 m above ground level. A vertical post 1.1 m high stands 3 m away and 5 m further away there is a vertical wall. How high up the wall does the shadow reach?



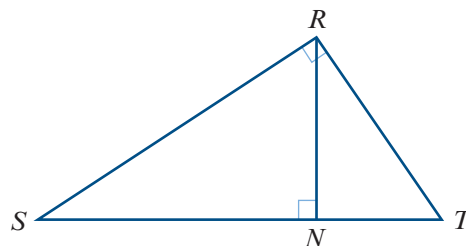
- 15** Consider the diagram on the right.
- Prove that $\triangle ABC \sim \triangle EDC$.
 - Find x .
 - Use Pythagoras' theorem to find y and z .
 - Verify that $y : z = ED : AB$.



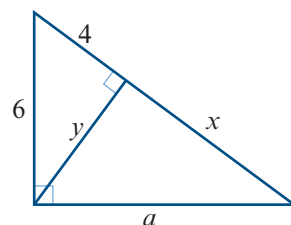
- 16** Find a .



- 17** A man who is 1.8 m tall casts a shadow of 0.76 m in length. If at the same time a telephone pole casts a 3 m shadow, find the height of the pole.
- 18** In the diagram shown, $RT = 4$ cm and $ST = 10$ cm. Find the length NT .



- 19** ABC is a triangular frame with $AB = 14$ m, $BC = 10$ m and $CA = 7$ m. A point P on AB , 1.5 m from A , is linked by a rod to a point Q on AC , 3 m from A . Calculate the length PQ .
- 20** Using this diagram, find a , x and y .



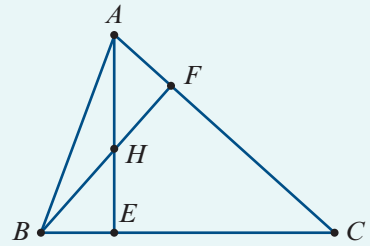
9G Proofs involving similarity

Example 16

The altitudes AE and BF of $\triangle ABC$ intersect at H .

Prove that

$$\frac{AE}{BF} = \frac{AC}{BC}$$



Solution

$$\angle CEA = \angle CFB \quad (\text{AE and BF are altitudes})$$

$$\angle ACE = \angle BCF \quad (\text{common})$$

$$\therefore \triangle CAE \sim \triangle CBF \quad (\text{AAA})$$

$$\therefore \frac{AE}{BF} = \frac{AC}{BC}$$

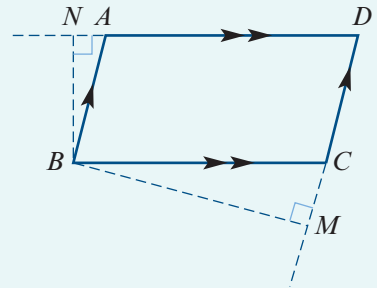


Example 17

$ABCD$ is a parallelogram with $\angle ABC$ acute.

BM is perpendicular to DC extended, and BN is perpendicular to DA extended. Prove that

$$DC \times CM = DA \times AN$$



Solution

$$\angle BCM = \angle ABC \quad (\text{alternate angles } AB \parallel CD)$$

$$\angle BAN = \angle ABC \quad (\text{alternate angles } BC \parallel AD)$$

$$\therefore \angle BCM = \angle BAN$$

$$\angle BNA = \angle BMC = 90^\circ \quad (\text{given})$$

$$\therefore \triangle BCM \sim \triangle BAN$$

$$\text{Hence } \frac{CM}{AN} = \frac{BC}{AB}$$

But $AB = CD$ and $BC = AD$, giving

$$\frac{CM}{AN} = \frac{AD}{CD}$$

Hence $DC \times CM = DA \times AN$.

Example 18

$ABCD$ is a trapezium with diagonals intersecting at O . A line through O , parallel to the base CD , meets BC at X . Prove that $BX \times DC = XC \times AB$.

Solution

$$\triangle ABC \sim \triangle OXC \quad (\text{AAA})$$

$$\triangle DCB \sim \triangle OXB \quad (\text{AAA})$$

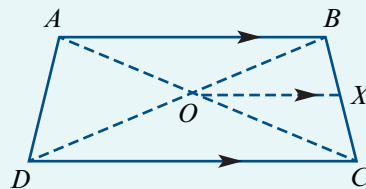
Thus
$$\frac{BX}{BC} = \frac{OX}{DC} \quad (1)$$

and
$$\frac{XC}{BC} = \frac{OX}{AB} \quad (2)$$

Divide (1) by (2):

$$\frac{BX}{XC} = \frac{AB}{DC}$$

$$\therefore BX \times DC = XC \times AB$$

**Exercise 9G****Skillsheet**

- 1** Let M be the midpoint of a line segment AB . Assume that AXB and MYB are equilateral triangles on opposite sides of AB and that XY cuts AB at Z . Prove that $\triangle AXZ \sim \triangle BYZ$ and hence prove that $AZ = 2ZB$.

Example 16

- 2** $ABCD$ is a rectangle. Assume that P , Q and R are points on AB , BC and CD respectively such that $\angle PQR$ is a right angle. Prove that $BQ \times QC = PB \times CR$.
- 3 a** AC is a diagonal of a regular pentagon $ABCDE$. Find the sizes of $\angle BAC$ and $\angle CAE$.
b AC , AD and BD are diagonals of a regular pentagon $ABCDE$, with AC and BD meeting at X . Prove that $(AB)^2 = BX \times BD$.

Example 17

- 4** $\triangle ABC$ has a right angle at A , and AD is the altitude to BC .
a Prove that $AD \times BC = AB \times AC$.
b Prove that $(DA)^2 = DB \times DC$.
c Prove that $(BA)^2 = BD \times BC$.

Example 18

- 5** $ABCD$ is a trapezium with AB one of the parallel sides. The diagonals meet at O . OX is the perpendicular from O to AB , and XO extended meets CD at Y .
 Prove that $\frac{OX}{OY} = \frac{OA}{OC} = \frac{AB}{CD}$.
- 6** P is the point on side AB of $\triangle ABC$ such that $AP : AB = 1 : 3$, and Q is the point on BC such that $CQ : CB = 1 : 3$. The line segments AQ and CP intersect at X . Prove that $AX : AQ = 3 : 5$.

SF

CF

- 7** P and Q are points on sides AB and AC respectively of $\triangle ABC$ such that $PQ \parallel BC$. The median AD meets PQ at M . Prove that $PM = MQ$.
- 8** $ABCD$ is a straight line and $AB = BC = CD$. An equilateral triangle $\triangle BCP$ is drawn with base BC . Prove that $(AP)^2 = AB \times AD$.
- 9** $ABCD$ is a quadrilateral such that $\angle BAD = \angle DBC$ and $\frac{DA}{AB} = \frac{DB}{BC}$. Prove that DB bisects $\angle ADC$.
- 10** $\triangle ABC$ has a right angle at C . The bisector of $\angle BCA$ meets AB at D , and DE is the perpendicular from D to AC . Prove that $\frac{1}{BC} + \frac{1}{AC} = \frac{1}{DE}$.
- 11** Proportions in a right-angled triangle
- a** Prove that, for a right-angled triangle, the altitude on its hypotenuse forms two triangles which are similar to the original triangle, and hence to each other.
 - b** Prove Pythagoras' theorem by using part **a** (or by using similar triangles directly).

9H Areas, volumes and similarity

In this section we look at the areas of similar shapes and the volumes of similar solids.

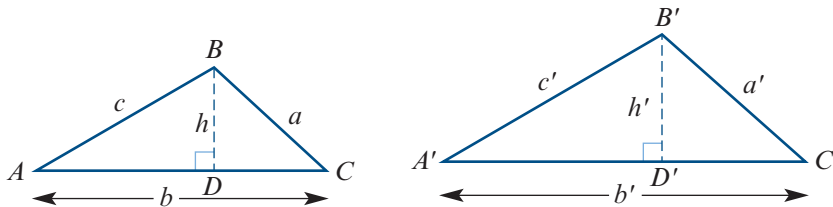
► Similarity and area

If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then

$$\text{area of similar shape} = k^2 \times \text{area of original shape}$$

For example, if triangles ABC and $A'B'C'$ are similar with $A'B' = kAB$, then

$$\text{area of } \triangle A'B'C' = k^2 \times \text{area of } \triangle ABC$$

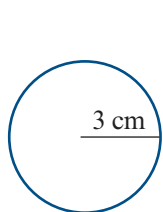


This can be shown by observing that, since $\triangle ABC \sim \triangle A'B'C'$, we have

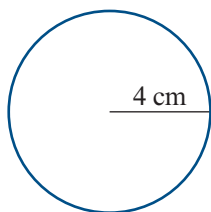
$$\begin{aligned} \text{area of } \triangle A'B'C' &= \frac{1}{2} b'h' \\ &= \frac{1}{2} (kb)(kh) \\ &= k^2 \left(\frac{1}{2} bh \right) \\ &= k^2 \times \text{area of } \triangle ABC \end{aligned}$$

Here are some more examples of similar shapes and the ratio of their areas.

Similar circles



Area = $9\pi \text{ cm}^2$

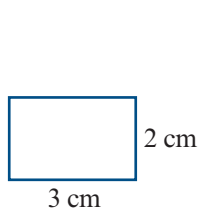


Area = $16\pi \text{ cm}^2$

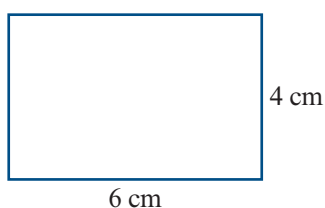
$$\text{Similarity factor} = \frac{4}{3}$$

$$\text{Ratio of areas} = \frac{16\pi}{9\pi} = \left(\frac{4}{3}\right)^2$$

Similar rectangles



Area = 6 cm^2

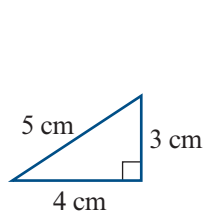


Area = 24 cm^2

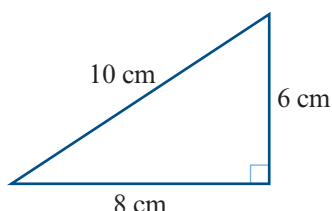
$$\text{Similarity factor} = 2$$

$$\text{Ratio of areas} = \frac{24}{6} = 4 = 2^2$$

Similar triangles



Area = 6 cm^2



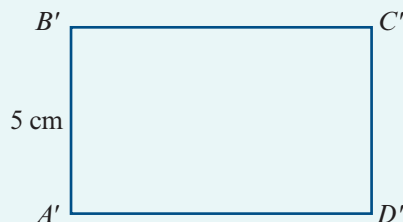
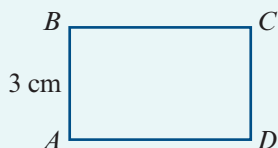
Area = 24 cm^2

$$\text{Similarity factor} = 2$$

$$\text{Ratio of areas} = \frac{24}{6} = 4 = 2^2$$

Example 19

The two rectangles shown below are similar. The area of rectangle $ABCD$ is 20 cm^2 . Find the area of rectangle $A'B'C'D'$.



Solution

The ratio of their side lengths is $\frac{A'B'}{AB} = \frac{5}{3}$.

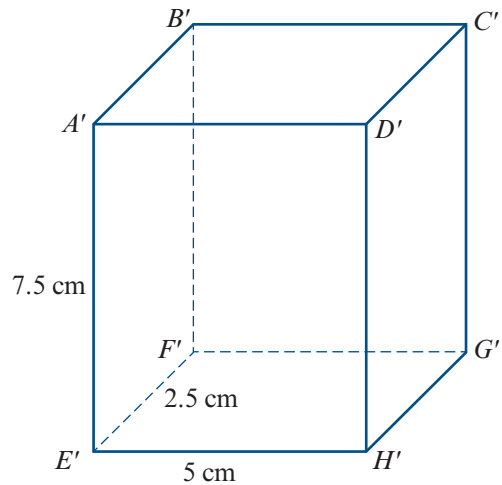
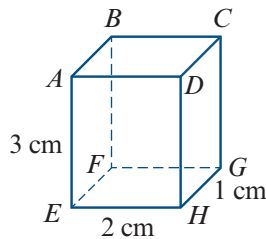
The ratio of their areas is $\frac{\text{Area of } A'B'C'D'}{\text{Area of } ABCD} = \left(\frac{5}{3}\right)^2 = \frac{25}{9}$.

$$\begin{aligned}\therefore \text{Area of } A'B'C'D' &= \frac{25}{9} \times 20 \\ &= 55\frac{5}{9} \text{ cm}^2\end{aligned}$$

► Similarity and volume

Two solids are considered to be similar if they have the same shape and the ratios of their corresponding linear dimensions are equal.

For example, the two cuboids $ABCDEFGH$ and $A'B'C'D'E'F'G'H'$ shown are similar, with similarity factor 2.5.



If two solids are similar and the similarity factor is k , then

$$\text{volume of similar solid} = k^3 \times \text{volume of original solid}$$

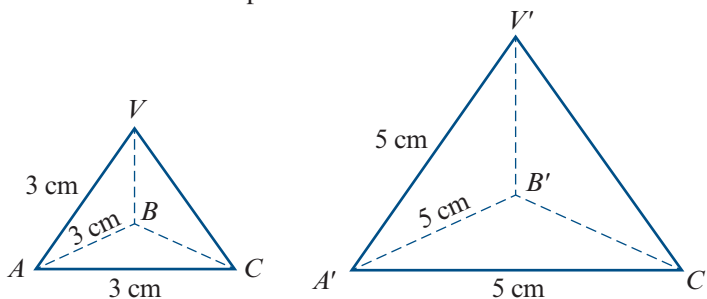
For example, for the two cuboids shown, we have

$$\text{Volume of } ABCDEFGH = 2 \times 1 \times 3 = 6 \text{ cm}^3$$

$$\text{Volume of } A'B'C'D'E'F'G'H' = 5 \times 2.5 \times 7.5 = 93.75 \text{ cm}^3$$

$$\therefore \text{Ratio of volumes} = \frac{93.75}{6} = 15.625 = 2.5^3$$

Here is another example:



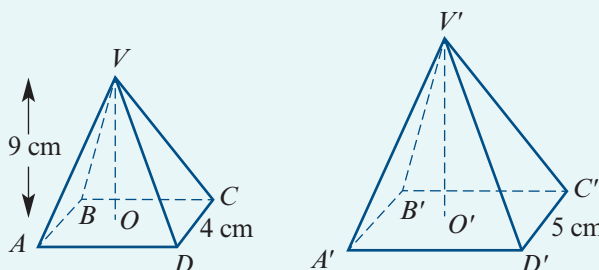
$$\text{Similarity factor} = \frac{5}{3}$$

$$\text{Ratio of volumes} = \left(\frac{5}{3}\right)^3$$

Example 20

The two square pyramids are similar and $VO = 9$ cm.

- a** Find the ratio of the lengths of their bases, and hence find the height $V'O'$ of pyramid $V'A'B'C'D'$.



- b** The volume of $VABCD$ is 48 cm^3 . Find the ratio of their volumes, and hence find the volume of $V'A'B'C'D'$.

Solution

- a** The ratio of the length of their bases is

$$\frac{C'D'}{CD} = \frac{5}{4}$$

$$\begin{aligned}\therefore V'O' &= \frac{5}{4} \times 9 \\ &= 11.25 \text{ cm}\end{aligned}$$

- b** The ratio of their volumes is

$$\frac{\text{Volume of } V'A'B'C'D'}{\text{Volume of } VABCD} = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$$

$$\begin{aligned}\therefore \text{Volume of } V'A'B'C'D' &= \frac{125}{64} \times 48 \\ &= 93.75 \text{ cm}^3\end{aligned}$$

Section summary

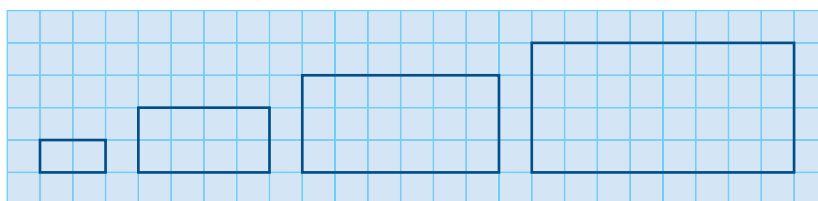
- If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then

area of similar shape $= k^2 \times$ area of original shape
- If two solids are similar and the similarity factor is k , then

volume of similar solid $= k^3 \times$ volume of original solid

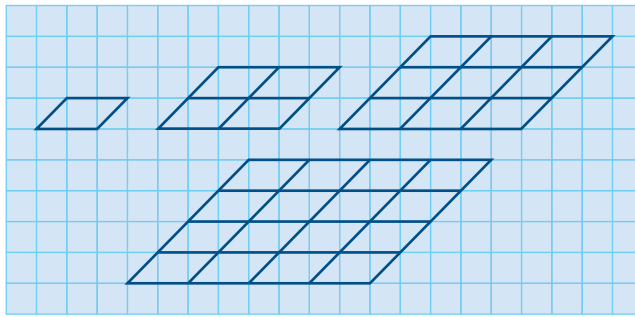
Exercise 9H**Skillsheet**

- 1** These four rectangles are similar:



- a** Write down the ratio of the lengths of their bases.
b By counting rectangles, write down the ratio of their areas.
c Is there a relationship between these two ratios?

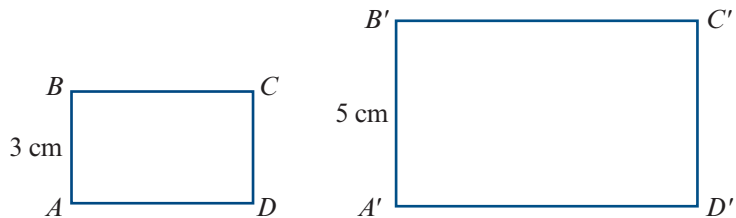
- 2 These four parallelograms are similar:



- Write down the ratio of the lengths of their bases.
- By counting parallelograms, write down the ratio of their areas.
- Is there a relationship between these two ratios?

Example 19

- 3 The two rectangles shown are similar. The area of rectangle $ABCD$ is 7 cm^2 . Find the area of rectangle $A'B'C'D'$.



- 4 Triangle ABC is similar to triangle XYZ with

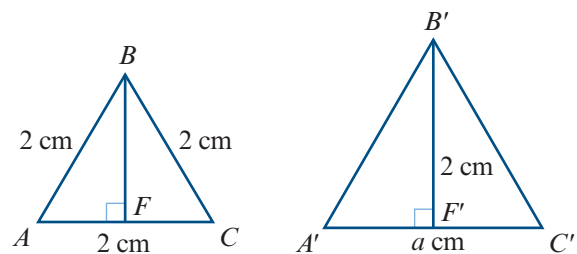
$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = 2.1$$

The area of triangle XYZ is 20 cm^2 . Find the area of triangle ABC .

- 5 Triangles ABC and $A'B'C'$ are equilateral triangles.

- Find the length of BF .
- Find a .
- Find the ratio

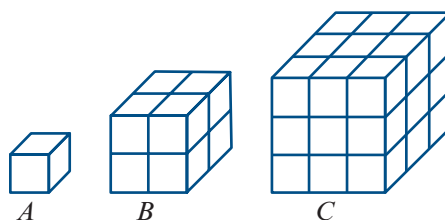
$$\frac{\text{Area of } \triangle A'B'C'}{\text{Area of } \triangle ABC}$$



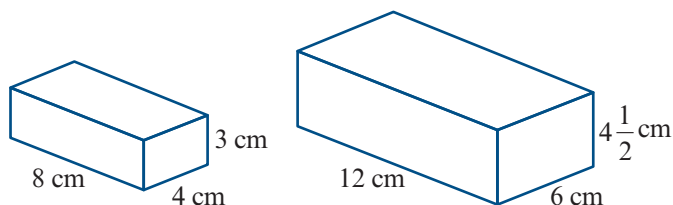
- The areas of two similar triangles are 16 and 25. What is the ratio of a pair of corresponding sides?
- The areas of two similar triangles are 144 and 81. If the base of the larger triangle is 30, what is the corresponding base of the smaller triangle?

8 These three solids are similar.

- Write down the ratio of the lengths of the bases.
- Write down the ratio of the lengths of the heights.
- By counting cuboids equal in shape and size to cuboid A, write down the ratio of the volumes.
- Is there a relationship between the answers to **a**, **b** and **c**?



9 These are two similar rectangular blocks.

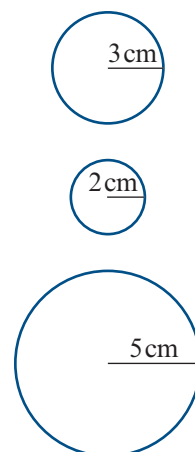


- Write down the ratio of their:
 - longest edges
 - depths
 - heights.
- By counting cubes of side length 1 cm, write down the ratio of their volumes.
- Is there any relationship between the ratios in **a** and **b**?

Example 20

10 These three solids are spheres.

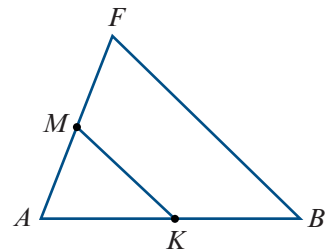
- Write down the ratio of the radii of the three spheres.
- The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.
Express the volume of each sphere as a multiple of π .
Hence write down the ratio of their volumes.
- Is there any relationship between the ratios found in **a** and **b**?



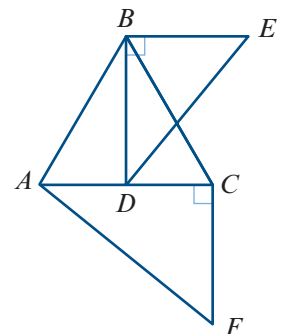
In each of Questions 11–20, the objects are mathematically similar.

- The sides of two cubes are in the ratio 2 : 1. What is the ratio of their volumes?
- The radii of two spheres are in the ratio 3 : 4. What is the ratio of their volumes?
- Two regular tetrahedrons have volumes in the ratio 8 : 27. What is the ratio of their sides?
- Two right cones have volumes in the ratio 64 : 27. What is the ratio of:
 - their heights
 - their base radii?
- Two similar bottles are such that one is twice as high as the other. What is the ratio of:
 - their surface areas
 - their capacities?

- 16** Each linear dimension of a model car is $\frac{1}{10}$ of the corresponding car dimension. Find the ratio of:
- a** the areas of their windscreens
 - b** the capacities of their boots
 - c** the widths of the cars
 - d** the number of wheels they have.
- 17** Three similar jugs have heights 8 cm, 12 cm and 16 cm. If the smallest jug holds $\frac{1}{2}$ litre, find the capacities of the other two.
- 18** Three similar drinking glasses have heights 7.5 cm, 9 cm and 10.5 cm. If the tallest glass holds 343 millilitres, find the capacities of the other two.
- 19** A toy manufacturer produces model cars which are similar in every way to the actual cars. If the ratio of the door area of the model to the door area of the car is 1 : 2500, find:
- a** the ratio of their lengths
 - b** the ratio of the capacities of their petrol tanks
 - c** the width of the model, if the actual car is 150 cm wide
 - d** the area of the rear window of the actual car if the area of the rear window of the model is 3 cm^2 .
- 20** The ratio of the areas of two similar labels on two similar jars of coffee is 144 : 169. Find the ratio of:
- a** the heights of the two jars
 - b** their capacities.
- 21** **a** In the figure, if M is the midpoint of AF and K is the midpoint of AB , then how many times larger is the area of $\triangle ABF$ than the area of $\triangle AKM$?
- b** If the area of $\triangle ABF$ is 15, find the area of $\triangle AKM$.



- 22** In the diagram, $\triangle ABC$ is equilateral, $\angle BDE = \angle CAF$ and D is the midpoint of AC . Find the ratio of the area of $\triangle BDE$ to the area of $\triangle ACF$.



- 23** The areas of two similar triangles are 144 cm^2 and 81 cm^2 . If the length of one side of the first triangle is 6 cm, what is the length of the corresponding side of the second?

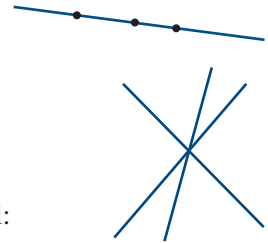
91 Geometric proofs using vectors

In this section we see how vectors can be used as a tool for proving geometric results.

We require the following two definitions.

Collinear points Three or more points are collinear if they all lie on a single line.

Concurrent lines Three or more lines are concurrent if they all pass through a single point.



Here are some properties of vectors from Chapter 6 that will be useful:

Parallel vectors

- For $k \in \mathbb{R}^+$, the vector $k\mathbf{a}$ is in the same direction as $\mathbf{a}$ and has magnitude $k|\mathbf{a}|$, and the vector $-\mathbf{ka}$ is in the opposite direction to $\mathbf{a}$ and has magnitude $k|\mathbf{a}|$.
- Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{b} = k\mathbf{a}$ for some $k \in \mathbb{R} \setminus \{0\}$.
- If $\mathbf{a}$ and $\mathbf{b}$ are parallel with at least one point in common, then $\mathbf{a}$ and $\mathbf{b}$ lie on the same straight line. For example, if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R} \setminus \{0\}$, then A , B and C are collinear.

Scalar product

- Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

Linear combinations of non-parallel vectors

- For two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ that are not parallel, if $m\mathbf{a} + n\mathbf{b} = p\mathbf{a} + q\mathbf{b}$, then $m = p$ and $n = q$.



Example 21

Three points P , Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $k(2\mathbf{p} + \mathbf{q})$ respectively, relative to a fixed origin O . The points O , P and Q are not collinear. Find the value of k if:

- a** $\overrightarrow{QR}$ is parallel to $\mathbf{p}$ **b** $\overrightarrow{PR}$ is parallel to $\mathbf{q}$ **c** P , Q and R are collinear.

Solution

a $\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

$$= -\mathbf{q} + k(2\mathbf{p} + \mathbf{q})$$

$$= 2k\mathbf{p} + (k-1)\mathbf{q}$$

If $\overrightarrow{QR}$ is parallel to $\mathbf{p}$, then there is some $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$2k\mathbf{p} + (k-1)\mathbf{q} = \lambda\mathbf{p}$$

This implies that

$$2k = \lambda \text{ and } k-1 = 0$$

Hence $k = 1$.

b $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$

$$= -\mathbf{p} + k(2\mathbf{p} + \mathbf{q})$$

$$= (2k-1)\mathbf{p} + k\mathbf{q}$$

If $\overrightarrow{PR}$ is parallel to $\mathbf{q}$, then there is some $m \in \mathbb{R} \setminus \{0\}$ such that

$$(2k-1)\mathbf{p} + k\mathbf{q} = m\mathbf{q}$$

This implies that

$$2k-1 = 0 \text{ and } k = m$$

Hence $k = \frac{1}{2}$.

Note: Since points O , P and Q are not collinear, the vectors $\mathbf{p}$ and $\mathbf{q}$ are not parallel.

- c** If points P , Q and R are collinear, then there exists $n \in \mathbb{R} \setminus \{0\}$ such that

$$n\overrightarrow{PQ} = \overrightarrow{QR}$$

$$\therefore n(-\mathbf{p} + \mathbf{q}) = 2k\mathbf{p} + (k-1)\mathbf{q}$$

This implies that

$$-n = 2k \quad \text{and} \quad n = k - 1$$

Therefore $3k - 1 = 0$ and so $k = \frac{1}{3}$.

Example 22

Suppose that $OABC$ is a parallelogram. Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{c} = \overrightarrow{OC}$.

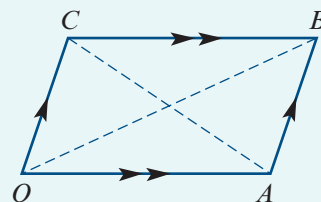
- a** Express each of the following in terms of $\mathbf{a}$ and $\mathbf{c}$:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{CA}$

- b** Find in terms of $\mathbf{a}$ and $\mathbf{c}$:

i $|\overrightarrow{OB}|^2$ **ii** $|\overrightarrow{CA}|^2$

- c** Hence, prove that if the diagonals of a parallelogram are of equal length, then the parallelogram is a rectangle.



Solution

a i $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
 $= \overrightarrow{OA} + \overrightarrow{OC}$
 $= \mathbf{a} + \mathbf{c}$

ii $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$
 $= \overrightarrow{OA} - \overrightarrow{OC}$
 $= \mathbf{a} - \mathbf{c}$

b i $|\overrightarrow{OB}|^2 = \overrightarrow{OB} \cdot \overrightarrow{OB}$
 $= (\mathbf{a} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{c})$
 $= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}$
 $= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{c} + |\mathbf{c}|^2$

ii $|\overrightarrow{CA}|^2 = \overrightarrow{CA} \cdot \overrightarrow{CA}$
 $= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{a} - \mathbf{c})$
 $= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}$
 $= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{c} + |\mathbf{c}|^2$

- c** Assume that the diagonals of the parallelogram $OABC$ are of equal length. Then $|\overrightarrow{OB}| = |\overrightarrow{CA}|$. This implies that

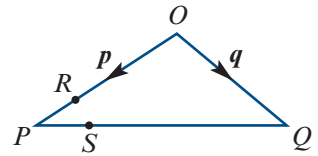
$$\begin{aligned} |\overrightarrow{OB}|^2 &= |\overrightarrow{CA}|^2 \\ |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{c} + |\mathbf{c}|^2 &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{c} + |\mathbf{c}|^2 \\ 4\mathbf{a} \cdot \mathbf{c} &= 0 \\ \therefore \mathbf{a} \cdot \mathbf{c} &= 0 \end{aligned}$$

We have shown that $\overrightarrow{OA} \cdot \overrightarrow{OC} = 0$. So $\angle COA = 90^\circ$. Hence the parallelogram $OABC$ is a rectangle.

Exercise 91

Skillsheet

- 1 In the diagram, $OR = \frac{4}{5}OP$, $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$ and $PS : SQ = 1 : 4$.



- a** Express each of the following in terms of $\mathbf{p}$ and $\mathbf{q}$:
- $\overrightarrow{OR}$
 - $\overrightarrow{RP}$
 - $\overrightarrow{PO}$
 - $\overrightarrow{PS}$
 - $\overrightarrow{RS}$
- b** What can be said about line segments RS and OQ ?
- c** What type of quadrilateral is $ORSQ$?
- d** The area of triangle PRS is 5 cm^2 . What is the area of $ORSQ$?
- 2 The position vectors of three points A , B and C relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $k\mathbf{a}$ respectively. The point P lies on AB and is such that $AP = 2PB$. The point Q lies on BC and is such that $CQ = 6QB$.

- a** Find in terms of $\mathbf{a}$ and $\mathbf{b}$:
- the position vector of P
 - the position vector of Q
- b** Given that OPQ is a straight line, find:
- the value of k
 - the ratio $\frac{OP}{PQ}$
- c** The position vector of a point R is $\frac{7}{3}\mathbf{a}$. Show that PR is parallel to BC .

Example 21

- 3 The position vectors of two points A and B relative to an origin O are $3\mathbf{i} + 3.5\mathbf{j}$ and $6\mathbf{i} - 1.5\mathbf{j}$ respectively.
- a**
- Given that $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$ and $\overrightarrow{AE} = \frac{1}{4}\overrightarrow{AB}$, write down the position vectors of D and E .
 - Hence find $|\overrightarrow{ED}|$.
- b** Given that OE and AD intersect at X and that $\overrightarrow{OX} = p\overrightarrow{OE}$ and $\overrightarrow{XD} = q\overrightarrow{AD}$, find the position vector of X in terms of:
- p
 - q
- c** Hence determine the values of p and q .
- 4 Points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$, with reference to an origin O , and M is the point on PQ such that
- $$\beta\overrightarrow{PM} = \alpha\overrightarrow{MQ}$$
- a** Prove that the position vector of M is given by $\mathbf{m} = \frac{\beta\mathbf{p} + \alpha\mathbf{q}}{\alpha + \beta}$.
- b** Write the position vectors of P and Q as $\mathbf{p} = k\mathbf{a}$ and $\mathbf{q} = \ell\mathbf{b}$, where k and ℓ are positive real numbers and $\mathbf{a}$ and $\mathbf{b}$ are unit vectors.
- Prove that the position vector of any point on the internal bisector of $\angle POQ$ has the form $\lambda(\mathbf{a} + \mathbf{b})$.
 - If M is the point where the internal bisector of $\angle POQ$ meets PQ , show that

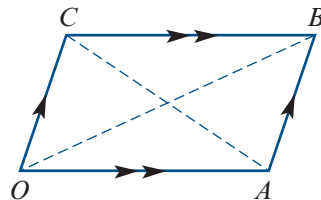
$$\frac{\alpha}{\beta} = \frac{k}{\ell}$$

Example 22

- 5 Suppose that $OABC$ is a parallelogram. Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{c} = \overrightarrow{OC}$.
- Express each of the following vectors in terms of $\mathbf{a}$ and $\mathbf{c}$:
 - $\overrightarrow{AB}$
 - $\overrightarrow{OB}$
 - $\overrightarrow{AC}$
 - Find $\overrightarrow{OB} \cdot \overrightarrow{AC}$.
 - Hence, prove that the diagonals of a parallelogram intersect at right angles if and only if it is a rhombus.
- 6 Suppose that $ORST$ is a parallelogram, where O is the origin. Let U be the midpoint of RS and let V be the midpoint of ST . Denote the position vectors of R, S, T, U and V by $\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}$ and $\mathbf{v}$ respectively.
- Express $\mathbf{s}$ in terms of $\mathbf{r}$ and $\mathbf{t}$.
 - Express $\mathbf{v}$ in terms of $\mathbf{s}$ and $\mathbf{t}$.
 - Hence, or otherwise, show that $4(\mathbf{u} + \mathbf{v}) = 3(\mathbf{r} + \mathbf{s} + \mathbf{t})$.
- 7 Prove that the midpoints of any quadrilateral are the vertices of a parallelogram.
- 8 Prove that the diagonals of a square are of equal length and bisect each other.
- 9 Prove that the diagonals of a parallelogram bisect each other.
- 10 Prove that the altitudes of a triangle are concurrent. That is, they meet at a point.
- 11 **Apollonius' theorem**
For $\triangle OAB$, the point C is the midpoint of side AB . Prove that:
- $4\overrightarrow{OC} \cdot \overrightarrow{OC} = OA^2 + OB^2 + 2\overrightarrow{OA} \cdot \overrightarrow{OB}$
 - $4\overrightarrow{AC} \cdot \overrightarrow{AC} = OA^2 + OB^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB}$
 - $2OC^2 + 2AC^2 = OA^2 + OB^2$
- 12 If P is any point in the plane of rectangle $ABCD$, prove that
- $$PA^2 + PC^2 = PB^2 + PD^2$$
- 13 Prove that the medians bisecting the equal sides of an isosceles triangle are equal.
- 14 **a** Prove that if $(\mathbf{c} - \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b} = 0$, then $(\mathbf{b} - \mathbf{a}) \cdot \mathbf{c} = 0$.
b Use part **a** to prove that the altitudes of a triangle meet at a point.
- 15 For a parallelogram $OABC$, prove that

$$OB^2 + AC^2 = 2OA^2 + 2OC^2$$

That is, prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.



9J The golden ratio

The golden ratio is the irrational number

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.618\ 033\ 988 \dots$$

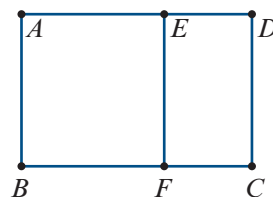
This number is mentioned in the books of Euclid, where it is used in the construction of a regular pentagon. (See Extended-response question 2.) The golden ratio also arises naturally in connection with the sequence of Fibonacci numbers.

► Golden rectangles

A **golden rectangle** is a rectangle that can be cut up into a square and a rectangle that is similar to the original one.

Let $ABCD$ be a rectangle with $AB < BC$.

Then there is a point E on AD and a point F on BC such that $ABFE$ is a square. We say that $ABCD$ is a golden rectangle if $FCDE$ is similar to $ABCD$.



Theorem

All golden rectangles are similar, with ratio of length to width given by

$$\frac{1 + \sqrt{5}}{2} : 1$$

Proof Assume that $ABCD$, as shown in the diagram above, is a golden rectangle.

Let $AD = \ell$ and $CD = w$. Then $ED = \ell - w$.

As the two rectangles are similar, we have

$$\frac{AD}{CD} = \frac{CD}{ED} = k$$

where k is the ratio that we want to find. Thus

$$\frac{\ell}{w} = \frac{w}{\ell - w} = k$$

and therefore

$$\ell^2 - \ell w = w^2$$

Substitute $\ell = kw$:

$$(kw)^2 - kw^2 = w^2$$

$$k^2 - k - 1 = 0$$

Using the quadratic formula gives $k = \frac{1 + \sqrt{5}}{2}$ since $k > 0$.

The **golden ratio** is denoted by φ and is given by

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

Alternatively, the golden ratio can be defined as the unique positive number φ such that

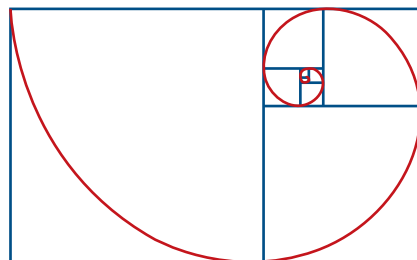
$$\varphi^2 - \varphi - 1 = 0$$

Note: A rectangle of length φ and width 1 is indeed a golden rectangle, since $\varphi^2 - \varphi - 1 = 0$ implies that $\frac{\varphi}{1} = \frac{1}{\varphi - 1}$.

A sequence of golden rectangles

Starting from any golden rectangle, we can remove a square to form another golden rectangle. Therefore we can remove another square to form yet another golden rectangle.

Continuing in this way, we can create the spiral shown.



► The golden ratio and the geometric mean

You are familiar with the **arithmetic mean** of two numbers a and b , defined as $\frac{a+b}{2}$.

The **geometric mean** of two positive numbers a and b is defined as $\sqrt{ab}$.

Note that the arithmetic mean of numbers a and b is the unique number c such that

$$c - a = b - c$$

Similarly, we can see that the geometric mean of positive numbers a and b is the unique positive number c such that

$$\frac{c}{a} = \frac{b}{c}$$

Again, consider a rectangle $ABCD$ with $AB < BC$.

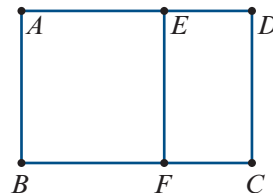
Choose points E and F as shown such that $ABFE$ is a square.

Then $CD = AE$.

Therefore $ABCD$ is a golden rectangle if and only if

$$\frac{AD}{AE} = \frac{AE}{ED}$$

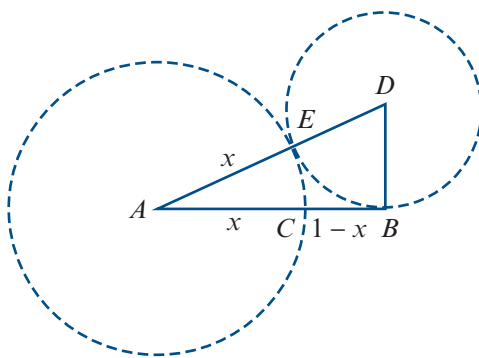
This is precisely the requirement that AE is the geometric mean of AD and ED .



► Construction of the golden ratio

We can construct the golden ratio as follows.

- 1 Start with a line segment AB of unit length.
- 2 Draw BD perpendicular to AB with length $\frac{AB}{2} = \frac{1}{2}$.
- 3 Draw line segment AD .
- 4 Draw a circle with centre D and radius DB , cutting AD at E .
- 5 Draw a circle with centre A and radius AE , cutting AB at C .
- 6 Use Pythagoras' theorem to show that $x = AE = \frac{\sqrt{5}-1}{2}$. Then $\frac{AB}{AC} = \frac{1}{x} = \varphi$.



► Irrationality of the golden ratio

One way to prove that the golden ratio is irrational is first to prove that $\sqrt{5}$ is irrational. However, we can give a more direct proof as follows.

Theorem

The golden ratio φ is irrational.

Proof Suppose that the golden ratio is rational. Then we can write

$$\varphi = \frac{m}{n} \quad \text{for some } m, n \in \mathbb{N}$$

We can assume that m and n have no common factors, and hence the numerator m is as small as possible. Note that $\varphi > 1$ and so $m > n$.

Since $\varphi^2 - \varphi - 1 = 0$, we have

$$\begin{aligned} \varphi &= \frac{1}{\varphi - 1} \\ &= \frac{1}{\frac{m}{n} - 1} \\ &= \frac{n}{m - n} \end{aligned}$$

As $m > n$, we have now expressed φ as a fraction with a numerator smaller than m . But this contradicts our initial assumption. Hence φ is irrational.

Section summary

- The **golden ratio** is $\varphi = \frac{1 + \sqrt{5}}{2}$.
- All golden rectangles are similar, with the ratio of length to width $\varphi : 1$.
- The golden ratio is the unique positive number φ such that $\varphi^2 - \varphi - 1 = 0$.

Exercise 9J

- 1 For the golden ratio φ show that:

a $\varphi - 1 = \frac{1}{\varphi}$

b $\varphi^3 = 2\varphi + 1$

c $2 - \varphi = (\varphi - 1)^2 = \frac{1}{\varphi^2}$

- 2 ABC is a right-angled triangle with the right angle at C , and CX is the altitude of the triangle from C .

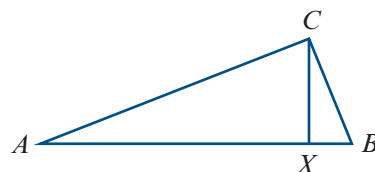
a Prove that $\frac{AX}{CX} = \frac{CX}{XB}$.

Note: This shows that the length CX is the geometric mean of lengths AX and XB .

- b** Find CX if:

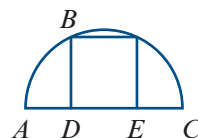
i $AX = 2$ and $XB = 8$

ii $AX = 1$ and $XB = 10$.



- 3 A square is inscribed in a semicircle as shown. Prove that

$$\frac{AD}{BD} = \frac{BD}{CD} = \varphi - 1$$



- 4 A regular decagon is inscribed in a circle with unit radius as shown.

- a** Find the magnitude of angle:

i AOB **ii** OAB

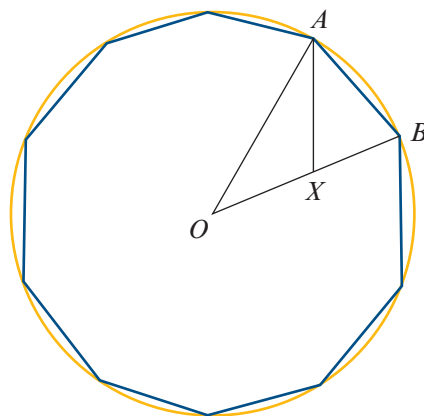
- b** The line AX bisects angle OAB . Prove that:

i triangle AXB is isosceles

ii triangle AXO is isosceles

iii triangle AOB is similar to triangle BXA .

- c** Find the length of AB , correct to two decimal places.



- 5 Calculate $\varphi^0, \varphi^1, \varphi^2, \varphi^3, \varphi^4$ and $\varphi^{-1}, \varphi^{-2}, \varphi^{-3}, \varphi^{-4}$. Show that each power of φ is equal to the sum of the two powers before it. That is, show that $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$.

- 6 The Fibonacci sequence $1, 1, 2, 3, 5, 8, \dots$ is defined by $t_1 = t_2 = 1$ and $t_{n+1} = t_{n-1} + t_n$. Consider the sequence

$$\frac{t_2}{t_1}, \frac{t_3}{t_2}, \frac{t_4}{t_3}, \frac{t_5}{t_4}, \dots$$

Show numerically that, as n gets very large, the ratio $\frac{t_{n+1}}{t_n}$ approaches φ .



Chapter summary

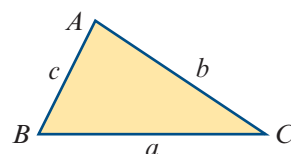


Parallel lines

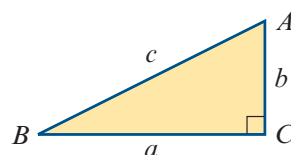
- If two parallel lines are crossed by a transversal, then:
 - alternate angles are equal
 - corresponding angles are equal
 - co-interior angles are supplementary.
- If two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Polygons

- The sum of the interior angles of an n -sided polygon is $(n - 2)180^\circ$.
- A **regular polygon** is a polygon in which all angles are equal and all sides are equal.
- **Triangle inequality**
In $\triangle ABC$: $a < b + c$, $b < c + a$ and $c < a + b$.



- **Pythagoras' theorem and its converse**
Let ABC be a triangle with side lengths a , b and c .
 - If $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.
 - If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.



- Classification of quadrilaterals:
 - A **trapezium** is a quadrilateral with at least one pair of opposite sides parallel.
 - A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.
 - A **rhombus** is a parallelogram with a pair of adjacent sides equal.
 - A **rectangle** is a parallelogram in which one angle is a right angle.
 - A **square** is a rectangle with a pair of adjacent sides equal.

Congruence

- **Congruent figures** have exactly the same shape and size.
- If triangle ABC is congruent to triangle XYZ , this can be written as $\triangle ABC \equiv \triangle XYZ$.
- Two triangles are congruent provided any one of the following four conditions holds:
 - SSS** the three sides of one triangle are equal to the three sides of the other triangle
 - SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
 - AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
 - RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

Similarity

- Two figures are **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.

- Matching lengths of similar figures are in the same ratio.
- Matching angles of similar figures are equal.

- If triangle ABC is similar to triangle XYZ , this can be written as $\triangle ABC \sim \triangle XYZ$.

- Two triangles are similar provided any one of the following four conditions holds:

AAA two angles of one triangle are equal to two angles of the other triangle

SAS the ratios of two pairs of matching sides are equal and the included angles are equal

SSS the sides of one triangle can be matched up with the sides of the other triangle so that the ratio of matching lengths is constant

RHS the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

- If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then

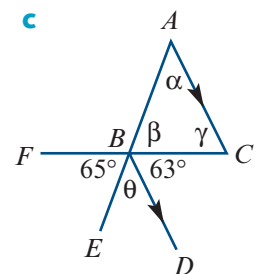
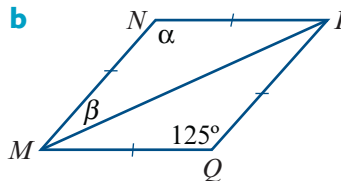
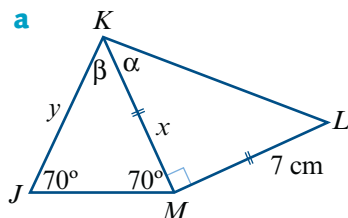
$$\text{area of similar shape} = k^2 \times \text{area of original shape}$$

- If two solids are similar and the similarity factor is k , then

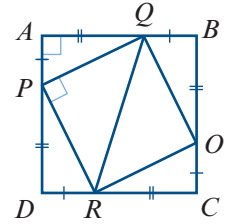
$$\text{volume of similar solid} = k^3 \times \text{volume of original solid}$$

Technology-free questions

- $ABCD$ is a rhombus with $AB = 16$ cm. The midpoints of its sides are joined to form a quadrilateral.
 - Describe the quadrilateral formed.
 - What is the length of the diagonal of this quadrilateral?
- Prove that a triangle with sides $x^2 - y^2$, $x^2 + y^2$ and $2xy$ is a right-angled triangle.
- Find the side length of a rhombus whose diagonals are 6 cm and 10 cm.
- Find the values of the unknowns (x , y , α , β , γ and θ) for each of the following:



- 5** A 25 m pole is leaning against a vertical wall with the foot of the pole 20 m from the wall. If its foot slips a further 4 m from the wall, find the distance that the top of the pole has slipped down the wall.
- 6** **a** Prove that $\triangle PAQ \equiv \triangle QBO$.
b Prove that $\triangle PQR \equiv \triangle ORQ$.

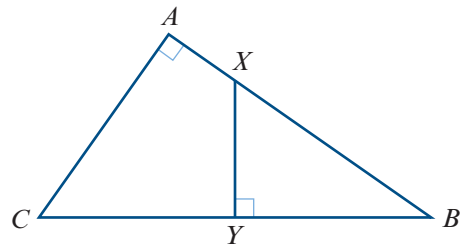


- 7** Let XYZ be a triangle with a point P on XY and a point Q on XZ such that PQ is parallel to YZ .
- a** Show that the two triangles XYZ and XPQ are similar.
b If $XY = 36$ cm, $XZ = 30$ cm and $XP = 24$ cm, find:
i XQ **ii** QZ
c Write down the values of $XP : PY$ and $PQ : YZ$.

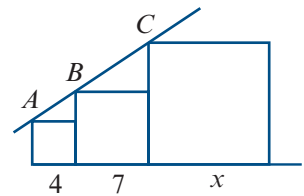
- 8** Triangles ABC and DEF are similar. If the area of triangle ABC is 12.5 cm^2 , the area of triangle DEF is 4.5 cm^2 and $AB = 5$ cm, find:
a the length of DE **b** the value of $AC : DF$ **c** the value of $EF : BC$.

- 9** If a 1 m stake casts a shadow 2.3 m long, find the height (in metres) of a tree which casts a shadow 21 m long.

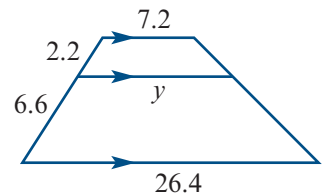
- 10** ABC is a right-angled triangle with $AB = 4$ and $AC = 3$. If the triangle is folded along the line XY , then vertex C coincides with vertex B . Find the length of XY .



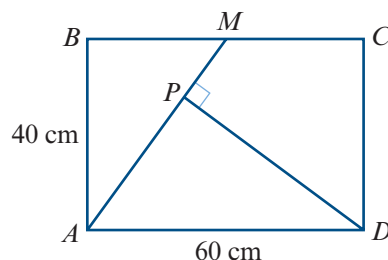
- 11** Points A , B and C lie on a straight line. The squares are adjacent and have side lengths 4, 7 and x . Find the value of x .



- 12** Find the value of y in the diagram on the right.

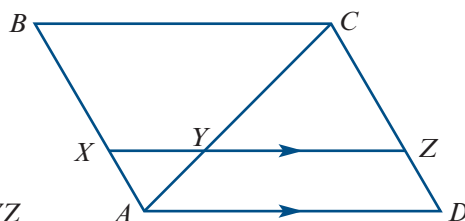


- 13** $ABCD$ is a rectangle in which $AB = 40$ cm and $AD = 60$ cm. The midpoint of BC is M , and DP is perpendicular to AM .

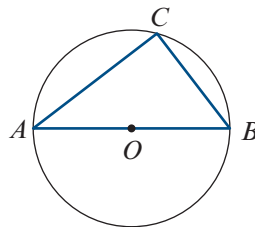


- Prove that the triangles BMA and PAD are similar.
 - Calculate the ratio of the areas of the triangles BMA and PAD .
 - Calculate the length of PD .
- 14** A sculptor is commissioned to create a bronze statue 2 m high. He begins by making a clay model 30 cm high.
- Express, in simplest form, the ratio of the height of the completed bronze statue to the height of the clay model.
 - If the surface area of the model is 360 cm^2 , find the surface area of the statue.
 - If the volume of the model is 1000 cm^3 , find the volume of the statue.
- 15** The radius of a spherical soap bubble increases by 1%. Find, correct to the nearest whole number, the percentage increase in:
- its surface area
 - its volume.

- 16** AC is the diagonal of a rhombus $ABCD$. The line XYZ is parallel to AD , and $AX = 3$ cm and $AB = 9$ cm. Find:



- $\frac{XY}{BC}$
 - $\frac{AY}{AC}$
 - $\frac{CY}{AC}$
 - $\frac{YZ}{AD}$
 - $\frac{\text{area } \triangle AXY}{\text{area } \triangle ABC}$
 - $\frac{\text{area } \triangle CYZ}{\text{area } \triangle ACD}$
- 17** AB and DC are parallel sides of a trapezium and $DC = 3AB$. The diagonals AC and DB intersect at O . Prove that $AO = \frac{1}{4}AC$.
- 18** Triangles ABC and PQR are similar. The medians AX and PY are drawn, where X is the midpoint of BC and Y is the midpoint of QR . Prove that:
- triangles ABX and PQY are similar
 - $\frac{AX}{PY} = \frac{BC}{QR}$
- 19** The figure shows a triangle ABC inscribed in a circle with centre O , where AB is a diameter of the circle. Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{c} = \overrightarrow{OC}$.
- Express the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.
 - Find $\overrightarrow{AC} \cdot \overrightarrow{BC}$.
 - Hence, show that the angle subtended by a diameter of a circle at the circumference is a right angle.



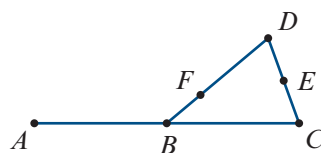
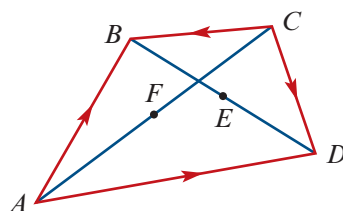
- 20** The points P and R have position vectors $\vec{OP} = 2\mathbf{a} + \mathbf{b}$ and $\vec{OR} = \mathbf{a} - 3\mathbf{b}$ respectively. Prove that if $OPQR$ is a square, then $|\mathbf{a}|^2 = 2|\mathbf{b}|^2$.

- 21** $ABCD$ is a quadrilateral, and E and F are the midpoints of BD and AC respectively. Prove that

$$\vec{AB} + \vec{CB} + \vec{CD} + \vec{AD} = 4\vec{FE}$$

- 22** In the diagram, points A , B and C lie on a straight line and $AB = BC = BD$. Point E is the midpoint of CD and $BF = \frac{1}{3}BD$.

Use vector methods to prove that points A , F and E are collinear and find the ratio $AF : FE$.



Multiple-choice questions



- 1** One angle of a triangle is twice the size of the second angle, and the third angle is 66° . The smallest angle is

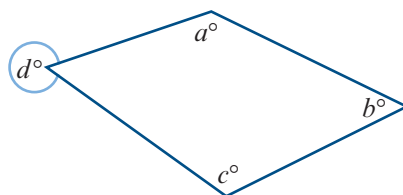
A 66° **B** 53° **C** 38° **D** 24° **E** 22°

- 2** Three of the angles of a pentagon are right angles. The other two angles are of equal size x° . The angle x° is equal to

A 45° **B** 135° **C** 120° **D** 150° **E** 108°

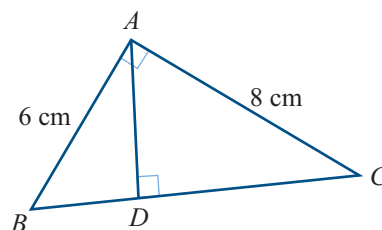
- 3** d is equal to

A $360 - (a + b + c)$ **B** $a + b + c$
C $b - a + c$ **D** $a - b + c$
E $a + b - c$



- 4** In the figure, $\angle BAC$ is a right angle and AD is perpendicular to BC . If $AB = 6$ cm and $AC = 8$ cm, then the length of AD is

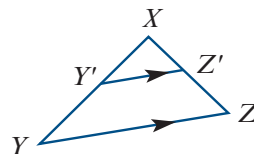
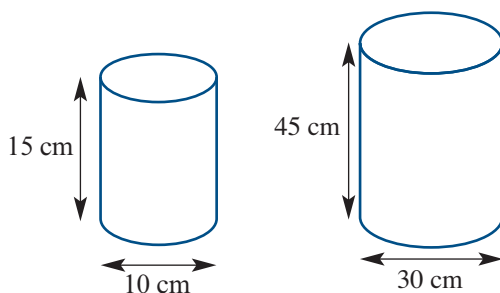
A 4 cm **B** $\frac{24}{5}$ cm **C** $\frac{17}{3}$ cm
D $\frac{13}{2}$ cm **E** 7 cm



- 5** Two sides of a triangle have lengths 14 and 18. Which of the following *cannot* be the length of the third side?

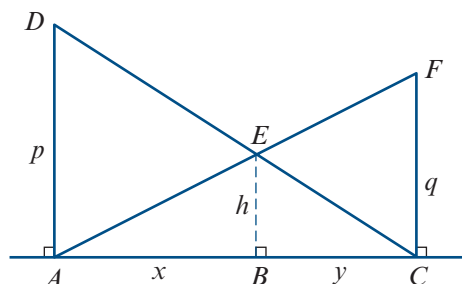
A 2 **B** 6 **C** 7 **D** 28 **E** 30

- 6** If $5 : 3 = 7 : x$, then x is equal to
A 12 **B** $\frac{35}{3}$ **C** 5 **D** $\frac{21}{5}$ **E** $\frac{5}{21}$
- 7** Brass is composed of a mixture of copper and zinc. If the ratio of copper to zinc is $85 : 15$, then the amount of copper in 400 kg of brass is
A 60 kg **B** 340 kg **C** 360 kg **D** 380 kg **E** 150 kg
- 8** If the total cost of P articles is Q dollars, then the cost of R articles of the same type is
A PQR **B** $\frac{P}{QR}$ **C** $\frac{PQ}{R}$ **D** $\frac{QR}{P}$ **E** $\frac{R}{PQ}$
- 9** A car is 3.2 m long. The length of a model of the car at the scale $1 : 100$ is
A 0.032 cm **B** 0.32 cm **C** 3.2 cm **D** 320 cm **E** 32 cm
- 10** An athlete runs 75 m in 9 s. If she were to maintain the same average speed for 100 m, then her time for 100 m would be
A 11.6 s **B** 12.0 s **C** 11.8 s **D** 12.2 s **E** 12.4 s
- 11** If 50 is divided into three parts in the ratio $1 : 3 : 6$, then the largest part is
A 5 **B** 15 **C** $\frac{50}{3}$ **D** 30 **E** 3
- 12** Two similar cylinders are shown.
 The ratio of the volume of the smaller cylinder to the volume of the larger cylinder is
A $1 : 3$ **B** $1 : 9$
C $1 : 27$ **D** $1 : 5$
E $2 : 9$
- 13** The radius of sphere A is $\frac{4}{5}$ times the radius of sphere B . Hence, the ratio of the volume of sphere A to the volume of sphere B is
A $16 : 25$ **B** $4 : 5$ **C** $5 : 4$ **D** $25 : 16$ **E** $64 : 125$
- 14** YZ is parallel to $Y'Z'$ and $Y'Y = \frac{1}{3}YX$. The area of triangle XYZ is 60 cm^2 . The area of triangle $XY'Z'$ is
A 20 cm^2 **B** 30 cm^2 **C** $\frac{20}{9} \text{ cm}^2$
D $\frac{20}{3} \text{ cm}^2$ **E** $\frac{80}{3} \text{ cm}^2$

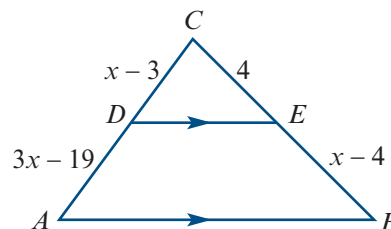
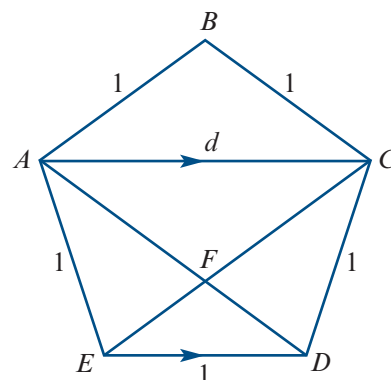


Extended-response questions

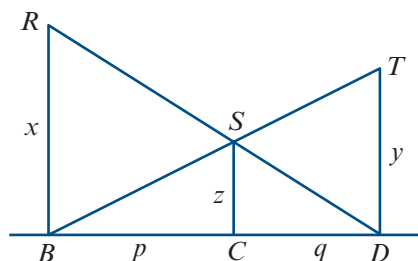
- 1 **a** In this diagram, which other triangle is similar to $\triangle DAC$?
- b** Explain why $\frac{h}{p} = \frac{y}{x+y}$.
- c** Use another pair of similar triangles to write down an expression for $\frac{h}{q}$ in terms of x and y .
- d** Explain why $h \cdot \left(\frac{1}{p} + \frac{1}{q}\right) = 1$.
- e** Calculate h when $p = 4$ and $q = 5$.



- 2 $ABCDE$ is a regular pentagon whose sides are each 1 unit long. Each diagonal is of length d units. In a regular pentagon, each diagonal is parallel to one of the sides of the pentagon.
 - a** What kind of shape is $ABCF$ and what is the length of CF ?
 - b** Explain why the length of EF is $d - 1$.
 - c** Which triangle is similar to $\triangle EFD$?
 - d** Use the pair of similar triangles to write an equation for d and show that the equation can be rewritten as $d^2 - d - 1 = 0$.
 - e** Find d .
- 3 Place conditions upon x such that DE is parallel to AB given that $CD = x - 3$, $DA = 3x - 19$, $CE = 4$ and $EB = x - 4$.



- 4 **a** If BR , CS and DT are perpendicular to BD , name the pairs of similar triangles.
- b** Which of the following is correct?
 $\frac{z}{y} = \frac{p}{q}$ or $\frac{z}{y} = \frac{p}{p+q}$
- c** Which of the following is correct?
 $\frac{z}{x} = \frac{q}{p}$ or $\frac{z}{x} = \frac{q}{p+q}$
- d** Show that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.



- 5 In the diagram, PQ is parallel to BC and PR is parallel to AC , with $AQ = 2$ cm, $QC = 6$ cm, $AP = 3$ cm and $PQ = 4$ cm.

a Calculate:

i PB

ii BR

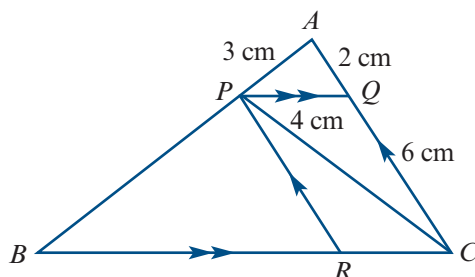
iii $\frac{\text{area } \triangle APQ}{\text{area } \triangle ABC}$

iv $\frac{\text{area } \triangle BPR}{\text{area } \triangle ABC}$

- b If the area of triangle APQ is a cm², express in terms of a :

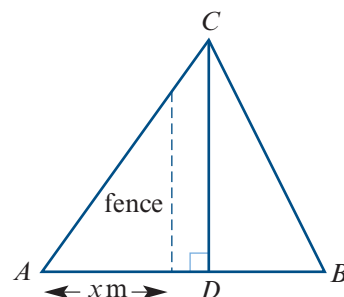
i $\text{area } \triangle ABC$

ii $\text{area } \triangle CPQ$



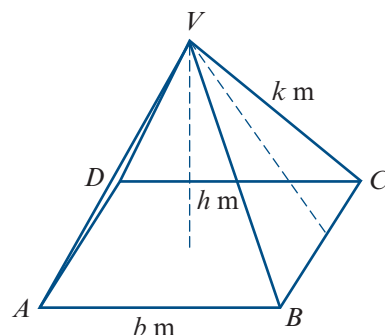
- 6 Construct a triangle ABC such that $BC = 10$ cm, $AC = 9$ cm and $AB = 6$ cm. Find a point D on AB and a point E on AC such that DE is parallel to BC and the area of $\triangle ADE$ is one-ninth the area of $\triangle ABC$.

- 7 A triangular lot has boundaries of lengths $AB = 130$ m, $BC = 40\sqrt{10}$ m and $CA = 150$ m. The length of CD is 120 m. A fence is to be erected which runs at right angles to AB . If the lot is to be divided into two equal areas, find x .



- 8 The Greek historian Herodotus wrote that the proportions of the great pyramid at Giza in Egypt were chosen so that the area of a square, for which the side lengths are equal to the height of the great pyramid, is equal to the area of one of the triangular faces.

Let h m be the height of the pyramid, let k m be the altitude of one of the face triangles, and let b m be the length of a side of the square base.



Show that Herodotus' definition gives $k : \frac{b}{2} = \varphi : 1$, where φ is the golden ratio.

10

Circle geometry

Objectives

- ▶ To establish the following results and use them to solve problems:
 - ▷ The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
 - ▷ Angles in the same segment of a circle are equal.
 - ▷ A tangent to a circle is perpendicular to the radius drawn from the point of contact.
 - ▷ The two tangents drawn from an external point to a circle are the same length.
 - ▷ The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.
 - ▷ A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180° .
 - ▷ If AB and CD are two chords of a circle that cut at a point P , then $PA \cdot PB = PC \cdot PD$.

The two basic figures of geometry in the plane are the triangle and the circle. We have considered triangles and their properties in Chapter 9, and we will use the results of that chapter in establishing results involving circles.

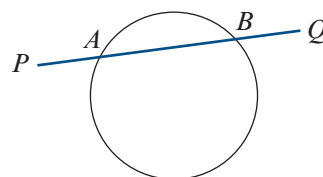
A **circle** is the set of all points in the plane at a fixed distance r from a point O . Circles with the same radius are congruent to each other (and are said to be equal circles). We have seen in the previous chapter that all circles are similar to each other.

You may have come across the Cartesian equation of the circle in Mathematical Methods Units 1 & 2. For example, the circle with radius 1 and centre the origin has equation $x^2 + y^2 = 1$. In this chapter we take a different approach to the study of circles.

The theorems and related results in this chapter can be investigated through a geometry package such as GeoGebra or Cabri Geometry.

10A Angle properties of circles

A line segment joining two points on a circle is called a **chord**. A line that cuts a circle at two distinct points is called a **secant**.



For example, in the diagram, the line PQ is a secant and the line segment AB is a chord.

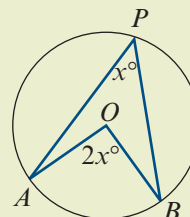
Suppose that we have a line segment or an arc AB and a point P not on AB . Then $\angle APB$ is the angle **subtended** by AB at the point P .

You should prove the following two results. The first proof uses the SSS congruence test and the second uses the SAS congruence test.

- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.

Theorem 1

The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



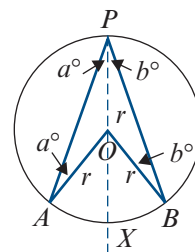
Proof Join points P and O and extend the line through O , as shown in the diagram on the right.

Note that $AO = BO = PO = r$, the radius of the circle. Therefore triangles PAO and PBO are isosceles.

Let $\angle APO = \angle PAO = a^\circ$ and $\angle BPO = \angle PBO = b^\circ$.

Then angle AOX is $2a^\circ$ (exterior angle of a triangle) and angle BOX is $2b^\circ$ (exterior angle of a triangle).

Hence $\angle AOB = 2a^\circ + 2b^\circ = 2(a + b)^\circ = 2\angle APB$.



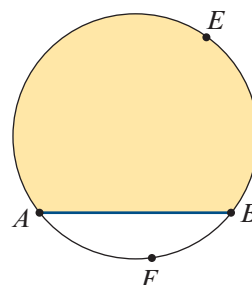
Note: In this proof, the centre O and point P are on the same side of chord AB . Slight variations of this proof can be used for other cases. The result is always true.

Converse of Theorem 1 Let A and B be points on a circle, centre O , and let P be a point on the same side of AB as O . If the angle APB is half the angle AOB , then P lies on the circle.

Segments

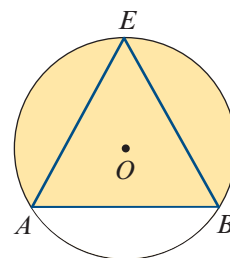
A **segment** of a circle is the part of the plane bounded by an arc and its chord. For example, in the diagram:

- Arc AEB and chord AB define a **major segment** (shaded).
- Arc AFB and chord AB define a **minor segment** (unshaded).



Angles in a segment

$\angle AEB$ is said to be an angle in segment AEB .



Theorem 2: Angles in the same segment

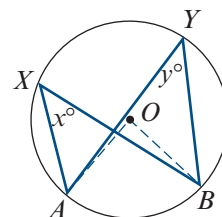
Angles in the same segment of a circle are equal.

Proof Let $\angle AXB = x^\circ$ and $\angle AYB = y^\circ$.

Then, by Theorem 1, $\angle AOB = 2x^\circ = 2y^\circ$.

Therefore $x = y$.

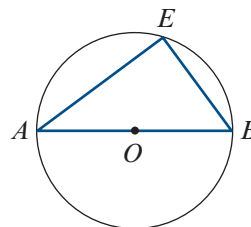
Note: A converse of this result is proved later in this section.



Theorem 3: Angle subtended by a diameter

The angle subtended by a diameter at the circumference is equal to a right angle (90°).

Proof The angle subtended at the centre is 180° , and so the result follows from Theorem 1.



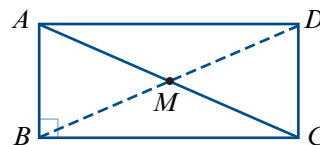
We can give a straightforward proof of a converse of this result.

Converse of Theorem 3 The circle whose diameter is the hypotenuse of a right-angled triangle passes through all three vertices of the triangle.

Proof In the diagram, triangle ABC has a right angle at B .
Let M be the midpoint of the hypotenuse AC .

We need to prove that $MA = MB = MC$.

Complete the rectangle $ABCD$.



In Question 12 of Exercise 9C, you proved that the diagonals of a rectangle are equal and bisect each other.

Hence $AC = BD$ and M is also the midpoint of BD . It follows that $MA = MB = MC$.

Cyclic polygons

- A set of points is said to be **concyclic** if they all lie on a common circle.
- A polygon is said to be **inscribed in a circle** if all its vertices lie on the circle. This implies that no part of the polygon lies outside the circle.
- A quadrilateral that can be inscribed in a circle is called a **cyclic quadrilateral**.

Theorem 4

The opposite angles of a quadrilateral inscribed in a circle sum to 180° .

That is, the opposite angles of a cyclic quadrilateral are supplementary.

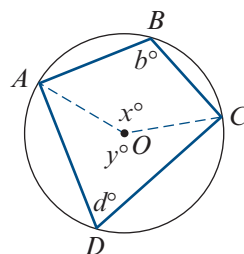
Proof In the diagram, the quadrilateral $ABCD$ is inscribed in a circle with centre O .

By Theorem 1, we have $x = 2d$ and $y = 2b$.

Now $x + y = 360$

and so $2b + 2d = 360$

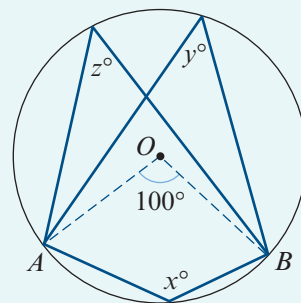
Hence $b + d = 180$



Converse of Theorem 4 If opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Example 1

Find the value of each of the pronumerals in the diagram, where O is the centre of the circle and $\angle AOB = 100^\circ$.



Solution

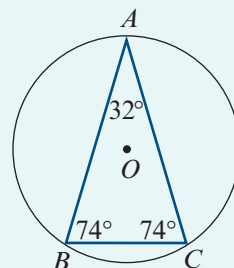
Theorem 1 gives $y = z = 50$.

The value of x can be found by observing either of the following:

- 1 Reflex angle AOB is 260° .
Therefore $x = 130$ (by Theorem 1).
- 2 We have $x + y = 180$ (by Theorem 4).
Therefore $x = 180 - 50 = 130$.

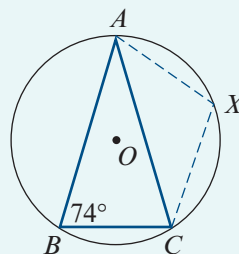
**Example 2**

An isosceles triangle is inscribed in a circle as shown. Find the angles in the three minor segments of the circle cut off by the sides of this triangle.

**Solution**

To find $\angle AXC$, form the cyclic quadrilateral $AXCB$. Then $\angle AXC$ and $\angle ABC$ are supplementary. Therefore $\angle AXC = 106^\circ$, and so all angles in the minor segment formed by AC have magnitude 106° .

Similarly, it can be shown that all angles in the minor segment formed by AB have magnitude 106° , and that all angles in the minor segment formed by BC have magnitude 148° .

**Example 3**

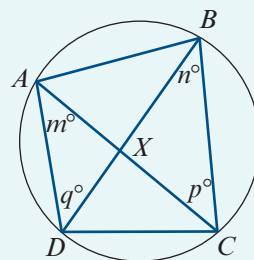
A, B, C and D are points on a circle. The diagonals of quadrilateral $ABCD$ meet at X . Prove that triangles ADX and BCX are similar.

Solution

$\angle DAC$ and $\angle DBC$ are in the same segment. Therefore $m = n$.

$\angle ADB$ and $\angle ACB$ are in the same segment. Therefore $q = p$.

Hence triangles ADX and BCX are similar (AAA).



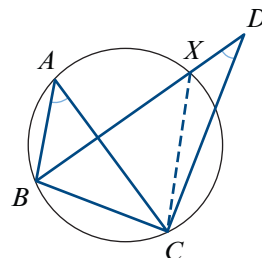
► The converse theorems

We only prove a converse of Theorem 2 here, but the proofs of the converses of Theorems 1 and 4 use similar techniques. Try them for yourself.

Converse of Theorem 2 If a line segment subtends equal angles at two points on the same side of the line segment, then the two points and the endpoints of the line segment are concyclic.

Proof A circle is drawn through points A, B and C . (This can be done with any three non-collinear points.)

Assume that $\angle BAC = \angle BDC$ and that D lies outside the circle. (There is another case to consider when D is inside, but the proof is similar. If D lies on the circle, then we are finished.)



Let X be the point of intersection of line BD with the circle. Then, by Theorem 2, $\angle BAC = \angle BXC$ and so $\angle BDC = \angle BXC$. But this is impossible. (You can use the equality of the angle sums of $\triangle BXC$ and $\triangle BDC$ to show this.)

Hence D lies on the same circle as A , B and C .

Section summary

- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.
- **Theorem 1** The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
- **Theorem 2** Angles in the same segment of a circle are equal.
- **Theorem 3** The angle subtended by a diameter at the circumference is 90° .
- **Theorem 4** Opposite angles of a cyclic quadrilateral sum to 180° .

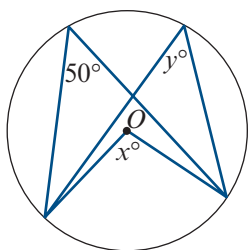
Exercise 10A

Skillsheet

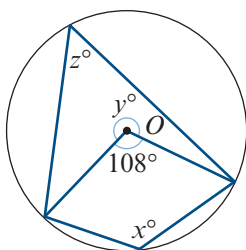
- 1 Find the values of the pronumerals for each of the following, where O denotes the centre of the given circle:

Example 1

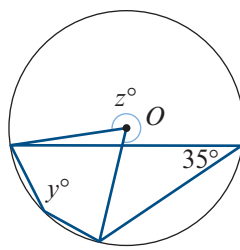
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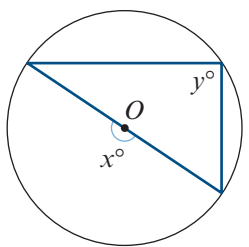
b



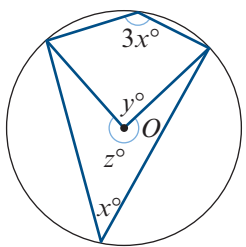
c



d

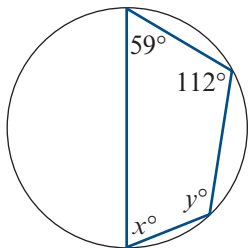


e

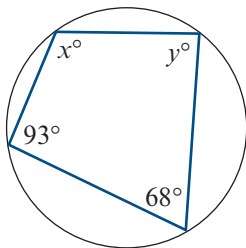


- 2 Find the values of the pronumerals for each of the following:

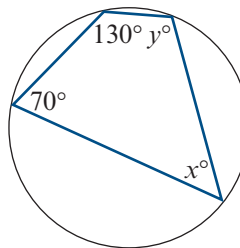
a



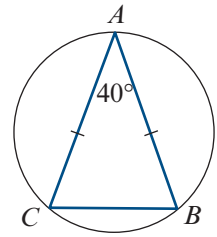
b



c



- Example 2** 3 An isosceles triangle ABC is inscribed in a circle. (Inscribed means that all the vertices of the triangle lie on the circle.) What are the angles in the three minor segments cut off by the sides of this triangle?

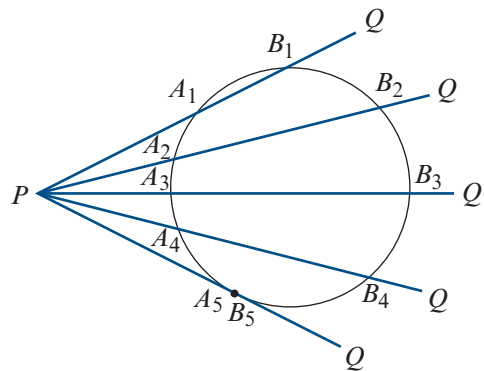


- 4 $ABCDE$ is a pentagon inscribed in a circle. If $AE = DE$, $\angle BDC = 20^\circ$, $\angle CAD = 28^\circ$ and $\angle ABD = 70^\circ$, find all the interior angles of the pentagon.

- Example 3** 5 Prove that if two opposite sides of a cyclic quadrilateral are equal, then the other two sides are parallel.
- 6 $ABCD$ is a parallelogram. The circle through the points A , B and C cuts CD (extended if necessary) at E . Prove that $AE = AD$.
- 7 $ABCD$ is a cyclic quadrilateral and O is the centre of the circle through A , B , C and D . If $\angle AOC = 120^\circ$, find the magnitude of $\angle ADC$.
- 8 $PQRS$ is a cyclic quadrilateral with $\angle SQR = 36^\circ$, $\angle PSQ = 64^\circ$ and $\angle RSQ = 42^\circ$. Find the interior angles of the quadrilateral.
- 9 Prove that if a parallelogram can be inscribed in a circle, then it must be a rectangle.
- 10 Prove that the bisectors of the four interior angles of a quadrilateral form a cyclic quadrilateral.

10B Tangents

Consider a point P outside a circle, as shown in the diagram. By rotating the secant PQ , with P as the pivot point, we obtain a sequence of pairs of points on the circle. As PQ moves towards the edge of the circle, the pairs of points become closer together, until they eventually coincide.



When PQ is in this final position (i.e. when the intersection points A and B coincide), it is called a **tangent** to the circle.

A tangent touches the circle at only one point, and this point is called the **point of contact**.

The **length of a tangent** from a point P outside the circle is the distance between P and the point of contact.

Theorem 5: Tangent is perpendicular to radius

A tangent to a circle is perpendicular to the radius drawn from the point of contact.

Proof This will be a proof by contradiction.

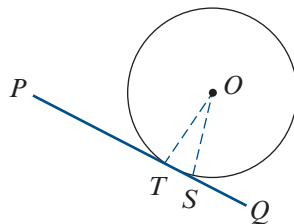
Let T be the point of contact of tangent PQ and suppose that $\angle OTP$ is not a right angle.

Let S be the point on PQ , not T , such that OSP is a right angle. Then triangle OST has a right angle at S .

Therefore $OT > OS$, as OT is the hypotenuse of triangle OST .

This implies that S is inside the circle, as OT is a radius.

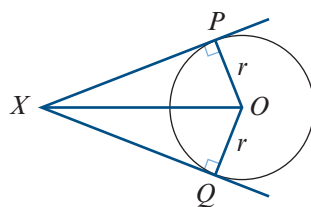
Thus the line through T and S must cut the circle again. But PQ is a tangent, and so this is a contradiction. Hence we have shown that $\angle OTP$ is a right angle.

**Theorem 6: Two tangents from the same point**

The two tangents drawn from an external point to a circle are the same length.

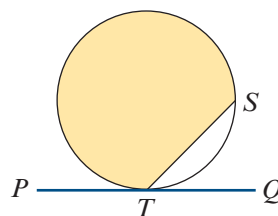
Proof We can see that $\triangle XPO$ is congruent to $\triangle XQO$ using the RHS test, as $\angle XPO = \angle XQO = 90^\circ$, the side XO is common and $OP = OQ$ (radii).

Therefore $XP = XQ$.

► **The alternate segment theorem**

In the diagram:

- The shaded segment is called the **alternate segment** in relation to $\angle STQ$.
- The unshaded segment is alternate to $\angle STP$.

**Theorem 7: Alternate segment theorem**

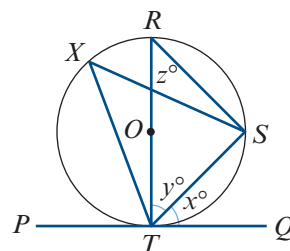
The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.

Proof Let $\angle STQ = x^\circ$, $\angle RTS = y^\circ$ and $\angle TRS = z^\circ$, where RT is a diameter.

Then $\angle RST = 90^\circ$ (Theorem 3, angle subtended by a diameter), and therefore $y + z = 90$.

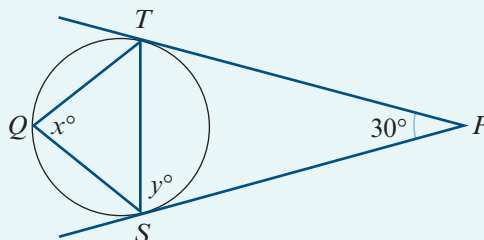
Also $\angle RTQ = 90^\circ$ (Theorem 5, tangent is perpendicular to radius), and therefore $x + y = 90$.

Thus $x = z$. But $\angle TXS$ is in the same segment as $\angle TRS$ and so $\angle TXS = x^\circ$.



Example 4

Find the magnitudes of the angles x and y in the diagram.

**Solution**

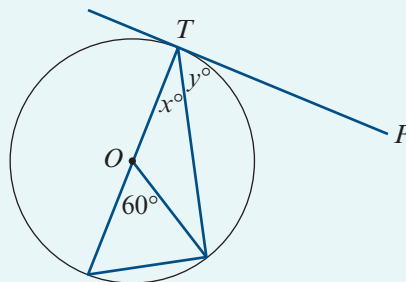
Triangle PST is isosceles (Theorem 6, two tangents from the same point).

Therefore $\angle PST = \angle PTS$ and so $y = 75$.

The alternate segment theorem gives $x = y = 75$.

**Example 5**

Find the values of x and y , where PT is tangent to the circle centre O .

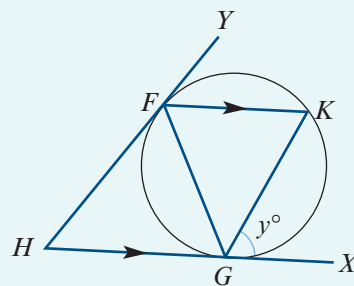
**Solution**

$x = 30$ as the angle at the circumference is half the angle subtended at the centre, and so

$y = 60$ as $\angle OTP$ is a right angle.

Example 6

The tangents to a circle at F and G meet at H . A chord FK is drawn parallel to HG . Prove that triangle FGK is isosceles.

**Solution**

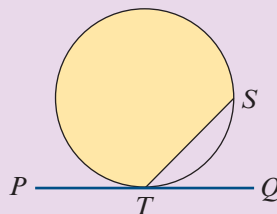
Let $\angle XGK = y^\circ$.

Then $\angle GFK = y^\circ$ (alternate segment theorem) and $\angle GKF = y^\circ$ (alternate angles).

Therefore triangle FGK is isosceles with $FG = KG$.

Section summary

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.
- The two tangents drawn from an external point to a circle are the same length.
- In the diagram, the **alternate segment** to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.
- **Alternate segment theorem** The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



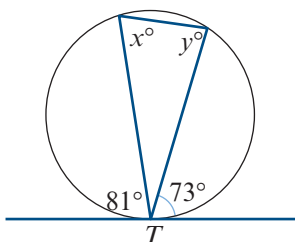
Exercise 10B

Skillsheet

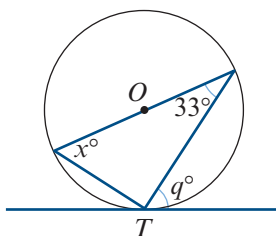
- 1 Find the values of the pronumerals for each of the following, where T is the point of contact of the tangent and O is the centre of the circle:

Example 4

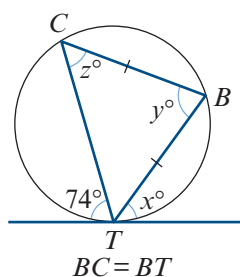
a



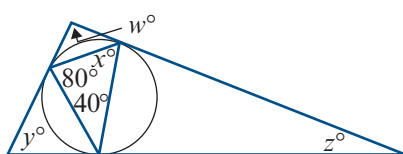
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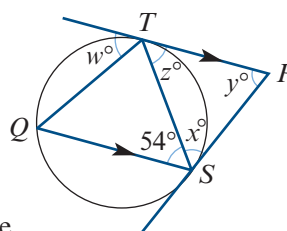
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d



e

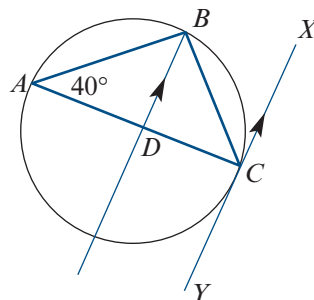


Note: In the diagram for part e, the two tangents from P have points of contact at S and T , and TP is parallel to QS .

Example 5

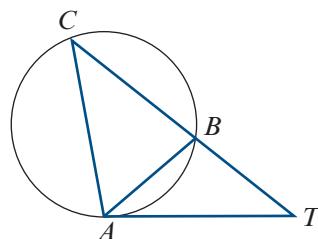
- 2 A triangle ABC is inscribed in a circle, and the tangent to the circle at C is parallel to the bisector of angle ABC .

- Find the magnitude of $\angle BCX$.
- Find the magnitude of $\angle CBD$, where D is the point of intersection of the bisector of angle ABC with AC .
- Find the magnitude of $\angle ABC$.



- 3 Assume that AB and AC are two tangents to a circle, touching the circle at B and C , and that $\angle BAC = 116^\circ$. Find the magnitudes of the angles in the two segments into which BC divides the circle.

- 4 AT is a tangent at A and TBC is a secant to the circle. Given that $\angle CTA = 30^\circ$ and $\angle CAT = 110^\circ$, find the magnitude of angles ACB , ABC and BAT .

**Example 6**

- 5 From a point A outside a circle, a secant ABC is drawn cutting the circle at B and C , and a tangent AD touching it at D . A chord DE is drawn equal in length to chord DB . Prove that triangles ABD and CDE are similar.
- 6 Assume that AB is a chord of a circle and that CT , the tangent at C , is parallel to AB . Prove that $CA = CB$.
- 7 Through a point T , a tangent TA and a secant TPQ are drawn to a circle APQ . The chord AB is drawn parallel to PQ . Prove that the triangles PAT and BAQ are similar.
- 8 PQ is a diameter of a circle and AB is a perpendicular chord cutting it at N . Prove that PN is equal in length to the perpendicular from P onto the tangent at A .

10C Chords in circles

Theorem 8

If AB and CD are two chords of a circle that cut at a point P (which may be inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.

Proof Case 1: The intersection point P is inside the circle.

Consider triangles APC and DPB :

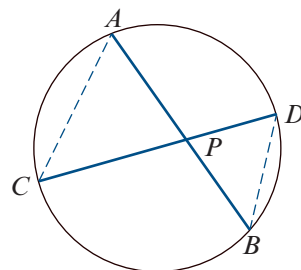
$$\angle APC = \angle DPB \quad (\text{vertically opposite})$$

$$\angle CAB = \angle BDC \quad (\text{angles in the same segment})$$

Thus triangle APC is similar to triangle DPB . This gives

$$\frac{PA}{PD} = \frac{PC}{PB}$$

$$\therefore PA \cdot PB = PC \cdot PD$$



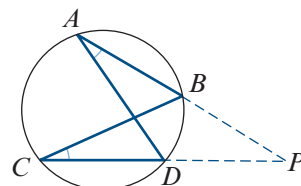
Case 2: The intersection point P is outside the circle.

Show that triangle APD is similar to triangle CPB .

This gives

$$\frac{PA}{PC} = \frac{PD}{PB}$$

$$\therefore PA \cdot PB = PC \cdot PD$$



A converse of Theorem 8 is:

If line segments AB and CD intersect at a point M and $AM \cdot BM = CM \cdot DM$, then the points A, B, C and D are concyclic.

This is proved in Extended-response question 2.

Theorem 9

If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant, then $PT^2 = PA \cdot PB$.

Proof Consider triangles PAT and PTB :

$$\angle ATP = \angle TBA \quad (\text{alternate segment theorem})$$

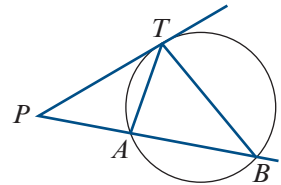
$$\angle PAT = \angle PTB \quad (\text{angle sum of a triangle})$$

Therefore triangle PAT is similar to triangle PTB .

This gives

$$\frac{PA}{PT} = \frac{PT}{PB}$$

$$\therefore PT^2 = PA \cdot PB$$



Example 7

The arch of a bridge is to be in the form of an arc of a circle. The span of the bridge is to be 25 m and the height in the middle 2 m. Find the radius of the circle.

Solution

Let r be the radius of the circle. Then $PQ = 2r - 2$.

Use Theorem 8 with the chords RQ and MN :

$$RP \cdot PQ = MP \cdot PN$$

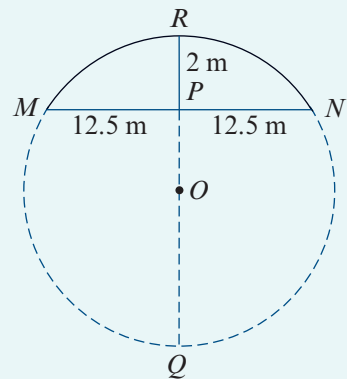
Therefore

$$2PQ = 12.5^2$$

$$PQ = \frac{12.5^2}{2}$$

$$2r - 2 = \frac{12.5^2}{2} \quad \text{as } PQ = 2r - 2$$

$$\begin{aligned} \therefore r &= \frac{1}{2} \left(\frac{12.5^2}{2} + 2 \right) \\ &= \frac{641}{16} \text{ m} \end{aligned}$$



**Example 8**

Let A be any point inside a circle with radius r and centre O . Show that, if CD is a chord through A , then $CA \cdot AD = r^2 - OA^2$.

Solution

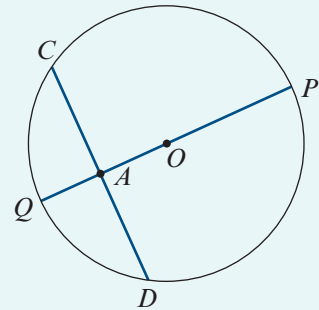
Let PQ be a diameter through A as shown.

By Theorem 8:

$$CA \cdot AD = QA \cdot AP$$

Since $QA = r - OA$ and $AP = r + OA$, this gives

$$\begin{aligned} CA \cdot AD &= (r - OA)(r + OA) \\ &= r^2 - OA^2 \end{aligned}$$

**Section summary**

- **Theorem 8** If AB and CD are two chords of a circle that cut at a point P (which may be inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.
- **Theorem 9** If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant, then $PT^2 = PA \cdot PB$.

Exercise 10C**Skillsheet**

- 1 Two chords AB and CD intersect at a point P within a circle.

Example 7

- a Given that $AP = 5$ cm, $PB = 4$ cm, $CP = 2$ cm, find PD .
- b Given that $AP = 4$ cm, $CP = 3$ cm, $PD = 8$ cm, find PB .

- 2 If AB is a chord and P is a point on AB such that $AP = 8$ cm, $PB = 5$ cm and P is 3 cm from the centre of the circle, find the radius.
- 3 If AB is a chord of a circle with centre O and P is a point on AB such that $BP = 4PA$, $OP = 5$ cm and the radius of the circle is 7 cm, find AB .

Example 8

- 4 Two circles intersect at points A and B . From any point P on the line AB , tangents PQ and PR are drawn to the circles. Prove that $PQ = PR$.
- 5 PQ is a variable chord of the smaller of two fixed concentric circles, and PQ extended meets the circumference of the larger circle at R . Prove that the product $RP \cdot RQ$ is constant for all positions and lengths of PQ .

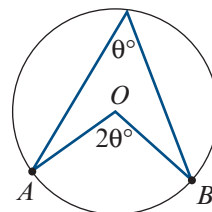


- 6 ABC is an isosceles triangle with $AB = AC$. A line through A meets BC at D and the circumcircle of the triangle at E . Prove that $AB^2 = AD \cdot AE$.

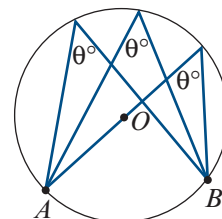
Chapter summary



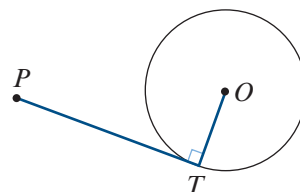
- The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



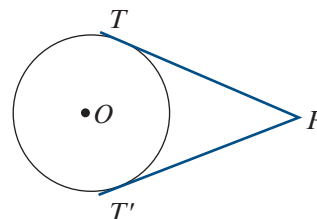
- Angles in the same segment of a circle are equal.



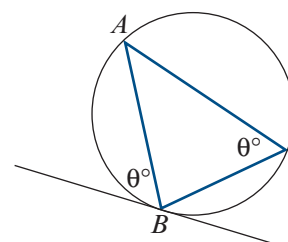
- A tangent to a circle is perpendicular to the radius drawn from the point of contact.



- The two tangents drawn from an external point are the same length, i.e. $PT = PT'$.



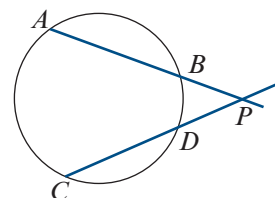
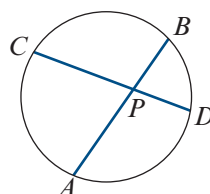
- The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



- A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180° .

- If AB and CD are two chords of a circle that cut at a point P , then

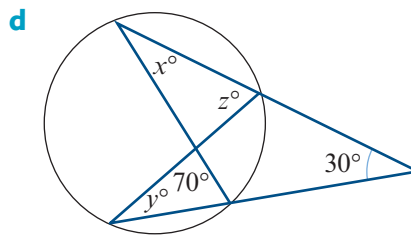
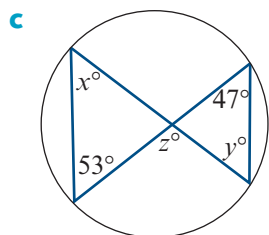
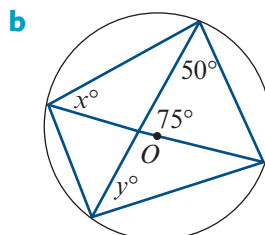
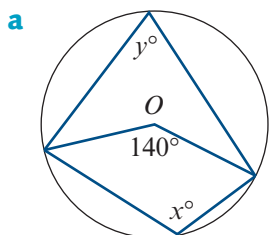
$$PA \cdot PB = PC \cdot PD$$



Technology-free questions

- 1 $\triangle ABC$ has $\angle A = 36^\circ$ and $\angle C = 90^\circ$. M is the midpoint of AB and CN is the altitude on AB . Find the size of $\angle MCN$.

- 2 Find the values of the pronumerals in each of the following:



- 3 Let OP be a radius of a circle with centre O . A chord BA is drawn parallel to OP . The lines OA and BP intersect at C . Prove that:

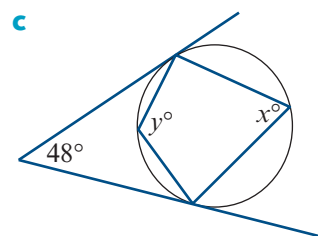
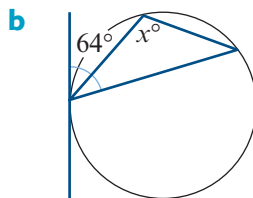
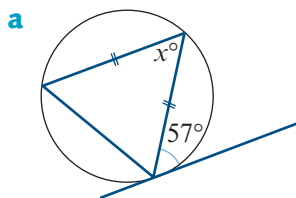
a $\angle CAB = 2\angle CBA$

b $\angle PCA = 3\angle PBA$

- 4 A chord AB of a circle, centre O , is extended to C . The straight line bisecting $\angle OAB$ meets the circle at E . Prove that EB bisects $\angle OBC$.

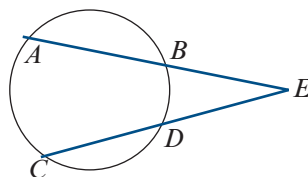
- 5 Two circles intersect at A and B . The tangent at B to one circle meets the second again at D , and a straight line through A meets the first circle at P and the second at Q . Prove that BP is parallel to DQ .

- 6 Find the values of the pronumerals for each of the following:



- 7 Two circles intersect at M and N . The tangent to the first circle at M meets the second circle at P , while the tangent to the second at N meets the first at Q . Prove that $MN^2 = NP \cdot QM$.

- 8 If $AB = 10$ cm, $BE = 5$ cm and $CE = 25$ cm, find DE .

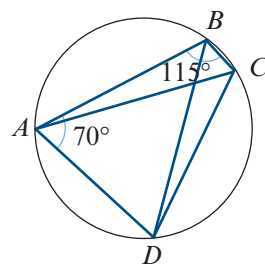


Multiple-choice questions



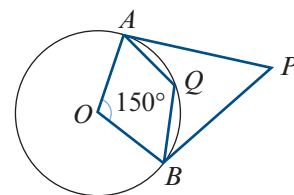
- 1 In the diagram, the points A , B , C and D lie on a circle, $\angle ABC = 115^\circ$, $\angle BAD = 70^\circ$ and $AB = AD$. The magnitude of $\angle ACD$ is

A 45° **B** 55° **C** 40° **D** 70° **E** 50°



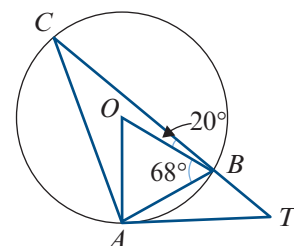
- 2 In the diagram, PA and PB are tangents to the circle centre O . Given that Q is a point on the minor arc AB and that $\angle AOB = 150^\circ$, the magnitudes of $\angle APB$ and $\angle AQB$ are

A $\angle APB = 30^\circ$ and $\angle AQB = 105^\circ$
B $\angle APB = 40^\circ$ and $\angle AQB = 110^\circ$
C $\angle APB = 25^\circ$ and $\angle AQB = 105^\circ$
D $\angle APB = 30^\circ$ and $\angle AQB = 110^\circ$
E $\angle APB = 25^\circ$ and $\angle AQB = 100^\circ$



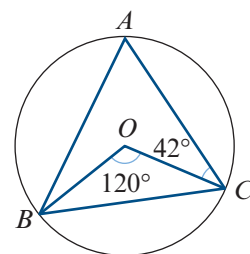
- 3 A circle with centre O passes through A , B and C . The line AT is the tangent to the circle at A , and CBT is a straight line. Given that $\angle ABO = 68^\circ$ and $\angle OBC = 20^\circ$, the magnitude of $\angle ATB$ is

A 60° **B** 64° **C** 65° **D** 70° **E** 66°

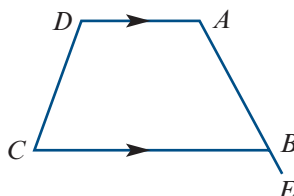


- 4 In the diagram, the points A , B and C lie on a circle centre O . If $\angle BOC = 120^\circ$ and $\angle ACO = 42^\circ$, then the magnitude of $\angle ABO$ is

A 18° **B** 20° **C** 22° **D** 24° **E** 26°



- 5 $ABCD$ is a cyclic quadrilateral with AD parallel to BC , and $\angle DCB = 65^\circ$. The magnitude of $\angle CBE$ is
- A** 100° **B** 110° **C** 115° **D** 120° **E** 122°



- 6** A chord AB of a circle subtends an angle of 50° at a point on the circumference of the circle. The acute angle between the tangents at A and B has magnitude

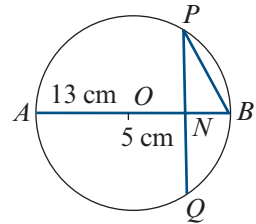
A 80° **B** 65° **C** 75° **D** 85° **E** 82°

- 7** Chords AB and CD of a circle intersect at P . If $AP = 12$ cm, $PB = 6$ cm and $CP = 2$ cm, then the length of PD in centimetres is

A 12 **B** 24 **C** 36 **D** 48 **E** 56

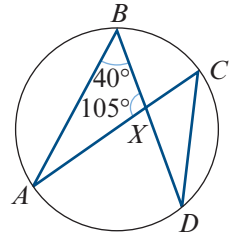
- 8** In the diagram, AB is the diameter of a circle with centre O and radius 13 cm. The chord PQ is perpendicular to AB , and N is the point of intersection of AB and PQ , with $ON = 5$ cm. The length of chord PB , in centimetres, is

A 12 **B** $4\sqrt{13}$ **C** $2\sqrt{13}$
D 14 **E** 8



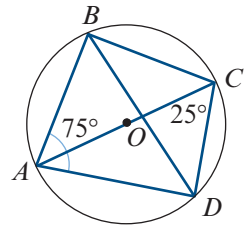
- 9** A, B, C and D are points on a circle, with $\angle ABD = 40^\circ$ and $\angle AXB = 105^\circ$. The magnitude of $\angle XDC$ is

A 35° **B** 40° **C** 45° **D** 50° **E** 55°



- 10** A, B, C and D are points on a circle, centre O , such that AC is a diameter of the circle. If $\angle BAD = 75^\circ$ and $\angle ACD = 25^\circ$, then the magnitude of $\angle BDC$ is

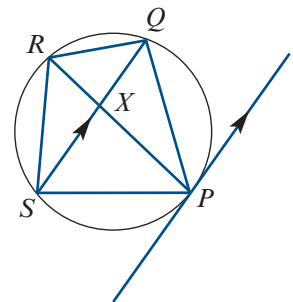
A 10° **B** 15° **C** 20° **D** 25° **E** 30°



Extended-response questions

- 1** The diagonals PR and QS of a cyclic quadrilateral $PQRS$ intersect at X . The tangent at P is parallel to QS . Prove that:

- a** $PQ = PS$
b PR bisects $\angle QRS$



- 2** If line segments AB and CD intersect at a point M and $AM \cdot BM = CM \cdot DM$, then the points A, B, C and D are concyclic.

To prove this claim, show that:

a $\frac{AM}{CM} = \frac{DM}{BM}$

b $\triangle AMC \sim \triangle DMB$

c $\angle CAM = \angle BDM$

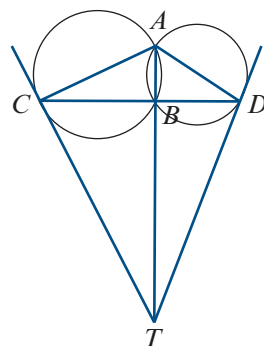
d $ABCD$ is cyclic

- 3** Two circles intersect at A and B . The tangents at C and D intersect at T on the extension of AB . Prove that, if CBD is a straight line, then:

a $TCAD$ is a cyclic quadrilateral

b $\angle TAC = \angle TAD$

c $TC = TD$.



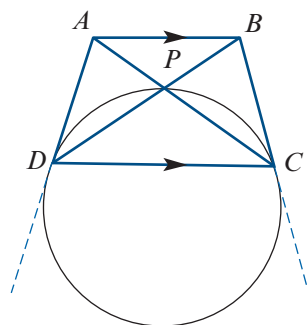
- 4** $ABCD$ is a trapezium in which AB is parallel to DC and the diagonals meet at P . The circle through D, P and C touches AD and BC at D and C respectively.

Prove that:

a $\angle BAC = \angle ADB$

b the circle through A, P and D touches BA at A

c $ABCD$ is a cyclic quadrilateral.



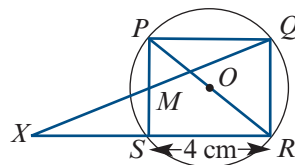
- 5** $PQRS$ is a square of side length 4 cm inscribed in a circle with centre O . The midpoint of the side PS is M . The line segments QM and RS are extended to meet at X .

a Prove that:

i $\triangle XPR$ is isosceles

ii PX is the tangent to the circle at P .

b Calculate the area of trapezium $PQRX$.



- 6** **a** An isosceles triangle ABC , with $AB = AC$, is inscribed in a circle. A chord AD intersects BC at E . Prove that

$$AB^2 - AE^2 = BE \cdot CE$$

b Diameter AB of a circle with centre O is extended to C , and from C a line is drawn tangent to the circle at P . The line PT is drawn perpendicular to AB at T . Prove that

$$CA \cdot CB - TA \cdot TB = CT^2$$

11

Revision of Chapters 6–10

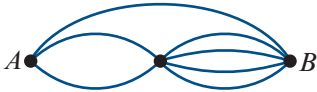
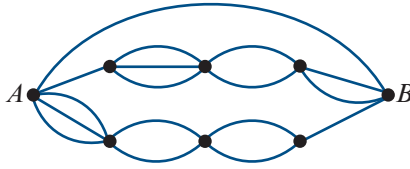
11A Technology-free questions

- 1 Let $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j}$.
 - a Find $|\mathbf{a}|$.
 - b Find the unit vector in the direction of $\mathbf{a}$.
 - c Write down a vector of magnitude 8 that has the same direction as $\mathbf{a}$.
 - d Write down a vector of magnitude 2 that has the opposite direction to $\mathbf{a}$.
- 2 Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c} = -3\mathbf{i} - 2\mathbf{j}$. Find:

a $\mathbf{a} \cdot \mathbf{a}$	b $\mathbf{b} \cdot \mathbf{b}$	c $\mathbf{c} \cdot \mathbf{c}$	d $\mathbf{a} \cdot \mathbf{b}$
e $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$	f $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{c})$	g $(\mathbf{a} + 2\mathbf{b}) \cdot (3\mathbf{c} - \mathbf{b})$	
- 3 The points A, B, C and D have position vectors $\overrightarrow{OA} = 4\mathbf{i} + 2\mathbf{j}$, $\overrightarrow{OB} = -\mathbf{i} + 7\mathbf{j}$, $\overrightarrow{OC} = 8\mathbf{i} + 6\mathbf{j}$ and $\overrightarrow{OD} = p\mathbf{i} - 2\mathbf{j}$.
 - a Find the values of m and n such that $m\overrightarrow{OA} + n\overrightarrow{BC} = 2\mathbf{i} + 10\mathbf{j}$.
 - b Find the value of p such that $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{CD}$.
 - c Find the values of p such that $|\overrightarrow{AD}| = \sqrt{17}$.
- 4 A particle starts at a point A . It moves with velocity $\mathbf{v}_1 = [4, 30^\circ]$ m/s for 2 seconds and then moves with velocity $\mathbf{v}_2 = [6, -60^\circ]$ m/s for 3 seconds, finishing at the point B .
 - a Express the velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in component form.
 - b Express the particle's displacement vector $\overrightarrow{AB}$ in component form.
- 5 A motorboat heads due east at 16 m/s across a river that flows due north at 9 m/s.
 - a What is the resultant velocity of the boat?
 - b If the river is 136 m wide, how long does it take the boat to cross the river?
 - c How far downstream is the boat when it reaches the other side of the river?

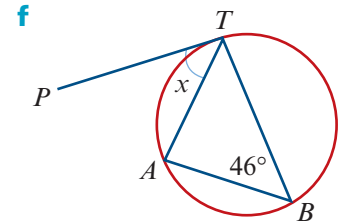
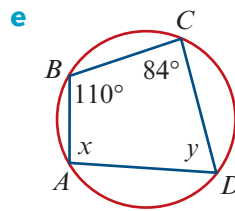
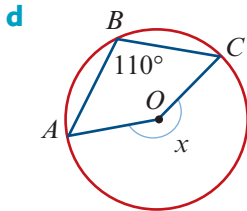
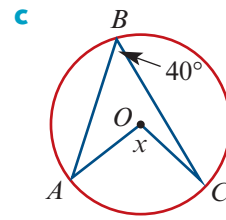
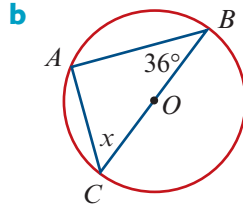
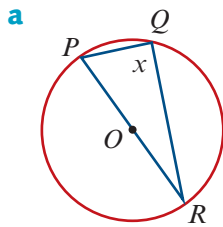
SF

CF

- 6** Two forces of equal magnitude F kg wt act on a particle and they have a resultant force of magnitude 6 kg wt. When one of the forces is doubled in magnitude, the resultant force is 11 kg wt. Find the value of F and the cosine of the angle between the two forces.
- 7** A particle in equilibrium is being acted on by three forces F_1 , F_2 and F_3 in the same plane. Given that $F_1 = [6, 45^\circ]$ kg wt and $F_2 = [8, -90^\circ]$ kg wt, find the square of the magnitude of F_3 .
- 8** A block of mass 10 kg is maintained at rest on a smooth plane inclined at 30° to the horizontal by a string. Calculate the tension in the string and the reaction of the plane if:
- a** the string is parallel to the plane **b** the string is horizontal.
- 9** A mass of 10 kg is suspended by two strings of lengths 5 cm and 12 cm that are attached to fixed points on the same horizontal level 13 cm apart. Find the tension in each string.
- 10** How many ways can four different books be arranged on a shelf?
- 11** How many ways can three teachers and three students be arranged in a row if a teacher must be at the start of the row?
- 12** How many different three-digit numbers can be formed using the digits 1, 3, 5, 7 and 9:
- a** as many times as you would like **b** at most once?
- 13** Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?
- a**
- 
- b**
- 
- 14** Evaluate each of the following:
- a** $4!$ **b** $\frac{6!}{4!}$ **c** $\frac{8!}{6! 2!}$ **d** ${}^{10}C_2$
- 15** How many ways can five children be arranged on a bench with space for:
- a** four children **b** five children?
- 16** A bookshelf has three different mathematics books and two different physics books. How many ways can these books be arranged:
- a** without restriction **b** if the mathematics books are kept together?
- 17** Using the digits 0, 1, 2, 3 and 4 without repetition, how many five-digit numbers can you form:
- a** without restriction **b** that are divisible by 10
- c** that are greater than 20 000 **d** that are even?

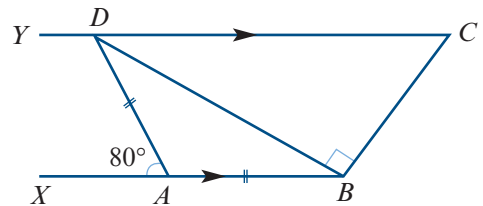
- 18** Asha has three identical 20 cent pieces and two identical 10 cent pieces. How many ways can she arrange these coins in a row? CF
- 19** How many ways can you select:
- a** three children from a group of six
 - b** two letters from the alphabet
 - c** four numbers from the set $\{1, 2, \dots, 10\}$
 - d** three sides of an octagon?
- 20** Consider the set of numbers $X = \{1, 2, \dots, 8\}$.
- a** How many subsets of X have exactly two elements?
 - b** How many subsets of X have exactly three elements, one of which is the number 8?
 - c** Find the total number of subsets of X .
- 21** A team of five students is selected randomly from a group of five boys and four girls. What is the probability that the team consists of three boys and two girls?
- 22** There are four Labor and five Liberal parliamentarians, from which four are to be selected to form a committee. If the committee must include at least one member from each party, how many ways can this be done?
- 23** There are 10 blue, 11 green and 12 red balls in a bag. How many balls must be chosen at random to be sure that at least three will have the same colour?
- 24** How many different natural numbers from 1 to 99 inclusive must be chosen at random to be sure there will be at least one pair of numbers that sums to 100?
- 25** How many integers from 1 to 120 inclusive are divisible by 2 or 3?
- 26** Suppose that n is odd. Prove that $n^2 + n$ is even. SF
- 27** Prove that if m and n are consecutive integers, then $n^2 - m^2 = n + m$.
- 28** Let $n \in \mathbb{Z}$. Consider the statement: If $5n + 3$ is even, then n is odd.
- a** Write down the converse statement.
 - b** Prove the converse.
 - c** Write down the contrapositive statement.
 - d** Prove the contrapositive.
- 29** Suppose the number x is irrational. Prove by contradiction that $x + 1$ is also irrational. CF
- 30** Prove by contradiction that 6 cannot be written as the difference of two perfect squares.
- 31** Let $n \in \mathbb{Z}$. Prove that $3n + 1$ is even if and only if n is odd.
- 32** Prove that each of the following statements is false by finding a counterexample:
- a** The sum of two prime numbers cannot be a prime number.
 - b** For all $x \in \mathbb{R}$, we have $x^3 > x^2$.
- 33** Show this statement is false: There exists $n \in \mathbb{N}$ such that $25n^2 - 9$ is a prime number.
- 34** Prove by mathematical induction that:
- a** $2 + 4 + \dots + 2n = n(n + 1)$
 - b** $11^n - 6$ is divisible by 5, for all $n \in \mathbb{N}$ SF

35 Find the values of x and y in each of the following diagrams, giving reasons.



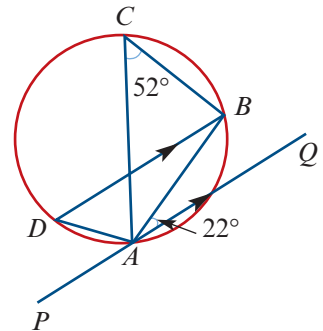
36 In the diagram, XB and YC are parallel. Given that $AD = AB$, $\angle XAD = 80^\circ$ and $\angle DBC = 90^\circ$, find:

- a** $\angle ABD$ **b** $\angle BDY$ **c** $\angle BCD$

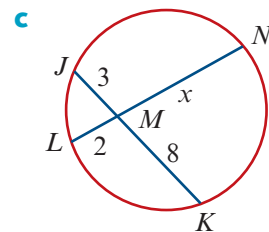
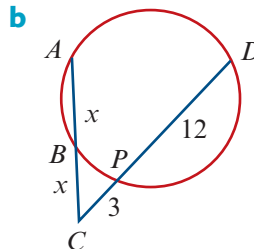
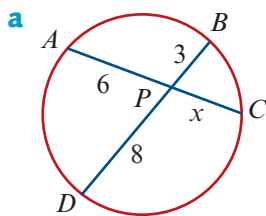


37 In the diagram, AC is a diameter of the circle, $PQ \parallel DB$, $\angle BCA = 52^\circ$ and $\angle BAQ = 22^\circ$. Find:

- a** $\angle CAB$ **b** $\angle PAD$ **c** $\angle CBD$



38 In each of the following, find the value of x :



39 In quadrilateral $ABCD$, $AB = 9$ cm, $BC = 6$ cm, $CD = 3$ cm, $DA = 6$ cm and $\angle CDA = 90^\circ$. Prove that $\angle ACB$ is a right angle.

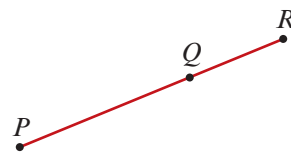
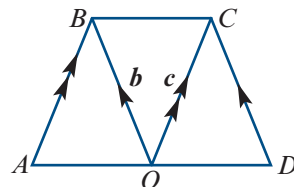
40 PN is an altitude of $\triangle PQR$, which has a right angle at P . Given that $PN = 60$ cm and $PR = 65$ cm, find:

- a** PQ **b** QN **c** NR

- 41** $ABCD$ is a square and X is a point outside the square such that $\triangle AXB$ is equilateral. Find the size of $\angle ACX$.
- 42** Prove that the sum of the interior angles of an 11-sided polygon is three times that of a pentagon.
- 43** $ABCD$ is a square and the bisector of $\angle ACD$ cuts BD at Q . Prove that $BQ = CD$.
- 44** The sides of a triangle are 28 cm, 45 cm and 54 cm. Is it an obtuse angle or not?
Hint: Consider the right-angled triangle with shorter sides 28 cm and 45 cm.

11B Multiple-choice questions

- 1** The magnitude of vector $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ is
A $\sqrt{29}$ **B** 29 **C** 21 **D** $\sqrt{21}$ **E** 7
- 2** The unit vector in the direction of vector $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ is
A $\mathbf{i} - \mathbf{j}$ **B** $\frac{1}{5}(3\mathbf{i} - 4\mathbf{j})$ **C** $\mathbf{i} + \mathbf{j}$ **D** $\frac{1}{25}(3\mathbf{i} - 4\mathbf{j})$ **E** $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
- 3** If $\vec{OA} = 2\mathbf{i} - 4\mathbf{j}$ and $\vec{OB} = 3\mathbf{i} + 4\mathbf{j}$, then $\vec{AB}$ equals
A $5\mathbf{i}$ **B** $-\mathbf{i} - 8\mathbf{j}$ **C** $-5\mathbf{i}$ **D** $\mathbf{i} + 8\mathbf{j}$ **E** $\mathbf{i}$
- 4** If $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j}$, then $\mathbf{a} - 2\mathbf{b}$ equals
A $8\mathbf{i}$ **B** $-4\mathbf{i} + 8\mathbf{j}$ **C** $-\mathbf{i} + 6\mathbf{j}$ **D** $5\mathbf{i} + 2\mathbf{j}$ **E** $\mathbf{i} - 6\mathbf{j}$
- 5** In the diagram, AB is parallel to OC , DC is parallel to OB ,
 $\mathbf{b} = \vec{OB}$, $\mathbf{c} = \vec{OC}$ and $AB = OB = OC = DC$.
 Vector $\vec{AD}$ is equal to
A $\mathbf{b} + \mathbf{c}$ **B** $2(\mathbf{c} - \mathbf{b})$ **C** $2(\mathbf{b} - \mathbf{c})$
D $2\mathbf{b} + 2\mathbf{c}$ **E** $|\mathbf{b} + \mathbf{c}|$
- 6** Expressing the vector $[7, 20^\circ]$ in component form, with values correct to two decimal places, gives
A $7\mathbf{i} + 20\mathbf{j}$ **B** $6.58\mathbf{i} + 2.39\mathbf{j}$ **C** $2.39\mathbf{i} + 6.58\mathbf{j}$
D $0.94\mathbf{i} + 0.34\mathbf{j}$ **E** $0.34\mathbf{i} + 0.94\mathbf{j}$
- 7** PQR is a straight line and $PQ = 2QR$. If $\vec{OQ} = 2\mathbf{i} - 3\mathbf{j}$ and $\vec{OR} = \mathbf{i} + 2\mathbf{j}$, then $\vec{OP}$ could be equal to
A $4\mathbf{i} - 13\mathbf{j}$ **B** $3\mathbf{i} - \mathbf{j}$ **C** $2\mathbf{i} - 10\mathbf{j}$
D $3\mathbf{i} + \mathbf{j}$ **E** $\mathbf{i} - 5\mathbf{j}$



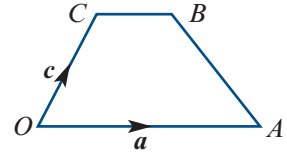
- 8** Let $\mathbf{u} = a\mathbf{i} - 5\mathbf{j}$ and $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j}$. Vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel when
A $a = -3$ **B** $a = \frac{5}{2}$ **C** $a = 3$ **D** $a = -\frac{5}{6}$ **E** $a = \frac{2}{5}$

- 9 Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{x} = \mathbf{i} + 5\mathbf{j}$. If $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$, then the scalars s and t are given by

A $s = -1$ and $t = -1$ **B** $s = -1$ and $t = 1$ **C** $s = 1$ and $t = -1$
D $s = 1$ and $t = 1$ **E** $s = 5$ and $t = 5$

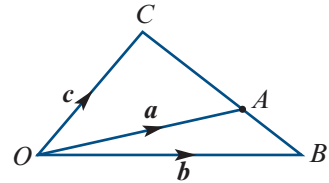
- 10 In this diagram, $OABC$ is a trapezium. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{c} = \overrightarrow{OC}$ and $\overrightarrow{OA} = 3\overrightarrow{CB}$, then $\overrightarrow{AB}$ equals

A $3\mathbf{c}$ **B** $\mathbf{c} - \frac{2}{3}\mathbf{a}$ **C** $3\mathbf{c} - 2\mathbf{a}$
D $\frac{2}{3}\mathbf{a} - \mathbf{c}$ **E** $\frac{4}{3}\mathbf{a} + \mathbf{c}$



- 11 In this diagram, $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$ and $AC : AB = 2 : 1$. The vector $\mathbf{c}$ is equal to

A $\mathbf{a} + 2\mathbf{b}$ **B** $3\mathbf{a} - 2\mathbf{b}$ **C** $2\mathbf{a} + \mathbf{b}$
D $2\mathbf{a} - \mathbf{b}$ **E** $3\mathbf{a} + \mathbf{b}$



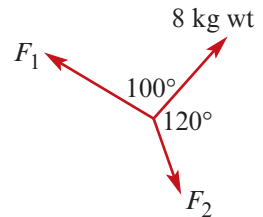
- 12 A bus and a car are on a straight level road that runs east–west. The bus is moving east at 20 m/s and the car is moving west at 20 m/s. If a man walks from the back to the front of the bus at 2 m/s, what is the velocity of the man relative to the car?
- A** 38 m/s east **B** 38 m/s west **C** 42 m/s east **D** 42 m/s west **E** 22 m/s east

Questions 13–14 refer to this system of forces, which is in equilibrium.

- 13 The magnitude of force F_1 is approximately
- A** 10.78 kg wt **B** 5.94 kg wt **C** 9.10 kg wt
D 12.26 kg wt **E** 7.04 kg wt

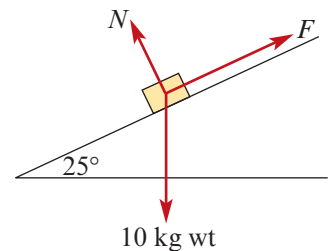
- 14 The magnitude of force F_2 is approximately

A 10.78 kg wt **B** 5.94 kg wt **C** 9.10 kg wt **D** 12.26 kg wt **E** 7.04 kg wt



Questions 15–16 refer to the following information:

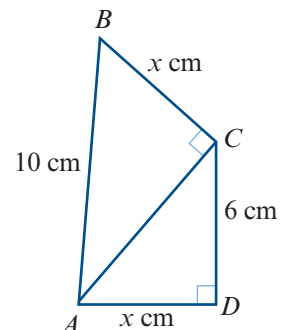
A 10 kg block is resting on a smooth plane inclined at 25° to the horizontal and is prevented from slipping down the plane by a string, as shown in the diagram.



- 15 The magnitude, N , of the normal reaction force is approximately
- A** 4.23 kg wt **B** 9.06 kg wt **C** 8.19 kg wt **D** 2.59 kg wt **E** 10 kg wt
- 16 The magnitude, F , of the tension in the string is approximately
- A** 4.23 kg wt **B** 9.06 kg wt **C** 8.19 kg wt **D** 2.59 kg wt **E** 10 kg wt

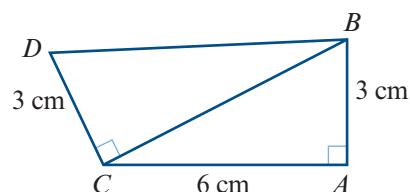
- 17** How many ways can five people be arranged in a line?
A $5!$ **B** 2^5 **C** 5^5 **D** ${}^5C_1$ **E** $5 + 4 + 3 + 2 + 1$
- 18** The letters of the word CAIRNS are arranged in a random order. What is the probability that the arrangement begins and ends with a vowel?
A $\frac{1}{3}$ **B** $\frac{1}{15}$ **C** $\frac{1}{30}$ **D** $\frac{1}{48}$ **E** $\frac{1}{720}$
- 19** How many four-digit numbers can be formed using the digits 1, 2, 3, 4 and 5 at most once?
A ${}^5C_4$ **B** $5 + 4 + 3 + 2$ **C** $5 \times 4 \times 3 \times 2$ **D** $4!$ **E** 5^4
- 20** The number of arrangements of the digits in the number 111222 is
A $\frac{6!}{3! \times 3!}$ **B** $6!$ **C** $\frac{6!}{3!}$ **D** $3! \times 3! \times 3!$ **E** $3! \times 3!$
- 21** Sam has n identical 10 cent pieces and n identical 20 cent pieces. How many ways can these coins be arranged in a row?
A $n! \times n!$ **B** $\frac{(2n)!}{(n!)^2}$ **C** $\frac{(n!)^2}{(2n)!}$ **D** $(2n)!$ **E** $2n!$
- 22** There are 10 flavours of ice-cream at a shop. Mark will select three flavours for his cone, one of which must be chocolate. The total number of different selections is
A ${}^{10}C_3$ **B** ${}^{10}C_2$ **C** ${}^9C_3$ **D** ${}^9C_2$ **E** ${}^8C_2$
- 23** There are four Labor and five Liberal parliamentarians, from which two of each are to be selected to form a committee. How many ways can this be done?
A ${}^9C_2$ **B** ${}^9C_4$ **C** ${}^9C_2 \times {}^9C_2$ **D** ${}^4C_2 \times {}^5C_2$ **E** ${}^4C_2 + {}^5C_2$
- 24** From 10 friends, you can invite any number of them to the movies. Assuming that you invite at least one friend, how many different selections can you make?
A 2^9 **B** $2^9 - 1$ **C** 2^{10} **D** $2^{10} - 1$ **E** ${}^{10}C_1$
- 25** An untidy kitchen drawer has a jumbled collection of eight knives, six forks and ten spoons. What is the smallest number of items that must be randomly chosen to ensure that at least four items of the same type are selected?
A 10 **B** 11 **C** 12 **D** 13 **E** 14
- 26** Whenever n integers are written on a whiteboard, at least six of them leave the same remainder when divided by 3. What is the smallest possible value of n ?
A 3 **B** 4 **C** 7 **D** 15 **E** 16
- 27** How many integers from 1 to 60 inclusive are multiples of 2 or 5?
A 32 **B** 36 **C** 40 **D** 44 **E** 50
- 28** Suppose that both m and n are odd. Which of the following statements is false?
A $m + n$ is even **B** $m - n$ is even **C** $3m + 5n$ is even
D $2m + n$ is odd **E** $mn + 1$ is odd

- 29** Suppose that m is divisible by 4 and n is divisible by 12. Which of the following statements might be false?
A $m \times n$ is divisible by 3 **B** $m \times n$ is divisible by 48 **C** $m + n$ is divisible by 4
D m^2n is divisible by 48 **E** n is divisible by m
- 30** Let m and n be integers. Which of the following statements is always true?
A If mn is even, then m is even.
B The number $m + n$ is even if and only if both m and n are even.
C If $m + n$ is odd, then mn is odd.
D If mn is odd, then $m + n$ is even.
E If $m + n$ is even, then $m - n$ is odd.
- 31** Consider the statement: If n is even, then $n + 3$ is odd. The converse of this statement is
A If $n + 3$ is even, then n is even. **B** If n is odd, then $n + 3$ is even.
C If $n + 3$ is odd, then n is even. **D** If $n + 3$ is odd, then n is odd.
E If $n + 3$ is even, then n is odd.
- 32** Assume that a and b are positive real numbers with $a > b$. Which of the following might be false?
A $\frac{1}{a-b} > 0$ **B** $\frac{a}{b} - \frac{b}{a} > 0$ **C** $a + b > 2b$ **D** $a + 3 > b + 2$ **E** $2a > 3b$
- 33** The number of pairs of integers (m, n) that satisfy $mn - n = 12$ is
A 2 **B** 3 **C** 4 **D** 6 **E** 12
- 34** Suppose that n is a positive integer. For how many values of n is the number $9n^2 - 4$ a prime?
A 0 **B** 1 **C** 2 **D** 3 **E** 4
- 35** If a, b, c and d are consecutive integers, then which of the following statements may be false?
A $a + b + c + d$ is divisible by 2 **B** $a + b + c + d$ is divisible by 4
C $a \times b \times c \times d$ is divisible by 3 **D** $a \times b \times c \times d$ is divisible by 8
E $a \times b \times c \times d$ is divisible by 24
- 36** In this diagram, angles ACB and ADC are right angles. If BC and AD each have a length of x cm, then x is equal to
A $2\sqrt{17}$ **B** 4 **C** 5
D $4\sqrt{2}$ **E** $5\sqrt{2}$



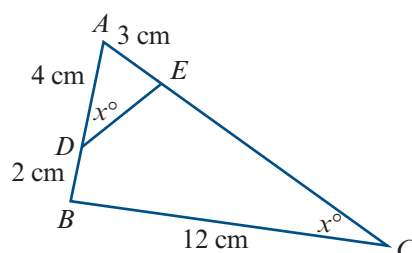
- 37** In this figure, the length of DB , in centimetres, is

A 6 **B** 9 **C** $3\sqrt{5}$
D $3\sqrt{6}$ **E** $3\sqrt{7}$



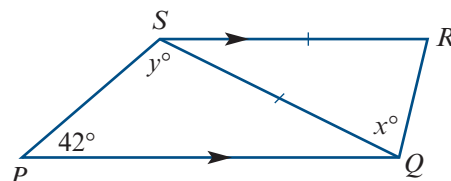
- 38** D and E are points on AB and AC , respectively. $AD = 4$ cm, $DB = 2$ cm, $AE = 3$ cm and $BC = 12$ cm. If $\angle ADE = \angle ACB$, then the length DE , in centimetres, is

A 6 **B** $\frac{9}{2}$ **C** 9 **D** 10 **E** 11



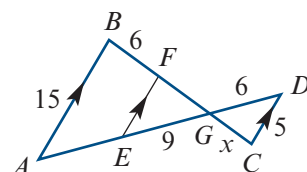
- 39** In this diagram, PQ and SR are parallel and $SR = SQ$. The angles x and y satisfy the equation

A $x = y$ **B** $x + y = 138$
C $2x + y = 42$ **D** $x = y + 42$
E $2x - y = 42$



- 40** In the figure, $AB = 15$, $CD = 5$, $BF = 6$, $GD = 6$ and $EG = 9$. The length x is equal to

A 3 **B** 4 **C** 4.5
D 4.75 **E** 5



- 41** A model car is 8 cm long and the real car is 3.2 m long. The scale factor is

A 1 : 8 **B** 1 : 32 **C** 1 : 24 **D** 1 : 400 **E** 1 : 40

- 42** A ladder rests against a wall, touching the wall at a height of 5.6 m. The bottom of the ladder is 2 m from the wall. The distance (to the nearest centimetre) that a person of height 1.6 m must be from the wall to just fit under the ladder is

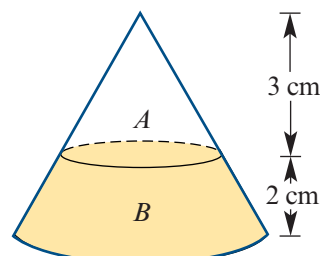
A 1.43 m **B** 0.57 m **C** 1.75 m **D** 0.25 m **E** 1.2 m

- 43** Let $\triangle ABC$ and $\triangle DEF$ be similar triangles such that $AB = 4$ cm and $DE = 10$ cm. If the area of $\triangle ABC$ is 24 cm², then the area of $\triangle DEF$ is

A 60 cm² **B** 240 cm² **C** 150 cm² **D** 96 cm² **E** none of these

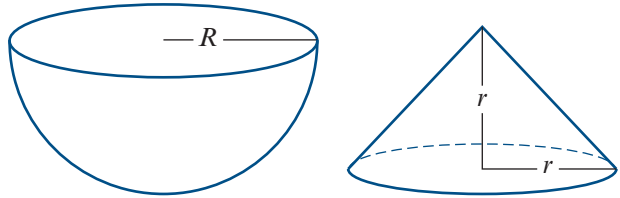
- 44** In the figure, the volume of the shaded solid B is 49 cm³. The volume of A is

A 19.5 cm³ **B** 17.3 cm³ **C** 13.5 cm³
D 12.5 cm³ **E** 10.5 cm³



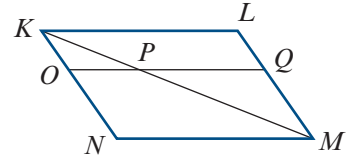
- 45** The ratio of the volume of the hemisphere to the volume of the right circular cone is $27 : 4$. Let R be the radius of the hemisphere and r the radius of the base of the cone. The ratio $R : r$ is equal to

A $1 : 2$ **B** $2 : 3$
C $3 : \sqrt{2}$ **D** $27 : 8$
E $3 : 2$



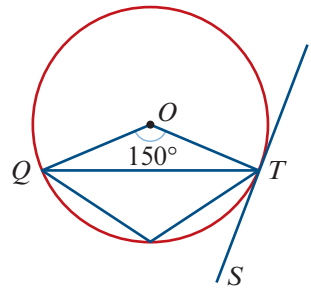
- 46** $KLMN$ is a parallelogram and OQ is parallel to KL . If O divides KN in the ratio $1 : 2$, then the ratio $\frac{\text{area } \triangle KOP}{\text{area } KLMN}$ is equal to

A $\frac{1}{4}$ **B** $\frac{1}{9}$ **C** $\frac{1}{12}$
D $\frac{1}{18}$ **E** $\frac{1}{20}$



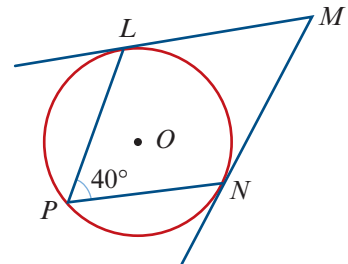
- 47** ST is a tangent at T to the circle with centre O . If $\angle QOT = 150^\circ$, then the magnitude of $\angle QTS$ is

A 70° **B** 75° **C** 95°
D 105° **E** 150°



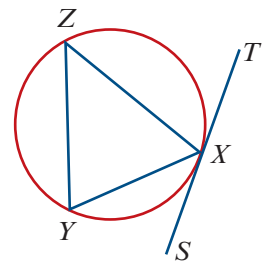
- 48** ML and MN are tangents to the circle at L and N . The magnitude of angle LMN is

A 80° **B** 90° **C** 100°
D 110° **E** 140°



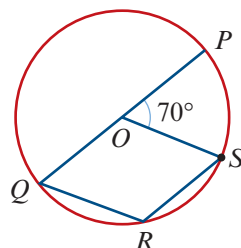
- 49** TS is a tangent at X and ZX bisects angle TXY . Given these facts, it can be proved that

A $YZ = XT$
B $YZ = XZ$
C $\angle YZX = \angle ZXT$
D $\angle SXY = \angle ZXY$
E $TX^2 = XY \cdot YZ$



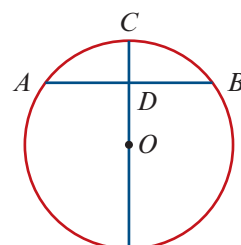
- 50** POQ is a diameter of the circle centre O . The size of angle QRS is

A 90° **B** 100° **C** 110°
D 125° **E** 160°



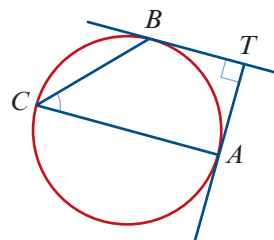
- 51** In the figure, O is the centre of the circle and D is the midpoint of AB . If $AB = 8$ cm and $CD = 2$ cm, the radius of the circle is

A 3 cm **B** 4 cm **C** 5 cm
D 6 cm **E** 7 cm



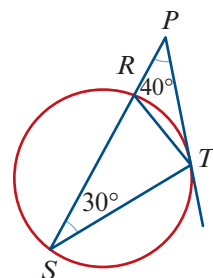
- 52** In the figure, TA and TB are tangents to the circle. If TA is perpendicular to TB and TA is perpendicular to AC , then the magnitude of $\angle BCA$ is

A 30° **B** 40° **C** 45°
D 55° **E** 65°



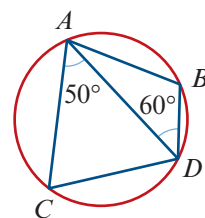
- 53** R , S and T are three points on the circumference of a circle, with $\angle RST$ equal to 30° . The tangent to the circle at T meets the line SR at P , and $\angle RPT$ is equal to 40° . The magnitude of $\angle RTS$ is

A 70° **B** 80° **C** 90°
D 100° **E** 110°



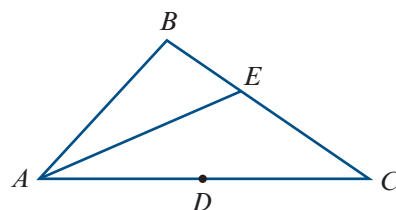
- 54** If $AB = AC$, $\angle ADB = 60^\circ$ and $\angle CAD = 50^\circ$, then $\angle ABD$ is equal to

A 80° **B** 90° **C** 100°
D 110° **E** 120°



11C Extended-response questions

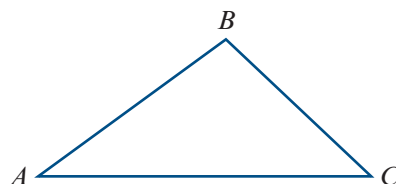
- 1 In the diagram, D is the midpoint of AC and E is the point on BC such that $BE : EC = 1 : t$, where $t > 0$. Suppose that DE is extended to a point F such that $DE : EF = 1 : 7$.



