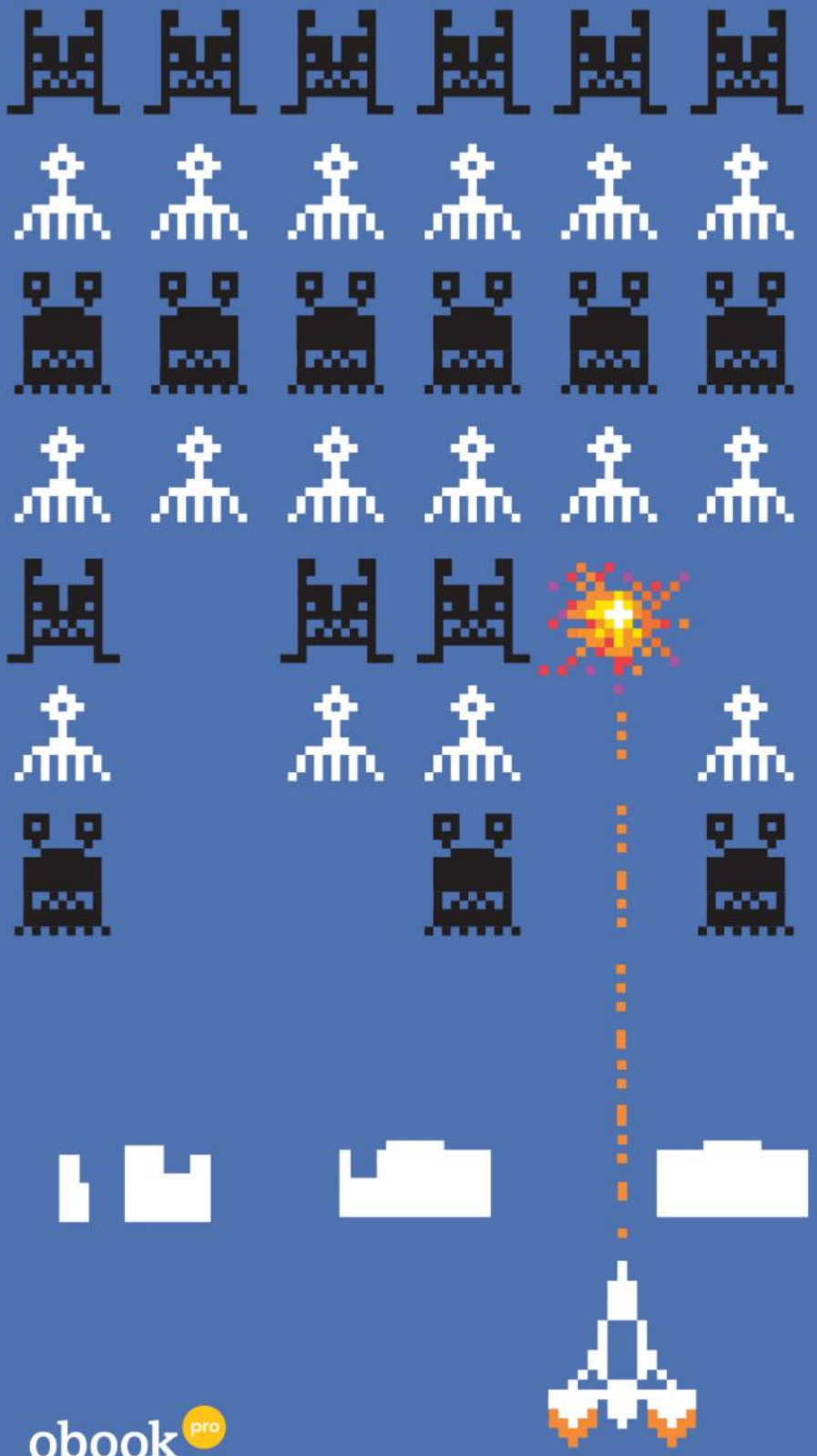
MATHEMATICS

OXFORD MATHS 10



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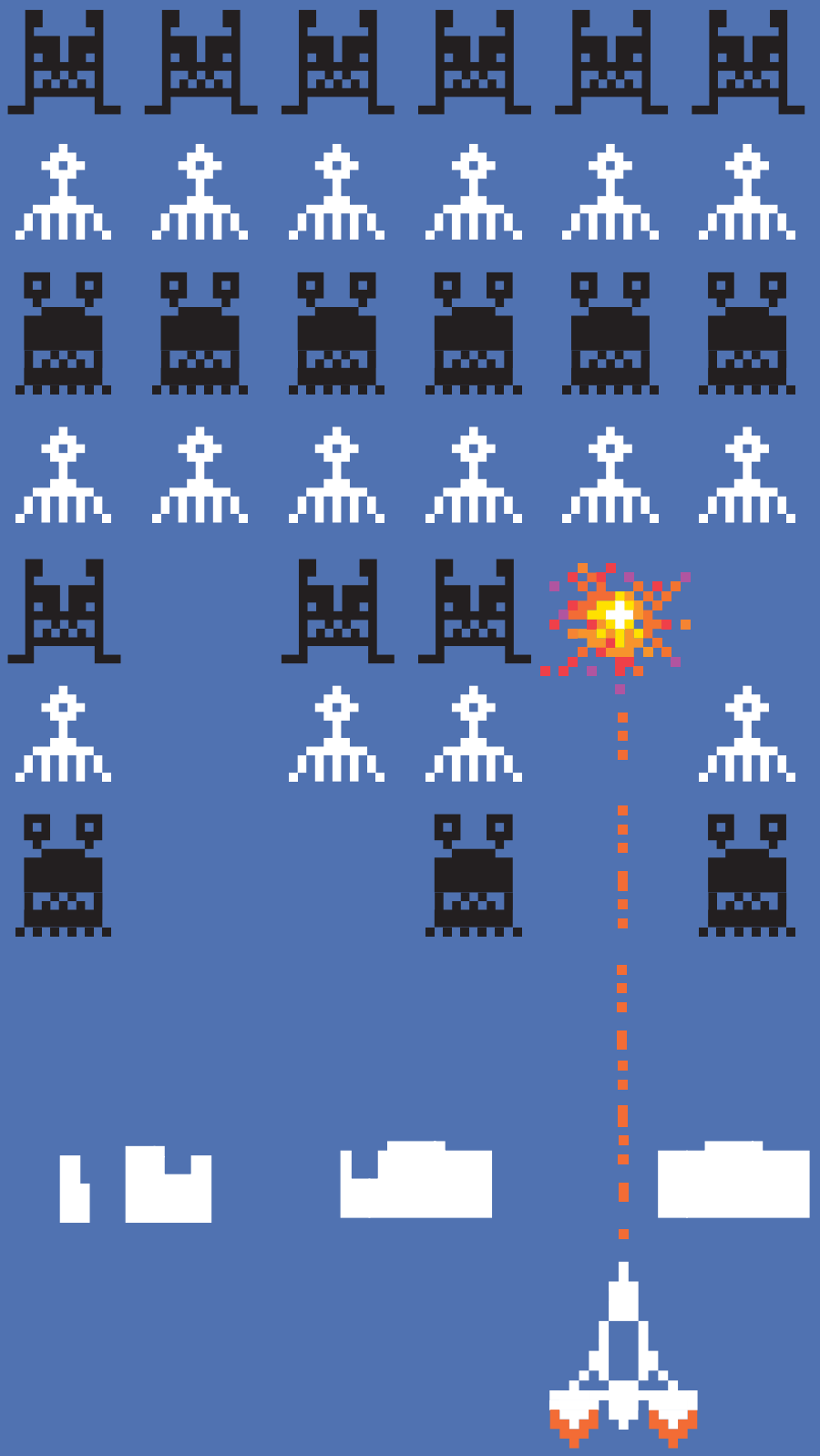

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Aboriginal and Torres Strait Islander peoples are advised that this publication may include images or names of people now deceased.

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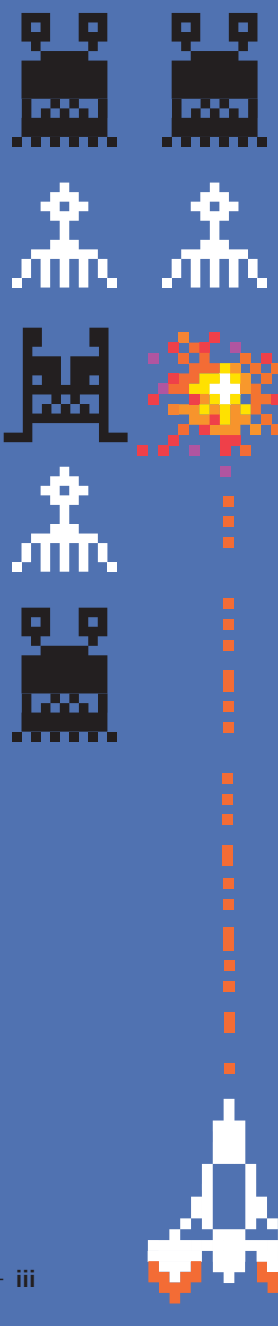

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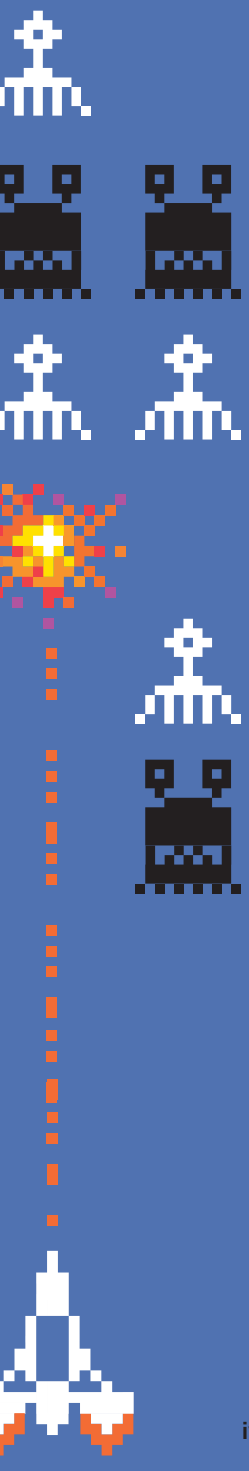

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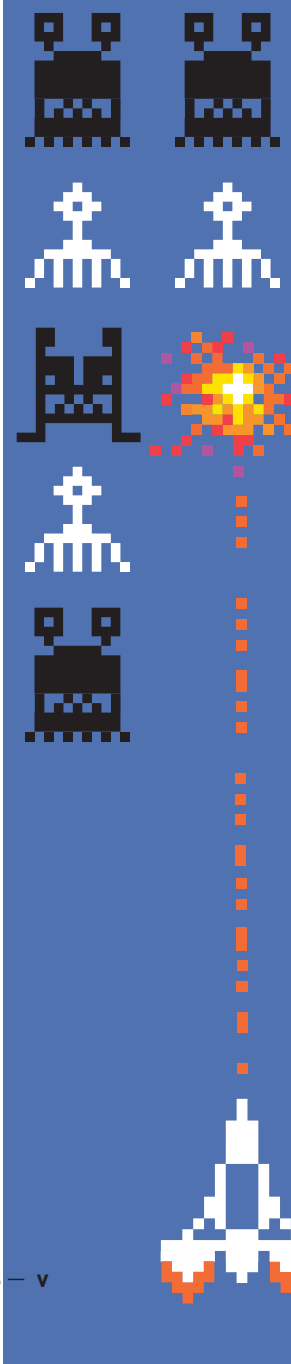


Introducing Oxford Maths 7–10

Oxford Maths 7–10 Australian Curriculum utilises an innovative suite of print and digital resources to guide students on a focused mathematics journey. The series makes maths accessible to students with differing levels of understanding, increasing engagement by giving learners the opportunity to achieve success at their own skill level while also providing comprehensive syllabus coverage.

Key features of Student Books

- Complete access to all digital resources available on Student obook pro.
- Australian Maths Trust (AMT) spreads offer unique questions designed to challenge students and build engagement.
- STEAM projects encourage inter-disciplinary thinking.
- Semester reviews provide an opportunity to revise key concepts from each semester.
- An illustrated glossary of newly introduced mathematical terms.

Each chapter opens with:

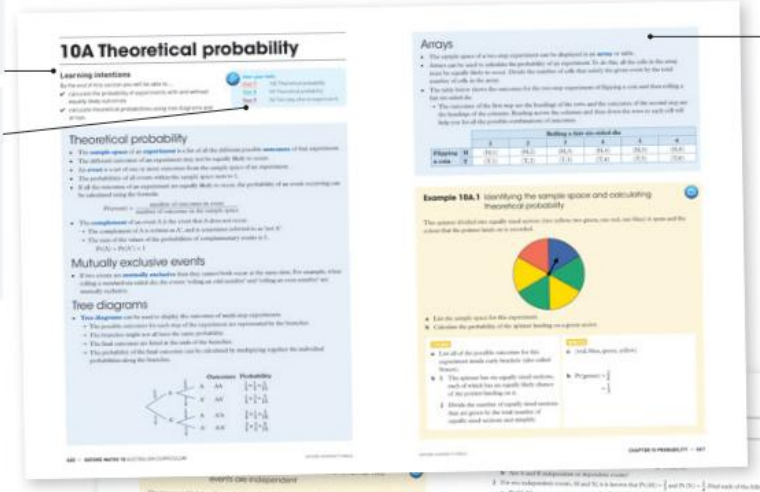
- **Prerequisite skills** with reference to an online diagnostic pre-test and interactive skillsheets.
- **Curriculum links** to all relevant content descriptions in the Australian Curriculum.
- **Materials** used to complete the exercises.

Learning intentions

- Signpost the foundational skills being developed in each section.

Inter-year links

- Provide easy access to support and extension material from each of the 7–10 Student Books as students build knowledge year on year.

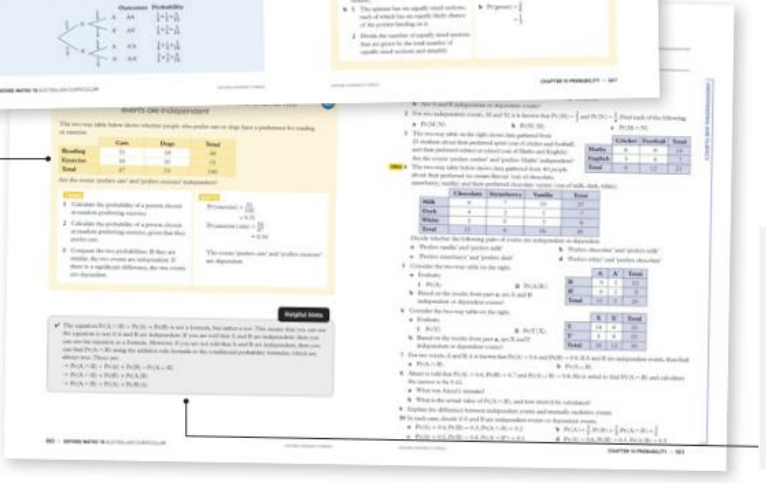


New theory

- Backed by the latest pedagogical research to promote engagement with the material.
- Filled with precise diagrams that bring key concepts to life, and aid understanding.

Worked examples

- Outline a step-by-step thought process for solving essential questions with direct reference to the exercises.



Helpful hints

- Provide additional strategies for tackling problems.
- Highlight important elements of the theory.
- Point out common misconceptions.

Differentiated learning pathways

- Each exercise is separated into three pathways, tailoring for students of all skill levels.
- Each pathway can be assigned based on results of the diagnostic pre-tests that are recommended at the beginning of every chapter.

Problem solving and reasoning

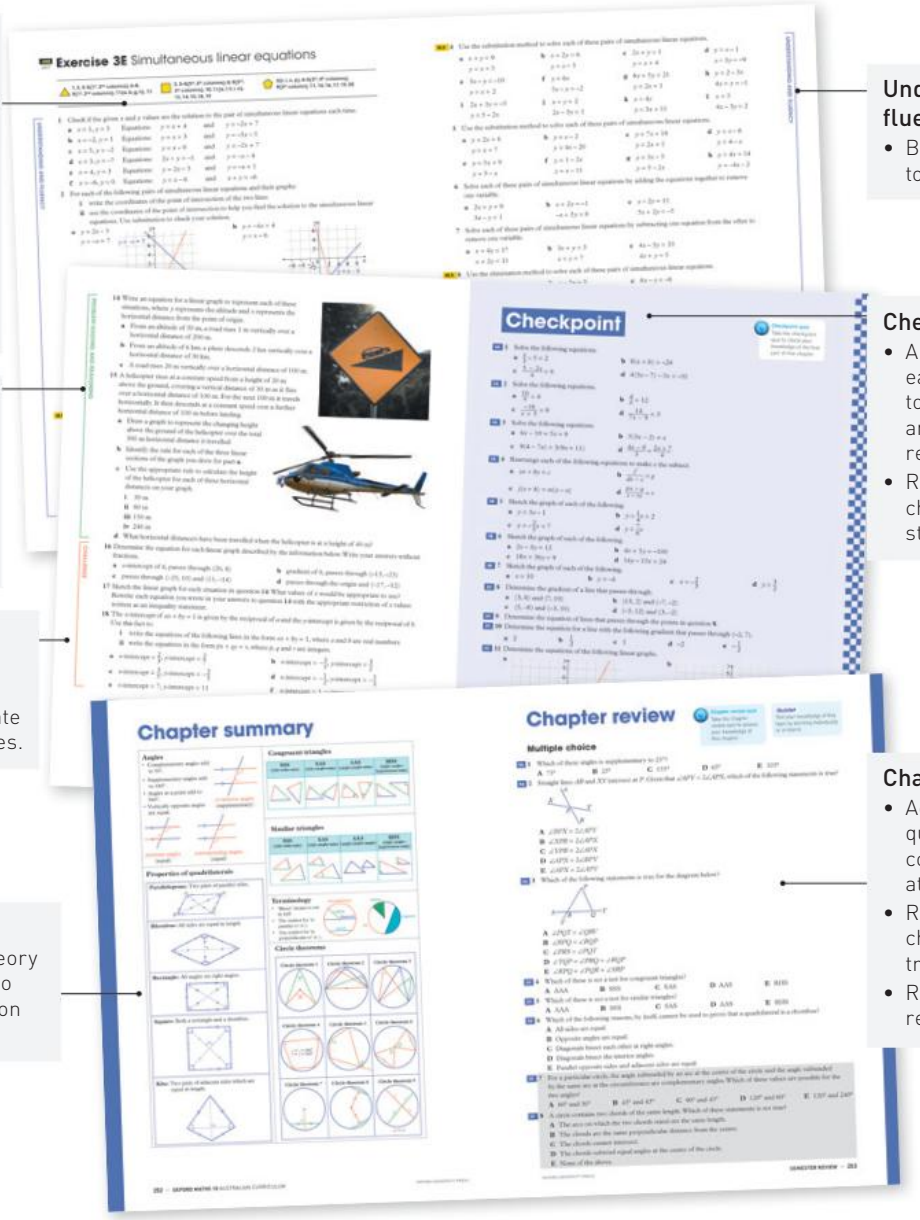
- Comprehensive exercises bring together new ideas and provide engaging contexts from real-world problems.

Challenge

- Advanced exercises designed to build engagment and anticipate future learning outcomes.

Chapter summary

- Condenses all the theory from each section into one accessible revision page.



Understanding and fluency

- Basic exercises dedicated to practising key concepts.

Checkpoint

- A section in the middle of each chapter dedicated to summarising key skills and encouraging memory retention.
- Reference to an online checkpoint quiz to gauge student progress.

Chapter reviews

- Additional practice questions to further consolidate understanding at the end of each chapter.
- Reference to an online chapter review quiz to track results.
- Reference to Quizlet test to revise new terminology.

Integrated STEAM projects

- Take the hard work out of cross-curricular learning with engaging STEAM projects. Two fully integrated projects are included at the end of each book in the series, and are scaffolded and mapped to the Science, Maths and Humanities curricula. The same projects also feature in the corresponding Oxford Humanities and Oxford Science series to assist cross-curricular learning.

Problem solving through design thinking

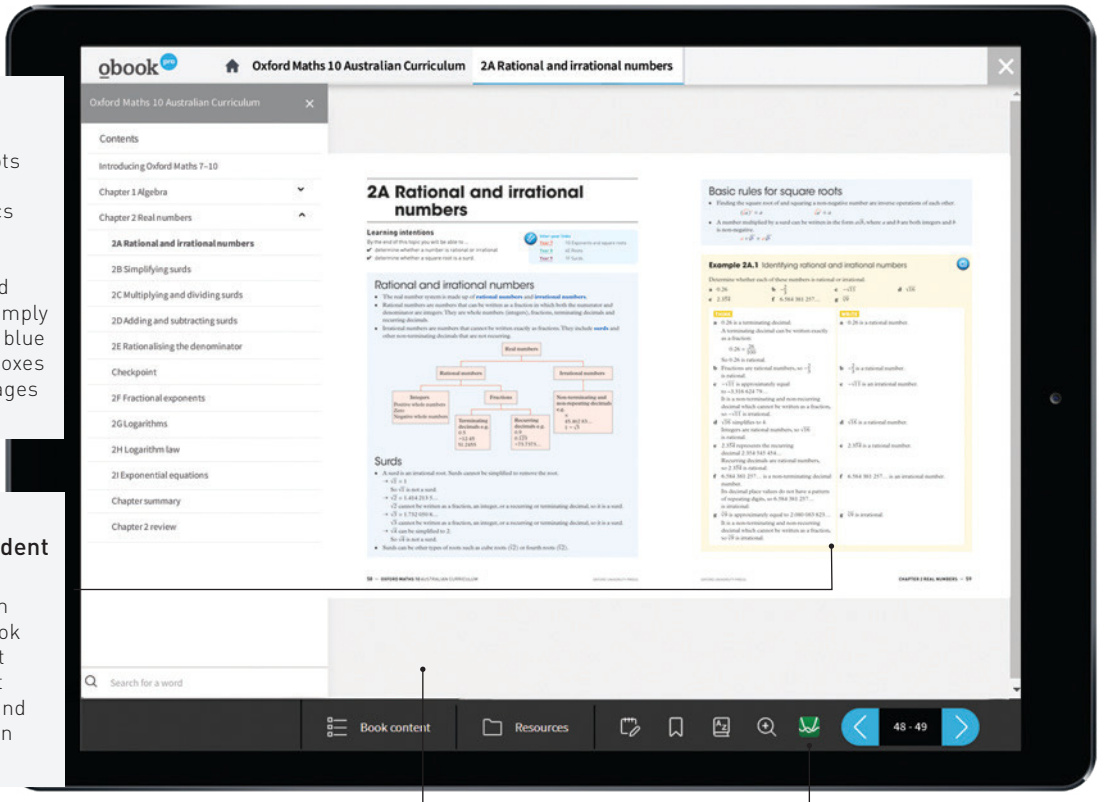
- Each STEAM project investigates a real-world problem that students are encouraged to problem-solve using design thinking.

Full digital support

- Each STEAM project is supported by a wealth of digital resources, including student booklets (to scaffold students through the design-thinking process of each project), videos to support key concepts and skills, and implementation and assessment advice for teachers.

Key features of Student obook pro

- Student obook pro is a completely digital product delivered via Oxford's online learning platform, **Oxford Digital**.
- It offers a complete digital version of the Student Book with interactive note-taking, highlighting and bookmarking functionality, allowing students to revisit points of learning.
- A complete ePDF of the Student Book is also available for download for offline use and read-aloud functionality.



Integrated digital resources

- Integrated hotspots allow students to access diagnostics tests, quizzes, interactive skill sheets, videos and inter-year links simply by clicking on the blue digital resource boxes throughout the pages of the book.

Complete digital version of the Student Book

- The digital version of the Student Book is true to the print version, making it easy to navigate and transition between print and digital.

Toolbar features

- Notes can be added and saved to the text by simply selecting and highlighting.
- Bookmarks can be saved to any page.
- *Australian Concise Oxford Dictionary* can provide immediate definitions to any word within the text.

Desmos integration

- Our partership with Desmos allows students to access a suite of calculator tools as they read through the text, providing convenient graphical support as well as the opportunity to investigate plane geometry and Cartesian coordinates.

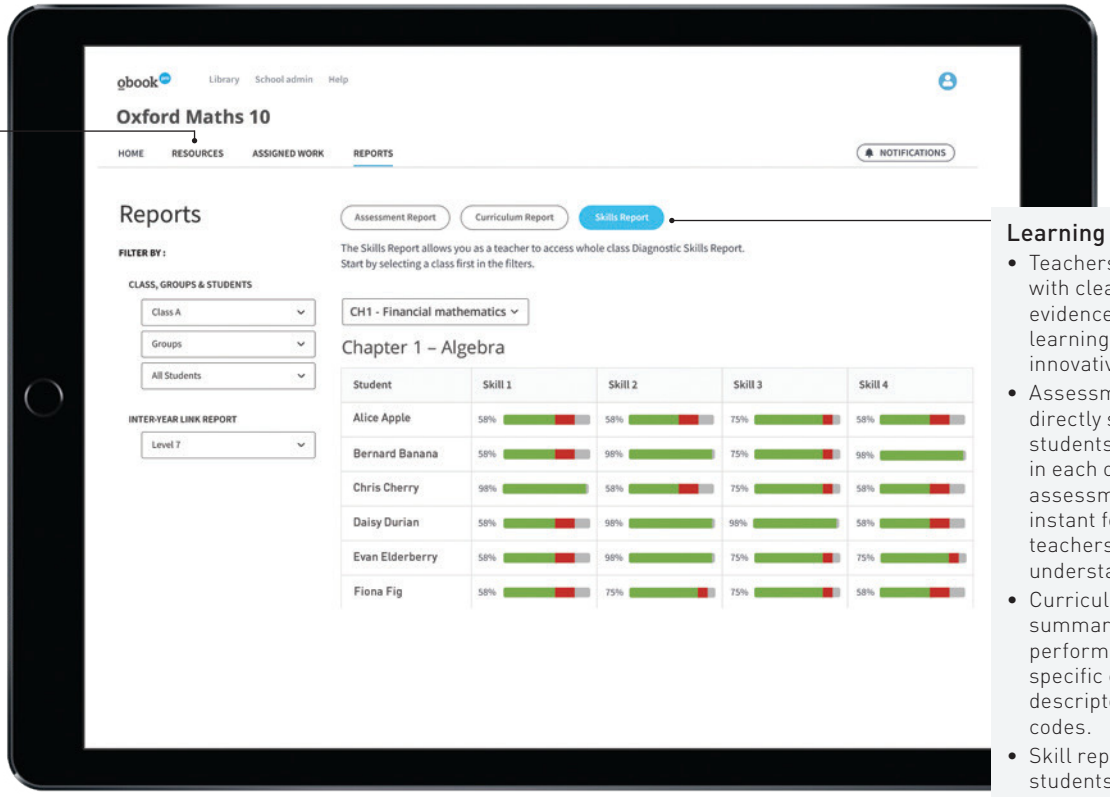
Benefits for students

- Integrated *Australian Concise Oxford Dictionary* look-up feature
- Targeted instructional videos for every worked example question
- Groundwork resources to support assumed knowledge
- Interactive assessments to consolidate understanding
- Auto-marked practice exam question sets
- Integrated Quizlet sets, including real-time online quizzes with live leaderboards
- Access to online assessment results to track progress.

Key features of Teacher obook pro

obook pro

- Teacher obook pro is a completely digital product delivered via Oxford's online learning platform, **Oxford Digital**.
- Each chapter and topic of the Student Book is accompanied by full teaching support, including assessment reporting, worked solutions, chapter tests, detailed teacher notes and lesson plans.
- Teachers can use their Teacher obook pro to share notes and easily assign resources or assessments to students, including due dates and email notifications.



Learning pathway reports

- Teachers are provided with clear and tangible evidence of student learning progress through innovative reports.
- Assessment reports directly show how students are performing in each online interactive assessment, providing instant feedback for teachers about areas of understanding.
- Curriculum reports summarise student performance against specific curriculum content descriptors and curriculum codes.
- Skill reports indicate the students' understanding of a specific skill in mathematics.

Additional resources

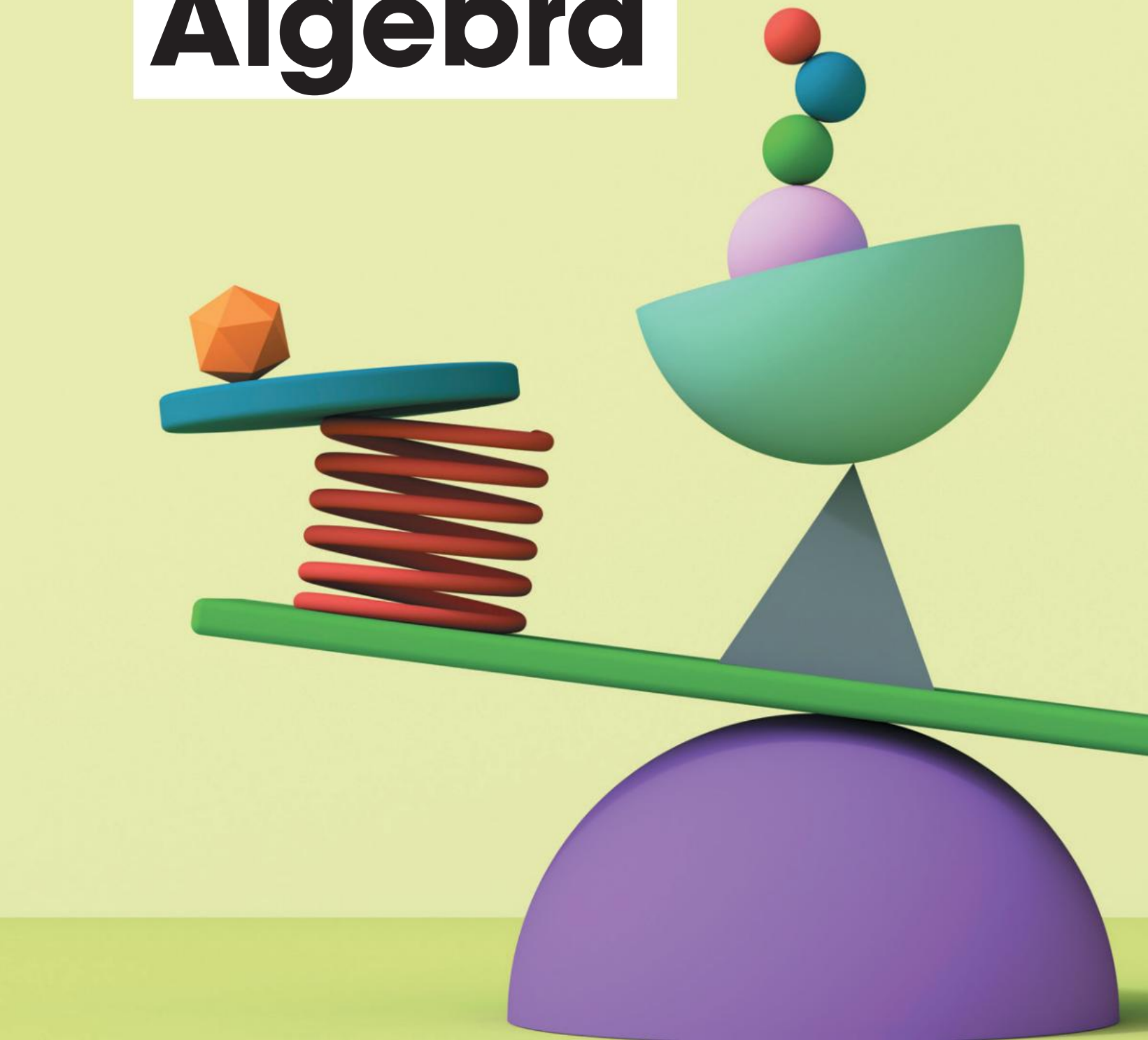
- Each chapter of the Student Book is accompanied by additional interactive skillsheets, worksheets, investigations and topic quizzes to help students progress.

Benefits for teachers

- Diagnostic pre-tests and chapter tests that track students' progress against Study Design key knowledge, providing detailed learning pathway reports that differentiate each student's ability in each skill
- Assign reading and assessments to students either individually, or in groups – administration is taken care of!
- Ability to set-up classes, monitor student progress and graph results
- Worked solutions for every Student Book question
- Detailed teacher notes, teaching programs and lesson plans.

1

Algebra



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1B Negative exponents
1C Simplifying
1D Expanding
1E Algebraic fractions
1F Factorising
1G Factorising quadratic expressions
1H Completing the square
1I Factorising non-monic quadratic expressions

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Exponents
- ✔ Roots
- ✔ Highest common factor
- ✔ Like terms
- ✔ Equivalent fractions

Curriculum links

- Expand, factorise and simplify expressions and solve equations algebraically, applying exponent laws involving products, quotients and powers of variables, and the distributive property (AC9M10A01)

Optional content

- Simplification of combinations of linear expressions with rational coefficients and the solution of related equations
- Algebraic representations of quadratic functions of the form $f(x) = ax^2 + bx + c$ where a, b and c are non-zero integers, and their transformation to the form $f(x) = a(x + h)^2 + k$ where h and k are non-zero rational numbers, and the solution of related equations

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1A Exponents

Learning intentions

By the end of this topic you will be able to ...

✔ use the exponent laws to simplify algebraic expressions.

Inter-year links

Year 7	1G Exponents and square roots
Year 8	4A Exponents
Year 9	1A Exponents

Exponent notation

- **Exponent notation**, or **exponent form**, is used for indicating the number of times a **base** number is multiplied by itself repeatedly. The number of repeated multiplications corresponds to the **exponent** (also called the **index**).

exponent/index

$3^4 = 3 \times 3 \times 3 \times 3 = 81$

base → exponent form | expanded form | basic numeral

exponent/index

$a^3 = a \times a \times a$

base → exponent form | expanded form

Exponent laws

- The **exponent laws** are used for simplifying expressions that include terms written in exponent form.
- The exponent laws for multiplying or dividing terms in exponent form only apply when both terms have the same base.
- Any non-zero base raised to the power of zero is equal to one.

Exponent law	Algebraic form	Example
Product of powers law	$a^m \times a^n = a^{m+n}$	$2^3 \times 2^5 = 2^{3+5} = 2^8$
Quotient of powers law	$a^m \div a^n = a^{m-n}$	$2^5 \div 2^3 = 2^{5-3} = 2^2$
Power of a power law	$(a^m)^n = a^{m \times n}$	$(2^3)^5 = 2^{3 \times 5} = 2^{15}$
The zero exponent	$a^0 = 1$	$2^0 = 1$

- We can use a combination of exponent laws to simplify complex expressions.

$$(a \times b)^m = \underbrace{a \times b \times \ldots \times a \times b}_{m \text{ times}} = a^m \times b^m \qquad (2 \times 3)^5 = 2^5 \times 3^5$$

$$\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b} \times \ldots \times \frac{a}{b}}_{m \text{ times}} = \frac{a^m}{b^m} \qquad \left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$$

Example 1A.1 Using the exponent laws

Use the appropriate exponent law to simplify each of these expressions.

a $(g^3)^4$ **b** $\left(\frac{w}{3}\right)^3$ **c** $3k^2 \times 6k^5$ **d** $8a^{12} \div 16a^7$

THINK

a Apply the power of a power law to expand the brackets by multiplying the exponents.

WRITE

a $(g^3)^4 = g^{3 \times 4}$

$= g^{12}$

b **1** Expand the brackets by applying the exponent to the numerator and the denominator.

2 Simplify.

c **1** Multiply 3 and 6.

2 Since the exponents have the same base, apply the product of powers law by adding the exponents.

3 Simplify.

d **1** Write the expression as a fraction.

2 Divide 8 and 16 by their highest common factor (HCF), which is 8.

3 Since the exponents have the same base, apply the quotient of powers law by subtracting the exponents.

4 Simplify.

b $\left(\frac{w}{3}\right)^3 = \frac{w^3}{3^3}$

$= \frac{w^3}{27}$

c $3k^2 \times 6k^5 = 18 \times k^2 \times k^5$

$= 18 \times k^{2+5}$

$= 18k^7$

d $8a^{12} \div 16a^7 = \frac{8a^{12}}{16a^7}$

$= \frac{\cancel{8}^{1}a^{12}}{\cancel{16}^{2}a^7}$

$= \frac{a^{12}}{2a^7}$

$= \frac{a^{12-7}}{2}$

$= \frac{a^5}{2}$

Example 1A.2 Combining exponent laws to simplify algebraic expressions

Use the appropriate exponent laws to simplify these expressions.

a $3a^4 \times 10a \div 6a^2$ **b** $(2z^7)^3$ **c** $\frac{x^6 \times 6x^2}{8(x^4)^2}$

THINK

a **1** Write the expression as a fraction.

2 Simplify the numerator by multiplying 3 and 10 and using the product of powers law to add the exponents.

3 Divide 30 and 6 by their HCF, which is 6.

4 Apply the quotient of powers law to subtract the exponents.

b **1** Expand the brackets.

2 Apply the power of a power law to multiply the exponents and simplify.

c **1** Simplify the numerator by applying product of powers law. Simplify the denominator by applying power of a power law.

2 Divide 6 and 8 by their HCF, which is 2.

3 Divide the numerator and denominator by the common factor x^8.

WRITE

a $3a^4 \times 10a \div 6a^2 = \frac{3a^4 \times 10a}{6a^2}$

$= \frac{30a^5}{6a^2}$

$= \frac{\cancel{30}^{5}a^5}{\cancel{6}^{1}a^2}$

$= \frac{5a^5}{a^2}$

$= 5a^{5-2}$

$= 5a^3$

b $(2z^7)^3 = 2^3 \times (z^7)^3$

$= 8 \times z^{7 \times 3}$

$= 8z^{21}$

c $\frac{x^6 \times 6x^2}{8(x^4)^2} = \frac{6x^8}{8x^8}$

$= \frac{\cancel{6}^{3}x^8}{\cancel{8}^{4}x^8}$

$= \frac{3x^8}{4x^8}$

$= \frac{3\cancel{x^8}}{4\cancel{x^8}}$

$= \frac{3}{4}$

Helpful hints

- ✔ Remember, the product and quotient of powers laws only apply if the expressions have the same base.
- ✔ Expanding brackets raised to an exponent only works if the brackets contain division and multiplication. You cannot apply the same method to expand brackets containing addition and subtraction!
  $(a + b)^m \neq a^m + b^m$ and $(a - b)^m \neq a^m - b^m$
  In order to expand these expressions, you will need to use the distributive law covered in section **1D**.
- ✔ Avoid errors in your working out by applying the exponent laws one at a time.

ANS p534

Exercise 1A Exponents

▲ 1–7, 8(a, b), 10(a–e), 11 ■ 1(e–h), 3(d, g–i), 4–6(2nd, 3rd columns), 7, 8(b, d), 9–10(a–e), 12, 17 ⬟ 6–7(f–i), 8, 9–10(f–i), 13–17

UNDERSTANDING AND FLUENCY

1 Use appropriate exponent laws to simplify these expressions.

a $a^7 \div a^3$ **b** $(b^4)^5$ **c** $c^9 \times c^6$ **d** $d^3 \div d^2$

e $y^5 \times y^5$ **f** $(m^7)^2$ **g** $n^{15} \div n$ **h** $h \times h^6$

i $\frac{x^{11}}{x^5}$ **j** $(p^6)^3$ **k** $\frac{w^4}{w}$ **l** $a^2 \times a \times a^7$

2 a What is the value of a non-zero term expressed in exponent form raised to the power of zero?

b Write the value of each of the following.

i 3^0 **ii** 100^0 **iii** m^0 **iv** $5^9 \div 5^9$

3 Use the property $a^0 = 1$ to simplify each of these expressions.

a $3a^0$ **b** $(3a)^0$ **c** $8m^0$ **d** $(8m)^0$

e $(-5x)^0$ **f** $a^0 + b^0$ **g** $j^0 - k^0$ **h** $p^0 + p^0$

i $2c^0 + 4g^0$ **j** $w^0 + x^0 - y^0$ **k** $(m + n)^0$ **l** x^0

4 Use the property $(a \times b)^m = a^m \times b^m$ to write each of these expressions without brackets.

a $(a \times b)^5$ **b** $(k \times p)^9$ **c** $(4 \times c)^3$

d $(7 \times y)^{13}$ **e** $(xy)^4$ **f** $(cd)^7$

g $(3w)^2$ **h** $(6g)^6$ **i** $(2p)^8$

5 Use $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ to write each of these expressions without brackets.

a $\left(\frac{a}{b}\right)^4$ **b** $\left(\frac{f}{g}\right)^7$ **c** $\left(\frac{k}{2}\right)^6$

d $\left(\frac{5}{x}\right)^6$ **e** $\left(\frac{d}{c}\right)^{-2}$ **f** $\left(\frac{m}{3}\right)^{-5}$

1A.1 **6** Use the appropriate exponent laws to simplify these expressions.

a $5x^2 \times x^3$ **b** $2a^2 \times 3a^5$ **c** $(3x)^2$

d $(2x^4)^3$ **e** $\frac{4a^5}{a^2}$ **f** $\frac{12x^8}{6x^3}$

g $\frac{4x^3}{2x^3}$ **h** $3x^6 \times 2x^2 \times 5x^4$ **i** $\frac{(6a^4)^2}{12a^5}$

1A.2 **7** Use the appropriate exponent laws to simplify these expressions.

a $\frac{4y^2 \times 12y^5}{6y^3}$ **b** $3(2x^3)^3$ **c** $(3k)^2 \times (2k)^3$

d $(3b^5)^2 \times 5b^2$ **e** $3a^3 \times (2a^4)^2 \times 5a^9$ **f** $15c^4 \times 6c^3 \div (3c)^2$

g $\frac{(2n^2)^2 \times 5n^3}{10n^5}$ **h** $\frac{3v^2}{4} \times \frac{8v^4}{6}$ **i** $\left(\frac{5m^5}{2}\right)^2 \times 4m^3$

8 Use the exponent laws to simplify each expression. Write your answers in exponent form.

a $\dfrac{p^4 \times (p^5)^3}{p^{10}}$ **b** $\dfrac{2(t^2)^8 \times (3t^5)^3}{3t^{11}}$

c $\dfrac{(5m^{11}n^{10})^8 \times (5mn^6)^2}{(5m^9n^7)^5 \times (5m^2n^3)^3}$ **d** $\dfrac{(8j^5p)^0 \times (j^0p^4)^3}{(jp^2)^6}$

9 Use the exponent laws to decide whether each of these statements is true or false. Explain your reasoning. For each false statement, change the right-hand side to make the statement true.

a $x^7 \times x \times x^7 = x^{14}$ **b** $(5a)^3 = 5 \times a^3$ **c** $-k^0 = -1$

d $a^2b^3 \times a^3b^4 = a^{12}b^{12}$ **e** $x^7y^4 \div y^6 = \dfrac{x^7}{y^2}$ **f** $\left(\dfrac{m}{n}\right)^5 = \dfrac{m^5}{n}$

g $\dfrac{w^6 \times w^6}{w^{12}} = 0$ **h** $\dfrac{(b^5)^4 \times b^2}{(b^3)^7} = b$ **i** $\dfrac{a^8b^5}{a^4b^3} \times \dfrac{a^2b^2}{a^9b^4} = a^3$

10 Find the value of x that makes the statement true.

a $a^x \times a^3 = a^{12}$ **b** $b^x = 1$ **c** $c^x \div c^6 = c^4$

d $(d^3)^x = d^{12}$ **e** $\left(\dfrac{m}{n}\right)^x = \dfrac{m^7}{n^7}$ **f** $\dfrac{w^x \times w^4}{w^7} = w^6$

g $(2a^x)^5 = 32a^{30}$ **h** $\dfrac{(n^3)^x \times n^4}{(n^2)^6} = n^7$ **i** $\dfrac{a^3b^6}{a^4b^x} \times \dfrac{a^7(b^2)^3}{a^5b^4} = a$

11 Write $\dfrac{8^5 \times 2^4}{4^7}$ in simplest exponent form with a base of 2.

12 Write $\dfrac{3^{4x} \times 9^{2x}}{27^x \times 3^{3x}}$ in simplest exponent form with a base of 3.

13 Use the appropriate exponent laws to simplify $\dfrac{25^{3x} \times 625^x}{125^{2x} \times 3^{3x}}$.

14 Find the value of x and y to make each of these statements true.

a $a^xb \times a^y(b^4)^y = a^{12}b^{33}$ **b** $\dfrac{a^xb^y}{a^2b^3} \times \dfrac{a^{x+y}b^y}{a^3b^7} = a^3b^{14}$

15 A square number is a number that can be written in the form n^2, where n is a whole number.

a Show that $\dfrac{2^{13} \times 3^{11} \times 5^9}{30^7}$ is a square number.

b Find the possible values of a if $\dfrac{5^5 \times 7^7}{a}$ is a square number.

c Describe the prime factorisation of a square number.

16 Evaluate $\dfrac{6^5 \times 5^7}{2^4 \times 15^5}$ without a calculator.

17 It is possible to apply an exponent to negative numbers, but it is necessary to include a pair of brackets. For example, -5^2 is negative 5^2, meaning $-5^2 = -25$. However, $(-5)^2$ is -5×-5 and therefore $(-5)^2 = 25$.

a Evaluate each the following. It may be helpful to write the expression in expanded form each time.

i 2^2 **ii** $(-2)^2$ **iii** 2^3

iv $(-2)^3$ **v** 10^5 **vi** $(-10)^5$

b Evaluate each of the following without a calculator.

i $(-2)^5$ **ii** $(-3)^4$ **iii** $(-5)^3$

c Decide if each of the following will evaluate to a positive number or a negative number. Explain your answer.

i $(-4)^{10}$ **ii** $(-7)^7$ **iii** $(-123)^{456}$

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Exponent laws

CAS instructions
Simplifying exponents

Topic quiz
1A

1B Negative exponents

Learning intentions

By the end of this topic you will be able to ...

- ✔ write algebraic terms with only positive exponents
- ✔ use the exponent laws to simplify expressions with negative exponents.

Inter-year links

Year 8 4D Combining the exponent laws

Year 9 1D Negative exponents

Negative exponents

- A **negative exponent** indicates the number of times that the base is divided repeatedly. This can be shown by extending the same pattern that is applied to positive exponents in the reverse direction.

multiply by the base →

$\ldots, 3^{-3} = \frac{1}{27}, 3^{-2} = \frac{1}{9}, 3^{-1} = \frac{1}{3}, 3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27, \ldots$

← divide by the base

- The **reciprocal** of a number can be expressed using an exponent of -1. The algebraic properties on the right apply for all non-zero values a and b.
- The exponent laws apply to negative exponents in the same way they do for positive exponents. For example, we can apply the power of a power law to generalise the properties above for any exponent.
- These properties allow for expressions containing negative exponents to be expressed using only positive exponents. For example,

$$3hr^{-2} = \frac{3h}{r^2}$$

- They can also be used to write fractions in exponent form. For example,

$$\frac{2b^3}{a} = 2a^{-1}b^3$$

Algebraic property	Example
$a^{-1} = \frac{1}{a}$	$2^{-1} = \frac{1}{2}$
$\frac{1}{a^{-1}} = a$	$\frac{1}{3^{-1}} = 3$
$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$	$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$

Algebraic property	Example
$a^{-m} = \frac{1}{a^m}$	$3^{-2} = \frac{1}{3^2}$ $= \frac{1}{9}$
$\frac{1}{a^{-m}} = a^m$	$\frac{1}{4^{-3}} = 4^3$ $= 64$
$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$	$\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^4$ $= \frac{81}{16}$

Example 1B.1 Writing terms with positive exponents

Write an equivalent term containing positive exponents for $\left(\frac{z}{3}\right)^{-2}$.

THINK

1. Apply $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ to write the expression with a positive exponent.
2. Expand the brackets by applying the exponent to the numerator and the denominator.
3. Simplify.

WRITE

$\left(\frac{z}{3}\right)^{-2} = \left(\frac{3}{z}\right)^2$

$= \frac{3^2}{z^2}$

$= \frac{9}{z^2}$

Example 1B.2 Combining exponent laws to simplify negative exponents

Use appropriate exponent laws to simplify these expressions. Write your answers using only positive exponents.

a $\left(\frac{x^2}{x^{-1}}\right)^{-3}$ **b** $6g^{-4} \times 2g \div 3g^{-2}$ **c** $\frac{3}{m^{-4}} \times (m^2)^{-2}$ **d** $\left(\frac{y}{x}\right)^{-2} \times \left(\frac{x}{y^{-2}}\right)^2$

THINK

a **1** Apply the quotient of powers law to subtract the exponents of the fraction inside the bracket.

2 Apply the power of a power law to multiply the exponents.

3 Apply $a^{-m} = \frac{1}{a^m}$ to write the expression with a positive exponent.

b **1** Write the expression as a fraction.

2 Simplify the numerator by multiplying 6 and 2 in the numerator and applying the product of powers law to add the exponents. Divide 12 and 3 by their HCF of 3.

3 Apply the quotient of powers law to subtract the exponents.

4 Apply $a^{-1} = \frac{1}{a}$ to write the number as a fraction with a positive exponent.

c **1** Separate the denominator from the numerator and write each factor with a positive exponent by applying:
$\frac{1}{a^{-m}} = a^m$ and $a^{-m} = \frac{1}{a^m}$.

2 Apply the power of a power law to multiply the exponents, then the quotient of powers law to subtract the exponents.

3 Simplify. Remember that any non-zero base raised to a zero exponent is equal to one.

d **1** Apply $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ and $\frac{1}{a^{-m}} = a^m$ to write the expression using positive exponents.

2 Apply the power of a power law and write the expression as a fraction.

3 Simplify the expression using the product and quotient of powers laws.

WRITE

a $\left(\frac{x^2}{x^{-1}}\right)^{-3} = (x^{2-(-1)})^{-3}$
$= (x^{2+1})^{-3}$
$= (x^3)^{-3}$
$= x^{3\times(-3)}$
$= x^{-9}$
$= \frac{1}{x^9}$

b $6g^{-4} \times 2g \div 3g^{-2} = \frac{6g^{-4} \times 2g}{3g^{-2}}$
$= \frac{12^4 g^{-4+1}}{3^1 g^{-2}}$
$= \frac{4g^{-3}}{g^{-2}}$
$= 4g^{-3-(-2)}$
$= 4g^{-3+2}$
$= 4g^{-1}$
$= 4 \times \frac{1}{g}$
$= \frac{4}{g}$

c $\frac{3}{m^{-4}} \times (m^2)^{-2} = 3 \times \frac{1}{m^{-4}} \times (m^2)^{-2}$
$= 3 \times m^4 \times \frac{1}{(m^2)^2}$
$= 3 \times m^4 \times \frac{1}{m^4}$
$= 3 \times m^{4-4}$
$= 3m^0$
$= 3$

d $\left(\frac{y}{x}\right)^{-2} \times \left(\frac{x}{y^{-2}}\right)^2 = \left(\frac{x}{y}\right)^2 \times (xy^2)^2$
$= \frac{x^2 \times x^2 y^{2\times 2}}{y^2}$
$= \frac{x^{2+2} y^4}{y^2}$
$= x^4 y^{4-2}$
$= x^4 y^2$

Helpful hints

- ✔ Don't confuse negative exponents with negative numbers.
  For example,
  $$2^{-3} = \frac{1}{2^3} \quad \text{and} \quad 2^{-3} \neq -(2^3)$$
- ✔ Make sure to always write your exponents clearly! Don't write $2^2 \times 4$ when you mean $2^{2\times 4}$!

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Exercise 1B Negative exponents

▲ 1–7(1st, 2nd columns), 8(a–c), 11(a, b) ■ 1–4(3rd column), 5–7(3rd, 4th columns), 8(c, e, f), 9(a–c, f), 11, 13(a) ⬟ 4–7(4th column), 8(d, g, h), 9(d–f), 10, 12–16

UNDERSTANDING AND FLUENCY

1B.1 **1** Write each of these terms using only positive exponents.

a x^{-5} **b** $c^{-9}d^4$ **c** j^5k^{-3} **d** $x^{-1}y^8$

e $2a^3b^{-6}$ **f** $3p^{-4}q^{-5}$ **g** $w^4x^{-5}y^7$ **h** $k^{-3}m^5n^{-8}$

i $6d^2e^{-7}f$ **j** $11a^{-3}b^{-5}c^{-9}$ **k** $\frac{1}{x^{-3}}$ **l** $\frac{b^4}{c^{-6}}$

m $\frac{4m^3}{p^{-7}}$ **n** $\frac{a^{-4}}{b^{-2}}$ **o** $\frac{8x^{-2}}{2y^{-5}}$ **p** $\frac{c^{-1}d^8}{e^{-4}}$

2 You have seen that the exponent laws also apply to terms with negative exponents. Simplify each of these expressions.

a $x^5 \times x^{-2}$ **b** $m^4 \div m^{-3}$ **c** $(k^{-6})^{-2}$ **d** $y^{-6} \times y^{10}$

e $a^{-4} \div a^{-9}$ **f** $(d^{-5})^3$ **g** $y^3 \times y^{-4} \times y^2$ **h** $n^{-7} \times n^{-3} \times n^{12}$

3 Write each of these expressions without brackets and using only positive exponents.

a $(x \times y)^{-4}$ **b** $(4 \times a)^{-6}$ **c** $(9 \times n)^{-1}$ **d** $(5p)^{-3}$

e $(3p)^{-2}$ **f** $(km)^{-5}$ **g** $\left(\frac{x}{4}\right)^{-3}$ **h** $\left(\frac{7}{d}\right)^{-1}$

4 Use the appropriate exponent laws to simplify these expressions.

a $2x^6 \times 7x^3$ **b** $x^7 \times 3x^{-5}$ **c** $x^3 \times x^8 \times x^4$ **d** $5x^9 \times x \times 4x^{-2}$

e $8x^6 \div 2x^3$ **f** $30x^9 \div 5x^3$ **g** $6x^5 \div 14x^4$ **h** $12x^{10} \div 8x^5$

i $(x^3)^5 \times x^2$ **j** $(x^6)^4 \times (x^5)^2$ **k** $x^7 \times 2x^8 \times (x^2)^4$ **l** $(x^4)^3 \times (x^5)^6 \times 5x^{-7}$

5 Simplify each of these expressions. Write your answers using only positive exponents.

a $x^3 \times x^{-5}$ **b** $2x^{-4} \times x^{-1}$ **c** $7x^{-3} \times 3x^6 \times 2x^{-2}$ **d** $4x^{-8} \times 5x^3 \times x$

e $x^{-11} \div x^{-9}$ **f** $6x^4 \div (3x^{-5})$ **g** $3x^{-7} \div (12x^2)$ **h** $9x^6 \div (15x^{10})$

i $(x^{-3})^2 \times x^{-5}$ **j** $(x^{-4})^6 \times 10x^{18}$ **k** $3x^{-6} \times (x^{-1})^7 \times (x^3)^2$ **l** $(x^{-2})^7 \times (x^{-5})^{-1} \times (x^{-3})^4$

1B.2 **6** Use the appropriate exponent laws to simplify each of these expressions. Write your answers using only positive exponents.

a $(x^3)^{-2}$ **b** $(2y^{-3})^2$ **c** $(w^{-3})^{-4}$ **d** $\left(\frac{2x^4}{x^{-3}}\right)^2$

e $\left(\frac{a^4}{3}\right)^{-2}$ **f** $(2m)^{-2} \times (2m)^3$ **g** $\left(\frac{h^{-4}}{2}\right)^{-3}$ **h** $3c^{-4} \times 5c^{-8} \times 2c^3$

i $(2d)^{-2} \times 8d^5$ **j** $(6t^2)^{-2} \times 72(t^4)^{-3}$ **k** $(4x^{-2})^2 \times (2x^{-3})^{-3} \div (2x^4)$ **l** $\left(\frac{a^2}{3}\right)^{-3} \times \left(\frac{2a^{-1}}{3}\right)^2$

7 Use the exponent laws to help you simplify each of these expressions. Write your answers using only positive exponents.

a $(ab)^4 \times a^7b^3$ **b** $(4p)^3 \times (6p)^2$ **c** $(2y^3)^5$ **d** $3(m^2n)^6$

e $\frac{d^5}{h^4} \times \left(\frac{d}{h}\right)^7$ **f** $\left(\frac{5w}{y}\right)^3$ **g** $\left(\frac{a^5}{b^{-2}}\right)^6$ **h** $\left(\frac{a^{-4}b^2}{c^7}\right)^4$

i $\left(\frac{wx^5}{u^3}\right)^4 \times \left(\frac{u^5}{wx}\right)^3$ **j** $\left(\frac{c^6}{a^6n^2}\right)^2 \times \frac{(5a^4)^3}{c^8n^{-5}}$ **k** $\frac{(m^5n^4)^3 \times (m^6n)^4}{(mn^3)^5}$ **l** $\frac{(2x^2)^6(y^4)^2}{(4x^4y^5)^3}$

8 Use the exponent laws to decide whether each of the following statements is true or false. Explain your reasoning. For each false statement, change the right-hand side to make the statement true.

a $\frac{a^5}{a^9} = a^4$ **b** $\frac{x^3}{x^6} = x^{\frac{1}{2}}$ **c** $\frac{y^3}{y^9} = \frac{1}{y^6}$

d $\left(\frac{3a^2b^{-1}}{c^3}\right)^2 = \frac{6a^4}{b^2c^6}$ **e** $(5m^4n^{-3})^{-2} = \frac{n^6}{25m^8}$ **f** $\frac{2a^{-4}b^3}{5^{-1}ab^2} = \frac{10b}{a^5}$

g $\frac{3(x^{-2}y^1)^3}{x^3y^{-2}z} = \frac{3y^5}{x^5z}$ **h** $(6abc)^{-2} \times (3a^{-1}b)^2 \div (4a^4c^2)^{-1} = 1$

9 Find the value of x that will make each of these statements true.

a $\frac{a^x}{a^4} = \frac{1}{a}$ **b** $\frac{3}{a^x} = 3a^4$ **c** $\left(\frac{2a^{-2}b}{c^3}\right)^x = \frac{a^4c^6}{4b^2}$

d $\frac{(ab)^x}{a^2b^4} = \frac{a}{b}$ **e** $\frac{a^3b^x}{a^{3x}b^2} = \frac{1}{b}$ **f** $\frac{y^7 \times y^x}{y^{2x}} = y^2$

10 Simplify each of these expressions.

a $x^{2a}y^{b+3} \times x^5y^{3b-2}$ **b** $\frac{m^{3x+4}n^{x-2}}{m^xn^{x-5}}$

11 Evaluate each of the following.

a 5^{-2} **b** 3^{-3} **c** $\frac{5^{-2}}{10^{-3}}$

d $\left(\frac{3^{-1}}{2^2}\right)^{-2}$ **e** $4^{-2} + 2^{-4}$ **f** $3^{-2} + 3^{-1} + 3^0$

12 Simplify each of these expressions.

a $x^{2a+1} \times x^{a-5}$ **b** $(y^m)^2 \times y^{m+5}$ **c** $\frac{a^{3x+3}}{a^{2x}}$ **d** $\frac{b^{2x}}{b^y} \times b^3 \times \frac{b^{y+1}}{b^x}$

13 Use exponent laws to simplify each of the following expressions. Write your answers with positive exponents only.

a $\frac{3a^2b^4c}{8a^5b^3} \times \frac{4a^5b^4}{b^3c^2}$ **b** $\frac{10ac^5}{5a^5b^9c^2} \times \frac{3a^4b^3c}{(2bc^3)^2}$ **c** $\frac{4(x^2y^{-1}z)^3}{10x^4y^2} \div \frac{2x^5z^2}{5x^3y^4z^{-1}}$

14 a Express $\left(\frac{3a^2b^3}{2c^4}\right)^{-3}$ in expanded form, using positive exponents only, by:

i raising each numeral and pronumeral to the power of −3 and then simplifying

ii flipping the fraction and then raising each numeral and pronumeral to the power of +3.

b Which method did you prefer and why?

15 Fully simplify $\left(\frac{c^7d^3}{4a^3b^{-6}}\right)^{-2} \times \left(\frac{2a^2b^{-4}}{c^5d^2}\right)^{-3}$.

16 The exponent '−1' is used in mathematics to refer to the 'multiplicative inverse' of a non-zero real number. That is, a^{-1} (read 'a inverse') is the number which, when multiplied by a, results in 1. For example, $2^{-1} = \frac{1}{2}$ because $2 \times \frac{1}{2} = 1$.

a Find the multiplicative inverses of:

i 3 **ii** 10 **iii** $\frac{1}{4}$ **iv** $\frac{5}{3}$

v −3 **vi** −10 **vii** $-\frac{1}{4}$ **viii** $-\frac{5}{3}$

b Complete the following sentences.

i If a number is positive, then its multiplicative inverse is _____.

ii If a number is negative, then its multiplicative inverse is _____.

c If $a \neq 0$, show that $(-a)^{-1} = -a^{-1}$

d Which two real numbers are equal to their multiplicative inverse?

e Why does 0 not have a multiplicative inverse?

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Negative exponents

Topic quiz
1B

1C Simplifying

Learning intentions

By the end of this topic you will be able to ...

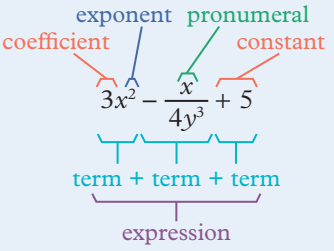
- ✔ evaluate algebraic expressions using substitution
- ✔ identify like terms to simplify algebraic expressions
- ✔ multiply and divide algebraic terms.

Inter-year links

Year 7	6E Substitution
Year 8	5C Adding and subtracting algebraic terms
Year 9	2A Simplifying

Evaluating algebraic expressions

- The terms in an algebraic expression are separated by a plus or minus sign.
- An algebraic term can be written as the product of a **coefficient** (number) multiplied by any combination of **pronumerals** raised to any exponent.
- If there are no pronumerals in a term, then it is a **constant term**. A constant term can also be considered as the coefficient of a variable with a zero exponent.
- A pronumeral can be referred to as either a **variable** or an unknown.



- When pronumerals are given numerical values, an algebraic expression can be evaluated by **substitution**.

  For example, if $x = 3$ and $y = 2$:

$$x^2 + 5y - 2\frac{x}{y} = 3^2 + 5(2) - 2\left(\frac{3}{2}\right)$$
$$= 9 + 10 - 3$$
$$= 16$$

Adding and subtracting terms

- **Like terms** are the terms of an algebraic expression that contain the same pronumerals raised to the same power. The order of the pronumerals does not matter.

  For example, the following are sets of like terms:

  → x^2 and $3x^2$

  → 3, 16 and 100

  → $2y^2z$ and $8zy^2$

  → $4w^3h^4t$, $2w^3th^4$ and $10th^4w^3$
- Like terms can be added or subtracted to simplify an algebraic expression.

  For example,

$$7x^2y + 2 + 3yx^2 - 8 = 10x^2y - 6$$

Multiplying and dividing terms

- To multiply algebraic terms, multiply the coefficients and combine the pronumerals, using exponent laws if necessary.
  For example,
  $$3a^3b \times -2a^2b^2 = 3 \times -2 \times a^{3+2} \times b^{1+2}$$
  $$= -6a^5b^3$$
- To divide algebraic terms, divide the coefficients and combine the pronumerals, using exponent laws if necessary.
  For example,
  $$\frac{6a^5b^3}{3a^3b} = \frac{{}^{2}\cancel{6} \times a^5 \times b^3}{{}^{1}\cancel{3} \times a^3 \times b}$$
  $$= 2 \times a^{5-3} \times b^{3-1}$$
  $$= 2a^2b^2$$
- If the terms being divided cannot be simplified using only positive exponents leave them in fraction form. For example,
  $$3t \div 6r^2 = \frac{t}{2r^2}$$

Example 1C.1 Using substitution to evaluate algebraic expressions

If $a = -2$, $b = 5$ and $c = -1$, evaluate:

a a^3bc^2 **b** $2b - ab + ac$

THINK

a Substitute the known values and evaluate.

b Substitute the known values and evaluate.

WRITE

a $a^3bc^2 = (-2)^3(5)(-1)^2$
$= -8 \times 5 \times 1$
$= -40$

b $2b - ab + ac = 2(5) - (-2)(5) + (-2)(-1)$
$= 10 + 10 + 2$
$= 22$

Example 1C.2 Adding and subtracting algebraic terms

Simplify each of these expressions by collecting like terms.

a $9a + 12b - 3 + 3a + 2a - 1$ **b** $2m^2n - 8n - 10n + n^{-2} + 3m + 4nm^2$

THINK

a 1 Rearrange the expression, grouping like terms together.
2 Simplify by combining like terms.

b 1 Rearrange the expression by grouping like terms together. Write each term with its pronumerals in alphabetical order so that the like terms can be seen clearly.
2 Simplify by combining like terms.

WRITE

a $9a + 12b - 3 + 3a + 2a - 1$
$= (9a + 3a + 2a) + (12b) + (-3 - 1)$
$= 14a + 12b - 4$

b $2m^2n - 8n - 10n + n^{-2} + 3m + 4nm^2$
$= (2m^2n + 4m^2n) + (n^{-2}) + (3m)$
$+ (-8n - 10n)$
$= 6m^2n + n^{-2} + 3m - 18n$

Example 1C.3 Using exponent laws to multiply and divide algebraic terms

Simplify these expressions by writing each one as a single term with positive exponents.

a $2ab^2 \times 6a^{-3}c$ **b** $12pqr^2 \div (4qr)$

THINK

a **1** Multiply the coefficients and apply the product of powers law to multiply the terms with the same base by adding their exponents.

2 Write the answer using positive exponents.

b **1** Write the expression as a fraction.

2 Simplify the coefficients by dividing them both by their HCF of 4.

3 Apply the quotient of powers law to divide the terms with the same base by subtracting their exponents.

4 Simplify. Remember that any non-zero base raised to a zero exponent is equal to 1.

WRITE

a $2ab^2 \times 6a^{-3}c = 12 \times a^{1+(-3)}b^2c$

$= 12a^{-2}b^2c$

$= \frac{12b^2c}{a^2}$

b $12pqr^2 \div (4qr) = \frac{12pqr^2}{4qr}$

$= \frac{\cancel{12}^{3}pqr^2}{\cancel{4}^{1}qr}$

$= \frac{3pqr^2}{qr}$

$= 3pq^{1-1}r^{2-1}$

$= 3pq^0r^1$

$= 3pr$

Helpful hints

✔ The order in which you write the coefficients and pronumerals in an algebraic term doesn't matter. But it is easier to identify like terms if you write the pronumerals in alphabetical order. For example,

$acb + cb - ba + 12bca + 8ab - 6bac = abc + 12abc - 6abc + bc - ab + 8ab$

$= 7abc + 7ab + bc$

✔ Recall the following rules for algebraic notation.

→ To simplify products, leave out the multiplication sign and place the number first. For example,

$7 \times x = 7x$

and:

$7 \times (x + 2) = 7(x + 2)$

→ When a pronumeral is multiplied by 1, the 1 does not need to be included. For example,

$1 \times x = x$

→ Division is represented using fractions. For example,

$x \div 7 = \frac{x}{7}$

and:

$(x + 2) \div 7 = \frac{x + 2}{7}$

→ Terms with coefficients that are fractions can be written in two ways. For example,

$\frac{1}{7}x$ is the same as $\frac{x}{7}$

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Exercise 1C Simplifying

1, 2, 3, 4(1st column), 5–7, 8(a, c, e), 10–12 | 2, 4, 7, 8, 10, 13, 14, 17(a–d) | 4, 7–9, 15–18

1 For each of the following terms:

i identify the coefficient **ii** write an example of a like term.

a $3abcd$ **b** $-4mn$ **c** xy^2 **d** $9k^2m^4p$

1C.1 **2** If $a = 1$, $b = 3$ and $c = -2$, evaluate:

a abc **b** $ab + bc$ **c** $ac - 2bc$ **d** a^4b^2

e $3ab^2c - ac$ **f** $7ac + 3bc - 4ac$ **g** $a^3b^2 + bc^3$ **h** $2a^2b + 3ac^2 - b^3$

1C.2 **3** Simplify each of these expressions by collecting like terms.

a $8x - 5x + 6x$ **b** $3ab - 4ab - 9ab$ **c** $4y^2 + y^2 + 5y^2$

d $7m^2n + 2m^2n - m^2n$ **e** $4a + 6d + 8a + 3d$ **f** $k - 3m + 5m + 7k$

g $4 + 8y - 2 - 11y$ **h** $a^2 + 6a - 5a^2 + a$ **i** $3xy + 2x^2 - xy + x^2$

4 Simplify each of these expressions by collecting like terms.

a $5a + 2b - a + 3b + 7a - 5b$ **b** $6mn - 3n - n + 2n^2 + m - 7mn$

c $x + 4x^2 - 8x + 1 + 5x - x^2$ **d** $2x^2y - 5x^2y - 4xy + 3 + 7xy - 8$

e $9p^2 - 3p + 2p^3 + 6p^2 - 9 + 3p$ **f** $a^2b + 4a^2 + 8ba^2 - b^2 + 3ab^2 - 5a^2$

5 Simplify each of the following expressions by writing each one as a single term.

a $5ab \times 2cd$ **b** $-7xy \times 3mn$ **c** $4kp \times k$

6 Simplify each of the following expressions by writing each one as a single term with positive exponents.

a $abc \div b$ **b** $6mnp \div mp$ **c** $8wx \div 4wy$

1C.3 **7** Simplify each of the following expressions by writing each one as a single term with positive exponents.

a $4x^2y^2 \times x^3y$ **b** $3ab^2 \times 2a^5b$ **c** $4m^5n^{-3} \times 6m^2n$

d $2x^4y^{-3}z^{-2} \times 5x^3y^5z^{-1}$ **e** $10xy^2 \div 5x^3$ **f** $(6x^2)^2 \div 9x^3$

g $12x^2y^{-4} \times 3x^3z^7$ **h** $(3x^{-2}y^4)^2 \times 2x^3y^3$ **i** $(5a^{-2}b^3)^{-2} \times (10b)^2$

8 If $a = 3$, $b = -2$ and $c = 4$, evaluate each of the following expressions. Hint: Simplify each expression first.

a $4a + 3b + 2c - 2a - c + b$ **b** $9ab + 7a - 10a + ab$ **c** $a^2b + ab^2 + a^2c - 2a^2b + 5a^2c$

d $3abc \times b^2c \times 6a$ **e** $12a^2b^2c \div 3bc^2$ **f** $ac^2 \times 2ab^3 \div 8a^2bc$

9 The surface area of this box can be calculated using the expression $2lw + 2lh + 2wh$. Calculate the surface area of a box with each of the following sets of dimensions.

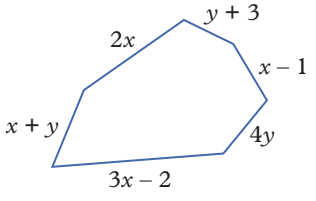
a $l = 25$ cm, $w = 15$ cm, $h = 10$ cm

b $l = 0.5$ m, $w = 0.3$ m, $h = 0.2$ m

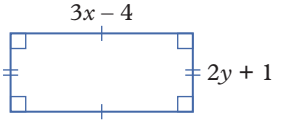
10 The formula for the area, A, of a circle with radius r is $A = \pi r^2$. Calculate the area when $r = 5$ cm. Leave your answer as an exact value containing π.

11 Write the perimeter of each of these shapes as an algebraic expression in simplest form.

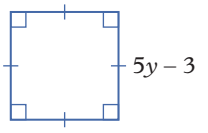
a



b



c



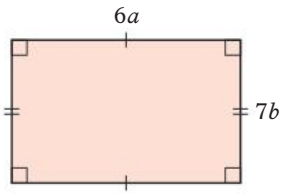
UNDERSTANDING AND FLUENCY

PROBLEM SOLVING AND REASONING

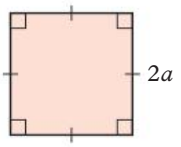
12 Calculate the perimeter of each shape in question **11** if $x = 7$ cm and $y = 2$ cm.

13 Write the area of each shaded shape below as an algebraic expression in simplest form.

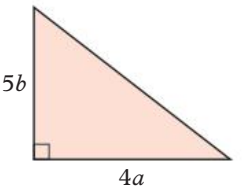
a



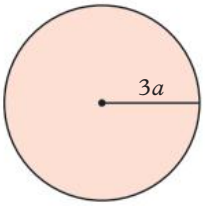
b



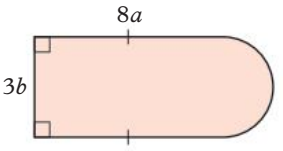
c



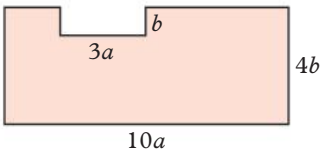
d Leave your answer as an exact value containing π.



e Leave your answer as an exact value containing π.

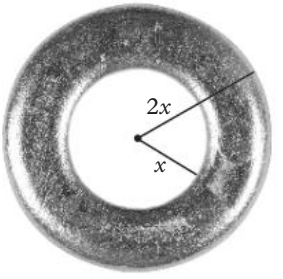


f



14 Calculate the area of each shape in question **13** if $a = 4$ m and $b = 3$ m. Leave your answers as exact values containing π.

15 Consider the top surface of a metal washer shaped like an annulus as shown on the right.



a Write an algebraic expression for the area of the top surface of one washer.

b List three possible values for x that will result in an area of between 400 mm^2 and 500 mm^2 for the surface area of the top of the washer.

c Write an algebraic expression for the total length of the inner and outer circular edges of one washer.

d Use the values you wrote for part **b** to calculate the total length of the inner and outer circular edges of one washer, correct to one decimal place.

16 Use the order of operations and exponent laws to simplify each of the following expressions.

a $2(3a + 5a) - 4a$

b $4b \times 3a + 5(ab + 3ab)$

c $2a \div 2 + 5a \times 3b - 4a$

d $19xy^2 + 8x \div 4 + 5x^2y^2 \div x - 3x$

e $3(2a)^2 + (5ab^3)^2 - 2(3a)^2$

f $(-6xy)^2 + 4x^2y^2 \div x^2 + (3y)^2 - 5xy^2 \times 4x$

17 Simplify each of the following expressions by collecting like terms.

a $5ab + 2ba$

b $3abc + 4bac$

c $6x^2y - 2xy + 5yx - yx^2$

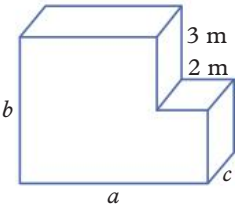
d $14map + 2amp + 3pam - 19mpa$

e $2a^2b \times 4b + 3ab^2 \times 5a$

f $\frac{6xy^3z^2}{2yx} + \frac{5y^3x^2z}{y^3} - \frac{14z^3x^5}{7x^3z^2} + \frac{(4zy)^2}{2}$

CHALLENGE

18 Consider the object on the right. Hint: All angles are right angles, meaning all opposite sides are parallel and of equal length.



a Write an expression for the volume of the shape in cubic metres (m^3) in terms of a, b and c in its simplest form.

b Write an expression for the surface area of the shape in square metres (m^2) in terms of a, b and c in its simplest form.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Multiplying terms

Interactive skillsheet
Dividing terms

Topic quiz
1C

1D Expanding

Learning intentions

By the end of this topic you will be able to ...

- ✔ expand algebraic expressions with one pair of brackets
- ✔ expand binomial products.

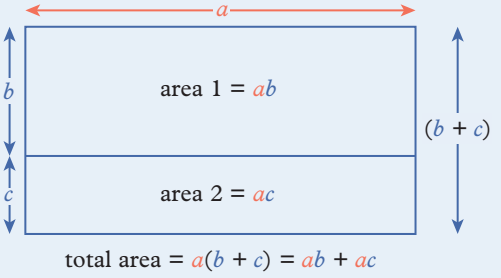
Inter-year links

Year 7	6D Order of operations in algebra
Year 8	5F Expanding
Year 9	2B Expanding

Expanding one pair of brackets

- The **distributive law** is used to distribute or expand products over addition and subtraction.

$a(b + c) = ab + ac$



total area = $a(b + c) = ab + ac$

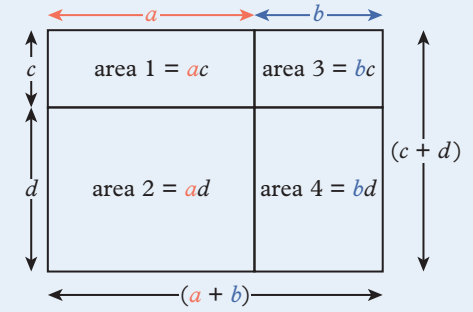
Expanding binomial products

- A **binomial** is an algebraic expression that contains two terms. Examples are:

  $x + 1$ $\quad a^3 - 3a^4$ $\quad 21m^2 + n$

- The distributive law can be used to expand the product of two binomials.

$(a + b)(c + d) = ac + ad + bc + bd$



total area = $(a + b)(c + d) = ac + ad + bc + bd$

- Expanding binomial products often requires simplification using the exponent laws and the identification of like terms.
- The **difference of two squares** is a specific form of expansion with the simplified rule:

  $(a + b)(a - b) = a^2 - b^2$

- The **expansion of a perfect square** is a specific form of expansion with the simplified rule:

$(a + b)^2 = a^2 + 2ab + b^2$ $\quad (a - b)^2 = a^2 - 2ab + b^2$

Example 1D.1 Expanding one pair of brackets

Expand each of these algebraic expressions to remove the brackets.

a $4(a + 11)$ **b** $5k(k - 6)$ **c** $6g^3(g^4 + 3g)$ **d** $cd(3d - c^2)$

THINK

a 1 Multiply each term inside the brackets by the term outside the brackets.
2 Simplify by performing the multiplications.

b 1 Multiply each term inside the brackets by the term outside the brackets.
2 Simplify by performing the multiplications. Remember to take care with the + and – signs when simplifying.

c 1 Multiply each term inside the brackets by the term outside the brackets.
2 Simplify by performing the multiplications. Apply the product of powers law, multiplying the terms with the same base by adding their exponents.

d 1 Multiply each term inside the brackets by the term outside the brackets.
2 Simplify by performing the multiplications. Apply the quotient of powers law, multiplying the terms with the same base by adding their exponents.

WRITE

a $4(a + 11) = 4 \times a + 4 \times 11$

$= 4a + 44$

b $5k(k - 6) = 5k \times k + 5k \times (-6)$

$= 5k^2 - 30k$

c $6g^3(g^4 + 3g) = 6g^3 \times g^4 + 6g^3 \times 3g$

$= 6g^7 + 18g^4$

d $cd(3d - c^2) = cd \times 3d + cd \times (-c^2)$

$= 3cd^2 - c^3d$

Example 1D.2 Expanding a binomial product

Expand each of these algebraic expressions to remove the brackets.

a $(x + 4)(y + 9)$ **b** $(-f + 3)(4f - 6)$ **c** $(2z^2 + z)(z + 8)$ **d** $(h^2g - 2)(h + 4g^3)$

THINK

a 1 Multiply each term inside the second pair of brackets, y and 9, by the first term in the first pair of brackets, x. Then multiply them by the second term in the first pair of brackets, 4.
2 Simplify each term.

b 1 Multiply each term inside the second pair of brackets, $4f$ and -6, by the first term in the first pair of brackets, $-f$. Then multiply them by the second term in the first pair of brackets, 3.
2 Simplify each term.
3 Simplify any like terms.

WRITE

a $(x + 4)(y + 9)$

$= x \times y + x \times 9 + 4 \times y + 4 \times 9$

$= xy + 9x + 4y + 36$

b $(-f + 3)(4f - 6)$

$= (-f) \times 4f + (-f) \times (-6) + 3 \times 4f + 3 \times (-6)$

$= -4f^2 + 6f + 12f - 18$

$= -4f^2 + 18f - 18$

c 1 Multiply each term inside the second pair of brackets, z and 8, by the first term in the first pair of brackets, $2z^2$. Then multiply them by the second term in the first pair of brackets, z.

2 Use the exponent laws to simplify each term.

3 Simplify any like terms.

d 1 Multiply each term inside the second pair of brackets, h and $4g^3$, by the first term in the first pair of brackets, h^2g. Then multiply them by the second term in the first pair of brackets, -2.

2 Use the exponent laws to simplify each term.

c
$$(2z^2 + z)(z + 8)$$
$$= 2z^2 \times z + 2z^2 \times 8 + z \times z + z \times 8$$
$$= 2z^3 + 16z^2 + z^2 + 8z$$
$$= 2z^3 + 17z^2 + 8z$$

d
$$(h^2g - 2)(h + 4g^3)$$
$$= h^2g \times h + h^2g \times 4g^3 + (-2) \times h + (-2) \times 4g^3$$
$$= gh^3 + 4g^4h^2 - 2h - 8g^3$$

Helpful hints

- ✔ It's important to be able to recognise when an expression is the difference of two squares, or a perfect square. These are both very useful in algebra!
- ✔ Remember to pay close attention to minus signs when you are expanding an expression! Using brackets around negative terms can make it clearer.
- ✔ To make sure you correctly expand the product of two binomials, it may help to include the following step:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$$

ANS p535

Exercise 1D Expanding

1–2(1st, 2nd columns), 3, 5–6(1st column), 7, 8 | 3, 4, 5–6(2nd column), 7, 8, 10, 12 | 4, 5–6(3rd column), 7–9, 11, 13

1 Expand each of these algebraic expressions to remove the brackets.

a $3(a + 7)$	**b** $5(b - 4)$	**c** $2(6 - c)$	**d** $4(1 + d)$
e $-6(e + 3)$	**f** $-7(f - 2)$	**g** $-3(g - 1)$	**h** $-9(5 - h)$
i $x(x + 2)$	**j** $2y(y - 4)$	**k** $-k(k + 5)$	**l** $-3p(p - 1)$
m $m(3n + k)$	**n** $4a(3b + 6c)$	**o** $-w(w - 8x)$	**p** $-2p(2 + 5p)$

1D.1 **2** Expand each of these algebraic expressions to remove the brackets.

a $3(a - 4)$	**b** $-5(b - 3)$	**c** $4(2x + 3)$	**d** $-7(2y + 4)$
e $5a(2 - a)$	**f** $2m(m^2 + 3)$	**g** $8ab(3a + 1)$	**h** $4w(2x - w)$
i $5k^2(3 - 2k^2)$	**j** $6x^2y(2x - y^3)$	**k** $5cd(3c - 2d)$	**l** $-2a^2c(4a - 3b)$

3 Expand and simplify each of these algebraic expressions.

a $2(x + 5) + 7x$	**b** $4(y - 7) + 15$	**c** $m(n + 6) - 3m$
d $-3x(2 - x) + 5x^2 + x$	**e** $7p^3 - 8p + 5p(1 - p^2)$	**f** $x^4(x^3 - 2x) - x^5$

4 Expand and simplify each of these algebraic expressions.

a $4(x - 2) + 2(x + 5)$	**b** $m(m + 8) + 6(m - 3)$	**c** $2a(3a + 1) - 3(2 - 4a)$
d $k^2(k^3 + 2) + 5(k^2 + 1)$	**e** $3p^4(p^2 - 2) + 2p(p^3 + 6)$	**f** $5y^3(2y^5 + 3) - 2y(7y^2 - 4)$

1D.2

5 Expand each of these algebraic expressions to remove the brackets.

a $(a + 4)(b + 2)$ **b** $(c + 3)(d + 6)$ **c** $(m + 7)(n + 5)$

d $(x + 6)(y + 8)$ **e** $(k + 5)(p - 1)$ **f** $(f - 2)(g + 3)$

g $(x - 9)(y - 4)$ **h** $(2m + 3)(n + 1)$ **i** $(4x + 5)(3y - 7)$

j $(3a - 2b)(5c - 4d)$ **k** $(x + 2)(x + 5)$ **l** $(y + 9)(y - 1)$

m $(m - 3)(m - 11)$ **n** $(p - 4)(p - 2)$ **o** $(5 - k)(6 - k)$

p $(4x + 7)(x + 3)$ **q** $(3a + 2)(a - 5)$ **r** $(8w - 7)(w - 2)$

s $(6p + 5)(2p + 7)$ **t** $(2x + 3)(4x - 1)$ **u** $(5y - 2)(3y - 4)$

6 Expand each of these algebraic expressions to remove the brackets.

a $(2a - b)(a + 3)$ **b** $(4t + u)(t - u)$ **c** $(2x + 3y)(x + 4y)$

d $(5 - x)(x + 4)$ **e** $(6x^2 + y)(x + 3y)$ **f** $(ab^2 + 1)(b^2 - 2a)$

g $(a + b)(b + c)$ **h** $(b + 3c)(b + d)$ **i** $(a + 2b)(3 - c)$

j $(a^3 + b^2)(3ab + 2)$ **k** $(x^4 - 3z)(5x - 2y)$ **l** $(c^2 + 4a^3b)(c^2 - 4a^3b)$

7 a Expand each of these binomial products.

i $(x + 2)(x - 2)$ **ii** $(k + 7)(k - 7)$ **iii** $(m + 8)(m - 8)$

iv $(w - 6)(w + 6)$ **v** $(y - 1)(y + 1)$ **vi** $(a + b)(a - b)$

b Describe any pattern or shortcut you can see in part **a**. What is special about the two binomial factors that were multiplied together in part **a**?

c The binomial products in this question are all examples of the 'difference of two squares'. Why do you think the rule is called the 'difference of two squares'?

d Does it matter whether the product is $(a + b)(a - b)$ or $(a - b)(a + b)$? Explain.

e Use the 'difference of two squares' rule to expand each of these binomial products.

i $(a - 3)(a + 3)$ **ii** $(x + 10)(x - 10)$ **iii** $(m + n)(m - n)$

iv $(3 + x)(3 - x)$ **v** $(1 + d)(1 - d)$ **vi** $(2x - 5)(2x + 5)$

vii $(2 - 3k)(2 + 3k)$ **viii** $(4g + h)(4g - h)$ **ix** $(5y + 2w)(5y - 2w)$

8 a Expand each of these binomial products.

i $(x + 3)(x + 3)$ **ii** $(y + 8)(y + 8)$ **iii** $(a + b)(a + b)$

iv $(k - 5)(k - 5)$ **v** $(p - 6)(p - 6)$ **vi** $(x - y)(x - y)$

b Describe any pattern or shortcut you can see in part **a**. What is special about the two binomial factors that were multiplied together in part **a**?

c These are all examples of 'perfect squares'. Why do you think the rule is called the 'perfect square' rule?

d Use the 'perfect square' rule to expand each of these algebraic expressions.

i $(a + 2)^2$ **ii** $(x + 5)^2$ **iii** $(p + 4)^2$

iv $(y + 10)^2$ **v** $(k + 9)^2$ **vi** $(m + n)^2$

vii $(5 + x)^2$ **viii** $(1 + 2d)^2$ **ix** $(3w + 7)^2$

e Why is $(a - b)^2 = a^2 - 2ab + b^2$ also a perfect square?

f Use the 'perfect square' rule to expand each of these algebraic expressions.

i $(a - 3)^2$ **ii** $(b - 4)^2$ **iii** $(x - 9)^2$

iv $(n - 11)^2$ **v** $(2 - x)^2$ **vi** $(g - h)^2$

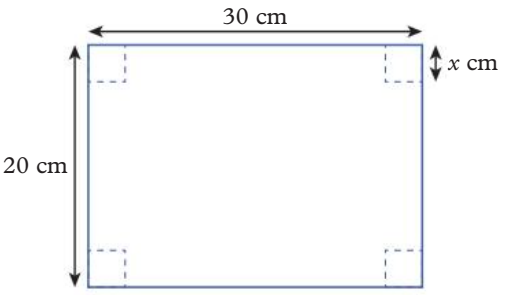
vii $(4w - 1)^2$ **viii** $(5 - 2y)^2$ **ix** $(3m - 2p)^2$

9 Use the 'difference of two squares' and 'perfect square' rules to expand and simplify each of these expressions.

a $(x^2 + 2)(x^2 - 2)$ **b** $(x^2 - 5)(x^2 + 5)$ **c** $(x^2 + y^2)(x^2 - y^2)$

d $(x^2 + 4)^2$ **e** $(x^2 - 3)^2$ **f** $(x^3 - 1)^2$

10 A rectangular piece of cardboard is 30 cm long and 20 cm wide. A square of side length x cm is cut from each corner so that the cardboard can be folded to form an open box.



a Write an expression containing x for:

i the length of the box **ii** the width of the box **iii** the height of the box.

b Draw a labelled diagram of the open box.

c Write an expression for:

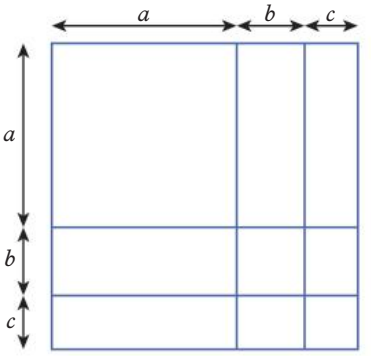
i the volume of space contained in the box **ii** the inner surface area of the box.

Simplify each of your expressions by expanding them to remove any brackets.

d If $x = 3$ cm, calculate:

i the volume of space contained in the box **ii** the inner surface area of the box.

11 It is possible to expand brackets with three terms in each bracket.



a Use the diagram on the right to help you expand $(a + b + c)^2$.

b Expand $(x + 2y + 3)^2$.

c Expand $(2a - 3b + c)(2a + 3b - c)$.

12 It is possible to expand products of more than two brackets.

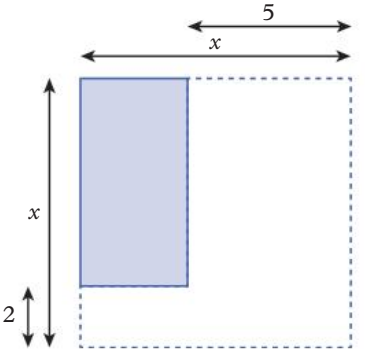
For example,

$$\begin{aligned}(a + b)^3 &= (a + b)(a + b)(a + b)\\ &= (a^2 + 2ab + b^2)(a + b)\\ &= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Expand each of the following:

a $(x + 1)^3$ **b** $(x - 3)^3$ **c** $(x + 2)^4$ **d** $(x + y)^2(x - y)^2$

13 In the diagram on the right, the area of the shaded rectangle is $(x - 2)(x - 5)$, which can be expanded to be expressed as $x^2 - 7x + 10$.



a Copy the diagram into your workbook and draw:

i the outline of a rectangle with area $2 \times x$ square units

ii the outline of a rectangle with area $5 \times x$ square units

iii the outline of a rectangle with area 10 square units.

b Explain why $+10$ appears in the expansion of $(x - 2)(x - 5)$, even though the rectangle with area 10 square units is not shaded.

Check your Student obook pro for these digital resources and more:

- **Interactive skillsheet** — Expanding over one pair of brackets
- **Investigation** — Binomial expansion and Pascal's Triangle
- **CAS instructions** — Expanding
- **Topic quiz** — 1D
- **Interactive skillsheet** — Expanding binomial products

1E Algebraic fractions

Learning intentions

By the end of this topic you will be able to ...

- ✔ simplify algebraic fractions by identifying common factors
- ✔ multiply and divide algebraic fractions
- ✔ add and subtract algebraic fractions.

Inter-year links

Year 7	3B Equivalent fractions
Year 8	5E Dividing algebraic terms
Year 9	2A Simplifying

Simplifying algebraic fractions

- **Algebraic fractions** are fractions that contain at least one pronumeral. They can be simplified just like any other fractions, by identifying common factors of the numerator and the denominator.
- The fastest way to simplify a fraction is to divide the numerator and the denominator by their highest common factor (HCF).

  For example, the HCF of $2x^2$ and $4x$ is $2x$, so to simplify $\frac{2x^2}{4x}$, we divide the numerator and denominator by $2x$, as shown below.

$$\frac{2x^2}{4x} \overset{\div 2x}{=} \frac{x}{2}$$

  This gives us $\frac{x}{2}$.

Multiplying and dividing algebraic fractions

- The product of two algebraic fractions is the product of the numerators over the product of the denominators. For example,

$$\frac{3x}{2} \times \frac{x}{x+1} = \frac{3x \times x}{2 \times (x+1)}$$
$$= \frac{3x^2}{2(x+1)}$$

- The **quotient** of two algebraic fractions is found by multiplying the fraction before the division sign by the reciprocal of the fraction after the division sign. For example,

$$\frac{2x}{y} \div \frac{5}{x} = \frac{2x}{y} \times \frac{x}{5}$$
$$= \frac{2x^2}{5y}$$

Adding and subtracting algebraic fractions

- Adding and subtracting algebraic fractions is no different from adding and subtracting any other fractions.
  1. Identify the lowest common denominator (LCD) by identifying the lowest common multiple (LCM) of the denominators.
  2. Write each fraction as an equivalent fraction with the LCD.

3 The sum of the algebraic fractions with a common denominator is then found by adding the numerators.
For example, $\frac{x+1}{4}+\frac{x}{6}$

The LCD of $\frac{x+1}{4}$ and $\frac{x}{6}$ is 12, so:

$$\begin{aligned}\frac{x+1}{4}+\frac{x}{6}&=\frac{x+1}{4}\times\frac{3}{3}+\frac{x}{6}\times\frac{2}{2}\\&=\frac{3(x+1)}{12}+\frac{2x}{12}\\&=\frac{3x+3+2x}{12}\\&=\frac{5x+3}{12}\end{aligned}$$

- The difference of algebraic fractions with a common denominator is found by subtracting the numerators. So, using the same algebraic fractions from the previous example, we have:

$$\begin{aligned}\frac{x+1}{4}-\frac{x}{6}&=\frac{3(x+1)}{12}-\frac{2x}{12}\\&=\frac{3x+3-2x}{12}\\&=\frac{x+3}{12}\end{aligned}$$

Example 1E.1 Simplifying algebraic fractions

Simplify each of these algebraic fractions.

a $\frac{7a}{7ab}$ **b** $\frac{8(2x+1)}{16}$ **c** $\frac{(x+9)(3x+1)}{(x+2)(x+9)}$

THINK

a 1 Look for factors common to both the numerator and the denominator to determine the highest common factor (HCF). The HCF is $7a$.
2 Divide both the numerator and the denominator by $7a$.

b 1 Look for factors common to both the numerator and the denominator to determine the HCF. The HCF is 8.
2 Divide both the numerator and the denominator by 8.

c 1 Look for factors common to both the numerator and the denominator to determine the HCF. The HCF is $(x+9)$.
2 Divide both the numerator and the denominator by $(x+9)$.

WRITE

a $\frac{7a}{7ab}=\frac{7a^{1}}{7a^{1}b}$

$=\frac{1}{b}$

b $\frac{8(2x+1)}{16}=\frac{8^{1}(2x+1)}{16^{2}}$

$=\frac{2x+1}{2}$

c $\frac{(x+9)(3x+1)}{(x+2)(x+9)}=\frac{(x+9)^{1}(3x+1)}{(x+2)(x+9)^{1}}$

$=\frac{3x+1}{x+2}$

Example 1E.2 Multiplying and dividing algebraic fractions

Write each of these expressions as one fraction in simplest form.

a $\frac{x(x-2)}{(x+3)(2x+1)} \times \frac{(x+1)(2x+1)}{(x-2)(x+3)}$ **b** $\frac{x+4}{x^2} \div \frac{3(x+4)}{x-9}$

THINK

a 1 Write the expression as a single fraction by multiplying the numerators together and the denominators together.

2 Look for common factors. Divide both the numerator and the denominator by $(x-2)$ and $(2x+1)$.

3 Simplify.

b 1 Write the expression as the product of two fractions.

2 Write the expression as a single fraction by multiplying the numerators together and the denominators together.

3 Look for common factors. Divide both the numerator and the denominator by $(x+4)$.

WRITE

a $\frac{x(x-2)}{(x+3)(2x+1)} \times \frac{(x+1)(2x+1)}{(x-2)(x+3)}$

$= \frac{x(x-2) \times (x+1)(2x+1)}{(x+3)(2x+1) \times (x-2)(x+3)}$

$= \frac{x\cancel{(x-2)}^{1}(x+1)\cancel{(2x+1)}^{1}}{(x+3)\cancel{(2x+1)}^{1}\cancel{(x-2)}^{1}(x+3)}$

$= \frac{x(x+1)}{(x+3)(x+3)}$

$= \frac{x(x+1)}{(x+3)^2}$

b $\frac{x+4}{x^2} \div \frac{3(x+4)}{x-9} = \frac{x+4}{x^2} \times \frac{x-9}{3(x+4)}$

$= \frac{(x+4) \times (x-9)}{x^2 \times 3(x+4)}$

$= \frac{\cancel{(x+4)}^{1}(x-9)}{3x^2\cancel{(x+4)}^{1}}$

$= \frac{x-9}{3x^2}$

Example 1E.3 Adding and subtracting algebraic fractions with numerical denominators

Write each of these expressions as one fraction in simplest form.

a $\frac{2x+3}{7} + \frac{x-1}{3}$ **b** $\frac{7x}{10} - \frac{x}{6}$

THINK

a 1 The LCM of 7 and 3 is 21. So write each fraction as an equivalent fraction with an LCD of 21.

2 Write the expression as a single fraction by adding the numerators.

3 Expand the brackets.

4 Simplify the numerator by adding and subtracting like terms.

WRITE

a $\frac{2x+3}{7} + \frac{x-1}{3} = \frac{2x+3}{7} \times \frac{3}{3} + \frac{x-1}{3} \times \frac{7}{7}$

$= \frac{3(2x+3)}{21} + \frac{7(x-1)}{21}$

$= \frac{3(2x+3)+7(x-1)}{21}$

$= \frac{6x+9+7x-7}{21}$

$= \frac{13x+2}{21}$

b 1 The LCM of 10 and 6 is 30. So write each fraction as an equivalent fraction with an LCD of 30.

2 Write the expression as a single fraction by subtracting the numerators.

3 Simplify by dividing the numerator and the denominator by their HCF of 2.

b $\frac{7x}{10} - \frac{x}{6} = \frac{7x}{10} \times \frac{3}{3} - \frac{x}{6} \times \frac{5}{5}$

$= \frac{21x}{30} - \frac{5x}{30}$

$= \frac{21x - 5x}{30}$

$= \frac{16x}{30}$

$= \frac{8x}{15}$

Example 1E.4 Adding and subtracting algebraic fractions with algebraic denominators

Write each of these expressions as one fraction in simplest form.

a $\frac{1}{3x} + \frac{4}{x}$ **b** $\frac{7x}{x+1} - \frac{x-3}{x}$

THINK

a 1 The LCM of $3x$ and x is $3x$. So write each fraction as an equivalent fraction with an LCD of $3x$.

2 Write the expression as a single fraction by adding the numerators.

b 1 The LCM of $(x + 1)$ and x is $x(x + 1)$. So write each fraction as an equivalent fraction with an LCD of $x(x + 1)$.

2 Write the expression as a single fraction by subtracting the numerators.

3 Expand the binomial product in the numerator.

4 Simplify the numerator by adding and subtracting like terms.

WRITE

a $\frac{1}{3x} + \frac{4}{x} = \frac{1}{3x} + \frac{4}{x} \times \frac{3}{3}$

$= \frac{1}{3x} + \frac{12}{3x}$

$= \frac{1 + 12}{3x}$

$= \frac{13}{3x}$

b $\frac{7x}{x+1} - \frac{x-3}{x} = \frac{7x}{x+1} \times \frac{x}{x} - \frac{x-3}{x} \times \frac{x+1}{x+1}$

$= \frac{7x \times x}{x(x+1)} - \frac{(x-3)(x+1)}{x(x+1)}$

$= \frac{7x^2 - (x-3)(x+1)}{x(x+1)}$

$= \frac{7x^2 - (x^2 - 3x + x - 3)}{x(x+1)}$

$= \frac{7x^2 - (x^2 - 2x - 3)}{x(x+1)}$

$= \frac{7x^2 - x^2 + 2x + 3}{x(x+1)}$

$= \frac{6x^2 + 2x + 3}{x(x+1)}$

Helpful hints

- ✔ You don't need to keep the brackets around a binomial factor if the other factors have been removed. For example,
$$\frac{2(x+3)}{4} = \frac{x+3}{2}$$
- ✔ Remember, if you want to turn an algebraic fraction into a sum of algebraic terms, you must divide *every* term in the numerator by the denominator.
$$\frac{x+9}{3} \neq x+3$$
- ✔ You don't need to find the LCD to add and subtract fractions! While the LCD is often the simplest to work with, any common denominator will do. The easiest way to find equivalent fractions with common denominators is to multiply each fraction by the denominator of the other fraction.
$$\frac{a}{c} + \frac{b}{d} = \frac{a}{c} \times \frac{d}{d} + \frac{b}{d} \times \frac{c}{c}$$
$$= \frac{ad}{cd} + \frac{bc}{dc}$$
$$= \frac{ad+bc}{cd}$$

ANS p537

Exercise 1E Algebraic fractions

△ 1–4, 5(a, c, e, g, i, j), 6(a, c), 7–9(1st, 2nd columns), 12(a–c)
□ 3–4(2nd column), 5(b, d, f, h, j, k), 6(b, d), 7–9(3rd, 4th columns), 10, 12(d, e, f), 14, 15
⬠ 3(3rd column), 5(h–l), 6(b, d), 7–9(3rd column), 11, 13, 16–18

UNDERSTANDING AND FLUENCY

1 Simplify each of these algebraic fractions.

a $\frac{3x}{6}$ **b** $\frac{9x}{9xy}$ **c** $\frac{8x}{2x}$ **d** $\frac{4xy}{6y}$

e $\frac{2(x+3)}{10}$ **f** $\frac{8(x-5)}{4}$ **g** $\frac{x(x-4)}{x}$ **h** $\frac{3x(x+1)}{3x}$

2 Simplify each of these algebraic fractions.

a $\frac{(x-2)(x+5)}{(x+1)(x-2)}$ **b** $\frac{(x-6)(x+4)}{(x-2)(x+4)}$ **c** $\frac{(x-1)(x+3)}{(x+3)(x+1)}$

d $\frac{(x-7)(x+2)}{(x-7)(x-8)}$ **e** $\frac{(x+4)(x-5)}{(x+4)(x-5)}$ **f** $\frac{(x-1)(x+1)}{(x+1)(x-1)}$

1E.1 **3** Simplify each of these algebraic fractions.

a $\frac{8x}{4xy}$ **b** $\frac{14abc}{7ac}$ **c** $\frac{5(a+1)}{a(a+1)}$

d $\frac{x(x+3)}{2(x+3)}$ **e** $\frac{5(m-2)}{15m(m-2)}$ **f** $\frac{(n+3)(n+7)}{2(n+3)}$

g $\frac{3y^2(y+x)}{y(x+y)}$ **h** $\frac{x^2(x-4)}{x(x+1)(x-4)}$ **i** $\frac{5(y-3)(y+4)}{(y+4)(y-4)}$

j $\frac{(t+2)(r+3)}{(r+3)(r+2)}$ **k** $\frac{(x+2)(x-1)(2x+7)}{(2x+7)(x-4)(x+2)}$ **l** $\frac{(p^2+2)(p+6)(p-2)}{3(p+6)(p+2)(p^2+2)}$

4 Simplify each of these expressions.

a $\frac{3x}{x+2} \times \frac{x+2}{6x}$ **b** $\frac{x-3}{x} \times \frac{5x}{x-3}$ **c** $\frac{x+4}{x+1} \times \frac{x+1}{x+4}$

d $\frac{6x}{x-5} \div \frac{2x}{x-5}$ **e** $\frac{x+7}{4x} \div \frac{x+7}{20x}$ **f** $\frac{8x}{2x+1} \div \frac{8}{2x+1}$

1E.2 **5** Write each of these expressions as a single fraction in simplest form.

a $\frac{a+2}{a-4} \times \frac{2a+1}{a+2}$

b $\frac{x(x+1)}{x+5} \times \frac{x+5}{x(x-3)}$

c $\frac{t+3}{4(t+5)} \div \frac{t(t+3)}{8(t+5)}$

d $\frac{m(m+2)}{(m-3)(3m+1)} \times \frac{3m+1}{m(m+7)}$

e $\frac{(x+3)(x-2)}{x(x-2)} \times \frac{x(x+5)}{(x-3)}$

f $\frac{x+4}{(x-2)(x+4)} \div \frac{x+5}{x-2}$

g $\frac{2y+3}{y-4} \div \frac{y+3}{3(y-4)}$

h $\frac{(m+1)(m-2)}{(m+4)(m-3)} \times \frac{m(m+4)}{(m-1)(m-2)}$

i $\frac{(c^2+6)(c-4)}{(c+2)(c^2+3)} \times \frac{(2c+1)(c^2+3)}{(c-4)(2c+1)}$

j $\frac{n^2(n+p)}{n+2p} \times \frac{(n-p)(n+2p)}{n^3(n+p)}$

k $\frac{x^2y^3(x+2y)}{y(x-2y)(x+y)} \div \frac{x^4(x+2y)(x+1)}{(x+y)(x-2y)}$

l $\frac{5(2a+b)}{(4-3a)(b+2)} \times \frac{(a+b)(b+2)}{(a-b)(a+b)} \div \frac{b+2a}{2(4-3a)(a^2+b^2)}$

6 Write each of these expressions as a single fraction in simplest form.

a $\frac{6(x+5)(x-5)}{(x-2)(x+2)} \times \frac{(x-1)(x+2)}{3(x+5)(x-1)}$

b $\frac{(x+3)(x+8)}{4x(x+2)(x-3)} \times \frac{4x(x+2)(x-6)}{(x+3)(x+6)}$

c $\frac{(2x-1)(3x+2)}{(3x-2)(x+1)} \div \frac{(x+2)(2x-1)}{(5x+1)(3x-2)}$

d $\frac{5x(3x-5)}{(2x+5)(7x-4)} \div \frac{10(3x-5)(x+8)}{(7x-4)(2x+5)}$

1E.3 **7** Write each of these expressions as a single fraction in simplest form.

a $\frac{x}{4} + \frac{x}{5}$

b $\frac{x}{2} - \frac{x}{7}$

c $\frac{x}{6} + \frac{x}{3}$

d $\frac{x}{8} - \frac{x}{10}$

e $\frac{2x}{3} + \frac{x}{2}$

f $\frac{3x}{4} - \frac{x}{8}$

g $\frac{4x}{5} + \frac{3x}{7}$

h $\frac{x^2}{2} - \frac{x^2}{9}$

i $\frac{5a}{3} + \frac{3a}{5}$

j $\frac{b^2}{2} - \frac{b}{4}$

k $\frac{c+1}{4} + \frac{c}{6}$

l $\frac{m-2}{3} + \frac{m+1}{4}$

m $\frac{x-2}{2} - \frac{x}{5}$

n $\frac{4k+1}{4} - \frac{2k}{5}$

o $\frac{5x+1}{8} + \frac{x-3}{10}$

p $\frac{y}{2} + \frac{y}{4} + \frac{y}{8}$

q $\frac{x}{6} - \frac{x-1}{3}$

r $\frac{b+4}{8} - \frac{b+1}{12}$

s $\frac{2a-1}{4} - \frac{3a+1}{6}$

t $\frac{x+3}{2} + \frac{2x-3}{5} + \frac{x}{4}$

8 Write each of these expressions as a single fraction in simplest form.

a $\frac{x+1}{2} + \frac{x+5}{3}$

b $\frac{x+2}{5} + \frac{x-4}{6}$

c $\frac{x-3}{4} + \frac{x-1}{8}$

d $\frac{x+4}{3} - \frac{x+2}{4}$

e $\frac{x-6}{8} - \frac{x-3}{12}$

f $\frac{x+1}{2} - \frac{x+4}{10}$

g $\frac{2x+3}{3} + \frac{x-7}{5}$

h $\frac{x+5}{6} + \frac{2x+1}{9}$

i $\frac{4x+7}{3} + \frac{5x-3}{2}$

j $\frac{3x-1}{4} - \frac{2x-5}{5}$

k $\frac{4-x}{7} - \frac{x-4}{3}$

l $\frac{x^2+2}{5} - \frac{x^2+3}{8}$

1E.4 **9** Write each of these expressions as a single fraction in simplest form.

a $\frac{1}{x} + \frac{1}{x}$

b $\frac{1}{3a} + \frac{1}{a}$

c $\frac{1}{b} - \frac{5}{b^2}$

d $\frac{1}{x} - \frac{x}{3}$

e $\frac{2}{m+1} + \frac{3}{m}$

f $\frac{k}{k^2+1} + \frac{3}{k}$

g $\frac{3}{y+2} + \frac{1}{y-3}$

h $\frac{x}{x+3} + \frac{2}{x}$

i $\frac{t}{t+5} + \frac{2}{t-1}$

j $\frac{5}{x} - \frac{2}{2x+1}$

k $\frac{5}{x+2} - \frac{3}{x+1}$

l $\frac{4}{2h-1} - \frac{3}{h+2}$

m $\frac{m}{m-3} - \frac{1}{2m+1}$

n $\frac{a^2}{3a+1} + \frac{4}{a}$

o $\frac{x}{2} - \frac{2x+3}{x+1}$

p $\frac{x+1}{x} - \frac{x}{x+1}$

PROBLEM SOLVING AND REASONING

10 Find the algebraic expression that is equal to a if $\frac{a}{2} + \frac{x+1}{4} = \frac{3x-5}{4}$.

11 Find the algebraic expression that is equal to b if $\frac{b}{5} - \frac{x-2}{2} = \frac{x+12}{10}$.

12 Simplify each of these expressions using the most appropriate method.

a $\frac{x}{3} + \frac{2x}{5} + \frac{x}{2}$ **b** $\frac{5x}{4} + \frac{x}{8} - \frac{x}{2}$ **c** $\frac{4x}{3} - \frac{5x}{6} - \frac{3x}{8}$

d $\frac{2x^2}{7} \times \frac{3}{10x} \times \frac{5}{x}$ **e** $\frac{4}{x} \div \frac{12}{x^2} - \frac{5x}{7}$ **f** $\frac{3x}{4} \times \frac{2x(x-7)}{6x^2} + \frac{x+2}{3}$

13 For each of the following, expand the brackets and write the final answer as a single fraction in simplest form. For some, it may be helpful to simplify before expanding.

a $\left(\frac{1}{x} + x\right)\left(\frac{1}{x} - x\right)$ **b** $\left(\frac{1}{x} + x\right)^2$

c $\frac{x}{x+1}\left(\frac{2}{x-1} - \frac{1}{x}\right)$ **d** $\left(\frac{2}{x+1} - \frac{1}{x}\right)\left(\frac{1}{x-1} - \frac{1}{2x}\right)$

14 Consider the expression $\frac{x}{x+1} - \frac{x-1}{x}$.

a Write this expression as a single fraction in simplest form.

b Evaluate:

i $\frac{6}{7} - \frac{5}{6}$ **ii** $\frac{9}{10} - \frac{8}{9}$

15 Consider the expression $\frac{x}{y} - \frac{x}{y+1}$.

a Write this expression as a single fraction in simplest form.

b Evaluate:

i $\frac{5}{6} - \frac{5}{7}$ **ii** $\frac{7}{10} - \frac{7}{11}$

CHALLENGE

16 Consider the following algebraic property: $x - y = -(y - x)$.

Use this property to simplify the following algebraic fractions.

a $\frac{a-b}{b-a}$ **b** $\frac{4(m-n)}{n-m}$ **c** $\frac{(x-1)(5-x)(x-3)}{(x+4)(1-x)(x-5)}$

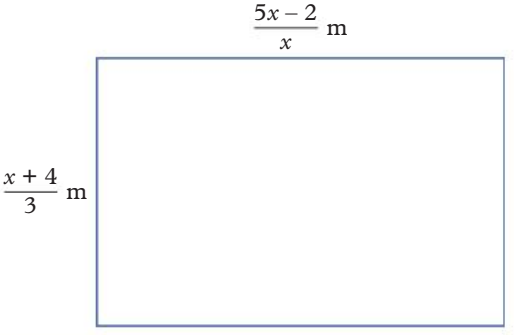
17 Write each of these expressions as a single fraction.

a $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ **b** $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$

18 Consider the rectangle below.

a Write an expression for the area of the rectangle.

b Write an expression for the perimeter of the rectangle, expressed as just a single fraction.



Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Multiplying and dividing algebraic fractions
- Investigation: The shoemaker's knife
- CAS instructions: Simplifying algebraic fractions
- Topic quiz: 1E
- Interactive skillsheet: Adding and subtracting algebraic fractions

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

1A **1** Use the appropriate exponent law to simplify each of these expressions.

a $x^2 \times x^7$ **b** $y^6 \div y^2$ **c** $\frac{2m^4}{m^3}$

d $(a^2)^5$ **e** $(3xy^2)^2$ **f** $\left(\frac{2a}{b^2}\right)^3$

1A **2** Use the property $a^0 = 1$ to simplify each of the following.

a x^0 **b** $2y^0$ **c** $(3a^2)^0$ **d** $x^0 + y^0 + z^0$

1B **3** Rewrite each of these terms with positive exponents.

a y^{-3} **b** $4x^{-2}$ **c** $\frac{a^{-7}}{6}$

d $\frac{2}{b^{-3}}$ **e** $\frac{a^{-2}}{2b^5}$ **f** $5c^{-3}d^4$

1B **4** Simplify the following expression, writing your answer using only positive exponents.

$$\frac{3a^2b^3}{a^3} \times \left(\frac{2a^{-3}}{b^2}\right)^2$$

1C **5** Simplify each of these expressions.

a $4x + 3x - x$ **b** $5m^2 + 3m^2$

c $3a + 5b + 4a$ **d** $c + 3d - 4c + 2d$

e $3a^2 - 4a + 2a^2 + 2a$ **f** $2x^2y + 3xy - 4x^2y + xy^2$

1C **6** For each of the following, find the missing term(s) that will make the equation true.

a $4ab + ? = 7ab$ **b** $5x^2y + 3xy + ? = 11xy + 5x^2y$

c $a^2b - b^2a + ? = 2ab^2 + a^2b$ **d** $2x^2 + 3x - 6 + ? = 2x^2 + 6$

1D **7** Expand and simplify each of these algebraic expressions.

a $3(a + 7)$ **b** $3(2y + 1) + 4$

c $2(x - 5) + x + 2$ **d** $-3(x + 2) - 4(x - 4)$

1D **8** Expand and simplify each of the following.

a $(a + 1)(a + 2)$ **b** $(x - 4)(x + 6)$

c $(k - 5)(k + 5)$ **d** $(2x - 3)(x + 2)$

1D **9** Expand this expression:

$$(x^2 - y)(x + 2y)$$

1E **10** Write each of these expressions as a single fraction in simplest form.

a $\frac{x}{2} + \frac{x}{3}$ **b** $\frac{y + 1}{3} + \frac{5y}{2}$

c $\frac{m^2 + 1}{4} - \frac{m}{5}$ **d** $\frac{2x + 1}{3} - \frac{x + 2}{7}$

1E **11** Simplify each of these algebraic fractions.

a $\frac{a(a + 4)}{3(a + 4)}$ **b** $\frac{(x + 4)(x - 3)}{5(x - 3)}$

c $\frac{(x + 4)(x - 3)}{(x + 7)(2x - 3)} \div \frac{(x - 3)(x + 4)}{2x - 3}$ **d** $\frac{m^2(m - 1)(m + 3)}{(m + 3)(m + 1)} \times \frac{(m + 1)(m - 4)}{m^3(m - 1)}$

1E **12** Simplify this expression:

$$\frac{x}{x - 2} - \frac{3}{x + 1}$$

1F Factorising

Learning intentions

By the end of this topic you will be able to ...

- ✔ factorise algebraic expressions by identifying the highest common factor
- ✔ factorise algebraic expressions by grouping terms
- ✔ factorise algebraic expressions using the 'difference of two' squares and the 'perfect square' rules.

Inter-year links

Year 7	2D Factors and the highest common factor
Year 8	5G Factorising
Year 9	2C Factorising using the HCF

Factorising using the highest common factor

- The distributive law can be used to factorise algebraic expressions by identifying the HCF of the terms in the expression.

$$ab + ac = a(b + c)$$

For example, the HCF of $2x^2$ and $-6x$ is $2x$, so:

$$2x^2 - 6x = 2x(x - 3)$$

expanded form factorised form

Factorising by grouping terms

- It is possible to find binomial factors of large algebraic expressions by grouping pairs of terms and factorising each pair using their HCF.

$$\begin{aligned} ac + ad + bc + bd &= a(c + d) + b(c + d) \\ &= (a + b)(c + d) \end{aligned}$$

For example, $x^2 + 2x - 3x + 6$ can be factorised into a binomial product by grouping the terms $x^2 + 2x$ and $-3x - 6$, and then factorising each of them with a common factor of $x + 2$:

$$\begin{aligned} x^2 + 2x - 3x - 6 &= x(x + 2) - 3(x + 2) \\ &= (x + 2)(x - 3) \end{aligned}$$

Factorising using the 'difference of two squares' rule

- Binomials that have the same terms but a different sign, such as $a + b$ and $a - b$, are conjugates of each other.
- If the expression is the difference of two squares, you can factorise it into the product of conjugate pairs using the 'difference of two squares' rule:

$$a^2 - b^2 = (a + b)(a - b)$$

- If the terms of a binomial expression have opposite signs, but both terms are not perfect squares, you can still factorise the expression using surds. For example,

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$$

Factorising using the 'perfect square' rule

- If the expression is of the same form as a perfect square, then you can factorise it using the 'perfect square' rule.

$$a^2 + 2ab + b^2 = (a + b)^2$$

For example,

$$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2$$
$$= (x + 3)^2.$$

Example 1F.1 Factorising using the highest common factor

Factorise each of these expressions.

a $4n + 36$ **b** $t^2 - 7t$ **c** $5x^5y - 13x^2y^3$ **d** $3klm + 9km - 6k$

THINK

a **1** Identify that the HCF is 4 and write each term as a product of the HCF and its other factor.

2 Use the distributive law. Write the HCF before a pair of brackets containing the sum of the remaining factors.

b **1** Identify that the HCF is t and write each term as a product of the HCF and its other factor.

2 Use the distributive law to factorise the expression.

c **1** Identify that the HCF is x^2y and write each term as a product of the HCF and its other factor.

2 Use the distributive law to factorise the expression.

d **1** Identify that the HCF is $3k$ and write each term as a product of the HCF and its other factor.

2 Use the distributive law. Write the HCF before a pair of brackets containing the sum of the remaining factors.

WRITE

a $4n + 36 = 4 \times n + 4 \times 9$

$= 4(n + 9)$

b $t^2 - 7t = t \times t + t \times (-7)$

$= t(t - 7)$

c $5x^5y - 13x^2y^3 = x^2y \times 5x^3 + x^2y \times (-13y^2)$

$= x^2y(5x^3 - 13y^2)$

d $3klm + 9km - 6k = 3k \times lm + 3k \times 3m + 3k \times (-2)$

$= 3k(lm + 3m - 2)$

Example 1F.2 Factorising by grouping terms

Factorise $a^2 + 3a - 15b - 5ab$ by grouping terms.

THINK

1. Check for the HCF of all four terms. In this case there isn't one. Pair the terms in groups of two so that each pair has a common factor.
2. Use the distributive law to factorise each pair of terms.
3. Factorise using the common binomial factor, which is $(a + 3)$. Put the common binomial factor in front of a new pair of brackets containing the sum of the other factors.

WRITE

$a^2 + 3a - 15b - 5ab = (a^2 + 3a) + (-15b - 5ab)$

$= a(a + 3) - 5b(a + 3)$

$= (a + 3)(a - 5b)$

Example 1F.3 Factorising using the 'difference of two squares' rule

Factorise each of these expressions.

a $16h^2 - 49$ **b** $12a^4 - 27b^2$ **c** $(r + 11)^2 - 1$

THINK

a 1. Check for the HCF of all the terms. In this case there isn't one. Write each term as a square in exponent form.
2. Factorise using the 'difference of two squares' rule.

b 1. Check for the HCF of all the terms. In this case, 3 is the HCF. Using the distributive law, factorise using the HCF.
2. Write each term in the binomial factor as a square in exponent form.
3. Factorise the binomial factor using the 'difference of two squares' rule.

c 1. Check for the HCF of all the terms. In this case, there isn't one. Write each term as a square in exponent form.
2. Factorise using the 'difference of two squares' rule.
3. Simplify each of the bracketed expressions.

WRITE

a $16h^2 - 49 = (4h)^2 - 7^2$

$= (4h + 7)(4h - 7)$

b $12a^4 - 27b^2 = 3(4a^4 - 9b^2)$

$= 3([2a^2]^2 - [3b]^2)$

$= 3(2a^2 + 3b)(2a^2 - 3b)$

c $(r + 11)^2 - 1 = (r + 11)^2 - 1^2$

$= (r + 11 + 1)(r + 11 - 1)$

$= (r + 12)(r + 10)$

Example 1F.4 Factorising using the 'difference of two squares' rule and surds

Factorise each of these expressions.

a $25x^2 - 7$ **b** $(y + 5)^2 - 12$

THINK

a 1 Check for the HCF of the two terms. In this case there isn't one. Write each term as a square in exponent form.

2 Factorise using the 'difference of two squares' rule.

b 1 Check for the HCF of the two terms. In this case there isn't one. Write each term as a square in exponent form.

2 Simplify the surd by factorising 12 so it is a product of 3 and 4, where 4 is a perfect square.

3 Factorise using the 'difference of two squares' rule.

WRITE

a $25x^2 - 7 = (5x)^2 - (\sqrt{7})^2$

$= (5x + \sqrt{7})(5x - \sqrt{7})$

b $(y + 5)^2 - 12 = (y + 5)^2 - (\sqrt{12})^2$

$= (y + 5)^2 - (\sqrt{4 \times 3})^2$
$= (y + 5)^2 - (\sqrt{4} \times \sqrt{3})^2$
$= (y + 5)^2 - (2\sqrt{3})^2$
$= (y + 5 + 2\sqrt{3})(y + 5 - 2\sqrt{3})$

Example 1F.5 Factorising using the 'perfect square' rule

Factorise:

a $t^2 - 18t + 81$ **b** $16x^2 - 56x + 49$

THINK

a 1 Check for the HCF of all three terms. In this case there isn't one. Identify that the square root of t^2 is t, the negative square root of 81 is -9, and $-18t$ is twice the product of both these roots: $2 \times (-9) \times t$.

2 Factorise using the 'perfect square' rule.

b 1 Check for the HCF of all three terms. In this case there isn't one. Identify that the square root of $16x^2$ is $4x$, the negative square root of 49 is -7, and $-56x$ is twice the product of both these roots: $2 \times (-7) \times (4x)$.

2 Factorise using the 'perfect square' rule.

WRITE

a $t^2 - 18t + 81 = t^2 + 2(-9)t + (-9)^2$

$= (t - 9)^2$

b $16x^2 - 56x + 49 = (4x)^2 + 2 \times (-7) \times (4x) + (-7)^2$

$= (4x - 7)^2$

Helpful hints

- ✔ It's important that you are able to recognise when an expression can be factorised by grouping terms, or by identifying it as the difference of two squares, or as a perfect square.
- ✔ Factorising an expression is the reverse process of expanding an expression. You can check if your factorised solution is correct by expanding it to see if you get the same expression that you started with.

expanding →

$7(a + 2) = 7a + 14$

← factorising

- ✔ Always check to see if the terms of an expression have any common factors. It is always best to start with common factors before you move on with any other factorising methods.

  For example, it is easier to identify that $3x^2 + 18x + 27$ can be solved using the 'perfect square' rule once you take out 3 as a common factor:

  $3(x^2 + 6x + 9) = 3(x + 3)^2$
- ✔ Remember that the difference of two squares requires one term to be positive and the other to be negative. For example, the following are NOT the difference of two squares:

  $x^2 + y^2$ and $-a^2 - 4$

ANS p538

Exercise 1F Factorising

△ 1–2(1st, 2nd columns), 3, 4(a, c, e, g), 5, 6(a, d, g, j), 7(a–f), 8, 9(1st, 2nd columns), 10(a, d, g), 11, 12, 13(a, e, g, h, j), 14, 15(a–c)

□ 1–2(3rd column), 4(b, e, h, i), 6–7(2nd column), 9(2nd column), 10, 13(3rd, 4th columns), 15(d, g, i), 17, 18

⬠ 2(4th column), 4(g–i), 6–7(3rd column), 9(j–r), 10, 13(4th column), 15(i–l), 16, 17, 19–21

UNDERSTANDING AND FLUENCY

1F.1

1 Factorise each of these expressions.

a $5a + 35$	**b** $30b + 6$	**c** $24 + 33c$	**d** $12d + 8$
e $8e - 18$	**f** $35f - 21$	**g** $24 + 16g$	**h** $10 - 40h$
i $9j + 9k$	**j** $16w - 28x$	**k** $m^2 + 2m$	**l** $n^2 - 6n$
m $5a + a^2$	**n** $9p - p^2$	**o** $3q^2 + 3q$	**p** $2r^2 - 8r$

2 Factorise each of these expressions.

a $3x - 12$	**b** $6y - 15$	**c** $8x^2 - 16x$	**d** $4k^2 + 6k$
e $5c + 11c^2$	**f** $8u - 4u^2$	**g** $6m^2 - 3m$	**h** $y^2 + 5y^3$
i $14t - 8t^2$	**j** $9p^3 + 3p^2$	**k** $4ab + 3a$	**l** $5t^2u - 3tu$
m $4m^2n + 2mn^2$	**n** $6x^2y + 3xy$	**o** $15c^2 + 21cd$	**p** $18a^2b^3c - 12a^4bc^2$

3 Complete the following factorisations so each expression is factorised using a negative HCF.

a $-2ab - 4a^2 = (-2a) \times __ + (-2a) \times 2a$

$= -2a(__ + 2a)$

b $-10x^3 + 35x = __ \times 2x^2 + __ \times (-7)$

$= __(2x^2 - 7)$

4 Factorise each of these expressions using a negative HCF.

a $-6mn - 18$	**b** $-3ab - 3bc$	**c** $-y^2 - 10y$
d $-18x^2 - 9x$	**e** $-4k + 8k^2$	**f** $-20a^3 - 30a$
g $-12x^2y + 14xy$	**h** $-w^5 + w^7$	**i** $-4b^3 + 2b^2 - 6b^4$

5 **a** Using the binomial factor $(x - 5)$ as the HCF, show that $y(x - 5) + 2(x - 5)$ is written in factorised form as $(x - 5)(y + 2)$.

b In the same way, show that $4a(3 + k^2) - 9(3 + k^2) = (3 + k^2)(4a - 9)$.

6 Factorise each of these expressions using an HCF that is a binomial factor.

a $a(x+3)+5(x+3)$	**b** $m(k-2)+4(k-2)$	**c** $y(y+5)+2(y+5)$
d $x(x-1)+9(x-1)$	**e** $k(k+6)-3(k+6)$	**f** $p(p-9)-6(p-9)$
g $7(4-a)+a(4-a)$	**h** $3n(2n-5)+4(2n-5)$	**i** $x(y^2+2)+8(y^2+2)$
j $a^2(a-4)+3(a-4)$	**k** $2x(y^2+1)-5(y^2+1)$	**l** $5x(7-x^3)+2(7-x^3)$

1F.2 **7** Factorise each of these expressions by grouping terms.

a $ab+4b+3a+12$	**b** $x^2-7x+2xy-14y$	**c** $mn+5m-6n-30$
d $a^2+2a+5a+10$	**e** $x^2-x+4x-4$	**f** $c^2+8c-3c-24$
g $6y-3y^2+8-4y$	**h** $p^3+2p^2+5p+10$	**i** $x^4+x^2+3x^2+3$
j $2m^3+4m-3m^2-6$	**k** $4+4k^2+k^5+k^3$	**l** $xy-15+5y-3x$

8 **a** Expand $(x+5)(x-5)$ using the 'difference of two squares' rule.

b Explain how you can factorise x^2-25.

1F.3 **9** Factorise each of these expressions.

a x^2-4	**b** a^2-36	**c** $100-y^2$
d $64b^2-9$	**e** $25-49p^2$	**f** $4a^2-81b^2$
g $3m^2-3$	**h** $8k^2-18$	**i** $(xy)^2-16w^2$
j m^6-n^2	**k** $4a^{10}-36b^6$	**l** $(x^2y)^2-w^4$
m $(h+3)^2-25$	**n** $(c-4)^2-9$	**o** $(2-x)^2-x^2$
p $1-(a+b)^2$	**q** $4-(y-5)^2$	**r** $(x+3)^2-(x-6)^2$

1F.4 **10** Factorise each of these expressions. Hint: Where needed, use a surd to write a term as a square.

a x^2-7	**b** a^2-13	**c** $19-y^2$
d $16k^2-5$	**e** $(p+8)^2-2$	**f** $(m-4)^2-6$
g $(x+7)^2-3$	**h** $(y-1)^2-19$	**i** $(d-17)^2-17$

11 **a** Expand $(x+5)^2$ using the 'perfect square' rule.

b Explain how you can factorise $x^2+10x+25$.

12 **a** Expand $(y-8)^2$ using the 'perfect square' rule.

b Explain how you can factorise $y^2-16y+64$.

1F.5 **13** Factorise each of these expressions using what you know from the 'perfect square' rule.

a $x^2+12x+36$	**b** $m^2+8m+16$	**c** $p^2+20p+100$	**d** y^2-6y+9
e $x^2-14x+49$	**f** g^2-4g+4	**g** $1+2a+a^2$	**h** $25-10x+x^2$
i $81-18b+b^2$	**j** $w^2+2wx+x^2$	**k** $k^2-2km+m^2$	**l** $9x^2+12x+4$

14 **a** Write an expression for the missing side length of each rectangular item below.

i The area of the welcome mat is (x^2-100) cm^2.

ii The area of the stained glass panel is $(2x^2+3x)$ cm^2.

b Write an expression in factorised form for the perimeter of each of the items above.

PROBLEM SOLVING AND REASONING

15 Simplify each of these algebraic fractions by factorising the numerator and denominator when possible.

a $\frac{a^2 + 2a}{ab + 2b}$ **b** $\frac{xy + 2x}{3y^2 + 6y}$ **c** $\frac{abc^2 - 4ab^2}{ab}$ **d** $\frac{2mn + m^2n^2}{mn + 2}$

e $\frac{x^2 - y^2}{2x + 2y}$ **f** $\frac{3x + 15}{x^2 + 5x}$ **g** $\frac{9x^2 - 15x}{12x - 20}$ **h** $\frac{m^2 - 49}{5m^2 - 35m}$

i $\frac{a^2 + 2ab + b^2}{a^2 - b^2}$ **j** $\frac{x^2 + 6x + 9}{x^2 - 9}$ **k** $\frac{16a^2 - 25}{8a^2 - 10a}$ **l** $\frac{4x^2 + 28x + 49}{49 - 4x^2}$

16 Simplify each of the following.

a $\frac{3x + 21}{2x^2 - 10x} \times \frac{4x^2 - 20x}{x^2 - 49}$ **b** $\frac{x^2 - 1}{10x^2 + 30x} \times \frac{15x + 45}{x^2 + 2x + 1} \times \frac{2x^2 + 2x}{3x - 3}$

17 Consider an odd number represented by the pronumeral n.

a Write expressions for the next two odd numbers.

b Write an expression for the sum of n and the next two odd numbers.

c Factorise your expression from part **b**. Explain how this relates to one of the three odd numbers.

d Use the shortcut you found in part **c** to find the sum of 423, 425 and 427 without adding.

e Investigate a shortcut for adding five consecutive odd numbers. Show your working.

f Use the shortcut you found in part **e** to find the sum of 2671, 2673, 2675, 2677 and 2679 without adding.

CHALLENGE

18 In question **17**, you found expressions for the sum of three consecutive odd numbers and five consecutive odd numbers.

a Investigate shortcuts for finding the sum of an even number of consecutive odd terms. Start with two consecutive odd numbers, then investigate for four consecutive odd numbers, then six consecutive odd numbers, where the first term is n each time.

b Without adding, find the sum of the numbers listed below.

73, 75, 77, 79, 81, 83, 85, 87, 89, 91

19 The difference between two numbers is 7 and the difference between the squares of the two numbers is 105. What is the sum of the two numbers?

20 Factorising using the 'difference of two squares' method can be useful for simplifying certain multiplications. Evaluate each of the following without a calculator by using the 'difference of two squares' factorisation method.

a 19×21 **b** 28×32 **c** 83×77 **d** 54×46

21 Legend has it that the famous mathematician Carl Friedrich Gauss was once punished for misbehaving in primary school by being told he had to find the sum of all the numbers from 1 to 100 before leaving class. Much to his teacher's surprise, he did so in seconds.

The shortcut Gauss probably used was to start by pairing the highest and lowest numbers together, like so:

$(1 + 100) + (2 + 99) + (3 + 98) + (4 + 97) + \ldots$

a What is the sum of each binomial pair in Gauss's method?

b How many binomial pairs will there be once all of the numbers from 1 to 100 are grouped together?

c Use your answers from parts **b** and **c** to write the sum of all the numbers from 1 to 100 as the product of two numbers. Evaluate this product to find Gauss's solution.

d This method can be generalised to write the sum of all the numbers from 1 to n, for any positive integer n. Write the sum of all the numbers from 1 to n as a factorised expression in terms of n.

Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Factorising using the HCF
- Interactive skillsheet: Factorising the difference of two squares
- CAS instructions: Factorising
- Topic quiz: 1F
- Interactive skillsheet: Factorising by grouping terms

1G Factorising quadratic expressions

Learning intentions

By the end of this topic you will be able to ...

✔ factorise quadratic expressions by considering each term.

Inter-year links

Year 7	2D Factors and the highest common factor
Year 8	5G Factorising
Year 9	2E Factorising quadratic expressions

Quadratic trinomials

- A quadratic expression is an algebraic expression that contains a squared pronumeral, with no exponents greater than 2 in the expression. The following are all quadratic expressions:

  $6t^2$ $\quad x^2 + 5$ $\quad k^2 + 14k + 30$ $\quad 4b^2 - a^2$

- A **trinomial** is an algebraic expression that contains three terms.
- A **quadratic trinomial** is an algebraic expression of the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

  $ax^2 + bx + c$

  a is the **leading coefficient**, b is the coefficient of the linear term, and c is the constant term that can be viewed as the coefficient of x^0.
- A **monic quadratic** is a quadratic in which the leading coefficient is equal to 1.

  $x^2 + bx + c$

Factorising monic quadratic trinomials

- Expanding a binomial product of the form $(x + m)(x + n)$ results in a monic quadratic trinomial:

$$\begin{aligned}(x + m)(x + n) &= x^2 + mx + nx + mn \\ &= x^2 + (m + n)x + mn \\ &= x^2 + bx + c \qquad \text{where } b = m + n \text{ and } c = m \times n\end{aligned}$$

- The process can be reversed to factorise a monic quadratic trinomial, $x^2 + bx + c$, by finding two numbers (m and n) that add to give b and multiply to give c.

  For example, in the quadratic trinomial $x^2 + 4x + 3$, the linear coefficient is 4 (that is, $b = 4$) and the constant is 3 (that is, $c = 3$). So the values $m = 3$ and $n = 1$ satisfy the conditions $b = m + n$ and $c = m \times n$.

$$\begin{aligned}x^2 + 4x + 3 &= x^2 + (3 + 1)x + (3 \times 1) \\ &= (x + 3)(x + 1)\end{aligned}$$

expanding ←

$x^2 + 4x + 3 = (x + 1)(x + 3)$

quadratic trinomial (expanded form) | binomial product (factorised form)

→ factorising

Example 1G.1 Factorising simple quadratic trinomials

Factorise each of these quadratic trinomials.

a $x^2 + 6x + 5$ **b** $x^2 + 9x + 18$

THINK

a **1** For $x^2 + 6x + 5$, list the factor pairs of the constant term, 5. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are two combinations of factor pairs with the same numerals.

2 Add the factor pairs together and identify which pair add to the linear coefficient, 6.

3 Write the expression in factorised form.

b **1** For $x^2 + 9x + 18$, list the factor pairs of the constant term, 18.

2 Add the factor pairs together and identify which pair add to the linear coefficient, 9.

3 Write the expression in factorised form.

WRITE

a $1 \times 5 = 5 \rightarrow \quad 1 + 5 = 6$
$(-1) \times (-5) = 5 \rightarrow -1 - 5 = -6$

$x^2 + 6x + 5 = (x + 1)(x + 5)$

b $1 \times 18 = 18 \rightarrow \quad 1 + 18 = 19$
$(-1) \times (-18) = 18 \rightarrow -1 - 18 = -19$
$2 \times 9 = 18 \rightarrow \quad 2 + 9 = 11$
$(-2) \times (-9) = 18 \rightarrow \quad -2 - 9 = -11$
$3 \times 6 = 18 \rightarrow \quad 3 + 6 = 9$
$(-3) \times (-6) = 18 \rightarrow \quad -3 - 6 = -9$

$x^2 + 9x + 18 = (x + 3)(x + 6)$

Example 1G.2 Factorising more complex quadratic trinomials

Factorise each of these quadratic trinomials.

a $x^2 + 2x - 3$ **b** $x^2 - 7x - 8$ **c** $x^2 - 5x + 6$

THINK

1 List the factor pairs of the constant term. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are always two combinations of factor pairs with the same numerals.

2 Add the factor pairs together and identify which pair add to the linear coefficient.

3 Write the expression in factorised form.

WRITE

a $1 \times -3 = -3 \rightarrow \quad 1 - 3 = -2$
$(-1) \times 3 = -3 \rightarrow -1 + 3 = +2$

$x^2 + 2x - 3 = (x + 3)(x - 1)$

b $1 \times (-8) = -8 \rightarrow \quad 1 - 8 = -7$
$(-1) \times 8 = -8 \rightarrow -1 + 8 = 7$
$2 \times (-4) = -8 \rightarrow \quad 2 - 4 = -2$
$(-2) \times 4 = -8 \rightarrow -2 + 4 = 2$

$x^2 - 7x - 8 = (x - 8)(x + 1)$

c $1 \times 6 = 6 \rightarrow \quad 1 + 6 = 7$
$(-1) \times (-6) = 6 \rightarrow -1 - 6 = -7$
$2 \times 3 = 6 \rightarrow \quad 2 + 3 = 5$
$(-2) \times (-3) = 6 \rightarrow -2 - 3 = -5$

$x^2 - 5x + 6 = (x - 3)(x - 2)$

Example 1G.3 Factorising quadratic trinomials by first taking out a common factor

Factorise these by first taking out a common factor.

a $2x^2 - 14x + 12$ **b** $-x^2 + 4x - 3$

THINK

1 If the coefficient of x^2 is not 1, look for a common factor of the three terms (–1 can be a common factor). Write the HCF before the brackets. Inside the brackets, write the results of dividing each term by the HCF.

2 List the factor pairs of the constant term each time. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are always two combinations of factor pairs with the same numerals.

3 Add the factor pairs together and identify which pair add to the linear coefficient.

4 Write the expression in factorised form.

WRITE

a $2x^2 - 14x + 12 = 2(x^2 - 7x + 6)$

$1 \times 6 = 6 \rightarrow \quad 1 + 6 = 7$

$(-1) \times (-6) = 6 \rightarrow -1 - 6 = -7$

$2 \times 3 = 6 \rightarrow \quad 2 + 3 = 5$

$(-2) \times (-3) = 6 \rightarrow -2 - 3 = -5$

$2x^2 - 14x + 12 = 2(x^2 - 7x + 6)$

$= 2(x - 6)(x - 1)$

b $-x^2 + 4x - 3 = -(x^2 - 4x + 3)$

$1 \times 3 = 3 \rightarrow \quad 1 + 3 = 4$

$(-1) \times (-3) = 3 \rightarrow -1 - 3 = -4$

$-x^2 + 4x - 3 = -(x^2 - 4x + 3)$

$= -(x - 3)(x - 1)$

Helpful hints

- ✔ You can always check your factorised quadratic by expanding your solution to see if you get the same expression that you started with.
- ✔ You don't need to list all the factor pairs of the constant term. Just write each new pair as you go and stop when you find the pair that add to give the linear term.
- ✔ You don't always need to consider all the factor pairs of the constant term, c, to factorise a quadratic of the form $x^2 + bx + c$.

  If the constant term, c, is positive, then both factors have the same sign. So:

  → if b is also positive, both factors are positive

  → if b is negative, both factors are negative.
- ✔ Remember, always look for common factors! You should start by checking to see if the leading coefficient is a common factor of all the terms.

ANS p539

Exercise 1G Factorising quadratic expressions

△ 1, 2, 3(1st column), 4, 5, 6(1st column), 7(a, d, g, h, k), 8(a, b, e, f), 9(a, c), 10

□ 3(2nd, 3rd columns), 6–7(2nd, 3rd columns), 8(e–h), 9(b, d), 12, 15

⬠ 3(3rd column), 6–7(2nd, 3rd columns), 8(e–h), 9(b, d), 11, 13–14, 16

1 For each of these pairs of numbers, identify the two numbers that multiply together to give the first number and add together to give the second number.

a 7, 8 **b** 16, 10 **c** 45, 14 **d** 60, 16 **e** 36, 13 **f** 24, 11

2 Use your results from question **1** to factorise each of these quadratic trinomials.

a $x^2 + 8x + 7$ **b** $x^2 + 10x + 16$ **c** $x^2 + 14x + 45$

d $x^2 + 16x + 60$ **e** $x^2 + 13x + 36$ **f** $x^2 + 11x + 24$

1G.1 **3** Factorise each of these quadratic trinomials.

a $x^2 + 3x + 2$ **b** $x^2 + 5x + 6$ **c** $x^2 + 11x + 10$

d $x^2 + 12x + 27$ **e** $x^2 + 7x + 10$ **f** $x^2 + 9x + 14$

g $x^2 + 6x + 9$ **h** $x^2 + 12x + 11$ **i** $x^2 + 11x + 18$

j $x^2 + 12x + 32$ **k** $x^2 + 16x + 63$ **l** $x^2 + 14x + 40$

m $x^2 + 11x + 30$ **n** $x^2 + 13x + 22$ **o** $x^2 + 15x + 36$

4 For each of these pairs of numbers, identify the two numbers that multiply together to give the first number and add together to give the second number.

a $-8, 7$ **b** $6, -7$ **c** $-35, -2$

d $-15, 2$ **e** $-8, -2$ **f** $30, -11$

5 Use your results from question **4** to factorise each of these quadratic trinomials.

a $x^2 + 7x - 8$ **b** $x^2 - 7x + 6$ **c** $x^2 - 2x - 35$

d $x^2 + 2x - 15$ **e** $x^2 - 2x - 8$ **f** $x^2 - 11x + 30$

1G.2 **6** Factorise each of these quadratic trinomials.

a $x^2 + 4x - 12$ **b** $x^2 - 8x - 20$ **c** $x^2 - 5x + 6$

d $x^2 + 6x - 27$ **e** $x^2 - 13x + 40$ **f** $x^2 - x - 20$

g $x^2 + 4x - 21$ **h** $x^2 - 14x + 33$ **i** $x^2 - 3x - 28$

j $x^2 + x - 12$ **k** $x^2 - 7x - 18$ **l** $x^2 - 15x + 56$

1G.3 **7** Factorise each of these quadratic trinomials by first taking out a common factor.

a $2x^2 + 18x + 16$ **b** $3x^2 + 15x + 18$ **c** $7x^2 + 14x - 56$

d $-5x^2 - 35x - 60$ **e** $-4x^2 + 28x - 40$ **f** $6 - 5x - x^2$

g $2x^2 + 14x - 36$ **h** $19x^2 - 38x - 57$ **i** $-3x^2 + 33x + 78$

j $6x - 2x^2 - 4$ **k** $\frac{1}{2}x^2 + \frac{3}{2}x - 2$ **l** $\frac{1}{3}x^2 + x - 6$

8 Simplify each of the following algebraic fractions.

a $\frac{x^2 + 6x + 8}{x + 2}$ **b** $\frac{x + 2}{x^2 - 4}$ **c** $\frac{x^2 + 6x}{x^2 + 4x - 12}$

d $\frac{x^2 - 8x + 15}{x^2 - 9}$ **e** $\frac{2x^2 - 50}{x^2 + 7x + 10}$ **f** $\frac{11x^2 - 22x}{11x^2 - 44}$

g $\frac{x^2 + 2x + 1}{1 - x^2}$ **h** $\frac{4x^2 + 8x - 60}{2x^2 + 18x + 40}$ **i** $\frac{3x^2 - 9x - 30}{6x^2 + 42x + 60}$

9 Simplify each of these expressions using the most appropriate method.

a $\frac{x^2 + x - 20}{x^2 - 6x + 8} \times \frac{x^2 - 4}{x^2 - 25} \times \frac{8x^2 - 40x}{4x^2 + 8x}$ **b** $\frac{x}{x + 7} \times \frac{x^2 - 49}{3x} \div \frac{x^2 - 5x - 14}{x^2 - 4x - 12}$

c $\frac{3x^2 + 3x}{x - 3} \times \frac{x^2 - 7x + 12}{6x^2 + 6x} + \frac{x + 5}{3}$ **d** $\frac{x^2 + 9x + 18}{2x + 6} \div \frac{3x + 18}{5x - 1} + \frac{3x - 5}{8}$

10 Choose the most appropriate method to factorise each of these expressions. Then factorise them.

a $x^2 + 15x + 44$ **b** $x^2 - 2x - 15$ **c** $x^2 + 4x + 4$

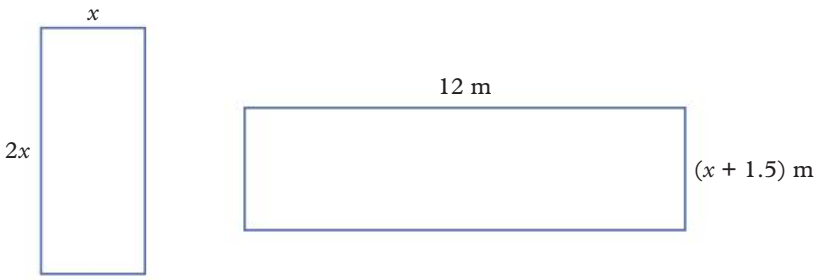
d $3x^2 - 18x + 24$ **e** $5x^2 - 45$ **f** $2x^2 - 8x$

11 Explain why each of these expressions cannot be factorised using the methods covered so far.

a $x^2 + 9$ **b** $(x - 3)^2 + 16$ **c** $(x + 1)^2 + 4$

d $x^2 - 4x + 5$ **e** $(x + 2)^2 - 4x$ **f** $x^2 + 12x + 40$

12 Consider the following two rectangles:



a Write an expression for the area of each rectangle in terms of x.

b Write an expression for the total area of the two rectangles in terms of x.

c The total area of the two rectangles can be combined and reshaped to form two equally sized squares. What is the side length of these squares, in terms of x?

13 The triangle below has a base of $(x + 2)$ metres and an area of $\left(\frac{1}{2}x^2 + 4x + 6\right)$ square metres. Find an expression, in terms of x, for the height of the triangle.

14 It is possible for an expression to be a quadratic with two variables.

a Expand each of the following:

i $(x + y)(x + y)$ **ii** $(x + 2y)(x + 3y)$

b Factorise each of the following:

i $x^2 + 10x + 16$ **ii** $x^2 + 10xy + 16y^2$

c Factorise each of the following:

i $x^2 + 7x + 10$ **ii** $x^2 + 7xy + 10y^2$

d Factorise each of the following:

i $x^2 - 6xy + 8y^2$ **ii** $y^2 - 9xy + 14x^2$ **iii** $4x^2 + 5xy + y^2$

15 If possible, factorise each of the following quadratic expressions, using a method already covered in this chapter.

a $x^2 + 6x$ **b** $x^2 + 6x + 1$ **c** $x^2 + 6x + 2$ **d** $x^2 + 6x + 3$

e $x^2 + 6x + 4$ **f** $x^2 + 6x + 5$ **g** $x^2 + 6x + 6$ **h** $x^2 + 6x + 7$

i $x^2 + 6x + 8$ **j** $x^2 + 6x + 9$

16 What values can k have for each quadratic trinomial below to be able to be factorised using the methods already covered in this chapter?

a $x^2 - 6x + k$

b $x^2 + 20x + k$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Factorising quadratic expressions

Topic quiz
1G

PROBLEM SOLVING AND REASONING

CHALLENGE

1H Completing the square

Learning intentions

By the end of this topic you will be able to ...

- ✔ add a term to a quadratic binomial to make a perfect square
- ✔ factorise quadratic expressions by completing the square.

Inter-year links

Year 8 5G Factorising

Year 9 2D Factorising the difference of two squares

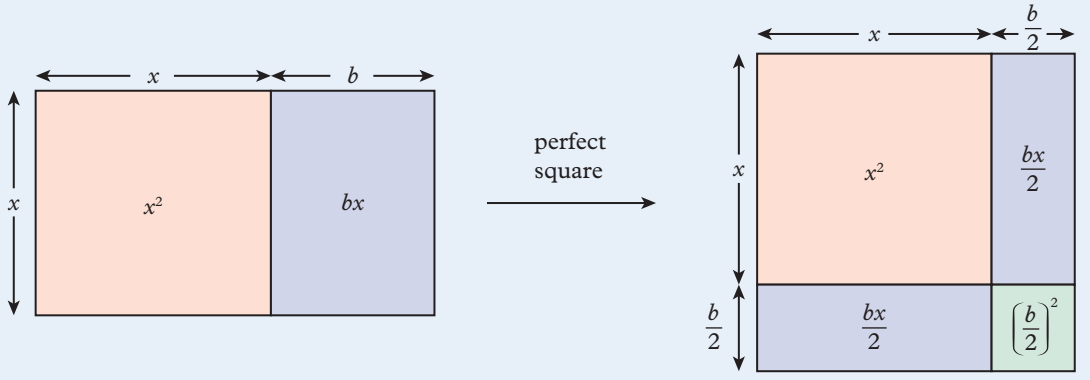
Adding a term to make a perfect square

- For any quadratic of the form $x^2 + bx$, a constant term of the form $\left(\frac{b}{2}\right)^2$ can be added to make it a perfect square.

$$x^2 + 2x\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

'perfect square' rule

$$a^2 + 2ab + b^2 = (a + b)^2$$

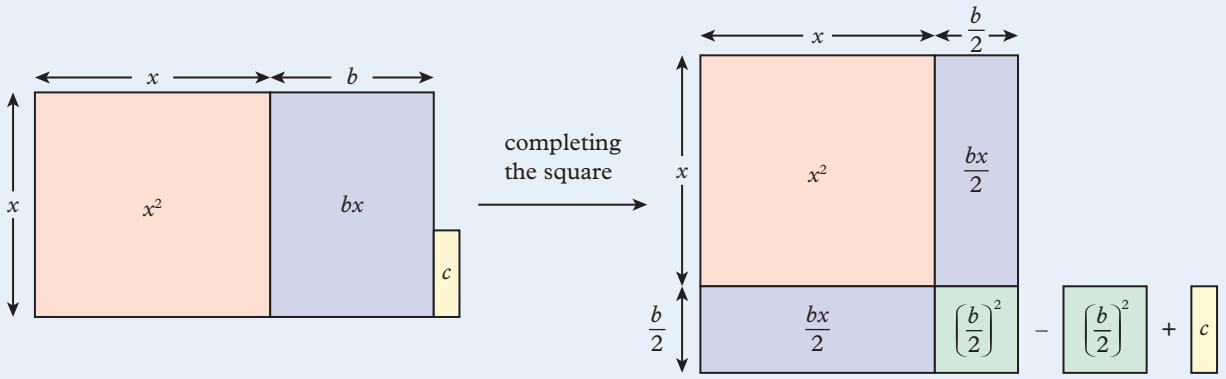


For example, consider the expression $x^2 + 6x$. The coefficient of the linear term is 6, so we can add $\left(\frac{6}{2}\right)^2 = 9$ to the binomial to make it a perfect square:

$$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2$$
$$= (x + 3)^2$$

Completing the square

- **Completing the square** is the process of changing a quadratic expression from the general form, $ax^2 + bx + c$, to the form $a(x - h)^2 + k$ by applying the 'perfect square' rule.
- To complete the square for a monic quadratic, $x^2 + bx + c$, the term $\left(\frac{b}{2}\right)^2$ must be *added and subtracted* from the expression to change it to the form $(x - h)^2 + k$.



- Just as zero can be added to any number without changing its value, an expression equivalent to zero can be added to an expression without changing its value. In this case the expression is:

$$+\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = 0$$

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + 0 + c \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= (x - h)^2 + k \end{aligned}$$

when $h = -\frac{b}{2}$ and $k = -\left(\frac{b}{2}\right)^2 + c$

For example, to complete the square for $x^2 + 8x + 1$, first find the constant term needed to make this a perfect square: $\left(\frac{b}{2}\right)^2 = \left(\frac{8}{2}\right)^2 = 16$

We then add and subtract this value to complete the square:

$$\begin{aligned} x^2 + 8x + 1 &= x^2 + 8x + 0 + 1 \\ &= x^2 + 8x + 16 - 16 + 1 \\ &= (x^2 + 8x + 16) - 16 + 1 \\ &= (x + 4)^2 - 15 \end{aligned}$$

Factorising using the 'difference of two squares' rule

- Once a monic quadratic is written in the form $(x - h)^2 + k$, the value of k determines whether it can be factorised and written as the product of two linear factors.
  - → If $k = 0$, then the quadratic is already factorised and is written as the square of a linear factor.
  - → If $k > 0$, then the quadratic cannot be factorised.
  - → If $k < 0$, the quadratic can be factorised using the 'difference of two squares' rule.
- For example, to factorise the expression $(x + 4)^2 - 15$, from above:

$$a^2 - b^2 = (a + b)(a - b)$$

$$\begin{aligned} (x + 4)^2 - 15 &= (x + 4)^2 - (\sqrt{15})^2 \\ &= (x + 4 + \sqrt{15})\ (x + 4 + \sqrt{15}). \end{aligned}$$

Example 1H.1 Adding a term to make a perfect square

Add a term to each of these expressions to make a new expression that is a perfect square in expanded form.

a $x^2 + 10x$ **b** $x^2 - 8x$

THINK

a **1** For $x^2 + 10x$, substitute the coefficient of the linear term, 10, into the expression $\left(\frac{b}{2}\right)^2$ to find the correct term.

2 Add the term to make a perfect square.

b **1** For $x^2 - 8x$, substitute the coefficient of the linear term, -8, into the expression $\left(\frac{b}{2}\right)^2$ to find the correct term.

2 Add the term to make a perfect square.

WRITE

a $\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2$

$= 5^2$

$= 25$

$x^2 + 10x + 25$

b $\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2$

$= (-4)^2$

$= 16$

$x^2 - 8x + 16$

Example 1H.2 Factorising by completing the square

Factorise each of these quadratic trinomials by completing the square.

a $x^2 + 12x + 15$ **b** $x^2 - 2x - 7$ **c** $x^2 + x - 3$

THINK

1 Substitute the coefficient of the linear term into the expression $\left(\frac{b}{2}\right)^2$ each time to find the term to make a perfect square.

2 Add and subtract the term from the quadratic so that the overall expression remains unchanged.

3 Factorise the perfect square expression using the 'perfect square' rule and simplify the constant terms at the end of the expression.

4 Write as the difference of two squares. Simplify the square root if possible.

5 Factorise using the 'difference of two squares' rule.

WRITE

a The coefficient of the linear term in $x^2 + 12x + 15$ is 12.

$\left(\frac{12}{2}\right)^2 = 6^2$

$= 36$

$x^2 + 12x + 15 = x^2 + 12x + 36 - 36 + 15$

$= (x^2 + 12x + 36) - 36 + 15$

$= (x^2 + 2[6]x + 6^2) - 21$

$= (x + 6)^2 - 21$

$= (x + 6)^2 - (\sqrt{21})^2$

$= (x + 6 + \sqrt{21})(x + 6 - \sqrt{21})$

b The coefficient of the linear term in $x^2 - 2x - 7$ is -2.

$$\left(\frac{-2}{2}\right)^2 = (-1)^2$$
$$= 1$$
$$x^2 - 2x - 7 = x^2 - 2x + 1 - 1 - 7$$
$$= (x^2 - 2x + 1) - 1 - 7$$
$$= (x^2 + 2[-1]x + [-1]^2) - 8$$
$$= (x - 1)^2 - 8$$
$$= (x - 1)^2 - (\sqrt{8})^2$$
$$= (x - 1)^2 - (2\sqrt{2})^2$$
$$= (x - 1 + 2\sqrt{2})(x - 1 - 2\sqrt{2})$$

c The coefficient of the linear term in $x^2 + x - 3$ is 1.

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
$$x^2 + x - 3 = x^2 + x + \frac{1}{4} - \frac{1}{4} - 3$$
$$= \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} - 3$$
$$= \left(x^2 + 2\left[\frac{1}{2}\right]x + \left[\frac{1}{2}\right]^2\right) - \frac{1}{4} - \frac{12}{4}$$
$$= \left(x + \frac{1}{2}\right)^2 - \frac{13}{4}$$
$$= \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{13}}{2}\right)^2$$
$$= \left(x + \frac{1 + \sqrt{13}}{2}\right)\left(x + \frac{1 - \sqrt{13}}{2}\right)$$

Helpful hints

- ✔ Remember to always simplify surds by factorising any square numbers.
- ✔ Remember that you must always add and also subtract the $\left(\frac{b}{2}\right)^2$ term for the expression to remain unchanged.
- ✔ Factorising using the 'difference of two squares' rule only works if $k < 0$. If $k > 0$, then the expression cannot be written as the product of two linear factors.

ANS p539

Exercise 1H Completing the square

△ 1–5, 6(1st column), 7, 8, 11 | □ 3–5, 6(2nd column), 7, 9, 10, 12, 13 | ⬠ 3, 4, 6(2nd column), 7, 10, 12, 14–16

1 a i Expand $(x + 3)^2$ using the 'perfect square' rule. **ii** Use your answer for part **i** to factorise $x^2 + 6x + 9$.

b i Expand $(x - 9)^2$ using the 'perfect square' rule. **ii** Use your answer for part **i** to factorise $x^2 - 18x + 81$.

c Factorise each of these quadratic trinomials.

i $x^2 + 8x + 16$ **ii** $x^2 + 4x + 4$ **iii** $x^2 - 12x + 36$

iv $x^2 + 16x + 64$ **v** $x^2 - 10x + 25$ **vi** $x^2 - 20x + 100$

UNDERSTANDING AND FLUENCY

2 Copy and complete each of these expressions to make a perfect square.

a $x^2 + 6x + __ = (x + 3)^2$ **b** $x^2 + 10x + __ = (x + 5)^2$

c $x^2 - 8x + __ = (x - 4)^2$ **d** $x^2 - 4x + __ = (x - 2)^2$

e $x^2 + 2x + __ = (x + 1)^2$ **f** $x^2 - 16x + __ = (x - 8)^2$

1H.1 **3** Add a term to each of these expressions to make a new perfect square in expanded form.

a $x^2 + 14x$ **b** $x^2 - 6x$

c $x^2 + 18x$ **d** $x^2 + 20x$

e $x^2 - 12x$ **f** $x^2 + 22x$

4 Factorise each perfect square expression you wrote for question **3**.

5 The quadratic trinomial $x^2 + 6x + 7$ can be written as $(x^2 + 6x + 9) - 2$. Explain how knowledge of the 'perfect square' rule and the 'difference of two squares' rule can be used to factorise it.

1H.2 **6** Factorise each of these quadratic trinomials by completing the square.

a $x^2 + 8x + 9$ **b** $x^2 - 10x + 14$

c $x^2 + 4x - 6$ **d** $x^2 + 2x - 5$

e $x^2 - 6x - 4$ **f** $x^2 - 12x + 13$

g $x^2 + 10x - 4$ **h** $x^2 - 14x + 46$

i $x^2 + 8x + 7$ **j** $x^2 + 3x + 1$

k $x^2 - 5x + 3$ **l** $x^2 + 11x + 9$

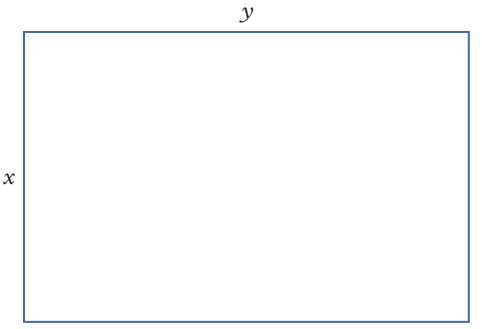
7 Factorise each of these quadratic trinomials by first taking out a common factor.

a $2x^2 + 8x + 2$ **b** $-x^2 - 6x + 2$

c $-5x^2 + 10x + 5$ **d** $3x^2 - 15x + 6$

PROBLEM SOLVING AND REASONING

8 To create a square from the rectangle on the right, other rectangles can be cut out or attached.



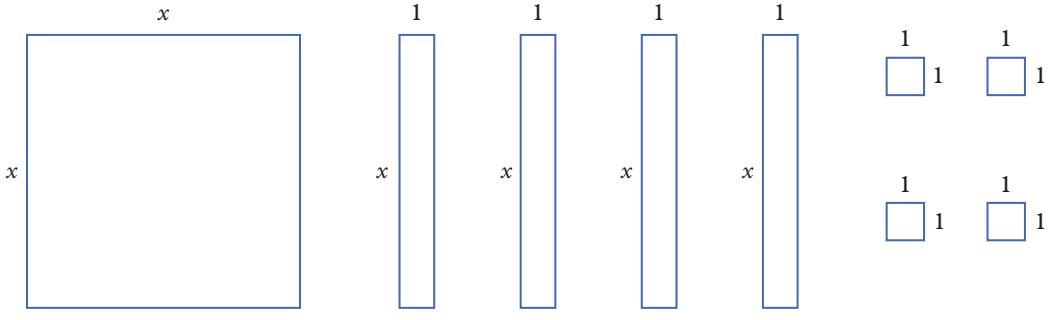
a What are the dimensions of the rectangle needing to be cut out to turn the rectangle into a square?

b If a rectangle is cut out to turn the rectangle into a square, write an expression for the area of the resulting square.

c What are the dimensions of the rectangle needing to be attached to turn the rectangle into a square?

d If a rectangle is attached to turn the rectangle into a square, write an expression for the area of the resulting square.

9 **a** Show how the following shapes can be arranged to form a square.



b Express the area of the square you arranged in part **a** as an expanded quadratic.

c Express the area of the square you arranged in part **a** as a factorised quadratic.

10 In the challenge section of Exercise **1G**, you were asked to factorise as many of the following quadratic expressions as possible using the methods that had been covered. Now that you know how to complete the square, factorise each of the following quadratics using the most appropriate method.

a $x^2 + 6x$ **b** $x^2 + 6x + 1$ **c** $x^2 + 6x + 2$ **d** $x^2 + 6x + 3$

e $x^2 + 6x + 4$ **f** $x^2 + 6x + 5$ **g** $x^2 + 6x + 6$ **h** $x^2 + 6x + 7$

i $x^2 + 6x + 8$ **j** $x^2 + 6x + 9$

11 a Factorise $x^2 + 7x + 6$ by completing the square.

b Look at your answer to part **a**. Is there an easier way of obtaining it? Explain.

12 Not all quadratic trinomials can be factorised.

If a quadratic cannot be factorised by completing the square, then it cannot be factorised at all. For example, a quadratic like $x^2 + 2x + 5$ becomes $(x + 1)^2 + 3$ after completing the square, and this expression cannot be factorised because it cannot be expressed as the difference of two squares. Complete the square for each of the following expressions to find the specified values.

a Find the values of c for which $x^2 + 8x + c$ cannot be factorised.

b Find the values of c for which $x^2 - 10x + c$ cannot be factorised.

c Find the values of b for which $x^2 + bx + 1$ cannot be factorised.

13 Explain why each of these expressions cannot be factorised using the methods covered in this chapter so far.

a $x^2 + 6x + 10$ **b** $x^2 - 8x + 21$ **c** $3x^2 + 7x + 2$

d $12x^2 + 64x + 45$ **e** $2x^2 + 4x + 4$

14 Factorise each of the following expressions, if possible.

a $2x^2 + 6x - 56$ **b** $x^2 - 3x + 5$ **c** $x^3 + 3x^2 - 18x$

d $12 + x - x^2$ **e** $98 - 2x^2$ **f** $4x^2 + 25$

15 Non-monic quadratics can also be factorised by completing the square. For example:

$$\begin{aligned} 2x^2 + 4x - 9 &= 2\left(x^2 + 2x - \frac{9}{2}\right) \\ &= 2\left[(x^2 + 2x + 1) - 1 - \frac{9}{2}\right] \\ &= 2\left[\left(x + 1\right)^2 - \frac{11}{2}\right] \\ &= 2\left(x + 1 - \sqrt{\frac{11}{2}}\right)\left(x + 1 + \sqrt{\frac{11}{2}}\right) \\ &= 2\left(x + 1 - \frac{\sqrt{22}}{2}\right)\left(x + 1 + \frac{\sqrt{22}}{2}\right) \end{aligned}$$

Factorise each of the following quadratics by completing the square:

a $2x^2 + 12x + 5$ **b** $2x^2 + 7x + 1$ **c** $3x^2 + 11x - 1$

16 a Complete the square to factorise the general monic quadratic trinomial $x^2 + bx + c$.

b Considering the fact that you cannot take the square root of a negative number, write an algebraic fraction in terms of b and c that must be *greater than or equal to zero* for $x^2 + bx + c$ to be factorised.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Factorising by completing the square

CAS instructions
Completing the square

Topic quiz
1H

OPTIONAL

CHALLENGE

11 Factorising non-monic quadratic expressions

Learning intentions

By the end of this topic you will be able to ...