Let $\mathbf{a} = \overrightarrow{AD}$ and $\mathbf{b} = \overrightarrow{AB}$.

- Express $\overrightarrow{AE}$ in terms of t , $\mathbf{a}$ and $\mathbf{b}$.
- Express $\overrightarrow{AE}$ in terms of $\mathbf{a}$ and $\overrightarrow{AF}$.
- Show that $\overrightarrow{AF} = \frac{9-7t}{1+t}\mathbf{a} + \frac{8t}{1+t}\mathbf{b}$.
- If A , B and F are collinear, find the value of t .

- 2 The vertices A , B and C of a triangle have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, relative to an origin in the plane ABC .



- Let P be an arbitrary point on the line segment AB . Show that the position vector of P can be written in the form

$$m\mathbf{a} + n\mathbf{b}, \quad \text{where } m \geq 0, n \geq 0 \text{ and } m + n = 1$$

Hint: Assume that P divides AB in the ratio $x : y$.

- Find $\overrightarrow{PC}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
- Let Q be an arbitrary point on the line segment PC . Show that the position vector of Q can be written in the form

$$\lambda\mathbf{a} + \mu\mathbf{b} + \gamma\mathbf{c}, \quad \text{where } \lambda \geq 0, \mu \geq 0, \gamma \geq 0 \text{ and } \lambda + \mu + \gamma = 1$$

Note: The triple of numbers (λ, μ, γ) are known as the **barycentric coordinates** of the point Q in the triangle ABC .

- 3 $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = \frac{4}{5}\overrightarrow{OA}$ and Q is the midpoint of AB .

- Express $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
- The line segments PQ and OB are extended to meet at R , with $\overrightarrow{QR} = n\overrightarrow{PQ}$ and $\overrightarrow{BR} = k\mathbf{b}$. Express the vector $\overrightarrow{QR}$ in terms of:
 - n , $\mathbf{a}$ and $\mathbf{b}$
 - k , $\mathbf{a}$ and $\mathbf{b}$
- Find the values of n and k .

- 4 a** A man walks north at a rate of 4 km/h and notices that the wind *appears* to blow from the west. He doubles his speed and now the wind appears to blow from the north-west. What is the velocity of the wind?
Note: Both the direction and the magnitude must be given.
- b** A river 400 m wide flows from east to west at a steady speed of 1 km/h. A swimmer, whose speed in still water is 2 km/h, starts from the south bank and heads north across the river. Find the swimmer's speed over the river bed and how far downstream he is when he reaches the north bank.
- c** To a motorcyclist travelling due north at 50 km/h, the wind appears to come from the north-west at 60 km/h. What is the true velocity of the wind?
- d** A dinghy in distress is 6 km on a bearing of 230° from a lifeboat and is drifting in a direction of 150° at 5 km/h. In what direction should the lifeboat travel to reach the dinghy as quickly as possible if the maximum speed of the lifeboat is 35 km/h?
- 5** A five-digit number is formed using the digits 0, 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done:
- | | |
|-------------------------------|---|
| a without restriction | b if the number is divisible by 10 |
| c if the number is odd | d if the number is even? |
- 6** Mike and Sonia belong to a group of eight coworkers. There are three men and five women in this group. A team of four workers is required to complete a project. How many ways can the team be selected:
- | |
|--|
| a without restriction |
| b if it must contain two men and two women |
| c if it must contain both Mike and Sonia |
| d if it must not contain both Mike and Sonia? |
- 7** A sailing boat has three identical black flags and three identical red flags. The boat can send signals to nearby boats by arranging flags along its mast.
- | |
|---|
| a How many ways can all six flags be arranged in a row? |
| b How many ways can all six flags be arranged in a row if no two black flags are adjacent? |
| c Using at least one flag, how many arrangements in a row are possible? |
- 8** Consider the letters in the word BAGGAGE.
- | |
|--|
| a How many arrangements of these letters are there? |
| b How many arrangements begin and end with a vowel? |
| c How many arrangements begin and end with a consonant? |
| d How many arrangements have all vowels together and all consonants together? |

- 9** There are 25 people at a party.
- a** If every person shakes hands with every other person, what is the total number of handshakes?
 - b** In fact, there are two rival groups at the party, so everyone only shakes hands with every other person in their group. If there are 150 handshakes, how many people are in each of the rival groups?
 - c** At another party, there are 23 guests. Explain why it is not possible for each person to shake hands with exactly three other guests.
- 10** On a clock's face, twelve points are evenly spaced around a circle.
- a** How many ways can you select four of these points?
 - b** How many ways can you select two points that are not diametrically opposite?
 - c** For every selection of two points that are not diametrically opposite, you can draw one rectangle on the face that has these two points as vertices. What are the other two vertices?
 - d** How many ways can you select four points that are the vertices of a rectangle?
Hint: Why must you divide the answer to part **b** by 4?
 - e** Four points are randomly selected. What is the probability that the four points are the vertices of a rectangle?
- 11** Let a , b and c be integers. Suppose you know that $a + b$ is even and $b + c$ is odd.
- a** Is it possible to work out whether a , b and c are even or odd?
 - b** What if you also know that $a + b + c$ is even?
- 12** **a** Find all positive integer values of a , b and c such that $a < b < c$ and
- $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$
- b** Find all positive integer values of a , b , c and d such that $a < b < c < d$ and
- $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 2$$
- 13** Let a , b and c be positive real numbers. Prove that if $b > a$, then $\frac{a+c}{b+c} > \frac{a}{b}$.
- 14** **a** Find the smallest value of $n \in \mathbb{N}$ such that $2^n > 10^3$.
- b** Hence, prove that 2^{100} has at least 31 digits.
 - c** Hence, explain why some digit in the decimal expansion of 2^{100} occurs at least four times. (**Hint:** There are 10 different digits: 0, 1, ..., 9.)

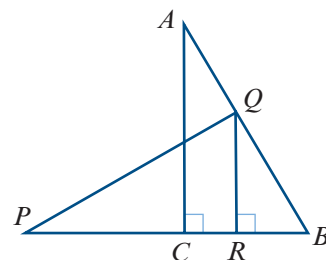
- 15** Take a close look at the following square numbers:

$$15^2 = 225, \quad 25^2 = 625, \quad 35^2 = 1225, \quad 45^2 = 2025, \quad 55^2 = 3025, \quad 65^2 = 4225$$

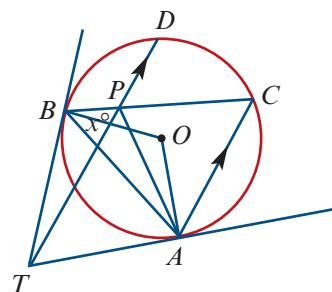
- Find and describe the pattern that you see in these square numbers.
 - Confirm that your pattern works for the number 75.
 - Prove that your pattern actually works. (*Hint:* Each number is of the form $10n + 5$.)
- 16** Heidi has 10 wooden cubes, with edges of length 1 cm through to 10 cm.
- Using all the cubes, can she build two towers of the same height?
 - Now Heidi has n wooden cubes, with edges of length 1 through to n . For what values of n can Heidi use all the cubes to build two towers of the same height?
- 17**
- Suppose that a is odd and b is odd. Prove that ab is odd.
 - Suppose that a is odd and $n \in \mathbb{N}$. Prove by induction that a^n is odd.
 - Hence, prove that if x satisfies $3^x = 2$, then x is irrational.
- 18**
- If $n^4 + 6n^3 + 11n^2 + 6n + 1 = (an^2 + bn + c)^2$, find the positive values of a , b and c .
 - Hence, prove that when 1 is added to the product of four consecutive integers, the result is always a perfect square.
 - Hence, write the number $5 \times 6 \times 7 \times 8 + 1$ as a product of prime numbers.

19 Proof of Pythagoras' theorem

In the diagram, the right-angled triangle ABC is congruent to the right-angled triangle PQR . The side PR is on the same line as BC , and the point Q is on AB . Let $AB = c$, $BC = a$ and $AC = b$.

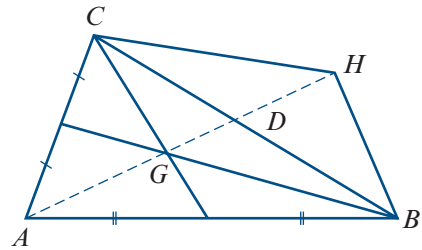


- Prove that $\triangle PBQ$ is similar to $\triangle ABC$.
 - Find PB in terms of a , b and c .
 - Prove that $\triangle QBR$ is similar to $\triangle ABC$.
 - Prove that $PB = b + \frac{a^2}{b}$.
 - Prove that $c^2 = a^2 + b^2$.
- 20** In the figure, O is the centre of a circle. TD and AC are parallel. TA and TB are tangents to the circle. Let $\angle BPT = x^\circ$.
- Prove that $TBOA$ is a cyclic quadrilateral.
 - Find $\angle BCA$, $\angle BOA$, $\angle TAB$ and $\angle TBA$ in terms of x .

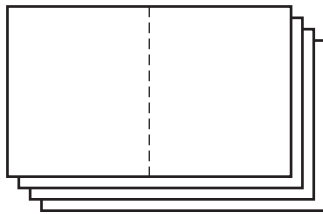


21 The medians of a triangle are concurrent.In $\triangle ABC$:

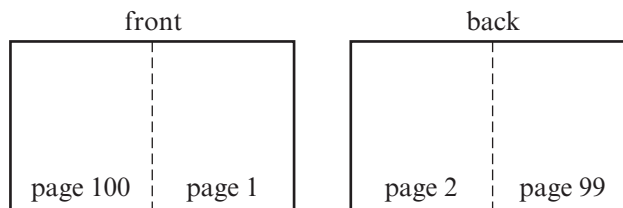
- E and F are the midpoints of AC and AB respectively
 - BE and CF intersect at G
 - the line segment AG is extended to H so that $AG = GH$
 - AH intersects BC at D .
- a Prove that $\triangle AFG$ is similar to $\triangle ABH$.
 - b Hence show $GC \parallel BH$.
 - c Prove that $GB \parallel CH$.
 - d Prove that $GBHC$ is a parallelogram and hence $BD = DC$.

**11D** Problem-solving and modelling questions

- 1** A stack of paper, printed on both sides, is folded in the middle to make a newspaper.



Each sheet contains four pages. The page numbers on the top sheet of Monday's newspaper are 1, 2, 99 and 100.



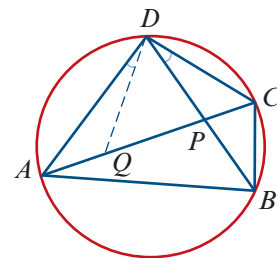
- a What are the page numbers on the bottom sheet of Monday's stack?
- b One of the sheets in Monday's newspaper has page numbers 7 and 8. What are its other two page numbers?
- c Suppose that a newspaper is made from n sheets of paper. Prove that the sum of the four page numbers on each sheet is a constant.
- d Tuesday's newspaper has a sheet whose pages are numbered 11, 12, 33 and 34. How many pages does this newspaper have?

- 2** Sam has 20 one-dollar coins and seven pockets. He wants to put coins into his pockets so that each pocket contains a different number of coins. (The number 0 is allowed.)
- Prove that this is impossible.
 - What is the minimum number of coins Sam would need to do this?
 - If Sam had 50 one-dollar coins, find the maximum number of pockets that he could fill, each with a different number of coins.
- 3** **Ptolemy's theorem:** *In a cyclic quadrilateral, the sum of the products of the opposite sides is equal to the product of the diagonals.*

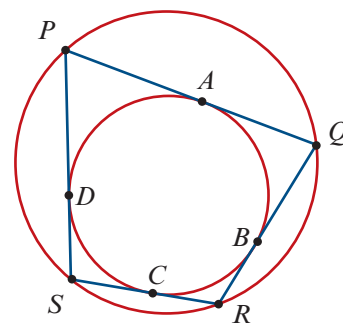
- a** To prove Ptolemy's theorem, we need to show that $BC \cdot AD + AB \cdot CD = AC \cdot BD$.

Line DQ is drawn such that $\angle BDC = \angle ADQ$.

- Prove that $\triangle ADQ \sim \triangle BDC$.
- Prove that $\triangle ADB \sim \triangle QDC$.
- Show that $AQ = \frac{BC \cdot AD}{BD}$ and $QC = \frac{AB \cdot CD}{BD}$.
- Complete the proof of Ptolemy's theorem.
(Hint: $AC = AQ + QC$.)



- b** Prove Pythagoras' theorem by applying Ptolemy's theorem to a rectangle.
- c** Use Ptolemy's theorem to prove that, in a regular pentagon with side length 1, the length of each diagonal is the golden ratio ϕ .
Hint: Draw the diagonals and take three sides of the pentagon and one diagonal to form a quadrilateral.
- d** Given an equilateral triangle inscribed in a circle and a point on the circle, use Ptolemy's theorem to prove that the distance from the point to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
- 4** The diagram shows a circle inscribed in a cyclic quadrilateral $PQRS$. The sides PQ , QR , RS and SP touch the incircle at the points A , B , C and D respectively.



- Prove that AC is perpendicular to BD .
 - Prove that the midpoints of AB , BC , CD and DA are concyclic.
- 5** Each of the following is true:
- $1 = 1^2$
 - $2 + 3 + 4 = 3^2$
 - $3 + 4 + 5 + 6 + 7 = 5^2$

Generalise these results and prove the generalisation.

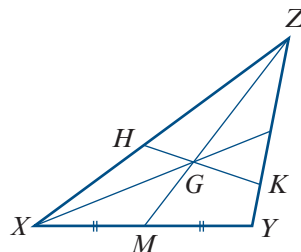
- 6** Let p be a prime number with $p \geq 5$.
- a** Prove that when p is divided by 6, the remainder is either 1 or 5.
 - b** Hence, prove that when p^2 is divided by 24, the remainder is 1.
 - c** Hence, prove that $p^2 + 2$ is divisible by 3.
- 7 a** Let points O, A, B and C be coplanar and let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. Assume that $\mathbf{a}$ and $\mathbf{b}$ are not parallel. If points A, B and C are collinear with

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{where } \alpha, \beta \in \mathbb{R}$$

show that $\alpha + \beta = 1$.

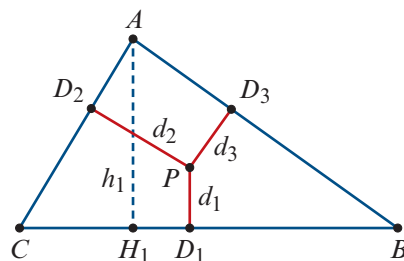
- b** In the figure, the point G is the centroid of a triangle (i.e. the point where the lines joining each vertex to the midpoint of the opposite side meet).

A line passing through G meets ZX and ZY at points H and K respectively, with $ZH = hZX$ and $ZK = kZY$.



- i** Prove that $\overrightarrow{ZG} = \frac{2}{3}\overrightarrow{ZM}$.
- ii** Express $\overrightarrow{ZG}$ in terms of $h, k, \overrightarrow{ZH}$ and $\overrightarrow{ZK}$.
- iii** Find the value of $\frac{1}{h} + \frac{1}{k}$. (Use the result from **a**.)
- iv** If $h = k$, find the value of h and describe geometrically what this implies.
- v** If the area of triangle XYZ is 1 cm^2 , what is the area of triangle HKZ when $h = k$?
- vi** If $k = 2h$, find the value of h and describe geometrically what this implies.
- vii** Describe the restrictions on h and k , and sketch the graph of h against k for suitable values of k .
- viii** Investigate the area, $A \text{ cm}^2$, of triangle HKZ as a ratio with respect to the area of triangle XYZ , as k varies. Sketch the graph of A against k . Be careful with the domain.

- 8** Consider a point P inside a triangle ABC with side lengths a, b and c . Let d_1, d_2 and d_3 denote the perpendicular distances from P to sides BC, CA and AB respectively. Let h_1, h_2 and h_3 denote the lengths of the altitudes from A, B and C respectively. (The diagram only shows the altitude from A .)



- a** Find expressions for the areas of $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$. Hence write the area of $\triangle ABC$ as the sum of these three areas.
- b** Find three expressions for the area of $\triangle ABC$ by using each of the three altitudes.
- c** Use these results to prove that

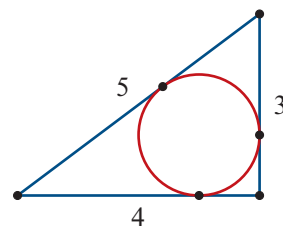
$$\frac{d_1}{h_1} + \frac{d_2}{h_2} + \frac{d_3}{h_3} = 1$$

11E Problem-solving and modelling investigations

For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

1 Circumcircles and incircles of right-angled triangles

- Find the radius of the circumcircle of any right-angled triangle with hypotenuse of length c .
- A right-angled triangle with sides 3, 4 and 5 is shown with its incircle. Find the radius of the circle.
- Find the radius of the incircle of a right-angled triangle with sides 5, 12 and 13.
- Find the radius of the incircle of a right-angled triangle with sides 7, 24 and 25.
- Generalise these results for any Pythagorean triple (a, b, c) such that $c = b + 1$, where c is the length of the hypotenuse.



- Motion of two particles** In this question, the unit of distance is metres and the unit of velocity is metres per second. Relative to the origin O , particles A and B have initial positions $0\mathbf{i} + 0\mathbf{j}$ (the origin) and $10\mathbf{i} + 4\mathbf{j}$ respectively. Both particles start moving at the same time. Particle A moves with constant velocity $3\mathbf{i} + 4\mathbf{j}$, and particle B moves with constant velocity $a\mathbf{i} + b\mathbf{j}$.
 - Find the position vectors of particles A and B at time t seconds after the start of motion.
 - Find a in terms of b given that the particles collide at some time $t \geq 0$.
 - Investigate collisions for different values of a and b .
 - Find a condition for which the paths of A and B cross (collision not necessary).
 - Find the values of a and b such that the particles collide and their paths are at right angles at the point of collision.
 - Investigate the velocity of A relative to B .
 - Investigate different starting points and constant velocities.
- Round-trip flight** An aircraft flies with speed v in still air. The aircraft starts at a point A , flies directly to a point B and then returns to point A . Throughout the flight, the wind is blowing with speed w at an angle of θ° to $\overrightarrow{AB}$.
 - Show that, for both the outbound trip and the return trip, the pilot must steer the aircraft at an angle of α° to the line AB , where $v \sin(\alpha^\circ) = w \sin(\theta^\circ)$.
 - Let d denote the distance AB , and let t_1 and t_2 denote the lengths of time taken for the outbound and return trips, respectively. Show that:
 - $d(t_1 + t_2) = 2vt_1t_2 \cos(\alpha^\circ)$
 - $d^2 = t_1t_2(v^2 - w^2)$
 - Complete several calculations for specific values of v , w , θ and d .

4 Stars and bars The technique of using stars and bars for combinatorics problems was introduced by William Feller (1906–1970).

- a** Suppose that 10 identical chocolates are to be distributed among three children, Amy, Ben and Clara. We will investigate the number of ways that this can be done. Let a , b and c denote the number of chocolates given to Amy, Ben and Clara respectively. We can represent allocations of the 10 chocolates using *stars and bars*. For example:

- The allocation $(a, b, c) = (2, 3, 5)$ is represented as

* * | * * * | * * * * *

- The allocation $(a, b, c) = (3, 0, 7)$ is represented as

* * * | | * * * * * *

- i** Using stars and bars, represent each of the following allocations:
- $(a, b, c) = (4, 5, 1)$
 - $(a, b, c) = (0, 6, 4)$
 - $(a, b, c) = (0, 0, 10)$
- ii** Notice that each allocation is represented by some arrangement of 10 stars and 2 bars. How many different ways can you allocate the 10 chocolates to the three children?
- iii** Using a similar technique, find the number of ways that eight chocolates can be distributed among four children.
- iv** How many ways can you distribute n chocolates among k children?
- v** If Amy, Ben and Clara are each to receive at least one of the 10 chocolates, how many ways can this be done?
- vi** How many ways can you distribute n chocolates among k children if each child is to receive at least one chocolate?

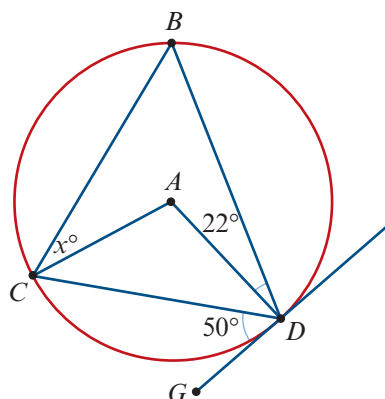
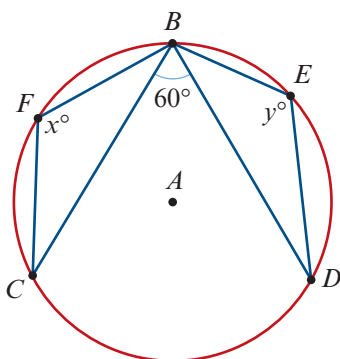
All of the following problems can be solved using this technique:

- b** How many ways can you distribute three identical balls into three different boxes?
- c** How many sequences of four non-negative integers are there that sum to 10?
- d** How many sequences of three odd positive integers are there that sum to 17?
- e** How many paths are there from the top-left corner to the bottom-right corner of an $m \times n$ grid if you can only travel right or down along the grid lines?
- f** List some other situations that can be considered in this way and analyse them using the technique of stars and bars.

11F Degree-of-difficulty classified questions

► Simple familiar questions

- Let $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j}$. Calculate:
a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $3\mathbf{a} - 4\mathbf{b}$ **d** $\mathbf{a} \cdot \mathbf{a}$ **e** $|\mathbf{a}|$ **f** $\mathbf{a} \cdot \mathbf{b}$
- For non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ that are not parallel, find the values of s and t such that $2\mathbf{b} + t(\mathbf{a} + \mathbf{b}) = 2\mathbf{a} + \mathbf{b} + s(\mathbf{a} + 3\mathbf{b})$.
- Resolve $4\mathbf{i} + 3\mathbf{j}$ into rectangular components, one of which is parallel to $\sqrt{3}\mathbf{i} - \mathbf{j}$.
- A particle starts at the point A with position vector $\overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}$, where the unit is metres. The particle begins moving with a constant velocity of $\mathbf{i} + \mathbf{j}$ m/s. Find the position vector of the particle after:
a 5 seconds **b** t seconds.
- A hiker goes from point O to point A by first walking east for 10 km and then walking on a bearing of 030° for 8 km. Describe the hiker's displacement vector $\overrightarrow{OA}$ by giving a distance and a bearing.
- a** Find the value of $x + y$. Give reasons. **b** In the diagram, the circle has centre A and line GD is tangent to the circle at D . Find the value of x .



- Let $\mathbf{a} = \sqrt{3}\mathbf{i} + \mathbf{j}$.
a Find the magnitude of $\mathbf{a}$.
b Find the unit vector in the direction of $\mathbf{a}$.
c Find the vector of magnitude 30 in the direction of $\mathbf{a}$.
- How many ways can:
a five children be arranged in a row
b five children be arranged on a bench with space for three
c three children be selected from a group of five?
- How many ways can two adults and three children be arranged along a park bench so that the three children are sitting together?

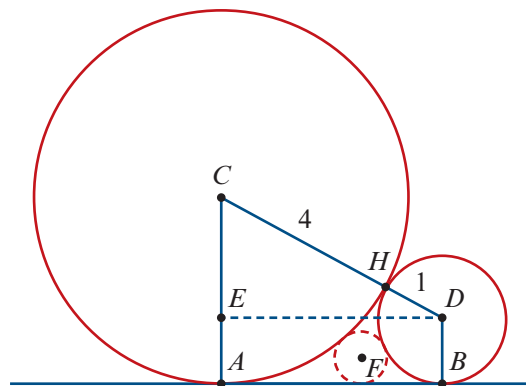
- 10** How many ways can the letters in the word COOLANGATTA be arranged?
- 11** How many ways can four letters be chosen from the set $\{A, B, C, D, E, F, G, H, I\}$:
- a** without restriction
 - b** if the letter B must be chosen
 - c** if at least one vowel must be chosen?
- 12 a** Prove that there are two pairs of prime numbers (p, q) for which
- $$15p^2 - 19pq + 6q^2 = 0$$
- b** Prove that there is one pair of prime numbers (p, q) for which
- $$10p^2 - 9pq + 2q^2 = 0$$
- c** Prove that there are no pairs of prime numbers (p, q) for which
- $$6p^2 - 5pq + q^2 = 0$$

► Complex familiar questions

- 1** Consider a triangle ABC . Let D be the point on side AC such that $AD : DC = 2 : 3$. Let E be the point on side BC such that DE is parallel to AB . Find the ratio of the area of triangle ADE to the area of triangle ABC .
- 2** Prove that there are no real numbers a and b such that $\frac{a+b}{b} = \frac{2a}{a+b}$.
- 3** Let AB and CD be perpendicular diameters of a circle with centre O . Let AM be a chord meeting CD at P . Prove that $AP \times AM = AO \times AB$.
- 4** Let $\vec{OA} = i + j$, $\vec{OB} = 2i + j$ and $\vec{OP} = \vec{OA} + t\vec{OB}$, where t is a real number. Determine the value of t such that:
- a** $\vec{OP}$ is perpendicular to $\vec{AB}$
 - b** $|\vec{OP}| = |\vec{AP}|$
- 5** A quadrilateral $ABCD$ is inscribed in a circle. Given $\angle ADB = 28^\circ$ and $\angle CBD = 62^\circ$, show that

$$\text{Arc length } AB + \text{Arc length } CD = \text{Arc length } BC + \text{Arc length } AD$$

- 6** Two circles, with centres C and D , touch at H . The circle with centre C has radius 4, and the circle with centre D has radius 1. A common tangent line touches the two circles at points A and B as shown. A third circle, centre F , touches the two other circles and the tangent line. Find the radius of the third circle.



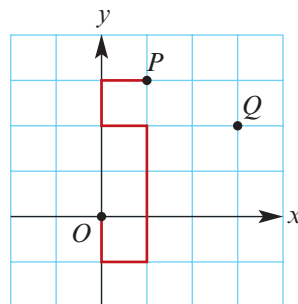
- 7** How many five-digit even numbers greater than 30 000 can be formed using the digits 0, 1, 2, 3 and 4 if:
- a** the digits can be repeated
 - b** the digits cannot be repeated?
- 8** Consider the letters in the word AUSTRALIA.
- a** How many arrangements of these letters are there?
 - b** How many arrangements begin and end with a consonant?
 - c** How many arrangements have all the vowels together?
- 9** Determine how many three-digit numbers have both of the following properties:
- none of the digits is 1
 - one digit is the product of the other two digits.
- 10** Let a and b be integers.
- a** Show that if d is a factor of both a and b , then d is also a factor of $a - b$.
 - b** Hence, explain why the highest common factor of 1 000 001 and 999 999 is 1.
 - c** Now assume that a and b are odd numbers that differ by a power of 2. Show that the highest common factor of a and b is 1.
- 11** Consider the sum
- $$S_n = \frac{1}{4 \cdot 1^2 - 1} + \frac{1}{4 \cdot 2^2 - 1} + \cdots + \frac{1}{4 \cdot n^2 - 1}$$
- a** Evaluate S_n for:
 - i** $n = 1$ **ii** $n = 2$ **iii** $n = 3$
 - b** Hence, guess a formula for S_n in terms of n .
 - c** Prove that your formula is correct using mathematical induction.

► Complex unfamiliar questions

- 1** An isosceles triangle ABC is inscribed in a circle. Given that $AB = AC$ and that chord AD intersects side BC at the point E , prove that $AB^2 - AE^2 = BE \times CE$.
- 2** Let M be the midpoint of the side AB of an equilateral triangle ABC . Let N be the point on side BC such that MN is perpendicular to BC , and let P be the point on side AC such that NP is perpendicular to AC . Find the ratio of the area of quadrilateral $AMNP$ to the area of triangle ABC .
- 3** Let x be a real number. Use the pigeon-hole principle to prove that, among the first 99 multiples of x (that is, $x, 2x, 3x, \dots, 99x$), there is at least one number that differs from an integer by no more than 0.01.

- 4** Points A and B have position vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j}$ respectively, relative to an origin O .
- a** Let P be a point on the line AB .
- Explain why the position vector of P can be written as $\overrightarrow{OP} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, for some real number t .
 - Show that, if $\overrightarrow{OP}$ makes an angle of 60° with $\overrightarrow{OA}$, then $3t^2 - 1 = 0$.
- b** Let P_1 and P_2 be two points on the line AB such that both $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ make an angle of 60° with $\overrightarrow{OA}$. Use part **a** to find $\overrightarrow{OP_1} \cdot \overrightarrow{OP_2}$.

- 5** An ant moves around a unit grid by walking, one unit at a time, in any one of the four directions: north (N), east (E), south (S) or west (W). Consider the three points $O(0, 0)$, $P(1, 3)$ and $Q(3, 2)$. The path from O to P shown in the diagram has length 8 and can be described by the string of letters SENNNWNE.



Find the number of paths of length 8 from O to:

- a** O **b** P **c** Q
- 6** **a** Let a and b be real numbers and let x and y be positive real numbers. Prove that
- $$\frac{(a+b)^2}{x+y} \leq \frac{a^2}{x} + \frac{b^2}{y}$$
- b** Now let a , b and c be real numbers and let x , y and z be positive real numbers. Use part **a** to prove that
- $$\frac{(a+b+c)^2}{x+y+z} \leq \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}$$
- c** Let x , y and z be positive real numbers for which $x + y + z = 3$. Prove that
- $$\frac{1}{x} + \frac{4}{y} + \frac{9}{z} \geq 12$$
- 7** Show that $\frac{(n+1)! + n!}{n! + (n-1)!} = \frac{n(n+2)}{n+1}$ for each $n \in \mathbb{N}$.

- 8** **a** Show that $k \cdot {}^nC_k = n \cdot {}^{n-1}C_{k-1}$.

b In Section 7G we showed that

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

Using this result and part **a**, prove that

$$0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + \cdots + (n-1) \cdot {}^nC_{n-1} + n \cdot {}^nC_n = n \cdot 2^{n-1}$$

- 9** Prove by mathematical induction that, for every $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

- 10** The sum of a set of positive integers is 20. Find the largest possible value of their product, and prove that you can't do any better than this value.

12

Graphing techniques

Objectives

- ▶ To understand the **modulus function** $y = |x|$.
- ▶ To sketch transformations of the graph of the modulus function $y = |x|$.
- ▶ To sketch the graphs of functions with rules of the form $y = |f(x)|$ and $y = f(|x|)$.
- ▶ To sketch the graphs of **reciprocals** of polynomial functions.
- ▶ To sketch the graphs of **rational functions**.

In this chapter we build on the study of functions in Mathematical Methods Units 1 & 2, extending the range of functions for which we can sketch graphs.

The modulus function

The modulus function is useful in many contexts. For example, it enables us to talk about the distance between two points on the real number line. We will generalise the modulus function to complex numbers in Chapter 15.

Reciprocal functions

We will see that, given the graph of a function $y = f(x)$, we can sketch the graph of $y = \frac{1}{f(x)}$. This function is called the reciprocal of f .

Rational functions

A rational function is the quotient of two polynomial functions. For example:

$$g(x) = \frac{2x^2 + 1}{x} \qquad h(x) = \frac{x}{x^2 - 1}$$

In this chapter we develop some techniques for sketching the graphs of simple rational functions. You will be able to analyse a wider range of rational functions after you have studied differentiation in Mathematical Methods Units 1 & 2.

12A The modulus function

The **modulus** or **absolute value** of a real number x is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$. For example: $|5| = 5$ and $|-5| = 5$.

Example 1

Evaluate each of the following:

a i $|-3 \times 2|$

ii $|-3| \times |2|$

b i $\left| \frac{-4}{2} \right|$

ii $\frac{|-4|}{|2|}$

c i $|-6 + 2|$

ii $|-6| + |2|$

Solution

a i $|-3 \times 2| = |-6| = 6$

ii $|-3| \times |2| = 3 \times 2 = 6$

Note: $|-3 \times 2| = |-3| \times |2|$

b i $\left| \frac{-4}{2} \right| = |-2| = 2$

ii $\frac{|-4|}{|2|} = \frac{4}{2} = 2$

Note: $\left| \frac{-4}{2} \right| = \frac{|-4|}{|2|}$

c i $|-6 + 2| = |-4| = 4$

ii $|-6| + |2| = 6 + 2 = 8$

Note: $|-6 + 2| \neq |-6| + |2|$

Properties of the modulus function

- $|ab| = |a||b|$ and $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$
- $|x| = a$ implies $x = a$ or $x = -a$
- $|a + b| \leq |a| + |b|$
- If a and b are both positive or both negative, then $|a + b| = |a| + |b|$.
- If $a \geq 0$, then $|x| \leq a$ is equivalent to $-a \leq x \leq a$.
- If $a \geq 0$, then $|x - k| \leq a$ is equivalent to $k - a \leq x \leq k + a$.

The following example uses the second property listed above to solve simple equations.

Example 2

Solve each of the following equations for x :

a $|x - 4| = 6$

b $|2x - 4| = 16$

Solution

a $|x - 4| = 6$

$\Rightarrow x - 4 = 6 \quad \text{or} \quad x - 4 = -6$

$\Rightarrow x = 10 \quad \text{or} \quad x = -2$

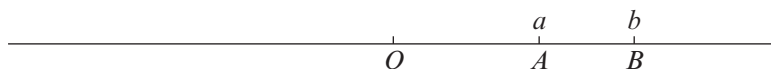
b $|2x - 4| = 16$

$\Rightarrow 2x - 4 = 16 \quad \text{or} \quad 2x - 4 = -16$

$\Rightarrow x = 10 \quad \text{or} \quad x = -6$

► The modulus function as a measure of distance

Consider two points A and B on a number line:



On a number line, the distance between points A and B is $|a - b| = |b - a|$.

For example:

- The statement $|x - 2| \leq 3$ can be read as ‘on the number line, the distance of x from 2 is less than or equal to 3’.
- The statement $|x| \leq 3$ can be read as ‘on the number line, the distance of x from the origin is less than or equal to 3’.

Note that $|x| \leq 3$ is equivalent to $-3 \leq x \leq 3$ or $x \in [-3, 3]$.

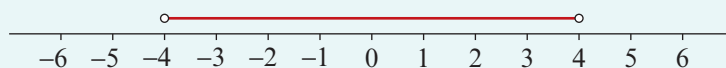
Example 3

Illustrate each of the following sets on a number line and represent the sets using interval notation:

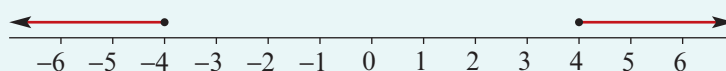
- a** $\{x : |x| < 4\}$ **b** $\{x : |x| \geq 4\}$ **c** $\{x : |x - 1| \leq 4\}$

Solution

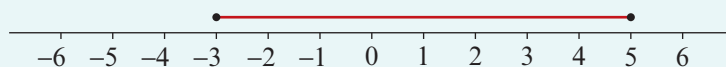
a $(-4, 4)$



b $(-\infty, -4] \cup [4, \infty)$



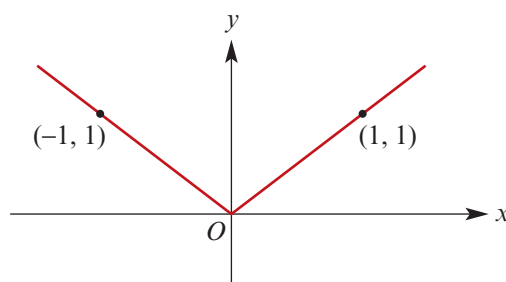
c $[-3, 5]$



► The graph of $y = |x|$

The graph of the function $y = |x|$ is shown on the right.

This graph is symmetric about the y -axis, since $|x| = |-x|$.



Example 4

For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 3| + 1$

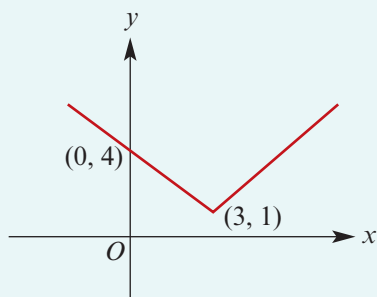
b $f(x) = -|x - 3| + 1$

Solution

Note that $|a - b| = a - b$ if $a \geq b$, and $|a - b| = b - a$ if $b \geq a$.

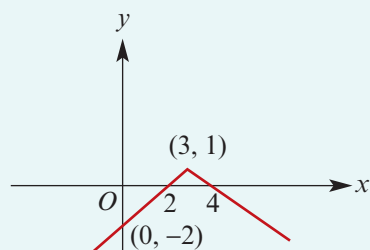
$$\begin{aligned} \mathbf{a} \quad f(x) &= |x - 3| + 1 = \begin{cases} x - 3 + 1 & \text{if } x \geq 3 \\ 3 - x + 1 & \text{if } x < 3 \end{cases} \\ &= \begin{cases} x - 2 & \text{if } x \geq 3 \\ 4 - x & \text{if } x < 3 \end{cases} \end{aligned}$$

Range = $[1, \infty)$



$$\begin{aligned} \mathbf{b} \quad f(x) &= -|x - 3| + 1 = \begin{cases} -(x - 3) + 1 & \text{if } x \geq 3 \\ -(3 - x) + 1 & \text{if } x < 3 \end{cases} \\ &= \begin{cases} -x + 4 & \text{if } x \geq 3 \\ -2 + x & \text{if } x < 3 \end{cases} \end{aligned}$$

Range = $(-\infty, 1]$

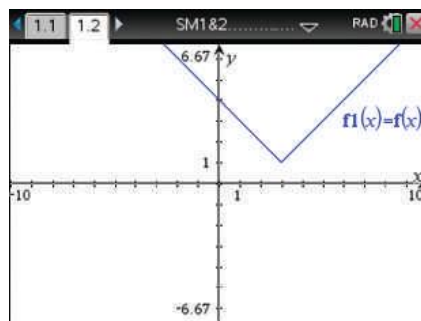
**Using the TI-Nspire CX non-CAS**

- Use **menu** > **Actions** > **Define** to define the function $f(x) = \text{abs}(x - 3) + 1$.

Note: The absolute value function can be obtained by typing **abs()** or using the 2D-template palette .

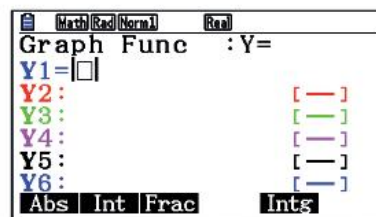
- Open a **Graphs** application (**ctrl** **I** > **Graphs**) and let $f1(x) = f(x)$.
- Press **enter** to obtain the graph.

Note: The expression $\text{abs}(x - 3) + 1$ could have been entered directly for $f1(x)$.



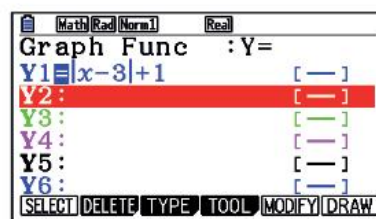
Using the Casio

- Press **MENU** **5** to select **Graph** mode.
- To obtain the absolute value function, go to the **Numeric** menu **OPTN** **F5**, then select **Abs** **F1**.



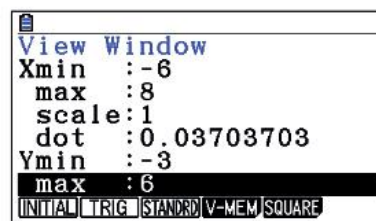
- Finish entering the rule $y = |x - 3| + 1$ in $Y1$:

X,**0**,**T** **-** **3** **▶** **+** **1** **EXE**

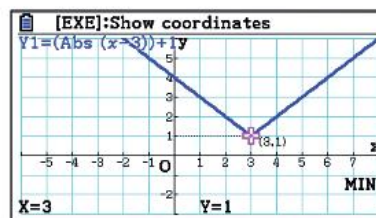


- To view the graph, select **Draw** **F6**.
- Adjust the View Window:

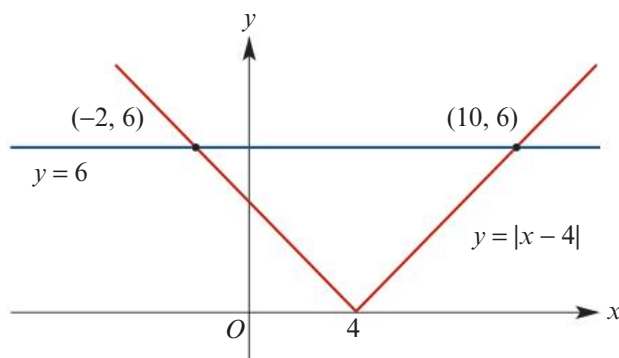
SHIFT **F3**
(-) **6** **EXE** **8** **EXE** **1** **EXE** ▼
(-) **3** **EXE** **6** **EXE** **1** **EXE**



- To return to the graph, press **EXIT**.
- To find the coordinates of the minimum point, go to the **G-Solve** menu **SHIFT** **F5**, then select **Minimum** **F3**.



Note: The solution of equations can be illustrated graphically. For example, the following graph shows the solutions of the equation $|x - 4| = 6$ from Example 2a.



► Functions with rules of the form $y = |f(x)|$ and $y = f(|x|)$

If the graph of $y = f(x)$ is known, then we can sketch the graph of $y = |f(x)|$ using the following observation:

$$|f(x)| = f(x) \text{ if } f(x) \geq 0 \quad \text{and} \quad |f(x)| = -f(x) \text{ if } f(x) < 0$$



Example 5

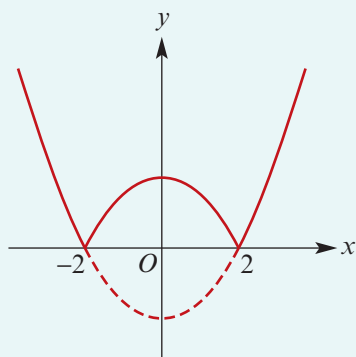
Sketch the graph of each of the following:

a $y = |x^2 - 4|$

b $y = |2^x - 1|$

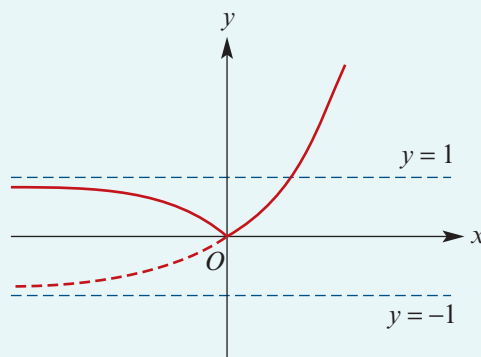
Solution

a



The graph of $y = x^2 - 4$ is drawn and the negative part reflected in the x -axis.

b



The graph of $y = 2^x - 1$ is drawn and the negative part reflected in the x -axis.

The graph of $y = f(|x|)$, for $x \in \mathbb{R}$, is sketched by reflecting the graph of $y = f(x)$, for $x \geq 0$, in the y -axis.



Example 6

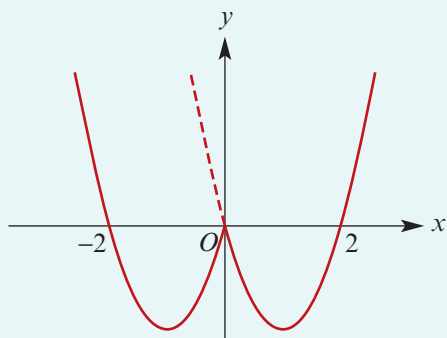
Sketch the graph of each of the following:

a $y = |x|^2 - 2|x|$

b $y = 2^{|x|}$

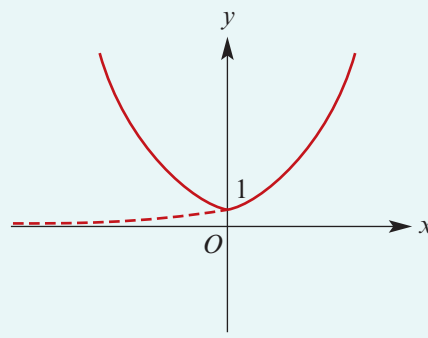
Solution

a



The graph of $y = x^2 - 2x$, $x \geq 0$, is reflected in the y -axis.

b



The graph of $y = 2^x$, $x \geq 0$, is reflected in the y -axis.

Example 7

- a** Solve the equation $|x^2 - 4x| = 3$ for x .
b Illustrate the solutions by graphing $y = |x^2 - 4x|$ and $y = 3$ on the same set of axes.

Solution

$$\begin{aligned} \mathbf{a} \quad & |x^2 - 4x| = 3 \\ \Rightarrow & x^2 - 4x = 3 \quad \text{or} \quad x^2 - 4x = -3 \\ \Rightarrow & x^2 - 4x - 3 = 0 \quad \text{or} \quad x^2 - 4x + 3 = 0 \\ \Rightarrow & x = 2 \pm \sqrt{7} \quad \text{or} \quad x = 1 \text{ or } x = 3 \end{aligned}$$

Therefore $x = 2 + \sqrt{7}$, $x = 2 - \sqrt{7}$, $x = 1$ or $x = 3$.

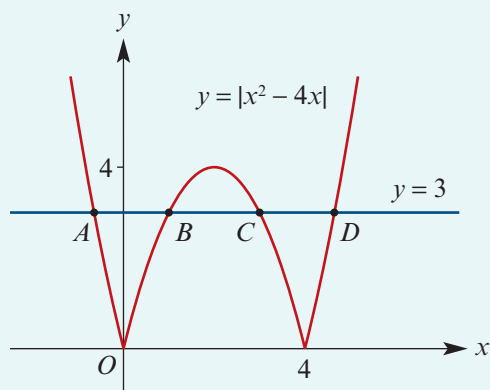
- b** The solutions correspond to the points of intersection of the two graphs:

$$A(2 - \sqrt{7}, 3)$$

$$B(1, 3)$$

$$C(3, 3)$$

$$D(2 + \sqrt{7}, 3)$$



Note: We can see from the graph of $y = |x^2 - 4x|$ that the equation $|x^2 - 4x| = 4$ has three solutions and the equation $|x^2 - 4x| = 5$ has two solutions.

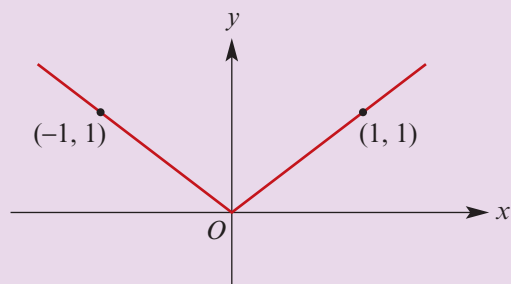
Section summary

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example: $|5| = 5$ and $|-5| = 5$.

- The graph of the modulus function $y = |x|$ is shown on the right.



- On the number line, the distance between a and b is given by $|a - b| = |b - a|$.
 For example: $|x - 2| < 5$ can be read as 'the distance of x from 2 is less than 5'.

Exercise 12A

Skillsheet

Example 1

1 Evaluate each of the following:

a $|-5| + 3$

b $|-5| + |-3|$

c $|-5| - |-3|$

d $|-5| - |-3| - 4$

e $|-5| - |-3| - |-4|$

f $|-5| + |-3| - |-4|$

Example 2

2 Solve each of the following equations for x :

a $|x - 1| = 2$

b $|2x - 3| = 4$

c $|5x - 3| = 9$

d $|x - 3| - 9 = 0$

e $|3 - x| = 4$

f $|3x + 4| = 8$

g $|5x + 11| = 9$

Example 3

3 For each of the following, illustrate the set on a number line and represent the set using interval notation:

a $\{x : |x| < 3\}$

b $\{x : |x| \geq 5\}$

c $\{x : |x - 2| \leq 1\}$

d $\{x : |x - 2| < 3\}$

e $\{x : |x + 3| \geq 5\}$

f $\{x : |x + 2| \leq 1\}$

Example 4

4 For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 4| + 1$

b $f(x) = -|x + 3| + 2$

c $f(x) = |x + 4| - 1$

d $f(x) = 2 - |x - 1|$

5 Solve each of the following inequalities for x :

a $|x| \leq 5$

b $|x| \geq 2$

c $|2x - 3| \leq 1$

d $|5x - 2| < 3$

e $|-x + 3| \geq 7$

f $|-x + 2| \leq 1$

6 Let $f(x) = 4 - x$.**a** Sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$.**b** State the set of values of x for which $|f(x)| = f(|x|)$.

Example 5

7 Sketch the graph of each of the following:

a $y = |x^2 - 9|$

b $y = |3^x - 3|$

c $y = |x^2 - x - 12|$

d $y = |x^2 - 3x - 40|$

e $y = |x^2 - 2x - 8|$

f $y = |2^x - 4|$

Example 6

8 Sketch the graph of each of the following:

a $y = |x|^2 - 4|x|$

b $y = 3^{|x|}$

c $y = |x|^2 - 7|x| + 12$

d $y = |x|^2 - |x| - 12$

e $y = |x|^2 + |x| - 12$

f $y = -3^{|x|} + 1$

Example 7

9 Solve each of the following equations for x :

a $|x^2 - 2x| = \frac{1}{2}$

b $|x^2 - 2x| = 1$

c $|x^2 - 2x| = 8$

d $|x^2 - 6x| = 8$

e $|x^2 - 6x| = 16$

f $|x^2 - 6x| = 9$

10 Solve each of the following equations for x :

a $|x - 4| - |x + 2| = 6$

b $|2x - 5| - |4 - x| = 10$

c $|2x - 1| + |4 - 2x| = 10$

11 If $f(x) = |x - a| + b$ with $f(3) = 3$ and $f(-1) = 3$, find the values of a and b .12 Prove each of the following for all $x, y, z \in \mathbb{R}$:

a $|x - y| \leq |x| + |y|$

b $|x| - |y| \leq |x - y|$

c $|x + y + z| \leq |x| + |y| + |z|$

12B Reciprocals of polynomial functions

You have learned in previous years that the **reciprocal** of a non-zero number a is $\frac{1}{a}$. Likewise, we have the following definition.

If $y = f(x)$ is a polynomial function, then its **reciprocal function** is defined by the rule

$$y = \frac{1}{f(x)}$$

For example, the reciprocal of the function $y = x^3$ is $y = \frac{1}{x^3}$.

In this section, we will find relationships between the graph of a function and the graph of its reciprocal. Let's consider some specific examples, from which we will draw general conclusions.

Example 8

Sketch the graphs of $y = x^3$ and $y = \frac{1}{x^3}$ on the same set of axes.

Solution

We first sketch the graph of $y = x^3$. This is shown in blue.

Horizontal asymptotes

If $x \rightarrow \pm\infty$, then $\frac{1}{x^3} \rightarrow 0$. Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

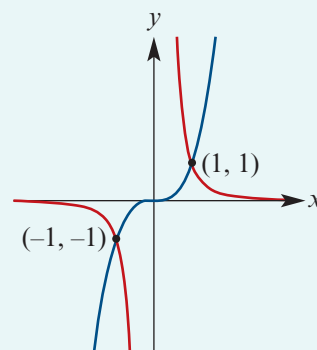
Vertical asymptotes

Notice that $x^3 = 0$ when $x = 0$.

If x is a small positive number, then $\frac{1}{x^3}$ is a large positive number.

If x is a small negative number, then $\frac{1}{x^3}$ is a large negative number.

Therefore the line $x = 0$ is a vertical asymptote of the reciprocal function.



Observations from the example

This example highlights behaviour typical of reciprocal functions:

- If $y = f(x)$ is a non-zero polynomial function, then the graph of $y = \frac{1}{f(x)}$ will have vertical asymptotes where $f(x) = 0$.
- The graphs of a function and its reciprocal are always on the same side of the x -axis.
- If the graphs of a function and its reciprocal intersect, then it must be where $f(x) = \pm 1$.

The following example is perhaps easier, because the reciprocal graph has no vertical asymptotes. This time we are interested in turning points.

Example 9

Consider the function $f(x) = x^2 + 2$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

Solution

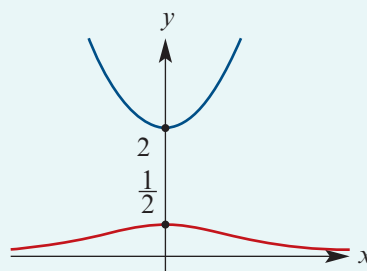
We first sketch $y = x^2 + 2$. This is shown in blue.

Horizontal asymptotes If $x \rightarrow \pm\infty$, then $\frac{1}{f(x)} \rightarrow 0$.

Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes There are no vertical asymptotes, as there is no solution to the equation $f(x) = 0$.

Turning points Notice that the graph of $y = x^2 + 2$ has a minimum at $(0, 2)$. The reciprocal function therefore has a maximum at $(0, \frac{1}{2})$.



- If the graph of $y = f(x)$ has a local minimum at $x = a$, then the graph of $y = \frac{1}{f(x)}$ will have a local maximum at $x = a$.
- If the graph of $y = f(x)$ has a local maximum at $x = a$, then the graph of $y = \frac{1}{f(x)}$ will have a local minimum at $x = a$.

Example 10

Consider the function $f(x) = 2(x - 1)(x + 1)$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

Solution

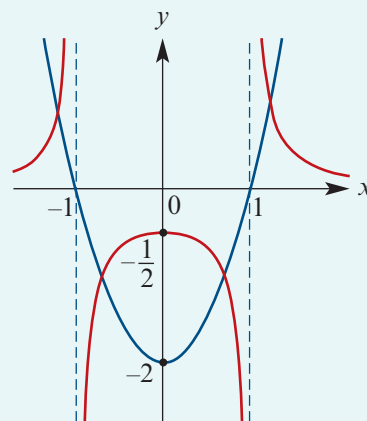
We first sketch $y = 2(x - 1)(x + 1)$. This is shown in blue.

Horizontal asymptotes If $x \rightarrow \pm\infty$, then $\frac{1}{f(x)} \rightarrow 0$.

Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes We have $f(x) = 0$ when $x = -1$ or $x = 1$. Therefore the lines $x = -1$ and $x = 1$ are vertical asymptotes of the reciprocal function.

Turning points The graph of $y = f(x)$ has a minimum at $(0, -2)$. Therefore the reciprocal has a local maximum at $(0, -\frac{1}{2})$.



Section summary

Given the graph of a continuous function $y = f(x)$, we can sketch the graph of $y = \frac{1}{f(x)}$ with the help of the following observations:

Function $y = f(x)$	Reciprocal function $y = \frac{1}{f(x)}$
x -axis intercept at $x = a$	vertical asymptote $x = a$
local maximum at $x = a$	local minimum at $x = a$
local minimum at $x = a$	local maximum at $x = a$
above the x -axis	above the x -axis
below the x -axis	below the x -axis
increasing over an interval	decreasing over the interval
decreasing over an interval	increasing over the interval
values approach ∞	values approach 0 from above
values approach $-\infty$	values approach 0 from below
values approach 0 from above	values approach ∞
values approach 0 from below	values approach $-\infty$

Exercise 12B

Skillsheet

- 1 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes:

Example 8, 9

a $f(x) = x + 3$

b $f(x) = x^2$

c $f(x) = x^2 + 4$

Example 10

d $f(x) = (x - 1)(x + 1)$

e $f(x) = 4 - x^2$

f $f(x) = (x - 1)^2 - 1$

g $f(x) = x^2 - 2x - 3$

h $f(x) = -x^2 - 2x + 3$

i $f(x) = x^3 + 1$

- 2 Sketch the graph of each of the following, labelling all axis intercepts and asymptotes:

a $y = \frac{1}{x^2 - 2x}$

b $y = \frac{1}{(x - 1)^2 + 1}$

c $y = \frac{1}{x^2 + 1}$

d $y = \frac{1}{x^2 - 1}$

e $y = \frac{1}{x^2 - x - 2}$

f $y = \frac{1}{4 + 3x - x^2}$

- 3 Consider the function $f(x) = 2^x - 1$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

- 4 Consider the quadratic function $f(x) = x^2 + 2x + 2$.

a By completing the square, find the turning point of the graph of $y = f(x)$.

b Hence sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

- 5** Sketch the graph of each of the following, labelling all axis intercepts, turning points and asymptotes:

a $f(x) = \frac{1}{9 - x^2}$

b $g(x) = \frac{1}{(x - 2)(3 - x)}$

c $h(x) = \frac{1}{x^2 + 2x + 4}$

d $f(x) = \frac{1}{x^2 + 2x + 1}$

- 6** Consider the quadratic function $f(x) = 5x(1 - x)$.

a Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

b Locate all points of intersection of the two graphs by solving $f(x) = 1$ and $f(x) = -1$.

- 7** Let $k \in \mathbb{R}$ and consider the function $f(x) = x^2 + 2kx + 1$.

a By completing the square, show that the graph of $y = f(x)$ has a minimum turning point at $(-k, 1 - k^2)$.

b For what values of k does the graph of $y = f(x)$ have:

i no x -axis intercept **ii** one x -axis intercept **iii** two x -axis intercepts?

c Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ when the graph of $y = f(x)$ has:

i no x -axis intercept **ii** one x -axis intercept **iii** two x -axis intercepts.

Hint: It helps to ignore the y -axis.

12C Rational functions

A rational function has a rule of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. There is a huge variety of different types of curves in this particular family of functions. An example of a rational function is

$$f(x) = \frac{x^2 + 2x + 3}{x^2 + 4x - 1}$$

The following are also rational functions, but are not given in the form used in the definition of a rational function:

$$g(x) = 1 + \frac{1}{x}$$

$$h(x) = x - \frac{1}{x^2 + 2}$$

Their rules can be rewritten as shown:

$$g(x) = \frac{x}{x} + \frac{1}{x} = \frac{x + 1}{x} \quad h(x) = \frac{x(x^2 + 2)}{x^2 + 2} - \frac{1}{x^2 + 2} = \frac{x^3 + 2x - 1}{x^2 + 2}$$

Note: In the previous section, we considered reciprocals of polynomial functions. These are examples of rational functions where $P(x) = 1$.

► Graphing rational functions

For sketching graphs, it is also useful to write rational functions in the alternative form, that is, with a division performed if possible. For example:

$$f(x) = \frac{8x^2 - 3x + 2}{x} = \frac{8x^2}{x} - \frac{3x}{x} + \frac{2}{x} = 8x - 3 + \frac{2}{x}$$

For this example, we can see that $\frac{2}{x} \rightarrow 0$ as $x \rightarrow \pm\infty$, so the graph of $y = f(x)$ will approach the line $y = 8x - 3$ as $x \rightarrow \pm\infty$.

We say that the line $y = 8x - 3$ is a **non-vertical asymptote** of the graph. This is a line or curve which the graph approaches as $x \rightarrow \pm\infty$.

Note: An asymptote that is a straight line can be vertical, horizontal or **oblique**.

Addition of ordinates

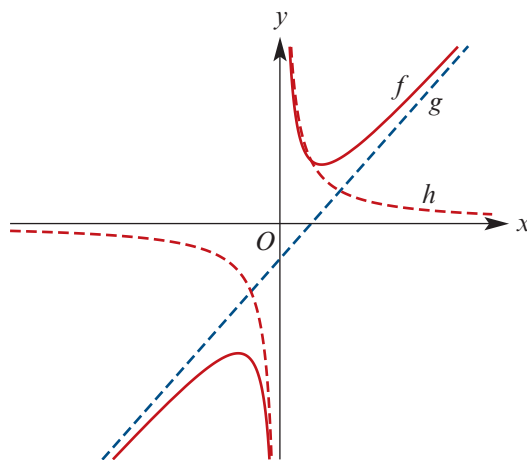
One method that can be useful for sketching the graph of a rational function is adding the y-coordinates (ordinates) of two simpler graphs.

For the function $f(x) = \frac{8x^2 - 3x + 2}{x}$ considered above, we can write

$$f(x) = g(x) + h(x), \quad \text{where } g(x) = 8x - 3 \text{ and } h(x) = \frac{2}{x}$$

We sketch the graphs of $y = g(x)$ and $y = h(x)$ on the same set of axes, and use them to help determine the shape of the graph of $y = f(x)$. A table of values may be useful.

x	$g(x)$	$h(x)$	$f(x)$
-10	-83	-0.2	-83.2
-1	-11	-2	-13
$\frac{3}{8}$	0	$\frac{16}{3}$	$\frac{16}{3}$
1	5	2	7
10	77	0.2	77.2



Key points for addition of ordinates

- When the two graphs have the same y-coordinate, the y-coordinate of the resultant graph will be double this.
- When the two graphs have opposite y-coordinates, the y-coordinate of the resultant graph will be zero (an x-axis intercept).
- When one of the two y-coordinates is zero, the resulting y-coordinate is equal to the other y-coordinate.

Turning points

There are several options for finding the coordinates of turning points:

- Calculus** Differential calculus can be used to find the exact coordinates of turning points. Once you have studied differentiation, this is the preferred method.
- Calculator** A graphics calculator will find approximate coordinates of turning points.
- Algebra** Some special types of rational functions can be analysed without calculus, using purely algebraic techniques.

In this section, we avoid the use of calculus and restrict to rational functions that can be analysed algebraically.

Example 11

Sketch the graph of $f(x) = \frac{4x^2 - 3x + 16}{x}$.

Solution

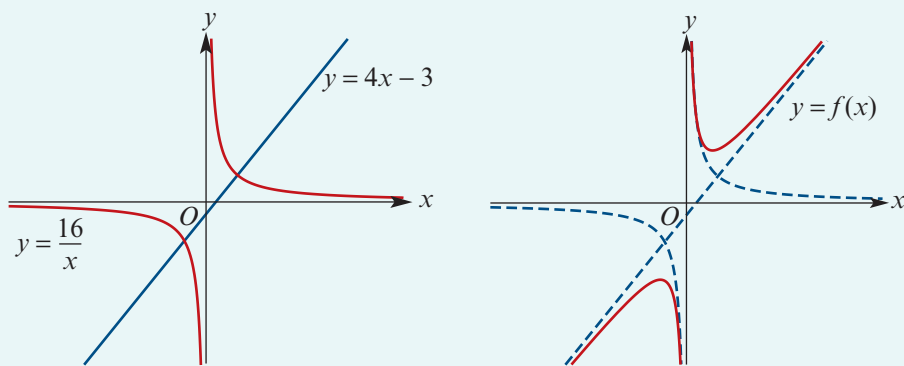
Asymptotes The vertical asymptote has equation $x = 0$, i.e. the y -axis.

Dividing through gives

$$f(x) = \frac{4x^2 - 3x + 16}{x} = \frac{4x^2}{x} - \frac{3x}{x} + \frac{16}{x} = 4x - 3 + \frac{16}{x}$$

Note that $\frac{16}{x} \rightarrow 0$ as $x \rightarrow \pm\infty$. Therefore the graph of $y = f(x)$ approaches the graph of $y = 4x - 3$ as $x \rightarrow \pm\infty$. The non-vertical asymptote has equation $y = 4x - 3$.

Addition of ordinates The graph of $y = f(x)$ can be obtained by adding the y -coordinates of the graphs of $y = 4x - 3$ and $y = \frac{16}{x}$.



Intercepts There is no y -axis intercept, as the domain of f is $\mathbb{R} \setminus \{0\}$. There are no x -axis intercepts, as the equation $\frac{4x^2 - 3x + 16}{x} = 0$ has no solutions.

Turning points We can write the rule for f in two different ‘turning point forms’. To do this, we complete the square in the numerator by working on the x term:

$$\begin{aligned} f(x) &= \frac{4x^2 - 3x + 16}{x} \\ &= \frac{4x^2 - 3x - 13x + 13x + 16}{x} \\ &= \frac{4x^2 - 16x + 16}{x} + 13 \\ &= \frac{4(x-2)^2}{x} + 13 \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{4x^2 - 3x + 16}{x} \\ &= \frac{4x^2 - 3x + 19x - 19x + 16}{x} \\ &= \frac{4x^2 + 16x + 16}{x} - 19 \\ &= \frac{4(x+2)^2}{x} - 19 \end{aligned}$$

By writing the rule in the form

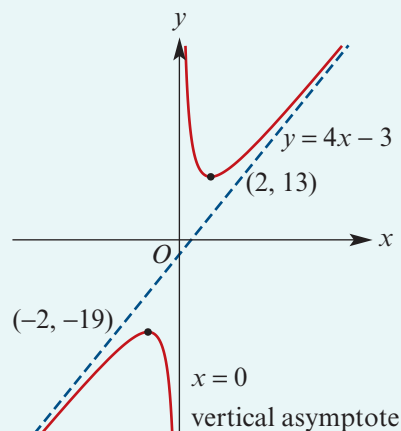
$$f(x) = \frac{4(x-2)^2}{x} + 13$$

we can see that $f(2) = 13$ and $f(x) \geq 13$ for all $x > 0$. Hence $(2, 13)$ is a local minimum.

By writing the rule in the form

$$f(x) = \frac{4(x+2)^2}{x} - 19$$

we can see that $f(-2) = -19$ and $f(x) \leq -19$ for all $x < 0$. Hence $(-2, -19)$ is a local maximum.



Using the TI-Nspire CX non-CAS

- In a **Graphs** application, plot the graph of

$$f1(x) = 4x - 3 + \frac{16}{x}$$

- Set an appropriate window using **menu** > **Window/Zoom**.

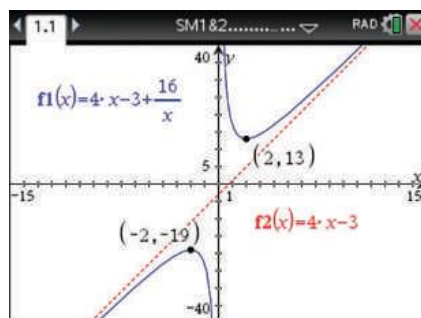
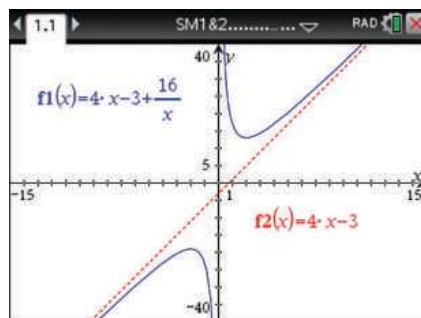
- Plot the non-vertical asymptote

$$f2(x) = 4x - 3$$

- Locate the turning points using **menu** > **Analyze Graph** > **Minimum** and **menu** > **Analyze Graph** > **Maximum**.

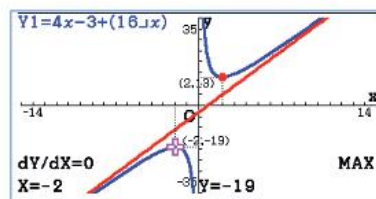
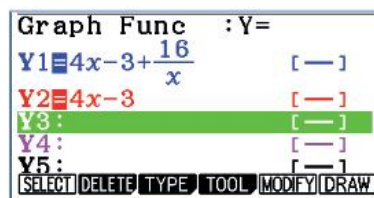
- There is a local minimum at $(2, 13)$ and a local maximum at $(-2, -19)$.

Note: Alternatively, the key points can be found using **menu** > **Trace** > **Graph Trace**.



Using the Casio

- Press **MENU** **5** to select **Graph** mode.
- Enter the function $y = 4x - 3 + \frac{16}{x}$ in Y1:
 $\boxed{4} \boxed{\times} \boxed{\theta, T} \boxed{-} \boxed{3} \boxed{+} \boxed{1} \boxed{6} \boxed{\div} \boxed{\times} \boxed{\theta, T} \boxed{\text{EXE}}$
- Enter the asymptote $y = 4x - 3$ in Y2:
 $\boxed{4} \boxed{\times} \boxed{\theta, T} \boxed{-} \boxed{3} \boxed{\text{EXE}}$
- Select **Draw** **F6**.
- Adjust the View Window **SHIFT** **F3** if required.
- To find the turning points, go to the **G-Solve** menu **SHIFT** **F5** and select **Minimum** **F3** and **Maximum** **F2**.



► Rational functions of the form $f(x) = \frac{ax + b}{cx + d}$

We start by considering rational functions with rules of the form

$$f(x) = \frac{ax + b}{cx + d}$$

where a , b , c and d are non-zero constants.

The graph of $y = f(x)$ can be obtained as a transformation of the graph of $y = \frac{1}{x}$.

Example 12

Sketch the graph of each of the following functions:

a $y = \frac{2x - 1}{x + 1}$

b $y = \frac{4x - 1}{2 - 2x}$

Solution

a We can write

$$y = \frac{2x - 1}{x + 1} = -\frac{3}{x + 1} + 2$$

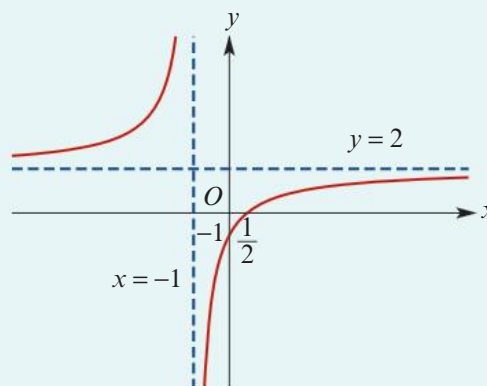
Asymptotes

The vertical asymptote is $x = -1$ and the horizontal asymptote is $y = 2$.

Intercepts

If $x = 0$, then $y = -1$.

If $y = 0$, then $\frac{2x - 1}{x + 1} = 0$, so $x = \frac{1}{2}$.



b We can write

$$y = \frac{4x-1}{2-2x} = -\frac{3}{2(x-1)} - 2$$

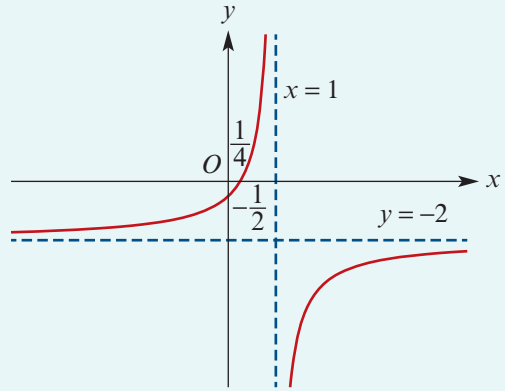
Asymptotes

The vertical asymptote is $x = 1$ and the horizontal asymptote is $y = -2$.

Intercepts

If $x = 0$, then $y = -\frac{1}{2}$.

If $y = 0$, then $\frac{4x-1}{2-2x} = 0$, so $x = \frac{1}{4}$.



► Rational functions of the form $f(x) = \frac{ax^2 + b}{x}$ where $a, b > 0$

We now consider rational functions with rules of the form

$$f(x) = \frac{ax^2 + b}{x} = ax + \frac{b}{x}$$

where a and b are positive constants. We can use the following algebraic technique to find the coordinates of the turning points.

Finding the turning points

Since both a and b are positive, we can complete the square in the numerator:

$$\begin{aligned} f(x) &= \frac{ax^2 + b}{x} \\ &= \frac{ax^2 - 2\sqrt{ab}x + b}{x} + 2\sqrt{ab} \\ &= \frac{(\sqrt{a}x - \sqrt{b})^2}{x} + 2\sqrt{ab} \end{aligned}$$

We see that $f\left(\sqrt{\frac{b}{a}}\right) = 2\sqrt{ab}$ and $f(x) \geq 2\sqrt{ab}$ for all $x > 0$. Therefore the point $\left(\sqrt{\frac{b}{a}}, 2\sqrt{ab}\right)$ is a local minimum on the graph of $y = f(x)$.

Similarly, we can write

$$f(x) = \frac{(\sqrt{a}x + \sqrt{b})^2}{x} - 2\sqrt{ab}$$

Therefore the point $\left(-\sqrt{\frac{b}{a}}, -2\sqrt{ab}\right)$ is a local maximum.

Note: After you have studied differentiation in Mathematical Methods Units 1 & 2, you will be able to find the two turning points more directly.

Example 13

Sketch the graph of $f(x) = \frac{x^2 + 1}{x}$.

Solution

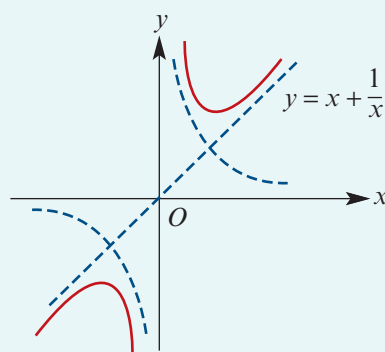
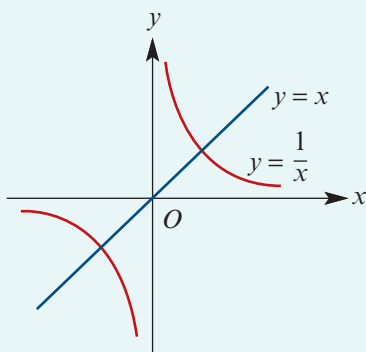
Asymptotes The vertical asymptote has equation $x = 0$, i.e. the y -axis.

Dividing through gives

$$f(x) = \frac{x^2 + 1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$$

Note that $\frac{1}{x} \rightarrow 0$ as $x \rightarrow \pm\infty$. Therefore the graph of $y = f(x)$ approaches the graph of $y = x$ as $x \rightarrow \pm\infty$. The non-vertical asymptote has equation $y = x$.

Addition of ordinates The graph of $y = f(x)$ can be obtained by adding the y -coordinates of the graphs of $y = x$ and $y = \frac{1}{x}$.



Intercepts There is no y -axis intercept, as the domain of f is $\mathbb{R} \setminus \{0\}$. There are no x -axis intercepts, as the equation $\frac{x^2 + 1}{x} = 0$ has no solutions.

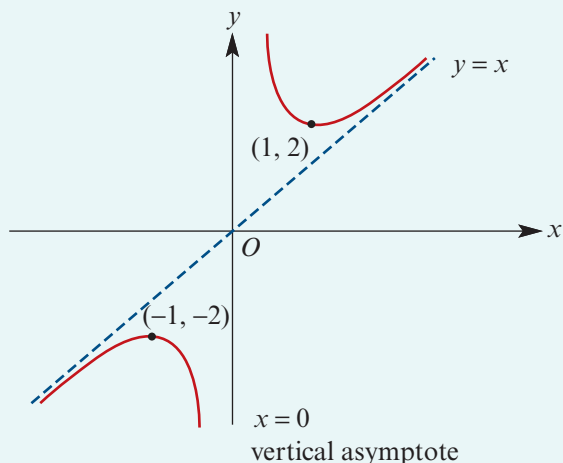
Turning points

We can write

$$\begin{aligned} f(x) &= \frac{x^2 + 1}{x} \\ &= \frac{x^2 - 2x + 1}{x} + 2 \\ &= \frac{(x - 1)^2}{x} + 2 \end{aligned}$$

Therefore $f(1) = 2$ and $f(x) \geq 2$ for all $x > 0$.

Hence $(1, 2)$ is a local minimum.



Similarly, we can write $f(x) = \frac{(x + 1)^2}{x} - 2$ to show that $(-1, -2)$ is a local maximum.

The reciprocal functions

Since we can sketch the graphs of functions of the form $f(x) = \frac{ax^2 + b}{x}$ where $a, b > 0$, we can also sketch the graphs of their reciprocals.

Example 14

Sketch the graph of $h(x) = \frac{x}{x^2 + 1}$.

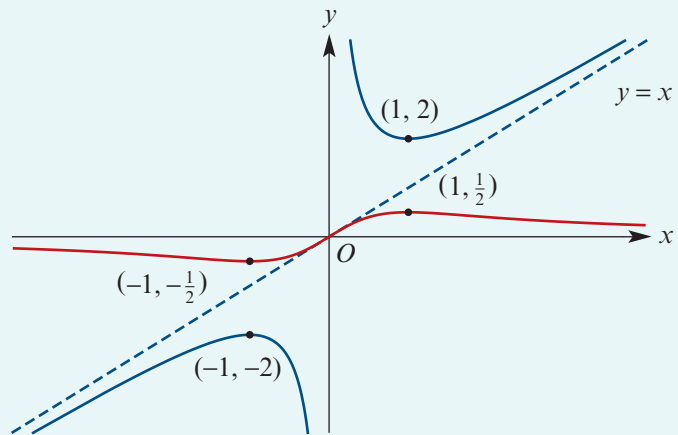
Solution

For $x \neq 0$, we have $h(x) = \frac{1}{\frac{x^2 + 1}{x}}$, where $f(x) = \frac{x^2 + 1}{x}$.

The graph of $y = f(x)$ from Example 13 is shown in blue.

By using properties of reciprocal functions we know that:

- The graph of $y = h(x)$ has a local maximum at $(1, \frac{1}{2})$ and a local minimum at $(-1, -\frac{1}{2})$.
- The graph of $y = h(x)$ is above the x -axis for $x > 0$ and below the x -axis for $x < 0$.
- Where the values of $f(x)$ approach ∞ , the values of $h(x)$ approach 0 from above.
- Where the values of $f(x)$ approach $-\infty$, the values of $h(x)$ approach 0 from below.



► Rational functions of the form $f(x) = \frac{ax^2 - b}{x}$ where $a, b > 0$

We now consider rational functions with rules of the form

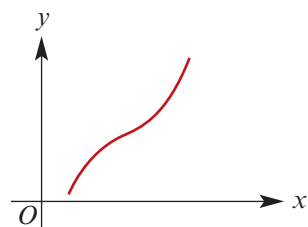
$$f(x) = \frac{ax^2 - b}{x} = ax - \frac{b}{x}$$

where a and b are positive constants. We can show that the graphs of these functions have no turning points.

We use the concept of a strictly increasing function, which is studied further in Mathematical Methods Units 1 & 2.

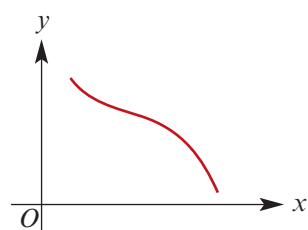
We say a function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

The graph opposite shows a strictly increasing function.



We say a function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

The graph opposite shows a strictly decreasing function.



Showing that there are no turning points

We can show that the function $f(x) = \frac{ax^2 - b}{x}$ is strictly increasing for $x > 0$.

Assume that $x_2 > x_1 > 0$. Then

$$\begin{aligned} f(x_2) - f(x_1) &= ax_2 - \frac{b}{x_2} - \left(ax_1 - \frac{b}{x_1}\right) \\ &= a(x_2 - x_1) + b\left(\frac{1}{x_1} - \frac{1}{x_2}\right) \\ &= a(x_2 - x_1) + \frac{b(x_2 - x_1)}{x_1x_2} > 0 \quad \text{since } x_2 > x_1 > 0 \end{aligned}$$

Therefore $f(x_2) > f(x_1)$, as required.

Similarly, we can show that f is strictly increasing for $x < 0$. Hence, the graph of $y = f(x)$ has no turning points.

Note: After you have studied differentiation in Mathematical Methods Units 1 & 2, you will be able to give an alternative proof that the function is strictly increasing.

Example 15

Sketch the graph of $f(x) = \frac{x^2 - 1}{x}$.

Solution

Asymptotes

The vertical asymptote has equation $x = 0$, i.e. the y-axis.

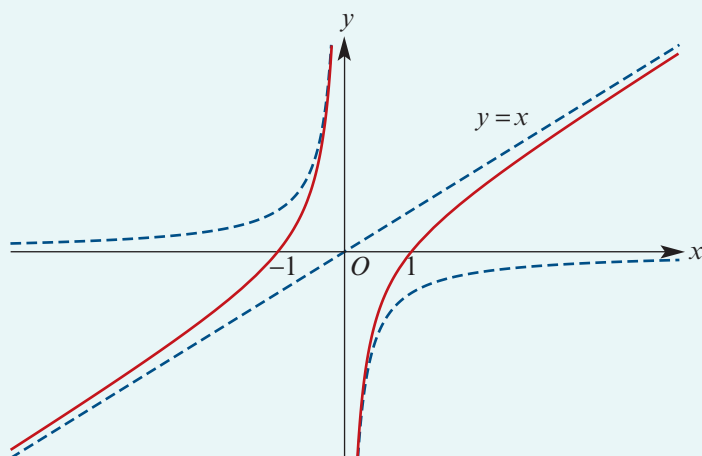
Dividing through gives

$$f(x) = \frac{x^2 - 1}{x} = x - \frac{1}{x}$$

Note that $\frac{1}{x} \rightarrow 0$ as $x \rightarrow \pm\infty$. Therefore the graph of $y = f(x)$ approaches the graph of $y = x$ as $x \rightarrow \pm\infty$. The non-vertical asymptote has equation $y = x$.

Addition of ordinates

The graph of $y = f(x)$ can be obtained by adding the y-coordinates of the graphs of $y = x$ and $y = -\frac{1}{x}$.

**Intercepts**

There is no y-axis intercept, as the domain of f is $\mathbb{R} \setminus \{0\}$. The x-axis intercepts are found by solving the equation $\frac{x^2 - 1}{x} = 0$. Hence the x-axis intercepts are at $(1, 0)$ and $(-1, 0)$.

Turning points

As discussed above, this function is strictly increasing for $x > 0$ and for $x < 0$. Hence there are no turning points.

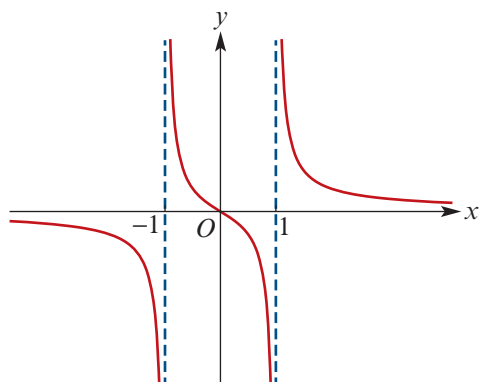
The reciprocal functions

Since we can sketch the graphs of functions of the form $f(x) = \frac{ax^2 - b}{x}$ where $a, b > 0$, we can also sketch the graphs of their reciprocals.

This is the graph of

$$h(x) = \frac{x}{x^2 - 1}$$

which is the reciprocal of the function from Example 15.



Section summary

- A rational function has a rule of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials.

- Important features of the graph of a rational function include:
 - asymptotes
 - axis intercepts
 - turning points.
- Turning points can be found using calculus or a calculator. For certain types of rational functions, the turning points can be found algebraically.
- Techniques for sketching the graph of a rational function:
 - addition of ordinates
 - reciprocals of ordinates.

Exercise 12C

Example 11

- 1 For each of the following, give the equations of the asymptotes of the graph of $y = f(x)$:

a $f(x) = \frac{2x+4}{x-3}$

b $f(x) = 5x + 3 + \frac{9}{x-2}$

c $f(x) = 4x - \frac{7}{x}$

d $f(x) = \frac{x}{4x^2+3}$

e $f(x) = 6x + 3 - \frac{5}{x}$

Example 12

- 2 Sketch the graph of each of the following rational functions:

a $f(x) = \frac{x+4}{x-1}$

b $f(x) = \frac{x-4}{x+1}$

c $f(x) = \frac{2x+1}{x-1}$

d $f(x) = \frac{2x-4}{x+2}$

e $f(x) = \frac{4-x}{x+1}$

f $f(x) = \frac{2x-4}{1-x}$

Example 13

- 3 Sketch the graph of each of the following rational functions:

a $f(x) = \frac{4x^2+9}{x}$

b $f(x) = \frac{x^2+4}{x}$

c $f(x) = \frac{x^2+5}{x}$

d $f(x) = 2x + \frac{9}{x}$

e $f(x) = 9x + \frac{4}{x}$

f $f(x) = 6x + \frac{3}{x}$

Example 14

- 4 Sketch the graph of each of the following rational functions:

a $f(x) = \frac{x}{x^2+9}$

b $f(x) = \frac{x}{4x^2+1}$

c $f(x) = \frac{x}{3x^2+9}$

d $f(x) = \frac{x}{x^2+5}$

e $f(x) = \frac{x}{x^2+4}$

f $f(x) = \frac{x}{6x^2+9}$

Example 15

- 5 Sketch the graph of each of the following rational functions:

a $f(x) = \frac{4x^2-9}{x}$

b $f(x) = \frac{x^2-4}{x}$

c $f(x) = \frac{x^2-5}{x}$

d $f(x) = 2x - \frac{9}{x}$

e $f(x) = 9x - \frac{4}{x}$

f $f(x) = 6x - \frac{3}{x}$

- 6** Sketch the graph of each of the following rational functions:

a $f(x) = \frac{x}{x^2 - 9}$

b $f(x) = \frac{x}{4x^2 - 1}$

c $f(x) = \frac{x}{3x^2 - 9}$

d $f(x) = \frac{x}{x^2 - 5}$

e $f(x) = \frac{x}{x^2 - 4}$

f $f(x) = \frac{x}{6x^2 - 9}$

- 7** Sketch the graph of $y = x + \frac{4}{x}$ for positive values of x , and find the least value of y .

Example 11

- 8** Let $f(x) = 2x + 3 + \frac{8}{x}$.

a Show that $f(x) \geq 11$ for all $x > 0$.

b Show that $f(x) \leq -5$ for all $x < 0$.

c Find x such that $f(x) = 11$.

d Find x such that $f(x) = -5$.

e Sketch the graph of $y = f(x)$.

- 9** Let $f(x) = 1 + 2x + \frac{4}{3 + 2x}$.

a Show that $f(x) \geq 2$ for all $x > -\frac{3}{2}$.

b Show that $f(x) \leq -6$ for all $x < -\frac{3}{2}$.

c Find x such that $f(x) = 2$.

d Find x such that $f(x) = -6$.

e Sketch the graph of $y = f(x)$.

- 10** Let $f(x) = 2x + 3 - \frac{8}{x}$.

a Show that the function f is strictly increasing for $x > 0$ and for $x < 0$.

b Sketch the graph of $y = f(x)$.

- 11** Let $f(x) = x - 5 - \frac{4}{x}$.

a Show that the function f is strictly increasing for $x > 0$ and for $x < 0$.

b Sketch the graph of $y = f(x)$.

- 12** Let $g(x) = \frac{(x-4)^2}{x}$.

a Show that $g(4) = 0$ and that $g(x) \geq 0$ for all $x > 0$.

b Show that $g(-4) = -16$ and that $g(x) \leq -16$ for all $x < 0$.

c Sketch the graph of $y = g(x)$.

- 13** Let $h(x) = \frac{2x^2 + x + 2}{x^2 + 1}$.

a Divide the denominator into the numerator.

b State the equation of the horizontal asymptote.

c Sketch the graph of $y = h(x)$.

- 14** Sketch the graphs of:

a $f(x) = \frac{x+2}{x-2}$

b $g(x) = \left| \frac{x+2}{x-2} \right|$

12D Graphing rational functions using calculus

In this section we continue our study of rational functions, but we use calculus to determine the turning points. You may wish to come back to this section later in the course, after you have studied differentiation in Mathematical Methods Units 1 & 2.

Example 16

Sketch the graph of $f(x) = \frac{x^4 + 2}{x^2}$.

Solution

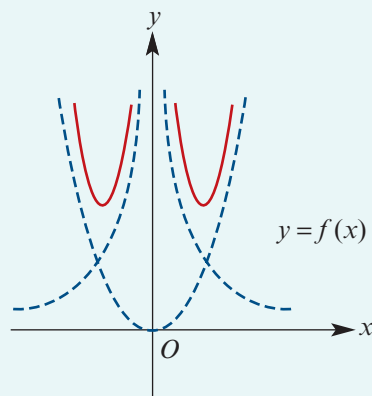
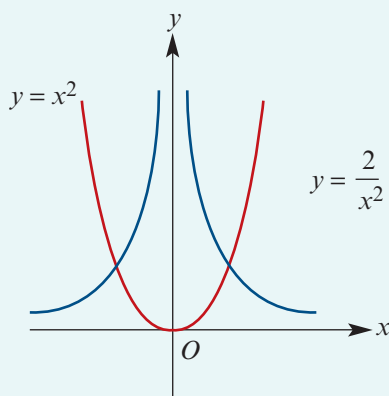
Asymptotes The vertical asymptote has equation $x = 0$.

Dividing through gives

$$f(x) = x^2 + \frac{2}{x^2}$$

The non-vertical asymptote has equation $y = x^2$.

Addition of ordinates



Intercepts There are no axis intercepts.

Stationary points

$$f(x) = x^2 + 2x^{-2}$$

$$\therefore f'(x) = 2x - 4x^{-3}$$

When $f'(x) = 0$,

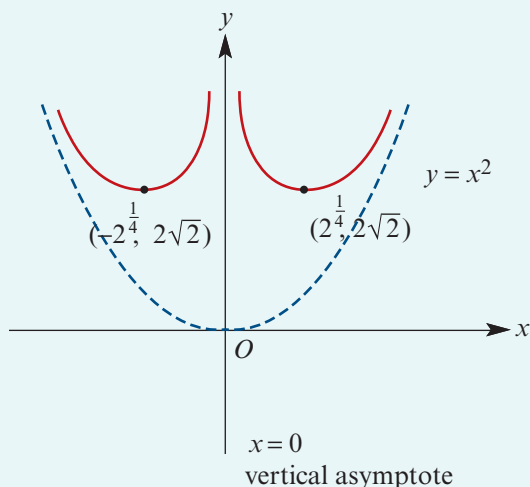
$$2x - \frac{4}{x^3} = 0$$

$$2x^4 - 4 = 0$$

$$\therefore x = \pm 2^{\frac{1}{4}}$$

The stationary points have coordinates

$$(2^{\frac{1}{4}}, 2\sqrt{2}) \text{ and } (-2^{\frac{1}{4}}, 2\sqrt{2}).$$



Example 17

Sketch the graph of $y = \frac{x^3 + 2}{x}$.

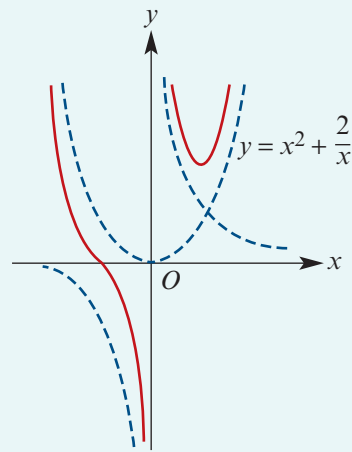
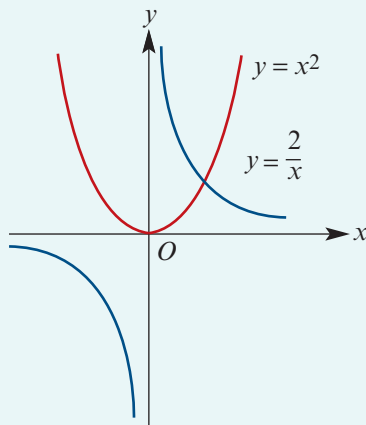
Solution**Asymptotes**

The vertical asymptote has equation $x = 0$.

Divide through to obtain

$$y = x^2 + \frac{2}{x}$$

The non-vertical asymptote has equation $y = x^2$.

Addition of ordinates**Intercepts**

Consider $y = 0$, which implies $x^3 + 2 = 0$, i.e. $x = -\sqrt[3]{2}$.

Stationary points

$$y = x^2 + 2x^{-1}$$

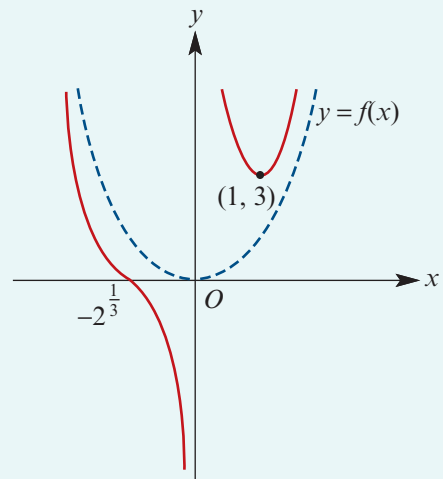
$$\therefore \frac{dy}{dx} = 2x - 2x^{-2}$$

$$\text{Thus } \frac{dy}{dx} = 0 \text{ implies } x - \frac{1}{x^2} = 0$$

$$x^3 = 1$$

$$\therefore x = 1$$

The turning point has coordinates $(1, 3)$.



Example 18

Sketch the graph of $y = \frac{4x^2 + 2}{x^2 + 1}$.

Solution**Asymptotes**

We can write

$$y = \frac{4x^2 + 2}{x^2 + 1} = 4 - \frac{2}{x^2 + 1}$$

The line $y = 4$ is a horizontal asymptote, since $\frac{2}{x^2 + 1} \rightarrow 0$ as $x \rightarrow \pm\infty$.

Intercepts

When $x = 0$, $y = 2$.

Since $\frac{4x^2 + 2}{x^2 + 1} > 0$ for all x , there are no x -axis intercepts.

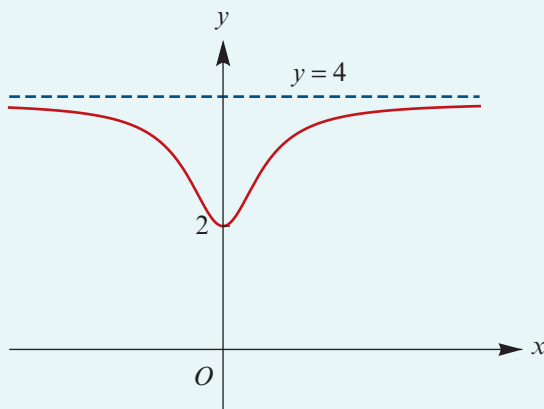
Stationary points

Using the quotient rule:

$$\frac{dy}{dx} = \frac{4x}{(x^2 + 1)^2}$$

Thus $\frac{dy}{dx} = 0$ implies $x = 0$.

There is a local minimum at $(0, 2)$.



Note: This example can also be completed by noting that $y = 4 - \frac{2}{x^2 + 1}$.

This is a transformation of $y = \frac{1}{x^2 + 1}$, which is the reciprocal of $y = x^2 + 1$.

Exercise 12D**Skillsheet**

- 1 Sketch the graph of each of the following, labelling all axis intercepts, turning points and asymptotes:

a $y = \frac{1}{x^2 - 2x}$

b $y = \frac{x^4 + 1}{x^2}$

c $y = \frac{x^2 - 1}{x}$

d $y = \frac{x^3 - 1}{x^2}$

e $y = \frac{x^2 + x + 1}{x}$

f $y = \frac{4x^3 - 8}{x}$

g $y = \frac{x^2}{x^2 + 1} = 1 - \frac{1}{x^2 + 1}$

- 2** For the curve $y = x - 5 + \frac{4}{x}$, find:

- a** the coordinates of the points of intersection with the axes
- b** the equations of all asymptotes
- c** the coordinates of all turning points.

Use this information to sketch the curve.

- 3 a** Find the coordinates of the stationary points of the curve $y = \frac{(x-3)^2}{x}$ and determine the nature of each stationary point.

- b** Sketch the graph of $y = \frac{(x-3)^2}{x}$.

- 4 a** Find the coordinates of the turning point(s) of the curve $y = 8x + \frac{1}{2x^2}$ and determine the nature of each point.

- b** Sketch the graph of $y = 8x + \frac{1}{2x^2}$.

- 5** Determine the asymptotes, intercepts and stationary points for the graph of the relation $y = \frac{x^3 + 3x^2 - 4}{x^2}$. Hence sketch the graph.

- 6** Consider the relation $y = \frac{4x^2 + 8}{2x + 1}$.

- a** State the maximal domain.
- b** Find $\frac{dy}{dx}$.
- c** Hence find the coordinates and nature of all stationary points.
- d** Find the equations of all asymptotes.
- e** State the range of this relation.

Example 18

- 7** Consider the function with rule $f(x) = \frac{x^2 + 4}{x^2 - 5x + 4}$.

- a** Find the equations of all asymptotes.
- b** Find the coordinates and nature of all stationary points.
- c** Sketch the graph of $y = f(x)$. Include the coordinates of the points of intersection of the graph with the horizontal asymptote.



Chapter summary

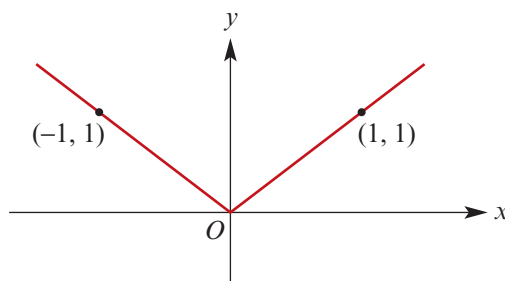


The modulus function

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example: $|5| = 5$ and $|-5| = 5$.



- The graph of the modulus function $y = |x|$ is shown on the right.
- On the number line, the distance between two numbers a and b is given by $|a - b| = |b - a|$.
For example: $|x - 2| < 5$ can be read as 'the distance of x from 2 is less than 5'.

Reciprocal functions

- If $y = f(x)$ is a function, then the **reciprocal function** is defined by the rule $y = \frac{1}{f(x)}$.
- To sketch the graph of $y = \frac{1}{f(x)}$, we first sketch the graph of $y = f(x)$.
- The x -axis intercepts of $y = f(x)$ will become vertical asymptotes of $y = \frac{1}{f(x)}$.
- Local maximums of $y = f(x)$ will become local minimums of $y = \frac{1}{f(x)}$, and vice versa.

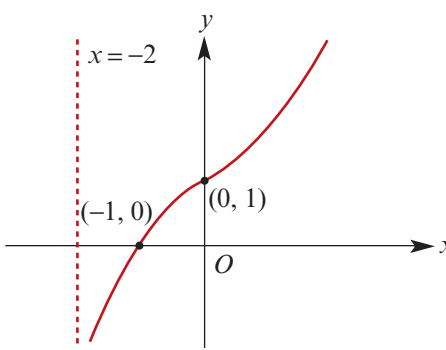
Rational functions

- A rational function has a rule of the form:

$$\begin{aligned} f(x) &= \frac{a(x)}{b(x)} && \text{where } a(x) \text{ and } b(x) \text{ are polynomials} \\ &= q(x) + \frac{r(x)}{b(x)} && \text{(quotient-remainder form)} \end{aligned}$$

- Vertical asymptotes occur where $b(x) = 0$.
- The non-vertical asymptote has equation $y = q(x)$.
- The x -axis intercepts occur where $a(x) = 0$.
- The y -axis intercept is $f(0) = \frac{a(0)}{b(0)}$, provided $b(0) \neq 0$.
- Turning points can be found using calculus or a calculator. For certain types of rational functions, the turning points can be found algebraically.
- Techniques that may help when sketching the graph of $y = f(x)$:
 - **Addition of ordinates**
First sketch the graphs of $y = q(x)$ and $y = \frac{r(x)}{b(x)}$. Then use addition of ordinates to sketch the graph of $y = f(x)$.
 - **Reciprocals of ordinates**
In some cases, it may be easier to sketch the graph of $y = \frac{1}{f(x)}$ and use the techniques for reciprocal functions to sketch the graph of $y = f(x)$.

Technology-free questions

- 1** State the value of each of the following without using the absolute value function in your answer:
a $|-9|$ **b** $\left| -\frac{1}{400} \right|$ **c** $|9 - 5|$ **d** $|5 - 9|$ **e** $|\pi - 3|$ **f** $|\pi - 4|$
- 2** Let $f(x) = |x^2 - 3x|$. Solve the equation $f(x) = x$.
- 3** For each of the following, sketch the graph of $y = f(x)$ and state the range of f :
a $f(x) = |x^2 - 4x|$ **b** $f(x) = |x^2 - 4x| - 3$ **c** $f(x) = 3 - |x^2 - 4x|$
- 4** **a** Find the four integer values of n such that $|n^2 - 9|$ is a prime number.
b Solve each equation for x :
i $x^2 + 5|x| - 6 = 0$ **ii** $x + |x| = 0$
c Solve the inequality $5 - |x| < 4$ for x .
- 5** For each of the following, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same axes:
a $f(x) = \frac{1}{2}(x^2 - 4)$ **b** $f(x) = (x + 1)^2 + 1$
- 6** This is the graph of $y = f(x)$.
 Sketch the graph of $y = \frac{1}{f(x)}$.
- 
- 7** Sketch the graph of $y = |x| + x + \frac{2}{x}$.
- 8** Let $f(x) = \frac{(x-2)^2}{x}$.
a Show that $f(2) = 0$ and that $f(x) \geq 0$ for all $x > 0$.
b Show that $f(-2) = -8$ and that $f(x) \leq -8$ for all $x < 0$.
c Sketch the graph of $y = f(x)$.
- 9** For each of the following, give the equations of the asymptotes (horizontal, vertical and oblique) of the graph of $y = f(x)$:
a $f(x) = \frac{2x+1}{x-3}$ **b** $f(x) = \frac{5x+1}{2x-3}$ **c** $f(x) = \frac{(x+2)^2}{x}$
d $f(x) = -3x - 1 + \frac{4}{(x+2)^2}$

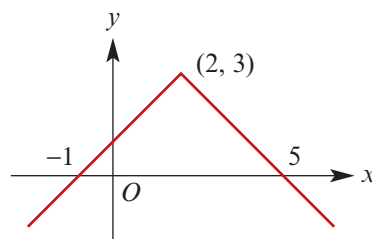
- 10** Let $g(x) = x^4 - x^2$.
- a** The graph of $y = g(x)$ has local minimums at the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{4}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{4}\right)$.
Use this information to help sketch the graph of $y = g(x)$.
- b** Sketch the graph of $y = \frac{1}{g(x)}$.
- 11** Let $h(x) = \frac{4x^2 + 4x + 6}{2x^2 + 3}$.
- a** Divide the denominator into the numerator.
- b** State the equation of the horizontal asymptote.
- c** Sketch the graph of $y = h(x)$.
- 12** Sketch the graphs of:
- a** $f(x) = \frac{2x - 1}{x + 1}$ **b** $g(x) = \left| \frac{2x - 1}{x + 1} \right|$

Multiple-choice questions



- 1** The rule for the graph shown is

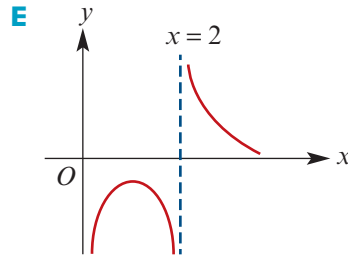
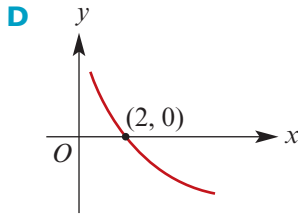
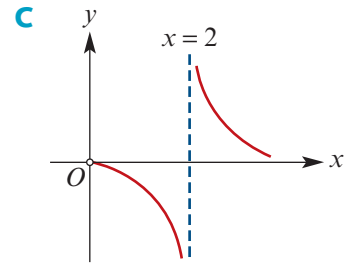
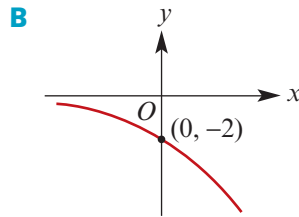
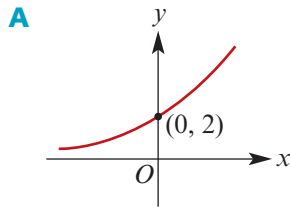
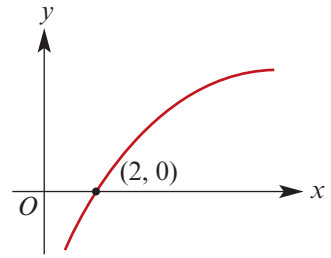
- A** $y = |x - 2| + 3$ **B** $y = |-x + 2| + 3$
C $y = |x + 2| + 3$ **D** $y = -|x - 2| + 3$
E $y = -|x + 2| + 3$



- 2** The curve with equation $y = \frac{1}{x^2 - 4}$ has
- A** a local minimum at $(0, -4)$ **B** a local minimum at $(0, -\frac{1}{4})$
C a local maximum at $(\frac{1}{4}, 0)$ **D** a local maximum at $(0, -4)$
E a local maximum at $(0, -\frac{1}{4})$
- 3** The graph of $y = \frac{1}{x^2 + 8x + k}$ will have two vertical asymptotes provided
- A** $k = 16$ **B** $k < 16$ **C** $k > 16$
D $k < -4$ or $k > 4$ **E** $-4 < k < 4$
- 4** The graph of $y = \frac{x^2 + x + 2}{x}$ has asymptotes with equations
- A** $y = x$ and $y = x^2 + x + 2$ **B** $y = x$ and $y = x + 1$
C $x = 0$ and $y = x^2 + x + 2$ **D** $x = 0$ and $y = x + 1$
E $y = \frac{2}{x}$ and $y = x + 1$

- 5 The graph of $y = f(x)$ is shown on the right.

Which one of the following best represents the graph of $y = \frac{1}{f(x)}$?



- 6 Let $f(x) = 2x^2 + 3x - 20$. The graph of $y = \frac{1}{f(x)}$ has

- A** x-axis intercepts at $x = \frac{5}{2}$ and $x = -4$
B vertical asymptotes at $x = \frac{5}{2}$ and $x = 4$
C vertical asymptotes at $x = -\frac{5}{2}$ and $x = 4$
D a local minimum at the point $\left(\frac{-3}{4}, \frac{-169}{8}\right)$
E a local maximum at the point $\left(\frac{-3}{4}, \frac{-8}{169}\right)$

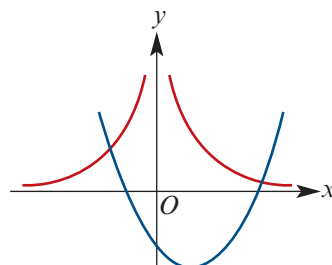
- 7 The graph of $y = \frac{x^2 + 5}{x^2 + 2x + 1}$ has asymptotes with equations

- A** $x = 1$, $x = -1$, $y = 1$ **B** $x = 1$, $x = -1$, $y = 5$ **C** $x = -1$, $y = 5$
D $x = -1$, $y = 1$ **E** $x = 1$, $y = 1$

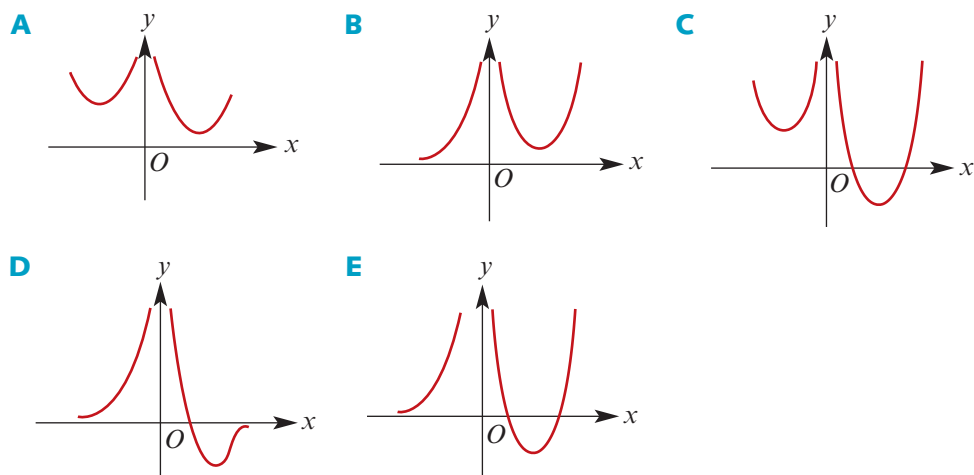
8 The axis intercepts of the graph of $y = \frac{x^2 - x - 12}{x + 6}$ are

- A** (4, 0), (-3, 0), (0, -2) **B** (-4, 0), (3, 0), (0, 16) **C** (4, 0), (-3, 0), (0, 2)
D (-4, 0), (3, 0), (0, -2) **E** (-12, 0), (1, 0), (0, -2)

9 The graphs of $y = f(x)$ and $y = g(x)$ are shown on the right.

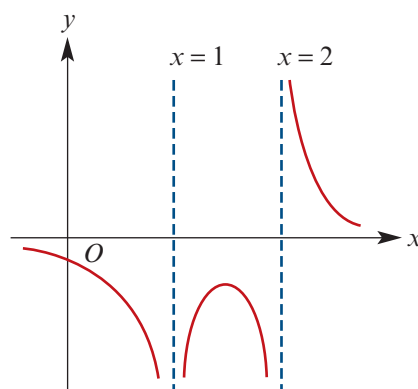


Which one of the following best represents the graph of $y = f(x) + g(x)$?



10 This is the graph of

- A** $y = \frac{1}{(x-1)(x-2)}$
B $y = \frac{x}{(x-1)(x-2)}$
C $y = \frac{(x-1)(x-2)}{x}$
D $y = \frac{1}{(x-2)(x-1)^2}$
E $y = \frac{1}{(x-1)(x-2)^2}$



- 11** Which function does not have a horizontal asymptote?

A $g(x) = \frac{x^2 + 2}{x - 6}$

B $g(x) = \frac{-3x^2 + 1}{x^2 - 1}$

C $g(x) = \frac{x - 9}{x + 3}$

D $g(x) = \frac{6x^2 - 5}{x^2 + 1}$

E $g(x) = \frac{2x - 9}{x + 3}$

- 12** Consider the equation $|x^2 + bx - c| = d$, where b , c and d are positive constants. This equation has four solutions when

A $d > \frac{b^2}{4} + c$

B $d > b^2 + 4c$

C $d < 0$

D $0 < d < \frac{b^2}{4} + c$

E $0 < d < b^2 + 4c$

Extended-response questions

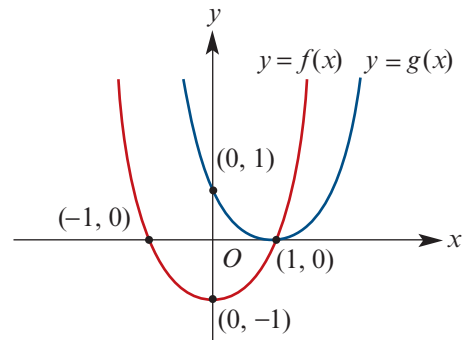
- 1** These are the graphs of $y = f(x)$ and $y = g(x)$, where f and g are quadratic functions.

- a** Sketch the graphs of:

i $y = f(x) + g(x)$

ii $y = \frac{1}{f(x) + g(x)}$

iii $y = \frac{1}{f(x)} + \frac{1}{g(x)}$



- b** Use the points given to determine the rules $y = f(x)$ and $y = g(x)$.

- c** Hence determine, in simplest form, the rules:

i $y = f(x) + g(x)$ **ii** $y = \frac{1}{f(x) + g(x)}$ **iii** $y = \frac{1}{f(x)} + \frac{1}{g(x)}$

- 2** Let $f(x) = |mx + 2|$, where $m > 0$.

- a** Find the x -axis intercept of the graph of f in terms of m .

- b** For what values of m is the x -axis intercept less than -2 ?

- c i** Find the equation of the line ℓ that is perpendicular to the graph of f at the point with coordinates $(0, 2)$.

- ii** For $m > 1$, find the coordinates of the other point of intersection of the line ℓ with the graph of f .

- iii** What happens for $m = 1$?

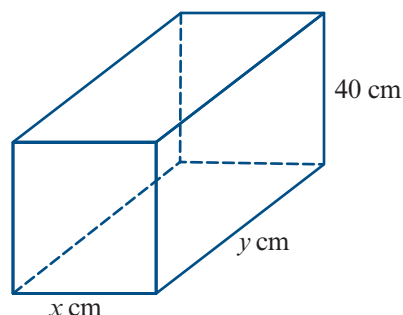
- iv** For what value of m does the line ℓ meet the graph of f where $x = -\frac{3}{2}$?

- d** Solve the equation $f(x) = \frac{1}{f(x)}$ for x in terms of m .

- 3 a** Consider the function with rule $f(x) = |x^2 - ax|$, where a is a constant.
- i** Sketch the graph of $y = f(x)$ for $a = 2$.
 - ii** For $a \neq 0$, find the x -axis intercepts of the graph of $y = f(x)$.
 - iii** For $a \neq 0$, find the coordinates of the local maximum on the graph of $y = f(x)$.
 - iv** Find the values of a for which the point $(-1, 4)$ lies on the graph of $y = f(x)$.
- b** Consider the function with rule $g(x) = |x|^2 - a|x|$, where a is a constant.
- i** Sketch the graph of $y = g(x)$ for $a = 2$.
 - ii** For $a > 0$, find the x -axis intercepts of the graph of $y = g(x)$.
 - iii** For $a > 0$, find the coordinates of the local minimums on the graph of $y = g(x)$.
 - iv** Find the value of a for which the point $(-1, 4)$ lies on the graph of $y = g(x)$.
- c** For $a > 0$, find the values of x such that $f(x) = g(x)$.
- d** For $a < 0$, find the values of x such that $f(x) = g(x)$.
- e** Find the rule for the function $h(x) = f(x) + g(x)$.
- f** Sketch the graph of $y = h(x)$ for $a = 2$.
- g** Sketch the graphs of $y = \frac{1}{f(x)}$ and $y = \frac{1}{g(x)}$ for $a = 2$.

- 4 a** Sketch the graph of $y = 80x + \frac{2000}{x}$. Give the equations of asymptotes and the coordinates of turning points.

- b** A box with a volume of 1000 cm^3 is to be made in the shape of a rectangular prism. It has a fixed height of 40 cm . The other dimensions are $x \text{ cm}$ and $y \text{ cm}$ as shown. The total surface area is $A \text{ cm}^2$.



- i** Express A in terms of x .
- ii** Sketch the graph of A against x .
- iii** Find the minimum surface area of the box and the dimensions of the box in this situation.
- iv** Find the minimum surface area of the box and the dimensions of the box if the height of the box is $k \text{ cm}$ (for a constant k) while the volume remains 1000 cm^3 .

- 5** Let $f(x) = 2x - \frac{1}{8x^2}$.

- a** Find the equations of the asymptotes of the graph of $y = f(x)$.
- b** Find the x -axis intercepts of the graph of $y = f(x)$.
- c** Show that the function f is strictly increasing for $x > 0$.
- d** On the one set of axes, sketch the graphs of $y = 2x$, $y = -\frac{1}{8x^2}$ and $y = f(x)$ by using addition of ordinates.
- e** Use calculus or a calculator to find the coordinates of the turning point.

13

Trigonometric functions

Objectives

- ▶ To review **radian measure**.
- ▶ To review the definitions of the trigonometric functions **sine**, **cosine** and **tangent**.
- ▶ To review the **symmetry properties** of trigonometric functions.
- ▶ To **solve equations** involving trigonometric functions.
- ▶ To **sketch graphs** of trigonometric functions.
- ▶ To define the reciprocal trigonometric functions **secant**, **cosecant** and **cotangent**.
- ▶ To sketch transformations of the graphs of the reciprocal trigonometric functions.
- ▶ To apply trigonometric functions to model periodic phenomena.

This chapter reviews and extends the study of the three trigonometric functions – sine, cosine and tangent – introduced in Mathematical Methods Units 1 & 2. This chapter also introduces the three reciprocal trigonometric functions – secant, cosecant and cotangent.

An important property of the trigonometric functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function f is **periodic** if there is a positive constant a such that $f(x + a) = f(x)$. The sine and cosine functions each have period 2π , while the tangent function has period π .

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

We will continue our study of trigonometric functions in Chapter 14, where we derive and apply several important trigonometric identities.

13A Defining the trigonometric functions

► Measuring angles in degrees and radians

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.

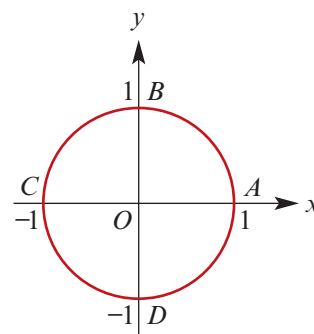
$$\begin{aligned}\text{The circumference of the unit circle} &= 2\pi \times 1 \\ &= 2\pi \text{ units}\end{aligned}$$

Thus, the distance in an anticlockwise direction around the circle from

$$A \text{ to } B = \frac{\pi}{2} \text{ units}$$

$$A \text{ to } C = \pi \text{ units}$$

$$A \text{ to } D = \frac{3\pi}{2} \text{ units}$$

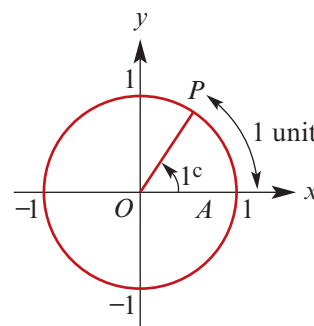


Definition of a radian

In moving around the circle a distance of 1 unit from A to P , the angle POA is defined. The measure of this angle is 1 radian.

One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.



Converting between degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \pi^c = 180^\circ$$

$$\therefore 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Example 1

a Convert 135° to radians.

b Convert $\frac{\pi^c}{6}$ to degrees.

Solution

a $135^\circ = \frac{135 \times \pi^c}{180} = \frac{3\pi^c}{4}$

b $\frac{\pi^c}{6} = \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ$

Note: Usually the symbol for radians, ^{c} , is omitted. Any angle is assumed to be measured in radians unless indicated otherwise.

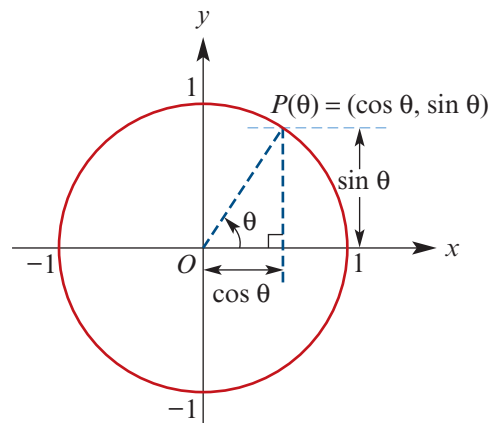
► Defining sine and cosine

Let $P(\theta)$ denote the point on the unit circle corresponding to an angle θ . Then:

- $\cos \theta$ is the x -coordinate of $P(\theta)$
- $\sin \theta$ is the y -coordinate of $P(\theta)$

Hence the coordinates of the point $P(\theta)$ are $(\cos \theta, \sin \theta)$.

Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos \theta$ and $\sin(2\pi + \theta) = \sin \theta$.



► The graphs of sine and cosine

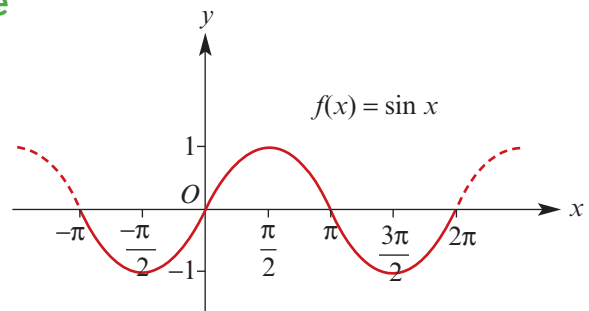
A sketch of the graph of

$$f(x) = \sin x, \quad x \in \mathbb{R}$$

is shown opposite.

As $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbb{R}$, the sine function is **periodic**.

The period is 2π . The amplitude is 1.



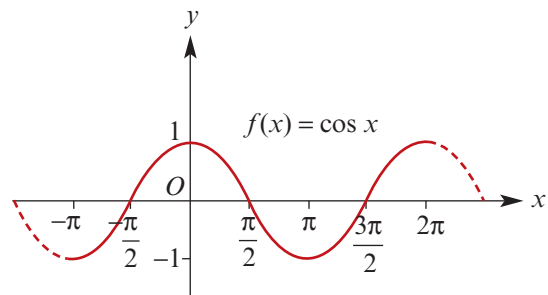
A sketch of the graph of

$$f(x) = \cos x, \quad x \in \mathbb{R}$$

is shown opposite.

The period of the cosine function is 2π .

The amplitude is 1.



Example 2

Evaluate each of the following:

a $\sin\left(\frac{3\pi}{2}\right)$

b $\cos\left(-\frac{\pi}{2}\right)$

c $\cos\left(\frac{21\pi}{2}\right)$

Solution

a $\sin\left(\frac{3\pi}{2}\right) = -1$

b $\cos\left(-\frac{\pi}{2}\right) = 0$

c $\cos\left(\frac{21\pi}{2}\right) = \cos\left(10\pi + \frac{\pi}{2}\right) = 0$

Explanation

since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.

since $P\left(-\frac{\pi}{2}\right)$ has coordinates $(0, -1)$.

since $P\left(\frac{\pi}{2}\right)$ has coordinates $(0, 1)$.

► Defining tangent

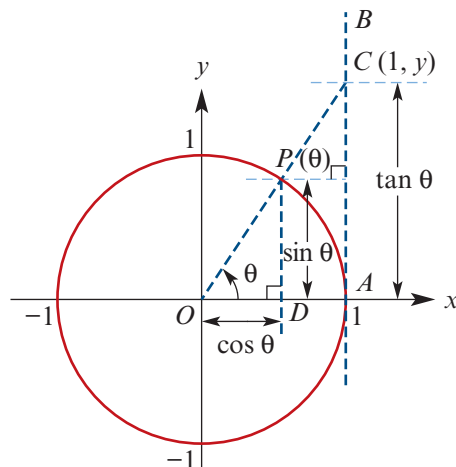
Again consider the unit circle.

If we draw a tangent to the unit circle at A , then the y -coordinate of C , the point of intersection of the line OP and the tangent, is called **tangent** θ (abbreviated to $\tan \theta$).

By considering the similar triangles OPD and OCA :

$$\frac{\tan \theta}{1} = \frac{\sin \theta}{\cos \theta}$$

$$\therefore \tan \theta = \frac{\sin \theta}{\cos \theta}$$



Note that $\tan \theta$ is undefined when $\cos \theta = 0$. So $\tan \theta$ is undefined for $\theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$

Note: Adding π to the angle does not change the line OP . Thus $\tan(\pi + \theta) = \tan \theta$.

► The trigonometric ratios

We have seen in Chapter 3 that, for acute angles, the unit-circle definition of sine and cosine is equivalent to the ratio definition.

For a right-angled triangle OBC , we can construct a similar triangle $OB'C'$ that lies in the unit circle.

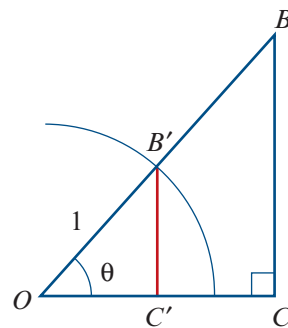
From the diagram:

$$B'C' = \sin \theta \quad \text{and} \quad OC' = \cos \theta$$

The similarity factor is the length OB , giving

$$BC = OB \sin \theta \quad \text{and} \quad OC = OB \cos \theta$$

$$\therefore \frac{BC}{OB} = \sin \theta \quad \text{and} \quad \frac{OC}{OB} = \cos \theta$$

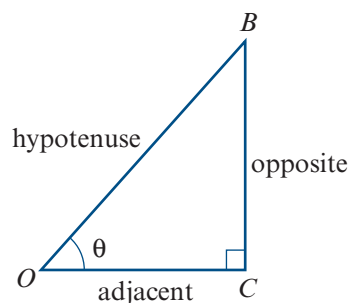


This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ is as shown.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

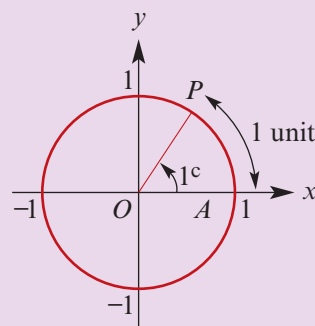


Section summary

■ Definition of a radian

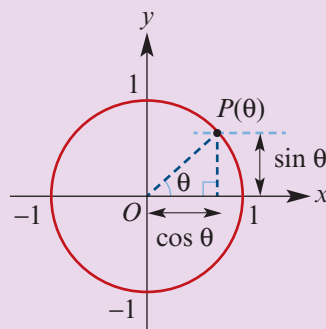
One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$



■ Sine and cosine functions

The point $P(\theta)$ on the unit circle corresponding to an angle θ has coordinates $(\cos \theta, \sin \theta)$.



Exercise 13A

Example 1a

1 Convert the following angles from degrees to exact values in radians:

a 720°

b 540°

c -450°

d 15°

e -10°

f -315°

Example 1b

2 Convert the following angles from radians to degrees:

a $\frac{5\pi}{4}$

b $-\frac{2\pi}{3}$

c $\frac{7\pi}{12}$

d $-\frac{11\pi}{6}$

e $\frac{13\pi}{9}$

f $-\frac{11\pi}{12}$

Example 2

3 Find the exact value of each of the following:

a $\cos\left(\frac{3\pi}{2}\right)$

b $\sin\left(-\frac{\pi}{2}\right)$

c $\cos(6\pi)$

d $\sin\left(\frac{15\pi}{2}\right)$

4 Find the exact value of each of the following:

a $\sin(270^\circ)$

b $\cos(-540^\circ)$

c $\sin(450^\circ)$

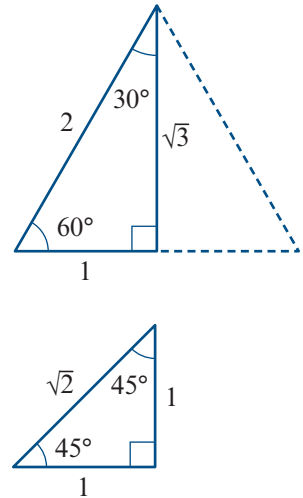
d $\cos(720^\circ)$

13B Symmetry properties and the Pythagorean identity

► Exact values of trigonometric functions

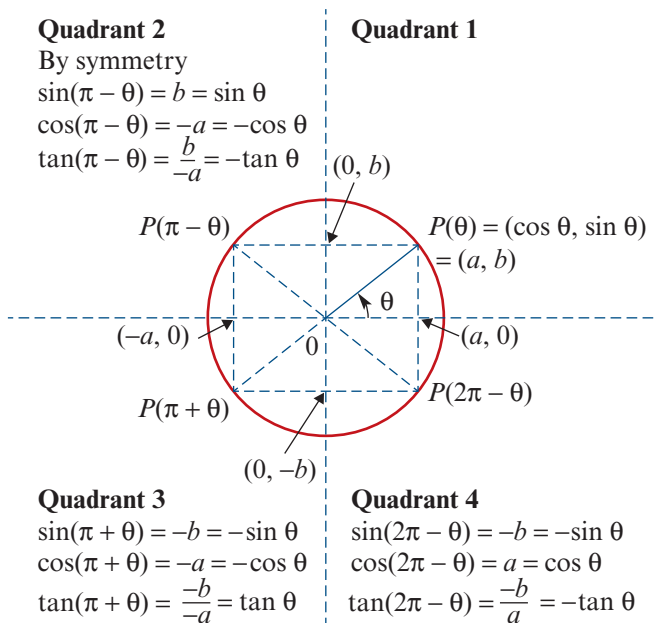
The values in this table for 30° , 45° and 60° can be determined from the two triangles shown.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined



► Symmetry properties

The coordinate axes divide the unit circle into four quadrants, numbered anticlockwise from the positive direction of the x -axis. Using symmetry, we can determine relationships between the trigonometric functions for angles in different quadrants:

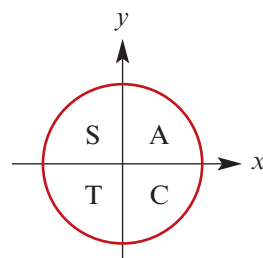


Note: $\sin(2\pi + \theta) = \sin \theta$
 $\cos(2\pi + \theta) = \cos \theta$
 $\tan(2\pi + \theta) = \tan \theta$

Signs of trigonometric functions

Using these symmetry properties, the signs of $\sin$, $\cos$ and $\tan$ for the four quadrants can be summarised as follows:

1st quadrant	all are positive	(A)
2nd quadrant	$\sin$ is positive	(S)
3rd quadrant	$\tan$ is positive	(T)
4th quadrant	$\cos$ is positive	(C)



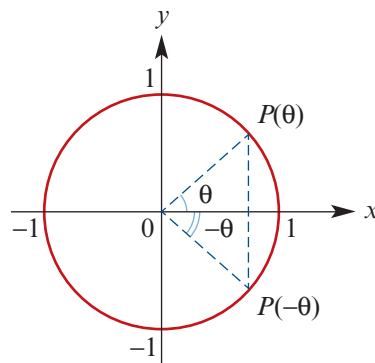
Negative angles

By symmetry:

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = \frac{-\sin \theta}{\cos \theta} = -\tan \theta$$



Example 3

Find the exact value of:

a $\sin\left(\frac{11\pi}{6}\right)$

b $\cos\left(\frac{-5\pi}{4}\right)$

Solution

$$\begin{aligned} \text{a } \sin\left(\frac{11\pi}{6}\right) &= \sin\left(2\pi - \frac{\pi}{6}\right) \\ &= -\sin\left(\frac{\pi}{6}\right) \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{b } \cos\left(\frac{-5\pi}{4}\right) &= \cos\left(\frac{5\pi}{4}\right) \\ &= \cos\left(\pi + \frac{\pi}{4}\right) \\ &= -\cos\left(\frac{\pi}{4}\right) \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

Example 4

Find the exact value of:

a $\sin(-150^\circ)$

b $\cos 585^\circ$

Solution

$$\begin{aligned} \text{a } \sin(-150^\circ) &= -\sin 150^\circ \\ &= -\sin(180^\circ - 30^\circ) \\ &= -\sin 30^\circ \\ &= -\frac{1}{2} \end{aligned}$$

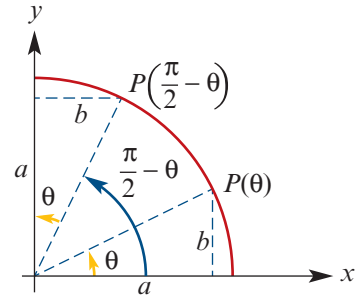
$$\begin{aligned} \text{b } \cos 585^\circ &= \cos(585^\circ - 360^\circ) \\ &= \cos 225^\circ \\ &= -\cos 45^\circ \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

► Complementary relationships

From the diagram to the right:

$$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos \theta$$

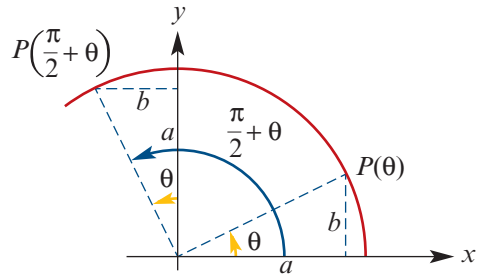
$$\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin \theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos \theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin \theta$$



Example 5

If $\sin \theta = 0.4$ and $\cos \alpha = 0.8$, find the value of:

a $\sin\left(\frac{\pi}{2} - \alpha\right)$

b $\cos\left(\frac{\pi}{2} + \theta\right)$

c $\sin(-\theta)$

Solution

a $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$
 $= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$
 $= -0.4$

c $\sin(-\theta) = -\sin \theta$
 $= -0.4$

► The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.

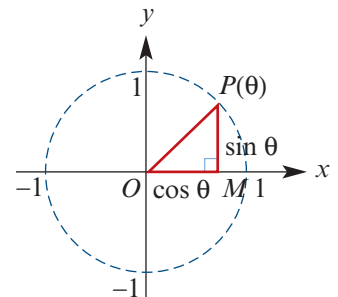
By Pythagoras' theorem:

$$OP^2 = OM^2 + MP^2$$

$$\therefore 1 = (\cos \theta)^2 + (\sin \theta)^2$$

Since this is true for all values of θ , it is called an identity.

We can write $(\cos \theta)^2$ and $(\sin \theta)^2$ as $\cos^2 \theta$ and $\sin^2 \theta$, and therefore we obtain:



Pythagorean identity

$$\cos^2 \theta + \sin^2 \theta = 1$$

Example 6

If $\sin x = 0.3$ and $0 < x < \frac{\pi}{2}$, find:

a $\cos x$

b $\tan x$

Solution

a $\cos^2 x + \sin^2 x = 1$

$$\cos^2 x + 0.09 = 1$$

$$\cos^2 x = 0.91$$

$$\therefore \cos x = \pm \sqrt{0.91}$$

Since the point $P(x)$ is in the 1st quadrant,
this gives

$$\begin{aligned}\cos x &= \sqrt{0.91} = \sqrt{\frac{91}{100}} \\ &= \frac{\sqrt{91}}{10}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \tan x &= \frac{\sin x}{\cos x} = \frac{0.3}{\sqrt{0.91}} \\ &= \frac{3}{\sqrt{91}} \\ &= \frac{3\sqrt{91}}{91}\end{aligned}$$

Section summary■ **Exact values**

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined

■ **Complementary relationships**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$$

■ **Pythagorean identity**

$$\cos^2 \theta + \sin^2 \theta = 1$$

Exercise 13B**Example 3**

1 Evaluate each of the following:

a $\cos\left(\frac{3\pi}{4}\right)$ **b** $\sin\left(\frac{5\pi}{4}\right)$ **c** $\sin\left(\frac{25\pi}{2}\right)$ **d** $\sin\left(\frac{15\pi}{6}\right)$ **e** $\cos\left(\frac{17\pi}{4}\right)$

f $\sin\left(-\frac{15\pi}{4}\right)$ **g** $\sin(27\pi)$ **h** $\sin\left(-\frac{17\pi}{3}\right)$ **i** $\cos\left(\frac{75\pi}{6}\right)$ **j** $\cos\left(-\frac{15\pi}{6}\right)$

k $\sin\left(-\frac{35\pi}{2}\right)$ **l** $\cos\left(-\frac{45\pi}{6}\right)$ **m** $\cos\left(\frac{16\pi}{3}\right)$ **n** $\sin\left(-\frac{105\pi}{2}\right)$ **o** $\cos(1035\pi)$

Example 4 2 Find the exact value of each of the following:

- a** $\sin(135^\circ)$ **b** $\cos(-300^\circ)$ **c** $\sin(480^\circ)$
d $\cos(240^\circ)$ **e** $\sin(-225^\circ)$ **f** $\sin(420^\circ)$

Example 5 3 If $\sin x = 0.3$ and $\cos \alpha = 0.6$, find the value of:

- a** $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\cos\left(\frac{\pi}{2} - x\right)$ **d** $\sin(-x)$
e $\cos\left(\frac{\pi}{2} + x\right)$ **f** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **g** $\sin\left(\frac{3\pi}{2} + \alpha\right)$ **h** $\cos\left(\frac{3\pi}{2} - x\right)$

Example 6 4 If $\sin x = 0.5$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

5 If $\cos x = -0.7$ and $\pi < x < \frac{3\pi}{2}$, find $\sin x$ and $\tan x$.

6 If $\sin x = -0.5$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

7 If $\sin x = -0.3$ and $\frac{3\pi}{2} < x < 2\pi$, find $\cos x$ and $\tan x$.

13C Solution of equations involving sine and cosine

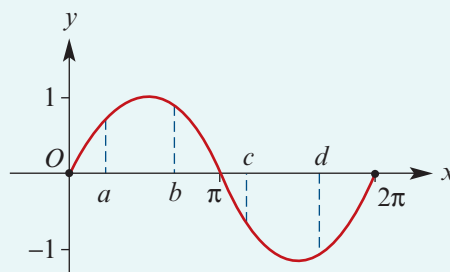
If a trigonometric equation has a solution, then it will have a corresponding solution in each 'cycle' of its domain. Such an equation is solved by using the symmetry of the graph to obtain solutions within one 'cycle' of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

Example 7

The graph shown is

$$y = \sin x, \quad 0 \leq x \leq 2\pi$$

For each pronumeral marked on the x -axis, find the other x -value which has the same y -value.