- ✔ factorise non-monic quadratic expressions by splitting the middle term
- ✔ factorise non-monic quadratic expressions by completing the square.

Inter-year links

Year 8 5G Factorising

Year 9 2E Factorising quadratic expressions

Non-monic quadratic trinomials

- A **non-monic quadratic** trinomial is a quadratic trinomial in which the leading coefficient is not equal to one. These trinomials are often more difficult to factorise.

  $ax^2 + bx + c$ when $a \neq 1$

- When a non-monic quadratic trinomial $ax^2 + bx + c$ can be factorised, it has the general form $(mx + n)(px + q)$ where m, n, p and q are non-zero constants.

  $ax^2 + bx + c = (mx + n)(px + q)$

Factorising by splitting the middle term

- Expanding the general form of the factorised non-monic quadratic trinomial gives:

  $$\begin{aligned}(mx + n)(px + q) &= mpx^2 + mqx + npx + nq \\ &= mpx^2 + (mq + np)x + nq \\ &= ax^2 + bx + c\end{aligned}$$

  So: $a = m \times p$ $b = mq + np$ $c = n \times q$

  This means there are two numbers, mq and np, which add to give b and multiply to give $a \times c$.

  $b = mq + np$ and $a \times c = (mq)(np)$

- If the middle term, bx, of a quadratic trinomial, $ax^2 + bx + c$, is split into the sum of two terms whose coefficients multiply to give $a \times c$, the quadratic can be factorised by grouping terms. This method of factorisation is called splitting the middle term.

  For example, $3x^2 + 10x + 8$ is in the form $ax^2 + bx + c$. We have $a = 3$, $b = 10$ and $c = 8$. To split the middle term, $10x$, we must find two numbers that add to give b (which is 10) and multiply to give $a \times c$ (which is $3 \times 8 = 24$).

  The numbers 4 and 6 satisfy these conditions:

  $4 + 6 = 10$ and $4 \times 6 = 24$

  So:
  $$\begin{aligned}3x^2 + 10x + 8 &= 3x^2 + 4x + 6x + 8 \\ &= x(3x + 4) + 2(3x + 4) \\ &= (3x + 4)(x + 2)\end{aligned}$$

- If you choose the correct values of mq and np, the order you write the two new terms when you split the middle term does not matter. The resulting factorisation is the same.

  For example, switching the $4x$ and the $6x$ above gives:

  $$\begin{aligned}3x^2 + 10x + 8 &= 3x^2 + 6x + 4x + 8 \\ &= 3x(x + 2) + 4(x + 2) \\ &= (x + 2)(3x + 4)\end{aligned}$$

Factorising by completing the square

- A non-monic quadratic trinomial can also be factorised by completing the square. This is done by taking out the leading coefficient as a factor.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a} + \frac{c}{a}\right)$$

Then we can complete the square using the normal method.

Example 11.1 Factorising by splitting the middle term

Factorise each of these quadratic trinomials by first splitting the middle term.

a $3x^2 + 8x + 4$ **b** $6x^2 + 7x - 3$ **c** $10x^2 - x - 2$

THINK

a
1. Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.
2. List the factor pairs of the product $a \times c = 12$. Remember to consider both positive and negative factors.
3. Add the factor pairs together and identify which pair add to the linear coefficient ($b = 8$).
4. Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.
5. Factorise by grouping terms.

b
1. Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.
2. List the factor pairs of the product $a \times c = -18$. Remember to consider both positive and negative factors.
3. Add the factor pairs together and identify which pair add to the linear coefficient ($b = 7$).
4. Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.
5. Factorise by grouping terms.

c
1. Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.
2. List the factor pairs of the product $a \times c = -20$. Remember to consider both positive and negative factors.
3. Add the factor pairs together and identify which pair add to the linear coefficient ($b = -1$).
4. Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.
5. Factorise by grouping terms.

WRITE

a $3x^2 + 8x + 4$

$b = 8$ and $a \times c = 3 \times 4 = 12$

$1 \times 12 = 12 \rightarrow 1 + 12 = 13$

$(-1) \times (-12) = 12 \rightarrow -1 - 12 = -13$

$2 \times 6 = 12 \rightarrow 2 + 6 = 8$

$(-2) \times (-6) = 12 \rightarrow -2 - 6 = -8$

$3 \times 4 = 12 \rightarrow 3 + 4 = 7$

$(-3) \times (-4) = 12 \rightarrow -3 - 4 = -7$

$$\begin{aligned} 3x^2 + 8x + 4 &= 3x^2 + 2x + 6x + 4 \\ &= x(3x + 2) + 2(3x + 2) \\ &= (3x + 2)(x + 2) \end{aligned}$$

b $6x^2 + 7x - 3$

$b = 7$ and $a \times c = 6 \times (-3) = -18$

$1 \times (-18) = -18 \rightarrow 1 + -18 = -17$

$(-1) \times 18 = -18 \rightarrow -1 + 18 = 17$

$2 \times (-9) = -18 \rightarrow 2 - 9 = -7$

$(-2) \times 9 = -18 \rightarrow -2 + 9 = 7$

$3 \times (-6) = -18 \rightarrow 3 - 6 = -3$

$(-3) \times 6 = -18 \rightarrow -3 + 6 = 3$

$$\begin{aligned} 6x^2 + 7x - 3 &= 6x^2 - 2x + 9x - 3 \\ &= 2x(3x - 1) + 3(3x - 1) \\ &= (3x - 1)(2x + 3) \end{aligned}$$

c $10x^2 - x - 2$

$b = -1$ and $a \times c = 10 \times (-2) = -20$

$1 \times (-20) = -20 \rightarrow 1 - 20 = -19$

$(-1) \times 20 = -20 \rightarrow -1 + 20 = 19$

$2 \times (-10) = -20 \rightarrow 2 - 10 = -8$

$(-2) \times 10 = -20 \rightarrow -2 + 10 = 8$

$4 \times (-5) = -20 \rightarrow 4 - 5 = -1$

$(-4) \times 5 = -20 \rightarrow -4 + 5 = 1$

$$\begin{aligned} 10x^2 - x - 2 &= 10x^2 + 4x - 5x - 2 \\ &= 2x(5x + 2) - 1(5x + 2) \\ &= (5x + 2)(2x - 1) \end{aligned}$$

Example 11.2 Factorising by completing the square

Factorise $5x^2 - 10x + 1$ by completing the square.

THINK

1. Factorise the trinomial by writing the expression as a product of the leading coefficient, which is 5, and a monic quadratic.
2. The linear term of the monic quadratic is -2, so add and subtract the term $\left(\frac{-2}{2}\right)^2 = 1$ from the quadratic so that the overall expression remains unchanged.
3. Factorise the perfect square expression using the 'perfect square' rule and simplify the constant terms at the end of the expression.
4. Write as the difference of two squares. Simplify the square root by identifying any perfect square factors and rationalising the denominator.
5. Factorise using the 'difference of two squares' rule.

WRITE

$$5x^2 - 10x + 1 = 5\left(x^2 - 2x + \frac{1}{5}\right)$$

$$= 5\left(x^2 - 2x + 1 - 1 + \frac{1}{5}\right)$$

$$= 5\left([x^2 - 2x + 1] - \frac{5}{5} + \frac{1}{5}\right)$$

$$= 5\left(\left[x - 1\right]^2 - \frac{4}{5}\right)$$

$$= 5\left(\left[x - 1\right]^2 - \left[\sqrt{\frac{4}{5}}\right]^2\right)$$

$$= 5\left(\left[x - 1\right]^2 - \left[\frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\right]^2\right)$$

$$= 5\left(\left[x - 1\right]^2 - \left[\frac{2\sqrt{5}}{5}\right]^2\right)$$

$$= 5\left(x - 1 + \frac{2\sqrt{5}}{5}\right)\left(x - 1 - \frac{2\sqrt{5}}{5}\right)$$

Helpful hints

✔ Factorising non-monic quadratic trinomials by completing the square often results in an expression containing fractions with surds.

ANS p540

Exercise 11 Factorising non-monic quadratic expressions

△ 1, 2, 3–5(1st, 2nd columns), 6(a, c, e), 7(a, b), 8(a, c, e, g), 12 ■ 3–6(2nd, 3rd columns), 7(c, d), 8(d, f, g, i), 9, 10, 13 ⬠ 4–6(2nd, 3rd columns), 7(e, f), 8(f–i), 11, 14–16

1 Each quadratic trinomial below is in the form $ax^2 + bx + c$. For each one:

- **i** identify ac and b
- **ii** find two numbers that multiply together to give ac and add together to give b
- **iii** use the two numbers you found in part **ii** to split the middle term into two terms.

a $2x^2 + 5x + 2$ **b** $3x^2 + 11x + 6$ **c** $4x^2 + 17x + 4$

d $3x^2 + 10x - 8$ **e** $6x^2 - 7x - 3$ **f** $5x^2 - 13x + 6$

2 Factorise each of these expressions by grouping terms.

a $3x^2 + 2x + 9x + 6$ **b** $4x^2 + x + 16x + 4$ **c** $2x^2 + x + 4x + 2$

d $6x^2 - 9x + 2x - 3$ **e** $3x^2 - 2x + 12x - 8$ **f** $5x^2 - 3x - 10x + 6$

11.1 **3** Factorise each of these quadratic trinomials by first splitting the middle term.

a $2x^2 + 7x + 6$ **b** $3x^2 + 5x + 2$ **c** $4x^2 + 12x + 5$

d $7x^2 + 9x + 2$ **e** $5x^2 + 14x + 8$ **f** $2x^2 + 9x + 9$

g $6x^2 + 7x + 2$ **h** $10x^2 + 17x + 3$ **i** $8x^2 + 14x + 5$

4 Factorise each of these quadratic trinomials by first splitting the middle term.

a $3x^2 - 8x - 3$ **b** $4x^2 + 5x - 6$ **c** $2x^2 - 9x + 10$

d $5x^2 - x - 4$ **e** $8x^2 + 10x - 3$ **f** $4x^2 + 4x - 15$

g $7x^2 - 18x + 8$ **h** $12x^2 + x - 1$ **i** $3x^2 - 11x - 20$

5 Factorise each of these quadratic trinomials. Where appropriate, take out any common factors first.
Hint: The common factor can be negative.

a $4x^2 + 10x + 6$ **b** $12x^2 + 27x + 6$ **c** $10x^2 + 38x - 8$

d $8x^2 + 22x + 15$ **e** $6x^2 + 3x - 30$ **f** $12x^2 - 8x - 4$

g $-4x^2 - 6x - 2$ **h** $-12x^2 - 21x + 6$ **i** $-6x^2 - 17x - 12$

6 Simplify each of these expressions. Remember to factorise first.

a $\dfrac{2x^2 + 15x + 18}{x + 6}$ **b** $\dfrac{3x + 4}{3x^2 + 13x + 12}$ **c** $\dfrac{4x^2 + 18x + 14}{2x + 2}$

d $\dfrac{6x^2 - 5x + 1}{3x^2 + 14x - 5}$ **e** $\dfrac{5x^2 - 28x - 12}{2x^2 - 15x + 18}$ **f** $\dfrac{-8x^2 + 4x + 40}{2x^2 - x - 10}$

7 Simplify each of these expressions.

a $\dfrac{5x^2 - 8x + 3}{x^2 - 1} \times \dfrac{2x^2 + x - 1}{2x^2 - 7x + 3}$ **b** $\dfrac{2x^2 + 13x + 21}{6x^2 - x - 2} \times \dfrac{3x^2 - 14x + 8}{x^2 - x - 12}$

c $\dfrac{4x^2 + 13x - 12}{2x^2 + 13x + 20} \div \dfrac{12x^2 - 5x - 3}{6x^2 + 17x + 5}$ **d** $\dfrac{4x^2 + 28x + 49}{4x^2 - 49} \div \dfrac{2x^2 - x - 28}{2x^2 + 15x + 28}$

e $\dfrac{3x^2 + 6x - 24}{3x^2 - 7x + 2} \times \dfrac{15x^2 + x - 2}{6x + 24}$ **f** $\dfrac{2x^3 - x^2 - 3x}{2x^2 + 9x - 18} \div \dfrac{x^3 - 9x}{x^2 + 9x + 18}$

11.2 **8** Factorise each of these quadratic trinomials by completing the square.

a $3x^2 + 18x + 21$ **b** $2x^2 - 20x + 40$ **c** $5x^2 + 10x - 30$

d $3x^2 + 12x + 3$ **e** $2x^2 - 12x - 3$ **f** $4x^2 + 8x - 1$

g $6x^2 - 24x - 5$ **h** $3x^2 + 24x + 1$ **i** $9x^2 - 18x + 2$

9 Simplify the following expression by first factorising each fraction.

$$\frac{2x^2 - 3x - 20}{x^3 - 16x} \times \frac{3x^2 + 16x + 16}{4x^2 - 25} \div \frac{9x^2 + 24x + 16}{70x - 28x^2}$$

10 a Factorise $4x^2 + 7x - 2$.

b Show that $-4x^2 - 7x + 2$ factorises to $(1 - 4x)(x + 2)$.

c Factorise each of these quadratic trinomials.

i $-2x^2 + x + 3$ **ii** $-3x^2 + 16x - 5$

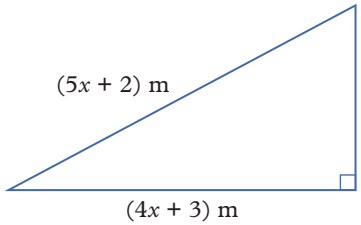
iii $-5x^2 - 3x + 14$ **iv** $-4x^2 + 5x + 6$

v $-6x^2 - x + 2$ **vi** $-25x^2 + 10x - 1$

11 Consider the right-angled triangle shown on the right.

a Find an expression for the length of the missing side, in terms of x.

b The expression you wrote for part **a** should include a square root symbol. Factorise the expression appearing under that square root symbol.



12 Factorise each of the following by first completing the square.

a $x^2 + 8x + 5$ **b** $2x^2 + 8x + 5$ **c** $3x^2 + 8x + 5$

13 Decide whether each of these expressions can be factorised using the methods covered in this chapter so far. Write those that can be factorised in factorised form.

a $2x^2 + 4x + 6$ **b** $15x^2 - 2x - 8$

c $3x^2 - 12x + 3$ **d** $4x^2 - 8x + 16$

e $-2x^2 + 12x - 22$ **f** $-6x^2 + 69x - 189$

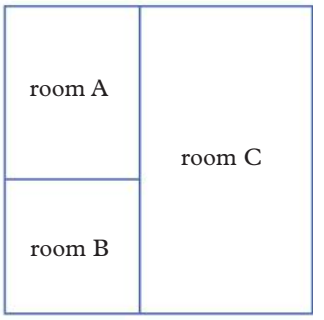
CHALLENGE

14 A karate instructor bought a gym with three training rooms: A, B and C. To make each room a safe place for training, the instructor plans to set down some large square mats of length x metres.

The dimensions of all the rooms are a number of mats plus or minus an integer number of metres. Training room A has an area of $(8x^2 + 2x - 15)$ m^2. The length of the square floor of training room B is the same as the smaller floor dimension of training room A. Assume $x < 4$ m.

Answer the questions about the karate gym.

karate gym



a Write expressions for the dimensions of the floors of training rooms A and B in terms of the length of the mats.

b Write an expression for the total floor area of training rooms A and B combined, in terms of the length of the mats.

c If the shape of the karate gym is a square, what are the dimensions of the gym?

d Write an expression for the total area of training rooms A and C combined, as the difference of two squares.

e Factorise your answer for part **d**.

f Divide the total area of the gym by the area of one mat to determine an expression for the maximum number of mats that the instructor will need to cover the floor of the gym. Give your answer as a trinomial.

g If $x = 3$ m, what is the maximum number of mats that can fit on the floor of the gym?

15 A quartic is an expression of the form $ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are constants. If a quartic does not include any exponents that are odd numbers (so the b and d coefficients are zero), it can be factorised using the same methods for factorising quadratics.

For example, $x^4 + 5x^2 + 6$ can be factorised to $(x^2 + 3)(x^2 + 2)$.

a Factorise:

i $x^4 + 7x^2 + 10$ **ii** $x^4 + 14x^2 + 24$ **iii** $x^4 + 4x^2 + 3$

b Write each of the following quartics as the product of one quadratic factor and two linear factors, by using the 'difference of two squares' method of factorisation.

i $x^4 - 1$ **ii** $x^4 - 16$ **iii** $x^4 - 81$

c Write each of the following quartics as the product of four linear factors.

i $x^4 - 13x^2 + 36$ **ii** $x^4 - 26x^2 + 25$ **iii** $x^4 - 11x^2 + 10$

16 a Complete the square for the general quadratic trinomial $ax^2 + bx + c$.

b Use your answer for part **a** to factorise the general quadratic trinomial $ax^2 + bx + c$.

c Considering the fact that you cannot find the square root of a negative number, write a simplified expression in terms of a, b and c that *must be non-negative* for the quadratic to be factorised.

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Factorising non-monic quadractic expressions

Topic quiz
1I

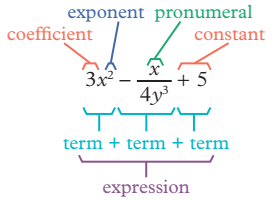
Chapter summary

Exponent laws and properties

$a^m \times a^n = a^{m+n}$	$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{m \times n}$	$(a \times b)^m = a^m \times b^m$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$
$a^0 = 1$	$a^{-1} = \frac{1}{a}$

Simplifying terms and expressions

- Like terms contain the same pronumerals raised to the same exponents.



- Like terms can be added and subtracted by adding and subtracting their cooefficients.

$7x^2y + 2 + 3yx^2 - 8 = 10x^2y - 6$

Expanding

- Use the distributive law $a(b + c) = ab + ac$ to expand brackets.

$(a + b)(c + d) = ac + ad + bc + bd$

Factorising using the HCF

expanding ←

$2x^2 - 6x = 2x(x - 3)$

expanded form factorised form

factorising →

Factorising by grouping terms

$x^2 + 2x - 3x - 6 = x(x + 2) - 3(x + 2)$

$= (x + 2)(x - 3)$

Factorising monic quadratics

$x^2 + bx + c$

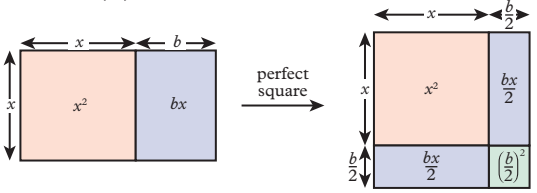
- Find numbers, m and n, that add to give b and multiply to give c.

$x^2 + (m + n)x + mn = (x + m)(x + n)$

- Substitute those two numbers into the factorised form (the binomial product).

Adding a term to make a perfect square

- Add $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to make it a perfect square.



$x^2 + 2x\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

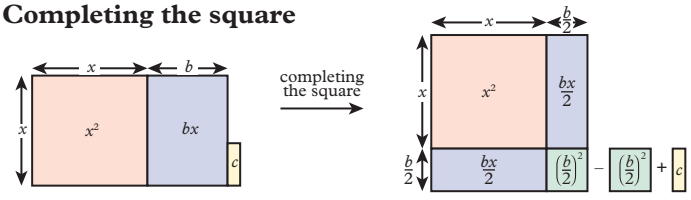
Factorising by splitting the middle term

$ax^2 + bx + c = mpx^2 + mqx + npx + nq$

- Find the unknowns that add to give b and multiply to give $a \times c$.
- Use these numbers to split the linear term.
- Factorise by grouping terms.

$mpx^2 + mqx + npx + nq = (mx + n)(px + q)$

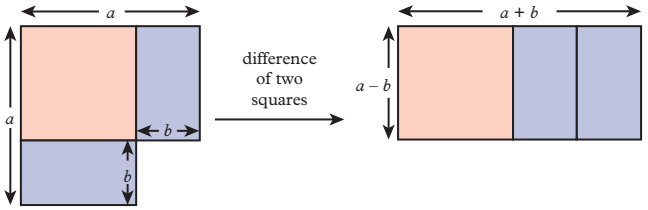
Completing the square



$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

Factorising using the difference of two squares



$a^2 - b^2 = (a + b)(a - b)$

$(x - 1)^2 - 2 = (x - 1)^2 - (\sqrt{2})^2$

$= (x - 1 + \sqrt{2})(x - 1 + \sqrt{2})$

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

1A **1** What is the simplified form of $x^2 \times (x^3)^4$?

A x^9 **B** x^{24} **C** x^{20} **D** x^{14} **E** x^5

1B **2** What is $\frac{36x^2y}{4x^3y^{-2}}$ when it is simplified?

A $\frac{9}{xy^3}$ **B** $\frac{9y^3}{x}$ **C** $9x^{-1}y^{-3}$ **D** $\frac{9}{xy}$ **E** $9xy^3$

1C **3** Which of the following expressions is equivalent to $a^2b - 2ab - ab^2 + 5ab$?

A $7ab$ **B** $a^2b + 3ab + ab^2$ **C** $3ab$
D $3a^2b^2$ **E** $a^2b + 3ab - ab^2$

1D **4** What is the expanded form of $6t^2w(4w^3 + 3t^4)$?

A $24t^2w^3 + 18t^6w$ **B** $24t^2w^4 + 18t^8w$ **C** $24t^2w^4 + 18t^6w$
D $24t^2w^3 + 18t^8w$ **E** $10t^2w^4 + 9t^6w$

1D **5** What is the expanded form of $(x - 4)(x + 3)$?

A $x^2 + x + 12$ **B** $x^2 + x - 12$ **C** $x^2 - x + 12$ **D** $x^2 - x - 12$ **E** $x^2 + 7x - 12$

1E **6** What does $\frac{2x}{3} + \frac{x}{4}$ equal?

A $\frac{3x}{7}$ **B** $\frac{11x}{12}$ **C** $\frac{3x}{12}$ **D** $\frac{11x}{7}$ **E** $\frac{x^2}{6}$

1F **7** What is the fully factorised form of $8x^2y^3 + 12x^3y$?

A $8x^2y(y^2 + 4x)$ **B** $4x^2y(2y^2 + 3x)$ **C** $4xy(2xy^2 + 3x^2)$
D $4xy(2y^2 + 3x)$ **E** $8x^3y^3(x + y^2)$

1F **8** What is the factorised form of $3(a - 2) - b(a - 2)$?

A $(a - 2)(3 - b)$ **B** $(a - 2)(b - 3)$ **C** $(a - 2)(3 + b)$
D $(a + 2)(3 + b)$ **E** $-3b(a - 2)$

1G **9** The quadratic trinomial $x^2 + 7x + c$ *cannot* be factorised for which of the following values of c?

A -18 **B** -8 **C** 10 **D** 12 **E** 18

1G **10** Which of these is a quadratic trinomial?

A $5x^2 + 4$ **B** $2x^3 + 4x^2 - 6$ **C** $x + 5x - 3$ **D** $4x - 2 + 3x^2$ **E** $x^2 - 3$

1H **11** Which of the following expressions is equivalent to $2x^2 + 6x + 7$?

A $2(x + 3)^2 - 2$ **B** $2\left(x + \frac{3}{2}\right)^2 - 2$ **C** $2(x + 3)^2 + 5$
D $2\left(x + \frac{3}{2}\right)^2 - \frac{5}{2}$ **E** $2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2}$

1I **12** What is the factorised form of $9x^2 + 21x + 10$?

A $(9x + 5)(x + 2)$ **B** $(9x + 10)(x + 1)$ **C** $(3x + 5)(3x + 2)$
D $(3x + 10)(3x + 1)$ **E** $(9x + 2)(x + 5)$

Short answer

1A **1** Simplify each of the following.

a $a^2 \times a^5$ **b** $(2x^2y)^3 \times 3xy^2$ **c** $3(x^2y)^0$ **d** $\frac{12ab^2c^5}{(2bc)^2}$

1B **2** Write each of these terms in simplified form with positive exponents.

a $\frac{xy^{-3}}{x^{-2}y}$ **b** $\frac{4a^2b^{-1}}{12a^{-2}b^4}$ **c** $\frac{18p^3q^{-1}r^5}{15p^{-1}q^4r^{-2}}$

1C **3** Simplify each of these expressions, writing the answers with positive exponents.

a $(x^2)^3 \times x^{-2} \div x^{-4}$ **b** $[(p^2q^{-3})^2 \div p^4q^{-3}] \times p^{-1}q$ **c** $(2a^2)^{-3} \times (3ab^{-1})^{-2} \div 12b^5$

1C **4** Simplify each of these expressions.

a $4xy - 10xy + x^2 - 4x^2 + 2x^2y - 5xy^2$ **b** $ab^2 + (ab)^2 - 3a^2b^2 + 2a^2b - 4b^2a$

c $pq - 5pq + p^2 - q^2 - (3p)^2 + (2q)^2$

1C **5** Name the like terms in each of these lists.

a $3abc^2,\ a^2b^2c^2,\ -2a^2b^2c,\ (2abc)^2$ **b** $5xyz^2,\ -2z^2xy,\ 3yz^2x,\ 4x^2y^2y^2$

c $2p^2q^2,\ -(pq)^2,\ p^2 \times q \times -q,\ (p \div q)^2$

1C **6** If $a = -2$, $b = -1$ and $c = 3$, evaluate each of these expressions.

a $a^2 + b^2 + c^2$ **b** $abc - ab^2c - c^3 \div b$ **c** $(a + b)^2 - (c - a)^2$

1D **7** Expand and simplify each of these expressions.

a $(x - 3)(x + 7)$ **b** $(x + 2)^2$ **c** $(x - 4)^2$

d $(2x + 1)(x - 4)$ **e** $(a - b)(a + b)$ **f** $(a^2 + b)(a + b^2)$

1D **8** Expand each of these expressions, giving your answers in simplest form.

a $(2x^2 - 4)(3x^2 + 5)$ **b** $(2x + 3y)(2x - 3y)$ **c** $(3x^2 - 4)^2$

1D **9** Expand and simplify each of these expressions.

a $3(x^2 - 2) - x(x + 5)$ **b** $-2x(3 - x) + 5x(2 - 3x)$ **c** $x(x + 1) - x(x - 1) + x(x - 3)$

1E **10** Simplify each of these expressions.

a $\frac{3x}{4} + \frac{4x}{3}$ **b** $\frac{x - 2}{4} - \frac{3 - x}{5}$ **c** $\frac{4x - 1}{12} - \frac{3 - 2x}{8}$ **d** $\frac{4}{3x} + \frac{5}{4x}$

1F **11** Factorise each of these expressions by grouping terms.

a $x^2 - 3x - 4x + 12$ **b** $x^3 + x^2 - x - 1$ **c** $16x + 6x^2 - 32 - 12x$

1F **12** Use the 'difference of two squares' rule to factorise each of these expressions. Write your answers in simplest form.

a $18x^2 - 8y^2$ **b** $(x - 3)^2 - x^2$ **c** $(x + 2)^2 - (x - 2)^2$

1F **13** Factorise each of these expressions, including surds in your answer if necessary.

a $x^2y^2 - 13$ **b** $(x + 1)^2 - 3$ **c** $7 - (a - 1)^2$

1G **14** Factorise each of these expressions.

a $x^2 + 8x + 12$ **b** $x^2 - 2x - 63$ **c** $-x^2 + 6x - 8$ **d** $-6x^2 - 6x + 36$

1G **15** Simplify each of these expressions.

a $\frac{x^2 - 4}{x^2 - x - 6} \times \frac{x^2 + x - 12}{x^2 + x - 6}$ **b** $\frac{x^2 - 5x + 6}{x^2 - 16} \div \frac{x^2 + x - 6}{x^2 - 3x - 4}$

1H **16** Factorise each of these expressions by first completing the square.

a $x^2 + 10x + 20$ **b** $x^2 - 8x - 1$

1I **17** Factorise each of these expressions.

a $3x^2 + 29x + 18$ **b** $8x^2 + 10x - 3$ **c** $2x^2 - 7x - 22$ **d** $12x^2 - 17x + 6$

1I **18** Factorise each of these expressions, if possible.

a $2x^2 + 12x - 4$ **b** $5x^2 - 20x + 30$ **c** $6x^2 + 21x - 90$ **d** $-4x^2 + 12x - 2$

Analysis

Sameera wants to know about factorising quadratic trinomials of the form $x^2 + bx + 12$, where b is an integer.

a Suppose $b = 0$. Explain why Sameera can't use the 'difference of two squares' rule.

b Is there a value of b that would allow the 'perfect square' rule to be used? If not, why not?

Sameera wants to find all values of b that allow the quadratic to be factorised into a product of two linear factors in the form $(x + c)$ where c is a non-zero integer.

c Complete the square of $x^2 + bx + 12$ and use your answer to show that $b^2 \geq 49$.

d Find all possible values of b by considering the factor pairs of 12 and the corresponding factorisations.

e Sameera next considers quadratics of the form $2x^2 + bx + 12$. Find all values of b for which the quadratic can be written in the form $(x + n)(px + q)$.

2

Real numbers

Index

2A Rational and irrational numbers
2B Simplifying surds
2C Multiplying and dividing surds
2D Adding and subtracting surds
2E Rationalising the denominator
2F Fractional exponents
2G Logarithms
2H Logarithm laws
2I Exponential equations

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Exponents
- ✔ Roots
- ✔ Multiplying terms
- ✔ Dividing terms
- ✔ Equivalent fractions

Curriculum links

- Recognise the effect of using approximations of real numbers in repeated calculations and compare the results when using exact representations (AC9M10N01)
- Use mathematical modelling to solve applied problems involving growth and decay, including financial contexts; formulate problems, choosing to apply linear, quadratic or exponential models; interpret solutions in terms of the situation; evaluate and modify models as necessary and report assumptions, methods and findings (AC9M10A04)

Optional content

- Operations on numbers involving fractional exponents and surds
- The inverse relationship between exponential functions and logarithmic functions and the solution of related equations

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Materials

- ✔ Calculator

2A Rational and irrational numbers

Learning intentions

By the end of this topic you will be able to ...

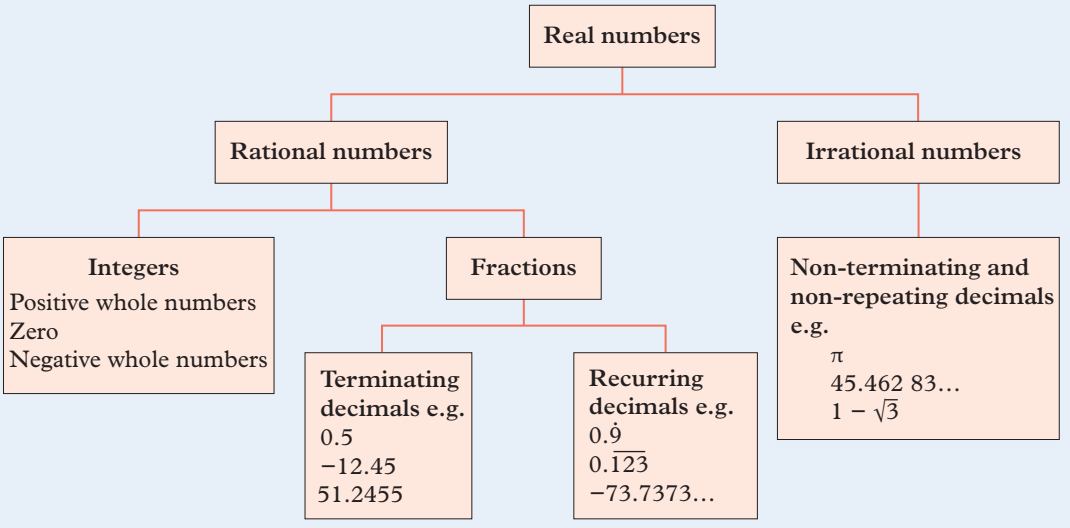
- ✔ determine whether a number is rational or irrational
- ✔ determine whether a square root is a surd.

Inter-year links

Year 7	1G Exponents and square roots
Year 8	4E Roots
Year 9	1F Surds

Rational and irrational numbers

- The real number system is made up of **rational numbers** and **irrational numbers**.
- Rational numbers are numbers that can be written as a fraction in which both the numerator and denominator are integers. They are whole numbers (integers), fractions, terminating decimals and recurring decimals.
- Irrational numbers are numbers that cannot be written exactly as fractions. They include **surds** and other non-terminating decimals that are not recurring.



Surds

- A surd is an irrational root. Surds cannot be simplified to remove the root.
  - → $\sqrt{1} = 1$
    So $\sqrt{1}$ is not a surd.
  - → $\sqrt{2} = 1.414\ 213\ 5\ldots$
    $\sqrt{2}$ cannot be written as a fraction, an integer, or a recurring or terminating decimal, so it is a surd.
  - → $\sqrt{3} = 1.732\ 050\ 8\ldots$
    $\sqrt{3}$ cannot be written as a fraction, an integer, or a recurring or terminating decimal, so it is a surd.
  - → $\sqrt{4}$ can be simplified to 2.
    So $\sqrt{4}$ is not a surd.
- Surds can be other types of roots such as cube roots ($\sqrt[3]{2}$) or fourth roots ($\sqrt[4]{2}$).

Basic rules for square roots

- Finding the square root of and squaring a non-negative number are inverse operations of each other.

$$(\sqrt{a})^2 = a \qquad \sqrt{a^2} = a$$

- A number multiplied by a surd can be written in the form $a\sqrt{b}$, where a and b are both integers and b is non-negative.

$$a \times \sqrt{b} = a\sqrt{b}$$

Example 2A.1 Identifying rational and irrational numbers

Determine whether each of these numbers is rational or irrational.

a 0.26 **b** $-\frac{2}{3}$ **c** $-\sqrt{11}$ **d** $\sqrt{16}$

e $2.3\overline{54}$ **f** 6.584 381 257… **g** $\sqrt[3]{9}$

THINK

a 0.26 is a terminating decimal.
A terminating decimal can be written exactly as a fraction:

$$0.26 = \frac{26}{100}$$

So 0.26 is rational.

b Fractions are rational numbers, so $-\frac{2}{3}$ is rational.

c $-\sqrt{11}$ is approximately equal to –3.316 624 79…
It is a non-terminating and non-recurring decimal which cannot be written as a fraction, so $-\sqrt{11}$ is irrational.

d $\sqrt{16}$ simplifies to 4.
Integers are rational numbers, so $\sqrt{16}$ is rational.

e $2.3\overline{54}$ represents the recurring decimal 2.354 545 454…
Recurring decimals are rational numbers, so $2.3\overline{54}$ is rational.

f 6.584 381 257… is a non-terminating decimal number.
Its decimal place values do not have a pattern of repeating digits, so 6.584 381 257… is irrational.

g $\sqrt[3]{9}$ is approximately equal to 2.080 083 823…
It is a non-terminating and non-recurring decimal which cannot be written as a fraction, so $\sqrt[3]{9}$ is irrational.

WRITE

a 0.26 is a rational number.

b $-\frac{2}{3}$ is a rational number.

c $-\sqrt{11}$ is an irrational number.

d $\sqrt{16}$ is a rational number.

e $2.3\overline{54}$ is a rational number.

f 6.584 381 257… is an irrational number.

g $\sqrt[3]{9}$ is irrational.

Example 2A.2 Identifying surds

Which of these numbers are surds?

a $\sqrt{9}$ **b** $2\sqrt{6}$ **c** $4\sqrt{25}$

THINK

a $\sqrt{9}$ can be simplified to 3, so $\sqrt{9}$ is not a surd.

b $2\sqrt{6} = 2 \times \sqrt{6}$
$\sqrt{6} = 2.449\,489\,743\ldots$ so $2\sqrt{6}$ is a surd.

c $4\sqrt{25} = 4 \times \sqrt{25}$
$\sqrt{25}$ can be simplified to 5.
$4\sqrt{25} = 4 \times 5 = 20$, which is rational.
So $4\sqrt{25}$ is not a surd.

WRITE

a $\sqrt{9}$ is not a surd.

b $2\sqrt{6}$ is a surd.

c $4\sqrt{25}$ is not a surd.

Helpful hints

✔ Some people think that decimals are more 'exact' than fractions. However, mathematicians prefer fractions because it is never possible to tell if a decimal has been rounded, truncated, or has only been measured to a certain number of decimal places. This uncertainty is not an issue with a fraction, which has integers for both its numerator and denominator.

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Exercise 2A Rational and irrational numbers

△ 1–12, 16, 17, 18, 21 ■ 2, 3, 5–9, 11(a, c, f, i, k), 13, 14, 17–22 ⬠ 2, 3, 6, 7, 9, 13–15, 22–26

UNDERSTANDING AND FLUENCY

1 Match each number below with the correct term from the list **A–F**.

a $\frac{3}{4}$	**A** recurring decimal
b 45.710 326 58…	**B** integer
c –9	**C** terminating decimal
d $\sqrt{23}$	**D** fraction
e 0.154 715 471 547…	**E** surd
f 6.213 764 29	**F** non-terminating and non-recurring decimal

2A.1 **2** Determine whether each of these numbers is rational or irrational.

a $\frac{4}{5}$	**b** –5	**c** $\sqrt{3}$	**d** $\sqrt{64}$	**e** 2.548 247 001…
f 5.684	**g** $-\frac{4}{9}$	**h** $-3\sqrt{5}$	**i** $5\frac{1}{6}$	**j** –1.5
k $\sqrt{9}$	**l** $\sqrt{8}$	**m** $\sqrt[3]{125}$	**n** 0.001 002 142…	**o** $\sqrt[4]{32}$

2A.2 **3** Which of these numbers are surds?

a $\sqrt{21}$	**b** $-\sqrt{16}$	**c** $\sqrt[4]{64}$	**d** $-\sqrt{6}$
e $\frac{1}{2}\sqrt{9}$	**f** $5\sqrt{5}$	**g** $-3\sqrt{27}$	**h** $-\sqrt[3]{27}$
i $\sqrt[3]{3}$	**j** $3\sqrt{3}$	**k** $3 + \sqrt{3}$	**l** $\sqrt{5} - 1$
m $\frac{\sqrt{100}}{5}$	**n** $\frac{\sqrt{6}}{5}$	**o** $\frac{\sqrt{6}}{\sqrt{36}}$	**p** $\frac{2\sqrt{2}}{4} - \sqrt{2}$

4 Use your calculator to help you write a decimal approximation, rounded to four decimal places, for each of the surds and irrational numbers in question **3**.

5 Which irrational numbers in question **2** are surds?

6 Calculate the value of each of these without using a calculator:

a $\sqrt{121}$ **b** $5\sqrt{36}$ **c** $-4\sqrt{64}$ **d** $\frac{\sqrt{16}}{4}$

e $4\sqrt{16} + \sqrt{64}$ **f** $\sqrt{36} + \sqrt{64}$ **g** $\sqrt{36 + 64}$ **h** $\sqrt{9 + 16}$

i $\sqrt{25 + 144}$ **j** $-\sqrt{9} - \frac{\sqrt{36}}{2}$ **k** $2\sqrt{100} - \frac{3\sqrt{144}}{4}$ **l** $\sqrt{9^2}$

7 Which of these are rational numbers and which are irrational?

a $-\sqrt{81}$ **b** $\frac{2}{3}$ **c** $\sqrt{0.64}$ **d** $\sqrt{0.036}$

e 3.358 294 721… **f** 8.4 **g** $-\frac{1}{7}$ **h** $-3\sqrt{0.004}$

i $\sqrt{1.369}$ **j** 4.435 353 535… **k** $\sqrt{20.25}$ **l** $\frac{3}{4}\sqrt{100}$

8 Which of the irrational numbers in question **7** are surds?

9 State whether each of these numbers is rational or irrational:

a $\sqrt{\frac{4}{16}}$ **b** $\sqrt{\frac{1}{12}}$ **c** $\frac{\sqrt{16}}{\sqrt{5}}$ **d** $\sqrt{\frac{1}{16}}$

e $\frac{2\sqrt{6}}{\sqrt{4}}$ **f** $\sqrt{\frac{1}{2}}$ **g** $\frac{\sqrt{6}}{\sqrt{25}}$ **h** $\sqrt{\frac{7}{49}}$

10 The Greek letter π (pi) is used as the symbol for a number commonly used in mathematics (equal to 3.141 592…). Is π a rational or irrational number? Provide a reason for your answer.

11 Use your calculator to help you find the value of each of the following expressions, correct to four decimal places.

a $\sqrt{5} + \sqrt{6}$ **b** $2\sqrt{3} + \sqrt{11}$ **c** $\sqrt{30} - \sqrt{15}$ **d** $4\sqrt{8} - 2\sqrt{10}$

e $\sqrt{7} \times \sqrt{5}$ **f** $4\sqrt{3} \times 3\sqrt{7}$ **g** $2\sqrt{24} \div \sqrt{8}$ **h** $(2\sqrt{7} - 4\sqrt{3})^2$

i $\frac{5\sqrt{6}}{\sqrt{7}}$ **j** $\frac{5}{4\sqrt{11}}$ **k** $\frac{4\sqrt{2}}{2 + \sqrt{3}}$ **l** $\frac{3 - \sqrt{5}}{\sqrt{3} + 5}$

12 Consider the two fractions $\sqrt{\frac{5}{8}}$ and $\frac{\sqrt{5}}{\sqrt{8}}$.

a Use your calculator to help you determine the approximate value for each of $\sqrt{\frac{5}{8}}$ and $\frac{\sqrt{5}}{\sqrt{8}}$, to six decimal places.

b Are the two fractions equivalent? Explain.

c Use your results to comment on the relationship between fractions written in the forms $\frac{\sqrt{a}}{\sqrt{b}}$ and $\sqrt{\frac{a}{b}}$.

d Is $\sqrt{\frac{5}{8}}$ a rational number or an irrational number? Provide a reason for your answer.

e Write each of these fractions in the form $\sqrt{\frac{a}{b}}$ to help you decide if it is a rational or irrational number.

i $\frac{\sqrt{32}}{\sqrt{2}}$ **ii** $\frac{\sqrt{15}}{\sqrt{3}}$ **iii** $\frac{\sqrt{7}}{\sqrt{28}}$

13 a Calculate:

i $\sqrt{4}$ **ii** $\sqrt{9}$

b Between which two consecutive integers does $\sqrt{5}$ lie?

c How does a knowledge of perfect squares help you decide which two consecutive integers a surd lies between?

d Between which two consecutive integer values do each of these surds lie?

i $\sqrt{11}$ **ii** $-\sqrt{20}$ **iii** $\sqrt{131}$

iv $-\sqrt{48}$ **v** $\sqrt{180}$ **vi** $\sqrt{300}$

14 Knowing which two integers a surd lies between can help us approximate values when more calculations are involved. We can apply the same operations to the two consecutive integers as are applied to the surd.

a Use the fact that $\sqrt{26}$ lies between 5 and 6 to determine which two integers the results of the following calculations lie between.

i $\sqrt{26} + 3$ **ii** $\sqrt{26} - 10$ **iii** $10 - \sqrt{26}$

iv $3\sqrt{26}$ **v** $3\sqrt{26} - 10$ **vi** $3(\sqrt{26} - 10)$

b Determine which two rational numbers each of the following lies between.

i $\frac{\sqrt{14}}{5}$ **ii** $\frac{5}{\sqrt{14}}$ **iii** $\frac{6 + \sqrt{51}}{2}$

iv $\frac{6 - \sqrt{51}}{2}$ **v** $-5 + \frac{\sqrt{91}}{6}$ **vi** $-5 + \frac{6}{\sqrt{91}}$

15 Determine which two rational numbers the answer to each of the following calculations lies between.

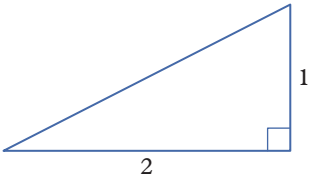
a $\sqrt[3]{11}$ **b** $\sqrt[4]{11}$ **c** $\sqrt[3]{36}$ **d** $\sqrt[5]{400\,000}$

e $7\sqrt[3]{25}$ **f** $7 - \sqrt[3]{70}$ **g** $3 - 5\sqrt[3]{144}$ **h** $\frac{6 + 2\sqrt[3]{300}}{3}$

16 You will have seen by now that the decimal value for a surd that is obtained using a calculator is only an approximation. But the surd represents an exact value.

a Use Pythagoras' theorem to show that the exact length of the hypotenuse of the triangle in this diagram is $\sqrt{5}$.

b Write a decimal approximation for the length of the hypotenuse, to two decimal places.



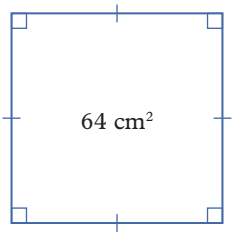
17 A square has an area of 64 cm^2.

a What is the length of each side of the square?

b Calculate the length of the diagonal of the square:

i as an exact value

ii rounded to two decimal places.



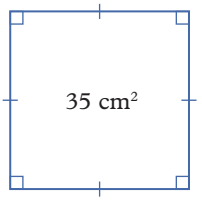
18 Find the side length of each of the squares:

i as an exact value

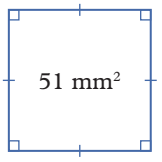
ii rounded to two decimal places

iii truncated to a number with two decimal places.

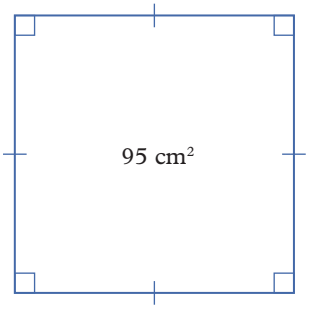
a



b



c



19 Find the perimeter of each of the squares in question **18** using the side length:

i expressed as an exact value, round your final answer to two decimal points

ii rounded to two decimal places

iii truncated to a number with two decimal places.

20 Explain why the perimeters you found for each square in question **19** varied.

21 Show that each of these decimals is a rational number by writing it as a fraction in simplest form.

a 0.4 **b** 0.65 **c** 0.49 **d** 1.6 **e** 2.57 **f** 0.851

22 You will have seen by now that recurring decimals are rational numbers. For example: $0.\dot{6}$ is equivalent to $\frac{2}{3}$.

You can verify that $\frac{2}{3} = 0.666\,666\ldots$ by using a calculator to find $2 \div 3$.

To write a recurring decimal as a fraction (and to show it is rational), there is a series of steps to follow.

a In the calculation below, match each step (**1–6**) with its corresponding line of working (**A–F**). Then write the lines of working in the correct order.

Steps:

1 Let x represent the recurring decimal.

2 Multiply the recurring decimal by 10 to write an equation for $10x$.

3 Form a new subtraction equation from the two equations for x and $10x$. Then carry out the subtraction on each side of that equation.

4 Write the simplified equation that results from the subtraction.

5 Divide both sides of the equation by 9 to obtain x on the left-hand side of the equation and a fraction on the right-hand side.

6 Write the recurring decimal as a fraction, in simplest form.

Lines of working:

A $0.\dot{6} = \frac{2}{3}$

B $x = \frac{6}{9}$

C $10x = 6.666\,666\ldots$

D $x = 0.666\,666\ldots$

E $10x - x = 6.666\,666\ldots - 0.666\,666\ldots$

F $9x = 6$

b Why did you multiply x by 10 in the calculation above? How did this help you obtain a whole number on the right-hand side of the equation in step **4**?

23 Use the method shown in question **22a** to write each of these recurring decimals as a fraction. (Use a calculator to check your answers.)

a $0.\dot{2}$ **b** $0.\dot{7}$ **c** $2.\dot{3}$ **d** $5.\dot{6}$

24 a Complete the working below to write $0.\overline{36}$ as a fraction.

$$\begin{aligned} \text{Let: } x &= 0.363\,636\ldots \\ 100x &= \underline{\qquad\qquad} \\ 100x - x &= \underline{\qquad\qquad} - 0.363\,636\ldots \\ 99x &= \underline{\qquad\qquad} \\ x &= \frac{\square}{99} \end{aligned}$$

So: $0.\overline{36} = \frac{\square}{11}$

b Explain why multiplying x by 100 was helpful in this case.

25 Write each of these recurring decimals as a fraction.

a $0.\overline{14}$ **b** $0.\dot{2}\dot{1}$ **c** $0.\overline{48}$ **d** $1.\dot{7}\dot{3}$

26 Write each of these recurring decimals as a fraction. Show all your working.

a $0.333\,333\,333\ldots$ **b** $0.\overline{24}$ **c** $0.\overline{785}$ **d** $0.\dot{4}\dot{1}$

e $0.5\dot{6}$ **f** $3.\overline{18}$ **g** $4.2\dot{5}$ **h** $3.5\overline{126}$

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Terminating, non-terminating and recurring decimals

Topic quiz
2A

2B Simplifying surds

Learning intentions

By the end of this topic you will be able to ...

- ✔ simplify surds in $\sqrt{a}$ form
- ✔ simplify surds in $a\sqrt{b}$ form.

Inter-year links

Year 7	1G Exponents and square roots
Year 8	4E Roots
Year 9	1F Surds

Basic rules for square roots

- The inverse operation of 'finding the square root' of a number is 'squaring' a number. When a number is squared, it is 'raised to the power of 2' (or the number has 'an exponent of 2').
  $(\sqrt{a})^2 = a$
- The inverse operation of 'squaring' a number is 'finding the square root' of the number.
  $\sqrt{a^2} = a$
- A number multiplied by a surd can be written in the form $a\sqrt{b}$, where a and b are both integers and b is non-negative.
  $a \times \sqrt{b} = a\sqrt{b}$
- The square root of a product can be written as the product of the square roots of its positive factors.
  $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- The square root of a negative number is not a real number. These rules only apply when the number under the square root is non-negative.

Simplifying square roots to surds

- A square root is simplified to a surd by taking out any square factors.
- To simplify a square root (in this case, $\sqrt{12}$):
  1. Rewrite the number under the root sign as a product of its factors, with the first factor being the highest possible perfect square.
  2. Write the square root as the product of its square root factors, using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.
  3. Simplify the square roots of any perfect squares.

$\sqrt{12} = \sqrt{4 \times 3}$
$= \sqrt{4} \times \sqrt{3}$
$= 2 \times \sqrt{3}$
$= 2\sqrt{3}$

Example 2B.1 Simplifying a surd in $\sqrt{a}$ form

Write $\sqrt{54}$ in its simplest form.

THINK	WRITE
1 Rewrite the number under the root sign as a product of its factors, with the first factor being the highest possible perfect square.	$\sqrt{54} = \sqrt{9 \times 6}$
2 Write the square root as the product of its square root factors, using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.	$= \sqrt{9} \times \sqrt{6}$
3 Simplify the square roots of any perfect squares.	$= 3 \times \sqrt{6}$ $= 3\sqrt{6}$

Example 2B.2 Simplifying a surd in $a\sqrt{b}$ form

Write $4\sqrt{72}$ in its simplest form.

THINK

1 Rewrite the number under the root sign as a product of its factors, with the first factor being the highest possible perfect square.

2 Simplify the square roots.

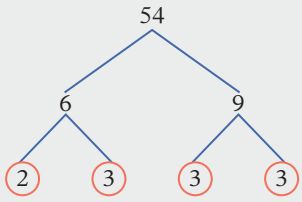
3 Multiply the coefficients and simplify the surd.

WRITE

$4\sqrt{72} = 4\sqrt{36 \times 2}$
$= 4 \times \sqrt{36} \times \sqrt{2}$
$= 4 \times 6 \times \sqrt{2}$
$= 24 \times \sqrt{2}$
$= 24\sqrt{2}$

Helpful hints

- ✔ Remember that factor trees are helpful for determining factors that are square numbers because they make it easy to see a number multiplied by itself to give a square number. For example, in the factor tree shown, $3 \times 3 = 3^2 = 9$ is a square number factor of 54.
- ✔ If there are no repeated prime factors of a root, then the root is a surd and cannot be simplified.
- ✔ Simplifying surds is a very important maths skill for Years 11 and 12.



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Exercise 2B Simplifying surds

△ 1, 2, 3–4(1st–3rd columns), 5, 6(1st–3rd columns), 8, 9, 13, 14

□ 3–4(3rd–5th columns), 6(3rd–5th columns), 8, 9(b, d, j, l), 10, 11, 13, 14, 15(b, d, h, j, m)

⬠ 3–4(4th, 5th columns), 6(4th, 5th columns), 8, 9(i–l), 10(a), 11, 12, 16, 17, 18(c, d)

1 Calculate the value of each of these.

a $\sqrt{16}$	**b** $\sqrt{81}$	**c** $\sqrt{49}$	**d** $\sqrt{4}$	**e** $\sqrt{100}$
f $\sqrt{25}$	**g** $3\sqrt{36}$	**h** $4\sqrt{121}$	**i** $\sqrt{1}$	**j** $\frac{1}{2}\sqrt{16}$

2 Simplify each of these expressions.

a $\sqrt{81} \times \sqrt{7}$	**b** $\sqrt{16} \times \sqrt{3}$	**c** $\sqrt{4} \times \sqrt{6}$
d $\sqrt{25} \times \sqrt{10}$	**e** $\sqrt{9} \times \sqrt{2}$	**f** $\sqrt{5} \times \sqrt{36}$
g $2 \times \sqrt{16} \times \sqrt{11}$	**h** $3 \times \sqrt{100} \times \sqrt{5}$	**i** $\frac{1}{2}\sqrt{144} \times \sqrt{10}$

2B.1 **3** Simplify the following roots.

a $\sqrt{28}$	**b** $\sqrt{50}$	**c** $\sqrt{24}$	**d** $\sqrt{27}$	**e** $\sqrt{48}$
f $\sqrt{75}$	**g** $\sqrt{63}$	**h** $\sqrt{60}$	**i** $\sqrt{56}$	**j** $\sqrt{45}$
k $\sqrt{128}$	**l** $\sqrt{108}$	**m** $\sqrt{180}$	**n** $\sqrt{147}$	**o** $\sqrt{162}$
p $\sqrt{125}$	**q** $\sqrt{450}$	**r** $\sqrt{288}$	**s** $\sqrt{270}$	**t** $\sqrt{567}$

2B.2 **4** Write each of these in simplest form.

a $5\sqrt{20}$	**b** $3\sqrt{8}$	**c** $6\sqrt{12}$	**d** $9\sqrt{18}$	**e** $4\sqrt{80}$
f $3\sqrt{90}$	**g** $12\sqrt{150}$	**h** $-3\sqrt{48}$	**i** $7\sqrt{96}$	**j** $-2\sqrt{245}$
k $8\sqrt{192}$	**l** $10\sqrt{252}$	**m** $-5\sqrt{112}$	**n** $7\sqrt{99}$	**o** $-12\sqrt{68}$
p $100\sqrt{300}$	**q** $\frac{1}{2}\sqrt{320}$	**r** $\frac{1}{2}\sqrt{216}$	**s** $8\sqrt{248}$	**t** $\frac{3}{4}\sqrt{176}$

5 Show that the surd $5\sqrt{27}$ simplifies to $15\sqrt{3}$. Hint: 'Show that' means to show all the necessary calculations.

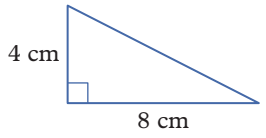
6 Write each of these in simplest form.

a $\sqrt{44}$	**b** $\sqrt{125}$	**c** $\sqrt{120}$	**d** $\sqrt{72}$	**e** $\sqrt{150}$
f $-\sqrt{216}$	**g** $\sqrt{126}$	**h** $\sqrt{240}$	**i** $-3\sqrt{18}$	**j** $\sqrt{245}$
k $\frac{1}{2}\sqrt{192}$	**l** $\frac{\sqrt{108}}{3}$	**m** $-2\sqrt{48}$	**n** $15\sqrt{80}$	**o** $3\sqrt{84}$
p $\frac{3}{5}\sqrt{450}$	**q** $-8\sqrt{800}$	**r** $-\frac{2}{3}\sqrt{900}$	**s** $-\sqrt{1000}$	**t** $\frac{\sqrt{288}}{4}$

PROBLEM SOLVING AND REASONING

7 Use Pythagoras' theorem to calculate the exact length of the unknown side in each triangle below. Where appropriate, write the answer in simplest surd form.

a



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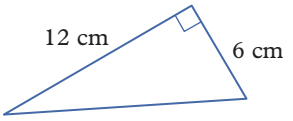

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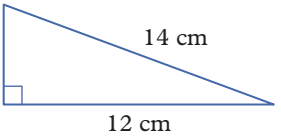

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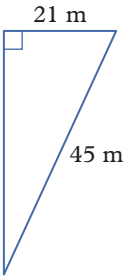


8 A surd in which the entire number is written beneath the root sign can be called an entire surd. For example, $\sqrt{6}$ can be classified as an entire surd, but $3\sqrt{6}$ cannot.

a Copy and complete the following to write $3\sqrt{6}$ as an entire surd.

$$\begin{aligned} 3\sqrt{6} &= \sqrt{3^2} \times \sqrt{6} \\ &= \sqrt{} \times \sqrt{6} \\ &= \sqrt{} \times 6 \\ &= \sqrt{} \end{aligned}$$

b Consider the first line of working in part **a**. Why do you think the coefficient has been written as a perfect square beneath the root sign?

c How do you think the surd we started with ($3\sqrt{6}$) and the answer (the entire surd) are related to each other?

d Briefly explain how you could check that your answer is correct when you write a surd as an entire surd.

e Follow the working shown in part **a** to write $4\sqrt{3}$ as an entire surd.

9 Write each of these as an entire surd.

a $3\sqrt{5}$	**b** $2\sqrt{3}$	**c** $4\sqrt{2}$	**d** $6\sqrt{7}$
e $4\sqrt{10}$	**f** $5\sqrt{11}$	**g** $8\sqrt{6}$	**h** $5\sqrt{5}$
i $4\sqrt{15}$	**j** $-8\sqrt{2}$	**k** $12\sqrt{7}$	**l** $-5\sqrt{3}$

10 a Write $a\sqrt{b}$ as an entire surd.

b Use the general form of the entire surd you wrote in part **a** and check that it produces the same answers you got for the surds in question **9**.

c In your own words, explain how a simplified surd is expressed as an entire surd.

11 Write each of these as an entire surd.

a $m\sqrt{n}$	**b** $7\sqrt{a}$	**c** $x\sqrt{5}$	**d** $10\sqrt{xy}$
e $d\sqrt{2c}$	**f** $3\sqrt{4abc}$	**g** $a\sqrt{ab^2}$	**h** $xy\sqrt{x^2yz^2}$

12 Write each of these in simplest form.

a $\sqrt{x^2y}$	**b** $\sqrt{x^2y^2}$	**c** $\sqrt{ab^2}$	**d** $\sqrt{x^2yz^2}$
e $\sqrt{x^4}$	**f** $\sqrt{x^4y}$	**g** $\sqrt{x^4y^2}$	**h** $\sqrt{a^4b^2c}$

13 An alternative method for simplifying a surd involves writing the number appearing under the root sign as a product of its prime factors. Consider the surd $\sqrt{12}$.

a Write 12 as a product of its prime factors in exponential form.

b Write the product of the prime factors of 12 under the root sign and show that it simplifies to $2\sqrt{3}$.

14 When using the method described in question **13**, you will notice that not all numbers under the root sign result in perfect squares. But, often, they can be written differently so that they do contain perfect squares. Consider $\sqrt{54}$.

a Write 54 as a product of its prime factors in exponential form.

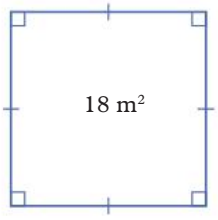
b Use your answer to part **a** to show that $\sqrt{3^2 \times 3 \times 2}$ is an equivalent form of $\sqrt{54}$.

c Simplify your answer and use your calculator to check that it is correct.

15 Use the methods from questions **13** and **14** to simplify each of the following.

a $\sqrt{18}$	**b** $\sqrt{28}$	**c** $\sqrt{32}$	**d** $\sqrt{27}$	**e** $\sqrt{24}$	**f** $\sqrt{50}$
g $3\sqrt{20}$	**h** $2\sqrt{128}$	**i** $5\sqrt{63}$	**j** $8\sqrt{40}$	**k** $7\sqrt{48}$	**l** $6\sqrt{45}$
m $\sqrt{180}$	**n** $5\sqrt{147}$	**o** $6\sqrt{162}$	**p** $\sqrt{125}$	**q** $2\sqrt{450}$	**r** $\sqrt{288}$

16 Consider this square with an area of 18 m^2.



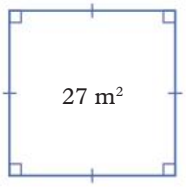
a Write the exact side length of the square as an entire surd and a simplified surd.

b The 18 m^2 square can be divided into nine 2 m^2 squares. Write the exact side length of each of the 2 m^2 squares.

c Describe the relationship between the side lengths of the two different sized squares from parts **a** and **b** and the number of squares the 18 m^2 square is divided into.

d Determine the side length of each of the following blue squares. Hint: In part **iii**, the area of each small square is 2 m^2; in part **iv**, the area of each small square is 3 m^2. Write your answer as an entire surd and a simplified surd each time.

i



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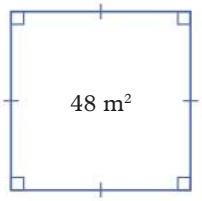

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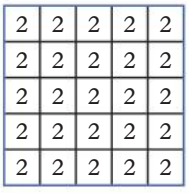

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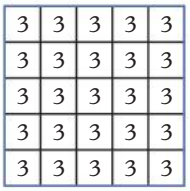


17 a Simplify each of the following surds.

i $\sqrt{8}$ **ii** $\sqrt{27}$ **iii** $\sqrt{125}$ **iv** $\sqrt{a^3}$, where $a > 0$

b Simplify each of the following surds.

i $\sqrt{32}$ **ii** $\sqrt{243}$ **iii** $\sqrt{3125}$ **iv** $\sqrt{a^5}$, where $a > 0$

c Simplify each of the following surds.

i $\sqrt[3]{16}$ **ii** $\sqrt[3]{81}$ **iii** $\sqrt[3]{625}$ **iv** $\sqrt[3]{a^4}$

d Simplify each of the following surds.

i $\sqrt[4]{32}$ **ii** $\sqrt[4]{243}$ **iii** $\sqrt[4]{3125}$ **iv** $\sqrt[4]{a^5}$, where $a > 0$

18 Arrange the surds in each of the following lists in ascending order.

a $\sqrt{46}, \sqrt{22}, \sqrt{42}, \sqrt{10}, \sqrt{27}, \sqrt{5}, \sqrt{21}, \sqrt{30}$

b $2\sqrt{3}, 3, 6, \sqrt{6}, \sqrt{39}, 2\sqrt{6}, \sqrt{11}, 4\sqrt{3}$

c $3\sqrt{3}, 2\sqrt{5}, 3\sqrt{6}, \sqrt{83}, \sqrt{22}, \sqrt{43}, 4, 2\sqrt{11}$

d $7\sqrt{10}, 12\sqrt{2}, 3\sqrt{43}, 2\sqrt{57}, 3\sqrt{14}, 20, 4\sqrt{5}, 2\sqrt{6}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Simplifying surds

CAS instructions Simplifying surds

Topic quiz 2B

2C Multiplying and dividing surds

Learning intentions

By the end of this topic you will be able to ...

- ✔ multiply two or more surds together
- ✔ divide a surd by a surd.

Inter-year links

Year 8	5D Multiplying algebraic terms
Year 9	2A Simplifying

Multiplying surds

- A square root of a number can be written as the product of the square roots of that number's positive factors.

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

- When multiplying surds that are in the form $a\sqrt{b}$, multiply the surd terms and multiply the coefficients.

$$a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$$

Dividing surds

- A square root divided by a square root can be written as the square root of the quotient.

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

- When dividing surds that are in the form $a\sqrt{b}$, divide the surd terms and divide the coefficients.

$$a\sqrt{b} \div c\sqrt{d} = \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \times \sqrt{\frac{b}{d}}$$

Example 2C.1 Multiplying surds

Simplify the following multiplication expressions that contain roots.

a $\sqrt{6} \times \sqrt{5}$ **b** $\sqrt{15} \times \sqrt{6}$ **c** $5\sqrt{3} \times 8\sqrt{11}$

THINK

a Multiply the square roots together.
Remember: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

b **1** Multiply the square roots together.
Remember: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

2 Simplify the square root.

WRITE

a $\sqrt{6} \times \sqrt{5} = \sqrt{6 \times 5}$
$= \sqrt{30}$

b $\sqrt{15} \times \sqrt{6} = \sqrt{15 \times 6}$
$= \sqrt{90}$
$= \sqrt{9 \times 10}$
$= \sqrt{9} \times \sqrt{10}$
$= 3 \times \sqrt{10}$
$= 3\sqrt{10}$

c **1** Multiply the square roots together and multiply the coefficients together. **2** Write the product of the coefficient first. Then write the product of the surd terms under the root sign.	**c** $5\sqrt{3} \times 8\sqrt{11} = 5 \times 8 \times \sqrt{3} \times \sqrt{11}$ $= 40 \times \sqrt{3 \times 11}$ $= 40 \times \sqrt{33}$ $= 40\sqrt{33}$

Example 2C.2 Dividing surds

Simplify the following division expressions that contain roots.

a $\sqrt{27} \div \sqrt{3}$ **b** $\frac{\sqrt{35}}{\sqrt{15}}$ **c** $\frac{36\sqrt{60}}{9\sqrt{12}}$

THINK	WRITE
a **1** Divide one square root by the other. Remember: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**a** $\sqrt{27} \div \sqrt{3} = \frac{\sqrt{27}}{\sqrt{3}}$
2 Simplify the fraction.	$= \sqrt{\frac{27}{3}}$ $= \sqrt{9}$
3 Simplify the square root.	$= 3$
b **1** Divide one square root by the other. Remember: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**b** $\frac{\sqrt{35}}{\sqrt{15}} = \sqrt{\frac{35}{15}}$
2 Simplify the fraction.	$= \sqrt{\frac{7}{3}}$
3 Simplify the square root. $\sqrt{7}$ and $\sqrt{3}$ are surds. Write in the form $\frac{\sqrt{a}}{\sqrt{b}}$.	$= \frac{\sqrt{7}}{\sqrt{3}}$
c **1** Divide the surd terms and divide the coefficients.	**c** $\frac{36\sqrt{60}}{9\sqrt{12}} = \frac{36 \times \sqrt{60}}{9 \times \sqrt{12}}$ $= \frac{36}{9} \times \frac{\sqrt{60}}{\sqrt{12}}$
2 Divide one square root by the other.	$= \frac{36}{9} \times \sqrt{\frac{60}{12}}$
3 Simplify the fractions. $\sqrt{5}$ is already in surd form.	$= 4 \times \sqrt{5}$ $= 4\sqrt{5}$

Helpful hints

- ✔ Make sure that the horizontal line that is part of the root symbol covers all relevant numbers or pronumerals.
- ✔ Your skills with fractions will be very useful to you in this chapter. Simplifying when dividing roots is very similar to multiplying and dividing fractions.

ANS p543

Exercise 2C Multiplying and dividing surds

1(e–l), 2, 3, 4(e–l), 5(e–l), 6–9, 10(a, c, e, g), 11, 12(a, d, e, f), 13, 14, 16, 21(a, b)	1(e–l), 2(a, b, e, f), 3, 4(g–l), 5(i–l), 7(e–h), 8, 10(c, d, g, h), 11(f, h, i, k), 12(d, f, g, h), 17, 18(c–f), 21(a, d, e), 23	4(k–p), 5(i–l), 10(f–i), 11(h–k), 12(g–i), 15, 19, 20, 21, 22, 24

UNDERSTANDING AND FLUENCY

Do not use a calculator for the questions in this exercise.

2C.1 **1** Evaluate the following multiplication expressions that contain roots.

a $\sqrt{7} \times \sqrt{5}$ **b** $\sqrt{3} \times \sqrt{11}$ **c** $\sqrt{2} \times (-\sqrt{7})$ **d** $-\sqrt{15} \times \sqrt{7}$

e $\sqrt{13} \times \sqrt{5}$ **f** $-\sqrt{2} \times \sqrt{11}$ **g** $\sqrt{7} \times \sqrt{3}$ **h** $\sqrt{19} \times \sqrt{5}$

i $\sqrt{7} \times 3\sqrt{11}$ **j** $5\sqrt{3} \times 4\sqrt{2}$ **k** $-4\sqrt{2} \times 7\sqrt{11}$ **l** $12\sqrt{6} \times (-3\sqrt{5})$

m $11\sqrt{2} \times 6\sqrt{17}$ **n** $8\sqrt{6} \times 8\sqrt{7}$ **o** $4\sqrt{3} \times (-6\sqrt{11})$ **p** $12\sqrt{3} \times 11\sqrt{5}$

2 Calculate the value of each of these.

a $\sqrt{7} \times \sqrt{7}$ **b** $\sqrt{3} \times \sqrt{3}$ **c** $\sqrt{2} \times \sqrt{2}$ **d** $\sqrt{6} \times \sqrt{6}$

e $\sqrt{5} \times \sqrt{5}$ **f** $-\sqrt{11} \times (-\sqrt{11})$ **g** $\sqrt{10} \times \sqrt{10}$ **h** $\sqrt{9} \times \sqrt{9}$

3 **a** What did you notice about each result in question **2**?

b Simplify $\sqrt{a} \times \sqrt{a}$, where a is a positive real number.

4 Calculate the value of each of these, writing your answer in surd form where appropriate.

a $3\sqrt{3} \times \sqrt{3}$ **b** $-\sqrt{11} \times 7\sqrt{11}$ **c** $4\sqrt{2} \times 5\sqrt{2}$ **d** $6\sqrt{5} \times 2\sqrt{5}$

e $\sqrt{5} \times 3\sqrt{2}$ **f** $15\sqrt{6} \times 2\sqrt{5}$ **g** $-4\sqrt{2} \times 8\sqrt{13}$ **h** $12\sqrt{3} \times \frac{1}{4}\sqrt{11}$

i $2\sqrt{12} \times 7\sqrt{3}$ **j** $\frac{1}{2}\sqrt{5} \times 3\sqrt{20}$ **k** $\frac{3}{4}\sqrt{13} \times -8\sqrt{2}$ **l** $\frac{3}{5}\sqrt{7} \times \frac{5}{3}\sqrt{15}$

m $-12\sqrt{6} \times (-6\sqrt{7})$ **n** $\frac{1}{2}\sqrt{15} \times \frac{3}{4}\sqrt{13}$ **o** $15\sqrt{10} \times \left(-\frac{1}{3}\sqrt{7}\right)$ **p** $-10\sqrt{17} \times 1.1\sqrt{5}$

2C.2 **5** Simplify the following expressions that contain roots.

a $\sqrt{34} \div \sqrt{2}$ **b** $\sqrt{24} \div \sqrt{8}$ **c** $\sqrt{15} \div \sqrt{15}$ **d** $\sqrt{15} \div (-\sqrt{3})$

e $\frac{\sqrt{42}}{\sqrt{7}}$ **f** $\frac{\sqrt{56}}{\sqrt{8}}$ **g** $4\sqrt{18} \div \sqrt{3}$ **h** $22\sqrt{40} \div 11\sqrt{8}$

i $\frac{-24\sqrt{35}}{8\sqrt{7}}$ **j** $\frac{16\sqrt{50}}{8\sqrt{10}}$ **k** $-12\sqrt{60} \div 3\sqrt{6}$ **l** $8\sqrt{55} \div 16\sqrt{11}$

6 Calculate the value of each of these.

a $\sqrt{32} \div \sqrt{2}$ **b** $\sqrt{24} \div \sqrt{6}$ **c** $\sqrt{54} \div \sqrt{6}$ **d** $\sqrt{40} \div \sqrt{10}$

7 Calculate the value of each of these, writing your answers in surd form.

a $12\sqrt{15} \div (-\sqrt{3})$ **b** $35\sqrt{24} \div 7\sqrt{8}$ **c** $-12\sqrt{48} \div 4\sqrt{24}$ **d** $\frac{18\sqrt{35}}{9\sqrt{7}}$

e $\frac{-32\sqrt{42}}{-8\sqrt{6}}$ **f** $\frac{35\sqrt{5}}{-15}$ **g** $\frac{-24}{-8\sqrt{3}}$ **h** $\sqrt{240} \div 5\sqrt{48}$

8 Show that the result of the calculation $6\sqrt{3} \times 3\sqrt{2}$ is $18\sqrt{6}$. Hint: 'Show that' means to show all the necessary calculations.

9 Consider $\frac{24\sqrt{40}}{8\sqrt{20}}$. By dividing the coefficients and then dividing the surd terms, show that the result is $3\sqrt{2}$.

10 Calculate the value of each of these expressions. Write your answer in fractional surd form where appropriate.

a $6\sqrt{15} \div \sqrt{15}$ (Remember: $6\sqrt{15} \div \sqrt{15}$ is equivalent to $\frac{6\sqrt{15}}{\sqrt{15}}$.)

b $\frac{12\sqrt{30}}{16\sqrt{6}}$ **c** $44\sqrt{34} \div 28\sqrt{42}$ **d** $\frac{-20\sqrt{75}}{25\sqrt{45}}$ **e** $\frac{36\sqrt{35}}{16\sqrt{7}}$

f $\frac{22\sqrt{5}}{-33\sqrt{5}}$ **g** $\frac{8\sqrt{3} \times 6}{-\sqrt{3}}$ **h** $\frac{1}{2}\sqrt{15} \div 4\sqrt{8}$ **i** $9\sqrt{32} \div 18\sqrt{8}$

11 Simplify each of these expressions.

a $\frac{\sqrt{54}}{\sqrt{6}}$ **b** $\sqrt{\frac{81}{100}}$ **c** $\sqrt{32} \times 3\sqrt{2}$

d $\frac{\sqrt{3}}{\sqrt{27}}$ **e** $5\sqrt{12} \times -2\sqrt{3}$ **f** $\sqrt{\frac{36}{9}}$

g $7\sqrt{8} \times \sqrt{18}$ **h** $4\sqrt{6} \times \sqrt{8} \times 6\sqrt{12}$ **i** $8\sqrt{45} \div 5\sqrt{5}$

j $\frac{\sqrt{32}}{16\sqrt{8}}$ **k** $\frac{-4\sqrt{75}}{15\sqrt{3}}$ **l** $\frac{-5\sqrt{343}}{15\sqrt{7}}$

12 Simplify each of these fractions, writing your answer in fraction form where appropriate.

a $\frac{3\sqrt{5} \times 4\sqrt{6}}{6\sqrt{3}}$ **b** $\frac{5\sqrt{7} \times \sqrt{6}}{25\sqrt{14}}$ **c** $\frac{12\sqrt{48}}{3\sqrt{2} \times 5\sqrt{6}}$

d $\frac{11\sqrt{132}}{4\sqrt{11} \times (-2\sqrt{6})}$ **e** $\frac{8\sqrt{3} \times 2\sqrt{8}}{6\sqrt{6} \times \sqrt{2}}$ **f** $\frac{4\sqrt{5} \times -3\sqrt{24}}{5\sqrt{10} \times 6\sqrt{2}}$

g $\frac{\sqrt{3} \times \sqrt{15} \times \sqrt{8}}{\sqrt{6} \times 9\sqrt{12}}$ **h** $\frac{4\sqrt{12} \times 3\sqrt{10} \times (-6\sqrt{7})}{5\sqrt{8} \times 8\sqrt{15}}$ **i** $\frac{-5\sqrt{6} \times 2\sqrt{28} \times (-\sqrt{44})}{3\sqrt{21} \times (-4\sqrt{18})}$

13 To square a number, we multiply the number by itself. Find the value of each of these expressions.

a $(\sqrt{5})^2$ **b** $(\sqrt{8})^2$ **c** $(\sqrt{2})^2$

d $(\sqrt{7})^2$ **e** $(\sqrt{9})^2$ **f** $(4\sqrt{5})^2$

g $(3\sqrt{6})^2$ **h** $(8\sqrt{3})^2$ **i** $(9\sqrt{11})^2$

14 Study your answer for each part of question **13**.

a Does the square of a surd expressed in the form $\sqrt{a}$ or $a\sqrt{b}$ result in a rational answer or an irrational answer?

b Show that your answer to part **a** will always be true.

c Write the surds in parts **f–i** as entire surds.

d Describe the connection between the number beneath the square root of an entire surd and the square of that surd.

15 Simplify each of these expressions.

a $(\sqrt{x})^2$ **b** $(\sqrt{ab})^2$

c $\sqrt{6y} \times \sqrt{6y}$ **d** $\sqrt{x+y} \times \sqrt{x+y}$

16 Find the value of each of these expressions.

a $\left(\frac{5}{\sqrt{7}}\right)^2$ **b** $\left(\frac{1}{\sqrt{5}}\right)^2$ **c** $\left(-\frac{3}{\sqrt{6}}\right)^2$ **d** $\left(\frac{\sqrt{10}}{\sqrt{13}}\right)^2$

17 Consider $3\sqrt{392} \times 5\sqrt{192}$.

a Calculate the value of the expression using each of these methods:

Method 1: Multiply the coefficients, then multiply the surd terms and write the surd in simplest form.

Method 2: Simplify each surd first and then perform the multiplication.

b Do both methods from part **a** produce the same answer?

c When do you think it is best to use:

i Method 1? **ii** Method 2?

18 Calculate the value of each of the following, writing each answer as a fraction in simplest surd form when necessary.

a $3\sqrt{48} \div \sqrt{6}$ **b** $\frac{2\sqrt{54}}{6\sqrt{3}}$ **c** $15\sqrt{104} \div 25\sqrt{8}$

d $\frac{54\sqrt{75}}{63\sqrt{15}}$ **e** $\frac{-21\sqrt{84}}{6\sqrt{7}}$ **f** $\frac{4\sqrt{300}}{3\sqrt{25}}$

g $\frac{8\sqrt{960}}{-\sqrt{48}}$ **h** $\sqrt{384} \div 5\sqrt{64}$ **i** $\frac{-\sqrt{504}}{\sqrt{54}}$

PROBLEM SOLVING AND REASONING

19 Calculate the value of each of these, writing your answer in simplest surd form.

a $\dfrac{22\sqrt{48}}{8\sqrt{2} \times 4\sqrt{3}}$ **b** $\dfrac{4\sqrt{10} \times 6\sqrt{24}}{2\sqrt{5} \times 3\sqrt{2}}$ **c** $\dfrac{3\sqrt{8} \times 7\sqrt{12}}{12\sqrt{6} \times (-8\sqrt{2})}$

d $\dfrac{2\sqrt{15} \times 4\sqrt{12} \times \sqrt{10}}{\sqrt{2} \times 5\sqrt{12}}$ **e** $\dfrac{3\sqrt{14} \times (-5\sqrt{12})}{4\sqrt{2} \times \sqrt{7}}$ **f** $\dfrac{2\sqrt{35} \times 3\sqrt{2} \times 6\sqrt{8}}{5\sqrt{5} \times 12\sqrt{14}}$

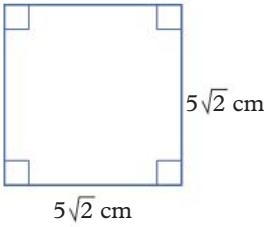
g $\dfrac{-4\sqrt{8} \times 3\sqrt{7} \times 12\sqrt{45}}{2\sqrt{14} \times 6\sqrt{15} \times 3\sqrt{3}}$ **h** $\dfrac{-6\sqrt{26} \times 12\sqrt{2} \times 8\sqrt{12} \times 16\sqrt{33}}{2\sqrt{11} \times 4\sqrt{13} \times 18\sqrt{3} \times 22\sqrt{8}}$

20 Simplify $\dfrac{a\sqrt{bc^2} \times b^2\sqrt{ac}}{c\sqrt{ab^2}}$, where a, b and c are positive real numbers.

21 For each of these shapes, calculate:

i the area, correct to two decimal places **ii** the exact area.

a



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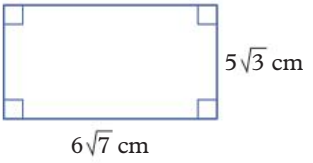

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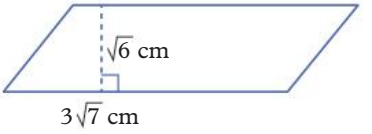

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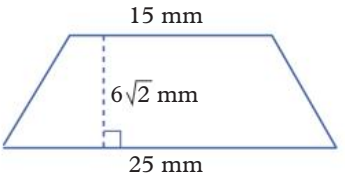

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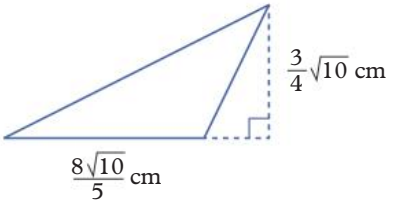

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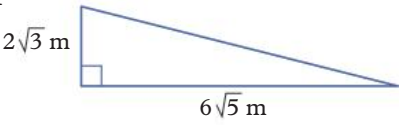

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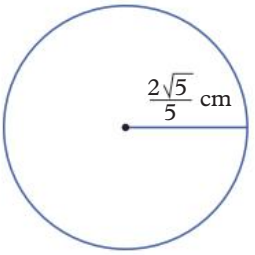

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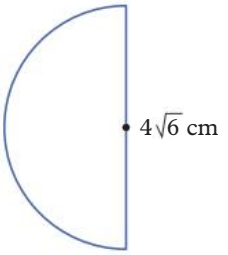

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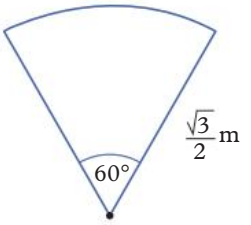


CHALLENGE

22 Consider a square with side length $\sqrt{a}$. Is it possible for the value representing the perimeter to be the same as the value representing the area of this square? Explain your answer.

23 Determine the exact simplified perimeter of a square with an area of:

a 5 m^2 **b** 17 m^2 **c** 18 m^2

d 20 m^2 **e** $\frac{3}{4}$ m^2 **f** $\frac{175}{36}$ m^2

24 For each of the following, determine the smallest surd, x, that will make the product an integer.

a $x \times \sqrt{180}$ **b** $x \times \sqrt{5400}$ **c** $x \times \sqrt{2835}$

d $x \times \sqrt{8064}$ **e** $x \times \sqrt{1911}$ **f** $x \times \sqrt{11\,154}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Multiplying and dividing surds

Topic quiz
2C

2D Adding and subtracting surds

Learning intentions

By the end of this topic you will be able to ...

- ✔ identify like surds
- ✔ add and subtract like surds
- ✔ add and subtract surds after first simplifying.

Inter-year links

Year 8	5C Adding and subtracting algebraic terms
Year 9	2A Simplifying

Adding and subtracting surds

- **Like surds** contain exactly the same surd term, regardless of the coefficient.
  For example, $2\sqrt{2}$ and $3\sqrt{2}$ are like surds but $2\sqrt{5}$ and $2\sqrt{3}$ are not like surds.
- To add or subtract like surds, add or subtract the coefficients.
  $$a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$$
- Sometimes seemingly unlike surds can be simplified to produce like surds. Simplify each surd in an expression by rewriting the number under each root sign as a product of its factors, with the first factor being the highest possible perfect square. Write surds in their simplest form before attempting to add or subtract.
  For example,
  $$\begin{aligned} 2\sqrt{27} - \sqrt{12} &= 2\sqrt{9 \times 3} - \sqrt{4 \times 3} \\ &= 2 \times \sqrt{9} \times \sqrt{3} - \sqrt{4} \times \sqrt{3} \\ &= 2 \times 3\sqrt{3} - 2\sqrt{3} \\ &= 6\sqrt{3} - 2\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

Example 2D.1 Identifying like surds

State whether the surds in each of these pairs are like surds or not.

a $2\sqrt{6}$ and $5\sqrt{6}$ **b** $7\sqrt{6}$ and $7\sqrt{3}$ **c** $5\sqrt{2}$ and $\sqrt{8}$

THINK

a The surds are like surds because they have exactly the same surd term.

b The surds are not like surds because they do not have exactly the same surd term.

c $\sqrt{8}$ needs to be simplified first.
$\sqrt{8} = 2\sqrt{2}$
$\sqrt{8}$ and $2\sqrt{2}$ are like surds because $2\sqrt{2}$ and $5\sqrt{2}$ have exactly the same surd term.

WRITE

a Yes, they are like surds.

b No, they are not like surds.

c Yes, they are like surds.

Example 2D.2 Adding and subtracting surds

Simplify each of these expressions.

a $7\sqrt{3} + 4\sqrt{3} - \sqrt{3}$ **b** $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5}$

THINK

a **1** Group the like surds together. All the surds in this expression are like surds.

2 Simplify the expression by adding and subtracting the coefficients.

b **1** Group the like surds together. There are two groups of like surds in this expression: $\sqrt{5}$ and $\sqrt{2}$

2 Simplify the like terms in the expression by adding and subtracting the coefficients.

WRITE

a $7\sqrt{3} + 4\sqrt{3} - \sqrt{3} = 7\sqrt{3} + 4\sqrt{3} - 1\sqrt{3}$

$= 10\sqrt{3}$

b $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5} = 6\sqrt{5} + 5\sqrt{5} - 3\sqrt{2} - 8\sqrt{2}$

$= 11\sqrt{5} - 11\sqrt{2}$

Example 2D.3 Adding and subtracting surds after first simplifying

Simplify the following expressions.

a $3\sqrt{2} + 4\sqrt{8}$ **b** $4\sqrt{48} - 2\sqrt{75}$

THINK

a **1** Simplify each surd in the expression. The first term, $3\sqrt{2}$, is already in its simplest form. Rewrite the number under the other root sign as a product of its factors, with the first factor being the highest possible perfect square.

2 Simplify the like terms in the expression by adding the coefficients.

b **1** Simplify each surd in the expression, rewriting the number under each root sign as a product of its factors, with the first factor being the highest possible perfect square.

2 Simplify the like terms in the expression by subtracting the coefficients.

WRITE

a $3\sqrt{2} + 4\sqrt{8} = 3\sqrt{2} + 4\sqrt{4 \times 2}$

$= 3\sqrt{2} + 4 \times \sqrt{4} \times \sqrt{2}$

$= 3\sqrt{2} + 4 \times 2 \times \sqrt{2}$

$= 3\sqrt{2} + 8\sqrt{2}$

$= 11\sqrt{2}$

b $4\sqrt{48} - 2\sqrt{75} = 4\sqrt{16 \times 3} - 2\sqrt{25 \times 3}$

$= 4 \times \sqrt{16} \times \sqrt{3} - 2 \times \sqrt{25} \times \sqrt{3}$

$= 4 \times 4 \times \sqrt{3} - 2 \times 5 \times \sqrt{3}$

$= 16\sqrt{3} - 10\sqrt{3}$

$= 6\sqrt{3}$

Helpful hints

- ✔ Pay attention to your surds! Do not attempt to simplify the sum or difference of unlike surds!
- ✔ Starting every question by asking yourself 'Can I simplify these surds?' is a great way to avoid common mistakes.

ANS p544

Exercise 2D Adding and subtracting surds

1, 2(1st column), 3–5, 6, 7(1st column), 8(a, c, e, g), 9(a–f), 10(a–c), 11, 12 | 2(2nd column), 4, 5, 6–10(2nd column), 13, 14, 15(a, b), 17 | 6–10(2nd column), 16, 18–21

2D.1 **1** State whether the terms in each of these pairs are like surds or not.

a $3\sqrt{5}$ and $\sqrt{5}$ **b** $4\sqrt{2}$ and $2\sqrt{4}$ **c** 3 and $\sqrt{3}$ **d** $-3\sqrt{6}$ and $6\sqrt{6}$

e $5\sqrt{5}$ and $2\sqrt{2}$ **f** $3\sqrt{7}$ and $3\sqrt{10}$ **g** $2\sqrt{2}$ and $-\sqrt{2}$ **h** $2\sqrt{2}$ and $\sqrt{8}$

2D.2 **2** Simplify each of these expressions by adding (or subtracting) the like terms, if possible. Hint: Look for like surds.

a $2\sqrt{3} + \sqrt{2} + 5\sqrt{3}$ **b** $4\sqrt{3} - 2\sqrt{2} + \sqrt{2}$

c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$ **d** $6\sqrt{13} + 4\sqrt{39} - \sqrt{3} + 2\sqrt{13}$

e $9\sqrt{2} - \sqrt{2} - 6\sqrt{5} + 5\sqrt{2}$ **f** $3\sqrt{6} + 4\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6} - 4$

g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$ **h** $7\sqrt{21} - 6\sqrt{3} + 5\sqrt{3} - 6\sqrt{21}$

i $6\sqrt{6} + 3\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6}$ **j** $-3\sqrt{15} - 2\sqrt{7} - \sqrt{7} - 4\sqrt{15}$

3 Write each surd in simplest form and then identify any like surds in each list.

a $2\sqrt{3}$ $\sqrt{2}$ $4\sqrt{2}$ $2\sqrt{5}$ $3\sqrt{2}$ **b** $3\sqrt{7}$ $14\sqrt{2}$ $7\sqrt{3}$ $2\sqrt{14}$

c $-3\sqrt{6}$ $3\sqrt{12}$ $\sqrt{3}$ $\sqrt{18}$ $2\sqrt{27}$ **d** $9\sqrt{32}$ $4\sqrt{8}$ $\sqrt{128}$ $5\sqrt{200}$

2D.3 **4** Simplify each of these expressions.

a $5\sqrt{2} + 4\sqrt{8}$ **b** $4\sqrt{36} - 2\sqrt{12}$ **c** $3\sqrt{24} + \sqrt{45}$

d $2\sqrt{28} + 8\sqrt{63}$ **e** $4\sqrt{72} + 5\sqrt{98}$ **f** $-2\sqrt{8} + 7\sqrt{16}$

5 Simplify each of these expressions.

a $5\sqrt{27} + \sqrt{12} - 8\sqrt{48}$ **b** $2\sqrt{20} - 4\sqrt{80} + 2\sqrt{45}$ **c** $\sqrt{28} - 3\sqrt{112} + 2\sqrt{63}$

d $5\sqrt{12} + \sqrt{27} - 8\sqrt{3}$ **e** $2\sqrt{8} + 4\sqrt{32} + 3\sqrt{2}$ **f** $\sqrt{7} - 3\sqrt{28} + 5\sqrt{7}$

6 Simplify each of these expressions.

a $3\sqrt{45} - \sqrt{5} - 2\sqrt{80} + 2\sqrt{20}$ **b** $\sqrt{108} - 4\sqrt{27} - 2\sqrt{12} + 2\sqrt{75}$

c $3\sqrt{24} - 4\sqrt{54} + 5\sqrt{24}$ **d** $3\sqrt{18} - 4\sqrt{50} - \sqrt{8} + 2\sqrt{98}$

e $3\sqrt{24} - 7\sqrt{54} - \sqrt{96} + 4\sqrt{20}$ **f** $4\sqrt{108} - 5\sqrt{75} - 6\sqrt{48} + \sqrt{8}$

g $\sqrt{128} - 5\sqrt{27} + 2\sqrt{32} - 3\sqrt{75}$ **h** $2\sqrt{75} - 6\sqrt{3} + 2\sqrt{48} - 6\sqrt{12}$

i $3\sqrt{12} - \sqrt{28} + 5\sqrt{8} - 2\sqrt{20}$ **j** $6\sqrt{54} + \sqrt{45} - 5\sqrt{96} - 2\sqrt{27}$

7 Simplify each of these expressions.

a $\sqrt{180} + 4\sqrt{125} - 3\sqrt{245}$ **b** $4\sqrt{243} + 3\sqrt{300} - 2\sqrt{432}$

c $3\sqrt{338} - 12\sqrt{147} - 2\sqrt{50} + \sqrt{48}$ **d** $3\sqrt{216} - 4\sqrt{125} - 2\sqrt{147} + 5\sqrt{45}$

e $\sqrt{63} + 7\sqrt{567} - 2\sqrt{96} + 8\sqrt{588}$ **f** $\sqrt{200} - 4\sqrt{180} + 5\sqrt{864} - 3\sqrt{160}$

g $-4\sqrt{320} - 25 + \sqrt{245} - 36 + \sqrt{27}$ **h** $12\sqrt{72} - 6\sqrt{192} + 2\sqrt{128} - 5\sqrt{507}$

8 Simplify each of these surd expressions containing pronumerals.

For example: $\sqrt{27x} + \sqrt{12x} = \sqrt{9 \times 3 \times x} + \sqrt{4 \times 3 \times x}$

$= 3\sqrt{3x} + 2\sqrt{3x}$

$= 5\sqrt{3x}$

a $\sqrt{x} + \sqrt{16x} - \sqrt{25x}$ **b** $3\sqrt{4x} + 2\sqrt{x} - 5\sqrt{36x}$

c $\sqrt{9x} + \sqrt{12x} + \sqrt{16x} - 2\sqrt{48x}$ **d** $\sqrt{8xy} - 2\sqrt{32xy} + \frac{1}{3}\sqrt{18xy}$

e $\sqrt{150xy} + 2\sqrt{384xy} - \sqrt{6xy}$ **f** $\sqrt{24x^2y} + 3x\sqrt{96y} - 5\sqrt{54x^2y}$

g $\sqrt{3xy^2} - 5y\sqrt{12x} + 7\sqrt{27xy^2}$ **h** $3\sqrt{100x^2y} - \sqrt{147x^2y} + \frac{1}{2}\sqrt{48x^2y} + \sqrt{16x^2y}$

i $2x\sqrt{80xy} + 5\sqrt{12xy} - \sqrt{125x^2y} - 2\sqrt{432x^2y}$

9 Remember the distributive law? It is used to expand expressions, as shown below.

$$a(b + c) = ab + ac$$

The distributive law can be applied to expressions involving surds.

Expand each of the expressions below. Where necessary, write all surds in simplest form.

a $6(2 + \sqrt{3})$ **b** $8(\sqrt{5} - 4)$

c $5(4 - 6\sqrt{2})$ **d** $4(3\sqrt{2} + 2\sqrt{3})$

e $2(5\sqrt{6} - 3\sqrt{5})$ **f** $-5\sqrt{7}(2\sqrt{6} - 3\sqrt{10} + 2)$

g $3\sqrt{6}(5\sqrt{3} + 2\sqrt{10})$ **h** $5\sqrt{7}(2\sqrt{7} - 12)$

i $6\sqrt{11}(2\sqrt{3} - 5\sqrt{7} + \sqrt{11})$ **j** $(3 - 2\sqrt{5} + 5\sqrt{15})\sqrt{5}$

k $-3\sqrt{3}(4\sqrt{6} - 2\sqrt{10} + 9\sqrt{15})$

10 Remember how to expand binomial products? An example is shown below.

$$(a + b)(c + d) = ac + ad + bc + bd$$

Binomial products can also contain surds. (Expressions such as '$\sqrt{5} + \sqrt{3}$' are referred to as binomial surds. Can you see why?)

Expand and simplify each of these expressions.

a $(5 + 3\sqrt{3})(2 + 4\sqrt{6})$ **b** $(3\sqrt{2} + 6)(5\sqrt{5} - 8)$

c $(7\sqrt{5} - 3\sqrt{2})(4\sqrt{2} + 8\sqrt{6})$ **d** $(\sqrt{6} + 2\sqrt{3})(6\sqrt{2} + 5\sqrt{3})$

e $(4\sqrt{8} - 5\sqrt{5})(3\sqrt{3} - 9\sqrt{8})$ **f** $(4\sqrt{10} + 3\sqrt{5})(7\sqrt{2} - 6\sqrt{12})$

11 Expand and simplify each of these perfect squares.

a $(5 + \sqrt{6})^2$ **b** $(\sqrt{7} - 3)^2$ **c** $(3\sqrt{5} + 2\sqrt{3})^2$ **d** $(4\sqrt{6} - 3\sqrt{7})^2$

12 You might have noticed that the results of the expansions in question **11** followed a pattern that you've seen before; that is, the 'perfect square' rule.

a Show that the expansion of $(a + b)^2$ will always produce the expression $a^2 + 2ab + b^2$.

b Show that the expansion of $(a - b)^2$ will always produce the expression $a^2 - 2ab + b^2$.

c Use these rules to help you check your answers to question **11**.

13 Expand and simplify each of these expressions.

a $(4 + \sqrt{6})(4 - \sqrt{6})$ **b** $(5\sqrt{5} - 3)(5\sqrt{5} + 3)$

c $(2\sqrt{3} + \sqrt{2})(2\sqrt{3} - \sqrt{2})$ **d** $(4\sqrt{5} + 7\sqrt{3})(4\sqrt{5} - 7\sqrt{3})$

14 Consider each expansion in question **13**.

a What do you notice about the terms inside each pair of brackets?

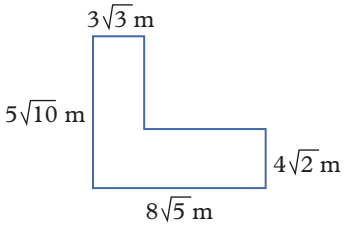
b Is the result of the expansion a rational number or an irrational number? Why do you think this is the case?

c You may have noticed that the results of the expansions in question **13** followed a pattern that you've seen before, the 'difference of two squares' rule. Show that the expansion of binomial products in the form $(a + b)(a - b)$ or $(a - b)(a + b)$ will always be $a^2 - b^2$.

d Use this rule to help you check your answers for question **13**.

15 Use the information provided to calculate the exact area of each of the following composite shapes.

a



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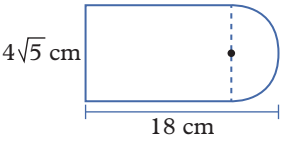

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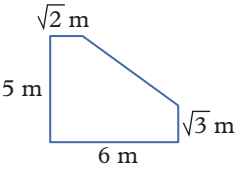
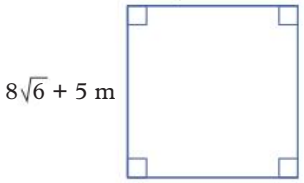


16 Calculate the exact area of each of these shapes. Write your answers in simplest surd form.

a

b



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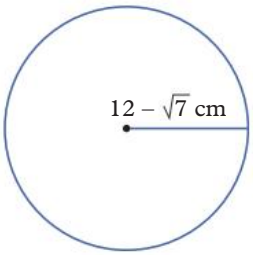
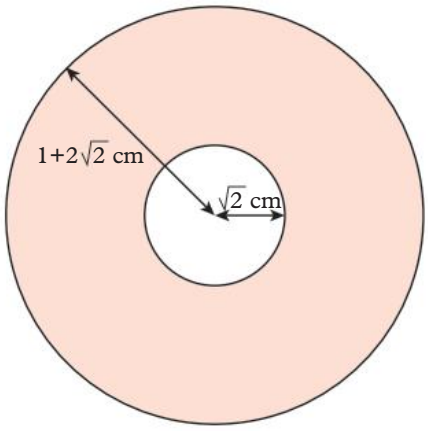


d



17 Calculate the exact perimeter of each shape in parts **a** and **b** of question **15**.

18 Evaluate each of the following.

a $\sqrt{3}(\sqrt{8} + \sqrt{72})$

b $-7\sqrt{15} + 5\sqrt{3} \times 2\sqrt{5}$

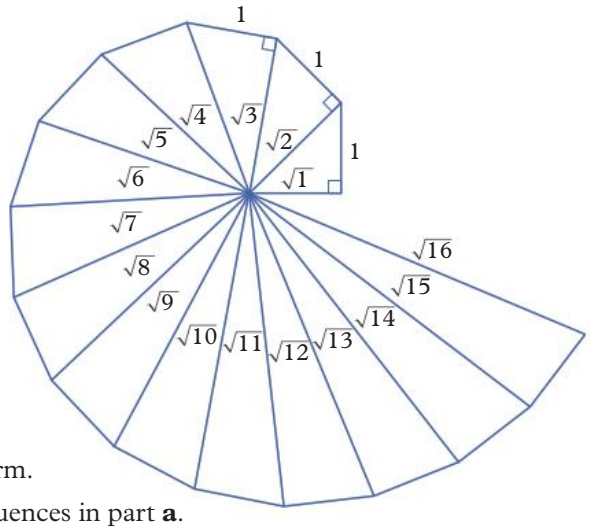
c $\dfrac{8\sqrt{30}}{3\sqrt{10}} + \dfrac{5\sqrt{42}}{6\sqrt{14}} + \dfrac{5\sqrt{105}}{9\sqrt{35}}$

d $\sqrt{50}\left(\dfrac{1}{\sqrt{10}} + \dfrac{1}{\sqrt{5}}\right) + \dfrac{\sqrt{10}}{2}(\sqrt{8} + \sqrt{18} + \sqrt{4})$

e $\dfrac{3\sqrt{55} - 5\sqrt{35}}{\sqrt{5}} + \dfrac{2\sqrt{77} - 2\sqrt{121}}{\sqrt{11}}$

f $\dfrac{6}{\sqrt{2}} \times \dfrac{6}{\sqrt{3}} + \left(\dfrac{4}{\sqrt{2}} + 2\sqrt{3}\right)\left(3\sqrt{2} - \dfrac{9}{\sqrt{3}}\right)$

19 The spiral of Theodorus is formed by consecutively forming a right-angled triangle on the hypotenuse of the previous right-angled triangle, so that the length of one of the shorter sides is always 1. Determine the total length of line segments drawn for this section of the spiral of Theodorus. Write your answer as a simplified expression.



20 a Describe the rule for generating the next term in each of the following sequences.

i $3\sqrt{5},\ 8\sqrt{5},\ 13\sqrt{5},\ 18\sqrt{5}, \ldots$

ii $2\sqrt{3},\ 3\sqrt{3} + 7,\ 4\sqrt{3} + 14,\ 5\sqrt{3} + 21, \ldots$

iii $3\sqrt{2},\ 6,\ 6\sqrt{2},\ 12, \ldots$

iv $\sqrt{5},\ 10,\ 20\sqrt{5},\ 200, \ldots$

b Continue each of the sequences in part **a** to the tenth term.

c Evaluate the sum of the first 10 terms in each of the sequences in part **a**.

21 For each of the following lists, calculate:

i the mean **ii** the median.

a $\sqrt{2},\ \sqrt{4},\ \sqrt{8},\ \sqrt{16},\ \sqrt{32},\ \sqrt{64},\ \sqrt{128},\ \sqrt{256}$

b $\sqrt{5},\ \sqrt{3},\ 5\sqrt{2},\ 2\sqrt{5},\ 3\sqrt{3},\ \sqrt{2}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Adding and subtracting surds

Investigation Tangrams and surds

Topic quiz 2D

2E Rationalising the denominator

Learning intentions

By the end of this topic you will be able to ...

- ✔ rationalise the denominator of simple fractions
- ✔ identify the conjugate of a surd expression
- ✔ rationalise the denominator of a fraction using the conjugate surd.

Inter-year links

Year 8 5D Multiplying algebraic terms

Year 9 2B Expanding

Rationalising the denominator of a simple fraction

- Fractions that contain surds are conventionally written with a rational denominator. In most cases, it is an integer denominator.
- To write a surd fraction with a rational denominator:

  Step 1: Identify the surd in the denominator.

  Step 2: Multiply the numerator and the denominator of the fraction by that surd.

  Step 3: Simplify the resulting fraction.

$$\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

Conjugate of a binomial expression

- The **conjugate** of a binomial expression with one surd has the same two terms but the surd term has a different sign. The conjugate of $a + \sqrt{b}$ is $a - \sqrt{b}$ and the conjugate of $a - \sqrt{b}$ is $a + \sqrt{b}$.
- The product of conjugate expressions is the difference of the two squares of the two terms.

$$(a + b)(a - b) = a^2 - b^2$$
$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

conjugates rational number

For example, the conjugate of $2 + \sqrt{3}$ is $2 - \sqrt{3}$ because:

$$\begin{aligned}(2 + \sqrt{3})(2 - \sqrt{3}) &= 2^2 + \sqrt{3} \times 2 - \sqrt{3} \times 2 - \sqrt{3}^2 \\ &= 2^2 - \sqrt{3}^2 \\ &= 4 - 3 \\ &= 1 \quad \text{which is a rational number.}\end{aligned}$$

Rationalising the denominator using the conjugate surd

- When the denominator of a fraction is a binomial including one surd term and one integer term, the denominator can be rationalised by multiplying the numerator and denominator of the fraction by the conjugate of the denominator.

$$\frac{1}{a + \sqrt{b}} \times \frac{a - \sqrt{b}}{a - \sqrt{b}} = \frac{a - \sqrt{b}}{a^2 - b}$$

Example 2E.1 Rationalising the denominator of a fraction

Rewrite each of these fractions with a rational denominator.

a $\frac{7}{2\sqrt{5}}$ **b** $\frac{2\sqrt{3}}{\sqrt{2}}$ **c** $\frac{1-\sqrt{3}}{\sqrt{2}}$

THINK

a **1** Multiply the numerator and the denominator by the surd term of the denominator. It is not necessary to include the coefficient of the denominator. It is already rational.

2 Simplify the fraction.

b **1** Multiply the numerator and the denominator by the surd term of the denominator.

2 Simplify the fraction.

c **1** Multiply the numerator and the denominator by the surd term of the denominator.

2 Expand the expression using the distributive law.

3 Simplify the fraction.

WRITE

a $\frac{7}{2\sqrt{5}} = \frac{7}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{7\sqrt{5}}{2\sqrt{25}}$

$= \frac{7\sqrt{5}}{2 \times 5}$

$= \frac{7\sqrt{5}}{10}$

b $\frac{2\sqrt{3}}{\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2\sqrt{6}}{\sqrt{4}}$

$= \frac{2^1\sqrt{6}}{2^1}$

$= \sqrt{6}$

c $\frac{1-\sqrt{3}}{\sqrt{2}} = \frac{1-\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{(1-\sqrt{3})\sqrt{2}}{\sqrt{4}}$

$= \frac{\sqrt{2}-\sqrt{6}}{\sqrt{4}}$

$= \frac{\sqrt{2}-\sqrt{6}}{2}$

Example 2E.2 Identifying the conjugate of an expression

State the conjugate for each of these expressions.

a $7+\sqrt{2}$ **b** $1-\sqrt{5}$

THINK

a To form a conjugate of a binomial expression with one surd, change the sign of the surd term.
So $7-\sqrt{2}$ is the conjugate of $7+\sqrt{2}$.

b To form a conjugate of a binomial expression with one surd, change the sign of the surd term.
So $1+\sqrt{5}$ is the conjugate of $1-\sqrt{5}$.

WRITE

a $7-\sqrt{2}$

b $1+\sqrt{5}$

Example 2E.3 Rationalising the denominator using the conjugate

Write $\frac{2}{3+\sqrt{6}}$ with a rational denominator.

THINK

1 Multiply the numerator and the denominator by the conjugate of the denominator. The conjugate of $3 + \sqrt{6}$ is $3 - \sqrt{6}$.

2 Simplify the fraction by first expanding the expression in the numerator and then the expression in the denominator using the 'difference of two squares' rule: $(a + b)(a - b) = a^2 - b^2$.

3 Check that your answer has a rational denominator.

WRITE

$$\frac{2}{3+\sqrt{6}} = \frac{2}{3+\sqrt{6}} \times \frac{3-\sqrt{6}}{3-\sqrt{6}}$$
$$= \frac{2(3-\sqrt{6})}{(3+\sqrt{6})(3-\sqrt{6})}$$
$$= \frac{6-2\sqrt{6}}{9-6}$$
$$= \frac{6-2\sqrt{6}}{3}$$

Helpful hints

✔ It can be helpful to work out the binominal multiplication before you multiply the denominator. Then you can easily see if your multiplication will rationalise the denominator.
For example, $(4 + \sqrt{3})(4 - \sqrt{3}) = 16 - 3 = 13$

then: $\frac{1}{4+\sqrt{3}} \times \frac{4-\sqrt{3}}{4-\sqrt{3}} = \frac{4-\sqrt{3}}{13}$

✔ To write a fraction containing a surd in the denominator in simplest form you need to rationalise the denominator.

ANS p545

Exercise 2E Rationalising the denominator

△ 1–3, 4–6(1st, 2nd columns), 7, 8(a), 9, 10(a, c), 13 ■ 3–6(3rd, 4th columns), 7–9, 11, 12(a, b, c), 13(e, f), 14 ⬟ 5–7(3rd, 4th columns), 7–9, 10(e, f), 11, 12(e, f), 15

UNDERSTANDING AND FLUENCY

1 a Find the answer for each of these multiplication calculations. Write your answers in simplest form.

i $\frac{1}{2} \times 1$ **ii** $\frac{3}{4} \times \frac{1}{1}$ **iii** $\frac{3}{4} \times \frac{4}{4}$ **iv** $\frac{2}{5} \times \frac{5}{5}$

b What did you notice about each of your results in part **a**?

c What is special about the second value in each of the multiplications in this question?

2 a Calculate results for these by multiplying the numerators together and multiplying the denominators together.

i $\frac{2}{\sqrt{3}} \times \frac{1}{1}$ **ii** $\frac{2}{\sqrt{3}} \times \frac{3}{3}$ **iii** $\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ **iv** $\frac{2}{\sqrt{2}} \times \frac{2}{2}$

b Which result in part **a** is a fraction with a rational denominator?

c What operation was performed so that a rational denominator was obtained?

2E.1 **3** Rewrite each of these fractions with a rational denominator.

a $\frac{3}{\sqrt{6}}$ **b** $\frac{2}{\sqrt{7}}$ **c** $\frac{3}{\sqrt{3}}$ **d** $\frac{4\sqrt{5}}{\sqrt{7}}$

e $\frac{5}{3\sqrt{2}}$ **f** $\frac{7}{4\sqrt{5}}$ **g** $\frac{2\sqrt{3}}{\sqrt{5}}$ **h** $-\frac{5}{2\sqrt{5}}$

i $\frac{1}{7\sqrt{7}}$ **j** $\frac{4\sqrt{11}}{\sqrt{10}}$ **k** $\frac{8\sqrt{5}}{3\sqrt{2}}$ **l** $\frac{9\sqrt{6}}{4\sqrt{5}}$

4 Rewrite each of these fractions with a rational denominator. Where appropriate, write the numerator in simplest surd form.

a $\frac{-\sqrt{6}}{\sqrt{2}}$ **b** $\frac{-\sqrt{15}}{\sqrt{6}}$ **c** $\frac{2\sqrt{8}}{3\sqrt{7}}$ **d** $\frac{2\sqrt{6}}{\sqrt{8}}$

e $\frac{6\sqrt{15}}{7\sqrt{3}}$ **f** $-\frac{3\sqrt{20}}{\sqrt{5}}$ **g** $\frac{5}{\sqrt{x}}$ **h** $\frac{x}{x\sqrt{y}}$

i $\frac{4x}{\sqrt{2y}}$ **j** $\frac{\sqrt{x}}{x\sqrt{y}}$ **k** $\frac{\sqrt{2y}}{2y\sqrt{x}}$ **l** $\frac{-3\sqrt{x}}{6x\sqrt{yz}}$

2E.2 **5** State the conjugate for each of these expressions.

a $3 + \sqrt{2}$ **b** $2 - \sqrt{3}$ **c** $-\sqrt{6} + 5$ **d** $-4 + 3\sqrt{5}$

e $3\sqrt{5} - 2$ **f** $-4\sqrt{7} - 1$ **g** $1 + 2\sqrt{3}$ **h** $\sqrt{2} + \sqrt{3}$

i $5 - \sqrt{3}$ **j** $-5\sqrt{6} + 2$ **k** $3\sqrt{7} - 2$ **l** $-5 - 2\sqrt{5}$

2E.3 **6** Rewrite each of these fractions with a rational denominator.

a $\frac{1}{2 + \sqrt{3}}$ **b** $\frac{3}{3 - \sqrt{2}}$ **c** $\frac{5}{6 + \sqrt{5}}$ **d** $\frac{3}{3 - 4\sqrt{5}}$

e $\frac{\sqrt{2}}{5 + 3\sqrt{2}}$ **f** $\frac{2\sqrt{3}}{1 + 2\sqrt{3}}$ **g** $\frac{3}{\sqrt{5} - 1}$ **h** $\frac{3\sqrt{7}}{2\sqrt{6} + 5}$

i $\frac{-4\sqrt{2}}{\sqrt{3} - 2}$ **j** $\frac{\sqrt{3}}{\sqrt{5} + 2}$ **k** $\frac{2\sqrt{5}}{3\sqrt{6} + 10}$ **l** $\frac{6\sqrt{10}}{5\sqrt{2} - 5}$

7 Rewrite each of these fractions with a rational denominator.

a $\frac{\sqrt{2} + 1}{2\sqrt{3} + 3}$ **b** $\frac{5 + 2\sqrt{3}}{4 + \sqrt{2}}$ **c** $\frac{1 + \sqrt{3}}{3 - \sqrt{5}}$ **d** $\frac{3 + \sqrt{5}}{\sqrt{6} + 1}$ **e** $\frac{2\sqrt{3} - 2}{3\sqrt{5} + 2}$ **f** $\frac{5\sqrt{2} - 1}{3 + 2\sqrt{5}}$

8 Write each of these fractions in simplest form, with a rational denominator.

a $\frac{x}{x - \sqrt{y}}$ **b** $\frac{\sqrt{x}}{\sqrt{x} - y}$ **c** $\frac{x + y}{x\sqrt{x} + y}$ **d** $\frac{x - \sqrt{y}}{x + \sqrt{y}}$

9 You can perform addition and subtraction with surd fractions in a way that is similar to how you do it with common fractions.

Consider $\frac{3}{\sqrt{2}} + \frac{1}{2}$.

a First write the fractions with a common denominator and add the two fractions. Then write your answer with a rational denominator.

b Repeat the addition, but this time rationalise the denominator of the first fraction *before* performing the addition.

10 Find the exact value of each of these expressions.

a $\frac{3}{\sqrt{5}} + \frac{1}{5}$ **b** $\frac{4}{\sqrt{3}} - \frac{1}{3}$ **c** $\frac{3\sqrt{2}}{2} + \frac{2}{\sqrt{3}}$

d $\frac{1}{4\sqrt{3}} + \frac{3}{2\sqrt{2}}$ **e** $\frac{5}{\sqrt{5}} - \frac{2}{3\sqrt{6}}$ **f** $\frac{2\sqrt{3}}{5\sqrt{2}} + \frac{\sqrt{5}}{4\sqrt{3}}$

11 Now consider the expression $\frac{1}{\sqrt{5} - 1} + \frac{3}{\sqrt{5} + 1}$.

a Write each fraction in the expression as an equivalent fraction with a rational denominator.

b Add your two fraction answers from part **a**, writing your answer in simplest form.

12 Simplify each of these expressions after first expressing each fraction with a rational denominator.

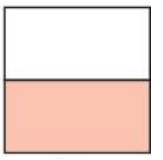
a $\frac{2}{\sqrt{6} - 2} + \frac{2}{\sqrt{6} + 2}$ **b** $\frac{2}{\sqrt{5} + 2} - \frac{2}{\sqrt{5} - 2}$ **c** $\frac{2}{\sqrt{6} - 1} + \frac{2}{\sqrt{3} + 1}$

d $\frac{5}{\sqrt{2} + 1} - \frac{4}{\sqrt{5} - 2}$ **e** $\frac{2\sqrt{3}}{2\sqrt{2} + 5} + \frac{\sqrt{3}}{3\sqrt{2} + 2}$ **f** $\frac{\sqrt{3} + 1}{\sqrt{3} + 4} - \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

13 Write the reciprocal of each of the following. Write your answers with rational denominators and simplified surds.

a $\sqrt{2}$ **b** $5\sqrt{3}$ **c** $\frac{\sqrt{28}}{7}$ **d** $\frac{\sqrt{27}}{4}$ **e** $\sqrt{6} + 2$ **f** $\frac{3 + \sqrt{5}}{3 - \sqrt{5}}$

14 Consider the square on the right, which has been divided into congruent rectangles.



a State the area of the whole square.

b State the area of the shaded rectangle.

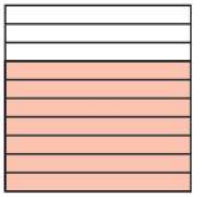
c The width of a rectangle can be determined by dividing its area by its length. Determine the width of the shaded rectangle. Leave your answer with an irrational denominator.

d The width of the shaded rectangle can also be determined by considering the fraction that the width of the shaded rectangle is of the square's side length. State what fraction the shaded rectangle's width is of the square's side length.

e State the width of the shaded rectangle in the form $\frac{a}{b}\sqrt{c}$, where a, b and c are integers.

f Describe the connection between the answers from parts **c** and **e** referring to 'rationalising the denominator'.

g Use a similar method to determine the width of the shaded region in the square on the right, with both a rational and irrational denominator.



CHALLENGE

15 When an expression contains two surd terms, the expression has two conjugates resulting from changing the sign on one of the surd terms. For example, the conjugates of $\sqrt{a}+\sqrt{b}$ are $\sqrt{a}-\sqrt{b}$ and $-\sqrt{a}+\sqrt{b}$. The conjugates of $-\sqrt{6}-\sqrt{5}$ are $\sqrt{6}-\sqrt{5}$ and $-\sqrt{6}+\sqrt{5}$.

a Write all of the conjugates of the following.

i $\sqrt{2}+\sqrt{3}$ **ii** $\sqrt{5}-\sqrt{3}$ **iii** $-2\sqrt{5}-5\sqrt{6}$ **iv** $-2\sqrt{5}+3\sqrt{7}$

b Evaluate the following products.

i $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ **ii** $(\sqrt{2}+\sqrt{3})(-\sqrt{2}+\sqrt{3})$ **iii** $(\sqrt{2}+\sqrt{3})(-\sqrt{2}-\sqrt{3})$

iv $(-\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ **v** $(\sqrt{2}-\sqrt{3})(-\sqrt{2}-\sqrt{3})$

c We can rationalise a denominator involving multiple surd terms by making use of the 'difference of two squares' rule. It is important that only the sign of one term is changed to create suitable conjugates.

Write the two conjugates which could be used to rationalise the denominator of each of the following.

i $\frac{1}{\sqrt{5}-\sqrt{3}}$ **ii** $\frac{1}{2\sqrt{2}+5\sqrt{5}}$ **iii** $\frac{1}{-\sqrt{11}+3\sqrt{2}}$ **iv** $\frac{1}{-5\sqrt{13}-4\sqrt{3}}$

d Use the conjugates from part **c** to rationalise the denominator for each fraction in part **c**.

e Lucy was rationalising the denominator of $\frac{4}{-\sqrt{2}+\sqrt{3}}$. Her working is shown.

$$\frac{4}{-\sqrt{2}+\sqrt{3}}\times\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}=\frac{4(\sqrt{2}-\sqrt{3})}{-2+\sqrt{6}+\sqrt{6}-3}$$
$$=\frac{4\sqrt{2}-4\sqrt{3}}{2\sqrt{6}-5}$$

i Explain why Lucy did not get a rational denominator.

ii Explain what she could do next to her last line to rationalise the denominator.

f Rationalise the denominator of each of the following fractions.

i $\frac{2}{\sqrt{3}+\sqrt{5}}$ **ii** $\frac{-4\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ **iii** $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{2}}$ **iv** $\frac{2\sqrt{5}}{3\sqrt{6}+2\sqrt{10}}$

v $\frac{6\sqrt{10}}{5\sqrt{2}-2\sqrt{5}}$ **vi** $\frac{5\sqrt{2}-1}{3\sqrt{3}+2\sqrt{5}}$ **vii** $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{5}-\sqrt{7}}$ **viii** $\frac{x+y}{x\sqrt{x}+\sqrt{y}}$

g Evaluate $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})$.

h Write $\frac{4}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ with a rational denominator.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Rationalising the denominator

Interactive skillsheet Rationalising the denominator using the conjugate surd

Topic quiz 2E

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

2A **1** State if the following numbers are rational or irrational.

a $\frac{4}{3}$ **b** -12.01 **c** $19.279\,102\,865\,725\ldots$

d $-2\frac{5}{7}$ **e** π **f** $\sqrt{2}$

2A **2** State if the following roots are surds or not.

a $\sqrt{15}$ **b** $\sqrt{16}$ **c** $\sqrt{100}$

d $\sqrt[3]{8}$ **e** $\sqrt{160}$ **f** $\sqrt{256}$

2B **3** Simplify the following surds.

a $\sqrt{8}$ **b** $\sqrt{45}$ **c** $\sqrt{175}$ **d** $\sqrt{600}$

2B **4** Simplify the following surds.

a $5\sqrt{12}$ **b** $7\sqrt{54}$ **c** $25\sqrt{72}$ **d** $5\sqrt{1280}$

2C **5** Simplify the following products.

a $-\sqrt{5} \times (-\sqrt{7})$ **b** $\sqrt{14} \times (-\sqrt{22})$ **c** $-6\sqrt{3} \times 5\sqrt{6}$ **d** $\sqrt{18} \times \sqrt{112}$

2C **6** Simplify the following quotients.

a $\frac{-\sqrt{60}}{-\sqrt{10}}$ **b** $\frac{-8\sqrt{34}}{12\sqrt{2}}$

c $\frac{\sqrt{864}}{-\sqrt{384}}$ **d** $\frac{9\sqrt{525}}{3\sqrt{252}}$

e $\frac{-5\sqrt{72}}{6\sqrt{15}}$ **f** $\frac{12\sqrt{338}}{13\sqrt{148}}$

2C **7** Simplify the following fractions.

a $\frac{\sqrt{54} \times \sqrt{12}}{\sqrt{18}}$ **b** $\frac{8\sqrt{5} \times 5\sqrt{6}}{10\sqrt{3} \times 6\sqrt{15}}$

2D **8** Simplify the following sums and differences.

a $3\sqrt{2} + 9\sqrt{2}$ **b** $3\sqrt{2} - 9\sqrt{2}$

c $5\sqrt{3} + 3\sqrt{5} + 15\sqrt{5} + 7\sqrt{3}$ **d** $21\sqrt{7} - 15\sqrt{11} - 23\sqrt{7} + 32\sqrt{11}$

2D **9** Simplify the following sums and differences.

a $\sqrt{8} + \sqrt{128} + \sqrt{18}$ **b** $\sqrt{75} - \sqrt{27}$

c $\sqrt{150} + \sqrt{108} + \sqrt{54} + \sqrt{192}$ **d** $\sqrt{4000} - \sqrt{242} - \sqrt{90} - \sqrt{72}$

2D **10** Simplify the following.

a $\frac{5\sqrt{6} + 13\sqrt{6}}{3\sqrt{2}}$ **b** $3\sqrt{3}(6\sqrt{2} - 2\sqrt{6})$

c $(\sqrt{5} - 2)(3 - \sqrt{7})$ **d** $(\sqrt{8} + \sqrt{27})(\sqrt{125} + \sqrt{48})$

e $(7\sqrt{12} + 6\sqrt{5})(5\sqrt{6} - 3\sqrt{40})$ **f** $(4\sqrt{32} + 2\sqrt{7})(5\sqrt{22} - \sqrt{72})$

2E **11** Rationalise the denominators of the following fractions.

a $\frac{5}{\sqrt{2}}$ **b** $\frac{12}{\sqrt{3}}$ **c** $\frac{\sqrt{20}}{\sqrt{7}}$ **d** $\frac{\sqrt{55}}{\sqrt{30}}$

2E **12** Rationalise the denominators of the following fractions.

a $\frac{5}{1 + \sqrt{2}}$ **b** $\frac{\sqrt{3}}{3 - \sqrt{3}}$

c $\frac{2\sqrt{5}}{3\sqrt{5} - 2}$ **d** $\frac{2 + \sqrt{7}}{-\sqrt{63} + 7}$

e $\frac{4 - \sqrt{5}}{-13 + \sqrt{5}}$ **f** $\frac{\sqrt{10} + 2\sqrt{3}}{2 - \sqrt{128}}$

2F Fractional exponents

Learning intentions

By the end of this topic you will be able to ...

✔ apply the exponent laws to fractional exponents.

Inter-year links

Year 8 — 4B Products and quotients of powers

Year 9 — 1B Products and quotients of powers

Fractional exponents

- Fractional exponents are an alternate notation for expressing roots and exponents together.

base → $a^{\frac{1}{3}}$ ← fractional exponent/index/power

→ The numerator of the fractional exponent is the number the base is raised to.

→ The denominator of the fractional exponent refers to the type of root applied to the base.

Numbers raised to a fractional exponent also have an equivalent surd form:

$$a^{\frac{2}{3}} = \left(\sqrt[3]{a}\right)^2$$

fractional exponent form — surd form

- Expressions that contain fractional exponents obey the usual exponent laws.

	Integer exponents	Fractional exponents
Zero exponent law	$a^0 = 1$	$a^0 = 1$
Product of powers rule	$a^5 \times a^3 = a^{5+3}$	$a^{\frac{1}{5}} \times a^{\frac{1}{3}} = a^{\frac{1}{5}+\frac{1}{3}}$
Quotient of powers rule	$a^5 \div a^3 = a^{5-3}$	$a^{\frac{1}{5}} \div a^{\frac{1}{3}} = a^{\frac{1}{5}-\frac{1}{3}}$
Power of a power rule	$\left(a^5\right)^3 = a^{5\times 3}$ $(ab)^3 = a^3b^3$ $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$	$\left(a^{\frac{1}{5}}\right)^{\frac{1}{3}} = a^{\frac{1}{5}\times\frac{1}{3}}$ $(ab)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}$ $\left(\frac{a}{b}\right)^{\frac{1}{3}} = \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

- Zero exponent law: Excluding 0, any number or variable with an exponent of 0 is equal to 1.
- Product of powers rule: When multiplying numbers or variables in exponential form that have the same base, add the exponents.
- Quotient of powers rule: When dividing numbers or variables in exponential form that have the same base, subtract the second exponent from the first exponent.
- Power of a power rule: When raising a number or variable to two exponents, multiply the exponents. Every base number or variable inside brackets should have its exponent multiplied by the exponent outside the brackets.

Example 2F.1 Applying the exponent laws to fractional exponents

Simplify each of these expressions.

a $x^{\frac{1}{4}} \times x^{\frac{2}{3}}$ **b** $\dfrac{x^{\frac{3}{5}}}{x^{\frac{1}{3}}}$ **c** $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}}$

THINK

a Add the exponents of the common base, x. To add the fraction exponents, convert them to equivalent fractions with a common denominator.

b Subtract the exponents of the common base, x. To subtract the fraction exponents, convert them to equivalent fractions with a common denominator.

c Multiply the exponent of the base inside the brackets by the exponent outside the brackets.

WRITE

a $x^{\frac{1}{4}} \times x^{\frac{2}{3}} = x^{\frac{1}{4}+\frac{2}{3}}$
$= x^{\frac{3}{12}+\frac{8}{12}}$
$= x^{\frac{11}{12}}$

b $\dfrac{x^{\frac{3}{5}}}{x^{\frac{1}{3}}} = x^{\frac{3}{5}-\frac{1}{3}}$
$= x^{\frac{9}{15}-\frac{5}{15}}$
$= x^{\frac{4}{15}}$

c $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}} = x^{\frac{1}{3}\times\frac{1}{2}}$
$= x^{\frac{1}{6}}$

Example 2F.2 Raising a term to a fractional exponent

Simplify $4\left(x^{\frac{2}{3}}y^3\right)^{\frac{5}{6}}$

THINK

1 Raise each term inside the brackets to the exponent outside the brackets.

2 Multiply the exponent of every base inside the brackets by the exponent outside the brackets.

3 Simplify the fractional exponents.

WRITE

$4\left(x^{\frac{2}{3}}y^3\right)^{\frac{5}{6}} = 4\left(x^{\frac{2}{3}}\right)^{\frac{5}{6}}\left(y^3\right)^{\frac{5}{6}}$
$= 4 \times x^{\frac{2}{3}\times\frac{5}{6}} \times y^{3\times\frac{5}{6}}$
$= 4 \times x^{\frac{10}{18}} \times y^{\frac{15}{6}}$
$= 4x^{\frac{5}{9}}y^{\frac{5}{2}}$

Example 2F.3 Multiplying terms with fractional exponents

Simplify $5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}}$.

THINK

1 Multiply the coefficients and multiply the terms with the same bases. Add the exponents of the common bases, x and y.

2 Simplify the fractional exponents.

WRITE

$5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}} = 5 \times 2 \times x^{\frac{1}{2}} \times x^{\frac{1}{5}} \times y^{\frac{1}{3}} \times y^{\frac{3}{5}}$
$= 10 \times x^{\frac{1}{2}+\frac{1}{5}} \times y^{\frac{1}{3}+\frac{3}{5}}$
$= 10 \times x^{\frac{7}{10}} \times y^{\frac{14}{15}}$
$= 10x^{\frac{7}{10}}y^{\frac{14}{15}}$

Helpful hints

- ✔ Take care when writing fractional exponents. They should be smaller than the base and sit high up on the shoulder of the base to avoid confusion between the base and the exponent. For example, $x^{\frac{1}{3}}$ not $x\frac{1}{3}$.
- ✔ Take care not to mix up the exponent laws.
  - → When multiplying numbers with the same base, add the exponents.
  - → When dividing numbers with the same base, subtract the exponents.
  - → If a number raised to a power is enclosed in brackets and then raised to another power, multiply the exponents.
- ✔ Keep the exponent laws and the rules for negative exponents written somewhere close until they become second nature to you.

$$a^{-1} = \frac{1}{a} \qquad a^{-m} = \frac{1}{a^m} \qquad \frac{1}{a^{-m}} = a^m$$

ANS p546

Exercise 2F Fractional exponents

▲ 1–9, 11, 12, 13(a–e), 14 ■ 1–4(e–h), 5–9, 11, 12, 13(g–l), 15, 18, 19, 21 ⬟ 5, 6, 10, 11, 15–17, 19–22

UNDERSTANDING AND FLUENCY

2F.1 **1** Use the appropriate exponent law to simplify each of these expressions.

a $x^{\frac{1}{5}} \times x^{\frac{2}{5}}$ **b** $x^{\frac{3}{10}} \times x^{\frac{1}{5}}$ **c** $x^{\frac{3}{4}} \times x^{\frac{1}{8}}$ **d** $x^{\frac{2}{5}} \times x^{\frac{1}{6}}$

e $x^{\frac{5}{6}} \times x^{\frac{1}{6}}$ **f** $x^{\frac{2}{3}} \times x^{\frac{1}{4}}$ **g** $x^{\frac{1}{4}} \times x^{\frac{1}{4}}$ **h** $x^{\frac{1}{3}} \times x^{\frac{1}{6}}$

2 Use the appropriate exponent law to simplify each of these expressions.

a $x^{\frac{13}{11}} \div x^{\frac{9}{11}}$ **b** $x^{\frac{3}{8}} \div x^{\frac{1}{8}}$ **c** $\dfrac{x^{\frac{1}{2}}}{x^{\frac{3}{7}}}$ **d** $\dfrac{x^{\frac{4}{5}}}{x^{\frac{3}{4}}}$

e $\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{6}}}$ **f** $x^{\frac{2}{5}} \div x^{\frac{3}{10}}$ **g** $\dfrac{x^{\frac{9}{4}}}{x^{\frac{1}{2}}}$ **h** $x^{\frac{6}{7}} \div x^{\frac{6}{7}}$

3 Use the appropriate exponent law to simplify each of these expressions.

a $\left(x^{\frac{3}{7}}\right)^{\frac{5}{4}}$ **b** $\left(x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **c** $\left(x^{\frac{3}{7}}\right)^{0}$ **d** $\left(x^{\frac{2}{5}}\right)^{\frac{3}{4}}$

e $\left(x^{\frac{1}{8}}\right)^{\frac{4}{9}}$ **f** $\left(x^{\frac{2}{3}}\right)^{\frac{3}{4}}$ **g** $\left(x^{\frac{27}{2}}\right)^{\frac{8}{9}}$ **h** $\left(x^{\frac{8}{45}}\right)^{\frac{9}{56}}$

4 Simplify each of these expressions.

a $5x^{\frac{1}{3}} \times 2x^{\frac{1}{4}}$ **b** $6x^{\frac{2}{5}} \times 5x^{\frac{3}{8}}$ **c** $9x^{\frac{3}{7}} \times -4x^{\frac{2}{9}}$ **d** $4x^{\frac{1}{4}} \div x^{\frac{1}{6}}$

e $12x^{\frac{3}{5}} \div 6x^{\frac{3}{8}}$ **f** $\dfrac{32x^{\frac{5}{6}}}{8x^{\frac{4}{5}}}$ **g** $3x^{\frac{1}{2}} \times 2x^{\frac{1}{3}}$ **h** $\dfrac{x^{\frac{3}{5}}}{4x^{\frac{1}{2}}}$

2F.2 **5** Simplify each of these expressions.

a $\left(5x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **b** $\left(16x^{\frac{2}{5}}\right)^{\frac{1}{4}}$ **c** $\left(27x^{\frac{3}{8}}\right)^{\frac{2}{3}}$ **d** $2\left(x^{\frac{2}{5}}y^{\frac{3}{4}}\right)^{\frac{1}{2}}$

e $4\left(x^{\frac{1}{5}}y^{\frac{3}{7}}\right)^{\frac{2}{3}}$ **f** $5\left(x^{\frac{2}{3}}y^{\frac{4}{5}}\right)^{\frac{3}{8}}$ **g** $5\left(x^{2}y^{\frac{3}{5}}\right)^{\frac{1}{2}}$ **h** $7\left(x^{\frac{1}{5}}y^{\frac{4}{11}}\right)^{\frac{5}{3}}$

2F.3 **6** Simplify each of these expressions.

a $4x^{\frac{1}{4}}y^{\frac{1}{5}} \times 7x^{\frac{1}{3}}y^{\frac{3}{4}}$ **b** $-6x^{\frac{2}{3}}y^{\frac{4}{7}} \times 12x^{\frac{5}{6}}y^{\frac{3}{7}}$ **c** $9x^{\frac{1}{6}}y^{\frac{2}{5}} \times 5x^{\frac{5}{8}}y$

d $25x^{\frac{3}{5}}y^{\frac{1}{4}} \div 5x^{\frac{1}{5}}y^{\frac{1}{12}}$ **e** $12x^{\frac{3}{8}}y^{\frac{5}{11}} \div 6x^{\frac{1}{5}}y^{\frac{1}{3}}$ **f** $\dfrac{7x^{\frac{1}{2}}y^{\frac{3}{7}}}{7x^{\frac{3}{7}}y^{\frac{2}{5}}}$

7 Consider the numerical expression $3^{\frac{1}{2}} \times 3^{\frac{1}{2}}$.

a Which one of the exponent laws would you use to simplify it?

b Use the exponent law you chose to simplify the expression. Then check your answer with a calculator.

c Now consider the surd expression $\sqrt{3} \times \sqrt{3}$. Multiply the two surds and write your answer in simplest form.

d Compare your answers for parts **b** and **c**. What do you notice?

e Repeat parts **b–d** for each of these pairs of expressions.

i $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ and $\sqrt{2} \times \sqrt{2}$

ii $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$ and $\sqrt{5} \times \sqrt{5}$

iii $10^{\frac{1}{2}} \times 10^{\frac{1}{2}}$ and $\sqrt{10} \times \sqrt{10}$

iv $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ and $\sqrt{x} \times \sqrt{x}$

f Using your results from part **e**, explain what you notice about the relationship between the fractional exponent of $\frac{1}{2}$ and the square root sign.

8 Now consider the numerical expression $3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{\frac{1}{3}}$.

a Use the appropriate exponent law to simplify the expression. Then check your answer with a calculator.

b Calculate $\sqrt[3]{3} \times \sqrt[3]{3} \times \sqrt[3]{3}$.

c Compare your answers for parts **a** and **b**. What do you notice?

d Repeat parts **a–c** for each of these pairs of expressions.

i $2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}}$ and $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2}$

ii $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}}$ and $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}$

iii $10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}}$ and $\sqrt[3]{10} \times \sqrt[3]{10} \times \sqrt[3]{10}$

iv $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}}$ and $\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x}$

e What are the differences between the surd expression in question **8b** and the surd expression in question **7c**?

f Use your results to explain what you notice about the relationship between the fractional exponent of $\frac{1}{3}$ and the cube root sign.

9 The fractional exponents you have seen so far and their corresponding surd forms have involved fractions with a numerator of 1. Now consider the term $x^{\frac{2}{3}}$.

a Show that this term has the equivalent forms of $\left(x^2\right)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2$.

b Show that both expressions in part **a** are equivalent to $\sqrt[3]{x^2}$ and $\left(\sqrt[3]{x}\right)^2$.

c Use a calculator to substitute each of the values below for x in $x^{\frac{2}{3}}$. Check that it produces the same results as substituting those values into each of the expressions from parts **a** and **b**.

i $x = 5$ **ii** $x = 2$ **iii** $x = 3$ **iv** $x = 10$ **v** $x = 12$ **vi** $x = 27$

10 Show that any fractional exponent of the form $\frac{m}{n}$ is equivalent to $\sqrt[n]{(\,)^m}$ or $\left(\sqrt[n]{\ }\right)^m$.

That is, show that $x^{\frac{m}{n}}$ is equivalent to $\sqrt[n]{x^m}$ or $\left(\sqrt[n]{x}\right)^m$.

11 Calculate each of these values, without using a calculator.

a $25^{\frac{1}{2}}$ **b** $36^{\frac{1}{2}}$ **c** $1^{\frac{1}{2}}$ **d** $144^{\frac{1}{2}}$ **e** $100^{\frac{1}{2}}$ **f** $196^{\frac{1}{2}}$

g $8^{\frac{1}{3}}$ **h** $27^{\frac{1}{3}}$ **i** $125^{\frac{1}{3}}$ **j** $1^{\frac{1}{3}}$ **k** $1^{\frac{1}{5}}$ **l** $32^{\frac{1}{5}}$

12 Look at each part of question **11**.

i If each fractional exponent was negative, briefly explain how the answers would change.

ii What would the answer for each part of question **11** be if the exponents were negative?

iii Which parts can have a negative base? Explain why.

iv What would the value of each of the parts you identified in part **iii** be if the bases were negative?

13 Write each of these values in simplest surd form.

a $5^{\frac{1}{2}}$ **b** $8^{\frac{1}{2}}$ **c** $24^{\frac{1}{2}}$ **d** $108^{\frac{1}{2}}$ **e** $152^{\frac{1}{2}}$ **f** $54^{\frac{1}{3}}$

g $128^{\frac{1}{3}}$ **h** $320^{\frac{1}{2}}$ **i** $243^{\frac{1}{2}}$ **j** $432^{\frac{1}{2}}$ **k** $5^{\frac{3}{2}}$ **l** $6^{\frac{3}{2}}$

14 Write each of these expressions using fractional exponents.

a $\sqrt{x^3}$ **b** $\sqrt{6}$ **c** $\sqrt{x^3 y^5}$ **d** $\sqrt{3xy^3}$ **e** $\sqrt{16x^{-1}y}$ **f** $\sqrt[3]{x^4 y^2}$

PROBLEM SOLVING AND REASONING

15 Simplify each of these expressions, writing your answers with fractional exponents.

a $12x^{\frac{1}{4}}y^{\frac{1}{5}} \times 9x^{\frac{1}{5}}y^{\frac{2}{3}}$ **b** $\dfrac{x^{\frac{1}{3}}y^{\frac{3}{4}}}{5x^{\frac{1}{9}}y^{\frac{2}{5}}}$ **c** $\left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$ **d** $\sqrt{4x^3y} \times 3\sqrt{x^5y^3}$

e $3\sqrt{x^4y^3} \div \sqrt{9xy^2}$ **f** $\left(\sqrt{x^3y^5}\right)^{\frac{1}{3}}$ **g** $\sqrt{8x^3yz^5} \times 2\sqrt{3xz^3}$ **h** $\left(\sqrt[3]{x^6y^{\frac{1}{3}}z}\right)^2$

16 Write each of these expressions in simplest form.

a $\sqrt{16x^2y}$ **b** $\sqrt{12x^{-3}y^7}$ **c** $\sqrt{\dfrac{72x^4y^3}{24xy}}$

d $\sqrt[3]{27xy^2}$ **e** $\sqrt[4]{16x^3y^2z}$ **f** $\sqrt{343x^{-1}y^5}$

17 Simplify each of these expressions, writing your answers with positive exponents where appropriate.

a $3x^{-\frac{1}{3}}y^{\frac{2}{3}} \times 6x^{\frac{3}{4}}y^{-\frac{1}{5}}$ **b** $12x^{\frac{1}{4}}y^{\frac{1}{3}} \times 6x^{-\frac{2}{3}}y^{-\frac{1}{4}}$ **c** $\dfrac{xy^2}{x^{\frac{1}{3}}y^{\frac{5}{2}}}$ **d** $\dfrac{3x^{\frac{1}{3}}y^{\frac{2}{5}}}{12x^{\frac{2}{5}}y^{\frac{3}{4}}}$

e $\left(\dfrac{x^{-\frac{1}{3}}}{y^3}\right)^{\frac{1}{4}}$ **f** $\left(\dfrac{2x^{-\frac{2}{5}}y^{\frac{3}{4}}}{y^2}\right)^4$ **g** $\dfrac{5x^{-\frac{1}{2}} \times 4x^{\frac{1}{3}}}{2x^{-\frac{2}{5}}}$ **h** $\sqrt{\dfrac{x^{-\frac{1}{3}}y^{\frac{2}{3}}}{3x^{-\frac{1}{4}}y^{-\frac{1}{5}}}}$

18 Consider the expression $\dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$.

a Use an appropriate combination of the exponent laws to simplify the expression.

b Use your working and your answer from part **a** to help you solve this equation for x:

$9^x = \dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$

19 Solve each of these equations for x.

a $25^{x+2} = \dfrac{1}{5}$ **b** $27^{x-3} = 9^{x+1}$ **c** $8^x = \dfrac{16^{-\frac{1}{4}} \times 2^{\frac{1}{2}}}{64^{\frac{1}{6}}}$

CHALLENGE

20 Consider the diagram on the right.

a Refer to the diagram to compare multiplying by a fraction and raising a number to a fractional exponent. Describe what is similar and what is different.

b Evaluate each of the following.

i $81^{\frac{1}{4}}$ **ii** $81^{\frac{3}{4}}$ **iii** $81^{\frac{5}{4}}$

iv $343^{\frac{7}{3}}$ **v** $81^{\frac{5}{2}}$ **vi** $256^{\frac{3}{8}}$

c Write the values appearing in the questions in part **b** in surd form.

d Describe how the denominator of an exponent and the root of a surd are related, as well as how the numerator of an exponent and the exponent of a surd are related. Your descriptions refer to the number of repeated factors the base has.

$6 = 2 + 2 + 2$

$\frac{1}{3} \times 6 = 2$

$\frac{2}{3} \times 6 = 2 + 2$

$\frac{4}{3} \times 6 = 2 + 2 + 2 + 2$

$8 = 2 \times 2 \times 2$

$8^{\frac{1}{3}} = 2$

$8^{\frac{2}{3}} = 2 \times 2$

$8^{\frac{4}{3}} = 2 \times 2 \times 2 \times 2$

21 **a** Use exponents to show that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, where a and b are positive integers.

b Use exponents to show that $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$, where a and b are positive integers.

22 Write each of the following in the form $\sqrt[a]{x^b}$, where a and b are positive integers.

a $10^{2.5}$ **b** $2^{4.25}$ **c** $3^{0.16}$ **d** $5^{5.555}$

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Interactive skillsheet Fractional exponents

Topic quiz 2F

2G Logarithms

Learning intentions

By the end of this topic you will be able to ...

- ✔ write equivalent expressions for logarithms
- ✔ evaluate logarithms and understand them as a numerical value
- ✔ solve simple logarithmic equations.

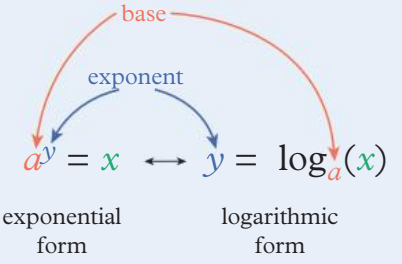
Inter-year links

Year 8 4A Exponents

Year 9 1A Exponents

Logarithms and exponentials

- Any equation in which the pronumeral appears as an exponent is called an exponential function.
  For example, $2^y = 8$ is an exponential equation with the solution $y = 3$.
- A **logarithm** is a function that asks, 'What exponent applied to a gives x?' It is the number to which the base is raised to produce a specific number.
  For example, in the equation above, the logarithm is $y = \log_2(8) = 3$. It is read as 'the logarithm, base 2, of 8 is equal to 3'.
  The diagram to the right shows the general relationship between logarithms and exponentials.



- To evaluate a logarithm, ask yourself, 'What exponent applied to a gives x?' In the diagram here, the answer is y.
- Logarithms and exponentials are the inverse operations of each other.
- Logarithmic identities include:

$$\log_a(a) = 1$$

$$\log_a(1) = 0$$

$$\log_a(a^x) = x$$

$$a^{\log_a(x)} = x$$

Example 2G.1 Converting between exponential form and logarithmic form

For each of these, write an equivalent equation in the form specified.

a $3^4 = 81$ (in logarithmic form)

b $\log_2(16) = 4$ (in exponential form)

THINK

a Remember that $a^y = x$ is the same as $\log_a(x) = y$. Identify the value of each variable and rewrite the equation.

b Recall that $\log_a(x) = y$ is the same as $a^y = x$. Identify the value of each variable and restate the equation.

WRITE

a $\log_3(81) = 4$

b $2^4 = 16$

Example 2G.2 Calculating the value of logarithms

Find the value of each of these logarithms.

a $\log_3(9)$ **b** $\log_2(64)$

THINK

a Determine the value of the logarithm. The term $\log_3(9)$ is asking, 'What exponent applied to 3 gives 9?' Since $3^2 = 9$, the value of the logarithm is 2.

b Determine the value of the logarithm. The term $\log_2(64)$ is asking, 'What exponent applied to 2 gives 64?'. Since $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$, the value of the logarithm is 6.

WRITE

a $\log_3(9) = 2$

b $\log_2(64) = 6$

Example 2G.3 Converting from exponential form with a negative exponent to logarithmic form

For each part:

i find the value of the term

ii write an equation in logarithmic form.

a 5^{-1} **b** $\left(\frac{1}{4}\right)^{-2}$ **c** 3^{-3}

THINK

a 1 Write an equivalent value with a positive exponent and simplify.

2 Write it in log form. (Remember that $a^y = x$ is the same as $\log_a(x) = y$.)

b 1 Write an equivalent value with a positive exponent and simplify.

2 Write it in log form. (Remember that $a^y = x$ is the same as $\log_a(x) = y$.)

c 1 Write an equivalent value with a positive exponent and simplify.

2 Write it in log form. (Remember that $a^y = x$ is the same as $\log_a(x) = y$.)

WRITE

a i $5^{-1} = \frac{1}{5^1}$

$= \frac{1}{5}$

ii $\log_5\left(\frac{1}{5}\right) = -1$

b i $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^2$

$= 16$

ii $\log_{\frac{1}{4}}(16) = -2$

c i $3^{-3} = \frac{1}{3^3}$

$= \frac{1}{27}$

ii $\log_3\left(\frac{1}{27}\right) = -3$

Helpful hints

- ✔ When the base of the logarithm and the number in the brackets are the same, the result is always 1. It doesn't matter how big or small the values are! For example, $\log_{238\,042}(238\,042) = 1$.
- ✔ Be careful when writing the base of the logarithms. The base needs to be written in a smaller size than the number in the brackets. For example, write $\log_2(3)$ rather than $\log 2(3)$.

ANS p547

Exercise 2G Logarithms

▲ 1–9 ■ 1–5(3rd, 4th columns), 6, 7, 10–13, 14(a) ⬟ 1–5(3rd, 4th columns), 12–17

2G.1 **1** Write each of these equations in logarithmic form.

a $3^3 = 27$ **b** $4^1 = 4$ **c** $5^4 = 625$ **d** $10^3 = 1000$

e $6^2 = 36$ **f** $9^3 = 729$ **g** $3^4 = 81$ **h** $2^7 = 128$

2 Write each of these equations in exponential form.

a $\log_2(32) = 5$ **b** $\log_3(9) = 2$ **c** $\log_5(25) = 2$ **d** $\log_7(49) = 2$

e $\log_4(16) = 2$ **f** $\log_2(1024) = 10$ **g** $\log_6(216) = 3$ **h** $\log_{10}(100) = 2$

2G.2 **3** Calculate each of these values.

a $\log_2(16)$ **b** $\log_4(256)$ **c** $\log_5(125)$ **d** $\log_{10}(1000)$

e $\log_8(64)$ **f** $\log_3(27)$ **g** $\log_9(81)$ **h** $\log_2(32)$

i $\log_{10}(10\,000)$ **j** $\log_5(625)$ **k** $\log_6(36)$ **l** $\log_3(81)$

m $\log_2(8)$ **n** $\log_4(64)$ **o** $\log_7(49)$ **p** $\log_3(243)$

2G.3 **4** **i** Find the value of each of the terms below.

ii Write an equation in logarithmic form for each term.

a 6^{-1} **b** 4^{-1} **c** 2^{-2} **d** 2^{-5}

e 5^{-2} **f** 10^{-1} **g** 10^{-3} **h** 3^{-3}

i $\left(\frac{1}{2}\right)^{-1}$ **j** $\left(\frac{1}{3}\right)^{-2}$ **k** $\left(\frac{1}{5}\right)^{-3}$ **l** $(0.01)^{-1}$

5 Calculate the value of each of these.

a $\log_2\left(\frac{1}{8}\right)$ **b** $\log_3\left(\frac{1}{27}\right)$ **c** $\log_{10}(0.01)$ **d** $\log_4\left(\frac{1}{4}\right)$

e $\log_3\left(\frac{1}{81}\right)$ **f** $\log_2\left(\frac{1}{64}\right)$ **g** $\log_4\left(\frac{1}{64}\right)$ **h** $\log_8\left(\frac{1}{64}\right)$

6 **a** Evaluate the following.

i 2^4 **ii** 2^{-4}

b Is there a real number such that 2 to the power of that real number is -16? (Find the value of $\log_2(-16)$.)

c Repeat part **b** for each of the following values.

i $\log_5(-25)$ **ii** $\log_3(-27)$ **iii** $\log_{10}(-1000)$

iv $\log_6(-36)$ **v** $\log_4(-64)$ **vi** $\log_8(-8)$

d What can you conclude about the logarithm of any negative number to any base?

7 **a** Now consider $\log_1(5)$. Is it possible for this to be written in an equivalent unique exponential form?

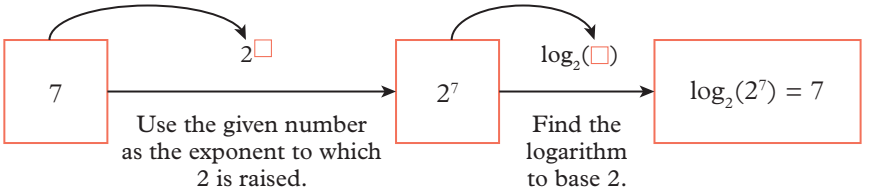
b Is it possible for any of the following to be written in an equivalent exponential form?

i $\log_1(3)$ **ii** $\log_1(10)$ **iii** $\log_1(6)$ **iv** $\log_1(4)$ **v** $\log_1(1)$

c What can you conclude about the logarithm of any number to the base 1?

8 Since the exponential and logarithmic operations are inverse operations, they 'undo' each other. If you raise a base to a power or exponent (which is called 'exponentiating'), then you find the logarithm to the same base of that result, and you will get the original number you raised the base to.

a Describe the process shown below.



b Evaluate each of the following.

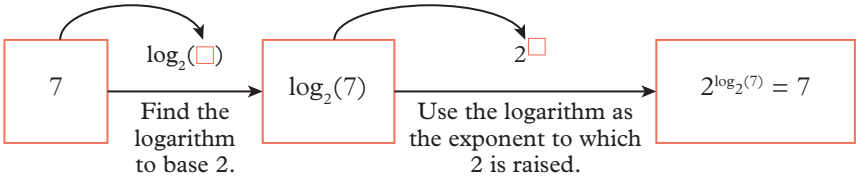
i $\log_5(5^7)$ **ii** $\log_7(7^5)$ **iii** $\log_{12}(12^9)$

iv $\log_2(2^{140})$ **v** $\log_{19}(19^{-6})$ **vi** $\log_{456}(456^{123})$

c Describe, in terms of how we evaluate logarithms, why $\log_a(a^x) = x$ for $a > 0$.

9 If we find the logarithm of a number to a given base, then exponentiate that with the same base, we get the original number back. (See question **8** if you don't know what 'exponentiate' means.)

a Describe the process shown below.



b Evaluate each of the following.

i 5^x, where $x = \log_5(7)$

ii 7^x, where $x = \log_7(5)$

iii 21^x, where $x = \log_{21}(11)$

iv 2^x, where $x = \log_2(531)$

v 101^x, where $x = \log_{101}(8)$

vi 963^x, where $x = \log_{963}(147)$

c Describe, in terms of how we evaluate exponential expressions, why $a^{\log_a(x)} = x$, when $a > 0$ and $x > 0$.

10 A knowledge of the inverse relationship between logarithms and exponential operations is useful when solving logarithmic equations.

Consider the equation $\log_x(64) = 2$, when $x > 0$.

a Copy and complete the following, using both sides of the equation as powers to which x is raised.

$\log_x(64) = 2$

$x^{\square} = x^{\square}$

$____ = x^{\square}$

$x = ____$

b What is the value of x? Explain why there is only one solution.

c Write the exponential equation in the last line of your answer for part **a** in the form $x^a = b^a$.

d Notice that, in the equation shown in part **c**, the two exponents are the same. If those exponents are the same, equating the bases will produce the solution. How does the answer found using this method compare with your answer to part **b**?

11 Use the methods discussed in question **10** to solve each of these logarithmic equations for x.

a $\log_x(64) = 6$ **b** $\log_x(125) = 3$

c $\log_x(16) = 2$ **d** $\log_x(8) = 3$

e $\log_x(128) = 7$ **f** $\log_x(81) = 4$

g $\log_x\left(\frac{1}{4}\right) = -1$ **h** $\log_x\left(\frac{1}{8}\right) = -3$

12 Consider the equation $\log_4(x) = 3$.

a Copy and complete the following, using both sides of the equation as powers to which 4 is raised.

$\log_4(x) = 3$

$4^{\square} = 4^{\square}$

$___ = 4^{\square}$

b Find the value of x.

c Briefly explain how using both sides of the equation as powers to which the base, 4, is raised expresses the logarithmic equation in exponential form.

13 Use the method from question **12** to solve each of these logarithmic equations for x.

a $\log_3(x) = 3$ **b** $\log_5(x) = 2$

c $\log_6(x) = 1$ **d** $\log_{10}(x) = 4$

e $\log_2(x) = 5$ **f** $\log_4(x) = 3$

g $\log_3(x) = -3$ **h** $\log_2(x) = -2$

14 a Use the exponent laws to show $(a^2)^{\frac{1}{2}} = a$ and $(a^{\frac{1}{2}})^2 = a$

b Use the rule in part **a** to complete the following:

$12^{\square} = 144$

$12 = 144^{\square}$

$\log_{144}(12) = \log_{144}(144^{\square})$

$\log_{144}(12) = \square$

c What is $\log_{a^x}(a)$?

15 Use the method from question **14** to find the value of each of the following:

a $\log_{729}(9)$ **b** $\log_{729}(3)$ **c** $\log_{625}(5)$ **d** $\log_{512}(2)$

16 Use the findings from question **8** and question **14** to find the value of the following:

a $\log_6(36)\log_{36}(6)$ **b** $\log_4(256)\log_{256}(4)$

c $\log_{\frac{1}{2}}(8)\log_8\left(\frac{1}{2}\right)$ **d** $\log_a(b)\log_b(a)$

17 Evaluate the following.

a $\log_2(\sqrt{2})$ **b** $\log_{\sqrt{2}}(2)$

c $\log_2\left(\frac{1}{\sqrt{8}}\right)$ **d** $\log_2\left(\sqrt[3]{128}\right)$

e $\log_3(3\sqrt{3})$ **f** $\log_5\left(\sqrt[8]{125}\right)$

g $\sqrt{\log_3(81)}$ **h** $\log_{\sqrt{125}}\left(\frac{\sqrt{5}}{5}\right)$

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Topic quiz 2G

PROBLEM SOLVING AND REASONING

CHALLENGE

2H Logarithm laws

Learning intentions

By the end of this topic you will be able to ...

- ✔ evaluate the logarithm of 1
- ✔ simplify logarithmic expressions involving addition and subtraction
- ✔ simplify logarithmic expressions involving exponents
- ✔ simplify expressions using the logarithm laws.

Inter-year links

Year 8 4A Exponents

Year 9 1C Raising exponents and the zero exponent

Logarithm laws

- The logarithm laws can be applied to expressions in logarithmic form $y = \log_a(x)$, where $a > 0$, $a \neq 1$, $x > 0$.
- The logarithm laws include:
  - → The **logarithm of a product** is equal to the sum of the logarithms of the factors.
    $$\log_a(xy) = \log_a(x) + \log_a(y)$$
  - → The **logarithm of a quotient** is equal to the difference of the logarithm of the dividend and the logarithm of the divisor.
    $$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$
  - → The **logarithm of a power** is equal to the product of the exponent of x and the logarithm of x.
    $$\log_a(x^n) = n \times \log_a(x)$$
- Recall that logarithmic identities include:
  $$\log_a(a) = 1$$
  $$\log_a(1) = 0$$

Example 2H.1 Calculating logarithmic expressions

Calculate each of these values.

a $\log_3(1)$ **b** $\log_4(4)$

THINK

a The logarithm of 1 is always equal to 0, no matter what the base is.
$1 = 3^0$, so $\log_3(1) = 0$.

b The logarithm of a number to a base equal to the number itself is always 1.
$4^1 = 4$, so $\log_4(4) = 1$.

WRITE

a $\log_3(1) = 0$

b $\log_4(4) = 1$

Example 2H.2 Adding and subtracting logarithms

Use the appropriate logarithm law to calculate:

a $\log_4(32) + \log_4(2)$ **b** $\log_5(50) - \log_5(2)$

THINK

a 1 Both logarithms in the sum contain the same base, so use the 'logarithm of a product' law:
$\log_a(x) + \log_a(y) = \log_a(xy)$

2 Which logarithm is the exponent of base 4 that will give an answer of 64?
The logarithm is 3.

b 1 Both logarithms in the expression contain the same base, so use the 'logarithm of a quotient' law:
$\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$

2 Which logarithm is the exponent of base 5 that will give an answer of 25?
The logarithm is 2.

WRITE

a $\log_4(32) + \log_4(2) = \log_4(32 \times 2)$
$= \log_4(64)$
$= \log_4(4^3)$
$= 3$

b $\log_5(50) - \log_5(2) = \log_5\left(\frac{50}{2}\right)$
$= \log_5(25)$
$= \log_5(5^2)$
$= 2$

Example 2H.3 Determining the logarithm of a number in exponential form

Use the appropriate logarithm law to calculate:

a $\log_2(4^3)$ **b** $\log_5\left(\frac{1}{5}\right)$

THINK

a 1 Use the 'logarithm of a power' law to simplify the logarithm:
$\log_a(x^n) = n \times \log_a(x)$

2 Calculate the logarithm and simplify.

b 1 Write the fraction as an integer with a negative exponent.

2 Use the 'logarithm of a power' law to simplify the logarithm:
$\log_a(x^n) = n \times \log_a(x)$

3 Use the logarithm identity $\log_a(a) = 1$ to simplify the expression.

WRITE

a $\log_2(4^3) = 3 \times \log_2(4)$
$= 3 \times \log_2(2^2)$
$= 3 \times 2$
$= 6$

b $\log_5\left(\frac{1}{5}\right) = \log_5(5^{-1})$
$= -1 \times \log_5(5)$
$= -1 \times 1$
$= -1$

Example 2H.4 Simplifying a logarithmic expression

Calculate $\log_6(48) + \log_6(3) - 2\log_6(2)$.

THINK

1 Use the 'logarithm of a power' law to rewrite the third term.

2 Use the 'logarithm of a product' law to simplify the first two terms.

3 Simplify the product.

4 Use the 'logarithm of a quotient' law to simplify the expression to one term.

5 Work out the value of the logarithm. (Remember: $6^2 = 36$)

WRITE

$\log_6(48) + \log_6(3) - 2\log_6(2)$

$= \log_6(48) + \log_6(3) - \log_6(2^2)$

$= \log_6(48 \times 3) - \log_6(2^2)$

$= \log_6(144) - \log_6(4)$

$= \log_6\left(\frac{144}{4}\right)$

$= \log_6(36)$

$= \log_6(6^2)$

$= 2$

Helpful hints

✔ Order of operations is important when solving logarithms. Recall your knowledge of BIDMAS. The I stands for indices, which can also be called exponents. Logarithms and exponential equations are in the same category as exponents. After brackets, you'll need to simplify or apply the logarithms and exponents.

✔ Remember that a logarithm cannot be applied to a negative number. This is important when solving logarithmic equations for which there might be two solutions.

For example, $2\log_2(x) = \log_2(9)$

$\log_2(x^2) = \log_2(9)$

$x^2 = 9$

$x = \pm 3$

Except, x cannot be negative, so $x = 3$ only.

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Exercise 2H Logarithm laws

△ 1–5, 6(a–f), 7–11, 12–13(1st column), 14(1st, 2nd columns), 15, 16

□ 3, 5, 6(g–l), 9–11, 12–13(2nd column), 14(3rd, 4th columns), 15, 17(a, c, e), 18, 19(a, c, e), 21

⬠ 6(g–l), 9, 11, 13(2nd column), 14(3rd, 4th columns), 17, 18, 19(d–f), 20, 22

2H.1 **1** Evaluate each of the following.

a $\log_5(5)$ **b** $\log_3(3)$ **c** $\log_{10}(10)$ **d** $4\log_5(5)$

e $3\log_3(3)$ **f** $\log_3(1)$ **g** $\log_2(1)$ **h** $4\log_5(1)$

2 Use the 'logarithm of a product' law, $\log_a(xy) = \log_a(x) + \log_a(y)$, to write each of the following expressions as the sum of two logarithms.

a $\log_{10}(3 \times 5)$ **b** $\log_2(6 \times 3)$ **c** $\log_5(2.5 \times 10)$ **d** $\log_2(xy)$

2H.2 **3** Write each of these expressions as the logarithm of a product.

a $\log_3(6) + \log_3(2)$ **b** $\log_{10}(5) + \log_{10}(3)$ **c** $\log_2(10) + \log_2(5)$

d $\log_4(6) + \log_4(12)$ **e** $\log_a(5) + \log_a(3)$ **f** $\log_2(x) + \log_2(y)$

4 Use the 'logarithm of a quotient' law, $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$, to write each of the following expressions as the difference of two logarithms.

a $\log_{10}\left(\frac{5}{7}\right)$ **b** $\log_2\left(\frac{5}{3}\right)$ **c** $\log_5\left(\frac{11}{20}\right)$ **d** $\log_2\left(\frac{a}{b}\right)$

5 Write each of these expressions as the logarithm of a quotient.

a $\log_3(12) - \log_3(5)$ **b** $\log_{10}(50) - \log_{10}(15)$ **c** $\log_2(25) - \log_2(10)$

d $\log_4(36) - \log_4(9)$ **e** $\log_a(3) - \log_a(15)$ **f** $\log_2(x) - \log_2(y)$

6 Use the appropriate logarithm law to evaluate each of the following.

a $\log_3(9) + \log_3(9)$ **b** $\log_6(108) - \log_6(3)$ **c** $\log_2(8) + \log_2(4)$

d $\log_{10}(500) + \log_{10}(20)$ **e** $\log_4(256) - \log_4(16)$ **f** $\log_3(243) - \log_3(9)$

g $\log_6(9) + \log_6(4)$ **h** $\log_5(175) - \log_5(7)$ **i** $\log_2(32) + \log_2(16)$

j $\log_{10}(25) + \log_{10}(4)$ **k** $\log_3(54) - \log_3(6)$ **l** $\log_4(768) - \log_4(3)$

7 Use the appropriate logarithm laws to calculate these values.

a $\log_6(27) + \log_6(4) - \log_6(3)$ **b** $\log_{10}(60) - \log_{10}\left(\frac{3}{5}\right)$ **c** $\log_3\left(\frac{3}{2}\right) + \log_3(18)$

8 Use the 'logarithm of a power' law, $\log_a(x^n) = n \times \log_a(x)$, to rewrite each of the following.

a $\log_3(7^2)$ **b** $\log_5(2^9)$ **c** $\log_4(8^3)$ **d** $\log_9(11^4)$

2H.3 **9** Use the appropriate logarithm law to evaluate each of the following.

a $\log_3(3^5)$ **b** $\log_4(4^2)$ **c** $\log_6(6^5)$ **d** $\log_3(9^4)$

e $3\log_3(27)$ **f** $4\log_4(64)$ **g** $5\log_5(5^2)$ **h** $\frac{1}{2}\log_4(4^4)$

10 Use the appropriate logarithm law to evaluate each of the following.

a $\log_{15}\left(\frac{1}{15^2}\right)$ **b** $\log_{12}\left(\frac{1}{12}\right)$ **c** $\log_{19}\left(\left[\frac{1}{19}\right]^2\right)$ **d** $\log_{103}\left(\frac{1}{103^{14}}\right)$

e $5\log_2\left(\frac{1}{64}\right)$ **f** $7\log_6\left(\frac{1}{216}\right)$ **g** $\frac{1}{9}\log_3\left(\frac{4}{108}\right)$ **h** $\frac{5}{2}\log_9\left(\frac{3}{243}\right)$

11 Write each of these in a simpler form and evaluate them.

a $2\log_9(3)$ **b** $8\log_4(2)$ **c** $2\log_3(9)$

d $\log_a(\sqrt{a})$ **e** $\log_3(5) - \log_3\left(\frac{5}{3}\right)$ **f** $2\log_x(\sqrt{xy}) - \log_x(y)$

2H.4 **12** Use the appropriate logarithm laws to evaluate each of the following.

a $\log_6(12) + \log_6(9) + \log_6(2)$ **b** $\log_2(24) + \log_2(6) - \log_2(36)$

c $\log_5(40) + \log_5(10) - \log_5(16)$ **d** $\log_3(15) - \log_3(45)$

e $\log_4(2) - \log_4(32)$ **f** $\log_{10}(20) + \log_{10}(45) + \log_{10}\left(\frac{10}{9}\right)$

13 Evaluate each of the following.

a $3\log_4(2) + \log_4(8)$ **b** $\log_5(225) - 2\log_5(3)$

c $2\log_3(9) + 2\log_3(3)$ **d** $4\log_{10}(5) + 4\log_{10}(2)$

e $\log_2(72) - 2\log_2(6)$ **f** $5\log_2(4) - \log_2(2^3)$

g $2\log_2(4) + \log_2(14) - \log_2(7)$ **h** $4\log_3(2) - \log_3(32) + \log_3(54)$

i $\log_5(15) + 2\log_5(5) + 3\log_5(2) - \log_5(24)$ **j** $3\log_{10}(5) + 2\log_{10}(3) + \log_{10}\left(\frac{8}{9}\right)$

14 Solve each of these logarithmic equations for x.

a $\log_x(27) = 3$ **b** $\log_4(64) = x$ **c** $\log_3(x) = 2$ **d** $\log_7(2401) = x$

e $\log_6(x) = 3$ **f** $\log_2(x) = 6$ **g** $\log_5\left(\frac{1}{125}\right) = x$ **h** $\log_x(6) = \frac{1}{2}$

i $\log_3(x) = 0$ **j** $\log_x\left(\frac{1}{16}\right) = 4$ **k** $\log_2(\sqrt{2}) = x$ **l** $\log_x\left(\frac{1}{16}\right) = -2$

PROBLEM SOLVING AND REASONING

15 Fractions containing logarithmic expressions in the numerator and the denominator can be simplified in a similar way to common fractions; that is, by cancelling any common factors in the numerator and denominator.