Solution

For $x = a$, the other value is $\pi - a$.

For $x = b$, the other value is $\pi - b$.

For $x = c$, the other value is $2\pi - (c - \pi) = 3\pi - c$.

For $x = d$, the other value is $\pi + (2\pi - d) = 3\pi - d$.

**Example 8**

Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$.

Solution

Let $\theta = 2x + \frac{\pi}{3}$. Note that

$$\begin{aligned} 0 \leq x \leq 2\pi &\Leftrightarrow 0 \leq 2x \leq 4\pi \\ &\Leftrightarrow \frac{\pi}{3} \leq 2x + \frac{\pi}{3} \leq \frac{13\pi}{3} \\ &\Leftrightarrow \frac{\pi}{3} \leq \theta \leq \frac{13\pi}{3} \end{aligned}$$

To solve $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$, we first solve $\sin \theta = \frac{1}{2}$ for $\frac{\pi}{3} \leq \theta \leq \frac{13\pi}{3}$.

Consider $\sin \theta = \frac{1}{2}$.

$$\therefore \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} \text{ or } \dots$$

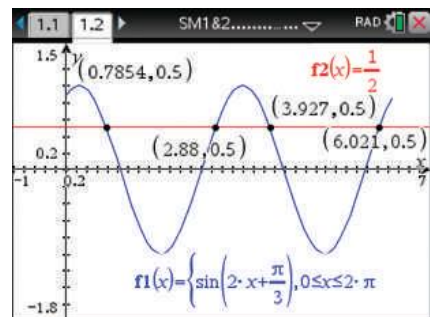
The solutions $\frac{\pi}{6}$ and $\frac{29\pi}{6}$ are not required, as they lie outside the restricted domain for θ .

For $\frac{\pi}{3} \leq \theta \leq \frac{13\pi}{3}$:

$$\begin{aligned} \theta &= \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6} \\ \therefore 2x + \frac{\pi}{6} &= \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6} \\ \therefore 2x &= \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6} \\ \therefore x &= \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12} \end{aligned}$$

**Using the TI-Nspire CX non-CAS**

- Ensure that the calculator is in radian mode.
- In a **Graphs** application, plot the graphs of:
 - $f1(x) = \sin\left(2x + \frac{\pi}{3}\right) \mid 0 \leq x \leq 2\pi$
 - $f2(x) = \frac{1}{2}$
- Use **menu** > **Geometry** > **Points & Lines** > **Intersection Point(s)**.



Note: To change the mode, go to **on** > **Settings** > **Document Settings**. The **Graphs** application has its own settings, which are accessed from a **Graphs** page using **menu** > **Settings**.

Using the Casio

Ensure that the angle setting is Radians:

- In **Run-Matrix** mode, go to the set-up screen by pressing **SHIFT** **MENU**.
- Use the cursor key $\blacktriangledown$ to move down to **Angle**.
- Press **F2** to change to **Radians**. Then press **EXIT**.

```
Input/Output:Math
Mode          :Comp
Frac Result   :d/c
Func Type     :Y=
Draw Type     :Connect
Derivative    :On
Angle         :Rad
Deg Rad Gra   [ ] [ ] [ ]
```

Method 1: Using the numerical solver

- In **Run-Matrix** mode, select the numerical solver (**Calculation** **OPTN** **F4**, **SolveN** **F5**).
- Complete the equation and domain by entering:

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}, x, 0, 2\pi$$

```
sin ( 2 X,0,T + a/b/c SHIFT EXP ▼ 3 ►
) SHIFT • a/b/c 1 ▼ 2 ► ,
X,0,T , 0 , 2 SHIFT EXP ) EXE EXIT
```

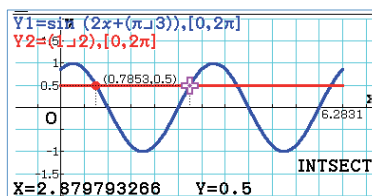
```
Math Rad Norm1 d/c Real
SolveN sin ( 2x + π/3 ) = 1/2
{ 1/4 π, 11/12 π, 5/4 π, 23/12 π }
Solve d/dx d²/dx² ∫ dx SolveN ►
```

Method 2: Using Graph mode

- In **Graph** mode, enter the rules for the curve and the line with the restricted domain, as shown.
- Adjust the View Window **SHIFT** **F3** for $-1 \leq x \leq 7$ and $-2 \leq y \leq 2$.
- Select **Draw** **F6**.
- To find the intersection points, go to the **G-Solve** menu **SHIFT** **F5** and select **Intersection** **F5**. Use the cursor key $\blacktriangleright$ to find the next point.

Note: Exact solutions cannot be obtained using the graphical method.

```
Math Rad Norm1 Real
Graph Func :Y=
Y1=sin ( 2x + π/3 ), [ 0, 2π ]
Y2=1/2, [ 0, 2π ] [ — ]
Y3: [ — ]
Y4: [ — ]
Y r Xt Yt X
```



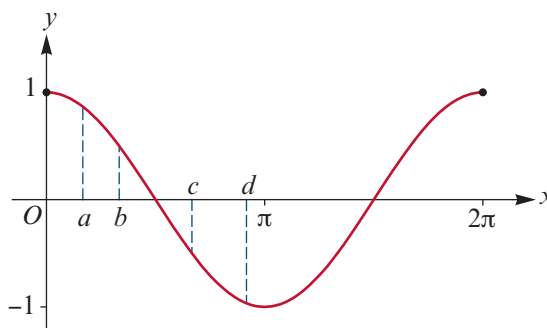
Exercise 13C

Skillsheet

1 The graph shown is

$$y = \cos x, \quad 0 \leq x \leq 2\pi$$

For each pronumeral marked on the x -axis, find the other x -value which has the same y -value.



SF

Example 8 2 Solve each of the following for $x \in [0, 2\pi]$:

a $\sin x = -\frac{\sqrt{3}}{2}$

b $\sin(2x) = -\frac{\sqrt{3}}{2}$

c $2 \cos(2x) = -1$

d $\sin\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}$

e $2 \cos\left(2\left(x + \frac{\pi}{3}\right)\right) = -1$

f $2 \sin\left(2x + \frac{\pi}{3}\right) = -\sqrt{3}$

3 Solve each of the following for $x \in [-\pi, \pi]$:

a $\sin x + \frac{1}{2} = 0$

b $\sin(3x) = 0$

c $\cos\left(\frac{x}{2}\right) = 1$

d $\sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

e $2 \cos\left(2\left(x + \frac{\pi}{6}\right)\right) = -1$

13D Transformations of the graphs of sine and cosine

The graphs of functions with rules of the form

$$f(x) = a \sin(n(x + \epsilon)) + b \quad \text{and} \quad f(x) = a \cos(n(x + \epsilon)) + b$$

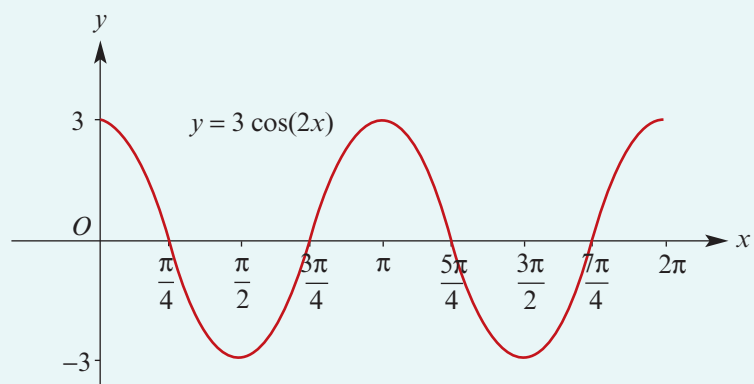
can be obtained from the graphs of $y = \sin x$ and $y = \cos x$ by transformations.**Example 9**

Sketch the graph of the function

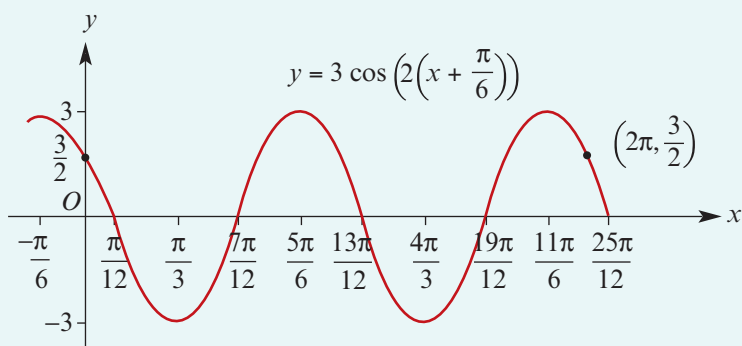
$$h(x) = 3 \cos\left(2x + \frac{\pi}{3}\right) + 1, \quad x \in [0, 2\pi]$$

SolutionWe can write $h(x) = 3 \cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 1$.The graph of $y = h(x)$ is obtained from the graph of $y = \cos x$ by:

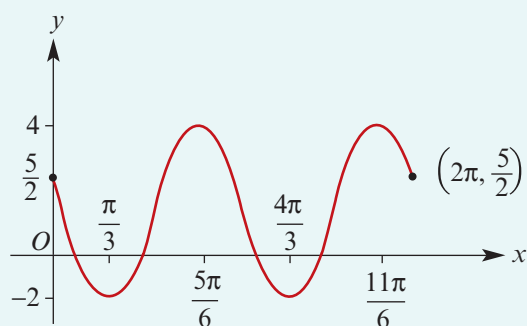
- a dilation of factor $\frac{1}{2}$ from the y -axis
- a dilation of factor 3 from the x -axis
- a translation of $\frac{\pi}{6}$ units in the negative direction of the x -axis
- a translation of 1 unit in the positive direction of the y -axis.

First apply the two dilations to the graph of $y = \cos x$.

Next apply the translation $\frac{\pi}{6}$ units in the negative direction of the x -axis.



Apply the final translation and restrict the graph to the required domain.



Section summary

For the graphs of $y = a \sin(nx)$ and $y = a \cos(nx)$, where $a > 0$ and $n > 0$:

- Period = $\frac{2\pi}{n}$
- Amplitude = a
- Range = $[-a, a]$

Exercise 13D

Skillsheet

1 Sketch the graph of each of the following for the stated domain:

a $f(x) = \sin(2x)$, $x \in [0, 2\pi]$

b $f(x) = \cos\left(x + \frac{\pi}{3}\right)$, $x \in \left[\frac{-\pi}{3}, \pi\right]$

c $f(x) = \cos\left(2\left(x + \frac{\pi}{3}\right)\right)$, $x \in [0, \pi]$

d $f(x) = 2 \sin(3x) + 1$, $x \in [0, \pi]$

e $f(x) = 2 \sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}$, $x \in [0, 2\pi]$

f $f(x) = \cos(2x)$, $x \in [-\pi, \pi]$

2 Sketch the graph of each of the following for the stated domain:

a $f(x) = \sin\left(x + \frac{\pi}{6}\right)$, $x \in [-\pi, \pi]$

b $f(x) = \sin\left(2\left(x + \frac{\pi}{4}\right)\right)$, $x \in [-\pi, \pi]$

c $f(x) = 2 \cos\left(\frac{x}{3}\right) + 1$, $x \in [0, 6\pi]$

d $f(x) = 2 \cos\left(x - \frac{\pi}{3}\right) + \sqrt{3}$, $x \in [0, 2\pi]$

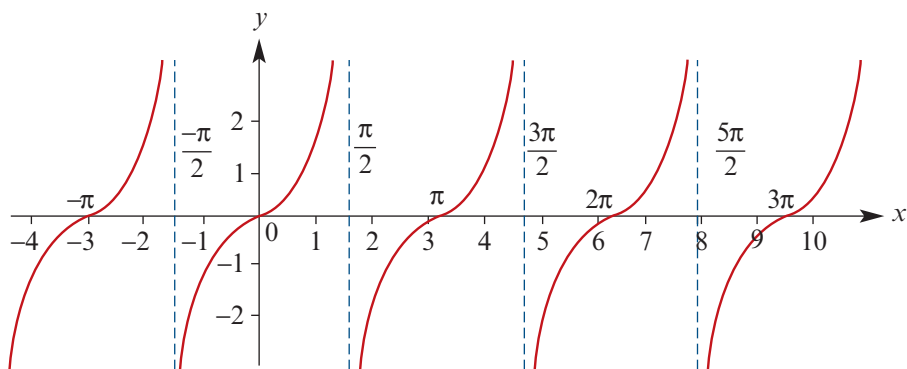
e $f(x) = \cos(\pi x)$, $x \in [-2, 2]$

f $f(x) = \cos\left(\frac{\pi x}{6}\right)$, $x \in [-12, 12]$

13E The tangent function

Recall that the tangent function is given by $\tan x = \frac{\sin x}{\cos x}$ for $\cos x \neq 0$.

The graph of $y = \tan x$ is shown below.



Properties of the tangent function

- The graph repeats itself every π units, i.e. the period of $\tan$ is π .
- The range of $\tan$ is $\mathbb{R}$.
- The vertical asymptotes have equations $x = \frac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $x = k\pi$ where $k \in \mathbb{Z}$.

► Symmetry properties of tangent

Using symmetry properties of sine and cosine, we have

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin \theta}{-\cos \theta} = -\tan \theta$$

Similarly, we obtain:

■ $\tan(\pi + \theta) = \tan \theta$	■ $\tan(2\pi - \theta) = -\tan \theta$	■ $\tan(-\theta) = -\tan \theta$
■ $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\cos \theta}{\sin \theta}$	■ $\tan\left(\frac{\pi}{2} + \theta\right) = -\frac{\cos \theta}{\sin \theta}$	

Note: We will see in Section 13H that $\frac{\cos \theta}{\sin \theta}$ can be written as $\cot \theta$.

Example 10

Find the exact value of:

a $\tan\left(\frac{4\pi}{3}\right)$

b $\tan 330^\circ$

Solution

a $\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right)$
 $= \tan\left(\frac{\pi}{3}\right)$
 $= \sqrt{3}$

b $\tan 330^\circ = \tan(360^\circ - 30^\circ)$
 $= -\tan 30^\circ$
 $= -\frac{1}{\sqrt{3}}$

► Solution of equations involving the tangent function

We now consider the solution of equations involving the tangent function, which can be applied to finding the x -axis intercepts for graphs of the tangent function.

The method is similar to that used for solving equations involving sine and cosine, except that only one solution needs to be found and then all other solutions are one period length apart.

Example 11

Solve the equation $3 \tan(2x) = \sqrt{3}$ for $x \in (0, 2\pi)$.

Solution

$$3 \tan(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore 2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6}$$

$$x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$

Explanation

Since we want solutions for x in $(0, 2\pi)$, we find solutions for $2x$ in $(0, 4\pi)$.

Once we have found one solution for $2x$, we can obtain all other solutions by adding and subtracting multiples of π .



Example 12

Solve the equation $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$ for $x \in [-2\pi, 2\pi]$.

Solution

Let $\theta = \frac{1}{2}\left(x - \frac{\pi}{4}\right)$. Note that

$$\begin{aligned} -2\pi \leq x \leq 2\pi &\Leftrightarrow -\frac{9\pi}{4} \leq x - \frac{\pi}{4} \leq \frac{7\pi}{4} \\ &\Leftrightarrow -\frac{9\pi}{8} \leq \frac{1}{2}\left(x - \frac{\pi}{4}\right) \leq \frac{7\pi}{8} \\ &\Leftrightarrow -\frac{9\pi}{8} \leq \theta \leq \frac{7\pi}{8} \end{aligned}$$

We first solve the equation $\tan \theta = -1$ for $-\frac{9\pi}{8} \leq \theta \leq \frac{7\pi}{8}$:

$$\theta = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{-\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore x = \frac{-\pi}{4} \text{ or } \frac{7\pi}{4}$$

► Graphing the tangent function

When graphing a transformation of the tangent function:

- Find the period.
- Find the equations of the asymptotes.
- Find the intercepts with the axes.

Example 13

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3 \tan(2x)$

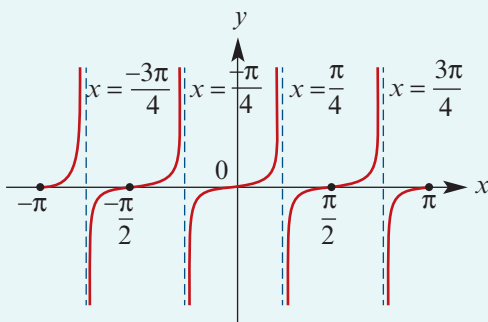
b $y = -2 \tan(3x)$

Solution

a Period = $\frac{\pi}{n} = \frac{\pi}{2}$

Asymptotes: $x = \frac{(2k+1)\pi}{4}, k \in \mathbb{Z}$

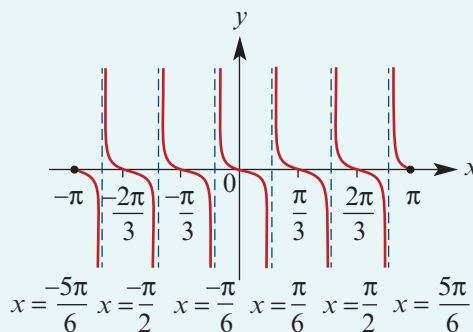
Axis intercepts: $x = \frac{k\pi}{2}, k \in \mathbb{Z}$



b Period = $\frac{\pi}{n} = \frac{\pi}{3}$

Asymptotes: $x = \frac{(2k+1)\pi}{6}, k \in \mathbb{Z}$

Axis intercepts: $x = \frac{k\pi}{3}, k \in \mathbb{Z}$



Section summary

- The tangent function is given by $\tan \theta = \frac{\sin \theta}{\cos \theta}$ for $\cos \theta \neq 0$.
- The graph of $y = a \tan(nx)$, where $a \neq 0$ and $n > 0$:
 - The period is $\frac{\pi}{n}$.
 - The vertical asymptotes have equations $x = \frac{(2k+1)\pi}{2n}$ where $k \in \mathbb{Z}$.
 - The axis intercepts are at $x = \frac{k\pi}{n}$ where $k \in \mathbb{Z}$.
- Useful symmetry properties:
 - $\tan(\pi + \theta) = \tan \theta$
 - $\tan(-\theta) = -\tan \theta$

Exercise 13E

Example 10

1 Find the exact value of each of the following:

a $\tan\left(\frac{5\pi}{4}\right)$

b $\tan\left(-\frac{2\pi}{3}\right)$

c $\tan\left(-\frac{29\pi}{6}\right)$

2 Find the exact value of each of the following:

a $\tan 240^\circ$

b $\tan(-150^\circ)$

c $\tan 315^\circ$

3 If $\tan x = \frac{1}{4}$ and $\pi \leq x \leq \frac{3\pi}{2}$, find the exact value of:

a $\sin x$

b $\cos x$

c $\tan(-x)$

d $\tan(\pi - x)$

4 If $\tan x = -\frac{\sqrt{3}}{2}$ and $\frac{\pi}{2} \leq x \leq \pi$, find the exact value of:

a $\sin x$

b $\cos x$

c $\tan(-x)$

d $\tan(x - \pi)$

Example 11

5 Solve each of the following equations for x in the stated interval:

a $\tan x = -1, x \in (0, 2\pi)$

b $\tan x = \sqrt{3}, x \in (0, 2\pi)$

c $\tan x = \frac{1}{\sqrt{3}}, x \in (0, 2\pi)$

d $\tan(2x) = 1, x \in (-\pi, \pi)$

e $\tan(2x) = \sqrt{3}, x \in (-\pi, \pi)$

f $\tan(2x) = -\frac{1}{\sqrt{3}}, x \in (-\pi, \pi)$

Example 12

6 Solve each of the following equations for x in the stated interval:

a $\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = 1, x \in (0, 2\pi)$

b $\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = -1, x \in (-\pi, \pi)$

c $\tan\left(3\left(x - \frac{\pi}{6}\right)\right) = \sqrt{3}, x \in (-\pi, \pi)$

d $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right) = -\frac{1}{\sqrt{3}}, x \in (-\pi, \pi)$

Example 13

7 Sketch the graph of each of the following:

a $y = \tan(2x)$

b $y = \tan(3x)$

c $y = -\tan(2x)$

d $y = 3 \tan x$

e $y = \tan\left(\frac{x}{2}\right)$

f $y = 2 \tan\left(x + \frac{\pi}{4}\right)$

g $y = 3 \tan x + 1$

h $y = 2 \tan\left(x + \frac{\pi}{2}\right) + 1$

i $y = 3 \tan\left(2\left(x - \frac{\pi}{4}\right)\right) - 2$

8 Sketch the graph of each of the following for the stated domain:

a $y = \tan\left(x + \frac{\pi}{3}\right) + \sqrt{3}$ for $x \in [0, 2\pi]$

b $y = \tan\left(\frac{x}{2}\right)$ for $x \in [0, 4\pi]$

c $y = \tan\left(\frac{\pi x}{2}\right)$ for $x \in [0, 4]$

13F General solution of trigonometric equations

We have seen how to solve equations involving trigonometric functions over a restricted domain. We now consider the general solutions of such equations over the maximal domain for each function.

By convention:

- $\cos^{-1}$ has range $[0, \pi]$
- $\sin^{-1}$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\tan^{-1}$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For example:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \quad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \quad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

If an equation involving a trigonometric function has one or more solutions in one 'cycle', then it will have corresponding solutions in each 'cycle' of its domain, i.e. there will be infinitely many solutions.

For example, consider the equation

$$\cos x = a$$

for some fixed $a \in [-1, 1]$. The solution in the interval $[0, \pi]$ is given by

$$x = \cos^{-1}(a)$$

By the symmetry properties of the cosine function, the other solutions are given by

$$-\cos^{-1}(a), \pm 2\pi + \cos^{-1}(a), \pm 2\pi - \cos^{-1}(a), \pm 4\pi + \cos^{-1}(a), \pm 4\pi - \cos^{-1}(a), \dots$$

In general, we have the following:

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

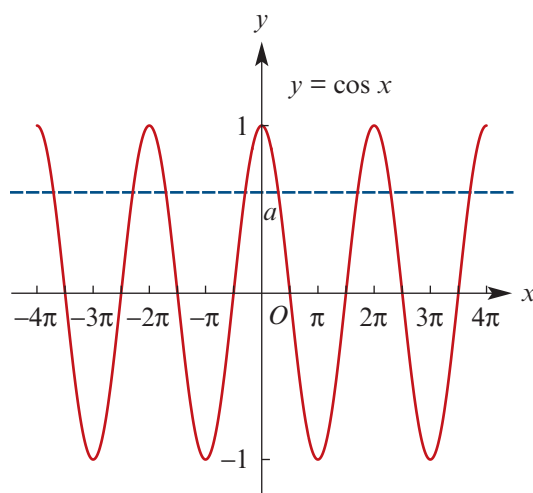
$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$



Note: An alternative and more concise way to express the general solution of $\sin x = a$ is

$$x = n\pi + (-1)^n \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}.$$

Example 14

Find the general solution of each of the following equations:

a $\cos x = 0.5$

b $\sqrt{3} \tan(3x) = 1$

c $2 \sin x = \sqrt{2}$

Solution

a $\cos x = 0.5$

$$\begin{aligned}
 x &= 2n\pi \pm \cos^{-1}(0.5) \\
 &= 2n\pi \pm \frac{\pi}{3} \\
 &= \frac{(6n \pm 1)\pi}{3}, \quad n \in \mathbb{Z}
 \end{aligned}$$

$$\begin{aligned}
 \textbf{b} \quad \tan(3x) &= \frac{1}{\sqrt{3}} \\
 3x &= n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\
 &= n\pi + \frac{\pi}{6} \\
 &= \frac{(6n + 1)\pi}{6} \\
 x &= \frac{(6n + 1)\pi}{18}, \quad n \in \mathbb{Z}
 \end{aligned}$$

c $\sin x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$$\begin{aligned}
 x &= 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \text{or} \quad x = (2n + 1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\
 &= 2n\pi + \frac{\pi}{4} \qquad \qquad \qquad = (2n + 1)\pi - \frac{\pi}{4} \\
 &= \frac{(8n + 1)\pi}{4}, \quad n \in \mathbb{Z} \qquad \qquad \qquad = \frac{(8n + 3)\pi}{4}, \quad n \in \mathbb{Z}
 \end{aligned}$$

**Example 15**

Find the first three positive solutions of each of the following equations:

a $\cos x = 0.5$

b $\sqrt{3} \tan(3x) = 1$

c $2 \sin x = \sqrt{2}$

Solution

a The general solution (from Example 14a) is given by $x = \frac{(6n \pm 1)\pi}{3}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \pm \frac{\pi}{3}$, and when $n = 1$, $x = \frac{5\pi}{3}$ or $x = \frac{7\pi}{3}$.

Thus the first three positive solutions of $\cos x = 0.5$ are $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$.

b The general solution (from Example 14b) is given by $x = \frac{(6n + 1)\pi}{18}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \frac{\pi}{18}$, and when $n = 1$, $x = \frac{7\pi}{18}$, and when $n = 2$, $x = \frac{13\pi}{18}$.

Thus the first three positive solutions of $\sqrt{3} \tan(3x) = 1$ are $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$.

c The general solution (from Example 14c) is $x = \frac{(8n+1)\pi}{4}$ or $x = \frac{(8n+3)\pi}{4}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$, and when $n = 1$, $x = \frac{9\pi}{4}$ or $x = \frac{11\pi}{4}$.

Thus the first three positive solutions of $2 \sin x = \sqrt{2}$ are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$.

Example 16

Find the general solution for each of the following:

a $\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b $\tan\left(2x - \frac{\pi}{3}\right) = 1$

Solution

a $\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$x - \frac{\pi}{3} = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore x = n\pi + (-1)^n \left(\frac{\pi}{3}\right) + \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

The solutions are $x = \frac{(3n+2)\pi}{3}$ for n even
and $x = n\pi$ for n odd.

b $\tan\left(2x - \frac{\pi}{3}\right) = 1$

$$2x - \frac{\pi}{3} = n\pi + \frac{\pi}{4}$$

$$2x = n\pi + \frac{\pi}{4} + \frac{\pi}{3}$$

$$\therefore x = \frac{1}{2}\left(n\pi + \frac{7\pi}{12}\right)$$

$$= \frac{(12n+7)\pi}{24}, \quad n \in \mathbb{Z}$$

Section summary

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Exercise 13F

Skillsheet

- 1** Evaluate each of the following for **i** $n = 1$ **ii** $n = 2$ **iii** $n = -2$:

a $2n\pi \pm \cos^{-1}(1)$

b $2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

Example 14

- 2** Find the general solution of each of the following equations:

a $\cos x = \frac{\sqrt{3}}{2}$

b $2 \sin(3x) = \sqrt{3}$

c $\sqrt{3} \tan x = 3$

Example 15

3 Find the first two positive solutions of each of the following equations:

a $\sin x = 0.5$

b $2 \cos(2x) = \sqrt{3}$

c $\sqrt{3} \tan(2x) = -3$

4 Given that a trigonometric equation has general solution $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-2\pi, 2\pi]$.

5 Given that a trigonometric equation has general solution $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-\pi, 2\pi]$.

Example 16

6 Find the general solution for each of the following:

a $\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = \frac{1}{2}$

b $2 \tan\left(2\left(x + \frac{\pi}{4}\right)\right) = 2\sqrt{3}$

c $2 \sin\left(x + \frac{\pi}{3}\right) = -1$

7 Find the general solution of $2 \cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$ and hence find all the solutions for x in the interval $(-2\pi, 2\pi)$.

8 Find the general solution of $\sqrt{3} \tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$ and hence find all the solutions for x in the interval $[-\pi, 0]$.

9 Find the general solution of $2 \sin(4\pi x) + \sqrt{3} = 0$ and hence find all the solutions for x in the interval $[-1, 1]$.

13G Reciprocals of trigonometric functions

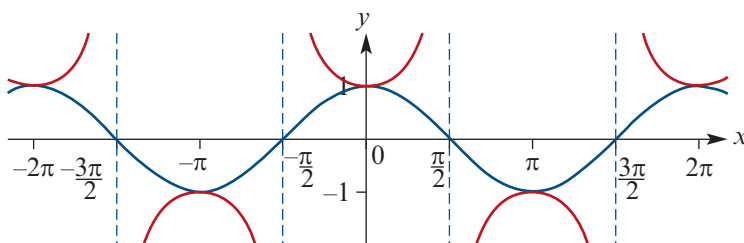
In Chapter 12 we found that, given the graph of a continuous function $y = f(x)$, we can sketch the graph of the reciprocal function $y = \frac{1}{f(x)}$.

Function $y = f(x)$	Reciprocal function $y = \frac{1}{f(x)}$
x -axis intercept at $x = a$	vertical asymptote $x = a$
local maximum at $x = a$	local minimum at $x = a$
local minimum at $x = a$	local maximum at $x = a$
above the x -axis	above the x -axis
below the x -axis	below the x -axis
increasing over an interval	decreasing over the interval
decreasing over an interval	increasing over the interval
values approach ∞	values approach 0 from above
values approach $-\infty$	values approach 0 from below
values approach 0 from above	values approach ∞
values approach 0 from below	values approach $-\infty$

The reciprocal functions of cosine and sine are called secant and cosecant. We investigate these two functions further in the next section.

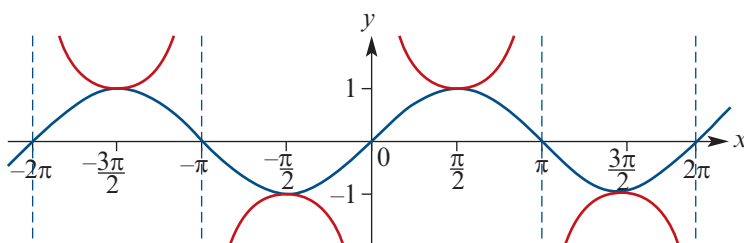
Secant

$$\sec x = \frac{1}{\cos x}$$



Cosecant

$$\operatorname{cosec} x = \frac{1}{\sin x}$$



The first graph shows $y = \cos x$ and $y = \sec x$; the second shows $y = \sin x$ and $y = \operatorname{cosec} x$.

Note: The x -axis intercepts become vertical asymptotes.

Local maximums become local minimums, and vice versa.



Example 17

Let $f(x) = 2 \cos x$ for $-2\pi \leq x \leq 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2 \cos x$ for $x \in [-2\pi, 2\pi]$. This is shown in blue.

Vertical asymptotes

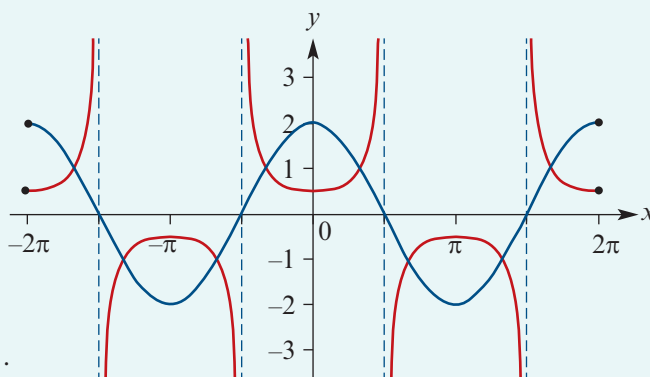
Vertical asymptotes of the reciprocal function will occur when $f(x) = 0$.

These are given by $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$.

Turning points

The points $(0, 2)$ and $(\pm 2\pi, 2)$ are local maximums of $y = f(x)$. Therefore the points $(0, \frac{1}{2})$ and $(\pm 2\pi, \frac{1}{2})$ are local minimums of the reciprocal.

The points $(\pm \pi, -2)$ are local minimums of $y = f(x)$. Therefore the points $(\pm \pi, -\frac{1}{2})$ are local maximums of the reciprocal.



The graph of the next function has no x -axis intercepts, and so its reciprocal has no vertical asymptotes.

Example 18

Let $f(x) = 0.5 \sin x + 1$ for $0 \leq x \leq 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

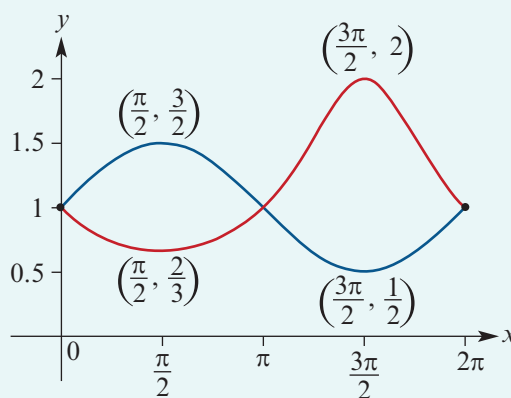
Solution

We first sketch $y = 0.5 \sin x + 1$ for $x \in [0, 2\pi]$. This is shown in blue.

Turning points

The point $(\frac{\pi}{2}, \frac{3}{2})$ is a local maximum of $y = f(x)$. Therefore the point $(\frac{\pi}{2}, \frac{2}{3})$ is a local minimum of the reciprocal.

The point $(\frac{3\pi}{2}, \frac{1}{2})$ is a local minimum of $y = f(x)$. Therefore the point $(\frac{3\pi}{2}, 2)$ is a local maximum of the reciprocal.



Section summary

Reciprocal functions

- If $y = f(x)$ is a function, then the reciprocal function is defined by the rule $y = \frac{1}{f(x)}$.
- To sketch the graph of $y = \frac{1}{f(x)}$, we first sketch the graph of $y = f(x)$.
- The x -axis intercepts of $y = f(x)$ will become vertical asymptotes of $y = \frac{1}{f(x)}$.
- Local maximums of $y = f(x)$ will become local minimums of $y = \frac{1}{f(x)}$, and vice versa.

Exercise 13G

Example 17, 18

- 1 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes. Label asymptotes, turning points and endpoints.

a $f(x) = \sin x$ for $0 \leq x \leq 2\pi$

b $f(x) = \cos x$ for $0 \leq x \leq 2\pi$

c $f(x) = -2 \cos x$ for $-\pi \leq x \leq \pi$

d $f(x) = \cos x + 1$ for $0 \leq x \leq 4\pi$

e $f(x) = -\sin x - 1$ for $-2\pi \leq x \leq 2\pi$

f $f(x) = \cos x - 2$ for $0 \leq x \leq 2\pi$

g $f(x) = -\sin x + 2$ for $0 \leq x \leq 2\pi$

h $f(x) = -2 \cos x + 3$ for $-\pi \leq x \leq \pi$

- 2 Sketch the graphs of $y = 2 \sin^2 x$ and $y = \frac{1}{2 \sin^2 x}$ on the same set of axes, over the interval $0 \leq x \leq 2\pi$.

13H Graphing the reciprocal trigonometric functions

We now consider the graphs of the **reciprocal trigonometric functions** in greater depth. In this section we will apply basic transformations to the graphs of these functions.

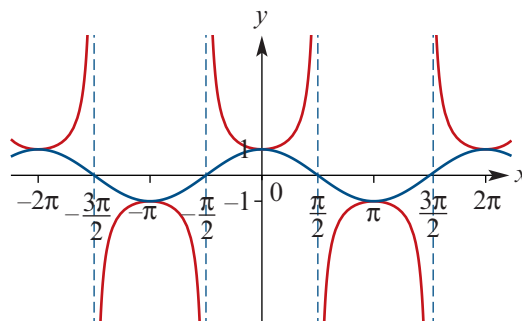
► The secant function

The secant function is defined by

$$\sec x = \frac{1}{\cos x}$$

provided $\cos x \neq 0$.

The graphs of $y = \cos x$ and $y = \sec x$ are shown here on the same axes.



The significant features of the two graphs are listed in the following table.

Function $y = \cos x$	Reciprocal function $y = \sec x$
x -axis intercepts at $x = \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$	vertical asymptotes at $x = \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$
domain = $\mathbb{R}$	domain = $\mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$
local maximums at $(2n\pi, 1), n \in \mathbb{Z}$	local minimums at $(2n\pi, 1), n \in \mathbb{Z}$
local minimums at $((2n+1)\pi, -1), n \in \mathbb{Z}$	local maximums at $((2n+1)\pi, -1), n \in \mathbb{Z}$
range = $[-1, 1]$	range = $(-\infty, -1] \cup [1, \infty)$

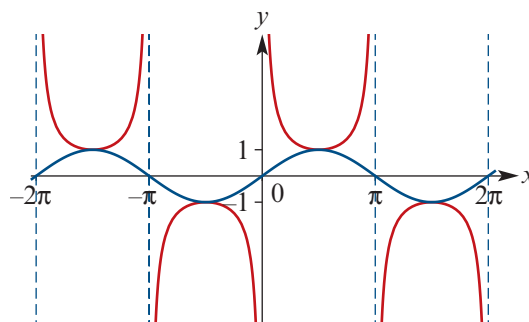
► The cosecant function

The cosecant function is defined by

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

provided $\sin x \neq 0$.

The graphs of $y = \sin x$ and $y = \operatorname{cosec} x$ are shown here on the same axes.



Function $y = \sin x$	Reciprocal function $y = \operatorname{cosec} x$
x -axis intercepts at $x = n\pi, n \in \mathbb{Z}$	vertical asymptotes at $x = n\pi, n \in \mathbb{Z}$
domain = $\mathbb{R}$	domain = $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$
local maximums at $\left(2n\pi + \frac{\pi}{2}, 1\right), n \in \mathbb{Z}$	local minimums at $\left(2n\pi + \frac{\pi}{2}, 1\right), n \in \mathbb{Z}$
local minimums at $\left(2n\pi - \frac{\pi}{2}, -1\right), n \in \mathbb{Z}$	local maximums at $\left(2n\pi - \frac{\pi}{2}, -1\right), n \in \mathbb{Z}$
range = $[-1, 1]$	range = $(-\infty, -1] \cup [1, \infty)$

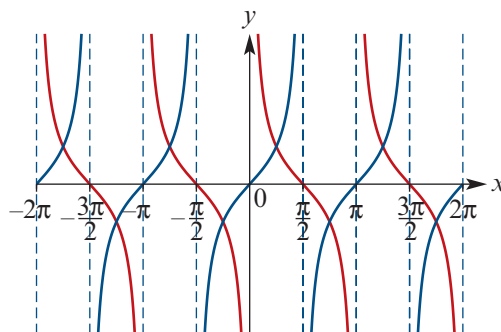
► The cotangent function

The cotangent function is defined by

$$\cot x = \frac{\cos x}{\sin x}$$

provided $\sin x \neq 0$.

This diagram shows the graph of $y = \tan x$ in blue and the graph of $y = \cot x$ in red.



Function $y = \tan x$	Function $y = \cot x$
x -axis intercepts at $x = n\pi$, $n \in \mathbb{Z}$	vertical asymptotes at $x = n\pi$, $n \in \mathbb{Z}$
vertical asymptotes at $x = \frac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$	x -axis intercepts at $x = \frac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$
domain = $\mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$	domain = $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$
range = $\mathbb{R}$	range = $\mathbb{R}$

Note the similarity between the graphs of $y = \cot x$ and $y = \tan x$. Using the complementary relationship between sine and cosine, we have

$$\cot x = \frac{\cos x}{\sin x} = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \tan\left(\frac{\pi}{2} - x\right) = \tan\left(-\left(x - \frac{\pi}{2}\right)\right) = -\tan\left(x - \frac{\pi}{2}\right)$$

Therefore the graph of $y = \cot x$ can be obtained from the graph of $y = \tan x$ by a reflection in the x -axis followed by a translation of $\frac{\pi}{2}$ units in the positive direction of the x -axis.

► Transformations of the reciprocal trigonometric functions

We now look at the effect of the basic transformations (dilations, reflections and translations) on the reciprocal trigonometric functions.

Example 19

Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}(2x)$

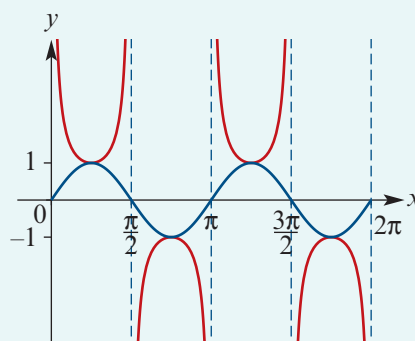
b $y = 2 \sec x$

c $y = -\cot x$

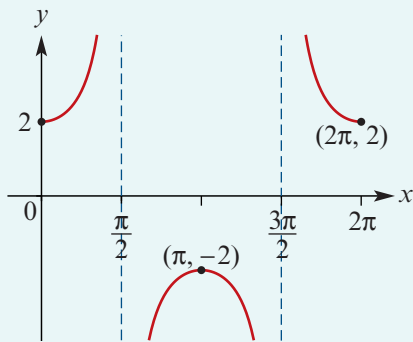
Solution

a The graph of $y = \operatorname{cosec}(2x)$ can be obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{2}$ from the y -axis.

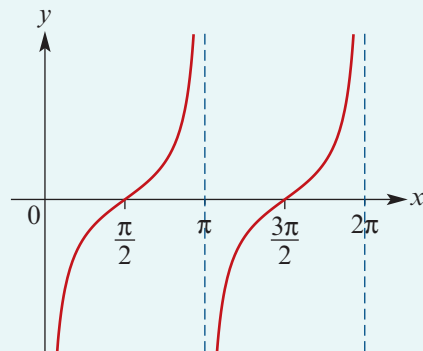
It is helpful to draw the graph of $y = \sin(2x)$ on the same axes.



- b** The graph of $y = 2 \sec x$ can be obtained from the graph of $y = \sec x$ by a dilation of factor 2 from the x -axis.



- c** The graph of $y = -\cot x$ can be obtained from the graph of $y = \cot x$ by a reflection in the x -axis.



Example 20

Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec\left(x + \frac{\pi}{3}\right)$

b $y = \operatorname{cosec}(x) - 2$

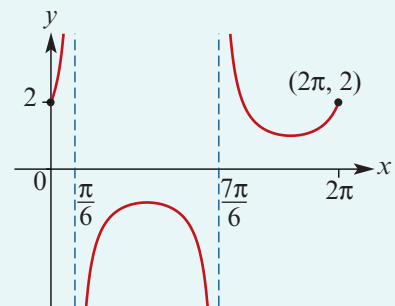
c $y = \cot\left(x - \frac{\pi}{4}\right)$

Solution

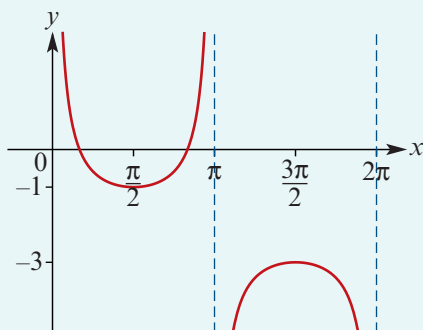
- a** The graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ can be obtained from the graph of $y = \sec x$ by a translation of $\frac{\pi}{3}$ units in the negative direction of the x -axis.

The y -axis intercept is $\sec\left(\frac{\pi}{3}\right) = 2$.

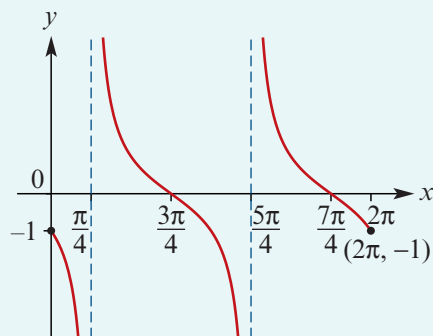
The asymptotes are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.



- b** The graph of $y = \operatorname{cosec}(x) - 2$ can be obtained from the graph of $y = \operatorname{cosec} x$ by a translation of 2 units in the negative direction of the y -axis.



- c** The graph of $y = \cot\left(x - \frac{\pi}{4}\right)$ can be obtained from the graph of $y = \cot x$ by a translation of $\frac{\pi}{4}$ units in the positive direction of the x -axis.



Example 21

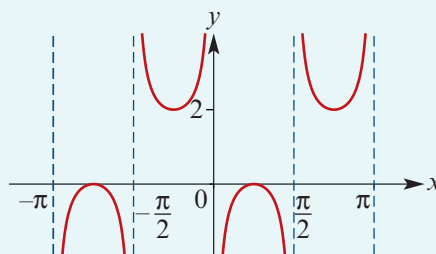
Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -\sec\left(2x - \frac{\pi}{2}\right) + 1$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

Solution

It helps to write the equation of the transformed graph as $y = -\sec\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

An appropriate sequence is:

- 1 reflection in the x -axis
- 2 dilation of factor $\frac{1}{2}$ from the y -axis
- 3 translation of $\frac{\pi}{4}$ units to the right and 1 unit up.

**Section summary****Reciprocal trigonometric functions**

■ $\sec x = \frac{1}{\cos x}$
(provided $\cos x \neq 0$)

■ $\operatorname{cosec} x = \frac{1}{\sin x}$
(provided $\sin x \neq 0$)

■ $\cot x = \frac{\cos x}{\sin x}$
(provided $\sin x \neq 0$)

Exercise 13H**Example 19**

- 1 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec(2x)$

b $y = \cot(2x)$

c $y = 3 \sec x$

d $y = 2 \operatorname{cosec} x$

e $y = -\operatorname{cosec} x$

f $y = -2 \sec x$

Example 20

- 2 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec\left(x - \frac{\pi}{2}\right)$

b $y = \cot\left(x + \frac{\pi}{4}\right)$

c $y = -\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$

d $y = 1 + \sec x$

e $y = 2 - \operatorname{cosec} x$

f $y = 1 + \cot\left(x + \frac{\pi}{4}\right)$

Example 21

- 3 Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -2 \sec\left(x - \frac{\pi}{2}\right)$. Sketch the transformed graph over the interval $[-\pi, \pi]$.
- 4 Describe a sequence of transformations that will take the graph of $y = \operatorname{cosec} x$ to the graph of $y = \operatorname{cosec}(-2x) + 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.
- 5 Describe a sequence of transformations that will take the graph of $y = \cot x$ to the graph of $y = -\cot\left(2x - \frac{\pi}{2}\right) - 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.
- 6 On the one set of axes, sketch the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$ over the interval $[0, 2\pi]$. Find and label the points of intersection.

131 Applications of trigonometric functions

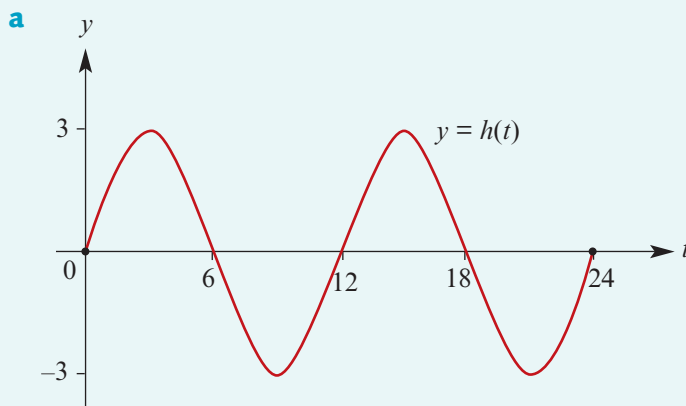
A **sinusoidal function** has a rule of the form $y = a \sin(nt + \epsilon) + b$ or, equivalently, of the form $y = a \cos(nt + \epsilon) + b$. Such functions can be used to model periodic phenomena.

Example 22

The height, $h(t)$ metres, of the tide above mean sea level at a harbour entrance over one day is given by the rule $h(t) = 3 \sin\left(\frac{\pi t}{6}\right)$, where t is the number of hours after midnight.

- Draw the graph of $y = h(t)$ for $0 \leq t \leq 24$.
- When was high tide?
- What was the height of the high tide?
- What was the height of the tide at 8 a.m.?
- A boat can only enter the harbour when the tide is at least 1 metre above mean sea level. When could the boat enter the harbour during this particular day?

Solution



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

- b** High tide occurs when $h(t) = 3$:

$$3 \sin\left(\frac{\pi t}{6}\right) = 3$$

$$\sin\left(\frac{\pi t}{6}\right) = 1$$

$$\frac{\pi t}{6} = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore t = 3, 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).

- c** The high tide has height 3 metres above the mean height.

d $h(8) = 3 \sin\left(\frac{8\pi}{6}\right) = 3 \sin\left(\frac{4\pi}{3}\right) = 3 \times \frac{-\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$

The water is $\frac{3}{2}\sqrt{3}$ metres below the mean height at 8 a.m.

- e** First consider when $h(t) = 1$:

$$3 \sin\left(\frac{\pi t}{6}\right) = 1$$

$$\sin\left(\frac{\pi t}{6}\right) = \frac{1}{3}$$

$$\therefore t = 0.649, 5.351, 12.649, 17.351$$

i.e. the water is at height 1 metre at 00:39, 05:21, 12:39, 17:21.

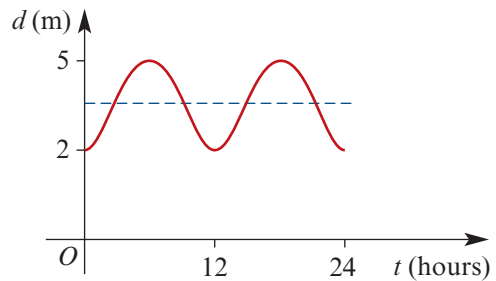
Thus the boat can enter the harbour between 00:39 and 05:21, and between 12:39 and 17:21.

Exercise 131

Example 22

- 1 The number of hours of daylight at a point on the Arctic Circle is given approximately by $d = 12 - 12 \cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$, where t is the number of months which have elapsed since 1 January.
 - a i Find d on 21 December ($t \approx 11.7$).
 - ii Find d on 21 June ($t \approx 5.7$).
 - b When will there be 5 hours of daylight?
- 2 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 12 + 4 \sin\left(\frac{\pi t}{6}\right)$, $0 \leq t \leq 24$.
 - a Sketch the graph of $D(t)$ for $0 \leq t \leq 24$.
 - b Find the values of t for which $D(t) \geq 12$.
 - c Boats which need a depth of w metres are permitted to enter the harbour only if the depth of the water at the entrance is at least w metres for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of w which satisfies this condition.
- 3 A particle moves on a straight line, OX , and its distance x metres from O at time t seconds is given by $x = 4 + 3 \sin(2\pi t)$.
 - a Find its greatest distance from O .
 - b Find its least distance from O .
 - c Find the times at which it is 7 metres from O for $0 \leq t \leq 2$.
 - d Find the times at which it is 1 metre from O for $0 \leq t \leq 2$.
 - e Describe the motion of the particle.
- 4 The temperature, $A^\circ\text{C}$, inside a house at t hours after 4 a.m. is given by the rule $A = 21 - 3 \cos\left(\frac{\pi t}{12}\right)$, for $0 \leq t \leq 24$. The temperature, $B^\circ\text{C}$, outside the house at the same time is given by $B = 22 - 5 \cos\left(\frac{\pi t}{12}\right)$, for $0 \leq t \leq 24$.
 - a Find the temperature inside the house at 8 a.m.
 - b Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.
 - c Sketch the graph of D for $0 \leq t \leq 24$.
 - d Determine when the inside temperature is less than the outside temperature.
- 5 The water level on a beach wall is given by $d(t) = 6 + 4 \cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$, where t is the number of hours after midnight and d is the depth of the water in metres.
 - a What is the earliest time of day at which the water is at its highest?
 - b When is the water 2 m up the wall?

- 6** The graph shows the distance, $d(t)$, of the tip of the hour hand of a large clock from the ceiling at time t hours.



- a** The function d is sinusoidal. Find:
- the amplitude
 - the period
 - the rule for $d(t)$
 - the length of the hour hand.
- b** At what times is the distance less than 3.5 metres from the ceiling?
- 7** In a tidal river, the time between high tide and low tide is 8 hours. The average depth of water at a point in the river is 4 metres; at high tide the depth is 5 metres.
- Sketch the graph of the depth of water at the point for the time interval from 0 to 24 hours if the relationship between time and depth is sinusoidal and there is a high tide at noon.
 - If a boat requires a depth of 4 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?
 - If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?
- 8** The population of a particular species of ant varies with time. The population, $N(t)$, at time t weeks after 1 January 2019 is given by

$$N(t) = 3000 \sin\left(\frac{\pi(t-10)}{26}\right) + 4000$$

- a** For the rule $N(t)$, state:
- the period
 - the amplitude
 - the range.
- b**
- State the values of $N(0)$ and $N(100)$.
 - Sketch the graph of $y = N(t)$ for $t \in [0, 100]$.
- c** Find the values of $t \in [0, 100]$ for which the population is:
- 7000
 - 1000
- d** Find the values of $t \in [0, 100]$ for which $N(t) > 5500$. That is, find the intervals of time during the first 100 weeks for which the population of ants is greater than 5500.
- e** A second population of ants also varies with time. The rule for the population, $M(t)$, at time t weeks after 1 January 2019 is of the form

$$M(t) = a \sin\left(\frac{\pi(t-c)}{b}\right) + d$$

where a , b , c and d are positive constants. Find a set of possible values for the constants a , b , c and d given that the population has the following properties:

- the maximum population is 40 000 and occurs at $t = 10$
- the minimum population is 10 000 and occurs at $t = 20$
- the maximum and minimum values do not occur between $t = 10$ and $t = 20$.



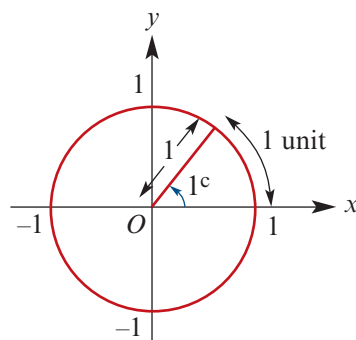
Chapter summary



■ Definition of a radian

One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

$$1^c = \frac{180^\circ}{\pi} \quad 1^\circ = \frac{\pi^c}{180}$$



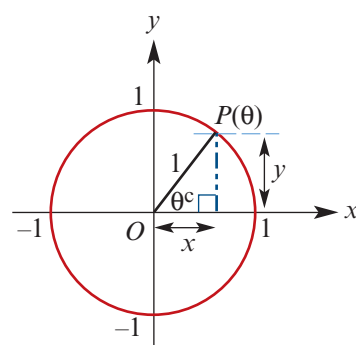
■ Sine and cosine functions

x -coordinate of $P(\theta)$ on unit circle:

$$x = \cos \theta, \quad \theta \in \mathbb{R}$$

y -coordinate of $P(\theta)$ on unit circle:

$$y = \sin \theta, \quad \theta \in \mathbb{R}$$



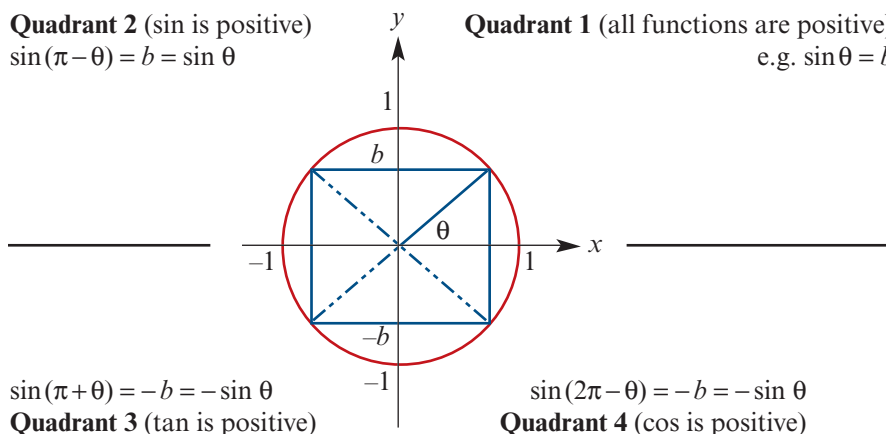
■ Tangent function

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \text{for } \cos \theta \neq 0$$

■ Symmetry properties of trigonometric functions

Quadrant 2 (sin is positive)
 $\sin(\pi - \theta) = b = \sin \theta$

Quadrant 1 (all functions are positive)
 e.g. $\sin \theta = b$



$$\sin(\pi + \theta) = -b = -\sin \theta$$

Quadrant 3 (tan is positive)

$$\sin(2\pi - \theta) = -b = -\sin \theta$$

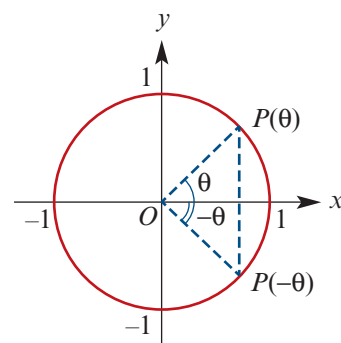
Quadrant 4 (cos is positive)

Negative angles:

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = \frac{-\sin \theta}{\cos \theta} = -\tan \theta$$



Complementary relationships:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta, \quad \sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta, \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$$

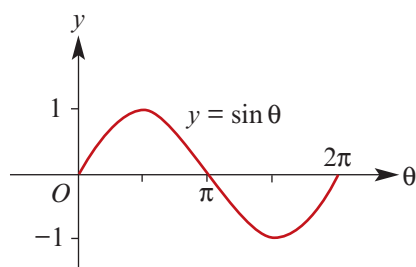
■ Pythagorean identity

$$\cos^2 \theta + \sin^2 \theta = 1$$

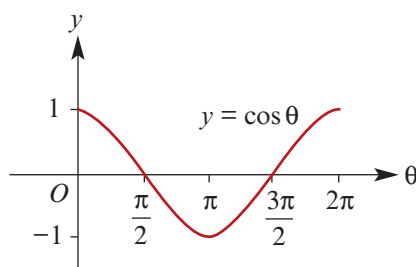
■ Exact values of trigonometric functions

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

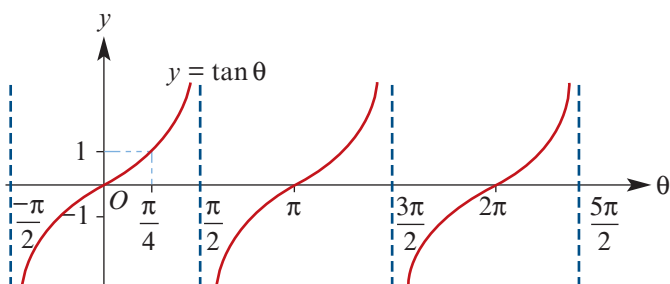
■ Graphs of trigonometric functions



Amplitude = 1
Period = 2π



Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

■ Transformations of the graphs of trigonometric functions

For the graphs of $y = a \sin(n(x + \varepsilon)) + b$ and $y = a \cos(n(x + \varepsilon)) + b$, where $a, n \in \mathbb{R}^+$:

- Period = $\frac{2\pi}{n}$
- Amplitude = a
- Range = $[-a + b, a + b]$

For the graph of $y = a \tan(n(x + \varepsilon)) + b$, where $n \in \mathbb{R}^+$:

- Period = $\frac{\pi}{n}$
- Asymptotes: $x = \frac{(2k+1)\pi}{2n} - \varepsilon$, where $k \in \mathbb{Z}$

■ General solution of trigonometric equations

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

■ Reciprocal trigonometric functions

- $\sec \theta = \frac{1}{\cos \theta}$
- $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$
- $\cot \theta = \frac{\cos \theta}{\sin \theta}$
 $= \frac{1}{\tan \theta} \quad (\text{if } \cos \theta \neq 0)$

Technology-free questions

- 1 Convert each of the following to radian measure in terms of π :

- a** 390° **b** 840° **c** 1110° **d** 1065° **e** 165°
f 450° **g** 420° **h** 390° **i** 40°

- 2 Convert each of the following to degree measure:

- a** $\frac{11\pi}{6}$ **b** $\frac{17\pi}{4}$ **c** $\frac{9\pi}{4}$ **d** $\frac{7\pi}{12}$ **e** $\frac{17\pi}{2}$
f $-\frac{11\pi}{4}$ **g** $-\frac{5\pi}{4}$ **h** $-\frac{13\pi}{4}$ **i** $\frac{23\pi}{4}$

- 3 Give the exact value of each of the following:

- a** $\sin\left(\frac{9\pi}{4}\right)$ **b** $\cos\left(\frac{-5\pi}{4}\right)$ **c** $\sin\left(\frac{3\pi}{2}\right)$ **d** $\cos\left(\frac{-3\pi}{2}\right)$
e $\cos\left(\frac{11\pi}{6}\right)$ **f** $\sin\left(\frac{21\pi}{6}\right)$ **g** $\tan\left(\frac{-25\pi}{3}\right)$ **h** $\tan\left(\frac{-15\pi}{4}\right)$

- 4 State the amplitude and period of each of the following:

- a** $4 \sin\left(\frac{\theta}{2}\right)$ **b** $-5 \sin(6\theta)$ **c** $\frac{1}{3} \sin(4\theta)$
d $-2 \cos(5x)$ **e** $-7 \sin\left(\frac{\pi x}{4}\right)$ **f** $\frac{2}{3} \sin\left(\frac{2\pi x}{3}\right)$

- 5** Find the maximum and minimum values of the function with rule:
- a** $3 + 2 \sin \theta$ **b** $4 - 5 \cos \theta$
- 6** Sketch the graph of each of the following (showing one cycle):
- a** $y = 2 \cos(2x)$ **b** $y = -3 \sin\left(\frac{x}{3}\right)$ **c** $y = -2 \cos(3x)$
- d** $y = 2 \cos\left(\frac{x}{3}\right)$ **e** $y = \cos\left(x - \frac{\pi}{4}\right)$ **f** $y = \cos\left(x + \frac{2\pi}{3}\right)$
- g** $y = 2 \sin\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3 \sin\left(x + \frac{5\pi}{6}\right)$
- 7** Solve each of the following equations:
- a** $\cos \theta = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$ **b** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$
- c** $\cos\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$ **d** $\cos\left(\theta + \frac{\pi}{3}\right) = -1$ for $\theta \in [0, 2\pi]$
- e** $\cos\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$
- 8** Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:
- a** $f(x) = 2 \cos(2x) + 1$ **b** $f(x) = 1 - 2 \cos(2x)$ **c** $f(x) = 3 \sin\left(x + \frac{\pi}{3}\right)$
- d** $f(x) = 2 - \sin\left(x + \frac{\pi}{3}\right)$ **e** $f(x) = 1 - 2 \cos(3x)$
- 9** Solve each of the following for $x \in [0, 2\pi]$:
- a** $\tan x = -\sqrt{3}$ **b** $\tan\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$
- c** $2 \tan\left(\frac{x}{2}\right) + 2 = 0$ **d** $3 \tan\left(\frac{\pi}{2} + 2x\right) = -3$
- 10** Sketch the graph of each of the following for $x \in [0, \pi]$, clearly labelling all intercepts with the axes and all asymptotes:
- a** $f(x) = \tan(2x)$ **b** $f(x) = \tan\left(x - \frac{\pi}{3}\right)$
- c** $f(x) = 2 \tan\left(2x + \frac{\pi}{3}\right)$ **d** $f(x) = 2 \tan\left(2x + \frac{\pi}{3}\right) - 2$
- 11** Find the values of $\theta \in [0, 2\pi]$ for which:
- a** $\sin^2 \theta = \frac{1}{4}$ **b** $\sin(2\theta) = \frac{1}{2}$ **c** $\cos(3\theta) = \frac{\sqrt{3}}{2}$ **d** $\sin^2(2\theta) = 1$
- 12** Solve the equation $\tan(\theta^\circ) = 2 \sin(\theta^\circ)$ for values of θ° from 0° to 360° .
- 13** Find the general solution for each of the following:
- a** $\sin(2x) = -1$ **b** $\cos(3x) = 1$ **c** $\tan x = -1$
- 14** For each of the following, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same axes:
- a** $f(x) = \cos(x) + 1$, $x \in [0, 2\pi]$ **b** $f(x) = \sin(x) + 2$, $x \in [0, 2\pi]$

- 15** Sketch the graph of each of the following over the interval $[0, 2\pi]$:
- a** $y = \cot(4x)$ **b** $y = 2 \operatorname{cosec}(2x)$ **c** $y = -3 \sec(2x)$
d $y = \operatorname{cosec}\left(x - \frac{\pi}{3}\right)$ **e** $y = \sec(x) - 3$ **f** $y = \cot\left(x + \frac{\pi}{3}\right)$
- 16** Describe a sequence of transformations that will take the graph of $y = \operatorname{cosec} x$ to the graph of $y = -3 \operatorname{cosec}(2x) + 1$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

Multiple-choice questions



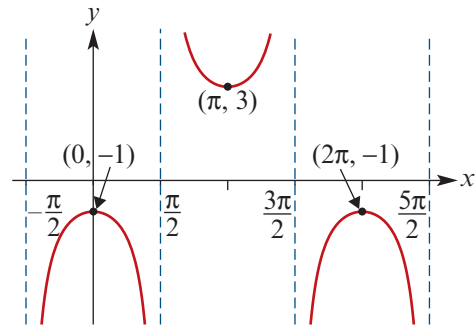
- 1** The period of the graph of $y = 2 \sin(3x - \pi) + 4$ is
A $\frac{2\pi}{3}$ **B** 2 **C** 3 **D** π **E** 2π
- 2** The amplitude of the graph of $y = -5 \cos(5x) + 3$ is
A -5 **B** -2 **C** 2 **D** 5 **E** 8
- 3** The number of solutions of $5 \sin(2x - \pi) + 2 = 0$ in the interval $[0, 2\pi]$ is
A 1 **B** 2 **C** 3 **D** 4 **E** 8
- 4** The solutions of $2 \sin(3x) + \sqrt{2} = 0$ in the interval $\left(\frac{5\pi}{12}, \frac{23\pi}{12}\right)$ are
A $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$ **B** 1.83, 3.40, 3.93, 5.50
C $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$ **D** $\frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}$
E none of the above
- 5** $\cos\left(-\frac{13\pi}{6}\right)$ is equal to
A $-\cos\left(\frac{13\pi}{6}\right)$ **B** $-\frac{\sqrt{3}}{2}$ **C** $\cos\left(-\frac{7\pi}{6}\right)$ **D** $-\frac{1}{2}$ **E** $\sin\left(\frac{2\pi}{3}\right)$
- 6** $\tan(180 - \theta)^\circ$ is equal to
A $\frac{\sin(90 + \theta)^\circ}{\cos(90 - \theta)^\circ}$ **B** $\frac{\cos(180 - \theta)^\circ}{\sin(180 - \theta)^\circ}$ **C** $\frac{\sin(90 - \theta)^\circ}{\cos(90 + \theta)^\circ}$
D $\frac{\cos(90 - \theta)^\circ}{\sin(90 + \theta)^\circ}$ **E** $\frac{\cos(90 + \theta)^\circ}{\sin(90 - \theta)^\circ}$
- 7** $\sin\left(\frac{\pi}{2} - x\right)$ is not equal to
A $\cos(2\pi - x)$ **B** $-\sin\left(\frac{3\pi}{2} + x\right)$ **C** $\sin x$
D $\cos(-x)$ **E** $\sin\left(\frac{\pi}{2} + x\right)$
- 8** Given that $\sin a^\circ = b$, which of the following is equal to b ?
A $\sin(90 + a)^\circ$ **B** $\sin(90 - a)^\circ$ **C** $\sin(180 + a)^\circ$ **D** $\sin(180 - a)^\circ$ **E** $\sin(360 - a)^\circ$

- 9 The period of the graph of $f(x) = 4 \sin(3\pi x) - 3 \cos(2\pi x)$ is

A 1 **B** 2 **C** 3 **D** 4 **E** $\frac{2}{3}$

- 10 An equation for the graph shown is

A $y = -2 \operatorname{cosec}(x) + 1$
B $y = -2 \sec(x) + 1$
C $y = 2 \operatorname{cosec}(x) + 1$
D $y = 2 \sec(x) + 1$
E $y = 2 \operatorname{cosec}(x) - 1$



Extended-response questions

- 1 The temperature, $T^\circ\text{C}$, in a small town in the mountains over a day is modelled by the function with rule

$$T = 15 - 8 \cos\left(\frac{\pi t}{12} + 6\right), \quad 0 \leq t \leq 24$$

where t is the time in hours after midnight.

- What is the temperature at midnight, correct to two significant figures?
 - What are the maximum and minimum temperatures reached?
 - At what times of the day, to the nearest minute, are temperatures warmer than 20°C ?
 - Sketch the graph for the temperatures over a day.
- 2 A particle oscillates back and forth, in a straight line, between points A and B about a point O . Its position, $x(t)$ metres, relative to O at time t seconds is given by the rule $x(t) = 3 \sin(2\pi t - a)$. The position of the particle when $t = 1$ is $x = -1.5$.



- If $a \in \left[0, \frac{\pi}{2}\right]$, find the value of a .
- Sketch the graph of $x(t)$ against t for $t \in [0, 2]$. Label the maximum and minimum points, the axis intercepts and the endpoints with their coordinates.
- How far from O is point A ?
- At what time does the particle first pass through A ?
- How long is it before the particle returns to A ?
- How long does it take for the particle to go from A to O ?
- How far does the particle travel in:
 - the first 2 seconds of its motion
 - the first 2.5 seconds of its motion?

- 3** The depth, D metres, of sea water in a bay t hours after midnight on a particular day may be represented by a function with rule

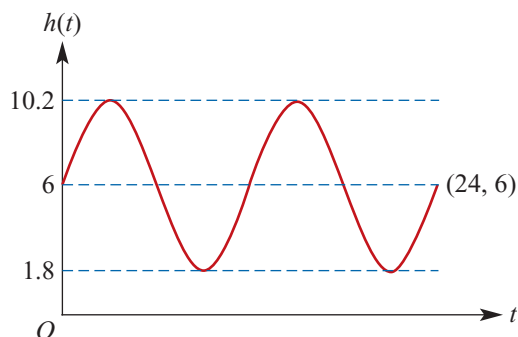
$$D(t) = a + b \cos\left(\frac{2\pi t}{k}\right)$$

where a , b and k are real numbers. The water is at a maximum depth of 15.4 metres at midnight and noon, and is at a minimum depth of 11.4 metres at 6 a.m. and 6 p.m.

- a** Find the value of:
- a
 - b
 - k
- b** Find the times when the depth of the water is 13.4 metres.
- c** Find the values of t for which the depth of the water is less than 14.4 metres.
- 4** The depth of water, $h(t)$ m, at a particular jetty in a harbour at time t hours after midnight is given by the rule

$$h(t) = p + q \sin\left(\frac{\pi t}{6}\right)$$

for constants p and q . The graph of $h(t)$ against t for $t \in [0, 24]$ is shown. The maximum depth is 10.2 m and the minimum depth is 1.8 m.



- a** Find the values of p and q .
- b** At what times during the time interval $[0, 24]$ is the depth of water at a maximum?
- c** What is the average depth of the water over the time interval $[0, 24]$?
- d** At what times during the time interval $[0, 24]$ is the depth of the water 3.9 m?
- e** For how long during the 24-hour period from midnight is the depth of the water more than 8.1 m?
- 5** Consider the function with rule $f(x) = 2 \sin(3x) + 1$ for $x \in [0, 2\pi]$.
- a** Find the values of k such that the equation $f(x) = k$ has:
- six solutions for $x \in [0, 2\pi]$
 - three solutions for $x \in [0, 2\pi]$
 - no solutions for $x \in [0, 2\pi]$.
- b** Find a sequence of transformations which takes the graph of $y = f(x)$ to the graph of $y = \sin x$.
- c** Find the values of $h \in [0, 2\pi]$ such that the graph of $y = f(x + h)$ has:
- a maximum at the point $\left(\frac{\pi}{3}, 3\right)$
 - a minimum at the point $\left(\frac{\pi}{3}, -1\right)$.

- 6** The population, N , of a particular species of ant varies with the seasons. The population is modelled by the equation

$$N = 3000 \sin\left(\frac{\pi(t-1)}{6}\right) + 4000$$

where t is the number of months after 1 January in a given year. The population, M , of a second species of ant also varies with time. Its population is modelled by the equation

$$M = 3000 \sin\left(\frac{\pi(t-3.5)}{5}\right) + 5500$$

where t is again the number of months after 1 January in a given year.

Use your calculator to plot the graphs of both equations over a period of one year on the same axes, and hence:

- a** find the maximum and minimum populations of both species and the months in which those maximums and minimums occur
 - b** find the months of the year during which the populations of both species are equal and find the population of each species at that time
 - c** by formulating a third equation, find when the combined population of species N and M is at a maximum and find this maximum value
 - d** by formulating a fourth equation, find when the difference between the two populations is at a maximum.
- 7** Passengers on a ferris wheel access their seats from a platform 5 m above the ground. As each seat is filled, the ferris wheel moves around so that the next seat can be filled. Once all seats are filled, the ride begins and lasts for 6 minutes. The height, h m, of Isobel's seat above the ground t seconds after the ride has begun is given by $h = 15 \sin(10t - 45)^\circ + 16.5$.
- a** Use a calculator to sketch the graph of h against t for the first 2 minutes of the ride.
 - b** How far above the ground is Isobel's seat at the commencement of the ride?
 - c** After how many seconds does Isobel's seat pass the access platform?
 - d** How many times will her seat pass the access platform in the first 2 minutes?
 - e** How many times will her seat pass the access platform during the entire ride?
- Due to a malfunction, the ferris wheel stops abruptly 1 minute 40 seconds into the ride.
- f** How far above the ground is Isobel stranded?
 - g** If Isobel's brother Hamish had a seat 1.5 m above the ground at the commencement of the ride, how far above the ground is Hamish stranded?

14

Trigonometric identities

Objectives

- ▶ To use the reciprocal trigonometric functions to obtain alternative forms of the **Pythagorean identity**.
- ▶ To evaluate simple trigonometric expressions using **trigonometric identities**.
- ▶ To prove simple trigonometric identities.
- ▶ To apply the **angle sum and difference identities** for sine and cosine.
- ▶ To apply the **double-angle identities** for sine and cosine.
- ▶ To simplify expressions of the form $a \cos x + b \sin x$.
- ▶ To sketch graphs of functions of the form $f(x) = a \cos x + b \sin x$.
- ▶ To solve equations of the form $a \cos x + b \sin x = c$.
- ▶ To apply the trigonometric identities for products of sines and cosines expressed as sums or differences, and vice versa.

In this chapter we build on our study of trigonometric functions from Chapter 13.

There are many interesting and useful relationships between the trigonometric functions. The most fundamental is the Pythagorean identity:

$$\sin^2 A + \cos^2 A = 1$$

Some of these identities were discovered a very long time ago. For example, the following two results were discovered by the Indian mathematician Bhāskara II in the twelfth century:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

They are of great importance in many areas of mathematics, including calculus.

14A Reciprocal trigonometric functions and the Pythagorean identity

In Section 13H we looked at the graphs of the reciprocal trigonometric functions. Here we use these functions in alternative forms of the Pythagorean identity.

► Reciprocal trigonometric functions

Recall that the trigonometric functions sine, cosine and tangent are used to form three new functions, called the reciprocal trigonometric functions.

Secant, cosecant and cotangent

$$\blacksquare \sec \theta = \frac{1}{\cos \theta}$$

(for $\cos \theta \neq 0$)

$$\blacksquare \operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

(for $\sin \theta \neq 0$)

$$\blacksquare \cot \theta = \frac{\cos \theta}{\sin \theta}$$

(for $\sin \theta \neq 0$)

Note: For $\cos \theta \neq 0$ and $\sin \theta \neq 0$, we have

$$\cot \theta = \frac{1}{\tan \theta} \quad \text{and} \quad \tan \theta = \frac{1}{\cot \theta}$$

Example 1

Find the exact value of each of the following:

a $\sec\left(\frac{2\pi}{3}\right)$

b $\cot\left(\frac{5\pi}{4}\right)$

c $\operatorname{cosec}\left(\frac{7\pi}{4}\right)$

Solution

$$\begin{aligned} \mathbf{a} \quad \sec\left(\frac{2\pi}{3}\right) &= \frac{1}{\cos\left(\frac{2\pi}{3}\right)} \\ &= \frac{1}{\cos\left(\pi - \frac{\pi}{3}\right)} \\ &= \frac{1}{-\cos\left(\frac{\pi}{3}\right)} \\ &= 1 \div \left(-\frac{1}{2}\right) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \cot\left(\frac{5\pi}{4}\right) &= \frac{\cos\left(\frac{5\pi}{4}\right)}{\sin\left(\frac{5\pi}{4}\right)} \\ &= \frac{\cos\left(\pi + \frac{\pi}{4}\right)}{\sin\left(\pi + \frac{\pi}{4}\right)} \\ &= \frac{-1}{\sqrt{2}} \div \left(\frac{-1}{\sqrt{2}}\right) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \operatorname{cosec}\left(\frac{7\pi}{4}\right) &= \frac{1}{\sin\left(2\pi - \frac{\pi}{4}\right)} \\ &= \frac{1}{-\sin\left(\frac{\pi}{4}\right)} \\ &= 1 \div \left(-\frac{1}{\sqrt{2}}\right) \\ &= -\sqrt{2} \end{aligned}$$

Note: In this example, we are using symmetry properties and exact values of trigonometric functions, which are revised in Section 13B.

Example 2Find the values of x between 0 and 2π for which:

a $\sec x = -2$

b $\cot x = -1$

Solution

a $\sec x = -2$

$$\frac{1}{\cos x} = -2$$

$$\cos x = \frac{-1}{2}$$

$$\therefore x = \pi - \frac{\pi}{3} \quad \text{or} \quad x = \pi + \frac{\pi}{3}$$

$$\therefore x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}$$

b $\cot x = -1$

$$\tan x = -1$$

$$\therefore x = \pi - \frac{\pi}{4} \quad \text{or} \quad x = 2\pi - \frac{\pi}{4}$$

$$\therefore x = \frac{3\pi}{4} \quad \text{or} \quad x = \frac{7\pi}{4}$$

ExplanationSince $\cos x$ is negative, the point $P(x)$ is in the 2nd or 3rd quadrant.Since $\tan x$ is negative, the point $P(x)$ is in the 2nd or 4th quadrant.**► The Pythagorean identity**

We introduced the Pythagorean identity in Section 13B. We can now derive two other forms of this identity.

Pythagorean identity

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$$

Proof Divide both sides of the Pythagorean identity by $\cos^2 \theta$:

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\therefore 1 + \tan^2 \theta = \sec^2 \theta$$

Divide both sides of the Pythagorean identity by $\sin^2 \theta$:

$$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$\therefore \cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$$

**Example 3**

a If $\operatorname{cosec} x = \frac{7}{4}$, find $\cos x$.

b If $\sec x = -\frac{3}{2}$ and $\frac{\pi}{2} \leq x \leq \pi$, find $\sin x$.

Solution

a Since $\operatorname{cosec} x = \frac{7}{4}$, we have $\sin x = \frac{4}{7}$.

$$\text{Now } \cos^2 x + \sin^2 x = 1$$

$$\cos^2 x + \frac{16}{49} = 1$$

$$\cos^2 x = \frac{33}{49}$$

$$\therefore \cos x = \pm \frac{\sqrt{33}}{7}$$

b Since $\sec x = -\frac{3}{2}$, we have $\cos x = -\frac{2}{3}$.

$$\text{Now } \cos^2 x + \sin^2 x = 1$$

$$\frac{4}{9} + \sin^2 x = 1$$

$$\therefore \sin x = \pm \frac{\sqrt{5}}{3}$$

But $\sin x$ is positive for $P(x)$ in the 2nd quadrant, and so $\sin x = \frac{\sqrt{5}}{3}$.

**Example 4**

Prove the identity $\frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta} = 2 \operatorname{cosec}^2 \theta$.

Solution

$$\text{LHS} = \frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta}$$

$$= \frac{1 + \cos \theta + 1 - \cos \theta}{1 - \cos^2 \theta}$$

$$= \frac{2}{1 - \cos^2 \theta}$$

$$= \frac{2}{\sin^2 \theta}$$

$$= 2 \operatorname{cosec}^2 \theta$$

$$= \text{RHS}$$

Section summary**■ Reciprocal trigonometric functions**

$$\sec \theta = \frac{1}{\cos \theta} \quad (\text{for } \cos \theta \neq 0)$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \quad (\text{for } \sin \theta \neq 0)$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (\text{for } \sin \theta \neq 0)$$

■ Pythagorean identity

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$$

Exercise 14A

Example 1

1 Find the exact value of each of the following:

a $\cot\left(\frac{3\pi}{4}\right)$

b $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$

c $\sec\left(\frac{5\pi}{6}\right)$

d $\operatorname{cosec}\left(\frac{\pi}{2}\right)$

e $\sec\left(\frac{4\pi}{3}\right)$

f $\operatorname{cosec}\left(\frac{13\pi}{6}\right)$

g $\cot\left(\frac{7\pi}{3}\right)$

h $\sec\left(\frac{5\pi}{3}\right)$

2 Without using a calculator, write down the exact value of each of the following:

a $\cot 135^\circ$

b $\sec 150^\circ$

c $\operatorname{cosec} 90^\circ$

d $\cot 240^\circ$

e $\operatorname{cosec} 225^\circ$

f $\sec 330^\circ$

g $\cot 315^\circ$

h $\operatorname{cosec} 300^\circ$

i $\cot 420^\circ$

Example 2

3 Find the values of x between 0 and 2π for which:

a $\operatorname{cosec} x = 2$

b $\cot x = \sqrt{3}$

c $\sec x + \sqrt{2} = 0$

d $\operatorname{cosec} x = \sec x$

Example 3

4 If $\sec \theta = \frac{-17}{8}$ and $\frac{\pi}{2} < \theta < \pi$, find:

a $\cos \theta$

b $\sin \theta$

c $\tan \theta$

5 If $\tan \theta = \frac{-7}{24}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\cos \theta$ and $\sin \theta$.6 Find the value of $\sec \theta$ if $\tan \theta = 0.4$ and θ is not in the 1st quadrant.7 If $\tan \theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$, evaluate $\frac{\sin \theta - 2 \cos \theta}{\cot \theta - \sin \theta}$.8 If $\cos \theta = \frac{2}{3}$ and θ is in the 4th quadrant, express $\frac{\tan \theta - 3 \sin \theta}{\cos \theta - 2 \cot \theta}$ in simplest surd form.

Example 4

9 Prove each of the following identities for suitable values of θ and φ :

a $(1 - \cos^2 \theta)(1 + \cot^2 \theta) = 1$

b $\cos^2 \theta \tan^2 \theta + \sin^2 \theta \cot^2 \theta = 1$

c $\frac{\tan \theta}{\tan \varphi} = \frac{\tan \theta + \cot \varphi}{\cot \theta + \tan \varphi}$

d $(\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 = 2$

e $\frac{1 + \cot^2 \theta}{\cot \theta \operatorname{cosec} \theta} = \sec \theta$

f $\sec \theta + \tan \theta = \frac{\cos \theta}{1 - \sin \theta}$

14B Angle sum and difference identities

Angle sum and difference identities for cosine

1 $\cos(A + B) = \cos A \cos B - \sin A \sin B$

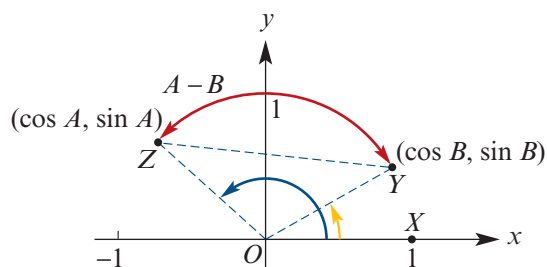
2 $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Proof Consider a unit circle as shown:

arc length $XY = B$ units

arc length $XZ = A$ units

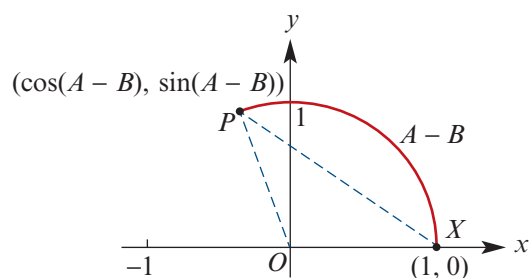
arc length $YZ = A - B$ units



Rotate $\triangle OZY$ so that Y is coincident with X . Then Z is moved to

$$P(\cos(A - B), \sin(A - B))$$

Since the triangles ZYO and PXO are congruent, we have $ZY = PX$.



Using the coordinate distance formula:

$$\begin{aligned} ZY^2 &= (\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\ &= 2 - 2(\cos A \cos B + \sin A \sin B) \end{aligned}$$

$$\begin{aligned} PX^2 &= (\cos(A - B) - 1)^2 + (\sin(A - B) - 0)^2 \\ &= 2 - 2\cos(A - B) \end{aligned}$$

Since $ZY = PX$, this gives

$$2 - 2\cos(A - B) = 2 - 2(\cos A \cos B + \sin A \sin B)$$

$$\therefore \cos(A - B) = \cos A \cos B + \sin A \sin B$$

We can now obtain the first identity from the second by replacing B with $-B$:

$$\begin{aligned} \cos(A + B) &= \cos(A - (-B)) \\ &= \cos A \cos(-B) + \sin A \sin(-B) \\ &= \cos A \cos B - \sin A \sin B \end{aligned}$$

Note: Here we used $\cos(-\theta) = \cos \theta$ and $\sin(-\theta) = -\sin \theta$.

We can use these two identities to find exact values for cosine.

Example 5

Evaluate:

a $\cos 75^\circ$

b $\cos 15^\circ$

Solution

a $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

b $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3} + 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Angle sum and difference identities for sine

1 $\sin(A + B) = \sin A \cos B + \cos A \sin B$

2 $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Proof We use the symmetry properties $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$ and $\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$:

$$\sin(A + B) = \cos\left(\frac{\pi}{2} - (A + B)\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - A\right) - B\right)$$

$$= \cos\left(\frac{\pi}{2} - A\right) \cos B + \sin\left(\frac{\pi}{2} - A\right) \sin B$$

$$= \sin A \cos B + \cos A \sin B$$

We can now obtain the second identity from the first by replacing B with $-B$:

$$\sin(A - B) = \sin(A + (-B))$$

$$= \sin A \cos(-B) + \cos A \sin(-B)$$

$$= \sin A \cos B - \cos A \sin B$$

Example 6

Evaluate:

a $\sin 75^\circ$

b $\sin 15^\circ$

Solution

a $\sin 75^\circ = \sin(30^\circ + 45^\circ)$

$$= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1 + \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

b $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 7Find $\sin(A - B)$ given that A and B are acute angles with $\cos A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$.**Solution**

Using the Pythagorean identity, we have

$$\sin^2 A = 1 - \cos^2 A \quad \text{and} \quad \sin^2 B = 1 - \cos^2 B$$

$$= 1 - \frac{9}{25}$$

$$= \frac{16}{25}$$

$$= 1 - \frac{144}{169}$$

$$= \frac{25}{169}$$

So $\sin A = \frac{4}{5}$ and $\sin B = \frac{5}{13}$, since A and B are acute. Hence

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}$$

$$= \frac{48 - 15}{65}$$

$$= \frac{33}{65}$$

► Double-angle identities

Using the angle sum identities for sine and cosine, we can easily derive useful expressions for $\sin(2A)$ and $\cos(2A)$.

Double-angle identities for cosine

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 && (\text{since } \sin^2 A = 1 - \cos^2 A) \\ &= 1 - 2\sin^2 A && (\text{since } \cos^2 A = 1 - \sin^2 A)\end{aligned}$$

Proof $\cos(A + A) = \cos A \cos A - \sin A \sin A$
 $= \cos^2 A - \sin^2 A$

Double-angle identity for sine

$$\sin(2A) = 2 \sin A \cos A$$

Proof $\sin(A + A) = \sin A \cos A + \cos A \sin A$
 $= 2 \sin A \cos A$

Example 8

If $\tan \theta = \frac{4}{3}$ and $0 < \theta < \frac{\pi}{2}$, evaluate:

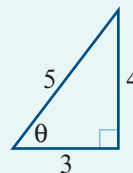
a $\sin(2\theta)$

b $\cos(2\theta)$

c $\tan(2\theta)$

Solution

Since θ is acute, we have $\sin \theta = \frac{4}{5}$ and $\cos \theta = \frac{3}{5}$.



a $\sin(2\theta) = 2 \sin \theta \cos \theta$ $= 2 \times \frac{4}{5} \times \frac{3}{5}$ $= \frac{24}{25}$	b $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$ $= \frac{9}{25} - \frac{16}{25}$ $= -\frac{7}{25}$	c $\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)}$ $= \frac{24}{25} \times \left(-\frac{25}{7}\right)$ $= -\frac{24}{7}$

**Example 9**

Prove each of the following identities:

a $\frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta} = \tan(2\theta)$

b $\frac{\sin \theta}{\sin \varphi} + \frac{\cos \theta}{\cos \varphi} = \frac{2 \sin(\theta + \varphi)}{\sin(2\varphi)}$

c $\frac{1}{\cos \theta + \sin \theta} + \frac{1}{\cos \theta - \sin \theta} = \tan(2\theta) \operatorname{cosec} \theta$

Solution

a LHS = $\frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}$
 $= \frac{\sin(2\theta)}{\cos(2\theta)}$
 $= \tan(2\theta)$
 $= \text{RHS}$

Note: Identity holds when $\cos(2\theta) \neq 0$.

b LHS = $\frac{\sin \theta}{\sin \varphi} + \frac{\cos \theta}{\cos \varphi}$
 $= \frac{\sin \theta \cos \varphi + \cos \theta \sin \varphi}{\sin \varphi \cos \varphi}$
 $= \frac{\sin(\theta + \varphi)}{\frac{1}{2} \sin(2\varphi)}$
 $= \frac{2 \sin(\theta + \varphi)}{\sin(2\varphi)}$
 $= \text{RHS}$

Note: Identity holds when $\sin(2\varphi) \neq 0$.

c LHS = $\frac{1}{\cos \theta + \sin \theta} + \frac{1}{\cos \theta - \sin \theta}$
 $= \frac{\cos \theta - \sin \theta + \cos \theta + \sin \theta}{\cos^2 \theta - \sin^2 \theta}$
 $= \frac{2 \cos \theta}{\cos(2\theta)}$
 $= \frac{2 \cos \theta}{\cos(2\theta)} \times \frac{\sin \theta}{\sin \theta}$
 $= \frac{\sin(2\theta)}{\cos(2\theta) \sin \theta}$
 $= \frac{\tan(2\theta)}{\sin \theta}$
 $= \tan(2\theta) \operatorname{cosec} \theta$
 $= \text{RHS}$

Note: Identity holds when $\cos(2\theta) \neq 0$ and $\sin \theta \neq 0$.

Sometimes the easiest way to prove that two expressions are equal is to simplify each of them separately. This is demonstrated in the following example.

Example 10

Prove that $(\sec A - \cos A)(\operatorname{cosec} A - \sin A) = \frac{1}{\tan A + \cot A}$.

Solution

$$\begin{aligned}\text{LHS} &= (\sec A - \cos A)(\operatorname{cosec} A - \sin A) \\ &= \left(\frac{1}{\cos A} - \cos A\right)\left(\frac{1}{\sin A} - \sin A\right) \\ &= \frac{1 - \cos^2 A}{\cos A} \times \frac{1 - \sin^2 A}{\sin A} \\ &= \frac{\sin^2 A \cos^2 A}{\cos A \sin A} \\ &= \cos A \sin A\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{\tan A + \cot A} \\ &= \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}} \\ &= \frac{\cos A \sin A}{\sin^2 A + \cos^2 A} \\ &= \cos A \sin A\end{aligned}$$

We have shown that LHS = RHS.

Section summary■ **Angle sum and difference identities**

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

■ **Double-angle identities**

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$= 2 \cos^2 A - 1$$

$$= 1 - 2 \sin^2 A$$

$$\sin(2A) = 2 \sin A \cos A$$

Note: Further multi-angle identities are investigated in Exercise 14B and Section 16C.

Exercise 14B**Skillsheet****Example 5, 6**

1 Using angle sum and difference identities, find exact values for the following:

a $\cos 165^\circ$

b $\cos 105^\circ$

c $\sin 165^\circ$

d $\sin 105^\circ$

2 Find the exact value of:

a $\cos\left(\frac{5\pi}{12}\right)$

b $\sin\left(\frac{11\pi}{12}\right)$

Example 7

3 If $\sin u = \frac{12}{13}$ and $\sin v = \frac{3}{5}$, evaluate $\sin(u + v)$. (**Note:** There is more than one answer.)

4 Simplify the following:

a $\sin\left(\theta + \frac{\pi}{6}\right)$

b $\cos\left(\varphi - \frac{\pi}{4}\right)$

c $\cos\left(\theta + \frac{\pi}{3}\right)$

d $\sin\left(\theta - \frac{\pi}{4}\right)$

5 Simplify:

a $\cos(u - v) \sin v + \sin(u - v) \cos v$

b $\sin(u + v) \sin v + \cos(u + v) \cos v$

Example 8 6 If $\sin \theta = \frac{-3}{5}$, with θ in the 3rd quadrant, and $\cos \varphi = \frac{-5}{13}$, with φ in the 2nd quadrant, evaluate each of the following without using a calculator:

- a** $\cos(2\theta)$ **b** $\sin(2\theta)$ **c** $\cos(2\theta)$ **d** $\sec(2\varphi)$
e $\sin(\theta + \varphi)$ **f** $\cos(\theta - \varphi)$ **g** $\operatorname{cosec}(\theta + \varphi)$ **h** $\cot(2\theta)$

7 If $\sin \alpha = \frac{3}{5}$ and $\sin \beta = \frac{24}{25}$, with $\frac{\pi}{2} < \beta < \alpha < \pi$, evaluate:

- a** $\cos(2\alpha)$ **b** $\sin(\alpha - \beta)$ **c** $\cos(\alpha + \beta)$ **d** $\sin(2\beta)$

8 If $\sin \theta = -\frac{\sqrt{3}}{2}$ and $\cos \theta = \frac{1}{2}$, evaluate:

- a** $\sin(2\theta)$ **b** $\cos(2\theta)$

9 Simplify each of the following expressions:

- a** $(\sin \theta - \cos \theta)^2$ **b** $\cos^4 \theta - \sin^4 \theta$

Example 9, 10 10 Prove the following identities:

- a** $\sqrt{2} \sin\left(\theta - \frac{\pi}{4}\right) = \sin \theta - \cos \theta$ **b** $\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos \theta$
c $\cos\left(\theta + \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{3} \cos \theta$ **d** $\frac{\sin(u + v)}{\cos u \cos v} = \tan v + \tan u$
e $\frac{\tan u + \tan v}{\tan u - \tan v} = \frac{\sin(u + v)}{\sin(u - v)}$ **f** $\frac{1 - \sin(2\theta)}{\sin \theta - \cos \theta} = \sin \theta - \cos \theta$

11 Angle sum and difference identities for tangent

a Show that

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

b Hence show that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \text{and} \quad \tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}$$

c Find the exact value of each of the following:

- i** $\tan 15^\circ$ **ii** $\tan 75^\circ$ **iii** $\tan 105^\circ$

12 Triple-angle identities Prove each of the following:

- a** $\cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta$ **b** $\sin(3\theta) = 3 \sin \theta - 4 \sin^3 \theta$

13 Quadruple-angle identities Prove each of the following:

- a** $\cos(4\theta) = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$ **b** $\cos(4\theta) = 1 - 8 \sin^2 \theta + 8 \sin^4 \theta$
c $\sin(4\theta) = 4 \sin \theta \cos \theta - 8 \sin^3 \theta \cos \theta$ **d** $\sin(4\theta) = 4 \sin \theta \cos^3 \theta - 4 \cos \theta \sin^3 \theta$

14 Prove the following identities:

- a** $8 \cos^4 \theta = 3 + 4 \cos(2\theta) + \cos(4\theta)$ **b** $8 \sin^4 \theta = 3 - 4 \cos(2\theta) + \cos(4\theta)$

14C Simplifying $a \cos x + b \sin x$

In this section, we see how to rewrite the rule of a function $f(x) = a \cos x + b \sin x$ in terms of a single trigonometric function.

$$a \cos x + b \sin x = r \cos(x - \alpha) \quad \text{where } r = \sqrt{a^2 + b^2}, \cos \alpha = \frac{a}{r} \text{ and } \sin \alpha = \frac{b}{r}$$

Proof Let $r = \sqrt{a^2 + b^2}$. Consider the point $P\left(\frac{a}{r}, \frac{b}{r}\right)$ and its distance from the origin O :

$$OP^2 = \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

Therefore the point P is on the unit circle, and so there is an angle α such that

$$\frac{a}{r} = \cos \alpha \quad \text{and} \quad \frac{b}{r} = \sin \alpha$$

We can now write

$$\begin{aligned} a \cos x + b \sin x &= r \left(\frac{a}{r} \cos x + \frac{b}{r} \sin x \right) \\ &= r (\cos \alpha \cos x + \sin \alpha \sin x) \\ &= r \cos(x - \alpha) \end{aligned}$$

Similarly, it may be shown that

$$a \cos x + b \sin x = r \sin(x + \beta) \quad \text{where } r = \sqrt{a^2 + b^2}, \sin \beta = \frac{a}{r}, \cos \beta = \frac{b}{r}$$

Example 11

- a** Express $\cos x - \sqrt{3} \sin x$ in the form $r \cos(x - \alpha)$.
- b** Hence find the range of the function f with rule $f(x) = \cos x - \sqrt{3} \sin x$, and find the maximum and minimum values of f .

Solution

- a** Here $a = 1$ and $b = -\sqrt{3}$. Therefore

$$r = \sqrt{1 + 3} = 2, \quad \cos \alpha = \frac{a}{r} = \frac{1}{2} \quad \text{and} \quad \sin \alpha = \frac{b}{r} = \frac{-\sqrt{3}}{2}$$

We see that $\alpha = -\frac{\pi}{3}$ and so

$$\cos x - \sqrt{3} \sin x = 2 \cos\left(x + \frac{\pi}{3}\right)$$

- b** From part **a** we have

$$f(x) = 2 \cos\left(x + \frac{\pi}{3}\right)$$

Thus the range of f is $[-2, 2]$, the maximum value is 2 and the minimum value is -2 .

Example 12

Solve $\cos x - \sqrt{3} \sin x = 1$ for $x \in [0, 2\pi]$.

Solution

From Example 11, we have

$$\cos x - \sqrt{3} \sin x = 2 \cos\left(x + \frac{\pi}{3}\right)$$

$$\therefore 2 \cos\left(x + \frac{\pi}{3}\right) = 1$$

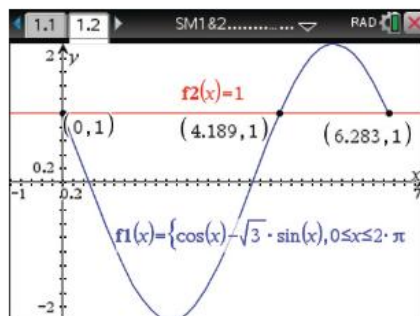
$$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3} \text{ or } \frac{7\pi}{3}$$

$$x = 0, \frac{4\pi}{3} \text{ or } 2\pi$$

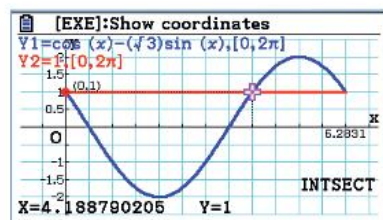
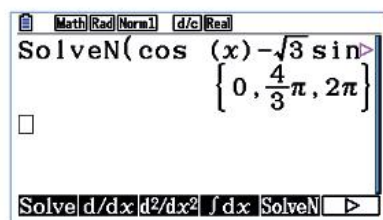
**Using the TI-Nspire CX non-CAS**

Since trigonometric equations often have multiple solutions, they are best solved graphically. Only approximate solutions are displayed.

**Using the Casio**

- Ensure that the angle setting is Radians.
- In **Run-Matrix** mode, select the numerical solver (**Calculation** **OPTN** **F4**, **SolveN** **F5**).
- Complete the equation and domain by entering:
 $\cos(x) - \sqrt{3} \sin(x) = 1, x, 0, 2\pi$

- Approximate solutions can also be found graphically.



**Example 13**

Express $\sqrt{3} \sin(2x) - \cos(2x)$ in the form $r \sin(2x + \alpha)$.

Solution

A slightly different technique is used. Assume that

$$\begin{aligned}\sqrt{3} \sin(2x) - \cos(2x) &= r \sin(2x + \alpha) \\ &= r (\sin(2x) \cos \alpha + \cos(2x) \sin \alpha)\end{aligned}$$

This is to hold for all x .

$$\text{For } x = \frac{\pi}{4}: \quad \sqrt{3} = r \cos \alpha \quad (1)$$

$$\text{For } x = 0: \quad -1 = r \sin \alpha \quad (2)$$

Squaring and adding (1) and (2) gives

$$\begin{aligned}r^2 \cos^2 \alpha + r^2 \sin^2 \alpha &= 4 \\ r^2 &= 4 \\ \therefore r &= \pm 2\end{aligned}$$

We take the positive solution. Substituting in (1) and (2) gives

$$\frac{\sqrt{3}}{2} = \cos \alpha \quad \text{and} \quad -\frac{1}{2} = \sin \alpha$$

Thus $\alpha = -\frac{\pi}{6}$ and hence

$$\sqrt{3} \sin(2x) - \cos(2x) = 2 \sin\left(2x - \frac{\pi}{6}\right)$$

Check: Expand the new expression using the angle difference identity for sine:

$$\begin{aligned}2 \sin\left(2x - \frac{\pi}{6}\right) &= 2 \left(\sin(2x) \cos\left(\frac{\pi}{6}\right) - \cos(2x) \sin\left(\frac{\pi}{6}\right) \right) \\ &= 2 \left(\sin(2x) \times \frac{\sqrt{3}}{2} - \cos(2x) \times \frac{1}{2} \right) \\ &= \sqrt{3} \sin(2x) - \cos(2x)\end{aligned}$$

Section summary

- $a \cos x + b \sin x = r \cos(x - \alpha)$ where $r = \sqrt{a^2 + b^2}$, $\cos \alpha = \frac{a}{r}$, $\sin \alpha = \frac{b}{r}$
- $a \cos x + b \sin x = r \sin(x + \beta)$ where $r = \sqrt{a^2 + b^2}$, $\sin \beta = \frac{a}{r}$, $\cos \beta = \frac{b}{r}$

Exercise 14C

Skillsheet

1 Find the maximum and minimum values of the following:

Example 11

a $4 \cos x + 3 \sin x$

b $\sqrt{3} \cos x + \sin x$

c $\cos x - \sin x$

d $\cos x + \sin x$

e $3 \cos x + \sqrt{3} \sin x$

f $\sin x - \sqrt{3} \cos x$

g $\cos x - \sqrt{3} \sin x + 2$

h $5 + 3 \sin x - 2 \cos x$

Example 12

2 Solve each of the following for $x \in [0, 2\pi]$ or for $\theta \in [0, 360]$:

a $\sin x - \cos x = 1$

b $\sqrt{3} \sin x + \cos x = 1$

c $\sin x - \sqrt{3} \cos x = -1$

d $3 \cos x - \sqrt{3} \sin x = 3$

e $4 \sin \theta^\circ + 3 \cos \theta^\circ = 5$

f $2\sqrt{2} \sin \theta^\circ - 2 \cos \theta^\circ = 3$

3 Write $\sqrt{3} \cos(2x) - \sin(2x)$ in the form $r \cos(2x + \alpha)$.

Example 13

4 Write $\cos(3x) - \sin(3x)$ in the form $r \sin(3x - \alpha)$.

5 Sketch the graph of each of the following, showing one cycle:

a $f(x) = \sin x - \cos x$

b $f(x) = \sqrt{3} \sin x + \cos x$

c $f(x) = \sin x + \cos x$

d $f(x) = \sin x - \sqrt{3} \cos x$

14D Sums and products of sines and cosines

In Section 14B, we derived the angle sum and difference identities for sine and cosine. We use them in this section to obtain new identities which allow us to rewrite products of sines and cosines as sums or differences, and vice versa.

► Expressing products as sums or differences

Product-to-sum identities

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

Proof We use the angle sum and difference identities for sine and cosine:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad (1)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad (2)$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad (3)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \quad (4)$$

The first product-to-sum identity is obtained by adding (2) and (1), the second identity is obtained by subtracting (1) from (2), and the third by adding (3) and (4).

Example 14

Express each of the following products as sums or differences:

a $2 \sin(3\theta) \cos(\theta)$

b $2 \sin 50^\circ \cos 60^\circ$

c $2 \cos\left(\theta + \frac{\pi}{4}\right) \cos\left(\theta - \frac{\pi}{4}\right)$

Solution

a Use the third product-to-sum identity:

$$\begin{aligned} 2 \sin(3\theta) \cos(\theta) &= \sin(3\theta + \theta) + \sin(3\theta - \theta) \\ &= \sin(4\theta) + \sin(2\theta) \end{aligned}$$

b Use the third product-to-sum identity:

$$\begin{aligned} 2 \sin 50^\circ \cos 60^\circ &= \sin 110^\circ + \sin(-10^\circ) \\ &= \sin 110^\circ - \sin 10^\circ \end{aligned}$$

c Use the first product-to-sum identity:

$$\begin{aligned} 2 \cos\left(\theta + \frac{\pi}{4}\right) \cos\left(\theta - \frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2}\right) + \cos(2\theta) \\ &= \cos(2\theta) \end{aligned}$$

► **Expressing sums and differences as products**

Sum-to-product identities

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$$

Proof Using the first product-to-sum identity, we have

$$\begin{aligned} 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) &= \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) \\ &= \cos B + \cos A \\ &= \cos A + \cos B \end{aligned}$$

The other three sum-to-product identities can be obtained similarly.

Example 15

Express each of the following as products:

a $\sin 36^\circ + \sin 10^\circ$

b $\cos 36^\circ + \cos 10^\circ$

c $\sin 36^\circ - \sin 10^\circ$

d $\cos 36^\circ - \cos 10^\circ$

Solution

a $\sin 36^\circ + \sin 10^\circ = 2 \sin 23^\circ \cos 13^\circ$

b $\cos 36^\circ + \cos 10^\circ = 2 \cos 23^\circ \cos 13^\circ$

c $\sin 36^\circ - \sin 10^\circ = 2 \cos 23^\circ \sin 13^\circ$

d $\cos 36^\circ - \cos 10^\circ = -2 \sin 23^\circ \sin 13^\circ$

Example 16

Prove that

$$\frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} = \tan(2\theta)$$

Solution

$$\begin{aligned}
 \text{LHS} &= \frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} \\
 &= \frac{-2 \sin(2\theta) \sin(-\theta)}{2 \sin(\theta) \cos(2\theta)} \\
 &= \frac{2 \sin(2\theta) \sin(\theta)}{2 \sin(\theta) \cos(2\theta)} \\
 &= \tan(2\theta) \\
 &= \text{RHS}
 \end{aligned}$$

Example 17Solve the equation $\sin(3x) + \sin(11x) = 0$ for $x \in [0, \pi]$.**Solution**

$$\begin{aligned}
 \sin(3x) + \sin(11x) &= 0 \\
 \Leftrightarrow 2 \sin(7x) \cos(4x) &= 0 \\
 \Leftrightarrow \sin(7x) = 0 \quad \text{or} \quad \cos(4x) &= 0 \\
 \Leftrightarrow 7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi \quad \text{or} \quad 4x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\
 \Leftrightarrow x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8} \quad \text{or} \quad \frac{7\pi}{8}
 \end{aligned}$$

Section summary

■ Product-to-sum identities

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

■ Sum-to-product identities

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$$

Exercise 14D

Example 14

1 Express each of the following products as sums or differences:

a $2 \sin(3\pi t) \cos(2\pi t)$

b $\sin 20^\circ \cos 30^\circ$

c $2 \cos\left(\frac{\pi x}{4}\right) \sin\left(\frac{3\pi x}{4}\right)$

d $2 \sin\left(\frac{A+B+C}{2}\right) \cos\left(\frac{A-B-C}{2}\right)$

2 Express $2 \sin(3\theta) \sin(2\theta)$ as a difference of cosines.

3 Use a product-to-sum identity to derive the expression for $2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$ as a difference of sines.

4 Show that $\sin 75^\circ \sin 15^\circ = \frac{1}{4}$.

Example 15

5 Express each of the following as products:

a $\sin 56^\circ + \sin 22^\circ$

b $\cos 56^\circ + \cos 22^\circ$

c $\sin 56^\circ - \sin 22^\circ$

d $\cos 56^\circ - \cos 22^\circ$

6 Express each of the following as products:

a $\sin(6A) + \sin(2A)$

b $\cos(x) + \cos(2x)$

c $\sin(4x) - \sin(3x)$

d $\cos(3A) - \cos(A)$

Example 16 7 Show that $\sin(A) + 2\sin(3A) + \sin(5A) = 4\cos^2(A)\sin(3A)$.

8 For any three angles α , β and γ , show that

$$\sin(\alpha + \beta)\sin(\alpha - \beta) + \sin(\beta + \gamma)\sin(\beta - \gamma) + \sin(\gamma + \alpha)\sin(\gamma - \alpha) = 0$$

9 Show that $\cos 70^\circ + \sin 40^\circ = \cos 10^\circ$.

10 Show that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

Example 17 11 Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\cos(5x) + \cos(x) = 0$

b $\cos(5x) - \cos(x) = 0$

c $\sin(5x) + \sin(x) = 0$

d $\sin(5x) - \sin(x) = 0$

12 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\cos(2\theta) - \sin(\theta) = 0$

b $\sin(5\theta) - \sin(3\theta) + \sin(\theta) = 0$

c $\sin(7\theta) - \sin(\theta) = \sin(3\theta)$

d $\cos(3\theta) - \cos(5\theta) + \cos(7\theta) = 0$

13 Prove that $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\frac{A+B}{2}\right)$.

14 Prove the identity:

$$4\sin(A+B)\sin(B+C)\sin(C+A) = \sin(2A) + \sin(2B) + \sin(2C) - \sin(2A+2B+2C)$$

15 Prove that $\frac{\cos(2A) - \cos(2B)}{\sin(2A - 2B)} = -\frac{\sin(A+B)}{\cos(A-B)}$.

16 Prove each of the following identities:

a $\frac{\sin(A) + \sin(3A) + \sin(5A)}{\cos(A) + \cos(3A) + \cos(5A)} = \tan(3A)$

b $\cos^2(A) + \cos^2(B) - 1 = \cos(A+B)\cos(A-B)$

c $\cos^2(A-B) - \cos^2(A+B) = \sin(2A)\sin(2B)$

d $\cos^2(A-B) - \sin^2(A+B) = \cos(2A)\cos(2B)$

17 Find the sum

$$\sin(x) + \sin(3x) + \sin(5x) + \cdots + \sin(99x)$$



Hint: First multiply this sum by $2\sin(x)$.

Chapter summary



■ Reciprocal trigonometric functions

$$\sec \theta = \frac{1}{\cos \theta} \quad \operatorname{cosec} \theta = \frac{1}{\sin \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} \quad (\text{if } \cos \theta \neq 0)$$

■ Pythagorean identity

$$\begin{aligned}\cos^2 \theta + \sin^2 \theta &= 1 \\ 1 + \tan^2 \theta &= \sec^2 \theta \\ \cot^2 \theta + 1 &= \operatorname{cosec}^2 \theta\end{aligned}$$

■ Angle sum and difference identities

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ \sin(A - B) &= \sin A \cos B - \cos A \sin B\end{aligned}$$

■ Double-angle identities

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1 \\ &= 1 - 2 \sin^2 A \\ \sin(2A) &= 2 \sin A \cos A\end{aligned}$$

■ Linear combinations

$$\begin{aligned}a \cos x + b \sin x &= r \cos(x - \alpha) \quad \text{where } r = \sqrt{a^2 + b^2}, \quad \cos \alpha = \frac{a}{r}, \quad \sin \alpha = \frac{b}{r} \\ a \cos x + b \sin x &= r \sin(x + \beta) \quad \text{where } r = \sqrt{a^2 + b^2}, \quad \sin \beta = \frac{a}{r}, \quad \cos \beta = \frac{b}{r}\end{aligned}$$

■ Product-to-sum identities

$$\begin{aligned}2 \cos A \cos B &= \cos(A - B) + \cos(A + B) \\ 2 \sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2 \sin A \cos B &= \sin(A + B) + \sin(A - B)\end{aligned}$$

■ Sum-to-product identities

$$\begin{aligned}\cos A + \cos B &= 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \\ \cos A - \cos B &= -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \\ \sin A + \sin B &= 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \\ \sin A - \sin B &= 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)\end{aligned}$$

Technology-free questions

- 1 Find the maximum and minimum values of each of the following:
a $3 + 2 \sin \theta$ **b** $1 - 3 \cos \theta$ **c** $4 \sin\left(\frac{3\theta}{2}\right)$ **d** $2 \sin^2\left(\frac{\theta}{2}\right)$ **e** $\frac{1}{2 + \cos \theta}$
- 2 Find the values of $\theta \in [0, 2\pi]$ for which:
a $\tan^2 \theta = \frac{1}{3}$ **b** $\tan(2\theta) = -1$ **c** $\sin(3\theta) = -1$ **d** $\sec(2\theta) = \sqrt{2}$
- 3 Prove each of the following identities:
a $\sec \theta + \operatorname{cosec} \theta \cot \theta = \sec \theta \operatorname{cosec}^2 \theta$ **b** $\sec \theta - \sin \theta = \frac{\tan^2 \theta + \cos^2 \theta}{\sec \theta + \sin \theta}$
- 4 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B are acute, find:
a $\cos(A + B)$ **b** $\sin(A + B)$ **c** $\tan(A + B)$
- 5 Find $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$.
- 6 If $A + B = \frac{\pi}{2}$, find the value of:
a $\sin A \cos B + \cos A \sin B$ **b** $\cos A \cos B - \sin A \sin B$
- 7 Prove each of the following:
a $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B$
b $\frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} = \frac{2}{\sin \theta}$ **c** $\frac{\sin \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - \cos \theta} = \tan \theta$
- 8 Given that $\sin A = \frac{\sqrt{5}}{3}$ and that A is obtuse, find the value of:
a $\cos(2A)$ **b** $\sin(2A)$ **c** $\sin(4A)$
- 9 Prove:
a $\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos(2A)$ **b** $\sqrt{2r^2(1 - \cos \theta)} = 2r \sin\left(\frac{\theta}{2}\right)$ for $r > 0$ and θ acute
- 10 Find $\tan 15^\circ$ in simplest surd form.
- 11 Solve each of the following equations for $x \in [0, 2\pi]$:
a $\sin x + \cos x = 1$ **b** $\sin\left(\frac{1}{2}x\right) \cos\left(\frac{1}{2}x\right) = -\frac{1}{4}$
c $3 \tan(2x) = 2 \tan x$ **d** $\sin^2 x = \cos^2 x + 1$
e $\sin(3x) \cos x - \cos(3x) \sin x = \frac{\sqrt{3}}{2}$ **f** $2 \cos\left(2x - \frac{\pi}{3}\right) = -\sqrt{3}$

SF

CF

- 12** Sketch the graph of:
- a** $y = 2 \cos^2 x$ **b** $y = 1 - 2 \sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$ **c** $f(x) = \tan(2x)$
- 13** Given that $\cos A = \frac{4}{5}$ and $\cos(A + \theta) = \frac{5}{13}$, where A and θ are acute, find the exact value of $\cos \theta$.
- 14** **a** Express $2 \cos \theta + 9 \sin \theta$ in the form $r \cos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.
- b** **i** Give the maximum value of $2 \cos \theta + 9 \sin \theta$.
- ii** Give the cosine of θ for which this maximum occurs.
- iii** Find the smallest positive solution of the equation $2 \cos \theta + 9 \sin \theta = 1$.
- 15** Solve each of the following equations for $\theta \in [0, \pi]$:
- a** $\sin(4\theta) + \sin(2\theta) = 0$ **b** $\sin(2\theta) - \sin(\theta) = 0$
- 16** Prove that $\frac{\cos A - \cos B}{\sin A + \sin B} = \tan\left(\frac{B - A}{2}\right)$.

Multiple-choice questions

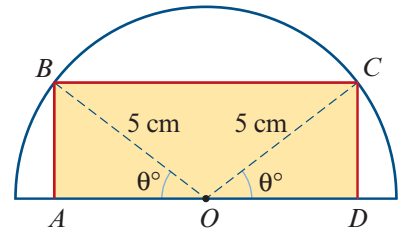


- 1** $\operatorname{cosec} x - \sin x$ is equal to
- A** $\cos x \cot x$ **B** $\operatorname{cosec} x \tan x$ **C** $1 - \sin^2 x$
- D** $\sin x \operatorname{cosec} x$ **E** $\frac{1 - \sin x}{\sin x}$
- 2** If $\cos x = \frac{-1}{3}$, then the possible values of $\sin x$ are
- A** $\frac{-2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}$ **B** $\frac{-2}{3}, \frac{2}{3}$ **C** $\frac{-8}{9}, \frac{8}{9}$
- D** $\frac{-\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$ **E** $\frac{1}{2}, \frac{-1}{2}$
- 3** If $\cos \theta = \frac{a}{b}$ and $0 < \theta < \frac{\pi}{2}$, then $\tan \theta$ is equal to
- A** $\frac{\sqrt{a^2 + b^2}}{b}$ **B** $\frac{\sqrt{b^2 - a^2}}{a}$ **C** $\frac{a}{\sqrt{b^2 - a^2}}$ **D** $\frac{a}{\sqrt{b^2 + a^2}}$ **E** $\frac{a}{b\sqrt{b^2 + a^2}}$
- 4** $\sin(\alpha + \beta) \cos(\beta) - \cos(\alpha + \beta) \sin(\beta)$ is equal to
- A** $\sin(\alpha) \cos^2(\beta) - \sin(\alpha) \sin^2(\beta)$ **B** $\sin(\alpha)$
- C** $2 \sin(\alpha) \cos(\alpha) \cos(\beta)$ **D** $2 \sin(\beta) \cos(\beta) (\sin(\alpha) - \cos(\alpha))$
- E** $\sin(\alpha - \beta)$

- 5** If $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$ with $\cos A = t$ and $\sin B = t$, then $\sin(B + A)$ equals
A 0 **B** 1 **C** $2t^2 - 1$ **D** $1 - 2t^2$ **E** -1
- 6** $\frac{\sin(2A)}{\cos(2A) - 1}$ is equal to
A $\cot(2A) - 1$ **B** $\sin(2A) + \sec(2A)$ **C** $\frac{\sin A}{\cos A - 1}$
D $\sin(2A) - \tan(2A)$ **E** $-\cot A$
- 7** $(1 + \cot x)^2 + (1 - \cot x)^2$ is equal to
A $2 + \cot(x) + 2 \cot(2x)$ **B** 2 **C** $-4 \cot x$
D $2 + \cot(2x)$ **E** $2 \operatorname{cosec}^2 x$
- 8** If $\sin(2A) = m$ and $\cos A = n$, then $\tan A$ is equal to
A $\frac{m}{2n^2}$ **B** $\frac{n}{m}$ **C** $\frac{2n}{m^2}$ **D** $\frac{2n}{m}$ **E** $\frac{2n^2}{m}$
- 9** Expressing $-\cos x + \sin x$ in the form $r \sin(x + \alpha)$, where $r > 0$, gives
A $\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$ **B** $-\sin\left(x + \frac{\pi}{4}\right)$ **C** $\sqrt{2} \sin\left(x + \frac{5\pi}{4}\right)$
D $\sqrt{2} \sin\left(x + \frac{7\pi}{4}\right)$ **E** $\sqrt{2} \sin\left(x + \frac{3\pi}{4}\right)$
- 10** The product $\sin 25^\circ \cos 75^\circ$ can be rewritten as
A $\sin 100^\circ - \sin 50^\circ$ **B** $2(\sin 100^\circ + \sin 50^\circ)$ **C** $2(\sin 100^\circ - \sin 50^\circ)$
D $\frac{1}{2}(\sin 100^\circ + \sin 50^\circ)$ **E** $\frac{1}{2}(\sin 100^\circ - \sin 50^\circ)$

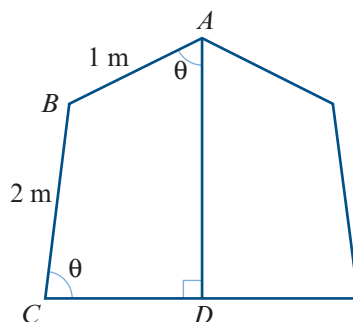
Extended-response questions

- 1** The diagram shows a rectangle $ABCD$ inside a semicircle, centre O and radius 5 cm, with $\angle BOA = \angle COD = \theta^\circ$.
- Show that the perimeter, P cm, of the rectangle is given by $P = 20 \cos \theta + 10 \sin \theta$.
 - Express P in the form $r \cos(\theta - \alpha)$ and hence find the value of θ for which $P = 16$.
 - Find the value of k for which the area of the rectangle is $k \sin(2\theta) \text{ cm}^2$.
 - Find the value of θ for which the area is a maximum.



CF

- 2** The diagram shows a vertical section through a tent in which $AB = 1$ m, $BC = 2$ m and $\angle BAD = \angle BCD = \theta$. The line CD is horizontal, and the diagram is symmetrical about the vertical AD .



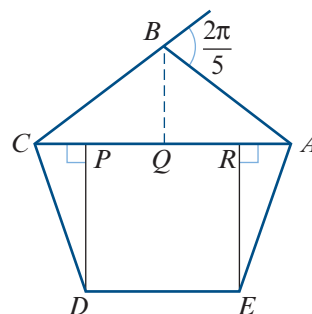
- a** Obtain an expression for AD in terms of θ .
- b** Express AD in the form $r \cos(\theta - \alpha)$, where r is positive.
- c** State the maximum length of AD and the corresponding value of θ .
- d** Given that $AD = 2.15$ m, find the value of θ for which $\theta > \alpha$.

- 3 a** Prove the identity

$$\cos(2\theta) = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

- i** Use the result of **a** to show that $1 + x^2 = \sqrt{2}x^2 - \sqrt{2}$, where $x = \tan(67\frac{1}{2})^\circ$.
- ii** Hence find the values of integers a and b such that $\tan(67\frac{1}{2})^\circ = a + b\sqrt{2}$.
- c** Find the value of $\tan(7\frac{1}{2})^\circ$.

- 4** $ABCDE$ is a regular pentagon with side length one unit. The exterior angles of a regular pentagon each have magnitude $\frac{2\pi}{5}$.



- i** Show that the magnitude of $\angle BCA$ is $\frac{\pi}{5}$.
- ii** Find the length of CA .
- i** Show the magnitude of $\angle DCP$ is $\frac{2\pi}{5}$.
- ii** Use the fact that $AC = 2CQ = 2CP + PR$ to show that $2 \cos\left(\frac{\pi}{5}\right) = 2 \cos\left(\frac{2\pi}{5}\right) + 1$.
- iii** Use the identity $\cos(2\theta) = 2 \cos^2 \theta - 1$ to form a quadratic equation in terms of $\cos\left(\frac{\pi}{5}\right)$.
- iv** Find the exact value of $\cos\left(\frac{\pi}{5}\right)$.

- 5 a** Prove each of the identities:

$$\text{i } \cos \theta = \frac{1 - \tan^2(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)} \quad \text{ii } \sin \theta = \frac{2 \tan(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$$

- b** Use the results of **a** to find the value of $\tan(\frac{1}{2}\theta)$, given that $8 \cos \theta - \sin \theta = 4$.

15

Complex numbers

Objectives

- ▶ To understand the **imaginary number** i and the set of **complex numbers** $\mathbb{C}$.
- ▶ To find the **real part** and the **imaginary part** of a complex number.
- ▶ To perform **addition**, **subtraction**, **multiplication** and **division** of complex numbers.
- ▶ To find the **conjugate** of a complex number.
- ▶ To represent complex numbers graphically on an **Argand diagram**.
- ▶ To work with complex numbers in **polar form**, and to understand the geometric interpretation of multiplication and division of complex numbers in this form.
- ▶ To factorise quadratic polynomials over $\mathbb{C}$.
- ▶ To solve quadratic equations over $\mathbb{C}$.
- ▶ To use **De Moivre's theorem** to find powers of complex numbers.

In this chapter we introduce a new set of numbers, called *complex numbers*. These numbers first arose in the search for solutions to polynomial equations.

In the sixteenth century, mathematicians including Girolamo Cardano began to consider square roots of negative numbers. Although these numbers were regarded as ‘impossible’, they arose in calculations to find real solutions of cubic equations.

For example, the cubic equation $x^3 - 15x - 4 = 0$ has three real solutions. Cardano's formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

which you can show equals 4.

Today complex numbers are widely used in physics and engineering, such as in the study of aerodynamics.

15A Starting to build the complex numbers

Mathematicians in the eighteenth century introduced the imaginary number i with the property that

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions, namely i and $-i$.

By declaring that $i = \sqrt{-1}$, we can find square roots of all negative numbers.

For example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)} \\ &= \sqrt{4} \times \sqrt{-1} \\ &= 2i\end{aligned}$$

Note: The identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds for positive real numbers a and b , but does not hold when both a and b are negative. In particular, $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

Now consider the equation $x^2 + 2x + 3 = 0$. Using the quadratic formula gives

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 12}}{2} \\ &= \frac{-2 \pm \sqrt{-8}}{2} \\ &= -1 \pm \sqrt{-2}\end{aligned}$$

This equation has no real solutions. However, using complex numbers we obtain solutions

$$x = -1 \pm \sqrt{2}i$$

► The set of complex numbers

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers.

The set of all complex numbers is denoted by $\mathbb{C}$. That is,

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

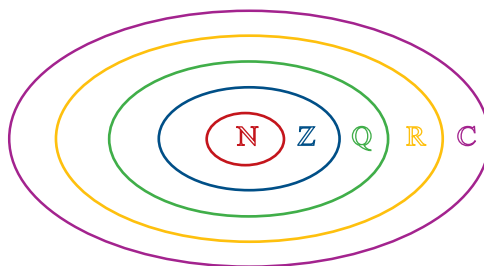
The letter often used to denote a complex number is z .

Therefore if $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$.

- If $a = 0$, then $z = bi$ is said to be an **imaginary number**.
- If $b = 0$, then $z = a$ is a **real number**.

The real numbers and the imaginary numbers are subsets of $\mathbb{C}$.

We can now extend the diagram from Chapter 2 to include the complex numbers.



Real and imaginary parts

For a complex number $z = a + bi$, we define

$$\operatorname{Re}(z) = a \quad \text{and} \quad \operatorname{Im}(z) = b$$

where $\operatorname{Re}(z)$ is called the **real part** of z and $\operatorname{Im}(z)$ is called the **imaginary part** of z .

For example, for the complex number $z = 2 + 5i$, we have $\operatorname{Re}(z) = 2$ and $\operatorname{Im}(z) = 5$.

Note: Both $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ are real numbers.

Equality of complex numbers

Two complex numbers are defined to be **equal** if both their real parts and their imaginary parts are equal:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Example 1

If $4 - 3i = 2a + bi$, find the real values of a and b .

Solution

$$2a = 4 \quad \text{and} \quad b = -3$$

$$\therefore a = 2 \quad \text{and} \quad b = -3$$

Example 2

Find the real values of a and b such that $(2a + 3b) + (a - 2b)i = -1 + 3i$.

Solution

$$2a + 3b = -1 \quad (1)$$

$$a - 2b = 3 \quad (2)$$

Multiply (2) by 2:

$$2a - 4b = 6 \quad (3)$$

Subtract (3) from (1):

$$7b = -7$$

Therefore $b = -1$ and $a = 1$.

► Operations on complex numbers

Addition and subtraction

Addition of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

The **zero** of the complex numbers can be written as $0 = 0 + 0i$.

If $z = a + bi$, then we define $-z = -a - bi$.

Subtraction of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = z_1 + (-z_2) = (a - c) + (b - d)i$$

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

$$\blacksquare z_1 + z_2 = z_2 + z_1 \quad \blacksquare (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \blacksquare z + 0 = z \quad \blacksquare z + (-z) = 0$$

Example 3

If $z_1 = 2 - 3i$ and $z_2 = -4 + 5i$, find:

a $z_1 + z_2$

b $z_1 - z_2$

Solution

$$\begin{aligned} \text{a } z_1 + z_2 &= (2 - 3i) + (-4 + 5i) \\ &= (2 + (-4)) + (-3 + 5)i \\ &= -2 + 2i \end{aligned}$$

$$\begin{aligned} \text{b } z_1 - z_2 &= (2 - 3i) - (-4 + 5i) \\ &= (2 - (-4)) + (-3 - 5)i \\ &= 6 - 8i \end{aligned}$$

Multiplication by a real constant

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = k(a + bi) = ka + kbi$$

For example, if $z = 3 - 6i$, then $3z = 9 - 18i$.

Powers of i

Successive multiplication by i gives the following:

$$\begin{array}{llll} \blacksquare i^0 = 1 & \blacksquare i^1 = i & \blacksquare i^2 = -1 & \blacksquare i^3 = -i \\ \blacksquare i^4 = (-1)^2 = 1 & \blacksquare i^5 = i & \blacksquare i^6 = -1 & \blacksquare i^7 = -i \end{array}$$

In general, for $n = 0, 1, 2, 3, \dots$

$$\blacksquare i^{4n} = 1 \quad \blacksquare i^{4n+1} = i \quad \blacksquare i^{4n+2} = -1 \quad \blacksquare i^{4n+3} = -i$$

Example 4

Simplify:

a i^{13}

b $3i^4 \times (-2i)^3$

Solution

$$\begin{aligned} \text{a } i^{13} &= i^{4 \times 3 + 1} \\ &= i \end{aligned}$$

$$\begin{aligned} \text{b } 3i^4 \times (-2i)^3 &= 3 \times (-2)^3 \times i^4 \times i^3 \\ &= -24i^7 \\ &= 24i \end{aligned}$$

Section summary

- The imaginary number i satisfies $i^2 = -1$.
- If a is a positive real number, then $\sqrt{-a} = i\sqrt{a}$.
- The set of **complex numbers** is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.
- For a complex number $z = a + bi$:
 - the **real part** of z is $\text{Re}(z) = a$
 - the **imaginary part** of z is $\text{Im}(z) = b$.
- Equality of complex numbers:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$
- If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i \quad \text{and} \quad z_1 - z_2 = (a - c) + (b - d)i$$
- When simplifying powers of i , remember that $i^4 = 1$.

Exercise 15A

- 1 State the values of $\text{Re}(z)$ and $\text{Im}(z)$ for each of the following:

a $z = 2 + 3i$

b $z = 4 + 5i$

c $z = \frac{1}{2} - \frac{3}{2}i$

d $z = -4$

e $z = 3i$

f $z = \sqrt{2} - 2\sqrt{2}i$

Example 1, 2

- 2 Find the real values of a and b in each of the following:

a $2a - 3bi = 4 + 6i$

b $a + b - 2abi = 5 - 12i$

c $2a + bi = 10$

d $3a + (a - b)i = 2 + i$

Example 3

- 3 Simplify:

a $(2 - 3i) + (4 - 5i)$

b $(4 + i) + (2 - 2i)$

c $(-3 - i) - (3 + i)$

d $(2 - \sqrt{2}i) + (5 - \sqrt{8}i)$

e $(1 - i) - (2i + 3)$

f $(2 + i) - (-2 - i)$

g $4(2 - 3i) - (2 - 8i)$

h $-(5 - 4i) + (1 + 2i)$

i $5(i + 4) + 3(2i - 7)$

j $\frac{1}{2}(4 - 3i) - \frac{3}{2}(2 - i)$

Example 4

- 4 Simplify:

a $\sqrt{-16}$

b $2\sqrt{-9}$

c $\sqrt{-2}$

d i^3

e i^{14}

f i^{20}

g $-2i \times i^3$

h $4i^4 \times 3i^2$

i $\sqrt{8}i^5 \times \sqrt{-2}$

- 5 Simplify:

a $i(2 - i)$

b $i^2(3 - 4i)$

c $\sqrt{2}i(i - \sqrt{2})$

d $-\sqrt{3}(\sqrt{-3} + \sqrt{2})$

15B Multiplication and division of complex numbers

In the previous section, we defined addition and subtraction of complex numbers. We begin this section by defining multiplication.

► Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\begin{aligned} z_1 \times z_2 &= (a + bi)(c + di) \\ &= ac + bci + adi + bdi^2 \\ &= (ac - bd) + (ad + bc)i \quad (\text{since } i^2 = -1) \end{aligned}$$

We carried out this calculation with an assumption that we are in a system where all the usual rules of algebra apply. However, it should be understood that the following is a *definition* of multiplication for $\mathbb{C}$.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

The multiplicative identity for $\mathbb{C}$ is $1 = 1 + 0i$.

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 z_2 = z_2 z_1$
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
- $z \times 1 = z$
- $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 5

If $z_1 = 3 - 2i$ and $z_2 = 1 + i$, find $z_1 z_2$.

Solution

$$\begin{aligned} z_1 z_2 &= (3 - 2i)(1 + i) \\ &= 3 - 2i + 3i - 2i^2 \\ &= 5 + i \end{aligned}$$

Explanation

Expand the brackets in the usual way.

Remember that $i^2 = -1$.

► The conjugate of a complex number

Let $z = a + bi$. The **conjugate** of z is denoted by $\bar{z}$ and is given by

$$\bar{z} = a - bi$$

For example, the conjugate of $-4 + 3i$ is $-4 - 3i$, and vice versa.

For a complex number $z = a + bi$, we have

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 + abi - abi - b^2i^2 \\ &= a^2 + b^2 \end{aligned} \quad \text{where } a^2 + b^2 \text{ is a real number}$$

The **modulus** of the complex number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$

The calculation above shows that

$$z\bar{z} = |z|^2$$

Note: In the case that z is a real number, this definition of $|z|$ agrees with the definition of the modulus of a real number given in Chapter 12.