Consider $\frac{\log_{10}(3)}{\log_{10}(9)}$.

a Use a calculator to find the value of this fraction.

b Show that an equivalent logarithmic expression for the denominator of the fraction is $2\log_{10}(3)$.

c Show that the fraction simplifies to the same answer you obtained in part **a**.

16 Three students attempt to simplify the fraction $\frac{\log_6(8)}{\log_6(2)}$.

Hailey says the correct answer is 3, Danielle thinks the answer is 4 and Joyce is certain that it must be $\log_6\left(\frac{8}{2}\right)$ or $\log_6(4)$. Which student do you think is correct and why?

17 Simplify each of these expressions.

a $\frac{\log_{10}(16)}{\log_{10}(2)}$

b $\frac{\log_6(216)}{\log_6(36)}$

c $\frac{\log_3(8)}{\log_3(2)}$

d $\frac{\log_4(64)}{\log_4(256)}$

e $\frac{\log_{10}(16)}{\log_{10}(64)}$

f $\frac{\log_5(125)}{\log_5(5)}$

18 You can use the logarithm laws to solve more logarithmic equations.

Consider the equation $\log_5(2x) + \log_5(3) = \log_5(12)$. The following parts of this question highlight two techniques that can be used to solve equations like this. Work through both parts to solve the equation in two different ways.

a i Subtract $\log_5(12)$ from both sides of the equation and show that the use of the logarithm laws can help you to simplify the equation to $\log_5\left(\frac{x}{2}\right) = 0$.

ii Write the simplified equation from part **i** in exponential form and solve it for x.

b i Look at the original equation. Apply the appropriate law to the left-hand side of the equation to show that it simplifies to $\log_5(6x) = \log_5(12)$.

ii If the same base appears on both sides, the number parts of each logarithm can be equated; that is, $6x = 12$. Show that this method produces the same value for x as you found in part **a**.

19 Use either method from question **18** to solve each of these logarithmic equations for x.

a $\log_3(x) + \log_3(5) = \log_3(15)$

b $\log_4(2x) + \log_4(8) = 3$

c $\log_5(5x) - \log_5(4) = 2$

d $\log_5(10) - \log_5(x) = \log_5(15)$

e $\log_2(2x + 1) - \log_2(x) = 4$

f $\log_{10}(2x) + \log_{10}(3) = 4 + \log_{10}(5)$

CHALLENGE

20 Solve this logarithmic equation for x: $\log_5(x^2 - 9x + 21) = 0$

21 Show that the solution to $\log_{10}(2^{2x+1}) = \log_{10}(4)$ is $\frac{1}{2}$.

22 If $2\log_x\left(\frac{7}{5}\right) + \log_x\left(\frac{704}{35}\right) - 3\log_x\left(\frac{4}{5}\right) = \log_x(y)$, find the value of y.

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2I Exponential equations

Learning intentions

By the end of this topic you will be able to ...

- ✔ solve simple exponential equations
- ✔ solve simple exponential equations describing population growth.

Inter-year links

Year 8	4A Exponents
Year 9	1C Raising exponents and the zero exponent

Solving exponential equations

- To solve exponential equations of the form $a^x = y$ for x, use one of the two methods below.

  Method 1: If y can be written in exponential form as a term with the base a, you can equate the exponents. That is, if $a^x = a^m$ then $x = m$.

  For example, $2^x = 8$ can be written as $2^x = 2^3$, so $x = 3$.

  In some cases, you might need to rewrite both sides of the equation so they have the same base.

  For example, $8^x = 4$ can be written as $2^{3x} = 2^2$, so $3x = 2$.

  Simplifying shows that $x = \frac{2}{3}$.

  Method 2: Take the logarithm with the same base of both sides and simplify.

  For example, $2^x = 8$ can be written as:

$$\begin{aligned} 2^x &= 8 \\ \log_2(2^x) &= \log_2(8) \\ \log_2(2^x) &= \log_2(2^3) \\ x &= 3 \end{aligned}$$

- Exponential relationships can be used to model many real situations involving growth and decay (for populations or investments, for example).

Example 2I.1 Solving exponential equations by writing with the same base

Solve each of these equations.

a $2^x = 32$ **b** $9^x = 27$

THINK

a 1 Write the equation so that both sides have the same base (in this case, 2).

2 Solve the equation by equating the exponents. The bases are the same and it is an equation, so the exponents are equal.

b 1 Write the equation so that both sides have the same base (in this case, 3).

2 Equate the exponents because it is an equation and the bases are the same.

3 Solve the linear equation for x.

WRITE

a $2^x = 32$

$2^x = 2^5$

$x = 5$

b $9^x = 27$

$(3^2)^x = 3^3$

$3^{2x} = 3^3$

$2x = 3$

$x = \frac{3}{2}$

Example 2I.2 Solving exponential equations using logarithmic form

Solve each of these equations. Give your answer as an exact value each time.

a $2^x = 5$ **b** $4 \times 3^x = 2$ **c** $5^{x-4} = 125$

THINK

a Write the equation in logarithmic form. The logarithm cannot be simplified so x is already in exact form.

b **1** Simplify by dividing both sides of the equation by 4.

2 Write the equation in logarithmic form.

3 Simplify the logarithm. The fraction can be simplified using negative exponents.

c **1** Write the equation in logarithmic form.

2 Solve for x by adding 4 to both sides.

3 Which exponent of base 5 will give an answer of 125? The answer is 3, then simplify.

WRITE

a $2^x = 5$

$x = \log_2(5)$

b $4 \times 3^x = 2$

$3^x = \frac{1}{2}$

$x = \log_3\left(\frac{1}{2}\right)$

$= \log_3(2^{-1})$

$= -\log_3(2)$

c $5^{x-4} = 125$

$x - 4 = \log_5(125)$

$x = \log_5(125) + 4$

$= 3 + 4$

$= 7$

Helpful hints

- ✔ Remember that logarithms 'undo' exponentials, and exponentials 'undo' logarithms. These are inverse operations.
- ✔ You'll need expert skills in negative exponents, factors and surds to effectively solve exponential equations!

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Exercise 2I Exponential equations

▲ 1–6, 7(a, c, e, g), 9, 11, 12 ■ 3–7, 10(a–c), 13, 15, 16 ⬟ 3–4, 6(c, e, f), 7(e–h), 8, 10(d–f), 14, 15, 17, 18

UNDERSTANDING AND FLUENCY

2I.1

1 Find x by equating the exponents to solve each of these equations.

a $2^x = 2^9$ **b** $3^x = 3^6$

c $4^x = 4^{-3}$ **d** $8^x = 8$

e $5^{x-2} = 5^6$ **f** $2^{x+4} = 2^8$

g $10^{2-x} = 10^{-5}$ **h** $2^{-x-7} = 2^3$

i $9^{2x} = 9^6$ **j** $6^{3x} = 6^{15}$

k $7^{2x-1} = 7^{11}$ **l** $3^{4x+5} = 3^5$

2 Solve each of these equations.

a $2^x = 8$ **b** $3^x = 9$

c $4^x = 1024$ **d** $10^x = 10\,000$

e $6^{x-5} = 36$ **f** $2^{x+3} = 64$

g $9^{2x-1} = 729$ **h** $3^{-x+4} = 243$

21.2 **3** Solve each of these equations. Give your answer as an exact value each time.

a $2^x = 11$ **b** $5^x = 92$ **c** $92^x = 5$

d $14^x = \frac{1}{4}$ **e** $6 \times 3^x = 3$ **f** $2 \times 7^x = 53$

g $5^{2x} = 17$ **h** $10^{3x} = 75$ **i** $8^x + 3 = 36$

j $11^{6x} - 7 = 48$ **k** $3 \times 17^x + 5 = 41$ **l** $9 \times 12^{-4x} - 10 = 36$

4 Solve each of these equations. Give your answer as an exact value in the form $a \log_b(c) + d$, where a, b, c and d are rational numbers.

a $8^{x-1} = 10$ **b** $3^{x+1} = 12$ **c** $5^{x+3} = 2$

d $2^{x-5} = 6$ **e** $2^{3-x} = 48$ **f** $9^{-x+2} = 20$

g $6^{3-x} = 4$ **h** $2^{6-x} = 30$ **i** $4^{2x+3} = 24$

j $3^{4x+1} = 87$ **k** $8 \times 7^{5x+2} = 24$ **l** $6 \times 10^{4x+3} + 11 = 20$

5 Solve the following equations.

a $2^{x^3} = 256$ **b** $2^{x^2} = 512$ **c** $3^{x^2} = 81$

6 Solve the following equations.

a $(3^x - 9)(2^x - 8) = 0$ **b** $(5^x - 125)(10^x - 10\,000) = 0$

c $(2^x - 64)(2^x - 16) = 0$ **d** $(3^x - 1)(3^x - 3)(3^x - 9) = 0$

e $\left(2^x - \frac{1}{32}\right)\left(4^x - \frac{1}{64}\right) = 0$ **f** $(4 \times 7^x - 28)\left(2^{3x-5} - 1\right) = 0$

7 The value of x in each of the exponential equations below is not a whole number. Between which two whole numbers will the value of x be for each of the equations? For example, $2^x = 10$ has a solution for x that is between 3 and 4, because $2^3 = 8$ and $2^4 = 16$.

a $2^x = 5$ **b** $3^x = 290$

c $5^x = 17$ **d** $10^x = 6850$

e $3^{2x} = 14$ **f** $7^{2x} = 3000$

g $4^{x+1} = 38$ **h** $6^{x-2} = 210$

8 Consider the exponential equation $9 \times 2^{6x-5} = 27$.

a Use exponent laws to write the equation in the form $b \times a^x = c$, where a, b and c are rational numbers.

b Solve the equation. Give your answer as an exact value.

9 Explain why the following equations have no real solutions.

a $3^x = -1$ **b** $3^x = 0$

10 Solve the following equations by first factorising the expression.

a $(2^x)^2 - 8 \times 2^x = 0$

b $(5^2)^x - 5 \times 5^x = 0$

c $10^{2x} - 15 \times 10^x = 0$

d $4^x - 6 \times 2^x + 8 = 0$

e $9^x - 5 \times 3^x - 14 = 0$

11 After the water boils in an electric kettle, the temperature of the water T (in °C) after m minutes follows the relationship $T = 20 + 80 \times 0.8^m$ as it cools.

a What was the temperature of the boiled water at the start? Hint: $m = 0$.

b What is the temperature of the water, to the nearest degree, after 3 minutes?

c How many minutes, correct to one decimal place, does it take for the temperature to be half the temperature it was at the start?

d What is the lowest temperature that this kettle of water can reach?

e Is the relationship discussed here showing growth or decay?

12 A population of rabbits can be modelled by the relationship $n = 9^t + 1$, where n is the number of rabbits after t months.

a How many rabbits were there to start with?

b How many rabbits are there after 2 months?

c How many months, correct to two decimal places, does it take for there to be 1000 rabbits?

d Is the model discussed here showing population growth or population decay?

13 The human population can be modelled using different exponential functions. One model uses $P = 2.6 \times 1.017^t$, where P is the population in billions and t is the number of years since 1950.

a Use this model to find a population figure for:

i 2020 **ii** 2040

b In what year does this model predict that the population will reach 15 billion?

c The population in 2010 was estimated to be 6.9 billion. How did this compare to the population calculated using this model? Give reasons why you think the population numbers might be different.

14 Bacteria grow by doubling every hour such that the number, N, of bacteria on a petri dish after t hours is given by $N = 45 \times 2^t$.

a State the number of bacteria on the petri dish initially.

b Determine the number of hours it would take for the number of bacteria to grow to 45 000:

i as an exact value

ii correct to the nearest hour.

c Initially, the petri dish was left for a day. Determine the number of bacteria on the petri dish after that day, correct to two significant figures.

d A virus is introduced to the petri dish that kills the bacteria. The number, N, of bacteria alive t hours after the virus is introduced is given by the equation $N = \frac{100}{1 - 0.99^{t+1}}$.

Determine the number of bacteria still alive one hour after the virus is introduced, correct to the nearest bacterium.

e Fill in the following table of values to determine the number of bacteria alive after 66, 67, 68, 69, 70 hours, correct your answer to the nearest bacterium.

t	66	67	68	69	70
N					

f After how many hours will the number of bacteria reduce to 200?

g Determine the number of hours it will take for the number of bacteria to reduce to 200 as an exact value.

15 Tristan fills a glass with tap water at room temperature, which is 25°C. He wants to cool it by placing it in the refrigerator which has a constant internal temperature of 4°C. The temperature, T°C, of the water in the glass t minutes after the glass is placed in the refrigerator is given by the equation $T = 21 \times 2.718^{-kt} + 4$, where k is a positive number.

a Tristan checks the temperature of the glass of water after 5 minutes to find it is 15°C. Determine the value of k, correct to two decimal places.

b Determine how much longer the water will take to cool to 9°C, correct to the nearest minute.

c Unhappy with how long the water will take to cool to 9°C, Tristan puts the glass in the freezer which has a constant internal temperature of −18°C.
The temperature, T°C, of the water in the glass t minutes after the glass is placed in the freezer is given by the equation $T = 33 \times 2.718^{-mt} - 18$, where m is a positive number.

Tristan assumes that $m = k$ while the glass is in the freezer. Determine how long, to the nearest 30 seconds, Tristan expects the glass to take to cool to 9°C.

d Tristan checks the temperature of the glass of water after 1 minute to find it is 7°C. Determine the actual value of m, correct to two decimal places.

16 Solve $\left(x^2 + x - 11\right)^{\left(x^2 - 5x - 14\right)} = 1$ for x.

17 Scientific calculators often have buttons for only two types of logarithms: the common (base 10) logarithm, usually labelled 'log'; and the natural logarithm with a special base, e, that is approximately 2.71, labelled 'ln'. To evaluate logarithms with other bases on these calculators, we need to change the base of the logarithm we are evaluating to 10 (or to the special base, e).

For example, to change $\log_2(54)$ to a base 10 logarithm, we change the base to 10, then divide by the base 10 logarithm of 2. That is, $\log_2(54) = \dfrac{\log_{10}(54)}{\log_{10}(2)}$.

a Convert the following logarithms to base 10.

i $\log_3(7)$ **ii** $\log_7(3)$ **iii** $\log_{11}(45)$

iv $\log_{101}(12)$ **v** $\log_{19}(24)$ **vi** $\log_a(b)$

b Write the following quotients as a single logarithm, for which the base is the value appearing inside the brackets of the logarithm in the denominator.

i $\dfrac{\log_{10}(8)}{\log_{10}(9)}$ **ii** $\dfrac{\log_{10}(9)}{\log_{10}(8)}$ **iii** $\dfrac{\log_2(18)}{\log_2(5)}$

iv $\dfrac{\log_e(6)}{\log_e(15)}$ **v** $\dfrac{\log_e(75)}{\log_e(2)}$ **vi** $\dfrac{\log_x(v)}{\log_x(t)}$

c Write $x = \log_2(54)$ in exponential form.

d Solve the exponential equation from part **c** using a base 10 logarithm to show that $x = \dfrac{\log_{10}(54)}{\log_{10}(2)}$.

e Use a similar method to show that $\log_b(a) = \dfrac{\log_x(a)}{\log_x(b)}$.

18 Solve each of these equations. Give your answers correct to three significant figures.

a $9^x = 10$ **b** $5^x = 30$ **c** $\frac{1}{4} \times 2^x = 2000$

d $3^{6x} = 120$ **e** $10^{\frac{1}{8}x + 5} = 40\,000$ **f** $7 \times 16^x + 9 = 130$

g $25^{3x} - 10 = 28$ **h** $6 \times 7^{2x - 7} + 1 = 1258$ **i** $\dfrac{100}{1 - 5^{2x}} = 200$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Solving exponential equations

Investigation The exponential growth of one grain of rice

Topic quiz 2I

Chapter summary

Surds

- $(\sqrt{a})^2 = a$
- $\sqrt{a^2} = a$

Adding and subtracting surds

- $a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$
- $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5}$

Rationalising the denominator

$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$

conjugates — rational number

- $\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$
- $\frac{1}{a + \sqrt{b}} \times \frac{a - \sqrt{b}}{a - \sqrt{b}} = \frac{a - \sqrt{b}}{a^2 - b}$

Multiplying surds

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$

Dividing surds

- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $a\sqrt{b} \div c\sqrt{d} = \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \times \sqrt{\frac{b}{d}}$

Negative exponents

- $a^{-1} = \frac{1}{a}$
- $\frac{1}{a^{-m}} = a^m$
- $a^{-m} = \frac{1}{a^m}$

Fractional exponents

base → $a^{\frac{1}{3}}$ ← fractional exponent/index/power

exponential form

$a^{\frac{2}{3}} = \left(\sqrt[3]{a}\right)^2$

fractional exponent form — surd form

The exponent laws

	Integer exponents	Fractional exponents
Zero exponent law	$a^0 = 1$	$a^0 = 1$
Product of powers rule	$a^5 \times a^3 = a^{5+3}$	$a^{\frac{1}{5}} \times a^{\frac{1}{3}} = a^{\frac{1}{5}+\frac{1}{3}}$
Quotient of powers rule	$a^5 \div a^3 = a^{5-3}$	$a^{\frac{1}{5}} \div a^{\frac{1}{3}} = a^{\frac{1}{5}-\frac{1}{3}}$
Power of a power rule	$\left(a^5\right)^3 = a^{5\times 3}$	$\left(a^{\frac{1}{5}}\right)^{\frac{1}{3}} = a^{\frac{1}{5}\times\frac{1}{3}}$

Logarithms and exponentials

- $\log_a(1) = 0$
- $\log_a(a) = 1$
- $\log_a(a^x) = x$
- $a^{\log_a(x)} = x$
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^n) = n \times \log_a(x)$

base, exponent

$a^y = x \leftrightarrow y = \log_a x$

exponential form — logarithmic form

Solving exponential equations

To solve exponential equations of the form $a^x = y$ for x, use one of the two methods below.
Method 1: If y can be written in exponential form as a term with the base a, you can equate the exponents. That is, if $a^x = a^m$ then $x = m$.
Method 2: Take the logarithm with the same base of both sides and simplify.

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

2A **1** Which of these numbers is irrational?

A $\sqrt{120}$ **B** $-\sqrt{16}$ **C** $3\sqrt{9}$ **D** $\sqrt[3]{8}$ **E** $\log_2(8)$

2B **2** Which of these numbers is the simplest form of $4\sqrt{32}$?

A $8\sqrt{8}$ **B** $4\sqrt{2}$ **C** $8\sqrt{2}$ **D** $16\sqrt{2}$ **E** $\sqrt{512}$

2C **3** Which of these surd expressions is equivalent to $\sqrt{35}$?

A $\sqrt{30} \times \sqrt{5}$ **B** $\sqrt{7} \times 5$ **C** $\sqrt{7} \times \sqrt{5}$ **D** $\sqrt{5} \times 5\sqrt{5}$ **E** $\sqrt{30} + \sqrt{5}$

2D **4** Which of these surd expressions does $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ simplify to?

A $8\sqrt{2}$ **B** $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ **C** $80\sqrt{2}$
D $16\sqrt{2}$ **E** $28\sqrt{2}$

2D **5** After expanding, what does $2\sqrt{6}(4\sqrt{3} + 5\sqrt{10})$ simplify to?

A $8\sqrt{18} + 10\sqrt{60}$ **B** $24\sqrt{2} + 20\sqrt{15}$ **C** $24\sqrt{2} + 5\sqrt{10}$
D $72\sqrt{2} + 40\sqrt{15}$ **E** $24\sqrt{2} + 5\sqrt{10}$

2E **6** Which of these is equivalent to $\frac{3\sqrt{5}}{2\sqrt{2}}$ written with a rational denominator?

A $\frac{3\sqrt{10}}{2}$ **B** $\frac{3\sqrt{10}}{4}$ **C** $\frac{3\sqrt{7}}{4}$ **D** $\frac{3\sqrt{10}}{8}$ **E** $\frac{15}{2\sqrt{10}}$

2F **7** Which of these expressions is *not* equivalent to $x^{\frac{3}{4}}$?

A $\sqrt[4]{x^3}$ **B** $\left(\sqrt[4]{x}\right)^3$ **C** $\left(x^6\right)^{\frac{1}{8}}$ **D** $\left(x^3\right)^{\frac{1}{4}}$ **E** $\left(\sqrt[3]{x}\right)^4$

2G **8** What is $x^y = z$ equivalent to in logarithmic form?

A $\log_x(z) = y$ **B** $\log_x(y) = z$ **C** $\log_z(x) = y$ **D** $\log_z(y) = x$ **E** $\log_y(z) = x$

2H **9** Which of these is an equivalent logarithmic expression for $\log_{10}(42)$?

A $\log_{10}(20) + \log_{10}(22)$ **B** $\log_{10}(6) + \log_{10}(7)$ **C** $\log_{10}(6) \times \log_{10}(7)$
D $\frac{\log_{10}(7)}{\log_{10}(6)}$ **E** $\frac{\log_{10}(84)}{\log_{10}(2)}$

2I **10** If $8^x = 32$, which of these is the solution to the equation?

A $x = 24$ **B** $x = \frac{1}{4}$ **C** $x = 4$ **D** $x = \frac{3}{5}$ **E** $x = \frac{5}{3}$

Short answer

2A **1** Determine whether each of these numbers is rational or irrational.

a $\frac{3}{7}$ **b** 0.256 371… **c** $-\sqrt{25}$
d $\frac{3\sqrt{5}}{2}$ **e** $4\sqrt{2} + 16$ **f** $0.\overline{215}$

2A **2** Determine which of these numbers are surds.

a $3\sqrt{8}$ **b** $\frac{1}{\sqrt{5}}$ **c** $6\sqrt{4}$ **d** $\sqrt{6} - 1$ **e** $\sqrt{\frac{9}{25}}$ **f** $\frac{\sqrt{6}}{4}$

2A **3** Use a calculator to find the value of each of these, correct to three decimal places.

a $2\sqrt{6}$ **b** $\frac{\sqrt{5} - 2}{4}$

2A **4** Show that each of these numbers is rational.

a 0.65 **b** 0.2175 **c** $0.5\dot{2}$ **d** $1.\overline{23}$

2B **5** Write each of these terms in simplest form.

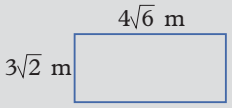
a $2\sqrt{24}$ **b** $3\sqrt{125}$ **c** $\frac{1}{2}\sqrt{108}$

2C **6** Calculate the value of each of these, writing your answers in simplest form.

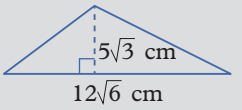
a $3\sqrt{12} \times 4\sqrt{6}$ **b** $-\sqrt{20} \times 5\sqrt{45}$

2C **7** Calculate the exact area of each of these shapes, in simplest surd form.

a



b



2C **8** Calculate the value of each of these.

a $\sqrt{11} \times \sqrt{7}$ **b** $3\sqrt{6} \times 9\sqrt{5}$

2C **9** Calculate the value of each of these.

a $\sqrt{24} \div \sqrt{8}$ **b** $\frac{4\sqrt{45}}{\sqrt{15}}$

2D **10** Simplify each of these expressions.

a $3\sqrt{2} - 6\sqrt{2} + \sqrt{2}$ **b** $3\sqrt{7} + 2\sqrt{5} - \sqrt{7} + 3\sqrt{5}$

c $2\sqrt{10} - \sqrt{2} + 4\sqrt{5} + \sqrt{2}$ **d** $5\sqrt{6} - 2\sqrt{6} + 5 + 3\sqrt{5} - 3$

2D **11** Simplify each of these expressions.

a $2\sqrt{24} - 2\sqrt{54} - 5\sqrt{96} + \sqrt{20}$ **b** $\sqrt{108} - 2\sqrt{75} + 2\sqrt{48} + 3\sqrt{8}$

c $4\sqrt{180} - 6\sqrt{125} + 3\sqrt{245}$ **d** $3\sqrt{200} + 4\sqrt{180} + \sqrt{160} - 3\sqrt{128}$

2D **12** Expand and simplify each of these expressions.

a $(2 + 4\sqrt{3})(3 - 2\sqrt{8})$ **b** $(2\sqrt{3} + 4\sqrt{5})^2$

2E **13** Write each of these fractions with a rational denominator, writing the numerator in simplest surd form where appropriate.

a $\frac{5}{2\sqrt{6}}$ **b** $\frac{3\sqrt{15}}{2\sqrt{5}}$ **c** $\frac{4\sqrt{5}}{\sqrt{3} + 2\sqrt{6}}$ **d** $\frac{2 + \sqrt{6}}{3\sqrt{7} - 5\sqrt{3}}$

2F **14** Use the appropriate exponent law to simplify each of these expressions. Write each answer:

i in fractional exponent form

ii in an appropriate surd form.

a $\left(x^{\frac{3}{5}}\right)^{\frac{5}{2}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{1}{3}}$ **c** $\frac{x^{\frac{2}{3}}}{9x^{\frac{1}{4}}}$

d $\sqrt{8x^3y^2} \times \sqrt{3xy}$ **e** $\left(9x^{\frac{1}{4}}y^{\frac{2}{5}}\right)^{\frac{1}{2}}$ **f** $\frac{\sqrt{3x^5y^2z}}{5\sqrt{8x^2yz^2}}$

2G **15** Write each of these equations in logarithmic form.

a $3^4 = 81$ **b** $6^3 = 216$ **c** $2^5 = 32$ **d** $x^y = w$

2G **16** Write each of these equations in exponential form.

a $\log_2(8) = 3$ **b** $\log_4(64) = 3$ **c** $\log_{10}\left(\frac{1}{10\,000}\right) = -4$ **d** $\log_x(z) = b$

2H **17** Use the appropriate logarithm law to calculate the value of each of these.

a $\log_5(5^3)$ **b** $\log_4(96) - \log_4(6)$

c $3\log_9(3) + \log_9(27)$ **d** $2\log_3(4) - \log_3(32) + \log_3(54)$

2H **18** Solve each of these logarithmic equations for x.

a $\log_x(32) = 5$ **b** $\log_5(x) = -3$ **c** $\log_4(256) = x$

d $\log_x\left(\frac{1}{8}\right) = 3$ **e** $\log_2(x) + \log_2(5) = \log_2(12)$ **f** $\log_5(4x) - \log_5(5) = 3$

2I **19** Solve each of these equations.

a $4^{x+2} = 16$ **b** $81^{x+1} = 27$ **c** $5^{3x} = 15$ **d** $6^{2x+1} = 42$

Analysis

1 A storage chest is the shape of a cube with edge lengths each measuring $7\sqrt{10} + 11\sqrt{7}$ cm.

- **a** Draw a labelled diagram to represent the chest.
- **b** Calculate the exact area of one of the faces of the chest.
- **c** Is the area you calculated in part **b** a rational number or an irrational number?
- **d** All the external faces of the chest need to be covered with a special coating to protect it against wear and tear. What is the total surface area that needs to be covered?
- **e** If the special coating is sold for $12.80 per square metre, how much will it cost to cover the chest?
- **f** An alternative protective paint offers the best protection when three coats are applied. It is advertised that 1 L of this protective paint will cover an area of 5 m^2. Show that a 1 L can of the protective paint is enough to provide best protection for the chest.
- **g** Calculate the percentage of the paint in the 1 L can from part **f** that will be used to cover the chest with three coats.
- **h** A narrow 80 cm long pole needs to fit inside the chest. Show that it is not possible for the pole to lie along the bottom of the chest.
- **i** Is there any way that the pole from part **h** can fit inside the chest? Use appropriate mathematics to show your working and to support your answer.
- **j** What assumptions did you need to consider for the calculations you have performed in this task?

2 Toby has begun beekeeping. He started with 10 000 bees in his four-frame hive. One month later, there are now 20 000 bees in the hive. The number of bees, B, in Toby's hive after t months can be modelled using the equation $B = a \times 10^{bt}$.

- **a** State the value of a.
- **b** Determine the value of b
  - **i** as an exact value
  - **ii** as a decimal correct to two decimal places.
- **c** Write the equation modelling the number of bees in the form $B = a \times d^t$.
- **d** Determine the expected number of bees after 6 months.
- **e** Toby's four-frame hive can support up to 1 000 000 bees.

  After how many months does Toby need to expand his hive to allow for the population of bees to continue growing? Write your answer:
  - **i** as an exact value
  - **ii** correct to the nearest month.
- **f** After some years, Toby's population of bees became relatively constant at 10 million. Toby began to notice his population of bees declining in number. After a year, there was approximately 9.1 million bees.

  Determine the equation for the number of bees, B, n years after Toby noticed the population was declining, in the form $B = p \times q^n$.
- **g** If Toby does not make any changes, how many years will it take for the number of bees to drop below his original 10 000 bees?

3

Linear relationships

Index

3A Solving linear equations
3B Sketching linear graphs
3C Determining linear equations
3D Solving linear inequalities
3E Simultaneous linear equations

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Number lines
- ✔ The Cartesian plane
- ✔ Plotting linear relationships
- ✔ Intercepts

Curriculum links

- Solve linear inequalities and simultaneous linear equations in 2 variables; interpret solutions graphically and communicate solutions in terms of the situation (AC9M10A02)

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Materials

- ✔ Ruler
- ✔ Graph paper

3A Solving linear equations

Learning intentions

By the end of this section you will be able to ...

- ✔ solve linear equations using inverse operations
- ✔ solve linear equations with an unknown on both sides
- ✔ rearrange linear equations.

Inter-year links

Year 7	6G Solving equations using inverse operations
Year 8	6C Solving equations with the unknown on both sides
Year 9	3A Solving linear equations

Linear equations

- A **linear equation** is an equation featuring pronumerals that are only raised to the power of 1. These pronumerals represent variable or unknown values.

Linear equations	Non-linear equations
$4x - 1 = 6$	$2x^2 = 32$
$y = x + 3$	$y = \frac{1}{x}$
$2a - \frac{3}{5}b = 10$	$a^3 + 6b = 0$

- If a linear equation has only one pronumeral, algebraic methods can be used to find a solution for the unknown value.
- If a linear equation has more than one pronumeral, then it has more than one solution for each variable. It cannot be solved for numerical solutions by itself and can only be rearranged.

Solving linear equations using inverse operations

- If the same operation is performed on both sides of an equation, the result is an equation equivalent to the original equation and the value(s) of the unknown(s) remains the same.
- Any operation can be undone by applying its inverse operation.
- Operations performed on an unknown in an equation can be undone by performing the appropriate inverse operation to both sides of the equation.

For example, $\frac{5x - 4}{2} = 8$ $(\times 2)$

$5x - 4 = 16$ $(+ 4)$

$5x = 20$ $(\div 5)$

$x = 4$

Operation	Inverse operation
$+ 3$	$- 3$
$- 3$	$+ 3$
$\times 3$	$\div 3$
$\div 3$	$\times 3$

Rearranging linear equations

- A literal equation is an equation containing two or more variables. For example, $A = \pi r^2$ and $d = st$.
- A pronumeral is the **subject of an equation** if it is by itself on one side of the equation (usually the left-hand side) and does not appear on the other side of the equation.
- Inverse operations can be used to rearrange a linear equation to make any pronumeral the subject.

For example, $\frac{x}{a} - b = c$ $(+ b)$

$\frac{x}{a} = b + c$ $(\times a)$

$x = a(b + c)$

Example 3A.1 Solving linear equations

Solve each of these equations by finding the value of x.

a $5x - 6 = -4$ **b** $2 = \frac{1 - x}{7}$

THINK	WRITE
a Identify the correct inverse operation and apply it to both sides of the equation until x is the subject of the equation.	**a** $5x - 6 = -4$ $(+ 6)$ $5x = 2$ $(\div 5)$ $x = \frac{2}{5}$
b **1** Identify the correct inverse operation and apply it to both sides of the equation until x is the subject of the equation. The numerator of a fraction is always grouped as if it is enclosed in brackets, so the inverse operations in the numerator must be applied last.	**b** $2 = \frac{1 - x}{7}$ $(\times 7)$ $14 = 1 - x$ $(- 1)$
2 Change the sign when dividing by -1 on both sides of the equation. Write the solution so that x is on the left-hand side (LHS).	$13 = -x$ $(\div (-1))$ $x = -13$

Example 3A.2 Solving linear equations with an unknown in the denominator

Solve $\frac{24}{x + 2} - 1 = 5$ by finding the value of x.

THINK	WRITE
1 Use inverse operations to rearrange the equation so that the fraction is by itself on one side.	$\frac{24}{x + 2} - 1 = 5$ $(+ 1)$
2 Multiply both sides of the equation by the denominator of the fraction.	$\frac{24}{x + 2} = 6$ $(\times (x + 2))$
3 Identify and apply the correct inverse operations to both sides of the equation.	$24 = 6(x + 2)$ $(\div 6)$ $4 = x + 2$ $(- 2)$ $2 = x$
4 Write the solution to the equation with x on the LHS.	$x = 2$

Example 3A.3 Solving linear equations with the unknown on both sides

Solve each of these equations by finding the value of x.

a $9x + 2 = 7x - 8$ **b** $5 - 2x = 6x - 1$

THINK

a **1** Remove the pronumeral term from one side of the equation by subtracting the pronumeral that has the smaller coefficient ($7x$) from both sides of the equation.

2 Solve the equation using inverse operations.

b **1** Remove the negative pronumeral term by adding $2x$ to both sides of the equation.

2 Solve the equation using inverse operations.

3 Simplify the solution and write x on the LHS.

WRITE

a $9x + 2 = 7x - 8 \quad (-7x)$

$2x + 2 = -8 \quad (-2)$

$2x = -10 \quad (\div 2)$

$x = -5$

b $5 - 2x = 6x - 1 \quad (+2x)$

$5 = 8x - 1 \quad (+1)$

$6 = 8x \quad (\div 8)$

$\frac{6}{8} = x$

$x = \frac{3}{4}$

Example 3A.4 Rearranging linear equations

Rearrange each equation to make x the subject on the left-hand side.

a $2x - m = 4n$ **b** $\frac{h+1}{x} = \frac{a}{b}$

THINK

a Identify the correct inverse operation and apply it to both sides of the equation until x is the subject of the equation.

b **1** Multiply both sides of the equation by the denominator containing x.

2 Identify the inverse operation and apply it to both sides of the equation until x is the subject of the equation.

WRITE

a $2x - m = 4n \quad (+m)$

$2x = 4n + m \quad (\div 2)$

$x = \frac{4n + m}{2}$

b $\frac{h+1}{x} = \frac{a}{b} \quad (\times x)$

$h + 1 = \frac{ax}{b} \quad (\times b)$

$b(h + 1) = ax \quad (\div a)$

$\frac{b(h+1)}{a} = x$

$x = \frac{b(h+1)}{a}$

or, in expanded form:

$x = \frac{bh + b}{a}$

Helpful hints

- ✔ Writing the inverse operation beside the appropriate line of working is a great way to keep track of your calculations.

  $\frac{x}{3} - 5 = 7 \qquad (+ 5)$

- ✔ Writing the pronumeral on the LHS of your solution is only a convention. Once you have isolated the unknown on one side of the equation, you have the solution.
  For example, $3 = x$ is the same as $x = 3$.
- ✔ When given a problem in words, define your pronumerals before writing the equation to represent the problem.
  For example, if you are told that a community garden grows twice as many zucchinis as tomatoes, you could define the pronumerals like this: t = the number of tomatoes, z = the number of zucchinis, and the formula can be written as: $2t = z$.

ANS p551

Exercise 3A Solving linear equations

△ 1(1st column), 2–4(1st, 2nd columns), 5–6(d, f, h, j, l), 7, 8(b, d, f, h), 9(a, c, e), 10(a, c, e, h, k, m), 11, 12 ▢ 1–4(3rd column), 5–10(2nd column), 13, 14, 16, 17 ⬠ 1–7(2nd column), 8–10(3rd column), 15–19

3A.1

1 Solve each of these equations.

a $3x - 4 = 15$ **b** $\frac{x}{5} + 6 = -2$ **c** $3 = 5 - 7x$

d $8 = \frac{5x}{3}$ **e** $\frac{x + 3}{10} = -5$ **f** $6 = 3(x + 5)$

g $3(x - 2) = -11$ **h** $1 - \frac{x}{2} = 5$ **i** $-4(5 - 3x) = 9$

j $\frac{2x - 5}{8} = -3$ **k** $\frac{7 - 6x}{5} = 10$ **l** $6 = \frac{5(x - 4)}{7}$

m $-4 = \frac{9x}{4} + 4$ **n** $12 - \frac{8x}{15} = 8$ **o** $8 = \frac{4(11 - 3x)}{9}$

2 Solve each of these equations by finding the value of x. Hint: You will need to perform operations that involve fractions and decimals.

a $x - \frac{1}{2} = -9$ **b** $2x + 3.6 = 7$ **c** $8.5x = 34$

d $\frac{x}{0.28} + 3.2 = -5.6$ **e** $\frac{3 - 0.5x}{4} = 0.45$ **f** $\frac{11}{3}x = \frac{5}{7}$

g $\frac{4}{3}x + \frac{5}{6} = \frac{3}{2}$ **h** $\frac{3}{4}\left(x + \frac{9}{5}\right) = \frac{11}{10}$ **i** $\frac{8}{7}\left(\frac{21}{2}x + \frac{56}{5}\right) = \frac{48}{35}$

3 Solve each of these equations.

a $2(x + 4) - 5 = -7$ **b** $3(x - 1) + 2 = 8$ **c** $4(x + 3) + 2x = 0$

d $9(x - 5) - x = 3$ **e** $2(3x + 4) + 7 = 13$ **f** $7(2x - 11) - 4x = -7$

g $5(2 - x) - 1 = -6$ **h** $4(1 - 3x) + 3x = 3$ **i** $-3(4 - x) - 2x + 5 = -1$

UNDERSTANDING AND FLUENCY

3A.2 **4** Solve each of these equations.

a $\frac{8}{x} = 4$ **b** $\frac{45}{x} = -9$ **c** $\frac{6}{x} = -1$

d $\frac{-12}{x} = 3$ **e** $\frac{-20}{x} = -5$ **f** $\frac{15}{x} = 2$

g $\frac{10}{3x} = 1$ **h** $\frac{-2}{5x} = -7$ **i** $\frac{6}{x} + 5 = 8$

j $\frac{11}{x} + 7 = -4$ **k** $\frac{7}{2x} - 1 = 13$ **l** $\frac{-24}{5x} - 2 = 0$

m $\frac{5}{x+2} = 8$ **n** $2 = \frac{21}{x-7}$ **o** $\frac{10}{2x-5} = -3$

3A.3 **5** Solve each of these equations.

a $5x + 3 = 2x + 9$ **b** $6x - 7 = 4x + 3$

c $3x + 4 = x - 2$ **d** $4x - 5 = 2x - 3$

e $3x + 8 = 2x + 1$ **f** $5x + 6 = 2 + 3x$

g $4x - 5 = 13 - 2x$ **h** $2x - 1 = 11 - x$

i $2x + 3 = 6 - 5x$ **j** $3x - 5 = 7x + 11$

k $2 - x = 4x - 8$ **l** $8x + 5 = 11x + 20$

6 Solve each of these equations by first expanding to remove the brackets.

a $3(x + 2) = 2x + 11$ **b** $8x - 9 = 5(x - 3)$

c $7x + 3 = 6(x + 1)$ **d** $5(2x + 1) = 7x + 2$

e $4(x - 7) = 2 - x$ **f** $x - 4 = 2(3 - 2x)$

g $3(x + 6) = 2(x + 7)$ **h** $9(x - 3) = 5(x + 1)$

i $4(x - 1) = 2(x + 2)$ **j** $6(x + 5) = 4(5 - x)$

k $2(3x - 4) = 11(x - 3)$ **l** $7(x - 12) = -3(x - 2)$

7 Solve each of these equations. Start by multiplying both sides of the equation by the common denominator.

a $\frac{2x+1}{5} = \frac{x-7}{5}$ **b** $\frac{4x-3}{3} = \frac{2x+5}{3}$

c $\frac{7x-2}{4} = \frac{3x-10}{4}$ **d** $\frac{x+5}{6} = \frac{11-2x}{6}$

e $\frac{9-2x}{11} = \frac{4-7x}{11}$ **f** $\frac{3(x-4)}{2} = \frac{4-x}{2}$

8 Solve each of these equations. Start by writing each fraction as an equivalent fraction with the same denominator as the other. Hint: Some solutions may not be integers.

a $\frac{x-2}{5} = \frac{x+4}{2}$ **b** $\frac{x-4}{3} = \frac{x+5}{4}$ **c** $\frac{x+2}{2} = \frac{x+7}{3}$

d $\frac{x+3}{4} = \frac{x+6}{7}$ **e** $\frac{x-8}{3} = \frac{x+1}{6}$ **f** $\frac{3x+5}{4} = \frac{x+9}{2}$

g $\frac{2x+1}{5} = \frac{x-2}{4}$ **h** $\frac{x-7}{3} = \frac{5-x}{4}$ **i** $\frac{2x-1}{6} = \frac{3x+2}{5}$

9 Solve each of these equations.

a $\frac{5x}{6} + \frac{4x}{3} = \frac{13}{2}$ **b** $\frac{3x}{4} - \frac{4x}{3} = \frac{21}{12}$ **c** $\frac{7x}{8} = \frac{1}{2} + \frac{5x}{6}$

d $\frac{4x}{9} + \frac{7}{3} = \frac{7}{4} - \frac{11x}{12}$ **e** $\frac{2}{5}\left(\frac{3x}{4} + \frac{7}{2}\right) = \frac{9x}{5}$ **f** $\frac{55}{6}\left(\frac{14x}{3} - \frac{2}{5}\right) = \frac{22}{15}\left(\frac{5}{6} - \frac{15x}{2}\right)$

3A.4 **10** Rearrange each of these equations to make x the subject on the left-hand side.

a $x + a = b$ **b** $k = x - p$ **c** $cx = d$

d $g + 3x = h$ **e** $y = 4x + 5$ **f** $\frac{a+x}{m} = n$

g $7x - 2w = 5$ **h** $b - \frac{x}{a} = c$ **i** $k = \frac{h}{x}$

j $6x - e = 4x + f$ **k** $2(x + y) = v$ **l** $\frac{x}{p} + q = w$

m $\frac{a}{x} - b = c$ **n** $y = \frac{kx - m}{2n}$ **o** $x + 3d = 5c - 7x$

11 Liam is saving to buy a tennis racquet that costs $219. He is able to save $24 per week and currently has $105. Follow the steps below to work out how many weeks it will take Liam to save enough for the racquet.

a Choose a pronumeral to represent the unknown quantity in the problem.

b Write an equation to represent the problem, using your chosen pronumeral for the unknown value.

c Solve the equation using inverse operations.

d Write your answer to the problem, in weeks.

12 For each of these problems, set up an equation and solve it using inverse operations.

a Edith bought four makeup kits online for a total cost of $387, which included the delivery charge of $29. What was the cost of each makeup kit?

b The perimeter of a rectangular car park is 88 m. If the length of the car park is 12 m more than its width, what are the car park's dimensions?

c Lisa, Sara and Bindi scored a combined total of 29 goals in a netball match. Lisa scored five more goals than Sara, and Bindi scored three goals less than Sara. How many goals did each girl score?

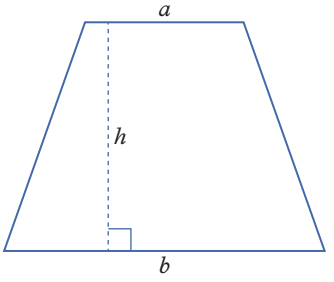
d Oliver has a budget of $1800 for his birthday party. The cost of hiring a local hall is $400 and catering is $27 per person. What is the maximum number of people that Oliver can afford to attend his party?

13 Shana and Paulo have the same amount of money. Shana buys six dumplings and has $3.80 left over. Paulo buys four dumplings and has $9.20 left over.

a Write an equation to represent this situation.

b Solve the equation to find the cost of one dumpling.

14 The formula $A = \frac{h(a + b)}{2}$ links the area, A, of a trapezium with its three length measurements a, b and h, as shown in the diagram below.



a Calculate the area, A, of a trapezium for which $a = 4$ cm, $b = 6$ cm and $h = 10$ cm.

b Calculate the height, h, of a trapezium for which $a = 15$ mm, $b = 19$ mm and $A = 119$ mm^2.
Hint: Substitute the values for a, b and A into the formula and then solve the resulting equation to find h.

c Calculate a when $b = 5$ m, $h = 3$ m and $A = 18$ m^2.

d Rearrange the formula to make a the subject.

e Calculate a when:

i $b = 37$ cm, $h = 24$ cm and $A = 1068$ cm^2

ii $b = 2.4$ m, $h = 6.1$ m and $A = 30.5$ m^2.

PROBLEM SOLVING AND REASONING

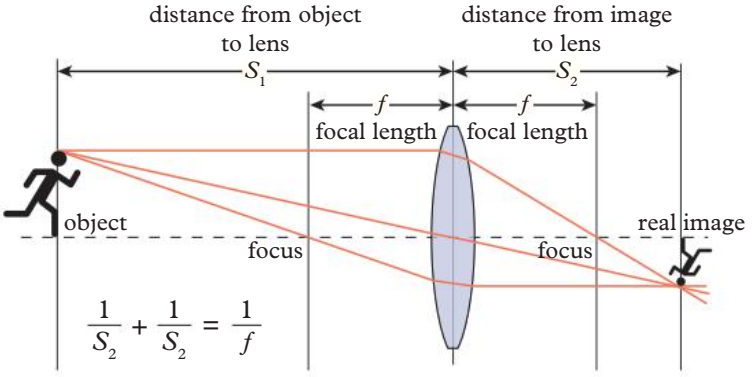
15 The optic equation, $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$, equates the sum of the reciprocals of two numbers with the reciprocal of a third. The optic equation is used for dealing with many systems, such as:

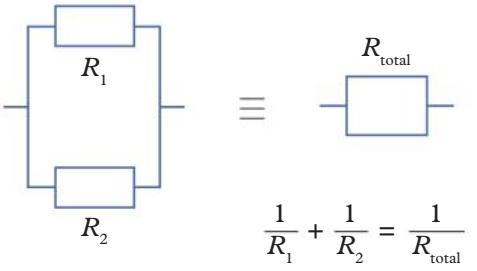
- the distances between an object and its image through a thin lens,

$$\frac{1}{\text{distance from object to lens}}+\frac{1}{\text{distance from image to lens}}=\frac{1}{\text{focal length}}$$

- the resistances in a parallel circuit,

$$\frac{1}{\text{resistance}_1}+\frac{1}{\text{resistance}_2}=\frac{1}{\text{total resistance}}$$





a Show that, for the optic equation, $c=\frac{ab}{a+b}$.

b Calculate:

i the focal length when the distance from the object to the lens is 4 cm and the distance from the image to the lens is 12 cm

ii the total resistance in a parallel circuit when the two resistances are 500 ohms and 2000 ohms.

c Rearrange the optic equation to make a the subject.

d Determine:

i the distance to the object when the distance from the image is 10 cm and the focal length is 8 cm

ii the resistance of a resistor that has been connected in parallel with a 16 ohm resistor, where the total resistance is 8 ohms.

CHALLENGE

16 The sum of five consecutive odd numbers is 50 more than triple the smallest number. Write an equation you can solve to find the five numbers. Hint: Let the smallest number be x.

17 Solve each of these equations.

a $\frac{2x+5}{x-4}=7$ **b** $\frac{9-4x}{3x+5}=10$ **c** $\frac{5}{6}=\frac{7x-3}{3-2x}$

18 Solve each of these equations. Use substitution to check your solutions.

a $\frac{3(2-x)}{4}+9=15$ **b** $\frac{4(2x-1)}{3}-5=4x-17$ **c** $\frac{2x-3}{5}+\frac{x+7}{3}=1$

d $\frac{x+5}{2}-\frac{x-7}{6}=8$ **e** $\frac{3(2x-3)}{4}+\frac{x+23}{10}=2x$ **f** $\frac{5x+6}{3}-\frac{1-2x}{4}=3x+4$

19 Make x the subject of each of the following.

a $\frac{mx+p}{nx+q}=r$ **b** $\frac{a+by}{b+ax}=\frac{b-ay}{a-bx}$ **c** $\frac{a+by}{b-ax}=\frac{b+ay}{a-bx}$

Check your Student obook pro for these digital resources and more:

- Interactive skillsheet — Solving equations using inverse operations
- Interactive skillsheet — Solving equations with the unknown on both sides
- Interactive skillsheet — Solving equations with the unknown in the denominator
- Interactive skillsheet — Rearranging linear equations
- Investigation — Calendar capers
- CAS instructions — Solving equations
- Topic quiz — 3A

3B Sketching linear graphs

Learning intentions

By the end of this section you will be able to ...

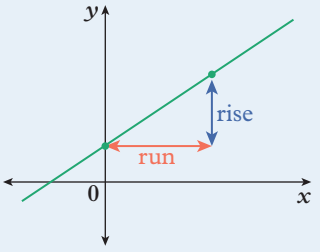
- ✔ sketch a linear graph using the gradient–intercept method
- ✔ sketch a linear graph using the x-and-y-intercept method
- ✔ sketch horizontal and vertical linear graphs.

Inter-year links

Support	The Cartesian plane
Year 7	5D The Cartesian plane
Year 8	6D Plotting linear relationships
Year 9	3E Sketching linear graphs using intercepts

Linear graphs

- A linear equation with two pronumerals describes a **linear relationship** between two variables.
- A linear relationship can be represented on the Cartesian plane as a straight line called a **linear graph**.
- An intercept is the point where a graph crosses an axis.
- The **gradient** is a numerical measure of the slope or steepness of a graph. It can be calculated as the ratio of the change in the y values (the **rise**) to the change in the x values (the **run**).



$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$

- If a linear graph is not vertical or horizontal, then two points must be plotted in order to sketch the graph.

The gradient–intercept method

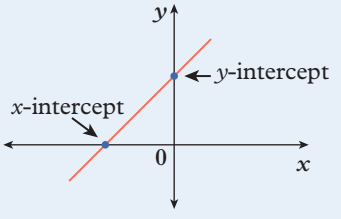
- The equations for all linear relationships can be expressed in the general **gradient–intercept form** shown on the right, where:
  - → m is the gradient of the line
  - → c is the y-coordinate of the y-intercept (the point where the line crosses the y-axis).
- The gradient–intercept form can be used to identify the gradient and y-intercept, enabling us to sketch the graph.

$$y = mx + c$$

gradient m; y-intercept c

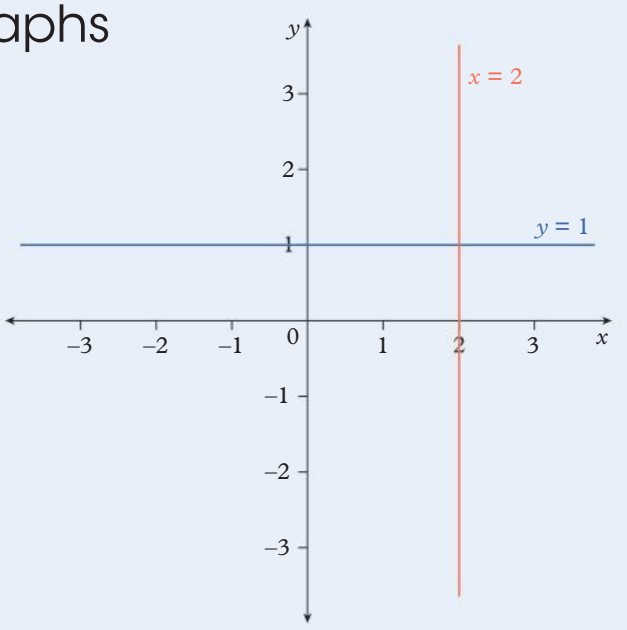
The x-and-y-intercept method

- The x-intercept is the point at which the graph crosses the x-axis. That is, where $y = 0$.
- The y-intercept is the point at which the graph crosses the y-axis. That is, where $x = 0$.
- The values $x = 0$ and $y = 0$ can be substituted into a linear equation in order to find the line's x- and y-intercepts and sketch the graph.



Horizontal and vertical graphs

- A horizontal graph has a gradient of zero, because the rise = 0. Horizontal graphs have equations of the form $y = c$, where c is any real number.
  For example, the blue graph line on the right shows the horizontal graph $y = 1$.
- A vertical graph has an undefined gradient because the run = 0. Vertical graphs have equations of the form $x = a$, where a is any real number.
  For example, the red graph line on the right shows the vertical graph $x = 2$.



Example 3B.1 The gradient–intercept method

Sketch the graph of $y = -\frac{1}{2}x + 3$.

THINK

1. Identify the gradient, m, and the y-intercept, c, by comparing this to the general gradient–intercept form of the equation: $y = mx + c$.
2. Write the gradient, m, and identify appropriate values for the rise and the run.
3. Plot a point at the y-intercept and label the coordinates (0, 3).
4. Add the rise to the y-coordinate and the run to the x-coordinate of the y-intercept to find a second point on the graph. Move two units to the right and one unit down to the point (2, 2).
5. Rule a straight line through the points. Label the graph with its equation.

WRITE

$y = -\frac{1}{2}x + 3$

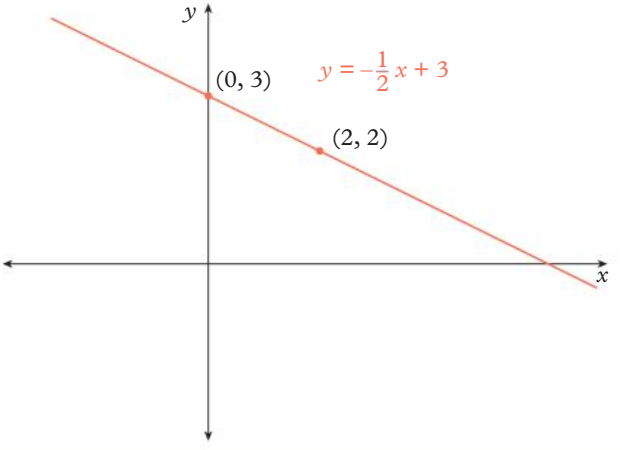
From $y = mx + c$:

$m = -\frac{1}{2}$

$c = 3$

$\frac{\text{rise}}{\text{run}} = -\frac{1}{2}$

So, rise = –1, run = 2



Example 3B.2 The *x*-and-*y*-intercept method

Sketch the graph of $4x - 3y = 12$ using the *x*-and-*y*-intercept method.

THINK

1 Determine the *x*-intercept by substituting $y = 0$ into the equation and solving for *x*.

2 Determine the *y*-intercept by substituting $x = 0$ into the equation and solving for *y*.

3 Plot and label the *x*- and *y*-intercepts on the Cartesian plane.

4 Rule a straight line through the points. Label the graph with its equation.

WRITE

For the *x*-intercept, $y = 0$:

$4x - 3y = 12$

$4x - 3(0) = 12$

$4x = 12$

$x = 3$

x-intercept is at $(3, 0)$

For the *y*-intercept, $x = 0$:

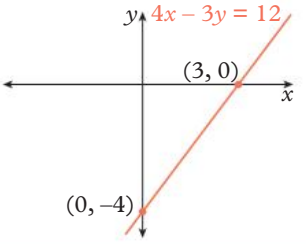
$4x - 3y = 12$

$4(0) - 3y = 12$

$-3y = 12$

$y = -4$

y-intercept is at $(0, -4)$



Example 3B.3 Horizontal and vertical graphs

For each linear equation below:

i list the coordinates of any intercepts

ii sketch the graph.

a $y = 3$ **b** $x = -1$

THINK

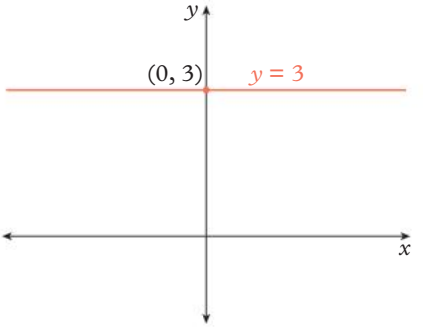
a i The equation tells us that *y* is 3 for any value of *x*, so the graph is a horizontal line passing through a *y*-intercept at $(0, 3)$.

ii Rule a straight horizontal line through $(0, 3)$. Label the graph with its equation.

WRITE

a i *y*-intercept at $(0, 3)$

ii



THINK

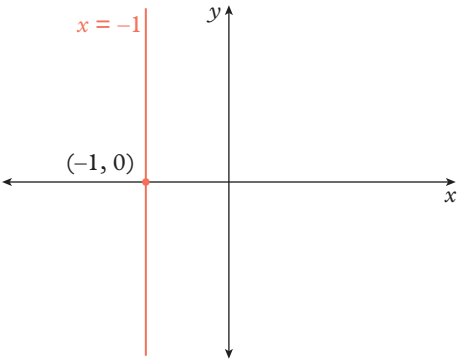
b i The equation tells us that x is -1 for any value of y, so the graph is a vertical line passing through an x-intercept at $(-1, 0)$.

ii Rule a straight vertical line through $(-1, 0)$. Label the graph with its equation.

WRITE

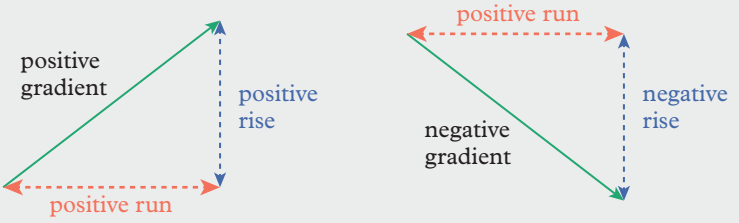
b i x-intercept at $(-1, 0)$

ii



Helpful hints

✔ Check that the slope of your graph matches the sign of the gradient. A positive gradient slopes upwards to the right and a negative gradient slopes downwards to the right.



✔ If you used the gradient–intercept method to sketch your graph, check the x- and y-intercepts of your graph by substituting $y = 0$ and $x = 0$ into the equation.

✔ If you used the x-and-y-intercept method to sketch your graph, rearrange the equation to the form $y = mx + c$ and check to see if the gradient, m, matches the gradient of your graph.

ANS p552

Exercise 3B Sketching linear graphs

△ 1(a, c, e, g, i), 2(1st column), 3–6, 7(1st, 2nd column), 8–10, 11(a, b), 12(a–d), 13(a–c, g), 15, 16(b–g), 17(a–d), 19

□ 1(b, c, f, g), 2(b, e, h, k), 4, 6, 7(3rd, 4th columns), 9, 10, 13(f–i), 14, 17(e–h), 18–22

⬠ 2(j–l), 9, 10, 17(g–j), 18, 20, 22–24

UNDERSTANDING AND FLUENCY

3B.1 **1** Sketch the graph of each of these linear relationships using the gradient–intercept method.

a $y = 2x - 4$ **b** $y = \frac{2}{3}x + 1$ **c** $y = -3x + 5$

d $y = \frac{5}{4}x - 2$ **e** $y = -5x - 1$ **f** $y = -\frac{1}{2}x + 3$

g $y = 3x - 4$ **h** $y = x + 6$ **i** $y = -x - 3$

2 Rearrange each of these equations to be in the form $y = mx + c$ and then:

i identify the gradient **ii** identify the y-intercept **iii** sketch the graph.

a $y = 1 + 2x$ **b** $y = 4 - 3x$ **c** $y = -3 - \frac{3}{4}x$

d $y = 3(x - 2)$ **e** $y = -2(x - 3)$ **f** $y - 4 = -(5x + 6)$

g $3x + y = 2$ **h** $-4x + y = -5$ **i** $6x + 3y = 9$

j $-4x + 5y = 15$ **k** $3x - 2y = -4$ **l** $x + 3y + 6 = 0$

3 **a** If $y = 3x$ is compared to the gradient–intercept form, $y = mx + c$, then identify the value of:

i m

ii c

b Sketch the graph of $y = 3x$ using the gradient–intercept method.

4 Sketch each of the following linear relationships using the gradient–intercept method.

a $y = 2x$ **b** $y = -5x$ **c** $y = -x$ **d** $y = \frac{2}{5}x$

5 **a** If $y = 4$ is compared to the gradient–intercept form, $y = mx + c$, identify the value of:

i m

ii c

b Sketch the graph of $y = 4$ using the gradient–intercept method.

6 Sketch each of these linear relationships using the gradient–intercept method.

a $y = 6$ **b** $y = -2$ **c** $y = \frac{1}{3}$ **d** $y = -\frac{7}{2}$

3B.2 **7** Sketch the graph of each of these linear relationships using the x-and-y-intercept method.

a $x + 2y = 2$ **b** $3x + y = -6$ **c** $2x + 4y = 8$ **d** $5x + 10y = 20$

e $4x + 3y = -12$ **f** $-6x + 2y = 6$ **g** $8x - y = -8$ **h** $-x + 4y = 2$

i $9x - 3y = -18$ **j** $-12x + 9y = 36$ **k** $7x + 4y = 21$ **l** $2x - y = 5$

8 For each of the linear relationships below:

i find the x-intercept

ii find the y-intercept

iii sketch the graph.

a $y = x + 5$ **b** $y = x - 4$ **c** $y = 3x + 6$

d $y = 2x - 8$ **e** $y = 2 - x$ **f** $y = -6x - 3$

9 Sketch the graph of each of these using the x-and-y-intercept method.

a $2y = 3x + 6$ **b** $4y = 4x - 8$ **c** $x + 3y - 3 = 0$

d $3x + 6y + 18 = 0$ **e** $5x - 2y - 10 = 0$ **f** $2x - y - 7 = 0$

10 Consider the linear equation $y = 2x$.

a Find:

i the x-intercept

ii the y-intercept.

b Do you have enough information from part **a** to sketch the graph? Explain.

c What other information do you need? Discuss this with a classmate.

d Another way of identifying a second point to plot is to find the matching y value for a chosen x value. Any x value can be used, but we often choose $x = 1$.

Find the value of y in $y = 2x$ when x is 1. Hint: Substitute $x = 1$ into the equation.

e List the coordinates of the two points you can now use to sketch the graph of $y = 2x$.

f Plot the two points and rule a straight line through them to produce the graph of $y = 2x$. Label your graph with its equation.

11 Use the approach used in question **10** to sketch linear graphs with these equations.

a $y = 4x$ **b** $y = -3x$ **c** $y = x$ **d** $y = -\frac{1}{2}x$

3B.3 **12** For each of the linear equations below:

i list the coordinates of any intercepts **ii** sketch the graph.

a $y = 8$ **b** $x = 3$ **c** $y = -5$ **d** $x = -1$

e $x = 7$ **f** $y = 0$ **g** $x = 0$ **h** $x = -4.5$

PROBLEM SOLVING AND REASONING

13 How many axis intercepts does the graph of each of these linear equations have? List them.

a $y = 2x - 4$ **b** $y = -7$ **c** $4x + 6y = 12$

d $y = 9x$ **e** $y = 5(x - 10)$ **f** $3x - 5y + 15 = 0$

g $x = 28$ **h** $x = 3y + 6$ **i** $y = 5.6$

14 a Write $x = py + q$ in the form:

i $y = a(x - b)$

ii $y = mx + c$.

b What do the values of p and q correspond with on the graph of $x = py + q$?

c Write $x = 3$ in the form $x = py + q$.

d Identify the value of:

i p

ii q

e i Write your answer from part **c** in the forms $y = a(x - b)$ and $x = py + q$.

ii Use your answers from parts **b** and **e i** to explain why the gradient of the lines $x = 3$ and, more generally, $x = q$ are undefined.

15 Describe which method you think would be the best to use to sketch:

a linear graphs with equations in the general form $y = mx + c$ (for example, $y = 3x + 2$)

b linear graphs with equations in the general form $ax + by = d$ (for example, $4x + 2y = 8$)

c linear graphs that pass through the origin (for example, $y = 3x$)

d linear graphs that are horizontal lines (for example, $y = 7$)

e linear graphs that are vertical lines (for example, $x = -2$).

16 Use the most appropriate method to sketch the graph of each of these linear relationships.

a $5x - 2y = -10$ **b** $y = 2x + 4$

c $x + y = 5$ **d** $y = 2x$

e $y = 5$ **f** $y = 8 - 4x$

g $x = -9$ **h** $y = 3(x - 2)$

i $2x + 4y - 8 = 0$ **j** $y = -3x$

k $y = -2x + 7$ **l** $y = 3 - 6x$

17 Sketch each of these linear relationship for the given x or y values. Label the endpoints with their coordinates.

a $y = 5x$ for $x \geq 0$ **b** $y = 2x - 4$ for $x \leq 3$

c $y = 7$ for $-2 \leq x \leq 4$ **d** $x = -3$ for $-1 \leq y \leq 1$

e $2x + 3y = 12$ for $0 \leq x \leq 6$ **f** $y = x$ for $x > 2$

g $y = 3 - x$ for $x < 3$ **h** $y = -3$ for $-1 < x < 4$

i $x - 4y = 2$ for $-2 \leq x < 6$ **j** $x = 6$ for $y < 3$

18 For a line segment between the points (x_1, y_1) and (x_2, y_2), the length is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

a Sketch each of these linear relationships on the same Cartesian plane.

i $x = -6$ for $3 \leq y \leq 13$

ii $y = 3$ for $-6 \leq x \leq 2$

iii $y = -\frac{5}{4}x + \frac{11}{2}$ for $-6 \leq x \leq 2$

b Calculate the perimeter of the shape formed, correct to two decimal places.

c Calculate the area of the shape formed.

19 A racing car accelerates from a constant speed of 20 m/s. Its speed, s, after t seconds of accelerating can be calculated using the formula $s = 20 + 5t$.

a Draw the graph of this linear relationship for $0 \leq t \leq 10$ (that is, t values from 0 to 10 seconds).

b What does the s-intercept represent on this graph?

c What does the gradient represent on this graph?

d From the graph, what is the speed of the racing car after:

i 3 seconds?

ii 5 seconds?

iii 7.5 seconds?

e Convert each speed in part **d** to kilometres per hour.

f From the graph, how long does it take the racing car to reach speeds of:

i 30 m/s?

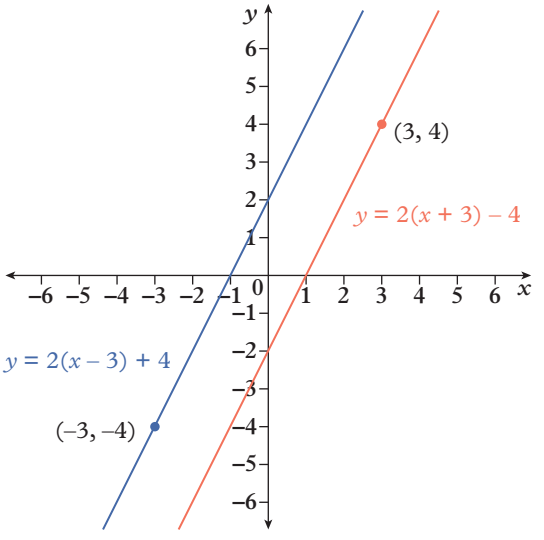
ii 55 m/s?

iii 65 m/s?

g Using the graph, how long does it take the racing car to reach a speed of 216 km/h?

h Using the graph, what is the highest speed, in kilometres per hour, reached by the racing car? How long does it take to reach this speed?

20 The gradient–intercept form $y = mx + c$ is a specific case of the gradient–point form $y = m(x - h) + k$, where (h, k) is a point on the line. For example, $y = 2(x - 3) + 4$ passes through the point $(3, 4)$ and $y = 2(x + 3) - 4$ passes through the point $(-3, -4)$.



a Substitute $(h, k) = (0, c)$ and show that the equation $y = m(x - h) + k$ becomes $y = mx + c$.

b Sketch the following lines by plotting the point (h, k) and then use the gradient to plot another point in the same way you would be using the gradient–intercept method.

i $y = -3(x - 2) + 6$

ii $y = 3(x - 2) + 6$

iii $y = \frac{5}{2}(x + 3) - 4$

iv $y = -\frac{3}{4}(x + 7) + 10$

c Determine the coordinates for the x- and y-intercepts of $y = m(x - h) + k$ in terms of m, h and k.

d **i** Substitute $(h, k) = (b, 0)$ to determine a new general form for writing a linear relationship in terms of the x-intercept.

ii Explain what you can do to the equation $y = mx + c$ to write it in this form.

iii Explain how you can use this form to determine the x-intercept.

e Use your method from part **d** to write the following equations in the new general form and determine their x-intercepts.

i $y = 3x - 12$

ii $y = 2x - 5$

iii $y = -3x - 12$

iv $y = 2x + 5$

CHALLENGE

21 a Sketch these linear relationships on the same Cartesian plane:

- $2x + 4y = 8$, for $4 \leq x \leq 8$
- $y = -2$, for $-2 \leq x \leq 8$
- $x = -2$, for $-2 \leq y \leq 2$
- $y = -x$, for $-2 \leq x \leq 0$
- $y = 0$, for $0 \leq x \leq 4$.

b Use Pythagoras' theorem to calculate the perimeter of the shape formed by the graph lines.

c Calculate the area of the shape formed.

22 a Sketch these linear relationships on the same Cartesian plane.

- $y = -4x + 10$, for $0 \leq x \leq 2$
- $y = 4x + 10$, for $-2 \leq x \leq 0$
- $x - 4y = -10$, for $-10 \leq x \leq -2$
- $4y = x - 10$, for $2 \leq x \leq 10$
- $y = 4x - 10$, for $0 \leq x \leq 2$
- $y = -4x - 10$, for $-2 \leq x \leq 0$
- $x + 4y = -10$, for $-10 \leq x \leq -2$
- $x + 4y = 10$, for $2 \leq x \leq 10$

b Calculate the perimeter of the shape formed by the graph lines.

c Calculate the area of the shape formed.

d Join the points $(-2, -2)$ and $(2, 2)$ to form a line segment. Write a rule to represent the line segment.

e Join the points $(-2, 2)$ and $(2, -2)$ to form a line segment. Write a rule to represent the line segment.

23 When an equation is written in the form $ax + by = c$, we can determine both of the intercepts simultaneously.

a Determine the coordinates of the x- and y-intercepts of $ax + by = c$ in terms of a, b and c.

b Divide both sides of the equation $ax + by = c$ by c.

c Explain the relationship between the coefficients of x and y in your answer for part **b** and the coordinates in part **a**.

d Use the method in part **b** to help you determine the intercepts of the following lines.

i $2x + 3y = 5$

ii $4x - 7y = 11$

iii $2x - 3y = 12$

iv $9y - 2x = 36$

v $8x - 4y = -32$

vi $20x + 50y = 10$

24 a Sketch each pair of graphs on the same Cartesian plane.

i $2x + 3y = 12$ and $2x - 3y = 12$

ii $2x + 3y = 12$ and $\frac{1}{2}x - \frac{1}{3}y = 12$

iii $2x + 3y = 12$ and $2x - 3y = \frac{1}{12}$

iv $5x + 2y = 20$ and $2x - 5y = 20$

v $5x + 2y = 20$ and $-2x + 5y = 20$

vi $5x + 2y = 20$ and $2x + 5y = -20$

b Determine which pairs of graphs are perpendicular.

c Write the equation of a line perpendicular to $ax + by = 1$ in the same form and in terms of a and b.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Sketching linear relationships using intercepts

Interactive skillsheet Sketching linear relationships using the gradient-intercept method

Worksheet Linear graphs

CAS instructions Graphing functions

Topic quiz 3B

3C Determining linear equations

Learning intentions

By the end of this section you will be able to ...

- ✔ determine the gradient of a line given two points
- ✔ determine linear equations given the gradient and a point or two points
- ✔ determine the linear equation of a graph.

Inter-year links

Support	The Cartesian plane
Year 7	5D The Cartesian plane
Year 8	6F Finding linear equations
Year 9	3F Determining linear equations

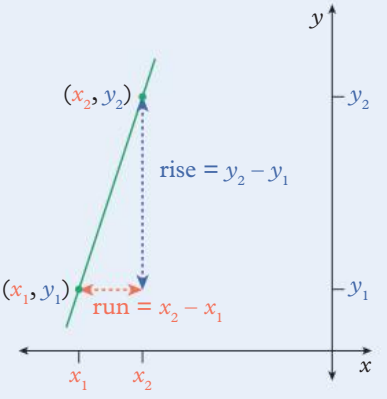
Determining the gradient

- The formula for the gradient, m, between any two points, (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, the gradient of a line that passes through $(1, 3)$ and $(2, -1)$ can be calculated as:

$$m = \frac{3 - (-1)}{1 - 2}$$
$$= \frac{4}{-1}$$
$$= -4$$



Determining linear equations

- To determine the equation of a line in the form $y = mx + c$:
  1. determine the gradient, m
  2. identify the coordinates of a point on the graph
  3. substitute the gradient and the coordinates into the formula $y = mx + c$ to determine c.
- If the y-coordinate of the y-intercept is easy to read from the graph, then it is quicker to substitute this value for c, rather than solve the equation algebraically.

Example 3C.1 Determining the gradient

Determine the gradient of a line that passes through $(3, -8)$ and $(-7, -10)$.

THINK

1. Substitute the given x- and y-coordinates into the gradient formula.
2. Calculate and simplify the gradient.

WRITE

Let $(x_1, y_1) = (3, -8)$ and $(x_2, y_2) = (-7, -10)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{(-10) - (-8)}{(-7) - 3}$$
$$= \frac{-10 + 8}{-7 - 3}$$
$$= \frac{-2}{-10}$$
$$= \frac{1}{5}$$

Example 3C.2 Determining the equation given the gradient

Determine the equation for a line with a gradient of 4 that passes through (–2, 7).

THINK

1 Substitute the values for the gradient and the coordinates into the gradient–intercept equation: $y = mx + c$.

2 Solve the equation for c and write the final equation in gradient–intercept form.

WRITE

From the question:

$m = 4 \qquad x = -2 \qquad y = 7$

$y = mx + c$

$7 = 4 \times (-2) + c$

$7 = -8 + c \qquad (+ 8)$

$c = 15$

So, $y = 4x + 15$.

Example 3C.3 Determining the equation given two points

Determine the equations for the lines that pass through these given pairs of points.

a (2, 6) and (–4, 3) **b** (0, 2) and (3, –7)

THINK

a 1 Substitute the x- and y-coordinates into the gradient formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

2 Substitute the values for the gradient and one of the sets of coordinates into the gradient–intercept equation: $y = mx + c$.

3 Solve for c and write the final equation in gradient–intercept form.

b 1 Substitute the x- and y-coordinates into the gradient formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

2 If one of the coordinates has an x value of 0, the point is a y-intercept, and the corresponding y value is the value of c in $y = mx + c$.

3 Write the final equation in gradient–intercept form.

WRITE

a Let $(x_1, y_1) = (2, 6)$ and $(x_2, y_2) = (-4, 3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 6}{-4 - 2} = \frac{-3}{-6} = \frac{1}{2}$$

Use: $x = 2 \qquad y = 6$

$y = mx + c$

$6 = \frac{1}{2} \times 2 + c$

$6 = 1 + c \qquad (- 1)$

$c = 5$

So, $y = \frac{1}{2}x + 5$.

b Let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (3, -7)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - 2}{3 - 0} = \frac{-9}{3} = -3$$

(0, 2) is the y-intercept.

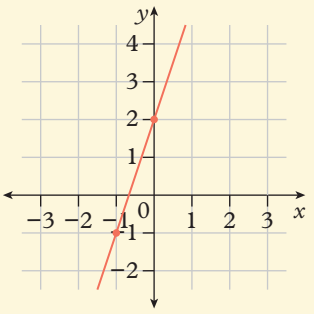
So, $c = 2$.

$y = -3x + 2$.

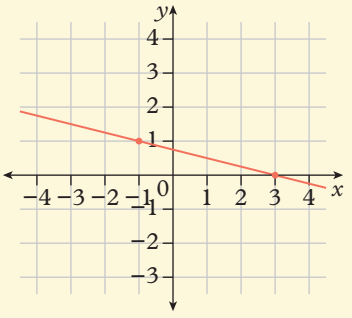
Example 3C.4 Determining the equation given the graph

Determine the equation of each of these linear graphs.

a



b



THINK

a 1 Identify two coordinates on the graph.

2 Determine the value of the gradient, m, using the chosen coordinates.

3 Determine the value of c, by identifying the y-coordinate of the y-intercept.

4 Write the equation of the linear graph by substituting m and c into the equation.

b 1 Identify two integer (whole number) coordinates on the graph.

2 Determine the value of the gradient, m, using the chosen coordinates.

3 The y-coordinate of the y-intercept is unclear on the graph, so choose a clearer coordinate to substitute into the gradient–intercept equation: $y = mx + c$.

4 Solve the equation for c and write the final equation in gradient–intercept form.

WRITE

a Let $(x_1, y_1) = (-1, -1)$ and $(x_2, y_2) = (0, 2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - (-1)}{0 - (-1)}$$
$$= 3$$
$$c = 2$$

$$y = mx + c$$
$$y = 3x + 2$$

b Let $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (3, 0)$.

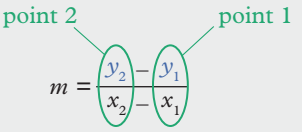
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{0 - 1}{3 - (-1)}$$
$$= -\frac{1}{4}$$

Use: $x = 3 \quad y = 0$

$$y = mx + c$$
$$0 = -\frac{1}{4} \times 3 + c$$
$$0 = -\frac{3}{4} + c \qquad \left(+\frac{3}{4}\right)$$
$$c = \frac{3}{4}$$
$$y = -\frac{1}{4}x + \frac{3}{4}$$

Helpful hints

✔ The order in which you substitute points into the formula for the gradient of a line won't affect the final value – you just need to make sure the x- and y-coordinates of a given point match up vertically when written as a fraction:



✔ Remember, if you know the y-intercept, $(0, c)$, all you need to do is determine the gradient, m, and substitute both values into the gradient–intercept form of the equation: $y = mx + c$.

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Exercise 3C Determining linear equations

△ 1–3, 4(e–h), 5(a–e), 6, 7, 8(1st column), 9, 10, 14, 16 ■ 1, 3, 4–5(e–h), 8(2nd column), 11, 12, 14, 17, 20 ⬠ 3, 4–5(e–h), 8(2nd column), 13–15, 17, 18

UNDERSTANDING AND FLUENCY

3C.1 **1** Determine the gradient of the line passing through each of these pairs of points.

a (2, 3) and (3, 7)
b (5, 4) and (10, –6)
c (4, 2) and (6, 3)
d (0, 5) and (4, 1)
e (–3, –2) and (–1, 8)
f (–1, –1) and (2, –10)

2 For each line passing through a pair of points given below:

i determine the gradient
ii determine the equation of the line by substituting the correct values into $y = mx + c$.

a (0, 2) and (2, 10)
b (0, –7) and (3, –4)
c (0, 5) and (4, 7)
d (–1, –4) and (0, 6)
e (–2, 1) and (0, –5)
f (–8, 6) and (0, 4)

3 Use the x- and y-intercepts given below to determine the equation of the linear graph on which they appear.

a x-intercept: (2, 0), y-intercept: (0, 4)
b x-intercept: (–3, 0), y-intercept: (0, 3)
c x-intercept: (3, 0), y-intercept: (0, –6)
d x-intercept: (1, 0), y-intercept: (0, 5)
e x-intercept: (5, 0), no y-intercept
f no x-intercept, y-intercept: (0, –2)

3C.2 **4** Use the given information to determine the equation for each of these lines.

a gradient of 3; passes through (1, 4)
b gradient of 2; passes through (3, 9)
c gradient of –4; passes through (–1, 2)
d gradient of –1; passes through (2, 7)
e gradient of 5; passes through (–3, –6)
f gradient of –3; passes through (–4, 11)
g gradient of $\frac{6}{5}$; passes through (15, 4)
h gradient of $-\frac{3}{4}$; passes through (8, –5)
i gradient of $\frac{5}{2}$; passes through (–3, –6)
j gradient of $-\frac{7}{9}$; passes through (–6, 4)

3C.3 **5** Determine the equations for the lines that pass through these given pairs of points.

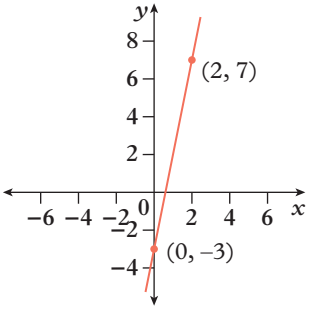
a (1, 3) and (3, 7) **b** (5, 2) and (6, 8) **c** (2, 4) and (4, –2)

d (3, –2) and (6, –8) **e** (–2, –3) and (2, 1) **f** (–4, –2) and (–1, –5)

g (1, –3) and (4, 4) **h** (–7, 5) and (–2, –2) **i** (–5, 3) and (1, 4)

6 Use $y = mx + c$ to help you determine the equation of the line passing through the points shown each time.

a



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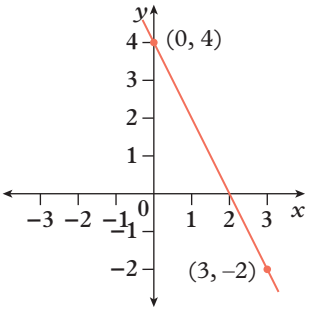

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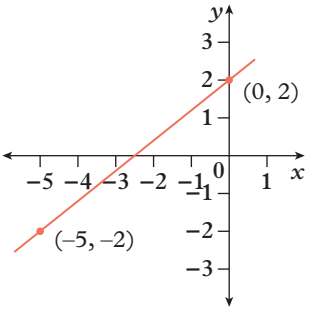

7 Determine the equation of the line passing through the points shown each time.

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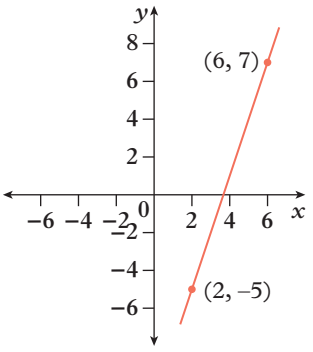

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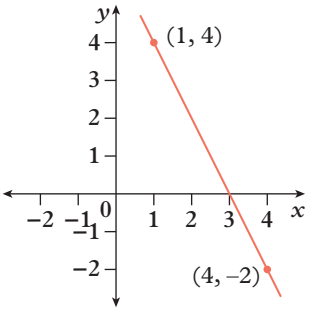

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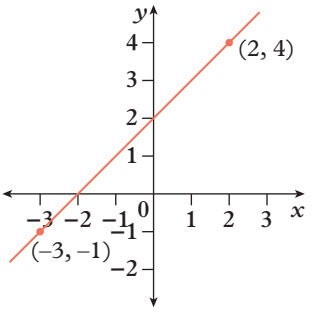

3C.4 **8** Determine the equation of each of these linear graphs.

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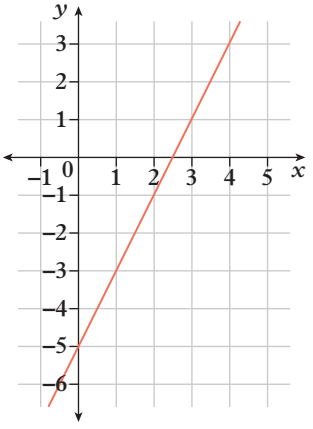

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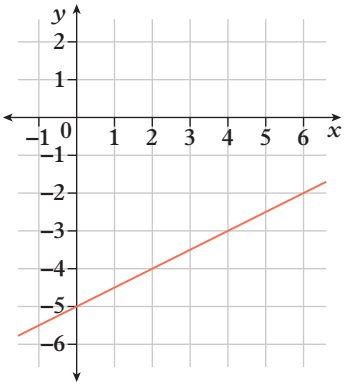

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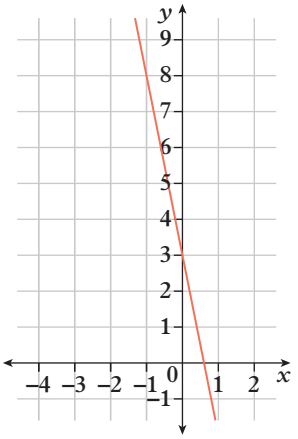

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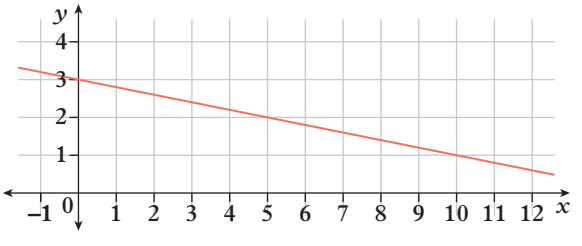

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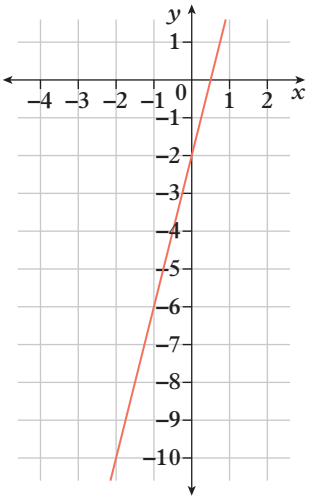
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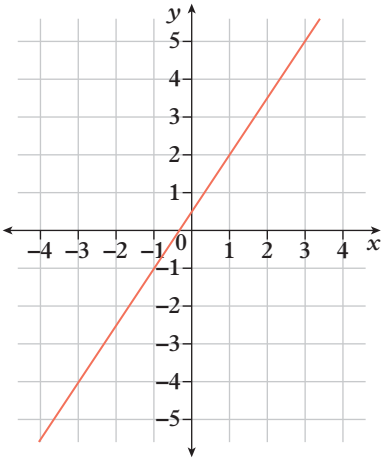
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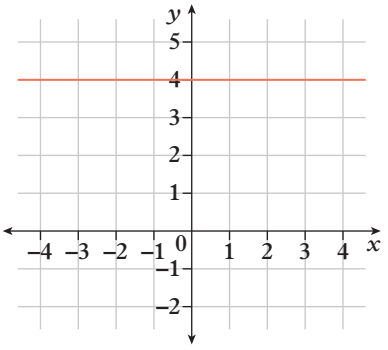
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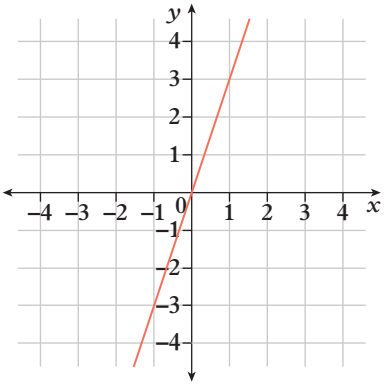
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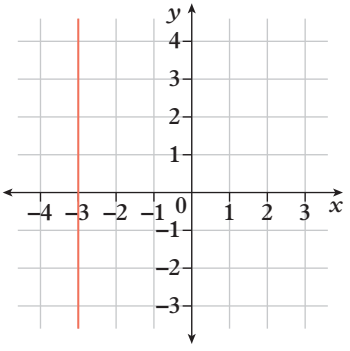
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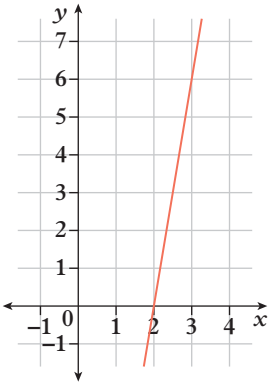
j


y
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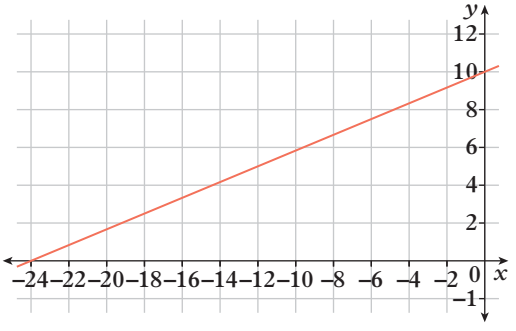
k


y
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8
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4
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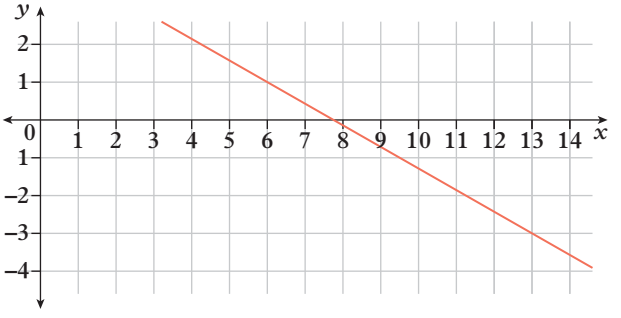
−24 −22 −20 −18 −16 −14 −12 −10 −8 −6 −4 −2 0

l


y
x
2
1
−1
−2
−3
−4
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

9 Liza and Tate are finding the gradient of a line that passes through (–2, 2) and (4, 5). Liza writes $m = \frac{5-2}{4-(-2)}$, while Tate writes $m = \frac{2-5}{-2-4}$.

a Show that Liza and Tate both find the gradient to be $m = \frac{1}{2}$.

b Juan attempts to find the gradient of the same line and writes $m = \frac{5-2}{-2-4}$. Explain what Juan has done wrong and how his answer will compare to the correct gradient found by Liza and Tate.

c Show that $m = \frac{y_1 - y_2}{x_1 - x_2}$ is equivalent to $m = \frac{y_2 - y_1}{x_2 - x_1}$.

10 Liza and Tate are now finding the equation of the line that passes through (–2, 2) and (4, 5). To find the y-intercept, c, Liza writes the equation $2 = \frac{1}{2}(-2) + c$ and Tate writes $5 = \frac{1}{2}(4) + c$.

a Show that Liza and Tate will both get the same value for c.

b Juan attempts to find the equation of the same line and says it is $y = 4x + 5$. Explain what Juan has done wrong.

11 Consider the equation $y = mx + c$.

a For a line passing through the point (h, k), the coordinates can be substituted into the equation $y = mx + c$ to produce $k = mh + c$. Rearrange this equation to make c the subject.

b Substitute the equation you wrote for c into $y = mx + c$.

c Rearrange your answer to part **b** and show that the equation can be expressed in gradient–point form: $y = m(x - h) + k$.

12 Determine the equations of the lines with the features described in **a**–**h** below:

i expressed in the gradient–point form $y = m(x - h) + k$ derived in question **11**, where m is the gradient and (h, k) is a point on the line

ii expressed in the gradient–intercept form $y = mx + c$.

a gradient of $\frac{3}{4}$; passes through (8, 2)

b gradient of $-\frac{7}{3}$; passes through (15, −6)

c (−3, 5) and (1, 6) lie on the line

d (−6, 5) and (−1, −3) lie on the line

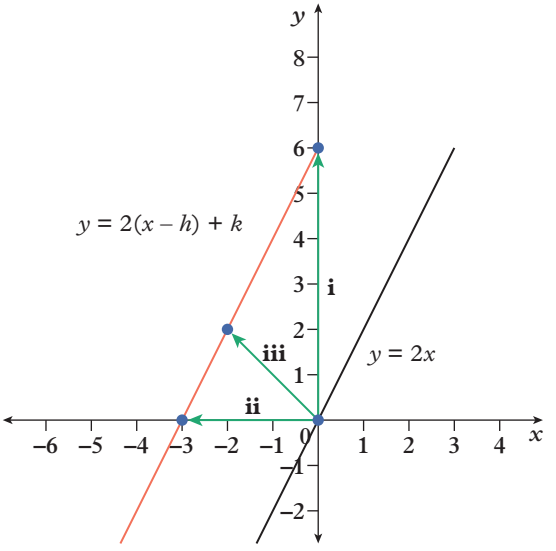
e (−4, 2) and (2, −1) lie on the line

f (−7, 3) and (4, 3) lie on the line

g x-intercept = 4, y-intercept = 3

h x-intercept = −5, y-intercept = 2

13 Consider the linear graphs $y = 2x$ and $y = 2(x - h) + k$ shown on the right. The line $y = 2(x - h) + k$ can be considered as a translation of the line $y = 2x$. Three possible translations are shown as arrows on the diagram.



a State the translation each arrow describes.

b Write the equation of the translated line in the form $y = 2(x - h) + k$ using the three points at the end of the arrows.

c Describe the translation required for the graph of the first equation to become the graph of the second equation each time:

i $y = 3x \rightarrow y = 3(x + 1) - 2$

ii $y = 3x \rightarrow y = 3(x - 1) + 2$

iii $y = -3x \rightarrow y = -3(x - 1) + 2$

iv $y = \frac{5}{2}x \rightarrow y = \frac{5}{2}(x + 4) + 6$

v $y = \frac{2}{5}x \rightarrow y = \frac{2}{5}\left(x - \frac{3}{2}\right) + \frac{4}{3}$

vi $y = 3x + 6 \rightarrow y = 3(x - 4) + 8$

PROBLEM SOLVING AND REASONING

14 Write an equation for a linear graph to represent each of these situations, where y represents the altitude and x represents the horizontal distance from the point of origin.

a From an altitude of 50 m, a road rises 1 m vertically over a horizontal distance of 200 m.

b From an altitude of 6 km, a plane descends 2 km vertically over a horizontal distance of 30 km.

c A road rises 20 m vertically over a horizontal distance of 100 m.

15 A helicopter rises at a constant speed from a height of 20 m above the ground, covering a vertical distance of 30 m as it flies over a horizontal distance of 100 m. For the next 100 m it travels horizontally. It then descends at a constant speed over a further horizontal distance of 100 m before landing.

a Draw a graph to represent the changing height above the ground of the helicopter over the total 300 m horizontal distance it travelled.

b Identify the rule for each of the three linear sections of the graph you drew for part **a**.

c Use the appropriate rule to calculate the height of the helicopter for each of these horizontal distances on your graph.

i 30 m

ii 80 m

iii 150 m

iv 240 m

d What horizontal distance/s have been travelled when the helicopter is at a height of 40 m?

CHALLENGE

16 Determine the equation for each linear graph described by the information below. Write your answers without fractions.

a x-intercept of 4, passes through (20, 8)

b gradient of 0, passes through (–15, –23)

c passes through (–25, 10) and (11, –14)

d passes through the origin and (–17, –12)

17 Sketch the linear graph for each situation in question **14**. What values of x would be appropriate to use? Rewrite each equation you wrote in your answers to question **14** with the appropriate restriction of x values written as an inequality statement.

18 The x-intercept of $ax + by = 1$ is given by the reciprocal of a and the y-intercept is given by the reciprocal of b. Use this fact to:

i write the equations of the following lines in the form $ax + by = 1$, where a and b are real numbers

ii write the equations in the form $px + qy = r$, where p, q and r are integers.

a x-intercept $= \frac{2}{3}$, y-intercept $= \frac{2}{7}$

b x-intercept $= -\frac{3}{5}$, y-intercept $= \frac{3}{2}$

c x-intercept $= \frac{3}{5}$, y-intercept $= -\frac{2}{5}$

d x-intercept $= -\frac{1}{2}$, y-intercept $= -\frac{1}{3}$

e x-intercept $= 7$, y-intercept $= 11$

f x-intercept $= 1$, y-intercept $= -1$

Check your Student obook for these digital resources and more:

Interactive skillsheet
Gradients

Interactive skillsheet
Determining linear equations

Topic quiz
3C

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

3A **1** Solve the following equations.

a $\frac{x}{3} - 5 = 2$ **b** $8(x + 6) = -24$

c $\frac{5 - 2x}{4} = 9$ **d** $4(3x - 7) - 5x = -31$

3A **2** Solve the following equations.

a $\frac{10}{x} = 4$ **b** $\frac{4}{x} = 12$

c $\frac{-18}{x + 5} = 9$ **d** $\frac{14}{7x - 9} = 3$

3A **3** Solve the following equations.

a $6x - 10 = 5x + 9$ **b** $5(3x - 2) = x$

c $9(4 - 7x) = 3(9x + 11)$ **d** $\frac{4x - 9}{3} = \frac{2x + 7}{4}$

3A **4** Rearrange each of the following equations to make x the subject.

a $ax + by = c$ **b** $\frac{f}{dx - e} = g$

c $j(x + k) = m(x - n)$ **d** $\frac{px - q}{s - tx} = r$

3B **5** Sketch the graph of each of the following.

a $y = 3x - 1$ **b** $y = \frac{1}{4}x + 2$

c $y = -\frac{2}{5}x + 7$ **d** $y = \frac{7}{6}x$

3B **6** Sketch the graph of each of the following.

a $2x - 3y = 12$ **b** $4x + 5y = -100$

c $18x + 36y = 9$ **d** $14y - 15x = 24$

3B **7** Sketch the graph of each of the following.

a $x = 10$ **b** $y = -4$ **c** $x = -\frac{2}{3}$ **d** $y = \frac{3}{2}$

3C **8** Determine the gradient of a line that passes through:

a (3, 8) and (7, 10) **b** (13, 2) and (–7, –2)

c (5, –8) and (–3, 10) **d** (–5, 12) and (3, –2)

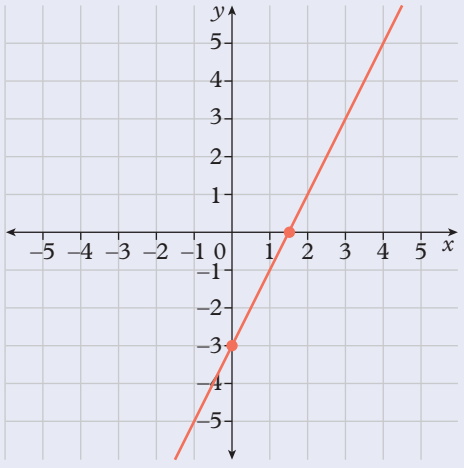
3C **9** Determine the equation of lines that passes through the points in question **8**.

3C **10** Determine the equation for a line with the following gradient that passes through (–2, 7).

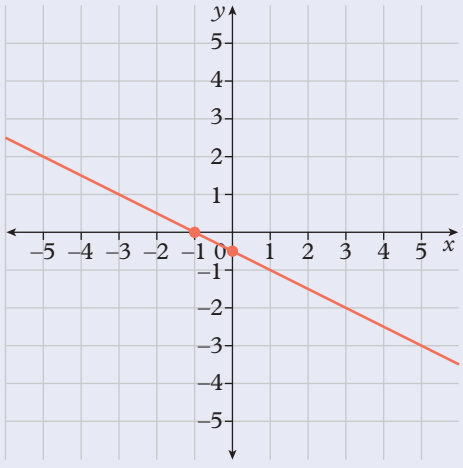
a 2 **b** $\frac{1}{2}$ **c** 1 **d** –2 **e** $-\frac{1}{2}$

3C **11** Determine the equations of the following linear graphs.

a



b



3D Solving linear inequalities

Learning intentions

By the end of this section you will be able to ...

- ✔ solve linear inequalities using inverse operations
- ✔ test if a point on the Cartesian plane is a solution to a linear inequality
- ✔ represent linear inequalities on the Cartesian plane.

Inter-year links

Year 7	6G Solving equations using inverse operations
Year 8	6G Solving linear inequalities
Year 9	3B Solving linear inequalities

Solving linear inequalities

- A **linear inequality** is an inequality containing pronumerals only raised to the power of 1.
- If the same operation is performed on both sides of an inequality sign, this results in an equivalent inequality of a different form. The possible values of the unknown remains the same.
- Inverse operations can be used to solve linear inequalities. However, the inequality sign needs to be reversed when both sides of the inequality are multiplied or divided by a negative number.

  For example, $5 - 2x \leq 1 \quad (-5)$

  $-2x \leq -4 \quad (\div (-2))$

  $x \geq 2$
- A linear inequality can also have two variables. For example, $y > 5 - 2x$ and $3x + 4y < 5$.
- Linear inequalities with two variables have infinitely many solutions that can be written as ordered pairs in the form (x, y).
- Each ordered pair would produce a true statement when the values of x and y are substituted into the inequality.

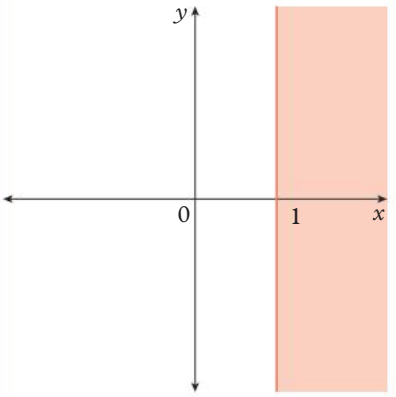
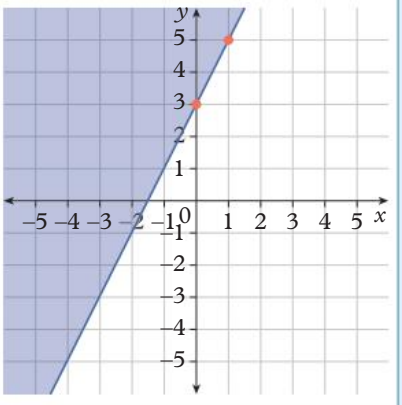
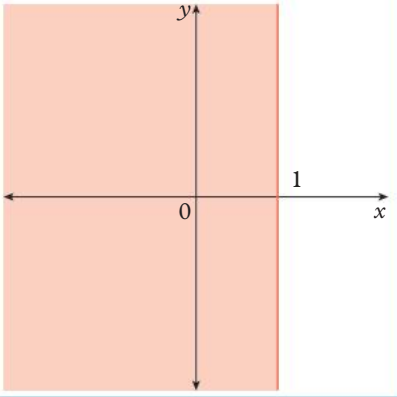
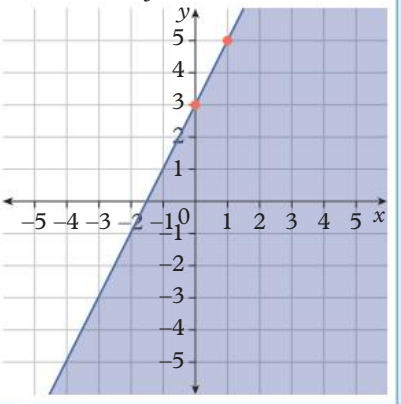
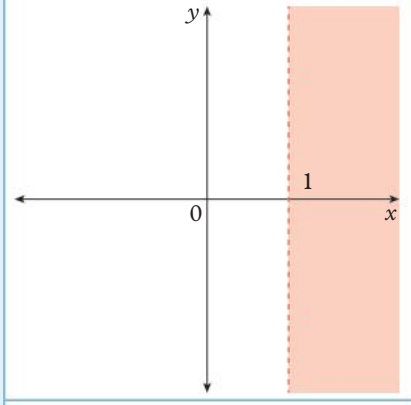
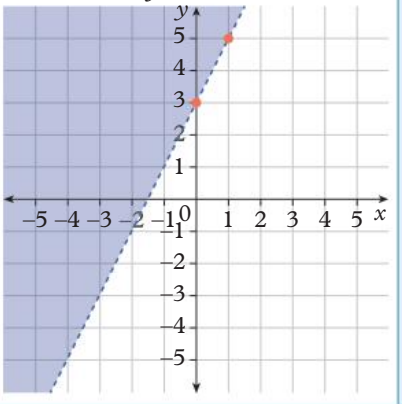
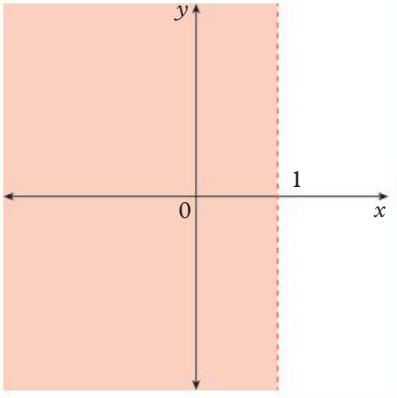
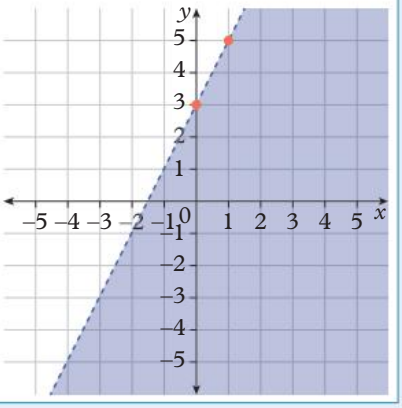
  For example, $(0, 0)$ is one of the solutions to $3x + 4y < 5$.

  Because $3 \times 0 + 4 \times 0 = 0$, $0 < 5$ is a true statement.

Representing linear inequalities with two variables on the Cartesian plane

- Linear inequalities with two variables can be represented on the Cartesian plane using a boundary line and shading.
  - → a solid boundary line indicates that the points on the boundary line are included, meaning one side of the inequality is either 'less than or equal to' (≤) or 'greater than or equal to' (≥) the other side of the inequality. All the points on the boundary line and in the shaded area are solutions to the inequality.
  - → a dashed line indicates that one side of the inequality is either 'less than' (<) or 'greater than' (>) the other side of the inequality. All the points in the shaded area are solutions to the linear inequality.
- The boundary line of linear inequalities on the Cartesian plane can be found by changing the inequality sign in the inequality to an equals sign.

  For example:
  - → the boundary line of $2x + 3 < 1$ can be found by solving $2x + 3 = 1$
  - → the boundary line of $2x + 3 < y$ is $2x + 3 = y$.
- The boundary line divides the Cartesian plane into two halves; points on either side of the boundary line can be used as test points to determine which half should be shaded.

Inequality with one variable	Inequality with two variables
$x \geq 1$ 	$y \geq 2x + 3$ 
$x \leq 1$ 	$y \leq 2x + 3$ 
$x > 1$ 	$y > 2x + 3$ 
$x < 1$ 	$y < 2x + 3$ 

Example 3D.1 Solving linear inequalities

Solve each of these inequalities.

a $5x + 3 < -2$ **b** $3 > -1 - \frac{x}{7}$ **c** $7x - 2 \geq 3x - 10$

THINK

a 1 Apply the inverse operations to both sides in the correct order.

2 Write the solution with x on the LHS.

3 Check your solution. Substitute $x = -1$ into the LHS of the inequality. The result should equal the RHS. Then substitute any value for x that is less than -1 into the original inequality to check the inequality sign. The result should be less than the RHS.

b 1 Apply the inverse operations to both sides in the correct order. Reverse the inequality sign when both sides of the inequality are multiplied or divided by a negative number.

2 Write the solution with x on the LHS, reverse the inequality sign if needed.

3 Check your solution. Substitute $x = -28$ into the RHS of the inequality. The result should equal the LHS. Then substitute any value for x that is greater than -28 into the original inequality to check the inequality sign. The result should be less than the LHS.

c 1 Use inverse operations to remove the pronumeral on the side where the coefficient is less than the coefficient on the other side.

2 Identify and apply the correct inverse operations to both sides of the inequality in the correct order.

3 Check your solution by substituting appropriate values for x into the original inequality to see if the result is a true statement.

WRITE

a $5x + 3 < -2 \quad (-3)$

$5x < -5 \quad (\div 5)$

$x < -1$

Check by substituting $x = -1$ into $5x + 3 < -2$.

LHS $= 5(-1) + 3$

$= -2$

$=$ RHS (For $x = -1$, LHS = RHS as required.)

Check by substituting a value less than -1 (for example, $x = -2$).

LHS $= 5(-2) + 3$

$= -7$

$-7 < -2$ (For $x < -1$, LHS < RHS as required.)

b $3 > -1 - \frac{x}{7} \quad (+1)$

$4 > -\frac{x}{7} \quad (\times (-7))$

$-28 < x$

$x > -28$

Check by substituting $x = -28$ into $3 > -1 - \frac{x}{7}$.

RHS $= -1 - \frac{-28}{7}$

$= -1 + 4$

$= 3$

$=$ LHS (For $x = -28$, LHS = RHS as required.)

Check by substituting a value greater than -28 (for example, $x = 0$).

RHS $= -1 - \frac{0}{7}$

$= -1$

$-1 < 3$ (For $x > -28$, LHS > RHS as required.)

c $7x - 2 \geq 3x - 10 \quad (-3x)$

$4x - 2 \geq -10 \quad (+2)$

$4x \geq -8 \quad (\div 4)$

$x \geq -2$

Check by substituting $x = -2$ into $7x - 2 \geq 3x - 10$.

LHS $= 7(-2) - 2$

$= -16$

RHS $= 3(-2) - 10$

$= -16$ (For $x = -2$, LHS = RHS as required.)

Check by substituting a value greater than −2 (for example, $x = 0$).

LHS = 7(0) − 2
= −2

RHS = 3(0) − 10
= −10 (For $x > -2$, LHS > RHS as required.)

Example 3D.2 Determining if a point is a solution to a linear inequality

Determine if the following number pairs are solutions of $3x + 2y \geq 23$.

a (0, 0) **b** (5, 4) **c** (4, 5) **d** (4, 6)

THINK

a 1 Substitute $x = 0$ and $y = 0$ into the LHS of the linear inequality.

2 Calculate the LHS of the linear inequality.

3 Compare the LHS and RHS of the linear inequality. LHS ≥ RHS is a false statement, so the number pair is not a solution to the given linear inequality.

b 1 Substitute $x = 5$ and $y = 4$ into the LHS of the linear inequality.

2 Calculate the LHS of the linear inequality.

3 Compare the LHS and RHS of the linear inequality. The LHS = RHS, it satisfies the inequality LHS ≥ RHS so the number pair is a solution to the given linear inequality.

c 1 Substitute $x = 4$ and $y = 5$ into the LHS of the linear inequality.

2 Calculate the LHS of the linear inequality.

3 Compare the LHS and RHS of the linear inequality. LHS ≥ RHS is a false statement, so the number pair is not a solution to the given linear inequality.

d 1 Substitute $x = 4$ and $y = 6$ into the LHS of the linear inequality.

2 Calculate the LHS of the linear inequality.

3 Compare the LHS and RHS of the linear inequality. The LHS > RHS, it satisfies the inequality LHS ≥ RHS so the number pair is a solution to the given linear inequality.

WRITE

a $3x + 2y = 3(0) + 2(0)$

LHS = 0
0 ≥ 23 is a false statement
(0, 0) is not a solution of $3x + 2y \geq 23$.

b $3x + 2y = 3(5) + 2(4)$

LHS = 15 + 8
= 23
23 ≥ 23 is a true statement
(5, 4) is a solution of $3x + 2y \geq 23$.

c $3x + 2y = 3(4) + 2(5)$

LHS = 12 + 10
= 22
22 ≥ 23 is a false statement
(4, 5) is not a solution of $3x + 2y \geq 23$.

d $3x + 2y = 3(4) + 2(6)$

LHS = 12 + 12
= 24
24 ≥ 23 is a true statement
(4, 6) is a solution of $3x + 2y \geq 23$.

Example 3D.3 Representing linear inequalities on the Cartesian plane

Represent the following linear inequalities on the Cartesian plane.

a $y > 2x + 4$ **b** $2y + x \leq 2$

THINK

a 1 Sketch $y = 2x + 4$ on the Cartesian plane by finding the x- and y-intercepts.

2 Connect the intercepts with a dashed line as the inequality sign is $>$ not $\geq$.

3 Use $(0, 0)$ as a test point to find out which side should be shaded.

4 As $(0, 0)$ is not a solution to $y > 2x + 4$, shade the area above the boundary line.

b 1 Sketch $2y + x = 2$ on the Cartesian plane by finding the x- and y- intercepts.

WRITE

a $y = 2x + 4$

When $x = 0$,

$y = 2(0) + 4$

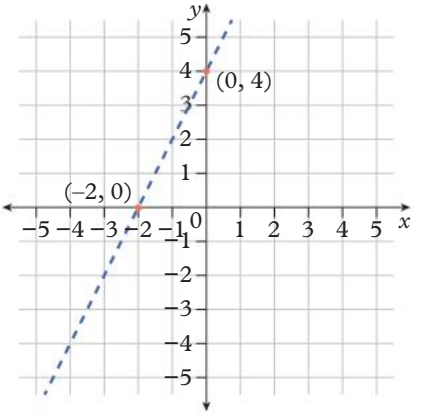
$= 4$

When $y = 0$,

$0 = 2x + 4$

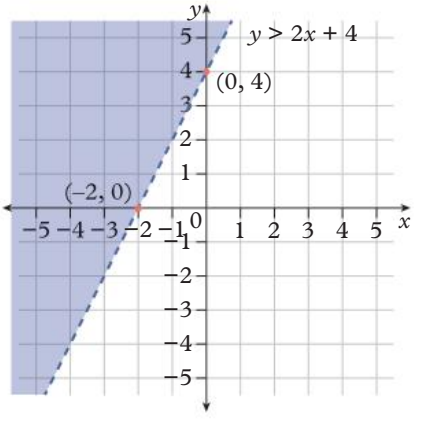
$x = -2$

The x- and y- intercepts are $(0, 4)$ and $(-2, 0)$.



Substitute $(0, 0)$ into the inequality.

$0 > 2(0) + 4$ is a false statement. $(0, 0)$ is not a solution of $y > 2x + 4$.



b $2y + x = 2$

When $x = 0$,

$2y + 0 = 2$

$y = 1$

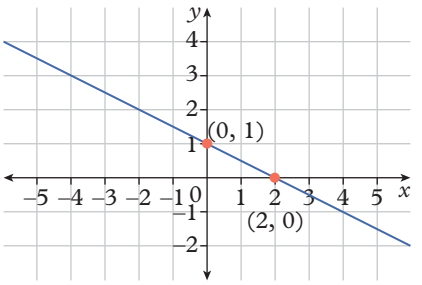
When $y = 0$,

$0 + x = 2$

$x = 2$

The x- and y-intercepts are $(0, 1)$ and $(2, 0)$.

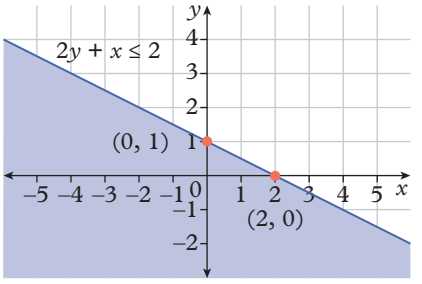
2 Connect the intercepts with a solid line.



3 Use (0, 0) as a test point to find out which side should be shaded.

Substitute (0, 0) into the inequality.
$2(0) + 0 \leq 2$ is a true statement. (0, 0) is a solution of $2y + x \leq 2$.

4 As (0, 0) is a solution to $2y + x \leq 2$, shade the area below the boundary line.



Helpful hints

- ✔ Be careful with the direction of the inequality sign when multiplying and dividing inequalities by negative numbers.
- ✔ If you need to swap the LHS and RHS of an inequality, remember to reverse the inequality sign so that it is still pointing towards the correct direction.
- ✔ When sketching the boundary line of a linear inequality, you can use either two intercepts or one intercept and the gradient.

ANS p564

Exercise 3D Solving linear inequalities

1, 2–4(1st, 2nd columns), 6–8, 13, 14 | 4(2nd column) 6, 7, 8–11, 15 | 4(3rd column), 7, 8, 12, 16

1 Determine which of the following are linear inequalities.

a $x = 1$ **b** $y = 2x + 1$ **c** $y + 2 < 2x + 1$ **d** $x < 2x$

e $x + y + x \geq 0$ **f** $y < x^2 + 1$ **g** $x^2 < 1$ **h** $y > \frac{1}{2}$

i $y < \frac{1}{x}$ **j** $y \leq 2x$ **k** $xy > 2$ **l** $y < \frac{x}{2}$

3D.1 **2** Solve the following linear inequalities.

a $x + 5 < 9$ **b** $2 \geq 2x + 1$ **c** $12 < 2x - 4$ **d** $2x < -20$

e $x + 12 \geq 0$ **f** $23 < x + 1$ **g** $11(x + 1) \leq 121$ **h** $\frac{(x - 3)}{2} > 72$

i $12(2x + 2) \leq 144$ **j** $100 \leq \frac{3}{5}x$ **k** $\frac{2x}{5} - 10 \geq 0$ **l** $21 < \frac{x}{7} + 7$

3 Solve the following linear inequalities.

a $-x + 3 < 9$ **b** $2 \geq -2x + 1$ **c** $-12x - 10 < 110$ **d** $-2x < -20$

e $-2x + 12 \geq 0$ **f** $23 < -x + 1$ **g** $11(2 - x) \leq 121$ **h** $\frac{(3 - x)}{2} > 72$

i $12(2 - 2x) \leq 144$ **j** $12 \leq 10 - \frac{3}{5}x$ **k** $7 - \frac{2x}{5} \geq 3$ **l** $21 < 7 - \frac{x}{7}$

4 Solve the following linear inequalities.

a $x + 3 < 9x - 2$ **b** $2x \geq 5x + 1$ **c** $12x - 10 < 2x + 110$

d $2 - 2x < x - 20$ **e** $-3x + 12 \geq 2x$ **f** $x + 23 < -x + 1$

g $2(2 - x) \leq 12x$ **h** $\frac{(3 - x)}{2} > 7$ **i** $12(2 - 2x) \leq 2x + 5$

j $12x \leq 10 - \frac{3}{5}x$ **k** $7 - \frac{2x}{5} \geq 3x$ **l** $2x - 1 < 7 - \frac{x}{7}$

5 **a** $\frac{x + 3}{12} < \frac{9}{12}$ **b** $\frac{x + 1}{3} < \frac{2}{3}$ **c** $\frac{3 - x}{2} < \frac{9 - x}{12}$

d $\frac{1 - x}{6} > \frac{x - 2}{12}$ **e** $\frac{x - 3}{21} < \frac{9 - x}{7}$ **f** $\frac{x + 3}{5} \leq \frac{9 - x}{7}$

3D.2 **6** Determine if the following points are solutions of the given inequalities.

a $2x + 1 < y$ **b** $y \geq 3 - x$ **c** $x - 5 \leq y$ **d** $y < 8 + 2x$

i (0, 0) **ii** (1, 1) **iii** (3, 5)

7 Determine if the following points are solutions of the given inequalities.

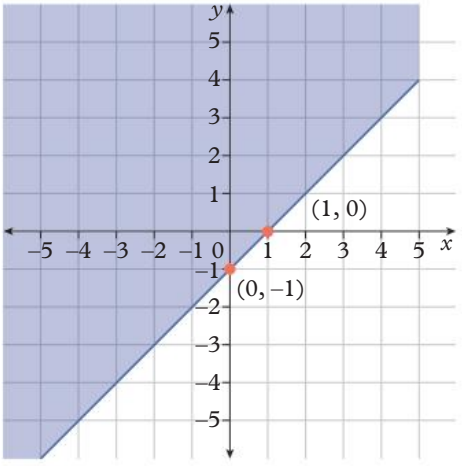
a $2x + 1 \leq 2y + 3$ **b** $3x + 2y \geq 3$ **c** $x + 2y \leq 8$ **d** $x + y \leq 3 + 2x$

i (0, 0) **ii** (2, 3) **iii** (1, 9)

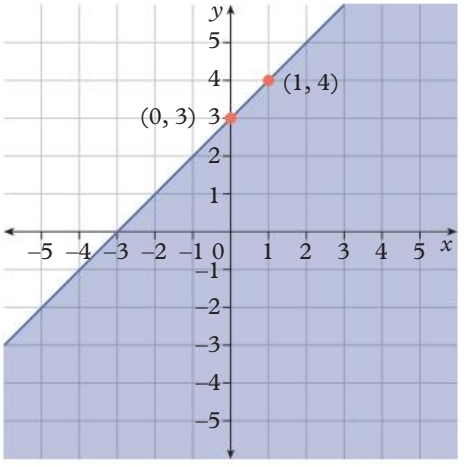
8 Use your answer to question **7** to match the inequalities with the graphs below.

a $2x + 1 \leq 2y + 3$ **b** $3x + 2y \geq 3$ **c** $x + 2y \leq 8$ **d** $x + y \leq 3 + 2x$

A



B



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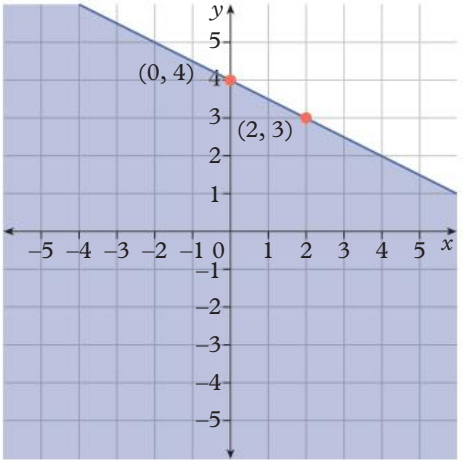

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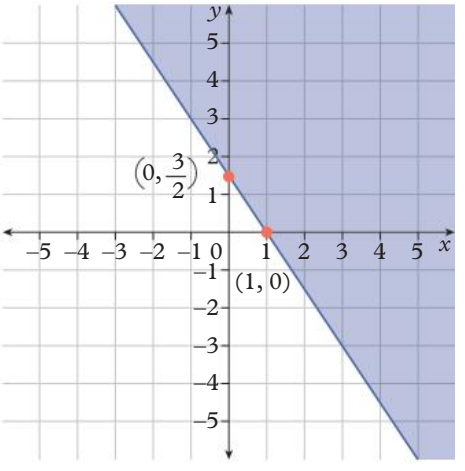


3D.3 **9** Represent the linear inequalities in question **6** on the Cartesian plane.

10 Represent the following linear inequalities on the Cartesian plane.

a $y + 2x \leq 5$ **b** $2y + 4x > 2$ **c** $3y + 2x < 2$

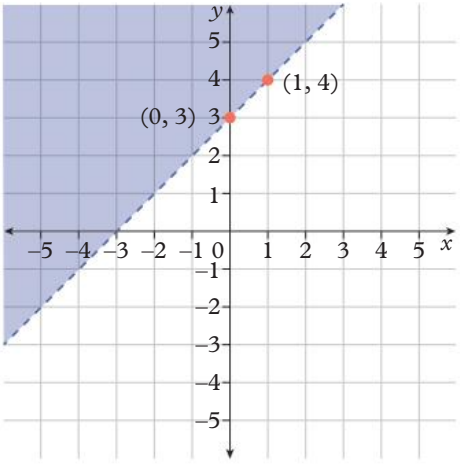
11 Represent the following linear inequalities on the Cartesian plane.

a $y + 2x \leq 5 - x$ **b** $y + 4x > 2 + 2y$ **c** $x + y + 1 \leq 2x$

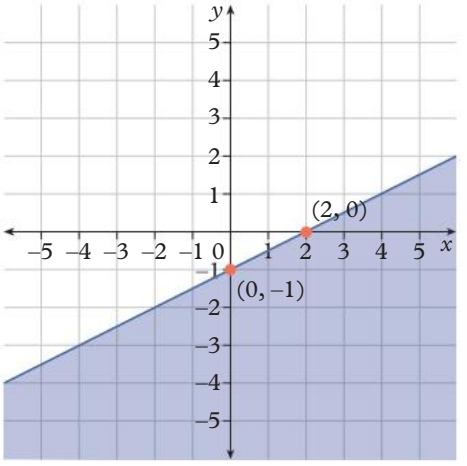
d $\frac{1-x}{6} < \frac{-2y}{12}$ **e** $\frac{x-3}{21} \geq \frac{9-y}{7}$ **f** $\frac{y+3}{5} > \frac{9-x}{7}$

12 Write the inequalities represented by the following graphs.

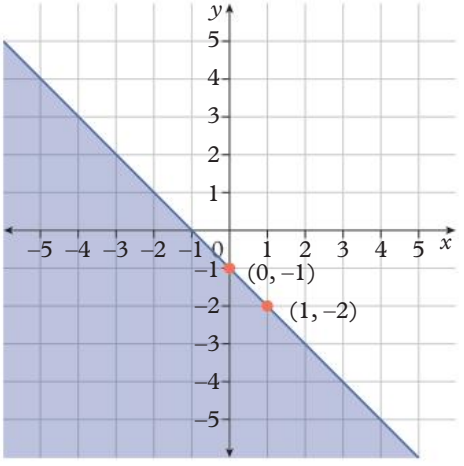
a



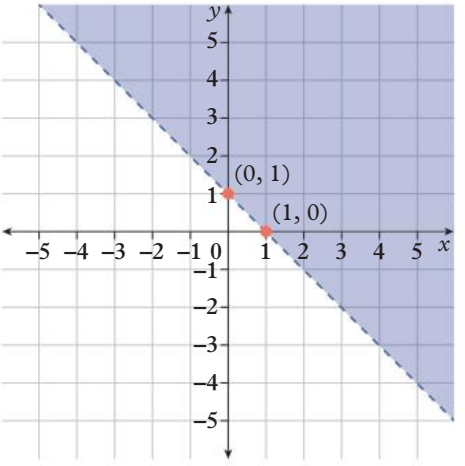
b



c



d



13 Write a linear inequality to represent each of the following situations.

a The width and length of a rectangle are w and l, and the perimeter of the rectangle is greater than 20 cm.

b Eric has \$100 to spend on fertilisers and pesticides, each bag of fertiliser costs \$22 and each bag of pesticide costs \$6.

c The capacity of a water tank is 2145 L, each shower uses 55 L of water, each bath uses 80 L of water.

d Emily has decided to buy p pizzas for g guests. Each pizza is cut into 8 pieces. Emily wants to eat 4 pieces and ensure each guest can have at least 3 pieces.

e Mark is making plans for a triathlon. He plans to swim for s hours and bike for b hours. The fastest he can complete the running course is in 3 hours 45 minutes and he aims to finish the whole race in 12 hours.

PROBLEM SOLVING AND REASONING

14 Ken runs a small business selling products A and B. He buys product A for $10 and product B for $20, respectively, and resells them later at a higher price.

a Write an inequality to represent the relationship between the numbers of product A, x, and product B, y, Ken can stock up on if he has $400.

b Ken prices product A and B at $15 and $25, respectively. Ken wants to make at least $100 in profit. Ken wrote the inequality $15x + 25y \geq 400 + 100$ to represent the amount of money paid by customers versus his costs and desired profit. Explain why this is incorrect.

c Write the correct inequality to represent Ken's profit.

d What is Ken's profit if he only stocks up on product B and manages to sell all the stock?

e Ken is confident that he can sell all the products in stock. How many of product A and product B should Ken buy if he wants to maximise his profit?

15 Ada is buying souvenirs for her friends in the airport. Each snow globe costs $10 and weighs 1 kg and each hoodie costs $20 and weighs 0.7 kg.

a Write an inequality to represent the relationship between the number of snow globes x, and the number of hoodies, y, Ada could afford if she has $50.

b Graph the inequality on the Cartesian plane.

c Explain why you only need to include the first quadrant in the graph for part **b**.

d Use your graph to find out if Ada can buy 2 snow globes and 2 hoodies.

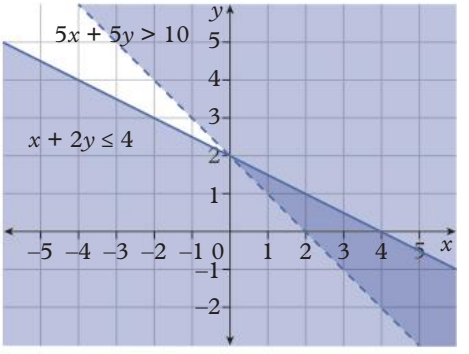
e List all the combinations of snow globes and hoodies Ada could buy with $50.

f The weight limit of Ada's carry-on luggage is 7 kg. Her laptop weighs 3 kg. Write an inequality to represent the relationship between the number of snow globes, x, and the number of hoodies, y, if Ada wants to fit all the souvenirs and her laptop into the luggage.

g Which combination(s) in part **e** satisfy the inequality in **f**?

CHALLENGE

16 When two or more inequalities are sketched on the same set of axes, the shaded areas overlap and form an intersecting region. Points in the intersecting region satisfy both inequalities. For example, the graph on the right shows $x + 2y \leq 4$ and $5x + 5y > 10$ on the same set of axes.



a Sketch the following inequalities on the same set of axes.

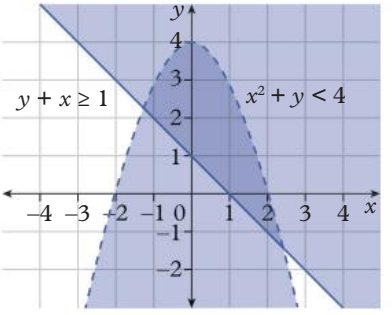
i $y \geq 2x + 4$ and $y < -x$

ii $y + 2x \geq 4$ and $y + x < 1$

iii $2x - y \leq 2$ and $y < 3 - x$

iv $y + 2x < 2$ and $y + x < 2$

b Use the graph on the right to find out if the given value pairs satisfy $x^2 + y < 4$ and $y + x \geq 1$



i $x = 1, y = 1$

ii $x = 2, y = 1$

iii $x = 2, y = -2$

iv $x = -1, y = 2$

v $x = 0, y = 4$

vi $x = -2, y = 3$

Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Representing linear inequalities on the Cartesian plane
- Interactive skillsheet: Solving linear inequalties
- Worksheet: Solving linear equations and inequalities
- Topic quiz: 3D

3E Simultaneous linear equations

Learning intentions

By the end of this section you will be able to ...

- ✔ solve simultaneous linear equations graphically
- ✔ solve simultaneous linear equations using substitution
- ✔ solve simultaneous linear equations using elimination.

Inter-year links

Year 8	6E Solving linear equations using graphs
Year 9	3E Sketching linear graphs using intercepts

Simultaneous linear equations

- If linear equations contain the same variables, we call them **simultaneous linear equations**.
- The solutions to a pair of simultaneous linear equations are the values for the variables that are solutions to both equations.

  For example, to check whether $x = 2$ and $y = 7$ are solutions to $y = 5x - 3$ and $y = -x + 9$, substitute the value $x = 2$ into both equations.

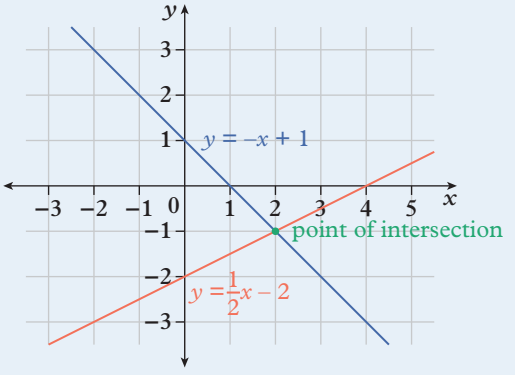
$y = 5x - 3$	$y = -x + 9$
$y = 5(2) - 3$	$y = -1(2) + 9$
$y = 10 - 3$	$y = -2 + 9$
$y = 7$	$y = 7$

  When $x = 2$, the result of both equations is $y = 7$, showing that the solution to the pair of simultaneous linear equations is $x = 2$ and $y = 7$.

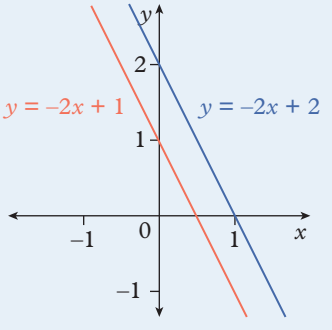
Solving simultaneous linear equations graphically

- If two linear graphs are not parallel, then they will have a **point of intersection**. The x and y value at the point of intersection is the solution to the simultaneous linear equations.

  For example, the pair of simultaneous linear equations $y = -x + 1$ and $y = \frac{1}{2}x - 2$ can be represented by the pair of linear graphs shown on the right.



- The point of intersection is on both linear graphs, so the x and y value at the point of intersection is a solution to both linear equations describing the pair of intersecting linear graphs.
- If the linear graphs are parallel, the simultaneous linear equations have no solution. The graphs do not intersect.



Solving simultaneous linear equations by substitution

- The **substitution method** involves substituting an expression from one equation into another equation.
- When at least one of the equations already has one variable as the subject, using the substitution method is the simplest.

  For example, if $y = 4 - 2x$ and $y + 11 = 3x$ are being solved simultaneously, substituting y from the first equation into the second equation will give us a value for x.

  Substitute $y = 4 - 2x$ into $y + 11 = 3x$:

$$\begin{aligned}(4 - 2x) + 11 &= 3x \\ 4 - 2x + 11 &= 3x \qquad (+ 2x) \\ 4 + 11 &= 5x \\ 15 &= 5x \qquad (\div 5) \\ x &= 3\end{aligned}$$

  Substituting the value $x = 3$ into either equation gives the result $y = -2$:

$$\begin{aligned}y &= 4 - 2x & y + 11 &= 3x \\ y &= 4 - 2(3) & y + 11 &= 3(3) \\ y &= 4 - 6 & y + 11 &= 9 \qquad (- 11) \\ y &= -2 & y &= -2\end{aligned}$$

  So, the solution to the pair of simultaneous equations is $x = 3$ and $y = -2$.

Solving simultaneous linear equations by elimination

- The LHS and RHS of two equations can be added or subtracted from each other to create a new equation.

  For example, the LHS and RHS of $6 = 1 + 5$ and $3 + 4 = 7$ can be added to get $6 + (3 + 4) = (1 + 5) + 7$.
- The **elimination method** uses this idea to 'add' or 'subtract' simultaneous linear equations to create a new equation in which one of the variables is eliminated.

  For example, the pair of simultaneous linear equations $x + y = 4$ and $-3x + y = -8$ can be solved by elimination when the second equation is 'subtracted' from the first.

$$\begin{aligned}& x + y = 4 \qquad ① \\ & -3x + y = -8 \qquad ② \\ ① - ②: \quad & (x + y) - (-3x + y) = (4) - (-8) \\ & x + 3x + y - y = 4 + 8 \\ & 4x = 12 \qquad (\div 4) \\ & x = 3\end{aligned}$$

  Substituting this value for x into either equation gives the result $y = 1$. The pair of simultaneous equations has been solved.
- One or both of the equations might need to be multiplied by a constant so that the coefficient of one of the variables becomes zero when the equations are added/subtracted.

Example 3E.1 Solving simultaneous linear equations graphically

Solve this pair of simultaneous linear equations using a graph.

$y = x + 2$ $y = 3x - 2$

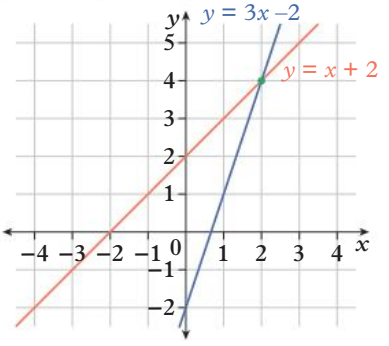
THINK

1 Sketch the graphs of $y = x + 2$ and $y = 3x - 2$ on the same Cartesian plane, using graph paper. Use a ruler to extend both lines until they intersect.

2 Use the coordinates of the point of intersection to determine a solution to the simultaneous linear equations.

3 Check the solution by substituting the value for x into both equations. If both equations result in the same y value, the solution is correct.

WRITE



The point of intersection is at (2, 4), so the solution to the simultaneous linear equations is $x = 2, y = 4$.

Check: $y = x + 2$ $y = 3x - 2$

$y = (2) + 2$ $y = 3(2) - 2$

$y = 4$ $y = 6 - 2$

$y = 4$

Example 3E.2 Solving simultaneous linear equations by substitution

Solve this pair of simultaneous linear equations using the substitution method.

$5x + 2y = 2$ $y = 7x + 20$

THINK

1 Number each equation for reference.

2 Equation ② has one variable (y) as its subject. Substitute the expression for y from equation ② into equation ①. Use brackets when substituting the expression.

3 Solve for x.

4 Substitute the solution for x into either equation and solve it for y.

WRITE

$5x + 2y = 2$ ①

$y = 7x + 20$ ②

Substituting $y = 7x + 20$ into ①:

$5x + 2y = 2$

$5x + 2(7x + 20) = 2$

$5x + 14x + 40 = 2$

$19x + 40 = 2$ $(- 40)$

$19x = -38$ $(\div 19)$

$x = -2$

Substitute $x = -2$ into ②:

$y = 7x + 20$

$y = 7(-2) + 20$

$y = 6$

The solution is $x = -2, y = 6$.

Example 3E.3 Solving simultaneous linear equations by elimination

Solve this pair of simultaneous linear equations using the elimination method.

$y - 3x = -10$ $\quad$ $3x + 2y = 16$

THINK

1 Number each equation for reference.

2 The coefficient for x in equation ① is the negative value of the coefficient for x in equation ②. So, the equations can be added to remove x.

3 Solve the resulting equation for y.

4 Substitute the solution for y into either equation to solve it for x.

WRITE

$y - 3x = -10$ ①

$3x + 2y = 16$ ②

① + ②:

$(y - 3x) + (3x + 2y) = (-10) + (16)$

$y - 3x + 3x + 2y = -10 + 16$

$3y = 6$ $\quad (\div 3)$

$y = 2$

Substitute $y = 2$ into ①:

$y - 3x = -10$

$(2) - 3x = -10$ $\quad (-2)$

$-3x = -12$ $\quad (\div -3)$

$x = 4$

The solution is $x = 4$, $y = 2$.

Example 3E.4 Solving simultaneous linear equations by elimination (multiplying by a constant)

Solve each of these pairs of simultaneous linear equations using the elimination method.

a $y - x = 11$ $\quad$ $3y - 2x = 27$

b $4x - 3y = -5$ $\quad$ $2y - 5x = 1$

THINK

a 1 Number each equation for reference.

2 The coefficient of x in equation ① is -1 and the coefficient of x in equation ② is -2. Equation ② can stay the same, multiply equation ① by 2.

3 Equation '2 × ①' and equation ② have the same coefficients of x, subtract the equations to remove x.

4 Solve the equation for y.

WRITE

a $y - x = 11$ ①

$3y - 2x = 27$ ②

2 × ①:

$y - x = 11$ $\quad (\times 2)$

$2y - 2x = 22$

(2 × ①) − ②:

$(2y - 2x) - (3y - 2x) = (22) - (27)$

$2y - 2x - 3y + 2x = 22 - 27$

$-y = -5$

$y = 5$

5 Substitute the solution for y into either equation and solve it for x.

Substituting $y = 5$ into ①:

$y - x = 11$

$(5) - x = 11 \quad (-5)$

$-x = 6$

$x = -6$

The solution is $x = -6$, $y = 5$.

b **1** Number each equation for reference.

$4x - 3y = -5$ ①

$2y - 5x = 1$ ②

2 The coefficient of y in equation ① is –3 and the coefficient of y in equation ② is 2. Multiply equation ① by 2 and equation ② by 3.

$2 \times$ ①:

$4x - 3y = -5 \quad (\times 2)$

$8x - 6y = -10$

$3 \times$ ②:

$2y - 5x = 1 \quad (\times 3)$

$6y - 15x = 3$

3 The coefficients of y in equation '$2 \times$ ①' and equation '$3 \times$ ②' can be added to remove y.

$(2 \times ①) + (3 \times ②)$:

$(8x - 6y) + (6y - 15x) = (-10) + (3)$

$8x - 6y + 6y - 15x = -10 + 3$

$-7x = -7$

4 Solve for x.

$x = 1$

5 Substitute the solution for x into either equation to solve it for y.

Substituting $x = 1$ into ①:

$4x - 3y = -5$

$4(1) - 3y = -5$

$4 - 3y = -5 \quad (-4)$

$-3y = -9 \quad (\div 3)$

$y = 3$

The solution is $x = 1$, $y = 3$.

Helpful hints

- ✔ Always number your equations for easy reference.
- ✔ Remember, you can always check your solution to a pair of simultaneous equations by substituting one of the values back into each equation to see if the result matches your solution for the other variable.
- ✔ When solving simultaneous linear equations graphically, make sure to choose a suitable size for the units of your x- and y-axes and use graph paper so that you can identify the intersection point with accuracy.
- ✔ When solving simultaneous linear equations using the substitution method, always make sure you substitute algebraic expressions enclosed in brackets, then remove the brackets by expanding in the next step.
- ✔ Remember, you can't eliminate a variable by adding or subtracting equations unless the magnitude of the coefficients are the same.

ANS p567

Exercise 3E Simultaneous linear equations

1, 2, 3–5(1st, 2nd columns), 6–8, 9(1st, 2nd columns), 11(a, b, g, h), 13	2, 3–5(3rd, 4th columns), 8, 9 (2nd, 3rd columns), 10, 11(e, f, h, i, o), 12, 14, 15, 18, 19	3(k, l, o, p), 4–5(3rd, 4th columns), 9(3rd column), 11, 14, 16, 17, 19, 20

UNDERSTANDING AND FLUENCY

1 Check if the given x and y values are the solution to the pair of simultaneous linear equations each time.

a $x = 1, y = 5$ Equations: $y = x + 4$ and $y = -2x + 7$

b $x = -2, y = 1$ Equations: $y = x + 3$ and $y = -3x - 5$

c $x = 5, y = -2$ Equations: $y = x - 9$ and $y = -2x + 7$

d $x = 3, y = -7$ Equations: $2x + y = -1$ and $y = -x - 4$

e $x = 4, y = 3$ Equations: $y = 2x - 5$ and $y = -x + 1$

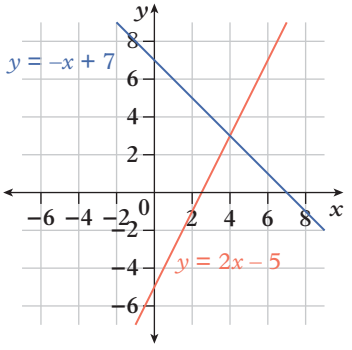
f $x = -6, y = 0$ Equations: $y = x - 6$ and $x + y = -6$

2 For each of the following pairs of simultaneous linear equations and their graphs:

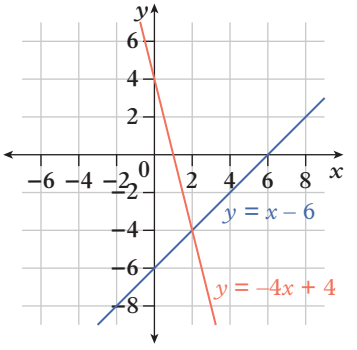
i write the coordinates of the point of intersection of the two lines

ii use the coordinates of the point of intersection to help you find the solution to the simultaneous linear equations. Use substitution to check your solution.

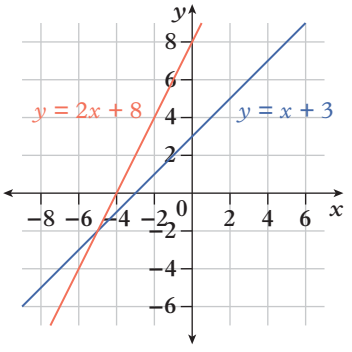
a $y = 2x - 5$
$y = -x + 7$



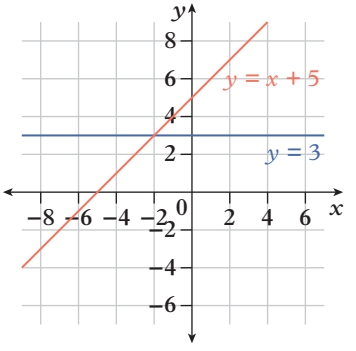
b $y = -4x + 4$
$y = x - 6$



c $y = 2x + 8$
$y = x + 3$



d $y = x + 5$
$y = 3$



3E.1 **3** Use graphs to help you solve these pairs of simultaneous linear equations (**a**–**p**). Use substitution to check your solutions.

a $y = 2x - 6$
$y = x - 1$

b $y = 3x$
$y = x + 4$

c $y = x - 2$
$y = 10 - x$

d $y = x + 7$
$y = -3x - 1$

e $y = 3x + 6$
$2x + y = 1$

f $y = -2x - 5$
$y = 2x + 7$

g $y = 2x - 4$
$y = x - 1$

h $2x + y = 6$
$y = x - 6$

i $y = 3x + 3$
$y = -x - 5$

j $2x + y = -4$
$x + y = 1$

k $x - y = 4$
$4x + y = 6$

l $3y = x + 12$
$y = x + 2$

m $y = \frac{7}{2}x + 13$
$-2x + 3y = 5$

n $y = -\frac{5}{4}x - \frac{11}{4}$
$3x - 2y = 11$

o $y = \frac{8}{5}x + \frac{23}{5}$
$9x + 16y = -134$

p $y = \frac{5}{7}x + \frac{19}{7}$
$7x + 4y = 1$

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3E.2 **4** Use the substitution method to solve each of these pairs of simultaneous linear equations.

a $x + y = 9$
$y = x + 5$

b $x + 2y = 6$
$y = x - 3$

c $2x + y = 1$
$y = x + 4$

d $y = x - 1$
$x - 3y = -9$

e $3x - y = -10$
$y = x + 2$

f $y = 6x$
$5x - y = -2$

g $6x + 5y = 21$
$y = 2x + 1$

h $y = 2 - 3x$
$4x + y = -1$

i $2x + 3y = -5$
$y = 5 - 2x$

j $x = y + 2$
$2x - 3y = 1$

k $x = 4y$
$y = 3x + 11$

l $x = 5$
$4x - 3y = 2$

5 Use the substitution method to solve each of these pairs of simultaneous linear equations.

a $y = 2x + 6$
$y = x + 7$

b $y = x - 2$
$y = 4x - 20$

c $y = 7x + 16$
$y = 2x + 1$

d $y = x - 6$
$y = 4 - x$

e $y = 5x + 9$
$y = 3 - x$

f $y = 1 - 2x$
$y = x - 11$

g $y = 3x - 5$
$y = 5 - 2x$

h $y = 4x + 14$
$y = -4x - 2$

6 Solve each of these pairs of simultaneous linear equations by adding the equations together to remove one variable.

a $2x + y = 9$
$3x - y = 1$

b $x + 2y = -1$
$-x + 5y = 8$

c $x - 2y = 11$
$5x + 2y = -5$

7 Solve each of these pairs of simultaneous linear equations by subtracting one equation from the other to remove one variable.

a $x + 4y = 17$
$x + 2y = 11$

b $3x + y = 3$
$x + y = 7$

c $4x - 3y = 33$
$4x + y = 5$

3E.3 **8** Use the elimination method to solve each of these pairs of simultaneous linear equations.

a $-3x + y = 10$
$3x - 4y = -4$

b $x - 2y = 2$
$3x - 2y = 14$

c $6x - y = -8$
$-6x + 5y = 16$

d $6x + 5y = 19$
$4x + 5y = 21$

e $4x + 3y = 29$
$2x - 3y = 1$

f $-5x + y = 11$
$-5x - 2y = 23$

3E.4 **9** Use the elimination method to solve each of these pairs of simultaneous linear equations.

a $5x + 3y = 21$
$2x - y = 4$

b $x + 4y = -2$
$4x - 3y = 11$

c $3x + y = -5$
$7x + 2y = -14$

d $6x - 7y = 9$
$-2x + 5y = 5$

e $4x + 3y = -24$
$5x - 12y = 33$

f $2x + 5y = 26$
$10x - 3y = -38$

g $4x + 3y = 16$
$3x - 2y = -5$

h $3x + 7y = -2$
$5x + 11y = -4$

i $-8x + 5y = -21$
$5x + 2y = 8$

j $3x - 10y = -32$
$-7x + 3y = 34$

k $-9x + 2y = -35$
$8x - 9y = -5$

l $-4x - 5y = 22$
$-3x + 7y = -5$

10 Consider the three methods you have used for solving simultaneous linear equations:

i graphical
ii substitution
iii elimination

a For each method, describe the situation for which its use is most appropriate.
b For each method, describe the situation for which its use is least appropriate.
c What could you do to make the least appropriate situation for each method more appropriate for its use?

11 Use the most appropriate method to solve each of these pairs of simultaneous equations. Hint: Not all solutions are integers.

a $y = 3x - 6$
$4x + 11y = 8$

b $y = 6x - 5$
$y = 3x - 4$

c $4x - 5y = 4$
$2x + 5y = 5$

d $y = -7$
$y = 4x + 29$

e $x = 5y - 6$
$x - y = 2$

f $2y = x - 4$
$3x - 2y = 0$

g $2x - y = -4$
$4x + y = 22$

h $x + 5y = -5$
$x - 2y = 9$

i $-x - 7y = 5$
$-2x - 7y = 3$

j $-3x + 4y = 14$
$3x - y = -8$

k $x + y = 11$
$x - y = 3$

l $5x + 2y = -18$
$-5x + 2y = 22$

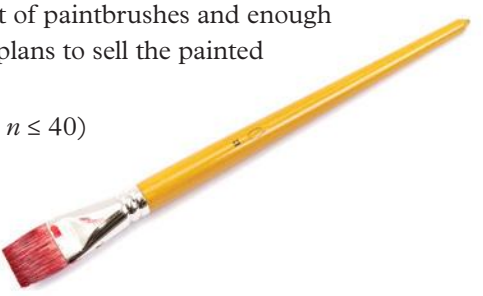
m $y = 2.5x + 4.3$
$y = 7.5x + 8.3$

n $4x = y - 6$
$8x + 3y = 13$

o $6x - 7y = -119$
$8x - 5y = 19$

12 Explain why there is no solution to the simultaneous linear equations $y = 2x + 5$ and $2x - y = 1$.

13 Sophie has decided to make hand-painted cards. She invests \$60 in a set of paintbrushes and enough paint to make 40 cards. The cost of each plain white card is \$2 and she plans to sell the painted cards for \$6 each.



a Explain how the amount of money, a, to produce n cards (where $0 \leq n \leq 40$) can be written as $a = 2n + 60$.

b Explain how the amount of money, a, received for selling n cards can be written as $a = 6n$.

c Draw graphs of the two equations on the same Cartesian plane. Label the horizontal axis as n and the vertical axis as a.

d What are the coordinates of the point of intersection of the two graphs? What does this tell you?

e How many cards should Sophie produce and then sell to break even (recover her costs but nothing extra)?

f How many cards should Sophie produce and sell to make a profit?

14 The perimeter around the top of a rectangular billiard table is 810 cm.

a Use l and w to write a linear equation for the perimeter of the table.

b Use l and w to write a linear equation describing the relationship between the length and width of the top of the table if the length is twice the width.

c Solve the simultaneous linear equations you wrote for parts **a** and **b** and find the dimensions of the top of the billiard table.

15 Explain why the pair of simultaneous linear equations $y = 4 - 3x$ and $6x + 2y = 8$ has an infinite number of solutions.

16 Consider this pair of simultaneous equations:

$7x - 6y = -14$ ①
$5x - 3y = -1$ ②

a Demonstrate that the same solution is obtained using each of the strategies described below.

Strategy A: Multiply equation ② by 2 to form equation ③ and then subtract equation ③ from equation ①.

Strategy B: Multiply equation ② by –2 to form equation ④ and then add equations ① and ④.

Strategy C: Multiply equation ① by 5 to form equation ⑤ and multiply equation ② by 7 to form equation ⑥. Subtract equation ⑥ from equation ⑤.

b Are there other strategies you could use? Explain.

17 Chloe is planning a party and obtains prices from two catering companies. Each caterer has a fixed price for delivery and set-up, and a cost per person for the food.
Angie's Catering: fixed price $200, $28 per person
Cool Food Club: fixed price $100, $32 per person
Chloe is unsure how many people she will invite.

a Write a linear equation to represent the cost of hiring Angie's Catering. Remember to define the two variables you are using.

b Write a linear equation to represent the cost of hiring Cool Food Club.

c Show the graphs of both linear equations on the same Cartesian plane.

d Which company is cheaper if catering for 18 people? Explain how you can see this from your graph.

e Which company is cheaper if catering for 36 people?

f How many people can be catered for so that the cost of using either company is the same? What is this cost?

g Write a summary advising Chloe on what her options are.

18 A linear graph has a y-intercept of –5. Another linear graph has a gradient of –3. The two linear graphs intersect at the point $(4, -6)$.

a Show the two graphs on the same Cartesian plane.

b Find the equation for each linear graph.

c Write the solution to the simultaneous linear equations.

d Use substitution to check that the solution satisfies both equations.

19 a Show that $x = 15$, $y = -5$ and $z = 5$ is a set of solutions to the simultaneous equations $x + y = 10$ and $x + z = 20$.

b Show that $x = 22$, $y = -12$ and $z = -2$ is a set of solutions to the simultaneous equations $x + y = 10$ and $x + z = 20$.

c Explain why we cannot solve the simultaneous equations $x + y = 10$ and $x + z = 20$ for one set of values of x, y and z.

20 a Show that the solution to the simultaneous equations $ax + by = c$ and $ax + qy = r$ is $x = \frac{br - cq}{ab - aq}$ and $y = \frac{c - r}{b - q}$.

b Solve the simultaneous equations $3x + 4y = 9$ and $3x + 6y = 7$ by substituting the values of a, b, c, q and r into the solution from part **a**.

c Show that the solution to the simultaneous equations $ax + by = c$ and $px + qy = r$ is $x = \frac{br - cq}{bp - aq}$ and $y = \frac{cp - ar}{bp - aq}$.

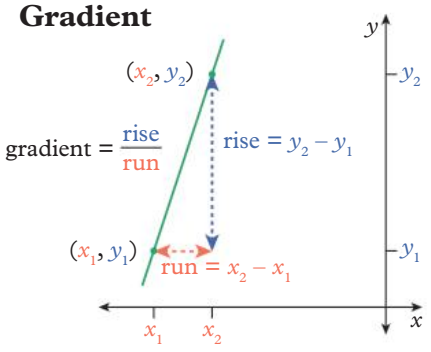
d Solve the simultaneous equations $3x + 4y = 9$ and $5x + 6y = 7$ by substituting the values of a, b, c, p, q and r into the solution from part **c**.

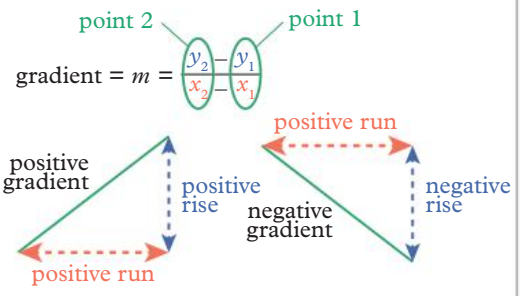
Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Solving simultaneous linear equations graphically
- Interactive skillsheet: Solving simultaneous linear equations by substitution
- Interactive skillsheet: Solving simultaneous linear equations by elimination
- Worksheet: Solving simultaneous linear equations
- Investigation: Drawing a circumference around a triangle
- CAS instructions: Solving simultaneous equations
- Topic quiz: 3E

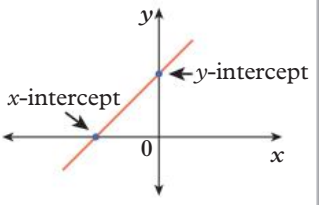
Chapter summary

Gradient





Intercepts



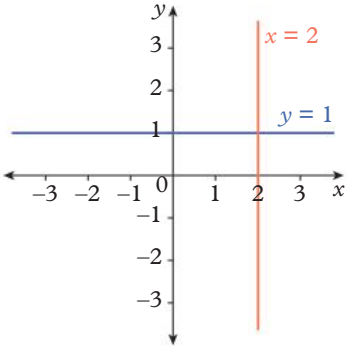
Gradient–intercept form

$$y = mx + c$$

m: gradient; c: y-intercept

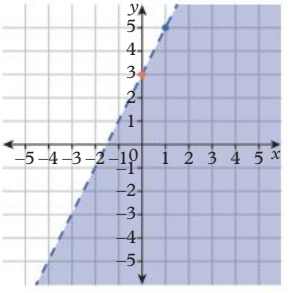
Horizontal and vertical lines

- Horizontal: $y = c$, gradient = 0
- Vertical: $x = c$, gradient is undefined

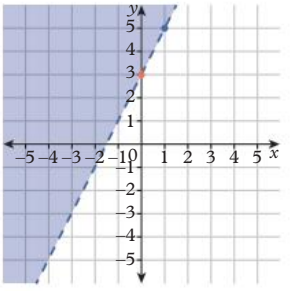


Linear inequality

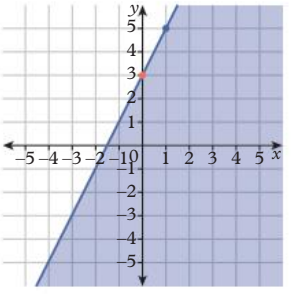
$y < 2x + 3$



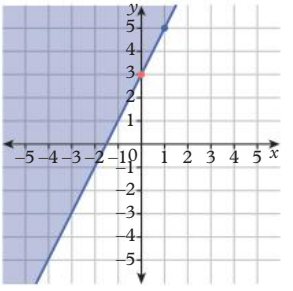
$y > 2x + 3$



$y \leq 2x + 3$



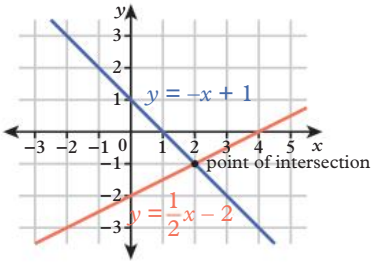
$y \geq 2x + 3$



Simultaneous linear equations

Solving graphically

- The coordinates for the point of intersection of the two graphs give the solution to the simultaneous linear equations



Solving by substitution

- Substitute an expression from one equation into the other equation to create a new equation containing only one variable
  For example:
  Substitute $y = 4 - 2x$ into
  $y + 11 = 3x$
  to get $4 - 2x + 11 = 3x$
- Solve for the remaining variable, then substitute the solution into one of the original equations to solve for the other variable.

Solving by elimination

- Multiply one or both of the equations by a constant so that the magnitudes of the coefficients for one of the variables are the same in both equations.
- Add or subtract the LHS and RHS of one equation from the other equation to eliminate one of the variables.

  For example, $2x - y = 4$ ①
  $-3x - 2y = -6$ ②

  $2 \times$ ① – ②:
  LHS = $2 \times (2x - y) - (-3x - 2y)$
  RHS = $2 \times (4) - (-6)$
  $7x = 14$
- Solve for the remaining variable, then substitute the solution into one of the original equations to solve for the other variable.

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

3A **1** The solution to $\frac{x-4}{2} = -3$ is:

A $x = 2$ **B** $x = -2$ **C** $x = 10$ **D** $x = -10$ **E** $x = -14$

3A **2** Which of these equations has the solution $x = -5$?

A $\frac{12}{x} = -2.4$ **B** $\frac{2x+3}{13} = 1$ **C** $3x - 7 = 8$ **D** $\frac{3}{2-x} = 1$ **E** $3x + 1 = 8(x + 6)$

3B **3** The y-intercept of the graph of $12x + 4y = 8$ is:

A 8 **B** −8 **C** 4 **D** −2 **E** 2

3B **4** The gradient of the line $12x + 4y = 8$ is:

A 3 **B** −3 **C** 12 **D** −12 **E** 4

3C **5** The equation for the linear graph with a gradient of $\frac{3}{4}$ and a y-intercept of $\frac{5}{8}$ is:

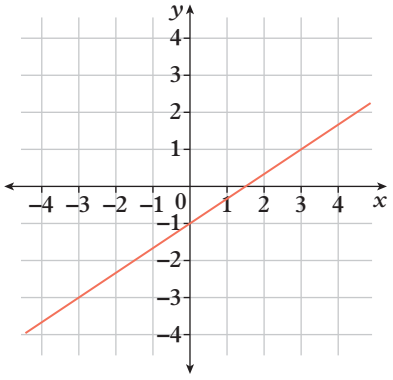
A $8x - 6y + 5 = 0$ **B** $8y = 6x - 5$ **C** $6x - 8y + 5 = 0$
D $6y = 8x + 5$ **E** $6x - 5y + 8 = 0$

3C **6** The gradient of the line passing through the points $(1, 8)$ and $(-3, -6)$ is:

A 1 **B** −1 **C** $\frac{2}{7}$ **D** $\frac{7}{2}$ **E** $-\frac{7}{2}$

3C **7** The equation of the linear graph below is:

A $y = \frac{2}{3}x + 1$ **B** $y = \frac{3}{2}x - 1$ **C** $y = -\frac{2}{3}x - 1$ **D** $y = \frac{3}{2}x + 1$ **E** $y = \frac{2}{3}x - 1$



3D **8** If $3x + 2 > 5x - 6$, which of the following could be a value for x?

A 10 **B** 8 **C** 6 **D** 4 **E** 2

3D **9** Which of the following value pairs could not be a solution to $3x + 2y < 11$?

A $(0, 5)$ **B** $(1, 2)$ **C** $(2, 3)$ **D** $(-1, 6)$ **E** $(-2, 5)$

3E **10** The pair of linear equations $y = x + 3$ and $y = -2x + 6$ intersect at which of these points?

A $(-3, 0)$ **B** $(3, 6)$ **C** $(1, 8)$ **D** $(1, 4)$ **E** $(0, 6)$

3E **11** The pair of linear equations $y = 3x + 1$ and $6x - 2y + 2 = 0$ have how many simultaneous solutions?

A none **B** one **C** two **D** three **E** many

3E **12** When $y = 3x - 4$ is substituted into the equation $x - 2y = 3$, the resulting equation is:

A $-5x - 8 = 3$ **B** $-5x + 8 = 3$ **C** $5x + 8 = 3$ **D** $5x - 8 = 3$ **E** $7x + 8 = 3$

3E **13** Consider the two equations:

$2x - y + 3 = 0$ ①
$4x - y + 5 = 0$ ②

If equation ② is subtracted from equation ①, the resulting equation will be:

A $-2x - 2 = 0$ **B** $-2x - 8 = 0$ **C** $2x + 2 = 0$ **D** $2x + 8 = 0$ **E** $2x - 2 = 0$

Short answer

3A **1** Solve each of these equations. Leave your answer as a fraction where appropriate.

a $3 - 2(4 - 5x) = -20$ **b** $4(3x + 5) = 3(5x - 3)$

3A **2** Solve each of these equations, leaving your answer as a fraction if appropriate.

a $\frac{4 + x}{3} = \frac{2x - 5}{3}$ **b** $\frac{5 - 3x}{2} = \frac{2x + 5}{3}$

3A **3** Rearrange each of these equations so that the coloured pronumeral in the brackets is the subject.

a $A = \frac{h(a + b)}{2}$ (b)

b $2(l + w)h$ (h)

3B **4** Sketch each of these linear equations using the gradient–intercept method.

a $y = -2x + 3$ **b** $x - 2y = 4$ **c** $5y - 10 = 15x$

3B **5** Sketch the graph of each of these equations using the x- and y-intercept method.

a $2x - 4y - 12 = 0$ **b** $x + 3y + 9 = 0$ **c** $7x + 3y + 21 = 0$

3B **6** Find the equation for the line passing through each of these pairs of points.

a $(0, -4)$ and $(5, 6)$ **b** $(3, 6)$ and $(-2, 6)$ **c** $(-3, 5)$ and $(-3, -3)$

3C **7** Find the equation for the line that has:

a a gradient of 2 and passes through $(1, 5)$

b a gradient of -2 and passes through $(1, 5)$.