Example 6

If $z_1 = 2 - 3i$ and $z_2 = -1 + 2i$, find:

a $\overline{z_1 + z_2}$ and $\overline{z_1} + \overline{z_2}$

b $\overline{z_1 \cdot z_2}$ and $\overline{z_1} \cdot \overline{z_2}$

Solution

We have $\overline{z_1} = 2 + 3i$ and $\overline{z_2} = -1 - 2i$.

a $z_1 + z_2 = (2 - 3i) + (-1 + 2i)$
 $= 1 - i$

$$\overline{z_1 + z_2} = 1 + i$$

$$\begin{aligned} \overline{z_1} + \overline{z_2} &= (2 + 3i) + (-1 - 2i) \\ &= 1 + i \end{aligned}$$

b $z_1 \cdot z_2 = (2 - 3i)(-1 + 2i)$
 $= 4 + 7i$

$$\overline{z_1 \cdot z_2} = 4 - 7i$$

$$\begin{aligned} \overline{z_1} \cdot \overline{z_2} &= (2 + 3i)(-1 - 2i) \\ &= 4 - 7i \end{aligned}$$

- The conjugate of a sum is equal to the sum of the conjugates:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

- The conjugate of a product is equal to the product of the conjugates:

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

► Division of complex numbers

Multiplicative inverse

We begin with some familiar algebra that will motivate the definition:

$$\frac{1}{a + bi} = \frac{1}{a + bi} \times \frac{a - bi}{a - bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2}$$

We can see that

$$(a + bi) \times \frac{a - bi}{a^2 + b^2} = 1$$

Although we have carried out this arithmetic, we have not yet defined what $\frac{1}{a+bi}$ means.

Multiplicative inverse of a complex number

If $z = a + bi$ with $z \neq 0$, then

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

Note: We can check that $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$.

Division

The formal definition of division in the complex numbers is via the multiplicative inverse:

Division of complex numbers

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1 \bar{z}_2}{|z_2|^2} \quad (\text{for } z_2 \neq 0)$$

Here is the procedure that is used in practice:

Assume that $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di}$$

Multiply the numerator and denominator by the conjugate of z_2 :

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \end{aligned}$$

Complete the division by simplifying. This process is demonstrated in the next example.

Example 7

If $z_1 = 2 - i$ and $z_2 = 3 + 2i$, find $\frac{z_1}{z_2}$.

Solution

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2 - i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i} \\ &= \frac{6 - 3i - 4i + 2i^2}{3^2 + 2^2} \\ &= \frac{4 - 7i}{13} \\ &= \frac{1}{13}(4 - 7i) \end{aligned}$$

**Example 8**

Solve for z in the equation $(2 + 3i)z = -1 - 2i$.

Solution

$$\begin{aligned}
 (2 + 3i)z &= -1 - 2i \\
 \therefore z &= \frac{-1 - 2i}{2 + 3i} \\
 &= \frac{-1 - 2i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i} \\
 &= \frac{-8 - i}{13} \\
 &= -\frac{1}{13}(8 + i)
 \end{aligned}$$

There is an obvious similarity between the process for expressing a complex number with a real denominator and the process for rationalising the denominator of a surd expression.

Example 9

If $z = 2 - 5i$, find z^{-1} and express with a real denominator.

Solution

$$\begin{aligned}
 z^{-1} &= \frac{1}{z} \\
 &= \frac{1}{2 - 5i} \\
 &= \frac{1}{2 - 5i} \times \frac{2 + 5i}{2 + 5i} \\
 &= \frac{2 + 5i}{29} \\
 &= \frac{1}{29}(2 + 5i)
 \end{aligned}$$

**Using the TI-Nspire CX non-CAS**

Set to complex mode using > **Settings** > **Document Settings**. Select **Rectangular** from the **Real or Complex** field.



Note: The square root of a negative number can be found only in complex mode. But most computations with complex numbers can also be performed in real mode.

Arithmetic operations

The results of the arithmetic operations $+$, $-$, $\times$ and $\div$ are illustrated using the two complex numbers $2 + 3i$ and $3 + 4i$.

Note: Do not use the text i for the imaginary constant. The symbol i is found using π or the Symbols palette (ctrl π).

Operation	Result
$2+3i + 3+4i$	$5+7i$
$2+3i - (3+4i)$	$-1-i$
$(2+3i) \cdot (3+4i)$	$-6+17i$
$\frac{2+3i}{3+4i}$	$\frac{18}{25} + \frac{1}{25}i$

Complex number tools

Special operations on complex numbers can be accessed using $\text{menu} > \text{Number} > \text{Complex}$

Number Tools:

- Select **Real Part** to find the real part.
- Select **Magnitude** to find the modulus.
- Select **Complex Conjugate** to find the conjugate.
- Select **Polar Angle** to find the argument. (The argument of a complex number is introduced in Section 15E.)

Operation	Result
$\text{real}(2+3i)$	2
$\text{real}\left(\frac{1}{2+3i}\right)$	$\frac{2}{13}$

Operation	Result
$(2+3i)^2$	$-5+12i$
$ (2+3i)^2 $	13
$\text{conj}((2+3i)^2)$	$-5-12i$
$\text{angle}(2+3i)$	0.9827937232

Hint: The commands $\text{real}()$, $\text{conj}()$ and $\text{angle}()$ can be typed in directly.

The modulus of a complex number can be also found by selecting $|z|$ from the 2D-template palette π or by typing $\text{abs}()$.

Using the Casio

Calculations with complex numbers can be performed in **Run-Matrix** mode using the **Complex numbers** menu OPTN F3 .

Arithmetic operations

To perform arithmetic calculations with complex numbers:

- Go to the **Complex numbers** menu OPTN F3 .
- Enter the complex numbers as shown. For the symbol i , select i F1 .

Note: You can also enter i by pressing SHIFT 0 .

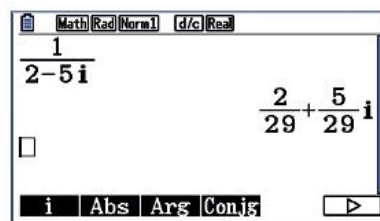
Operation	Result
$(2-3i) + (-4+5i)$	$-2+2i$
$(2-3i) - (-4+5i)$	$6-8i$
$(3-2i)(1+i)$	$5+i$

Operation	Result
$\frac{2-i}{3+2i}$	$\frac{4}{13} - \frac{7}{13}i$

To find the multiplicative inverse of a complex number:

- Go to the **Complex numbers** menu (OPTN) (F3).
- Enter the reciprocal number as shown.

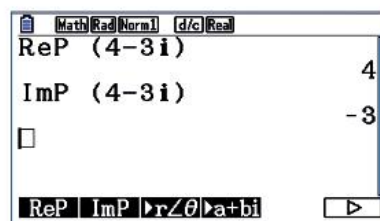
Note: Entering $(2 - 5i)^{-1}$ will give a decimal approximation only.



Complex number operations

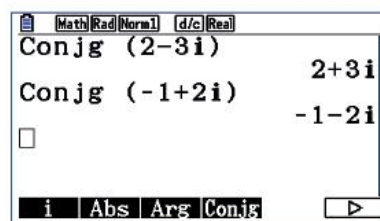
To find the real and imaginary parts of a complex number:

- Go to the **Complex numbers** menu (OPTN) (F3).
- For the real part, select **ReP** (F6) (F1) and enter the complex number as shown. (For the symbol i , select **i** (F6) (F1).)
- For the imaginary part, use **ImP** (F6) (F2).



To find the conjugate of a complex number:

- Go to the **Complex numbers** menu (OPTN) (F3).
- Select **Conjg** (F4) and enter the complex number as shown.



Section summary

- **Multiplication** To find a product $(a + bi)(c + di)$, expand the brackets in the usual way, remembering that $i^2 = -1$.
- **Conjugate** If $z = a + bi$, then $\bar{z} = a - bi$.
- **Division** To perform a division, start with

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2}\end{aligned}$$

and then simplify.

- **Multiplicative inverse** To find z^{-1} , calculate $\frac{1}{z}$.

Exercise 15B

Example 5 1 Expand and simplify:

a $(4 + i)^2$

b $(2 - 2i)^2$

c $(3 + 2i)(2 + 4i)$

d $(-1 - i)^2$

e $(\sqrt{2} - \sqrt{3}i)(\sqrt{2} + \sqrt{3}i)$

f $(5 - 2i)(-2 + 3i)$

2 Write down the conjugate of each of the following complex numbers:

a $2 - 5i$

b $-1 + 3i$

c $\sqrt{5} - 2i$

d $-5i$

Example 6 3 If $z_1 = 2 - i$ and $z_2 = -3 + 2i$, find:

a $\bar{z}_1$

b $\bar{z}_2$

c $z_1 \cdot z_2$

d $\overline{z_1 \cdot z_2}$

e $\bar{z}_1 \cdot \bar{z}_2$

f $z_1 + z_2$

g $\overline{z_1 + z_2}$

h $\bar{z}_1 + \bar{z}_2$

4 If $z = 2 - 4i$, express each of the following in the form $x + yi$:

a $\bar{z}$

b $z\bar{z}$

c $z + \bar{z}$

d $z(z + \bar{z})$

Example 9 **e** $z - \bar{z}$

f $i(z - \bar{z})$

g z^{-1}

h $\frac{z}{i}$

5 Find the real values of a and b such that $(a + bi)(2 + 5i) = 3 - i$.

Example 7 6 Express in the form $x + yi$:

a $\frac{2 - i}{4 + i}$

b $\frac{3 + 2i}{2 - 3i}$

c $\frac{4 + 3i}{1 + i}$

d $\frac{2 - 2i}{4i}$

e $\frac{1}{2 - 3i}$

f $\frac{i}{2 + 6i}$

7 Find the real values of a and b if $(3 - i)(a + bi) = 6 - 7i$.

Example 8 8 Solve each of the following for z :

a $(2 - i)z = 42i$

b $(1 + 3i)z = -2 - i$

c $(3i + 5)z = 1 + i$

d $2(4 - 7i)z = 5 + 2i$

e $z(1 + i) = 4$

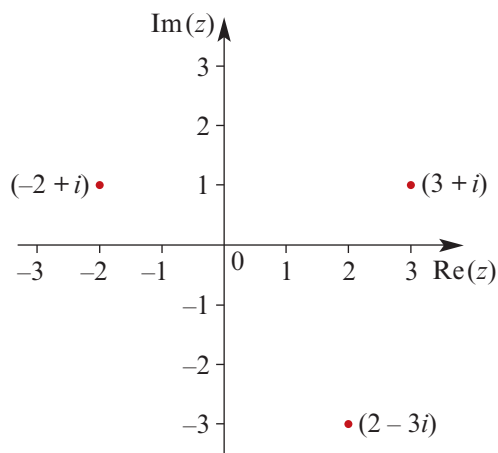
15C Argand diagrams

An **Argand diagram** is a geometric representation of the set of complex numbers. A complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent $\mathbb{C}$.

An Argand diagram is drawn with two perpendicular axes. The horizontal axis represents $\text{Re}(z)$, for $z \in \mathbb{C}$, and the vertical axis represents $\text{Im}(z)$, for $z \in \mathbb{C}$.

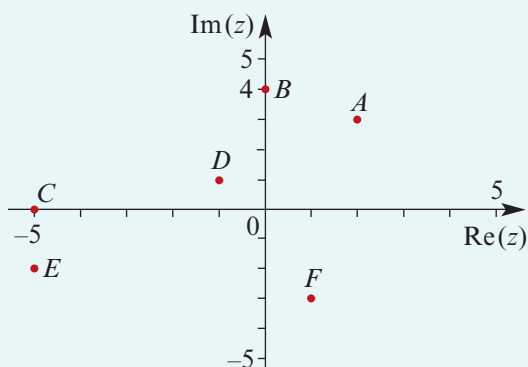
Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point (a, b) on the equivalent Cartesian axes, as shown by the examples in this figure.

A complex number written as $a + bi$ is said to be in **Cartesian form**.



Example 10

Write down the complex number represented by each of the points shown on this Argand diagram.

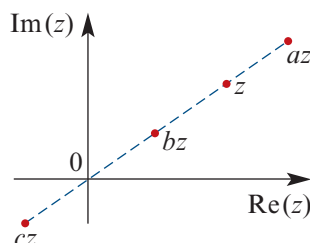
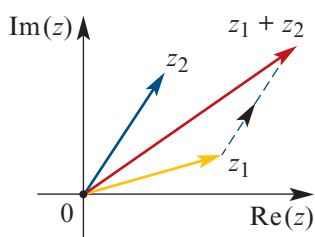
**Solution**

A $2 + 3i$ **B** $4i$ **C** -5
D $-1 + i$ **E** $-5 - 2i$ **F** $1 - 3i$

► **Geometric representation of the basic operations on complex numbers**

We can associate each complex number z with the position vector $\overrightarrow{OP}$ of the point P that represents z on an Argand diagram.

- Addition of complex numbers corresponds to vector addition. This is shown in the diagram on the left.
- Multiplying a complex number by a real number corresponds to scalar multiplication. This is shown in the diagram on the right.



$a > 1$
 $0 < b < 1$
 $c < 0$

The difference $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

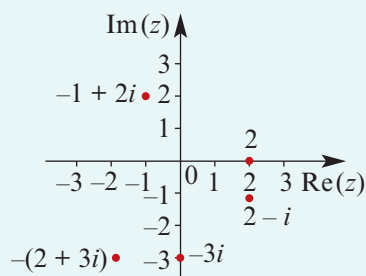
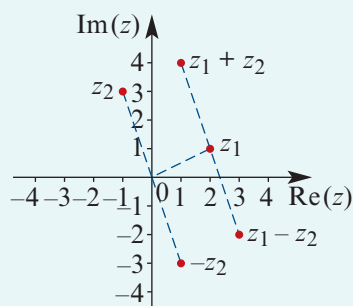
Example 11

a Represent the following complex numbers as points on an Argand diagram:

i 2 **ii** $-3i$ **iii** $2 - i$ **iv** $-(2 + 3i)$ **v** $-1 + 2i$

b Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.

Represent the complex numbers z_1 , z_2 , $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.

Solution**a****b**

$$z_1 + z_2 = (2 + i) + (-1 + 3i) = 1 + 4i$$

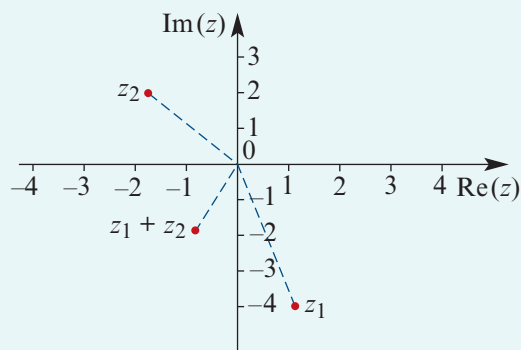
$$z_1 - z_2 = (2 + i) - (-1 + 3i) = 3 - 2i$$

Example 12

Let $z_1 = 1 - 4i$ and $z_2 = -2 + 2i$. Find $z_1 + z_2$ algebraically and illustrate $z_1 + z_2$ on an Argand diagram.

Solution

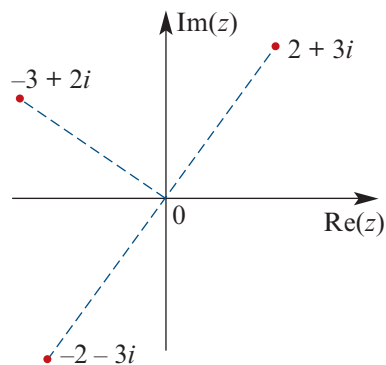
$$\begin{aligned} z_1 + z_2 &= (1 - 4i) + (-2 + 2i) \\ &= -1 - 2i \end{aligned}$$

**Rotation about the origin**

When the complex number $2 + 3i$ is multiplied by -1 , the result is $-2 - 3i$. This is achieved through a rotation of 180° about the origin.

When the complex number $2 + 3i$ is multiplied by i , we obtain

$$\begin{aligned} i(2 + 3i) &= 2i + 3i^2 \\ &= 2i - 3 \\ &= -3 + 2i \end{aligned}$$

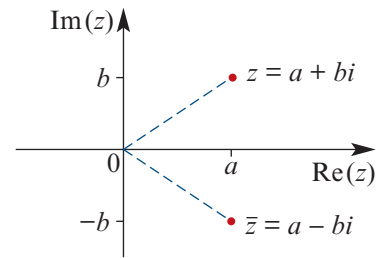


The result is achieved through a rotation of 90° anticlockwise about the origin.

If $-3 + 2i$ is multiplied by i , the result is $-2 - 3i$. This is again achieved through a rotation of 90° anticlockwise about the origin.

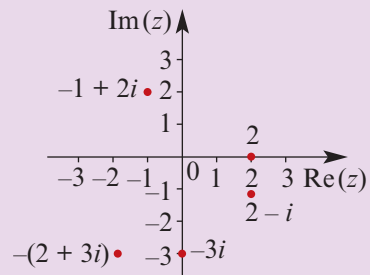
Reflection in the horizontal axis

The conjugate of a complex number $z = a + bi$ is $\bar{z} = a - bi$. Therefore $\bar{z}$ is the reflection of z in the horizontal axis of an Argand diagram.



Section summary

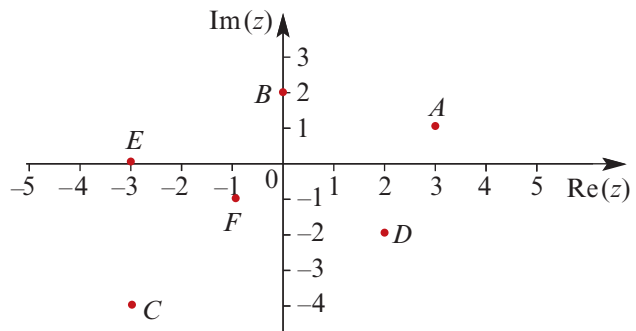
- An **Argand diagram** is a geometric representation of the set of complex numbers.
- The horizontal axis represents $\text{Re}(z)$ and the vertical axis represents $\text{Im}(z)$, for $z \in \mathbb{C}$.
- The operations of addition, subtraction and multiplication by a real constant all have geometric interpretations on an Argand diagram.
- Multiplication of a complex number by i corresponds to a rotation of 90° anticlockwise about the origin.
- Complex conjugate corresponds to reflection in the horizontal axis.



Exercise 15C

Example 10

- 1 Write down the complex numbers represented on this Argand diagram.



Example 11

- 2 Represent each of the following complex numbers as points on an Argand diagram:

a $3 - 4i$ **b** $-4 + i$ **c** $4 + i$ **d** -3 **e** $-2i$ **f** $-5 - 2i$

Example 12

- 3 If $z_1 = 6 - 5i$ and $z_2 = -3 + 4i$, represent each of the following on an Argand diagram:

a $z_1 + z_2$ **b** $z_1 - z_2$

- 4 If $z = 1 + 3i$, represent each of the following on an Argand diagram:

a z **b** $\bar{z}$ **c** z^2 **d** $-z$ **e** $\frac{1}{z}$

- 5 If $z = 2 - 5i$, represent each of the following on an Argand diagram:

a z **b** zi **c** zi^2 **d** zi^3 **e** zi^4

15D Solving equations over the complex numbers

Quadratic equations with a negative discriminant have no real solutions. The introduction of complex numbers enables us to solve such quadratic equations.

► Sum of two squares

Since $i^2 = -1$, we can rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned} z^2 + a^2 &= z^2 - (ai)^2 \\ &= (z + ai)(z - ai) \end{aligned}$$

This allows us to solve equations of the form $z^2 + a^2 = 0$.

Example 13

Solve the equations:

a $z^2 + 16 = 0$

b $2z^2 + 6 = 0$

Solution

a
$$\begin{aligned} z^2 + 16 &= 0 \\ z^2 - 16i^2 &= 0 \\ (z + 4i)(z - 4i) &= 0 \\ \therefore z &= \pm 4i \end{aligned}$$

b
$$\begin{aligned} 2z^2 + 6 &= 0 \\ z^2 + 3 &= 0 \\ z^2 - 3i^2 &= 0 \\ (z + \sqrt{3}i)(z - \sqrt{3}i) &= 0 \\ \therefore z &= \pm \sqrt{3}i \end{aligned}$$

► Solution of quadratic equations

To solve quadratic equations which have a negative discriminant, we can use the quadratic formula in the usual way.



Example 14

Solve the equation $3z^2 + 5z + 3 = 0$.

Solution

Using the quadratic formula:

$$\begin{aligned} z &= \frac{-5 \pm \sqrt{25 - 36}}{6} \\ &= \frac{-5 \pm \sqrt{-11}}{6} \\ &= \frac{1}{6}(-5 \pm \sqrt{11}i) \end{aligned}$$

► Factorisation of quadratics

To factorise a quadratic over the complex numbers, we can complete the square in the usual way. We may then need to rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned}(z+a)^2 + b^2 &= (z+a)^2 - (bi)^2 \\ &= (z+a+bi)(z+a-bi)\end{aligned}$$

Example 15

Factorise each quadratic over the complex numbers:

a $z^2 + z + 3$

b $2z^2 - z + 1$

Solution

a By completing the square, we have

$$\begin{aligned}z^2 + z + 3 &= \left(z^2 + z + \frac{1}{4}\right) + 3 - \frac{1}{4} \\ &= \left(z + \frac{1}{2}\right)^2 + \frac{11}{4} \\ &= \left(z + \frac{1}{2}\right)^2 - \frac{11}{4}i^2 \\ &= \left(z + \frac{1}{2} + \frac{\sqrt{11}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{11}}{2}i\right)\end{aligned}$$

b By completing the square, we have

$$\begin{aligned}2z^2 - z + 1 &= 2\left(z^2 - \frac{1}{2}z + \frac{1}{2}\right) \\ &= 2\left(\left(z^2 - \frac{1}{2}z + \frac{1}{16}\right) + \frac{1}{2} - \frac{1}{16}\right) \\ &= 2\left(\left(z - \frac{1}{4}\right)^2 + \frac{7}{16}\right) \\ &= 2\left(\left(z - \frac{1}{4}\right)^2 - \frac{7}{16}i^2\right) \\ &= 2\left(z - \frac{1}{4} + \frac{\sqrt{7}}{4}i\right)\left(z - \frac{1}{4} - \frac{\sqrt{7}}{4}i\right)\end{aligned}$$

Note: From this example, we can see that given any quadratic $P(z) = az^2 + bz + c$, where the coefficients a , b and c are real numbers, we can rewrite $P(z)$ as a product of two linear factors over the complex numbers:

$$az^2 + bz + c = a(z - \alpha)(z - \beta)$$

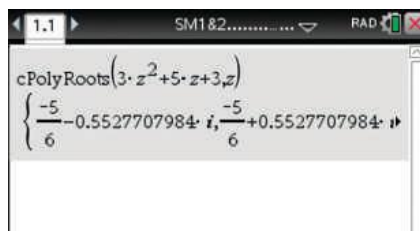
for some complex numbers α and β . Moreover, if $\alpha, \beta \in \mathbb{C} \setminus \mathbb{R}$, then α and β will be conjugates of each other.



Using the TI-Nspire CX non-CAS

To solve a polynomial equation over the complex numbers:

- Use **[menu]** > **Algebra** > **Polynomial Tools** > **Find Roots of Polynomial**.
- Select **Complex** in the **Roots** field.
- The solutions will not be given in exact form.



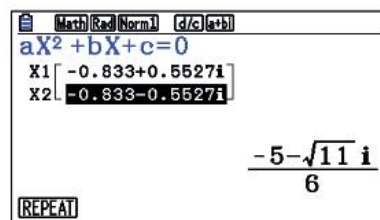
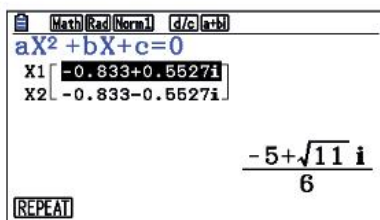
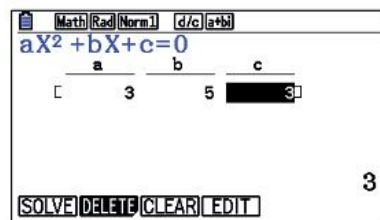
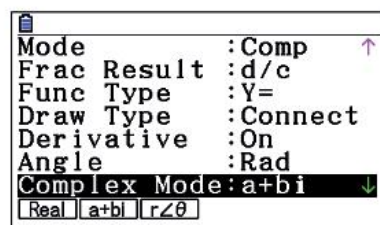
Note: Alternatively, use **[menu]** > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**. Do not include '=' when entering the expression.

Using the Casio

To solve the quadratic equation $3z^2 + 5z + 3 = 0$ over the complex numbers:

- Select **Equation** mode (**MENU** **ALPHA** **X,θ,T**).
- Ensure that the Complex Mode setting is $a + bi$:
 - Go to the set-up screen **SHIFT** **MENU**.
 - Use the cursor key **▼** to move down to **Complex Mode**.
 - Press **F2** to select $a + bi$. Then press **EXIT**.
- Select **Polynomial** **F2**; choose degree 2 **F1**.
- Enter the coefficients of the equation:

3 **EXE** **5** **EXE** **3** **EXE**
- Select **Solve** **F1**.



Section summary

- Quadratic equations can be solved over the complex numbers using the same techniques as for the real numbers.
- Two properties of complex numbers that are useful when solving equations:
 - $z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$
 - $\sqrt{-a} = i\sqrt{a}$, where a is a positive real number.

Exercise 15D

Skillsheet

Example 13, 14

- 1 Solve each of the following equations over $\mathbb{C}$:

a $z^2 + 4 = 0$

b $2z^2 + 18 = 0$

c $3z^2 = -15$

d $(z - 2)^2 + 16 = 0$

e $(z + 1)^2 = -49$

f $z^2 - 2z + 3 = 0$

g $z^2 + 3z + 3 = 0$

h $2z^2 + 5z + 4 = 0$

i $3z^2 = z - 2$

j $2z = z^2 + 5$

k $2z^2 - 6z = -10$

l $z^2 - 6z = -14$

Example 15

- 2 Factorise each of the following quadratics over $\mathbb{C}$:

a $z^2 + 9$

b $z^2 + 3$

c $3z^2 + 12$

d $z^2 + 2z + 5$

e $z^2 - 3z + 6$

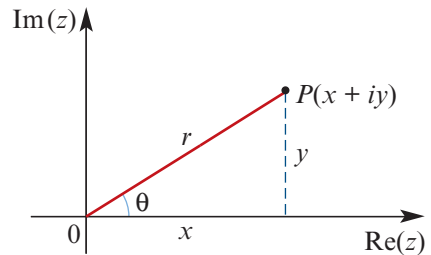
f $2z^2 + 2z + 1$

15E Polar form of a complex number

Polar form for vectors was introduced in Chapter 6. Similarly, each complex number may be described by an angle and a distance from the origin. In this section, we will see that this is a very useful way to describe complex numbers.

The diagram shows the point P corresponding to the complex number $z = x + yi$. We see that $x = r \cos \theta$ and $y = r \sin \theta$, and so we can write

$$\begin{aligned} z &= x + yi \\ &= r \cos \theta + (r \sin \theta) i \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$



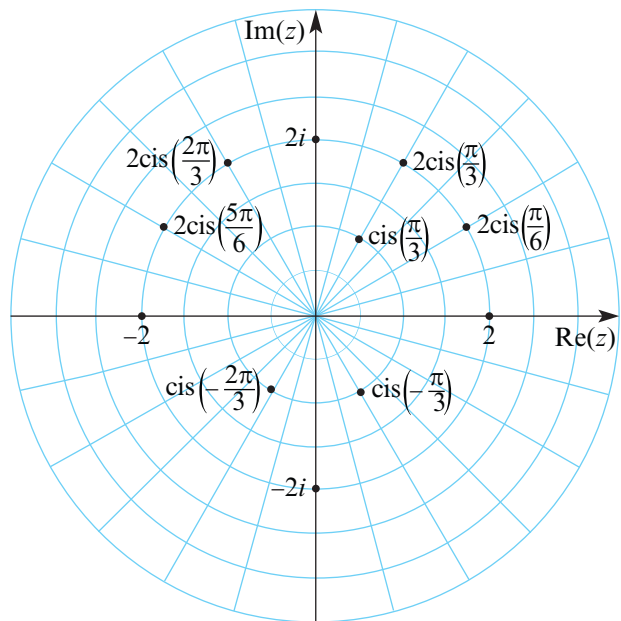
This is called the **polar form** of the complex number. The polar form is abbreviated to

$$z = r \operatorname{cis} \theta$$

- The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
- The angle θ , measured anticlockwise from the horizontal axis, is called the **argument** of z .

Polar form for complex numbers is also called **modulus–argument form**.

This Argand diagram uses a polar grid with rays at intervals of $\frac{\pi}{12} = 15^\circ$.



Non-uniqueness of polar form

Each complex number has more than one representation in polar form.

Since $\cos \theta = \cos(\theta + 2n\pi)$ and $\sin \theta = \sin(\theta + 2n\pi)$, for all $n \in \mathbb{Z}$, we can write

$$z = r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2n\pi) \quad \text{for all } n \in \mathbb{Z}$$

The convention is to use the angle θ such that $-\pi < \theta \leq \pi$. This value of θ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$. That is,

$$-\pi < \operatorname{Arg} z \leq \pi$$

**Example 16**

Express each of the following complex numbers in polar form:

a $z = 1 + \sqrt{3}i$

b $z = 2 - 2i$

Solution**a** We have $x = 1$ and $y = \sqrt{3}$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1 + 3} \\ &= 2 \end{aligned}$$

The point $z = 1 + \sqrt{3}i$ is in the 1st quadrant, and so $0 < \theta < \frac{\pi}{2}$.

We know that

$$\cos \theta = \frac{x}{r} = \frac{1}{2}$$

$$\text{and } \sin \theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

Hence $\theta = \frac{\pi}{3}$ and therefore

$$\begin{aligned} z &= 1 + \sqrt{3}i \\ &= 2 \operatorname{cis}\left(\frac{\pi}{3}\right) \end{aligned}$$

b We have $x = 2$ and $y = -2$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4 + 4} \\ &= 2\sqrt{2} \end{aligned}$$

The point $z = 2 - 2i$ is in the 4th quadrant, and so $-\frac{\pi}{2} < \theta < 0$.

We know that

$$\cos \theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$$

$$\text{and } \sin \theta = \frac{y}{r} = \frac{-1}{\sqrt{2}}$$

Hence $\theta = -\frac{\pi}{4}$ and therefore

$$\begin{aligned} z &= 2 - 2i \\ &= 2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right) \end{aligned}$$

Example 17Express $z = 2 \operatorname{cis}\left(\frac{-2\pi}{3}\right)$ in Cartesian form.**Solution**

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta \\ &= 2 \cos\left(\frac{-2\pi}{3}\right) & &= 2 \sin\left(\frac{-2\pi}{3}\right) \\ &= 2 \times \left(-\frac{1}{2}\right) & &= 2 \times \left(\frac{-\sqrt{3}}{2}\right) \\ &= -1 & &= -\sqrt{3} \end{aligned}$$

Hence

$$\begin{aligned} z &= 2 \operatorname{cis}\left(\frac{-2\pi}{3}\right) \\ &= -1 - \sqrt{3}i \end{aligned}$$

► Multiplication and division in polar form

We can give a simple geometric interpretation of multiplication and division of complex numbers in polar form.

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad (\text{multiply the moduli and add the angles})$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \quad (\text{divide the moduli and subtract the angles})$$

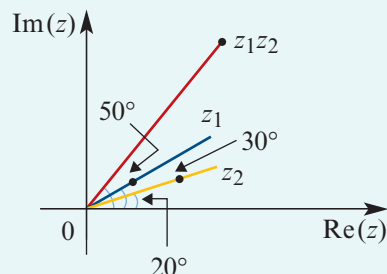
These two results can be proved using the angle sum and difference identities for sine and cosine established in Chapter 14.

Example 18

Let $z_1 = 2 \operatorname{cis} 30^\circ$ and $z_2 = 4 \operatorname{cis} 20^\circ$. Find the product $z_1 z_2$ and represent it on an Argand diagram.

Solution

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= 2 \times 4 \operatorname{cis}(30^\circ + 20^\circ) \\ &= 8 \operatorname{cis} 50^\circ \end{aligned}$$



Example 19

Let $z_1 = 3 \operatorname{cis}\left(\frac{\pi}{2}\right)$ and $z_2 = 2 \operatorname{cis}\left(\frac{5\pi}{6}\right)$. Find the product $z_1 z_2$.

Solution

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= 6 \operatorname{cis}\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) \\ &= 6 \operatorname{cis}\left(\frac{4\pi}{3}\right) \end{aligned}$$

$$\therefore z_1 z_2 = 6 \operatorname{cis}\left(\frac{-2\pi}{3}\right) \quad \text{since } -\pi < \operatorname{Arg} z \leq \pi$$



Example 20

If $z_1 = -\sqrt{3} + i$ and $z_2 = 2\sqrt{3} + 2i$, find the quotient $\frac{z_1}{z_2}$ and express in Cartesian form.

Solution

First express z_1 and z_2 in polar form:

■ $|z_1| = \sqrt{3 + 1} = 2$

Let $\theta_1 = \text{Arg } z_1$. Then $\cos \theta_1 = \frac{-\sqrt{3}}{2}$ and $\sin \theta_1 = \frac{1}{2}$, giving $\theta_1 = \frac{5\pi}{6}$.

■ $|z_2| = \sqrt{12 + 4} = 4$

Let $\theta_2 = \text{Arg } z_2$. Then $\cos \theta_2 = \frac{\sqrt{3}}{2}$ and $\sin \theta_2 = \frac{1}{2}$, giving $\theta_2 = \frac{\pi}{6}$.

Hence $z_1 = 2 \text{cis}\left(\frac{5\pi}{6}\right)$ and $z_2 = 4 \text{cis}\left(\frac{\pi}{6}\right)$. Therefore

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2) \\ &= \frac{2}{4} \text{cis}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) \\ &= \frac{1}{2} \text{cis}\left(\frac{2\pi}{3}\right)\end{aligned}$$

In Cartesian form:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{1}{2} \cos\left(\frac{2\pi}{3}\right) + \frac{1}{2} \sin\left(\frac{2\pi}{3}\right) i \\ &= \frac{1}{2} \left(\frac{-1}{2}\right) + \frac{1}{2} \left(\frac{\sqrt{3}}{2}\right) i \\ &= -\frac{1}{4}(1 - \sqrt{3}i)\end{aligned}$$

Section summary

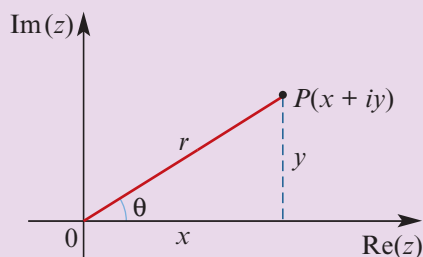
■ Polar form

A complex number in Cartesian form

$$z = x + yi$$

can be written in polar form as

$$\begin{aligned}z &= r(\cos \theta + i \sin \theta) \\ &= r \text{cis } \theta\end{aligned}$$



- The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
- The angle θ , measured anticlockwise from the horizontal axis, is called the **argument** of z .

- The polar form of a complex number is not unique. The argument θ of z such that $-\pi < \theta \leq \pi$ is called the **principal value** of the argument of z and is denoted by $\text{Arg } z$.

■ Multiplication and division in polar form

If $z_1 = r_1 \text{cis } \theta_1$ and $z_2 = r_2 \text{cis } \theta_2$, then

$$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$$

Exercise 15E

Skillsheet

- 1** Express each of the following in polar form $r \text{cis } \theta$ with $-\pi < \theta \leq \pi$:

Example 16

- a** $1 + \sqrt{3}i$ **b** $1 - i$ **c** $-2\sqrt{3} + 2i$
d $-4 - 4i$ **e** $12 - 12\sqrt{3}i$ **f** $-\frac{1}{2} + \frac{1}{2}i$

Example 17

- 2** Express each of the following in the form $x + yi$:

- a** $3 \text{cis}\left(\frac{\pi}{2}\right)$ **b** $\sqrt{2} \text{cis}\left(\frac{\pi}{3}\right)$ **c** $2 \text{cis}\left(\frac{\pi}{6}\right)$
d $5 \text{cis}\left(\frac{3\pi}{4}\right)$ **e** $12 \text{cis}\left(\frac{5\pi}{6}\right)$ **f** $3\sqrt{2} \text{cis}\left(\frac{-\pi}{4}\right)$
g $5 \text{cis}\left(\frac{4\pi}{3}\right)$ **h** $5 \text{cis}\left(\frac{-2\pi}{3}\right)$

Example 18, 19

- 3** Simplify the following and express the answers in Cartesian form:

- a** $2 \text{cis}\left(\frac{\pi}{6}\right) \cdot 3 \text{cis}\left(\frac{\pi}{12}\right)$ **b** $4 \text{cis}\left(\frac{\pi}{12}\right) \cdot 3 \text{cis}\left(\frac{\pi}{4}\right)$
c $\text{cis}\left(\frac{\pi}{4}\right) \cdot 5 \text{cis}\left(\frac{5\pi}{12}\right)$ **d** $12 \text{cis}\left(\frac{-\pi}{3}\right) \cdot 3 \text{cis}\left(\frac{2\pi}{3}\right)$
e $12 \text{cis}\left(\frac{5\pi}{6}\right) \cdot 3 \text{cis}\left(\frac{\pi}{2}\right)$ **f** $(\sqrt{2} \text{cis } \pi) \cdot \sqrt{3} \text{cis}\left(\frac{-3\pi}{4}\right)$

Example 20

- g** $\frac{10 \text{cis}\left(\frac{\pi}{4}\right)}{5 \text{cis}\left(\frac{\pi}{12}\right)}$ **h** $\frac{12 \text{cis}\left(\frac{-\pi}{3}\right)}{3 \text{cis}\left(\frac{2\pi}{3}\right)}$
i $\frac{12\sqrt{8} \text{cis}\left(\frac{3\pi}{4}\right)}{3\sqrt{2} \text{cis}\left(\frac{\pi}{12}\right)}$ **j** $\frac{20 \text{cis}\left(\frac{-\pi}{6}\right)}{8 \text{cis}\left(\frac{5\pi}{6}\right)}$

15F De Moivre's theorem

De Moivre's theorem allows us to readily simplify expressions of the form z^n when z is expressed in polar form.

De Moivre's theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta), \text{ where } n \in \mathbb{Z}$$

Proof This result is usually proved by mathematical induction, but can be explained by a simple inductive argument.

$$\text{Let } z = \operatorname{cis} \theta$$

$$\text{Then } z^2 = \operatorname{cis} \theta \times \operatorname{cis} \theta = \operatorname{cis}(2\theta) \quad \text{by the multiplication rule}$$

$$z^3 = z^2 \times \operatorname{cis} \theta = \operatorname{cis}(3\theta)$$

$$z^4 = z^3 \times \operatorname{cis} \theta = \operatorname{cis}(4\theta)$$

Continuing in this way, we see that $(\operatorname{cis} \theta)^n = \operatorname{cis}(n\theta)$, for each positive integer n .

To obtain the result for negative integers, again let $z = \operatorname{cis} \theta$. Then

$$z^{-1} = \frac{1}{z} = \bar{z} = \operatorname{cis}(-\theta)$$

For $k \in \mathbb{N}$, we have

$$z^{-k} = (z^{-1})^k = (\operatorname{cis}(-\theta))^k = \operatorname{cis}(-k\theta)$$

using the result for positive integers.



Example 21

Simplify:

a $\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^9$

b $\frac{\operatorname{cis}\left(\frac{7\pi}{4}\right)}{\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^7}$

Solution

$$\begin{aligned} \text{a } \left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^9 &= \operatorname{cis}\left(9 \times \frac{\pi}{3}\right) \\ &= \operatorname{cis}(3\pi) \\ &= \operatorname{cis} \pi \\ &= \cos \pi + i \sin \pi \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{b } \frac{\operatorname{cis}\left(\frac{7\pi}{4}\right)}{\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^7} &= \operatorname{cis}\left(\frac{7\pi}{4}\right) \left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^{-7} \\ &= \operatorname{cis}\left(\frac{7\pi}{4}\right) \operatorname{cis}\left(\frac{-7\pi}{3}\right) \\ &= \operatorname{cis}\left(\frac{7\pi}{4} - \frac{7\pi}{3}\right) \\ &= \operatorname{cis}\left(\frac{-7\pi}{12}\right) \end{aligned}$$

**Example 22**

Simplify $\frac{(1+i)^3}{(1-\sqrt{3}i)^5}$.

Solution

First convert to polar form:

$$1+i = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$1-\sqrt{3}i = 2 \operatorname{cis}\left(\frac{-\pi}{3}\right)$$

Therefore

$$\begin{aligned} \frac{(1+i)^3}{(1-\sqrt{3}i)^5} &= \frac{\left(\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^3}{\left(2 \operatorname{cis}\left(\frac{-\pi}{3}\right)\right)^5} \\ &= \frac{2\sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right)}{32 \operatorname{cis}\left(\frac{-5\pi}{3}\right)} && \text{by De Moivre's theorem} \\ &= \frac{\sqrt{2}}{16} \operatorname{cis}\left(\frac{3\pi}{4} - \left(\frac{-5\pi}{3}\right)\right) \\ &= \frac{\sqrt{2}}{16} \operatorname{cis}\left(\frac{29\pi}{12}\right) \\ &= \frac{\sqrt{2}}{16} \operatorname{cis}\left(\frac{5\pi}{12}\right) \end{aligned}$$

Section summary**De Moivre's theorem**

We can use De Moivre's theorem to simplify powers of complex numbers expressed in polar form:

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \quad \text{where } n \in \mathbb{Z}$$

Exercise 15F

Skillsheet

- 1 Simplify each of the following using De Moivre's theorem. Give your answers in both polar form and Cartesian form.

a $(1+i)^2$ **b** $(1+i)^3$ **c** $(1+i)^4$ **d** $(1+i)^5$ **e** $(1+i)^8$

- 2 Simplify each of the following using De Moivre's theorem. Give your answers in both polar form and Cartesian form.

a $(1+\sqrt{3}i)^2$ **b** $(1+\sqrt{3}i)^3$ **c** $(1+\sqrt{3}i)^4$ **d** $(1+\sqrt{3}i)^5$
e $(1+\sqrt{3}i)^6$ **f** $(1+\sqrt{3}i)^9$ **g** $(1+\sqrt{3}i)^{12}$

- 3 Simplify each of the following using De Moivre's theorem. Give your answers in both polar form and Cartesian form.

a $\left(2 \operatorname{cis}\left(\frac{5\pi}{6}\right)\right)^2$ **b** $\left(3 \operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^3$ **c** $\left(3 \operatorname{cis}\left(-\frac{2\pi}{3}\right)\right)^4$ **d** $\left(2 \operatorname{cis}\left(-\frac{3\pi}{4}\right)\right)^4$

Example 21

- 4 Simplify each of the following:

a $2 \operatorname{cis}\left(\frac{5\pi}{6}\right) \times \left(\sqrt{2} \operatorname{cis}\left(\frac{7\pi}{8}\right)\right)^4$

b $\frac{1}{\left(\frac{3}{2} \operatorname{cis}\left(\frac{5\pi}{8}\right)\right)^3}$

c $\left(\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^8 \times \left(\sqrt{3} \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^6$

d $\left(\frac{1}{2} \operatorname{cis}\left(\frac{\pi}{2}\right)\right)^{-5}$

e $\left(2 \operatorname{cis}\left(\frac{3\pi}{2}\right) \times 3 \operatorname{cis}\left(\frac{\pi}{6}\right)\right)^3$

f $\left(\frac{1}{2} \operatorname{cis}\left(\frac{\pi}{8}\right)\right)^{-6} \times \left(4 \operatorname{cis}\left(\frac{\pi}{3}\right)\right)^2$

g $\frac{\left(6 \operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^3}{\left(\frac{1}{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^{-5}}$

Example 22

- 5 Simplify each of the following, giving your answer in polar form $r \operatorname{cis} \theta$, with $r > 0$ and $\theta \in (-\pi, \pi]$:

a $(1+\sqrt{3}i)^6$

b $(1-i)^{-5}$

c $i(\sqrt{3}-i)^7$

d $(-3+\sqrt{3}i)^{-3}$

e $\frac{(1+\sqrt{3}i)^3}{i(1-i)^5}$

f $\frac{(-1+\sqrt{3}i)^4(-\sqrt{2}-\sqrt{2}i)^3}{\sqrt{3}-3i}$

g $(-1+i)^5 \left(\frac{1}{2} \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^3$

h $\frac{\left(\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^3}{(1-\sqrt{3}i)^2}$

i $\left((1-i) \operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^7$



Chapter summary

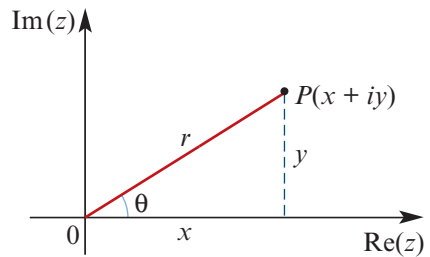


- The imaginary number i has the property $i^2 = -1$.
- The set of **complex numbers** is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.
- For a complex number $z = a + bi$:
 - the **real part** of z is $\text{Re}(z) = a$
 - the **imaginary part** of z is $\text{Im}(z) = b$.
- Complex numbers z_1 and z_2 are equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- An **Argand diagram** is a geometric representation of $\mathbb{C}$.
- The **modulus** of z , denoted by $|z|$, is the distance from the origin to the point representing z in an Argand diagram. Thus $|a + bi| = \sqrt{a^2 + b^2}$.
- The **argument** of z is an angle measured anticlockwise about the origin from the positive direction of the x -axis to the line joining the origin to z .
- The **principal value** of the argument, denoted by $\text{Arg } z$, is the angle in the interval $(-\pi, \pi]$.
- The complex number $z = x + yi$ can be expressed in **polar form** as

$$z = r(\cos \theta + i \sin \theta) \\ = r \text{cis } \theta$$

$$\text{where } r = |z| = \sqrt{x^2 + y^2}, \quad \cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}.$$

This is also called modulus–argument form.



- The **complex conjugate** of $z = a + bi$ is given by $\bar{z} = a - bi$. Note that $z\bar{z} = |z|^2$.
- Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

- Multiplication and division in polar form:
Let $z_1 = r_1 \text{cis } \theta_1$ and $z_2 = r_2 \text{cis } \theta_2$. Then

$$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$$

- **De Moivre's theorem**

$$(r \text{cis } \theta)^n = r^n \text{cis}(n\theta) \quad \text{where } n \in \mathbb{Z}$$

Technology-free questions

- 1 For $z_1 = m + ni$ and $z_2 = p + qi$, express each of the following in the form $a + bi$:

a $2z_1 + 3z_2$

b $\bar{z}_2$

c $z_1 \bar{z}_2$

d $\frac{z_1}{z_2}$

e $z_1 + \bar{z}_1$

f $(z_1 + z_2)(z_1 - z_2)$

g $\frac{1}{z_1}$

h $\frac{z_2}{z_1}$

i $\frac{3z_1}{z_2}$

- 2** Let $z = 1 - \sqrt{3}i$. For each of the following, express in the form $a + bi$ and mark on an Argand diagram:

a z **b** z^2 **c** z^3 **d** $\frac{1}{z}$ **e** $\bar{z}$ **f** $\frac{1}{\bar{z}}$

- 3** Write each of the following in polar form:

a $1 + i$ **b** $1 - \sqrt{3}i$ **c** $2\sqrt{3} + i$
d $3\sqrt{2} + 3\sqrt{2}i$ **e** $-3\sqrt{2} - 3\sqrt{2}i$ **f** $\sqrt{3} - i$

- 4** Write each of the following in Cartesian form:

a $-2 \operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $3 \operatorname{cis}\left(\frac{\pi}{4}\right)$ **c** $3 \operatorname{cis}\left(\frac{3\pi}{4}\right)$
d $-3 \operatorname{cis}\left(\frac{-3\pi}{4}\right)$ **e** $3 \operatorname{cis}\left(\frac{-5\pi}{6}\right)$ **f** $\sqrt{2} \operatorname{cis}\left(\frac{-\pi}{4}\right)$

- 5** Let $z = \operatorname{cis}\left(\frac{\pi}{3}\right)$. On an Argand diagram, carefully plot:

a z^2 **b** $\bar{z}$ **c** $\frac{1}{z}$ **d** $\operatorname{cis}\left(\frac{2\pi}{3}\right)$

- 6** Let $z = \operatorname{cis}\left(\frac{\pi}{4}\right)$. On an Argand diagram, carefully plot:

a iz **b** $\bar{z}$ **c** $\frac{1}{z}$ **d** $-iz$

- 7** Let $w = \operatorname{cis}\left(\frac{2\pi}{3}\right)$ and $z = \operatorname{cis}\left(\frac{3\pi}{4}\right)$. Find:

a wz **b** $\frac{w}{z}$ **c** $\frac{z}{w}$ **d** $\frac{w^3}{z}$ **e** $\frac{w^2}{z^3}$ **f** $w^2 z^3$

- 8** Express $\sqrt{3} + i$ in polar form. Hence find $(\sqrt{3} + i)^7$ and express in Cartesian form.

Multiple-choice questions

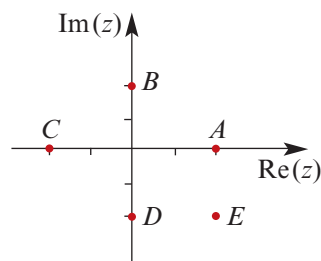


- 1** If $u = 1 + i$, then $\frac{1}{2 - u}$ is equal to

A $-\frac{1}{2} - \frac{1}{2}i$ **B** $\frac{1}{5} + \frac{2}{5}i$ **C** $\frac{1}{2} + \frac{1}{2}i$ **D** $-\frac{1}{2} + \frac{1}{5}i$ **E** $1 + 5i$

- 2** The point C on the Argand diagram represents the complex number z . Which point represents the complex number $i \times z$?

A A **B** B **C** C
D D **E** E



- 3** If $|z| = 5$, then $\left|\frac{1}{z}\right| =$

A $\frac{1}{\sqrt{5}}$ **B** $-\frac{1}{\sqrt{5}}$ **C** $\frac{1}{5}$ **D** $-\frac{1}{5}$ **E** $\sqrt{5}$

- 4** If $(x + yi)^2 = -32i$ for real values of x and y , then
A $x = 4, y = 4$ **B** $x = -4, y = 4$
C $x = 4, y = -4$ **D** $x = 4, y = -4$ or $x = -4, y = 4$
E $x = 4, y = 4$ or $x = -4, y = -4$
- 5** The linear factors of $z^2 + 6z + 10$ over $\mathbb{C}$ are
A $(z + 3 + i)^2$ **B** $(z + 3 - i)^2$ **C** $(z + 3 + i)(z - 3 + i)$
D $(z + 3 - i)(z + 3 + i)$ **E** $(z + 3 + i)(z - 3 - i)$
- 6** Let $z = \frac{1}{1-i}$. If $r = |z|$ and $\theta = \text{Arg } z$, then
A $r = 2$ and $\theta = \frac{\pi}{4}$ **B** $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ **C** $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$
D $r = \frac{1}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$ **E** $r = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$
- 7** The solution of the equation $\frac{z-2i}{z-(3-2i)} = 2$, where $z \in \mathbb{C}$, is
A $z = 6 + 2i$ **B** $z = 6 - 2i$ **C** $z = -6 - 6i$
D $z = 6 - 6i$ **E** $z = -6 + 2i$
- 8** Let $z = a + bi$, where $a, b \in \mathbb{R}$. If $z^2(1+i) = 2 - 2i$, then the Cartesian form of one value of z could be
A $\sqrt{2}i$ **B** $-\sqrt{2}i$ **C** $-1 - i$ **D** $-1 + i$ **E** $\sqrt{-2}$
- 9** The value of the discriminant for the quadratic expression $(2 + 2i)z^2 + 8iz - 4(1 - i)$ is
A -32 **B** 0 **C** 64 **D** 32 **E** -64
- 10** If $\text{Arg}(ai + 1) = \frac{\pi}{6}$, then the real number a is
A $\sqrt{3}$ **B** $-\sqrt{3}$ **C** 1 **D** $\frac{1}{\sqrt{3}}$ **E** $-\frac{1}{\sqrt{3}}$

Extended-response questions

- 1** **a** Find the exact solutions in $\mathbb{C}$ for the equation $z^2 - 2\sqrt{3}z + 4 = 0$.
b **i** Plot the two solutions from part **a** on an Argand diagram.
ii Find the equation of the circle, with centre the origin, which passes through these two points.
iii Find the value of $a \in \mathbb{Z}$ such that the circle passes through $(0, \pm a)$.
- 2** Let z be a complex number with $|z| = 6$. Let A be the point representing z and let B be the point representing $(1 + i)z$.
a **i** Find $|(1 + i)z|$. **ii** Find $|(1 + i)z - z|$.
b Prove that OAB is a right-angled isosceles triangle.

- 3** Let $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.
- a** On an Argand diagram, the points O, A, Z, P and Q represent the complex numbers $0, 1, z, 1 + z$ and $1 - z$ respectively. Show these points on a diagram.
- b** Prove that the magnitude of $\angle POQ$ is $\frac{\pi}{2}$. Find the ratio $\frac{OP}{OQ}$.
- 4** Let z_1 and z_2 be two complex numbers. Prove the following:
- a** $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1z_2$
- b** $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\bar{z}_2 + \bar{z}_1z_2)$
- c** $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$
- State a geometric theorem from the result of **c**.
- 5** Let z_1 and z_2 be two complex numbers.
- a** Prove the following:
- i** $\overline{z_1z_2} = z_1\bar{z}_2$
- ii** $z_1\bar{z}_2 + \bar{z}_1z_2$ is a real number
- iii** $z_1\bar{z}_2 - \bar{z}_1z_2$ is an imaginary number
- iv** $(z_1\bar{z}_2 + \bar{z}_1z_2)^2 - (z_1\bar{z}_2 - \bar{z}_1z_2)^2 = 4|z_1z_2|^2$
- b** Use the results from part **a** and Question 4 to prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.
Hint: Show that $(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 \geq 0$.
- c** Hence prove that $|z_1 - z_2| \geq |z_1| - |z_2|$.
- 6** Assume that $|z| = 1$ and that the argument of z is θ , where $0 < \theta < \pi$. Find the modulus and argument of:
- a** $z + 1$ **b** $z - 1$ **c** $\frac{z - 1}{z + 1}$
- 7** The quadratic expression $ax^2 + bx + c$ has real coefficients.
- a** Find the discriminant of $ax^2 + bx + c$.
- b** Find the condition in terms of a, b and c for which the equation $ax^2 + bx + c = 0$ has no real solutions.
- c** If this condition is fulfilled, let z_1 and z_2 be the complex solutions of the equation and let P_1 and P_2 the corresponding points on an Argand diagram.
- i** Find $z_1 + z_2$ and $|z_1|$ in terms of a, b and c .
- ii** Find $\cos(\angle P_1OP_2)$ in terms of a, b and c .
- 8** Let z_1 and z_2 be the solutions of the quadratic equation $z^2 + z + 1 = 0$.
- a** Find z_1 and z_2 .
- b** Prove that $z_1 = z_2^2$ and $z_2 = z_1^2$.
- c** Find the modulus and the principal value of the argument of z_1 and z_2 .
- d** Let P_1 and P_2 be the points on an Argand diagram corresponding to z_1 and z_2 . Find the area of triangle P_1OP_2 .

16

Revision of Chapters 12–15

16A Technology-free questions

- 1 Solve each of the following equations:

a $\sin\left(\frac{x}{3}\right) = \frac{1}{2}$ for $x \in [0, 12\pi]$

b $\sqrt{2} \cos\left(2x + \frac{\pi}{6}\right) + 1 = 0$ for $x \in [-\pi, \pi]$

- 2 Solve the equation $\cos x = \frac{\sqrt{3}}{2}$, giving the general solution.

- 3 Given that $\sin A = \frac{3}{5}$, where A is acute, and that $\cos B = -\frac{1}{2}$, where B is obtuse, find the exact values of:

a $\sec A$

b $\cot A$

c $\cot B$

d $\operatorname{cosec} B$

- 4 Given that $\cos A = \frac{1}{3}$, find the possible values of $\cos\left(\frac{A}{2}\right)$.

- 5 Prove the identity $\frac{1}{1 + \sin A} + \frac{1}{1 - \sin A} = 2 \sec^2 A$.

- 6 **a** Write the product $\cos(3x) \sin(x)$ as a sum or a difference.

- b** Find the general solution of the equation $\cos(3\theta) + \cos(\theta) = 0$.

- 7 Prove each of the following:

a $\frac{\sin(3x) + \sin(x)}{\cos(3x) + \cos(x)} = \tan(2x)$

b $\frac{\sin(x) + \sin(2x)}{1 + \cos(x) + \cos(2x)} = \tan(x)$

- 8 If $w = 3 + 2i$ and $z = 3 - 2i$, express each of the following in the form $a + bi$, where a and b are real numbers:

a $w + z$

b $w - z$

c wz

d $w^2 + z^2$

e $(w + z)^2$

f $(w - z)^2$

g $w^2 - z^2$

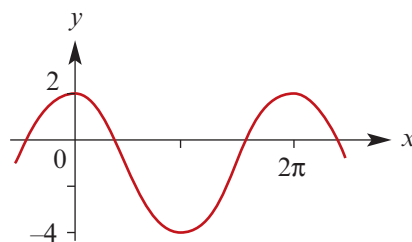
h $(w - z)(w + z)$

- 9** If $w = 1 - 2i$ and $z = 2 - 3i$, express each of the following in the form $a + bi$, where a and b are real numbers:
- a** $w + z$ **b** $w - z$ **c** wz **d** $\frac{w}{z}$ **e** iw **f** $\frac{i}{w}$
g $\frac{w}{i}$ **h** $\frac{z}{w}$ **i** $\frac{w}{w+z}$ **j** $(1+i)w$ **k** $\frac{w}{1+i}$ **l** w^2
- 10** Write each polynomial as a product of linear factors:
a $z^2 + 49$ **b** $z^2 - 2z + 10$ **c** $9z^2 - 6z + 5$ **d** $4z^2 + 12z + 13$
- 11** **a** Find the two square roots of $3 + 4i$.
b Use the quadratic formula to solve $(2 - i)z^2 + (4 + 3i)z + (-1 + 3i) = 0$ for z .
- 12** For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes:
a $f(x) = x^2 + 3x + 2$ **b** $f(x) = (x - 1)^2 + 1$
c $f(x) = \sin(x) + 1, x \in [0, 2\pi]$ **d** $f(x) = \cos(x) + 2, x \in [0, 2\pi]$
- 13** Solve each of the following equations for x :
a $|3x - 7| = 11$ **b** $|6x + 4| = 36$ **c** $|x^2 - 4x| = 21$
d $|x - 2| + |x - 3| = 5$ **e** $|x^2 - 6x| = 16$ **f** $|2x^2 - 2x - 8| = 2$
- 14** Sketch the graph of each of the following functions:
a $y = |x|^2 - 2|x| - 4$ **b** $y = |x^2 - 2x - 4|$ **c** $y = |x - 1| + |x - 2|$
- 15** Sketch the graph of each of the following functions:
a $y = \frac{x - 2}{x + 1}$ **b** $y = 5x + \frac{4}{x}$ **c** $y = \frac{2x}{x^2 + 4}$

16B Multiple-choice questions

- 1** The graph shown has amplitude

A 2 **B** 3 **C** 4
D 6 **E** 2π

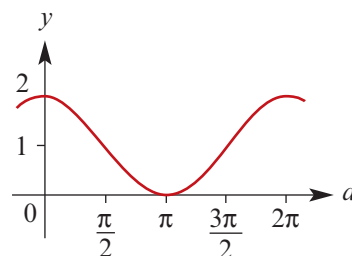


- 2** If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B are acute, then $\sin(A - B)$ is given by

A $\frac{140}{221}$ **B** $-\frac{21}{221}$ **C** $\frac{220}{221}$ **D** $-\frac{107}{140}$ **E** $\frac{107}{140}$

3 The graph shown is best described by

- A** $y = \sin(a)$ **B** $y = 2 \cos(a)$
C $y = \sin(a) + 1$ **D** $y = \cos(2a)$
E $y = \cos(a) + 1$



4 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B acute, then $\cos(A - B)$ is given by

- A** $\frac{140}{221}$ **B** $-\frac{21}{221}$ **C** $\frac{220}{221}$ **D** $-\frac{171}{140}$ **E** $\frac{171}{140}$

5 If $2 \sin\left(x - \frac{\pi}{6}\right) = \sqrt{3}$ and $0 \leq x \leq 2\pi$, then x is equal to

- A** $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ **B** $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ **C** $\frac{\pi}{6}$ or $\frac{\pi}{2}$ **D** $\frac{\pi}{2}$ or $\frac{5\pi}{6}$ **E** $\frac{\pi}{3}$ or π

6 If $\cos \theta = c$ and θ is acute, then $\cot \theta$ can be expressed in terms of c as

- A** $c\sqrt{1-c^2}$ **B** $\sqrt{1-c^2}$ **C** $\frac{1}{\sqrt{1-c^2}}$ **D** $\frac{c}{\sqrt{1-c^2}}$ **E** $2c\sqrt{1-c^2}$

7 A trigonometric graph has the following properties:

- period is 120° ■ amplitude is 3 ■ range is $[-4, 2]$ ■ $y = -1$ when $a = 0$

This graph would be described by the equation

- A** $y = 3 \sin(a)^\circ + 1$ **B** $y = 3 \cos(3a)^\circ - 1$ **C** $y = -3 \sin(3a)^\circ - 1$
D $y = 3 \sin(3a)^\circ + 1$ **E** $y = \cos(3a)^\circ - 2$

8 Compared with the graph of $y = \sin \theta$, the graph of $y = \sin\left(\frac{1}{2}\theta\right)$ has

- A** the same amplitude, but double the period
B the same amplitude, but half the period
C double the amplitude, but the same period
D half the amplitude, but the same period
E the same amplitude, but shifted $\frac{1}{2}$ unit to the left.

9 If $A + B = \frac{\pi}{2}$, then the value of $\cos A \cos B - \sin A \sin B$ is

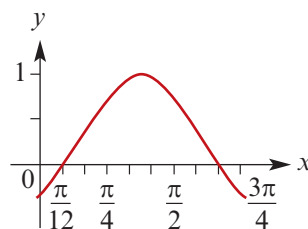
- A** -2 **B** 1 **C** -1 **D** 0 **E** 2

10 Given that $\sin A = \frac{\sqrt{5}}{3}$ and that A is obtuse, the value of $\sin(2A)$ is

- A** $\frac{16\sqrt{5}}{243}$ **B** $-\frac{1}{9}$ **C** $-\frac{8\sqrt{5}}{27}$ **D** $\frac{5}{9}$ **E** $-\frac{4\sqrt{5}}{9}$

- 11 A possible equation for the graph shown is

A $y = \sin\left(2\left(x - \frac{\pi}{12}\right)\right)$ **B** $y = \cos\left(2\left(x - \frac{\pi}{12}\right)\right)$
C $y = \sin\left(2\left(x + \frac{\pi}{12}\right)\right)$ **D** $y = \cos\left(2\left(x + \frac{\pi}{12}\right)\right)$
E $y = -\sin\left(2\left(x - \frac{\pi}{12}\right)\right)$



- 12 Which of the following statements are true for $f(x) = -2 \tan(3x)^\circ$?

I The period is 60° . II The amplitude is 2. III The period is 30° .

IV The graph is a reflection of the graph of $h(x) = 2 \tan(3x)^\circ$ in the x -axis.

V The graph is a reflection of the graph of $g(x) = \tan(x)^\circ$ in the y -axis.

- A** I and IV only **B** I, II and IV only **C** I, IV and V only
D II and IV only **E** II and IV only

- 13 If $\cos \theta = c$ and θ is acute, then $\sin(2\theta)$ can be expressed in terms of c as

A $c\sqrt{1-c^2}$ **B** $\sqrt{1-c^2}$ **C** $\frac{1}{\sqrt{1-c^2}}$ **D** $\frac{c}{\sqrt{1-c^2}}$ **E** $2c\sqrt{1-c^2}$

- 14 The angles between 0° and 360° which satisfy the equation $4 \cos x - 3 \sin x = 1$, given correct to two decimal places, are

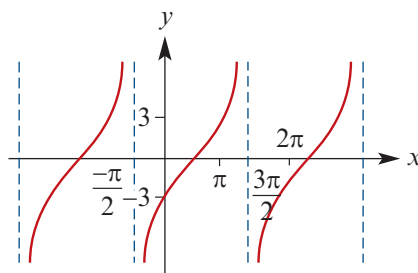
- A** 53.13° and 126.87° **B** 48.41° and 205.33° **C** 41.59° and 244.67°
D 131.59° and 334.67° **E** 154.67° and 311.59°

- 15 The expression $8 \sin \theta \cos^3 \theta - 8 \sin^3 \theta \cos \theta$ is equal to

- A** $8 \sin \theta \cos \theta$ **B** $\sin(8\theta)$ **C** $2 \sin(4\theta)$
D $4 \cos(2\theta)$ **E** $2 \sin(2\theta) \cos(2\theta)$

- 16 A possible equation for the graph shown is

A $y = \tan\left(\frac{1}{2}x - \frac{\pi}{4}\right) + 3$
B $y = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right) - 3$
C $y = 3 \tan\left(\frac{1}{2}x - \frac{\pi}{4}\right)$
D $y = 3 \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$
E $y = \tan(3x)$



- 17 If v , w and z are complex numbers such that $v = 4 \operatorname{cis}(-0.3\pi)$, $w = 5 \operatorname{cis}(0.6\pi)$ and $z = v\bar{w}$, then $\operatorname{Arg} z$ is equal to

- A** 0.9π **B** -0.9π **C** 0.3π **D** -0.3π **E** 1.8π

- 18** The complex number $2 \operatorname{cis}\left(\frac{2\pi}{3}\right)$ is written in Cartesian form as
A $\sqrt{3} - i$ **B** $-\sqrt{3} + i$ **C** $1 - \sqrt{3}i$ **D** $-1 + \sqrt{3}i$ **E** $\frac{1}{3} - \frac{\sqrt{3}}{2}i$
- 19** If $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, then $\operatorname{Arg} z$ is equal to
A $\frac{4\pi}{3}$ **B** $\frac{7\pi}{6}$ **C** $-\frac{\pi}{6}$ **D** $-\frac{2\pi}{3}$ **E** $-\frac{5\pi}{6}$
- 20** The imaginary part of the complex number $-2 - 3i$ is
A -3 **B** $-3i$ **C** 3 **D** -2 **E** $3i$
- 21** If $u = 3 \operatorname{cis}\left(\frac{\pi}{2}\right)$ and $v = 5 \operatorname{cis}\left(\frac{2\pi}{3}\right)$, then uv is equal to
A $15 \operatorname{cis}\left(\frac{\pi}{3}\right)$ **B** $15 \operatorname{cis}\left(\frac{\pi^2}{3}\right)$ **C** $15 \operatorname{cis}\left(\frac{-5\pi}{6}\right)$ **D** $8 \operatorname{cis}\left(\frac{\pi^2}{3}\right)$ **E** $8 \operatorname{cis}\left(\frac{7\pi}{6}\right)$
- 22** The modulus of $12 - 5i$ is
A 169 **B** 7 **C** 13 **D** $\sqrt{119}$ **E** $\sqrt{7}$
- 23** Let $z = x + yi$, where x and y are real numbers which are not both zero. Which one of the following expressions does not necessarily represent a real number?
A z^2 **B** $z\bar{z}$ **C** $z^{-1}z$ **D** $\operatorname{Im}(z)$ **E** $z + \bar{z}$
- 24** If $z = -14 - 7i$, then the complex conjugate of z is equal to
A $7 - 14i$ **B** $14 + 7i$ **C** $-14 + 7i$ **D** $14 - 7i$ **E** $-7 + 14i$
- 25** The expression $3z^2 + 9$ is factorised over $\mathbb{C}$. Which one of the following is a factor?
A $3z$ **B** $z + 3$ **C** $z + 3i$ **D** $z - 3i$ **E** $z + \sqrt{3}i$
- 26** $(1 + 2i)^2$ is equal to
A -3 **B** $-3 + 2i$ **C** $-3 + 4i$ **D** $-1 + 4i$ **E** $5 + 4i$
- 27** Which of the following equations has the graph shown?

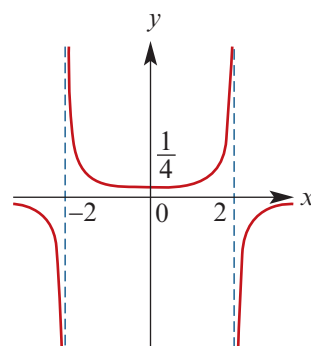
A $y = \frac{1}{4 - x^2}$

B $y = \frac{1}{x^2 - 4}$

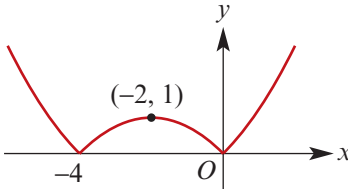
C $y = \frac{1}{2 - x^2}$

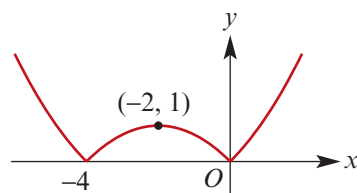
D $y = \frac{1}{x^2 - 2}$

E $y = \frac{1}{(x - 2)^2}$



- 28** If a and b are positive real numbers, then the graph of the reciprocal of $y = a \sin(x) + b$, where $0 \leq x \leq 2\pi$, will have two vertical asymptotes provided
A $b > a$ **B** $a > b$ **C** $b > -a$ **D** $a > -b$ **E** $a > 0$

- 29** The graph of $f(x) = \sec(2x)$, for $-\pi \leq x \leq \pi$, has its local minimum points at
A $x = 0$ **B** $x = -\pi, \pi$ **C** $x = -\pi, 0, \pi$
D $x = -\frac{\pi}{2}, \frac{\pi}{2}$ **E** $x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$
- 30** The inequality $|3 - 2x| > 5$ holds for
A $x > 1$ or $x < -1$ **B** $-1 < x < 4$ **C** $-1 < x < 1$
D $x < -4$ or $x > -1$ **E** $x < -1$ or $x > 4$
- 31** The rule for the graph shown could be
A $y = |x(x + 4)|$ **B** $y = |x|(|x| + 4)$
C $y = |\frac{1}{4}x^2 + x|$ **D** $y = \frac{1}{4}|x|^2 + |x|$
E $y = |1 - (x + 2)^2|$
- 
- 32** Which one of the following statements is *false* for graphs of the form $y = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials?
A It is possible to have both a vertical asymptote and a horizontal asymptote.
B It is possible to have both an oblique asymptote and a horizontal asymptote.
C It is possible to have both a vertical asymptote and an oblique asymptote.
D It is not possible to have an oblique asymptote, a horizontal asymptote and a vertical asymptote.
E It is not possible to have two horizontal asymptotes.



16C Extended-response questions

- 1** A particle oscillates along a straight line. Its position, x m, relative to a point O at time t s is given by $x = 5 + 3 \sin\left(\frac{\pi}{6}t\right)$.
 - a** Find its position at time:
 - i** $t = 0$
 - ii** $t = 3$
 - b** Sketch the graph of x against t for $t \in [0, 24]$, labelling clearly all turning points.
 - c**
 - i** State the maximum distance of the particle from O .
 - ii** State the minimum distance of the particle from O .
 - d** Correct to two decimal places, at what times $t \in [0, 24]$ is the particle:
 - i** 5 m from O
 - ii** 6 m from O ?
- 2** Consider the function $f(x) = 2 \sin(x) + k$ for $x \in [0, 2\pi]$, where k is a real number.
 - a** Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ when:
 - i** $k = 1$
 - ii** $k = 3$
 - b** For what positive value of k does the graph of $y = \frac{1}{f(x)}$ have only one vertical asymptote?
 - c** For this value of k , sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$.

3 Let $f(x) = \frac{x}{2} + \frac{1}{2x}$ for $x \neq 0$.

a Show that there are turning points on the graph of $y = f(x)$ at $x = 1$ and $x = -1$.

b Sketch the graph of $y = f(x)$.

c Sketch the graph of $y = g(x)$, where

$$g(x) = \begin{cases} \frac{1}{f(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

d Sketch the graph of $y = |f(x)|$.

e Sketch the graph of $y = f(|x|)$.

f Let $h(x) = |f(x)| + k$, where k is a real constant.

i For $k = -5$, find the turning points and x -axis intercepts of the graph of $y = h(x)$.

ii Find the value of k for which the graph of $y = h(x)$ has two x -axis intercepts.

Triple-angle identities for sine and cosine

4 In the figure, $AE = BE = BD = 1$ unit and $\angle BCD$ is a right angle.

a Show that the magnitude of $\angle BDE$ is 2θ .

b Use the cosine rule in triangle BDE to show that $DE = 2 \cos(2\theta)$.

c Show that:

i $DC = \sin(3\theta)$ **ii** $AD = \frac{\sin(3\theta)}{\sin \theta}$

d Use the results of **b** and **c** to show that $\sin(3\theta) = 3 \sin \theta - 4 \sin^3 \theta$.

5 a Use the angle sum and double-angle identities to prove each of the following:

i $\cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta$ **ii** $\sin(3\theta) = 3 \sin \theta - 4 \sin^3 \theta$

Now let $\alpha = \cos\left(\frac{\pi}{9}\right)$.

b Use an identity from part **a** to show that $x = \alpha$ is a solution of the equation $f(x) = 0$, where $f(x) = x^3 - \frac{3}{4}x - \frac{1}{8}$.

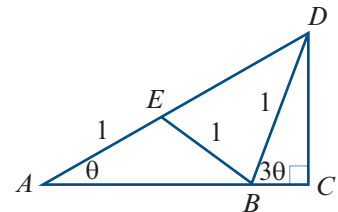
c Use your calculator to plot the graph of $y = f(x)$. Hence determine that $x = \alpha$ is the unique positive real solution of the cubic equation $f(x) = 0$.

For a cubic equation of the form $z^3 + pz + q = 0$, we can use Cardano's formula to find a solution over $\mathbb{C}$:

$$z = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

d Use Cardano's formula to find a solution of the equation $f(z) = 0$ over $\mathbb{C}$. Express your answer in the form $z = a(\sqrt[3]{1+bi} + \sqrt[3]{1-bi})$, for $a, b \in \mathbb{R}^+$.

Note: It can be shown that an expression of the form obtained in part **d** must be a positive real number. Thus this expression is equal to $\alpha = \cos\left(\frac{\pi}{9}\right)$.



- 6 a** Use the rule for multiplying complex numbers in polar form to show that:

i $(\text{cis } \theta)^2 = \text{cis}(2\theta)$ **ii** $(\text{cis } \theta)^3 = \text{cis}(3\theta)$

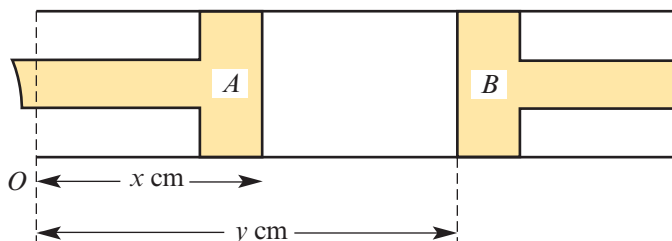
- b** From part **a ii**, we have the equation

$$(\cos \theta + i \sin \theta)^3 = \cos(3\theta) + i \sin(3\theta)$$

Expand and simplify the left-hand side of this equation. Then, by equating real and imaginary parts, derive the two triple-angle identities from Question 5.

16D Problem-solving and modelling questions

- 1** Two pistons *A* and *B* move backwards and forwards in a cylinder as shown.



The distance, x cm, of the right-hand end of piston *A* from the point *O* at time t seconds is modelled by the rule

$$x = 4 \sin(3t) + 4$$

The distance, y cm, of the left-hand end of piston *B* from the point *O* at time t seconds is modelled by the rule

$$y = 2 \sin\left(2t - \frac{\pi}{6}\right) + 10$$

The two pistons are set in motion at time $t = 0$.

- a** State the value of x and the value of y when $t = 0$.
- b**
 - i** State the amplitude of the motion of piston *A*.
 - ii** State the amplitude of the motion of piston *B*.
- c**
 - i** State the maximum and minimum values of x .
 - ii** State the maximum and minimum values of y .
- d**
 - i** State the period of the motion of piston *A*.
 - ii** State the period of the motion of piston *B*.
- e** Find the time(s) in the first cycle of *A* that its distance from *O* is a maximum.
- f** Find the next four values of t for which x takes its maximum value.
- g** Find the values of $t \in [0, 4\pi]$ for which y attains its minimum value.
- h** On one set of axes, sketch the graphs of $x = 4 \sin(3t) + 4$ and $y = 2 \sin\left(2t - \frac{\pi}{6}\right) + 10$ over the interval $[0, \pi]$.
 - i** State the time when the pistons first touch each other.
 - j** How many seconds are there between the first and second times the pistons touch?

- 2** The pistons A and B (from Question 1) are adjusted so that the distance, x cm, of the right-hand end of piston A from the point O at time t seconds is modelled by the rule

$$x = a \sin(nt) + b$$

and the distance, y cm, of the left-hand end of piston B from the point O at time t seconds is modelled by the rule

$$y = c \sin(mt) + d$$

The pistons meet every second at a point 8 cm from O . The right-hand end of piston A cannot go to the left of the point O .

- a** Find one possible set of values of a, b, n and c, d, m and explain your solution.
- b** Using the set of values found in part **a**, sketch the graphs of x against t and y against t on the one set of axes.
- 3** Let $u = a + bi$ and $v = c + di$ be complex numbers, where $a, b, c, d \in \mathbb{R}$.
- a** Show that $\overline{u + v} = \bar{u} + \bar{v}$.
- b** Show that $\overline{u \cdot v} = \bar{u} \cdot \bar{v}$.
- c** Show that $\overline{k \cdot u} = k \cdot \bar{u}$ for each real number k .
- d** **Conjugate root theorem** Consider a cubic polynomial $P(z) = az^3 + bz^2 + cz + d$ such that the coefficients a, b, c and d are all real numbers. Using the previous results, prove that if u is a complex number with $P(u) = 0$, then $P(\bar{u}) = 0$.
- e** Show that the result from part **d** may not be true if we do not assume that the coefficients a, b, c and d are all real numbers.
- 4 a** Let w and z be complex numbers such that the angles $\text{Arg}(w)$ and $\text{Arg}(z)$ are acute. Using the rule for multiplying complex numbers in polar form, show that

$$\text{Arg}(wz) = \text{Arg}(w) + \text{Arg}(z)$$

- b** Find the exact values of:
- i** $\text{Arg}(2 + i)$ **ii** $\text{Arg}(3 + i)$ **iii** $\text{Arg}(5 + 5i)$
- c** Show that $(2 + i)(3 + i) = 5 + 5i$.
- d** By taking the argument of both sides of the equation from part **c**, prove that

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

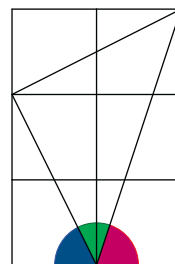
- e** By first evaluating $(3 + i)^2(7 + i)$, prove that

$$2 \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$$

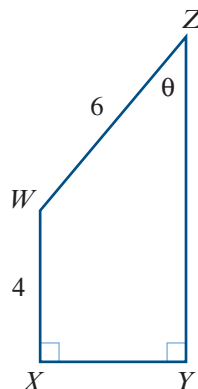
- f** By first evaluating $(1 + i)(1 + 2i)(1 + 3i)$, prove that

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$$

- g** Explain how the result from part **f** can also be shown using the diagram on the right. The grid is composed of unit-length squares.



- 5** The diagram shows a trapezium $XYZW$ such that $XW = 4$, $WZ = 6$, $\angle YXW = \angle XYZ = \frac{\pi}{2}$ and $\angle YZW = \theta$, where $0 < \theta < \frac{\pi}{2}$.



- a** Show that the area, A , of the trapezium is given by

$$A = 24 \sin \theta + 9 \sin(2\theta)$$

- b** Show that the perimeter, P , of the trapezium is given by

$$P = 14 + 6(\sin \theta + \cos \theta)$$

- c** Find the maximum value of P and the corresponding value of θ .

- d** Find the values of θ for which $P = 21$.

- e** Find the exact value of A when $\theta = \frac{\pi}{4}$.

- f** Use your calculator to find the maximum value of A (correct to two decimal places) and the value of θ for which this occurs (correct to one decimal place).

- g** Sketch the graph of A against θ .

- h** **Change of variable** Let $x = XY$. Show that

$$A = 4x + \frac{1}{2}x\sqrt{36 - x^2}$$

Use your calculator to find the maximum value of A by graphing A against x .

16E Problem-solving and modelling investigations

For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

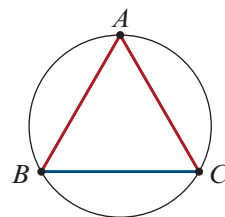
1 Graphs involving the modulus function

- a**
 - i** Sketch the graphs of the functions $y = |x - 2| + |x - 3|$ and $y = |2x - 3| + |x - 4|$.
 - ii** Investigate graphs of the form $y = |ax + b| + |cx + d|$ and $y = |ax + b| - |cx + d|$.
- b**
 - i** Sketch the graph of the relation $|2x + y| = 2$.
 - ii** Investigate graphs of the form $|ax + by| = c$.
- c**
 - i** Sketch the region of the plane defined by $|x + y| \leq 2$.
 - ii** Investigate graphs of the form $|ax + by| \leq c$.
- d**
 - i** Sketch the graph of the relation $|x| + |y| = 2$. Find the area of the enclosed region.
 - ii** Investigate graphs of the form $|ax| + |by| = c$ and the corresponding areas.
- e** Sketch the region of the plane defined by $|x| + |y| + |x + y| \leq 2$ and find the area of this region.
- f** Investigate other families of graphs involving the modulus function. For example, consider the graphs of $|xy| = 1$ and $|xy| \leq 2$.

2 Products of lengths

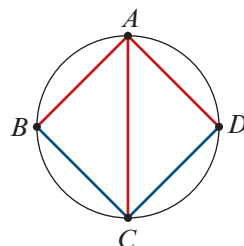
- a** The vertices A , B and C of an equilateral triangle lie on a circle of radius 1. Find the value of

$$AB \times AC$$



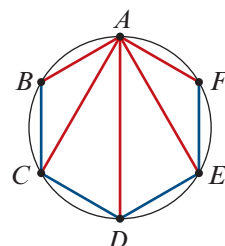
- b** The vertices A , B , C and D of a square lie on a circle of radius 1. Find the value of

$$AB \times AC \times AD$$



- c** The vertices A , B , C , D , E and F of a regular hexagon lie on a circle of radius 1. Find the value of

$$AB \times AC \times AD \times AE \times AF$$



- d** We now aim to prove the corresponding result for a regular pentagon. In the figure on the right, $\triangle BAC$ is isosceles. Moreover, $\angle A = \theta$, $\angle B = 2\theta$ and $AD = DC = 1$.

- i** Show that $BC = 1$.

Hint: First find $\angle BDC$ in terms of θ .

- ii** Prove that $\triangle BAC$ is similar to $\triangle DCB$.

- iii** Let $x = BD$. Use similar triangles to show that $x^2 + x - 1 = 0$.

- iv** Hence, show that $x = \frac{-1 + \sqrt{5}}{2}$.

- v** Explain why $\theta = \frac{\pi}{5}$.

- vi** Let E be the midpoint of BC . Using $\triangle BAE$, show that $\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4}$.

- vii** Hence, prove that $\sin\left(\frac{2\pi}{5}\right) = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$.

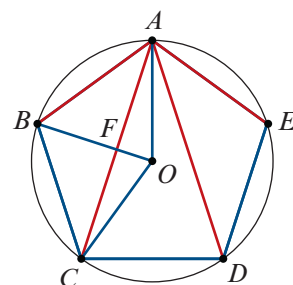
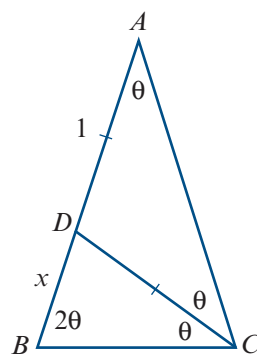
- e** The vertices A , B , C , D and E of a regular pentagon lie on a circle of radius 1 and centre O .

- i** Find the length of diagonal AC .

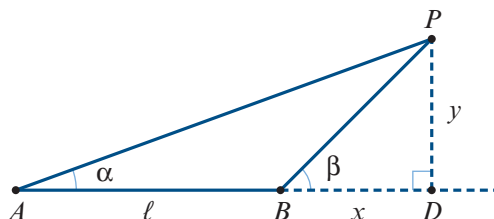
Hint: First find $\angle FOA$ and then find AF .

- ii** Hence, find the value of

$$AB \times AC \times AD \times AE$$



- 3 Measurement errors** A practical use of trigonometry is in determining the position of an inaccessible object. The technique used in this activity is called *triangulation*.



In this diagram: The point P is inaccessible. The distance $\ell = AB$ is known. The distances $x = BD$ and $y = PD$ are unknown. The angles α and β are measured.

- a** Note that $\tan \beta = \frac{y}{x}$ and $\tan \alpha = \frac{y}{x + \ell}$. Show that $x = \frac{\ell \tan \alpha}{\tan \beta - \tan \alpha}$.

- b** Hence, show that

$$x = \frac{\ell \sin \alpha \cos \beta}{\sin(\beta - \alpha)} \quad \text{and} \quad y = \frac{\ell \sin \alpha \sin \beta}{\sin(\beta - \alpha)}$$

- c** Investigate the effect that measurement errors in the angles α and β have on the values of x and y calculated using the formulas from part **b**. In particular, consider the effect for different magnitudes of $\beta - \alpha$. For example, what is the effect of a 1° error in α when the measured values are $\alpha = 40^\circ$ and $\beta = 45^\circ$?
- d** Take actual measurements in real-world situations and investigate your errors.
- 4 Complex quadratics** In this question, we use the quadratic formula to help solve polynomial equations over the complex numbers. Recall that the solutions of a quadratic equation $az^2 + bz + c = 0$ are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- a i** Solve the quadratic equation $z^2 + az + a^2 = 0$, where a is a positive real number.
- ii** Hence, solve the cubic equation $z^3 = a^3$, where a is a positive real number.
Hint: Use the factorisation $z^3 - a^3 = (z - a)(z^2 + az + a^2)$.
- iii** For the case $a = 1$, write the solutions of the cubic equation in polar form. Plot them on an Argand diagram with the unit circle.
- iv** Add the solutions for the cases $a = 2$ and $a = 3$ to your Argand diagram.
- v** Summarise what you have found.
- b** Repeat part **a** for $z^2 - az + a^2 = 0$ and $z^3 = -a^3$, where a is a positive real number.
Hint: Use the factorisation $z^3 + a^3 = (z + a)(z^2 - az + a^2)$.
- c** Repeat part **a** for $z^2 + aiz - a^2 = 0$ and $z^3 = -a^3i$, where a is a positive real number.
- d** Repeat part **a** for $z^2 - aiz - a^2 = 0$ and $z^3 = a^3i$, where a is a positive real number.
- e** There is more to consider in this way. For example, look at the equation $z^4 + 1 = 0$. Factorise the left-hand side by considering $z^4 + 2z^2 + 1 - 2z^2$.

5 Reciprocals of quadratics Consider the family of functions with rules of the form

$$f(x) = \frac{1}{ax^2 + bx + c}$$

where a , b and c are real constants with $a \neq 0$. It is a good idea to graph some examples of functions from this family before starting the following.

- a** Find the coordinates of the turning point on the graph of $y = f(x)$, and state whether it is a local maximum or a local minimum. Consider the two cases $a > 0$ and $a < 0$ separately.
- b** For $a > 0$, sketch the graph of $y = f(x)$, giving the equations of all asymptotes. Consider the following three cases separately:
- i** $b^2 - 4ac < 0$ **ii** $b^2 - 4ac = 0$ **iii** $b^2 - 4ac > 0$
- c** For $a < 0$, sketch the graph of $y = f(x)$, giving the equations of all asymptotes. Consider the following three cases separately:
- i** $b^2 - 4ac < 0$ **ii** $b^2 - 4ac = 0$ **iii** $b^2 - 4ac > 0$

6 More rational functions without calculus The algebraic methods for finding turning points from Section 12C can be applied to other families of rational functions. For example:

$$\begin{aligned} f(x) &= x^2 + \frac{1}{x^2} \\ &= \frac{x^4 + 1}{x^2} \\ &= \frac{x^4 - 2x^2 + 1 + 2x^2}{x^2} \\ &= \frac{(x^2 - 1)^2}{x^2} + 2 \end{aligned}$$

Therefore the graph of $y = f(x)$ has local minimums at $(1, 2)$ and $(-1, 2)$.

- a** Sketch the graph of $f(x) = x^2 + \frac{1}{x^2}$.
- b** Consider the family of curves with equations

$$y = x^{2m} + \frac{a}{x^{2m}} \quad \text{where } m \in \mathbb{N} \text{ and } a > 0$$

State the equations of the asymptotes and the coordinates of the turning points.

- c** Consider the family of curves with equations

$$y = \frac{x^{2m}}{a + x^{4m}} \quad \text{where } m \in \mathbb{N} \text{ and } a > 0$$

Sketch the curve for $m = a = 1$. Describe the properties of this family of curves.

- d** Investigate $y = x^{2m-1} + \frac{a}{x^{2m-1}}$, where $m \in \mathbb{N}$ and $a > 0$.
- e** Investigate $y = \frac{x^{2m-1}}{a + x^{4m-2}}$, where $m \in \mathbb{N}$ and $a > 0$.

16F Degree-of-difficulty classified questions

► Simple familiar questions

- 1 Let $z = 1 + 2i$ and $w = 3 - 5i$. Express each of the following complex numbers in Cartesian form:

a $z + w$	b $z - w$	c $2z - w$	d zw
e $\overline{zw}$	f $\frac{z}{w}$	g $\frac{1}{\overline{z}}$	
- 2 Solve the equation $z^2 - 2z + 10 = 0$ over $\mathbb{C}$.
- 3 Let $f(x) = x^2 - 4$. Find the range of the function given by $g(x) = \frac{1}{f(x)}$ for $-2 < x < 2$.
- 4 Write each of the following complex numbers in polar form:

a $3\sqrt{2} + 3\sqrt{2}i$	b $3 - 3\sqrt{3}i$	c $5\sqrt{3} - 5i$
- 5 Solve the equation $2 \cos(3x) = \sqrt{3}$ for $0 \leq x \leq \pi$.
- 6 Solve the equation $\frac{3z - (1 + i)}{z - 4} = 2$ for z .
- 7 Solve each of the following equations for $0 \leq \theta \leq 2\pi$:

a $1 + \cos \theta = \cos\left(\frac{\theta}{2}\right)$	b $\sin^2 \theta + 2 \cos^2 \theta = \sin \theta$
c $(5 + \cos \theta) \cot \theta + \sin \theta = 0$	
- 8 Let $f(x) = \sin x + \frac{1}{2} \cos x$. Rewrite this rule in the form $f(x) = r \sin(x + \alpha)$. Hence, find the maximum and minimum values of $f(x)$ for $0 \leq x \leq \frac{\pi}{2}$.
- 9 Sketch the graph of each of the following functions:

a $y = 2 3 - 4x $	b $y = x^2 - 9x $	c $y = \cos x $
- 10 Sketch the graph of each of the following functions:

a $y = 3 + \frac{2}{x-2}$	b $y = \frac{4x-5}{x-2}$	c $y = \frac{2-3x}{x-2}$
- 11 **a** Write $1 + \sqrt{3}i$ and $1 - i$ in polar form.
b Hence, express $\frac{(1 + \sqrt{3}i)^4}{(1 - i)^6}$ in Cartesian form.
- 12 **a** Represent $z_1 = -\sqrt{3} + i$ and $z_2 = \sqrt{2} + \sqrt{2}i$ on an Argand diagram.
b Find $z_1 z_2$ and $\frac{z_1}{z_2}$ in polar form, and represent them on the same diagram.
- 13 State the range of the function $f(x) = 3 \operatorname{cosec}(4x) + 3$ defined for the implied domain.
- 14 Sketch the graph of $y = \frac{4x^2 + 3}{x}$. State the equations of the asymptotes.

► **Complex familiar questions**

- 1 Prove that $\frac{(\sin x - 2 \cos x)^2}{(1 - \sin x)(1 + \sin x)} = 3 - 4 \tan x + \sec^2 x$.
- 2 Solve the following equation for $0 \leq x \leq 2\pi$:

$$\sin x + \cos x + \sec x + \operatorname{cosec} x = 0$$
- 3 Solve the equation $3z^2 - (1 + i)z - i = 0$ for z .
- 4 a Solve each of the following equations for x :
 i $|2x - 1| + |4 - x| = 5$ ii $|2x - 1| - |4 - x| = 7$ iii $|2x - 1| - |4 - x| = -\frac{7}{2}$
 b State the range of the function given by $f(x) = |2x - 1| - |4 - x|$ for $x \in \mathbb{R}$.
- 5 Find the values of x that satisfy the inequality $|x + 3a| > |3x - 2a|$, where a is a positive constant.
- 6 a Write $4x^2 - 24x + 32$ in the form $a(x - h)^2 + k$.
 b Sketch the graph of $y = \frac{1}{4x^2 - 24x + 32}$.
- 7 Suppose that $z = 1 + \sqrt{5}i$ is a solution of the equation $z^2 - 2z + a = 0$, where a is a real number. Find the value of a .
- 8 Solve the equation $z^2 = 21 - 20i$.
- 9 Sketch the graph of $y = 2 \sec(2x) + 1$ for $0 \leq x \leq 2\pi$.
- 10 Solve the equation $2 \sec\left(x - \frac{\pi}{4}\right) = 4$ for $0 \leq x \leq 2\pi$.

► **Complex unfamiliar questions**

- 1 Prove each of the following identities:
 a $\frac{\tan(2\theta) + \sec(2\theta) - 1}{\tan(2\theta) - \sec(2\theta) + 1} = \frac{1 + \tan \theta}{1 - \tan \theta}$ b $\frac{\cos(4\theta) + \cos(2\theta)}{\sin(4\theta) - \sin(2\theta)} + \tan \theta = 2 \operatorname{cosec}(2\theta)$
- 2 Solve the equation $3 \tan^3 x - 3 \tan^2 x - \tan x + 1 = 0$ for $0 < x < \pi$.
- 3 Solve the equation $z^2 = \bar{z}$ for z .
- 4 Suppose that $z = 5 - i$ is a solution of the equation $z^2 - (a + bi)z + 4 + 20i = 0$, where a and b are real numbers.
 a Find the values of a and b .
 b Find the other solution of this equation given that it is of the form $z = ki$, where k is a real number.
- 5 Show that if $\cos(2A) = \tan^2 B$, then $\cos(2B) = \tan^2 A$.

6 For real numbers α and β , let $k = \frac{\cos(\frac{1}{2}(\alpha + \beta))}{\cos(\frac{1}{2}(\alpha - \beta))}$. Prove that $\frac{1 - k^2}{1 + k^2} = \frac{\sin \alpha \sin \beta}{1 + \cos \alpha \cos \beta}$.

7 a Prove the identity

$$(\cot x + \operatorname{cosec} x)^2 = \frac{1 + \cos x}{1 - \cos x}$$

b Obtain a similar identity by putting $x = \frac{\pi}{2} + y$.

8 Let $z = \operatorname{cis} \theta$. Show that $\frac{z - 1}{z + 1} = i \tan\left(\frac{\theta}{2}\right)$.

9 Let z_1 and z_2 be the solutions of the quadratic equation $z^2 + 4z + 16 = 0$.

a Find z_1 and z_2 .

b Write z_1 and z_2 in polar form.

c Calculate $(z_1)^3$ and $(z_2)^3$.

d Plot z_1 , z_2 and $z_3 = 4 + 0i$ on an Argand diagram.

e Calculate $|z_1 - z_3|$ and $|z_2 - z_3|$. Comment on your answer.

f Investigate the quadratic equation $z^2 - 4z + 16 = 0$ in the same way.

g Investigate the family of quadratic equations $z^2 + az + a^2 = 0$, where a is an integer.

10 Consider triples of real numbers (a, b, c) such that

$$a \leq b \leq c, \quad |a| + |b| + |c| = 14, \quad |a + b + c| = 2 \quad \text{and} \quad |abc| = 72$$

a Explain why such triples must satisfy

$$a \leq 0 \leq b \leq c \quad \text{or} \quad a \leq b \leq 0 \leq c$$

b Determine all such triples.

11 In this question, we consider the geometry of complex numbers represented on an Argand diagram.

a For a complex number z , explain why $|z| = 1$ if and only if z lies on the circle of radius 1 centred at the origin.

b Similarly, explain why $|z| = 2$ if and only if z lies on the circle of radius 2 centred at the origin.

c Explain why $|z_1 - z_2|$ is the distance between the complex numbers z_1 and z_2 .

d Find the distance between z_1 and z_2 if:

i $z_1 = 2 + i$ and $z_2 = 3 - 4i$

ii $z_1 = 2 - i$ and $z_2 = 4 - 4i$

iii $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$

iv $z_1 = -2 - 3i$ and $z_2 = -4 + i$

e Find the complex number z , with $1 \leq |z| \leq 2$, such that:

i the distance $|z - (-1 - 3i)|$ is minimised

ii the distance $|z - (1 + 3i)|$ is minimised

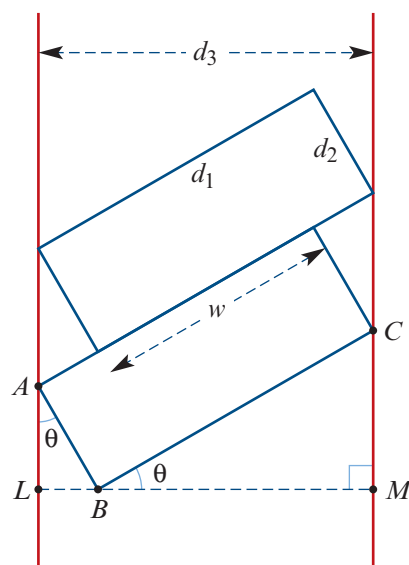
iii the distance $\left|z - \left(\frac{1}{2} + \frac{1}{2}i\right)\right|$ is minimised.

- 12** Let $z = \frac{1}{2}(1 + \sqrt{3}i)$.
- a** Calculate z^2 , z^3 , z^4 , z^5 and z^6 .
 - b** Show that $z + z^2 + z^3 + z^4 + z^5 + z^6 = 0$.
- 13** The motion of a particle moving in a straight line is modelled by the equation

$$x = 5 - 2 \cos^2 t \quad \text{for } t \geq 0$$

where x m is the distance of the particle from a point O on the line at time t seconds.

- a** Find the particle's distance from O when:
 - i** $t = 0$ **ii** $t = \frac{\pi}{2}$ **iii** $t = \pi$ **iv** $t = 2\pi$
 - b** Write the equation of motion in the form $x = a + b \cos(2t)$.
 - c** Determine:
 - i** the centre of the motion
 - ii** the period of the motion
 - iii** the amplitude of the motion.
 - d** Find the first time that the particle is 5 m from O .
 - e** Find the first time that the particle is 3.5 m from O .
- 14** The diagram illustrates part of a tiling pattern that is formed by placing rectangular tiles (shown in blue) between two parallel lines (shown in red). Each tile has size $d_1 \times d_2$ and the two parallel lines are a distance of d_3 apart.
- a** Prove that $\angle LAB = \angle CBM$. Let θ be the magnitude of this angle.
 - b** Find d_3 in terms of d_1 , d_2 and θ .
 - c** Given that $d_1 = 6$, $d_2 = 2$ and $d_3 = 4$, determine each of the following correct to two decimal places:
 - i** the angle θ in degrees
 - ii** the value of w , the overlap of the tiles.
 - d** Investigate by considering other values for d_1 , d_2 and d_3 .



17

Matrices

Objectives

- ▶ To identify when two matrices are **equal**.
- ▶ To **add** and **subtract** matrices of the same size.
- ▶ To multiply a matrix by a real number.
- ▶ To identify when the multiplication of two given matrices is possible.
- ▶ To perform **multiplication** of two suitable matrices.
- ▶ To find the **inverse** of a 2×2 matrix.
- ▶ To find the **determinant** of a 2×2 matrix.
- ▶ To solve **simultaneous linear equations** by using an inverse matrix.
- ▶ To use technology to find inverses and determinants of $n \times n$ matrices, where $n \geq 3$.

A **matrix** is a rectangular array of numbers. An example of a matrix is

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

Matrix algebra was first studied in England in the middle of the nineteenth century. Matrices are now used in many areas of science: for example, in physics, medical research, encryption and internet search engines.

In this chapter we will show how addition and multiplication of matrices can be defined and how matrices can be used to solve simultaneous linear equations. In Chapter 18 we will see how they can be used to study transformations of the plane.

17A Matrix notation

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.

The following are examples of matrices:

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 5 & 6 \end{bmatrix} \quad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \quad \begin{bmatrix} 5 \end{bmatrix}$$

► The size of a matrix

Matrices vary in size. The **size** of the matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines) that occur in the matrix.

The sizes of the above matrices are, in order:

$$3 \times 2, \quad 1 \times 4, \quad 3 \times 3, \quad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

An $m \times n$ matrix has m rows and n columns.

Example 1

Write down the sizes of the following matrices:

$$\text{a } \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{b } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{c } \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$$

Solution

$$\text{a } 2 \times 3 \quad \text{b } 4 \times 1 \quad \text{c } 1 \times 3$$

► Storing information in matrices

The use of matrices to store information is demonstrated by the following example.

Four exporters A , B , C and D sell refrigerators (r), dishwashers (d), microwave ovens (m) and televisions (t). The sales in a particular month can be represented by a 4×4 array of numbers. This array of numbers is called a matrix.

	r	d	m	t	
A	120	95	370	250	row 1
B	430	380	950	900	row 2
C	60	50	150	100	row 3
D	200	100	470	50	row 4
	column 1	column 2	column 3	column 4	

From this matrix it can be seen that:

- Exporter A sold 120 refrigerators, 95 dishwashers, 370 microwave ovens, 250 televisions.
- Exporter B sold 430 refrigerators, 380 dishwashers, 950 microwave ovens, 900 televisions.

The entries for the sales of refrigerators are in column 1.

The entries for the sales of exporter A are in row 1.

Example 2

A minibus has four rows of seats, with three seats in each row. If 0 indicates that a seat is vacant and 1 indicates that a seat is occupied, write down a matrix to represent:

- a the 1st and 3rd rows are occupied, but the 2nd and 4th rows are vacant
- b only the seat at the front-left corner of the minibus is occupied.

Solution

$$\mathbf{a} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{b} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 3

There are four clubs in a local football league:

- Club A has 2 senior teams and 3 junior teams.
- Club B has 2 senior teams and 4 junior teams.
- Club C has 1 senior team and 2 junior teams.
- Club D has 3 senior teams and 3 junior teams.

Represent this information in a matrix.

Solution

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$$

Explanation

The rows represent clubs A, B, C, D and the columns represent the number of senior and junior teams.

► Entries and equality

We will use uppercase letters **A**, **B**, **C**, ... to denote matrices.

If **A** is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of **A**. Thus a 3×4 matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Two matrices **A** and **B** are **equal**, and we can write $\mathbf{A} = \mathbf{B}$, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

For example:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 1 & -1 \\ 1-1 & 1 & \frac{6}{2} \end{bmatrix}$$

Example 4

If matrices **A** and **B** are equal, find the values of x and y .

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ x & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -3 & y \end{bmatrix}$$

Solution

$$x = -3 \text{ and } y = 4$$

Although a matrix is made from a set of numbers, it is important to think of a matrix as a single entity, somewhat like a ‘super number’.

Section summary

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.
- Two matrices **A** and **B** are equal when:
 - they have the same number of rows and the same number of columns, and
 - they have the same entry at corresponding positions.

Exercise 17A

Example 1

- 1 Write down the sizes of the following matrices:

a $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

b $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$

c $\begin{bmatrix} a & b & c & d \end{bmatrix}$

d $\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$

Example 2

- 2 There are 25 seats arranged in five rows and five columns. Using 0 to indicate that a seat is vacant and 1 to indicate that a seat is occupied, write down a matrix to represent the situation when:

- a** only the seats on the two diagonals are occupied
- b** all seats are occupied.

- 3** Seating arrangements are again represented by matrices, as in Question 2. Describe the seating arrangement represented by each of the following matrices:

- a** the entry a_{ij} is 1 if $i = j$, but 0 if $i \neq j$
- b** the entry a_{ij} is 1 if $i > j$, but 0 if $i \leq j$
- c** the entry a_{ij} is 1 if $i = j + 1$, but 0 otherwise.

Example 3

- 4** At a certain school there are 200 girls and 110 boys in Year 7. The numbers of girls and boys in the other year levels are 180 and 117 in Year 8, 135 and 98 in Year 9, 110 and 89 in Year 10, 56 and 53 in Year 11, and 28 and 33 in Year 12. Summarise this information in a matrix.

Example 4

- 5** From the following, select those pairs of matrices which could be equal, and write down the values of x and y which would make them equal:

a $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ x \end{bmatrix}$, $\begin{bmatrix} 0 & x \end{bmatrix}$, $\begin{bmatrix} 0 & 4 \end{bmatrix}$

b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$, $\begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 4 & x & 1 & -2 \end{bmatrix}$

c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$

- 6** Find the values of the pronumerals so that matrices **A** and **B** are equal:

a $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} x & 1 & -1 \\ 0 & 1 & y \end{bmatrix}$

b $\mathbf{A} = \begin{bmatrix} x \\ 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 \\ y \end{bmatrix}$

c $\mathbf{A} = \begin{bmatrix} -3 & x \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} y & 4 \end{bmatrix}$

d $\mathbf{A} = \begin{bmatrix} 1 & y \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$

- 7** The statistics for five members of a basketball team are recorded as follows:

Player A points 21, rebounds 5, assists 5

Player B points 8, rebounds 2, assists 3

Player C points 4, rebounds 1, assists 1

Player D points 14, rebounds 8, assists 60

Player E points 0, rebounds 1, assists 2

Express this information in a 5×3 matrix.

17B Addition, subtraction and multiplication by a real number

Addition of matrices

If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same size, then the sum $\mathbf{A} + \mathbf{B}$ is the matrix obtained by adding together the corresponding entries of the two matrices.

For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{bmatrix}$$

Multiplication of a matrix by a real number

If $\mathbf{A}$ is any matrix and k is a real number, then the product $k\mathbf{A}$ is the matrix obtained by multiplying each entry of $\mathbf{A}$ by k .

For example:

$$3 \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Note: If a matrix is added to itself, then the result is twice the matrix, i.e. $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$.

Similarly, for any natural number n , the sum of n matrices each equal to $\mathbf{A}$ is $n\mathbf{A}$.

If $\mathbf{B}$ is any matrix, then $-\mathbf{B}$ denotes the product $(-1)\mathbf{B}$.

Subtraction of matrices

If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size, then $\mathbf{A} - \mathbf{B}$ is defined to be the sum

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} + (-1)\mathbf{B}$$

For two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, the difference $\mathbf{A} - \mathbf{B}$ can be found by subtracting corresponding entries.

Example 5

Find:

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

Solution

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & -1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Zero matrix

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**, and will be denoted by **O**.

For any $m \times n$ matrix **A** and the $m \times n$ zero matrix **O**, we have

$$\mathbf{A} + \mathbf{O} = \mathbf{A} \quad \text{and} \quad \mathbf{A} + (-\mathbf{A}) = \mathbf{O}$$

**Example 6**

Let $\mathbf{X} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$.

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $3\mathbf{A} + \mathbf{B}$.

Solution

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$2\mathbf{X} = 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4\mathbf{Y} + \mathbf{X} = 4 \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$-3\mathbf{A} = -3 \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix}$$

$$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

Example 7

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, find the matrix **X** such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Solution

If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$. Therefore

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2 \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times (-1) & 8 - 2 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix} \end{aligned}$$



Using the TI-Nspire CX non-CAS

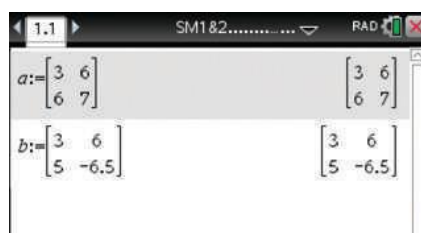
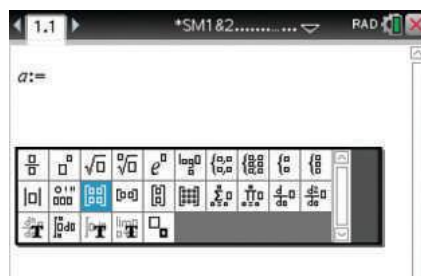
The matrix template

Matrices can be assigned (or stored) as variables for further computations.

Assign matrix $A = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as follows:

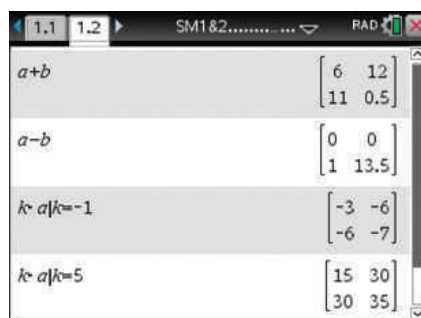
- In a **Calculator** page, type $a :=$ and then enter the matrix. (Access the assignment symbol $:=$ using $\text{(ctrl)} \text{(=)}$.)
- The simplest way to enter a 2×2 matrix is using the 2×2 matrix template as shown. (Access the templates using either (=) or $\text{(ctrl)} \text{(menu)} > \text{Math Templates.}$)
- Notice that there is also a template for entering $m \times n$ matrices.
- Use the touchpad arrows (or (tab)) to move between the entries of the matrix.

Assign the matrix $B = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$ similarly.



Operations on matrices

Once A and B are defined as above, the matrices $A + B$, $A - B$ and kA can easily be determined.



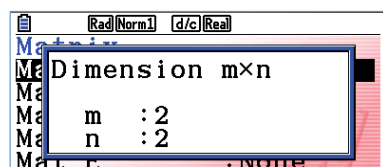
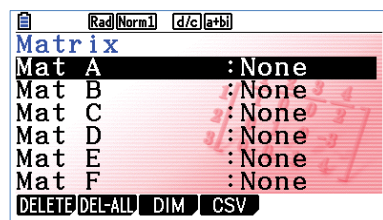
Using the Casio

Storing matrices

To store matrices $A = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$:

- In **Run-Matrix** mode, go to the Matrix Editor screen by selecting $\blacktriangleright \text{Mat (F3)}$.
- Press (EXE) to select Matrix A.
- Specify the size of Matrix A:

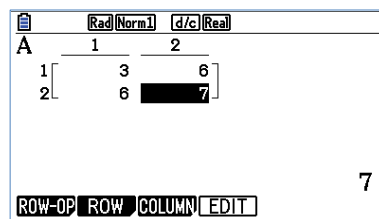
$\text{(2)} \text{(EXE)} \text{(2)} \text{(EXE)} \text{(EXE)}$



- Input the entries of Matrix A:

(3) (EXE) (6) (EXE) (6) (EXE) (7) (EXE)

- Press (EXIT) to return to the Matrix Editor screen.
- Use the cursor key $\blacktriangledown$ to move down to Matrix B, and then store the matrix **B** similarly.
- Press (EXIT).

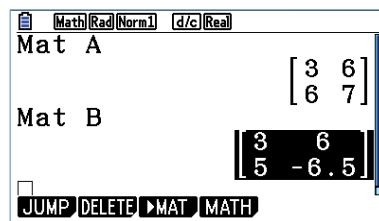


Accessing stored matrices

To access a stored matrix, use **Mat** (SHIFT) (2) followed by the matrix name.

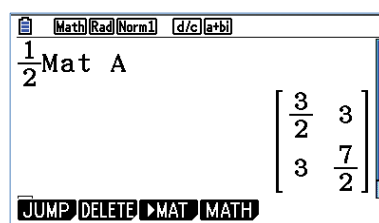
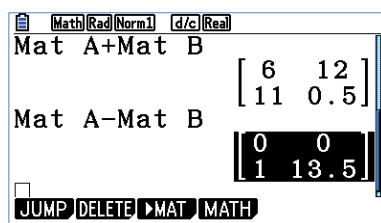
- For example, to view Matrix A, press:

(SHIFT) (2) (ALPHA) (X,θ,T) (EXE)



Operations on stored matrices

The matrices $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$ and $\frac{1}{2}\mathbf{A}$ can now be found as shown.



Section summary

- If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size, then:
 - the matrix $\mathbf{A} + \mathbf{B}$ is obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$
 - the matrix $\mathbf{A} - \mathbf{B}$ is obtained by subtracting the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$.
- If $\mathbf{A}$ is any matrix and k is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of $\mathbf{A}$ by k .

Exercise 17B

Example 6

- 1 Let $\mathbf{X} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$.

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $-3\mathbf{A} + \mathbf{B}$.

- 2 Let $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Find $2\mathbf{A}$, $-3\mathbf{A}$ and $-6\mathbf{A}$.

- 3 For $m \times n$ matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, is it always true that:

a $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

b $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

4 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix}$. Calculate:

a $2\mathbf{A}$

b $3\mathbf{B}$

c $2\mathbf{A} + 3\mathbf{B}$

d $3\mathbf{B} - 2\mathbf{A}$

5 Let $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$. Calculate:

a $\mathbf{P} + \mathbf{Q}$

b $\mathbf{P} + 3\mathbf{Q}$

c $2\mathbf{P} - \mathbf{Q} + \mathbf{R}$

Example 7

6 If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -10 \\ -2 & 17 \end{bmatrix}$, find matrices $\mathbf{X}$ and $\mathbf{Y}$ such that $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$ and $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$.

7 Matrices $\mathbf{X}$ and $\mathbf{Y}$ show the production of four models of cars a, b, c, d at two factories P, Q in successive weeks. Find $\mathbf{X} + \mathbf{Y}$ and describe what this sum represents.

$$\text{Week 1: } \mathbf{X} = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} P \\ Q \end{matrix} & \begin{bmatrix} 150 & 90 & 100 & 50 \\ 100 & 0 & 75 & 0 \end{bmatrix} \end{matrix}$$

$$\text{Week 2: } \mathbf{Y} = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} P \\ Q \end{matrix} & \begin{bmatrix} 160 & 90 & 120 & 40 \\ 100 & 0 & 50 & 0 \end{bmatrix} \end{matrix}$$

17C Multiplication of matrices

Multiplication of a matrix by a real number has been discussed in the previous section. The definition for multiplication of matrices is less straightforward. The procedure for multiplying two 2×2 matrices is shown first.

$$\text{Let } \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}.$$

$$\begin{aligned} \text{Then } \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and } \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

Example 8

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, find $\mathbf{AB}$.

Solution

$\mathbf{A}$ is a 2×2 matrix and $\mathbf{B}$ is a 2×1 matrix. Therefore the product $\mathbf{AB}$ is defined and will be a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$



Example 9

Matrix $\mathbf{X}$ shows the number of cars of models a and b bought by four dealers A, B, C, D . Matrix $\mathbf{Y}$ shows the cost in dollars of cars a and b . Find $\mathbf{XY}$ and explain what it represents.

$$\mathbf{X} = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{matrix} \quad \mathbf{Y} = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix} \begin{matrix} a \\ b \end{matrix}$$

Solution

$\mathbf{X}$ is a 4×2 matrix and $\mathbf{Y}$ is a 2×1 matrix. Therefore $\mathbf{XY}$ is a 4×1 matrix.

$$\begin{aligned} \mathbf{XY} &= \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{matrix} \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix} \begin{matrix} a \\ b \end{matrix} \\ &= \begin{bmatrix} 3 \times 26\,000 + 1 \times 32\,000 \\ 2 \times 26\,000 + 2 \times 32\,000 \\ 1 \times 26\,000 + 4 \times 32\,000 \\ 1 \times 26\,000 + 1 \times 32\,000 \end{bmatrix} = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix} \end{aligned}$$

The matrix $\mathbf{XY}$ shows that dealer A spent \$110 000, dealer B spent \$116 000, dealer C spent \$154 000 and dealer D spent \$58 000.

Example 10

For $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$, find $\mathbf{AB}$.

Solution

$\mathbf{A}$ is a 2×3 matrix and $\mathbf{B}$ is a 3×2 matrix. Therefore $\mathbf{AB}$ is a 2×2 matrix.

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 4 + 3 \times 1 + 4 \times 0 & 2 \times 0 + 3 \times 2 + 4 \times 3 \\ 5 \times 4 + 6 \times 1 + 7 \times 0 & 5 \times 0 + 6 \times 2 + 7 \times 3 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 26 & 33 \end{bmatrix} \end{aligned}$$

Section summary

- If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

- The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

Exercise 17C**Skillsheet**

- 1 Let $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 8, 10

Find the products $\mathbf{AX}$, $\mathbf{BX}$, $\mathbf{AY}$, $\mathbf{IX}$, $\mathbf{AC}$, $\mathbf{CA}$, $(\mathbf{AC})\mathbf{X}$, $\mathbf{C}(\mathbf{BX})$, $\mathbf{AI}$, $\mathbf{IB}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{A}(\mathbf{CA})$ and $\mathbf{A}^2\mathbf{C}$.

- Which of the following products of matrices from Question 1 are defined?
 $\mathbf{AY}$, $\mathbf{YA}$, $\mathbf{XY}$, $\mathbf{X}^2$, $\mathbf{CI}$, $\mathbf{XI}$
- If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$, find $\mathbf{AB}$.
- Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices and let $\mathbf{O}$ be the 2×2 zero matrix. Is the following argument correct?
'If $\mathbf{AB} = \mathbf{O}$ and $\mathbf{A} \neq \mathbf{O}$, then $\mathbf{B} = \mathbf{O}$.'
- Find a matrix $\mathbf{A}$ such that $\mathbf{A} \neq \mathbf{O}$ but $\mathbf{A}^2 = \mathbf{O}$.
- If $\mathbf{L} = \begin{bmatrix} 2 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find $\mathbf{LX}$ and $\mathbf{XL}$.

- 7 Assume that both **A** and **B** are $m \times n$ matrices. Are **AB** and **BA** defined and, if so, how many rows and columns do they have?
- 8 Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- a Show that $ad - bc = 1$.
- b What is the product matrix if the order of multiplication on the left-hand side is reversed?
- 9 Using the result of Question 8, write down a pair of matrices **A** and **B** such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- 10 Choose any three 2×2 matrices **A**, **B** and **C**. Find $\mathbf{A}(\mathbf{B} + \mathbf{C})$, $\mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\mathbf{A}$.
- 11 Find matrices **A** and **B** such that $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Example 9

- 12 It takes John 5 minutes to drink a milk shake which costs \$2.50, and 12 minutes to eat a banana split which costs \$3.00.
- a Find the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in fast-food economics.
- b Two friends join John. Find $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.
- 13 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$ and use your answer to find $\mathbf{A}^4$ and $\mathbf{A}^8$.
- 14 Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$. Write down a formula for $\mathbf{A}^n$.

17D Identities, inverses and determinants for 2×2 matrices

► Identities

A matrix with the same number of rows and columns is called a **square matrix**. For square matrices of a given size (e.g. 2×2), a multiplicative identity **I** exists.

For 2×2 matrices, the **identity matrix** is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For example, if $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, and this result holds for any square matrix multiplied by the appropriate multiplicative identity.

For 3×3 matrices, the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

► Inverses

Given a 2×2 matrix $\mathbf{A}$, is there a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$?

For example, consider $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $\begin{bmatrix} 2x + 3u & 2y + 3v \\ x + 4u & y + 4v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \quad \begin{array}{lcl} 2x + 3u = 1 & \text{and} & 2y + 3v = 0 \\ x + 4u = 0 & & y + 4v = 1 \end{array}$

These simultaneous equations can be solved to find x, y, u, v and hence $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

In general:

If $\mathbf{A}$ is a square matrix and if a matrix $\mathbf{B}$ can be found such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA}$$

then $\mathbf{A}$ is said to be **invertible** and $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$.

We leave it as an exercise to show that the inverse of an invertible matrix is unique.

We will denote the inverse of $\mathbf{A}$ by $\mathbf{A}^{-1}$.

For an invertible matrix $\mathbf{A}$, we have

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

The inverse of a general 2×2 matrix

Now consider $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $\begin{bmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \quad \begin{array}{lcl} ax + bu = 1 & \text{and} & ay + bv = 0 \\ cx + du = 0 & & cy + dv = 1 \end{array}$

These form two pairs of simultaneous equations, the first for x, u and the second for y, v .

The first pair of equations gives

$$(ad - bc)x = d \quad (\text{eliminating } u)$$

$$(bc - ad)u = c \quad (\text{eliminating } x)$$

These two equations can be solved for x and u provided $ad - bc \neq 0$:

$$x = \frac{d}{ad - bc} \quad \text{and} \quad u = \frac{c}{bc - ad} = \frac{-c}{ad - bc}$$

In a similar way, we obtain

$$y = \frac{-b}{ad - bc} \quad \text{and} \quad v = \frac{-a}{bc - ad} = \frac{a}{ad - bc}$$

We have established the following result.

Inverse of a 2×2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{provided } ad - bc \neq 0)$$

► The determinant

The quantity $ad - bc$ that appears in the formula for $\mathbf{A}^{-1}$ has a name: the **determinant** of $\mathbf{A}$. This is denoted $\det(\mathbf{A})$.

Determinant of a 2×2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

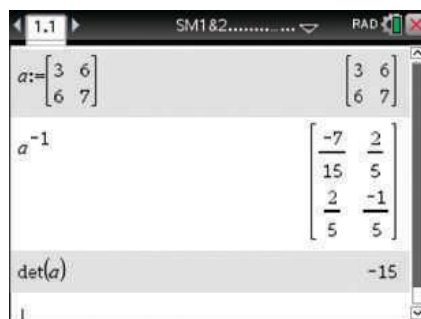
A 2×2 matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.



Using the TI-Nspire CX non-CAS

- The inverse of a matrix is obtained by raising the matrix to the power of -1 .
- The determinant command (> **Matrix and Vector** > **Determinant**) is used as shown.

Hint: You can also type in $\det(a)$.



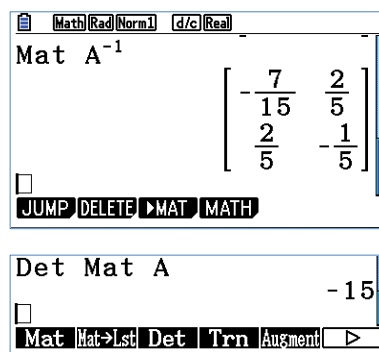
Using the Casio

- To find the inverse of a stored matrix, raise it to the power of -1 . For example, to find A^{-1} :

(SHIFT) (2) (ALPHA) (X,θ,T) (SHIFT) () (EXE)

- To find the determinant of a stored matrix, use the **Matrix operations** menu (OPTN) (F2). Select **Det** (F3); select **Mat** (F1); then enter the matrix name.

(Here $A = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ has been stored as Matrix A.)



Example 11

For the matrix $A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, find:

a $\det(A)$

b A^{-1}

Solution

a $\det(A) = 5 \times 1 - 2 \times 3 = -1$

b $A^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$



Example 12

For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, find:

a $\det(A)$

b A^{-1}

c X , if $AX = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

d Y , if $YA = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Solution

a $\det(A) = 3 \times 6 - 2 = 16$

b $A^{-1} = \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$

c $AX = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

d $YA = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the left) by A^{-1} .

Multiply both sides (on the right) by A^{-1} .

$$A^{-1}AX = A^{-1} \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$YAA^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} A^{-1}$$

$$\begin{aligned} \therefore IX = X &= \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} \\ &= \frac{1}{16} \begin{bmatrix} 16 & 32 \\ 16 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \therefore YI = Y &= \frac{1}{16} \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix} \\ &= \frac{1}{16} \begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

Section summary

■ For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

- the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{if } ad - bc \neq 0)$$

- the determinant of $\mathbf{A}$ is given by

$$\det(\mathbf{A}) = ad - bc$$

■ A 2×2 matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

Exercise 17D

Skillsheet

1 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, find:

- a** $\det(\mathbf{A})$ **b** $\mathbf{A}^{-1}$ **c** $\det(\mathbf{B})$ **d** $\mathbf{B}^{-1}$

2 Find the inverse of each of the following invertible matrices (where k is any non-zero real number):

- a** $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ **d** $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

3 If the matrix $\mathbf{A}$ is invertible, show that the inverse is unique.

4 Let $\mathbf{A}$ and $\mathbf{B}$ be the invertible matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

- a** Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.
b Find $\mathbf{AB}$ and hence find, if possible, $(\mathbf{AB})^{-1}$.
c From $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, find the products $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$. What do you notice?

Example 12

5 Let $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

- a** Find $\mathbf{A}^{-1}$. **b** If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{X}$. **c** If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{Y}$.

6 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$.

- a** Find $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$. **b** Find $\mathbf{Y}$ such that $\mathbf{YA} + \mathbf{B} = \mathbf{C}$.

7 Assume that $\mathbf{A}$ is a 2×2 matrix such that $a_{12} = a_{21} = 0$, $a_{11} \neq 0$ and $a_{22} \neq 0$. Show that $\mathbf{A}$ is invertible and find $\mathbf{A}^{-1}$.

8 Let $\mathbf{A}$ be an invertible 2×2 matrix, let $\mathbf{B}$ be a 2×2 matrix and assume that $\mathbf{AB} = \mathbf{O}$. Show that $\mathbf{B} = \mathbf{O}$.

- 9** Find all 2×2 matrices such that $\mathbf{A}^{-1} = \mathbf{A}$.
- 10** For what values of a does the matrix $\mathbf{A} = \begin{bmatrix} a & 1 \\ 2 & a \end{bmatrix}$ not have an inverse?
- 11** Let n be a natural number and let

$$\mathbf{A} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+2} \end{bmatrix}$$

Show that all the entries of the inverse matrix $\mathbf{A}^{-1}$ are integers.

SF

CF

17E Solution of simultaneous equations using matrices

Inverse matrices can be used to solve some systems of simultaneous linear equations.

Simultaneous equations with a unique solution

For example, consider the pair of simultaneous equations

$$3x - 2y = 5$$

$$5x - 3y = 9$$

This can be written as a matrix equation:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$. The determinant of $\mathbf{A}$ is $3(-3) - (-2)5 = 1$.

Since the determinant is non-zero, the inverse matrix exists:

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}$$

Now multiply both sides of the original matrix equation on the left by $\mathbf{A}^{-1}$:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{I} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix} \quad \text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This is the solution to the simultaneous equations. Check by substituting $x = 3$ and $y = 2$ into the two equations.

Simultaneous equations without a unique solution

If a pair of simultaneous linear equations in two variables corresponds to two parallel lines, then a non-invertible matrix results.

For example, the following pair of simultaneous equations has no solution:

$$\begin{aligned}x + 2y &= 3 \\ -2x - 4y &= 6\end{aligned}$$

The associated matrix equation is

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The determinant of the matrix $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is $1(-4) - 2(-2) = 0$, so the matrix has no inverse.

Example 13

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution

If $\mathbf{AX} = \mathbf{K}$, then

$$\begin{aligned}\mathbf{X} &= \mathbf{A}^{-1}\mathbf{K} \\ &= \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\end{aligned}$$

Example 14

Solve the following simultaneous equations:

$$\begin{aligned}3x - 2y &= 6 \\ 7x + 4y &= 7\end{aligned}$$

Solution

The matrix equation is

$$\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}$. Then $\mathbf{A}^{-1} = \frac{1}{26} \begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}$.

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 38 \\ -21 \end{bmatrix}$$

Exercise 17E

Skillsheet

- 1 Let $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Example 13

Solve the system $\mathbf{AX} = \mathbf{K}$, where:

a $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

b $\mathbf{K} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Example 14

- 2 Use matrices to solve each of the following pairs of simultaneous equations:

a $-2x + 4y = 6$

b $-x + 2y = -1$

$3x + y = 1$

$-x + 4y = 2$

- 3 Use matrices to find the point of intersection of the lines given by the equations $2x - 3y = 7$ and $3x + y = 5$.

- 4 Two children spend their pocket money buying some books and some DVDs. One child spends \$120 and buys four books and four DVDs. The other child spends \$114 and buys three DVDs and five books. Set up a system of simultaneous equations and use matrices to find the cost of a single book and a single DVD.

- 5 Consider the system

$$2x - 3y = 3$$

$$4x - 6y = 6$$

- a Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.
 b Is $\mathbf{A}$ an invertible matrix?
 c Can any solutions be found for this system of equations?
 d How many pairs does the solution set contain?
- 6 Suppose that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{X}$ are 2×2 matrices and that both $\mathbf{A}$ and $\mathbf{B}$ are invertible. Solve the following for $\mathbf{X}$:

a $\mathbf{AX} = \mathbf{C}$

b $\mathbf{ABX} = \mathbf{C}$

c $\mathbf{AXB} = \mathbf{C}$

d $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{C}$

e $\mathbf{AX} + \mathbf{B} = \mathbf{C}$

f $\mathbf{XA} + \mathbf{B} = \mathbf{A}$

17F Inverses and determinants for $n \times n$ matrices

In the next two sections, we see how the theory that has been developed for 2×2 matrices can be extended to $n \times n$ matrices, where $n \geq 3$. Much of the work in these two sections will be completed with the use of technology.

An $n \times n$ matrix $\mathbf{A}$ can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

Here a_{ij} is the entry in row i and column j of $\mathbf{A}$.

We will concentrate on 3×3 matrices, but the techniques used for larger square matrices are similar.

► Identities

For 3×3 matrices, the identity matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each 3×3 matrix $\mathbf{A}$, we have $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$.

Similarly, for 4×4 matrices, the identity matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In general, the $n \times n$ identity matrix has 1s along the main diagonal (top-left to bottom-right) and 0s everywhere else.

► Inverses

Recall that, if $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA}$$

then $\mathbf{B}$ is called the inverse of $\mathbf{A}$. When it exists, the inverse of a square matrix $\mathbf{A}$ is unique and is denoted by $\mathbf{A}^{-1}$.

We can use the following useful fact to help find inverses.

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{AB} = \mathbf{I}$, then it follows that $\mathbf{BA} = \mathbf{I}$ and so $\mathbf{B} = \mathbf{A}^{-1}$.

In this course, you are expected to use technology to find the inverse of a 3×3 matrix. However, the inverse can also be found by hand, as shown in the next example. (There are more efficient methods for finding inverses, but they are beyond the scope of this course.)

Example 15

Without using a calculator, find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

Solution

We want to find a matrix $\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ such that $\mathbf{AB} = \mathbf{I}$.

That is:

$$\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a - 3d + 2g & b - 3e + 2h & c - 3f + 2i \\ -3a + 3d - g & -3b + 3e - h & -3c + 3f - i \\ 2a - d & 2b - e & 2c - f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We first solve the three equations from the left-hand columns for a , d and g :

$$a - 3d + 2g = 1 \quad (1)$$

$$-3a + 3d - g = 0 \quad (2)$$

$$2a - d = 0 \quad (3)$$

From (3), we have $d = 2a$. Substitute into (1) and (2):

$$-5a + 2g = 1 \quad (1')$$

$$3a - g = 0 \quad (2')$$

We obtain $a = 1$, $d = 2$ and $g = 3$.

Solving the three equations from the middle columns gives $b = 2$, $e = 4$ and $h = 5$.

Solving the three equations from the right-hand columns gives $c = 3$, $f = 5$ and $i = 6$.

We obtain $\mathbf{A}^{-1} = \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Check: $\mathbf{AA}^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$

Not every 3×3 matrix has an inverse. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is non-invertible.

Example 16

Let $\mathbf{A} = \begin{bmatrix} 8 & 8 & 7 \\ 1 & 0 & 1 \\ 9 & 9 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 9 & 1 & -8 \\ -1 & -1 & 1 \\ -9 & 0 & 8 \end{bmatrix}$.

Find the product $\mathbf{AB}$, and hence find $\mathbf{A}^{-1}$.

Solution

$$\mathbf{AB} = \begin{bmatrix} 8 & 8 & 7 \\ 1 & 0 & 1 \\ 9 & 9 & 8 \end{bmatrix} \begin{bmatrix} 9 & 1 & -8 \\ -1 & -1 & 1 \\ -9 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

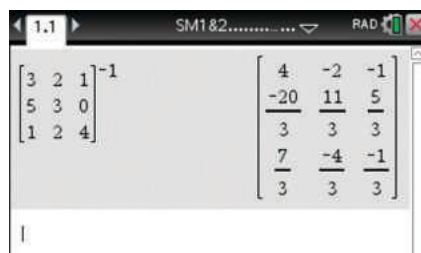
Hence $\mathbf{A}^{-1} = \mathbf{B}$.

Example 17

Using your calculator, find the inverse of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 5 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$.

**Using the TI-Nspire CX non-CAS**

The inverse of a matrix is obtained by raising the matrix to the power of -1 .

**Using the Casio**

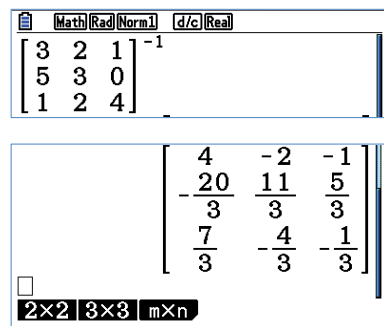
- In **Run-Matrix** mode, you can access a template for entering a 3×3 matrix by selecting **Math** (F4), **Matrix** (F1), then **3x3** (F2).

- Input the entries of the matrix:

$\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$

- Raise the matrix to the power of -1 :

(SHIFT) () (EXE)



Note: Matrices entered in this way are not stored for future calculations.

► The determinant

In Section 17D, we defined the determinant of a 2×2 matrix. The definition was motivated by the formula for the inverse of a 2×2 matrix. We saw that a 2×2 matrix has an inverse if and only if its determinant is non-zero.

In fact, the determinant is defined for all square matrices. In this course, you are expected to use technology to find the determinant of an $n \times n$ matrix when $n \geq 3$. However, we will also see how to find the determinant of a 3×3 matrix by hand.

The determinant of a 3×3 matrix

Consider a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of $\mathbf{A}$ can be defined as follows:

$$\begin{aligned} \det(\mathbf{A}) &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

This formula comes from working through the first row of $\mathbf{A}$:

- The first 2×2 matrix is obtained by deleting the row and column containing a_{11} .
- The second 2×2 matrix is obtained by deleting the row and column containing a_{12} .
- The third 2×2 matrix is obtained by deleting the row and column containing a_{13} .

Example 18

Find the determinant of $\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

Solution

$$\begin{aligned} \det(\mathbf{A}) &= 3 \times \det \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} - 2 \times \det \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} + 0 \times \det \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \\ &= 3(4 \times 2 - 1 \times 1) - 2(3 \times 2 - 1 \times 2) + 0 \\ &= 3 \times 7 - 2 \times 4 \\ &= 13 \end{aligned}$$

We can obtain equivalent formulas for the determinant by using any row or column of $\mathbf{A}$ in a similar way. For example, working through the first column of $\mathbf{A}$:

$$\det(\mathbf{A}) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Check that this formula gives the same result.

The sign of the a_{ij} term in a formula for the determinant is determined by $(-1)^{i+j}$.

For example:

- For a_{11} , the sign is given by $(-1)^{1+1} = 1$.
- For a_{12} , the sign is given by $(-1)^{1+2} = -1$.

These signs can also be determined using the following array:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For example, working through the second row of $\mathbf{A}$:

$$\det(\mathbf{A}) = -a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{22} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{23} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Note: When finding the determinant of a 3×3 matrix $\mathbf{A}$ by hand, it helps to work through the row or column of $\mathbf{A}$ that has the most 0 entries.

The determinant of an $n \times n$ matrix

We have seen that the determinant of a 3×3 matrix is defined using 2×2 matrices. Similarly, the determinant of a 4×4 matrix is defined using 3×3 matrices, and so on. You can use your calculator to find the determinant of large square matrices.

The determinant has the following important property.

Determinant of an $n \times n$ matrix

An $n \times n$ matrix $\mathbf{A}$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

Example 19

Using a calculator, find the determinant of:

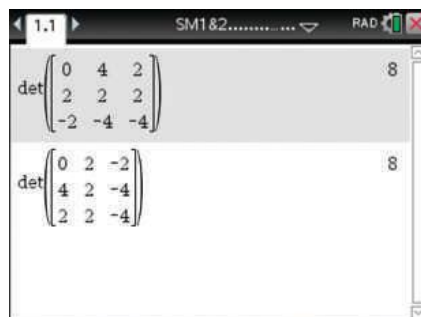
a $\begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$

b $\begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & -4 \\ 2 & 2 & -4 \end{bmatrix}$



Using the TI-Nspire CX non-CAS

The determinant command (**menu** > **Matrix and Vector** > **Determinant**) is used as shown. Alternatively, type $\det($.

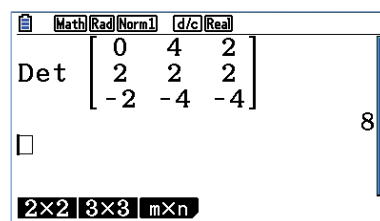


Using the Casio

- a** ■ In **Run-Matrix** mode, go to the **Matrix operations** menu (OPTN) (F2).
- Select **Det** (F3). Press (EXIT) twice.
- To access the 3×3 matrix template, select **Math** (F4), **Matrix** (F1), then **3×3** (F2).
- Input the entries of the matrix as shown:

(0) ► (4) ► (2) ►
 (2) ► (2) ► (2) ►
 (−) (2) ► (−) (4) ► (−) (4) ► EXE

- b** Similarly, we can find $\det \begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & -4 \\ 2 & 2 & -4 \end{bmatrix} = 8$.



Section summary

Identity matrix

- For each natural number n , there is an $n \times n$ identity matrix **I**. This matrix satisfies $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for all $n \times n$ matrices **A**.

- The 3×3 identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Inverse matrices

- If **A** is a square matrix and there exists a matrix **B** such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then **B** is called the inverse of **A**.
- When it exists, the inverse of a square matrix **A** is unique and is denoted by $\mathbf{A}^{-1}$.
- Let **A** and **B** be $n \times n$ matrices. If $\mathbf{AB} = \mathbf{I}$, then it follows that $\mathbf{BA} = \mathbf{I}$ and so $\mathbf{B} = \mathbf{A}^{-1}$.

Determinant

- The determinant is defined for all square matrices.
- A square matrix has an inverse if and only if its determinant is non-zero.
- For a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the determinant of **A** can be defined by

$$\det(\mathbf{A}) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Exercise 17F

SF

Example 15 1 Without using a calculator, find the inverse matrix of each of the following:

a $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

b $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

2 Show that the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 7 & 5 \\ 1 & 0 & 0 \end{bmatrix}$ is the matrix $\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 5 & -2 & 3 \\ -7 & 3 & -5 \end{bmatrix}$.

Example 16

3 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -4 \\ -2 & 5 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 19 & -17 & -11 \\ 6 & -5 & -2 \\ 8 & -9 & -5 \end{bmatrix}$.

Find the product $\mathbf{AB}$, and hence find $\mathbf{A}^{-1}$.

4 Let $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$. Find $\mathbf{A}^2$, and hence find $\mathbf{A}^{-1}$.

5 Let $\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$. Find $\mathbf{A}^2$, and hence find $\mathbf{A}^{-1}$.

Example 17

6 Use your calculator to find the inverse of each of the following matrices:

a $\begin{bmatrix} 5 & 2 & 3 \\ 1 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

b $\begin{bmatrix} 5 & 8 & 3 \\ 3 & 6 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

c $\begin{bmatrix} 9 & 1 & 2 & 5 \\ 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

d $\begin{bmatrix} 9 & 1 & 3 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ 5 & 1 & 0 & 2 \end{bmatrix}$

Example 18

7 Without using a calculator, find the determinant of:

a $\begin{bmatrix} 9 & 1 & 3 \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 7 & 5 \\ 1 & 0 & 0 \end{bmatrix}$

Example 19

8 Using a calculator, find the determinant of:

a i $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

ii $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

b i $\begin{bmatrix} 2 & 4 & 8 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

ii $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 4 & 4 \\ 6 & 4 & 2 \end{bmatrix}$

9 a Without using a calculator, find the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 2 & p \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ in terms of p .

b Hence find the value of p for which $\mathbf{A}$ does not have an inverse.

10 a Without using a calculator, find the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 2 & p \\ 2 & 2 & 2 \\ p & 2 & p \end{bmatrix}$ in terms of p .

b Hence find the values of p for which $\mathbf{A}$ does not have an inverse.

17G Simultaneous linear equations with more than two variables

In this section, we use inverse matrices to solve some systems of simultaneous linear equations. In Specialist Mathematics Units 3 & 4, you will learn a general technique that can be used to solve any system of simultaneous linear equations.

► Linear equations in three variables

Consider the general system of three linear equations in three variables:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

This can be written as a matrix equation:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Define the matrices

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then the matrix equation becomes

$$\mathbf{AX} = \mathbf{B}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, we can multiply both sides on the left by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B}$$

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B} \quad (\text{where } \mathbf{I} \text{ is the } 3 \times 3 \text{ identity matrix})$$

$$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Hence, if the inverse matrix $\mathbf{A}^{-1}$ exists, then the system of simultaneous equations has a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. You can use your calculator to find the inverse matrix $\mathbf{A}^{-1}$.

Example 20

Use matrix methods to solve the following system of three equations in three variables:

$$2x + y + z = -1$$

$$3y + 4z = -7$$

$$6x + z = 8$$

Solution

Define the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 4 \\ 6 & 0 & 1 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 \\ -7 \\ 8 \end{bmatrix}$$

Then the system of equations can be written as a matrix equation:

$$\mathbf{AX} = \mathbf{B}$$

Multiply both sides on the left by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$$

$$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Use your calculator to find $\mathbf{A}^{-1}\mathbf{B}$:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$$

The solution is $x = 1$, $y = -5$ and $z = 2$.

You can also use your calculator to solve a system of three linear equations directly, without finding an inverse matrix.

Example 21

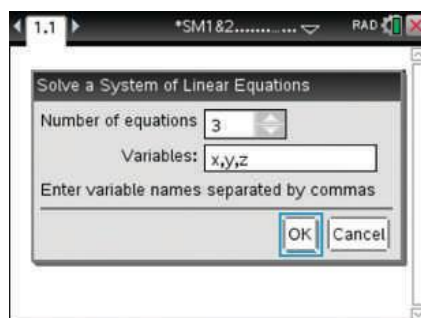
Solve the following simultaneous linear equations for x , y and z :

$$x - y + z = 6, \quad 2x + z = 4, \quad 3x + 2y - z = 6$$

**Using the TI-Nspire CX non-CAS**

Simultaneous linear equations can be solved in a **Calculator** application.

- Use **[menu]** > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen as shown.



- Enter the three equations:

$$x - y + z = 6$$

$$2x + z = 4$$

$$3x + 2y - z = 6$$

- Hence $x = \frac{14}{3}$, $y = -\frac{20}{3}$ and $z = -\frac{16}{3}$.

linSolve $\begin{pmatrix} x-y+z=6 \\ 2x+z=4 \\ 3x+2y-z=6 \end{pmatrix}, \{x,y,z\}$

$\left\{ \frac{14}{3}, -\frac{20}{3}, -\frac{16}{3} \right\}$

Using the Casio

- In **Equation** mode, select **Simultaneous** (F1), then select three unknowns (F2).
- Enter the coefficients of the three equations

$$x - y + z = 6$$

$$2x + z = 4$$

$$3x + 2y - z = 6$$

as shown:

1 [EXE] (-) 1 [EXE] 1 [EXE] 6 [EXE]
 2 [EXE] 0 [EXE] 1 [EXE] 4 [EXE]
 3 [EXE] 2 [EXE] (-) 1 [EXE] 6 [EXE]

- Select **Solve** (F1).
- Use the cursor key ▼ to move down the solution matrix and view the exact values.
- The solution is $x = \frac{14}{3}$, $y = -\frac{20}{3}$ and $z = -\frac{16}{3}$.

Equation

Select Type
 F1:Simultaneous
 F2:Polynomial
 F3:Solver
SIMUL POLY SOLVER

$a_n X + b_n Y + c_n Z = d_n$

	a	b	c	d
1	1	-1	1	6
2	2	0	1	4
3	3	2	-1	6

SOLVE DELETE CLEAR EDIT

$a_n X + b_n Y + c_n Z = d_n$

X 4.6666
 Y -6.6666
 Z -5.3333

REPEAT

Simultaneous equations without a unique solution

Just as for two linear equations in two variables, there is a geometric interpretation for three linear equations in three variables. There is only a unique solution if the three equations represent three planes intersecting at a point.

There are three possible cases for a system of three linear equations in three variables:

- a unique solution
- no solution
- infinitely many solutions.

These three cases and their geometric interpretations are studied in Specialist Mathematics Units 3 & 4.

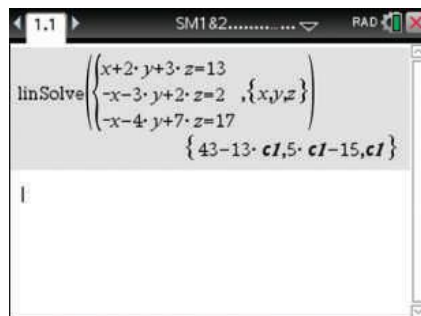
Example 22

Use your calculator to solve the simultaneous equations

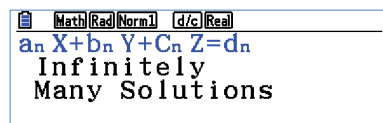
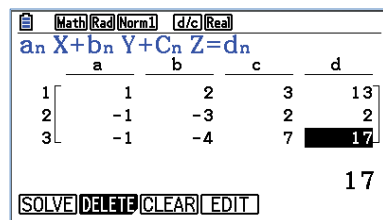
$$x + 2y + 3z = 13, \quad -x - 3y + 2z = 2, \quad -x - 4y + 7z = 17$$

**Using the TI-Nspire CX non-CAS**

- In a **Calculator** application, use **(menu) > Algebra > Solve System of Linear Equations**.
- Complete the pop-up screen and then enter the three equations as shown.
- The solutions are described in terms of a parameter $c1$. Using λ for the parameter, we can write the solutions as $x = 43 - 13\lambda$, $y = 5\lambda - 15$ and $z = \lambda$, for $\lambda \in \mathbb{R}$.

**Using the Casio**

- In **Equation** mode, select **Simultaneous** (F1), then select three unknowns (F2).
- Enter the coefficients of the three equations.
- Select **Solve** (F1).
- The system of simultaneous equations has infinitely many solutions.

**► Linear equations in more than three variables**

More generally, we can consider a system of n linear equations in n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Such a system of equations can be written as a matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If the $n \times n$ matrix has an inverse, then the system of equations has a unique solution.

Section summary

Matrix method for solving simultaneous linear equations

- A system of three linear equations in three variables has the form:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

- This can be written as a matrix equation $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- If the inverse matrix $\mathbf{A}^{-1}$ exists, then the system of simultaneous equations has a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

Exercise 17G

Example 20

- 1 Use matrix methods to solve each of the following systems of simultaneous equations:

a $2x + 3y - z = 12$

$$2y + z = 7$$

$$2y - z = 5$$

b $x + 2y + 3z = 13$

$$-x - y + 2z = 2$$

$$-x + 3y + 4z = 26$$

c $x + y = 5$

$$y + z = 7$$

$$z + x = 12$$

d $x - y - z = 0$

$$5x + 20z = 50$$

$$10y - 20z = 30$$

e $x + y - z = 3$

$$x - z + w = 0$$

$$2x - y - z + 3w = 1$$

$$-4x + 2y + 3z - 4w = 0$$

- 2 Consider the following system of simultaneous equations:

$$x + 2y + 3z = 13 \quad (1)$$

$$-x - 3y + 2z = 2 \quad (2)$$

$$-x - 4y + 7z = 17 \quad (3)$$

- a** Write this system as a matrix equation $\mathbf{AX} = \mathbf{B}$.
b Find $\det(\mathbf{A})$. Is the matrix $\mathbf{A}$ invertible?
c This system of simultaneous equations has infinitely many solutions. Express the solutions in terms of a parameter λ by following these steps:
i Add equation (2) to equation (1) and subtract equation (2) from equation (3).
ii Comment on the equations obtained in part **i**.
iii Let $z = \lambda$ and find y in terms of λ .
iv Substitute for z and y in terms of λ in equation (1) to find x in terms of λ .



Chapter summary



- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
 - they have the same number of rows and the same number of columns, and
 - they have the same entry at corresponding positions.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.
- Addition is defined for two matrices only when they have the same size. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Subtraction is performed in a similar way.

- If **A** is any matrix and k is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of **A** by k .

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note that the product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.

- If **A** is a square matrix and if a matrix **B** can be found such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then **A** is said to be **invertible** and **B** is called the **inverse** of **A**.

- For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

- the inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{if } ad-bc \neq 0)$$

- the determinant of **A** is given by

$$\det(\mathbf{A}) = ad - bc$$

- A square matrix **A** has an inverse if and only if $\det(\mathbf{A}) \neq 0$.
- Simultaneous equations can sometimes be solved using inverse matrices. For example, the system of equations

$$ax + by = c$$

$$dx + ey = f$$

can be written as $\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$ and solved using $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} \begin{bmatrix} c \\ f \end{bmatrix}$.

Technology-free questions

- 1** If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, find:
- a** $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **b** $\mathbf{A}^2 - \mathbf{B}^2$
- 2** Find all possible matrices $\mathbf{A}$ which satisfy the equation $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$.
- 3** Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.
- a** State whether or not each of the following products exists: $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{CD}$, $\mathbf{BE}$.
b Find $\mathbf{DA}$ and $\mathbf{A}^{-1}$.
- 4** If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $\mathbf{AB}$ and $\mathbf{C}^{-1}$.
- 5** Find the 2×2 matrix $\mathbf{A}$ such that $\mathbf{A} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$.
- 6** If $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, find $\mathbf{A}^2$ and hence find $\mathbf{A}^{-1}$.
- 7** If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ does not have an inverse, find the value of x .
- 8** **a** If $\mathbf{M} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, find:
- i** $\mathbf{MM} = \mathbf{M}^2$ **ii** $\mathbf{MMM} = \mathbf{M}^3$ **iii** $\mathbf{M}^{-1}$
- b** Find x and y , given that $\mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Multiple-choice questions



- 1** The matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -2 & 3 \\ 3 & 0 \end{bmatrix}$ has size

A 8**B** 4×2 **C** 2×4 **D** 1×4 **E** 3×4

2 If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & -3 & -1 \end{bmatrix}$, then $\mathbf{A} + \mathbf{B} =$

A $\begin{bmatrix} 3 & -3 \\ -2 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ **E** undefined

3 If $\mathbf{C} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -1 \end{bmatrix}$, then $\mathbf{D} - \mathbf{C} =$

A $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -6 & 4 \\ -2 & 0 & -4 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

D $\begin{bmatrix} 1 & -6 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ **E** undefined

4 If $\mathbf{M} = \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$, then $-\mathbf{M} =$

A $\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -4 \\ -6 & -2 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 0 \\ -2 & -6 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 6 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$

5 If $\mathbf{M} = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, then $2\mathbf{M} - 2\mathbf{N} =$

A $\begin{bmatrix} 0 & 0 \\ -9 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -2 \\ -6 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 12 & -2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 \\ 6 & -1 \end{bmatrix}$

6 If both $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, where $m \neq n$, then $\mathbf{A} + \mathbf{B}$ is

A an $m \times n$ matrix **B** an $m \times m$ matrix **C** an $n \times n$ matrix

D a $2m \times 2n$ matrix **E** not defined

7 If $\mathbf{P}$ is an $m \times n$ matrix and $\mathbf{Q}$ is an $n \times p$ matrix, where $m \neq p$, then $\mathbf{QP}$ is

A an $n \times n$ matrix **B** an $m \times p$ matrix **C** an $n \times p$ matrix

D an $m \times n$ matrix **E** not defined

8 The determinant of the matrix $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ is

A 4 **B** 0 **C** -4 **D** 1 **E** 2

9 The inverse of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ is

A -1 **B** $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

10 If $\mathbf{M} = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$, then $\mathbf{NM} =$

A $\begin{bmatrix} 0 & -4 \\ -9 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} -4 & -2 \\ 2 & -8 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 4 \\ 9 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} -6 & 2 \\ -3 & -5 \end{bmatrix}$ **E** $\begin{bmatrix} 6 & -2 \\ -3 & -5 \end{bmatrix}$

Extended-response questions

- 1 a** Consider the system of equations

$$2x - 3y = 3$$

$$4x + y = 5$$

- i** Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.
- ii** Find $\det(\mathbf{A})$ and $\mathbf{A}^{-1}$.
- iii** Solve the system of equations.
- iv** Interpret your solution geometrically.

- b** Consider the system of equations

$$2x + y = 3$$

$$4x + 2y = 8$$

- i** Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.
- ii** Find $\det(\mathbf{A})$ and explain why $\mathbf{A}^{-1}$ does not exist.

- c** Interpret your findings in part **b** geometrically.

- 2** The final grades for Physics and Chemistry are made up of three components: tests, practical work and exams. Each semester, a mark out of 100 is awarded for each component. Wendy scored the following marks in the three components for Physics:

Semester 1 tests 79, practical work 78, exam 80

Semester 2 tests 80, practical work 78, exam 82

- a** Represent this information in a 2×3 matrix.

To calculate the final grade for each semester, the three components are weighted: tests are worth 20%, practical work is worth 30% and the exam is worth 50%.

- b** Represent this information in a 3×1 matrix.

- c** Calculate Wendy's final grade for Physics in each semester.

Wendy also scored the following marks in the three components for Chemistry:

Semester 1 tests 86, practical work 82, exam 84

Semester 2 tests 81, practical work 80, exam 70

- d** Calculate Wendy's final grade for Chemistry in each semester.

Students who gain a total score of 320 or more for Physics and Chemistry over the two semesters are awarded a Certificate of Merit in Science.

- e** Will Wendy be awarded a Certificate of Merit in Science?

She asks her teacher to re-mark her Semester 2 Chemistry exam, hoping that she will gain the necessary marks to be awarded a Certificate of Merit.

- f** How many extra marks on the exam does she need?

- 3** A company runs computing classes and employs full-time and part-time teaching staff, as well as technical staff, catering staff and cleaners. The number of staff employed depends on demand from term to term.

In one year the company employed the following teaching staff:

Term 1 full-time 10, part-time 2

Term 2 full-time 8, part-time 4

Term 3 full-time 8, part-time 8

Term 4 full-time 6, part-time 10

- a** Represent this information in a 4×2 matrix.

Full-time teachers are paid \$70 per hour and part-time teachers are paid \$60 per hour.

- b** Represent this information in a 2×1 matrix.

- c** Calculate the cost per hour to the company for teaching staff for each term.

In the same year the company also employed the following support staff:

Term 1 technical 2, catering 2, cleaning 1

Term 2 technical 2, catering 2, cleaning 1

Term 3 technical 3, catering 4, cleaning 2

Term 4 technical 3, catering 4, cleaning 2

- d** Represent this information in a 4×3 matrix.

Technical staff are paid \$60 per hour, catering staff are paid \$55 per hour and cleaners are paid \$40 per hour.

- e** Represent this information in a 3×1 matrix.

- f** Calculate the cost per hour to the company for support staff for each term.

- g** Calculate the total cost per hour to the company for teaching and support staff for each term.

- 4** Bronwyn and Noel have a clothing warehouse in Summerville. They are supplied by three contractors: Brad, Flynn and Lina.

The matrix shows the number of dresses, pants and shirts that one worker, for each of the contractors, can produce in a week.

	Brad	Flynn	Lina
Dresses	5	6	10
Pants	3	4	5
Shirts	2	6	5

The number produced varies because of the different equipment used by the contractors. The warehouse requires 310 dresses, 175 pants and 175 shirts in a week. How many workers should each contractor employ to meet the requirement exactly?

- 5** Suppose that **A** and **B** are 2×2 matrices.

- a** Prove that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

- b** Hence prove that if both **A** and **B** are invertible, then **AB** is invertible.

18

Transformations of the plane

Objectives

- ▶ To define **linear transformations**.
- ▶ To represent a linear transformation as a 2×2 matrix.
- ▶ To study the effect of important transformations, including **dilations**, **reflections**, **rotations**, **shears** and **translations**.
- ▶ To investigate **compositions** and **inverses** of transformations.
- ▶ To investigate the connection between the **determinant** of a transformation matrix and area.
- ▶ To investigate the effect of transformations on regions of the plane, including points, shapes and graphs.

Modern animations are largely created with the use of computers. Many basic visual effects can be understood in terms of simple transformations of the plane.

For example, suppose that an animator wants to give the car below a sense of movement. This can be achieved by gradually tilting the car so that it leans forwards. We will see later how this can easily be done using a transformation called a **shear**.



Aside from computer graphics, linear transformations play an important role in many diverse fields such as mathematics, physics, engineering and economics.

18A Linear transformations

Each point in the plane can be denoted by an ordered pair (x, y) . The set of all ordered pairs is often called the **Cartesian plane**: $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

A **transformation** of the plane maps each point (x, y) in the plane to a new point (x', y') . We say that (x', y') is the **image** of (x, y) .

We will mainly be concerned with **linear transformations**, which have rules of the form

$$(x, y) \rightarrow (ax + by, cx + dy)$$

Example 1

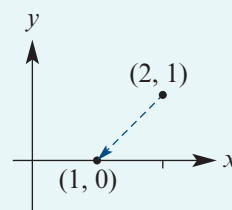
Find the image of the point $(2, 1)$ under the transformation with rule

$$(x, y) \rightarrow (3x - 5y, 2x - 4y)$$

Solution

We let $x = 2$ and $y = 1$, giving

$$(2, 1) \rightarrow (3 \times 2 - 5 \times 1, 2 \times 2 - 4 \times 1) = (1, 0)$$



► Matrices and linear transformations

Each ordered pair can also be written as a 2×1 matrix, which we call a **column vector**:

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

This is a very useful observation, since we can now easily perform the linear transformation $(x, y) \rightarrow (ax + by, cx + dy)$ by using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Example 2

- Find the matrix of the linear transformation with rule $(x, y) \rightarrow (x - 2y, 3x + y)$.
- Use the matrix to find the image of the point $(2, 3)$ under the transformation.

Solution

$$\mathbf{a} \quad \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{b} \quad \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 - 2 \times 3 \\ 3 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$$

Therefore the image of $(2, 3)$ is $(-4, 9)$.

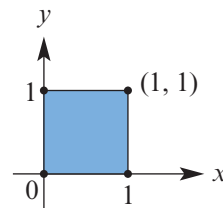
Explanation

The rows of the matrix are given by the coefficients of x and y .

We write the point $(2, 3)$ as a column vector and multiply by the transformation matrix.

► Transforming the unit square

The **unit square** is the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The effect of a linear transformation can often be demonstrated by studying its effect on the unit square.



Example 3

A linear transformation is represented by the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

- Find the image of the unit square under this transformation.
- Sketch the unit square and its image.

Solution

- We could find the images of the four vertices of the square one at a time:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

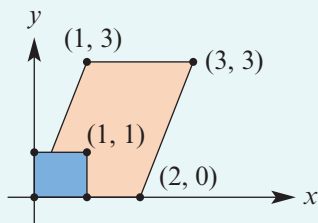
However, this can be done in a single step by multiplying the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex of the square:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

The columns of the result give the images of the vertices:

$$(0, 0), \quad (2, 0), \quad (1, 3), \quad (3, 3)$$

- The unit square is shown in blue and its image in red.



► Mapping the standard unit vectors

Let's express the points $(1, 0)$ and $(0, 1)$ as column vectors:

$$(1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These are called the **standard unit vectors** in $\mathbb{R}^2$.

We now consider the images of these points under the transformation with matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \text{first column of the matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \text{second column of the matrix}$$

To find the matrix of a linear transformation:

- The first column is the image of $(1, 0)$, written as a column vector.
- The second column is the image of $(0, 1)$, written as a column vector.

This observation allows us to write down the matrix of a linear transformation given just two pieces of information.

Example 4

A linear transformation maps the points $(1, 0)$ and $(0, 1)$ to the points $(1, 1)$ and $(-2, 3)$ respectively.

- a** Find the matrix of the transformation.
- b** Find the image of the point $(-3, 4)$.

Solution

$$\mathbf{a} \quad \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\mathbf{b} \quad \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \end{bmatrix}$$

Therefore $(-3, 4) \rightarrow (-11, 9)$.

Explanation

The image of $(1, 0)$ is $(1, 1)$, and the image of $(0, 1)$ is $(-2, 3)$. We write these images as the columns of a matrix.

Write the point $(-3, 4)$ as a column vector and multiply by the transformation matrix.

Section summary

- A **transformation** maps each point (x, y) in the plane to a new point (x', y') .
- A **linear transformation** is defined by a rule of the form $(x, y) \rightarrow (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- The **unit square** has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The effect of a linear transformation can be seen by looking at the image of the unit square.
- In the matrix of a linear transformation:
 - the first column is the image of $(1, 0)$, written as a column vector
 - the second column is the image of $(0, 1)$, written as a column vector.

Exercise 18A

Skillsheet

- 1 Find the image of the point $(2, -4)$ under the transformation with rule:

Example 1

- a** $(x, y) \rightarrow (x + y, x - y)$ **b** $(x, y) \rightarrow (2x + 3y, 3x - 4y)$
c $(x, y) \rightarrow (3x - 5y, x)$ **d** $(x, y) \rightarrow (y, -x)$

Example 2

- 2 Find the image of the point $(2, 3)$ under the linear transformation with matrix:

- a** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

- 3 Find the matrix of the linear transformation defined by the rule:

- a** $(x, y) \rightarrow (2x + 3y, 4x + 5y)$ **b** $(x, y) \rightarrow (11x - 3y, 3x - 8y)$
c $(x, y) \rightarrow (2x, x - 3y)$ **d** $(x, y) \rightarrow (y, -x)$

Example 3

- 4 Find and sketch the image of the unit square under the linear transformation represented by the matrix:

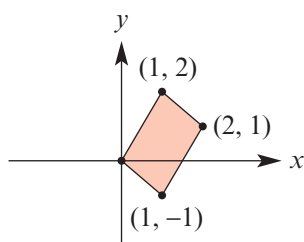
- a** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}$

- 5 Find the image of the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$ under the linear transformation represented by the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

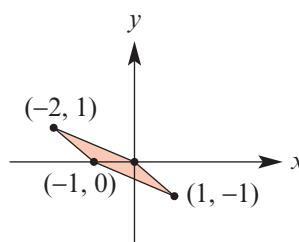
Example 4

- 6 Find the matrix of the linear transformation that maps the points $(1, 0)$ and $(0, 1)$ to the points $(3, 4)$ and $(5, 6)$ respectively. Hence find the image of the point $(-2, 4)$.
- 7 Find the matrix of the linear transformation that maps the points $(1, 0)$ and $(0, 1)$ to the points $(-3, 2)$ and $(1, -1)$ respectively. Hence find the image of the point $(2, 3)$.
- 8 Find a matrix that transforms the unit square to each of the following parallelograms.
- Note:** There are two possible answers for each part.

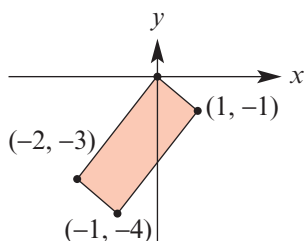
a



b



c



18B Geometric transformations

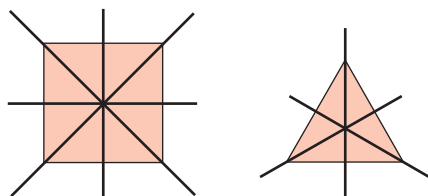
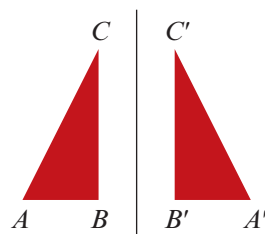
We now look at various important transformations that are geometric in nature.

► Reflections

A **reflection** in a line ℓ maps each point in the plane to its mirror image on the other side of the line. The point A and its image A' are the same distance from ℓ and the line AA' is perpendicular to ℓ .

These transformations are important for studying figures with **reflective symmetry**, that is, figures that look the same when reflected in a **line of symmetry**.

A square has four lines of symmetry, while an equilateral triangle has just three.



Note: A reflection is an example of a transformation that does not change lengths. Such a transformation is called an **isometry**.

Reflection in the x -axis

A reflection in the x -axis is defined by

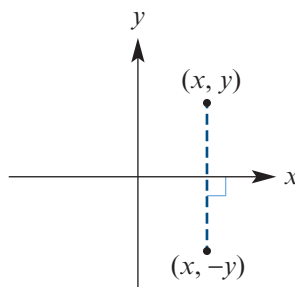
$$(x, y) \rightarrow (x, -y)$$

So if (x', y') is the image of the point (x, y) , then

$$x' = x \quad \text{and} \quad y' = -y$$

This transformation can also be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Reflection in the y -axis

A reflection in the y -axis is defined by

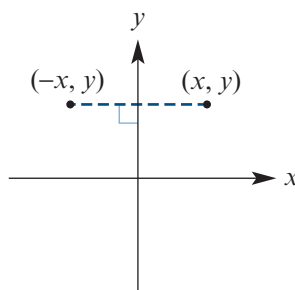
$$(x, y) \rightarrow (-x, y)$$

So if (x', y') is the image of the point (x, y) , then

$$x' = -x \quad \text{and} \quad y' = y$$

Once again, this transformation can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



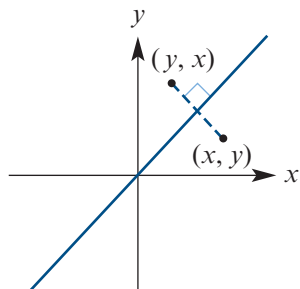
Reflection in the line $y = x$

If the point (x, y) is reflected in the line $y = x$, then it is mapped to the point (y, x) . So if (x', y') is the image of (x, y) , then

$$x' = y \quad \text{and} \quad y' = x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



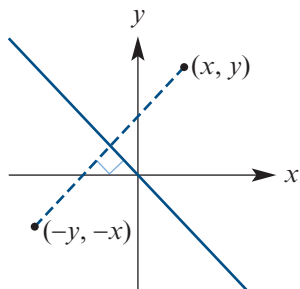
Reflection in the line $y = -x$

If the point (x, y) is reflected in the line $y = -x$, it is mapped to $(-y, -x)$. So if (x', y') is the image of (x, y) , then

$$x' = -y \quad \text{and} \quad y' = -x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Transformation	Rule	Matrix
Reflection in the x -axis	$x' = 1x + 0y$ $y' = 0x - 1y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y -axis	$x' = -1x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$x' = 0x + 1y$ $y' = 1x + 0y$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection in the line $y = -x$	$x' = 0x - 1y$ $y' = -1x + 0y$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

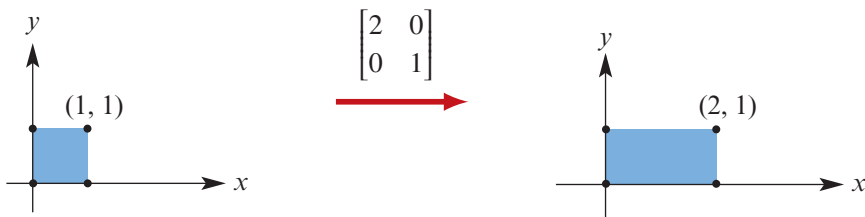
► Dilations

Dilation from the y -axis

A dilation from the y -axis is a transformation of the form

$$(x, y) \rightarrow (cx, y)$$

where $c > 0$. The x -coordinate is scaled by a factor of c , but the y -coordinate is unchanged.

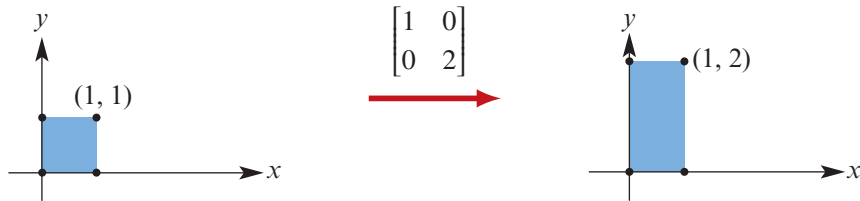


Dilation from the x -axis

Likewise, a dilation from the x -axis is a transformation of the form

$$(x, y) \rightarrow (x, cy)$$

where $c > 0$. The y -coordinate is scaled by a factor of c , but the x -coordinate is unchanged.



Dilation from the x - and y -axes

We can also simultaneously scale along the x - and y -axes using the transformation

$$(x, y) \rightarrow (cx, dy)$$

with scale factors $c > 0$ and $d > 0$.

Transformation	Rule	Matrix
Dilation from the y -axis	$x' = cx + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$
Dilation from the x -axis	$x' = 1x + 0y$ $y' = 0x + cy$	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Dilation from the x - and y -axes	$x' = cx + 0y$ $y' = 0x + dy$	$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

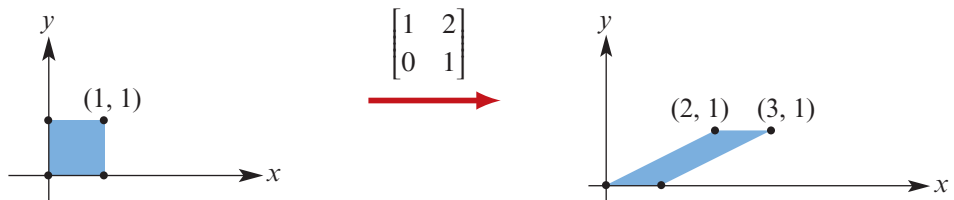
► Shears

Shear parallel to the x -axis

A shear parallel to the x -axis is a transformation of the form

$$(x, y) \rightarrow (x + cy, y)$$

Notice that each point is moved in the x -direction by an amount proportional to the distance from the x -axis. This means that the unit square is tilted in the x -direction.



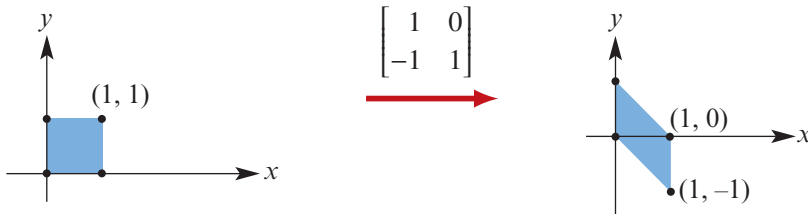
Shear parallel to the y -axis

A shear parallel to the y -axis is a transformation of the form

$$(x, y) \rightarrow (x, cx + y)$$

Here, each point is moved in the y -direction by an amount proportional to the distance from the y -axis. Now the unit square is tilted in the y -direction.

Note that if $c < 0$, then we obtain a shear in the negative direction.



Transformation	Rule	Matrix
Shear parallel to the x -axis	$x' = 1x + cy$ $y' = 0x + 1y$	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$
Shear parallel to the y -axis	$x' = 1x + 0y$ $y' = cx + 1y$	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$

► Projections

The transformation defined by

$$(x, y) \rightarrow (x, 0)$$

will project the point (x, y) onto the x -axis.

Likewise, the transformation defined by

$$(x, y) \rightarrow (0, y)$$

will project the point (x, y) onto the y -axis.

Transformation	Rule	Matrix
Projection onto the x -axis	$x' = 1x + 0y$ $y' = 0x + 0y$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the y -axis	$x' = 0x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projections are an important class of transformations. For example, the image on a television screen is the projection of a three-dimensional scene onto a two-dimensional surface.

Example 5

Find the image of the point $(3, 4)$ under each of the following transformations:

a reflection in the y -axis

b dilation of factor 2 from the y -axis

c shear of factor 4 parallel to the x -axis

d projection onto the y -axis

Solution

$$\mathbf{a} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$(3, 4) \rightarrow (-3, 4)$$

$$\mathbf{b} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$(3, 4) \rightarrow (6, 4)$$

$$\mathbf{c} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix}$$

$$(3, 4) \rightarrow (19, 4)$$

$$\mathbf{d} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$(3, 4) \rightarrow (0, 4)$$

► **Translations**

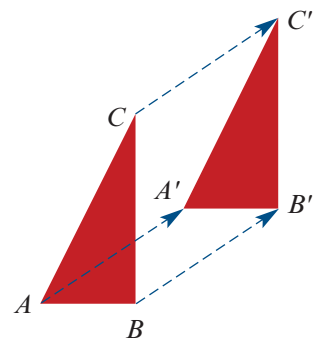
A **translation** moves a figure so that every point in the figure moves in the same direction and over the same distance.

A translation of a units in the x -direction and b units in the y -direction is defined by the rule

$$(x, y) \rightarrow (x + a, y + b)$$

This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$



Note: Translations cannot be represented using matrix multiplication. To see this, note that matrix multiplication will always map the point $(0, 0)$ to itself. Therefore, there is no matrix that will translate the point $(0, 0)$ to (a, b) , unless $a = b = 0$.

Example 6

Find the rule for a translation of 2 units in the x -direction and -1 units in the y -direction, and sketch the image of the unit square under this translation.

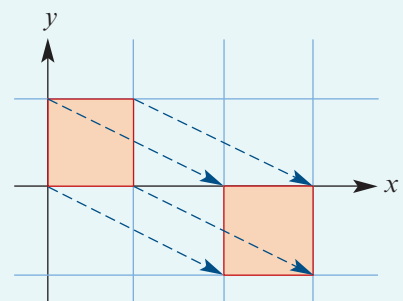
Solution

Using vector addition, this translation can be defined by the rule

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x + 2 \\ y - 1 \end{bmatrix}$$

or equivalently

$$x' = x + 2 \quad \text{and} \quad y' = y - 1$$



Section summary

- Important geometric transformation matrices are summarised in the table below.

Transformation	Matrix	Transformation	Matrix
Reflection in the x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y -axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y -axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x -axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y -axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y -axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- A translation of a units in the x -direction and b units in the y -direction is defined by the rule $(x, y) \rightarrow (x + a, y + b)$. This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Exercise 18B

Skillsheet

- 1 For each of the transformations described below:

Example 5

- find the matrix of the transformation
 - sketch the image of the unit square under this transformation.
- dilation of factor 2 from the x -axis
 - dilation of factor 3 from the y -axis
 - shear of factor 3 parallel to the x -axis
 - shear of factor -1 parallel to the y -axis
 - reflection in the x -axis
 - reflection in the line $y = -x$

Example 6

- 2 For each of the translations described below:

- find the rule for the translation using column vectors
 - sketch the image of the unit square under this translation.
- translation of 2 units in the x -direction
 - translation of -3 units in the y -direction
 - translation of -2 units in the x -direction and -4 units in the y -direction
 - translation by the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$
 - translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

SF

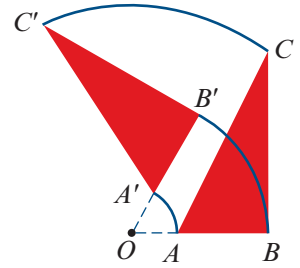
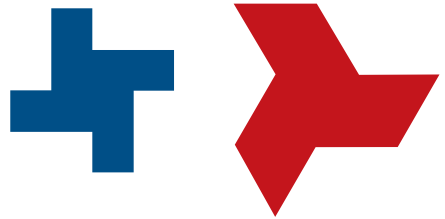
18C Rotations and general reflections

► Rotations

A **rotation** turns an object about a point, but keeps its distance to the point fixed. A rotation does not change lengths, and so is another example of an isometry.

Rotations are important for studying figures with **rotational symmetry**, that is, figures that look the same when rotated through a certain angle.

These two figures have rotational symmetry, but no reflective symmetry.



Finding the rotation matrix

Consider the transformation that rotates each point in the plane about the origin by angle θ anticlockwise. We will show that this is a linear transformation and find its matrix.

Let O be the origin and let $P(x, y)$ be a point in the plane.

Then we can write

$$x = r \cos \varphi \quad \text{and} \quad y = r \sin \varphi$$

where r is the distance OP and φ is the angle between OP and the positive direction of the x -axis.

Now let $P'(x', y')$ be the image of $P(x, y)$ under a rotation about O by angle θ anticlockwise.

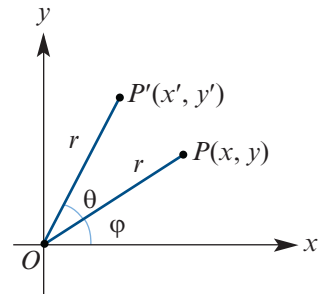
As $OP' = r$, we can use the angle sum identities to show that

$$\begin{aligned} x' &= r \cos(\varphi + \theta) \\ &= r \cos \varphi \cos \theta - r \sin \varphi \sin \theta \\ &= x \cos \theta - y \sin \theta \end{aligned}$$

$$\begin{aligned} \text{and} \quad y' &= r \sin(\varphi + \theta) \\ &= r \sin \varphi \cos \theta + r \cos \varphi \sin \theta \\ &= y \cos \theta + x \sin \theta \end{aligned}$$

Writing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Example 7

Find the matrix that represents a rotation of the plane about the origin by:

- a** 90° anticlockwise
b 45° clockwise.

Solution

$$\mathbf{a} \quad \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{b} \quad \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Explanation

An anticlockwise rotation means that we let $\theta = 90^\circ$ in the formula for the rotation matrix.

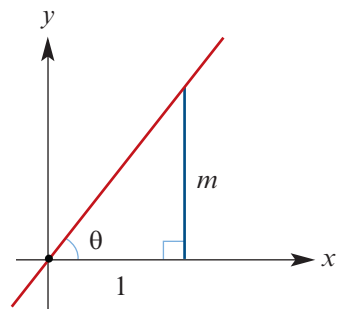
A clockwise rotation means that we let $\theta = -45^\circ$ in the formula for the rotation matrix.

► **Reflection in the line $y = mx$**

Reflection in a line that passes through the origin is also a linear transformation. We will find the matrix that will reflect the point (x, y) in the line $y = mx$.

Let's suppose that the angle between the positive direction of the x -axis and the line $y = mx$ is θ . Then $\tan \theta = m$ and so

$$y = mx = x \tan \theta$$

**Finding the reflection matrix**

We will use the fact that the first column of the required matrix will be the image A of $C(1, 0)$, written as a column vector, and the second column will be the image B of $D(0, 1)$, written as a column vector.

Since $\angle AOC = 2\theta$, we have

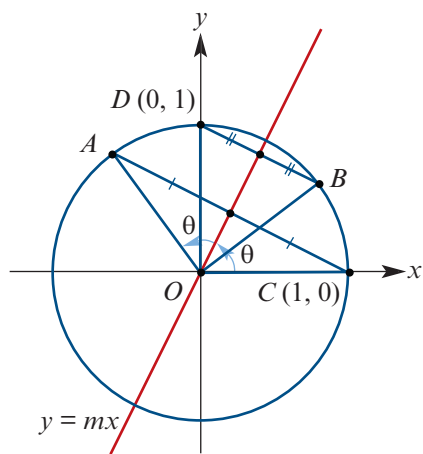
$$(1, 0) \rightarrow (\cos(2\theta), \sin(2\theta))$$

Moreover, since $\angle BOC = 2\theta - 90^\circ$, we have

$$\begin{aligned} (0, 1) &\rightarrow (\cos(2\theta - 90^\circ), \sin(2\theta - 90^\circ)) \\ &= (\sin(2\theta), -\cos(2\theta)) \end{aligned}$$

Writing these images as column vectors gives the reflection matrix:

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$





Example 8

- a** Find the matrix that will reflect the point (x, y) in the line through the origin at an angle of 30° to the positive direction of the x -axis.
- b** Find the matrix that will reflect the point (x, y) in the line $y = 2x$.

Solution

- a** We simply let $\theta = 30^\circ$, and so the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

- b** Since $\tan \theta = 2 = \frac{2}{1}$, we draw a right-angled triangle with opposite and adjacent lengths 2 and 1 respectively.

Pythagoras' theorem gives the hypotenuse as $\sqrt{5}$. Therefore

$$\cos \theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \sin \theta = \frac{2}{\sqrt{5}}$$

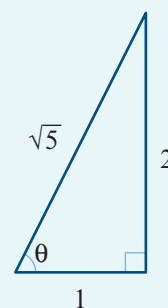
We then use the double-angle identities to show that

$$\cos(2\theta) = 2 \cos^2 \theta - 1 = 2 \left(\frac{1}{\sqrt{5}} \right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta = 2 \times \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}} = \frac{4}{5}$$

Therefore the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$



Section summary

Rotation matrix

The matrix that will rotate the plane about the origin by angle θ anticlockwise is

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Reflection matrix

The matrix that will reflect the plane in the line $y = mx = x \tan \theta$ is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Exercise 18C

Example 7

- 1** Find the matrix for each of the following rotations about the origin:

- a** 270° anticlockwise **b** 30° anticlockwise
c 60° clockwise **d** 135° clockwise

If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

Example 9

Find the matrix that corresponds to:

- a** a reflection in the x -axis and then a rotation about the origin by 90° anticlockwise
- b** a rotation about the origin by 90° anticlockwise and then a reflection in the x -axis.

Solution

$$\mathbf{a} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{b} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Explanation

A reflection in the x -axis has matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

An anticlockwise rotation by 90° has matrix

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We then multiply these two matrices together in the correct order.

Note: In this example, we get a different matrix when the same two transformations take place in reverse order. This should not be a surprise, as matrix multiplication is not commutative in general.

► Compositions involving translations

Example 10

- a** Find the rule for the transformation that will reflect (x, y) in the x -axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.
- b** Find the rule for the transformation if the translation takes place before the reflection.

Solution

$$\begin{aligned} \mathbf{a} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} x-3 \\ -y+4 \end{bmatrix} \end{aligned}$$

Therefore the transformation is
 $(x, y) \rightarrow (x-3, -y+4)$.

$$\begin{aligned} \mathbf{b} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x-3 \\ y+4 \end{bmatrix} \\ &= \begin{bmatrix} x-3 \\ -y-4 \end{bmatrix} \end{aligned}$$

Therefore the transformation is
 $(x, y) \rightarrow (x-3, -y-4)$.

Section summary

If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

The order is important, as matrix multiplication is not commutative in general.

Exercise 18D

Skillsheet

Example 9

- 1 Find the matrix that represents a reflection in the y -axis followed by a dilation of factor 3 from the x -axis.
- 2 Find the matrix that represents a rotation about the origin by 90° anticlockwise followed by a reflection in the x -axis.
- 3
 - a Find the matrix that represents a reflection in the x -axis followed by a reflection in the y -axis.
 - b Show that this matrix corresponds to a rotation about the origin by 180° .
- 4 Consider these two transformations:
 - T_1 : A reflection in the x -axis.
 - T_2 : A dilation of factor 2 from the y -axis.
 - a Find the matrix of T_1 followed by T_2 .
 - b Find the matrix of T_2 followed by T_1 .
 - c Does the order of transformation matter in this instance?
- 5 Consider these two transformations:
 - T_1 : A rotation about the origin by 90° clockwise.
 - T_2 : A reflection in the line $y = x$.
 - a Find the matrix of T_1 followed by T_2 .
 - b Find the matrix of T_2 followed by T_1 .
 - c Does the order of transformation matter in this instance?

Example 10

- 6 Consider these two transformations:
 - T_1 : A reflection in the y -axis.
 - T_2 : A translation of -3 units in the x -direction and 5 units in the y -direction.
 - a Find the rule for T_1 followed by T_2 .
 - b Find the rule for T_2 followed by T_1 .
 - c Does the order of transformation matter in this instance?
- 7 Express each of the following transformation matrices as the product of a dilation matrix and a reflection matrix:

a $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$

c $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

d $\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$

- 8 a** Find the matrix for the transformation that is a reflection in the x -axis followed by a reflection in the line $y = x$. SF
- b** Show that these two reflections can be achieved with one rotation.
- 9** Suppose that matrix **A** gives a rotation about the origin by angle θ anticlockwise and that matrix **B** gives a reflection in the line $y = x$. If $\mathbf{AB} = \mathbf{BA}$, find the angle θ . CF
- 10** Suppose that matrix **A** rotates the plane about the origin by angle θ anticlockwise.
- a** Through what angle will the matrix $\mathbf{A}^2$ rotate the plane?
- b** Evaluate $\mathbf{A}^2$.
- c** Hence find formulas for $\cos(2\theta)$ and $\sin(2\theta)$.
- 11** A transformation T consists of a reflection in the line $y = x$ followed by a translation by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- a** Find the rule for the transformation T .
- b** Show that the transformation T can also be obtained by a translation and then a reflection in the line $y = x$. Find the translation vector.
- 12 a** Find the rotation matrix for an angle of 60° anticlockwise.
- b** Find the rotation matrix for an angle of 45° clockwise.
- c** By multiplying these two matrices, find the rotation matrix for an angle of 15° anticlockwise.
- d** Hence write down the exact values of $\sin 15^\circ$ and $\cos 15^\circ$.
- 13** A transformation consists of a reflection in the line $y = x \tan \varphi$ and then in the line $y = x \tan \theta$. Show that this is equivalent to a single rotation. CU

18E Inverse transformations

If transformation T maps the point (x, y) to the point (x', y') , then the **inverse transformation** T^{-1} maps the point (x', y') to the point (x, y) .

For a linear transformation T , we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{X}' = \mathbf{AX}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, then we have

$$\mathbf{A}^{-1}\mathbf{X}' = \mathbf{A}^{-1}\mathbf{AX}$$

$$\mathbf{A}^{-1}\mathbf{X}' = \mathbf{IX}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}'$$

Therefore $\mathbf{A}^{-1}$ is the matrix of the inverse transformation T^{-1} .

If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse exists if and only if $\det(\mathbf{A}) = ad - bc \neq 0$.



Example 11

Find the inverse of the transformation with rule $(x, y) \rightarrow (3x + 2y, 5x + 4y)$.

Solution

Since the matrix of this linear transformation is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

the inverse transformation will have matrix

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3 \times 4 - 2 \times 5} \begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix} \end{aligned}$$

Therefore the rule of the inverse transformation is $(x, y) \rightarrow (2x - y, -\frac{5}{2}x + \frac{3}{2}y)$.

Example 12

Find the matrix of the linear transformation such that $(4, 3) \rightarrow (9, 10)$ and $(2, 1) \rightarrow (5, 6)$.

Solution

We need to find a matrix $\mathbf{A}$ such that

$$\mathbf{A} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix} \quad \text{and} \quad \mathbf{A} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

This can be written as a single equation:

$$\mathbf{A} \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}$$

Therefore

$$\mathbf{A} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

► Inverses of important transformations

For important geometric transformations, it is often obvious what the inverse transformation should be.



Example 13

Let $\mathbf{R}$ be the matrix corresponding to a rotation of the plane by angle θ anticlockwise. Show that $\mathbf{R}^{-1}$ corresponds to a rotation by angle θ clockwise.

Solution

$$\begin{aligned} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1} &= \frac{1}{\cos^2 \theta + \sin^2 \theta} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \end{aligned}$$

This matrix corresponds to a rotation of the plane by angle θ clockwise.

Explanation

We find the inverse matrix using the formula

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We also use the symmetry properties:

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

The following table summarises the important geometric transformations along with their inverses. You will demonstrate some of these results in the exercises.

Transformation	Matrix $\mathbf{A}$	Inverse matrix $\mathbf{A}^{-1}$	Inverse transformation
Dilation from the y -axis	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the y -axis
Dilation from the x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$	Dilation from the x -axis
Shear parallel to the x -axis	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$	Shear parallel to the x -axis
Rotation by θ anticlockwise	$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$	$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$	Rotation by θ clockwise
Reflection in the x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the x -axis
Reflection in the y -axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Reflection in the y -axis
Reflection in the line $y = mx$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	Reflection in the line $y = mx$

Section summary

If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Exercise 18E

- 1 Find the inverse matrix of each of the following transformation matrices:

a $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$

b $\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$

c $\begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$

d $\begin{bmatrix} -1 & 3 \\ -4 & 5 \end{bmatrix}$

Example 11

- 2 For each of the following transformations, find the rule for their inverse:

a $(x, y) \rightarrow (5x - 2y, 2x - y)$

b $(x, y) \rightarrow (x - y, x)$

- 3 Find the point (x, y) that is mapped to $(1, 1)$ by the transformation with matrix:

a $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

b $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

Example 12

- 4 Find the matrix of the linear transformation such that $(1, 2) \rightarrow (2, 1)$ and $(2, 3) \rightarrow (1, 1)$.

- 5 Find the vertices of the rectangle that is mapped to the unit square by the transformation with matrix $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

Example 13

- 6 Consider a dilation of factor k from the y -axis, where $k > 0$.

a Write down the matrix of this transformation.

b Show that the inverse matrix corresponds to a dilation of factor $\frac{1}{k}$ from the y -axis.

- 7 Consider a shear of factor k parallel to the x -axis.

a Write down the matrix of this transformation.

b Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the x -axis.

- 8 Consider the transformation that reflects each point in the x -axis.

a Write down the matrix $\mathbf{A}$ of this transformation.

b Show that $\mathbf{A}^{-1} = \mathbf{A}$, and explain why you should expect this result.

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- 9 Consider the transformation that reflects each point in the line $y = mx = x \tan \theta$.
- Write down the matrix $\mathbf{B}$ of this transformation.
 - Show that $\mathbf{B}^{-1} = \mathbf{B}$, and explain why you should expect this result.

18F Transformations of straight lines and other graphs

We have considered the effect of various transformations on points and figures in the plane. We will now turn our attention to graphs.

Here, we will aim to find the equations of transformed graphs. We will also investigate the effects of linear transformations on straight lines. You will study this application in much greater detail in Mathematical Methods.

► Linear transformations of straight lines

We will first investigate the effect of linear transformations on straight lines.

Example 14

Find the equation of the image of the line $y = 2x + 3$ under a reflection in the x -axis followed by a dilation of factor 2 from the y -axis.

Solution

The matrix of the combined transformation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

If (x', y') are the coordinates of the image of (x, y) , then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix}$$

Therefore

$$x' = 2x \quad \text{and} \quad y' = -y$$

Rearranging gives

$$x = \frac{x'}{2} \quad \text{and} \quad y = -y'$$

Therefore the equation $y = 2x + 3$ becomes

$$-y' = 2\left(\frac{x'}{2}\right) + 3$$

$$-y' = x' + 3$$

$$y' = -x' - 3$$

We now ignore the dashes, and so the equation of the image is simply

$$y = -x - 3$$

Example 15

Consider the graph of $y = x + 1$. Find the equation of its image under the linear transformation $(x, y) \rightarrow (x + 2y, y)$.

Solution

Let (x', y') be the coordinates of the image of (x, y) . Then this transformation can be written in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x' - 2y' \\ y' \end{bmatrix}$$

and so $x = x' - 2y'$ and $y = y'$.

The equation $y = x + 1$ becomes

$$y' = x' - 2y' + 1$$

$$3y' = x' + 1$$

$$y' = \frac{x'}{3} + \frac{1}{3}$$

The equation of the image is $y = \frac{x}{3} + \frac{1}{3}$.

In the previous two examples, you will have noticed that the image of each straight line was another straight line. In fact, linear transformations get their name in part from the following fact, which is proved in the exercises.

The image of any straight line under an invertible linear transformation is a straight line.

**Example 16**

Find a matrix that transforms the line $y = x + 2$ to the line $y = -2x + 4$.

Solution

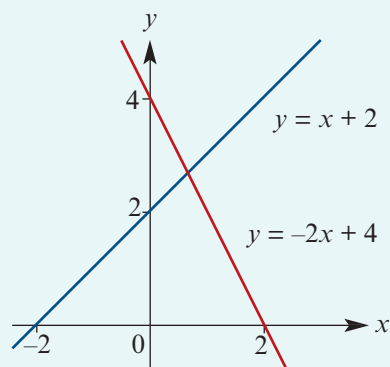
Let's find the matrix that maps the x -axis intercept of the first line to the x -axis intercept of the second line, and likewise for the y -axis intercepts.

We want

$$(-2, 0) \rightarrow (2, 0) \quad \text{and} \quad (0, 2) \rightarrow (0, 4)$$

This can be achieved by a reflection in the y -axis and then a dilation of factor 2 from the x -axis.

This transformation has the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.



► Ellipses as transformations of circles

If the unit circle is dilated by factor a from the x -axis and factor b from the y -axis, we obtain the curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

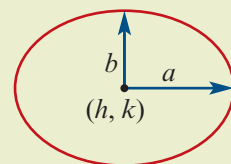
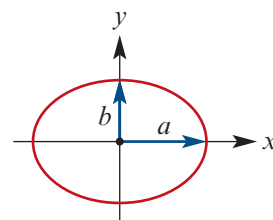
This is an **ellipse** with x -axis intercepts $\pm a$ and y -axis intercepts $\pm b$.

Applying the translation defined by $(x, y) \rightarrow (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k) .



Note: Ellipses are studied further in Chapter 20.

Example 17

Sketch the graph of each ellipse:

a $\frac{x^2}{9} + \frac{y^2}{4} = 1$

b $4x^2 + 9y^2 = 1$

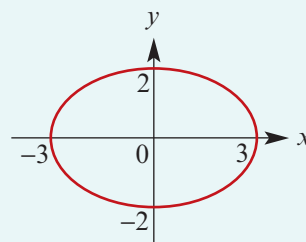
c $\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{9} = 1$

Solution

a The equation can be written as

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

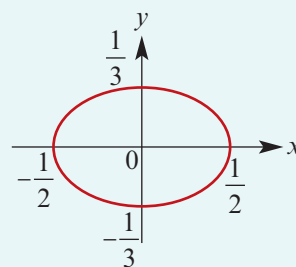
This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm 3$ and $y = \pm 2$.



b The equation can be written as

$$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$$

This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm \frac{1}{2}$ and $y = \pm \frac{1}{3}$.



c This is an ellipse with centre $(1, -2)$.

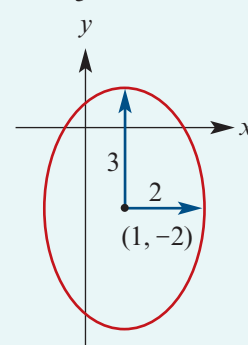
To find the x -axis intercepts, let $y = 0$. Then solving for x gives

$$x = \frac{3 \pm 2\sqrt{5}}{3}$$

Likewise, to find the y -axis intercepts, let $x = 0$.

This gives

$$y = \frac{-4 \pm 3\sqrt{3}}{2}$$



Example 18

Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 2 from the y-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

Solution

The dilation matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

The rotation matrix is $\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

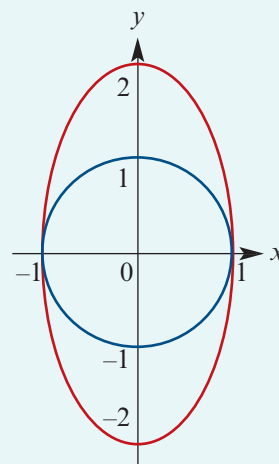
Let (x', y') be the image of (x, y) . Then the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ 2x \end{bmatrix}$$

Thus $x' = -y$ and $y' = 2x$, giving $y = -x'$ and $x = \frac{y'}{2}$.

The equation $x^2 + y^2 = 1$ becomes $\left(\frac{y'}{2}\right)^2 + (-x')^2 = 1$.

Hence the image is the ellipse with equation $x^2 + \frac{y^2}{2^2} = 1$.



► **Transformations of other graphs**

The method for finding the image of a straight line or circle can be used for other graphs.

Example 19

Find the image of the graph of $y = x^2 + 1$ under a translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ followed by a reflection in the y-axis.

Solution

Let (x', y') be the image of (x, y) . Then the transformation is given by

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x+2 \\ y-1 \end{bmatrix} = \begin{bmatrix} -x-2 \\ y-1 \end{bmatrix} \end{aligned}$$

Therefore $x' = -x - 2$ and $y' = y - 1$.

This gives $x = -x' - 2$ and $y = y' + 1$.

The equation $y = x^2 + 1$ becomes

$$\begin{aligned} y' + 1 &= (-x' - 2)^2 + 1 \\ y' &= (-x' - 2)^2 \\ &= (x' + 2)^2 \end{aligned}$$

The equation of the image is $y = (x + 2)^2$.

Exercise 18F

Example 14

- 1 Find the equation of the image of the graph of $y = 3x + 1$ under:
- a** a reflection in the x -axis
 - b** a dilation of factor 2 from the y -axis
 - c** a dilation of factor 3 from the x -axis and factor 2 from the y -axis
 - d** a reflection in the x -axis and then in the y -axis
 - e** a reflection in the y -axis and then a dilation of factor 3 from the x -axis
 - f** a rotation about the origin by 90° anticlockwise
 - g** a rotation about the origin by 90° clockwise and then a reflection in the x -axis.

Example 15

- 2 Find the image of $y = 2 - 3x$ under each of the following transformations:

- a** $(x, y) \rightarrow (2x, 3y)$
- b** $(x, y) \rightarrow (-y, x)$
- c** $(x, y) \rightarrow (x - 2y, y)$
- d** $(x, y) \rightarrow (3x + 5y, x + 2y)$

Example 16

- 3 Find a matrix that transforms the line $x + y = 1$ to the line $x + y = 2$.
- 4 Find a matrix that transforms the line $y = x + 1$ to the line $y = 6 - 2x$.

Example 17

- 5 Sketch the graph of each ellipse, labelling the centre and the axis intercepts:
- a** $\frac{x^2}{9} + \frac{y^2}{64} = 1$
 - b** $\frac{x^2}{100} + \frac{y^2}{25} = 1$
 - c** $25x^2 + 9y^2 = 225$
 - d** $\frac{(x-3)^2}{9} + \frac{(y-4)^2}{16} = 1$
 - e** $\frac{(x+3)^2}{9} + \frac{(y+4)^2}{25} = 1$

Example 18

- 6 Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 3 from the x -axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

Example 19

- 7 Find the equation of the image of the graph of $y = x^2 - 1$ under a translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and then a reflection in the x -axis.

- 8 Find the equation of the image of the graph of $y = (x - 1)^2$ under a reflection in the y -axis and then a translation by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

- 9 Consider any invertible linear transformation

$$(x, y) \rightarrow (ax + by, cx + dy)$$

Show that the image of the straight line $px + qy = r$ is a straight line.

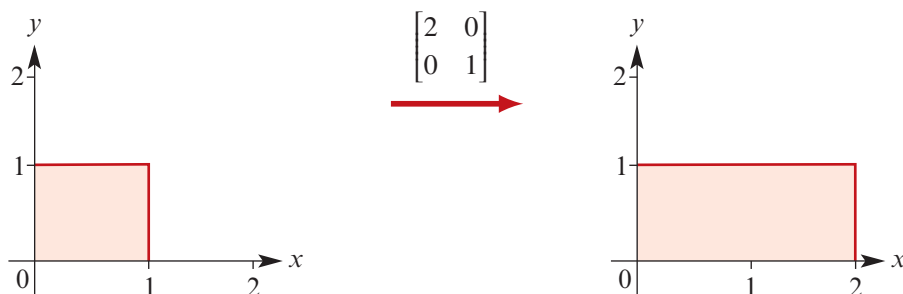
- 10 Rotate the graph of $y = \frac{1}{x}$ by 45° anticlockwise. Show that the equation of the image is $y^2 - x^2 = 2$.

Note: This shows that the two curves are congruent hyperbolas.

18G Area and determinant

If we apply a linear transformation to some region of the plane, then the area may change.

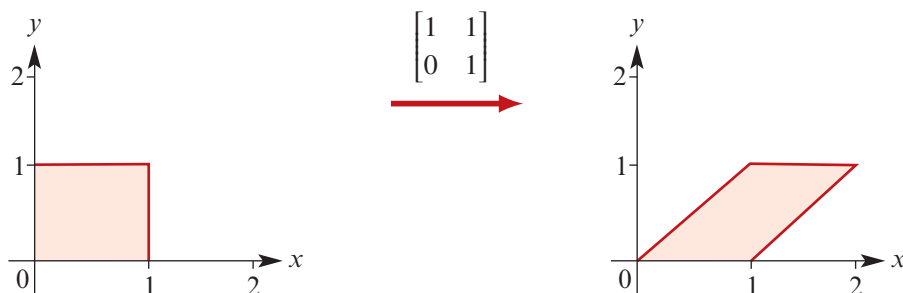
For example, if we dilate the unit square by a factor of 2 from the y -axis, then the area increases by a factor of 2.



Notice that this increase corresponds to the determinant of the transformation matrix:

$$\det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

On the other hand, if we shear the unit square by a factor of 1 parallel to the x -axis, then the area is unchanged.



Notice that the determinant of this transformation matrix is

$$\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

More generally, we can prove the following remarkable result.

If a region of the plane is transformed by matrix $\mathbf{B}$, then

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

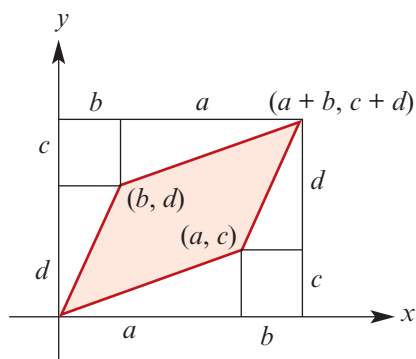
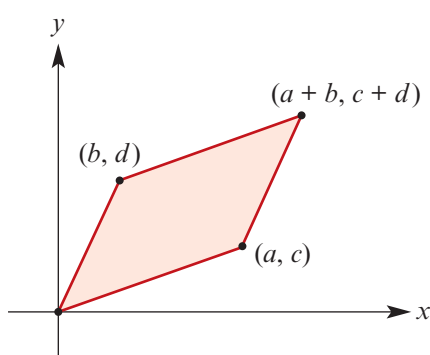
Proof We will prove the result when the unit square is transformed by matrix

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The result can be extended to other regions by approximating them by squares.

We will assume that a , b , c and d are all positive and that $\det(\mathbf{B}) > 0$. The proof can easily be adapted if we relax these assumptions.

The image of the unit square under transformation **B** is a parallelogram.



To find the area of the image, we draw a rectangle around it as shown, and subtract the area of the two small rectangles and four triangles from the total area:

$$\begin{aligned}
 \text{Area of image} &= (a+b)(c+d) - bc - bc - \frac{ac}{2} - \frac{ac}{2} - \frac{bd}{2} - \frac{bd}{2} \\
 &= (a+b)(c+d) - 2bc - ac - bd \\
 &= ac + ad + bc + bd - 2bc - ac - bd \\
 &= ad - bc
 \end{aligned}$$

This is equal to the determinant of matrix **B**.



Example 20

The triangular region with vertices $(1, 1)$, $(2, 1)$ and $(1, 2)$ is transformed by the rule $(x, y) \rightarrow (-x + 2y, 2x + y)$.

- Find the matrix of the linear transformation.
- On the same set of axes, sketch the region and its image.
- Find the area of the image.

Solution

a The matrix is given by $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

b The region is shown in blue and its image in red.

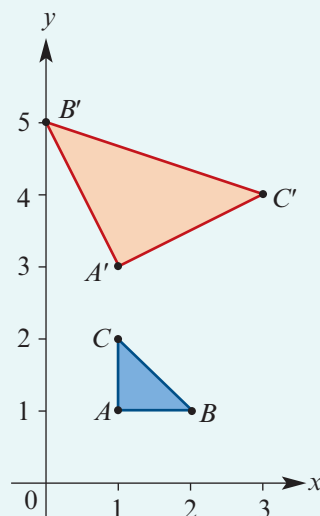
c The area of the original region is $\frac{1}{2}$.

The determinant of the transformation matrix is

$$\det \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = (-1) \times 1 - 2 \times 2 = -5$$

Therefore

$$\begin{aligned}
 \text{Area of image} &= |\det(\mathbf{B})| \times \text{Area of region} \\
 &= |-5| \times \frac{1}{2} = \frac{5}{2}
 \end{aligned}$$



Example 21

The unit square is mapped to a parallelogram of area 3 by the matrix

$$\mathbf{B} = \begin{bmatrix} m & 2 \\ m & m \end{bmatrix}$$

Find the possible values of m .

Solution

The original area is 1. Therefore

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

$$3 = |ad - bc| \times 1$$

$$3 = |m^2 - 2m|$$

Therefore either $m^2 - 2m = 3$ or $m^2 - 2m = -3$.

Case 1:

$$m^2 - 2m = 3$$

$$m^2 - 2m - 3 = 0$$

$$(m + 1)(m - 3) = 0$$

$$m = -1 \text{ or } m = 3$$

Case 2:

$$m^2 - 2m = -3$$

$$m^2 - 2m + 3 = 0$$

This quadratic equation has no solutions, since the discriminant is

$$\Delta = b^2 - 4ac$$

$$= (-2)^2 - 4(1)(3) = 4 - 12 < 0$$

The connection between area and determinant has many important applications. In the next example, we see how it can be used to find the area of an ellipse. Alternative approaches to finding this area are much more sophisticated.

Example 22

The circle with equation $x^2 + y^2 = 1$ is mapped to an ellipse by the rule $(x, y) \rightarrow (ax, by)$, where both a and b are positive.

- Find the equation of the ellipse and sketch its graph.
- Find the area of the ellipse.

Solution

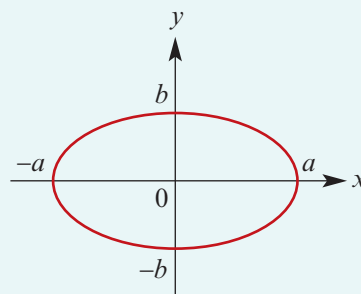
- a** We have $x' = ax$ and $y' = by$.

$$\text{This gives } x = \frac{x'}{a} \text{ and } y = \frac{y'}{b}.$$

The equation $x^2 + y^2 = 1$ becomes

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

Hence the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



- b** The area of the original circle of radius 1 is π .

The determinant of the transformation matrix is

$$\det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \times b - 0 \times 0 = ab$$

Therefore the area of the ellipse is πab .

Note: When $a = b = r$, this formula gives the area of a circle of radius r .

Section summary

If a region of the plane is transformed by matrix **B**, then

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

Exercise 18G

- 1** Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

a $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

b $\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$

c $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

d $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

Example 20

- 2** The matrix $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ maps the triangle with vertices (0, 1), (1, 1) and (0, 0) to a new triangle.

a Sketch the original triangle and its image.

b Find the areas of both triangles.

- 3** The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ maps the triangle with vertices (-1, 1), (1, 1) and (1, 0) to a new triangle.

a Sketch the original triangle and its image.

b Find the areas of both triangles.

Example 21

- 4** The matrix $\begin{bmatrix} m & 2 \\ -1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 6 square units. Find the value(s) of m .

- 5** The matrix $\begin{bmatrix} m & m \\ 1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 2 square units. Find the value(s) of m .

- 6 a** By evaluating a determinant, show that each of the following transformations will not change the area of any region:

- i** a shear of factor k parallel to the x -axis
- ii** an anticlockwise rotation about the origin by angle θ
- iii** a reflection in any straight line through the origin

Note: We say that each of these transformations **preserves area**.

- b** Let $k > 0$. A linear transformation has matrix $\begin{bmatrix} k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$.

- i** Describe the geometric effect of the transformation.
- ii** Show that this transformation preserves area.

- 7** For each $x \in \mathbb{R}$, the matrix $\begin{bmatrix} x & 1 \\ -2 & x+2 \end{bmatrix}$ maps the unit square to a parallelogram.

a Show that the area of the parallelogram is $(x+1)^2 + 1$.

b For what value of x is the area of the parallelogram a minimum?

- 8** For what values of m does the matrix $\begin{bmatrix} m & 2 \\ 3 & 4 \end{bmatrix}$ map the unit square to a parallelogram of area greater than 2?

- 9** Find all matrices that will map the unit square to rhombus of area $\frac{1}{2}$ with one vertex at $(0, 0)$ and another at $(1, 0)$.

- 10 a** Find a matrix that transforms the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ to the triangle with vertices $(0, 0)$, (a, c) and (b, d) .

b Hence show that the area of the triangle with vertices $(0, 0)$, (a, c) and (b, d) is given by the formula

$$A = \frac{1}{2}|ad - bc|$$

c Hence prove that if a , b , c and d are rational numbers, then the area of this triangle is rational.

d A **rational point** has coordinates (x, y) such that both x and y are rational numbers. Prove that no equilateral triangle can be drawn in the Cartesian plane so that all three of its vertices are rational points.

Hint: You can assume that the vertices of the triangle are $(0, 0)$, (a, c) and (b, d) .

Find another expression for the area of the triangle using Pythagoras' theorem.

You can also assume that $\sqrt{3}$ is irrational.

18H General transformations

Earlier in this chapter we considered rotations about the origin. But what if we want to rotate a figure about a point that is not the origin? In this section we will see how a more complicated transformation can be achieved by a sequence of simpler transformations.

► Rotation about the point (a, b)

If we want to rotate the plane about the point (a, b) by angle θ anticlockwise, we can do this in a sequence of three steps:

Step 1 Translate the plane so that the centre of rotation is now the origin, by adding $\begin{bmatrix} -a \\ -b \end{bmatrix}$.

Step 2 Rotate the plane through angle θ anticlockwise, by multiplying by $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.

Step 3 Translate the plane back to its original position, by adding $\begin{bmatrix} a \\ b \end{bmatrix}$.

Chaining these three transformations together gives the overall transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Example 23

- Find the transformation that rotates the plane by 90° anticlockwise about the point $(1, 1)$.
- Check your answer by showing that $(0, 1)$ is mapped to the correct point.

Solution

- We do this in a sequence of three steps, starting with the initial point (x, y) :

Initial point	Translate	Rotate 90° anticlockwise	Translate back
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This gives the overall transformation

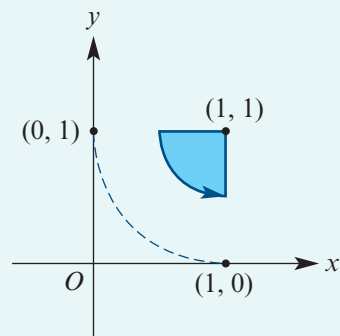
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -y + 2 \\ x \end{bmatrix}$$

- We check our answer by finding the image of $(0, 1)$.

Let $x = 0$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 + 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore $(0, 1) \rightarrow (1, 0)$, as expected.



► Reflection in the line $y = x \tan \theta + c$

To reflect the plane in a line $y = x \tan \theta + c$ that does not go through the origin, we can also do this in a sequence of three steps:

Step 1 Translate the plane so that the line passes through the origin, by adding $\begin{bmatrix} 0 \\ -c \end{bmatrix}$.

Step 2 Reflect the plane in the line $y = x \tan \theta$, by multiplying by $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Step 3 Translate the plane back to its original position, by adding $\begin{bmatrix} 0 \\ c \end{bmatrix}$.

Example 24

- a** Find the transformation that reflects the plane in the line $y = -x + 1$.
b Check your answer by finding the image of the point $(1, 1)$.

Solution

- a** We do this in a sequence of three steps, starting with the initial point (x, y) . The first step translates the line $y = -x + 1$ so that it passes through the origin.

Initial point	Translate	Reflect in line $y = -x$	Translate back
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x \\ y - 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y - 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y - 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

This gives the overall transformation

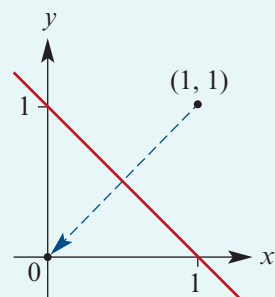
$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y - 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -y + 1 \\ -x + 1 \end{bmatrix} \end{aligned}$$

- b** We check our answer by finding the image of $(1, 1)$.

Let $x = 1$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 + 1 \\ -1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $(1, 1) \rightarrow (0, 0)$, as expected.



Section summary

More difficult transformations can be achieved by combining simpler transformations.

- To rotate the plane about the point (a, b) :
 - 1 Translate the plane so that the origin is the centre of rotation.
 - 2 Rotate the plane about the origin.
 - 3 Translate the plane back to its original position.
- To reflect the plane in the line $y = mx + c$:
 - 1 Translate the plane so that the line passes through the origin.
 - 2 Reflect the plane in the line $y = mx$.
 - 3 Translate the plane back to its original position.

Exercise 18H

Example 23

- 1 Find the transformation that rotates the plane by 90° clockwise about the point $(2, 2)$. Check your answer by showing that the point $(2, 1)$ is mapped to the correct point.
- 2 Find the transformation that rotates the plane by 180° anticlockwise about the point $(-1, 1)$. Check your answer by showing that the point $(-1, 0)$ is mapped to the correct point.

Example 24

- 3 Find the transformation that reflects the plane in each of the following lines. Check your answer by showing that the point $(0, 0)$ is mapped to the correct point.

a $y = x - 1$	b $y = -x - 1$
c $y = 1$	d $x = -2$
- 4 **a** Write down the matrix **A** for a rotation about the origin by angle θ clockwise.
b Write down the matrix **B** for a dilation of factor k from the x -axis.
c Write down the matrix **C** for a rotation about the origin by angle θ anticlockwise.
d Hence find the matrix that increases the perpendicular distance from the line $y = x \tan \theta$ by a factor of k .
- 5 Find the transformation matrix that projects the point (x, y) onto the line $y = x \tan \theta$.
Hint: First rotate the plane clockwise by angle θ .
- 6 Consider these two transformations:
 - T_1 : A reflection in the line $y = x + 1$.
 - T_2 : A reflection in the line $y = x$.
 Show that T_1 followed by T_2 is a translation.



Chapter summary



- A **linear transformation** is defined by a rule of the form $(x, y) \rightarrow (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The point (x', y') is called the **image** of the point (x, y) .

- The matrix of a composition of two linear transformations can be found by multiplying the two transformation matrices in the correct order.
- If **A** is the matrix of a linear transformation, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.
- If a region of the plane is transformed by matrix **B**, then

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

- Difficult transformations can be achieved by combining simpler transformations.

Transformation	Matrix	Transformation	Matrix
Reflection in the x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y -axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y -axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x -axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y -axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y -axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Rotation by θ anticlockwise	$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$	Reflection in the line $y = x \tan \theta$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Technology-free questions

- The rule for a transformation is $(x, y) \rightarrow (2x + y, -x + 2y)$.
 - Find the image of the point $(2, 3)$.
 - Find the matrix of this transformation.
 - Sketch the image of the unit square and find its area.
 - Find the rule for the inverse transformation.

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- 2** Find the matrix corresponding to each of the following linear transformations:
- a** reflection in the y -axis **b** dilation of factor 5 from the x -axis
c shear of factor -3 parallel to the x -axis **d** projection onto the x -axis
e rotation by 30° anticlockwise **f** reflection in the line $y = x$
- 3** **a** Find the matrix that will reflect the plane in the line $y = 3x$.
b Find the image of the point $(2, 4)$ under this transformation.
- 4** Find the transformation matrix that corresponds to:
- a** a reflection in the x -axis and then a reflection in the line $y = -x$
b a rotation about the origin by 90° anticlockwise and then a dilation of factor 2 from the x -axis
c a reflection in the line $y = x$ and then a shear of factor 2 parallel to the y -axis.
- 5** **a** Find the rule for the transformation that will reflect (x, y) in the x -axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.
b Find the rule for the transformation if the translation takes place before the reflection.
- 6** **a** Write down the matrix for a shear of factor k parallel to the y -axis.
b Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the y -axis.
- 7** Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.
- a** $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$
- 8** **a** Find the rule for the transformation that rotates the plane about the point $(1, -1)$ by 90° anticlockwise. (*Hint:* Translate the point $(1, -1)$ to the origin, rotate the plane, and then translate the point back to its original position.)
b Find the image of the point $(2, -1)$ under this transformation.
c Sketch the unit square and its image under this transformation.

Multiple-choice questions



- 1** The image of the point $(2, -1)$ under the transformation $(x, y) \rightarrow (2x - 3y, -x + 4y)$ is
A $(1, -6)$ **B** $(7, -6)$ **C** $(7, 6)$ **D** $(7, 2)$ **E** $(1, 2)$
- 2** The matrix that will reflect the plane in the line $y = -x$ is
A $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- 3** The matrix that corresponds to a dilation of factor 2 from the y -axis followed by a reflection in the x -axis is

A $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

- 4** The matrix $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ will

- A** rotate the plane by 30° **B** rotate the plane by -30°
C rotate the plane by -60° **D** reflect the plane in the line $y = x \tan 30^\circ$
E reflect the plane in the line $y = x \tan 60^\circ$

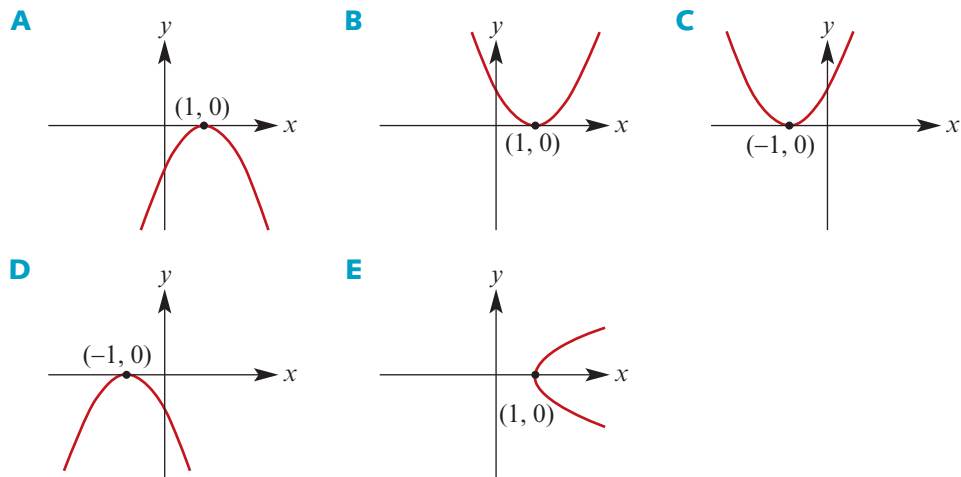
- 5** The transformation that translates (x, y) by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and then rotates the result about the origin by 90° anticlockwise is given by

A $(x, y) \rightarrow (y - 3, x + 2)$ **B** $(x, y) \rightarrow (-y + 3, -x - 2)$
C $(x, y) \rightarrow (-y + 3, x + 2)$ **D** $(x, y) \rightarrow (-x - 2, -y + 3)$
E $(x, y) \rightarrow (x + 2, -y - 3)$

- 6** The matrix $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ corresponds to

- A** a rotation by 180° and then a dilation of factor 2 from the x -axis
B a rotation by 90° clockwise and then a dilation of factor 2 from the x -axis
C a rotation by 180° and then a dilation of factor 2 from the y -axis
D a reflection in the y -axis and then a dilation of factor 2 from the y -axis
E a reflection in the x -axis and then a dilation of factor 2 from the x -axis

- 7** Which of the following shows the image of the graph of $y = (x - 1)^2$ under the linear transformation with matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$?

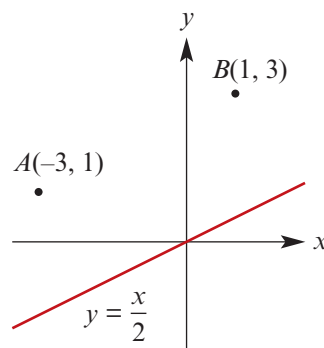


- 8 Which of these matrices maps the unit square to a parallelogram of area 2 square units?
- A** $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$
- 9 The matrix **R** will rotate the plane through angle 40° . The smallest value of m such that $\mathbf{R}^m = \mathbf{I}$, where **I** is the identity matrix, is
- A** 6 **B** 7 **C** 8 **D** 9 **E** 10

Extended-response questions

- 1 **a** Find the matrix that will rotate the plane by 45° anticlockwise.
b Find the matrix that will rotate the plane by 30° anticlockwise.
c Hence find the matrix that will rotate the plane by 75° anticlockwise.
d Hence deduce exact values for $\cos 75^\circ$ and $\sin 75^\circ$.
- 2 The triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$ is transformed by the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.
a Sketch the triangle and its image on the same set of axes.
b Find the area of the triangle and its image.
c The image of the triangle is revolved around the y -axis to create a three-dimensional solid. Find the volume of this solid.
- 3 Consider the transformation with rule $(x, y) \rightarrow (x + y, y)$.
a Write down the matrix of this transformation.
b What name is given to this type of transformation?
c Find the images of the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ under this transformation.
d Hence sketch the graph of $y = x^2$ and its image under this transformation.
- 4 A square with vertices $(\pm 1, \pm 1)$ is rotated about the origin by 45° anticlockwise.
a Find the coordinates of the vertices of its image.
b Sketch the square and its image on the same set of axes.
c When these two squares are combined, the resulting figure is called a Star of Lakshmi. Find its area.
- 5 In this chapter we investigated two important transformation matrices. These were the rotation and reflection matrices, which we will now denote by
- $$\text{Rot}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{and} \quad \text{Ref}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$
- a** Using matrix multiplication and an application of trigonometric identities, prove the following four matrix equations:
- i** $\text{Rot}(\theta)\text{Rot}(\varphi) = \text{Rot}(\theta + \varphi)$ **ii** $\text{Ref}(\theta)\text{Ref}(\varphi) = \text{Rot}(2\theta - 2\varphi)$
iii $\text{Rot}(\theta)\text{Ref}(\varphi) = \text{Ref}(\varphi + \frac{1}{2}\theta)$ **iv** $\text{Ref}(\theta)\text{Rot}(\varphi) = \text{Ref}(\varphi - \frac{1}{2}\theta)$
- b** Explain in words what each of the above four equations shows.
- c** Using these identities, find the matrix $\text{Rot}(60^\circ)\text{Ref}(60^\circ)\text{Ref}(60^\circ)\text{Rot}(60^\circ)$.

- 6** An ant is at point $A(-3, 1)$. His friend is at point $B(1, 3)$. The ant wants to walk from A to B , but first wants to visit the straight line $y = \frac{1}{2}x$. Being an economical ant, he wants the total length of his path to be as short as possible.



- Find the matrix that will reflect the plane in the line $y = \frac{1}{2}x$.
- Find the image A' of the point A when reflected in the line $y = \frac{1}{2}x$.
- Find the distance from point A' to point B .
- The straight line $A'B$ intersects the line $y = \frac{1}{2}x$ at the point C . What type of triangle is ACA' ?
- Suppose that D is any other point on the line $y = \frac{1}{2}x$. Show that

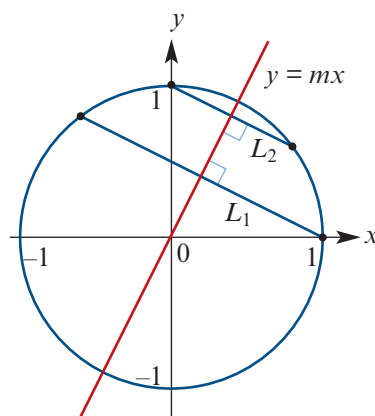
$$AD + DB \geq AC + CB$$
- Hence find the shortest possible distance travelled by the ant.

- 7** A rectangle R_1 has vertices $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b) , where a and b are positive real numbers.
- Sketch the rectangle R_1 .
 - The rectangle R_1 is rotated about the origin by angle θ anticlockwise, where $0^\circ \leq \theta \leq 90^\circ$. The image is another rectangle R_2 . Find the coordinates of the vertices of rectangle R_2 in terms of a , b and θ .
 - The vertices of R_2 lie on another rectangle R_3 that has edges parallel to the coordinate axes. Show that the area of rectangle R_3 is

$$A = \frac{1}{2}(a^2 + b^2) \sin(2\theta) + ab$$

- Hence show that the maximum area of rectangle R_3 is $\frac{1}{2}(a + b)^2$, which occurs when $\theta = 45^\circ$.

- 8** The graphs of the unit circle $x^2 + y^2 = 1$ and the line $y = mx$ are shown. Lines L_1 and L_2 are perpendicular to the line $y = mx$ and go through the points $(1, 0)$ and $(0, 1)$ respectively.



- Find the equation of the line L_1 , and find where it intersects the unit circle in terms of m .
- Find the equation of the line L_2 , and find where it intersects the unit circle in terms of m .
- Hence deduce the formula for the matrix that reflects the point (x, y) in the line $y = mx$.

Hint: Recall that the columns of the matrix will be the images of the standard unit vectors.

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Revision of Chapters 17–18

19A Technology-free questions

- 1** Let $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & -1 & -2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$.
- a** State whether or not each of the following products exists: $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{CD}$, $\mathbf{BE}$
b Evaluate $\mathbf{DA}$ and $\mathbf{A}^{-1}$.
- 2** If $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix}$, find:
- a** $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **b** $\mathbf{A}^2 - \mathbf{B}^2$
- 3** If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ is non-invertible, find the value of x .
- 4** Find all possible matrices $\mathbf{A}$ which satisfy the equation $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.
- 5** If $\mathbf{A} = \begin{bmatrix} -1 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -4 \\ -1 & -6 \\ -3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$, evaluate $\mathbf{AB}$ and $\mathbf{C}^{-1}$.
- 6** Suppose that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{X}$ are $n \times n$ matrices and that both $\mathbf{A}$ and $\mathbf{B}$ are invertible. Denote the $n \times n$ zero matrix by $\mathbf{O}$ and the $n \times n$ identity matrix by $\mathbf{I}$. Solve each of the following matrix equations for $\mathbf{X}$:
- a** $\mathbf{B} + \mathbf{XA} = \mathbf{C}$ **b** $\mathbf{B}(\mathbf{X} + \mathbf{A}) = \mathbf{C}$ **c** $\mathbf{AX} + \mathbf{BA} = \mathbf{A}$
d $\mathbf{X} + \mathbf{A} = \mathbf{O}$ **e** $2\mathbf{X} - \mathbf{B} = \mathbf{O}$ **f** $\mathbf{AX} + \mathbf{I} = \mathbf{A}$

- 7** Assume that **A** and **B** are invertible $n \times n$ matrices. Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- 8** A transformation has rule $(x, y) \rightarrow (2x + y, -x - 2y)$.
- Find the image of the point $(2, 3)$.
 - Find the matrix of this transformation.
 - Sketch the image of the unit square and find its area.
 - Find the rule for the inverse transformation.
- 9** Find the matrix corresponding to each of the following linear transformations:
- reflection in the x -axis
 - dilation of factor 3 from the y -axis
 - shear of factor 2 parallel to the y -axis
 - projection onto the y -axis
 - rotation by 45° anticlockwise
 - rotation by 30° clockwise
 - reflection in the line $y = -x$
 - reflection in the line $y = x \tan 30^\circ$
- 10** **a** Find the matrix that will reflect the plane in the line $y = 4x$.
b Find the image of the point $(2, 4)$ under this transformation.
- 11** Find the transformation matrix that corresponds to:
- a reflection in the y -axis and then a dilation of factor 2 from the x -axis
 - a rotation by 90° anticlockwise and then a reflection in the line $y = x$
 - a reflection in the line $y = -x$ and then a shear of factor 2 parallel to the x -axis
- 12** **a** Find the rule for the transformation that will reflect (x, y) in the y -axis then translate the result by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
b Find the rule for the transformation if the translation takes place before the reflection.
- 13** Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.
- $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$
 - $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$
- 14** **a** Find the rule for the transformation that will reflect the plane in the line $y = x - 1$.
Hint: Translate the plane 1 unit in the y -direction, reflect in the line $y = x$, and then translate the plane back to its original position.
b Find the image of the point $(0, 0)$ under this transformation.
c Sketch the unit square and its image under this transformation.

19B Multiple-choice questions

- 1 If $\mathbf{P}^2 = 4\mathbf{I}$, then $\mathbf{P}^{-1}$ equals
A $\frac{1}{4}\mathbf{P}$ **B** $\frac{1}{2}\mathbf{P}$ **C** $\frac{1}{2}\mathbf{I}$ **D** $2\mathbf{P}$ **E** $4\mathbf{P}$
- 2 If $\mathbf{R} = \begin{bmatrix} 5 & 3 & 1 \end{bmatrix}$ and $\mathbf{S} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, then $\mathbf{RS}$ is
A undefined **B** $\begin{bmatrix} -1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 \\ -5 & -3 & -1 \\ 10 & 6 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -3 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$
- 3 If $\mathbf{A} = \begin{bmatrix} 9 & 8 \\ -11 & 5 \end{bmatrix}$, then $\det(\mathbf{A})$ equals
A -43 **B** $-\frac{1}{43}$ **C** $\frac{1}{333}$ **D** 17 **E** 133
- 4 If $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 6 & 4 \end{bmatrix}$, then $\mathbf{BA}$ has size
A 1×1 **B** 3×1 **C** 1×3 **D** 3×3 **E** 3×2
- 5 Let $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$. If $\mathbf{AX} + \mathbf{B} = \mathbf{C}$, then $\mathbf{X}$ equals
A $\frac{1}{20} \begin{bmatrix} -2 & 19 \\ -2 & 6 \end{bmatrix}$ **B** $\begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} -2 & 19 \\ -2 & 6 \end{bmatrix}$
D $\begin{bmatrix} 3 & -10 \\ -4 & 10 \end{bmatrix}$ **E** $\frac{1}{20} \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$
- 6 Let $\mathbf{P} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$.
If $\mathbf{X} = \mathbf{PQR}$, then the number of zero entries of $\mathbf{X}$ is
A 0 **B** 1 **C** 2 **D** 3 **E** 4
- 7 If $\mathbf{X} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$, then $\mathbf{X}^{-1}$ is
A $\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} \frac{1}{3} & \frac{1}{5} \\ -1 & -\frac{1}{2} \end{bmatrix}$ **D** $\begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & -1 \\ -5 & -2 \end{bmatrix}$

- 8** The determinant of the matrix $\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$ is
A 16 **B** 4 **C** -16 **D** $\frac{1}{4}$ **E** -4
- 9** If $S = \begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$, then S^{-1} is
A $-\begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & -7 \\ -2 & 5 \end{bmatrix}$ **C** $-\frac{1}{4}\begin{bmatrix} -2 & 7 \\ 2 & -5 \end{bmatrix}$
D $\frac{1}{4}\begin{bmatrix} -2 & 7 \\ 2 & -5 \end{bmatrix}$ **E** $\frac{1}{4}\begin{bmatrix} -2 & -7 \\ -2 & -5 \end{bmatrix}$
- 10** The point $(5, -2)$ is reflected in the line $y = x$. The coordinates of its image are
A $(5, -2)$ **B** $(-5, 2)$ **C** $(2, -5)$ **D** $(-2, 5)$ **E** $(-5, -2)$
- 11** The point $(2, -6)$ is reflected in the line $y = -x$. The coordinates of its image are
A $(2, -6)$ **B** $(-2, 6)$ **C** $(6, -2)$ **D** $(-6, 2)$ **E** $(-2, -6)$
- 12** Let P be the point $(5, -4)$. After a translation by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and then a reflection in the line $y = 1$, the coordinates of the image of P are
A $(7, 9)$ **B** $(7, 9)$ **C** $(-5, -7)$ **D** $(7, 11)$ **E** $(7, 10)$
- 13** The point (a, b) is reflected in the line with equation $x = m$. The image has coordinates
A $(2m - a, b)$ **B** $(a, 2m - b)$ **C** $(a - m, b)$ **D** $(a, b - m)$ **E** $(2m + a, b)$
- 14** The image of the line $\{(x, y) : x + y = 4\}$ under a dilation of factor $\frac{1}{2}$ from the y -axis followed by a reflection in the line $x = 4$ is
A $\{(x, y) : y = 2x\}$ **B** $\{(x, y) : y + 2 = 0\}$ **C** $\{(x, y) : y + 2x - 16 = 0\}$
D $\{(x, y) : x + y = 0\}$ **E** $\{(x, y) : y = 2x - 12\}$
- 15** The image of $\{(x, y) : y = x^2\}$ under a translation by the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ followed by a reflection in the x -axis is
A $\{(x, y) : y = (x - 3)^2 + 2\}$ **B** $\{(x, y) : -(x - 3)^2 = y + 2\}$
C $\{(x, y) : y = (x + 3)^2 + 2\}$ **D** $\{(x, y) : -y + 2 = (x - 3)^2\}$
E none of these
- 16** Consider these two transformations:
 ■ T_1 : A reflection in the line with equation $x = 2$.
 ■ T_2 : A translation by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
 The rule for T_1 followed by T_2 is given by
A $(x, y) \rightarrow (2 - x, y + 3)$ **B** $(x, y) \rightarrow (-x, y + 3)$ **C** $(x, y) \rightarrow (x + 2, y + 3)$
D $(x, y) \rightarrow (6 - x, y + 3)$ **E** none of these

- 17** The image of the graph of $y = 2^x$ under a dilation of factor 2 from the x -axis followed by a dilation of factor $\frac{1}{3}$ from the y -axis has the equation

A $y = \frac{1}{3} \times 2^{3x}$ **B** $y = 3 \times 2^{\frac{x}{2}}$ **C** $y = 2 \times 2^{3x}$ **D** $y = 2 \times 2^{\frac{x}{3}}$ **E** none of these

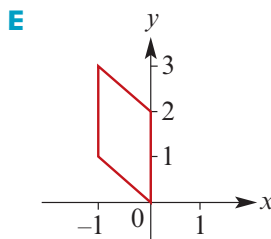
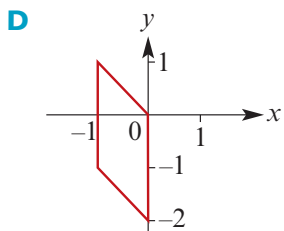
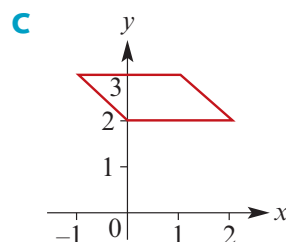
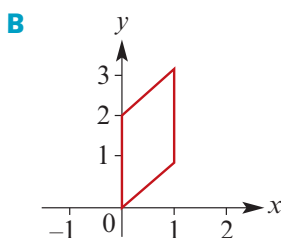
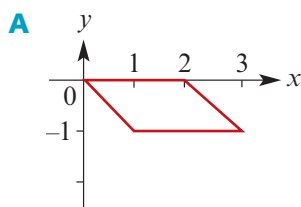
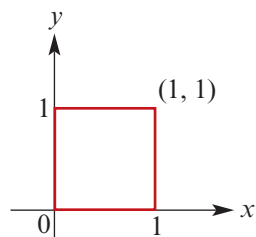
- 18** A transformation has rule $(x, y) \rightarrow (4x + 3y, 5x + 4y)$. The rule for the inverse transformation is

A $(x, y) \rightarrow (3x + 4y, 5x + 4y)$ **B** $(x, y) \rightarrow (3x - 4y, 5x - 4y)$
C $(x, y) \rightarrow (4x + 3y, 5x + 4y)$ **D** $(x, y) \rightarrow (4x - 3y, -5x + 4y)$
E $(x, y) \rightarrow (-4x + 3y, 5x - 4y)$

- 19** The unit square is subject to two successive linear transformations:

- the first transformation has matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- the second transformation has matrix $\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$.

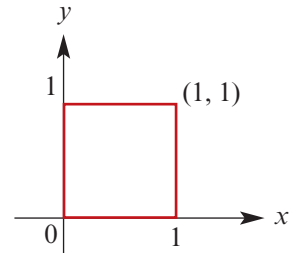
Which one of the following shows the image of the unit square under these two transformations?



- 20** Transformation T rotates the plane about the origin by 35° clockwise. Transformation S rotates the plane about the origin by 15° anticlockwise. The matrix of T followed by S is

A $\begin{bmatrix} \cos 50^\circ & -\sin 50^\circ \\ \sin 50^\circ & \cos 50^\circ \end{bmatrix}$ **B** $\begin{bmatrix} \cos 20^\circ & -\sin 20^\circ \\ \sin 20^\circ & \cos 20^\circ \end{bmatrix}$ **C** $\begin{bmatrix} \cos 50^\circ & \sin 50^\circ \\ -\sin 50^\circ & \cos 50^\circ \end{bmatrix}$
D $\begin{bmatrix} \cos 20^\circ & \sin 20^\circ \\ -\sin 20^\circ & \cos 20^\circ \end{bmatrix}$ **E** $\begin{bmatrix} \cos 50^\circ & \sin 50^\circ \\ \sin 50^\circ & -\cos 50^\circ \end{bmatrix}$

- 21** The linear transformation determined by the matrix $\begin{bmatrix} \cos 40^\circ & -\sin 40^\circ \\ \sin 40^\circ & \cos 40^\circ \end{bmatrix}$ is
- A** clockwise rotation by 40° **B** anticlockwise rotation by 40°
C reflection in the line $y = x \tan 40^\circ$ **D** reflection in the line $y = x \tan 20^\circ$
E anticlockwise rotation by 20°
- 22** The unit square is transformed by the linear transformation with matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. The area of the image is
- A** 1 **B** 2 **C** 3 **D** 4 **E** 5



19C Extended-response questions

- 1** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $b \neq 0$ and $c \neq 0$.

a i Find $\mathbf{A}^2$. **ii** Find $3\mathbf{A}$.

b If $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$, show that:

i $a + d = 3$ **ii** $\det(\mathbf{A}) = 1$

c Assume that $\mathbf{A}$ has the properties:

■ $a + d = 3$ ■ $\det(\mathbf{A}) = 1$

Show that $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$.

- 2** The coordinates of A , B and C are $(-4, 1)$, $(-2, 1)$ and $(-2, 5)$ respectively.

a Find the rule of the transformation that maps triangle ABC to triangle $A'B'C'$.

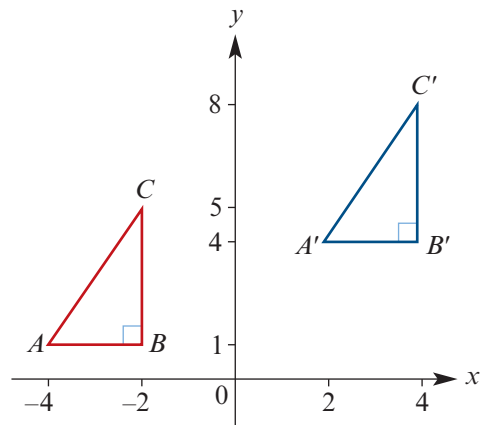
b On graph paper, draw triangle ABC and its image under a reflection in the x -axis.

c On the same set of axes, draw the image of ABC under a dilation of factor 2 from the y -axis.

d Find the image of the parabola $y = x^2$ under a dilation of factor 2 from the x -axis followed by a translation by the vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

e Find the rule for the transformation that maps the graph of $y = x^2$ to the graph of $y = -2(x - 3)^2 + 4$.

f Let $f(x) = x^3 - 2x$. Use a calculator to help sketch the graph of $y = 3f(x - 2) + 4$.



CF

- 3** Let transformation D be a dilation of factor 4 from the y -axis.
- a** Find the image of the point $(1, 1)$ under dilation D .
 - b**
 - i** Describe the image of the square with vertices $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, $E(1, 0)$ under the dilation D .
 - ii** Find the area of the square $ABCE$.
 - iii** Find the area of the image of $ABCE$.
 - iv** If the dilation had been of factor k , what would be the area of the image?
 - c** State the rule for the dilation D .
 - d**
 - i** Find the equation of the image of the graph of $y = x^2$ under the dilation D .
 - ii** Find the equation of the image of the graph of $y = x^2$ under the dilation D followed by the translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
 - iii** Sketch the graph of $y = x^2$ and its image defined in **d ii** on the one set of axes. State the coordinates of the vertex and the axis intercepts of the image.
 - e** State the rule for the transformation that maps the graph of $y = 5(x + 2)^2 - 3$ to the graph of $y = x^2$.

- 4** A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

- a** Show that this transformation is a rotation.
 - b** Let C be the circle that passes through the origin and has its centre at $(0, 1)$.
 - i** Find the equation of C .
 - ii** Find the equation of C' , the image of C under the transformation defined by $\mathbf{M}$.
 - c** Find the coordinates of the points of intersection of C and C' .
- 5** A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

- a** Find the image of the point $(-2, 5)$ under this transformation.
- b** Find the inverse of $\mathbf{M}$.
- c** Given that the point $(11, 13)$ is the image of the point (a, b) , find the values of a and b .
- d** Find the coordinates of the image of the point (a, a) in terms of a .
- e** Given that $a \neq 0$ and $b \neq 0$ with

$$\mathbf{M} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix}$$

find the possible values of λ and the relationship between a and b for each of these values.

- 6** Let $\mathbf{R}$ be the transformation matrix for a rotation about the origin by $\frac{\pi}{4}$ anticlockwise.
- a** Give the 2×2 matrix $\mathbf{R}$.
 - b** Find the inverse of this matrix.
 - c** If the image of (a, b) is $(1, 1)$, find the values of a and b .
 - d** If the image of (c, d) is $(1, 2)$, find the values of c and d .
 - e**
 - i** If $(x, y) \rightarrow (x', y')$ under this transformation, use the result of **b** to find x and y in terms of x' and y' .
 - ii** Find the image of $y = x^2$ under this transformation.

- 7** Consider lines $y = x$ and $y = 2x$.
- a** Sketch these two lines on the same set of axes.
 - b** The acute angle between the two lines, θ radians, can be written in the form $\theta = \tan^{-1}(a) - b$. What are the values of a and b ?
 - c** Hence find a rotation matrix that will rotate the line $y = x$ to the line $y = 2x$. You will need to use the angle difference identities for sine and cosine.

- 8** Let M be the transformation that reflects the plane in the line $y = x$.
- a**
 - i** Find the image of the point $A(1, 3)$ under this transformation.
 - ii** The image of the triangle with vertices $A(1, 3)$, $B(1, 5)$ and $C(3, 3)$ is another triangle. Find the coordinates of the vertices of the image.
 - iii** Sketch triangle ABC and its image on a set of axes, with both axes from -5 to 5 .
 - b**
 - i** Show that the equation of the image of the graph of $y = x^2 - 2$ under the transformation M is $x = y^2 - 2$.
 - ii** Find the coordinates of the points of intersection of $y = x^2 - 2$ and the line $y = x$.
 - iii** Show that the x -coordinates of the points of intersection of $y = x^2 - 2$ and its image may be determined by the equation $x^4 - 4x^2 - x + 2 = 0$.
 - iv** Two solutions of the equation $x^4 - 4x^2 - x + 2 = 0$ are

$$x = \frac{1}{2}(-1 + \sqrt{5}) \quad \text{and} \quad x = \frac{1}{2}(-1 - \sqrt{5})$$

Use this result and the result of **b ii** to find the coordinates of the points of intersection of $y = x^2 - 2$ and its image under M .

- 9** The **trace** of a square matrix $\mathbf{A}$ is defined to be the sum of the entries along the main diagonal of $\mathbf{A}$ (from top-left to bottom-right) and is denoted by $\text{Tr}(\mathbf{A})$.

For example, if $\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 2 & 2 \end{bmatrix}$, then $\text{Tr}(\mathbf{A}) = 6 + 2 = 8$.

- a** Prove each of the following for all 2×2 matrices $\mathbf{X}$ and $\mathbf{Y}$:
 - i** $\text{Tr}(\mathbf{X} + \mathbf{Y}) = \text{Tr}(\mathbf{X}) + \text{Tr}(\mathbf{Y})$
 - ii** $\text{Tr}(-\mathbf{X}) = -\text{Tr}(\mathbf{X})$
 - iii** $\text{Tr}(\mathbf{XY}) = \text{Tr}(\mathbf{YX})$
- b** Use the results of **a** to show that $\mathbf{XY} - \mathbf{YX} \neq \mathbf{I}$ for all 2×2 matrices $\mathbf{X}$ and $\mathbf{Y}$.

19D Problem-solving and modelling questions

- 1** A square matrix $\mathbf{A}$ is said to be **idempotent** if $\mathbf{A}^2 = \mathbf{A}$.
- a** Show that each of the following 2×2 matrices is idempotent:
- i** $\begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$ **iv** $\frac{1}{2} \begin{bmatrix} 1 - \cos \theta & \sin \theta \\ \sin \theta & 1 + \cos \theta \end{bmatrix}$
- b** Show that each of the following 3×3 matrices is idempotent:
- i** $\frac{1}{2} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$ **ii** $\frac{1}{6} \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$
- c** Show that a product of two idempotent matrices is not necessarily idempotent.
- d** Show that for every idempotent matrix $\mathbf{A}$, either $\det(\mathbf{A}) = 0$ or $\det(\mathbf{A}) = 1$.
Note: You can use the result that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.
- e** Show that if an idempotent matrix $\mathbf{A}$ has an inverse, then $\mathbf{A} = \mathbf{I}$.
- f** Show that for every idempotent matrix $\mathbf{A}$, the matrix $\mathbf{I} - \mathbf{A}$ is also idempotent.
- g** Describe all 2×2 idempotent matrices.

- 2** The Fibonacci sequence 1, 1, 2, 3, 5, 8, ... is defined by the recurrence relation

$$f_{n+2} = f_n + f_{n+1} \quad \text{and} \quad f_1 = f_2 = 1$$

a Let $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Find $\mathbf{Q}^2$, $\mathbf{Q}^3$ and $\mathbf{Q}^4$. Deduce the entries of $\mathbf{Q}^n$, for $n \geq 2$. If you have studied mathematical induction, then you can prove your claim.

- b** Find $\det(\mathbf{Q})$, $\det(\mathbf{Q}^2)$, $\det(\mathbf{Q}^3)$ and $\det(\mathbf{Q}^4)$. Use these to deduce the result

$$f_{n+1}f_{n-1} - (f_n)^2 = (-1)^n$$

Note: You can use the result that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

- c** From the equation $\mathbf{Q}^{n+1}\mathbf{Q}^n = \mathbf{Q}^{2n+1}$, prove that $(f_{n+1})^2 + (f_n)^2 = f_{2n+1}$.
- d** From the equation $\mathbf{Q}^m\mathbf{Q}^{n-1} = \mathbf{Q}^{m+n-1}$, prove that $f_{m+n} = f_{m+1}f_n + f_m f_{n-1}$.
- e** Solve the equation $\det(\mathbf{Q} - x\mathbf{I}) = 0$ for x .

- 3** A square matrix $\mathbf{A}$ is said to be **involutory** if $\mathbf{A}^2 = \mathbf{I}$, that is, if it is its own inverse.

- a** Show that each of the following matrices is involutory:

i $\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$ **ii** $\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix}$

- b** Show that for every involutory matrix $\mathbf{A}$, either $\det(\mathbf{A}) = 1$ or $\det(\mathbf{A}) = -1$.
- c** Describe all 2×2 involutory matrices.
- d** Let $\mathbf{A}$ be a square matrix. Prove that $\mathbf{A}$ is involutory if and only if the matrix $\frac{1}{2}(\mathbf{A} + \mathbf{I})$ is idempotent. (See Question 1 for the definition of idempotent.)

19E Problem-solving and modelling investigations

For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

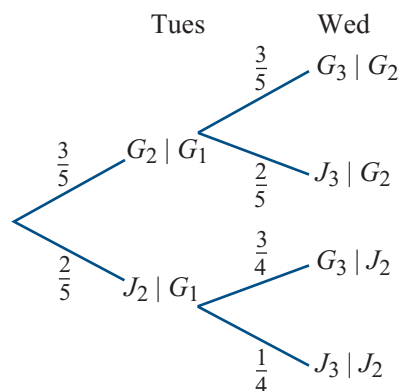
- 1 Transition matrices** Olivia drinks either green tea or jasmine tea every day. If she drinks green tea one day, then she drinks jasmine tea the next day with probability $\frac{2}{5}$. If she drinks jasmine tea one day, she drinks green tea the next day with probability $\frac{3}{4}$.

- a** Olivia drinks green tea on Monday (day 1).

Let G_n be the event ‘Green tea on day n ’.

Let J_n be the event ‘Jasmine tea on day n ’.

Using the tree diagram, find $\Pr(G_3)$. That is, find the probability that Olivia will drink green tea on Wednesday.



We can represent the probabilities in this example using a **transition matrix**, $\mathbf{T}$, and a sequence of **state vectors**, $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \dots$, defined as follows:

$$\mathbf{T} = \begin{bmatrix} \Pr(G_n | G_{n-1}) & \Pr(G_n | J_{n-1}) \\ \Pr(J_n | G_{n-1}) & \Pr(J_n | J_{n-1}) \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad \mathbf{S}_n = \begin{bmatrix} \Pr(G_n) \\ \Pr(J_n) \end{bmatrix}$$

- b i** Explain why $\mathbf{S}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **ii** Explain why $\mathbf{S}_n = \mathbf{T}\mathbf{S}_{n-1}$ for all $n \geq 2$.
- c** Use part **b** to find $\mathbf{S}_2$ and $\mathbf{S}_3$. Check against your answer to part **a**.
- d** Explain why $\mathbf{S}_n = \mathbf{T}^{n-1}\mathbf{S}_1$ for all $n \geq 2$.
- e** Use your calculator to find $\mathbf{S}_{20}$. Hence find the probability that Olivia will drink green tea on day 20.
- f** Use your calculator to find $\mathbf{S}_{200}$. Hence find the probability that Olivia will drink green tea on day 200.
- g** Find $\mathbf{S} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that $a + b = 1$ and $\mathbf{S} = \mathbf{T}\mathbf{S}$. Compare with your answer for part **f**.

Now consider another example. A computer system operates in two different modes. Every hour, it remains in the same mode or switches to the other mode, according to the following transition matrix:

$$\mathbf{T} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

- h** If the system is in mode 1 at 5:30 p.m., what is the probability that it will be in mode 1 at 8:30 p.m. on the same day?
- i** Construct similar examples based on switching between two alternatives. Investigate what happens to the state vector $\mathbf{S}_n$ as n gets larger and larger.

- 2 Matrix powers and walks** The graph G on the right represents towns (shown as vertices) and the routes between the towns (shown as edges).

For example, there are two direct routes from v_1 to v_3 . Each of these is a **walk of length 1**. You can also go from v_1 to v_3 via v_2 . This is a **walk of length 2**.

We start by counting the number of different walks of length 2 from v_1 to v_1 .

- You can go from v_1 to v_3 in two ways, and then from v_3 to v_1 in two ways. Therefore there are four walks from v_1 to v_1 via v_3 .
- Alternatively, you can go from v_1 to v_2 to v_1 .
- In total, there are five walks of length 2 from v_1 to v_1 .

The **adjacency matrix**, $\mathbf{A}$, describes the graph G by giving the number of edges between each pair of vertices:

$$\mathbf{A} = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix} \quad \text{and} \quad \mathbf{A}^2 = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{pmatrix} 5 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 6 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix} \end{matrix}$$

You can see that, when you multiply the first row of $\mathbf{A}$ by the first column of $\mathbf{A}$, you get

$$0 \times 0 + 1 \times 1 + 2 \times 2 + 0 \times 0 = 5$$

This replicates our calculation above for the number of walks of length 2 from v_1 to v_1 . The entries of $\mathbf{A}^2$ give the number of walks of length 2 between each pair of vertices. More generally:

If $\mathbf{A}$ is the adjacency matrix of a graph G , then the entry of $\mathbf{A}^n$ in row i and column j gives the number of walks of length n from vertex v_i to vertex v_j .

- a** Give the adjacency matrix of the graph H on the right.

- b** Find the number of walks of length 2 between:

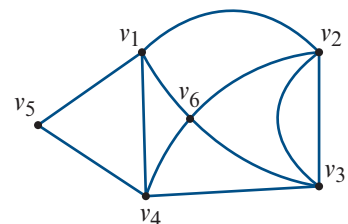
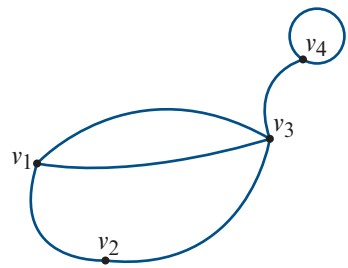
i v_1, v_2 **ii** v_3, v_4 **iii** v_1, v_1 **iv** v_5, v_6

- c** Find the number of walks of length 3 between:

i v_1, v_2 **ii** v_3, v_4 **iii** v_1, v_1 **iv** v_5, v_6

- d** A **triangle** in a graph is a cycle of three edges that join three vertices. For example, in the graph H shown above, there are two triangles with vertices v_2, v_3 and v_6 .

- i** List all the triangles in the graph H .
- ii** Draw a graph with five vertices and five edges that has one triangle.
- iii** Draw a graph with five vertices and five edges that has no triangles.
- iv** Assume that a graph G has no loops (that is, no edges joining a vertex to itself). How could you determine the number of triangles in G by looking at $\mathbf{A}^3$, where $\mathbf{A}$ is the adjacency matrix of G ?



19F Degree-of-difficulty classified questions

► Simple familiar questions

- 1 Express each of the following as a single matrix:

$$\mathbf{a} \begin{bmatrix} 3 \\ -2 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{b} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{c} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$$

- 2 Given that $\mathbf{A} = \begin{bmatrix} w & 2w+5 \\ -1 & w+1 \end{bmatrix}$ and $\det(\mathbf{A}) = 15$, find the possible values of w .

- 3 Determine the value of x if

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ 5 \end{bmatrix} = \begin{bmatrix} x & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- 4 Determine the values of a and b if

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

- 5 Find the matrix corresponding to each of the following linear transformations:

- $\mathbf{a}$ dilation of factor 4 from the x -axis $\mathbf{b}$ dilation of factor $\frac{1}{2}$ from the y -axis
 $\mathbf{c}$ rotation by 60° clockwise $\mathbf{d}$ projection onto the x -axis
 $\mathbf{e}$ shear of factor $\frac{1}{2}$ parallel to the y -axis $\mathbf{f}$ reflection in the line $y = \frac{1}{\sqrt{3}}x$

- 6 $\mathbf{a}$ Find the single transformation matrix that represents a reflection in the y -axis followed by a dilation of factor 2 from the x -axis.

- $\mathbf{b}$ Find the single transformation matrix that represents a rotation by 90° anticlockwise followed by a reflection in the line $y = x$.

- 7 The unit square is mapped to a parallelogram by the linear transformation with matrix $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. Find the area of the parallelogram.

► Complex familiar questions

- 1 Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & a \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 \\ b & 2 \end{bmatrix}$.

- $\mathbf{a}$ Find $\mathbf{AB}$.

- $\mathbf{b}$ If $\mathbf{B} = \mathbf{A}^{-1}$, determine the values of a and b .

- 2 Find all triples of real numbers (a, b, c) such that $\mathbf{AB} = \mathbf{O}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- 3 a** Describe in words a sequence of transformations of the plane that will take the graph of $y = x^2$ to the graph of $y = p(x - h)^2 + k$, where $p > 0$.
- b** Find a rule for a single transformation T that will achieve the same result. Write the rule for T in the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

- c** Now find a rule for a transformation S that will take the graph of $y = q(x - m)^2 + n$ to the graph of $y = x^2$, where $q > 0$. Write the rule for S in the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \right)$$

- d** Finally, find a rule for a single transformation of the plane that will take the graph of $y = q(x - m)^2 + n$ to the graph of $y = p(x - h)^2 + k$, where $p, q > 0$.
- 4** In this question, we consider shears of the plane parallel to a line through the origin.
- a** Write down the matrix that represents each of the following transformations:
- i** T_1 : A rotation by angle θ clockwise.
 - ii** T_2 : A shear of factor k parallel to the x -axis.
 - iii** T_3 : A rotation by angle θ anticlockwise.
- b** Hence, find the single transformation matrix that represents T_1 followed by T_2 followed by T_3 .
- c** Give a brief geometric description of the transformation represented by this matrix.

► Complex unfamiliar questions

- 1** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Consider the matrix $\mathbf{A} - m\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix.
- a** Find $\det(\mathbf{A} - m\mathbf{I})$, writing your answer as a quadratic polynomial in m .
- b** If $m = \lambda_1$ and $m = \lambda_2$ are the solutions of the quadratic equation $\det(\mathbf{A} - m\mathbf{I}) = 0$, show that $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1\lambda_2 = \det(\mathbf{A})$.
- c** Suppose that $a + b = c + d = 1$. Show that $m = 1$ is a solution of the quadratic equation, and find the other solution in terms of a and c .
- d i** Suppose that $\mathbf{A} = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}$. Solve the equation $\det(\mathbf{A} - m\mathbf{I}) = 0$ for m .
- ii** The equation $\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ has infinitely many solutions. Describe them.
- iii** If $m = 1$ and $m = \lambda_2$ are the two solutions from part **i**, describe the solutions of the equation $\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_2 \begin{bmatrix} x \\ y \end{bmatrix}$.
- e** Now consider examples of matrices $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a + b = c + d = 6$.
More generally, consider examples with $a + b = c + d = k$, where k is an integer.

- 2** A **glide reflection** is a transformation of the plane that consists of a reflection in a mirror line ℓ followed by a translation parallel to the mirror line.

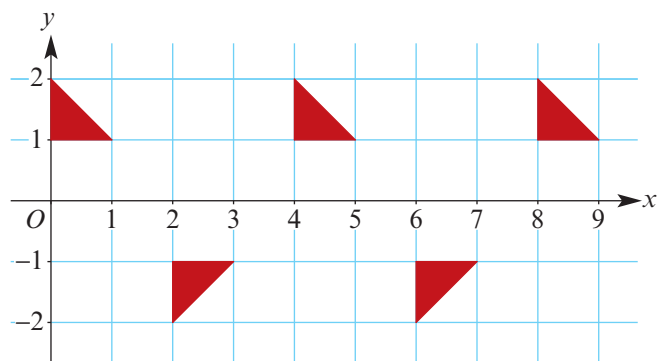
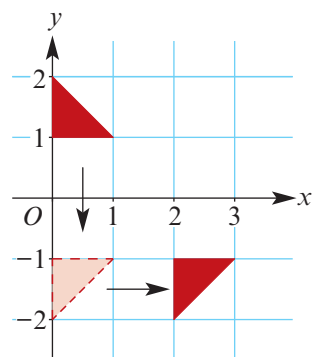
For example, the glide reflection shown on the right consists of:

- a reflection in the x -axis
- followed by a translation of 2 units in the x -direction.

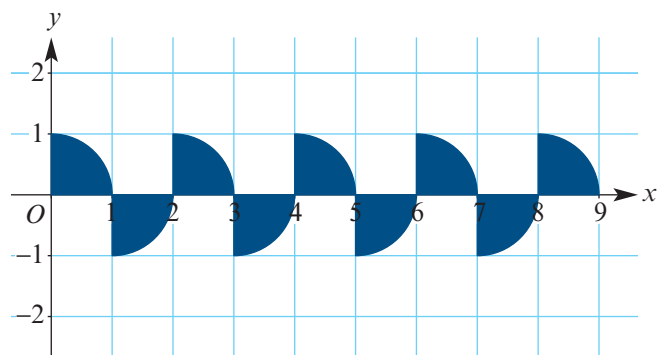
This glide reflection can be described by the rule

$$(x, y) \rightarrow (x + 2, -y)$$

If this glide reflection is applied many times, then the pattern shown below is formed.

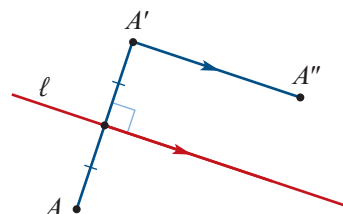


- a** Find the rule for the glide reflection that consists of:
- a reflection in the y -axis
 - followed by a translation of 3 units in the y -direction.
- b** Describe in words the glide reflection that will generate the pattern shown below, starting from the blue quarter disc at the origin.
- c** Find the rule for the glide reflection described in part **b**.



- d** Find the rule for the glide reflection that is obtained by reflecting in the line $y = x$ and then translating by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- 3** A glide reflection reflects a point A in a mirror line ℓ to point A' and then translates the result parallel to the line ℓ to the point A'' .



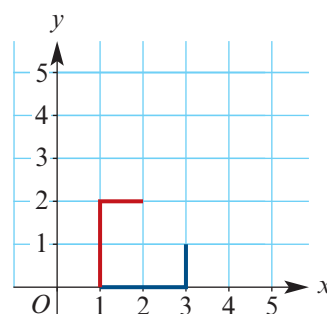
- a** Let M be the point where the line segment AA'' intersects ℓ . Show that M is the midpoint of AA'' .

Hint: If AA' intersects ℓ at B , then $AB = A'B$.

- b** Suppose that a glide reflection maps the point $A(1, 2)$ to the point $A''(3, 4)$, and maps the point $C(2, 7)$ to the point $C''(6, 5)$. Find the equation of the mirror line ℓ for this glide reflection.

Hint: Using part **a**, we know that the midpoint of AA'' lies on the line ℓ , and likewise for CC'' .

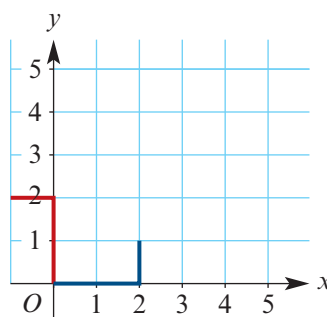
- 4 a** For the diagram on the right, find the equation of the mirror line through which the blue figure is reflected to obtain the red figure.



- b** In the diagram on the right, the blue figure can be transformed to the red figure by a sequence of two reflections. Find the equations of the two mirror lines.

Note: There is more than one possible answer.

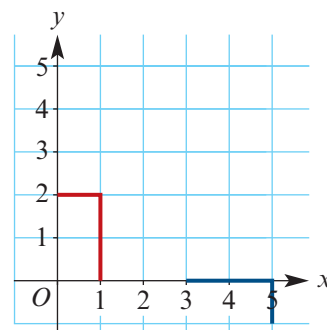
- c** Briefly explain why the blue figure in part **b** cannot be transformed to the red figure using only one reflection.



- d** In the diagram on the right, the blue figure can be transformed to the red figure by a sequence of three reflections. Find the equations of the three mirror lines.

Note: There is more than one possible answer.

- e** Briefly explain why the blue figure in part **d** cannot be transformed to the red figure using only two reflections.



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Further graphing techniques

Objectives

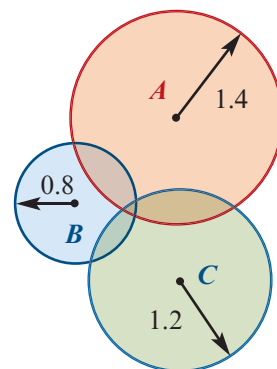
- ▶ To give **locus definitions** of lines, circles, parabolas, ellipses and hyperbolas, and to find the **Cartesian equations** of these curves.
- ▶ To use **parametric equations** to describe curves in the plane.
- ▶ To understand **polar coordinates** and their relationship to **Cartesian coordinates**.
- ▶ To sketch graphs in polar form.

This chapter is provided as preparation for Units 3 and 4.

The extensive use of mobile phones has led to an increased awareness of potential threats to the privacy of their users. For example, a little basic mathematics can be employed to track the movements of someone in possession of a mobile phone.

Suppose that there are three transmission towers within range of your mobile phone. By measuring the time taken for signals to travel between your phone and each transmission tower, it is possible to estimate the distance from your phone to each tower.

In the diagram, there are transmission towers at points A , B and C . If it is estimated that a person is no more than 1.4 km from A , no more than 0.8 km from B and no more than 1.2 km from C , then the person can be located in the intersection of the three circles.



In this chapter, we will look at different ways of describing circles and various other interesting figures.

20A Locus of points

Until now, all the curves we have studied have been described by an algebraic relationship between the x - and y -coordinates, such as $y = x^2 + 1$. In this section, we are interested in sets of points described by a geometric condition. A set described in this way is often called a **locus**. Many of these descriptions will give curves that are already familiar.

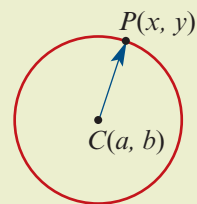
► Circles

Circles have a very simple geometric description.

Locus definition of a circle

A **circle** is the locus of a point $P(x, y)$ that moves so that its distance from a fixed point $C(a, b)$ is constant.

Note: The constant distance is called the **radius** and the fixed point $C(a, b)$ is called the **centre** of the circle.



This definition can be used to find the equation of a circle.

Recall that the distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let r be the radius of the circle. Then

$$\begin{aligned} CP &= r \\ \sqrt{(x - a)^2 + (y - b)^2} &= r \\ (x - a)^2 + (y - b)^2 &= r^2 \end{aligned}$$

The circle with radius r and centre $C(a, b)$ has equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Example 1

- Find the locus of points $P(x, y)$ whose distance from $C(2, -1)$ is 3.
- Find the centre and radius of the circle with equation $x^2 + 2x + y^2 - 4y = 1$.

Solution

- We know that the point $P(x, y)$ satisfies

$$\begin{aligned} CP &= 3 \\ \sqrt{(x - 2)^2 + (y + 1)^2} &= 3 \\ (x - 2)^2 + (y + 1)^2 &= 3^2 \end{aligned}$$

This is a circle with centre $(2, -1)$ and radius 3.

b We must complete the square in both variables. This gives

$$\begin{aligned}x^2 + 2x + y^2 - 4y &= 1 \\(x^2 + 2x + 1) - 1 + (y^2 - 4y + 4) - 4 &= 1 \\(x + 1)^2 + (y - 2)^2 &= 6\end{aligned}$$

Therefore the centre of the circle is $(-1, 2)$ and its radius is $\sqrt{6}$.

► Straight lines

You have learned in previous years that a straight line is the set of points (x, y) satisfying

$$ax + by = c$$

for some constants a, b, c with $a \neq 0$ or $b \neq 0$.

Lines can also be described geometrically as follows.

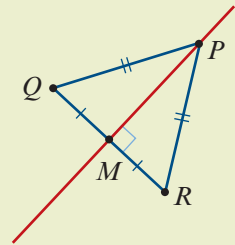
Locus definition of a straight line

Suppose that points Q and R are fixed.

A **straight line** is the locus of a point P that moves so that its distance from Q is the same as its distance from R . That is,

$$QP = RP$$

We can say that point P is **equidistant** from points Q and R .



Note: This straight line is the **perpendicular bisector** of line segment QR . To see this, we note that the midpoint M of QR is on the line. If P is any other point on the line, then

$$QP = RP, \quad QM = RM \quad \text{and} \quad MP = MP$$

and so $\triangle QMP$ is congruent to $\triangle RMP$. Therefore $\angle QMP = \angle RMP = 90^\circ$.



Example 2

- Find the locus of points $P(x, y)$ that are equidistant from the points $Q(1, 1)$ and $R(3, 5)$.
- Show that this is the perpendicular bisector of line segment QR .

Solution

a We know that the point $P(x, y)$ satisfies

$$\begin{aligned}QP &= RP \\ \sqrt{(x-1)^2 + (y-1)^2} &= \sqrt{(x-3)^2 + (y-5)^2} \\ (x-1)^2 + (y-1)^2 &= (x-3)^2 + (y-5)^2 \\ x + 2y &= 8 \\ y &= -\frac{1}{2}x + 4\end{aligned}$$

- b** This line has gradient $-\frac{1}{2}$. The line through $Q(1, 1)$ and $R(3, 5)$ has gradient $\frac{5-1}{3-1} = 2$.
Because the product of the two gradients is -1 , the two lines are perpendicular.
- We also need to check that the line $y = -\frac{1}{2}x + 4$ passes through the midpoint of QR , which is $(2, 3)$. When $x = 2$, $y = -\frac{1}{2} \times 2 + 4 = 3$. Thus $(2, 3)$ is on the line.

Section summary

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R .

Exercise 20A

Skillsheet

Example 1

- 1 Find the locus of points $P(x, y)$ whose distance from $Q(1, -2)$ is 4.
- 2 Find the locus of points $P(x, y)$ whose distance from $Q(-4, 3)$ is 5.

Example 2

- 3 **a** Find the locus of points $P(x, y)$ that are equidistant from $Q(-1, -1)$ and $R(1, 1)$.
b Show that this is the perpendicular bisector of line segment QR .
- 4 **a** Find the locus of points $P(x, y)$ that are equidistant from $Q(0, 2)$ and $R(1, 0)$.
b Show that this is the perpendicular bisector of line segment QR .
- 5 Point P is equidistant from points $Q(0, 1)$ and $R(2, 3)$. Moreover, its distance from point $S(3, 3)$ is 3. Find the possible coordinates of P .
- 6 Point P is equidistant from points $Q(0, 1)$ and $R(2, 0)$. Moreover, it is also equidistant from points $S(-1, 0)$ and $T(0, 2)$. Find the coordinates of P .
- 7 A valuable item is buried in a forest. It is 10 metres from a tree stump located at coordinates $T(0, 0)$ and 2 metres from a rock at coordinates $R(6, 10)$. Find the possible coordinates of the buried item.
- 8 Consider the three points $R(4, 5)$, $S(6, 1)$ and $T(1, -4)$.
 - a** Find the locus of points $P(x, y)$ that are equidistant from the points R and S .
 - b** Find the locus of points $P(x, y)$ that are equidistant from the points S and T .
 - c** Hence find the point that is equidistant from the points R , S and T .
 - d** Hence find the equation of the circle through the points R , S and T .

- 9 Given two fixed points $A(0, 1)$ and $B(2, 5)$, find the locus of P if the gradient of AB equals that of BP .
- 10 A triangle OAP has vertices $O(0, 0)$, $A(4, 0)$ and $P(x, y)$, where $y > 0$. The triangle has area 12 square units. Find the locus of P .
- 11 a Determine the locus of a point $P(x, y)$ that moves so that its distance from the origin is equal to the sum of its x - and y -coordinates.
b Determine the locus of a point $P(x, y)$ that moves so that the *square* of its distance from the origin is equal to the sum of its x - and y -coordinates.
- 12 $A(0, 0)$ and $B(3, 0)$ are two vertices of a triangle ABP . The third vertex P is such that $AP : BP = 2$. Find the locus of P .
- 13 Find the locus of the point P that moves so that its distance from the line $y = 3$ is always 2 units.
- 14 A steel pipe is too heavy to drag, but can be lifted at one end and rotated about its opposite end. How many moves are required to rotate the pipe into the parallel position indicated by the dotted line? The distance between the parallel lines is less than the length of the pipe.



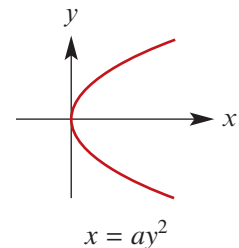
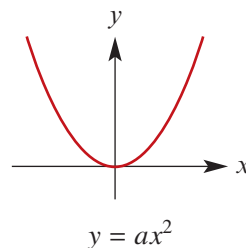
20B Parabolas

The parabola has been studied since antiquity and is admired for its range of applications, one of which we will explore at the end of this section.

The standard form of a parabola is $y = ax^2$.

Rotating the figure by 90° gives a parabola with equation $x = ay^2$.

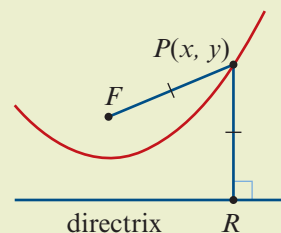
The parabola can also be defined geometrically.



Locus definition of a parabola

A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.

Note: The fixed point is called the **focus** and the fixed line is called the **directrix**.



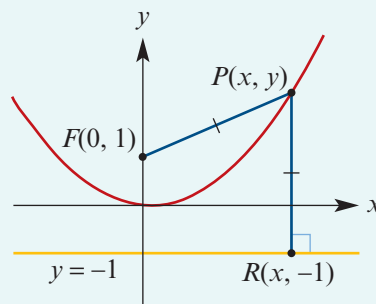
Example 3

Verify that the set of all points $P(x, y)$ that are equidistant from the point $F(0, 1)$ and the line $y = -1$ is a parabola.