3C **8** Find the equation of the line that passes through both the given points in each of the following pairs. Write your answers in the form $y = mx + c$.

a $(1, 3)$ and $(2, 4)$ **b** $(-1, 5)$ and $(3, -7)$ **c** $(-4, -4)$ and $(5, 5)$

3D **9** Solve each of these inequalities.

a $5 - 2x < 7$ **b** $\frac{2 - 3x}{5} > 2$

3D **10** Represent the following linear inequalities on the Cartesian plane with boundary lines and shading.

a $y + x \leq 3$ **b** $3y + 6x > 3$

c $x + 2y \leq 5 - y$ **d** $x + 7y \leq y + 2x$

3E **11** Use a graph to help you solve each of these pairs of simultaneous equations.

a $y = 4x - 2$
$y = -3x + 5$

b $2x - y + 5 = 0$
$3x - y = 1$

3E **12** Use the substitution method to solve each of these pairs of simultaneous equations.

a $3x + 4y = -1$
$y = x - 2$

b $y = 4x + 3$
$x - 2y = 8$

c $7x - y = 4$
$y = 4x + 2$

3E **13** Use the elimination method to solve each of these pairs of simultaneous equations.

a $2x + y = 8$
$4x - y = 4$

b $x + 5y = 13$
$x + 2y = 4$

c $2x - y = 7$
$3x + 5y = 4$

Analysis

1 For a hot food stall, Sophie buys and then sells sausage rolls and party pies. The sausage rolls cost 71 cents each and the party pies cost 26 cents each. Sophie sells the sausage rolls for \$1.20 each and the party pies for 50 cents each. Sophie has a budget of \$60 per day. Let s be the number of sausage rolls Sophie buys and p be the number of party pies Sophie buys.

a Write an inequality that describes how Sophie can spend her budget each day.

b If Sophie buys 50 sausage rolls, write an inequality that describes the number of party pies she can buy.

c If Sophie buys 150 party pies, write an inequality that describes the number of sausage rolls she can buy.

d Assume Sophie is able to sell all the sausage rolls and party pies she buys on a particular day. Let P be the profit Sophie will make that day.

Write an equation that describes the profit Sophie will make.

e On another day, Sophie made $50 profit.

Sketch the graph of the number of sausage rolls against the number of party pies Sophie could have bought and sold. Write the coordinates of the intercepts, correct to two decimal places. Hint: Put sausage rolls on the vertical axis.

f State the maximum number of party pies Sophie could have bought and sold on the day she made $50 profit.

g On the day Sophie made $50 profit, the price she paid for the sausage rolls and party pies was $59.90. Determine the number of sausage rolls and party pies Sophie bought and sold that day.

2 a A triangle has vertices $X(4, -4)$, $Y(-2, 2)$ and $Z(1, 5)$. Draw this triangle on a Cartesian plane.

b Write an equation for each of the line segments XY, YZ and ZX and limit the values of x for each equation with an appropriate inequality statement.

c Let the length of the side opposite vertex X be represented by x, the side opposite vertex Y be represented by y, and the side opposite Z be represented by z. Calculate the length of x, y and z. Leave your answers as exact values.

d Use the lengths from your answer to part **c** to demonstrate Pythagoras' theorem.

e Determine the coordinates of the midpoint, M, of ZX. Join M to vertex Y.

f Calculate the length of MY.

g The line segment MY divides $\triangle XYZ$ into two smaller triangles. Considering the side lengths and angles, describe the shape of these triangles.

h Draw the perpendicular height of the triangle from vertex Y to the base XZ, meeting the base at H.

The equation for the line segment YH is $y = \frac{x + 8}{3}$. Solve a pair of simultaneous equations to find the coordinates of H.

i Calculate the length of YH.

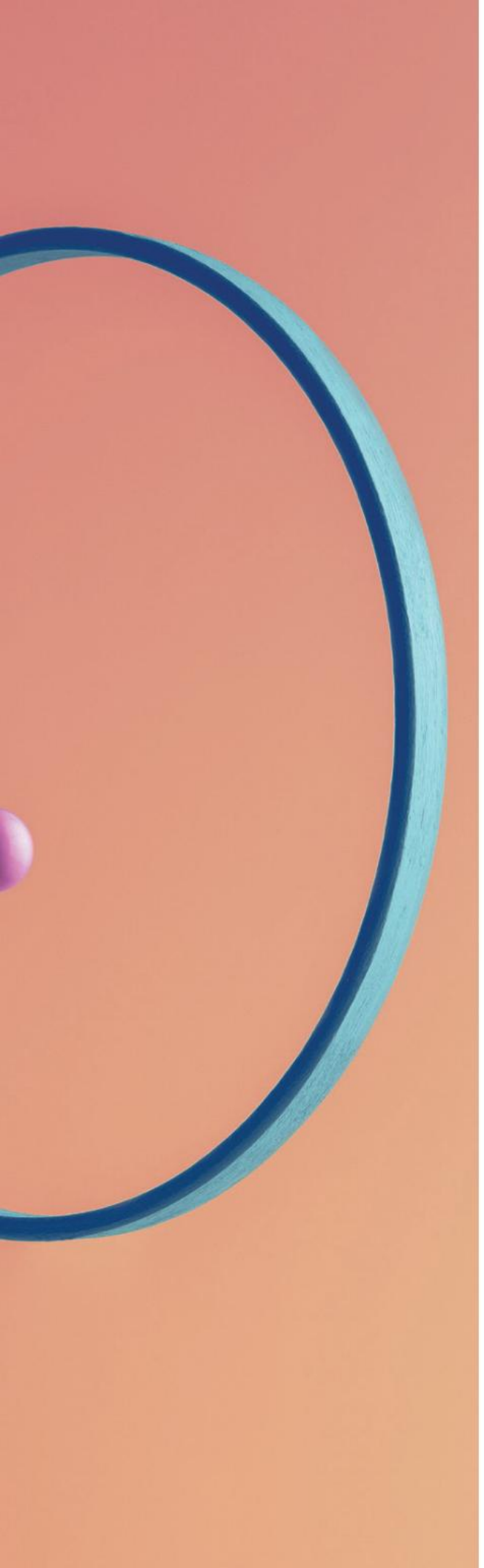
j Use two sets of different measurements to calculate the area of $\triangle XYZ$. Show all your working and give your answer correct to one decimal place.

k Show that the length of $YH = \frac{xz}{y}$.

l Explain how a semi-circle can be drawn to enclose $\triangle XYZ$. What does this show about the angle in a semi-circle?

4

Non-linear relationships



Index

4A Solving quadratic equations
4B The quadratic formula
4C Graphing parabolas using intercepts
4D Graphing parabolas using transformations
4E Graphing circles
4F Graphing exponential relationships
4G Rates of change

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Solving equations using inverse operations
- ✔ Factorising quadratic expressions
- ✔ Factorising the difference of two squares
- ✔ The Cartesian plane
- ✔ Non-linear relationships

Curriculum links

- Recognise the connection between algebraic and graphical representations of exponential relations and solve related exponential equations, using digital tools where appropriate (AC9M10A03)
- Experiment with functions and relations using digital tools, making and testing conjectures and generalising emerging patterns (AC9M10A05)

Optional content

- Algebraic representations of quadratic functions of the form $f(x) = ax^2 + bx + c$ where a, b and c are non-zero integers, and their transformation to the form $f(x) = a(x + h)^2 + k$ where h and k are non-zero rational numbers, and the solution of related equations
- The effect of increasingly small changes in the value of variables on the average rate of change and in relation to limiting values

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Materials

- ✔ Calculator
- ✔ Graph paper

4A Solving quadratic equations

Learning intentions

By the end of this section you will be able to ...

- ✔ solve quadratic equations using the Null Factor Law
- ✔ solve quadratic equations with irrational solutions
- ✔ solve quadratic equations by first rearranging the equation.

Inter-year links

Year 7	6C Terms, expressions and equations
Year 8	6A Equations
Year 9	4A Solving quadratic equations

Quadratic equations

- The general form of a quadratic equation is:
  $ax^2 + bx + c = 0$ where a, b and c are constants.
- The factor form of a quadratic equation is: $(x - p)(x - q) = 0$ where p and q are constants.

The Null Factor Law

- The **Null Factor Law** states that, if the product of two factors is 0, one or both of the factors must be 0; that is:
  If $p \times q = 0$, then $p = 0$ or $q = 0$, or both $p = 0$ and $q = 0$.
- Some quadratic equations can be solved by making one side equal to zero and factorising the non-zero expression on the other side.
- When we apply the Null Factor Law to quadratic equations in factor form, $(x - p)(x - q) = 0$, we get:
  $x - p = 0$ or $x - q = 0$ then $x = p$ or $x = q$.
- Solutions can be written in a shorthand way using a ± ('plus or minus') sign. For example:
  $x = -2$ or $x = 2$ can be written as $x = \pm 2$.

Example 4A.1 Solving quadratic equations using the Null Factor Law

Solve each of these equations.

a $(x + 5)(x - 4) = 0$ **b** $x^2 - 4x - 12 = 0$ **c** $x^2 - 16 = 0$ **d** $5x^2 = -35x$

THINK

a 1 Check that the quadratic equation is in factor form and one side of the equation is equal to 0.
2 Apply the Null Factor Law by equating each factor to 0.
3 Solve each linear equation.

b 1 Check that one side of the quadratic equation is equal to 0.
2 Factorise the quadratic equation.
3 Apply the Null Factor Law.
4 Solve each linear equation.

WRITE

a $(x + 5)(x - 4) = 0$

$x + 5 = 0$ or $x - 4 = 0$

$x = -5$ or $x = 4$

b $x^2 - 4x - 12 = 0$

$(x - 6)(x + 2) = 0$
$x - 6 = 0$ or $x + 2 = 0$
$x = 6$ or $x = -2$

c 1 Check that one side of the equation is equal to 0.
2 Factorise the quadratic expression using the 'difference of two squares' rule.
3 Apply the Null Factor Law.
4 Solve each linear equation. Use ± to write the solutions.

d 1 Rearrange the equation so one side of the equation is equal to 0.
2 Factorise the quadratic expression by taking out a common factor.
3 Apply the Null Factor Law.
4 Solve each linear equation.

c $x^2 - 16 = 0$
$x^2 - 4^2 = 0$
$(x + 4)(x - 4) = 0$
$x + 4 = 0$ or $x - 4 = 0$
$x = -4$ or $x = 4$
$x = \pm 4$

d $5x^2 = -35x$
$5x^2 + 35x = 0$
$5x(x + 7) = 0$

$5x = 0$ or $x + 7 = 0$
$x = 0$ or $x = -7$

Example 4A.2 Solving quadratic equations with irrational solutions

Solve each of these equations. Write the solutions as exact values.
a $x^2 - 5 = 0$ **b** $-2x^2 + 24 = 0$

THINK

a 1 Check if the quadratic expression is in factor form and that the other side of the equation is equal to 0.
2 Factorise the quadratic expression using the 'difference of two squares' rule.
3 Apply the Null Factor Law.
4 Solve each linear equation, write the solutions using ±.

b 1 Check that one side of the quadratic equation is equal to 0.
2 Take out a common factor (in this case, –2).
3 Divide both sides of the equation by –2.
4 Factorise the quadratic expression using the 'difference of two squares' rule.
5 Simplify the surd. $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$
6 Apply the Null Factor Law.
7 Solve each linear equation, write the solutions using ±.

WRITE

a $x^2 - 5 = 0$

$x^2 - (\sqrt{5})^2 = 0$

$(x + \sqrt{5})(x - \sqrt{5}) = 0$
$x + \sqrt{5} = 0$ or $x - \sqrt{5} = 0$
$x = -\sqrt{5}$ or $x = \sqrt{5}$
$x = \pm\sqrt{5}$

b $-2x^2 + 24 = 0$

$-2(x^2 - 12) = 0$
$x^2 - 12 = 0$
$x^2 - (\sqrt{12})^2 = 0$
$(x + \sqrt{12})(x - \sqrt{12}) = 0$
$(x + 2\sqrt{3})(x - 2\sqrt{3}) = 0$
$x + 2\sqrt{3} = 0$ or $x - 2\sqrt{3} = 0$
$x = -2\sqrt{3}$ or $x = 2\sqrt{3}$
$x = \pm 2\sqrt{3}$

Example 4A.3 Solving quadratic equations by first rearranging the equation

Solve $2x^2 + 6x + 3 = x^2 + 2x + 15$ by first rearranging the equation.

THINK

1 Apply inverse operations so that one side of the equation is equal to 0. Subtract terms on the right hand side x^2, $2x$ and 15, so the coefficient of the leading term would be positive after the rearrangement.

2 Simplify the equation by combining like terms.

3 Factorise the quadratic expression.

4 Apply the Null Factor Law.

5 Solve each linear equation.

WRITE

$$2x^2 + 6x + 3 = x^2 + 2x + 15$$

$$2x^2 + 6x + 3 - x^2 - 2x - 15 = 0$$

$$x^2 + 4x - 12 = 0$$

$$(x + 6)(x - 2) = 0$$

$$x + 6 = 0 \text{ or } x - 2 = 0$$

$$x = -6 \text{ or } x = 2$$

Helpful hints

✔ Watch for changes in sign when solving quadratic equations for x.

$$(x + 6)(x - 2) = 0$$
$$x + 6 = 0 \text{ or } x - 2 = 0$$
$$x = -6 \text{ or } x = +2$$

✔ Although it is possible to add, subtract and multiply by x, you must not divide both sides of an equation by x, in the case of $x = 0$.

$$\frac{1}{0} = \text{undefined}$$

✔ The Null Factor Law can only be used to solve equations if one side of the equation is equal to zero. You cannot use the rule if the equation is equal to any other number.
For example, if $x(x - 2) = 1$, it does not imply that $x = 1$ or $x - 2 = 1$.
But, if $x(x - 2) = 0$, two possible solutions are $x = 0$ or $x - 2 = 0$.

ANS p572

Exercise 4A Solving quadratic equations

△ 1(1st column), 2–6, 7(1st column), 8–10, 11(1st column), 12 ■ 1(2nd column), 2, 5, 6, 7(2nd column), 8, 9, 11(1st column), 13, 14, 16(a–f) ⬠ 2(2nd column), 6, 7(i–l), 8, 9, 13, 15–17

4A.1

1 Solve each of the following equations. Show all steps of your working.

a $(x + 2)(x - 4) = 0$ **b** $(x + 7)(x + 1) = 0$ **c** $(x - 3)(x - 6) = 0$

d $(x + 9)(x - 9) = 0$ **e** $x(x - 5) = 0$ **f** $x(2x - 1) = 0$

g $2x(x + 1) = 0$ **h** $7x(2x + 4) = 0$ **i** $\frac{1}{2}x(4x - 3) = 0$

j $(x + \sqrt{2})(x - \sqrt{2}) = 0$ **k** $(x - \sqrt{5})(x + \sqrt{5}) = 0$ **l** $(x + 4\sqrt{3})(x - 4\sqrt{3}) = 0$

m $(2x + 3)(x - 4) = 0$ **n** $(3x - 1)(3x + 1) = 0$ **o** $(7x + 2)(4x - 5) = 0$

2 Solve each of these equations.

a $x^2 + 7x + 12 = 0$ **b** $x^2 + 3x - 10 = 0$

c $x^2 - 9x + 8 = 0$ **d** $x^2 - 25 = 0$

e $x^2 - 81 = 0$ **f** $4 - x^2 = 0$

g $x^2 - 3x = 0$ **h** $x^2 + x = 0$

i $2x^2 + 8x = 0$ **j** $10x^2 - 4x = 0$

k $3x^2 = 18x$ **l** $7x^2 = -70x$

3 Use substitution to check that your solutions to the first four quadratic equations in question **2** are correct.

4 Consider $3(x + 6)(x - 8) = 0$.

a Why is this equivalent to $(x + 6)(x - 8) = 0$?

b Solve $(x + 6)(x - 8) = 0$. Then write the solutions for $3(x + 6)(x - 8) = 0$.

c Use substitution to check that you have the correct solutions.

5 Solve each of these quadratic equations.

a $2(x + 5)(x - 6) = 0$ **b** $7(x - 2)(x - 4) = 0$ **c** $-5(x + 8)(x + 1) = 0$

d $-4(x + 9)(x - 3) = 0$ **e** $-11(x - 4)(x + 4) = 0$ **f** $-x(x + 7) = 0$

g $-6x(x - 2) = 0$ **h** $-2x(3x + 5) = 0$ **i** $-8(2x - 3)(x + 10) = 0$

6 Solve each of these quadratic equations. Hint: Take out a common factor first.

a $2x^2 + 6x + 4 = 0$ **b** $3x^2 + 6x - 45 = 0$ **c** $6x^2 - 30x + 24 = 0$

d $-5x^2 - 5x + 30 = 0$ **e** $-4x^2 + 16x + 48 = 0$ **f** $-x^2 - x + 56 = 0$

g $11x^2 - 44 = 0$ **h** $-7x^2 + 63 = 0$ **i** $-16x^2 + 8x = 0$

4A.2 **7** Solve each of these equations. Write the solutions as exact values.

a $x^2 - 3 = 0$ **b** $x^2 - 11 = 0$

c $2x^2 - 14 = 0$ **d** $x^2 - 8 = 0$

e $x^2 - 27 = 0$ **f** $3x^2 - 6 = 0$

g $-x^2 + 5 = 0$ **h** $-4x^2 + 24 = 0$

i $-2x^2 + 64 = 0$ **j** $5x^2 = 95$

k $8x^2 = 160$ **l** $156 - 3x^2 = 0$

4A.3 **8** Solve each of these quadratic equations, leave your answer as exact values. Hint: First rearrange the equation into the form $ax^2 + bx + c = 0$.

a $x^2 + 4x = 12$ **b** $x^2 + 4 = 5x$

c $6x^2 = 54$ **d** $x^2 - 3x - 20 = 20$

e $x^2 - 6x = 3x$ **f** $x^2 + 1 = 4x$

g $5x - x^2 = x$ **h** $x^2 + 21 = 11x + 3$

i $2x^2 - 3x = x^2 + 18$ **j** $x(x + 4) = 21$

k $(x - 5)^2 = 1$ **l** $x^2 + 3x + 10 = 1 - 3x$

9 Explain why you can divide both sides of $3(x - 2) = 0$ by 3 but you cannot divide both sides of $x(x - 2) = 0$ by x.

10 How many real solutions does each of these quadratic equations have?

a $(x - 3)(x - 5) = 0$ **b** $(x + 2)(x + 2) = 0$

c $(x - 4)(x + 4) = 0$ **d** $(x - 7)^2 = 0$

e $x(x - 6) = 0$ **f** $x^2 + 9 = 0$

11 Find solution/s for each of these equations, if real solutions exist. Hint: Some of these equations cannot be solved.

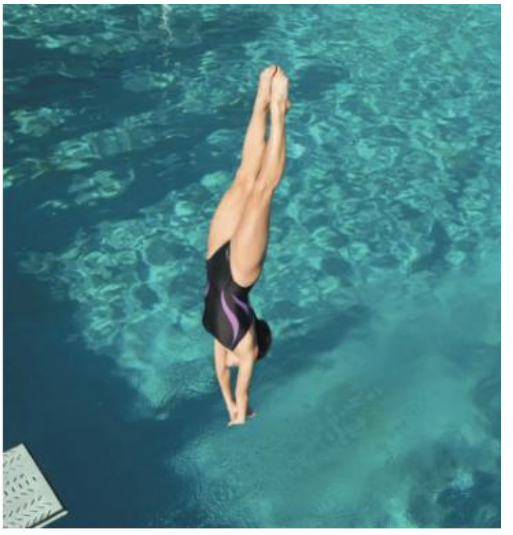
a $x^2 - 9 = 0$ **b** $x^2 + 4x - 77 = 0$

c $x^2 - 8x + 16 = 0$ **d** $x^2 + 4 = 0$

e $x^2 + 4x = 0$ **f** $x^2 - 6x - 1 = 0$

PROBLEM SOLVING AND REASONING

12 Nutsa dives with her arms outstretched from a platform into a diving pool. The height of her fingertips above the pool surface can be represented by the quadratic relationship $h = -2t^2 + 4t + 6$, where h is the height of her fingertips above the water, in metres, after t seconds.



a What is the height of Nutsa's fingertips above the water after:

i 1 second?

ii 2 seconds?

b What is the height of Nutsa's fingertips when she starts the dive?

c How long does the dive take? Hint: How long does it take for Nutsa's fingertips to hit the surface of the water?

d Explain why there is only one time value for your answer to part **c** even though you have solved a quadratic equation that has two solutions.

13 Oliver buys rice to serve with stir-fried meals over the coming weeks. As time passes, the amount of rice Oliver has left can be represented by the quadratic relationship $y = x^2 - 10x + 25$, where y is the amount left, in kilograms, after x weeks.

a How much rice did Oliver buy?

b How much rice is left after 2 weeks?

c How long does it take for all of the rice to be used?

d Did Oliver use the same amount of rice each week? Explain.

14 The area of a rectangular vegetable plot is 70 m^2. The length is 3 m longer than the width.

a Write a quadratic equation to represent this. Hint: Let x represent the width of the vegetable plot, in metres.

b Solve the quadratic equation.

c Write the dimensions of the vegetable plot.

15 The width of a mobile phone screen is 5 cm less than its length. Write an equation to represent the situation if the area of the screen is 66 cm^2. Then solve your equation to find the dimensions of the screen.

CHALLENGE

16 Solve each of these quadratic equations by completing the square.

a $x^2 + 2x - 6 = 0$

b $x^2 - 4x + 2 = 0$

c $x^2 + 8x + 11 = 0$

d $x^2 - 12x + 28 = 0$

e $x^2 + 20x + 25 = 0$

f $x^2 - x - 1 = 0$

g $3x^2 + 18x + 21 = 0$

h $4x^2 + 8x - 16 = 0$

i $-x^2 + 4x - 1 = 0$

j $-6x^2 - 48x - 36 = 0$

k $2x^2 - 12x - 3 = 0$

l $5x^2 + 5x - 2 = 0$

17 Solve each of these quadratic equations.

a $2x^2 + 13x + 15 = 0$

b $5x^2 - 22x + 8 = 0$

c $6x^2 + x - 2 = 0$

d $4x^2 + 25x - 21 = 0$

e $3x^2 + 2x = 1$

f $7x^2 = 11x + 6$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Solving quadratic equations

Investigation
Beware of irreversible operations

Topic quiz
4A

4B The quadratic formula

Learning intentions

By the end of this section you will be able to ...

- ✔ identify the number of solutions to a quadratic equation using the discriminant
- ✔ solve quadratic equations using the quadratic formula.

Inter-year links
Year 7 6E Substitution
Year 8 5B Substitution

The quadratic formula

- The **quadratic formula** can be used to solve equations of the general form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.
- The solutions for $ax^2 + bx + c = 0$ are:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

This can be written more simply as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant

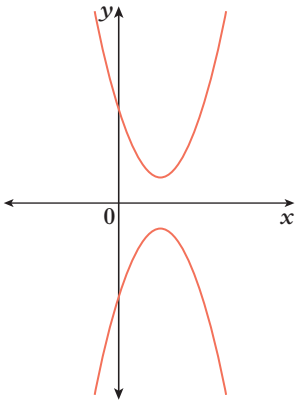
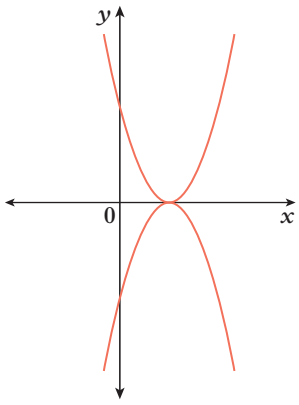
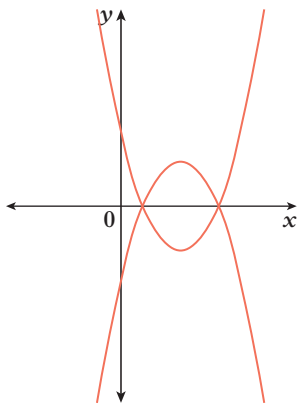
- The discriminant is $\Delta = b^2 - 4ac$.

Evaluating this expression after substituting values for a, b and c allows us to identify the number of real solutions for the quadratic equation.

→ If $\Delta < 0$, there are no real solutions.

→ If $\Delta = 0$, there is one real solution. It is $x = \frac{-b}{2a}$ and the quadratic is a perfect square.

→ If $\Delta > 0$, there are two real solutions. They are $x = \frac{-b \pm \sqrt{\Delta}}{2a}$.

Discriminant	$\Delta < 0$ (negative)	$\Delta = 0$ (zero)	$\Delta > 0$ (positive)
Number of solutions	No real solutions	One real solution	Two real solutions
Graph			

Example 4B.1 Identifying the number of solutions to a quadratic equation

Using the discriminant, identify the number of solutions to each of these quadratic equations.

a $3x^2 + 7x + 2 = 0$

b $x^2 + 16 = 8x$

c $-2x^2 + 3x - 5 = 0$

THINK

1. If necessary, rearrange each equation so that the right-hand side is 0.
2. Identify a, b and c.
3. Write the discriminant formula and substitute values for a, b and c into it. Simplify the result.
4. Identify the number of solutions.
5. Use digital tools to verify the number of solutions by graphing each quadratic relationship.

WRITE

a $3x^2 + 7x + 2 = 0$

$a = 3, b = 7, c = 2$

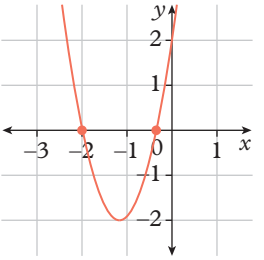
$\Delta = b^2 - 4ac$

$= 7^2 - 4 \times 3 \times 2$

$= 49 - 24$

$= 25$

Since $\Delta > 0$ (is positive), the equation has two solutions.



b $x^2 + 16 = 8x$

$x^2 - 8x + 16 = 0$

$a = 1, b = -8, c = 16$

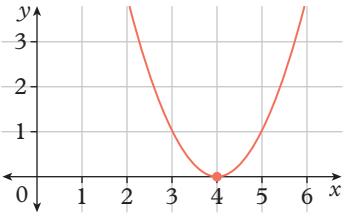
$\Delta = b^2 - 4ac$

$= (-8)^2 - 4 \times 1 \times 16$

$= 64 - 64$

$= 0$

Since $\Delta = 0$, the equation has one solution.



c $-2x^2 + 3x - 5 = 0$

$a = -2, b = 3, c = -5$

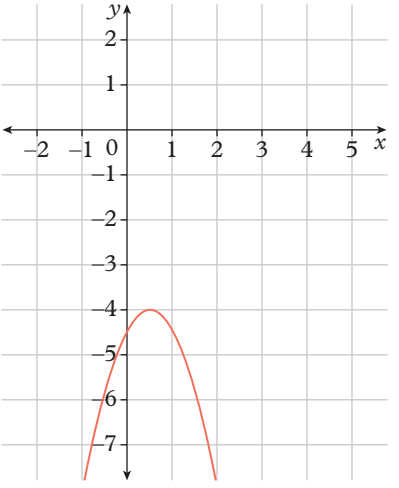
$\Delta = b^2 - 4ac$

$= 3^2 - 4 \times (-2) \times (-5)$

$= 9 - 40$

$= -31$

Since $\Delta < 0$, the equation has no solutions.



Example 4B.2 Using the quadratic formula to solve quadratic equations

Use the quadratic formula to solve each of these equations. If a solution exists, write the solutions as:

i exact values

ii approximate values, correct to two decimal places.

a $2x^2 - 5x + 1 = 0$

b $9x^2 + 12x + 4 = 0$

c $4x^2 = 3x - 1$

THINK

1. If necessary, rearrange the equation so that one side of the equation equals 0.
2. Identify a, b and c.
3. Determine the discriminant and identify the number of solutions.
4. Write the quadratic formula and substitute values for a, b and c into it. Simplify the result.
5. Write the solution as exact values.
6. Use a calculator to help you rewrite the values correct to two decimal places.

WRITE

a $2x^2 - 5x + 1 = 0$

$a = 2, b = -5, c = 1$

$\Delta = b^2 - 4ac$

$= (-5)^2 - 4 \times 2 \times 1$

$= 25 - 8$

$= 17$

$\Delta > 0$. The equation has two solutions.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-(-5) \pm \sqrt{17}}{2(2)}$

$= \dfrac{5 \pm \sqrt{17}}{4}$

i $x = \dfrac{5 - \sqrt{17}}{4}$ or $\dfrac{5 + \sqrt{17}}{4}$

ii $x \approx 0.22$ or 2.28

b $9x^2 + 12x + 4 = 0$

$a = 9, b = 12, c = 4$

$\Delta = b^2 - 4ac$

$= (12)^2 - 4 \times 9 \times 4$

$= 144 - 144$

$= 0$

$\Delta < 0$. The equation has one solution.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-12 \pm \sqrt{0}}{2(9)}$

$= \dfrac{-12}{18}$

$= -\dfrac{2}{3}$

i $x = -\dfrac{2}{3}$

ii $x \approx -0.67$

c $4x^2 = 3x - 1$

$4x^2 - 3x + 1 = 0$

$a = 4, b = -3, c = 1$

$\Delta = b^2 - 4ac$

$= (-3)^2 - 4 \times 4 \times 1$

$= 9 - 16$

$= -7$

$\Delta < 0$. The equation has no solutions.

Helpful hints

- ✔ If the discriminant is positive, don't forget to evaluate both possible solutions using the quadratic formula.

  Remember that the quadratic formula is equal to $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so:

  $$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- ✔ Be careful when identifying the values of a, b and c. Make sure one side of the equation equals zero before identifying a, b and c.
- ✔ Be careful when substituting the values of a, b and c into the formula. You need to include the sign of each number.

  For example, if $-2x^2 - 6x - 4 = 0$, then $a = -2$, $b = -6$ and $c = -4$.

  So, the solutions are $x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(-2)(-4)}}{2(-2)}$.
- ✔ If you have already calculated the discriminant of a quadratic equation, you can substitute it into the quadratic formula.

  For example, $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$.

ANS p573

Exercise 4B The quadratic formula

▲ 1, 2, 3–4(1st, 2nd columns), 5, 6, 8, 10, 12 ■ 2, 3–4(2nd columns), 5–9, 14, 15 ⬟ 3–4(2nd columns), 5–7, 11, 13, 14, 16

UNDERSTANDING AND FLUENCY

4B.1 **1** Using the discriminant, identify the number of solutions to each of these quadratic equations.

a $x^2 + 5x + 7 = 0$ **b** $3x^2 - 8x - 4 = 0$ **c** $x^2 - 9 = 0$

d $2x^2 + 3x + 4 = 0$ **e** $12x - x^2 = 36$ **f** $6x^2 - 5 = 7x$

2 Complete the following for each quadratic equation below.

i Calculate the discriminant.

ii Identify the number of solutions.

iii Solve, by first factorising and then using the Null Factor Law.

iv Check whether your answer for part **iii** confirms your result for part **ii** each time.

a $x^2 - 2x - 15 = 0$ **b** $x^2 + 4x + 4 = 0$ **c** $x^2 - 49 = 0$

d $3x^2 - 18x = 0$ **e** $x^2 - 6x + 7 = 0$ **f** $x^2 + 2x + 3 = 0$

4B.2 **3** Use the quadratic formula to solve each of these equations. If a solution exists, write the solutions as:

i exact values

ii approximate values, correct to two decimal places.

a $x^2 - 7x + 5 = 0$ **b** $3x^2 - x - 1 = 0$

c $2x^2 + x - 4 = 0$ **d** $-3x^2 - x + 9 = 0$

e $-6x^2 - 5x + 3 = 0$ **f** $x^2 - 2x - 1 = 0$

g $x^2 + 8x + 4 = 0$ **h** $4x^2 - 12x + 1 = 0$

i $-x^2 + 4x + 2 = 0$ **j** $5x^2 + 10x - 3 = 0$

k $2x^2 = 11 - 3x$ **l** $x^2 + 2x = 7$

4 Use the quadratic formula to find any solutions to each of the following quadratic equations. If solutions exist, write them as exact values.

a $x^2 + 3x = 1$ **b** $x^2 + 6x + 12 = 0$ **c** $\frac{1}{2}x^2 + 3x + 2 = 0$

d $x^2 + 3x + 5 = 0$ **e** $2x^2 + 5x + 1 = 0$ **f** $x^2 + 16 = 8x$

g $x^2 + 2x = 4$ **h** $x^2 = 3 - 7x$ **i** $10 = x^2 + 8x$

j $x^2 - 5 = x$ **k** $x^2 + 5 = x$ **l** $x^2 + \sqrt{5}x = 1$

5 Use the quadratic formula to find the solutions to each of the following quadratic equations.

a $m^2 + 3m - 7 = 0$ **b** $u^2 + 4u = 1$ **c** $r^2 + 9r + 1 = 0$

d $y^2 - 3 = 2y$ **e** $3h^2 + 7h = 1$ **f** $w^2 = 5 + 6w$

6 Solve $\frac{1}{x} = x - 3$ by first multiplying both sides by x to create a quadratic equation.

7 Copy and complete the lines of working below to derive the quadratic formula. You will complete the square to factorise the left-hand side and then use the Null Factor Law to find the solutions for $ax^2 + bx + c = 0$ (where $a \neq 0$).

THINK	**WRITE**
1 Write the quadratic equation.	$ax^2 + bx + c = 0$
2 Factorise by taking out the coefficient of x^2.	$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$
3 Divide both sides by a.	$x^2 + \frac{b}{a}x + \frac{\square}{\square} = 0$
4 Add a new third term to complete the square and compensate by subtracting the same term.	$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{\square} = 0$
5 Form a perfect square using the first three terms and simplify the fourth term.	$\left(x + \frac{b}{2a}\right)^2 - \frac{\square}{4a^2} + \frac{c}{a} = 0$
6 Group the last two terms together.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0$
7 Combine the last two terms using a common denominator.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\square - 4ac}{4a^2}\right) = 0$
8 Write the left-hand side as the difference of two squares.	$\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 = 0$
9 Simplify the second square.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0$
10 Factorise the left-hand side using the 'difference of two squares' rule.	$\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \square\right) = 0$
11 Use the Null Factor Law.	$x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \square = 0$
12 Solve each linear equation.	$x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = -\frac{b}{2a} + \square$
13 Simplify each solution.	$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

The quadratic formula for finding the solutions to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

8 Consider the quadratic equation $x^2 - 5x + 6 = 0$.

a Solve the equation by factorising the left-hand side and using the Null Factor Law.

b Now solve the equation again, using the quadratic formula.

c Which method was quicker or easier to use?

d The solutions to any quadratic equation can be found using the quadratic formula. Why might you try using the method described in part **a** first?

PROBLEM SOLVING AND REASONING

9 The profit, p (in dollars), made from selling n soft toys is given by the relationship $p = -0.1n^2 + 17n + 200$.

a What is the profit when 30 toys are sold?

b What is the smallest number of toys that need to be sold to make a profit of at least $850?

10 A rectangular paddock has an area of 5000 m^2; the length is 20 m greater than the width. Solve a quadratic equation to find the dimensions of the paddock, correct to one decimal place.

11 The perimeter of a framed painting is 100 cm. Within the 2 cm frame, the area of the painting that can be viewed is 300 cm^2. Find the dimensions of the framed painting, correct to the nearest millimetre.

12 When answering each of these, consider $\sqrt{b^2 - 4ac}$ in the quadratic formula.

a Explain why there are no real solutions to a quadratic equation when the discriminant is negative.

b Explain why there is only one real solution when the discriminant is zero.

c Explain why there are two real solutions when the discriminant is positive.

13 The discriminant of a quadratic equation can be used to find the number of solutions, but it can also be used to find the nature of the solutions, if they exist. If a quadratic equation has integer coefficients and the discriminant is a square number, then the solutions are rational (that is, whole numbers or fractions).

a Consider each of the following equations with integer coefficients. Find the discriminant and use it to determine whether the solutions are rational or irrational.

i $0 = x^2 + 5x - 2$

ii $0 = 2x^2 + 3x - 2$

iii $0 = x^2 - 7x + 10$

iv $0 = 5x^2 + 11x + 4$

v $0 = 3x^2 - x - 4$

b Explain with reference to the quadratic formula why, if the discriminant is a square number, the solutions are guaranteed to be rational, assuming all coefficients are integers.

14 a Find the coordinates of the x-intercepts, if they exist, of the graphs of each of the following equations. You do not need to sketch the graph.

i $y = x^2 + 7x + 10$

ii $y = x^2 - 6x + 8$

iii $y = x^2 + 3x + 5$

iv $y = x^2 - 8x + 16$

v $y = -2x^2 + 6x - 7$

b Explain how the discriminant of a quadratic relationship relates to the number of x-intercepts of its graph.

CHALLENGE

15 a Find the value of k so that $kx^2 + 6x + 3 = 0$ has only one real solution.

b For what values of k does $kx^2 + 6x + 3 = 0$ have:

i two real solutions

ii no real solutions?

16 For each of these equations, find the value of k that will give the number of real solutions shown in square brackets.

a $x^2 + 2x + k = 0$ [one real solution]

b $2x^2 - 4x + k = 0$ [two real solutions]

c $kx^2 + 6x - 1 = 0$ [no real solutions]

d $kx^2 - 3x + 1 = 0$ [one real solution]

e $3x^2 + kx + 3 = 0$ [two real solutions]

f $x^2 + kx + 25 = 0$ [no real solutions]

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
The quadratic formula

Worksheet
Solving quadratic equations

Topic quiz
4B

4C Graphing parabolas using intercepts

Learning intentions

By the end of this section you will be able to ...

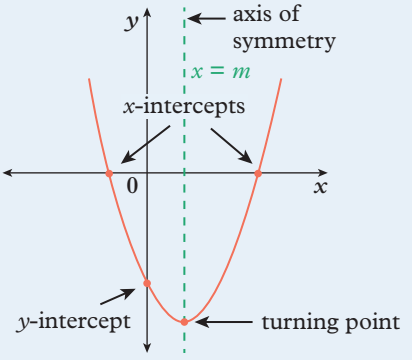
- ✔ find the x- and y-intercepts of a quadratic relationship
- ✔ find the coordinates of the turning point for a quadratic relationship with two x-intercepts
- ✔ graph parabolas using the intercepts and turning point.

Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution
Year 9	4C Sketching parabolas using intercepts

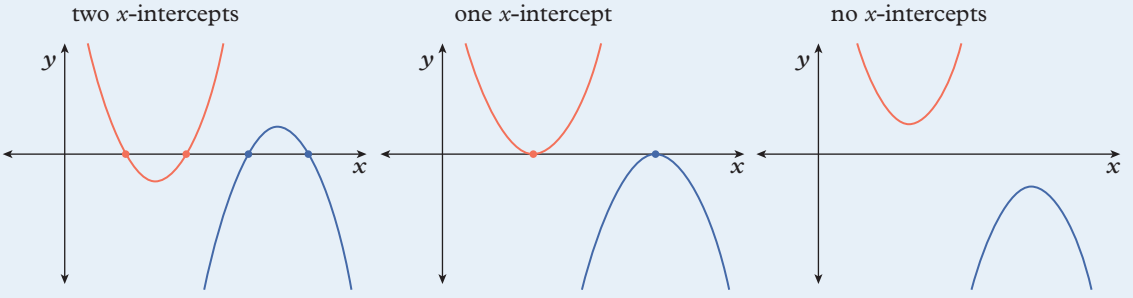
Parabolas

- The graph of a quadratic relationship is called a **parabola**.
- The key features of a parabola include:
  - → x- and y-intercepts (parabolas can have two, one or no x-intercepts)
  - → a **turning point** or vertex
  - → an **axis of symmetry**, $x = m$, where m is the x-coordinate of the turning point.

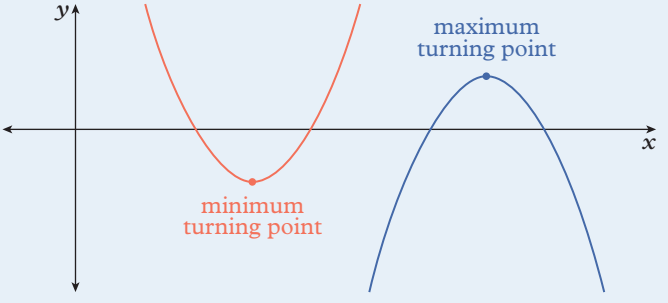


Sketching parabolas

- One way of sketching a parabola is to use the x- and y-intercepts.
  - → The y-intercept is found by substituting $x = 0$ into the equation and simplifying.
  - → The x-intercept(s) are found by substituting $y = 0$ into the equation and solving for x. A parabola can have two, one or no x-intercepts.

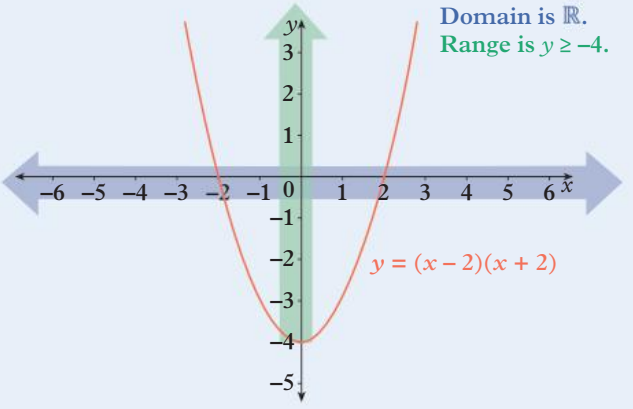


- A **minimum turning point** is the turning point of a graph that opens upwards or is 'upright'.
- A **maximum turning point** is the turning point of a graph that opens downwards or is 'inverted'.
- Turning points can be identified using the axis of symmetry or the halfway point between the x-intercepts (or any two x values with the same y value).



Domain and range

- The **domain** of a graph is the complete set of possible x values that output real y values.
  For example, the domain of the parabola $y = (x - 2)(x + 2)$ is all real x values, which can be represented as $\mathbb{R}$.
- The **range** is the y values that would result from substituting the complete set of possible x values.
  For example, the range of the parabola $y = (x - 2)(x + 2)$ is all real y values greater than or equal to –4, which can be represented as $y \geq -4$.



Example 4C.1 Finding the coordinates of the x- and y-intercepts

For each quadratic relationship below, find the coordinates of the:

i x-intercepts **ii** y-intercept.

a $y = x^2 - 6x$ **b** $y = x^2 + 7x + 10$

THINK

a i 1 To find the x-coordinates of the x-intercepts, substitute $y = 0$ into the quadratic relationship.

2 Factorise the quadratic expression.

3 Solve the equation using the Null Factor Law.

ii To find the y-coordinate of the y-intercept, substitute $x = 0$ into the quadratic relationship and solve for y.

b i 1 To find the x-coordinates of the x-intercepts, substitute $y = 0$ into the quadratic relationship.

2 Factorise the quadratic equation.

3 Solve the equation using the Null Factor Law.

ii To find the y-coordinate of the y-intercept, substitute $x = 0$ into the quadratic relationship and solve for y.

WRITE

a i $y = x^2 - 6x$

For the x-intercepts, $y = 0$:

$x^2 - 6x = 0$

$x(x - 6) = 0$

$x = 0$ or $(x - 6) = 0$

$x = 0$ or $x = 6$

The x-intercepts are $(0, 0)$ and $(6, 0)$.

ii For the y-intercept, $x = 0$:

$y = (0)^2 - 6(0)$

$= 0$

The y-intercept is $(0, 0)$.

b i $y = x^2 + 7x + 10$

For the x-intercepts, $y = 0$:

$x^2 + 7x + 10 = 0$

$(x + 2)(x + 5) = 0$

$x + 2 = 0$ or $x + 5 = 0$

$x = -2$ or $x = -5$

The x-intercepts are $(-2, 0)$ and $(-5, 0)$.

ii For the y-intercept, $x = 0$:

$y = (0)^2 + 7(0) + 10$

$= 10$

The y-intercept is $(0, 10)$.

Example 4C.2 Finding the coordinates of the turning point using the *x*-intercepts

Find the coordinates of the turning point for $y = x^2 - 6x$.

THINK	WRITE
1 Find the *x*-intercepts by substituting $y = 0$ into the relationship. Solve for x using the Null Factor Law.	$y = x^2 - 6x$ $0 = x(x - 6)$ $x = 0$ or $x = 6$ The *x*-intercepts are $(0, 0)$ and $(6, 0)$.
2 Calculate the *x*-coordinate of the turning point by finding the midpoint of the *x*-intercepts.	$x = \frac{0 + 6}{2}$ $= \frac{6}{2}$ $= 3$
3 Calculate the *y*-coordinate of the turning point by substituting the *x*-coordinate into the quadratic relationship.	When $x = 3$: $y = (3)^2 - 6(3)$ $= 9 - 18$ $= -9$
4 Write the coordinates of the turning point.	The turning point is $(3, -9)$.

Example 4C.3 Sketching a parabola using the intercepts

Sketch the graph of $y = -x^2 + 4x + 5$ by first finding the *x*- and *y*-intercepts. Label the turning point with its coordinates. List the domain and range of the graph.

THINK	WRITE
1 Find the *x*-intercepts by substituting $y = 0$ into the relationship and solving it for x. Factorise so that the Null Factor Law can be used.	For the *x*-intercepts, $y = 0$: $-x^2 + 4x + 5 = 0$ $-(x^2 - 4x - 5) = 0$ $x^2 - 4x - 5 = 0$ $(x + 1)(x - 5) = 0$ $x + 1 = 0$ or $x - 5 = 0$ $x = -1$ or $x = 5$ The *x*-intercepts are $(-1, 0)$ and $(5, 0)$.
2 Find the *y*-intercept by substituting $x = 0$ into the relationship and simplifying.	For the *y*-intercept, $x = 0$: $y = -(0)^2 + 4(0) + 5$ $= 5$ The *y*-intercept is $(0, 5)$.

3 Find the coordinates of the turning point. First, calculate the x-coordinate by finding the midpoint of the two x-intercepts, then substitute that value into the equation to find the corresponding y value. Use both values to write down the coordinates of the turning point.

For the turning point:

$$x = \frac{-1 + 5}{2}$$
$$= \frac{4}{2}$$
$$= 2$$

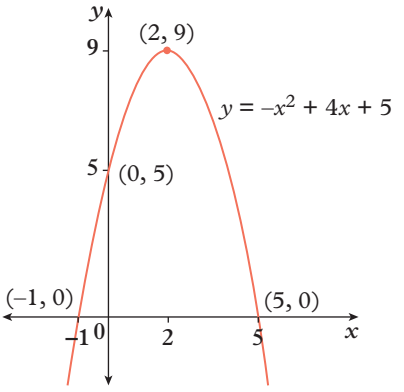
When $x = 2$:

$$y = -(2)^2 + 4(2) + 5$$
$$= -4 + 8 + 5$$
$$= 9$$

The turning point is $(2, 9)$.

4 Plot the intercepts and the turning point on the Cartesian plane.

5 Draw an inverted parabola through the points and label the graph with the equation.



6 List the domain by identifying the x values used to produce the graph. The set of real numbers is represented by $\mathbb{R}$.

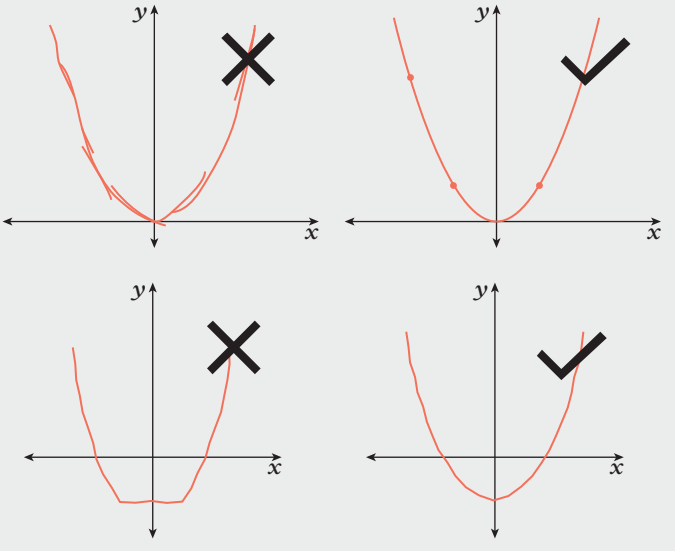
Domain is $\mathbb{R}$.

7 List the range by identifying the y values used to produce the graph.

Range is $y \leq 9$.

Helpful hints

- ✔ Rather than drawing messy inaccurate parabolas, try starting at the turning point and drawing a smooth curve up or down through the other plotted points. Do not connect the dots with straight lines.
- ✔ Label your working out so that it is clear when you are calculating the x-intercepts, the y-intercept and the turning point. This can make it easier to check your calculations when you are doing a test.
- ✔ Make sure that your parabola has a clearly defined turning point. If your graph has a flat bottom or top where the turning point should be, you may lose marks when doing a test.



ANS p574

Exercise 4C Graphing parabolas using intercepts

1–5, 6(a–e), 7, 8, 10 | 1–3, 5, 6(c–g), 9, 10, 11(a, b), 12(a, b) | 6(d–i), 8–12

4C.1 **1** For each quadratic relationship below, find the coordinates of:

i the x-intercepts **ii** the y-intercept.

a $y = x^2 + 6x$ **b** $y = -x^2 + 10x$ **c** $y = x^2 - 6x + 5$

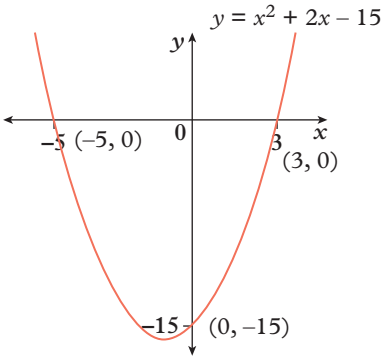
d $y = x^2 - 1$ **e** $y = -x^2 - 4x + 12$ **f** $y = 9 - x^2$

2 For each quadratic relationship in question **1**, state whether its graph is an upright parabola or an inverted parabola.

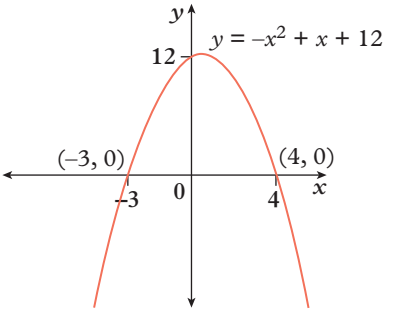
4C.2 **3** Find the coordinates of the turning point for each quadratic relationship in question **1**.

4 Find the coordinates of the turning point for each of these parabolas.

a



b



5 Sketch the graph of each quadratic relationship in question **1**, use your answers for questions **1–3** to help you.

4C.3 **6** Use intercepts to help you sketch the graph of each quadratic relationship below. Label each turning point with its coordinates. List the domain and range of each graph.

a $y = x^2 - 6x + 8$ **b** $y = -x^2 + 10x - 16$ **c** $y = x^2 + 4x - 12$

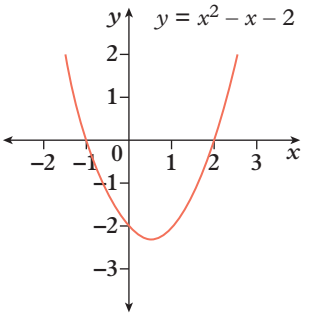
d $y = 4 - x^2$ **e** $y = x^2 + 2x$ **f** $y = -x^2 + 10x - 25$

g $y = x^2 - 4x - 21$ **h** $y = x^2 - x - 6$ **i** $y = x^2 + 7x + 10$

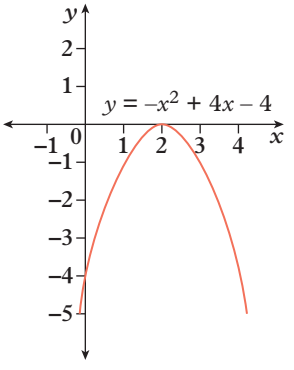
7 Consider the graphs of the three quadratic relationships shown below.

a How many x-intercepts does each parabola have?

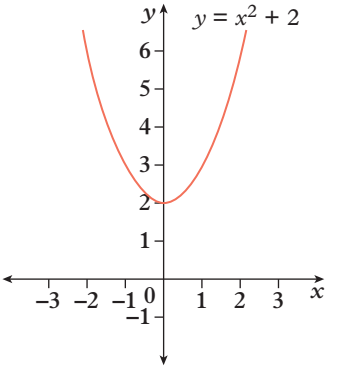
i



ii



iii



b Use the graphs to help you solve the following:

i $x^2 - x - 2 = 0$ **ii** $-x^2 + 4x - 4 = 0$ **iii** $x^2 + 2 = 0$

PROBLEM SOLVING AND REASONING

8 Consider the graphs of $y = (x + 5)(x - 1)$, $y = 2(x + 5)(x - 1)$ and $y = -(x + 5)(x - 1)$.

a Find the coordinates of the x-intercepts for each graph.

b Explain why the graphs of these equations will be different even though all parabolas have the same x-intercepts.

c Find the y-intercept for each parabola.

d Find the coordinates of the turning point for each parabola.

e Sketch all three parabolas on the same Cartesian plane.

9 Thomas throws a discus. The position of the discus can be represented by $h = -0.01(d + 4)(d - 72)$, where h is the height of the discus above the ground in metres and d is the horizontal distance from where the discus was thrown, in metres.

a Sketch the graph of this relationship by finding the intercepts.

b Thomas is standing on a platform. At what height off the ground was the discus thrown?

c What horizontal distance did the discus travel before hitting the ground?

d What is the greatest height that the discus reached?

10 A ball was kicked off the ground. Its path can be represented by $y = -0.05x^2 + 1.6x$, where x is the horizontal distance (in metres) and y is the vertical distance (in metres).

a Sketch the graph of this relationship.

b What was the maximum height the ball reached? What horizontal distance had it travelled when it reached its maximum height?

c What horizontal distance did the ball travel before hitting the ground?

d If the ball was caught at a height of 1.8 m above the ground as it was moving downwards, what horizontal distance did the ball travel?

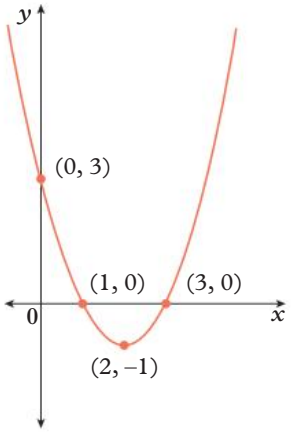
CHALLENGE

11 Use the intercepts to help you sketch the graph of each of these quadratic relationships. Label the turning point with its coordinates.

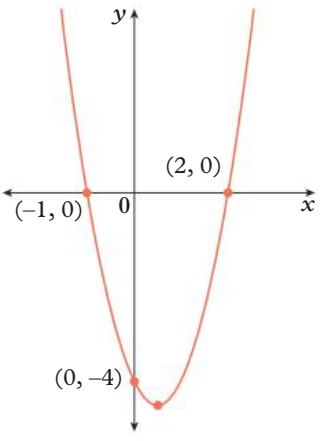
a $y = x^2 + 2x - 7$ **b** $y = -x^2 + 4x + 2$ **c** $y = x^2 - 2x - 1$

12 The following parabolas are all graphs of quadratic relationships in the form $y = a(x - m)(x - n)$. Find the equation for each graph in factor form.

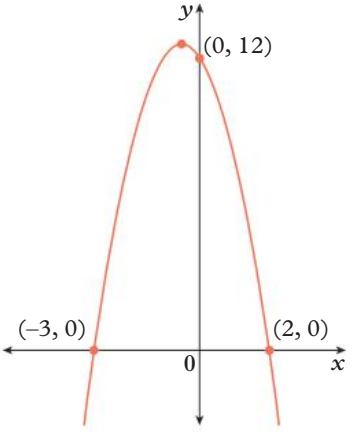
a



b



c



Check your Student obook pro for these digital resources and more:

Interactive skillsheet Sketching parabolas using intercepts

Investigation Appoximating the location of an intercept

Topic quiz 4C

4D Graphing parabolas using transformations

Learning intentions

By the end of this section you will be able to ...

- ✔ sketch quadratic graphs of the form $y = a(x - h)^2 + k$ using reflections, dilations, vertical translations and horizontal translations.

Inter-year links

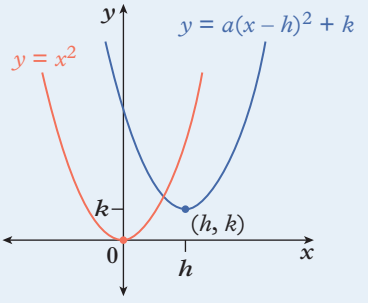
Support	Transformations
Year 7	8E Rotations and reflections
Year 8	7E Transformations
Year 9	4D Sketching parabolas using transformations

Turning point form

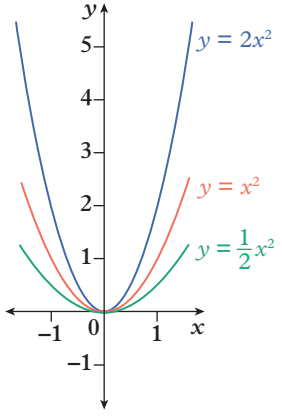
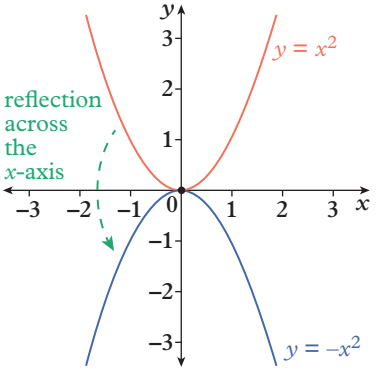
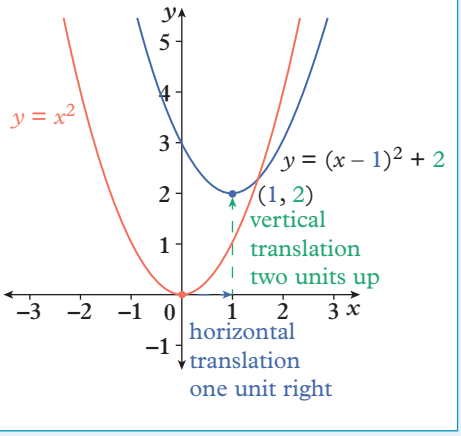
- Quadratic relationships can be written in **turning point form**, as shown below where the coordinates of the turning point are (h, k):

$$y = a(x - h)^2 + k$$

- k: Vertical translation of k units. If $k > 0$, up. If $k < 0$, down.
- a: Dilate in the y-direction. If $a > 0$, upright. If $a < 0$, inverted.
- h: Horizontal translation of h units. If $h > 0$, right. If $h < 0$, left.



- The following **transformations** can be performed on the graph of $y = x^2$ to produce the graph of $y = a(x - h)^2 + k$:
  - → a **dilation** of a factor of a in the y-direction. If a is greater than 1 or less than −1, the parabola becomes thinner. If a is between 1 and −1, the parabola becomes wider.
  - → a **reflection** across the x-axis if $a < 0$
  - → a horizontal **translation** of h units right or left
  - → a vertical translation of k units up or down.

Dilation	Reflection	Translation
		

- A combination of transformations can be performed on the graph of $y = x^2$ to produce the graph of $y = a(x - h)^2 + k$.

Example 4D.1 Sketching a parabola using vertical and horizontal translations

Sketch the graph of $y = x^2$ on a Cartesian plane, then perform a translation to sketch the graph of each quadratic relationship below. Label the turning point and y-intercept on each parabola with their coordinates.

a $y = x^2 + 3$ **b** $y = (x + 4)^2$

THINK

a **1** Identify the transformation. The equation has the form $x^2 + k$ where $k = 3$, so the graph of $y = x^2$ undergoes a vertical translation of 3 units up, resulting in a turning point of $(h, k) = (0, 3)$.

2 Sketch the graphs of $y = x^2$ and $y = x^2 + 3$, and label their turning points. The x-coordinates of both turning points are 0, so the y-intercept and the turning point is the same point for both relationships.

b **1** Identify the transformation. The equation has the form $(x - h)^2$ where $h = -4$, so the graph of $y = x^2$ undergoes a horizontal translation of 4 units to the left, resulting in a turning point of $(h, k) = (-4, 0)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

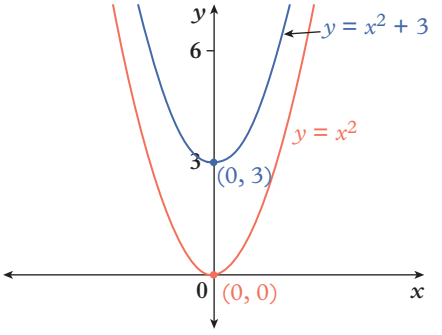
3 Sketch the graphs of $y = x^2$ and $y = (x + 4)^2$, and label their turning points and intercepts.

WRITE

a $y = x^2 + 3$

Vertical translation of three units up

The turning point is $(0, 3)$.



b $y = (x + 4)^2$

Horizontal translation of four units to the left

The turning point is $(-4, 0)$.

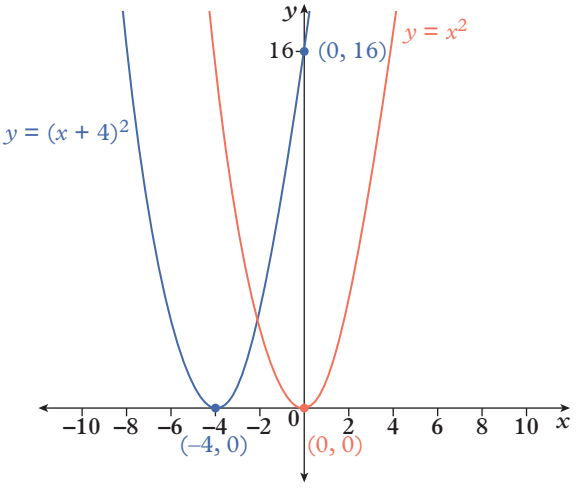
For the y-intercept, $x = 0$:

$y = (x + 4)^2$

$= (0 + 4)^2$

$= 16$

The y-intercept is $(0, 16)$.



Example 4D.2 Sketching a parabola using two transformations

Sketch the graph of each of the following parabolas by performing transformations on $y = x^2$. Label the y-intercept and the turning point each time.

a $y = -(x + 3)^2$ **b** $y = (x - 2)^2 - 1$

THINK

a **1** Identify the transformations. The equation has the form $a(x - h)^2$ where $a = -1$, so the graph of $y = x^2$ is reflected in the x-axis. Because $h = -3$, the graph of $y = x^2$ also undergoes a horizontal translation of three units to the left, resulting in a turning point of $(h, k) = (-3, 0)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

3 Sketch the graphs of $y = x^2$ and $y = -(x + 3)^2$, and label their turning points and intercepts.

b **1** Identify the transformations. The equation has the form $(x - h)^2 + k$ where $h = 2$ and $k = -1$, so the graph of $y = x^2$ undergoes a horizontal translation of two units to the right and a vertical translation of one unit down, giving a turning point of $(h, k) = (2, -1)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

WRITE

a

$y = -(x + 3)^2$

Reflection in the x-axis

Horizontal translation of three units to the left

The turning point is $(-3, 0)$.

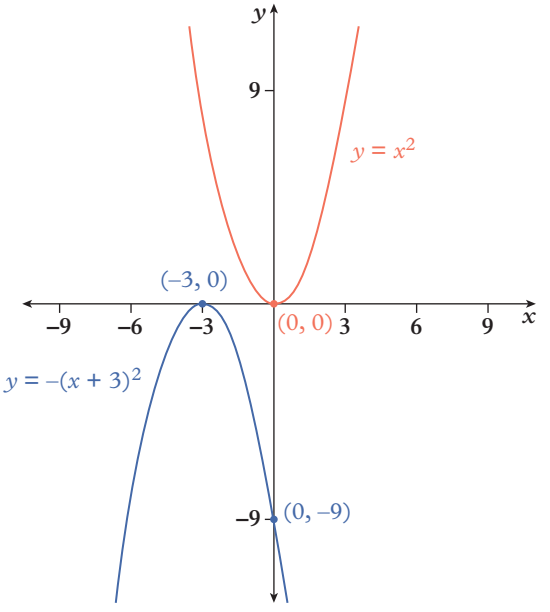
For the y-intercept, $x = 0$:

$y = -(x + 3)^2$

$= -(0 + 3)^2$

$= -9$

The y-intercept is $(0, -9)$.



b

$y = (x - 2)^2 - 1$

Horizontal translation of two units to the right

Vertical translation of one unit down

The turning point is $(2, -1)$.

For the y-intercept, $x = 0$:

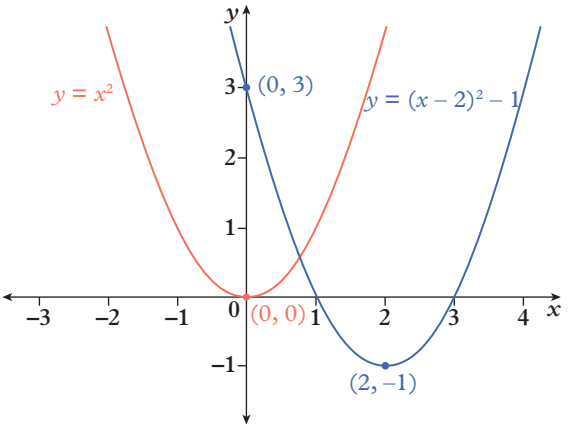
$y = (x - 2)^2 - 1$

$= (0 - 2)^2 - 1$

$= 3$

The y-intercept is $(0, 3)$.

3 Sketch the graphs of $y = x^2$ and $y = (x - 2)^2 - 1$, and label their turning points and intercepts.



Example 4D.3 Sketching a parabola using multiple transformations

Sketch the graph of $y = x^2$ on a Cartesian plane, then perform transformations on $y = x^2$ to sketch the graph of $y = -3(x - 1)^2 + 2$. For both graphs, label the y-intercept and the turning point.

THINK

1 Identify the transformations. The equation has the form $a(x - h)^2 + k$ where $a = -3$, $h = 1$ and $k = 2$, so the graph of $y = x^2$ is:
- reflected in the x-axis
- dilated in the y-direction by a factor of 3
- translated horizontally 1 unit to the right
- translated vertically 2 units up.

resulting in a turning point of $(h, k) = (1, 2)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

3 Sketch the graphs of $y = x^2$ and $y = -3(x - 1)^2 + 2$, and label their turning points and intercepts.

WRITE

$y = -3(x - 1)^2 + 2$

Reflection in the x-axis and dilated in the y-direction by a factor of 3

Horizontal translation of one unit to the right

Vertical translation of two units up

The turning point is $(1, 2)$.

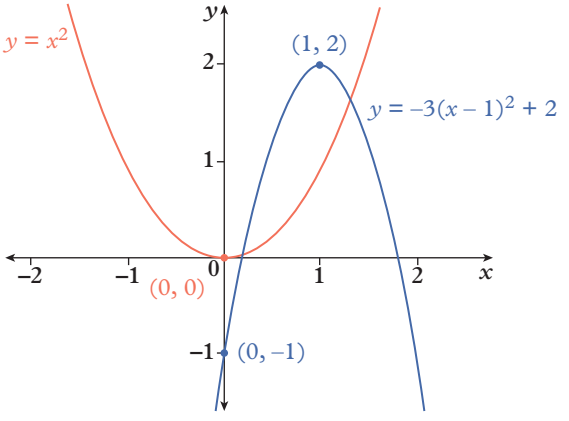
For the y-intercept, $x = 0$:

$y = -3(x - 1)^2 + 2$

$= -3(0 - 1)^2 + 2$

$= -1$

The y-intercept is $(0, -1)$.



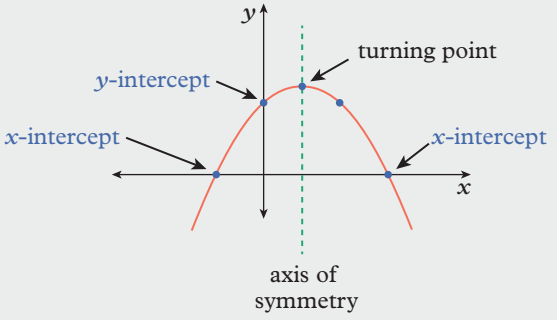
Helpful hints

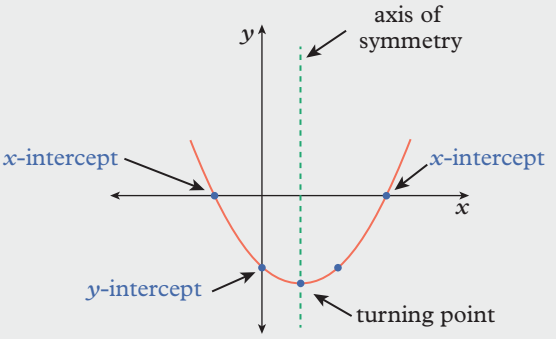
- ✔ In turning point form:
  - → the size and direction of vertical translations are determined by the constant outside the brackets
  - → the size and direction of horizontal translations are determined by the constant that is inside the brackets with x.

$$y = a(x - h)^2 + k$$

Horizontal translation of h units Vertical translation of k units

- ✔ Always label key features when sketching a graph. These include the intercepts and the turning points.
- ✔ If you need help drawing a symmetrical curve, it can help to draw a dashed line of symmetry.





- ✔ For $y = a(x - h)^2 + k$, if $-1 < a < 1$, the graph of $y = x^2$ has undergone a 'dilation' in the direction of the y-axis which actually compresses the graph.

  For example, if $y = x^2$ becomes $y = \frac{1}{2}x^2$ then it has been 'dilated' by a factor of $\frac{1}{2}$ in the direction of the y-axis, which really means it has been compressed by a factor of 2 in the direction of the y-axis and has become wider.

ANS p576

Exercise 4D Graphing parabolas using transformations

1–7, 8(a, b, d, e), 9–11, 12, 13, 17(a) 2–7, 8(a, c, d, f), 9–11, 13, 16(a, b) 2–3(b, d, f), 5, 6(d–f), 8, 9, 12, 14, 16, 17

1 For each set of quadratic rules below:

- **i** draw the graphs on the same Cartesian plane
- **ii** compare each parabola with the graph of $y = x^2$
- **iii** identify the transformation (or transformations) performed. Describe the effect of different coefficients for x^2 and any constants that are added or subtracted.

a	**b**	**c**	**d**
$y = x^2$	$y = x^2$	$y = x^2$	$y = x^2$
$y = 2x^2$	$y = x^2 + 1$	$y = (x - 1)^2$	$y = -x^2$
$y = 4x^2$	$y = x^2 + 2$	$y = (x - 2)^2$	$y = -x^2 + 2$
$y = \frac{1}{2}x^2$	$y = x^2 - 1$	$y = (x + 1)^2$	$y = -2x^2$
$y = \frac{1}{4}x^2$	$y = x^2 - 2$	$y = (x + 2)^2$	$y = \frac{1}{2}x^2$

UNDERSTANDING AND FLUENCY

4D.1 **2** Perform transformations on the graph of $y = x^2$ to help you sketch the graph of each quadratic equation below. Clearly show the coordinates of the turning point each time.

a $y = x^2 + 3$ **b** $y = x^2 + 2$ **c** $y = x^2 - 5$

d $y = x^2 + 6$ **e** $y = x^2 - 4$ **f** $y = x^2 - 7.5$

3 Perform a transformation on the graph of $y = x^2$ to help you sketch the graph of each quadratic equation below. Clearly show the y-intercept and the coordinates of the turning point each time.

a $y = (x - 4)^2$ **b** $y = (x + 3)^2$ **c** $y = (x - 5)^2$

d $y = (x + 2)^2$ **e** $y = (x - 8)^2$ **f** $y = (x + 1.5)^2$

4 Match each equation below with a description (**A–F**) of the transformation/s that must be performed on the graph of $y = x^2$ to produce its graph.

a $y = x^2 - 9$ **A** dilated in the y-direction by a factor of 9

b $y = (x + 9)^2$ **B** translated 9 units down

c $y = 9x^2$ **C** reflected in the x-axis and then translated 9 units right

d $y = -9x^2$ **D** translated 9 units left

e $y = -x^2 + 9$ **E** reflected in the x-axis and then translated 9 units up

f $y = -(x - 9)^2$ **F** reflected in the x-axis and dilated by a factor of 9 in the y-direction

5 Write the rule for the parabola that would be produced by performing each transformation (or combination of transformations) below on the graph of $y = x^2$.

a dilated in the y-direction by a factor of 6

b reflected in the x-axis

c translated 5 units right

d translated 2 units down

e dilated in the y-direction by a factor of $\frac{1}{4}$ and reflected in the x-axis

f reflected in the x-axis and translated 3 units up

4D.2 **6** Sketch the graph of each of the following quadratic relationships by performing transformations on $y = x^2$. Label the y-intercept and the turning point each time.

a $y = (x - 1)^2 + 2$ **b** $y = (x + 5)^2 - 3$ **c** $y = -(x - 2)^2 + 7$

d $y = -(x + 4)^2 - 6$ **e** $y = (x + 7)^2 + 4$ **f** $y = -(x - 6)^2 - 1$

7 Identify intervals on the real number line over which the quadratic relationships in question **6** is positive.

4D.3 **8** For each quadratic relationship:

i identify whether the graph will be narrower or wider than the graph of $y = x^2$

ii identify whether the parabola will be upright or inverted

iii identify the coordinates of the turning point

iv sketch the graph.

a $y = 2(x - 4)^2 - 8$ **b** $y = -3(x + 1)^2 + 12$ **c** $y = 5(x - 3)^2$

d $y = -4x^2 + 1$ **e** $y = \frac{1}{3}(x + 6)^2 - 3$ **f** $y = -\frac{1}{2}(x - 2)^2 - 9$

9 Write the equation for the parabola produced after performing each set of transformations below on the graph of $y = x^2$.

a Dilated in the y-direction by a factor of 2 then translated 5 units right.

b Reflected in the x-axis then translated 3 units down and 4 units left.

c Dilated in the y-direction by a factor of $\frac{1}{3}$ and reflected in the x-axis.

d Dilated in the y-direction by a factor of 5, reflected in the x-axis then translated 7 units up.

10 What is the range of $y = (x + 2)^2 + 4$? Explain.

11 What is the range of $y = -(x - 6)^2 - 1$? Explain.

12 Riley tries to kick a goal during a game of football, but accidentally hits the post, scoring a behind. The height of the ball above the ground can be represented by the relationship $h = -4(t-1)^2 + 5$, where h is the height in metres after t seconds.

a What are the coordinates of the turning point of the graph of this relationship?

b Sketch the graph, from $t = 0$ to $t = 1.5$, and list the domain and range.

c Use the graph to find:

i the height at which the ball made contact with Riley's boot

ii the maximum height of the ball during Riley's kick

iii the height of the football when it hits the post, 1.5 seconds after it was kicked.

d If the ball had not hit the post, how long would it have taken for the ball to hit the ground after Riley kicked it? Give your answer correct to one decimal place.

13 A soccer ball is kicked from the ground into the air. The path of the ball follows the relationship $y = -\frac{1}{40}(x-20)^2 + 10$, where y is the height above the ground of the ball for a horizontal distance x from where the ball was kicked. Both x and y are in metres.

a Sketch the graph of this relationship from when the ball was kicked to when it landed. List its domain and range.

b What was the maximum height of the soccer ball?

c What horizontal distance did the ball travel before it hit the ground?

14 The equation $H = -x^2 + 6x + 1$ models the path of a poorly hit baseball, where H represents the height above the ground of the ball (in metres) and x represents the horizontal distance travelled by the ball (in metres).

a Substitute $x = 0$ into the equation to find the height above the ground of the ball when it was hit by the batter.

b Complete the square for $-x^2 + 6x + 1$.

c Find the maximum height reached by the ball.

d Find the horizontal distance the ball travelled before hitting the ground. Give your answer in metres rounded to two decimal places.

e Use the information found in parts **a–d** to help you sketch the graph of H versus x, labelling all relevant points.

15 Sketch each of the following parabolas by first completing the square for the quadratic, labelling all relevant points as exact values.

a $y = x^2 + 6x + 9$ **b** $y = x^2 - 2x + 3$ **c** $y = x^2 - 4x + 1$ **d** $y = 3x^2 - 6x + 5$

16 From the given information, write an equation for each of these quadratic relationships in the turning point form.

a The graph is an upright parabola, with turning point at $(2, 6)$ and y-intercept of 10.

b The graph is an upright parabola, with turning point at $(-4, -12)$ and y-intercept of 20.

c The graph is an inverted parabola, with turning point at $(-1, 7)$ and y-intercept of 4.

d The graph is an inverted parabola, with turning point at $(3, -5)$ and y-intercept of -8.

17 a Show that $(2-x)^2$ is equivalent to $(x-2)^2$ by expanding each expression.

b Sketch each of the following graphs, labelling all relevant points.

i $y = (2-x)^2 + 1$ **ii** $y = (1-x)^2 - 4$

iii $y = -\frac{1}{3}(3-x)^2 - 1$ **iv** $y = -(-x-2)^2 - 1$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Sketching parabolas using transformations

Investigation
The connection between linear and quadratic graphs

Topic quiz
4D

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

4A **1** Use the Null Factor Law to solve each of the following equations.

a $x(x-2)=0$ **b** $(x+3)(x-4)=0$

c $(x+1)(x-1)=0$ **d** $(2x+3)(x-5)=0$

4A **2** Solve each of the following quadratics by first factorising and then applying the Null Factor Law.

a $x^2+5x=0$ **b** $2x^2-12x=0$

c $x^2+6x+8=0$ **d** $x^2-22x+40=0$

e $x^2+16x-17=0$ **f** $x^2+x-6=0$

4A **3** Solve each of the following equations and give your answers as exact values.

a $x^2-4=0$ **b** $x^2-7=0$

c $x^2=22$ **d** $5x^2=45$

e $3x^2-15=0$ **f** $-2x^2+12=0$

4B **4** Find the value of the discriminant for each of the following quadratic equations.

a $x^2+5x+7=0$ **b** $x^2+4x=2$

c $2x^2+6x+5=0$ **d** $3x^2+7x=10$

4B **5** Use the discriminant to identify the number of solutions for each of the following quadratic equations.

a $x^2-5x+3=0$ **b** $x^2-6x=-9$

c $2x^2+7x=10$ **d** $2x^2+5x+4=0$

4B **6** Use the quadratic formula to solve the following quadratic equations, leave your answers as exact values.

a $x^2+3x+1=0$ **b** $x^2-2x+1=0$

c $3x+1=x^2$ **d** $5x^2+x=2$

e $4x^2+12x+9=0$ **f** $\frac{1}{2}x^2+5x=3$

4C **7** Find the x-intercepts and the y-intercept for the parabolas with the following equations. You do not need to sketch the graphs.

a $y=x(x-4)$ **b** $y=(x+3)(x-1)$

c $y=x^2-x-2$ **d** $y=x^2+6x-16$

4C **8** Sketch the graph of $y=(x-1)(x-5)$, labelling all relevant points.

4C **9** Sketch the graph of $y=-2(x+1)(x-2)$, labelling all relevant points.

4D **10** Sketch the graphs of each of the following by transformations, labelling all relevant points.

a $y=x^2+1$ **b** $y=x^2-4$

c $y=(x-2)^2$ **d** $y=(x+3)^2$

4D **11** Sketch the graphs of each of the following by transformations, labelling all relevant points with exact values.

a $y=(x-3)^2+1$ **b** $y=2(x+1)^2+2$

c $y=-x^2+1$ **d** $y=-(x+1)^2-1$

e $y=-(x-2)^2+1$ **f** $y=(x-3)^2-7$

4D **12** Write the rule for each parabola resulting from performing each of these sets of transformations on the graph of $y=x^2$.

a a translation 2 units right and 1 unit up

b a reflection in the x-axis, then a translation 3 units left and 1 unit up

c a dilation in the y-direction by a factor of 2, then a translation 3 units down

d a dilation in the y-direction by a factor of 3 and a reflection in the x-axis, then a translation 4 units right and 2 units down

4E Graphing circles

Learning intentions

By the end of this section you will be able to ...

- ✔ sketch simple circles of the form $x^2 + y^2 = r^2$
- ✔ sketch circles of the form $(x - h)^2 + (y - k)^2 = r^2$ using vertical translations and horizontal translations
- ✔ determine the equation of a circle from a description or a graph.

Inter-year links

Year 7	8D Translations
Year 8	7E Transformations
Year 9	4E Other non-linear relationships

Circles

- The graph of $x^2 + y^2 = 1$ is a circle with centre at $(0, 0)$ and a radius of $r = 1$ unit.
- The general equation of a circle graph is: $(x - h)^2 + (y - k)^2 = r^2$

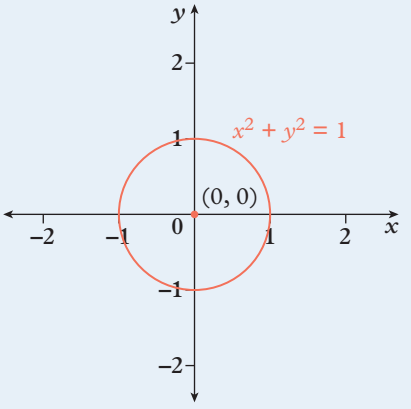
  where:
  - → the circle has a radius of r units
  - → the coordinates of the centre of the circle are (h, k).

$(x - h)^2 + (y - k)^2 = r^2$

Radius of r units

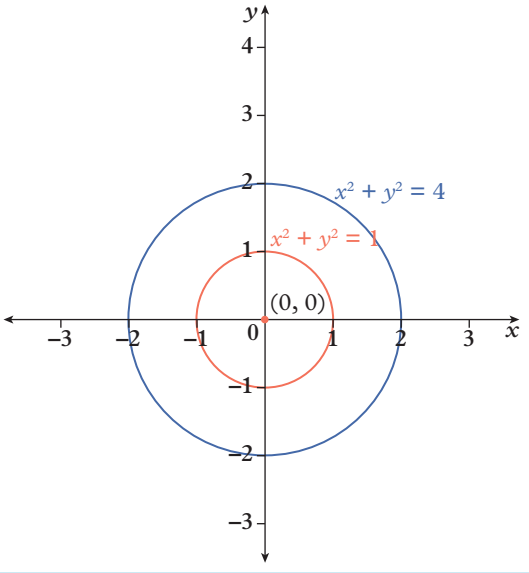
Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

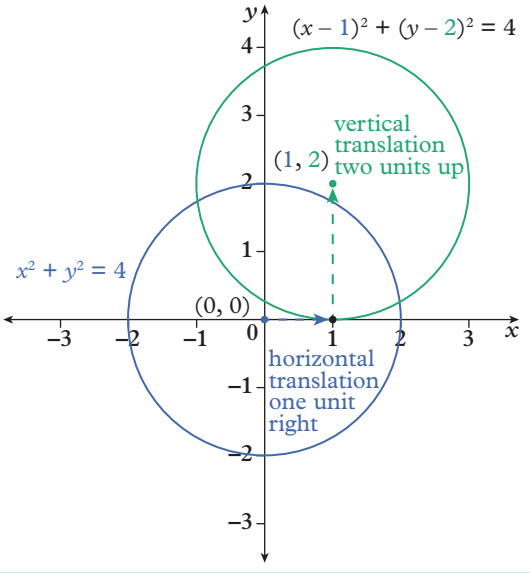


- The following transformations can be performed on the graph of $x^2 + y^2 = r^2$ to produce the graph of $(x - h)^2 + (y - k)^2 = r^2$:
  - → a dilation in all directions as r increases
  - → a horizontal translation of h units right or left
  - → a vertical translation of k units up or down.

Dilation



Translation



Example 4E.1 Sketching simple circles

Sketch the graph of $x^2 + y^2 = 36$ by first finding the horizontal and vertical extremes of the circle.