Solution

We know that the point $P(x, y)$ satisfies

$$\begin{aligned}
 FP &= RP \\
 \sqrt{x^2 + (y - 1)^2} &= \sqrt{(y - (-1))^2} \\
 x^2 + (y - 1)^2 &= (y + 1)^2 \\
 x^2 + y^2 - 2y + 1 &= y^2 + 2y + 1 \\
 x^2 - 2y &= 2y \\
 x^2 &= 4y \\
 y &= \frac{x^2}{4}
 \end{aligned}$$



Therefore the set of points is the parabola with equation $y = \frac{x^2}{4}$.

**Example 4**

- Find the equation of the parabola with focus $F(0, c)$ and directrix $y = -c$.
- Hence find the focus of the parabola with equation $y = 2x^2$.

Solution

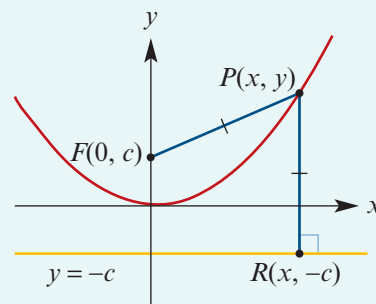
- A point $P(x, y)$ on the parabola satisfies

$$\begin{aligned}
 FP &= RP \\
 \sqrt{x^2 + (y - c)^2} &= \sqrt{(y - (-c))^2} \\
 x^2 + (y - c)^2 &= (y + c)^2 \\
 x^2 + y^2 - 2cy + c^2 &= y^2 + 2cy + c^2 \\
 x^2 - 2cy &= 2cy \\
 x^2 &= 4cy
 \end{aligned}$$

The parabola has equation $4cy = x^2$.

- Since $\frac{y}{2} = x^2$, we solve $\frac{1}{2} = 4c$, giving $c = \frac{1}{8}$.

Hence the focus is $F\left(0, \frac{1}{8}\right)$.



In the previous example, we proved the following result:

The parabola with focus $F(0, c)$ and directrix $y = -c$ has equation $4cy = x^2$.

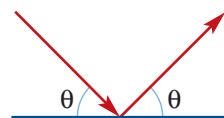
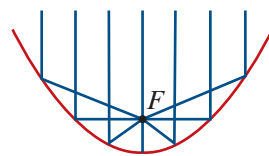
► A remarkable application

Parabolas have a remarkable property that makes them extremely useful. Light travelling parallel to the axis of symmetry of a reflective parabola is always reflected to its focus.

Parabolas can therefore be used to make reflective telescopes. Low intensity signals from outer space will reflect off the dish and converge at a receiver located at the focus.

To see how this works, we require a simple law of physics:

- When light is reflected off a surface, the angle between the ray and the tangent to the surface is preserved after reflection.



Reflective property of the parabola

Any ray of light parallel to the axis of symmetry of the parabola that reflects off the parabola at point P will pass through the focus at F .

Proof Since point P is on the parabola, the distance to the focus F is the same as the distance to the directrix. Therefore $FP = RP$, and so $\triangle FPR$ is isosceles.

Let M be the midpoint of FR . Then $\triangle FMP$ is congruent to $\triangle RMP$ (by SSS). Therefore MP is the perpendicular bisector of FR and

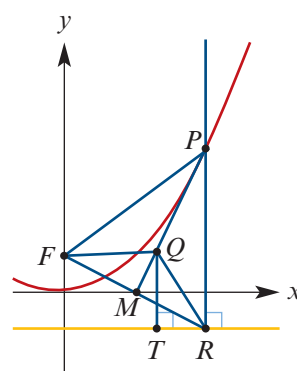
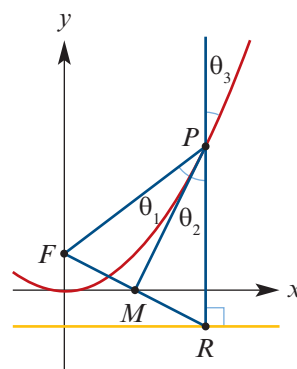
$$\begin{aligned}\theta_1 &= \theta_2 && (\text{as } \triangle FMP \equiv \triangle RMP) \\ &= \theta_3 && (\text{vertically opposite angles})\end{aligned}$$

However, we also need to ensure that line MP is tangent to the parabola. To see this, we will show that point P is the only point common to the parabola and line MP .

Take any other point Q on line MP . Suppose that point T is the point on the directrix closest to Q . Then

$$FQ = RQ > TQ$$

and so point Q is not on the parabola.



Section summary

- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- The fixed point is called the **focus** and the fixed line is called the **directrix**.
- The parabola with equation $4cy = x^2$ has focus $F(0, c)$ and directrix $y = -c$.

Exercise 20B

Example 3

- Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(0, 3)$ is equal to the perpendicular distance to the line with equation $y = -3$.
- Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(0, -4)$ is equal to the perpendicular distance to the line with equation $y = 2$.
- Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(2, 0)$ is equal to the perpendicular distance to the line with equation $x = -4$.

Example 4

- Find the equation of the parabola with focus $F(c, 0)$ and directrix $x = -c$.
 - Hence find the focus of the parabola with equation $x = 3y^2$.
- Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(a, b)$ is equal to the perpendicular distance to the line with equation $y = c$.
 - Hence find the equation of the parabola with focus $(1, 2)$ and directrix $y = 3$.
- A parabola goes through the point $P(7, 9)$ and its focus is $F(1, 1)$. The axis of symmetry of the parabola is $x = 1$. Find the equation of its directrix.
Hint: The directrix will be a horizontal line, $y = c$. Expect to find two answers.
- A parabola goes through the point $(1, 1)$, its axis of symmetry is the line $x = 2$ and its directrix is the line $y = 3$. Find the coordinates of its focus.
Hint: The focus must lie on the axis of symmetry.

20C Ellipses

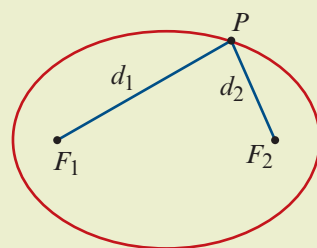
A ball casts a shadow that looks like a squashed circle. This figure – called an ellipse – is of considerable geometric significance. For instance, the planets in our solar system have elliptic orbits.

Locus definition of an ellipse

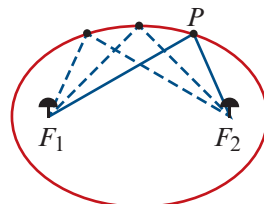
An **ellipse** is the locus of a point P that moves so that the sum of its distances from two fixed points F_1 and F_2 is a constant. That is,

$$F_1P + F_2P = k$$

Note: Points F_1 and F_2 are called the **foci** of the ellipse.



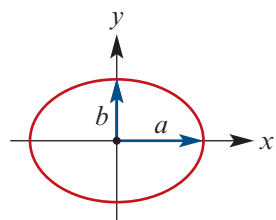
Drawing an ellipse An ellipse can be drawn by pushing two pins into paper. These will be the foci. A string of length k is tied to each of the two pins and the tip of a pen is used to pull the string taut and form a triangle. The pen will trace an ellipse if it is moved around the pins while keeping the string taut.



► Cartesian equations of ellipses

In Chapter 18, we obtained ellipses as transformations of circles. The standard form of the Cartesian equation of an ellipse centred at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



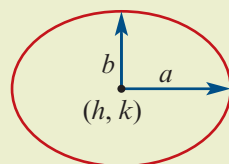
This ellipse has x -axis intercepts $\pm a$ and y -axis intercepts $\pm b$.

Applying the translation defined by $(x, y) \rightarrow (x + h, y + k)$, we obtain:

The graph of

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k) .



Example 5

Sketch the graph of each ellipse:

a $\frac{x^2}{9} + \frac{y^2}{4} = 1$

b $4x^2 + 9y^2 = 1$

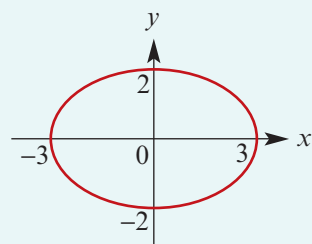
c $\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{9} = 1$

Solution

a The equation can be written as

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

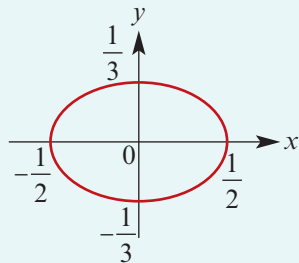
This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm 3$ and $y = \pm 2$.



b The equation can be written as

$$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$$

This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm \frac{1}{2}$ and $y = \pm \frac{1}{3}$.



c This is an ellipse with centre $(1, -2)$.

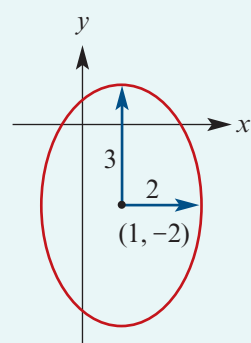
To find the x -axis intercepts, let $y = 0$. Then solving for x gives

$$x = \frac{3 \pm 2\sqrt{5}}{3}$$

Likewise, to find the y -axis intercepts, let $x = 0$.

This gives

$$y = \frac{-4 \pm 3\sqrt{3}}{2}$$



► Using the locus definition

Example 6

Consider points $A(-2, 0)$ and $B(2, 0)$. Find the equation of the locus of points P satisfying $AP + BP = 8$.

Solution

Let (x, y) be the coordinates of point P . If $AP + BP = 8$, then

$$\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 8$$

and so
$$\sqrt{(x+2)^2 + y^2} = 8 - \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$(x+2)^2 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2$$

$$x^2 + 4x + 4 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2$$

$$x - 8 = -2\sqrt{(x-2)^2 + y^2}$$

Square both sides again:

$$x^2 - 16x + 64 = 4(x^2 - 4x + 4 + y^2)$$

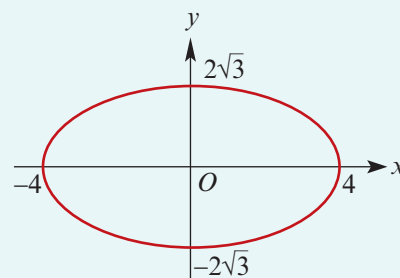
Simplifying yields

$$3x^2 + 4y^2 = 48$$

i.e.
$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

This is an ellipse with centre the origin and axis intercepts at $x = \pm 4$ and $y = \pm 2\sqrt{3}$.

Every point P on the ellipse satisfies $AP + BP = 8$.

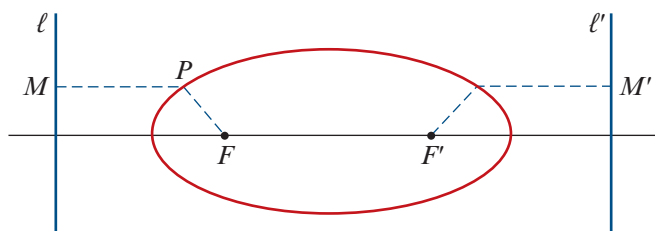


Note: You might like to consider the general version of this example with $A(-c, 0)$, $B(c, 0)$ and $AP + BP = 2a$, where $a > c > 0$.

It can also be shown that an ellipse is the locus of points $P(x, y)$ satisfying

$$FP = eMP$$

where F is a fixed point, $0 < e < 1$ and MP is the perpendicular distance from P to a fixed line ℓ . From the symmetry of the ellipse, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eM'P$ defines the same locus, where $M'P$ is the perpendicular distance from P to ℓ' .



Example 7

Find the equation of the locus of points $P(x, y)$ if the distance from P to the point $F(1, 0)$ is half the distance MP , the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = \frac{1}{2}MP$.

Solution

Let (x, y) be the coordinates of point P .

If $FP = \frac{1}{2}MP$, then

$$\sqrt{(x-1)^2 + y^2} = \frac{1}{2}\sqrt{(x+2)^2}$$

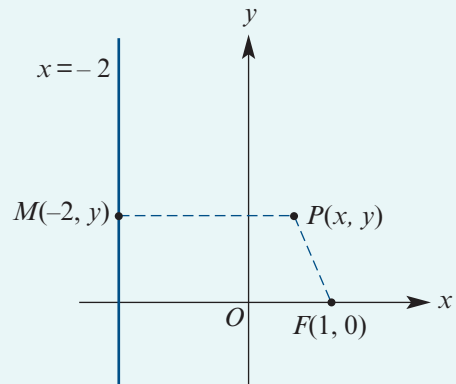
Square both sides:

$$\begin{aligned}(x-1)^2 + y^2 &= \frac{1}{4}(x+2)^2 \\ 4(x^2 - 2x + 1) + 4y^2 &= x^2 + 4x + 4 \\ 3x^2 - 12x + 4y^2 &= 0\end{aligned}$$

Complete the square:

$$\begin{aligned}3(x^2 - 4x + 4) + 4y^2 &= 12 \\ 3(x-2)^2 + 4y^2 &= 12 \quad \text{or equivalently} \quad \frac{(x-2)^2}{4} + \frac{y^2}{3} = 1\end{aligned}$$

This is an ellipse with centre $(2, 0)$.

**Section summary**

- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 (called the **foci**) is equal to a fixed positive constant.
- The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k) .

Exercise 20C**Skillsheet**

- 1 Sketch the graph of each ellipse, labelling the axis intercepts:

a $\frac{x^2}{9} + \frac{y^2}{64} = 1$ **b** $\frac{x^2}{100} + \frac{y^2}{25} = 1$ **c** $\frac{y^2}{9} + \frac{x^2}{64} = 1$ **d** $25x^2 + 9y^2 = 225$

Example 5

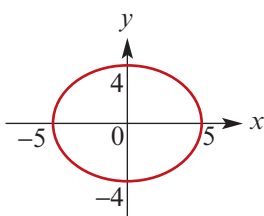
- 2 Sketch the graph of each ellipse, labelling the centre and the axis intercepts:

a $\frac{(x-3)^2}{9} + \frac{(y-4)^2}{16} = 1$ **b** $\frac{(x+3)^2}{9} + \frac{(y+4)^2}{25} = 1$

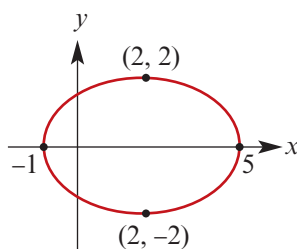
c $\frac{(y-3)^2}{16} + \frac{(x-2)^2}{4} = 1$ **d** $25(x-5)^2 + 9y^2 = 225$

- 3 Find the Cartesian equations of the following ellipses:

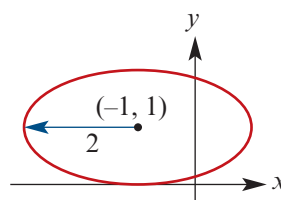
a



b



c



Example 6

- 4 Find the locus of the point P as it moves such that the sum of its distances from two fixed points $A(1, 0)$ and $B(-1, 0)$ is 4 units.

- 5 Find the locus of the point P as it moves such that the sum of its distances from two fixed points $A(0, 2)$ and $B(0, -2)$ is 6 units.

Example 7

- 6 Find the equation of the locus of points $P(x, y)$ such that the distance from P to the point $F(2, 0)$ is half the distance MP , the perpendicular distance from P to the line with equation $x = -4$. That is, $FP = \frac{1}{2}MP$.
- 7 A circle has equation $x^2 + y^2 = 1$. It is then dilated by a factor of 3 from the x -axis and by a factor of 5 from the y -axis. Find the equation of the image and sketch its graph.

20D Hyperbolas

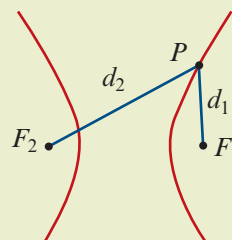
Hyperbolas are defined analogously to ellipses, but using the difference instead of the sum.

Locus definition of a hyperbola

A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant. That is,

$$|F_2P - F_1P| = k$$

Note: Points F_1 and F_2 are called the **foci** of the hyperbola.



The standard form of the Cartesian equation of a hyperbola centred at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Applying the translation defined by $(x, y) \rightarrow (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k) .

Note: Interchanging x and y in this equation produces another hyperbola (rotated by 90°).

► Asymptotes of the hyperbola

We now investigate the behaviour of the hyperbola as $x \rightarrow \pm\infty$. We first show that the hyperbola with equation

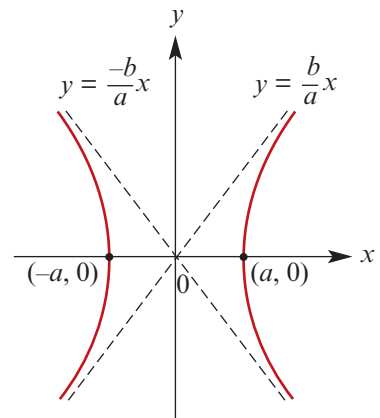
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

To see why this should be the case, we rearrange the equation of the hyperbola as follows:

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ \frac{y^2}{b^2} &= \frac{x^2}{a^2} - 1 \\ y^2 &= \frac{b^2 x^2}{a^2} - b^2 \\ &= \frac{b^2 x^2}{a^2} \left(1 - \frac{a^2}{x^2}\right) \end{aligned}$$



If $x \rightarrow \pm\infty$, then $\frac{a^2}{x^2} \rightarrow 0$. Therefore $y^2 \rightarrow \frac{b^2 x^2}{a^2}$ as $x \rightarrow \pm\infty$. That is,

$$y \rightarrow \pm \frac{bx}{a} \quad \text{as} \quad x \rightarrow \pm\infty$$

Applying the translation defined by $(x, y) \rightarrow (x + h, y + k)$, we obtain the following result:

The hyperbola with equation

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm \frac{b}{a}(x - h)$$

Example 8

For each of the following equations, sketch the graph of the corresponding hyperbola. Give the coordinates of the centre, the axis intercepts and the equations of the asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{y^2}{9} - \frac{x^2}{4} = 1$

c $(x - 1)^2 - (y + 2)^2 = 1$

d $\frac{(y - 1)^2}{4} - \frac{(x + 2)^2}{9} = 1$

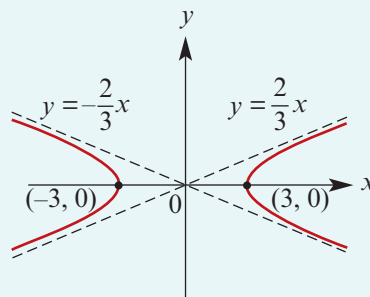
Solution

- a** Since $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we have

$$y^2 = \frac{4x^2}{9} \left(1 - \frac{9}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm \frac{2}{3}x$.

If $y = 0$, then $x^2 = 9$ and so $x = \pm 3$. The x -axis intercepts are $(3, 0)$ and $(-3, 0)$. The centre is $(0, 0)$.



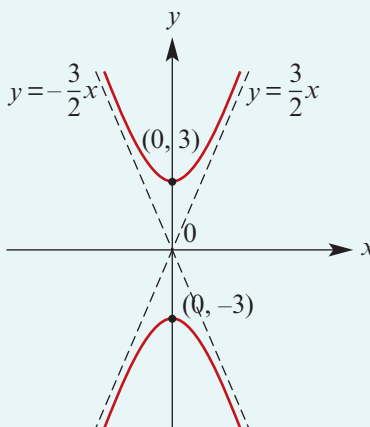
- b** Since $\frac{y^2}{9} - \frac{x^2}{4} = 1$, we have

$$y^2 = \frac{9x^2}{4} \left(1 + \frac{4}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm \frac{3}{2}x$.

The y -axis intercepts are $(0, 3)$ and $(0, -3)$.

The centre is $(0, 0)$.



- c** First sketch the graph of $x^2 - y^2 = 1$. The asymptotes are $y = x$ and $y = -x$. The centre is $(0, 0)$ and the axis intercepts are $(1, 0)$ and $(-1, 0)$.

Note: This hyperbola is called a **rectangular hyperbola**, as its asymptotes are perpendicular.

Now to sketch the graph of

$$(x - 1)^2 - (y + 2)^2 = 1$$

we apply the translation $(x, y) \rightarrow (x + 1, y - 2)$.

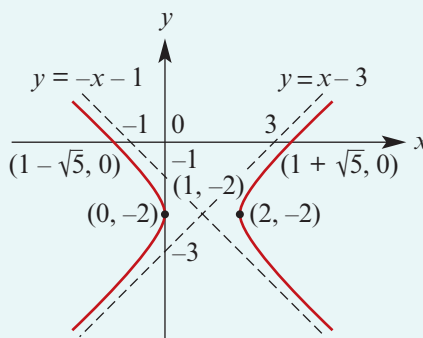
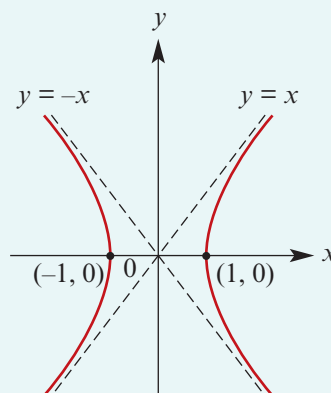
The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm(x - 1)$. That is, $y = x - 3$ and $y = -x - 1$.

Axis intercepts

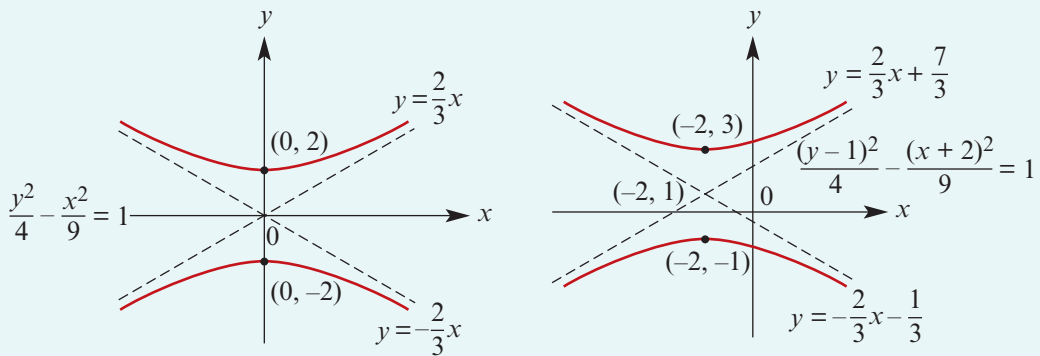
If $x = 0$, then $y = -2$.

If $y = 0$, then $(x - 1)^2 = 5$ and so $x = 1 \pm \sqrt{5}$.

Therefore the axis intercepts are $(0, -2)$ and $(1 \pm \sqrt{5}, 0)$.



- d The graph of $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$ is obtained from the hyperbola $\frac{y^2}{4} - \frac{x^2}{9} = 1$ through the translation $(x, y) \rightarrow (x-2, y+1)$. Its centre will be $(-2, 1)$.



► Using the locus definition



Example 9

Consider the points $A(-2, 0)$ and $B(2, 0)$. Find the equation of the locus of points P satisfying $AP - BP = 3$.

Solution

Let (x, y) be the coordinates of point P .

If $AP - BP = 3$, then

$$\sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2} = 3$$

and so
$$\sqrt{(x+2)^2 + y^2} = 3 + \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$\begin{aligned}(x+2)^2 + y^2 &= 9 + 6\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2 \\ x^2 + 4x + 4 + y^2 &= 9 + 6\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2 \\ 8x - 9 &= 6\sqrt{(x-2)^2 + y^2}\end{aligned}$$

Note that this only holds if $x \geq \frac{9}{8}$. Squaring both sides again gives

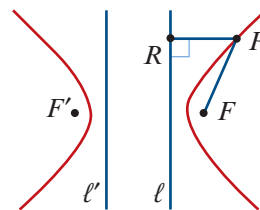
$$\begin{aligned}64x^2 - 144x + 81 &= 36(x^2 - 4x + 4 + y^2) \\ 28x^2 - 36y^2 &= 63 \\ \frac{4x^2}{9} - \frac{4y^2}{7} &= 1 \quad \text{for } x \geq \frac{3}{2}\end{aligned}$$

This is the right branch of a hyperbola with centre the origin and x -axis intercept $\frac{3}{2}$.

It can also be shown that a hyperbola is the locus of points $P(x, y)$ satisfying

$$FP = eRP$$

where F is a fixed point, $e > 1$ and RP is the perpendicular distance from P to a fixed line ℓ .



From the symmetry of the hyperbola, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eR'P$ defines the same locus, where $R'P$ is the perpendicular distance from P to ℓ' .

Example 10

Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance from P to the point $F(1, 0)$ is twice the distance MP , the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = 2MP$.

Solution

Let (x, y) be the coordinates of point P .

If $FP = 2MP$, then

$$\sqrt{(x-1)^2 + y^2} = 2\sqrt{(x+2)^2}$$

Squaring both sides gives

$$(x-1)^2 + y^2 = 4(x+2)^2$$

$$x^2 - 2x + 1 + y^2 = 4(x^2 + 4x + 4)$$

$$3x^2 + 18x - y^2 + 15 = 0$$

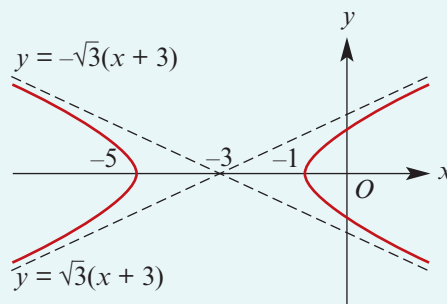
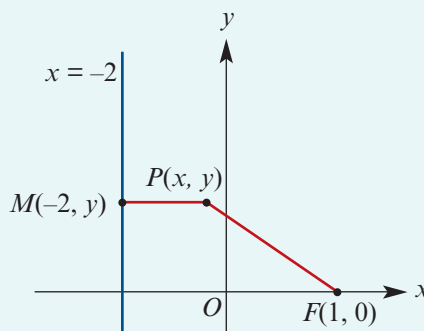
By completing the square, we obtain

$$3(x^2 + 6x + 9) - 27 - y^2 + 15 = 0$$

$$3(x+3)^2 - y^2 = 12$$

$$\frac{(x+3)^2}{4} - \frac{y^2}{12} = 1$$

This is a hyperbola with centre $(-3, 0)$.



Section summary

- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 (called the **foci**) is a constant. That is, $|F_2P - F_1P| = k$.

- The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k) . The asymptotes are $y - k = \pm \frac{b}{a}(x - h)$.

Exercise 20D

Skillsheet

- 1 Sketch the graph of each of the following hyperbolas. Label axis intercepts and give the equations of the asymptotes.

Example 8

a $\frac{x^2}{4} - \frac{y^2}{9} = 1$

b $x^2 - \frac{y^2}{4} = 1$

c $\frac{y^2}{25} - \frac{x^2}{100} = 1$

d $25x^2 - 9y^2 = 225$

- 2 Sketch the graph of each of the following hyperbolas. State the centre and label axis intercepts and asymptotes.

a $(x - 1)^2 - (y + 2)^2 = 1$

b $\frac{(x + 1)^2}{4} - \frac{(y - 2)^2}{16} = 1$

c $\frac{(y - 3)^2}{9} - (x - 2)^2 = 1$

d $25(x - 4)^2 - 9y^2 = 225$

e $x^2 - 4y^2 - 4x - 8y - 16 = 0$

f $9x^2 - 25y^2 - 90x + 150y = 225$

Example 9

- 3 Consider the points $A(4, 0)$ and $B(-4, 0)$. Find the equation of the locus of points P satisfying $AP - BP = 6$.

- 4 Find the equation of the locus of points $P(x, y)$ satisfying $AP - BP = 4$, given coordinates $A(-3, 0)$ and $B(3, 0)$.

Example 10

- 5 Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance to P from the point $F(5, 0)$ is twice the distance MP , the perpendicular distance to P from the line with equation $x = -1$. That is, $FP = 2MP$.

- 6 Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance to P from the point $F(0, -1)$ is twice the distance MP , the perpendicular distance to P from the line with equation $y = -4$. That is, $FP = 2MP$.

20E Parametric equations

A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

The variable t is called the **parameter**, and for each choice of t we get a point in the plane $(f(t), g(t))$. The set of all such points will be a curve in the plane.

It is sometimes useful to think of t as being *time*, so that the equations $x = f(t)$ and $y = g(t)$ give the position of an object at time t . Points on the curve can be plotted by substituting various values of t into the two equations.

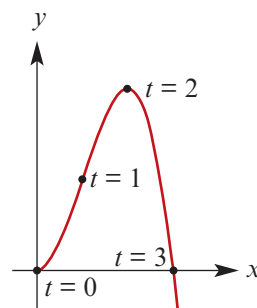
For instance, we can plot points on the curve defined by the parametric equations

$$x = t \quad \text{and} \quad y = 3t^2 - t^3$$

by letting $t = 0, 1, 2, 3$.

In this instance, it is possible to eliminate the parameter t to obtain a Cartesian equation in x and y alone. Substituting $t = x$ into the second equation gives $y = 3x^2 - x^3$.

t	0	1	2	3
x	0	1	2	3
y	0	2	4	0



► Lines

Example 11

- a** Find the Cartesian equation for the curve defined by the parametric equations

$$x = t + 2 \quad \text{and} \quad y = 2t - 3$$

- b** Find parametric equations for the line through the points $A(2, 3)$ and $B(4, 7)$.

Solution

- a** Substitute $t = x - 2$ into the second equation to give

$$\begin{aligned} y &= 2(x - 2) - 3 \\ &= 2x - 7 \end{aligned}$$

Thus every point lies on the straight line with equation $y = 2x - 7$.

- b** The gradient of the straight line through points $A(2, 3)$ and $B(4, 7)$ is

$$m = \frac{7 - 3}{4 - 2} = 2$$

Therefore the line has equation

$$\begin{aligned} y - 3 &= 2(x - 2) \\ y &= 2x - 1 \end{aligned}$$

We can simply let $x = t$ and so $y = 2t - 1$.

Note: There are infinitely many pairs of parametric equations that describe the same curve.

In part **b**, we could also let $x = 2t$ and $y = 4t - 1$. These parametric equations describe exactly the same set of points. As t increases, the point moves along the same line twice as fast.

► Parabolas

Example 12

Find the Cartesian equation of the parabola defined by the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 + 1$$

Solution

Substitute $t = x + 1$ into the second equation to give $y = (x + 1)^2 + 1$.

► Circles

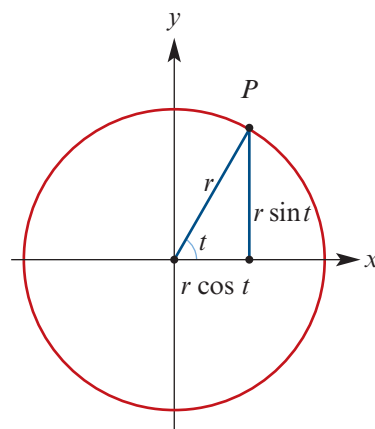
We have seen that the circle with radius r and centre at the origin can be written in Cartesian form as

$$x^2 + y^2 = r^2$$

We now introduce the parameter t and let

$$x = r \cos t \quad \text{and} \quad y = r \sin t$$

As t increases from 0 to 2π , the point $P(x, y)$ travels from $(r, 0)$ anticlockwise around the circle and returns to its original position.



To demonstrate that this parameterises the circle, we evaluate

$$\begin{aligned} x^2 + y^2 &= r^2 \cos^2 t + r^2 \sin^2 t \\ &= r^2 (\cos^2 t + \sin^2 t) \\ &= r^2 \end{aligned}$$

where we have used the Pythagorean identity $\cos^2 t + \sin^2 t = 1$.



Example 13

- a** Find the Cartesian equation of the circle defined by the parametric equations

$$x = \cos t + 1 \quad \text{and} \quad y = \sin t - 2$$

- b** Find parametric equations for the circle with Cartesian equation

$$(x + 1)^2 + (y + 3)^2 = 4$$

Solution

- a** We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$x - 1 = \cos t \quad \text{and} \quad y + 2 = \sin t$$

Using the Pythagorean identity:

$$(x - 1)^2 + (y + 2)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the circle with equation $(x - 1)^2 + (y + 2)^2 = 1$.

- b** We let

$$\cos t = \frac{x + 1}{2} \quad \text{and} \quad \sin t = \frac{y + 3}{2}$$

giving

$$x = 2 \cos t - 1 \quad \text{and} \quad y = 2 \sin t - 3$$

We can easily check that these equations parameterise the given circle.

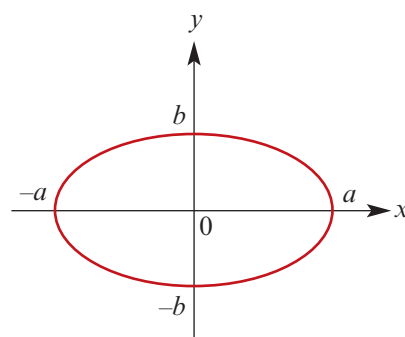
► Ellipses

An ellipse can be thought of as a squashed circle.

This is made apparent from the parametric equations for an ellipse:

$$x = a \cos t \quad \text{and} \quad y = b \sin t$$

As with the circle, we see the sine and cosine functions, but these are now scaled by different constants, giving different dilations from the x - and y -axes.



We can turn this pair of parametric equations into one Cartesian equation as follows:

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{b} = \sin t$$

giving

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

which is the standard form of an ellipse centred at the origin with axis intercepts at $x = \pm a$ and $y = \pm b$.

Example 14

- a** Find the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3 \cos t + 1 \quad \text{and} \quad y = 2 \sin t - 1$$

- b** Find parametric equations for the ellipse with Cartesian equation

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$$

Solution

- a** We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$\frac{x-1}{3} = \cos t \quad \text{and} \quad \frac{y+1}{2} = \sin t$$

Using the Pythagorean identity:

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the ellipse with equation $\frac{(x-1)^2}{3^2} + \frac{(y+1)^2}{2^2} = 1$.

- b** We let

$$\cos t = \frac{x-1}{3} \quad \text{and} \quad \sin t = \frac{y+1}{2}$$

giving

$$x = 3 \cos t + 1 \quad \text{and} \quad y = 2 \sin t - 1$$

► Hyperbolas

We can parameterise a hyperbola using the equations

$$x = a \sec t \quad \text{and} \quad y = b \tan t$$

From these two equations, we can find the more familiar Cartesian equation:

$$\frac{x}{a} = \sec t \quad \text{and} \quad \frac{y}{b} = \tan t$$

giving

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

which is the standard form of a hyperbola centred at the origin.

Example 15

- a** Find the Cartesian equation of the hyperbola defined by the parametric equations

$$x = 3 \sec t - 1 \quad \text{and} \quad y = 2 \tan t + 2$$

- b** Find parametric equations for the hyperbola with Cartesian equation

$$\frac{(x+2)^2}{4} - \frac{(y-3)^2}{16} = 1$$

Solution

- a** We rearrange each equation to isolate $\sec t$ and $\tan t$ respectively. This gives

$$\frac{x+1}{3} = \sec t \quad \text{and} \quad \frac{y-2}{2} = \tan t$$

and therefore

$$\left(\frac{x+1}{3}\right)^2 - \left(\frac{y-2}{2}\right)^2 = \sec^2 t - \tan^2 t = 1$$

So each point on the graph lies on the hyperbola with equation $\frac{(x+1)^2}{3^2} - \frac{(y-2)^2}{2^2} = 1$.

- b** We let

$$\sec t = \frac{x+2}{2} \quad \text{and} \quad \tan t = \frac{y-3}{4}$$

giving

$$x = 2 \sec t - 2 \quad \text{and} \quad y = 4 \tan t + 3$$

► Parametric equations with restricted domains

Example 16

Eliminate the parameter to determine the graph of the parameterised curve

$$x = t - 1, \quad y = t^2 - 2t + 1 \quad \text{for } 0 \leq t \leq 2$$

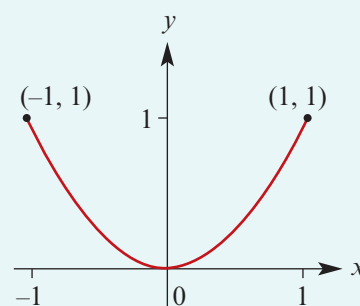
Solution

Substitute $t = x + 1$ from the first equation into the second equation, giving

$$\begin{aligned} y &= (x + 1)^2 - 2(x + 1) + 1 \\ &= x^2 + 2x + 1 - 2x - 2 + 1 \\ &= x^2 \end{aligned}$$

Since $0 \leq t \leq 2$, it follows that $-1 \leq x \leq 1$.

Therefore, as t increases from 0 to 2, the point travels along the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.



► Intersections of curves defined parametrically

It is often difficult to find the intersection of two curves defined parametrically. This is because, although the curves may intersect, they might do so for different values of the parameter t .

In many instances, it is easiest to find the points of intersection using the Cartesian equations for the two curves.

Example 17

Find the points of intersection of the circle and line defined by the parametric equations:

circle $x = 5 \cos t$ and $y = 5 \sin t$

line $x = t - 3$ and $y = 2t - 8$

Solution

The Cartesian equation of the circle is $x^2 + y^2 = 25$.

The Cartesian equation of the line is $y = 2x - 2$.

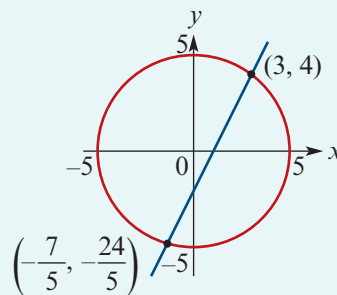
Substituting the second equation into the first gives

$$\begin{aligned} x^2 + (2x - 2)^2 &= 25 \\ x^2 + 4x^2 - 8x + 4 &= 25 \\ 5x^2 - 8x - 21 &= 0 \\ (x - 3)(5x + 7) &= 0 \end{aligned}$$

This gives solutions $x = 3$ and $x = -\frac{7}{5}$.

Substituting these into the equation $y = 2x - 2$ gives $y = 4$ and $y = -\frac{24}{5}$ respectively.

The points of intersection are $(3, 4)$ and $(-\frac{7}{5}, -\frac{24}{5})$.



► Using a graphics calculator with parametric equations

Example 18

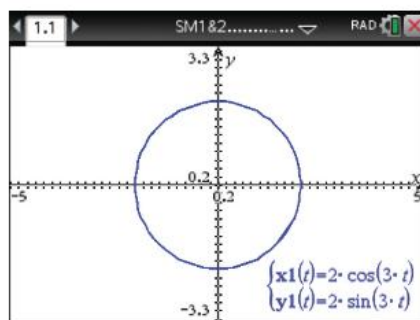
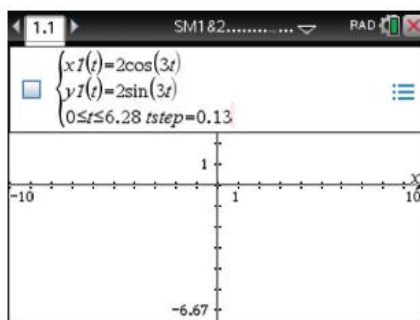
Plot the graph of the parametric curve given by

$$x = 2 \cos(3t) \quad \text{and} \quad y = 2 \sin(3t)$$



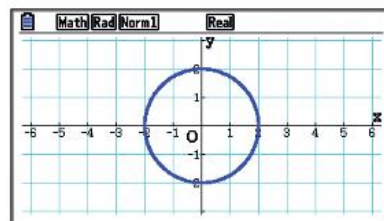
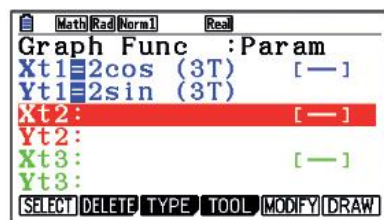
Using the TI-Nspire CX non-CAS

- Open a **Graphs** application (on > **New Document** > **Add Graphs**).
- Use > **Graph Entry/Edit** > **Parametric** to show the entry line for parametric equations.
- Enter $x1(t) = 2 \cos(3t)$ and $y1(t) = 2 \sin(3t)$ as shown.



Using the Casio

- Press to select **Graph** mode.
- To specify the graph type, go to **Type** () and choose **Parametric** ()
- Enter the rule $x = 2 \cos(3t)$ in $Xt1$:
 $\langle 2 \rangle \langle \cos \rangle \langle (\rangle \langle 3 \rangle \langle X, \theta, T \rangle \langle) \rangle \langle \text{EXE} \rangle$
- Enter the rule $y = 2 \sin(3t)$ in $Yt1$:
 $\langle 2 \rangle \langle \sin \rangle \langle (\rangle \langle 3 \rangle \langle X, \theta, T \rangle \langle) \rangle \langle \text{EXE} \rangle$
- Adjust the View Window:
 $\langle \text{SHIFT} \rangle \langle \text{F3} \rangle \langle \text{F1} \rangle \langle \text{EXE} \rangle$
- Select **Draw** ()



Section summary

- A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is called the **parameter** of the curve. For example:

	Cartesian equation	Parametric equations
Circle	$x^2 + y^2 = r^2$	$x = r \cos t$ and $y = r \sin t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a \cos t$ and $y = b \sin t$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a \sec t$ and $y = b \tan t$

- We can sometimes find the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x .

Exercise 20E

Skillsheet

- 1 Consider the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 - 1$$

Example 11

- a** Find the Cartesian equation of the curve described by these equations.
b Sketch the curve and label the points on the curve corresponding to $t = 0, 1, 2$.

Example 12

- 2 For each of the following pairs of parametric equations, find the Cartesian equation and sketch the curve:

a $x = t + 1$ and $y = 2t + 1$

b $x = t - 1$ and $y = 2t^2 + 1$

c $x = t^2$ and $y = t^6$

d $x = t + 2$ and $y = \frac{1}{t + 1}$

Example 13

- 3 **a** Find the Cartesian equation of the circle defined by the parametric equations

$$x = 2 \cos t \quad \text{and} \quad y = 2 \sin t$$

Example 14

- b** Find the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3 \cos t - 1 \quad \text{and} \quad y = 2 \sin t + 2$$

- c** Find parametric equations for the circle with Cartesian equation

$$(x + 3)^2 + (y - 2)^2 = 9$$

- d** Find parametric equations for the ellipse with Cartesian equation

$$\frac{(x + 2)^2}{9} + \frac{(y - 1)^2}{4} = 1$$

- 4 Find parametric equations for the line through the points $A(-1, -2)$ and $B(1, 4)$.

SF

CF

Example 16

- 5 a** Eliminate the parameter t to determine the equation of the parameterised curve

$$x = t - 1 \quad \text{and} \quad y = -2t^2 + 4t - 2 \quad \text{for } 0 \leq t \leq 2$$

- b** Sketch the graph of this curve over an appropriate domain.

Example 17

- 6** Find the points of intersection of the circle and line defined by the parametric equations:

circle $x = \cos t \quad \text{and} \quad y = \sin t$

line $x = 3t + 6 \quad \text{and} \quad y = 4t + 8$

- 7** A curve is parameterised by the equations

$$x = \sin t \quad \text{and} \quad y = 2 \sin^2 t + 1 \quad \text{for } 0 \leq t \leq 2\pi$$

- a** Find the curve's Cartesian equation. **b** What is the domain of the curve?
c What is the range of the curve? **d** Sketch the graph of the curve.

- 8** A curve is parameterised by the equations

$$x = 2^t \quad \text{and} \quad y = 2^{2t} + 1 \quad \text{for } t \in \mathbb{R}$$

- a** Find the curve's Cartesian equation. **b** What is the domain of the curve?
c What is the range of the curve? **d** Sketch the graph of the curve.

- 9** Eliminate the parameter to determine the graph of the parameterised curve

$$x = \cos t \quad \text{and} \quad y = 1 - 2 \sin^2 t \quad \text{for } 0 \leq t \leq 2\pi$$

- 10** Consider the parametric equations

$$x = 2^t + 2^{-t} \quad \text{and} \quad y = 2^t - 2^{-t}$$

- a** Show that the Cartesian equation of the curve is $\frac{x^2}{4} - \frac{y^2}{4} = 1$ for $x \geq 2$.
b Sketch the graph of the curve.

- 11** Consider the circle with Cartesian equation $x^2 + (y - 1)^2 = 1$.

- a** Sketch the graph of the circle.
b Show that the parametric equations $x = \cos t$ and $y = \sin t + 1$ define the same circle.
c A different parameterisation of the circle can be found without the use of the cosine and sine functions. Suppose that t is any real number and let $P(x, y)$ be the point of intersection of the line $y = 2 - tx$ with the circle. Solve for x and y in terms of t , assuming that $x \neq 0$.
d Verify that the equations found in part **c** parameterise the same circle.

- 12** The curve with parametric equations $x = \frac{t}{2\pi} \cos t$ and $y = \frac{t}{2\pi} \sin t$ is called an **Archimedean spiral**.

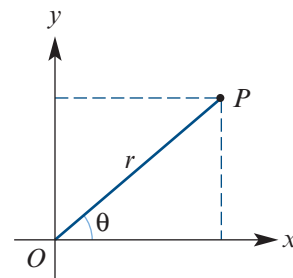
- a** With the help of your calculator, sketch the curve over the interval $0 \leq t \leq 6\pi$.
b Label the points on the curve corresponding $t = 0, 1, 2, 3, 4, 5, 6$.

20F Polar coordinates

We have already used polar form for vectors (in Section 6C) and polar form for complex numbers (in Section 15E). We now apply the same idea to points in the Cartesian plane. We will see in the following two sections that polar coordinates can be very convenient for describing interesting curves in the plane.

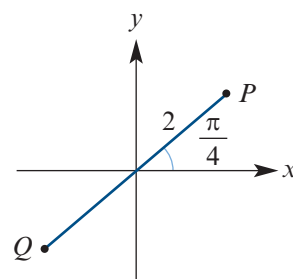
Using **polar coordinates**, every point P in the plane is described by a pair of numbers (r, θ) .

- The number r is the distance from the origin O to P .
- The number θ measures the angle between the positive direction of the x -axis and the ray OP , as shown.



For example, the diagram on the right shows the point P with polar coordinates $\left(2, \frac{\pi}{4}\right)$.

We can even make sense of polar coordinates such as $Q\left(-2, \frac{\pi}{4}\right)$:
Go to the direction $\frac{\pi}{4}$ and then move a distance of 2 in the opposite direction.



Note: This is different from polar form for vectors and complex numbers, where we require that $r \geq 0$.

Converting between polar and Cartesian coordinates

- If a point P has polar coordinates (r, θ) , then its Cartesian coordinates (x, y) satisfy

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

- If a point P has Cartesian coordinates (x, y) , then its polar coordinates (r, θ) satisfy

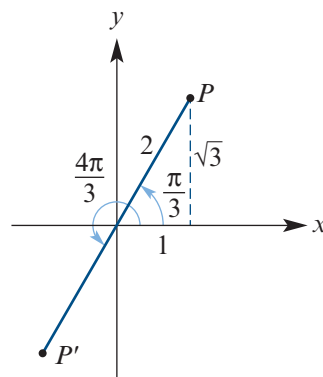
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x} \quad (\text{if } x \neq 0)$$

Non-uniqueness of polar coordinates

Polar coordinates differ from Cartesian coordinates in that each point in the plane has more than one representation in polar coordinates.

For example, the following polar coordinates all represent the same point:

$$\left(2, \frac{\pi}{3}\right), \quad \left(-2, \frac{4\pi}{3}\right) \quad \text{and} \quad \left(2, \frac{7\pi}{3}\right)$$



The point $P(r, \theta)$ can be described in infinitely many ways:

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n + 1)\pi) \quad \text{for all } n \in \mathbb{Z}$$

Example 19

Convert polar coordinates $\left(2, \frac{5\pi}{6}\right)$ into Cartesian coordinates.

Solution

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta \\ &= 2 \cos\left(\frac{5\pi}{6}\right) & &= 2 \sin\left(\frac{5\pi}{6}\right) \\ &= -\sqrt{3} & &= 1 \end{aligned}$$

The Cartesian coordinates are $(-\sqrt{3}, 1)$.

Example 20

For each pair of Cartesian coordinates, find two representations using polar coordinates, one with $r > 0$ and the other with $r < 0$.

a $(3, 3)$ **b** $(1, -\sqrt{3})$ **c** $(-5, 0)$ **d** $(0, 3)$ **Solution**

a $r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$$\theta = \tan^{-1}(1) = \frac{\pi}{4}$$

The point has polar coordinates $\left(3\sqrt{2}, \frac{\pi}{4}\right)$.

We could also let $r = -3\sqrt{2}$ and add π to the angle, giving $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$.

b $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

The point has polar coordinates $\left(2, -\frac{\pi}{3}\right)$.

We could also let $r = -2$ and add π to the angle, giving $\left(-2, \frac{2\pi}{3}\right)$.

c $r = 5$ and $\theta = \pi$

The point has polar coordinates $(5, \pi)$.

We could also let $r = -5$ and subtract π from the angle, giving $(-5, 0)$.

d $r = 3$ and $\theta = \frac{\pi}{2}$

The point has polar coordinates $\left(3, \frac{\pi}{2}\right)$.

We could also let $r = -3$ and subtract π from the angle, giving $\left(-3, -\frac{\pi}{2}\right)$.

Section summary

- Each point P in the plane can be represented using polar coordinates (r, θ) , where:
 - r is the distance from the origin O to P
 - θ is the angle between the positive direction of the x -axis and the ray OP .
- If a point P has polar coordinates (r, θ) , then its Cartesian coordinates (x, y) satisfy

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

- If a point P has Cartesian coordinates (x, y) , then all its polar coordinates (r, θ) satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x} \quad (\text{if } x \neq 0)$$

- Each point in the plane has more than one representation in polar coordinates.
For example, the coordinates $\left(2, \frac{\pi}{4}\right)$, $\left(2, \frac{9\pi}{4}\right)$ and $\left(-2, \frac{5\pi}{4}\right)$ all represent the same point.

Exercise 20F

Example 19

- 1 Plot the points with the following polar coordinates and then find their Cartesian coordinates:

a $A\left(1, \frac{\pi}{2}\right)$ **b** $B\left(2, \frac{3\pi}{4}\right)$ **c** $C\left(3, \frac{-\pi}{2}\right)$ **d** $D\left(-2, \frac{\pi}{4}\right)$ **e** $E(-1, \pi)$
f $F\left(0, \frac{\pi}{4}\right)$ **g** $G\left(4, -\frac{5\pi}{6}\right)$ **h** $H\left(-2, \frac{2\pi}{3}\right)$ **i** $I\left(-2, -\frac{\pi}{4}\right)$

Example 20

- 2 For each of the following pairs of Cartesian coordinates, find two representations using polar coordinates, one with $r > 0$ and the other with $r < 0$:

a $(1, -1)$ **b** $(1, \sqrt{3})$ **c** $(2, -2)$
d $(-\sqrt{2}, -\sqrt{2})$ **e** $(3, 0)$ **f** $(0, -2)$

- 3 Two points have polar coordinates $P\left(2, \frac{\pi}{6}\right)$ and $Q\left(3, \frac{\pi}{2}\right)$ respectively. Find the exact length of line segment PQ .
- 4 Two points have polar coordinates $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$. Find a formula for the length of PQ .

20G Graphing using polar coordinates

Polar coordinates are useful for describing and sketching curves in the plane, especially in situations that involve symmetry with respect to the origin. Suppose that f is a function. The graph of f in polar coordinates is simply the set of all points (r, θ) such that $r = f(\theta)$.

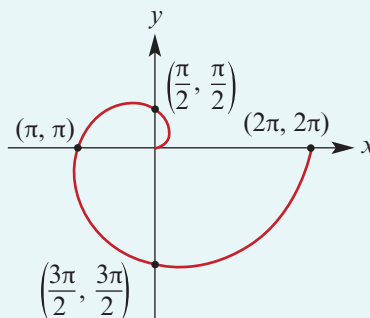
Example 21

Sketch the spiral with polar equation $r = \theta$, for $0 \leq \theta \leq 2\pi$.

Solution

The distance r from the origin exactly matches the angle θ . So as the angle increases, so too does the distance from the origin.

Note that the coordinates on the graph are in polar form.



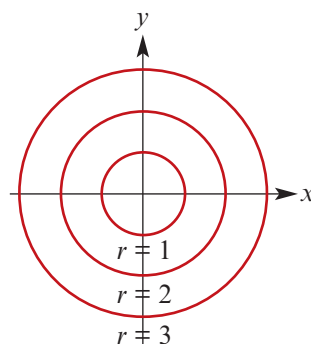
► Circles

If a circle is centred at the origin, then its polar equation could not be simpler.

A circle of radius a centred at the origin has polar equation

$$r = a$$

That is, the distance r from the origin is constant, having no dependence on the angle θ . This illustrates rather forcefully the utility of polar coordinates: they simplify situations that involve symmetry with respect to the origin.



For circles not centred at the origin, the polar equations are less obvious.

Example 22

A curve has polar equation $r = 2 \sin \theta$. Show that its Cartesian equation is $x^2 + (y - 1)^2 = 1$.

Solution

The trick here is to first multiply both sides of the polar equation by r to get

$$r^2 = 2r \sin \theta$$

Since $r^2 = x^2 + y^2$ and $r \sin \theta = y$, this equation becomes

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + (y^2 - 2y + 1) - 1 = 0 \quad (\text{completing the square})$$

$$x^2 + (y - 1)^2 = 1$$

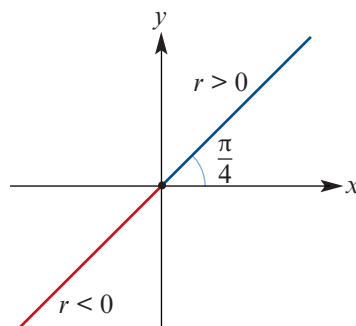
This is a circle with centre $(0, 1)$ and radius 1.

► Lines

For a straight line through the origin, the angle θ is fixed and the distance r varies. Because we have allowed negative values of r , the straight line goes in both directions.

The straight line shown has equation

$$\theta = \frac{\pi}{4}$$



For a straight line that does not go through the origin, the equation is more complicated.

A line in Cartesian form $ax + by = c$ can be converted into polar form by substituting $x = r \cos \theta$ and $y = r \sin \theta$.

Example 23

- a** Express $x + y = 1$ in polar form. **b** Express $r = \frac{2}{3 \cos \theta - 4 \sin \theta}$ in Cartesian form.

Solution

- a** Since $x = r \cos \theta$ and $y = r \sin \theta$, the equation $x + y = 1$ becomes **b** Since $\frac{x}{r} = \cos \theta$ and $\frac{y}{r} = \sin \theta$, the equation becomes

$$r \cos \theta + r \sin \theta = 1$$

$$r(\cos \theta + \sin \theta) = 1$$

Therefore the straight line has polar equation

$$r = \frac{1}{\cos \theta + \sin \theta}$$

$$r = \frac{2}{\frac{3x}{r} - \frac{4y}{r}}$$

$$r = \frac{2r}{3x - 4y}$$

$$1 = \frac{2}{3x - 4y}$$

Therefore the Cartesian equation is

$$3x - 4y = 2$$

Section summary

- For a function f , the graph of f in polar coordinates is the set of all points (r, θ) such that $r = f(\theta)$.
- Polar coordinates are useful for describing graphs that are symmetric about the origin. For example:
 - The circle centred at the origin with radius a has equation $r = a$.
 - The line through the origin at angle α to the positive x -axis has equation $\theta = \alpha$.
- To convert between the polar form and the Cartesian form of an equation, substitute $x = r \cos \theta$ and $y = r \sin \theta$.

Exercise 20G**Skillsheet****Example 21**

- 1** Sketch the spiral with polar equation $r = \frac{\theta}{2\pi}$, for $0 \leq \theta \leq 4\pi$.

- 2** Express each of the following Cartesian equations in polar form:

a $x = 4$

b $xy = 1$

c $y = x^2$

d $x^2 + y^2 = 9$

e $x^2 - y^2 = 1$

f $2x - 3y = 5$

- 3** Express each of the following polar equations in Cartesian form:

a $r = \frac{2}{\cos \theta}$

b $r = 2$

c $\theta = \frac{\pi}{4}$

d $r = \frac{4}{3 \cos \theta - 2 \sin \theta}$

Example 22

- 4** By finding the Cartesian equation, show that each of the following polar equations describes a circle:

a $r = 6 \cos \theta$

b $r = 4 \sin \theta$

c $r = -6 \cos \theta$

d $r = -8 \sin \theta$

- 5 Show that the graph of $r = 2a \cos \theta$ is a circle of radius a centred at $(a, 0)$.

Example 23

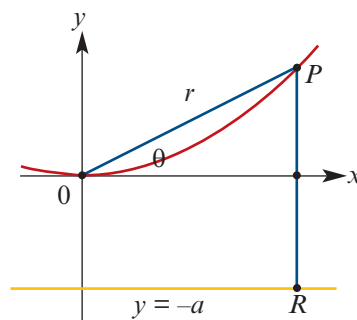
- 6 a Show that the graph of $r = \frac{a}{\cos \theta}$ is a vertical line.

b Find the polar form of the horizontal line $y = a$.

- 7 A set of points $P(x, y)$ is such that the distance from P to the origin O is equal to the perpendicular distance from P to the line $y = -a$, where $a > 0$. This set of points is a parabola. Suppose that point P has polar coordinates (r, θ) .

a Show that the distance from P to the line is $a + r \sin \theta$.

b Conclude that the equation for the parabola can be written as $r = \frac{a}{1 - \sin \theta}$.



20H Further graphing using polar coordinates

Various geometrically significant and beautiful figures are best described using polar coordinates.

► Cardioids

The name **cardioid** comes from the Greek word for heart. A cardioid is the curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius.

Example 24

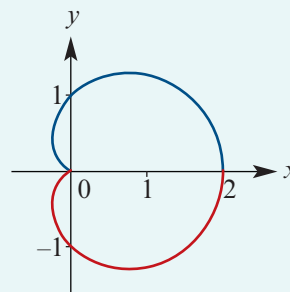
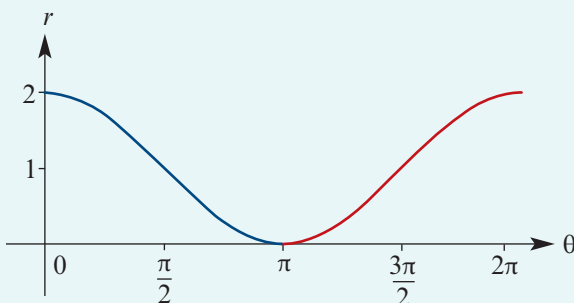
Graph the cardioid with equation $r = 1 + \cos \theta$, for $\theta \in [0, 2\pi]$.

Solution

To help sketch this curve, we first graph the function $r = 1 + \cos \theta$ using Cartesian coordinates, as shown on the left. This allows us to see how r changes as θ increases.

- As the angle θ increases from 0 to π , the distance r decreases from 2 to 0.
- As the angle θ increases from π to 2π , the distance r increases from 0 to 2.

This gives the graph of the cardioid shown on the right.

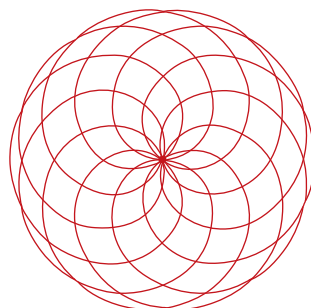


Roses

This impressive curve is fittingly called a **rose**. It belongs to the family of curves with polar equations of the form

$$r = \cos(n\theta)$$

For the example shown, $n = \frac{5}{8}$.



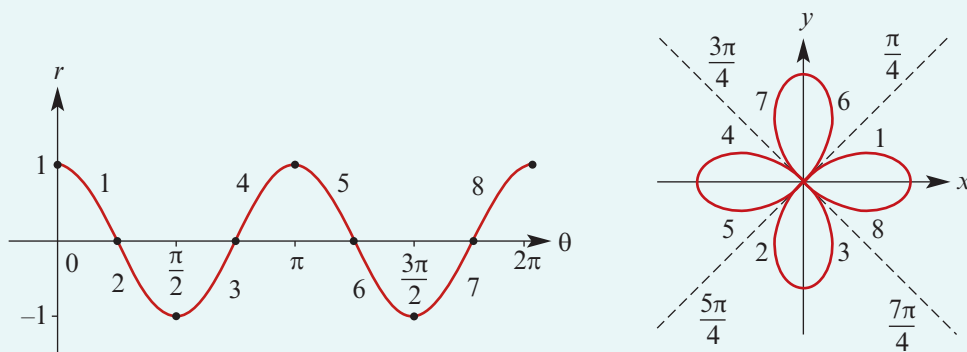
Example 25

A curve has polar equation $r = \cos(2\theta)$.

- Sketch the graph of the curve.
- Show that its Cartesian equation is $(x^2 + y^2)^3 = (x^2 - y^2)^2$.

Solution

- To help sketch this curve, we first graph the function $r = \cos(2\theta)$ using Cartesian coordinates, as shown on the left. This allows us to see how r changes as θ increases. Using numbers, we have labelled how each section of this graph corresponds to a section of the rose shown on the right.



- Using the double-angle identity $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$, we have

$$\begin{aligned} r &= \cos(2\theta) \\ r &= \cos^2 \theta - \sin^2 \theta \\ r &= \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \\ r^3 &= x^2 - y^2 \\ (\sqrt{x^2 + y^2})^3 &= x^2 - y^2 \\ (x^2 + y^2)^3 &= (x^2 - y^2)^2 \end{aligned}$$

Note: This example further illustrates how polar coordinates can give more pleasing equations than their Cartesian counterparts.

The curve in this example is a **four-leaf rose**. More generally, the equations $r = \cos(n\theta)$ and $r = \sin(n\theta)$ give $2n$ -leaf roses if n is even, and give n -leaf roses if n is odd.

► Using a graphics calculator with polar coordinates

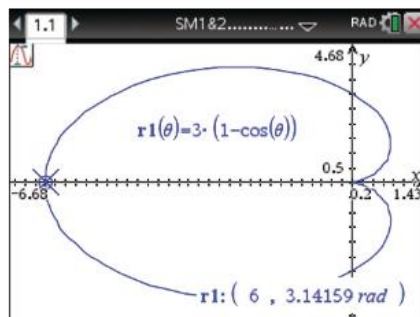
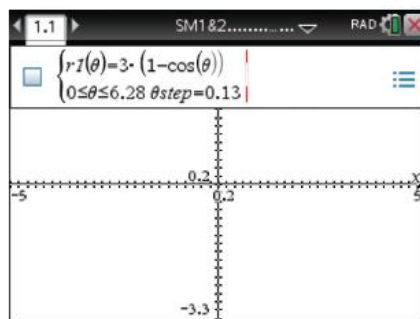
Example 26

Plot the graph of $r = 3(1 - \cos \theta)$.



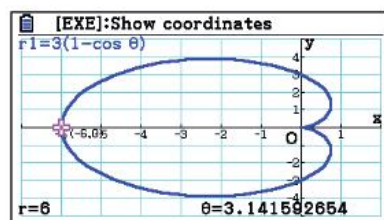
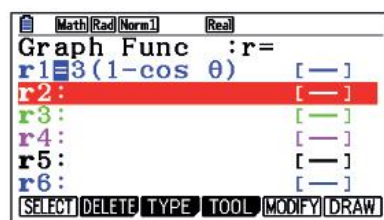
Using the TI-Nspire CX non-CAS

- Open a **Graphs** application (on) > **New Document** > **Add Graphs** and set to polar using > **Graph Entry/Edit** > **Polar**.
- Enter $r1(\theta) = 3(1 - \cos(\theta))$ as shown. The variable θ is entered using or the Symbols palette ().
- Note:** The domain and the step size can be adjusted in this window.
- Set the scale using > **Window/Zoom** > **Zoom – Fit**.
- You can see the polar coordinates (r, θ) of points on the graph using > **Trace** > **Graph Trace**.
- To go to the point where $\theta = \pi$, simply type π and then press . The cursor will move to the point $(r, \theta) = (6, \pi)$ as shown.



Using the Casio

- Press to select **Graph** mode.
- To specify the graph type, go to **Type** and choose **r=** for polar coordinates.
- Enter the rule $r = 3(1 - \cos \theta)$ in $r1$:
- Adjust the View Window:
- Select **Draw** .
- You can view the polar coordinates of points on the graph. For example, to go to the point where $\theta = \pi$, use **Trace** and enter the angle π .



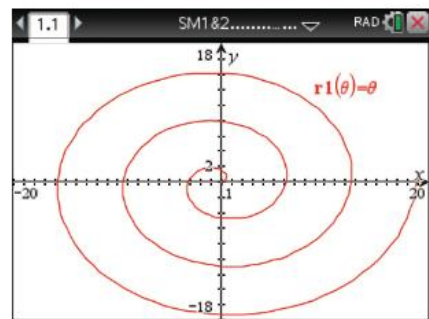
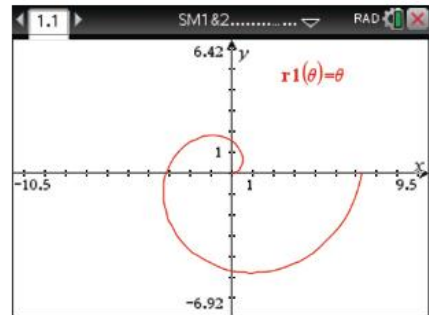
Example 27

Plot the graph of $r = \theta$ for $0 \leq \theta \leq 6\pi$.

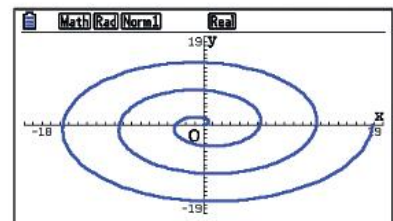
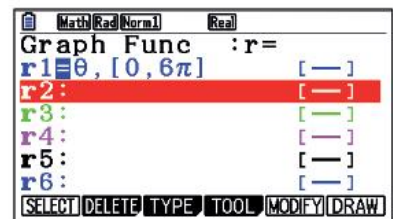
**Using the TI-Nspire CX non-CAS**

- Open a **Graphs** application (on) > **New Document** > **Add Graphs**) and set to polar using > **Graph Entry/Edit** > **Polar**.
- Enter $r1(\theta) = \theta$.
- The graph is shown.
- If the domain is extended, the graph continues to spiral out. This can be observed by extending the domain to $0 \leq \theta \leq 6\pi$.

Note: Change the domain in the graph entry line; it is not the window setting.

**Using the Casio**

- Press **5** to select **Graph** mode.
- To specify the graph type, go to **Type** () and choose **r=** () for polar coordinates.
- Enter the rule $r = \theta$ and domain $[0, 6\pi]$ in $r1$:
 X, θ, T ,
+ **0** , **6** **EXP** **-** **EXE**
- Adjust the View Window **F3** for $-20 \leq x \leq 20$ and $-20 \leq y \leq 20$.
- Select **Draw** ()



Section summary

- To sketch the curve $r = f(\theta)$ in polar coordinates, it helps to first sketch the graph in Cartesian coordinates.

Exercise 20H

- 1 Consider the polar equation

$$r = \frac{e}{1 + e \sin \theta}$$

When $0 < e < 1$, it can be shown that this equation describes an ellipse.

- With the help of your calculator, sketch the graphs of this polar equation when $e = 0.7, 0.8, 0.9$ on the same set of axes.
- The number e is called the **eccentricity** of the ellipse. What happens to the shape of the ellipse as the value of e is increased?

Example 24

- 2 The curve with polar equation $r = 1 - \sin \theta$ is a cardioid.

- Sketch the graph of the cardioid.
- Show that its Cartesian equation is $(x^2 + y^2 + y)^2 = x^2 + y^2$.

Example 25

- 3 Sketch the graphs of the roses with the following polar equations:

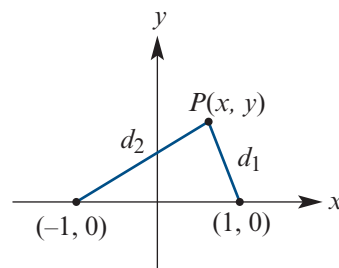
- $r = \cos(3\theta)$
- $r = \sin(3\theta)$

- 4 The polar equation $r = \sin(2\theta)$ defines a four-leaf rose.

- Sketch the graph of the rose.
- Using a suitable double-angle identity, show that its Cartesian equation is $(x^2 + y^2)^3 = 4x^2y^2$.

- 5 a With the help of your calculator, sketch the graph of the **lemniscate**, which has polar equation $r = \sqrt{2 \cos(2\theta)}$.

- Using an appropriate double-angle identity, find its equation in Cartesian coordinates.
- Show that the lemniscate can also be described as the set of all points $P(x, y)$ in the diagram that satisfy $d_1 d_2 = 1$.



Chapter summary



Parabolas, ellipses and hyperbolas

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R .
- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 is a constant.
- The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k) .

- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant.
- The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k) . The asymptotes are $y - k = \pm \frac{b}{a}(x - h)$.

Parametric curves

- A **parametric curve** in the plane is a pair of functions

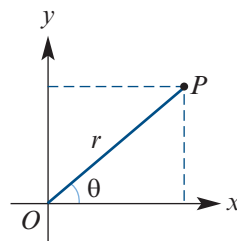
$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is called the **parameter** of the curve.

- It can be helpful to think of the parameter t as describing time. Parametric curves are then useful for describing the motion of an object.
- We can sometimes find the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x .

Polar coordinates

- Each point P in the plane can be represented using polar coordinates (r, θ) , where:
 - r is the distance from the origin O to P
 - θ is the angle between the positive direction of the x -axis and the ray OP .
- If f is a function, then the graph of f in polar coordinates is the set of all points (r, θ) such that $r = f(\theta)$.



Technology-free questions

- 1** The point $P(x, y)$ moves so that it is equidistant from $Q(2, -1)$ and $R(1, 2)$. Find the locus of the point P . SF
- 2** Find the locus of the point $P(x, y)$ such that $AP = 6$, given point $A(3, 2)$. CF
- 3** A circle has equation $x^2 + 4x + y^2 - 8y = 0$. Find the coordinates of the centre and the radius of the circle.
- 4** Sketch the graph of each ellipse and find the coordinates of its axis intercepts:
a $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **b** $\frac{(x-2)^2}{4} + \frac{(y+1)^2}{9} = 1$
- 5** An ellipse has equation $x^2 + 4x + 2y^2 = 0$. Find the coordinates of the centre and the axis intercepts of the ellipse.
- 6** Sketch the graph of each hyperbola and write down the equations of its asymptotes:
a $x^2 - \frac{y^2}{4} = 1$ **b** $\frac{(y-1)^2}{16} - \frac{(x+2)^2}{4} = 1$
- 7** A point $P(x, y)$ moves so that its distance from the point $K(-2, 5)$ is twice its distance from the line $x = 1$. Find its locus. SF
- 8** For each of the following pairs of parametric equations, find the corresponding Cartesian equation:
a $x = 2t - 1$ and $y = 6 - 4t$ **b** $x = 2 \cos t$ and $y = 2 \sin t$
c $x = 3 \cos t + 1$ and $y = 5 \sin t - 1$ **d** $x = \cos t$ and $y = 3 \sin^2 t - 2$
- 9** A curve has parametric equations $x = t - 1$ and $y = 2t^2 - 1$ for $0 \leq t \leq 2$.
a Find the curve's Cartesian equation. **b** What is the domain of the curve?
c What is the range of the curve? **d** Sketch the graph of the curve.
- 10** Convert the polar coordinates $\left(2, \frac{3\pi}{4}\right)$ into Cartesian coordinates.
- 11** The point P has Cartesian coordinates $(2, -2\sqrt{3})$. Find two representations of P using polar coordinates, one with $r > 0$ and the other with $r < 0$.
- 12** Find an equation for the straight line $2x + 3y = 5$ using polar coordinates.
- 13** Show that the graph of $r = 6 \sin \theta$ is a circle of radius 3 centred at $(0, 3)$.
Hint: First multiply both sides of the equation by r .
- 14** Convert the following polar equations into Cartesian equations:
a $r = 5$ **b** $\theta = \frac{\pi}{3}$ **c** $r = \frac{3}{\sin \theta}$
d $r = \frac{2}{3 \sin \theta + 4 \cos \theta}$ **e** $r^2 = \frac{1}{\sin(2\theta)}$

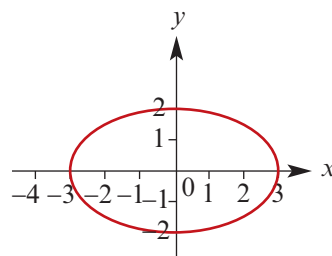
Multiple-choice questions



- 1 The locus of points $P(x, y)$ which satisfy the property that $AP = BP$, given points $A(2, -5)$ and $B(-4, 1)$, is described by the equation
A $y = x - 1$ **B** $y = x - 6$ **C** $y = -x - 3$ **D** $y = x + 1$ **E** $y = 3 - x$
- 2 A parabola has focus $(0, 2)$ and directrix $y = -2$. Which of the following is not true about the parabola?
A Its axis of symmetry is the line $x = 0$. **B** It passes through the origin.
C It contains no point below the x -axis. **D** The point $(2, 1)$ lies on the parabola.
E The point $(4, 2)$ lies on the parabola.

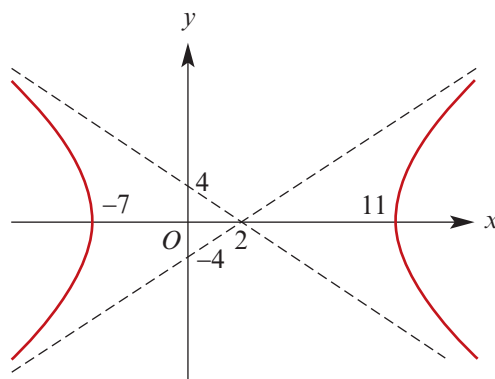
- 3 The equation of the graph shown is

- A** $\frac{x^2}{3} + \frac{y^2}{2} = 1$ **B** $\frac{x^2}{9} - \frac{y^2}{4} = 1$
C $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **D** $\frac{x^2}{3} - \frac{y^2}{2} = 1$
E $3x^2 + 2y^2 = 1$



- 4 The equation of the graph shown is

- A** $\frac{(x+2)^2}{27} - \frac{y^2}{108} = 1$
B $\frac{(x-2)^2}{9} - \frac{y^2}{34} = 1$
C $\frac{(x+2)^2}{81} - \frac{y^2}{324} = 1$
D $\frac{(x-2)^2}{81} - \frac{y^2}{324} = 1$
E $\frac{(x+2)^2}{9} - \frac{y^2}{36} = 1$

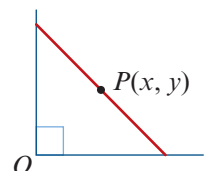


- 5 The asymptotes of the hyperbola with equation $\frac{(y-2)^2}{9} - \frac{(x+3)^2}{4} = 1$ intersect at the point
A $(3, 2)$ **B** $(3, -2)$ **C** $(-3, 2)$ **D** $(2, -3)$ **E** $(-2, 3)$
- 6 An ellipse is parameterised by the equations $x = 4 \cos t + 1$ and $y = 2 \sin t - 1$. The coordinates of its x -axis intercepts are
A $(1 - 3\sqrt{2}, 0), (1 + 3\sqrt{2}, 0)$ **B** $(-3, 0), (5, 0)$
C $(1 - 2\sqrt{3}, 0), (1 + 2\sqrt{3}, 0)$ **D** $(0, -3), (0, 5)$
E $(0, 1 - 2\sqrt{3}), (0, 1 + 2\sqrt{3})$

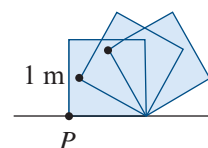
- 7** Which of the following pairs of polar coordinates represent the same point?
- A** $\left(2, \frac{\pi}{4}\right)$ and $\left(2, \frac{3\pi}{4}\right)$ **B** $\left(3, \frac{\pi}{2}\right)$ and $\left(-3, \frac{\pi}{2}\right)$ **C** $\left(2, \frac{\pi}{3}\right)$ and $\left(-2, \frac{2\pi}{3}\right)$
- D** $\left(3, \frac{\pi}{4}\right)$ and $\left(3, \frac{5\pi}{4}\right)$ **E** $\left(1, \frac{\pi}{6}\right)$ and $\left(-1, \frac{7\pi}{6}\right)$
- 8** A curve has polar equation $r = 1 + \cos \theta$. Its equation in Cartesian coordinates is
- A** $xy = x^2 + y^2$ **B** $(x^2 + y^2 - x)^2 = x^2 + y^2$ **C** $x = x^2 + y^2$
- D** $(x^2 + y^2 - y)^2 = x^2 + y^2$ **E** $y = x^2 + y^2$
- 9** A curve is parameterised by the equations $x = 2t - 3$ and $y = t^2 - 3t$. Which of the following points does the curve pass through?
- A** (5, 1) **B** (5, 2) **C** (5, 3) **D** (5, 4) **E** (5, 5)

Extended-response questions

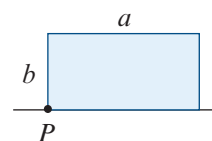
- 1** Consider points $A(0, 3)$ and $B(6, 0)$. Find the locus of the point $P(x, y)$ given that:
- a** $AP = BP$
- b** $AP = 2BP$
- 2** Find the equation of the locus of points $P(x, y)$ which satisfy the property that the distance to P from the point $F(0, 4)$ is equal to:
- a** MP , the perpendicular distance from the line with equation $y = -2$
- b** half the distance MP , the perpendicular distance from the line $y = -2$
- c** twice the distance MP , the perpendicular distance from the line $y = -2$.
- 3** A ball is thrown into the air. The position of the ball at time $t \geq 0$ is given by the parametric equations $x = 10t$ and $y = 20t - 5t^2$.
- a** Find the Cartesian equation of the ball's flight.
- b** Sketch the graph of the ball's path.
- c** What is the maximum height reached by the ball?
- A second ball is thrown into the air. Its position at time $t \geq 0$ is given by the parametric equations $x = 60 - 10t$ and $y = 20t - 5t^2$.
- d** Find the Cartesian equation of the second ball's flight.
- e** Sketch the graph of the second ball's path on the same set of axes.
- f** Find the points of intersection of the two paths.
- g** Do the balls collide?
- 4** A ladder of length 6 metres stands against a vertical wall. The ladder then slides along on the floor until it lies flat. Show that the midpoint $P(x, y)$ of the ladder moves along a circular path.



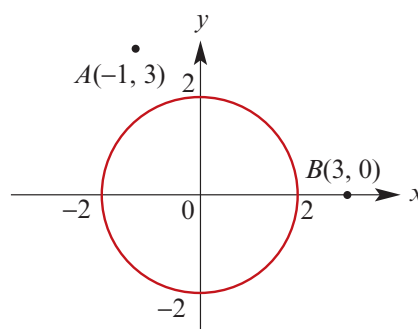
- 5** A square box of side length 1 metre is too heavy to lift, but can be rolled along the flat ground, using each edge as a pivot. The box is rolled one full revolution.



- Sketch the full path of the point P .
- Find the total distance travelled by the point P .
- A second rectangular box has length a metres and width b metres. Sketch the path taken by the point P when the box is rolled one full revolution, and find the total distance travelled by this point.
- For the second box, find the area between the path taken by P and the ground.



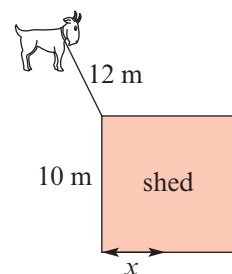
- 6** The circle with equation $x^2 + y^2 = 4$ is shown. We will say that point A is **visible** to point B if the line AB does not intersect the circle.



- Consider points $A(-1, 3)$ and $B(3, 0)$. Show that the equations

$$x = 4t - 1 \quad \text{and} \quad y = 3 - 3t$$
 parameterise the line AB .
- Show that A is not visible to B by showing that there are two values of t for which the line AB intersects the circle.
- Find parametric equations for the line that goes through points $C(-1, 4)$ and $B(3, 0)$.
- Show that C is visible to B by showing that there is no value of t for which the line CB intersects the circle.
- Find the range of values k for which the point $D(-1, k)$ is visible to B .

- 7** A shed has a square base of side length 10 metres. A goat is tied to a corner of the shed by a rope of length 12 metres. As the goat pulls tightly on the rope and walks around the shed in both directions, a path is traced by the goat.



- Sketch the shed and the path described above.
- Find the size of the area over which the goat can walk.
- The goat is now tied to a point on the shed x metres from the corner, where $0 \leq x \leq 5$. Find a formula for the area A over which the goat can walk, in terms of x .
Hint: Consider the two cases $0 \leq x \leq 2$ and $2 < x \leq 5$.
- Sketch the graph of A against x for $0 \leq x \leq 5$.
- Where should the goat be tied if the area is to be:
 - a maximum
 - a minimum?

Glossary

A

Absolute value function [p. 365] The absolute value of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *modulus function*

Addition of complex numbers [p. 464]

If $z_1 = a + bi$ and $z_2 = c + di$, then
 $z_1 + z_2 = (a + c) + (b + d)i$.

Addition of ordinates [p. 376] a method for sketching the graph of a function by adding the y-coordinates (ordinates) of two simpler graphs

Addition of vectors [p. 137]

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then
 $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$.

Addition principle [p. 188] Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

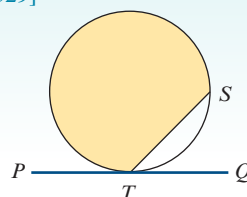
Airspeed [p. 167] the speed of an aircraft relative to air. The actual velocity of an aircraft is the vector sum of the velocity of the aircraft relative to air and the velocity of the wind.

Alternate angles [p. 270] In each diagram, the two marked angles are alternate angles. (They are on alternate sides of the transversal.)



Alternate segment [p. 329]

The alternate segment to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.



Altitude of a triangle [p. 275] a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side

Amplitude of trigonometric functions [p. 400]

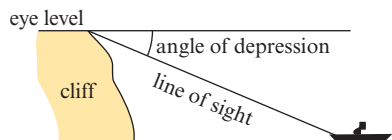
The distance between the mean position and the maximum position is called the amplitude. The graph of $y = a \sin x$ has an amplitude of $|a|$.

Angle between two vectors [p. 154] can be found using the scalar product:

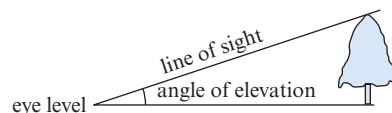
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

Angle of depression [p. 79] the angle between the horizontal and a direction below the horizontal



Angle of elevation [p. 79] the angle between the horizontal and a direction above the horizontal



Angle sum and difference identities [p. 442]

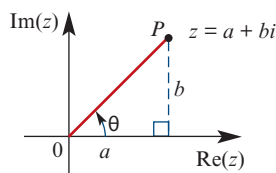
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Angle sum of a polygon [p. 276] The sum of the interior angles of a simple n -sided polygon is $(n - 2)180^\circ$.

Arc [p. 323] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

Area of image [p. 572] If a linear transformation (with matrix **B**) is applied to a region of the plane, then Area of image = $|\det(\mathbf{B})| \times$ Area of region.

Argand diagram [p. 473] a geometric representation of the set of complex numbers

**Argument of a complex number** [p. 480]

- The argument of z is an angle θ from the positive direction of the x -axis to the line joining the origin to z .
- The *principal value* of the argument, denoted by $\text{Arg } z$, is the angle in the interval $(-\pi, \pi]$.

Arrangement [p. 191] see permutation

Asymptote [p. 376] A straight line or curve is an asymptote of the graph of a function $y = f(x)$ if the graph of $y = f(x)$ gets arbitrarily close to the straight line or curve. A straight-line asymptote can be vertical, horizontal or oblique.

Asymptotes of hyperbolas [p. 612]

The hyperbola with equation

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm \frac{b}{a}(x - h)$$

Asymptotes of rational functions [p. 376]

The rule of a rational function can be written in quotient-remainder form as $f(x) = q(x) + \frac{r(x)}{b(x)}$.

- The vertical asymptotes occur where $b(x) = 0$.
- The non-vertical asymptote is given by $y = q(x)$.

B

Bearing [p. 80] the compass bearing; the direction measured from north clockwise

C

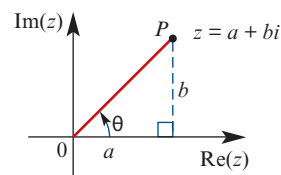
$\mathbb{C}$ [p. 463] the set of complex numbers:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Cartesian equation [p. 601] an equation that describes a curve in the plane by giving the relationship between the x - and y -coordinates of the points on the curve; e.g. $y = x^2 + 1$

Cartesian form of a complex number [p. 463]

A complex number is expressed in Cartesian form as $z = a + bi$, where a is the real part of z and b is the imaginary part of z .



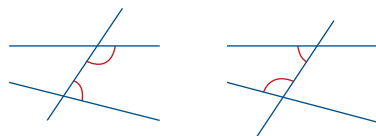
Chord [p. 323] a line segment with endpoints on a circle

Circle, general Cartesian equation [p. 601]

The circle with radius r and centre (h, k) has equation $(x - h)^2 + (y - k)^2 = r^2$.

cis θ [p. 480] $\cos \theta + i \sin \theta$

Co-interior angles [p. 270] In each diagram, the two marked angles are co-interior angles. (They are on the same side of the transversal.)



Collinear points [p. 306] Three or more points are collinear if they all lie on a single line.

Column vector [pp. 136, 547] an $n \times 1$ matrix.

A column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be used to represent a vector in the plane, an ordered pair, a point in the Cartesian plane or a translation of the plane.

Combination [p. 202] a selection where order is not important. The number of combinations of n objects taken r at a time is given by

$${}^nC_r = \frac{n!}{r!(n - r)!}$$

An alternative notation for ${}^nC_r$ is $\binom{n}{r}$.

Compass bearing [p. 80] the direction measured from north clockwise

Complement of a set [p. 36] The complement of a set A , written A' , is the set of all elements of ξ that are not elements of A .

Complementary angles [p. 268] two angles whose sum is 90°

Complementary relationships [p. 405]

$$\begin{aligned} \blacksquare \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \blacksquare \sin\left(\frac{\pi}{2} + \theta\right) &= \cos \theta \\ \blacksquare \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta & \blacksquare \cos\left(\frac{\pi}{2} + \theta\right) &= -\sin \theta \end{aligned}$$

Complex conjugate, $\bar{z}$ [p. 467]

- If $z = a + bi$, then $\bar{z} = a - bi$.
- If $z = r \operatorname{cis} \theta$, then $\bar{z} = r \operatorname{cis}(-\theta)$.

Complex conjugate, properties [p. 468]

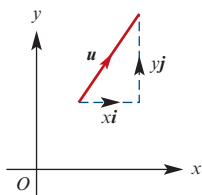
$$\begin{aligned} \blacksquare z + \bar{z} &= 2 \operatorname{Re}(z) & \blacksquare z\bar{z} &= |z|^2 \\ \blacksquare \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 & \blacksquare \overline{z_1 \cdot z_2} &= \bar{z}_1 \cdot \bar{z}_2 \end{aligned}$$

Complex number [p. 463] an expression of the form $a + bi$, where a and b are real numbers

Complex plane [p. 473] *see* Argand diagram

Component form [p. 145] Each vector $\mathbf{u}$ in the plane can be written in component form as $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$, where

- $\mathbf{i}$ is the unit vector in the positive direction of the x -axis
- $\mathbf{j}$ is the unit vector in the positive direction of the y -axis.



Composite [p. 47] A natural number m is a composite number if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m .

Concurrent lines [p. 306] Three or more lines are concurrent if they all pass through a single point.

Conditional statement [p. 233]

a statement of the form 'If P is true, then Q is true', which can be abbreviated to $P \Rightarrow Q$

Congruence tests [pp. 63, 279] Two triangles are congruent if one of the following conditions holds:

- **SSS** the three sides of one triangle are equal to the three sides of the other triangle
- **SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
- **AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
- **RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

Congruent figures [pp. 63, 279] have exactly the same shape and size

Constant velocity [p. 161] If a particle moves with a constant velocity of v m/s for t seconds, then its displacement vector, s m, is given by $s = tv$.

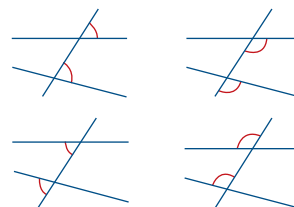
Contrapositive [p. 239] The contrapositive of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P). The contrapositive is equivalent to the original statement.

Converse [p. 246] The converse of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Conversion between Cartesian and polar forms [pp. 149, 480, 625]

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2$$

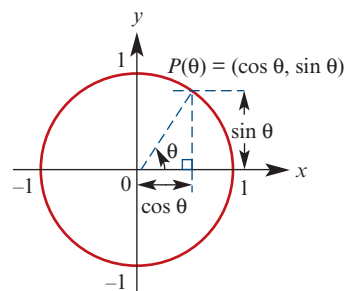
Corresponding angles [p. 270] In each diagram, the two marked angles are corresponding angles.



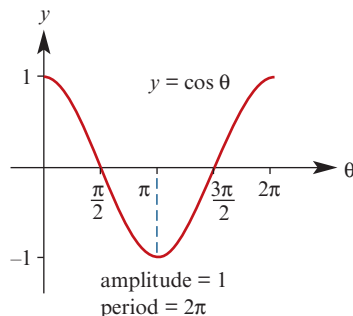
Cosecant function [pp. 422, 438]

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \text{ for } \sin \theta \neq 0$$

Cosine function [p. 400] cosine θ is defined as the x -coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x -axis.

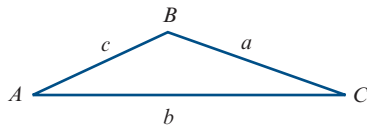


Cosine function, graph [p. 400]



Cosine rule [p. 76] For triangle ABC :

$$a^2 = b^2 + c^2 - 2bc \cos A$$



Cotangent function [pp. 423, 438] $\cot \theta = \frac{\cos \theta}{\sin \theta}$
for $\sin \theta \neq 0$

Counterexample [p. 250] an example that shows that a universal statement is false. For example, the number 2 is a counterexample to the claim 'Every prime number is odd.'

Cyclic quadrilateral [p. 325] a quadrilateral such that all its vertices lie on a circle. Opposite angles of a cyclic quadrilateral are supplementary.

D

De Moivre's theorem [p. 485]

$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta)$, where $n \in \mathbb{Z}$

De Morgan's laws [p. 238]

- 'not (P and Q)' is '(not P) or (not Q)'
- 'not (P or Q)' is '(not P) and (not Q)'

Degree of a polynomial [p. 91] given by the highest power of x with a non-zero coefficient; e.g. the polynomial $2x^5 - 7x^2 + 4$ has degree 5

Determinant of a matrix [pp. 523, 532]

Associated with each square matrix $\mathbf{A}$, there is a real number called the determinant of $\mathbf{A}$, which is denoted by $\det(\mathbf{A})$. A square matrix $\mathbf{A}$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

Diameter [p. 323] a chord of a circle that passes through the centre

Dilation [p. 552] A dilation scales the x - or y -coordinate of each point in the plane.

- Dilation from the x -axis: $(x, y) \rightarrow (x, cy)$
- Dilation from the y -axis: $(x, y) \rightarrow (cx, y)$

Direct proof [p. 234] To give a direct proof of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Discriminant, Δ , of a quadratic [p. 96]

the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:

- If $b^2 - 4ac > 0$, there are two real solutions.
- If $b^2 - 4ac = 0$, there is one real solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint sets [p. 35] Sets A and B are said to be disjoint if they have no elements in common, i.e. if $A \cap B = \emptyset$.

Displacement [p. 160] the change in position. If a particle moves from point A to point B , then its displacement is described by the vector $\vec{AB}$.

Division of complex numbers [pp. 468, 482]

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Dot product [p. 153] see scalar product

Double-angle identities [p. 445]

$$\begin{aligned} \cos(2A) &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1 \\ &= 1 - 2 \sin^2 A \end{aligned}$$

$$\sin(2A) = 2 \sin A \cos A$$

E

Ellipse [pp. 569, 607] The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k) .

Equality of complex numbers [p. 464]

$a + bi = c + di$ if and only if $a = c$ and $b = d$

Equally likely outcomes [p. 223] If the outcomes of a random experiment are all equally likely, then the probability of an event A is given by

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}$$

Equilibrium [p. 170] A particle is said to be in equilibrium if the resultant force acting on it is zero; the particle will remain at rest or continue moving with constant velocity.

Equivalence of vectors [p. 145]

Let $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j}$ and $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j}$. If $\mathbf{a} = \mathbf{b}$, then $a_1 = b_1$ and $a_2 = b_2$.

Equivalent statements [p. 247]

Statements P and Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$; this is abbreviated to $P \Leftrightarrow Q$.

For equivalent statements P and Q , we also say ' P is true if and only if Q is true'.

Event [p. 223] a subset of the sample space. It may consist of a single outcome, or it may consist of several outcomes.

Existence statement [pp. 249, 251] a statement claiming that a property holds for some member of a given set. Such a statement can be written using the quantifier ‘there exists’.

F

Factor [p. 47] A natural number a is a factor of a natural number b if there exists a natural number k such that $b = ak$.

Factorial notation [p. 191] The notation $n!$ (read as ‘ n factorial’) is an abbreviation for the product of all the integers from n down to 1:
 $n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$

Force [p. 170] causes a change in motion; e.g. gravitational force, tension force, normal reaction force. Force is a vector quantity.

Formula [p. 19] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length $\times$ width). The value of A , the subject of the formula, can be found by substituting given values of ℓ and w .

G

Geometric mean [p. 311] For $a, b, c \in \mathbb{R}^+$, if $\frac{c}{a} = \frac{b}{c}$, then c is the geometric mean of a and b .

Golden ratio, φ [p. 310] $\varphi = \frac{1 + \sqrt{5}}{2}$

Golden rectangle [p. 310] a rectangle with ratio of length to width $\varphi : 1$

H

Highest common factor [p. 48] The highest common factor of two natural numbers a and b , denoted by $\text{HCF}(a, b)$, is the largest natural number that is a factor of both a and b .

Hyperbola [p. 611] The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k) ; the asymptotes are given by

$$y - k = \pm \frac{b}{a}(x - h)$$

I

Imaginary number i [p. 463] $i^2 = -1$

Imaginary part of a complex number [p. 464]

If $z = a + bi$, then $\text{Im}(z) = b$. Note that $\text{Im}(z)$ is a real number.

Implication [p. 233] *see* conditional statement

Inclusion–exclusion principle [p. 218] allows us to count the number of elements in a union of sets. In the case of two sets: $|A \cup B| = |A| + |B| - |A \cap B|$

Index laws [p. 2]

■ $a^m \times a^n = a^{m+n}$	■ $a^m \div a^n = a^{m-n}$
■ $(a^m)^n = a^{mn}$	■ $(ab)^n = a^n b^n$
■ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	■ $a^{-n} = \frac{1}{a^n}$
■ $\frac{1}{a^{-n}} = a^n$	■ $a^0 = 1$
■ $a^{\frac{1}{n}} = \sqrt[n]{a}$	■ $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Integers [p. 38] the elements of

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

Intersection of sets [p. 36] The intersection of two sets A and B , written $A \cap B$, is the set of all elements common to A and B .

Interval [p. 41] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞) , etc.

Irrational number [p. 38] a real number that is not rational; e.g. π and $\sqrt{2}$

K

Kilogram weight, kg wt [p. 170] a unit of force. If an object on the surface of the Earth has a mass of 1 kg, then the gravitational force acting on this object is 1 kg wt.

L

Like surds [p. 44] surds with the same irrational factor; e.g. $2\sqrt{7}$ and $9\sqrt{7}$

Linear equation [p. 8] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear transformation [p. 547] a transformation of the plane with a rule of the form

$$(x, y) \rightarrow (ax + by, cx + dy)$$

Each linear transformation can be represented by a 2×2 matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear transformation, inverse [p. 563]

If $\mathbf{A}$ is the matrix of a linear transformation and $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.

Linear transformations, composition [p. 560]

If $\mathbf{A}$ and $\mathbf{B}$ are the matrices of two different linear transformations, then the product $\mathbf{BA}$ is the matrix of the transformation $\mathbf{A}$ followed by $\mathbf{B}$.

Literal equation [p. 25] an equation for the variable x in which the coefficients of x , including the constants, are pronumerals; e.g. $ax + b = c$

Locus [p. 601] a set of points described by a geometric condition; e.g. the locus of points P that satisfy $PO = 3$, where O is the origin, is the circle of radius 3 centred at the origin

Lowest common multiple [p. 50] The lowest common multiple of two natural numbers a and b , denoted by $\text{LCM}(a, b)$, is the smallest natural number that is a multiple of both a and b .

M

Magnitude of a vector [p. 145] the length of a directed line segment corresponding to the vector. If $\mathbf{u} = xi + yj$, then $|\mathbf{u}| = \sqrt{x^2 + y^2}$.

Mass [p. 170] The mass of an object is the amount of matter it contains, and can be measured in kilograms. Mass is not the same as weight.

Mathematical induction [p. 253] a proof technique for showing that a statement is true for all natural numbers; uses the *principle of mathematical induction*

Matrices, addition [p. 514] Addition is defined for two matrices of the same size (same number of rows and same number of columns). The sum is found by adding corresponding entries. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

Matrices, equal [p. 512] Two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal, and we write $\mathbf{A} = \mathbf{B}$, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

Matrices, multiplication [p. 518] The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is only defined if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Matrix [p. 510] a rectangular array of numbers

Matrix, dimensions [p. 510] A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

Matrix, identity [pp. 521, 529]

For square matrices of a given size (e.g. 2×2), a multiplicative identity $\mathbf{I}$ exists.

For 2×2 matrices, the identity is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ for each 2×2 matrix $\mathbf{A}$.

Matrix, inverse [pp. 522, 529] If $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then $\mathbf{B}$ is called the inverse of $\mathbf{A}$. When it exists, the inverse of a square matrix $\mathbf{A}$ is unique and is denoted by $\mathbf{A}^{-1}$.

$$\text{If } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $ad - bc \neq 0$.

Matrix, invertible [p. 522] A square matrix is said to be invertible if its inverse exists.

Matrix, multiplication by a scalar [p. 514]

If $\mathbf{A}$ is an $m \times n$ matrix and k is a real number, then $k\mathbf{A}$ is an $m \times n$ matrix whose entries are k times the corresponding entries of $\mathbf{A}$. For example:

$$3 \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Matrix, non-invertible [p. 522] A square matrix is said to be non-invertible if it does not have an inverse.

Matrix, size [p. 510] A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

Matrix, square [p. 521] A matrix with the same number of rows and columns is called a square matrix; e.g. a 2×2 matrix.

Matrix, zero [p. 515] The $m \times n$ matrix with all entries equal to zero is called the zero matrix and is usually denoted by $\mathbf{O}$.

Matrix algebra [pp. 514–521] Some properties of arithmetic operations on $n \times n$ matrices:

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ commutative law
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ associative law
- $\mathbf{A} + \mathbf{O} = \mathbf{A}$ zero matrix
- $\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$ additive inverse
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ associative law
- $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ identity matrix
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ distributive law
- $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ distributive law

Note: Matrix multiplication is not commutative.

Median of a triangle [p. 275] a line segment from a vertex to the midpoint of the opposite side

Midpoint of a line segment [p. 140] If M is the midpoint of line segment AB , then

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

Modulus–argument form of a complex number

[p. 480] *see* polar form of a complex number

Modulus function [p. 365] The modulus of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *absolute value function*

Modulus of a complex number, $|z|$ [p. 480]

the distance of the complex number from the origin. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

Modulus, properties [p. 482]

For complex numbers z_1 and z_2 :

- $|z_1 z_2| = |z_1| |z_2|$ (the modulus of a product is the product of the moduli)
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ (the modulus of a quotient is the quotient of the moduli)

Multiple [p. 50] A natural number a is a multiple of a natural number b if there exists a natural number k such that $a = kb$.

Multiplication of a complex number by i

[pp. 475, 482] corresponds to a rotation about the origin by 90° anticlockwise. If $z = a + bi$, then $iz = i(a + bi) = -b + ai$.

Multiplication of a vector by a scalar

[p. 138] If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $m \in \mathbb{R}$, then $m\mathbf{a} = ma_1\mathbf{i} + ma_2\mathbf{j}$.

Multiplication of complex numbers

[pp. 467, 482] If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 z_2 = (ac - bd) + (ad + bc)i$$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Multiplication principle [p. 187] If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

N

$n!$ [p. 191] *see* factorial notation

Natural numbers [p. 38] the elements of

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Negation [p. 238] The negation of a statement P is the opposite statement, called ‘not P ’. For example, the negation of $x \geq 1$ is $x < 1$.

Normal reaction force [p. 171] A mass placed on a surface (horizontal or inclined) experiences a force perpendicular to the surface, called the normal force.

O

Ordered pair [p. 39] An ordered pair, denoted (x, y) , is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

P

Parallelogram [pp. 140, 280] a quadrilateral with both pairs of opposite sides parallel

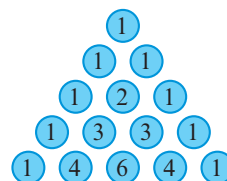
Parametric equations [p. 616] a pair of equations $x = f(t)$ and $y = g(t)$ describing a curve in the plane, where t is called the *parameter* of the curve

Partial fractions [p. 105] Some rational functions may be expressed as a sum of partial fractions; e.g.

$$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{ex^2+fx+g}$$

Particle model [p. 170] an object is considered as a point. This can be done when the size of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

Pascal’s triangle [p. 211] a triangular pattern of numbers formed by the binomial coefficients ${}^nC_r$. Each entry of Pascal’s triangle is the sum of the two entries immediately above.



Period of a function [p. 400] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x+a) = f(x)$ for all x . The smallest such a is called the period of f . For example, the period of the sine function is 2π , as $\sin(x+2\pi) = \sin x$.

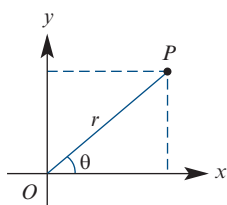
Permutation [p. 191] an ordered arrangement of objects. The number of permutations of n objects taken r at a time is given by

$${}_n P_r = \frac{n!}{(n-r)!}$$

Pigeonhole principle [p. 214] If $n+1$ or more objects are placed into n holes, then some hole contains at least two objects.

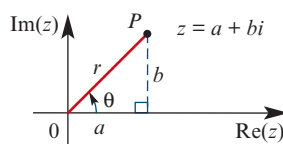
Polar coordinates [p. 625] A point P in the plane has polar coordinates (r, θ) , where

- r is the distance from the origin O to P
- θ is the angle between the positive direction of the x -axis and the ray OP .



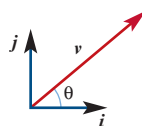
Polar form of a complex number [p. 480]

A complex number is expressed in polar form as $z = r \operatorname{cis} \theta$, where r is the modulus of z and θ is an argument of z . This is also called *modulus–argument form*.



Polar form of a vector [p. 149] Each vector v in the plane can be written in polar form as $v = [r, \theta]$, where

- r is the magnitude of v
- θ describes the angle that v makes with the i -direction, measured anticlockwise.



Polynomial function [p. 91] A polynomial has a rule of the type

$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $n \in \mathbb{N} \cup \{0\}$ where $a_0, a_1, \dots, a_n$ are numbers called coefficients.

Position vector [p. 139] A position vector, $\vec{OP}$, indicates the position in space of the point P relative to the origin O .

Prime [p. 47] A natural number greater than 1 is a prime number if its only factors are itself and 1.

Prime decomposition [p. 47] expressing a composite number as a product of powers of prime numbers; e.g. $500 = 2^2 \times 5^3$

Principle of mathematical induction [p. 253] used to prove that a statement is true for all natural numbers

Probability [p. 223] a numerical value assigned to the likelihood of an event occurring. If the event A is impossible, then $\Pr(A) = 0$; if the event A is certain, then $\Pr(A) = 1$; otherwise $0 < \Pr(A) < 1$.

Product-to-sum identities [p. 452]

- $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$
- $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$
- $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

Projection [p. 554] A projection maps each point in the plane onto an axis.

- Projection onto the x -axis: $(x, y) \rightarrow (x, 0)$
- Projection onto the y -axis: $(x, y) \rightarrow (0, y)$

Proof by contradiction [p. 242] a proof that begins by assuming the negation of what is to be proved

Pythagoras' theorem [p. 284] For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides:

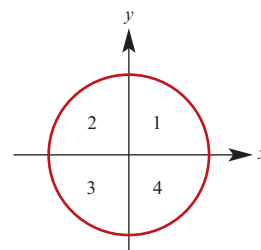
$$(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$$

Pythagorean identity [pp. 405, 439]

- $\cos^2 \theta + \sin^2 \theta = 1$
- $1 + \tan^2 \theta = \sec^2 \theta$
- $\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$

Q

Quadrants [p. 83] The coordinate axes divide the unit circle into four quadrants, which are numbered anticlockwise as shown.



Quadratic formula [p. 95] An equation of the form $ax^2 + bx + c = 0$, with $a \neq 0$, may be solved quickly by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic function [p. 95] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a , b and c are constants and $a \neq 0$.

Quadratic surd [p. 43] a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number

Quantifier [p. 249] see existence statement, universal statement

R

$\mathbb{R}^+$ [p. 42] $\{x : x > 0\}$, positive real numbers

$\mathbb{R}^-$ [p. 42] $\{x : x < 0\}$, negative real numbers

$\mathbb{R} \setminus \{0\}$ [p. 42] the set of real numbers excluding 0

$\mathbb{R}^2$ [p. 39] $\{(x, y) : x, y \in \mathbb{R}\}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [p. 399] One radian (written 1°) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^\circ = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^\circ}{180}$$

Random experiment [p. 223] an experiment, such as the rolling of a die, in which the outcome of a single trial is uncertain but observable

Rate [p. 100] describes how a certain quantity changes with respect to the change in another quantity (often time)

Rational function [pp. 105, 375] a function with a rule of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials

Rational number [p. 38] a number that can be written as $\frac{p}{q}$, for some integers p and q with $q \neq 0$

Real part of a complex number [p. 464]

If $z = a + bi$, then $\text{Re}(z) = a$.

Reciprocal function [p. 372] The reciprocal of the function $y = f(x)$ is defined by $y = \frac{1}{f(x)}$.

Reciprocal trigonometric functions

[pp. 422, 438] the secant, cosecant and cotangent functions

Recurrence relation [p. 257] a rule which enables each subsequent term of a sequence to be found from previous terms; e.g. $t_1 = 1$, $t_n = t_{n-1} + 2$

Reflection [p. 551] A reflection in a line ℓ maps each point in the plane to its mirror image on the other side of the line.

- Reflection in the x -axis: $(x, y) \rightarrow (x, -y)$
- Reflection in the y -axis: $(x, y) \rightarrow (-x, y)$
- Reflection in the line $y = x$: $(x, y) \rightarrow (y, x)$
- Reflection in the line $y = -x$: $(x, y) \rightarrow (-y, -x)$

Reflection matrix [p. 558] A reflection in the line $y = mx = x \tan \theta$ is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Regular polygon [p. 274] a polygon in which all the angles are equal and all the sides are equal

Resultant force [p. 170] the vector sum of the forces acting at a point

Rhombus [p. 280] a parallelogram with all sides of equal length

Rotation matrix [p. 557] A rotation about the origin by angle θ anticlockwise is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

S

Sample space [p. 223] the set of all possible outcomes for a random experiment

Scalar [p. 138] a real number; name used when working with vectors or matrices

Scalar product [p. 153] The scalar product of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ is given by $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$

Scalar product, properties [p. 154]

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$
- $\mathbf{a} \cdot \mathbf{0} = 0$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

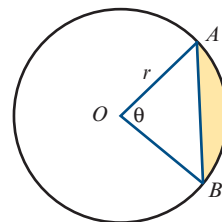
Scalar quantity [p. 160] a quantity determined by its magnitude; e.g. distance, time, speed, mass

Scientific notation [p. 5] A number is in standard form when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34} .

Secant function [pp. 422, 438] $\sec \theta = \frac{1}{\cos \theta}$ for $\cos \theta \neq 0$

Secant of a circle [p. 323] a line that cuts a circle at two distinct points

Segment [p. 323] Every chord divides the interior of a circle into two regions called segments; the smaller is the *minor segment* (shaded), and the larger is the *major segment*.



Selection [p. 202] *see* combination

Sequence [p. 257] a list of numbers, with the order being important; e.g. 1, 1, 2, 3, 5, 8, 13, ... The numbers of a sequence are called its *terms*, and the n th term is often denoted by t_n .

Set notation [p. 35]

- $\in$ means 'is an element of'
- $\notin$ means 'is not an element of'
- $\subseteq$ means 'is a subset of'
- $\cap$ means 'intersection'
- $\cup$ means 'union'
- $\emptyset$ is the empty set, containing no elements
- ξ is the universal set, containing all elements being considered
- A' is the complement of a set A
- $|A|$ is the number of elements in a finite set A

Sets of numbers [pp. 38, 463]

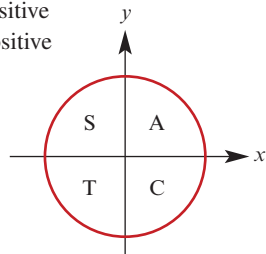
- $\mathbb{N}$ is the set of natural numbers
- $\mathbb{Z}$ is the set of integers
- $\mathbb{Q}$ is the set of rational numbers
- $\mathbb{R}$ is the set of real numbers
- $\mathbb{C}$ is the set of complex numbers

Shear [p. 553] A shear moves each point in the plane by an amount proportional to its distance from an axis.

- Shear parallel to the x -axis: $(x, y) \rightarrow (x + cy, y)$
- Shear parallel to the y -axis: $(x, y) \rightarrow (x, cx + y)$

Signs of trigonometric functions [p. 404]

- 1st quadrant all are positive
- 2nd quadrant sin is positive
- 3rd quadrant tan is positive
- 4th quadrant cos is positive



Similar figures [p. 290] Two figures are similar if we can enlarge one figure so that its enlargement is congruent to the other figure.

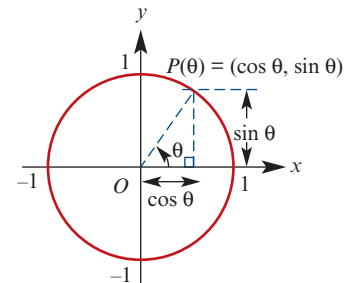
Similarity tests [p. 291] Two triangles are similar if one of the following conditions holds:

- **AAA** two angles of one triangle are equal to two angles of the other triangle
- **SAS** the ratios of two pairs of matching sides are equal and the included angles are equal
- **SSS** the ratios of matching sides are equal
- **RHS** the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

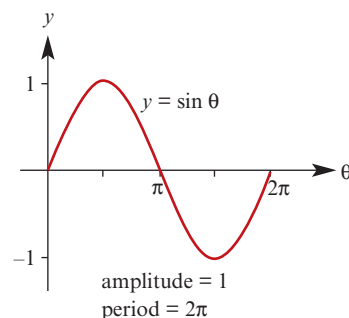
Simplest form [p. 43] a surd $\sqrt{a}$ is in simplest form if the number under the square root has no factors which are squares of a rational number

Simultaneous equations [pp. 8, 111, 526] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

Sine function [p. 400] $\sin \theta$ is defined as the y -coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x -axis.

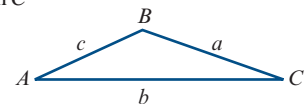


Sine function, graph [p. 400]



Sine rule [p. 72] For triangle ABC :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Speed [p. 161] the magnitude of velocity

Standard form [p. 5] *see* scientific notation

Strictly decreasing [p. 383] A function f is strictly decreasing on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

Strictly increasing [p. 383] A function f is strictly increasing on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Subtraction of complex numbers [p. 465]

If $z_1 = a + bi$ and $z_2 = c + di$, then
 $z_1 - z_2 = (a - c) + (b - d)i$.

Subtraction of vectors [p. 138]

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then
 $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}$.

Sum-to-product identities [p. 453]

- $\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$
- $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$
- $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$
- $\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$

Supplementary angles [p. 268] two angles whose sum is 180°

Surd of order n [p. 43] a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect n th power

Surd, quadratic [p. 43] a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number

T

Tangent function [pp. 401, 412] $\tan \theta = \frac{\sin \theta}{\cos \theta}$
 for $\cos \theta \neq 0$

Tangent to a circle [p. 328] a line that touches the circle at exactly one point, called the *point of contact*

Tension force [p. 171] the pulling force exerted by a string that connects two objects. The forces at each end of the string have equal magnitude.

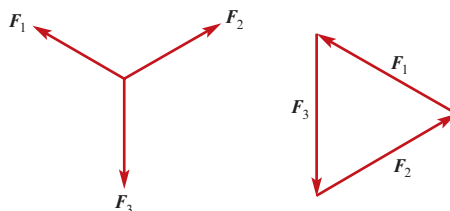
Transformation [p. 547] A transformation of the plane maps each point (x, y) in the plane to a new point (x', y') . We say that (x', y') is the *image* of (x, y) .

Translation [p. 555] a transformation that moves each point in the plane in the same direction and over the same distance: $(x, y) \rightarrow (x + a, y + b)$

Trapezium [p. 280] a quadrilateral with at least one pair of opposite sides parallel

Triangle inequality [p. 275] If a , b and c are the side lengths of a triangle, then $a < b + c$, $b < c + a$ and $c < a + b$.

Triangle of forces [p. 171] If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.

**Trigonometric equations** [p. 416]

- For $a \in [-1, 1]$, the general solution of the equation $\cos(x) = a$ is $x = 2n\pi \pm \cos^{-1}(a)$, where $n \in \mathbb{Z}$.
- For $a \in [-1, 1]$, the general solution of the equation $\sin(x) = a$ is $x = 2n\pi + \sin^{-1}(a)$ or $x = (2n + 1)\pi - \sin^{-1}(a)$, where $n \in \mathbb{Z}$.
- For $a \in \mathbb{R}$, the general solution of the equation $\tan(x) = a$ is $x = n\pi + \tan^{-1}(a)$, where $n \in \mathbb{Z}$.

Trigonometric functions [pp. 400, 401] the sine, cosine and tangent functions

Trigonometric functions, exact values
[pp. 69, 403]

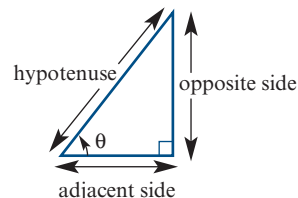
θ°	θ°	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	undefined

Trigonometric ratios [p. 64]

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$



U

Union of sets [p. 35] The union of two sets A and B , written $A \cup B$, is the set of all elements which are in A or B or both.

Unit vector [p. 146] a vector of magnitude 1. The unit vectors in the positive directions of the x - and y -axes are $\mathbf{i}$ and $\mathbf{j}$ respectively. The unit vector in the direction of $\mathbf{a}$ is given by

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \mathbf{a}$$

Universal statement [pp. 249, 250] a statement claiming that a property holds for all members of a given set. Such a statement can be written using the quantifier ‘for all’.

V

Vector [p. 136] a set of equivalent directed line segments

Vector quantity [p. 160] a quantity determined by its magnitude and direction; e.g. position, displacement, velocity, force

Vectors, parallel [p. 139] Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} = k\mathbf{b}$ for some $k \in \mathbb{R} \setminus \{0\}$.

Vectors, perpendicular [p. 154] Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Vectors, properties [pp. 136–138]

■ $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	commutative law
■ $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$	associative law
■ $\mathbf{a} + \mathbf{0} = \mathbf{a}$	zero vector
■ $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$	additive inverse
■ $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$	distributive law

Vectors, resolution [p. 157] A vector $\mathbf{a}$ is resolved into rectangular components by writing it as a sum of two vectors, one parallel to a given vector $\mathbf{b}$ and the other perpendicular to $\mathbf{b}$.

The *vector resolute* of $\mathbf{a}$ in the direction of $\mathbf{b}$ is given by

$$\mathbf{u} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}$$

Velocity [p. 161] the rate of change of position with respect to time. Velocity is a vector quantity.

Velocity, relative [p. 166] The relative velocity of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B .

Velocity, resultant [p. 165] the sum of two or more velocity vectors. For example, if a train is travelling north at 60 km/h and a passenger walks at 3 km/h towards the back of the train, then the passenger’s resultant velocity is 57 km/h north.

Velocity, true [p. 166] the velocity of an object measured relative to Earth

W

Weight [p. 170] On the Earth’s surface, a mass of m kg has a force of m kg wt acting on it; this force is known as the weight.

Z

Zero vector, 0 [p. 138] a line segment of zero length with no direction

Answers

Chapter 1

Exercise 1A

- 1 a x^7 b a^2 c x^3 d y^{-4}
 e x^{12} f p^{-7} g $a^{-\frac{1}{6}}$ h a^{-8}
 i y^{14} j x^{15} k a^{-12} l x^2
 m n^2 n $8x^{\frac{7}{2}}$ o a p x^4
 q $\frac{1}{2n^6}$ r $-8x^2$ s $a^{-2}b^5$ t 1
- 2 a 5 b 4 c $\frac{4}{3}$ d $\frac{1}{4}$
 e $\frac{6}{7}$ f 3 g 12 h 16
 i 27 j $\frac{3}{2}$ k 1 l 8
- 3 a 18.92 b 79.63 c 5.89
 d 125 000 e 0.9 f 1.23
 g 0.14 h 1.84 i 0.4
- 4 a a^4b^7 b $64a^4b^7$ c b
 d a^6b^9 e $2a^4b^7$ f $\frac{a^2b^5}{128}$
- 5 2^{2n-4}
- 6 6^{3x}
- 7 a $\left(\frac{1}{2}\right)^{\frac{1}{6}}$ b $a^{\frac{11}{20}}$ c $2^{\frac{5}{6}}$
 d $2^{\frac{19}{6}}$ e $2^{\frac{3}{5}}$
- 8 a $a^{\frac{1}{3}}b$ b $a^{\frac{5}{2}}b^{\frac{1}{2}}$ c $ab^{\frac{1}{5}}$
 d $\left(\frac{b}{a}\right)^{\frac{1}{2}}$ e $a^{\frac{5}{2}}b^{\frac{1}{2}}c^{-4}$ f $a^{\frac{1}{5}}b^{\frac{3}{5}}$
 g $a^{-4}b^{\frac{7}{2}}c^5$

Exercise 1B

- 1 a 4.78×10 b 6.728×10^3
 c 7.923×10 d 4.358×10^4
 e 2.3×10^{-3} f 5.6×10^{-7}
 g $1.200\ 034 \times 10$ h 5.0×10^7
 i 2.3×10^{10} j 1.3×10^{-9}
 k 1.65×10^5 l 1.4567×10^{-5}
- 2 a 2.99×10^{-23} b 3.1536×10^7
 c 3.057×10^{21}
- 3 a 1 390 000 000 b 0.000 0075
 c 0.000 000 000 000 0056
- 4 a 4.569×10^2 b 3.5×10^4
 c 5.6791×10^3 d 4.5×10^{-2}
 e 9.0×10^{-2} f 4.5682×10^3
- 5 a 1 b 4.8×10^{-2}
 c 2.16×10^{11} d 2.5×10^{-3}
 e 2×10^5 f 2×10^7
- 6 a 11.8 b 4.76×10^7

Exercise 1C

- 1 a $x = \frac{8}{3}$ b $x = 48$ c $x = -\frac{20}{3}$
 d $x = 63$ e $x = -0.7$ f $x = 2.4$
 g $x = 0.3$ h $x = -6$ i $x = -\frac{15}{92}$
 j $x = -\frac{21}{17}$
- 2 a $x = \frac{160}{9}$ b $x = 19.2$ c $x = -4$
 d $x = \frac{80}{51}$ e $x = 6.75$ f $x = -\frac{85}{38}$
 g $x = \frac{487}{13}$ h $x = \frac{191}{91}$

$$\begin{array}{ll} \mathbf{3 a} & x = \frac{18}{13}, y = -\frac{14}{13} \\ \mathbf{b} & x = \frac{16}{11}, y = -\frac{18}{11} \\ \mathbf{c} & x = 12, y = 17 \\ \mathbf{d} & x = 8, y = 2 \\ \mathbf{e} & x = 0, y = 2 \\ \mathbf{f} & x = 1, y = 6 \end{array}$$

Exercise 1D

- 1 a** $4(x - 2) = 60$; $x = 17$
- b** $\left(\frac{2x+7}{4}\right)^2 = 49$; $x = 10.5$
- c** $x - 5 = 2(12 - x)$; $x = \frac{29}{3}$
- d** $y = 6x - 4$
- e** $Q = np$
- f** $R = 1.1pS$
- g** $\frac{60n}{5} = 2400$
- h** $a = \frac{\pi}{3}(x + 3)$
- 2** \$2500
- 3** Eight dresses and three handbags
- 4** 8.375 m by 25.125 m
- 5** \$56.50
- 6** Nine
- 7** 20, 34 and 17
- 8** Annie 165, Belinda 150, Cassie 189
- 9** 15 km/h
- 10** 2.04×10^{-23} g
- 11** 30 pearls
- 12** Oldest \$48, Middle \$36, Youngest \$12
- 13** 98%
- 14** 25 students
- 15** After 20 minutes
- 16 a** 40 minutes
- b** 90 minutes
- c** 20 minutes
- 17** 200 km
- 18** 39 km/h

Exercise 1E

- 1** 140.625 km
- 2** 50
- 3** 10 000 adults
- 4** Men \$220; boys \$160
- 5** 127 and 85
- 6** 500 adults, 1100 students
- 7** 252 litres 40% and 448 litres 15%
- 8** 120 and 100; 60
- 9** \$370 588

Exercise 1F

$$\begin{array}{llll} \mathbf{1 a} & 25 & \mathbf{b} & 330 \\ \mathbf{c} & 340.47 & \mathbf{d} & 1653.48 \\ \mathbf{e} & 612.01 & \mathbf{f} & 77.95 \\ \mathbf{g} & 2.42 & \mathbf{h} & 2.1 \\ \mathbf{i} & 9.43 & \mathbf{j} & 9.54 \end{array}$$

$$\begin{array}{ll} \mathbf{2 a} & a = \frac{v-u}{t} \\ \mathbf{b} & \ell = \frac{2S}{n} - a \\ \mathbf{c} & b = \frac{2A}{h} \\ \mathbf{d} & I = \pm \sqrt{\frac{P}{R}} \\ \mathbf{e} & a = \frac{2(s-ut)}{t^2} \\ \mathbf{f} & v = \pm \sqrt{\frac{2E}{m}} \\ \mathbf{g} & h = \frac{Q^2}{2g} \\ \mathbf{h} & x = \frac{-z}{y} \\ \mathbf{i} & x = \frac{-b(c+y)}{a-c} \\ \mathbf{j} & x = \frac{-b(c+1)}{m-c} \end{array}$$

$$\begin{array}{ll} \mathbf{3 a} & 82.4^\circ\text{F} \\ \mathbf{b} & C = \frac{5(F-32)}{9}; 57.22^\circ\text{C} \\ \mathbf{4 a} & 1080^\circ \\ \mathbf{b} & n = \frac{S}{180} + 2; 9 \text{ sides} \\ \mathbf{5 a} & 115.45 \text{ cm}^3 \\ \mathbf{b} & 12.53 \text{ cm} \\ \mathbf{c} & 5.00 \text{ cm} \\ \mathbf{6 a} & 66.5 \\ \mathbf{b} & 4 \\ \mathbf{c} & 11 \end{array}$$

Exercise 1G

$$\begin{array}{llll} \mathbf{1 a} & \frac{13x}{6} & \mathbf{b} & \frac{5a}{4} \\ \mathbf{c} & \frac{-h}{8} & \mathbf{d} & \frac{5x-2y}{12} \\ \mathbf{e} & \frac{3y+2x}{xy} & \mathbf{f} & \frac{7x-2}{x(x-1)} \\ \mathbf{g} & \frac{5x-1}{(x-2)(x+1)} & \mathbf{h} & \frac{-7x^2-36x+27}{2(x+3)(x-3)} \\ \mathbf{i} & \frac{4x+7}{(x+1)^2} & \mathbf{j} & \frac{5a^2+8a-16}{8a} \\ \mathbf{k} & \frac{4(x^2+1)}{5x} & \mathbf{l} & \frac{2x+5}{(x+4)^2} \\ \mathbf{m} & \frac{3x+14}{(x-1)(x+4)} & \mathbf{n} & \frac{x+14}{(x-2)(x+2)} \\ \mathbf{o} & \frac{7x^2+28x+16}{(x-2)(x+2)(x+3)} & \mathbf{p} & \frac{(x-y)^2-1}{x-y} \\ \mathbf{q} & \frac{4x+3}{x-1} & \mathbf{r} & \frac{3-2x}{x-2} \end{array}$$

$$\begin{array}{llll} \mathbf{2 a} & 2xy^2 & \mathbf{b} & \frac{xy}{8} \\ \mathbf{c} & \frac{2}{x} & \mathbf{d} & \frac{x}{y^2} \\ \mathbf{e} & \frac{a}{3} & \mathbf{f} & \frac{1}{2x} \\ \mathbf{g} & \frac{x-1}{x+4} & \mathbf{h} & x+2 \\ \mathbf{i} & \frac{x-1}{x} & \mathbf{j} & \frac{a}{4b} \\ \mathbf{k} & \frac{2x}{x+2} & \mathbf{l} & \frac{x-1}{4x} \\ \mathbf{m} & \frac{x+1}{2x} & \mathbf{n} & \frac{1}{3}x(x+3) \\ \mathbf{o} & \frac{x-2}{3x(3x-2)(x+5)} \end{array}$$

$$\begin{array}{ll} \mathbf{3 a} & \frac{3}{x-3} \\ \mathbf{b} & \frac{4x-14}{x^2-7x+12} \\ \mathbf{c} & \frac{5x-1}{x^2+x-12} \\ \mathbf{d} & \frac{2x^2+10x-6}{x^2+x-12} \\ \mathbf{e} & \frac{2x-9}{x^2-10x+25} \\ \mathbf{f} & \frac{5x-8}{(x-4)^2} \\ \mathbf{g} & \frac{1}{3-x} \\ \mathbf{h} & \frac{23-3x}{x^2+x-12} \\ \mathbf{i} & \frac{5x^2-3x}{x^2-9} \\ \mathbf{j} & \frac{11-2x}{x^2-10x+25} \\ \mathbf{k} & \frac{12}{(x-6)^3} \\ \mathbf{l} & \frac{9x-25}{x^2-7x+12} \end{array}$$

$$\begin{array}{lll} \mathbf{4 a} & \frac{3-x}{\sqrt{1-x}} & \mathbf{b} & \frac{2\sqrt{x-4}+6}{3\sqrt{x-4}} \\ \mathbf{c} & \frac{5}{\sqrt{x+4}} \\ \mathbf{d} & \frac{x+7}{\sqrt{x+4}} & \mathbf{e} & \frac{12x^2}{\sqrt{x+4}} \\ \mathbf{f} & \frac{9x^2(x+2)}{2\sqrt{x+3}} \\ \mathbf{5 a} & \frac{6x-4}{(6x-3)^{\frac{2}{3}}} & \mathbf{b} & \frac{3}{(2x+3)^{\frac{2}{3}}} \\ \mathbf{c} & \frac{3-3x}{(x-3)^{\frac{2}{3}}} \end{array}$$

Exercise 1H

- 1 **a** $x = \frac{m-n}{a}$ **b** $x = \frac{b}{b-a}$ **c** $x = -\frac{bc}{a}$
d $x = \frac{5}{p-q}$ **e** $x = \frac{m+n}{n-m}$ **f** $x = \frac{ab}{1-b}$
g $x = 3a$ **h** $x = -mn$ **i** $x = \frac{a^2-b^2}{2ab}$
j $x = \frac{p-q}{p+q}$ **k** $x = \frac{3ab}{b-a}$ **l** $x = \frac{1}{3a-b}$
m $x = \frac{p^2+p^2t+t^2}{q(p+t)}$ **n** $x = -\frac{5a}{3}$
- 4 **a** $x = \frac{d-bc}{1-ab}, y = \frac{c-ad}{1-ab}$
b $x = \frac{a^2+ab+b^2}{a+b}, y = \frac{ab}{a+b}$
c $x = \frac{t+s}{2a}, y = \frac{t-s}{2b}$ **d** $x = a+b, y = a-b$
e $x = c, y = -a$ **f** $x = a+1, y = a-1$
- 5 **a** $s = a(2a+1)$ **b** $s = \frac{2a^2}{1-a}$
c $s = \frac{a^2+a+1}{a(a+1)}$ **d** $s = \frac{a}{(a-1)^2}$
e $s = 3a^3(3a+1)$ **f** $s = \frac{3a}{a+2}$
g $s = 2a^2 - 1 + \frac{1}{a^2}$ **h** $s = \frac{5a^2}{a^2+6}$

Chapter 1 review

Technology-free questions

- 1 **a** x^{12} **b** y^{-9} **c** $-15x^{\frac{11}{2}}$ **d** x^{-1}
2 3.22×10^{11}
3 **a** $\frac{2x+y}{10}$ **b** $\frac{4y-7x}{xy}$
c $\frac{7x-1}{(x+2)(x-1)}$ **d** $\frac{7x+20}{(x+2)(x+4)}$
e $\frac{13x^2+2x+40}{2(x+4)(x-2)}$ **f** $\frac{3(x-4)}{(x-2)^2}$
- 4 **a** $\frac{2}{x}$ **b** $\frac{x-4}{4x}$ **c** $\frac{x^2-4}{3}$ **d** $4x^2$
- 5 **a** 2×10^6 photos, i.e. 2 million photos
b 2×10^5 seconds (≈ 55.6 hours)
- 6 12
- 7 88 classical, 80 blues, 252 heavy metal
- 8 **a** $300\pi \text{ cm}^3$ **b** $h = \frac{V}{\pi r^2}; \frac{117}{5\pi} \text{ cm}$
c $r = \sqrt{\frac{V}{\pi h}}; \sqrt{\frac{128}{\pi}} \text{ cm} = \frac{8\sqrt{2}}{\sqrt{\pi}} \text{ cm}$
- 9 **a** $x = \frac{b}{a+y}$ **b** $x = \frac{a+b}{c}$
c $x = \frac{2ab}{b-a}$ **d** $x = \frac{ab+b^2d-d^2}{d(a+b)}$

- 10 **a** $\frac{p^2+q^2}{p^2-q^2}$ **b** $\frac{x+y}{x(y-x)}$
c $(x-2)(2x-1)$ **d** $\frac{2}{a}$
- 11 A 36; B 12; C 2
- 12 **a** $a = 8, b = 18$ **b** $x = p+q, y = 2q$
- 13 $x = 3.5$
- 14 **a** $4n^2k^2$ **b** $\frac{40cx^2}{ab^2}$
- 15 $x = -1$

Multiple-choice questions

- 1 A 2 A 3 C 4 A 5 B
6 E 7 B 8 B 9 B 10 B

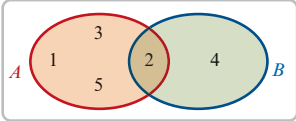
Extended-response questions

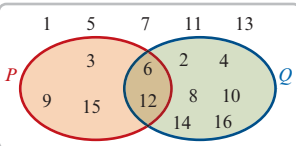
- 1 **a** $\frac{5x}{4}$ hours **b** $\frac{4x}{7}$ hours **c** $\frac{19x}{28}$ hours
d **i** $x = \frac{14}{19} \approx 0.737$
ii Jack $\frac{140}{19} \approx 7$ km; Benny $\frac{560}{19} \approx 29$ km
- 2 **a** 18 000 cm³ per minute
b $V = 18\,000t$ **c** $h = \frac{45t}{4\pi}$ **d** After
- 3 **a** Thomas a ; George $\frac{3a}{2}$; Sally $a-18$;
Zeb $\frac{a}{3}$; Henry $\frac{5a}{6}$
b $\frac{3a}{2} + a - 18 + \frac{a}{3} = a + \frac{5a}{6} + 6$
c $a = 24$; Thomas 24; Henry 20; George 36;
Sally 6; Zeb 8
- 4 **a** $1.9 \times 10^{-8} \text{ N}$ **b** $m_1 = \frac{Fr^2 10^{11}}{6.67m_2}$
c $9.8 \times 10^{24} \text{ kg}$
- 5 **a** $V = (1.8 \times 10^7)d$ **b** $5.4 \times 10^8 \text{ m}^3$
c $k = 9.81 \times 10^3$ **d** $1.325 \times 10^{15} \text{ J}$
e 1202 days (to the nearest day)
- 6 $\frac{10\sqrt{3}}{3} \text{ cm}$
- 7 -40°
- 8 $\frac{240}{11} \text{ km/h}$
- 9 **a** $h = 20 - r$
b **i** $V = \left(20r^2 - \frac{r^3}{3}\right)\pi$
ii $r = 5.94 \text{ cm}$; $h = 14.06 \text{ cm}$
- 10 **a** $\frac{2}{3}$ litre from A; $\frac{1}{3}$ litre from B
b 600 mL from A; 400 mL from B
c $\frac{(p-q)(n+m)}{2(np-qm)}$ litres from A,
 $\frac{(n-m)(p+q)}{2(np-qm)}$ litres from B,
where $\frac{n}{m} \neq \frac{q}{p}$ and one of $\frac{n}{m}$ or $\frac{q}{p}$ is ≥ 1
and the other is ≤ 1

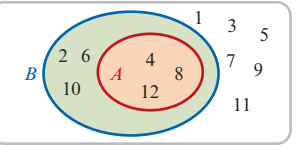
- 11 a** $h = 2(10 - r)$ **b** $V = 2\pi r^2(10 - r)$
c $r = 3.4986, h = 13.0029$ or
 $r = 9.0224, h = 1.9551$

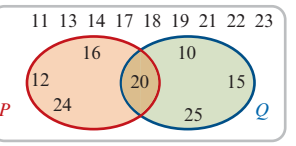
Chapter 2

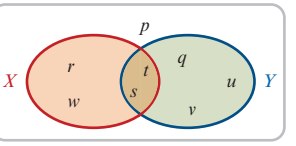
Exercise 2A

- 1** ξ 

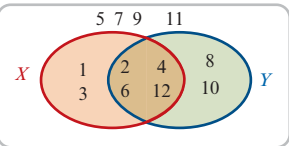
a {4} **b** {1, 3, 5} **c** {1, 2, 3, 4, 5} = ξ
d $\emptyset$ **e** $\emptyset$
- 2** ξ 

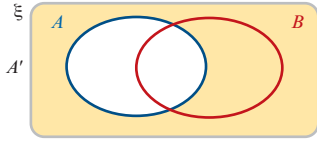
a {1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16}
b {1, 3, 5, 7, 9, 11, 13, 15}
c {2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16}
d {1, 5, 7, 11, 13} **e** {1, 5, 7, 11, 13}
- 3** ξ 

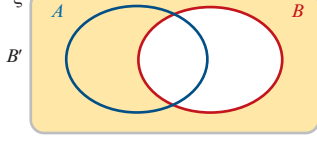
a {1, 2, 3, 5, 6, 7, 9, 10, 11}
b {1, 3, 5, 7, 9, 11} **c** {2, 4, 6, 8, 10, 12}
d {1, 3, 5, 7, 9, 11} **e** {1, 3, 5, 7, 9, 11}
- 4** ξ 

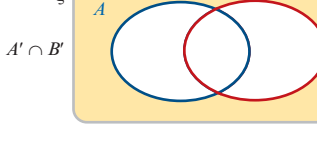
a {10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25}
b {11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24}
c {10, 12, 15, 16, 20, 24, 25}
d {11, 13, 14, 17, 18, 19, 21, 22, 23}
e {11, 13, 14, 17, 18, 19, 21, 22, 23}
- 5** ξ 

a {p, q, u, v} **b** {p, r, w} **c** {p}
d {p, q, r, u, v, w} **e** {q, r, s, t, u, v, w} **f** {p}

- 6** ξ 

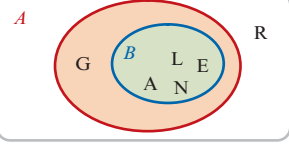
a {5, 7, 8, 9, 10, 11} **b** {1, 3, 5, 7, 9, 11}
c {1, 3, 5, 7, 8, 9, 10, 11}
d {1, 3, 5, 7, 8, 9, 10, 11}
e {1, 2, 3, 4, 6, 8, 10, 12} **f** {5, 7, 9, 11}
- 7 a** ξ 

b ξ 

c ξ 

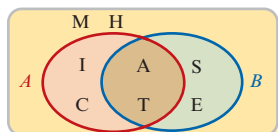
d ξ 

e ξ 

f ξ 
- 8** ξ 

a {R} **b** {G, R} **c** {L, E, A, N}
d {A, N, G, E, L} **e** {R} **f** {G, R}

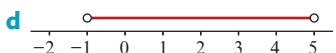
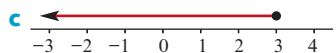
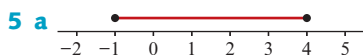
9 ξ



- a** {E, H, M, S} **b** {C, H, I, M}
c {A, T} **d** {H, M} **e** {C, E, H, I, M, S}
f {H, M}

Exercise 2B

- 1 a** Yes **b** Yes **c** Yes
2 a No **b** No **c** No
3 a $\frac{9}{20}$ **b** $\frac{3}{11}$ **c** $\frac{3}{25}$ **d** $\frac{2}{7}$ **e** $\frac{4}{11}$ **f** $\frac{2}{9}$
4 a 0.285714 **b** 0.45 **c** 0.35
d 0.307692 **e** 0.0588235294117647



- 6 a** $(-\infty, 3)$ **b** $[-3, \infty)$ **c** $(-\infty, -3]$
d $(5, \infty)$ **e** $[-2, 3)$ **f** $[-2, 3]$
g $(-2, 3]$ **h** $(-5, 3)$

Exercise 2C

- 1 a** $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $3\sqrt{3}$ **d** $5\sqrt{2}$
e $3\sqrt{5}$ **f** $11\sqrt{10}$ **g** $7\sqrt{2}$ **h** $6\sqrt{3}$
i 5 **j** $5\sqrt{3}$ **k** $16\sqrt{2}$
2 a $3\sqrt{2}$ **b** $6\sqrt{3}$ **c** $4\sqrt{7}$
d $5\sqrt{10}$ **e** $28\sqrt{2}$ **f** 0
3 a $11\sqrt{3} + \sqrt{14}$ **b** $5\sqrt{7}$
c 0 **d** $\sqrt{2} + \sqrt{3}$
e $5\sqrt{2} + 15\sqrt{3}$ **f** $\sqrt{2} + \sqrt{5}$
4 a $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{7}}{7}$ **c** $\frac{-\sqrt{2}}{2}$
d $\frac{2\sqrt{3}}{3}$ **e** $\frac{\sqrt{6}}{2}$ **f** $\frac{\sqrt{2}}{4}$
g $\sqrt{2} - 1$ **h** $2 + \sqrt{3}$ **i** $\frac{4 + \sqrt{10}}{6}$
j $\sqrt{6} - 2$ **k** $\frac{\sqrt{5} + \sqrt{3}}{2}$ **l** $3(\sqrt{6} + \sqrt{5})$
m $3 + 2\sqrt{2}$
5 a $6 + 4\sqrt{2}$ **b** $9 + 4\sqrt{5}$ **c** $-1 + \sqrt{2}$
d $4 - 2\sqrt{3}$ **e** $\frac{2\sqrt{3}}{9}$ **f** $\frac{8 + 5\sqrt{3}}{11}$
g $\frac{3 + \sqrt{5}}{2}$ **h** $\frac{6 + 5\sqrt{2}}{14}$

- 6 a** $4a - 4\sqrt{a} + 1$
b $3 + 2x + 2\sqrt{(x+1)(x+2)}$
7 a $5 - 3\sqrt{2}$ **b** $7 - 2\sqrt{6}$
8 a $\frac{3}{\sqrt{2}}$ **b** $\frac{\sqrt{5}}{2}$ **c** $\frac{\sqrt{5}}{5}$ **d** $\frac{8}{\sqrt{3}}$
9 a $b = 0, c = -3$ **b** $b = 0, c = -12$
c $b = -2, c = -1$ **d** $b = -4, c = 1$
e $b = -6, c = 1$
f $b = -7 + 5\sqrt{5}, c = -58 - 13\sqrt{5}$
10 $\frac{3\sqrt{2} + 2\sqrt{3} - \sqrt{30}}{12}$

- 11 b** $-1 - 2^{\frac{1}{3}} - 2^{\frac{2}{3}}$

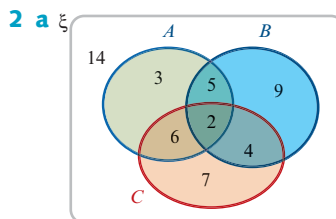
Exercise 2D

- 1 a** $2^2 \times 3 \times 5$ **b** $2^2 \times 13^2$
c $2^2 \times 3 \times 19$ **d** $2^2 \times 3^2 \times 5^2$
e $2^2 \times 3^2 \times 7$ **f** $2^2 \times 3^2 \times 5^2 \times 7$
g $2^5 \times 3 \times 5 \times 11 \times 13$
h $2^5 \times 3 \times 7 \times 11 \times 13$
i $2^5 \times 7 \times 11 \times 13$
j $2^5 \times 7 \times 11 \times 13 \times 17$
2 a 1 **b** 27 **c** 5 **d** 31 **e** 6
3 a 18: 1, 2, 3, 6, 9, 18; 36: 1, 2, 3, 4, 6, 9, 18, 36
b 36 is a square number ($36 = 6 \times 6$)
c 121 has factors 1, 11 and 121
4 5, 14 and 15 **5** 121
6 15 **7** 105
8 8 **9** 4
10 1:12 p.m.
11 600 and 108 000; 2400 and 27 000;
3000 and 21 600; 5400 and 12 000

Exercise 2E



- b i** 19 **ii** 9 **iii** 23



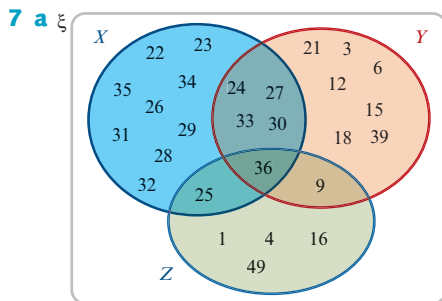
- b i** 23 **ii** 37 **iii** 9

- 3** 20%

- 4 a** 5 **b** 10

- 5** 45

- 6 a** $x = 5$ **b** 16 **c** 0



- b i $X \cap Y \cap Z = \{36\}$ ii $|X \cap Y| = 5$
 8 31 students; 15 black, 12 green, 20 red
 9 $|M \cap F| = 11$ 10 1
 11 $x = 6$; 16 students 12 102 students

Chapter 2 review

Technology-free questions

- 1 a $\frac{7}{90}$ b $\frac{5}{11}$ c $\frac{1}{200}$
 d $\frac{81}{200}$ e $\frac{4}{15}$ f $\frac{6}{35}$
 2 $2^3 \times 3^2 \times 7$
 3 a $\frac{2\sqrt{6} - \sqrt{2}}{2}$ b $4\sqrt{5} + 9$ c $2\sqrt{6} + 5$
 4 $-23 - 12\sqrt{3}$
 5 a $2\sqrt{6} + 6$ b $\frac{a - \sqrt{a^2 - b^2}}{b}$
 6 a 15 b 15
 7 a 1 b 22 c 22
 8 5 9 2 cm^2
 10 $-15\sqrt{7}$ 11 $x = \pm 2$
 12 $\sqrt{5} - \sqrt{6}$ 13 $\frac{51\sqrt{3}}{5}$
 14 a 57 b 3 c 32
 15 $2\sqrt{2} + 3$
 17 Tom is 36 and Fred is 27

Multiple-choice questions

- 1 A 2 D 3 D 4 D 5 C 6 D
 7 B 8 B 9 C 10 A 11 D 12 B

Extended-response questions

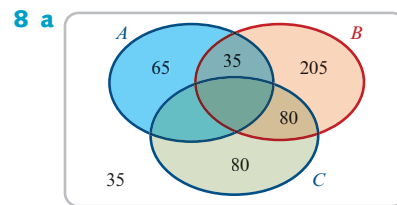
- 1 c i $\sqrt{11} + \sqrt{3}$
 ii $2\sqrt{2} - \sqrt{7}$ or $\sqrt{7} - 2\sqrt{2}$
 iii $3\sqrt{3} - 2\sqrt{6}$ or $2\sqrt{6} - 3\sqrt{3}$
 2 a $a = 6$, $b = 5$ b $p = 26$, $q = 16$
 c $a = -1$, $b = \frac{2}{3}$
 d i $\frac{12\sqrt{3} - 19}{71}$ ii $3 \pm \sqrt{3}$ iii $\frac{1 \pm \sqrt{3}}{2}$
 e $\mathbb{Q} = \{a + 0\sqrt{3} : a \in \mathbb{Q}\}$
 3 d $x = \pm 2$

- 5 a $b = -4$, $c = 1$ b $2 + \sqrt{3}$

- 6 a (20, 21, 29)

- 7 a i Region 8
 ii Male, red hair, blue eyes
 iii Male, not red hair, blue eyes

- b i 5 ii 182



$$|A \cap C| = 0$$

- b 160 c 65 d 0

Chapter 3

Exercise 3A

- 1 a A and C (SAS) b All of them (AAS)
 c A and B (SSS)
 2 a Yes (SAS) b Yes (SSS) c No
 d Yes (RHS) e No f Yes (AAS)
 3 a 4.10 b 0.87 c 2.94
 d 4.08 e 33.69° f 11.92°
 4 23 m
 5 a 9.59° b $\sqrt{35}$ m
 6 a 6.84 m b 6.15 m
 7 12.51° 8 182.7 m 9 1451 m
 10 66.42° , 66.42° , 47.16°
 11 a $5\sqrt{2}$ cm b 90°
 12 3.07 cm 13 37.8 cm 14 4.38 m
 15 $x = 4.08$, $y = 3.43$, $z = 5.33$

Exercise 3B

- 1 a $-\frac{1}{\sqrt{2}}$ b $\frac{1}{\sqrt{2}}$ c $-\frac{1}{2}$ d $\frac{1}{2}$
 2 a $x = 5\sqrt{3}$, $y = 5$ b $x = \frac{16\sqrt{3}}{3}$, $y = 8\sqrt{3}$
 c $x = 6$, $y = 3\sqrt{3}$ d $x = y = 4\sqrt{2}$
 3 $\frac{100\sqrt{3}}{3}$ m 4 $\frac{40\sqrt{3}}{3}$ cm
 5 $(6 + 4\sqrt{3})$ m
 6 a 60° b $10\sqrt{3}$ m
 7 $x = 5(\sqrt{3} - 1)$

Exercise 3C

- 1 a 8.15 b 3.98 c 11.75 d 9.46
 2 a 56.32° b 36.22° c 49.54°
 d 98.16° or 5.84°

- 3 a** $A = 48^\circ$, $b = 13.84$, $c = 15.44$
b $a = 7.26$, $C = 56.45^\circ$, $c = 6.26$
c $B = 19.8^\circ$, $b = 4.66$, $c = 8.27$
d $C = 117^\circ$, $b = 24.68$, $c = 34.21$
e $C = 30^\circ$, $a = 5.40$, $c = 15.56$
- 4 a** $B = 59.12^\circ$, $A = 72.63^\circ$, $a = 19.57$ or $B = 120.88^\circ$, $A = 10.87^\circ$, $a = 3.87$
b $C = 26.69^\circ$, $A = 24.31^\circ$, $a = 4.18$
c $B = 55.77^\circ$, $C = 95.88^\circ$, $c = 17.81$ or $B = 124.23^\circ$, $C = 27.42^\circ$, $c = 8.24$
- 5** 554.26 m **6** 35.64 m
7 1659.86 m
8 a 26.60 m **b** 75.12 m

Exercise 3D

- 1** 5.93 cm
2 $\angle ABC = 97.90^\circ$, $\angle ACB = 52.41^\circ$
3 a 26 **b** 11.74 **c** 49.29° **d** 73
e 68.70 **f** 47.22° **g** 7.59 **h** 38.05°
4 2.626 km **5** 3.23 km **6** 55.93 cm
7 a 8.23 cm **b** 3.77 cm
8 a 87.61 m **b** 67.7 m

Exercise 3E

- 1** 400.10 m **2** 34.77 m **3** 575.18 m
4 16.51 m **5** 056°
6 a 034° **b** 214°
7 a 3583.04 m **b** 353°
8 027° **9** 113° **10** 22.01°
11 a $\angle BAC = 49^\circ$ **b** 264.24 km
12 10.63 km

Exercise 3F

1	sin	cos	1	sin	cos
a	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	b	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
c	0	-1	d	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
e	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	f	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
g	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	h	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
i	0	1	j	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
k	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	l	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
m	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	n	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
o	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	p	0	1

- 2 a** -0.4 **b** -0.6 **c** -0.4 **d** -0.6
e -0.4 **f** 0.6 **g** 0.6 **h** 0.4
3 a 120 **b** 240 **c** -60 **d** 120
e 240 **f** 300

Chapter 3 review

Technology-free questions

- 1** $\frac{1}{8}$ **2** $\frac{2}{5}$ **3** $10\sqrt{3}$ cm
4 143° **5** $\frac{17}{28}$ **6** $\frac{3\sqrt{93}}{31}$
7 $\sqrt{181}$ km
8 $AC = \frac{12\sqrt{3}}{5}$ km, $BC = 2.4$ km

Multiple-choice questions

- 1** D **2** C **3** C **4** D **5** B
6 C **7** A **8** B

Extended-response questions

- 1 a** $\angle ACB = 12^\circ$, $\angle CBO = 53^\circ$, $\angle CBA = 127^\circ$
b 189.33 m **c** 113.94 m
2 a 184.78 m **b** 199.71 m **c** 14.93 m
3 a 370.17 m **b** 287.94 m **c** 185.08 m

Chapter 4

Exercise 4A

- 1** $a = 10$, $b = 0$, $c = -7$
2 $a = 1$, $b = -2$
3 $a = 2$, $b = -1$, $c = 7$
4 $a = 2$, $b = 1$, $c = 3$
5 $(x+2)^2 - 4(x+2) + 4$
6 $(x+1)^3 - 3(x+1)^2 + 3(x+1) - 1$
7 $a = 1$, $b = -2$, $c = -1$
8 a It is impossible to find a , b and c such that $a = 3$, $3ab = -9$, $3ab^2 = 8$ and $ab^3 + c = 2$
b $a = 3$, $b = -1$, $c = 5$
9 $a = 1$, $b = -6$, $c = 7$, $d = -1$
10 a If $a = -\frac{5}{3}b$ and $a = -3b$, then both a and b are zero, but then $a + b = 1$ is not satisfied
b $(n+1)(n+2) - 3(n+1) + 1$
11 a $ax^2 + 2abx + ab^2 + c$
b $a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$
13 $a = -3$, $b = -\frac{1}{3}$, $c = 3$ or $a = -\frac{1}{3}$, $b = -3$, $c = 3$
14 $a = 3$, $b = -3$, $c = 1$
15 If $c = 5$, then $a = 1$ and $b = -5$;
if $c = -27$, then $a = -3$ and $b = 3$

Exercise 4B

- 1 a** $x = -1$ **b** $x = 3$
c $x = 1 \pm \frac{\sqrt{30}}{5}$ **d** $x = 1 \pm \frac{\sqrt{2}}{2}$
e $x = -1 \pm \frac{3\sqrt{2}}{2}$ **f** $x = \frac{-13 \pm \sqrt{145}}{12}$
2 a $m > \frac{9}{4}$ **b** $m < \frac{25}{4}$ **c** $m = -\frac{25}{32}$
d $m \leq -6$ or $m \geq 6$ **e** $-4 < m < 4$
f $m = 0$ or $m = -16$
3 a $x = \frac{1 \pm \sqrt{32t+1}}{4}$, $t \geq -\frac{1}{32}$
b $x = \frac{-1 \pm \sqrt{t+3}}{2}$, $t \geq -3$
c $x = \frac{-2 \pm \sqrt{5t-46}}{5}$, $t \geq \frac{46}{5}$
d $x = -2 \pm \frac{\sqrt{5t(t-2)}}{t}$, $t < 0$ or $t \geq 2$
4 a $x = \frac{-p \pm \sqrt{p^2+64}}{2}$ **b** $p = 0, 6$
5 a $\Delta = (3p-4)^2$ **b** $p = \frac{4}{3}$
c i $x = 1$ or $x = \frac{1}{2}$ **ii** $x = 1$ or $x = 2$
iii $x = 1$ or $x = -\frac{5}{2}$
6 a $\Delta = 16(2p-3)^2$ **b** $p = \frac{3}{2}$
c i $x = \frac{3}{2}$ or $x = \frac{1}{2}$ **ii** $x = \frac{1}{2}$ or $x = \frac{3}{10}$
iii $x = \frac{1}{2}$ or $x = -\frac{3}{14}$
7 a $x = 4$ or $x = 36$ **b** $x = 16$
c $x = 49$ **d** $x = 1$ or $x = 512$
e $x = 27$ or $x = -8$ **f** $x = 16$ or $x = 625$
8 $x = 2$
9 Side length 37.5 cm
10 $a = 3$, $b = -\frac{5}{6}$, $c = -\frac{13}{12}$; Minimum $-\frac{13}{12}$
12 $x = 1$ or $x = \frac{a-b}{b-c}$
13 $m = 8$
14 a $\Delta = 16((a-c)^2 + 2b^2) \geq 0$
b $a = c$ and $b = 0$
15 $-8 < k < 0$
16 $p = 10$

Exercise 4C

- 1 a** $\frac{18}{x(x+3)}$ **b** $x = -6$ or $x = 3$
2 $x = -30$ or $x = 25$
3 17 and 19
4 a $\frac{40}{x}$ hours **b** $\frac{40}{x-2}$ hours **c** 10 km/h

5 a Car $\frac{600}{x}$ km/h; Plane $(\frac{600}{x} + 220)$ km/h

b Car 80 km/h; Plane 300 km/h

6 $x = 20$

7 6 km/h

8 a $x = 50$ **b** 72 minutes

9 Slow train 30 km/h; Express train 50 km/h

10 60 km/h

11 Small pipe 25 minutes; Large pipe 20 minutes

12 Each pipe running alone takes 14 minutes

13 Rail 43 km/h; Sea 18 km/h

14 22 km

15 10 litres

16 32.23 km/h, 37.23 km/h

17 a $a + \sqrt{a^2 - 24a}$ minutes,
 $a - 24 + \sqrt{a^2 - 24a}$ minutes

b i 84 minutes, 60 minutes

ii 48 minutes, 24 minutes

iii 36 minutes, 12 minutes

iv 30 minutes, 6 minutes

18 a 120 km **b** 20 km/h, 30 km/h

Exercise 4D

- 1 a** $\frac{2}{x-1} + \frac{3}{x+2}$ **b** $\frac{1}{x+1} - \frac{2}{2x+1}$
c $\frac{2}{x+2} + \frac{1}{x-2}$ **d** $\frac{1}{x+3} + \frac{3}{x-2}$
e $\frac{3}{5(x-4)} - \frac{8}{5(x+1)}$
2 a $\frac{2}{x-3} + \frac{9}{(x-3)^2}$
b $\frac{4}{1+2x} + \frac{2}{1-x} + \frac{3}{(1-x)^2}$
c $\frac{-4}{9(x+1)} + \frac{4}{9(x-2)} + \frac{2}{3(x-2)^2}$
3 a $\frac{-2}{x+1} + \frac{2x+3}{x^2+x+1}$ **b** $\frac{x+1}{x^2+2} + \frac{2}{x+1}$
c $\frac{x-2}{x^2+1} - \frac{1}{2(x+3)}$
4 $3 + \frac{3}{x-1} + \frac{2}{x-2}$
5 It is impossible to find A and C such that
 $A = 0$, $C - 2A = 2$ and $A + C = 10$
6 a $\frac{1}{2(x-1)} - \frac{1}{2(x+1)}$ **b** $\frac{2}{5(x-2)} + \frac{3}{5(x+3)}$
c $\frac{1}{x-2} + \frac{2}{x+5}$
d $\frac{2}{5(2x-1)} - \frac{1}{5(x+2)}$
e $\frac{3}{3x-2} - \frac{1}{2x+1}$ **f** $\frac{2}{x-1} - \frac{2}{x}$

$$\begin{array}{ll} \text{g } \frac{1}{x} + \frac{3-x}{x^2+1} & \text{h } \frac{2}{x} + \frac{x}{x^2+4} \\ \text{i } \frac{1}{4(x-4)} - \frac{1}{4x} & \text{j } \frac{7}{4(x-4)} - \frac{3}{4x} \\ \text{k } x + \frac{1}{x} - \frac{1}{x-1} & \text{l } -x-1 - \frac{3}{x} - \frac{1}{2-x} \\ \text{m } \frac{2}{3(x+1)} + \frac{x-4}{3(x^2+2)} & \\ \text{n } \frac{2}{3(x-2)} + \frac{1}{3(x+1)} - \frac{1}{(x+1)^2} & \\ \text{o } \frac{2}{x} + \frac{1}{x^2+4} & \text{p } \frac{8}{2x+3} - \frac{5}{x+2} \\ \text{q } \frac{26}{9(x+2)} + \frac{1}{9(x-1)} - \frac{1}{3(x-1)^2} & \\ \text{r } \frac{16}{9(2x+1)} - \frac{8}{9(x-1)} + \frac{4}{3(x-1)^2} & \\ \text{s } x-2 + \frac{1}{4(x+2)} + \frac{3}{4(x-2)} & \\ \text{t } x - \frac{1}{x+1} + \frac{2}{x-1} & \text{u } \frac{3}{x+1} - \frac{7}{3x+2} \end{array}$$

Exercise 4E

- 1 a (1, 1), (0, 0) b (0, 0), $(\frac{1}{2}, \frac{1}{2})$
c $(\frac{3+\sqrt{13}}{2}, 4+\sqrt{13})$, $(\frac{3-\sqrt{13}}{2}, 4-\sqrt{13})$
- 2 a (13, 3), (3, 13) b (10, 5), (5, 10)
c (-8, -11), (11, 8) d (9, 4), (4, 9)
e (9, 5), (-5, -9)
- 3 a (11, 17), (17, 11) b (37, 14), (14, 37)
c (14, 9), (-9, -14)
- 4 (0, 0), (2, 4)
- 5 $(\frac{5+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2})$, $(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2})$
- 6 $(\frac{15}{2}, \frac{5}{2})$, (3, 1)
- 7 $(\frac{-130+80\sqrt{2}}{41}, \frac{60+64\sqrt{2}}{41})$,
 $(\frac{-130-80\sqrt{2}}{41}, \frac{60-64\sqrt{2}}{41})$
- 8 $(\frac{1+\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2})$, $(\frac{1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2})$
- 9 $(\frac{4}{9}, 2)$
- 10 $(\frac{-6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5})$
- 11 $(-2, \frac{1}{2})$
- 12 (0, -1), (3, 2)
- 13 a $(\frac{2}{3}, -\frac{7}{9})$ b $(-\frac{1}{2}, 0)$, (1, 0)
c $(-\frac{3}{2}, \frac{7}{4})$ d (-1, 4), (0, 2)

Chapter 4 review

Technology-free questions

- 1 a = 3, b = 2, c = 1
- 2 $(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$
- 5 a x = -4 or x = 3 b x = -1 or x = 2
c x = -2 or x = 5 d $x = \frac{2 \pm \sqrt{2}}{2}$
e $x = \frac{1 \pm \sqrt{3t-14}}{3}$ f $x = \frac{t \pm \sqrt{t^2-16t}}{2t}$
- 6 $x = \frac{-3 \pm \sqrt{73}}{2}$
- 7 a $\frac{-1}{x-3} - \frac{2}{x+2}$ b $\frac{3}{x+2} + \frac{4}{x-2}$
c $\frac{1}{2(x-3)} - \frac{3}{2(x+5)}$ d $\frac{1}{x-5} + \frac{2}{x+1}$
e $\frac{13}{x+2} - \frac{13}{x+3} - \frac{10}{(x+2)^2}$
f $\frac{4}{x+4} + \frac{2}{x-1} - \frac{3}{(x-1)^2}$
g $\frac{1}{x+1} - \frac{6}{x^2+2}$ h $\frac{1}{x-1} - \frac{x+3}{x^2+x+1}$
i $\frac{1}{3-x} - \frac{3}{x+4}$ j $\frac{2}{7(x-3)} - \frac{16}{7(x+4)}$
- 8 a $\frac{1}{x-3} - \frac{x-10}{x^2+x+2}$
b $\frac{1}{4(x+1)} - \frac{x-2}{4(x^2-x+2)}$
c $3x+15 + \frac{64}{x-4} - \frac{1}{x-1}$
- 9 a (0, 0), (-1, 1) b (0, 4), (4, 0)
c (1, 4), (4, 1)
- 10 (-4, -1), (2, 1)
- 11 a $t = \frac{135}{x}$ b $t = \frac{135}{x-15}$ c x = 60
d 60 km/h, 45 km/h

Multiple-choice questions

- 1 C 2 D 3 D 4 C 5 E
6 E 7 C 8 D 9 B 10 B

Extended-response questions

- 1 a 24 km/h
b Speed = $\frac{a + \sqrt{a(a+480)}}{2}$, $a > 0$;
When $a = 60$, speed = 120 km/h, which is a very fast constant speed for a train. If we choose this as the upper limit for the speed, then $0 < a < 60$ and $0 < \text{speed} < 120$

a	1	8	14	22	34	43	56	77	118
speed	16	20	24	30	40	48	60	80	120

- 2 a** $\frac{a + \sqrt{a^2 + 4abc}}{2ac}$
b e.g. $a = 3$, $b = 1$, $c = \frac{4}{3}$
3 a $k = -2$, $k = 1$ **b** $-10 < c < 10$
c $p = 5$

Chapter 5

Technology-free questions

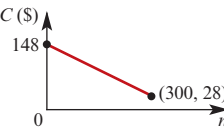
- 1 a** $-\frac{\sqrt{2}+3}{7}$ **b** $\frac{3(\sqrt{5}+1)}{4}$ **c** $\frac{4\sqrt{2}+2}{7}$
d $\frac{3(\sqrt{5}+\sqrt{3})}{2}$ **e** $\frac{\sqrt{7}+\sqrt{2}}{5}$ **f** $\frac{2\sqrt{5}+\sqrt{3}}{17}$
2 b i $x = \frac{-1-\sqrt{5}}{2}$ **ii** $x = \frac{-1+\sqrt{5}}{2}$
3 a $a = -7$, $b = -5$, $c = 1$
5 a $576 = 2^6 \times 3^2$, $\sqrt{576} = 24$
b $1225 = 5^2 \times 7^2$, $\sqrt{1225} = 35$
c $1936 = 2^4 \times 11^2$, $\sqrt{1936} = 44$
d $1296 = 6^4$, $\sqrt{1296} = 36$
6 a 21.80° **b** 3.06 **c** 9.97
7 $x = -b - c$
8 $x = \frac{2ab}{a+b}$
9 a $\sqrt{21}$ cm **b** $\frac{2\sqrt{7}}{7}$
10 12 cm
11 a $5\sqrt{3} \pm \sqrt{11}$ **b** $\sin^{-1}\left(\frac{5}{6}\right)$ or $180^\circ - \sin^{-1}\left(\frac{5}{6}\right)$
12 a $-\frac{1}{3}$, $b = -2$, $\lambda = -\frac{4}{3}$;
 $a = -\frac{1}{2}$, $b = -1$, $\lambda = -\frac{3}{2}$
13 a $x = \frac{(y-3)^2+1}{2}$ **b** $x = \frac{1}{3}\left(\frac{4}{(y+2)^2}-1\right)$
14 150 minutes
15 a $x = \frac{51}{25}$, $y = \frac{32}{25}$
b $x = \frac{a(b^2+1)}{a^2+b^2}$, $y = \frac{b(a^2-1)}{a^2+b^2}$
16 a 3 **b** 12 **c** 8
17 a $\Delta = 4a(a-1)$
b i $a = 1$ **ii** $a > 1$ or $a < 0$
iii $0 < a < 1$

Multiple-choice questions

- 1** E **2** B **3** C **4** C **5** A
6 C **7** C **8** A **9** B **10** B
11 A **12** D **13** B **14** B **15** A
16 A **17** C **18** A **19** C **20** B
21 B **22** E **23** C **24** C **25** E
26 D **27** A **28** A **29** D **30** D
31 C **32** B **33** B **34** C

Extended-response questions

- 1 a** 8 **b** 7.7 **c** 6 cm **d** 15 cm
2 a $a = -0.4$, $b = 148$
b C (\$) **c** \$68 **d** 248



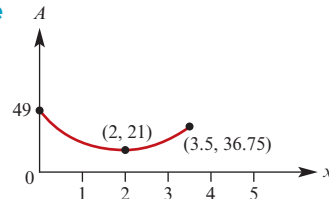
- 3 a i** 178 **ii** 179 **iii** 179.5 **iv** 179.95
b i 180 **ii** Circle
c 20 **d** $n = \frac{360}{180-A}$ **e** Square

- 4 a** Volume of hemisphere = $\frac{2}{3}\pi r^3$,
 Volume of cylinder = $\pi r^2 s$,
 Volume of cone = $\frac{1}{3}\pi r^2 w$

- b i** 6 : 2 : 3 **ii** 54π cubic units

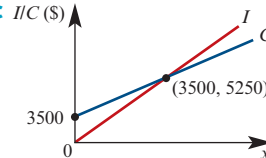
- 5 a** $a = 6000$, $b = -15\,000$ **b** \$57 000
c 2016

- 6 a** $8x$ cm **b** $28 - 8x$ cm **c** $7 - 2x$ cm
e



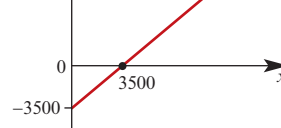
- f** $A = 21$ when $x = 2$

- 7 a** $C = 3500 + 0.5x$ **b** $I = 1.5x$
c I/C (\$) **d** 3500



- e** 5500

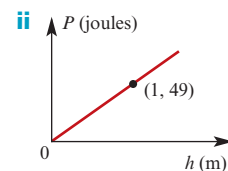
- f** P (\$) **P** represents profit



- 8 b i** $x = \frac{1}{24}$ **ii** $x = \frac{25}{24}$

- 9 c** 11, 24 and 39

- 10 a i** $g = 9.8$



- iii** 7 kg

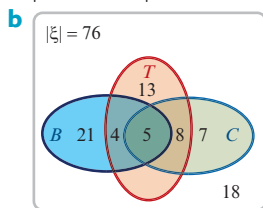
- b i** Doubled **ii** Halved

- c i** 14 m/s **ii** 42 m/s

- d** 4

11 a 1 hour 35 minutes b 2.5 km

12 a $|B' \cap C' \cap T| = |C \cap T|$,
 $|B \cap C' \cap T'| = 3|B' \cap C \cap T'|$,
 $|B \cap C' \cap T| = 4$



c i 5 ii 0

13 a $\angle TAB = 3^\circ$, $\angle ABT = 97^\circ$, $\angle ATB = 80^\circ$

b 2069.87 m c 252.25 m

14 a 78 m to C, 49 m to D

b 60 m c 279°

Problem-solving and modelling

See solutions supplement

Degree-of-difficulty classified questions

Simple familiar questions

1 a $\frac{5}{9}$ b $\frac{6}{11}$ c $\frac{605}{999}$ d $\frac{7}{3}$ e $\frac{145\,223}{33\,333}$

2 a $2 \times 7 \times 11 \times 13$ b $3 \times 5 \times 37$
c $7^2 \times 11 \times 13$ d $2^4 \times 5^4$

3 $\frac{1}{m+p}$

4 a $\sqrt{6} - \sqrt{3} - \sqrt{2} + 3$ b $21\sqrt{2} + 33$
c $4x - 12\sqrt{x} + 9$ d $-6\sqrt{x-2} + x + 7$

5 a $A = 1$, $B = 0$, $C = -1$
b $A = 20$, $B = 2$ and $C = 1$, or
 $A = 20$, $B = -2$ and $C = -1$

6 $(\frac{21}{5}, \frac{7}{5})$, (12, 4) 7 (1, -4), (-1, 4)

8 $x = -3$ 9 64 km/h 10 $\sqrt{6} - 2$

11 a $\frac{1}{3(x-2)} + \frac{1}{3(x+2)}$ b $\frac{1}{x+2} + \frac{1}{x+3}$

c $\frac{2}{x+2} + \frac{3x-2}{x^2+4}$

d $\frac{1}{x+1} + \frac{1}{x-1} - \frac{2}{(x-1)^2}$

e $\frac{x}{x^2+1} + \frac{1}{x-3}$ f $\frac{x+1}{x^2+4} + \frac{2}{x-2}$

12 $\angle A = 81.8^\circ$, $\angle B = 54.9^\circ$, $\angle C = 43.3^\circ$

13 $\angle ACB = 32.4^\circ$, $\angle ABC = 105.6^\circ$

Complex familiar questions

1 a 12 b 142

2 $x = \frac{5}{3}$ 3 $m = -\frac{1}{2}$ or $m = -\frac{11}{2}$

4 $x = \frac{5 \pm \sqrt{155}}{5}$

5 Maths 111, Physics 74, Chemistry 63

6 6.21 km 7 4 8 $a = 16$, $b = 192$

9 a $n(n-2)(n-1)(n+1)(n+2)$

10 1449.8 m

11 43.59 km on a bearing of 083.4°

12 7 children

Complex unfamiliar questions

1 (4, 12), (6, 6), (12, 4)

2 8 ways

3 a 2 b 5, 2

4 $(2^5 \times 3^8)^5$

5 a 90 km b 70 km/h

6 a 91 b 42 857

7 a 0, ± 6 , ± 15 b ± 9 , ± 12 , ± 21

c 11, 20, 27, 32, 35, 36

9 a $c = \frac{b^2 - 4}{4}$ b $c = \frac{2b^2}{9} + 3$

c $b = \pm 12$, $c = 35$

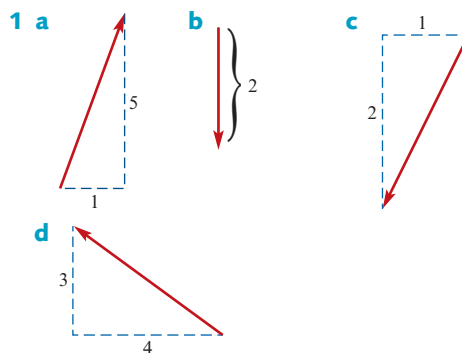
10 a $m = 5$, $n = 2$ b $b = 4$, $c = -1$

12 a $y - b = m(x - a)$

c $y = 2$, $y = -\frac{12}{5}x - \frac{26}{5}$

Chapter 6

Exercise 6A



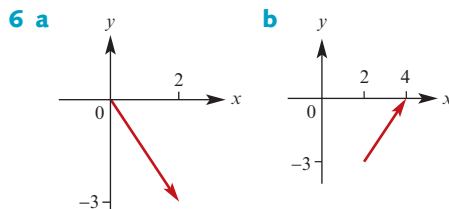
2 $a = 5$, $b = 1$

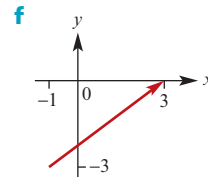
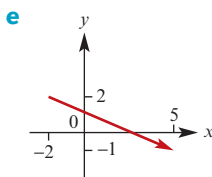
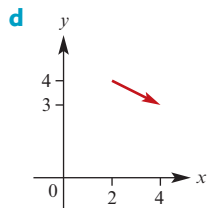
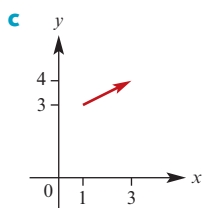
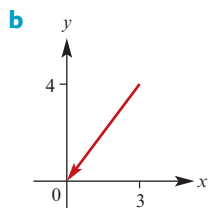
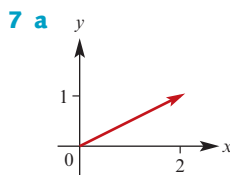
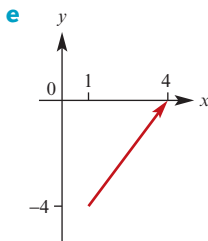
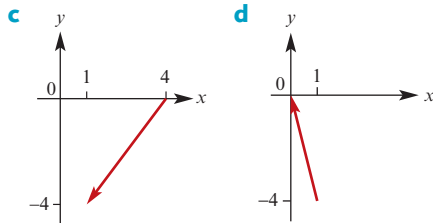
3 $a = 3$, $b = -15$

4 a $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ b $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ c $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ d $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ e $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

5 a i $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ii $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ iii $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

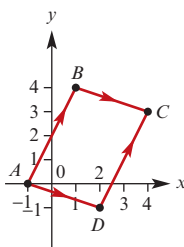
b $a + b = -c$





8 a and c

9 a b



d Parallelogram

10 $m = -11$, $n = 7$

11 a i $b - \frac{1}{2}a$ **ii** b

b $\overrightarrow{MN} = \overrightarrow{AD}$

12 a $\overrightarrow{CB} = a - b$, $\overrightarrow{MN} = \frac{1}{2}(b - a)$

b $\overrightarrow{CB} = -2\overrightarrow{MN}$

13 a a **b** b **c** $2a$ **d** $2b$

e $-a$ **f** $b - a$ **g** $a + b$

14 a a **b** $-b$ **c** $a + b$

d $-a - b$ **e** $b - a$

15 a $a - b$ **b** $\frac{1}{3}(b - a)$ **c** $\frac{1}{3}(a + 2b)$

d $\frac{1}{9}(a + 2b)$ **e** $\frac{1}{9}(4a - b)$

16 a $u + v$ **b** $v + w$ **c** $u + v + w$

17 a $\overrightarrow{OB} = u + v$, $\overrightarrow{OM} = u + \frac{1}{2}v$ **b** $u - \frac{1}{2}v$

c $\frac{2}{3}(u - \frac{1}{2}v)$

d $\overrightarrow{OP} = \frac{2}{3}(u + v) = \frac{2}{3}\overrightarrow{OB}$ **e** $2 : 1$

Exercise 6B

1 $2i - 7j$

2 a $5i + 6j$ **b** $-5i + 6j$ **c** $5i - 6j$

3 $7i + \frac{5}{2}j$

4 a 5 **b** 2 **c** 5 **d** 13

5 a 13 **b** $x = 2$, $y = -7$

6 a i $\frac{2}{5}i$ **ii** $-\frac{2}{5}i + j$ **iii** $\frac{1}{6}(-\frac{2}{5}i + j)$

iv $\frac{1}{3}i + \frac{1}{6}j$ **v** $2i + j$

b i $\overrightarrow{ON} = \frac{1}{6}\overrightarrow{OA}$ **ii** $1 : 5$

7 $4\sqrt{2}$ units

8 a $k = \frac{3}{2}$, $\ell = \frac{1}{2}$ **b** $x = 6$, $y = 2$

c $x = 3$, $y = 3$ **d** $k = -\frac{1}{3}$, $\ell = -\frac{5}{3}$

9 $3i - 2j$, $\sqrt{13}$

10 a $-2i + 4j$ **b** $-6i + j$ **c** 5

11 a $D(-6, 3)$ **b** $F(4, -3)$ **c** $G(\frac{3}{2}, -\frac{3}{2})$

12 $A(-1, -4)$, $B(-2, 2)$, $C(0, 10)$

13 a i $2i - j$ **ii** $-5i + 4j$ **iii** $i + 7j$

iv $6i + 3j$ **v** $6i + 3j$

b $D(8, 2)$

14 a $\overrightarrow{OP} = 12i + 5j$, $\overrightarrow{PQ} = 6i + 8j$ **b** 13 , 10

15 a i $\sqrt{29}$ **ii** $\sqrt{116}$ **iii** $\sqrt{145}$

b $(\sqrt{29})^2 + (\sqrt{116})^2 = (\sqrt{145})^2$

16 a i $-i - 3j$ **ii** $4i + 2j$ **iii** $-3i + j$

b i $\sqrt{10}$ **ii** $2\sqrt{5}$ **iii** $\sqrt{10}$

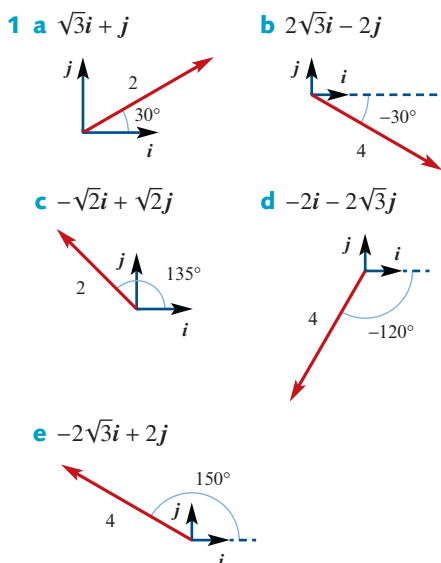
17 a i $-3i + 2j$ **ii** $7j$

iii $-3i - 5j$ **iv** $-2i - \frac{10}{3}j$

b $M(-2, \frac{11}{3})$

18 a $\frac{1}{5}(3i + 4j)$ b $\frac{1}{\sqrt{10}}(3i - j)$
 c $\frac{1}{\sqrt{2}}(-i + j)$ d $\frac{1}{\sqrt{2}}(i - j)$
 e $\frac{6}{\sqrt{13}}\left(\frac{1}{2}i + \frac{1}{3}j\right)$ f $\frac{1}{\sqrt{13}}(3i - 2j)$

Exercise 6C



2 a $4i + 4\sqrt{3}j$ b $5\sqrt{3}i + 5j$
 c $-3\sqrt{3}i + 3j$ d $4\sqrt{2}i - 4\sqrt{2}j$
 e $-6\sqrt{3}i - 6j$
 3 a $[4\sqrt{2}, -45^\circ]$ b $[3\sqrt{2}, 135^\circ]$
 c $[4, 150^\circ]$ d $[5, 53.13^\circ]$
 e $[13, -112.62^\circ]$
 4 a $[2, 30^\circ]$ b $[\sqrt{5}, 63.43^\circ]$
 c $[\sqrt{2}, -135^\circ]$ d $[2, -30^\circ]$
 e $[\sqrt{2}, 135^\circ]$
 5 a $5.36i + 4.50j$ b $8.19i + 5.74j$
 c $-9.97i + 4.65j$ d $7.37i - 5.16j$
 e $-6.89i - 4.83j$
 6 a $[20, 60^\circ]$ b $[1, -165^\circ]$
 c $[18, 160^\circ]$
 7 a $[24.18, 78.07^\circ]$ b $[19.83, 52.50^\circ]$
 c $[9.18, -5.00^\circ]$
 8 a $(6 + 4\sqrt{3})i + (4 + 6\sqrt{3})j$
 b $5\sqrt{2}i + (5\sqrt{2} - 10)j$ c $4.80i + 7.28j$

Exercise 6D

1 a 17 b 13 c 8 d -10
 e -4 f 3 g -58
 2 a 5 b 13 c 8
 d -5 e 13
 3 a $15\sqrt{2}$ b $-15\sqrt{2}$

4 a $|a|^2 + 4|b|^2 + 4a \cdot b$ b $4a \cdot b$
 c $|a|^2 - |b|^2$ d $|a|$
 5 a $-3i + j$ b $\sqrt{10}$ c 116.57°
 6 $\sqrt{66}$
 7 a $-\frac{11}{2}$ b $\frac{10}{3}$ c -1
 d $\frac{-2 \pm \sqrt{76}}{6}$
 8 a $-a + qb$ b $\frac{22}{29}$ c $\left(\frac{44}{29}, \frac{110}{29}\right)$
 9 a 139.40° b 71.57° c 26.57°
 d 126.87°
 11 a $25\sqrt{3}$ b 20 c $60\sqrt{3}$
 d -36 e $7\sqrt{3}$ f 20
 12 a $\frac{3}{2}i$ b 45° c 116.57°
 13 a i $\frac{3}{2}i + 2j$ ii $\frac{1}{2}i + 3j$
 b 27.41°
 c 55.30°

Exercise 6E

1 a $\frac{1}{\sqrt{10}}(i + 3j)$ b $\frac{1}{\sqrt{2}}(i + j)$ c $\frac{1}{\sqrt{2}}(i - j)$
 2 a $\hat{a} = \frac{1}{5}(3i + 4j)$, $|b| = \sqrt{2}$
 b $\frac{\sqrt{2}}{5}(3i + 4j)$
 3 a $\hat{a} = \frac{1}{5}(3i + 4j)$, $\hat{b} = \frac{1}{13}(5i + 12j)$
 b $\frac{1}{\sqrt{65}}(4i + 7j)$
 4 a $-\frac{11}{17}(i - 4j)$ b $\frac{13}{17}(i - 4j)$ c $4i$
 5 a $a = u + w$ where $u = 2i$ and $w = j$
 b $a = u + w$ where $u = 2i + 2j$ and $w = i - j$
 c $a = u + w$ where $u = 0$ and $w = -i + j$
 6 a 2 b $\frac{1}{\sqrt{5}}$ c $\frac{2\sqrt{3}}{\sqrt{7}}$ d $\frac{-1 - 4\sqrt{5}}{\sqrt{17}}$
 7 a $2i + 2j$ b $\frac{1}{\sqrt{2}}(-i + j)$
 8 a $\frac{3}{2}(i - j)$ b $\frac{5}{2}(i + j)$ c $\frac{5\sqrt{2}}{2}$
 9 a i $i - j$ ii $i - 5j$
 b $\frac{3}{13}(i - 5j)$ c $\frac{2\sqrt{26}}{13}$ d 2

Exercise 6F

1 a $-i - 11j$ b $5i - 6j$ c $i + 5j$
 d $-11j$ e $4i$ f $6i + 11j$
 2 12.58 km on a bearing of 341.46°
 3 7.74 km on a bearing of 071.17°

- 4 **a** $\sqrt{41}$ m/s **b** 5 m/s **c** $\sqrt{17}$ m/s
d $2\sqrt{10}$ m/s **e** 13 m/s **f** $\sqrt{170}$ m/s
5 **a** $24i + 62j$ **b** $(5t - 1)i + (12t + 2)j$
6 $-4i + 4j$ m/s
7 **a** **i** $26i + 99j$
ii $(7t - 2)i + (24t + 3)j$
b **i** 102.36 m
ii $\sqrt{(7t - 2)^2 + (24t + 3)^2}$ m
8 **a** $-i - \frac{1}{2}j$ m/s **b** $\frac{\sqrt{5}}{2}$ m/s
9 After $\frac{12(6 + \sqrt{5})}{31}$ seconds;
position vector $\frac{12(6 + \sqrt{5})}{31}(i + 2j)$
10 **a** $8i + 4j$ **b** $2i - 4j$ m/s
11 **a** $20i + 10j$ **b** j m/s

Exercise 6G

- 1 **a** On a bearing of 143.13°
b 5 km/h
2 100.08 km/h on a bearing of 357.71°
3 **a** 20 km/h west **b** 180 km/h west
4 47 m/s north
5 10 m/s
6 **a** 20 km/h north **b** 20 km/h south
7 252.98 km/h on a bearing of 018.43°
8 100 km/h on a bearing of 053.13°
9 42.5 km/h on a bearing of 221.73°
10 **a** $i - 4j$ m/s **b** 4.12 m/s
11 10.36 m/s
12 196.83 km/h on a bearing of 345.44°
13 **a** Bearing 210.67°
b 243.28 km/h
14 **a** Upstream at an angle of 48.59° to his desired path
b 45.36 seconds

Exercise 6H

- 1 $T_1 = T_2 = \frac{5\sqrt{2}}{2}$ kg wt
2 90°
3 $T_1 = 14.99$ kg wt, $T_2 = 12.10$ kg wt
4 28.34 kg wt, $W48.5^\circ S$
5 $T = 40$ kg wt, $N = 96$ kg wt
6 $F = 6.39$ kg wt
7 **a** No **b** Yes
8 146.88° , 51.32° , 161.8°
9 **a** 7.5 kg wt **b** 9.64 kg wt **c** 7.62 kg wt
10 32.97 kg wt, 26.88 kg wt, 39.29 kg wt,
 $W = 39.29$ kg

- 11 13.05 kg wt
12 5.74 kg wt
13 3.73 kg wt, 8.83 kg wt
14 4.13 kg wt
15 6.93 kg wt
16 31.11 kg, 23.84 kg wt
17 44.10 kg, 22.48° to the vertical
18 6.43 kg wt, 7.66 kg wt, 11.92 kg
19 3.24 kg wt

Chapter 6 review

Technology-free questions

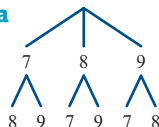
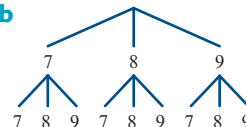
- 1 **a** $\frac{12}{7}$ **b** ± 9
2 $A(2, -1)$, $B(5, 3)$, $C(3, 8)$, $D(0, 4)$
3 $k = -\frac{3}{4}$
4 **a** $\sqrt{26}$ **b** $\frac{1}{\sqrt{26}}(i - 5j)$
5 **a** $\frac{1}{5}(4i + 3j)$ **b** $\frac{16}{25}(4i + 3j)$
6 **a** **i** $a + b$ **ii** $\frac{1}{3}(a + b)$ **iii** $b - a$
iv $\frac{1}{3}(2a - b)$ **v** $\frac{2}{3}(2a - b)$
b $\vec{TR} = 2\vec{PT}$
7 $s = -7$, $t = 9$
8 $\sqrt{109}$ units
9 **a** $11i - 2j$ **b** $\sqrt{29}$
c $\frac{1}{\sqrt{29}}(5i + 2j)$ **d** $2i + 4j$
10 **a** $(-1, 10)$ **b** $h = 3$, $k = -2$
11 **a** $b = a + c$ **b** $b = \frac{2}{5}a + \frac{3}{5}c$
12 $[4, 150^\circ]$
13 **a** 13 **b** 10 **c** 8 **d** -11
e -9 **f** 0 **g** -27
15 **a** $\frac{6}{5}$ **b** $\pm \frac{3}{\sqrt{2}}$ **c** $\frac{7}{3}$
16 **a** **i** $\vec{AB} = -i$ **ii** $\vec{AC} = -5j$
b 0
c 1
17 **a** 2 m/s **b** 30 seconds
c 36 m downstream of her starting point
18 9 kg wt, 12 kg wt
19 $14\sqrt{5}$ kg wt, $28\sqrt{5}$ kg wt
20 $5\sqrt{3}$ kg wt
- Multiple-choice questions**
- 1 C 2 C 3 A 4 B 5 B
6 A 7 C 8 D 9 D 10 A
11 B 12 E 13 C 14 B 15 B

Extended-response questions

- 1 **a** $\begin{bmatrix} -31 \\ -32 \end{bmatrix}$ **b** $\begin{bmatrix} -15 \\ -20 \end{bmatrix}$ **c** $|\vec{OR}| = 25$
 2 **a** $\sqrt{34}$ **b** $\sqrt{10} - \sqrt{20}$ **c** $r = i - 9j$
 3 **a** $(25, -7)$, $\begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **b** $\begin{bmatrix} -20 \\ 15 \end{bmatrix}$
 4 **a** $(12, 4)$ **b** $\begin{bmatrix} k-12 \\ -4 \end{bmatrix}$
c $\sqrt{160}$, k , $\sqrt{(k-12)^2 + 16}$, $k = \frac{40}{3}$
d 34.7°
 5 **a** 2.5 hours
b 011.54° , 36.7 minutes
c 168.46°
 6 **a** $4i + (16 - \alpha)j$ km/h
b i $\alpha = 16$ **ii** 2.5 hours

Chapter 7

Exercise 7A

- 1 45
 2 8
 3 120
 4 **a**  **b** 
 5 **a** 27 **b** 6
 6 30
 7 **a** 6 **b** 18 **c** 20 **d** 15
 8 BB, BR, BG, RB, RG, GB, GR, GG
 9 12
 10 9
 11 **a** 6 **b** 13
 12 16

Exercise 7B

- 1 1, 1, 2, 6, 24, 120, 720, 5040, 40 320, 362 880, 3 628 800
 2 **a** 5 **b** 90 **c** 66 **d** $\frac{161\,700}{(n+1)!}$
 3 **a** $n+1$ **b** $n+2$ **c** $n(n-1)$ **d** $\frac{n+2}{(n+1)!}$
 4 1, 4, 12, 24, 24
 5 DOG, DGO, ODG, OGD, GOD, GDO
 6 120
 7 362 880
 8 FR, FO, FG, RF, RO, RG, OF, OR, OG, GF, GR, GO
 9 **a** 720 **b** 720 **c** 360

- 10 **a** 120 **b** 120 **c** 60
 11 20 160
 12 **a** 125 **b** 60
 13 **a** 120 **b** 360 **c** 720
 14 60
 15 **a** 17 576 000 **b** 11 232 000
 16 $(m, n) = (6, 0), (6, 1), (5, 3)$
 17 $(n^2 - n) \cdot (n - 2)! = n \cdot (n - 1) \cdot (n - 2)! = n!$
 18 **a** 384 **b** 3072
 19 30

Exercise 7C

- 1 **a** 120 **b** 72 **c** 24 **d** 96
 2 **a** 120 **b** 48 **c** 72 **d** 12
 3 **a** 360 **b** 144 **c** 144 **d** 72
 4 **a** 1152 **b** 1152
 5 **a** 600 **b** 108 **c** 431 **d** 52
 6 **a** 720 **b** 48 **c** 144 **d** 96 **e** 48
 7 **a** 900 **b** 900
 8 84
 9 32
 10 **a** 480 **b** 192
 11 144

Exercise 7D

- 1 35 **2** 34 650
 3 4 989 600 **4** 56
 5 27 720
 6 **a** 420 **b** 105 **c** 90 **d** 12 **e** 105
 7 35
 8 **a** 15 **b** $\frac{(m+n)!}{m! \cdot n!}$
 9 **a** 52! **b** $\frac{104!}{(2!)^{52}}$ **c** $\frac{(52n)!}{(n!)^{52}}$
 10 4900
 11 89

Exercise 7E

- 1 1, 5, 10, 10, 5, 1
 2 **a** 7 **b** 6 **c** 66 **d** 56 **e** 100
f 499 500
 3 **a** n **b** $\frac{n(n-1)}{2}$ **c** n **d** $n+1$
e $\frac{(n+2)(n+1)}{2}$ **f** $\frac{n(n+1)}{2}$
 4 **a** 720 **b** 120
 5 2 598 960
 6 **a** 10 **b** 45 **c** 45 **d** 10
 7 45 379 620
 8 56
 9 **a** 45 **b** 16

10 15

$$\begin{aligned} 11 \quad {}^nC_{n-r} &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= \frac{n!}{(n-r)!r!} = {}^nC_r \end{aligned}$$

12 Each diagonal is obtained by choosing 2 vertices from n vertices. This can be done in ${}^nC_2$ ways. But n of these choices define a side of the polygon, not a diagonal. Therefore there are ${}^nC_2 - n$ diagonals.

13 There are ${}^{10}C_5$ ways to choose 5 students for team A. The remaining 5 students will belong to team B. However, the labelling of the teams does not matter, so we must divide by 2.

14 462

$$\begin{aligned} 15 \quad {}^{n-1}C_{r-1} + {}^{n-1}C_r &= \frac{(n-1)!}{(r-1)!(n-1-(r-1))!} + \frac{(n-1)!}{r!(n-1-r)!} \\ &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!} \left(\frac{1}{n-r} + \frac{1}{r} \right) \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!} \cdot \frac{n}{r(n-r)} \\ &= \frac{n!}{r!(n-r)!} = {}^nC_r \end{aligned}$$

16 a 2300 b 152 c 2148

Exercise 7F

- 1 153 2 126
 3 1176 4 140
 5 a 1716 b 700 c 980 d 1568
 6 a 25 200 b 4200
 7 a 1 392 554 592 b 5 250 960
 8 a 15 504 b 10 800 c 15 252
 9 a 21 b 10 c 11
 10 2100
 11 a 204 490 b 7 250 100
 12 a 48 b 210
 13 1440 14 3600
 15 14 400 16 150
 17 3744

Exercise 7G

- 1 ${}^7C_2 = 21$, ${}^6C_2 = 15$, ${}^6C_1 = 6$
 2 1, 7, 21, 35, 35, 21, 7, 1; ${}^7C_2 = 21$, ${}^7C_4 = 35$
 3 1, 8, 28, 56, 70, 56, 28, 8, 1;
 ${}^8C_4 = 70$, ${}^8C_6 = 28$
 4 $2^6 = 64$
 5 $2^5 = 32$
 6 $2^{10} = 1024$
 7 $2^6 - 1 = 63$

$$8 \quad 2^8 - {}^8C_1 - {}^8C_0 = 247$$

$$9 \quad 2^8 = 256$$

$$10 \quad 2^4 - 1 = 15$$

$$11 \quad \text{a } 128 \quad \text{b } 44$$

Exercise 7H

1 4

2 Label 26 holes from A to Z. Put each of the 27 words into the hole labelled by its first letter. Some hole contains at least two words.

3 Label 4 holes by 0, 1, 2, 3. Put each of the 5 numbers into the hole labelled by its remainder when divided by 4. Some hole contains at least two numbers.

4 a 3

b 5

c 14

5 Divide $[0, 1]$ into 10 subintervals: $[0, 0.1]$, $[0.1, 0.2]$, ..., $[0.9, 1]$. Some interval contains at least two of the 11 numbers.

6 Divide into 4 equilateral triangles of side length 1 unit as shown. Some triangle contains at least two of the 5 points.



7 Divide the rectangle into squares of size 2×2 . There are 12 squares and 13 points, so some square contains at least two points. The distance between two points in the same square cannot exceed the length of the square's diagonal, $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

8 a For two-digit numbers, the possible digital sums are 1, 2, ..., 18. Since $19 > 18$, some digital sum occurs at least twice.

b For three-digit numbers, the possible digital sums are 1, 2, ..., 27. Since $82 = 3 \times 27 + 1$, some digital sum occurs at least 4 times.

9 Label 4 holes by 0, 1, 2, 3. Place each number into the hole labelled by its remainder when divided by 4. Since $13 = 3 \times 4 + 1$, some hole contains at least 4 numbers.

10 Two teams can be chosen in ${}^8C_2 = 28$ ways. Since there are 29 games, some pair of teams play each other at least twice.

11 At least 26 students. To show that 26 numbers suffice, label 25 holes by (1 or 49), (2 or 48), ..., (24 or 26), (25). To show that 25 numbers do not, consider 1, 2, 3, ..., 25.

12 Label the chairs 1, 2, ..., 14. There are 14 groups of three consecutive chairs: $\{1, 2, 3\}$, $\{2, 3, 4\}$, ..., $\{13, 14, 1\}$, $\{14, 1, 2\}$. Each of the 10 people belongs to 3 groups, so there are 30 people to be allocated to 14 groups. Since $30 \geq 2 \times 14 + 1$, some group contain at least 3 people.

- 13** Draw a diameter through one of the 4 points. This creates 2 half circles. One half circle contains at least two of the 3 remaining points (and the chosen point).
- 14** There are 195 possible sums: 3, 4, ..., 197. There are ${}^{35}C_2 = 595$ ways to choose a pair of players. Since $595 \geq 3 \times 195 + 1$, at least 4 pairs have the same sum.
- 15** Label the chairs 1, 2, ..., 12. There are 6 pairs of opposite seats:
{1, 7}, {2, 8}, {3, 9}, {4, 10}, {5, 11}, {6, 12}
Some pair contains two of the 7 boys.
- 16** Label n holes by 0, 1, 2, ..., $n - 1$. Place each guest in the hole labelled by the number of hands they shake. The first or last hole must be empty. (If a guest shakes 0 hands, then no guest shakes n hands. If a guest shakes n hands, then no guest shakes 0 hands.) This leaves $n - 1$ holes, so some hole contains at least two guests.

Exercise 7I

- 1 a** {1, 3, 4} **b** {1, 3, 4, 5, 6} **c** {4}
d {1, 2, 3, 4, 5, 6} **e** 3
f $\emptyset$, {4}, {5}, {6}, {4, 5}, {4, 6}, {5, 6}, {4, 5, 6}
- 2** 36 **3** 4
- 4** 150
- 5 a** 64 **b** 32
- 6 a** 72 **b** 72 **c** 36 **d** 108
- 7 a** 12 **b** 38
- 8** 88 **9** 80
- 10** 4
- 11 a** 756 **b** 700 **c** 360 **d** 1096
- 12** 1 452 555 **13** 3417
- 14** 5

Exercise 7J

- 1 a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$
- 2 a** $\frac{1}{4}$ **b** $\frac{3}{4}$
- 3** $\frac{3}{8}$
- 4 a** $\frac{1}{5}$ **b** $\frac{3}{5}$ **c** $\frac{3}{10}$
- 5 a** $\frac{68}{609}$ **b** $\frac{374}{1015}$
- 6** $\frac{1}{5}$
- 7** $\frac{329}{858}$
- 8 a** $\frac{2^7}{2^8 - 1} \approx 0.502$ **b** $\frac{56}{255}$ **c** $\frac{73}{85}$

- 9 a** $\frac{5}{204}$ **b** $\frac{35}{136}$
- 10 a** $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{17}{21}$
- 11 a** $\frac{2}{5}$ **b** $\frac{3}{5}$ **c** $\frac{2}{5}$ **d** $\frac{1}{10}$ **e** $\frac{3}{5}$
- 12 a** $\frac{24}{49}$ **b** $\frac{25}{49}$ **c** $\frac{3}{7}$
- 13** $\frac{4}{5}$
- 14 a** $\frac{1}{2}$ **b** $\frac{3}{14}$
- 15 a** 0.659 **b** 0.341 **c** 0.096
- 16 a** $\frac{5}{42}$ **b** $\frac{20}{21}$
- 17** $\frac{5}{24}$
- 18** $\frac{5}{11}$

Chapter 7 review

Technology-free questions

- 1 a** 20 **b** 190 **c** 300 **d** 4950
- 2** 11
- 3 a** 27 **b** 6
- 4** 120 **5** 60
- 6** 18 **7** 31
- 8** 10 **9** 3
- 10** 12 **11** 192
- 12** $\frac{1}{10}$ **13** $\frac{1}{2}$

Multiple-choice questions

- 1** C **2** B **3** A **4** D **5** B
6 B **7** C **8** D **9** C **10** C
11 A **12** A **13** E

Extended-response questions

- 1 a** 120 **b** 360 **c** 72 **d** 144
- 2 a** 20 **b** 80 **c** 60
- 3 a** 210 **b** 84 **c** 90 **d** 195
- 4 a** 420 **b** 15 **c** 105 **d** 30
- 5 a i** 20 **ii** 10 **iii** 64
b 8
- 6 a** 210 **b** 10 **c** 80
- 7 a** 676 **b** 235 **c** 74
- 8 a** 24 **b** 4 **c** 24 **d** $\frac{3}{4}$
- 9 a** 924
b There are at least $365 \times 3 = 1095$ days in three years and there are 924 different paths, so some path is taken at least twice.
c i 6 **ii** 70 **iii** 420
d 624

- 10 $\frac{1400}{6561}$
- 11 a $\frac{4 \cdot 48! \cdot 47!}{52! \cdot 44!}$ b $\frac{4 \cdot 48! (52 - k)!}{52! (49 - k)!}$
- c $1 - \frac{48! (52 - k)!}{52! (48 - k)!}$
- 12 196

Chapter 8

See solutions supplement

Chapter 9

Exercise 9A

- 1 a i Obtuse ii Straight
iii Acute iv Right
- b i $\angle HFB$ ii $\angle BFE$
iii $\angle HFG$ iv $\angle BFE$
- c i $\angle CBD, \angle BFE, \angle ABF, \angle HFG$
ii $\angle CBA, \angle BFH, \angle DBF, \angle EFG$
- 2 a $a = 65^\circ, b = 65^\circ$
b $x = 40^\circ, y = 130^\circ$
c $a = 60^\circ, b = 70^\circ, c = 50^\circ, d = 60^\circ,$
 $e = 50^\circ, f = 130^\circ$
d $\alpha = 60^\circ, \beta = 120^\circ$
e $\alpha = 90^\circ, \beta = 93^\circ$
f $\alpha = 108^\circ, \beta = 90^\circ, \theta = 108^\circ$
- 4 a $\angle B = \angle D = 180^\circ - \alpha$ b $\angle C = \alpha$
- 9 a $\theta = 107^\circ$ b $\theta = 55^\circ$

Exercise 9B

- 1 a Yes b Yes c Yes d No
- 2 a Scalene b Isosceles c Equilateral
- 3 Must be greater than 10 cm
- 4 a 6, 6.5, 7 b No
- 6 a $\theta = 46^\circ$, straight angle;
 $\beta = 70^\circ$, supplementary to $\angle EBC$;
 $\gamma = 64^\circ$, alternate angles ($\angle CBD$);
 $\alpha = 46^\circ$, corresponding angles ($\angle EBD$)
- b $\gamma = 80^\circ$, angle sum of triangle;
 $\beta = 80^\circ$, vertically opposite (γ);
 $\theta = 100^\circ$, supplementary to β ;
 $\alpha = 40^\circ$, alternate angles ($\angle BAD$)
- c $\alpha = 130^\circ$, supplementary to $\angle ADC$;
 $\beta = 65^\circ$, co-interior angles $\angle CDA$;
 $\gamma = 65^\circ$, co-interior angles $\angle CDA$
- d $\alpha = 60^\circ$, equilateral triangle
- e $\alpha = 60^\circ$, straight angle;
 $\beta = 60^\circ$, angle sum of triangle

- f $a = 55^\circ$, straight angle;
 $b = 55^\circ$, corresponding angles (a);
 $g = 45^\circ$, vertically opposite;
 $c = 80^\circ$, angle sum of triangle;
 $e = 100^\circ$, straight angle;
 $f = 80^\circ$, corresponding angles (c)

- g $m = 68^\circ$, corresponding angles;
 $n = 60^\circ$, angle sum of triangle;
 $p = 52^\circ$, straight angle;
 $q = 60^\circ$, alternate angles (n);
 $r = 68^\circ$, alternate angles (m)

- 7 a Sum = 720° ; Angles = 120°
b Sum = 1800° ; Angles = 150°
c Sum = 3240° ; Angles = 162°
- 8 a Together they form 10 straight angles
b 360°

- 10 10

Exercise 9C

- 1 a A and C (SAS)
b All of them (AAS)
c A and B (SSS)
- 2 a $\triangle ABC \equiv \triangle CDA$ (SSS)
b $\triangle CBA \equiv \triangle CDE$ (SAS)
c $\triangle CAD \equiv \triangle CAB$ (SAS)
d $\triangle ADC \equiv \triangle CBA$ (RHS)
e $\triangle DAB \equiv \triangle DCB$ (SSS)
f $\triangle DAB \equiv \triangle DBC$ (SAS)
- 6 a $a = b = c = d = 60^\circ$
- 7 a $a = 108^\circ, b = 36^\circ, c = 72^\circ, d = 36^\circ,$
 $e = 36^\circ, f = 36^\circ$

Exercise 9D

- 1 16.58 m 2 41 m
3 13.9 cm 4 18.38 cm
5 a 50 cm^2 b 32 cm^2
6 $2\sqrt{5} \text{ cm}$ 7 2 cm^2
8 $2\sqrt{6} \approx 4.9 \text{ cm}$ 9 b, d
12 13.86 cm 14 $XY = 2.8 \text{ cm}$
15 $x = 1.375, y = 2.67$ 16 $3\sqrt{2} \text{ cm}$

Exercise 9E

- 1 2 parts = 2000, 7 parts = 7000
2 1 part = 3000, 2 parts = 6000
3 3.6 4 264
5 22.5 6 $60^\circ, 50^\circ, 70^\circ$
7 \$14 8 30 g zinc, 40 g tin
9 16 white beads, 8 green beads
10 5.625 km 11 \$1200
12 $\frac{3}{5}$ 13 $\pi : 1$
14 1 : 1 15 6 : 7
16 8.75 km

Exercise 9F

- 1 **a** AAA, 11.25 cm **b** AAA, $11\frac{2}{3}$ cm
 c AAA, 3 cm **d** AAA, 7.5 cm
- 2 **a** AAA, 6 cm **b** AAA, $1\frac{1}{3}$ cm
 c AAA, $2\frac{2}{3}$ cm **d** AAA, 7.5 cm
- 3 $AC = 17.5$, $AE = 16$, $AB = 20$
- 4 4.42 m 5 7.5 m
- 6 15 m 7 22.5 m
- 8 $10\frac{10}{31}$ m 9 $x = 6\frac{2}{3}$
- 10 83.6 m 11 $x = \frac{39}{46}$
- 12 $40\frac{1}{7}$ m 13 7.2 m
- 14 $1\frac{14}{15}$ m
- 15 **b** $x = 10$ **c** $y = 2\sqrt{5}$, $z = 5\sqrt{5}$
- 16 $a = \frac{36}{7}$ 17 7.11 m
- 18 1.6 cm 19 $2\frac{1}{7}$ m
- 20 $a = 3\sqrt{5}$, $x = 5$, $y = 2\sqrt{5}$

Exercise 9G

See solutions supplement

Exercise 9H

- 1 **a** 1 : 2 : 3 : 4 **b** 1 : 4 : 9 : 16
 c Yes, second ratio is the square of the first
- 2 **a** 1 : 2 : 3 : 4 **b** 1 : 4 : 9 : 16
 c Yes, second ratio is the square of the first
- 3 $19\frac{4}{9}$ cm²
- 4 4.54 cm²
- 5 **a** $\sqrt{3}$ cm **b** $\frac{4\sqrt{3}}{3}$ cm **c** $\frac{4}{3}$
- 6 4 : 5
- 7 22.5
- 8 **a** 1 : 2 : 3 **b** 1 : 2 : 3 **c** 1 : 8 : 27
 d Yes, the third ratio is the cube of the first
- 9 **a** **i** 2 : 3 **ii** 2 : 3 **iii** 2 : 3
 b 8 : 27
 c Yes, the ratios in **a** are cubed to form the ratios in **b**
- 10 **a** 3 : 2 : 5
 b Volumes (in cm³) are 36π , $\frac{32}{3}\pi$ and $\frac{500}{3}\pi$;
 Ratio of volumes is 27 : 8 : 125
 c Yes, the ratios in **a** are cubed to form the ratios in **b**
- 11 8 : 1
- 12 27 : 64
- 13 2 : 3
- 14 **a** 4 : 3 **b** 4 : 3
- 15 **a** 4 : 1 **b** 8 : 1
- 16 **a** 1 : 100 **b** 1 : 1000 **c** 1 : 10
 d 1 : 1

- 17 $\frac{27}{16}$ litres, 4 litres
- 18 125 mL, 216 mL
- 19 **a** 1 : 50 **b** 1 : 125 000
 c 3 cm **d** 7500 cm²
- 20 **a** 12 : 13 **b** 1728 : 2197
- 21 **a** 4 **b** 3.75
- 22 3 : 4 23 4.5 cm

Exercise 9I

- 1 **a** **i** $\frac{4}{5}p$ **ii** $\frac{1}{5}p$ **iii** $-p$ **iv** $\frac{1}{5}(q-p)$ **v** $\frac{1}{5}q$
 b RS and OQ are parallel
 c $ORSQ$ is a trapezium
 d 120 cm²
- 2 **a** **i** $\frac{1}{3}a + \frac{2}{3}b$ **ii** $\frac{k}{7}a + \frac{6}{7}b$
 b **i** 3 **ii** $\frac{7}{2}$
- 3 **a** **i** $\vec{OD} = 2i - \frac{1}{2}j$, $\vec{OE} = \frac{15}{4}i + \frac{9}{4}j$
 ii $\frac{\sqrt{170}}{4}$
- b** **i** $p\left(\frac{15}{4}i + \frac{9}{4}j\right)$
 ii $(q+2)i + \left(4q - \frac{1}{2}\right)j$
 c $p = \frac{2}{3}$, $q = \frac{1}{2}$
- 5 **a** **i** $\vec{AB} = c$ **ii** $\vec{OB} = a + c$ **iii** $\vec{AC} = c - a$
 b $|c|^2 - |a|^2$
- 6 **a** $r + t$ **b** $\frac{1}{2}(s + t)$

Exercise 9J

- 2 **b** **i** 4 **ii** $\sqrt{10}$
- 4 **a** **i** 36° **ii** 72°
 c 0.62
- 5 $\varphi^0 = 1$, $\varphi^1 = \frac{1+\sqrt{5}}{2}$, $\varphi^2 = \frac{3+\sqrt{5}}{2}$,
 $\varphi^3 = 2 + \sqrt{5}$, $\varphi^4 = \frac{7+3\sqrt{5}}{2}$, $\varphi^{-1} = \frac{\sqrt{5}-1}{2}$,
 $\varphi^{-2} = \frac{3-\sqrt{5}}{2}$, $\varphi^{-3} = \sqrt{5}-2$, $\varphi^{-4} = \frac{7-3\sqrt{5}}{2}$

Chapter 9 review

Technology-free questions

- 1 **a** Rectangle **b** 16 cm
- 3 $\sqrt{34}$ cm
- 4 **a** $x = 7$ cm, $y = 7$ cm, $\alpha = 45^\circ$, $\beta = 40^\circ$
 b $\alpha = 125^\circ$, $\beta = 27.5^\circ$
 c $\theta = 52^\circ$, $\alpha = 52^\circ$, $\beta = 65^\circ$, $\gamma = 63^\circ$
- 5 8 m

- 7 b i** 20 cm **ii** 10 cm
c $XP : PY = 2 : 1$, $PQ : YZ = 2 : 3$
8 a 3 cm **b** 5 : 3 **c** 3 : 5
9 $\frac{210}{23}$ m **10** $\frac{15}{8}$
11 12.25 **12** 12
13 b 25 : 36 **c** 48 cm
14 a 20 : 3 **b** 1.6 m² **c** $\frac{8}{27}$ m³
15 a 2% **b** 3%
16 a $\frac{1}{3}$ **b** $\frac{1}{3}$ **c** $\frac{2}{3}$ **d** $\frac{2}{3}$ **e** $\frac{1}{9}$ **f** $\frac{4}{9}$
19 a $\vec{AC} = \mathbf{c} - \mathbf{a}$, $\vec{BC} = \mathbf{c} + \mathbf{a}$
b $\vec{AC} \cdot \vec{BC} = |\mathbf{c}|^2 - |\mathbf{a}|^2 = 0$
22 $AF : FE = 2 : 1$

Multiple-choice questions

- 1** C **2** B **3** B **4** B **5** A
6 D **7** B **8** D **9** C **10** B
11 D **12** C **13** E **14** E

Extended-response questions

- 1 a** $\triangle EBC$ **c** $\frac{h}{q} = \frac{x}{x+y}$ **e** $\frac{20}{9}$
2 a Rhombus; $CF = 1$
c $\triangle ACF$, $\triangle ABC$ and $\triangle AED$ **e** $\frac{1+\sqrt{5}}{2}$
3 $x = 8$ or $x = 11$
4 a $\triangle BDR$ and $\triangle CDS$; $\triangle BDT$ and $\triangle BCS$;
 $\triangle RSB$ and $\triangle DST$
b $\frac{z}{y} = \frac{p}{p+q}$ **c** $\frac{z}{x} = \frac{q}{p+q}$
5 a i 9 cm **ii** 12 cm **iii** $\frac{1}{16}$ **iv** $\frac{9}{16}$
b i $16a$ cm² **ii** $3a$ cm²
7 $15\sqrt{26}$ m

Chapter 10

Exercise 10A

- 1 a** $x = 100$, $y = 50$
b $x = 126$, $y = 252$, $z = 54$
c $y = 145$, $z = 290$
d $x = 180$, $y = 90$
e $x = 45$, $y = 90$, $z = 270$
2 a $x = 68$, $y = 121$ **b** $x = 112$, $y = 87$
c $x = 50$, $y = 110$
3 110° , 110° , 140°
4 $\angle ABC = 98^\circ$, $\angle BCD = 132^\circ$, $\angle CDE = 117^\circ$,
 $\angle DEA = 110^\circ$, $\angle EAB = 83^\circ$
7 60° or 120°
8 $\angle P = 78^\circ$, $\angle Q = 74^\circ$, $\angle R = 102^\circ$, $\angle S = 106^\circ$

Exercise 10B

- 1 a** $x = 73$, $y = 81$ **b** $x = 57$, $q = 57$
c $x = 53$, $y = 74$, $z = 53$
d $x = 60$, $y = 60$, $z = 20$, $w = 100$
e $w = 54$, $x = 54$, $y = 72$, $z = 54$
2 a 40° **b** 40° **c** 80°
3 32° and 148°
4 $\angle ACB = 40^\circ$, $\angle ABC = 70^\circ$, $\angle BAT = 40^\circ$

Exercise 10C

- 1 a** 10 cm **b** 6 cm
2 7 cm
3 $5\sqrt{6}$ cm

Chapter 10 review

Technology-free questions

- 1** $\angle MCN = 18^\circ$
2 a $x = 110$, $y = 70$ **b** $x = 35$, $y = 35$
c $x = 47$, $y = 53$, $z = 100$
d $x = 40$, $y = 40$, $z = 70$
6 a $x = 66$ **b** $x = 116$ **c** $x = 66$, $y = 114$
8 3 cm

Multiple-choice questions

- 1** B **2** A **3** E **4** A **5** C
6 A **7** C **8** B **9** A **10** A

Extended-response questions

- 5 b** 24 cm²

Chapter 11

Technology-free questions

- 1 a** $2\sqrt{10}$ **b** $\frac{1}{\sqrt{10}}(i+3j)$
c $\frac{8}{\sqrt{10}}(i+3j)$ **d** $-\frac{2}{\sqrt{10}}(i+3j)$
2 a 13 **b** 13 **c** 13 **d** -13
e -13 **f** 0 **g** -13
3 a $m = \frac{46}{11}$, $n = -\frac{18}{11}$ **b** $p = -48$
c $p = 3, 5$
4 a $\mathbf{v}_1 = 2\sqrt{3}\mathbf{i} + 2\mathbf{j}$ m/s, $\mathbf{v}_2 = 3\mathbf{i} - 3\sqrt{3}\mathbf{j}$ m/s
b $(4\sqrt{3}+9)\mathbf{i} + (4-9\sqrt{3})\mathbf{j}$
5 a $\sqrt{337}$ m/s on a bearing of $\tan^{-1}\left(\frac{16}{9}\right)$
b 8.5 seconds
c 76.5 m
6 $F = 7$ kg wt, $\cos \theta = -\frac{31}{49}$
7 $(F_3)^2 = 100 - 48\sqrt{2}$

- 8 a** $T = 5 \text{ kg wt}$, $N = 5\sqrt{3} \text{ kg wt}$
b $T = \frac{10\sqrt{3}}{3} \text{ kg wt}$, $N = \frac{20\sqrt{3}}{3} \text{ kg wt}$
- 9** $\frac{50}{13} \text{ kg wt}$, $\frac{120}{13} \text{ kg wt}$
- 10** 24
- 11** 360
- 12 a** 125 **b** 60
- 13 a** 9 **b** 25
- 14 a** 24 **b** 30 **c** 28 **d** 45
- 15 a** 120 **b** 120
- 16 a** 120 **b** 36
- 17 a** 96 **b** 24 **c** 72 **d** 60
- 18** 10
- 19 a** 20 **b** 325 **c** 210 **d** 56
- 20 a** 28 **b** 21 **c** $2^8 = 256$
- 21** $\frac{10}{21}$
- 22** 120
- 23** 7
- 24** 51
- 25** 80
- 28 a** If n is odd, then $5n + 3$ is even.
c If n is even, then $5n + 3$ is odd.
- 35 a** 90° **b** 54° **c** 80° **d** 220°
e $x = 96^\circ$, $y = 70^\circ$ **f** 46°
- 36 a** 40° **b** 140° **c** 50°
- 37 a** 38° **b** 52° **c** 68°
- 38 a** 4 **b** $\frac{3\sqrt{10}}{2}$ **c** 12
- 40 a** 156 **b** 144 **c** 25
- 41** 30°
- 44** Yes

Multiple-choice questions

- 1** A **2** B **3** D **4** B **5** B
6 B **7** A **8** B **9** C **10** B
11 B **12** C **13** A **14** D **15** B
16 A **17** A **18** B **19** C **20** A
21 B **22** D **23** D **24** D **25** A
26 E **27** B **28** E **29** E **30** D
31 C **32** E **33** E **34** B **35** B
36 D **37** D **38** A **39** E **40** B
41 E **42** A **43** C **44** C **45** E
46 D **47** B **48** C **49** B **50** D
51 C **52** C **53** B **54** D

Extended-response questions

- 1 a** $\vec{AE} = \frac{1}{t+1}(2\vec{a} + t\vec{b})$
b $\vec{AE} = \frac{1}{8}(7\vec{a} + \vec{AF})$ **d** $t = \frac{9}{7}$

- 2 b** $(n-1)\vec{a} - n\vec{b} + \vec{c}$
- 3 a** $\vec{AB} = \vec{b} - \vec{a}$, $\vec{PQ} = \frac{-3}{10}\vec{a} + \frac{1}{2}\vec{b}$
b i $n\left(\frac{-3}{10}\vec{a} + \frac{1}{2}\vec{b}\right)$ **ii** $\left(k + \frac{1}{2}\right)\vec{b} - \frac{1}{2}\vec{a}$
c $n = \frac{5}{3}$, $k = \frac{1}{3}$
- 4 a** $4\sqrt{2} \text{ km/h}$ blowing from the south-west
b $\sqrt{5} \text{ km/h}$; 200 m downstream
c 43.1 km/h on a bearing of 080° **d** 222°
- 5 a** 2160 **b** 360 **c** 900 **d** 1260
- 6 a** 70 **b** 30 **c** 15 **d** 55
- 7 a** 20 **b** 4 **c** 68
- 8 a** 420 **b** 60 **c** 120 **d** 24
- 9 a** 300 **b** 10 and 15
- 10 a** 495 **b** 60
c The two points diametrically opposite
d 15 **e** $\frac{1}{33}$
- 11 a** No
b Yes; both a and b are odd, and c is even
- 12 a** $(a, b, c) = (2, 3, 6)$
b $(a, b, c, d) = (1, 2, 3, 4)$ or
 $(a, b, c, d) = (1, 2, 3, 5)$
- 14 a** 10
- 16 a** No **b** $n = 4k$ or $n = 4k - 1$
- 18 a** $a = 1$, $b = 3$, $c = 1$ **c** 41^2
- 20 b** $\angle BCA = x^\circ$, $\angle BOA = 2x^\circ$, $\angle TAB = x^\circ$,
 $\angle TBA = x^\circ$

Problem-solving and modelling

See solutions supplement

Degree-of-difficulty classified questions

Simple familiar questions

- 1 a** $5\vec{i} + \vec{j}$ **b** $\vec{i} + 9\vec{j}$ **c** $\vec{i} + 31\vec{j}$ **d** 34
e $\sqrt{34}$ **f** -14
- 2** $s = \frac{3}{2}$, $t = \frac{7}{2}$
- 3** $4\vec{i} + 3\vec{j} = \vec{u} + \vec{w}$ where $\vec{u} = \left(\sqrt{3} - \frac{3}{4}\right)(\sqrt{3}\vec{i} - \vec{j})$
 and $\vec{w} = \left(\frac{3\sqrt{3}}{4} + 1\right)\vec{i} + \left(\sqrt{3} + \frac{9}{4}\right)\vec{j}$
- 4 a** $8\vec{i} + 9\vec{j}$ **b** $(3 + t)\vec{i} + (4 + t)\vec{j}$
- 5** $2\sqrt{61} \text{ km}$ on a bearing of 063.67°
- 6 a** $x + y = 240^\circ$ **b** $x = 28^\circ$
- 7 a** 2 **b** $\frac{1}{2}(\sqrt{3}\vec{i} + \vec{j})$ **c** $15(\sqrt{3}\vec{i} + \vec{j})$
- 8 a** 120 **b** 60 **c** 10
- 9** 36
- 10** 1 663 200
- 11 a** 126 **b** 56 **c** 111

Complex familiar questions

1 $6 : 25$

4 a $t = -\frac{1}{2}$ b $t = -\frac{1}{3}$

6 $\frac{4}{9}$

7 a 749 b 30

8 a 60 480 b 10 080 c 2400

9 18

11 a i $\frac{1}{3}$ ii $\frac{2}{5}$ iii $\frac{3}{7}$

b $S_n = \frac{n}{2n+1}$

Complex unfamiliar questions

2 $19 : 32$

4 b -4

5 a 4900 b 1568 c 0

10 1458

Chapter 12

Exercise 12A

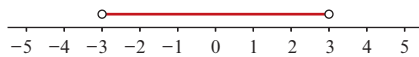
1 a 8 b 8 c 2 d -2

e -2 f 4

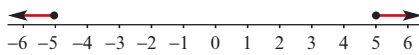
2 a 3, -1 b $\frac{7}{2}, -\frac{1}{2}$ c $\frac{12}{5}, -\frac{6}{5}$ d 12, -6

e -1, 7 f $\frac{4}{3}, -4$ g $-\frac{2}{5}, -4$

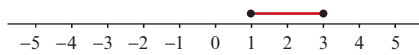
3 a (-3, 3)



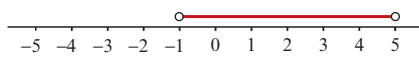
b $(-\infty, -5] \cup [5, \infty)$



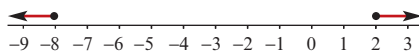
c [1, 3]



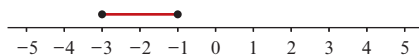
d (-1, 5)



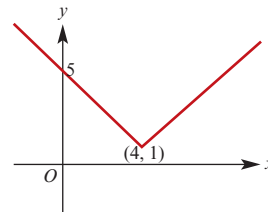
e $(-\infty, -8] \cup [2, \infty)$



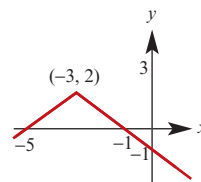
f [-3, -1]



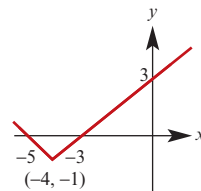
4 a Range $[1, \infty)$



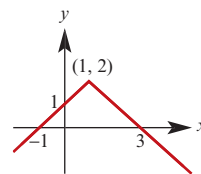
b Range $(-\infty, 2]$



c Range $[-1, \infty)$



d Range $(-\infty, 2]$



5 a $-5 \leq x \leq 5$

b $x \leq -2$ or $x \geq 2$

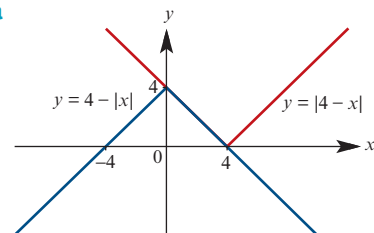
c $1 \leq x \leq 2$

d $-\frac{1}{5} < x < 1$

e $x \leq -4$ or $x \geq 10$

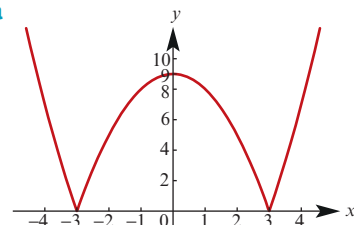
f $1 \leq x \leq 3$

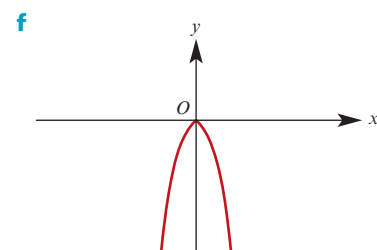
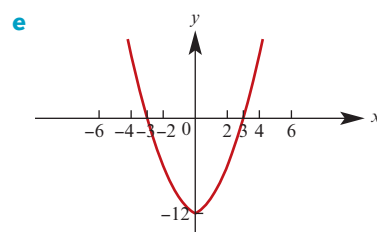
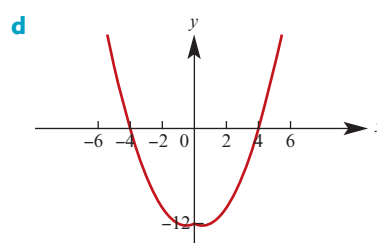
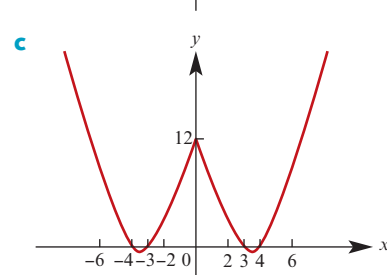
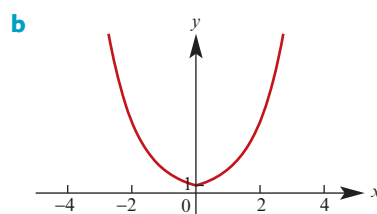
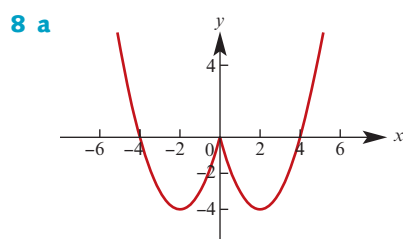
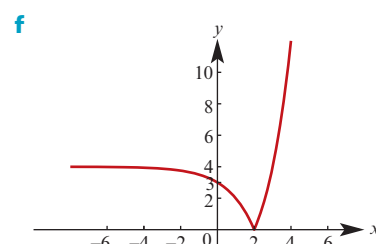
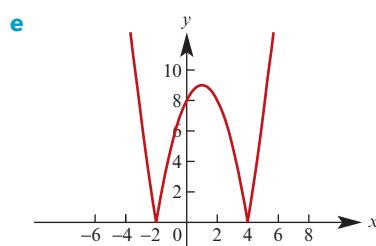
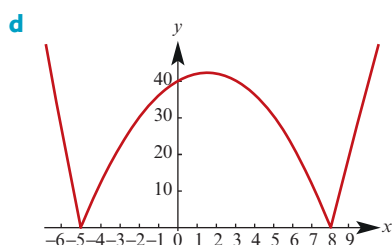
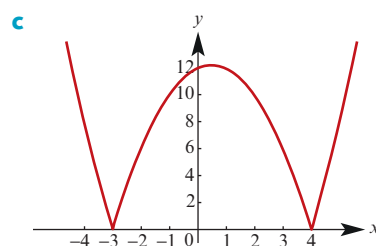
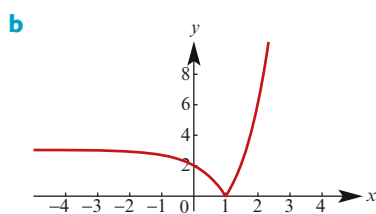
6 a



b $0 \leq x \leq 4$

7 a





9 a $\frac{2 - \sqrt{6}}{2}, \frac{2 + \sqrt{6}}{2}, \frac{2 - \sqrt{2}}{2}, \frac{2 + \sqrt{2}}{2}$

b $1 - \sqrt{2}, 1 + \sqrt{2}, 1$ **c** $-2, 4$

d $3 - \sqrt{17}, 3 + \sqrt{17}, 2, 4$ **e** $-2, 8$

f $3 - 3\sqrt{2}, 3 + 3\sqrt{2}, 3$

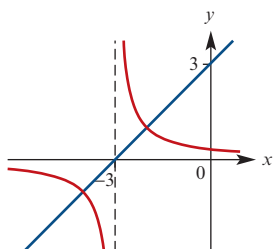
10 a $x \leq -2$ **b** $x = -9$ or $x = 11$

c $x = -\frac{5}{4}$ or $x = \frac{15}{4}$

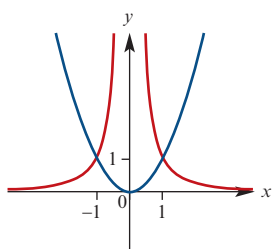
11 $a = 1, b = 1$

Exercise 12B

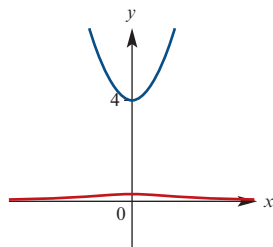
1 a



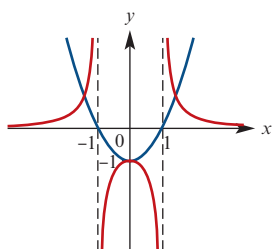
b



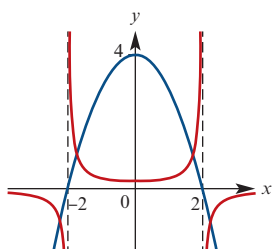
c



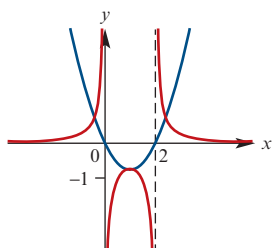
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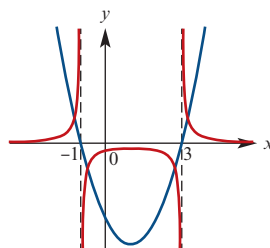
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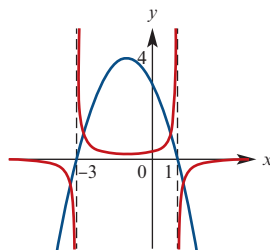
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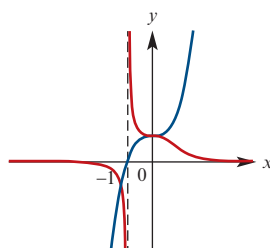
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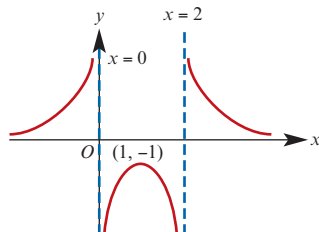
h



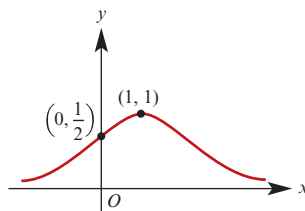
i



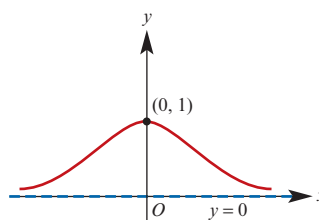
2 a

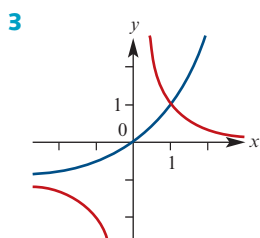
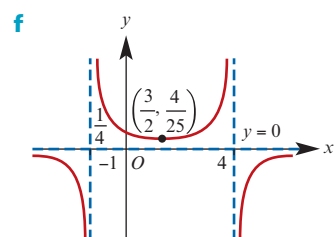
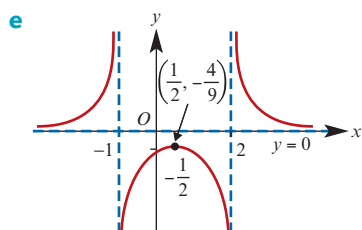
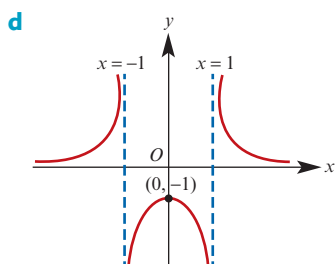


b

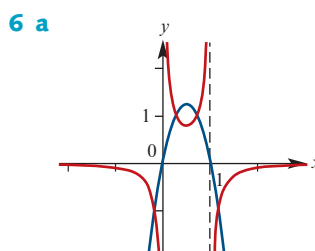
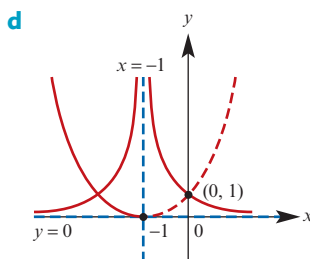
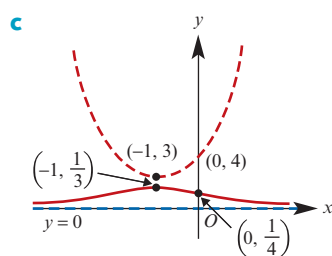
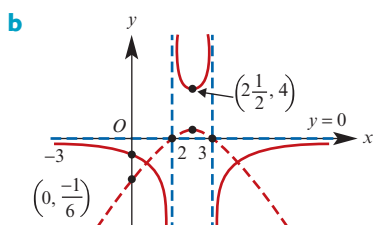
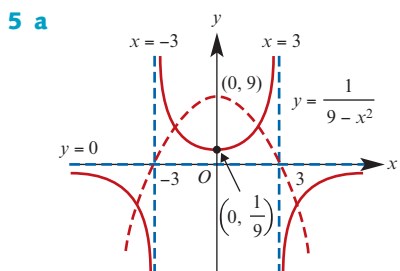
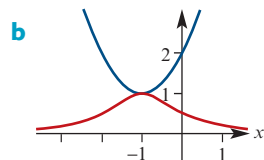


c





4 a $(-1, 1)$



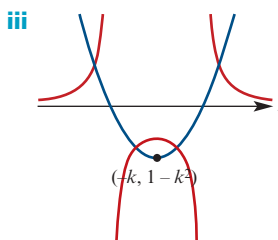
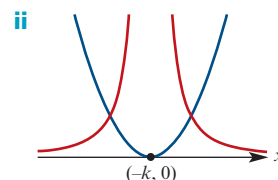
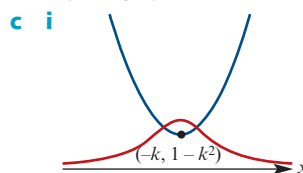
b $\left(\frac{5 \pm 3\sqrt{5}}{10}, -1\right), \left(\frac{5 \pm \sqrt{5}}{10}, 1\right)$

7 a $f(x) = (x + k)^2 + 1 - k^2$

b i $-1 < k < 1$

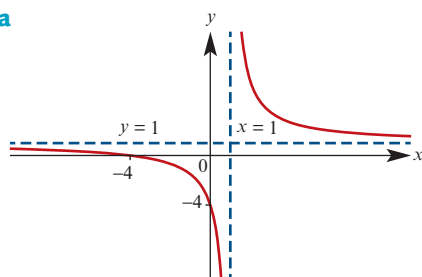
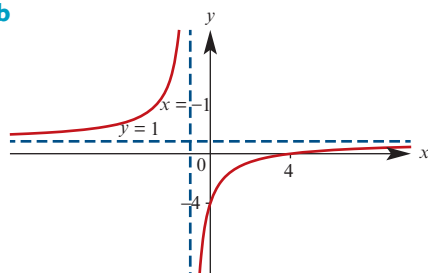
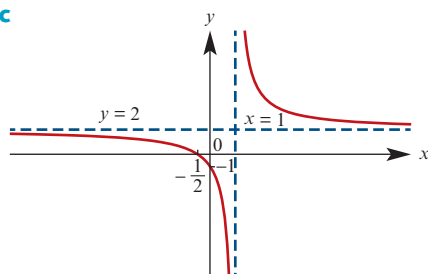
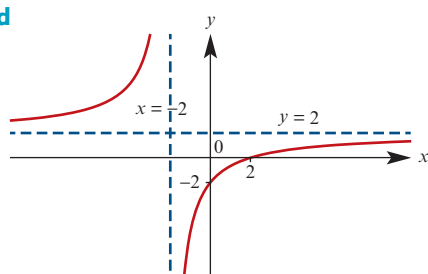
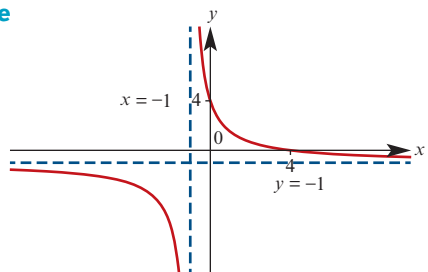
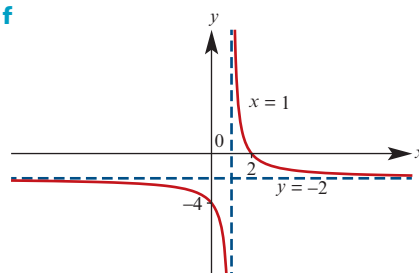
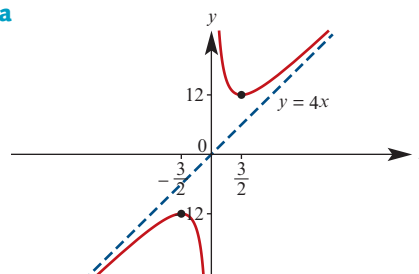
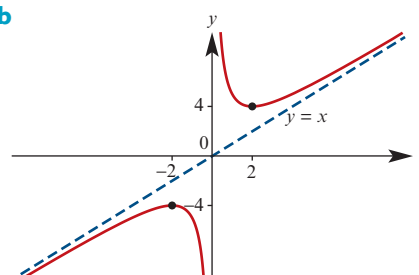
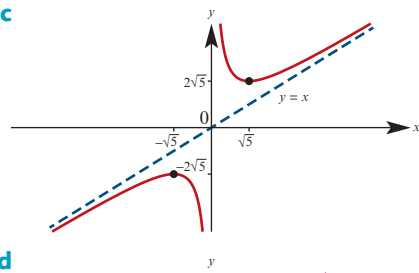
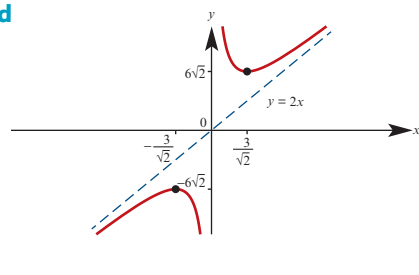
ii $k = \pm 1$

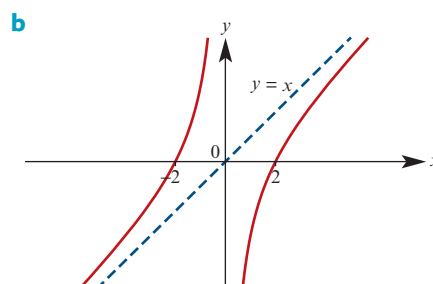
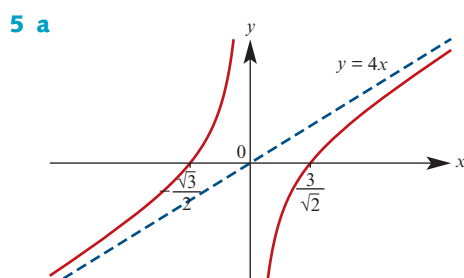
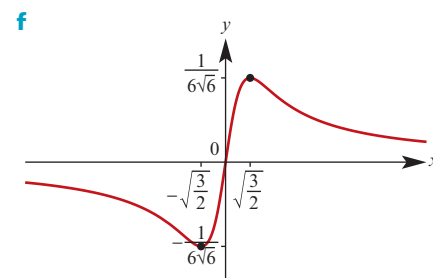
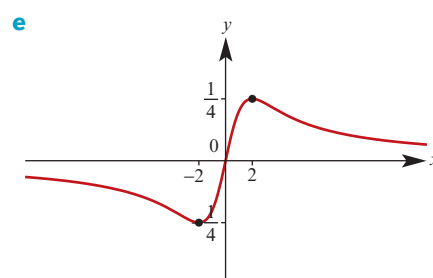
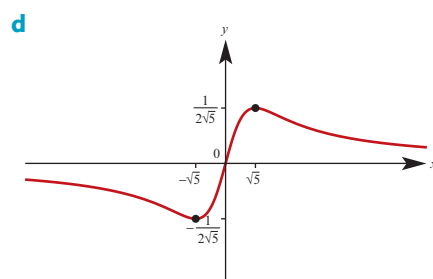
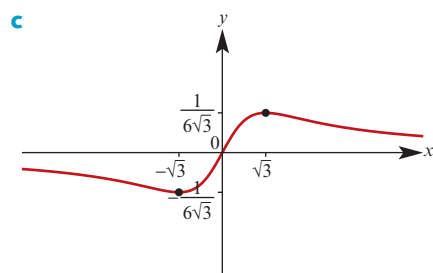
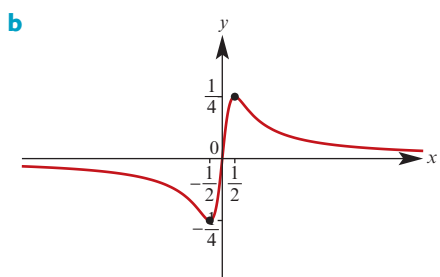
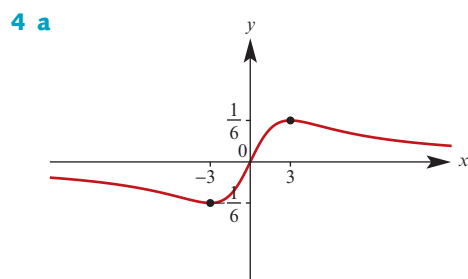
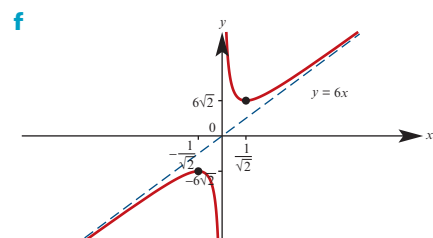
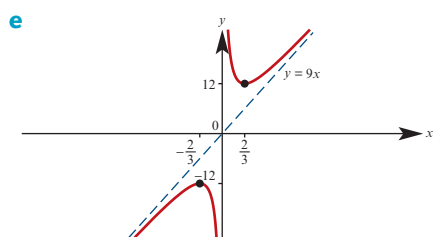
iii $k > 1$ or $k < -1$

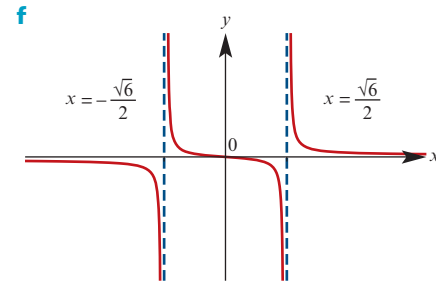
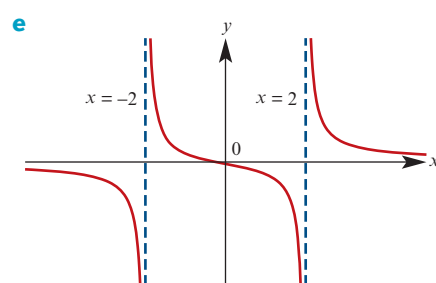
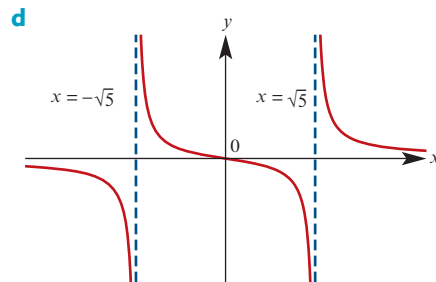
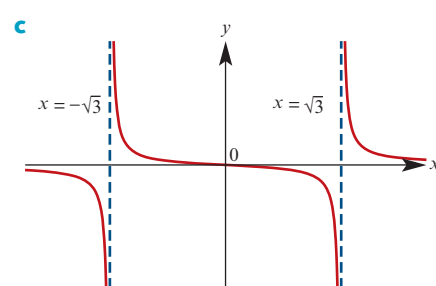
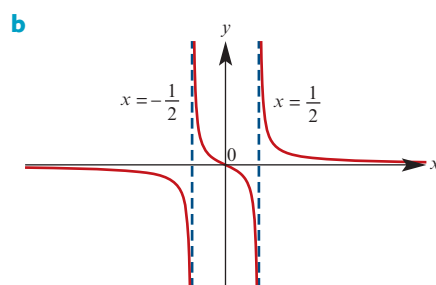
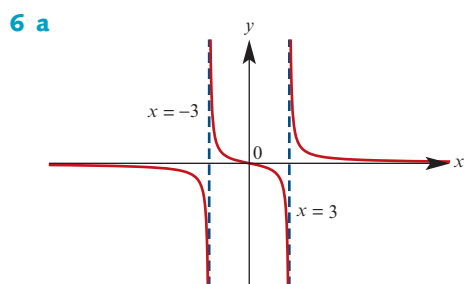
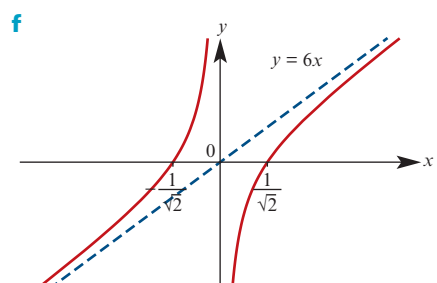
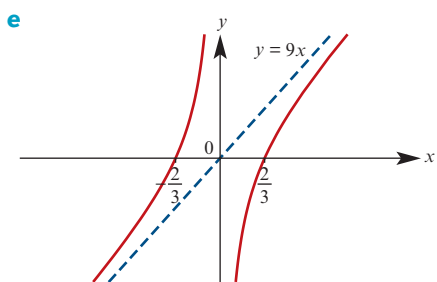
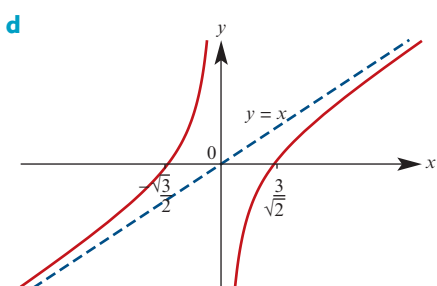
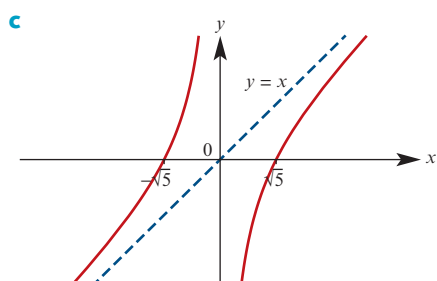


Exercise 12C

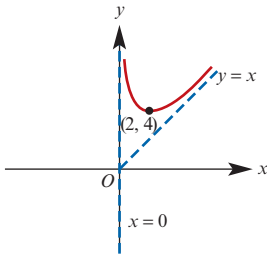
- 1 a** $x = 3, y = 2$ **b** $x = 2, y = 5x + 3$
c $x = 0, y = 4x$ **d** $y = 0$
e $x = 0, y = 6x + 3$

2 a

b

c

d

e

f

3 a

b

c

d




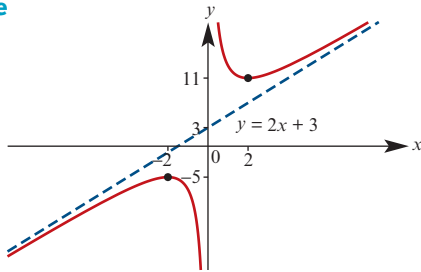


7 Least value = 4



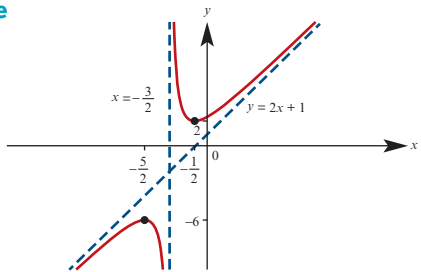
8 c 2
e

d -2

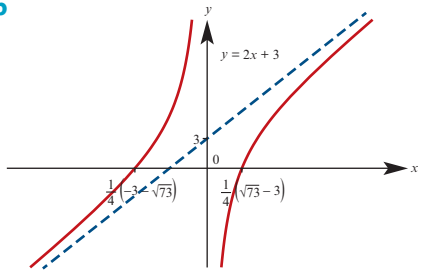


9 c $-\frac{1}{2}$
e

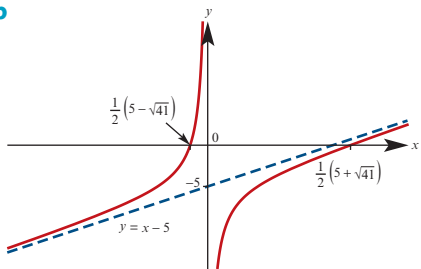
d $-\frac{5}{2}$



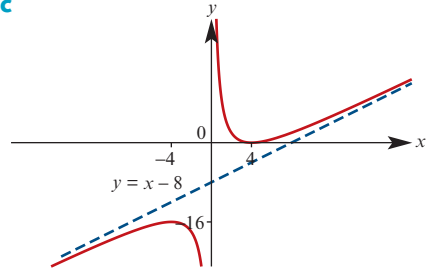
10 b



11 b

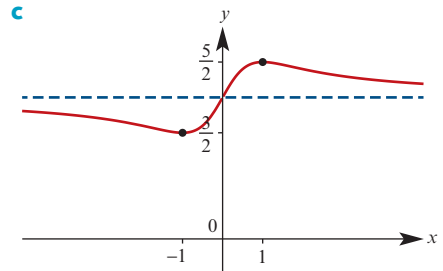


12 c

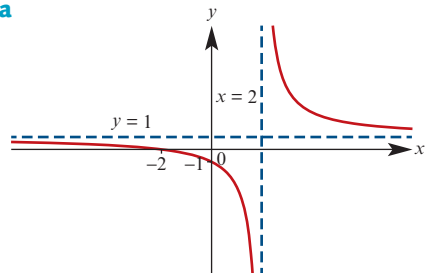


13 a $2 + \frac{x}{x^2 + 1}$

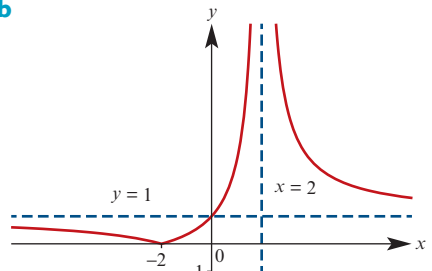
b y = 2



14 a

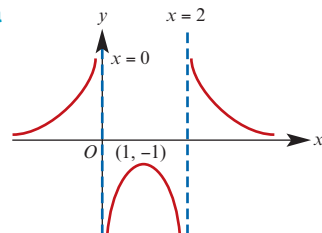


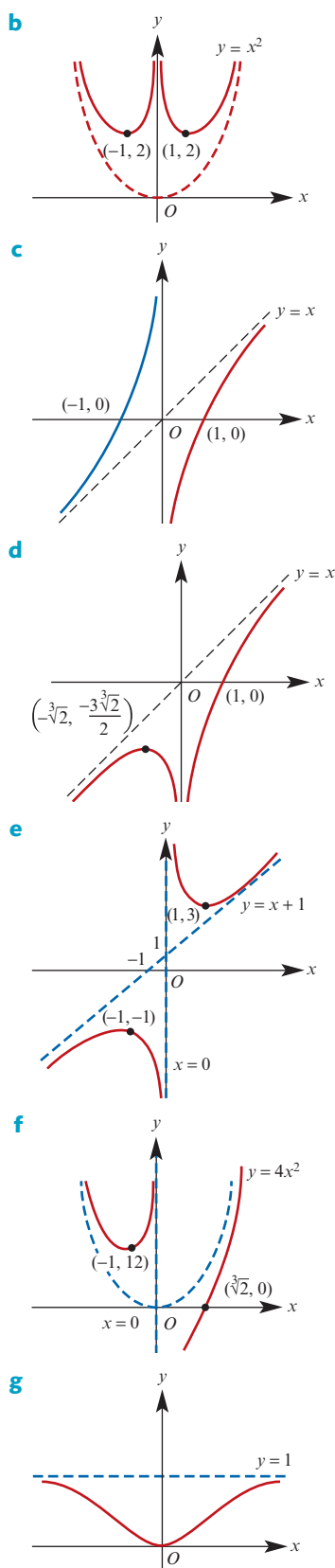
b



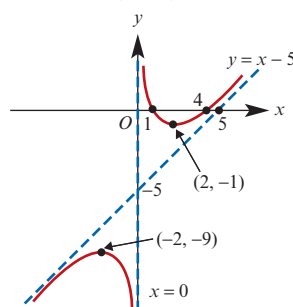
Exercise 12D

1 a

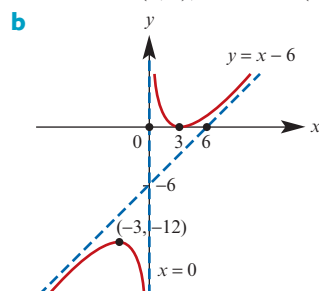




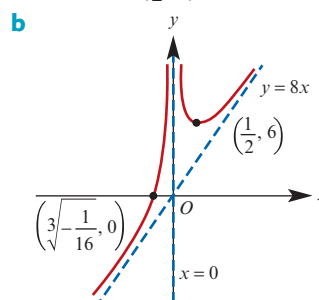
- 2 a** $(1, 0), (4, 0)$ **b** $x = 0, y = x - 5$
c Local min $(2, -1)$; local max $(-2, -9)$



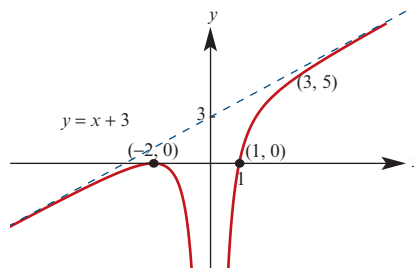
- 3 a** Local min $(3, 0)$; local max $(-3, -12)$



- 4 a** Local min $(\frac{1}{2}, 6)$



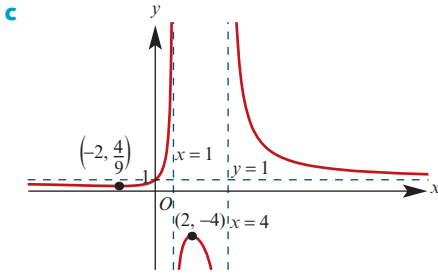
- 5** Asymptotes: $y = x + 3, x = 0$;
 Axis intercepts: $(-2, 0), (1, 0)$;
 Stationary points: local max $(-2, 0)$



- 6 a** $\mathbb{R} \setminus \{-\frac{1}{2}\}$ **b** $\frac{8(x^2 + x - 2)}{(2x + 1)^2}$
c Local min $(1, 4)$; local max $(-2, -8)$
d $x = -\frac{1}{2}, y = 2x - 1$ **e** $\mathbb{R} \setminus (-8, 4)$

7 a $x = 4$, $x = 1$, $y = 1$

b Local max $(2, -4)$; local min $(-2, \frac{4}{9})$



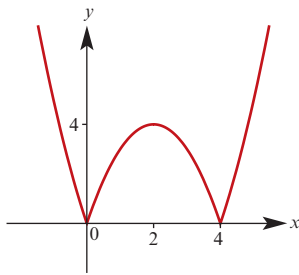
Chapter 12 review

Technology-free questions

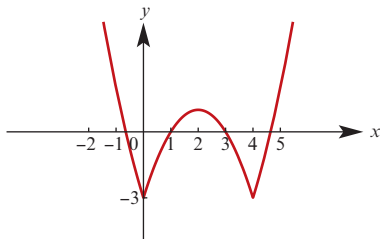
- 1 a 9 b $\frac{1}{400}$ c 4
d 4 e $\pi - 3$ f $4 - \pi$

2 $x = 0$ or $x = 2$ or $x = 4$

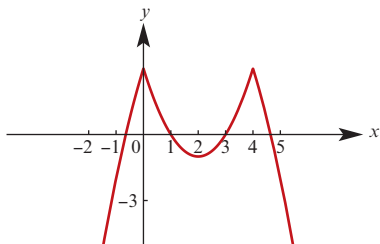
3 a Range $[0, \infty)$



b Range $[-3, \infty)$



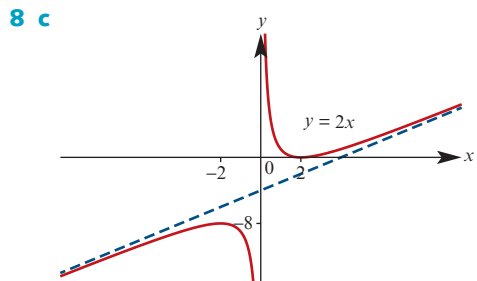
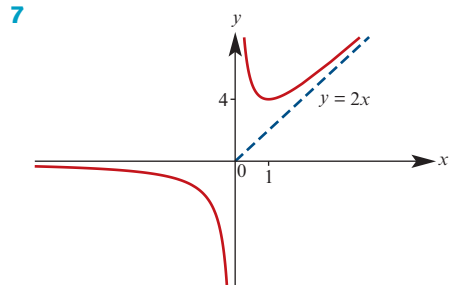
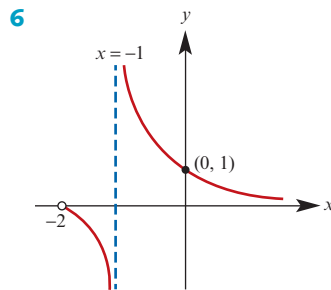
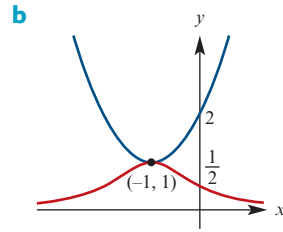
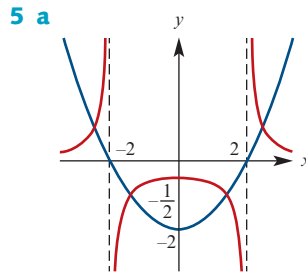
c Range $(-\infty, 3]$



4 a $n = \pm 2$ or $n = \pm 4$

b i $x = \pm 1$ ii $x \leq 0$

c $x < -1$ or $x > 1$



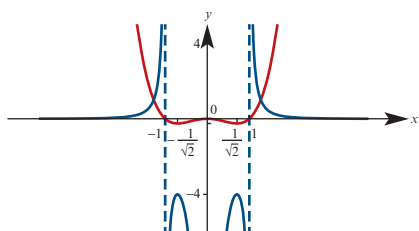
9 a $x = 3$, $y = 2$

b $x = \frac{3}{2}$, $y = \frac{5}{2}$

c $x = 0$, $y = x + 4$

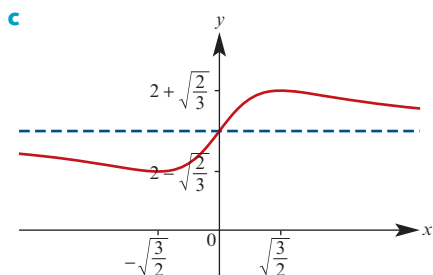
d $x = -2$, $y = -3x - 1$

10

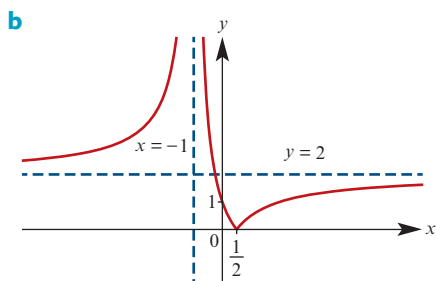
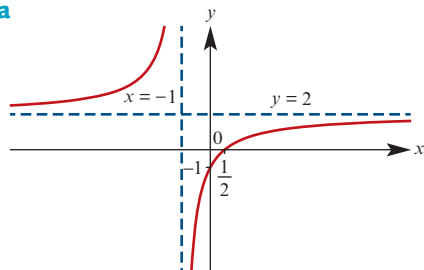


11 a $2 + \frac{4x}{2x^2 + 3}$

b $y = 2$



12 a

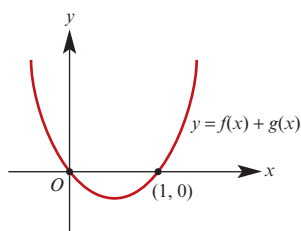


Multiple-choice questions

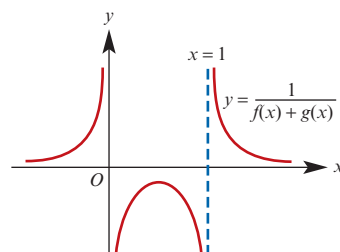
- 1 D 2 E 3 B 4 D 5 C 6 E
7 D 8 A 9 C 10 D 11 A 12 D

Extended-response questions

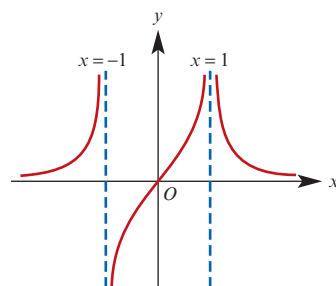
1 a i



ii



iii



b $f(x) = x^2 - 1, g(x) = (x - 1)^2$

c i $f(x) + g(x) = 2x^2 - 2x$

ii $\frac{1}{f(x) + g(x)} = \frac{1}{2x^2 - 2x}$

iii $\frac{1}{f(x)} + \frac{1}{g(x)} = \frac{2x}{(x - 1)^2(x + 1)}$

2 a $-\frac{2}{m}$

b $0 < m < 1$

c i $y = -\frac{1}{m}x + 2$

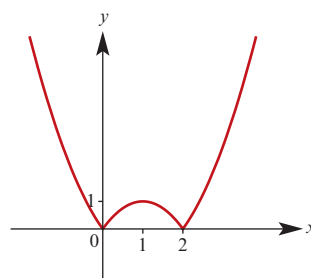
ii $\left(\frac{4m}{1 - m^2}, \frac{2(m^2 + 1)}{m^2 - 1}\right)$

 iii The line ℓ is parallel to the graph of $y = f(x)$ for $x < -2$

iv $m = 3$

d $x = -\frac{1}{m}$ or $x = -\frac{3}{m}$

3 a i

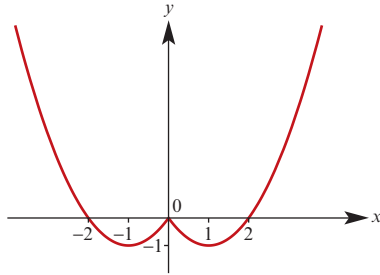


ii $(0, 0), (a, 0)$

iii $\left(\frac{a}{2}, \frac{a^2}{4}\right)$

iv $a = 3$ or $a = -5$

b i



ii $(0, 0), (a, 0), (-a, 0)$

iii $\left(\frac{a}{2}, -\frac{a^2}{4}\right), \left(-\frac{a}{2}, -\frac{a^2}{4}\right)$

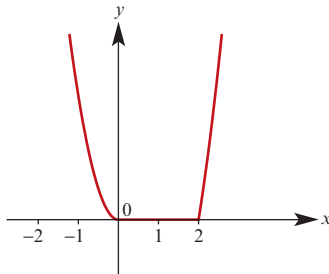
iv $a = -3$

c $x = 0$ or $x \geq a$

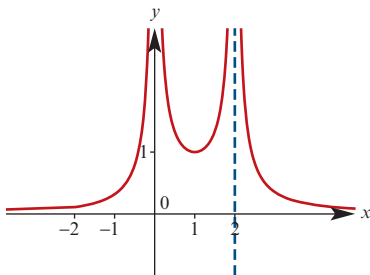
d $x \geq 0$

e $h(x) = |x^2 - ax| + |x|^2 - a|x|$

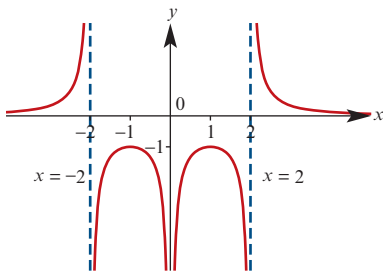
f



g $y = \frac{1}{f(x)}$

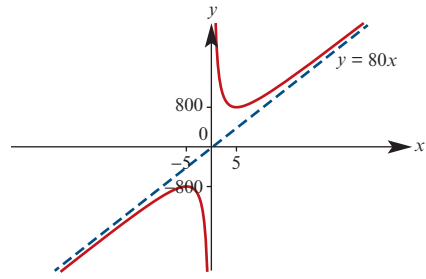


$y = \frac{1}{g(x)}$



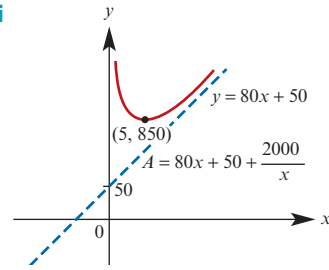
4 a Asymptotes: $x = 0, y = 80x$

Turning points: $(5, 800), (-5, -800)$



b i $A = 80x + 50 + \frac{2000}{x}$

ii

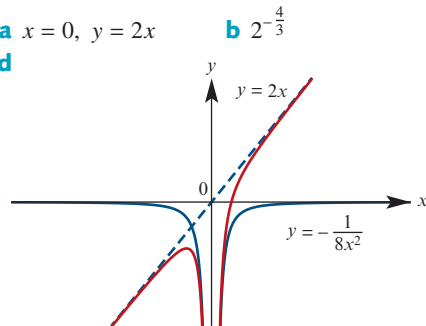


iii Min surface area = 850 cm^2 ,
 $x = 5, y = 5$

iv Min surface area = $\frac{2000}{k} + 40\sqrt{10k} \text{ cm}^2$,
 $x = y = \frac{10\sqrt{10k}}{k}$

5 a $x = 0, y = 2x$

d



e $\left(-\frac{1}{2}, -\frac{3}{2}\right)$

Chapter 13

Exercise 13A

1 a 4π

b 3π

c $-\frac{5\pi}{2}$

d $\frac{\pi}{12}$

e $-\frac{\pi}{18}$

f $-\frac{7\pi}{4}$

2 a 225°

b -120°

c 105°

d -330°

e 260°

f -165°

- 3 a** 0 **b** -1 **c** 1 **d** -1
4 a -1 **b** -1 **c** 1 **d** 1

Exercise 13B

- 1 a** $-\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** 1 **d** 1 **e** $\frac{1}{\sqrt{2}}$
f $\frac{1}{\sqrt{2}}$ **g** 0 **h** $\frac{\sqrt{3}}{2}$ **i** 0 **j** 0
k 1 **l** 0 **m** $-\frac{1}{2}$ **n** -1 **o** -1

- 2 a** $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $\frac{\sqrt{3}}{2}$ **d** $-\frac{1}{2}$

- e** $\frac{1}{\sqrt{2}}$ **f** $\frac{\sqrt{3}}{2}$

- 3 a** 0.6 **b** 0.6 **c** 0.3 **d** -0.3
e -0.3 **f** 0.6 **g** -0.6 **h** -0.3

4 $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = -\frac{1}{\sqrt{3}}$

5 $\sin x = -\frac{\sqrt{51}}{10}$, $\tan x = \frac{\sqrt{51}}{7}$

6 $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = \frac{1}{\sqrt{3}}$

7 $\cos x = \frac{\sqrt{91}}{10}$, $\tan x = -\frac{3\sqrt{91}}{91}$

Exercise 13C

1 $2\pi - a$, $2\pi - b$, $2\pi - c$, $2\pi - d$

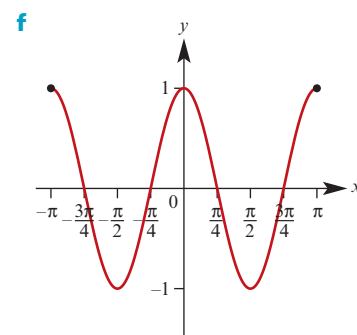
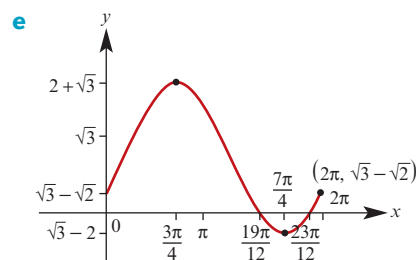
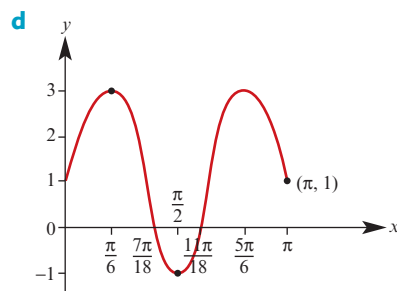
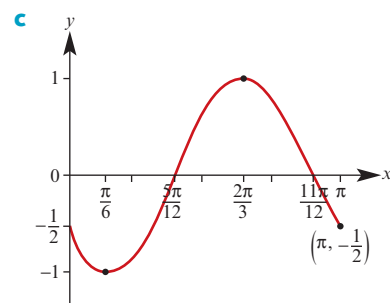
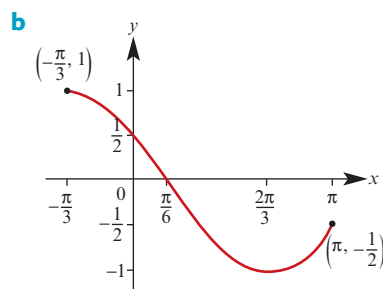
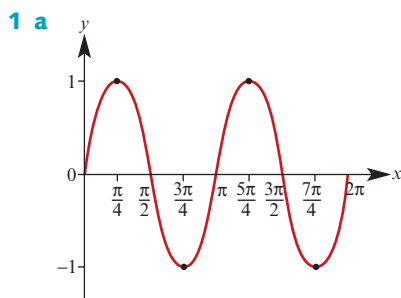
2 a $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ **b** $\frac{2\pi}{3}$, $\frac{5\pi}{6}$, $\frac{5\pi}{3}$, $\frac{11\pi}{6}$

c $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ **d** $\frac{5\pi}{6}$, $\frac{3\pi}{2}$

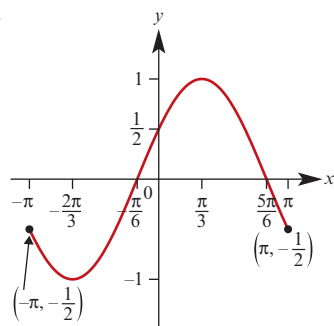
e 0, $\frac{\pi}{3}$, π , $\frac{4\pi}{3}$, 2π **f** $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{2}$, $\frac{5\pi}{3}$

3 a $-\frac{5\pi}{6}$, $-\frac{\pi}{6}$ **b** 0, $-\frac{2\pi}{3}$, $-\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $-\pi$, π

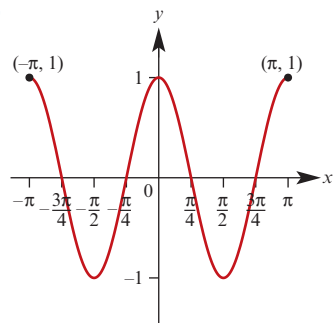
c 0 **d** 0, $-\frac{2\pi}{3}$ **e** $-\frac{5\pi}{6}$, $-\frac{\pi}{2}$, $\frac{\pi}{6}$, $\frac{\pi}{2}$

Exercise 13D


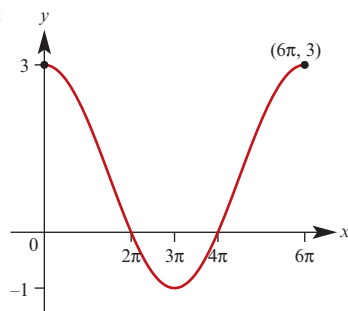
2 a



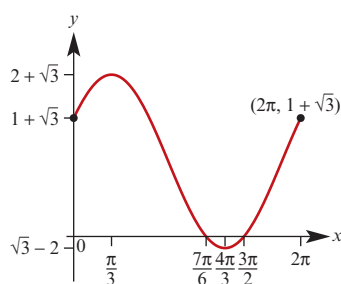
b



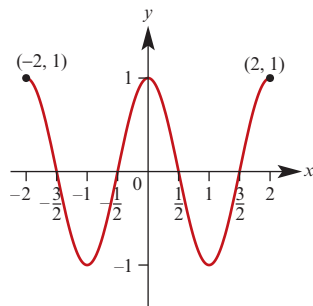
c



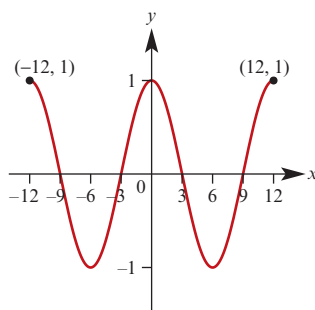
d



e



f



Exercise 13E

1 a 1

b $\sqrt{3}$

c $\frac{1}{\sqrt{3}}$

2 a $\sqrt{3}$

b $\frac{1}{\sqrt{3}}$

c -1

3 a $\frac{-\sqrt{17}}{17}$

b $\frac{-4\sqrt{17}}{17}$

c $\frac{-1}{4}$

d $\frac{-1}{4}$

4 a $\frac{\sqrt{21}}{7}$

b $\frac{-2\sqrt{7}}{7}$

c $\frac{\sqrt{3}}{2}$

d $\frac{-\sqrt{3}}{2}$

5 a $\frac{3\pi}{4}, \frac{7\pi}{4}$

b $\frac{\pi}{3}, \frac{4\pi}{3}$

c $\frac{\pi}{6}, \frac{7\pi}{6}$

d $-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$

e $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

f $-\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}$

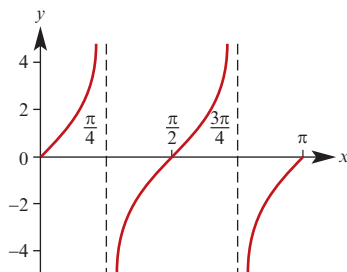
6 a $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

b $-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$

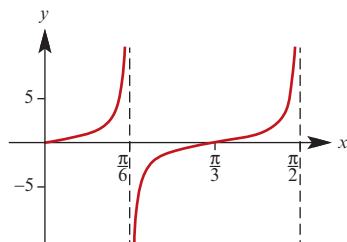
c $-\frac{13\pi}{18}, -\frac{7\pi}{18}, -\frac{\pi}{18}, \frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}$

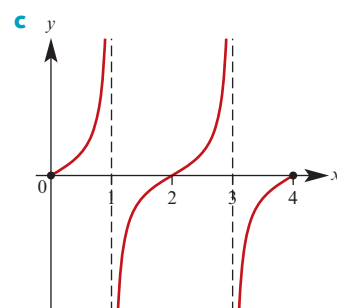
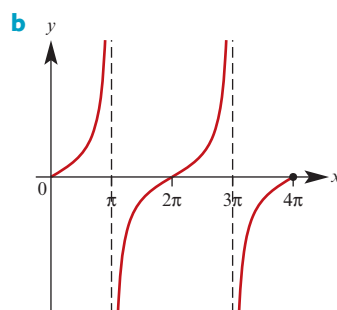
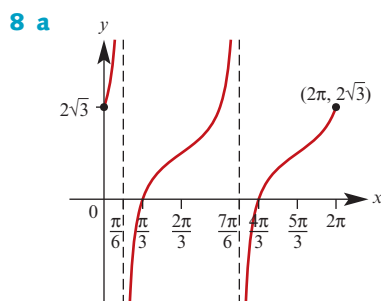
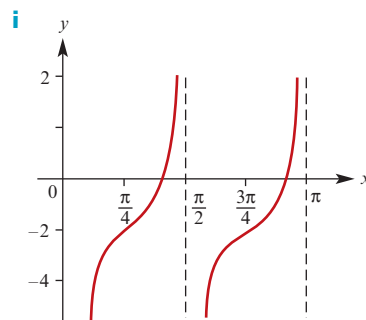
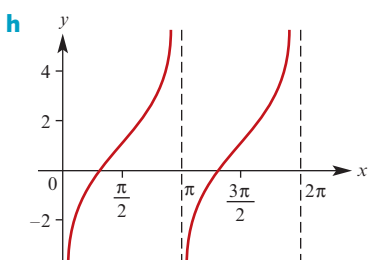
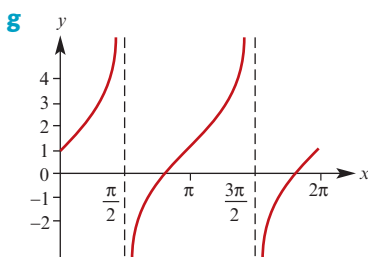
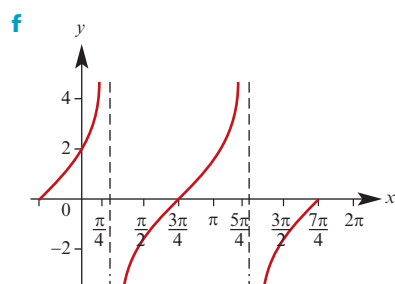
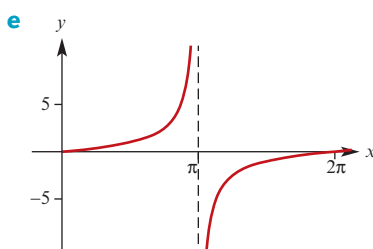
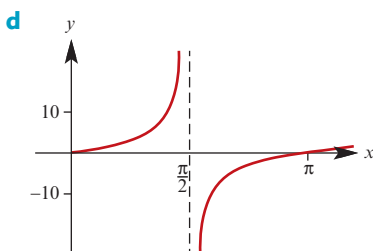
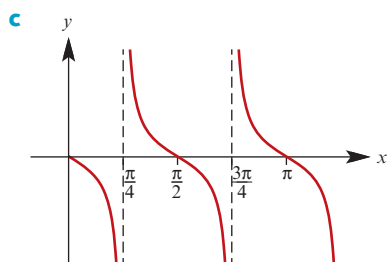
d $-\frac{\pi}{6}$

7 a



b





Exercise 13F

- 1 a i** 2π **ii** 4π **iii** -4π
b i $\frac{4\pi}{3}, \frac{8\pi}{3}$ **ii** $\frac{14\pi}{3}, \frac{10\pi}{3}$ **iii** $\frac{-14\pi}{3}, \frac{-10\pi}{3}$
- 2 a** $x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$
b $x = \frac{2n\pi}{3} + \frac{\pi}{9}$ or $x = \frac{2n\pi}{3} + \frac{2\pi}{9}, n \in \mathbb{Z}$
c $x = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$
- 3 a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{12}, \frac{11\pi}{12}$ **c** $\frac{\pi}{3}, \frac{5\pi}{6}$

4 $\frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

5 $\frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

6 a $x = n\pi - \frac{\pi}{6}$ or $x = n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

b $x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in \mathbb{Z}$

c $x = 2n\pi + \frac{5\pi}{6}$ or $x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

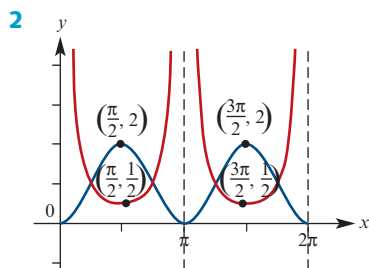
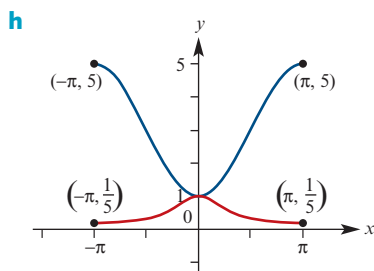
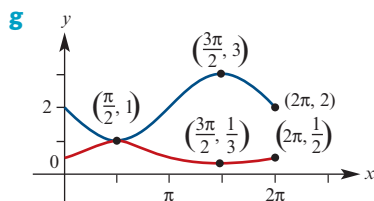
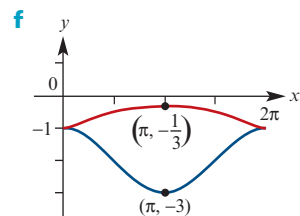
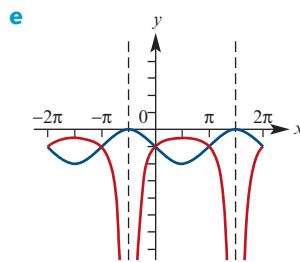
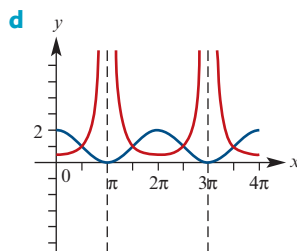
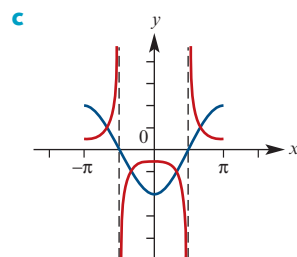
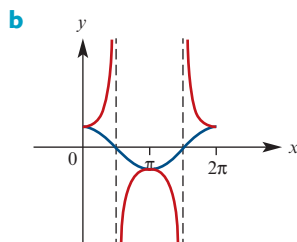
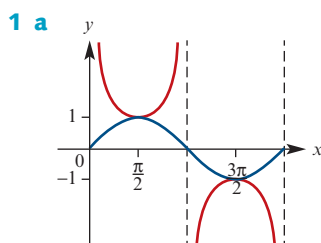
7 $x = \frac{(4n-1)\pi}{4}$ or $x = n\pi, n \in \mathbb{Z};$

$x = \frac{-5\pi}{4}, -\pi, \frac{\pi}{4}, 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$

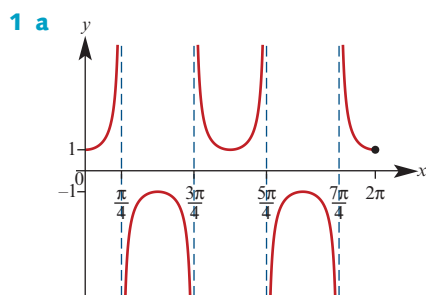
8 $x = \frac{n\pi}{3}, n \in \mathbb{Z}; \quad x = -\pi, \frac{-2\pi}{3}, \frac{-\pi}{3}, 0$

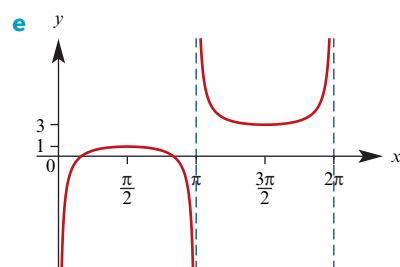
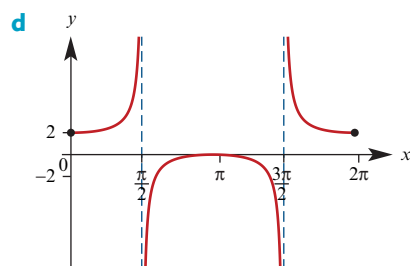
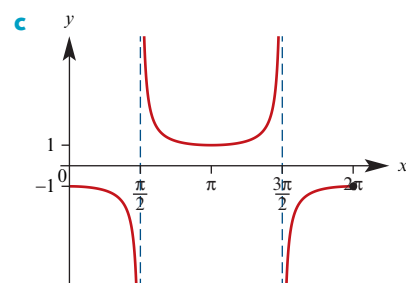
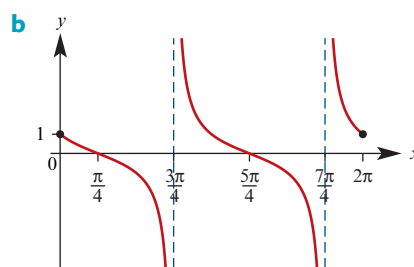
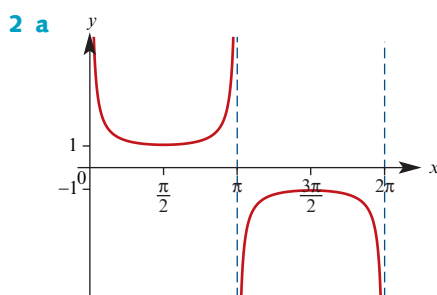
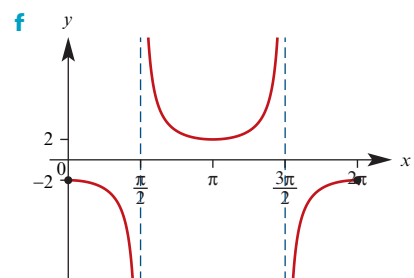
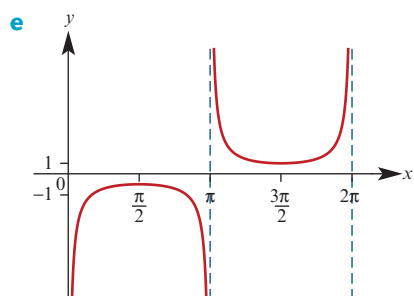
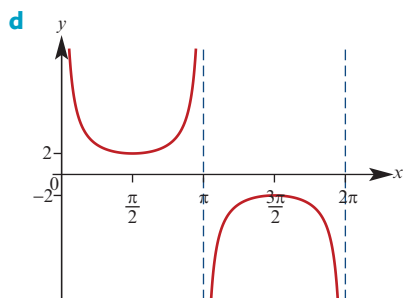
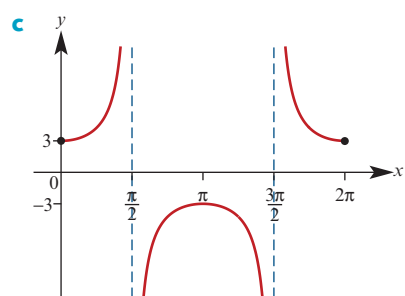
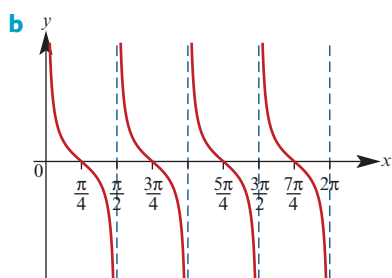
9 $x = \frac{6n-1}{12}$ or $x = \frac{3n+2}{6}, n \in \mathbb{Z};$
 $x = \frac{-2}{3}, \frac{-7}{12}, \frac{-1}{6}, \frac{-1}{12}, \frac{1}{3}, \frac{5}{12}, \frac{5}{6}, \frac{11}{12}$

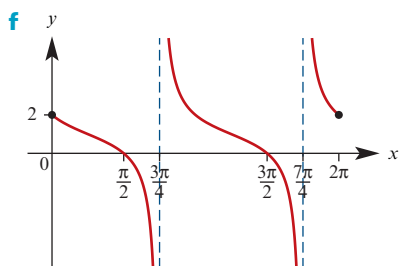
Exercise 13G



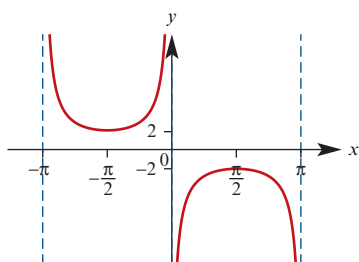
Exercise 13H



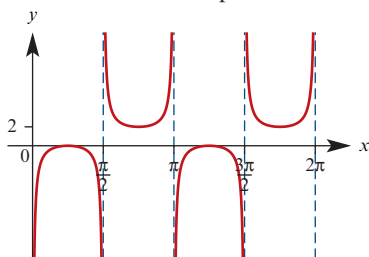




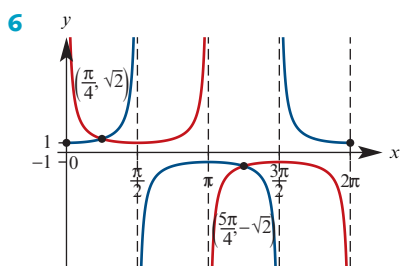
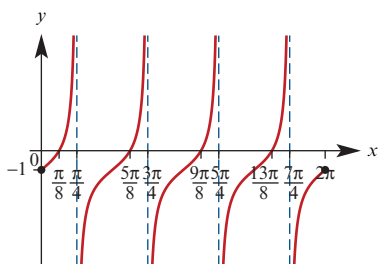
- 3** ■ Reflection in the x -axis
 ■ Dilation of factor 2 from the x -axis
 ■ Translation $\frac{\pi}{2}$ units to the right



- 4** ■ Reflection in the y -axis
 ■ Dilation of factor $\frac{1}{2}$ from the y -axis
 ■ Translation 1 unit up

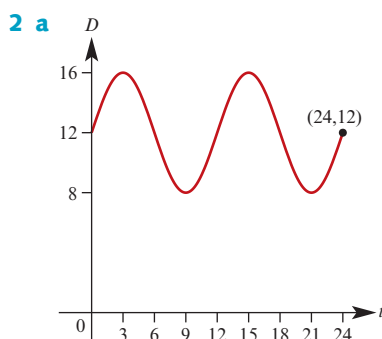


- 5** ■ Reflection in the x -axis
 ■ Dilation of factor $\frac{1}{2}$ from the y -axis
 ■ Translation $\frac{\pi}{4}$ units to the right and 1 unit down



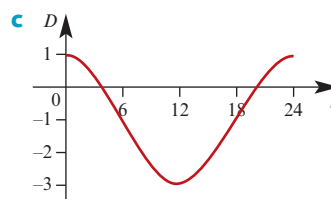
Exercise 131

- 1 a i** 0.00 hours **ii** 24.00 hours
b 14 February ($t = 1.48$),
 24 October ($t = 9.86$)

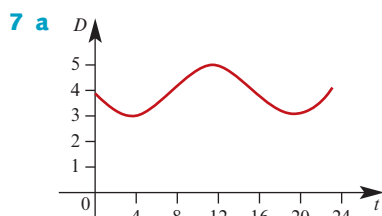


- b** $0 \leq t \leq 6$, $12 \leq t \leq 18$
c 15.9 m
3 a 7 metres **b** 1 metre
c $t = \frac{1}{4}$ or $t = \frac{5}{4}$ **d** $t = \frac{3}{4}$ or $t = \frac{7}{4}$
e Particle oscillates between $x = 1$ and $x = 7$

- 4 a** 19.5°C
b $D = -1 + 2 \cos\left(\frac{\pi t}{12}\right)$

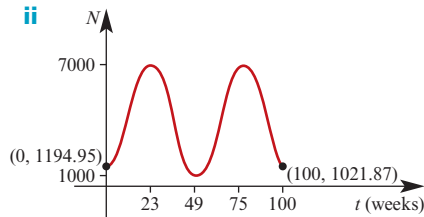


- d** $4 < t < 20$
5 a 2 a.m.
b 8 a.m. and 8 p.m.
6 a i $\frac{3}{2}$ **ii** 12 **iii** $d(t) = \frac{7}{2} - \frac{3}{2} \cos\left(\frac{\pi}{6}t\right)$
iv 1.5 m
b Between 9 p.m. and 3 a.m. and between
 9 a.m. and 3 p.m. each day



- b** The boat can enter at 8 a.m. and must leave
 by 4 p.m.
c The boat can enter at 6:40 a.m. and must
 leave by 5:20 p.m.

- 8 a i 52 weeks ii 3000 iii [1000, 7000]
 b i $N(0) = 1194.95$, $N(100) = 1021.87$

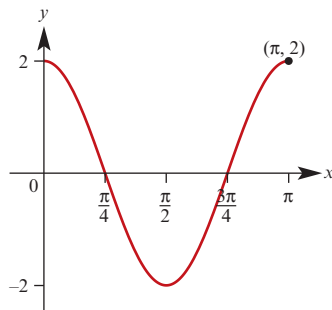


- c i $t = 23, 75$ ii $t = 49$
 d $14\frac{1}{3} < t < 31\frac{2}{3}$, $66\frac{1}{3} < t < 83\frac{2}{3}$
 e $d = 25\,000$, $a = 15\,000$, $b = 10$, $c = 5$

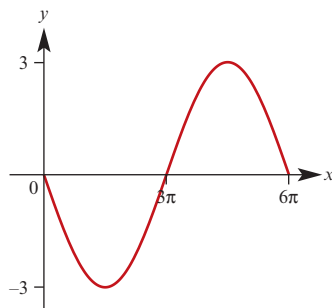
Chapter 13 review

Technology-free questions

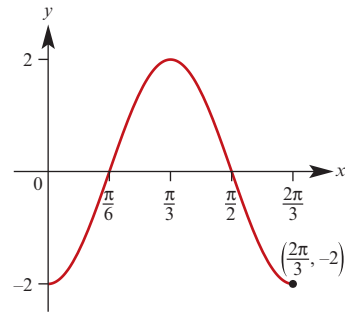
- 1 a $\frac{13\pi}{6}$ b $\frac{14\pi}{3}$ c $\frac{37\pi}{6}$ d $\frac{71\pi}{12}$ e $\frac{11\pi}{12}$
 f $\frac{5\pi}{2}$ g $\frac{7\pi}{3}$ h $\frac{13\pi}{6}$ i $\frac{2\pi}{9}$
 2 a 330° b 765° c 405° d 105°
 e 1530° f -495° g -225° h -585°
 i 1035°
 3 a $\frac{1}{\sqrt{2}}$ b $-\frac{1}{\sqrt{2}}$ c -1 d 0 e $\frac{\sqrt{3}}{2}$
 f -1 g $-\sqrt{3}$ h 1
 4 a $4, 4\pi$ b $5, \frac{\pi}{3}$ c $\frac{1}{3}, \frac{\pi}{2}$ d $2, \frac{2\pi}{5}$
 e $7, 8$ f $\frac{2}{3}, 3$
 5 a $5, 1$ b $9, -1$
 6 a $y = 2\cos(2x)$



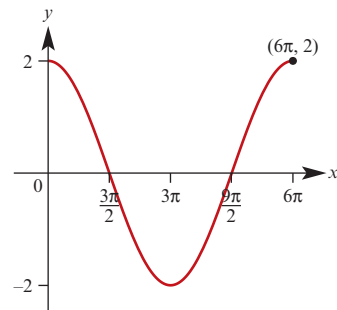
b $y = -3\sin\left(\frac{x}{3}\right)$



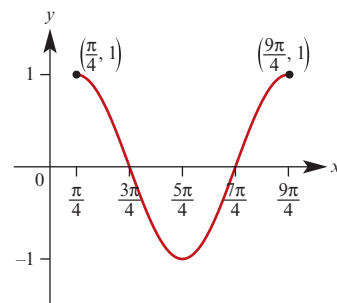
c $y = -2\cos(3x)$



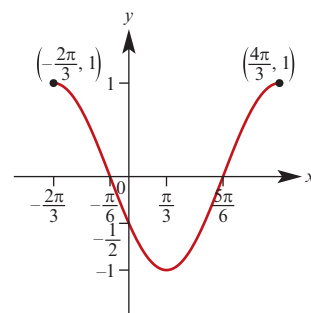
d $y = 2\cos\left(\frac{x}{3}\right)$



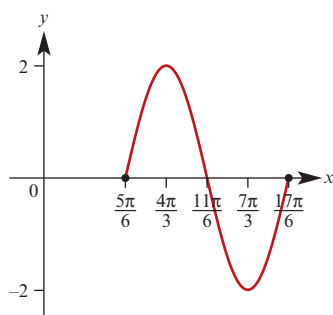
e $y = \cos\left(x - \frac{\pi}{4}\right)$



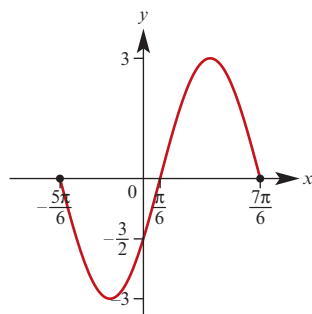
f $y = \cos\left(x + \frac{2\pi}{3}\right)$



g $y = 2 \sin\left(x - \frac{5\pi}{6}\right)$



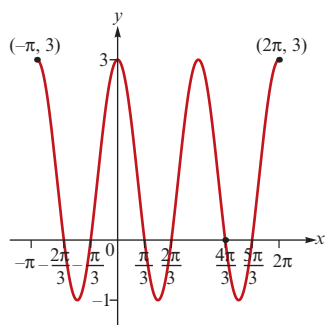
h $h = -3 \sin\left(x + \frac{5\pi}{6}\right)$



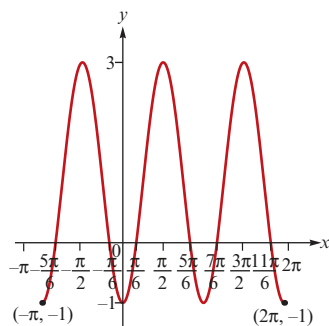
7 a $-\frac{5\pi}{6}, \frac{5\pi}{6}$ **b** $-\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$

c $\pi, \frac{5\pi}{3}$ **d** $\frac{2\pi}{3}$ **e** $\pi, \frac{5\pi}{3}$

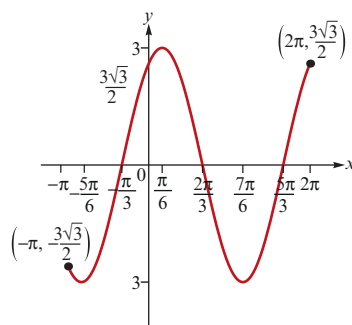
8 a $f(x) = 2 \cos(2x) + 1$



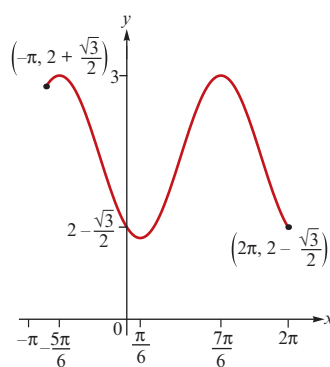
b $f(x) = 1 - 2 \cos(2x)$



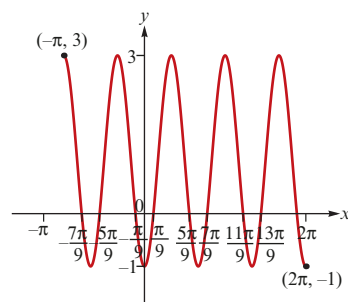
c $f(x) = 3 \sin\left(x + \frac{\pi}{3}\right)$



d $f(x) = 2 - \sin\left(x + \frac{\pi}{3}\right)$

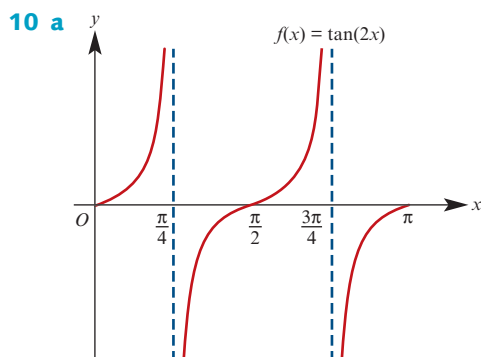


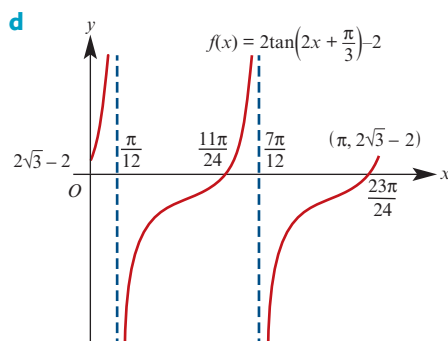
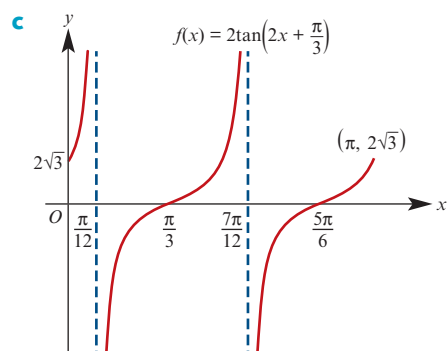
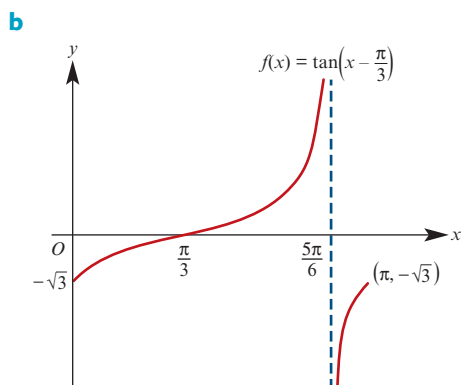
e $f(x) = 1 - 2 \cos(3x)$



9 a $\frac{2\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

c $\frac{3\pi}{2}$ **d** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$





11 a $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

b $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

c $\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$

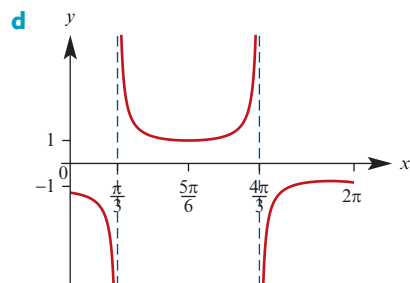
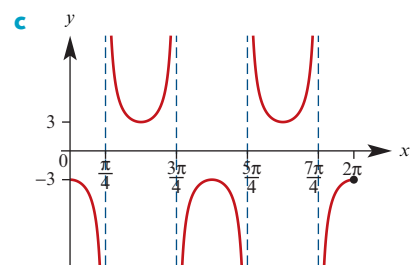
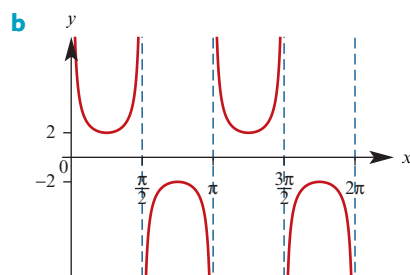
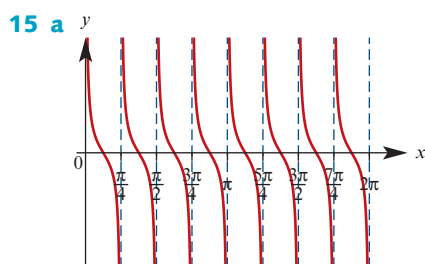
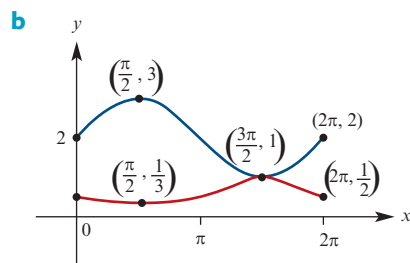
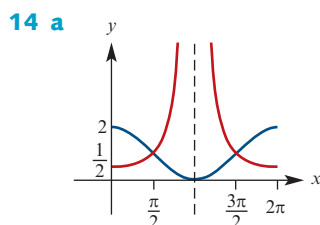
d $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

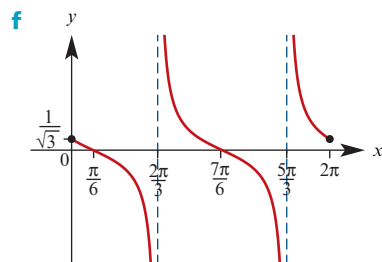
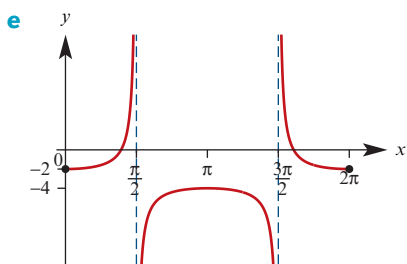
12 $0^\circ, 60^\circ, 180^\circ, 300^\circ, 360^\circ$

13 a $x = n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$

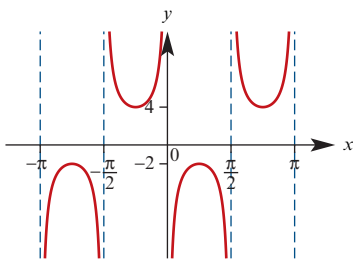
b $x = \frac{2n\pi}{3}, n \in \mathbb{Z}$

c $x = n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$





- 16** ■ Reflection in the x -axis
 ■ Dilation of factor 3 from the x -axis
 ■ Dilation of factor $\frac{1}{2}$ from the y -axis
 ■ Translation 1 unit up

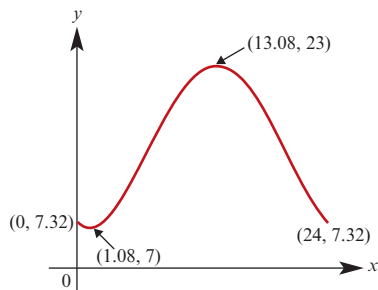


Multiple-choice questions

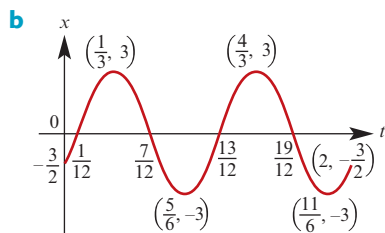
- 1** A **2** D **3** D **4** D **5** E
6 E **7** C **8** D **9** B **10** B

Extended-response questions

- 1** **a** 7.3°C
b Min = 7°C ; max = 23°C
c Between 9:40 a.m. and 4:30 p.m.
d



2 **a** $a = \frac{\pi}{6}$



c 3 m **d** $\frac{5}{6}$ s **e** 1 s **f** $\frac{1}{4}$ s

g i 24 m **ii** 30 m

3 **a** i $a = 13.4$ **ii** $b = 2$ **iii** $k = 12$

b 3 a.m., 9 a.m., 3 p.m., 9 p.m.

c $2 < t < 10$, $14 < t < 22$

4 **a** $p = 6$, $q = 4.2$

b 3 a.m., 3 p.m.

c 6 m

d 7 a.m., 11 a.m., 7 p.m., 11 p.m.

e 8 hours

5 **a** **i** $-1 < k < 1$ or $1 < k < 3$

ii $k = -1$ or $k = 3$

iii $k < -1$ or $k > 3$

b A translation of 1 unit in the negative direction of the y -axis, followed by a dilation of factor $\frac{1}{2}$ from the x -axis and a dilation of factor 3 from the y -axis

c **i** $h = \frac{\pi}{2}$, $\frac{7\pi}{6}$ **ii** $h = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{3\pi}{2}$

6 **a** For N : Max = 7000, occurs in April ($t = 4$);
 Min = 1000, occurs in October ($t = 10$).

For M : Max = 8500, occurs in June ($t = 6$);
 Min = 2500, occurs at end of January and November ($t = 1$ and $t = 11$)

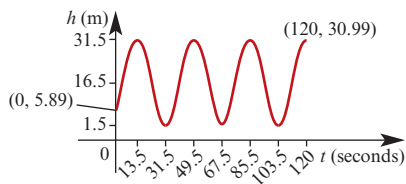
b $t = 4.31$ (April), population is 6961;

$t = 0.24$ (January), population is 2836

c 14 556 in May ($t = 5.19$)

d $t = 7.49$ (July)

7 **a**



b 5.89 m

c 27.51 s

d 6 times

e 20 times

f 4.21 m

g 13.9 m

Chapter 14

Exercise 14A

- 1 a** -1 **b** $-\sqrt{2}$ **c** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **d** 1
e -2 **f** 2 **g** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **h** 2
2 a -1 **b** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **c** 1
d $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **e** $-\sqrt{2}$ **f** $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
g -1 **h** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **i** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
3 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{6}, \frac{7\pi}{6}$ **c** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **d** $\frac{\pi}{4}, \frac{5\pi}{4}$
4 a $-\frac{8}{17}$ **b** $\frac{15}{17}$ **c** $-\frac{15}{8}$
5 $\cos \theta = \frac{24}{25}, \sin \theta = -\frac{7}{25}$
6 $-\frac{\sqrt{29}}{5}$
7 $\frac{8}{31}$
8 $\frac{15}{4(9+\sqrt{5})} = \frac{15(6-\sqrt{5})}{124}$

Exercise 14B

- 1 a** $-\frac{\sqrt{2}+\sqrt{6}}{4}$ **b** $\frac{\sqrt{2}-\sqrt{6}}{4}$
c $\frac{\sqrt{6}-\sqrt{2}}{4}$ **d** $\frac{\sqrt{2}+\sqrt{6}}{4}$
2 a $\frac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\frac{\sqrt{6}+\sqrt{2}}{4}$
3 For $u, v \in (0, \frac{\pi}{2})$, $\sin(u+v) = \frac{63}{65}$;
 For $u, v \in (\frac{\pi}{2}, \pi)$, $\sin(u+v) = -\frac{63}{65}$;
 For $u \in (0, \frac{\pi}{2})$, $v \in (\frac{\pi}{2}, \pi)$, $\sin(u+v) = -\frac{33}{65}$;
 For $u \in (\frac{\pi}{2}, \pi)$, $v \in (0, \frac{\pi}{2})$, $\sin(u+v) = \frac{33}{65}$
4 a $\frac{1}{2}(\sqrt{3}\sin\theta + \cos\theta)$ **b** $\frac{1}{\sqrt{2}}(\cos\varphi + \sin\varphi)$
c $\frac{1}{2}(\cos\theta - \sqrt{3}\sin\theta)$ **d** $\frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$
5 a $\sin u$ **b** $\cos u$
6 a $-\frac{119}{169}$ **b** $\frac{24}{25}$ **c** $\frac{7}{25}$ **d** $-\frac{169}{119}$
e $-\frac{33}{65}$ **f** $-\frac{16}{65}$ **g** $-\frac{65}{33}$ **h** $\frac{7}{24}$

7 a $\frac{7}{25}$ **b** $\frac{3}{5}$ **c** $-\frac{44}{125}$ **d** $-\frac{336}{625}$

8 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{2}$

9 a $1 - \sin(2\theta)$ **b** $\cos(2\theta)$

11 c i $2 - \sqrt{3}$ **ii** $2 + \sqrt{3}$ **iii** $-2 - \sqrt{3}$

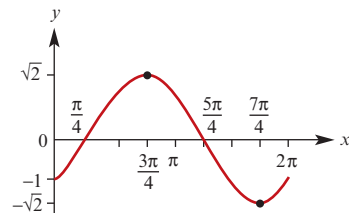
Exercise 14C

- 1 a** $5, -5$ **b** $2, -2$ **c** $\sqrt{2}, -\sqrt{2}$
d $\sqrt{2}, -\sqrt{2}$ **e** $2\sqrt{3}, -2\sqrt{3}$
f $2, -2$ **g** $4, 0$ **h** $5 + \sqrt{13}, 5 - \sqrt{13}$
2 a $\frac{\pi}{2}, \pi$ **b** $0, \frac{2\pi}{3}, 2\pi$
c $\frac{\pi}{6}, \frac{3\pi}{2}$ **d** $0, \frac{5\pi}{3}, 2\pi$
e 53.13° **f** $95.26^\circ, 155.26^\circ$

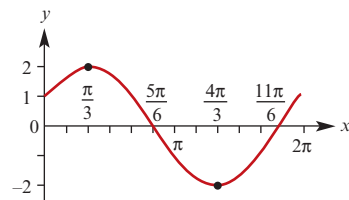
3 $2\cos(2x + \frac{\pi}{6})$

4 $\sqrt{2}\sin(3x - \frac{5\pi}{4})$

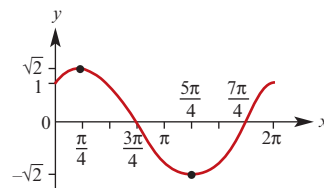
5 a $f(x) = \sin x - \cos x = \sqrt{2}\cos(x - \frac{3\pi}{4})$
 $= \sqrt{2}\sin(x + \frac{7\pi}{4}) = \sqrt{2}\sin(x - \frac{\pi}{4})$



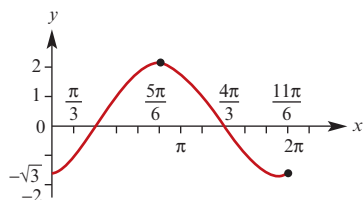
b $f(x) = \sqrt{3}\sin x + \cos x$
 $= 2\cos(x - \frac{\pi}{3}) = 2\sin(x + \frac{\pi}{6})$



c $f(x) = \sin x + \cos x$
 $= \sqrt{2}\cos(x - \frac{\pi}{4}) = \sqrt{2}\sin(x + \frac{\pi}{4})$



$$\begin{aligned} \text{d } f(x) &= \sin x - \sqrt{3} \cos x = 2 \cos\left(x - \frac{5\pi}{6}\right) \\ &= 2 \sin\left(x + \frac{5\pi}{3}\right) = 2 \sin\left(x - \frac{\pi}{3}\right) \end{aligned}$$



Exercise 14D

- 1 a $\sin(5\pi t) + \sin(\pi t)$ b $\frac{1}{2}(\sin 50^\circ - \sin 10^\circ)$
c $\sin(\pi x) + \sin\left(\frac{\pi x}{2}\right)$ d $\sin(A) + \sin(B + C)$
- 2 $\cos(\theta) - \cos(5\theta)$
- 3 $\sin A - \sin B$
- 5 a $2 \sin 39^\circ \cos 17^\circ$ b $2 \cos 39^\circ \cos 17^\circ$
c $2 \cos 39^\circ \sin 17^\circ$ d $-2 \sin 39^\circ \sin 17^\circ$
- 6 a $2 \sin(4A) \cos(2A)$ b $2 \cos\left(\frac{3x}{2}\right) \cos\left(\frac{x}{2}\right)$
c $2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{7x}{2}\right)$ d $-2 \sin(2A) \sin(A)$
- 11 a $-\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$
b $-\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$
c $-\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$
d $-\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$
- 12 a $\frac{\pi}{6}, \frac{5\pi}{6}$
b $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$
c $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$
d $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$
- 17 $\frac{1 - \cos(100x)}{2 \sin(x)}$

Chapter 14 review

Technology-free questions

- 1 a 5, 1 b 4, -2 c 4, -4 d 2, 0 e $1, \frac{1}{3}$
- 2 a $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ b $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$
c $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ d $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

$$4 \text{ a } \frac{140}{221} \quad \text{b } \frac{171}{221} \quad \text{c } \frac{171}{140}$$

$$5 \quad \frac{1}{2}$$

$$6 \text{ a } 1 \quad \text{b } 0$$

$$8 \text{ a } \frac{-1}{9} \quad \text{b } \frac{-4\sqrt{5}}{9} \quad \text{c } \frac{8\sqrt{5}}{81}$$

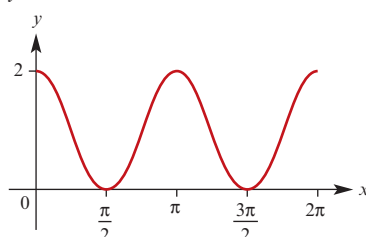
$$10 \quad 2 - \sqrt{3}$$

$$11 \text{ a } 0, \frac{\pi}{2}, 2\pi \quad \text{b } \frac{7\pi}{6}, \frac{11\pi}{6} \quad \text{c } 0, \pi, 2\pi$$

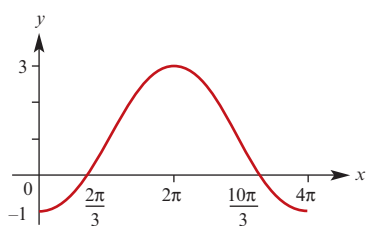
$$\text{d } \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{e } \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

$$\text{f } \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$$

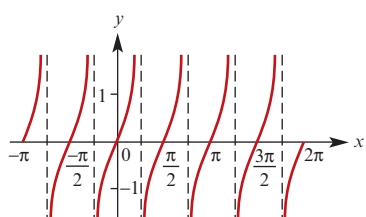
$$12 \text{ a } y = 2 \cos^2 x$$



$$\text{b } y = 1 - 2 \sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$$



$$\text{c } f(x) = \tan(2x)$$



$$13 \quad \frac{56}{65}$$

$$14 \text{ a } \sqrt{85} \cos(\theta - \alpha) \text{ where } \alpha = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$$

$$\text{b i } \sqrt{85} \quad \text{ii } \frac{2}{\sqrt{85}}$$

$$\text{iii } \theta = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$$

$$15 \text{ a } 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi \quad \text{b } 0, \frac{\pi}{3}, \pi$$

Multiple-choice questions

- 1 A 2 A 3 B 4 B 5 C
 6 E 7 E 8 A 9 D 10 E

Extended-response questions

- 1 **b** $P = 10\sqrt{5} \cos(\theta - \alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$;
 $\theta = 70.88^\circ$
c $k = 25$ **d** $\theta = 45^\circ$
- 2 **a** $AD = \cos \theta + 2 \sin \theta$
b $AD = \sqrt{5} \cos(\theta - \alpha)$ where
 $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63^\circ$
c Max length of AD is $\sqrt{5}$ m when $\theta = 63^\circ$
d $\theta = 79.38^\circ$
- 3 **b** **ii** $a = 1, b = 1$
c $\frac{1 + \sqrt{2} - \sqrt{3}}{1 + \sqrt{3} + \sqrt{6}} = \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1}$
 $= \sqrt{6} + \sqrt{2} - \sqrt{3} - 2$
- 4 **a** **ii** $2 \cos\left(\frac{\pi}{5}\right)$
b **iii** $4 \cos^2\left(\frac{\pi}{5}\right) - 2 \cos\left(\frac{\pi}{5}\right) - 1 = 0$
iv $\frac{1 + \sqrt{5}}{4}$
- 5 **b** $-\frac{2}{3}$ or $\frac{1}{2}$

Chapter 15

Exercise 15A

- 1

	Re(z)	Im(z)
a	2	3
c	$\frac{1}{2}$	$-\frac{3}{2}$
e	0	3

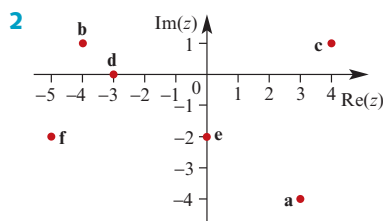
	Re(z)	Im(z)
b	4	5
d	-4	0
f	$\sqrt{2}$	$-2\sqrt{2}$
- 2 **a** $a = 2, b = -2$
b $a = 3, b = 2$ or $a = 2, b = 3$
c $a = 5, b = 0$ **d** $a = \frac{2}{3}, b = -\frac{1}{3}$
- 3 **a** $6 - 8i$ **b** $6 - i$ **c** $-6 - 2i$
d $7 - 3\sqrt{2}i$ **e** $-2 - 3i$ **f** $4 + 2i$
g $6 - 4i$ **h** $-4 + 6i$ **i** $-1 + 11i$
j -1
- 4 **a** $4i$ **b** $6i$ **c** $\sqrt{2}i$
d $-i$ **e** -1 **f** 1
g -2 **h** -12 **i** -4
- 5 **a** $1 + 2i$ **b** $-3 + 4i$
c $-\sqrt{2} - 2i$ **d** $-\sqrt{6} - 3i$

Exercise 15B

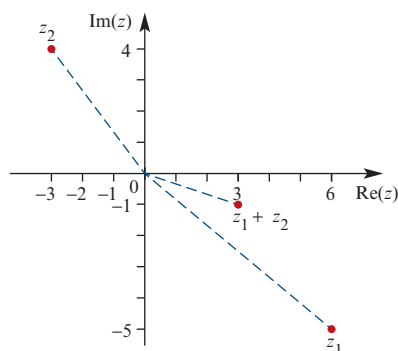
- 1 **a** $15 + 8i$ **b** $-8i$ **c** $-2 + 16i$
d $2i$ **e** 5 **f** $-4 + 19i$
- 2 **a** $2 + 5i$ **b** $-1 - 3i$
c $\sqrt{5} + 2i$ **d** $5i$
- 3 **a** $2 + i$ **b** $-3 - 2i$ **c** $-4 + 7i$
d $-4 - 7i$ **e** $-4 - 7i$ **f** $-1 + i$
g $-1 - i$ **h** $-1 - i$
- 4 **a** $2 + 4i$ **b** 20 **c** 4
d $8 - 16i$ **e** $-8i$ **f** 8
g $\frac{1}{10}(1 + 2i)$ **h** $-4 - 2i$
- 5 $a = \frac{1}{29}, b = -\frac{17}{29}$
- 6 **a** $\frac{7}{17} - \frac{6}{17}i$ **b** i **c** $\frac{7}{2} - \frac{1}{2}i$
d $-\frac{1}{2} - \frac{1}{2}i$ **e** $\frac{2}{13} + \frac{3}{13}i$ **f** $\frac{3}{20} + \frac{1}{20}i$
- 7 $a = \frac{5}{2}, b = -\frac{3}{2}$
- 8 **a** $-\frac{42}{5}(1 - 2i)$ **b** $-\frac{1}{2}(1 - i)$
c $\frac{1}{17}(4 + i)$ **d** $\frac{1}{130}(6 + 43i)$
e $2 - 2i$

Exercise 15C

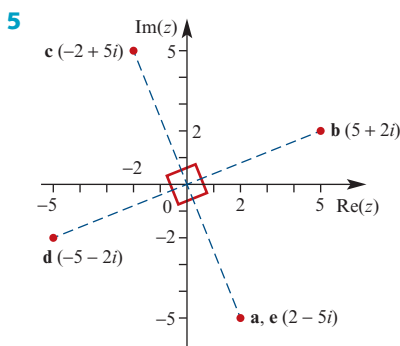
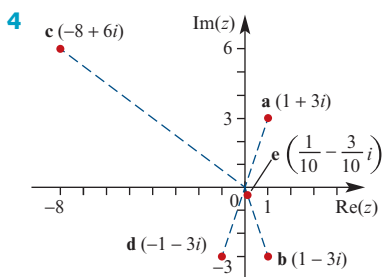
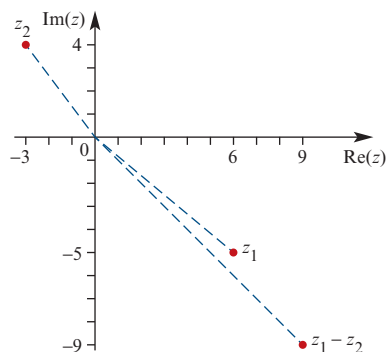
- 1 $A = 3 + i, B = 2i, C = -3 - 4i$
 $D = 2 - 2i, E = -3, F = -1 - i$



- 3 **a** $z_1 + z_2 = 3 - i$



b $z_1 - z_2 = 9 - 9i$



Exercise 15D

- 1** **a** $\pm 2i$ **b** $\pm 3i$
c $\pm \sqrt{5}i$ **d** $2 \pm 4i$
e $-1 \pm 7i$ **f** $1 \pm \sqrt{2}i$
g $\frac{1}{2}(-3 \pm \sqrt{3}i)$ **h** $\frac{1}{4}(-5 \pm \sqrt{7}i)$
i $\frac{1}{6}(1 \pm \sqrt{23}i)$ **j** $1 \pm 2i$
k $\frac{1}{2}(3 \pm \sqrt{11}i)$ **l** $3 \pm \sqrt{5}i$
- 2** **a** $(z + 3i)(z - 3i)$ **b** $(z + \sqrt{3}i)(z - \sqrt{3}i)$
c $3(z + 2i)(z - 2i)$
d $(z + 1 + 2i)(z + 1 - 2i)$
e $\left(z - \frac{3}{2} + \frac{\sqrt{15}}{2}i\right)\left(z - \frac{3}{2} - \frac{\sqrt{15}}{2}i\right)$
f $2\left(z + \frac{1}{2} + \frac{1}{2}i\right)\left(z + \frac{1}{2} - \frac{1}{2}i\right)$

Exercise 15E

- 1** **a** $2 \operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $\sqrt{2} \operatorname{cis}\left(\frac{-\pi}{4}\right)$
c $4 \operatorname{cis}\left(\frac{5\pi}{6}\right)$ **d** $4\sqrt{2} \operatorname{cis}\left(\frac{-3\pi}{4}\right)$
e $24 \operatorname{cis}\left(\frac{-\pi}{3}\right)$ **f** $\frac{1}{\sqrt{2}} \operatorname{cis}\left(\frac{3\pi}{4}\right)$
- 2** **a** $3i$ **b** $\frac{1}{\sqrt{2}}(1 + \sqrt{3}i) = \frac{\sqrt{2}}{2}(1 + \sqrt{3}i)$
c $\sqrt{3} + i$ **d** $\frac{-5}{\sqrt{2}}(1 - i) = -\frac{5\sqrt{2}}{2}(1 - i)$
e $-6(\sqrt{3} - i)$ **f** $3(1 - i)$
g $-\frac{5}{2}(1 + \sqrt{3}i)$ **h** $-\frac{5}{2}(1 + \sqrt{3}i)$
- 3** **a** $3\sqrt{2}(1 + i)$ **b** $6(1 + \sqrt{3}i)$
c $-\frac{5}{2}(1 - \sqrt{3}i)$ **d** $18(1 + \sqrt{3}i)$
e $-18(1 + \sqrt{3}i)$ **f** $\sqrt{3}(1 + i)$
g $\sqrt{3} + i$ **h** -4
i $-4(1 - \sqrt{3}i)$ **j** $-\frac{5}{2}$

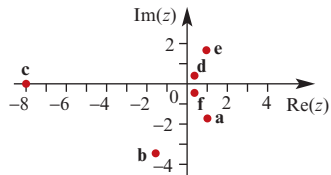
Exercise 15F

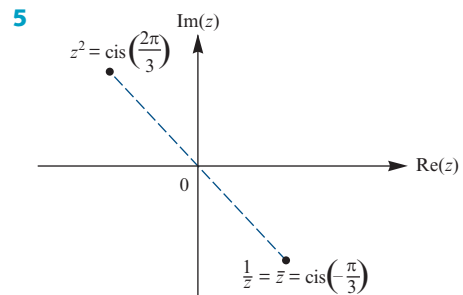
- 1** **a** $2 \operatorname{cis}\left(\frac{\pi}{2}\right) = 2i$ **b** $2\sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right) = -2 + 2i$
c $4 \operatorname{cis} \pi = -4$ **d** $4\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4}\right) = -4 - 4i$
e $16 \operatorname{cis} 0 = 16$
- 2** **a** $4 \operatorname{cis}\left(\frac{2\pi}{3}\right) = -2 + 2\sqrt{3}i$
b $8 \operatorname{cis} \pi = -8$
c $16 \operatorname{cis}\left(-\frac{2\pi}{3}\right) = -8 - 8\sqrt{3}i$
d $32 \operatorname{cis}\left(-\frac{\pi}{3}\right) = 16 - 16\sqrt{3}i$
e $64 \operatorname{cis} 0 = 64$ **f** $512 \operatorname{cis} \pi = -512$
g $4096 \operatorname{cis} 0 = 4096$
- 3** **a** $4 \operatorname{cis}\left(-\frac{\pi}{3}\right) = 2 - 2\sqrt{3}i$
b $27 \operatorname{cis} 0 = 27$
c $81 \operatorname{cis}\left(-\frac{2\pi}{3}\right) = -\frac{81}{2}(1 + \sqrt{3}i)$
d $16 \operatorname{cis} \pi = -16$
- 4** **a** $8 \operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $\frac{8}{27} \operatorname{cis}\left(\frac{\pi}{8}\right)$
c $27 \operatorname{cis}\left(\frac{5\pi}{6}\right)$ **d** $-32i$
e -216 **f** $1024 \operatorname{cis}\left(-\frac{\pi}{12}\right)$
g $\frac{27}{4} \operatorname{cis}\left(-\frac{\pi}{20}\right)$

- 5 a** $64 \operatorname{cis} 0 = 64$ **b** $\frac{\sqrt{2}}{8} \operatorname{cis}\left(-\frac{3\pi}{4}\right)$
c $128 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$
d $\frac{\sqrt{3}}{72} \operatorname{cis}\left(-\frac{\pi}{2}\right) = -\frac{\sqrt{3}}{72}i$
e $\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$ **f** $\frac{64\sqrt{3}}{3} \operatorname{cis}\left(\frac{3\pi}{4}\right)$
g $\frac{\sqrt{2}}{2} \operatorname{cis}\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}i$ **h** $\frac{1}{4} \operatorname{cis}\left(-\frac{2\pi}{15}\right)$
i $8\sqrt{2} \operatorname{cis}\left(\frac{11\pi}{12}\right)$

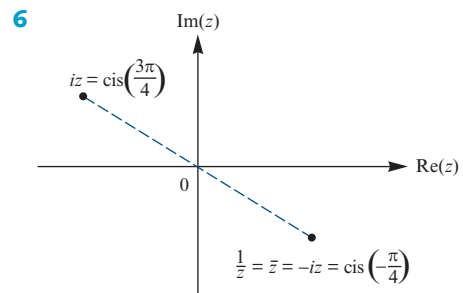
Chapter 15 review

Technology-free questions

- 1 a** $(2m + 3p) + (2n + 3q)i$ **b** $p - qi$
c $(mp + nq) + (np - mq)i$
d $\frac{(mp + nq) + (np - mq)i}{p^2 + q^2}$ **e** $2m$
f $(m^2 - n^2 - p^2 + q^2) + (2mn - 2pq)i$
g $\frac{m - ni}{m^2 + n^2}$
h $\frac{(mp + nq) + (mq - np)i}{m^2 + n^2}$
i $\frac{3((mp + nq) + (np - mq)i)}{p^2 + q^2}$
- 2** 
- a** $1 - \sqrt{3}i$ **b** $-2 - 2\sqrt{3}i$ **c** -8
d $\frac{1}{4}(1 + \sqrt{3}i)$ **e** $1 + \sqrt{3}i$ **f** $\frac{1}{4}(1 - \sqrt{3}i)$
- 3 a** $\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$ **b** $2 \operatorname{cis}\left(\frac{-\pi}{3}\right)$
c $\sqrt{13} \operatorname{cis}\left(\tan^{-1}\left(\frac{\sqrt{3}}{6}\right)\right)$ **d** $6 \operatorname{cis}\left(\frac{\pi}{4}\right)$
e $6 \operatorname{cis}\left(\frac{-3\pi}{4}\right)$ **f** $2 \operatorname{cis}\left(\frac{-\pi}{6}\right)$
- 4 a** $-1 - \sqrt{3}i$ **b** $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$
c $\frac{-3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$ **d** $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$
e $\frac{-3\sqrt{3}}{2} - \frac{3}{2}i$ **f** $1 - i$



- a** $z^2 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$ **b** $\bar{z} = \operatorname{cis}\left(-\frac{\pi}{3}\right)$
c $\frac{1}{z} = \operatorname{cis}\left(-\frac{\pi}{3}\right)$ **d** $\operatorname{cis}\left(\frac{2\pi}{3}\right)$



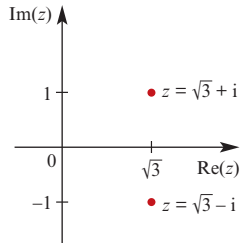
- a** $iz = \operatorname{cis}\left(\frac{3\pi}{4}\right)$ **b** $\bar{z} = \operatorname{cis}\left(-\frac{\pi}{4}\right)$
c $\frac{1}{z} = \operatorname{cis}\left(-\frac{\pi}{4}\right)$ **d** $-iz = \operatorname{cis}\left(-\frac{\pi}{4}\right)$
- 7 a** $\operatorname{cis}\left(-\frac{7\pi}{12}\right)$ **b** $\operatorname{cis}\left(-\frac{\pi}{12}\right)$ **c** $\operatorname{cis}\left(\frac{\pi}{12}\right)$
d $\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ **e** $\operatorname{cis}\left(-\frac{11\pi}{12}\right)$ **f** $\operatorname{cis}\left(-\frac{5\pi}{12}\right)$

8 $128 \operatorname{cis}\left(-\frac{5\pi}{6}\right) = -64(\sqrt{3} + i)$

Multiple-choice questions

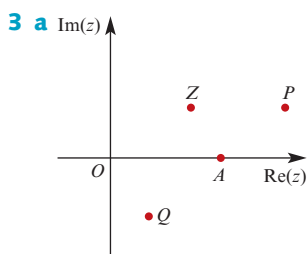
- 1** C **2** D **3** C **4** D **5** D
6 E **7** D **8** D **9** B **10** D

Extended-response questions

- 1 a** $z = \sqrt{3} + i$ or $z = \sqrt{3} - i$
b i 

ii $x^2 + y^2 = 4$ **iii** $a = 2$

- 2 a i** $6\sqrt{2}$ **ii** 6



- 3 a** $\sqrt{2} + 1$
- 6 a** $|z + 1| = \sqrt{2 + 2\cos\theta} = 2\cos\left(\frac{\theta}{2}\right)$,
 $\text{Arg}(z + 1) = \frac{\theta}{2}$
- b** $|z - 1| = \sqrt{2 - 2\cos\theta} = 2\sin\left(\frac{\theta}{2}\right)$,
 $\text{Arg}(z - 1) = \frac{\pi + \theta}{2}$
- c** $\left|\frac{z - 1}{z + 1}\right| = \tan\left(\frac{\theta}{2}\right)$, $\text{Arg}\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{2}$
- 7 a** $\Delta = b^2 - 4ac$
- b** $b^2 < 4ac$
- c i** $-\frac{b}{a}$, $\frac{\sqrt{4ac}}{2a}$ **ii** $\frac{b^2}{2ac} - 1$
- 8 a** $z_1 = \frac{1}{2}(-1 + \sqrt{3}i)$, $z_2 = \frac{1}{2}(-1 - \sqrt{3}i)$
- c** $|z_1| = 1$, $\text{Arg}(z_1) = \frac{2\pi}{3}$;
 $|z_2| = 1$, $\text{Arg}(z_2) = -\frac{2\pi}{3}$
- d** $\frac{\sqrt{3}}{4}$

Chapter 16

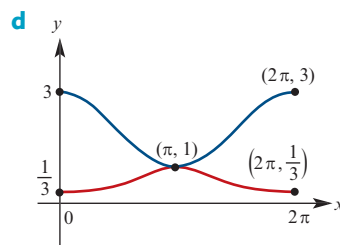
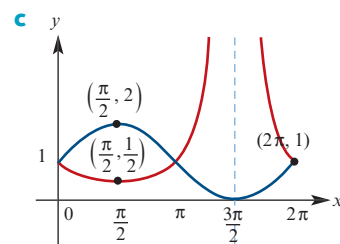
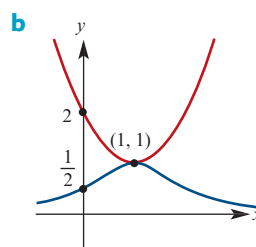
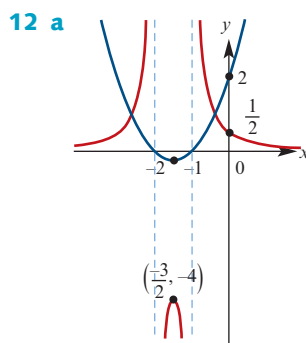
Technology-free questions

- 1 a** $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$
- b** $\frac{-17\pi}{24}, \frac{-11\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}$
- 2** $x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$
- 3 a** $\frac{5}{4}$ **b** $\frac{4}{3}$ **c** $-\frac{\sqrt{3}}{3}$ **d** $\frac{2\sqrt{3}}{3}$
- 4** $\pm \frac{\sqrt{6}}{3}$
- 6 a** $\frac{1}{2}(\sin(4x) - \sin(2x))$
- b** $\theta = \frac{(2n+1)\pi}{2}$ or $\theta = \frac{(2n+1)\pi}{4}, n \in \mathbb{Z}$
- 8 a** 6 **b** $4i$ **c** 13 **d** 10
e 36 **f** -16 **g** $24i$ **h** $24i$

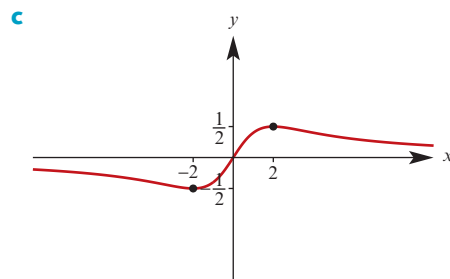
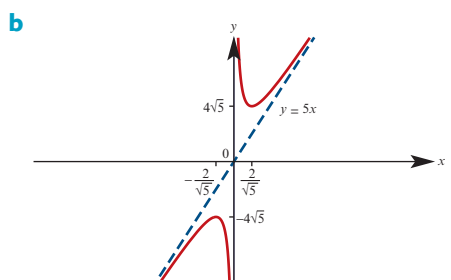
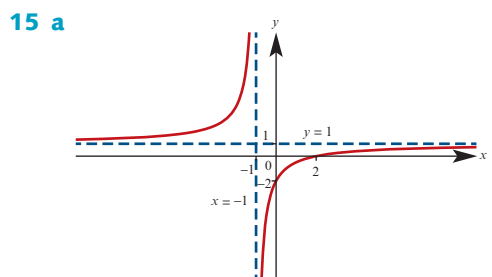
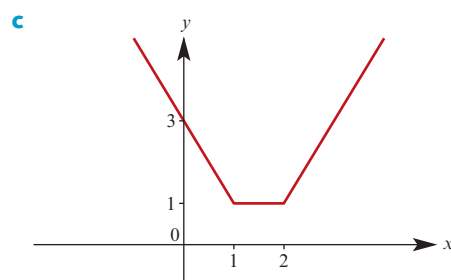
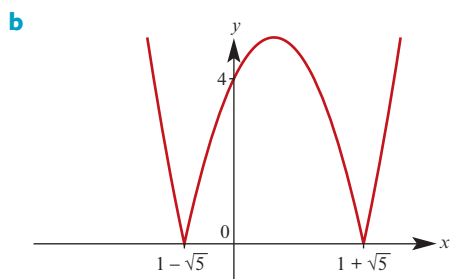
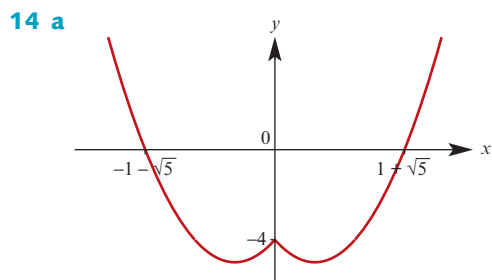
- 9 a** $3 - 5i$ **b** $-1 + i$ **c** $-4 - 7i$
- d** $\frac{8 - i}{13}$ **e** $2 + i$ **f** $\frac{-2 + i}{5}$
- g** $-2 - i$ **h** $\frac{8 + i}{5}$ **i** $\frac{13 - i}{34}$
- j** $3 - i$ **k** $\frac{-1 - 3i}{2}$ **l** $-3 - 4i$

- 10 a** $(z - 7i)(z + 7i)$
- b** $(z - 1 - 3i)(z - 1 + 3i)$
- c** $9\left(z - \frac{1}{3} - \frac{2}{3}i\right)\left(z - \frac{1}{3} + \frac{2}{3}i\right)$
- d** $4\left(z + \frac{3}{2} - i\right)\left(z + \frac{3}{2} + i\right)$

- 11 a** $2 + i, -2 - i$
- b** $z = -1 - i$ or $z = -i$



13 a $-\frac{4}{3}, 6$ **b** $-\frac{20}{3}, \frac{16}{3}$ **c** $-3, 7$
d $0, 5$ **e** $-2, 8$ **f** $\frac{1 \pm \sqrt{21}}{2}, \frac{1 \pm \sqrt{13}}{2}$

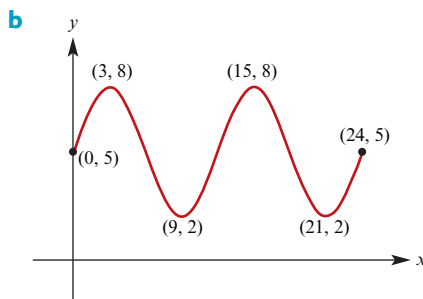


Multiple-choice questions

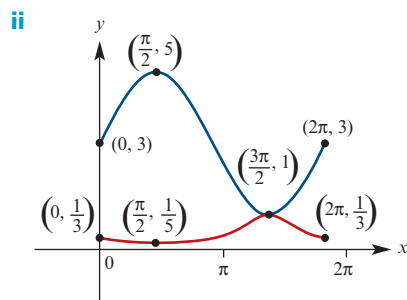
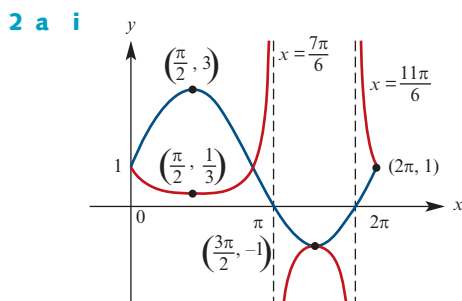
- 1 B** **2 B** **3 E** **4 C** **5 D**
6 D **7 C** **8 A** **9 D** **10 E**
11 A **12 A** **13 E** **14 C** **15 C**
16 C **17 B** **18 D** **19 E** **20 A**
21 C **22 C** **23 A** **24 C** **25 E**
26 C **27 A** **28 B** **29 C** **30 E**
31 C **32 B**

Extended-response questions

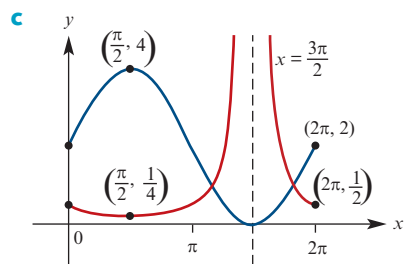
- 1 a i** 5 m **ii** 8 m



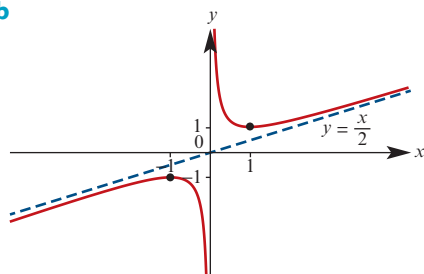
- c i** 8 m **ii** 2 m
d i 0, 6, 12, 18, 24
ii 0.65, 5.35, 12.65, 17.35



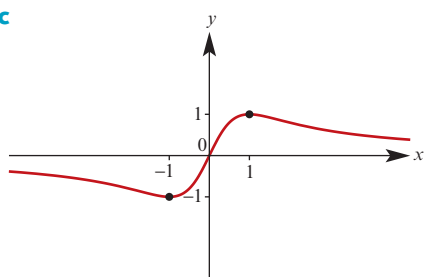
b $k = 2$



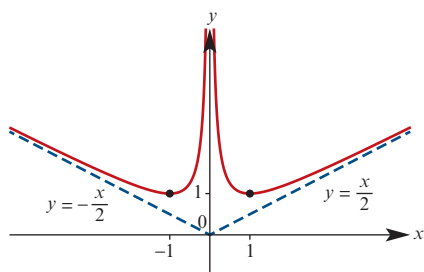
3 b



c



d e $|f(x)| = f(|x|)$



f i Turning points: $(1, -4)$, $(-1, -4)$;
Intercepts: $(5 \pm 2\sqrt{6}, 0)$, $(-5 \pm 2\sqrt{6}, 0)$

ii $k = -1$

5 d $z = \frac{1}{\sqrt[3]{16}} \left(\sqrt[3]{1 + \sqrt{3}i} + \sqrt[3]{1 - \sqrt{3}i} \right)$

Problem-solving and modelling

See solutions supplement

Degree-of-difficulty classified questions

Simple familiar questions

1 a $4 - 3i$ **b** $-2 + 7i$ **c** $-1 + 9i$ **d** $13 + i$
e $13 - i$ **f** $-\frac{7}{34} + \frac{11}{34}i$ **g** $\frac{1}{5} + \frac{2}{5}i$

2 $z = 1 + 3i$ or $z = 1 - 3i$

3 Range = $\left(-\infty, -\frac{1}{4}\right]$

4 a $6 \operatorname{cis}\left(\frac{\pi}{4}\right)$ **b** $6 \operatorname{cis}\left(-\frac{\pi}{3}\right)$ **c** $10 \operatorname{cis}\left(-\frac{\pi}{6}\right)$

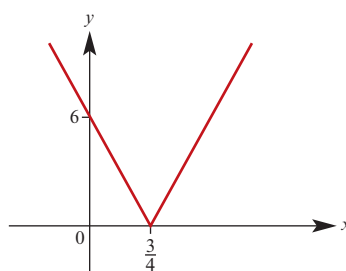
5 $\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}$

6 $z = -7 + i$

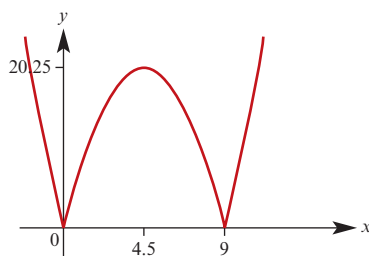
7 a $\pi, \frac{2\pi}{3}$ **b** $\frac{\pi}{2}$
c $\pi - \cos^{-1}\left(\frac{1}{5}\right), \pi + \cos^{-1}\left(\frac{1}{5}\right)$

8 $f(x) = \frac{\sqrt{5}}{2} \sin(x + \alpha)$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$;
Maximum = $\frac{\sqrt{5}}{2}$, Minimum = $\frac{1}{2}$

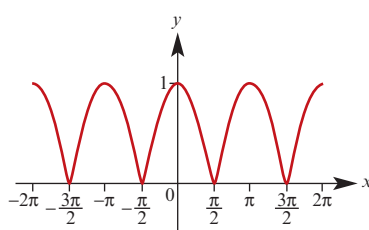
9 a



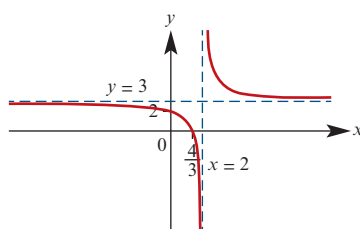
b

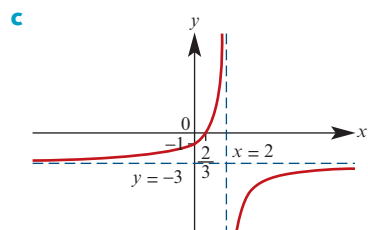
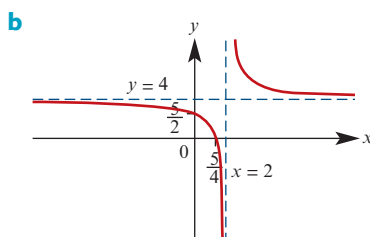


c

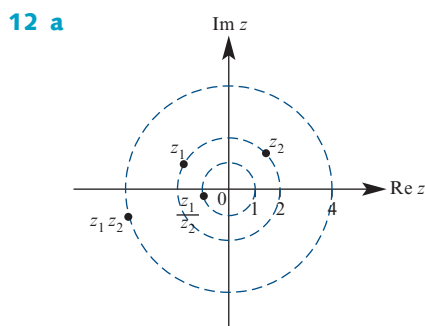


10 a





11 a $2 \operatorname{cis}\left(\frac{\pi}{3}\right), \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$ **b** $-\sqrt{3} + i$

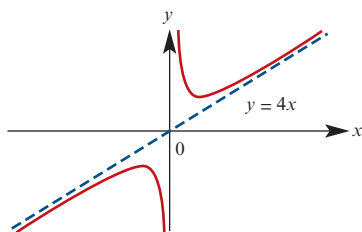


b $z_1 z_2 = 4 \operatorname{cis}\left(-\frac{11\pi}{12}\right), \frac{z_1}{z_2} = \operatorname{cis}\left(\frac{7\pi}{12}\right)$

13 $\mathbb{R} \setminus (0, 6)$

14 Asymptotes: $x = 0, y = 4x$

Turning points: $\left(\frac{\sqrt{3}}{2}, 4\sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, -4\sqrt{3}\right)$



Complex familiar questions

2 $x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$

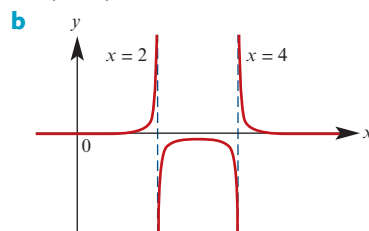
3 $z = \frac{\sqrt{7} + 1}{6}(1 + i)$ or $z = \frac{1 - \sqrt{7}}{6}(1 + i)$

4 a i 0, 2 **ii** -10, 4 **iii** $\frac{1}{2}$

b $\left[-\frac{7}{2}, \infty\right)$

5 $-\frac{a}{4} < x < \frac{5a}{2}$

6 a $4(x - 3)^2 - 4$

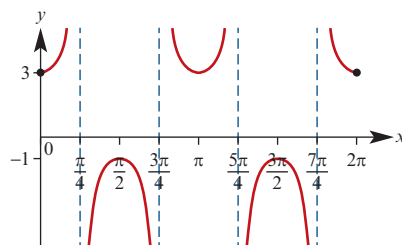


Local maximum at $\left(3, -\frac{1}{4}\right)$

7 $a = 6$

8 $z = -5 + 2i$ or $z = 5 - 2i$

9



10 $x = \frac{7\pi}{12}$ or $x = \frac{23\pi}{12}$

Complex unfamiliar questions

2 $\frac{\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$

3 $z = 1, z = \frac{1}{2}(-1 + \sqrt{3}i)$ or $z = \frac{1}{2}(-1 - \sqrt{3}i)$

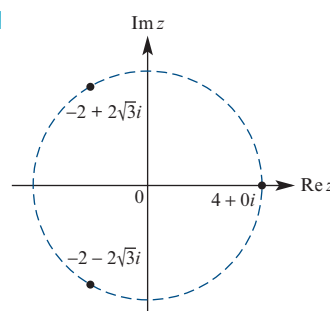
4 a $a = 5, b = 3$ **b** $z = 4i$

7 b $(\tan x - \sec x)^2 = \frac{1 - \sin x}{1 + \sin x}$

9 a $-2 + 2\sqrt{3}i, -2 - 2\sqrt{3}i$

b $4 \operatorname{cis}\left(\frac{2\pi}{3}\right), 4 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$ **c** 64, 64

d



e $4\sqrt{3}, 4\sqrt{3}$

10 b $(-6, 2, 6), (-6, -2, 6), (-8, 3, 3), (-3, -3, 8)$

11 d i $\sqrt{26}$ **ii** $\sqrt{13}$ **iii** $2\sqrt{2}$ **iv** $2\sqrt{5}$

e i $\frac{\sqrt{10}}{5}(-1 - 3i)$ **ii** $\frac{\sqrt{10}}{5}(1 + 3i)$

iii $\frac{\sqrt{2}}{2}(1 + i)$

- 12 a** $z^2 = \frac{1}{2}(-1 + \sqrt{3}i)$, $z^3 = -1$,
 $z^4 = \frac{1}{2}(-1 - \sqrt{3}i)$, $z^5 = \frac{1}{2}(1 - \sqrt{3}i)$,
 $z^6 = 1$
- 13 a** i 3 m ii 5 m iii 3 m iv 3 m
b $x = 4 - \cos(2t)$
c i 4 m from O ii π s iii 1 m
d $t = \frac{\pi}{2}$ s
e $t = \frac{\pi}{6}$ s
- 14 b** $d_3 = d_1 \cos \theta + d_2 \sin \theta$
c i $\theta = 69.20^\circ$ ii $w = 0.73$

Chapter 17

Exercise 17A

- 1 a** 2×2 **b** 2×3
c 1×4 **d** 4×1
- 2 a** $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$
- 3 a** $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$
- c** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$
- 4** $\begin{bmatrix} 200 & 180 & 135 & 110 & 56 & 28 \\ 110 & 117 & 98 & 89 & 53 & 33 \end{bmatrix}$
- 5 a** $\begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$ if $x = 4$
b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$ if $x = 4$
c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$ if $x = 0, y = 2$
- 6 a** $x = 2, y = 3$ **b** $x = 3, y = 2$
c $x = 4, y = -3$ **d** $x = 3, y = -2$
- 7** $\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 60 \\ 0 & 1 & 2 \end{bmatrix}$

Exercise 17B

- 1** $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $2\mathbf{X} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $4\mathbf{Y} + \mathbf{X} = \begin{bmatrix} 13 \\ -2 \end{bmatrix}$
 $\mathbf{X} - \mathbf{Y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ -6 & -9 \end{bmatrix}$
 $-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -7 & -7 \end{bmatrix}$
- 2** $2\mathbf{A} = \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ 0 & -6 \end{bmatrix}$
 $-6\mathbf{A} = \begin{bmatrix} -6 & 6 \\ 0 & -12 \end{bmatrix}$
- 3 a** Yes **b** Yes
- 4 a** $\begin{bmatrix} 6 & 4 \\ -4 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -9 \\ 12 & 3 \end{bmatrix}$
c $\begin{bmatrix} 6 & -5 \\ 8 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} -6 & -13 \\ 16 & 7 \end{bmatrix}$
- 5 a** $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 3 \\ 6 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 3 \\ -1 & 7 \end{bmatrix}$
- 6** $\mathbf{X} = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} -9 & -23 \\ 2 & 11 \end{bmatrix}$
- 7** $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 310 & 180 & 220 & 90 \\ 200 & 0 & 125 & 0 \end{bmatrix}$
represents the total production at two factories
in two successive weeks
- ### Exercise 17C
- 1** $\mathbf{AX} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ $\mathbf{BX} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\mathbf{AY} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$
 $\mathbf{IX} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\mathbf{AC} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$
 $\mathbf{CA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $(\mathbf{AC})\mathbf{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
 $\mathbf{C}(\mathbf{BX}) = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ $\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$
 $\mathbf{IB} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 $\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$
 $\mathbf{B}^2 = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$ $\mathbf{A}(\mathbf{CA}) = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$
 $\mathbf{A}^2\mathbf{C} = \begin{bmatrix} -2 & -5 \\ 3 & 7 \end{bmatrix}$
- 2** Defined: $\mathbf{AY}, \mathbf{CI}$;
Not defined: $\mathbf{YA}, \mathbf{XY}, \mathbf{X}^2, \mathbf{XI}$
- 3** $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- 4** No
- 5** One possible answer is $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$$6 \text{ } \mathbf{LX} = [7], \quad \mathbf{XL} = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

7 $\mathbf{AB}$ and $\mathbf{BA}$ are not defined unless $m = n$

$$8 \text{ } \mathbf{b} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

9 One possible answer is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

10 One possible answer is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix},$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}, \quad \mathbf{AB} + \mathbf{AC} = \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix},$$

$$(\mathbf{B} + \mathbf{C})\mathbf{A} = \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$$

11 For example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

12 a $\begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$, John took 29 minutes to eat food costing \$8.50

$$\mathbf{b} \begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix},$$

John's friends took 22 and 12 minutes to eat food costing \$8.00 and \$3.00 respectively

$$13 \text{ } \mathbf{A}^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}, \quad \mathbf{A}^4 = \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix},$$

$$\mathbf{A}^8 = \begin{bmatrix} -527 & 336 \\ -336 & -527 \end{bmatrix}$$

$$14 \text{ } \mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Exercise 17D

$$1 \text{ } \mathbf{a} \ 1 \quad \mathbf{b} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \quad \mathbf{c} \ 2 \quad \mathbf{d} \ \frac{1}{2} \begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$$

$$2 \text{ } \mathbf{a} \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \quad \mathbf{b} \begin{bmatrix} \frac{2}{7} & \frac{-1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$$

$$\mathbf{c} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix} \quad \mathbf{d} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$4 \text{ } \mathbf{a} \ \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\mathbf{b} \ \mathbf{AB} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}, \quad (\mathbf{AB})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

$$\mathbf{c} \ \mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix},$$

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}, \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$5 \text{ } \mathbf{a} \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix} \quad \mathbf{b} \begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix} \quad \mathbf{c} \begin{bmatrix} \frac{5}{2} & \frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$$

$$6 \text{ } \mathbf{a} \begin{bmatrix} -\frac{3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix} \quad \mathbf{b} \begin{bmatrix} -\frac{11}{16} & \frac{17}{16} \\ \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$$

$$7 \begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{bmatrix}$$

$$9 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix}, k \in \mathbb{R},$$

$$\begin{bmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{bmatrix}, b \neq 0$$

$$10 \ a = \pm\sqrt{2}$$

Exercise 17E

$$1 \text{ } \mathbf{a} \begin{bmatrix} 3 \\ 10 \end{bmatrix} \quad \mathbf{b} \begin{bmatrix} 5 \\ 17 \end{bmatrix}$$

$$2 \text{ } \mathbf{a} \ x = -\frac{1}{7}, y = \frac{10}{7} \quad \mathbf{b} \ x = 4, y = 1.5$$

$$3 \ (2, -1)$$

$$4 \text{ Book } \$12, \text{ DVD } \$18$$

$$5 \text{ } \mathbf{a} \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\mathbf{b} \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \text{ is non-invertible}$$

$\mathbf{c}$ System has solutions (not a unique solution)

$\mathbf{d}$ Solution set contains infinitely many pairs

$$6 \text{ } \mathbf{a} \ \mathbf{A}^{-1}\mathbf{C} \quad \mathbf{b} \ \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{C} \quad \mathbf{c} \ \mathbf{A}^{-1}\mathbf{CB}^{-1}$$

$$\mathbf{d} \ \mathbf{A}^{-1}\mathbf{C} - \mathbf{B} \quad \mathbf{e} \ \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$$

$$\mathbf{f} \ (\mathbf{A} - \mathbf{B})\mathbf{A}^{-1} = \mathbf{I} - \mathbf{BA}^{-1}$$

Exercise 17F

$$1 \text{ } \mathbf{a} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \quad \mathbf{b} \begin{bmatrix} 1 & -2 & \frac{1}{5} \\ 0 & \frac{1}{2} & \frac{-3}{10} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$3 \ \mathbf{AB} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{1}{7}\mathbf{B}$$

$$4 \ \mathbf{A}^2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{1}{9}\mathbf{A}$$

$$5 \ \mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{1}{4}\mathbf{A}$$

$$\begin{aligned} 6 \text{ a } & \begin{bmatrix} 2 & 1 & -10 \\ 3 & 2 & -17 \\ -5 & -3 & 28 \end{bmatrix} \\ \text{b } & \frac{1}{29} \begin{bmatrix} 8 & -13 & 14 \\ 2 & 4 & -11 \\ -9 & 11 & 6 \end{bmatrix} \\ \text{c } & \frac{1}{37} \begin{bmatrix} 6 & 4 & -7 & -17 \\ -13 & -21 & 46 & 43 \\ 8 & 30 & -34 & -35 \\ -4 & -15 & 17 & 36 \end{bmatrix} \\ \text{d } & \frac{1}{37} \begin{bmatrix} 6 & -13 & 8 & -4 \\ 4 & -21 & 30 & -15 \\ -7 & 46 & -34 & 17 \\ -17 & 43 & -35 & 36 \end{bmatrix} \end{aligned}$$

7 a -36 b 1

8 a i -2 ii -2
b i -4 ii -16

9 a $\det(\mathbf{A}) = -2p + 6$ b $p = 3$

10 a $\det(\mathbf{A}) = -2(p - 2)(p - 1)$
b $p = 2$ or $p = 1$

Exercise 17G

1 a $x = 2, y = 3, z = 1$ b $x = -3, y = 5, z = 2$
c $x = 5, y = 0, z = 7$ d $x = 6, y = 5, z = 1$
e $x = 5, y = 2, z = 4, w = -1$

2 a $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \\ -1 & -4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 17 \end{bmatrix}$

b $\det(\mathbf{A}) = 0$, so $\mathbf{A}$ is non-invertible

c i $-y + 5z = 15$, $-y + 5z = 15$

ii The two equations are the same

iii $y = 5\lambda - 15$

iv $x = 43 - 13\lambda$

Chapter 17 review

Technology-free questions

1 a $\begin{bmatrix} 0 & 0 \\ 12 & 8 \end{bmatrix}$ b $\begin{bmatrix} 0 & 0 \\ 8 & 8 \end{bmatrix}$

2 $\begin{bmatrix} a \\ 2 - \frac{3}{4}a \end{bmatrix}$, $a \in \mathbb{R}$

3 a Exist: **AC, CD, BE**; Does not exist: **AB**

b $\mathbf{DA} = \begin{bmatrix} 14 & 0 \end{bmatrix}$, $\mathbf{A}^{-1} = \frac{1}{7} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

4 $\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$, $\mathbf{C}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5 $\begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$

6 $\mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

7 8

8 a i $\begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$ ii $\begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix}$ iii $\frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

b $x = 2, y = 1$

Multiple-choice questions

1 B 2 E 3 C 4 E 5 C
6 A 7 E 8 A 9 E 10 D

Extended-response questions

1 a i $\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

ii $\det(\mathbf{A}) = 14$, $\mathbf{A}^{-1} = \frac{1}{14} \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$

iii $\frac{1}{7} \begin{bmatrix} 9 \\ -1 \end{bmatrix}$

iv $\left(\frac{9}{7}, -\frac{1}{7}\right)$ is the point of intersection of the two lines

b i $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

ii $\det(\mathbf{A}) = 0$, so $\mathbf{A}$ is non-invertible

c Equations of two parallel lines

2 a $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$ b $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$

c Semester 1: 79.2; Semester 2: 80.4

d Semester 1: 83.8; Semester 2: 75.2

e No, total score is 318.6

f 3 marks

3 a $\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix}$ b $\begin{bmatrix} 70 \\ 60 \end{bmatrix}$

c Term 1: \$820; Term 2: \$800;
Term 3: \$1040; Term 4: \$1020

d $\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}$ e $\begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix}$

f Term 1: \$270; Term 2: \$270;
Term 3: \$480; Term 4: \$480

g Term 1: \$1090; Term 2: \$1070;
Term 3: \$1520; Term 4: \$1500

4 Brad 20; Flynn 10; Lina 15

Chapter 18

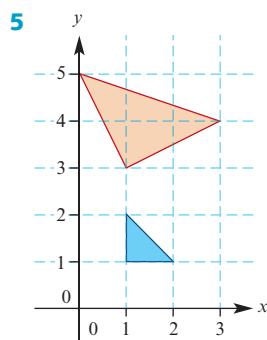
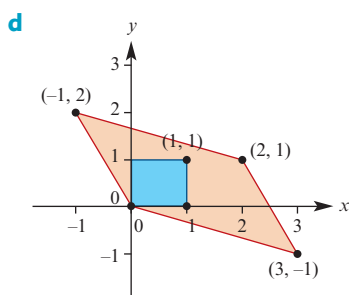
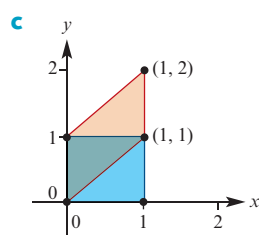
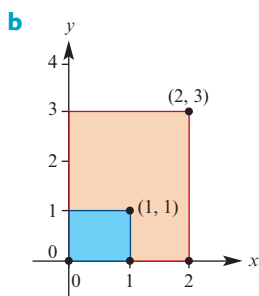
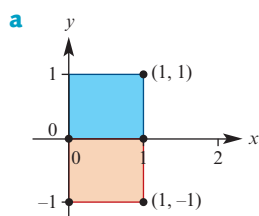
Exercise 18A

1 a (-2, 6) b (-8, 22)
c (26, 2) d (-4, -2)

2 a (3, 2) b (-4, 9)
c (8, 3) d (7, 11)

3 a $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 11 & -3 \\ 3 & -8 \end{bmatrix}$
c $\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4 Unit square is blue; image is red



6 $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$

7 $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

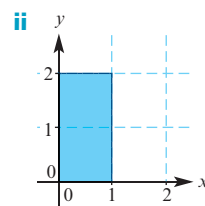
8 a $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

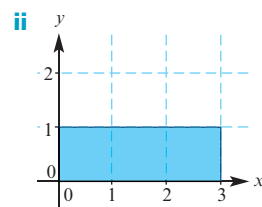
c $\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix}$

Exercise 18B

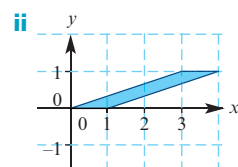
1 a i $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$



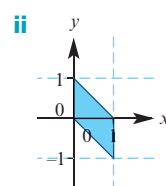
b i $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$



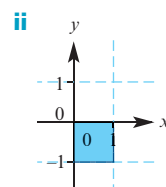
c i $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$



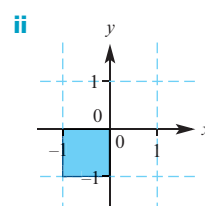
d i $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$



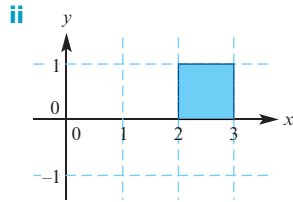
e i $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



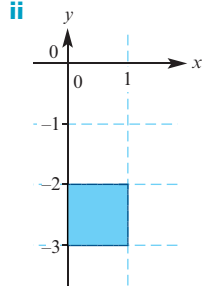
f i $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$



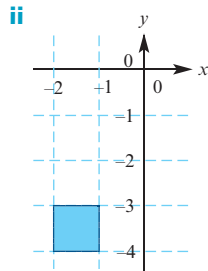
2 a i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$



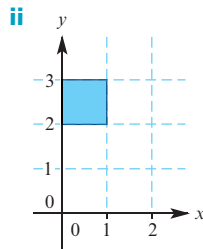
b i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y-3 \end{bmatrix}$



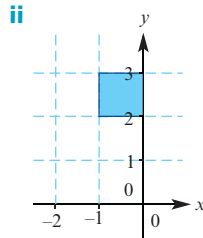
c i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} x-2 \\ y-4 \end{bmatrix}$



d i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y+2 \end{bmatrix}$



e i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$



Exercise 18C

1 a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ b $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ d $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

2 a $(-3, 2)$ b $\left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3 a $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ b $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ d $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

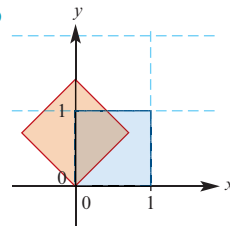
4 a $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ b $\begin{bmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}$

c $\begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}$ d $\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$

5 a $\begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$ b $\left(\frac{-23}{37}, \frac{47}{37}\right)$

6 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

b $\sqrt{2} - 1$



7 a $C\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), B\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b Equilateral

c $y = -\sqrt{3}x, y = 0, y = \sqrt{3}x$

Exercise 18D

1 $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ 2 $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

- 3 a** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
b $\begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
4 a $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **c** No
5 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **c** Yes
6 a $(x, y) \rightarrow (-x - 3, y + 5)$
b $(x, y) \rightarrow (-x + 3, y + 5)$ **c** Yes
7 a $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
c $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
8 a $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
b $\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
9 $\theta = 180^\circ k$, where $k \in \mathbb{Z}$
10 a 20
b $\begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}$
c $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$
 $\sin(2\theta) = 2 \sin \theta \cos \theta$
11 a $x' = y + 1$ **b** $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
 $y' = x + 2$
12 a $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$
c $\begin{bmatrix} \frac{\sqrt{2} + \sqrt{6}}{4} & \frac{\sqrt{2} - \sqrt{6}}{4} \\ \frac{\sqrt{6} - \sqrt{2}}{4} & \frac{\sqrt{6} + \sqrt{2}}{4} \end{bmatrix}$
d $\cos 15^\circ = \frac{\sqrt{2} + \sqrt{6}}{4}$, $\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$
13 $\begin{bmatrix} \cos(2\theta - 2\varphi) & -\sin(2\theta - 2\varphi) \\ \sin(2\theta - 2\varphi) & \cos(2\theta - 2\varphi) \end{bmatrix}$
 rotation matrix for angle $2\theta - 2\varphi$

Exercise 18E

- 1 a** $\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{14} \end{bmatrix}$
c $\begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}$ **d** $\begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$

- 2 a** $(x, y) \rightarrow (x - 2y, 2x - 5y)$
b $(x, y) \rightarrow (y, -x + y)$
3 a $(-1, 1)$ **b** $(-\frac{1}{2}, 1)$

4 $\begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}$

- 5** $(0, 0), (-1, -2), (1, 1), (0, -1)$

6 a $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ **b** $A^{-1} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$

7 a $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ **b** $A^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$

8 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b Reflecting twice in the same axis will return any point (x, y) to its original position

9 a $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

b Reflecting twice in the same line will return any point (x, y) to its original position

Exercise 18F

- 1 a** $y = -3x - 1$ **b** $y = \frac{3x}{2} + 1$ **c** $y = \frac{9x}{2} + 3$
d $y = 3x - 1$ **e** $y = -9x + 3$ **f** $y = \frac{-x - 1}{3}$

g $y = \frac{x - 1}{3}$

2 a $y = 6 - \frac{9x}{2}$

b $y = \frac{x + 2}{3}$

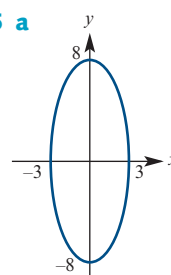
c $y = \frac{2 - 3x}{7}$

d $y = \frac{5x - 2}{12}$

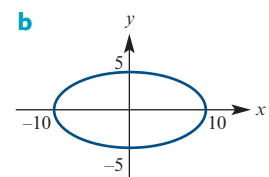
3 $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

4 $\begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix}$

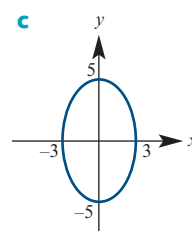
5 a



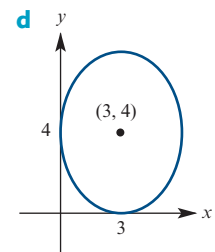
b

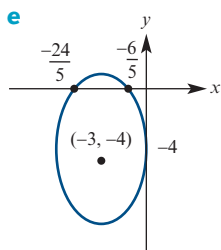


c

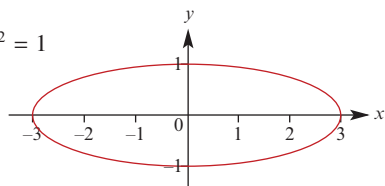


d





6 $\frac{x^2}{3^2} + y^2 = 1$

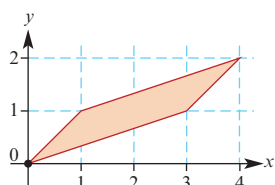


7 $y = -(x+1)^2 - 1$

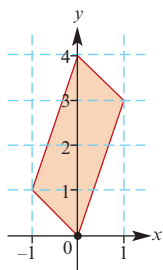
8 $y = (x-1)^2 - 3$

Exercise 18G

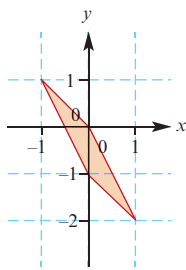
1 a Area = 2



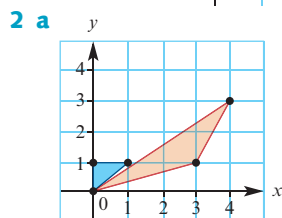
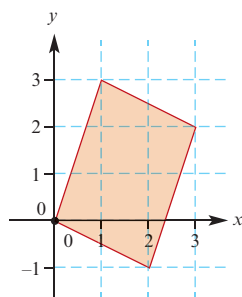
b Area = 4



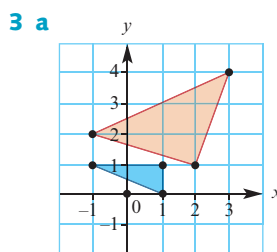
c Area = 1



d Area = 7



b Original area = $\frac{1}{2}$; image area = $\frac{5}{2}$



b Original area = 1; image area = 5

4 $m = \pm 2$

5 $m = -1, 2$

6 a i $\det \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = 1$

ii $\det \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = 1$

iii $\det \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = -1$

b i Dilation of factor k from the y -axis and dilation of factor $\frac{1}{k}$ from the x -axis

ii Determinant of matrix is 1

7 b $x = -1$

8 $m > 2$ or $m < 1$

9 $\begin{bmatrix} 1 & \pm \frac{\sqrt{3}}{2} \\ 0 & \pm \frac{1}{2} \end{bmatrix}$

10 a $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Exercise 18H

1 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x+4 \end{bmatrix}$

2 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x-2 \\ -y+2 \end{bmatrix}$

3 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}$

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y-1 \\ -x-1 \end{bmatrix}$

c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ -y+2 \end{bmatrix}$

d $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x-4 \\ y \end{bmatrix}$

4 a $\mathbf{A} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

b $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

c $\mathbf{C} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

d $\mathbf{CBA} =$

$\begin{bmatrix} \cos^2 \theta + k \sin^2 \theta & \cos \theta \sin \theta - k \sin \theta \cos \theta \\ \cos \theta \sin \theta - k \sin \theta \cos \theta & \sin^2 \theta + k \cos^2 \theta \end{bmatrix}$

5 $\begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$

6 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$

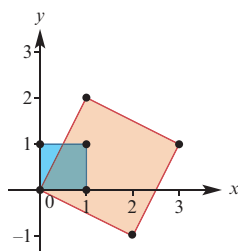
Chapter 18 review

Technology-free questions

1 a (7, 4)

b $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

c Area = 5



d $(x, y) \rightarrow \left(\frac{2}{5}x - \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y\right)$

2 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

c $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

e $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

f $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3 a $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$

b $\left(\frac{4}{5}, \frac{22}{5}\right)$

4 a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

b $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

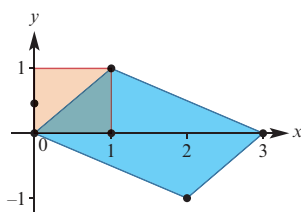
5 a $(x, y) \rightarrow (x - 3, -y + 4)$

b $(x, y) \rightarrow (x - 3, -y - 4)$

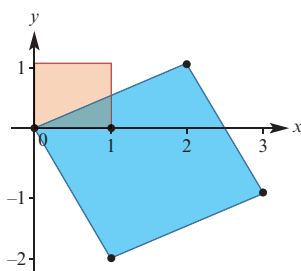
6 a $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

b $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$

7 a Area of image = 3 square units

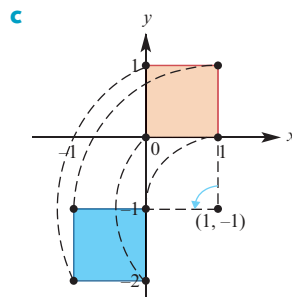


b Area of image = 5 square units



8 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y \\ x - 2 \end{bmatrix}$

b (1, 0)



Multiple-choice questions

1 B

2 D

3 A

4 D

5 C

6 A

7 D

8 E

9 D

Extended-response questions

1 a $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

b $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

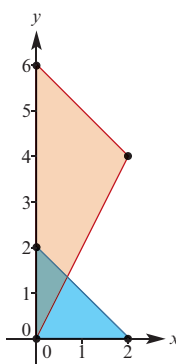
c Product of these two matrices:

$$\begin{bmatrix} \frac{-1 + \sqrt{3}}{2\sqrt{2}} & \frac{-1 + \sqrt{3}}{2\sqrt{2}} \\ \frac{1 + \sqrt{3}}{2\sqrt{2}} & \frac{-1 + \sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

d $\cos 75^\circ = \frac{-1 + \sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2} + \sqrt{6}}{4}$

$$\sin 75^\circ = \frac{1 + \sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

2 a

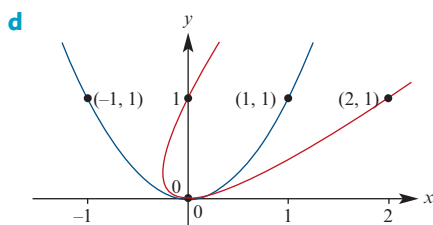

 b Original area = 2 square units;
Image area = 6 square units

 c 8π cubic units

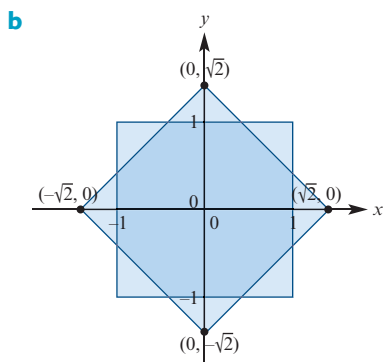
3 a $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

 b Shear of factor 1 parallel to the x -axis

c (0, 0), (2, 1), (0, 1)



4 a $(0, \sqrt{2}), (\sqrt{2}, 0), (0, -\sqrt{2}), (-\sqrt{2}, 0)$



c $13 - 8\sqrt{2}$ square units

5 b i The composition of two rotations is a rotation

ii The composition of two reflections is a rotation

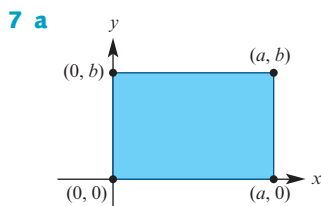
iii The composition of a reflection followed by a rotation is a reflection

iv The composition of a rotation followed by a reflection is a reflection

c $\begin{bmatrix} 1 & \sqrt{3} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \sqrt{3} & 1 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

6 a $\begin{bmatrix} 3 & 4 \\ 5 & 5 \\ 4 & 3 \\ 5 & -5 \end{bmatrix}$ **b** $A'(-1, -3)$ **c** $2\sqrt{10}$

d Isosceles **f** $2\sqrt{10}$



b $O(0, 0), A(a \cos \theta, a \sin \theta), B(-b \sin \theta, b \cos \theta), C(a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$

8 a $y = \frac{1}{m} - \frac{x}{m}; (1, 0), \left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$

b $y = 1 - \frac{x}{m}; (0, 1), \left(\frac{2m}{1+m^2}, \frac{m^2-1}{1+m^2}\right)$

c $\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{bmatrix}$

Chapter 19

Technology-free questions

1 a All defined except **AB**

b $DA = \begin{bmatrix} 6 & -12 \end{bmatrix}, A^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{2}{9} & -\frac{1}{9} \end{bmatrix}$

2 a $\begin{bmatrix} -2 & 4 \\ 18 & -24 \end{bmatrix}$ **b** $\begin{bmatrix} -10 & -19 \\ 7 & -16 \end{bmatrix}$

3 8

4 $A = \begin{bmatrix} t \\ 3t-5 \end{bmatrix}, t \in \mathbb{R}$

5 $AB = \begin{bmatrix} -9 & -8 \\ -15 & 10 \end{bmatrix}, C^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & \frac{1}{2} \end{bmatrix}$

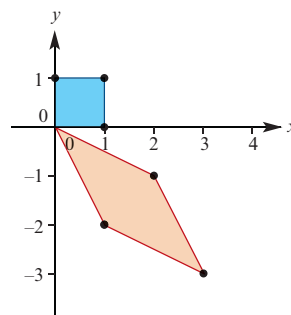
6 a $(C-B)A^{-1}$ **b** $B^{-1}C-A$

c $I - A^{-1}BA$ **d** $-A$

e $\frac{1}{2}B$ **f** $I - A^{-1}$

8 a $(7, -8)$ **b** $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

c Area = 3



d $(x, y) \rightarrow \left(\frac{2}{3}x + \frac{1}{3}y, -\frac{1}{3}x - \frac{2}{3}y\right)$

9 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

$$\mathbf{f} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \quad \mathbf{g} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{h} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

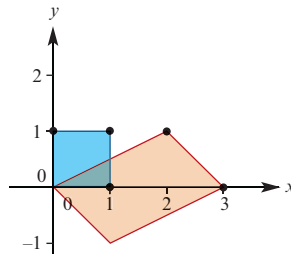
$$10 \mathbf{a} \begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix} \quad \mathbf{b} \left(\frac{2}{17}, \frac{76}{17} \right)$$

$$11 \mathbf{a} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad \mathbf{b} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{c} \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$$

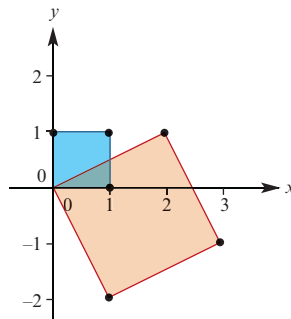
$$12 \mathbf{a} (x, y) \rightarrow (-x + 2, y - 1)$$

$$\mathbf{b} (x, y) \rightarrow (-x - 2, y - 1)$$

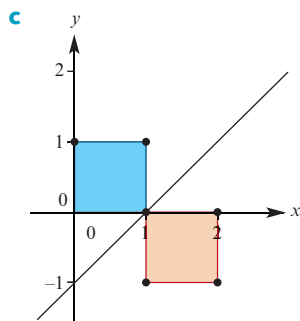
$$13 \mathbf{a} \text{ Area} = 3$$



$$\mathbf{b} \text{ Area} = 5$$



$$14 \mathbf{a} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y + 1 \\ x - 1 \end{bmatrix} \quad \mathbf{b} (0, 0) \rightarrow (1, -1)$$



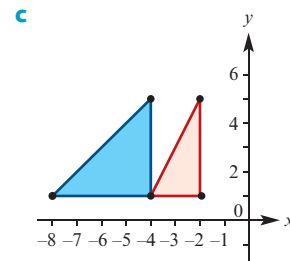
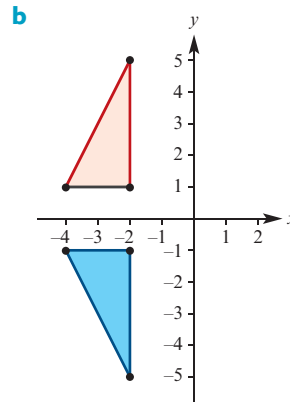
Multiple-choice questions

- 1 A 2 B 3 E 4 A 5 B
6 C 7 A 8 B 9 D 10 D
11 C 12 B 13 A 14 E 15 B
16 D 17 C 18 D 19 E 20 D
21 B 22 B

Extended-response questions

$$1 \mathbf{a} \mathbf{i} \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & d^2 + bc \end{bmatrix} \quad \mathbf{ii} \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

$$2 \mathbf{a} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 6 \\ y + 3 \end{bmatrix}$$



$$\mathbf{d} y = 2(x + 3)^2 + 2 \quad \mathbf{e} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 3 \\ -2y + 4 \end{bmatrix}$$

$$3 \mathbf{a} (4, 1)$$

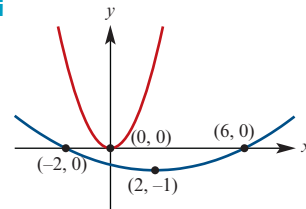
$$\mathbf{b} \mathbf{i} \text{ Rectangle with vertices } A'(0, 0), B'(0, 1), C'(4, 1), E'(4, 0)$$

$$\mathbf{ii} 1 \quad \mathbf{iii} 4 \quad \mathbf{iv} k$$

$$\mathbf{c} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4x \\ y \end{bmatrix}$$

$$\mathbf{d} \mathbf{i} y = \frac{1}{16}x^2 \quad \mathbf{ii} y = \frac{1}{16}(x - 2)^2 - 1$$

$$\mathbf{iii}$$



$$\mathbf{e} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2 \\ \frac{1}{5}(y + 3) \end{bmatrix}$$

$$4 \mathbf{b} \mathbf{i} x^2 + (y - 1)^2 = 1$$

$$\mathbf{ii} \left(x + \frac{4}{5} \right)^2 + \left(y - \frac{3}{5} \right)^2 = 1$$

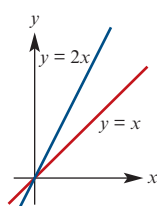
$$\mathbf{c} (0, 0), \left(-\frac{4}{5}, \frac{8}{5} \right)$$

5 a $(-3, 11)$ b $\frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$
 c $a = 2, b = 3$ d $(5a, 5a)$
 e $\lambda = 2, b = -2a; \lambda = 5, b = a$

6 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ b $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$
 c $a = \sqrt{2}, b = 0$ d $c = \frac{3\sqrt{2}}{2}, d = \frac{\sqrt{2}}{2}$

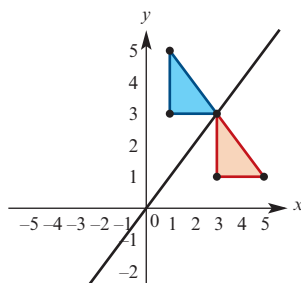
e i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \\ -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \end{bmatrix}$
 ii $\sqrt{2}(y - x) = (x + y)^2$

7 a b $a = 2, b = \frac{\pi}{4}$



c $\begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

8 a i $(3, 1)$ ii $A'(3, 1), B'(5, 1), C'(3, 3)$
 iii



b ii $(-1, -1), (2, 2)$
 iv $(-1, -1), (2, 2),$
 $\left(\frac{1}{2}(-1 + \sqrt{5}), \frac{1}{2}(-1 - \sqrt{5})\right),$
 $\left(\frac{1}{2}(-1 - \sqrt{5}), \frac{1}{2}(-1 + \sqrt{5})\right)$

Problem-solving and modelling

See solutions supplement

Degree-of-difficulty classified questions

Simple familiar questions

1 a $\begin{bmatrix} -1 \\ -10 \end{bmatrix}$ b $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ c $\begin{bmatrix} 4 & 24 \\ 7 & 4 \end{bmatrix}$
 2 $w = -5$ or $w = 2$
 3 $x = 3$
 4 $a = 0, b = -1$

5 a $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ b $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ c $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$
 d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ e $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$ f $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$
 6 a $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ b $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

7 5

Complex familiar questions

1 a $\begin{bmatrix} 1 - 2b & 0 \\ ab & 2a \end{bmatrix}$ b $a = \frac{1}{2}, b = 0$
 2 $(0, 0, 0), (1, 0, -1)$
 3 a ■ Dilation of factor p from the x -axis
 ■ Translation of h units in the x -direction and k units in the y -direction

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$
 c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{q} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -m \\ -n \end{bmatrix} \right)$
 d $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{p}{q} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -m \\ -n \end{bmatrix} \right) + \begin{bmatrix} h \\ k \end{bmatrix}$

4 a i $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ ii $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
 iii $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

b $\begin{bmatrix} 1 - k \sin \theta \cos \theta & k \cos^2 \theta \\ -k \sin^2 \theta & 1 + k \sin \theta \cos \theta \end{bmatrix}$

c A shear of factor k parallel to the line $y = x \tan \theta$. (The positive direction is given by the vector $[1, \theta]$ and the positive side of the line is given by the vector $[1, \frac{\pi}{2} + \theta]$.)

Complex unfamiliar questions

1 a $m^2 - (a + d)m + ad - bc$
 c $m = a - c$
 d i $m = 1$ or $m = -9$ ii $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}, k \in \mathbb{R}$
 iii $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ -\frac{3k}{2} \end{bmatrix}, k \in \mathbb{R}$

2 a $(x, y) \rightarrow (-x, y + 3)$
 b Reflection in the x -axis followed by a translation of 1 unit in the x -direction
 c $(x, y) \rightarrow (x + 1, -y)$
 d $(x, y) \rightarrow (y + 1, x + 1)$
 3 b $y = \frac{3}{2}x$
 4 a $y = x - 1$ b 1st: $x = 0$; 2nd: $y = -x$
 d 1st: $x = 2$; 2nd: $y = 0$; 3rd: $y = -x + 3$

Chapter 20

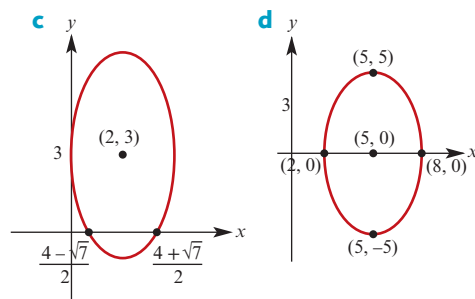
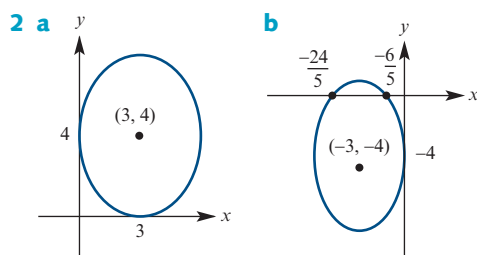
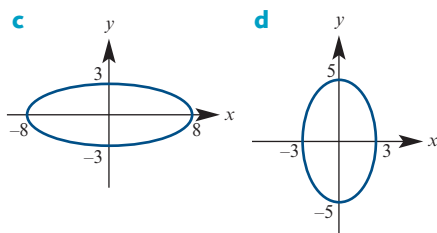
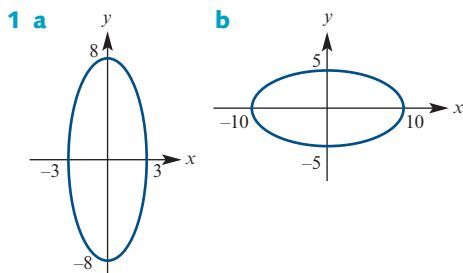
Exercise 20A

- 1 $(x-1)^2 + (y+2)^2 = 4^2$
- 2 $(x+4)^2 + (y-3)^2 = 5^2$
- 3 a $y = -x$
- 4 a $y = \frac{x}{2} + \frac{3}{4}$
- 5 $(0, 3)$ or $(3, 0)$
- 6 $(\frac{9}{10}, \frac{3}{10})$
- 7 $(6, 8)$ or $(\frac{72}{17}, \frac{154}{17})$
- 8 a $2y - x = 1$ b $x + y = 2$ c $P(1, 1)$
- d $(x-1)^2 + (y-1)^2 = 5^2$
- 9 $y = 2x + 1$
- 10 $y = 6$
- 11 a The lines $x = 0$ and $y = 0$
- b $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$
- 12 $(x-4)^2 + y^2 = 4$
- 13 The lines $y = 1$ and $y = 5$
- 14 3

Exercise 20B

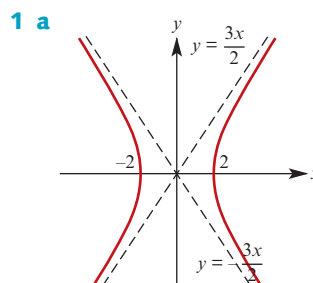
- 1 $y = \frac{x^2}{12}$
- 2 $y = -\frac{x^2}{12} - 1$
- 3 $x = \frac{y^2}{12} - 1$
- 4 a $x = \frac{y^2}{4c}$ b $(\frac{1}{12}, 0)$
- 5 a $y = \frac{1}{2b-2c}(x^2 - 2ax + a^2 + b^2 - c^2)$
- b $y = -\frac{1}{2}(x^2 - 2x - 4)$
- 6 $y = -1$ or $y = 19$
- 7 $(2, 1 + \sqrt{3})$ or $(2, 1 - \sqrt{3})$

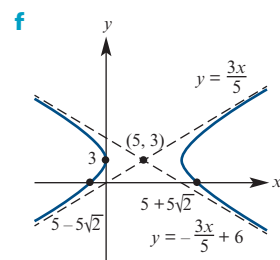
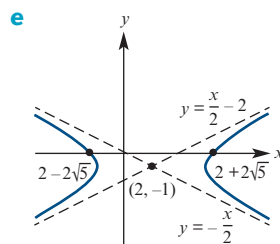
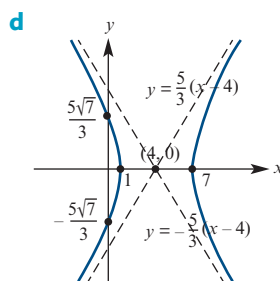
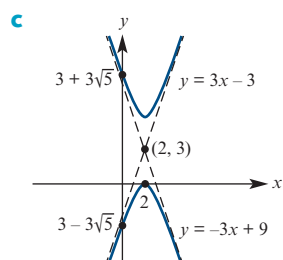
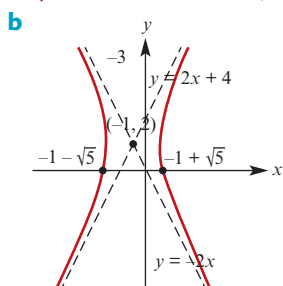
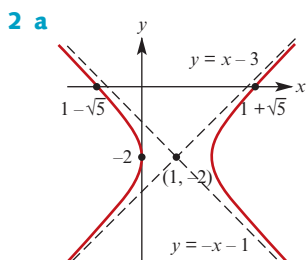
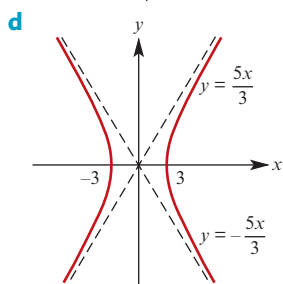
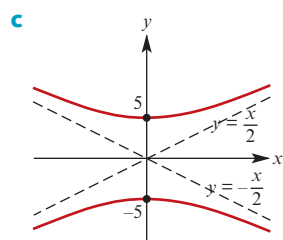
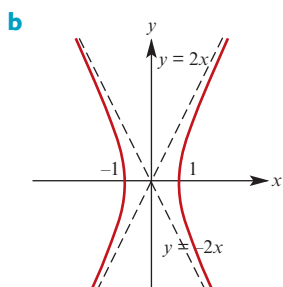
Exercise 20C



- 3 a $\frac{x^2}{25} + \frac{y^2}{16} = 1$ b $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$
- c $\frac{(x+1)^2}{4} + (y-1)^2 = 1$
- 4 $\frac{x^2}{4} + \frac{y^2}{3} = 1$
- 5 $\frac{x^2}{5} + \frac{y^2}{9} = 1$
- 6 $\frac{(x-4)^2}{16} + \frac{y^2}{12} = 1$
- 7 $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Exercise 20D





3 $\frac{x^2}{9} - \frac{y^2}{7} = 1$

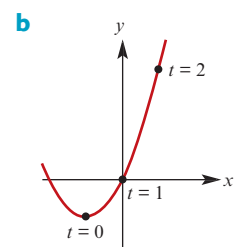
4 $5x^2 - 4y^2 = 20$

5 $\frac{(x+3)^2}{16} - \frac{y^2}{48} = 1$

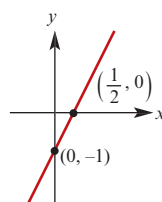
6 $\frac{(y+5)^2}{4} - \frac{x^2}{12} = 1$

Exercise 20E

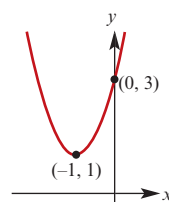
1 a $y = x^2 + 2x$

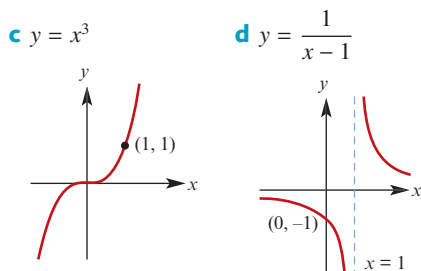


2 a $y = 2x - 1$



b $y = 2(x+1)^2 + 1$

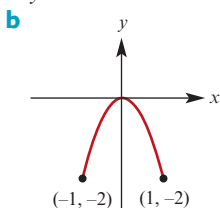




- 3 a** $x^2 + y^2 = 2^2$
b $\frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{2^2} = 1$
c $x = 3 \cos t - 3$ and $y = 3 \sin t + 2$
 (other answers are possible)
d $x = 3 \cos t - 2$ and $y = 2 \sin t + 1$
 (other answers are possible)

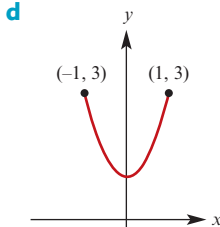
- 4** $x = t$ and $y = 3t + 1$
 (other answers are possible)

- 5 a** $y = -2x^2$ where $-1 \leq x \leq 1$

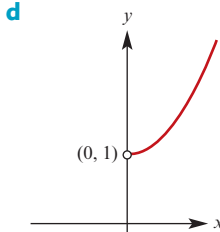


- 6** $\left(-\frac{3}{5}, -\frac{4}{5}\right), \left(\frac{3}{5}, \frac{4}{5}\right)$

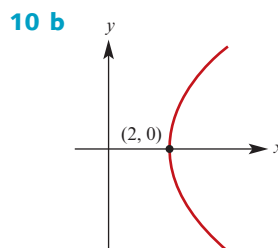
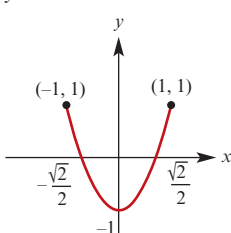
- 7 a** $y = 2x^2 + 1$ **b** $-1 \leq x \leq 1$ **c** $1 \leq y \leq 3$



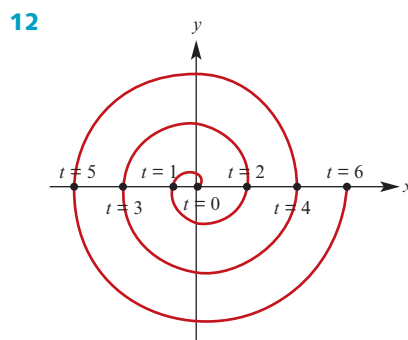
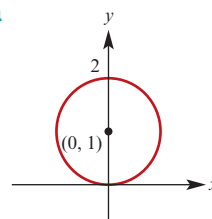
- 8 a** $y = x^2 + 1$ **b** $x > 0$ **c** $y > 1$



- 9** $y = -1 + 2x^2$ where $-1 \leq x \leq 1$

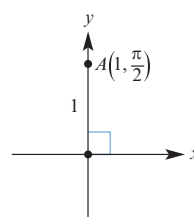


- 11 a**
- c** $x = \frac{2t}{t^2 + 1}$
 $y = \frac{2}{t^2 + 1}$

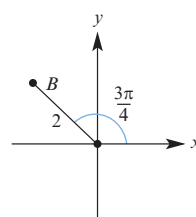


Exercise 20F

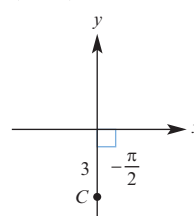
- 1 a** $(0, 1)$



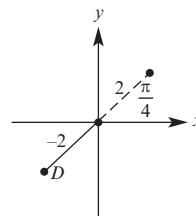
- b** $(-\sqrt{2}, \sqrt{2})$



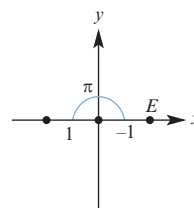
- c** $(0, -3)$



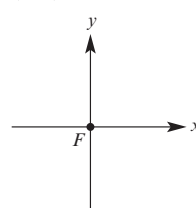
- d** $(-\sqrt{2}, -\sqrt{2})$



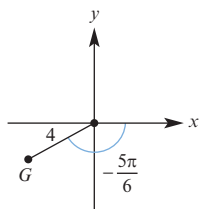
- e** $(1, 0)$



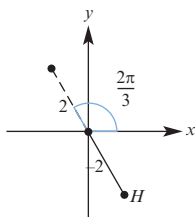
- f** $(0, 0)$



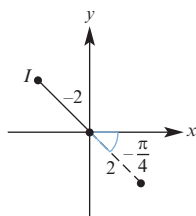
g $(-2\sqrt{3}, -2)$



h $(1, -\sqrt{3})$



i $(-\sqrt{2}, \sqrt{2})$



2 a $(\sqrt{2}, -\frac{\pi}{4}), (-\sqrt{2}, \frac{3\pi}{4})$

b $(2, \frac{\pi}{3}), (-2, \frac{4\pi}{3})$

c $(2\sqrt{2}, -\frac{\pi}{4}), (-2\sqrt{2}, \frac{3\pi}{4})$

d $(2, -\frac{3\pi}{4}), (-2, \frac{\pi}{4})$

e $(3, 0), (-3, \pi)$

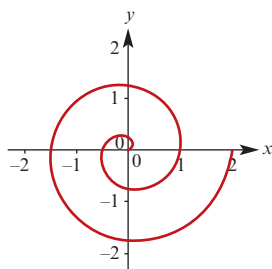
f $(2, -\frac{\pi}{2}), (-2, \frac{\pi}{2})$

3 $\sqrt{7}$

4 $PQ = \sqrt{(r_1)^2 + (r_2)^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$

Exercise 20G

1



2 a $r = \frac{4}{\cos \theta}$

c $r = \tan \theta \sec \theta$

e $r^2 = \frac{1}{\cos(2\theta)}$

3 a $x = 2$

c $y = x$

4 a $(x-3)^2 + y^2 = 9$

c $(x+3)^2 + y^2 = 9$

5 $(x-a)^2 + y^2 = a^2$

6 a Equation $x = a$

b $r^2 = \frac{1}{\cos \theta \sin \theta}$

d $r = 3$ or $r = -3$

f $r = \frac{5}{2 \cos \theta - 3 \sin \theta}$

b $x^2 + y^2 = 2^2$

d $3x - 2y = 4$

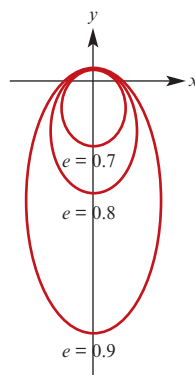
b $x^2 + (y-2)^2 = 4$

d $x^2 + (y+4)^2 = 16$

b $r = \frac{a}{\sin \theta}$

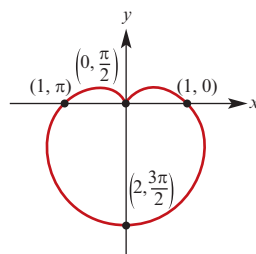
Exercise 20H

1 a

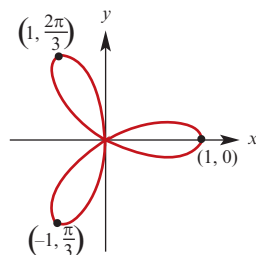


b Ellipse becomes larger and narrower

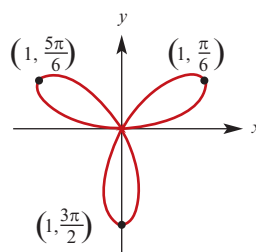
2 a



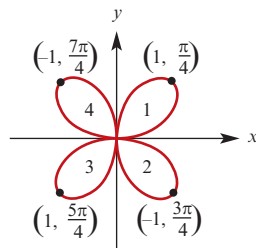
3 a



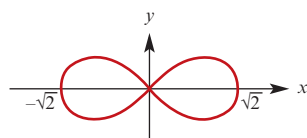
b



4 a



5 a



b $(x^2 + y^2)^2 = 2x^2 - 2y^2$

Chapter 20 review

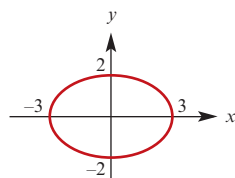
Technology-free questions

1 $y = \frac{x}{3}$

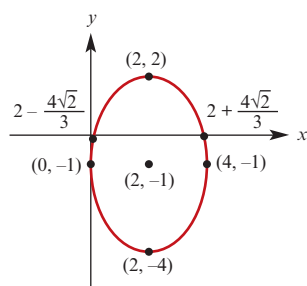
2 $(x-3)^2 + (y-2)^2 = 6^2$

3 $C(-2, 4), r = \sqrt{20}$

4 a

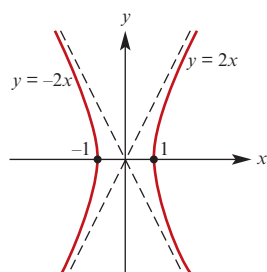


b

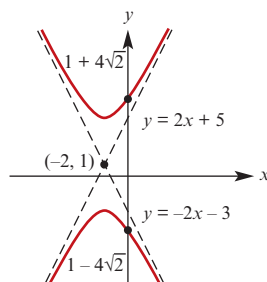


5 $C(-2, 0)$; Intercepts $(0, 0), (-4, 0)$

6 a



b



7 $\frac{(x-2)^2}{4} - \frac{(y-5)^2}{12} = 1$

8 a $y = 4 - 2x$ b $x^2 + y^2 = 2^2$

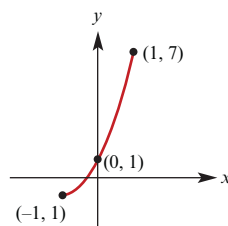
c $\frac{(x-1)^2}{3^2} + \frac{(y+1)^2}{5^2} = 1$

d $y = 1 - 3x^2$ where $-1 \leq x \leq 1$

9 a $y = 2(x+1)^2 - 1$ b $-1 \leq x \leq 1$

c $-1 \leq y \leq 7$

d



10 $(-\sqrt{2}, \sqrt{2})$

11 $(4, -\frac{\pi}{3}), (-4, \frac{2\pi}{3})$

12 $r = \frac{5}{2 \cos \theta + 3 \sin \theta}$

13 $x^2 + (y-3)^2 = 9$

14 a $x^2 + y^2 = 5^2$ b $y = \sqrt{3}x$ c $y = 3$

d $3y + 4x = 2$ e $y = \frac{1}{2x}$

Multiple-choice questions

1 A 2 D 3 C 4 D 5 C

6 C 7 E 8 B 9 D

Extended-response questions

1 a $y = 2x - \frac{9}{2}$

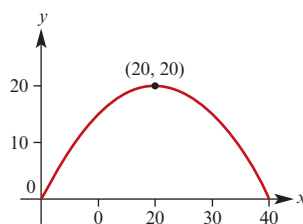
b $(x-8)^2 + (y+1)^2 = 20$

2 a $y = \frac{x^2}{12} + 1$ b $\frac{x^2}{12} + \frac{(y-6)^2}{16} = 1$

c $\frac{(y+4)^2}{16} - \frac{x^2}{48} = 1$

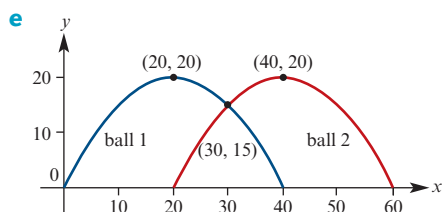
3 a $y = \frac{1}{20}x(40-x)$

b



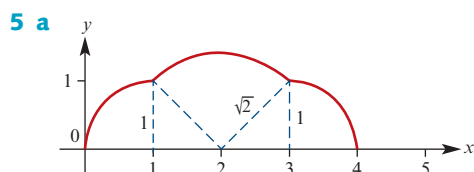
c 20 metres

d $y = -\frac{1}{20}(x-20)(x-60)$

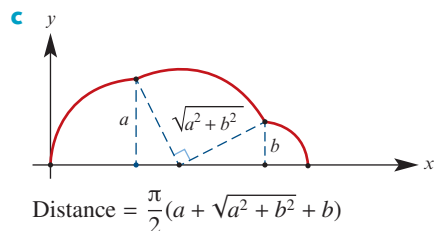


f (30, 15)

g Yes (same position at same time)



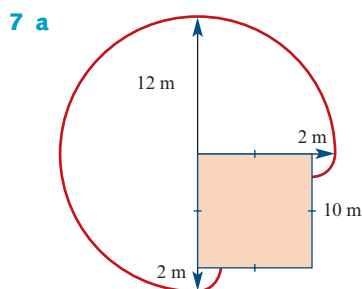
b Distance = $\frac{\pi}{2}(2 + \sqrt{2})$



d Area = $\frac{\pi}{2}(a^2 + b^2) + ab$

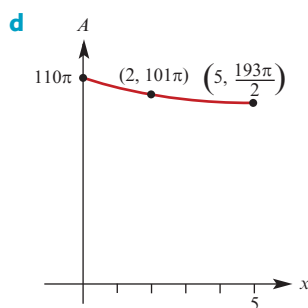
6 c $x = t$ and $y = -t + 3$

e $k > \frac{8}{\sqrt{5}}$ or $k < -\frac{8}{\sqrt{5}}$



b $110\pi \text{ m}^2$

c $A(x) = \begin{cases} \frac{3\pi x^2}{4} - 6\pi x + 110\pi, & 0 \leq x \leq 2 \\ \frac{\pi x^2}{2} - 5\pi x + 109\pi, & 2 < x \leq 5 \end{cases}$



e i $x = 0$ **ii** $x = 5$