THINK

1. Identify the centre of the circle. The equation has the form $x^2 + y^2 = r^2$, so $h = 0$ and $k = 0$, and the centre of the circle is at the origin $(0, 0)$.
2. Identify the radius. The equation has the form $x^2 + y^2 = r^2$ so $r^2 = 36$. A radius cannot be negative, so ignore the negative value.
3. Identifying the horizontal and vertical extremes that are 6 units left, right, below and above the centre $(0, 0)$.
4. Sketch the graph. Label the graph with its equation.

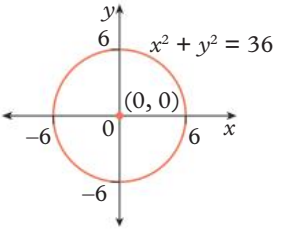
WRITE

Centre: $(0, 0)$

$r^2 = 36$
$r = \pm\sqrt{36}$
$= \pm 6$
The radius is 6 units.

Horizontal extremes: $(-6, 0)$ and $(6, 0)$

Vertical extremes: $(0, -6)$ and $(0, 6)$



Example 4E.2 Sketching circles using translations

Sketch the graph of $(x + 2)^2 + (y - 4)^2 = 9$ using transformations.

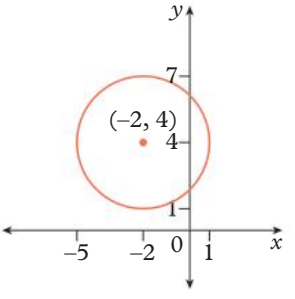
THINK

1. Identify the translations. The equation has the form $y = (x - h)^2 + (y - k)^2 = r^2$ where $h = -2$ and $k = 4$, so the coordinates of the centre are $(h, k) = (-2, 4)$.
2. Identify the radius. $r^2 = 9$. A radius cannot be negative, so ignore the negative value.
3. Sketch the graph by marking the centre of the circle at $(-2, 4)$, then identifying the horizontal and vertical extremes that are 3 units left, right, below and above the centre: $(-5, 4)$, $(1, 4)$, $(-2, 1)$ and $(-2, 7)$.

WRITE

Centre: $(-2, 4)$

$r^2 = 9$
$r = \pm\sqrt{9}$
$= \pm 3$
The radius is 3 units.

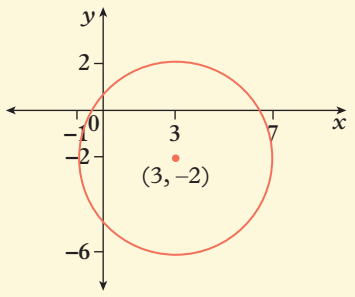


Example 4E.3 Determining the equation of a circle

Determine the equation of the circle in each case below.

a a circle with a radius of 6 units and a centre at (–2, 5)

b



THINK

The general equation of a circle is $y = (x - h)^2 + (y - k)^2 = r^2$. Identify the values of h, k and r, substitute them into the equation and simplify.

WRITE

a The centre is $(-2, 5) = (h, k)$.

$h = -2$

$k = 5$

Radius: $r = 6$

$(x - h)^2 + (y - k)^2 = r^2$

$(x - [-2])^2 + (y - 5)^2 = 6^2$

$(x + 2)^2 + (y - 5)^2 = 36$

b The centre is $(3, -2) = (h, k)$

$h = 3$

$k = -2$

Radius: $r = 4$

$(x - h)^2 + (y - k)^2 = r^2$

$(x - 3)^2 + (y - [-2])^2 = 4^2$

$(x - 3)^2 + (y + 2)^2 = 16$

Helpful hints

- ✔ Remember that the values of h and k have the opposite sign to the one appearing with them in the equation.
  For example, in the equation $(x + 2)^2 + (y - 4)^2 = 9$, $h = -2$ and $k = +4$
- ✔ Make sure you sketch your circles on a set of axes drawn to the same scale, because scales that don't match will make your circle look like an oval or an ellipse.
- ✔ Remember that the value of r is the positive square root of the right-hand side of the equation.
  For example, the radius of $(x - 1)^2 + (y - 3)^2 = 4$ is 2, not 4.
- ✔ Make sure that you mark and label the centre of your circles!

ANS p583

Exercise 4E Graphing circles

1–9, 13, 16, 17 | 1(e, f), 3–7, 9, 10, 13–18 | 1(e, f), 3(b, d, f, h), 5(b, d, f, h), 6, 7, 10, 11, 14, 15, 18, 19

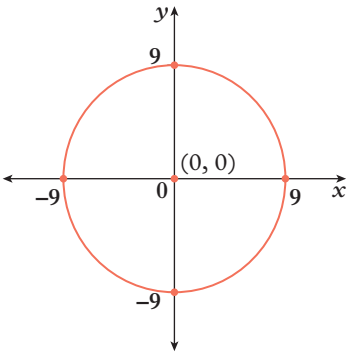
4E.1 **1** Sketch the graph of each of these equations.

a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 1$ **c** $x^2 + y^2 = 64$

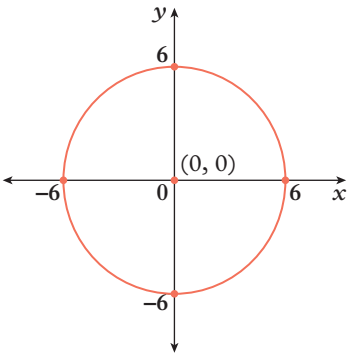
d $4x^2 + 4y^2 = 100$ **e** $15x^2 + 15y^2 = 60$ **f** $\frac{x^2}{16} + \frac{y^2}{16} = 1$

2 Write the equation for each of these circles.

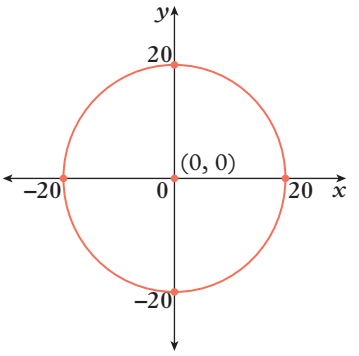
a



b



c



4E.2 **3** Sketch the graph of each of these equations. List the centre and radius.

a $(x-3)^2 + (y-2)^2 = 9$

b $(x-5)^2 + (y-7)^2 = 25$

c $(x+1)^2 + (y-3)^2 = 64$

d $(x-6)^2 + (y+6)^2 = 36$

e $(x-2)^2 + y^2 = 16$

f $x^2 + (y+5)^2 = 4$

g $(x+3)^2 + (y+5)^2 = 49$

h $(x+4)^2 + (y+3)^2 = 81$

4 Use digital technology to reproduce the first four graphs from question **3**. Compare those graphs to the ones you sketched.

4E.3 **5** Use the information provided to write the equation for each of these circles.

a a circle with radius 12 units and centre at (0, 0)

b a circle with radius 15 units and centre at (0, 0)

c a circle with radius 1 unit and centre at (4, –2)

d a circle with radius 7 units and centre at (–5, –6)

e a circle with radius 3 units and centre at (4, 4)

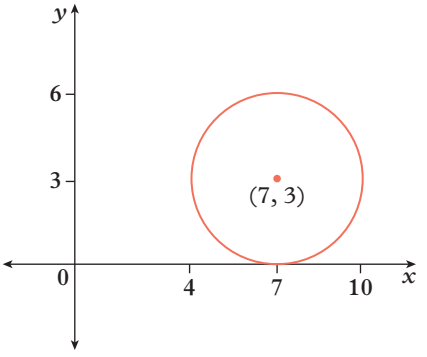
f a circle with radius 6 units and centre at (0, –5)

g a circle with radius 1.5 units and centre at (–1, 2.5)

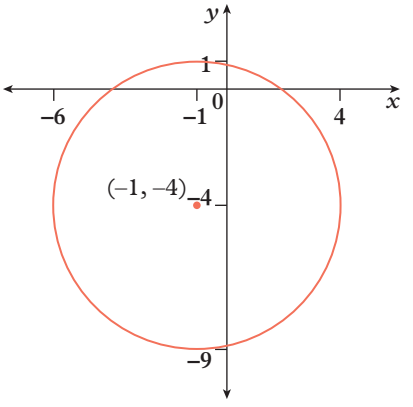
h a circle with radius $\sqrt{5}$ units and centre at (2, –3)

6 Write the equation for each of these circles.

a



b



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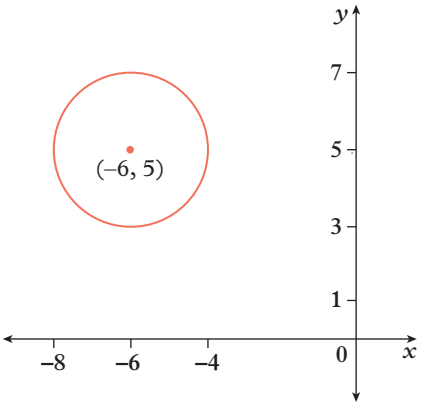
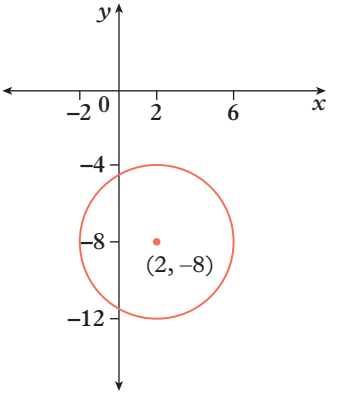


d

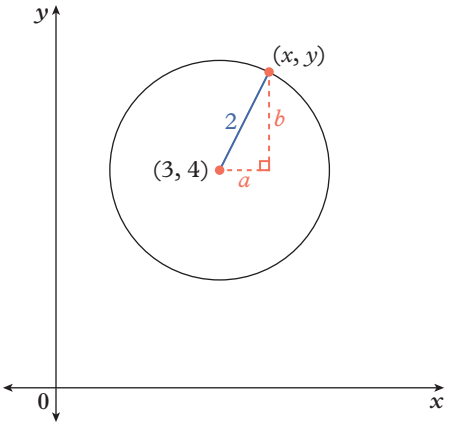


7 **a** The circle with equation $x^2 + y^2 = 1$ is dilated in all directions by a factor of 3, then translated 2 units right and 1 unit down. What is the equation of the transformed circle?

b The circle with equation $x^2 + y^2 = 1$ is transformed to become a circle with equation $(x + 7)^2 + (y - 4)^2 = 25$. Describe the transformations that have occurred. Hint: You must describe the dilation before the translations.

8 A circle is a set of points that are equidistant from a single point. This geometric description of a circle, along with Pythagoras' theorem, explains the Cartesian equation of a circle.

Consider the diagram on the right, where (x, y) is generalised to be any point on the circle.



a Explain why $a = x - 3$ and $b = y - 4$.

b Use Pythagoras' theorem to explain why the Cartesian equation of this circle is $(x - 3)^2 + (y - 4)^2 = 2^2$.

9 Consider $x^2 + y^2 = 25$.

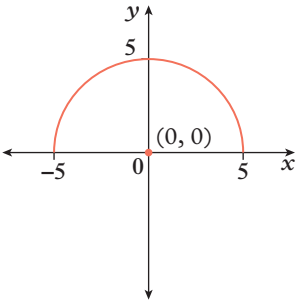
a Draw a graph of the equation.

b Rearrange the equation to make y the subject.

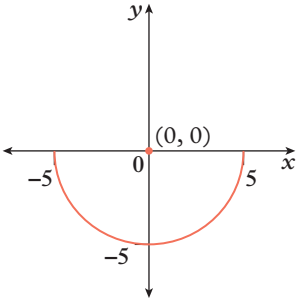
c Match each of these equations to one of the graphs shown below (**A** and **B**). Explain your reasoning.

i $y = -\sqrt{25 - x^2}$ **ii** $y = \sqrt{25 - x^2}$

A



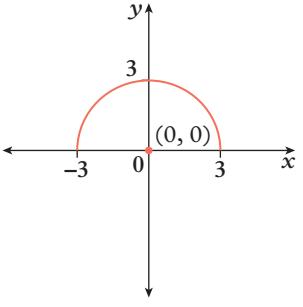
B



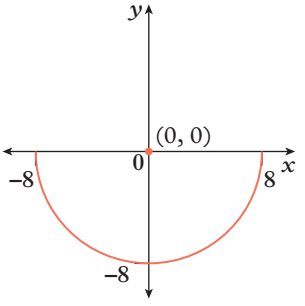
d Explain how the two half circles **A** and **B** are related to the graph of $x^2 + y^2 = 25$.

10 Write the equation for each of these half circles.

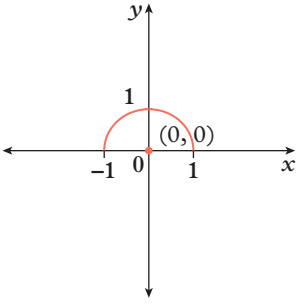
a



b



c



11 Sketch the graph of each of these half circles. List the domain and range of each graph.

a $y = -\sqrt{49 - x^2}$

b $y = \sqrt{16 - x^2}$

c $y = -\sqrt{100 - x^2}$

12 **a** Rearrange $x^2 + y^2 = 25$ to make x the subject of the equation.

b Your answer to part **a** produces two equations. What would the graph of each equation look like?

PROBLEM SOLVING AND REASONING

13 An arch is modelled using the relationship $(x-3)^2 + y^2 = 9$, where x is the horizontal distance from the place the left-hand side of the arch meets the ground and y is the corresponding height at that distance. Both x and y are measured in metres, $x \geq 0$ and $y \geq 0$.

a Sketch the graph of this relationship.

b What is the horizontal distance along the ground from one side of the arch to the other?

c What is the maximum height of the arch?

d Describe the position on the ground below a point where the arch is at half its maximum height.

14 The outline of a circular spa is mapped onto a Cartesian plane representing a 15 m long and 10 m wide outdoor area of a seaside resort, using the relationship $(x-6.2)^2 + (y-4.8)^2 = 2.25$.

a Sketch the graph of this relationship.

b What is the minimum distance from the water to the edge of the outdoor area?

c Calculate the area of the spa to the nearest square metre.

d What percentage of the outdoor area is taken up by the spa?

CHALLENGE

15 a Complete the square for each of these quadratics:

i $x^2 + 6x + 2$

ii $y^2 - 4y + 7$

b Sketch the circle with equation $x^2 + 6x + 2 + y^2 - 4y + 7 = 0$.

16 Use a graphical method to find the points of intersection of the graphs with equations $x^2 + y^2 = 25$ and $y = x + 7$.

17 Use an algebraic method to find the points of intersection of the graphs with equations:

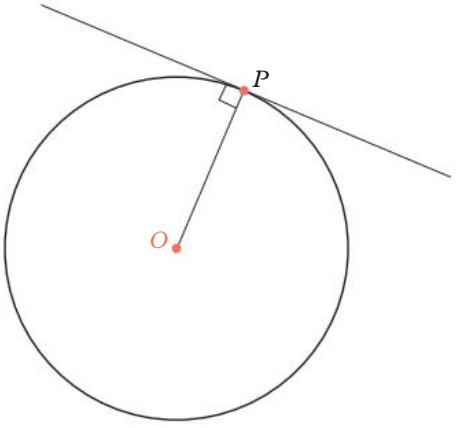
a $x^2 + y^2 = 9$ and $y = x - 3$

b $(x+3)^2 + (y-8)^2 = 4$ and $y = x + 9$.

18 Find the equation for a circle that passes through the four points $(-10, 5)$, $(-4, 11)$, $(2, 5)$ and $(-4, -1)$.

19 A tangent to a circle is a straight line which just touches the circle once, at one point on its circumference. The radius of the circle at the point of tangency is perpendicular to the tangent line.

Consider the circle with equation $(x-1)^2 + (y-2)^2 = 5$.



a What is the centre of this circle?

b Show that $(3, 3)$ is a point on the circle.

c Find the gradient of the radius that connects the centre of the circle to the point $(3, 3)$.

d Find the gradient of the line perpendicular to the radius referred to in part **c**.

e Use the gradient you found in part **d** to find the equation of the tangent to the circle at $(3, 3)$. Sketch the graphs of the circle and your line on a graphing calculator to confirm that the straight line is a tangent to the circle.

Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Circles
- Worksheet: Circles on the Cartesian plane
- CAS instructions: Graphing circles
- Topic quiz: 4E

4F Graphing exponential relationships

Learning intentions

By the end of this section you will be able to ...

- ✔ sketch simple exponential relationships using a table of values
- ✔ sketch exponential relationships using reflections and translations
- ✔ sketch exponential relationships using dilations.

Inter-year links

Year 7	8D Translations
Year 8	7E Transformations
Year 9	4E Other non-linear relationships

Exponential relationships

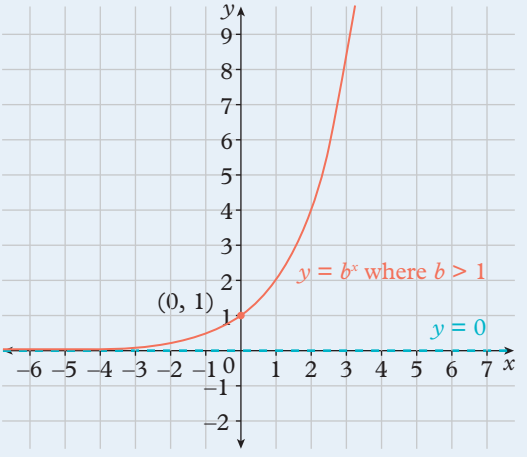
- Exponential relationships can be used to model many real situations involving growth and decay; for example, situations involving population growth or investments.
- The basic form of an **exponential relationship** has the following equation:

exponent/index

$$y = b^x$$

base

where $b > 1$.



- The graph has:
  - → a y-intercept of 1
  - → and a horizontal asymptote along the x-axis.
- An **asymptote** is a boundary or line that a curve approaches but never reaches. The asymptote for $y = b^x$ is the horizontal line with equation $y = 0$.

Graphing exponential equations by transformations

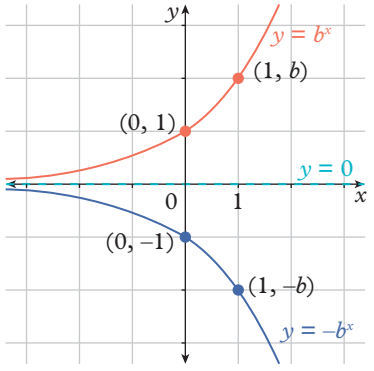
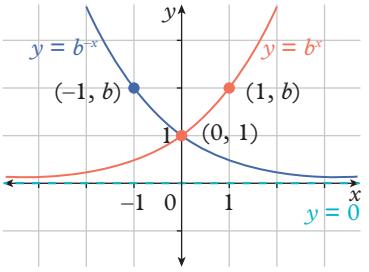
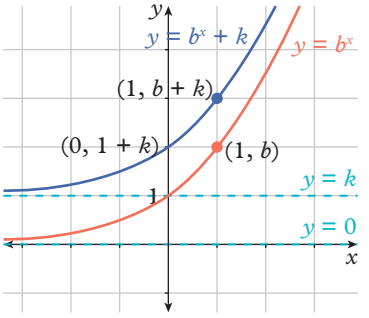
- The general equation of an exponential relationship is: $y = A \times b^{nx} + k$
  where the graph has:
  - → an asymptote at $y = k$
  - → a y-intercept at $(0, A + k)$.
- The following transformations can be performed on the graph of $y = b^x$ to produce a sketch of $y = A \times b^{nx} + k$:
  - → a dilation in the y-direction as A varies
  - → a vertical reflection across the x-axis if $A < 0$
  - → a dilation in the x-direction as n varies
  - → a horizontal reflection across the y-axis if $n < 0$
  - → a vertical translation of k units up or down.

If $n > 0$, upright
If $n < 0$, horizontal reflection

$$y = A \times b^{nx} + k$$

If $A > 0$, upright
If $A < 0$, vertical reflection

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Vertical reflection across x-axis	Horizontal reflection across y-axis	Vertical translation
		

Example 4F.1 Graphing exponential equations from a table of values

Sketch $y = \frac{3}{2} \times 2^{-x}$ by first completing a table of values for integer values of x from -2 to 2.

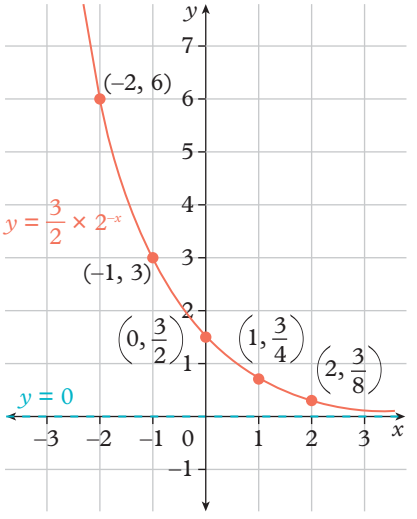
THINK

1. Construct a table of values for x from -2 to 2. Substitute each value of x into the equation to find the corresponding values of y.
2. Write out the coordinates of each point listed in the table.
3. The asymptote for $y = A \times b^{nx} + k$ is the horizonal line with $y = k$. In this case the asymptote is $y = 0$ as $k = 0$.
4. Plot the points on a Cartesian plane, mark the asymptote, and join the points with a smooth curve.

WRITE

x	-2	-1	0	1	2
$y = \frac{3}{2} \times 2^{-x}$	6	3	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$

$(-2, 6), (-1, 3), (0, \frac{3}{2}), (1, \frac{3}{4}), (2, \frac{3}{8})$

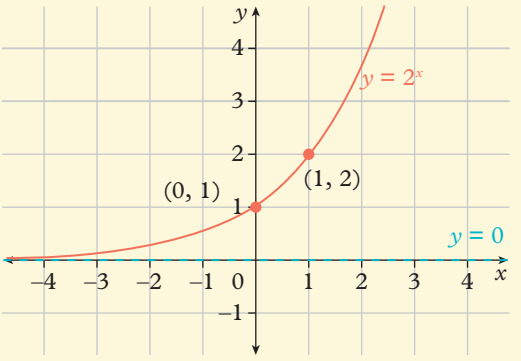


Example 4F.2 Sketching the graph of a simple exponential relationship by performing reflections

On the same Cartesian plane, sketch the graphs of the following exponential relationships by reflecting the graph of $y = 2^x$. Label the y-intercepts and asymptotes.

a $y = -2^x$

b $y = 2^{-x}$



THINK

1 Compare both equations with $y = 2^x$ to identify the transformations. Both equations are in the form $y = A \times 2^{nx} + k$.

In $y = -2^x$, $A = -1$, $n = 1$, $k = 0$.

In $y = 2^{-x}$, $A = 1$, $n = -1$, $k = 0$.

When $A = -1$, the graph is being reflected across the x-axis and when $n = -1$, the graph is being reflected across the y-axis. In both graphs, the asymptote remains $y = 0$ because $k = 0$.

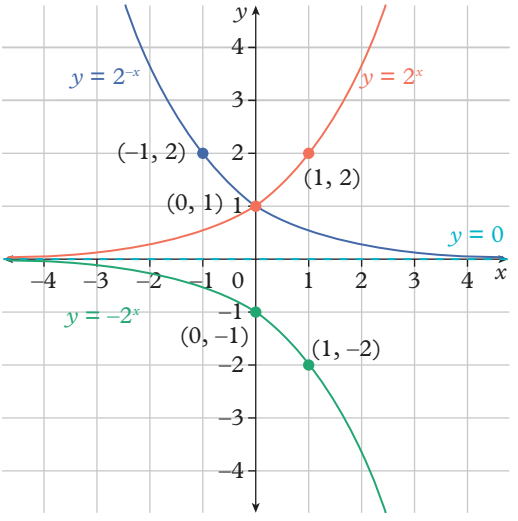
2 Find the corresponding coordinates of (1, 2) and relevant intercepts after the reflection, use these points as reference when sketching the curve.

3 Perform the reflection, sketch the curves through the y-intercepts and reference points, using the horizontal asymptote as a boundary. Label the y-intercepts and asymptotes.

WRITE

a Asymptote $y = 0$, y-intercept from (0, −1), curve passes through (1, −2).

b Asymptote $y = 0$, y-intercept from (0, 1), curve passes through (−1, 2).

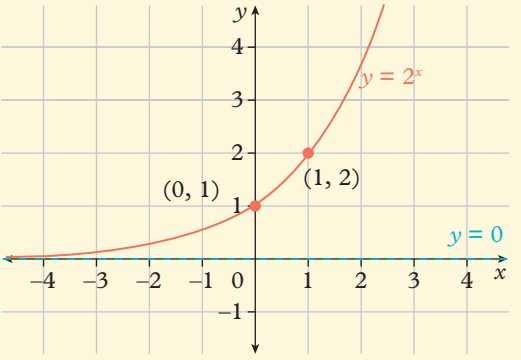


Example 4F.3 Sketching the graph of an exponential relationship by performing vertical translations

On the same Cartesian plane, sketch the graph of the following exponential relationship by transforming the graph of $y = 2^x$. Label the y-intercepts and asymptotes.

a $y = 2^x + 1$

b $y = 2^x - 3$



THINK

1 Compare both equations with $y = 2^x$ to identify the transformations. Both equations are in the form $y = 2^x + k$.

In **a**, $k = 1$, the asymptote of the graph is $y = 1$, translate the graph of $y = 2^x$ one unit up.

In **b**, $k = -3$, the asymptote of the graph is $y = -3$, translate the graph of $y = 2^x$ three units down.

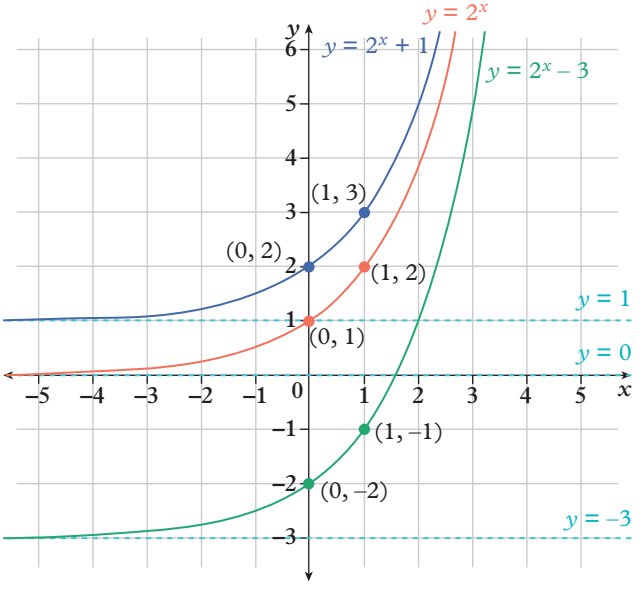
2 Find the corresponding coordinates of $(1, 2)$ and relevant intercepts after the translation, use these points as reference when sketching the curve.

3 Perform the translation, sketch the curve through the y-intercept and reference point, using the horizontal asymptote as a boundary. Label the y-intercepts and asymptotes.

WRITE

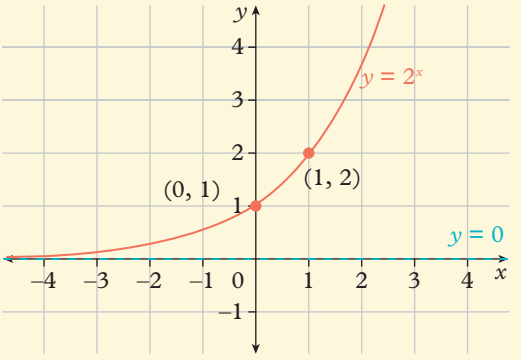
a Asymptote: $y = 1$, y-intercept $(0, 2)$, curve passes through $(1, 3)$.

b Asymptote: $y = -3$, y-intercept $(0, -2)$, curve passes through $(1, -1)$.



Example 4F.4 Sketching the graph of a simple exponential relationship using vertical dilations

On the same Cartesian plane, sketch the graph of $y = 2 \times 2^x$ by transforming the graph of $y = 2^x$. Label the y-intercept and asymptote.



THINK

1 Compare the equation with $y = 2^x$ to identify the transformations. The equation is in the form $y = A \times 2^{nx} + k$, where $k = 0$. The asymptote for the graph is $y = 0$.

2 $A = 2$, $n = 1$, dilate $y = 2^x$ in the y-direction. A point (x, y) on $y = 2^x$ becomes (x, Ay) after the dilation. Since $A = 2$, the y-intercept changes from $(0, 1)$ to $(0, 2)$:

$(0, 1) \rightarrow (0, A \times 1) \rightarrow (0, 2 \times 1) \rightarrow (0, 2)$.

The reference point changes from $(1, 2)$ to $(1, 4)$:

$(1, 2) \rightarrow (1, A \times 2) \rightarrow (1, 2 \times 2) \rightarrow (1, 4)$.

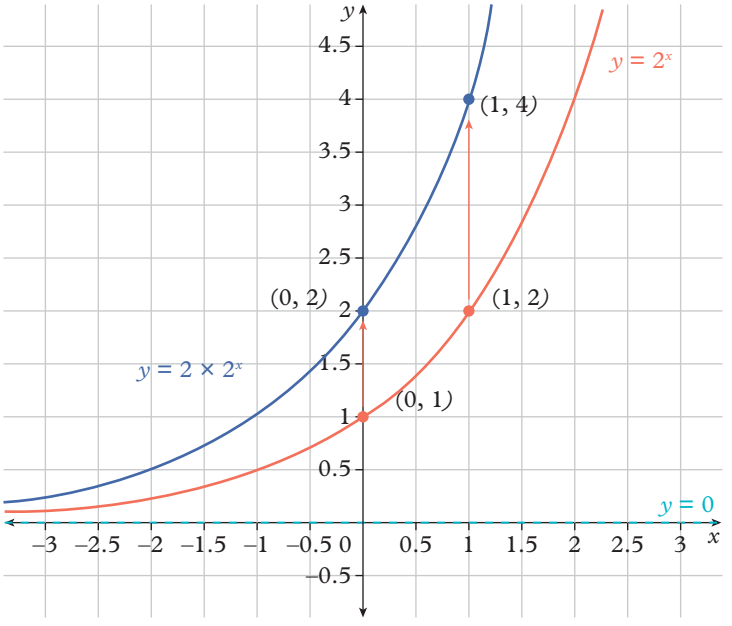
3 Sketch the curve for an exponential relationship through the y-intercept and the reference point, using the horizontal asymptote as a boundary. Label the y-intercept and asymptote.

WRITE

Asymptote: $y = 0$

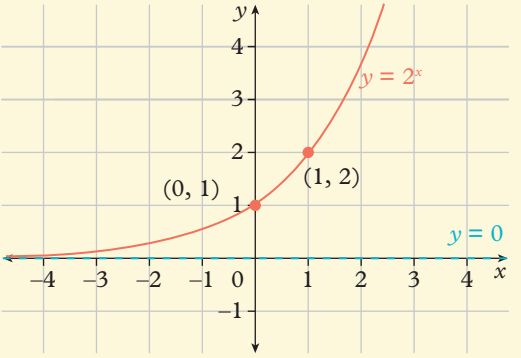
y-intercept: $(0, 2)$

curve passes through $(1, 4)$.



Example 4F.5 Sketching the graph of a simple exponential relationship using horizontal dilations

On the same Cartesian plane, sketch the graph of $y = 2^{2x}$ by transforming the graph of $y = 2^x$. Label the y-intercept and asymptote.



THINK

1 Compare the equations with $y = 2^x$ to identify the transformations. The equation is in the form $y = A \times 2^{nx} + k$, where, $k = 0$. The asymptote for the graph is $y = 0$.

2 $A = 1$, $n = 2$, dilate $y = 2^x$ in the x-direction. A point (x, y) on $y = 2^x$ becomes $\left(\frac{x}{n}, y\right)$ after the dilation. Since $n = 2$, the y-intercept remains the same at $(0, 1)$:

$(0, 1) \rightarrow \left(\frac{0}{n}, 1\right) \rightarrow \left(\frac{0}{2}, 1\right) \rightarrow (0, 1)$.

The reference point changes from $(1, 2)$ to $\left(\frac{1}{2}, 2\right)$:

$(1, 2) \rightarrow \left(\frac{1}{n}, 2\right) \rightarrow \left(\frac{1}{2}, 2\right) \rightarrow \left(\frac{1}{2}, 2\right)$.

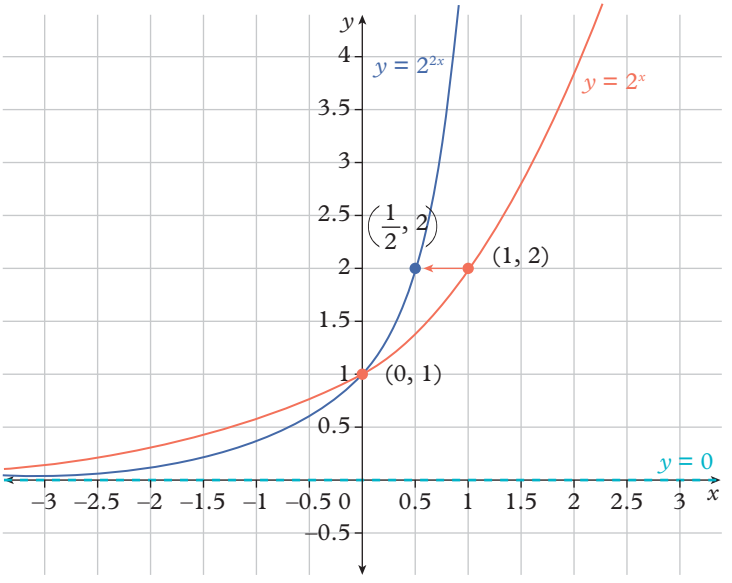
3 Sketch the curve for an exponential relationship through the y-intercept and the reference point, using the horizontal asymptote as a boundary. Label the y-intercept and asymptote.

WRITE

Asymptote $y = 0$

y-intercept: $(0, 1)$

curve passes through $\left(\frac{1}{2}, 2\right)$.



Helpful hints

- ✔ Using your graphing calculator to alter the exponential function will help you to better understand the transformations. You can start by moving a negative sign to different places in the same relationship to increase or decrease values, or you can compare the graphs for the following equations:
  - $y = -2^x$
  - $y = 2^{-x}$
  - $y = -2^{-x}$
  - $y = 2^x + 1$
  - $y = 2^x - 1$
- ✔ The graph of an exponential relationship can get increasingly closer to the asymptote but can never cross or touch it.
- ✔ Remember that every exponential graph has an asymptote and you must always include it, even when it is $y = 0$ (which is on the x-axis).

ANS p586

Exercise 4F Graphing exponential relationships

△ 1–11, 13, 15 □ 2–7, 9–12, 14, 16, 19 ⬠ 1(e, f), 3–7(d–f), 9, 10, 14, 17–19

4F.1 **1** Plot the graph of each of these exponential relationships by first completing a table of values for integer values of x from –2 to 2. Is there a pattern in the y value?

a $y = 3 \times 2^{-x}$ **b** $y = \frac{3}{5} \times 3^x$ **c** $y = 2 \times 4^x$

2 **a** Draw graphs of $y = 2^x$ and $y = 3^x$ for x values from –3 to 3 on the same Cartesian plane. You can plot the graphs using a table of values or digital technology.

b Compare the two curves. How are they different?

c Which features are the same?

d Write the y-intercept for each graph.

e Write the rule for the asymptote for each graph.

f Predict how the graph of $y = 5^x$ would compare to the two curves you have drawn.

3 **a** Draw graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{3}\right)^x$ for x values from –3 to 3 on the same Cartesian plane. You can plot the graphs using a table of values or digital technology.

b Compare the two curves. How are they different?

c Which features are the same?

d Write the y-intercept for each graph.

e Write the equation for the asymptote of each graph.

f Predict how the graph of $y = \left(\frac{1}{5}\right)^x$ would compare to the two curves you have drawn.

4 Draw the graphs of $y = 2^x$ and $y = 2^{-x}$ for x values from –3 to 3 on the same Cartesian plane and compare the two curves. You can plot the graphs using a table of values or digital technology.

a How are they different?

b Which features are the same?

c Write the y-intercept for each graph.

d Write the rule for the asymptote for each graph.

e Describe the transformation that has been performed on the basic graph of $y = 2^x$ to produce $y = 2^{-x}$.

f What transformations are needed to produce the graph of $y = -2^{-x}$ from the graph of $y = 2^x$?

4F.2 **5** Sketch the graph of each exponential relationship by performing a reflection.

a $y = -3^x$ **b** $y = 3^{-x}$ **c** $y = -3^{-x}$

d $y = \left(\frac{1}{2}\right)^{-x}$ **e** $y = -\left(\frac{1}{2}\right)^x$ **f** $y = -\left(\frac{1}{2}\right)^{-x}$

UNDERSTANDING AND FLUENCY

4F.3 **6** Sketch the graph of each exponential relationship by performing a vertical translation.

a $y = 2^x + 2$ **b** $y = 2^x - 2$ **c** $y = 2^x + 1$

d $y = \left(\frac{1}{3}\right)^x + 1$ **e** $y = \left(\frac{1}{3}\right)^x + 2$ **f** $y = \left(\frac{1}{3}\right)^x - 2$

4F.4 **7** Sketch the graph of each exponential relationship by performing a vertical dilation.

a $y = 2 \times 3^x$ **b** $y = 3 \times 3^x$ **c** $y = 0.5 \times 3^x$

d $y = 2 \times \left(\frac{1}{2}\right)^x$ **e** $y = 3 \times \left(\frac{1}{2}\right)^x$ **f** $y = 0.5 \times \left(\frac{1}{2}\right)^x$

4F.5 **8** Sketch the graph of each exponential relationship by performing a horizontal dilation.

a $y = 3^{2x}$ **b** $y = 3^{\frac{x}{2}}$ **c** $y = 3^{\frac{x}{3}}$

d $y = \left(\frac{1}{2}\right)^{2x}$ **e** $y = \left(\frac{1}{2}\right)^{\frac{x}{2}}$ **f** $y = \left(\frac{1}{2}\right)^{\frac{x}{3}}$

9 What is the rule obtained when the graph of $y = b^x$ is reflected in:

a the x-axis? **b** the y-axis? **c** the x-axis and the y-axis?

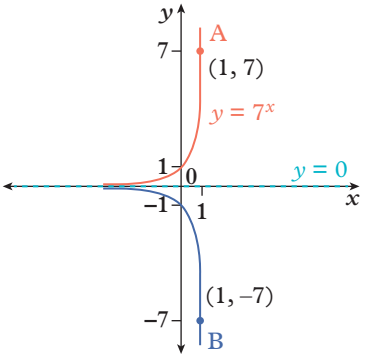
10 Add a row to the table of values you created in question **1** to find the ratio between consecutive values of the dependent variable. What do you notice? Use the exponent laws to explain your finding.

11 For each pair of graphs shown on a Cartesian plane below:

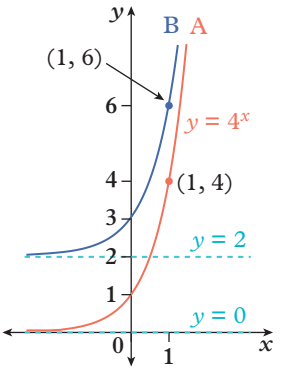
i identify the transformation performed on graph **A** to produce graph **B**

ii write the rule for graph **B**.

a



b



12 Rewrite the following equations in the form $y = A \times 2^{nx} + k$ using the exponent laws, and then sketch the graphs on the same Cartesian plane using dilations.

a $y = 2^{x+3}$ **b** $y = 2^{x+2}$ **c** $y = 2^{x+1}$

d $y = 2^{x-1}$ **e** $y = 2^{x-2}$ **f** $y = 2^{x-3}$

13 Create the graphs in question **12** using digital technology. Describe the horizontal translation needed to produce the graphs of parts **a–f** from the graph of $y = 2^x$.

PROBLEM SOLVING AND REASONING

14 Sienna invests \$2000 at 10% p.a. interest compounded annually. The amount, A (in dollars), of the investment after n years can be represented by $A = 2000 \times (1.1)^n$.

a What will the graph of this relationship look like? Sketch the graph showing the key features.

b Describe what happens to the value of A as n increases.

c Produce the graph of $A = 2000 \times (1.1)^n$ for $0 \leq n \leq 10$, using a table of values or digital technology to help you. Use the graph to find:

i the amount of the investment after 4 years

ii the number of years it takes for the investment amount to be more than \$4000.

15 The value of Gabriel's work tools depreciates at the rate of 20% p.a. compounded annually. The asset value, V (in dollars), after n years can be represented by $V = 3500 \times (1.25)^{-n}$.

a What will the graph of this relationship look like? Sketch the graph showing the key features.

b What was the initial value of Gabriel's tools?

c Describe what happens to the value of V as n increases.

d Produce the graph of $V = 3500 \times (1.25)^{-n}$ for $0 \leq n \leq 10$, using a table of values or digital technology to help you. Use the graph to find the number of years it takes for the asset value of the tools to be less than $1000.

16 A common mistake with exponential relationships is writing $a \times b^x$ as $(ab)^x$. For example, incorrectly writing 3×2^x as 6^x.

a Show that 3×2^x is not equivalent to 6^x by substituting $x = 2$ into each expression.

b Find, by inspection or by trial and error, the value of x for which 3×2^x is equal to 6^x.

c Considering your answers from parts **a** and **b**, explain the difference between the descriptions 'equal to' and 'equivalent to' in this context.

d Considering your answers from parts **a** and **b**, explain why checking the value of one variable is not enough to decide whether two expressions are equivalent.

17 The equation $x^2 = 2^x$ has three solutions.

a Complete the following table to find two of the solutions.

x	−2	−1	0	1	2	3	4	5
x^2								
2^x								

b Use the table to explain why the third solution must be between −1 and 0.

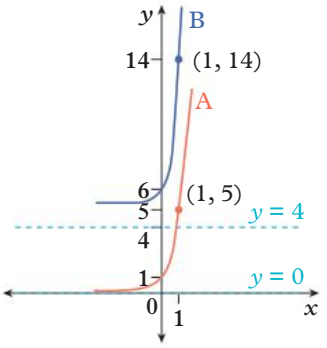
c Sketch the graphs of $y = x^2$ and $y = 2^x$ using appropriate technology, and find the value of the third solution to $x^2 = 2^x$, rounded to three decimal places.

18 For each pair of graphs shown on the Cartesian planes below:

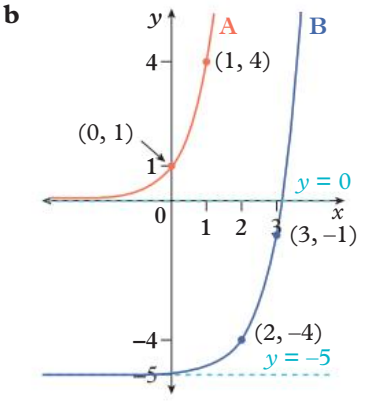
i identify the transformations performed on graph **A** to produce graph **B**

ii write the rules for graphs **A** and **B**.

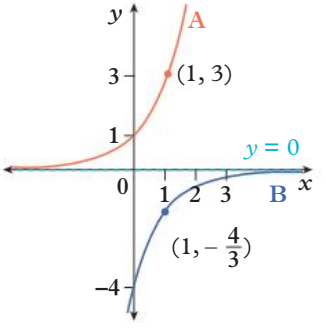
a



b



c



19 Describe the transformations performed on the graph of $y = 2^x$ to produce the graph of $y = -3 \times 2^{x+1} - 2$. Sketch the graph.

Check your Student obook pro for these digital resources and more:

- **Interactive skillsheet** — Exponential relationships
- **Investigation** — Intersection of two non-linear relationships
- **Topic quiz** — 4F

4G Rates of change

Learning intentions

By the end of this section you will be able to ...

- ✔ find the average rate of change and the gradient of the secant between two points
- ✔ approximate the instantaneous rate of change and the gradient of the tangent at a point.

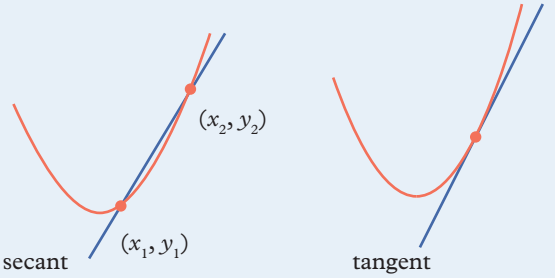
Inter-year links

Year 8 6F Finding linear equations

Year 9 3D Gradients and intercepts

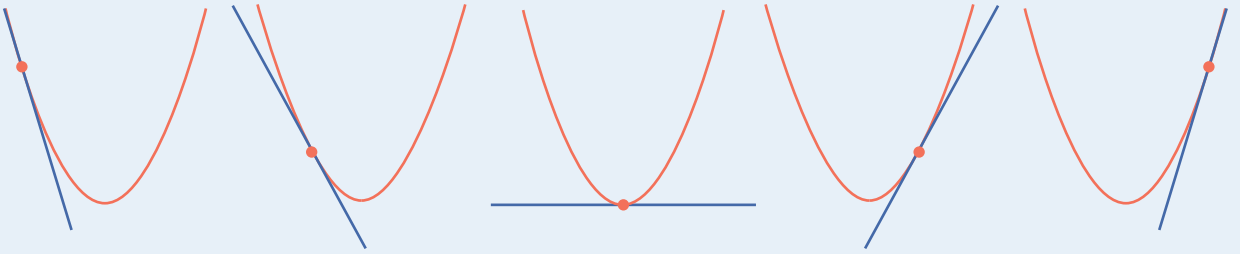
Secants and tangents

- A line that passes through at least two points of a non-linear graph is called a **secant**.
- The **tangent** of a curve at a point is the straight line that just touches, but does not cross, the curve at the point.

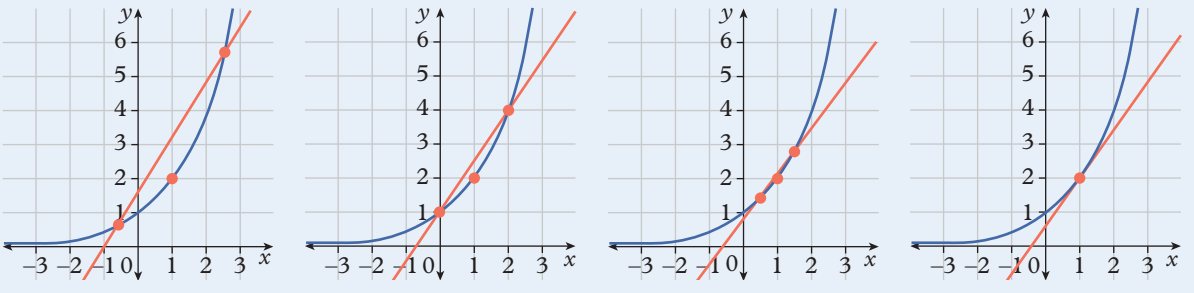


Gradient

- The gradient, m, of a linear relationship $y = mx + c$ is a constant, where $m = \frac{y_2 - y_1}{x_2 - x_1}$.
- The gradient of a non-linear relationship at a point is equal to the gradient of the tangent line at the point. The gradient of a non-linear relation depends on the value of x.



- Determining the gradient of a curve at a point is not easy if the tangent line is not given. The gradient of the tangent can be approximated by successively calculating the gradient of nearby secants that have increasingly smaller intervals between the two points that close in on the point of interest. The gradients of the secants approach the gradient of the tangent.



Rates of change

- The **instantaneous rate of change** at a point, m_T, is the gradient at that point.
- The **average rate of change** over an interval is the gradient of the line connecting the two endpoints of the interval.
  - → For linear relationships: the average rate of change is equal to the gradient of the line, which is constant.
  - → For non-linear relationships: the average rate of change from $A(x_1, y_1)$ to $B(x_2, y_2)$ is the gradient of the secant that passes through A and B: $\frac{y_2 - y_1}{x_2 - x_1}$.

Example 4G.1 Calculating the average rate of change

Calculate the average rate of change of $y = x^2$ from $x = 3$ to $x = 4$.

THINK

1 Determine the corresponding y-coordinates of the two points by substituting the x-coordinates into the equation.

2 Write the coordinates of the two points.

3 The average rate of change is the gradient of the secant between the two points.

WRITE

For $x_1 = 3$,
$y_1 = 3^2$
$= 9$
For $x_2 = 4$,
$y_2 = 4^2$
$= 16$
(3, 9), (4, 16)
$m = \frac{y_2 - y_1}{x_2 - x_1}$167
$= \frac{16 - 9}{4 - 3}$
$= \frac{7}{1}$
$= 7$

Example 4G.2 Approximating the instantaneous rate of change

Approximate the instantaneous rate of change of $y = x^2$ at $x = 3$ by evaluating the gradient of the secants between $x = 3$ and $x = 3.5$, $x = 3$ and $x = 3.1$, and $x = 3$ and $x = 3.01$.

THINK

1 Calculate the corresponding y-coordinates for each of the x-coordinates.

2 Calculate the gradient of the secants between the given points.

3 Approximate the instantaneous rate of change by estimating what value the gradients of the secants are approaching.

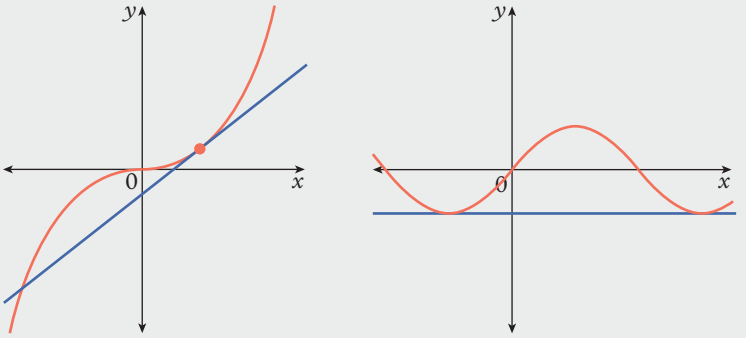
WRITE

x	3	3.5	3.1	3.01
y	9	12.25	9.61	9.0601

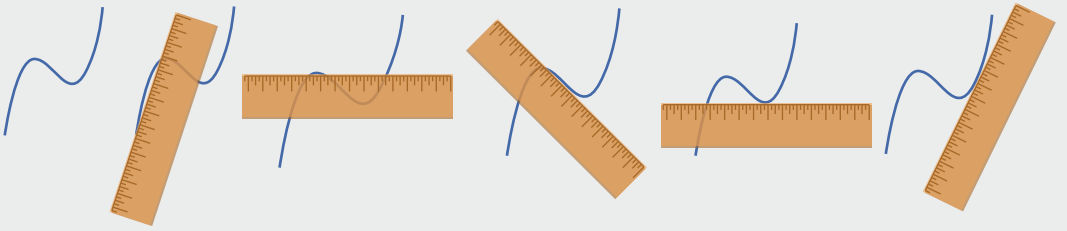
$m_1 = \frac{12.25 - 9}{3.5 - 3} = 6.5$
$m_2 = \frac{9.61 - 9}{3.1 - 3} = 6.1$
$m_3 = \frac{9.0601 - 9}{3.01 - 3} = 6.01$
The gradient of the secants is approaching 6, therefore, $m_T \approx 6$.

Helpful hints

✔ The tangent line at a point does not cross the curve in the nearby vicinity of the point but can touch or cross the graph again elsewhere. For example:



✔ The tangent line can be easily visualized by sliding a ruler along the graph of a non-linear relationship following the graph's shape.



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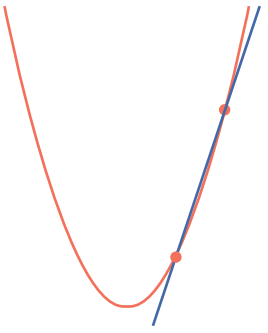
Exercise 4G Rates of change

▲ 1–4, 6, 8, 9 ■ 3–5, 7, 8, 10–12 ⬟ 3–4, 7, 10, 11, 13

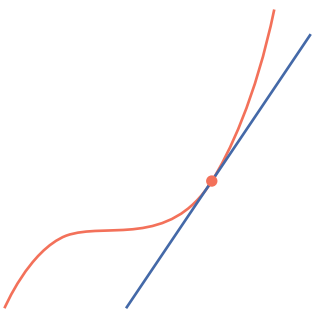
UNDERSTANDING AND FLUENCY

1 Determine which of the following is a tangent line.

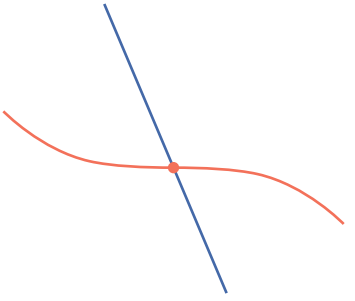
a



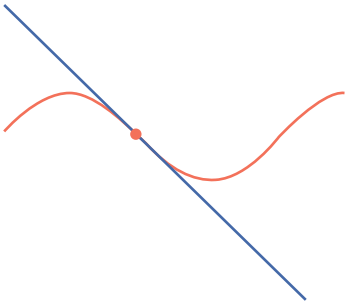
b



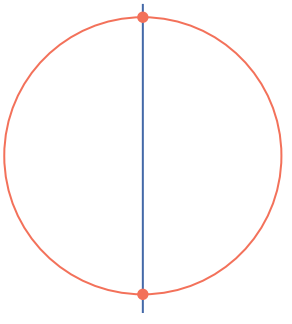
c



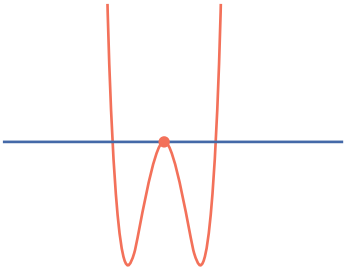
d



e

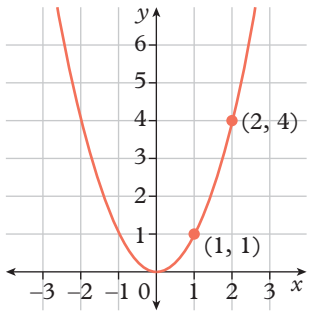


f

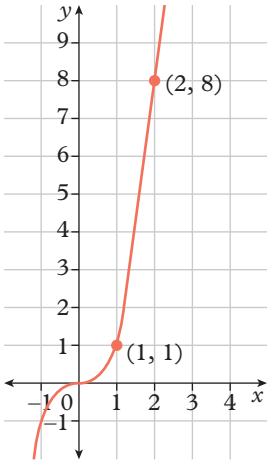


2 Calculate the average rate of change between the points on the following graphs.

a



b



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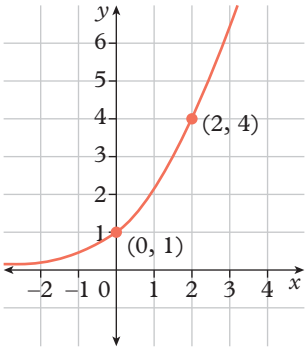

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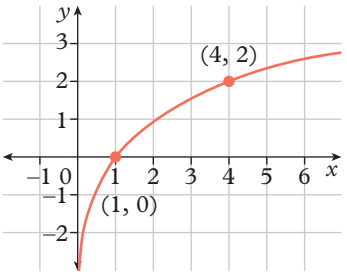
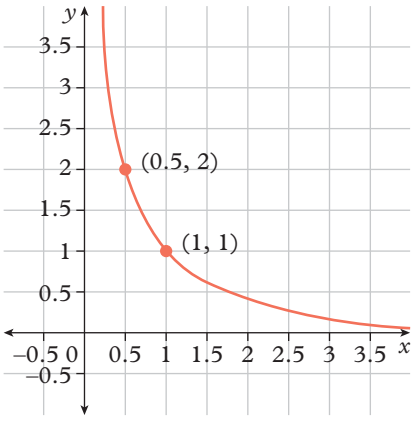
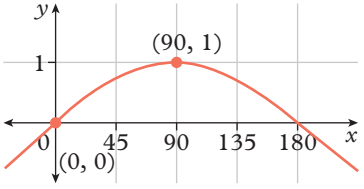


e



f



4G.1 **3** Calculate the average rate of change between the given x values of the following relationships:

a $y = x^2$ from $x = -3$ to $x = -2$

b $y = x^4$ from $x = 2$ to $x = 2.5$

c $y = 10^x$ from $x = -1$ to $x = 1$

d $y = x^3 - 3x^2$ from $x = 1$ to $x = 2$.

4G.2 **4** Approximate the instantaneous rate of change at the given x-coordinate by evaluating the gradients of the secants in the following intervals:

a $y = x^2$ at $x = -3$ between $x = -3$ to $x = -2.9$, $x = -3$ to $x = -2.99$, and $x = -3$ to $x = -2.999$

b $y = x^4$ at $x = 2$ between $x = 2$ to $x = 2.1$, $x = 2$ to $x = 2.01$, and $x = 2$ to $x = 2.001$

c $y = 10^x$ at $x = 0$ between $x = -1$ to $x = 1$, $x = -0.5$ to $x = 0.5$, and $x = -0.1$ to $x = 0.1$

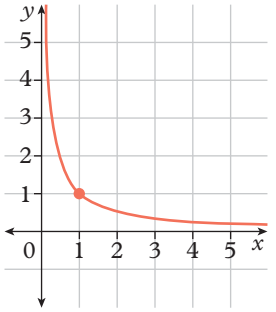
d $y = x^3 - 3x^2$ at $x = 1$ between $x = 1$ to $x = 2$, $x = 1$ to $x = 1.1$, and $x = 1$ to $x = 1.01$.

5 For each of the following graphs:

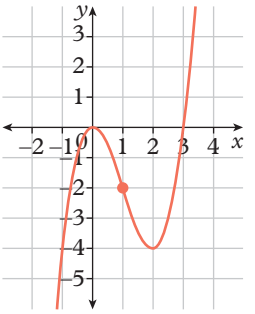
i use a ruler or straight edge to visualise the tangent at the shown point and write down the y-intercept of that tangent, if it exists, to the nearest integer

ii calculate the gradient of the tangent using your answer to part **i** and the point shown.

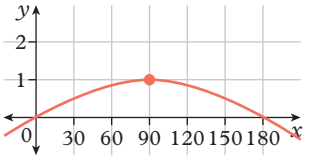
a



b



c



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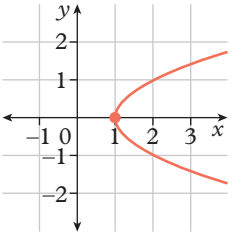
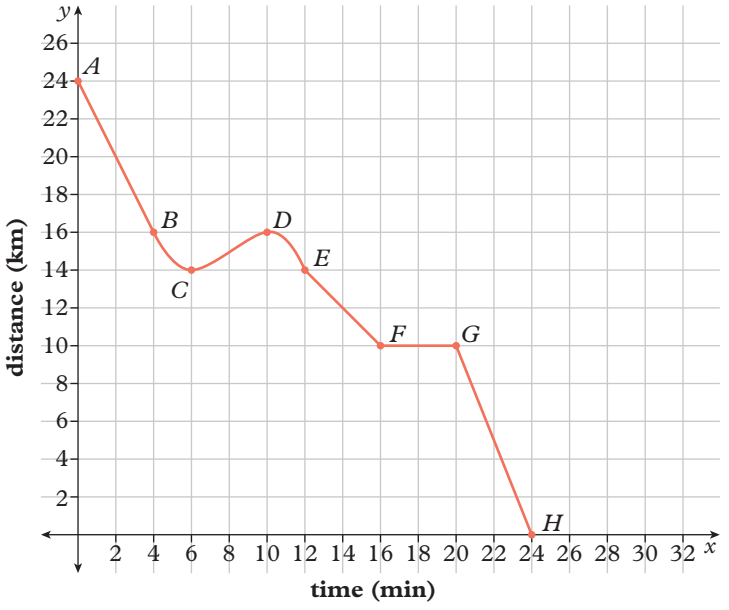


6 Consider the distance–time graph below.



Determine the average velocity in km/min:

a from A to B **b** from B to C **c** from C to D **d** from D to E **e** from A to H.

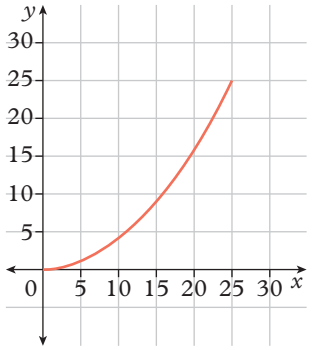
7 **a** Approximate the instantaneous rate of change of $y = x^2 - 5x + 6$ at $x = 1$ by evaluating the gradient of the secants between $x = 1$ and $x = 1.5$, $x = 1$ and $x = 1.1$, and $x = 1$ and $x = 1.01$.

b Sketch the graph of $y = x^2 - 5x + 6$ and the tangent to the graph at $x = 1$.

8 Joseph measured the instantaneous rate of change of a graph at three places and they were all equal to 5. Is the graph linear? Explain.

9 Describe how the gradient changes as x increases from 0.

a



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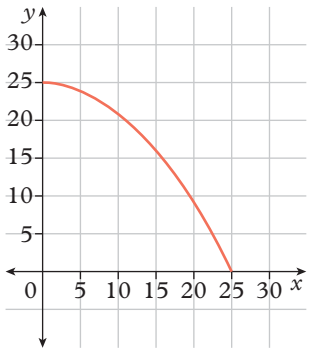

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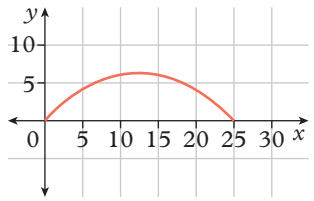

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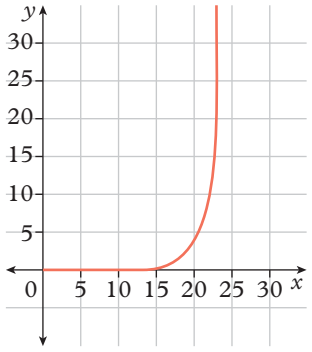

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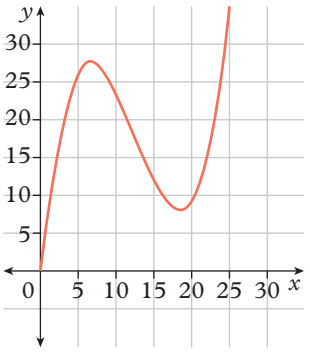

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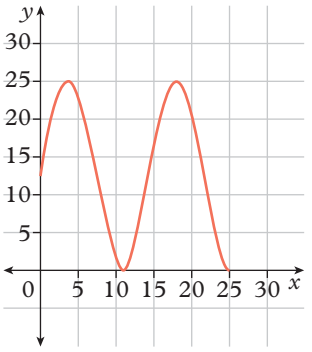
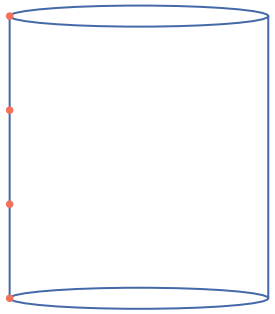
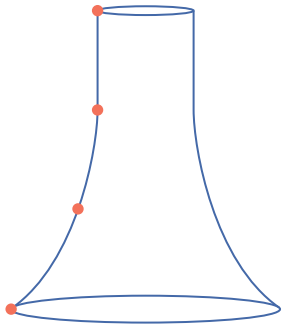


10 Water is being poured into each of the following vases at a constant rate. Sketch a graph with the height of the water on the vertical axis against the time on the horizontal axis. Note: each red dot on the vase marks an increase of 1 unit in height.

a



b



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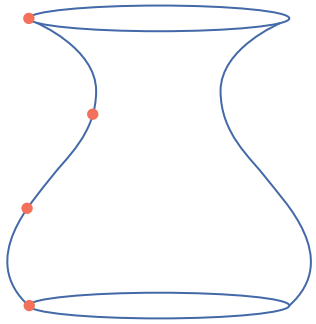

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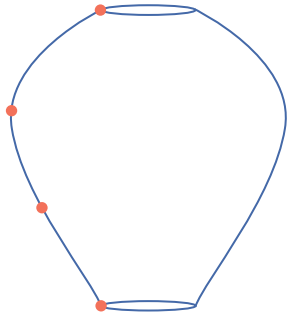

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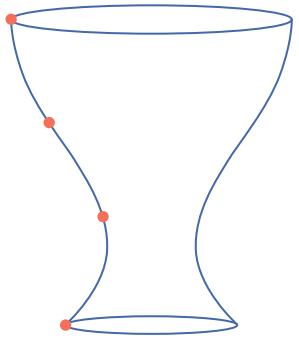

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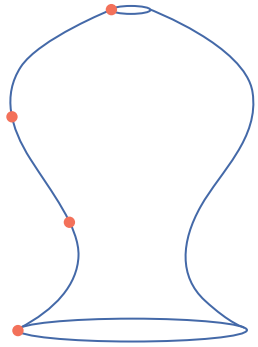


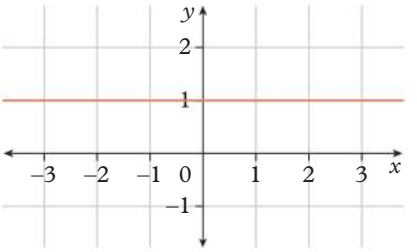
11 When approximating the gradient of a curve at $x = a$, we calculate the gradient of the secant connecting $x = a$ and $x = a + h$ for an decreasing values of h. Write an expression for the gradient of the following relationship at the given point using $x = a$ and $x = a + h$.

a $y = x^2$ at $x = 3$ **b** $y = x^3$ at $x = 1$ **c** $y = 3x^2 + 5x + 1$ at $x = 0$

12 Alternatively, we can use two points on either side of $x = a$: $x = a - h$ and $x = a + h$ for decreasing values of h. Write an expression for the gradient of the relationships in question **11** using $x = a - h$ and $x = a + h$.

CHALLENGE

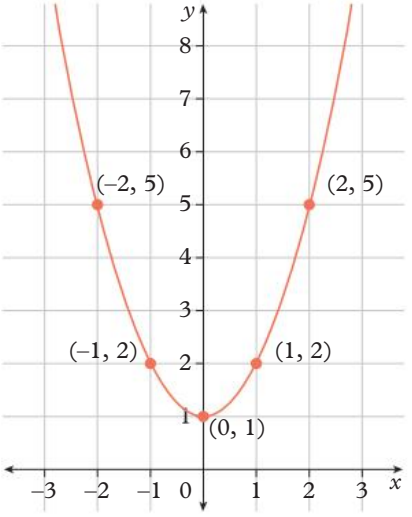
13 The graph on the right shows the gradient of $y = x + 1$. The equation of the graph is $y = 1$ because the gradient of $y = x + 1$ is always equal to 1.



a Sketch a graph to show the gradient of $y = x + 2$.

b If the graph of the gradient of an equation is a horizontal line, is the equation necessarily unique? Explain.

c Determine the gradient of the secants connecting adjacent points in the following graph.



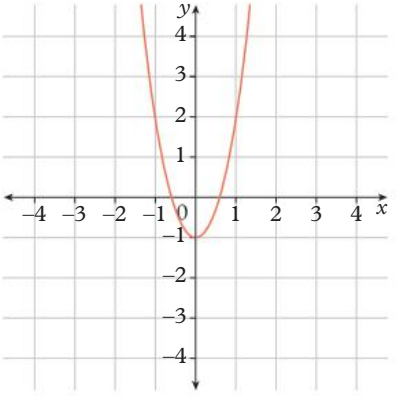
d Use your answers to part **c** to approximate the gradient of $y = x^2 + 1$ at the following points

x	−1.5	−0.5	0	0.5	1.5
Gradient					

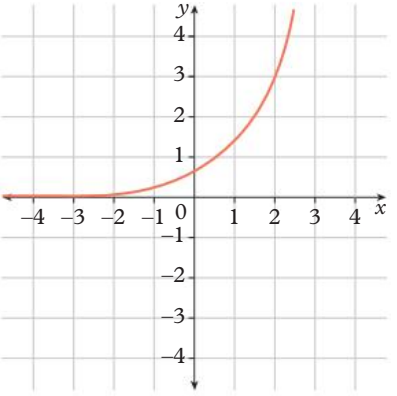
e Use the table of values from part **d** to sketch the graph of the gradient of $y = x^2 + 1$.

f Which of the following graphs shows the gradient of $y = x^2 - 1$?

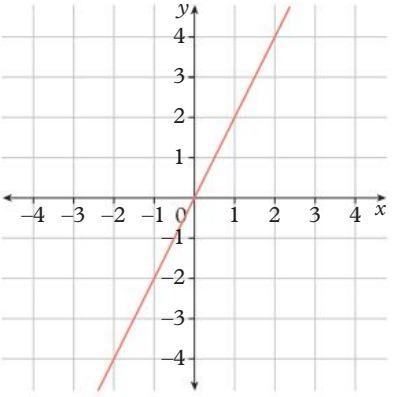
A



B



C



g If the graph of the gradient of an equation is a straight line, is the original equation necessarily unique? Explain.

Check your Student obook pro for these digital resources and more:

- **Interactive skillsheet** — Rates of change
- **Investigation** — Rates of change in linear, quadratic and exponential relationships
- **Topic quiz** — 4G

Chapter summary

The Null Factor Law

If $(x - p)(x - q) = 0$, then $x - p = 0$ or $x - q = 0$.

Quadratic formula

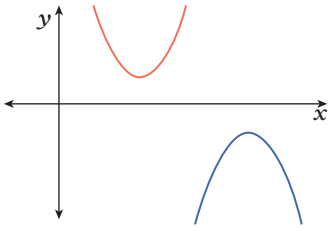
$ax^2 + bx + c = 0$ $\qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The discriminant

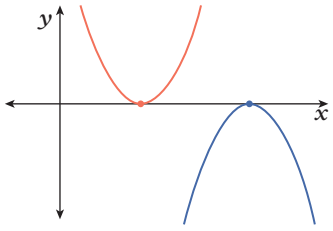
$\Delta = b^2 - 4ac$

- If $\Delta < 0$, there are no real solutions.
- If $\Delta = 0$, there is one real solution.
- If $\Delta > 0$, there are two real solutions.

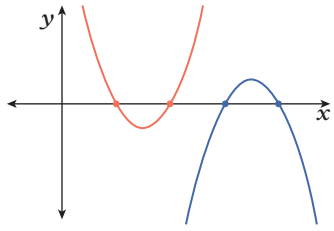
no x-intercepts



one x-intercept



two x-intercepts



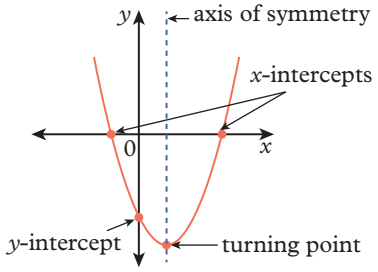
Parabolas

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

$y = a(x - h)^2 + k$

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

Stretch in the y-direction
If $a > 0$, vertical stretch up
If $0 < a < 1$, horizontal stretch outward
But if $a < 0$, invert



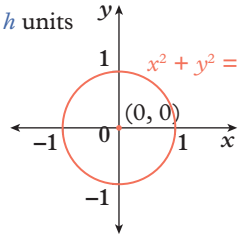
Circles

$(x - h)^2 + (y - k)^2 = r^2$

Radius of r units

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left



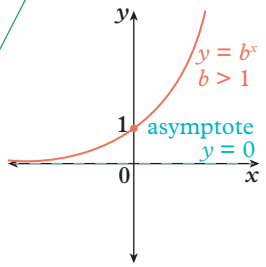
Exponential relationships

If $n > 0$, upright
If $n < 0$, horizontal reflection

$y = A \times b^{nx} + k$

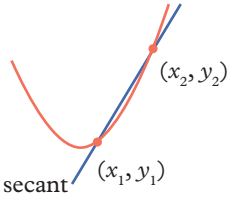
If $A > 0$, upright
If $A < 0$, vertical reflection

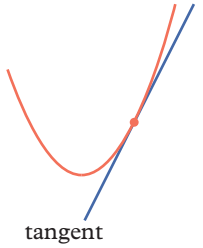
Vertical translation of k units
If $k > 0$, up
If $k < 0$, down



Rate of change

- The average rate of change over an interval is the gradient of the line connecting the two endpoints of the interval.
- The instantaneous rate of change at a point, m_T, is the gradient at that point.





Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

4A **1** Which of these is a quadratic equation?

A $y - 2 = x^2 + 4$ **B** $x^2 + y^2 = 9$ **C** $y = \frac{5}{x}$ **D** $3^x = 9$ **E** $y = 4x + 1$

4B **2** For the quadratic equation $2x - 4x^2 = 3$, which of these is the value of the discriminant?

A 52 **B** -8 **C** -44 **D** 44 **E** 0

4C **3** The coordinates of the x-intercepts of the graph of $y = x^2 - 6x + 5$ are:

A $(1, 0), (5, 0)$ **B** $(0, 1), (0, 5)$ **C** $(-3, 0), (2, 0)$
D $(-1, 0), (-5, 0)$ **E** $(0, -1), (0, -5)$

4C **4** Which of the following statements about the graph of $y = x^2 + 2x - 3$ is *incorrect*?

A It is upright. **B** The graph has two x-intercepts.
C The turning point has coordinates $(-4, -1)$. **D** The graph shows a minimum.
E The y-intercept is $(0, -3)$.

4D **5** Compared with the graph of $y = x^2$, the graph of $y = (x - 2)^2$ is:

A stretched by a factor of 2 in the y-direction. **B** reflected in the x-axis.
C translated 2 units up. **D** translated 2 units right.
E translated 2 units left.

4D **6** The coordinates of the turning point of the graph of $y = (x - 2)^2 - 5$ are:

A $(-2, -5)$ **B** $(2, -5)$ **C** $(-2, 5)$ **D** $(2, 5)$ **E** $(0, -1)$

4E **7** The coordinates of the centre of the circle $(x - 2)^2 + (y + 3)^2 = 16$ are:

A $(2, -3)$ **B** $(-2, 3)$ **C** $(-2, -3)$ **D** $(2, 3)$ **E** $(4, 4)$

4F **8** Which of these equations represents an exponential relationship?

A $y = 3x^2 + 4$ **B** $x^2 + y^2 = 4$ **C** $y = 3^x + 4$ **D** $y = \frac{1}{x}$ **E** $y = x^3$

4F **9** What is the rule obtained when the graph of $y = 6 \times 2^{3x} + 3$ is reflected across the x-axis?

A $y = 6 \times 2^{-3x} + 3$ **B** $y = -6 \times 2^{3x} + 3$ **C** $y = -6 \times 2^{-3x} - 3$
D $y = 6 \times 2^{\frac{x}{3}} + 3$ **E** $y = 6 \times 2^{3x} - 3$

4G **10** What is the average rate of change of $y = x^3 - 1$ between $x = -1$ and $x = 1$?

A 0 **B** 1 **C** 2 **D** -1 **E** -2

Short answer

4A **1** Solve each of these equations. Write your solutions as exact values.

a $(x - 2)^2 = 0$ **b** $(4x - 3)(3x + 4) = 0$ **c** $(\sqrt{2} - x)(\sqrt{2} + x) = 0$
d $3x^2 - 6x - 24 = 0$ **e** $5x^2 - 7 = 0$ **f** $3x^2 - 12x + 12 = 0$

4A **2** Solve each of these equations by completing the square. Write your solutions as exact values.

a $x^2 + 6x + 4 = 0$ **b** $x^2 - 3x - 3 = 0$

4B **3** Use the discriminant to identify the number of real solutions for each of these quadratic equations.

a $x^2 - 4x + 4 = 0$ **b** $3x^2 = 2x - 4$ **c** $5x - 4x^2 = 2$
d $4 - 3x^2 = 8x$ **e** $1 - x - x^2 = 0$ **f** $6x^2 + 4x = -1$

4B **4** Use the quadratic formula to solve each of these equations. Write your solutions as:

i exact values
ii approximate values, to two decimal places.

a $x^2 + 3x = 10$ **b** $5x^2 = 3x + 1$ **c** $x^2 = 5x - 2$
d $x^2 + 5x + 2 = 0$ **e** $2x^2 = x + 2$ **f** $3x^2 + 7x + 3 = 0$

4C **5** For each of these quadratic relationships, find the coordinates of the:

i x-intercept(s) **ii** y-intercept.

a $y = -x^2 + 25$ **b** $y = x^2 - 3x - 4$ **c** $y = 2x^2 - 8x + 8$ **d** $y = -x^2 - x + 6$

4C **6** Use intercepts to help you sketch the graph of each relationship below. Label the turning point with its coordinates each time.

a $y = x^2 - 4x + 3$ **b** $y = -x^2 - 4x + 5$

4D **7** For each of the quadratic relationships below:

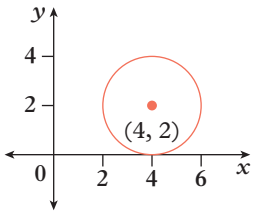
i describe the translations performed on $y = x^2$ **ii** write the coordinates of the turning point.

a $y = (x - 5)^2 + 2$ **b** $y = (x + 6)^2 - 4$

c $y = (x - 1)^2 - 1$ **d** $y = (x + 3)^2 + 5$

e $y = (x - 4)^2 + 2$ **f** $y = 3(x + 7)^2 - 3$

4E **8** Write the equation for the circle on the right.



4F **9** **a** Sketch the graph of $y = 5^x$.

b Describe the transformations performed on the graph of $y = 5^x$ to produce the graph of:

i $y = 5^{x+3}$ **ii** $y = 5^x + 3$

c Sketch all three graphs from parts **a** and **b** on the same Cartesian plane.

4G **10** Approximate the instantaneous rate of change at the given x-coordinate by evaluating the gradients of the secant between the following intervals:

a $y = x^3$ at $x = -3$ between $x = -3$ to $x = -2.9$, $x = -3$ to $x = -2.99$, and $x = -3$ to $x = -2.999$.

b $y = x^2 - x$ at $x = 2$ between $x = 2.1$ to $x = 2.5$, $x = 2$ to $x = 2.01$, and $x = 2$ to $x = 2.001$.

Analysis

1 The parabolas you have looked at so far are vertical parabolas, because they each have a vertical axis of symmetry. There are also horizontal parabolas, for which the axis of symmetry is horizontal. Similar to the way we have used $y = x^2$ as the basic rule for a vertical parabola, we can use $x = y^2$ as the basic rule for a horizontal parabola.

A horizontal parabola can be represented by the general equation $x - h = a(y - k)^2$, where (h, k) is the turning point of the parabola, and a is a constant.

a Consider the graph of $x + 4 = (y + 1)^2$.

i Provide values for a, h and k.

ii What are the coordinates of the turning point?

iii There is one x-intercept. Calculate its coordinates.

iv There are two y-intercepts. Calculate the coordinates of both points.

v Use the coordinates of the turning point and the intercepts to sketch the graph on a Cartesian plane.

vi Describe the transformations that need to be performed on the graph of $x = y^2$ to produce the same graph you sketched in part **v**.

b Follow the procedure in part **a** to sketch the graph of $x - 4 = -(y + 1)^2$ on the same Cartesian plane as the previous graph.

c Compare the two graphs from parts **a** and **b**. What is the effect of the negative value of a?

2 By now, you are familiar with the quadratic formula expressed in the form $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

An alternative form of the quadratic formula is $x = \dfrac{2c}{-b \pm \sqrt{b^2 - 4ac}}$.

a Show that both quadratic formulas give the same solution for the quadratic equation $x^2 - x - 3 = 0$. Give your solutions correct to one decimal place.

b Start with the first formula given above. Show how algebraic methods can be used to transform the standard quadratic formula into the second formula. Hint: Take the positive value of the square root, then rationalise the numerator.

Algorithmic Thinking

Algorithmic thinking: Number and algebra

An algorithm is a set of precise instructions that when followed will achieve a goal or solve a problem. Once the actions in an algorithm have been identified, it may be possible to encode the algorithm in a coding language so it can be automated by a computer or machine.

The term iteration refers to repetitive actions in algorithms, as shown in Example 1 and 2. Iterations can be done within other iterations in an algorithm. This is referred to as nested iteration.

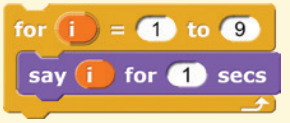
Some algorithms contain actions that are only done if certain conditions are true; these are called conditional actions. Conditions and iterations can be combined to create more intelligent algorithms, as shown in Example 3.

Examples 1 to 3 are written in pseudocode and have been coded using the block coding language Snap!

Example 1

This algorithm shows iteration.

For each positive integer ≤ 9:
    print the integer



Example 2

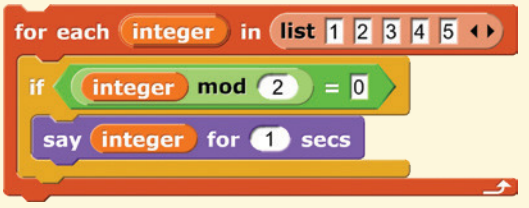
This algorithm shows nested iteration.

For each integer in the set {1, 3, 6}:
    For each value in the set {5, 10}:
        print (integer × value)

Example 3

This algorithm combines conditions and iterations.

For each integer in the set {1, 2, 3, 4, 5}:
    If the integer is even then:
        print the integer



Notice that indentation is used in Examples 1–3 to group the actions that belong to the iterations and conditions. These correspond to the blocks in Snap!

Task 1

1 What are the algorithms in Examples 1–3 printing out in each case?

2 Using some of the ideas for structuring iteration and conditions shown in Examples 1–3, write an algorithm using iteration and conditional actions that will find all the factors of 12.

3 What are the iterative actions of your algorithm?

4 What are the conditional actions of your algorithm?

5 **CHALLENGE:** Code your algorithm into a coding language such as Snap! and try it out.

Algorithms can be used to solve algebraic problems, such as factorising expressions and equations.
Example 4 shows an algorithm for factorising a quadratic expression $x^2 + bx + c$, where the coefficients b and c are integers. The restrictions on the values of b and c within this algorithm are called pre-conditions.

Example 4

Here are the steps for factorising the expression $x^2 + bx + c$, expressed as an algorithm.

1 If b, c are not integers then stop.

2 Find the integer factors of c that add to get b. Call them d and e, so $c = de$ and $b = d + e$. If no integer factors of c that add to b are found then stop.

3 Split the bx term in the quadratic using the integer factors found in step 2:

$x^2 + bx + c = x^2 + dx + ex + de$.

4 Group the four terms in pairs using $b = d + e$ and $c = de$:

$= x(x + d) + e(x + d)$

5 Factorise by grouping using the common factor:

$= (x + d)(x + e)$

Example 5

This example demonstrates running the algorithm defined in Example 4 for the quadratic expression $x^2 + 7x + 10$. Note that $b = 7$ and $c = 10$.

Step 1: Since b and c are integers, continue.

Step 2: $2 \times 5 = 10$ and $2 + 5 = 7$. Therefore $d = 2$ and $e = 5$, continue.

Step 3: $x^2 + 7x + 10 = x^2 + 2x + 5x + (5 \times 2)$

Step 4: $= x(x + 2) + 5(x + 2)$

Step 5: $= (x + 2)(x + 5)$

Task 2

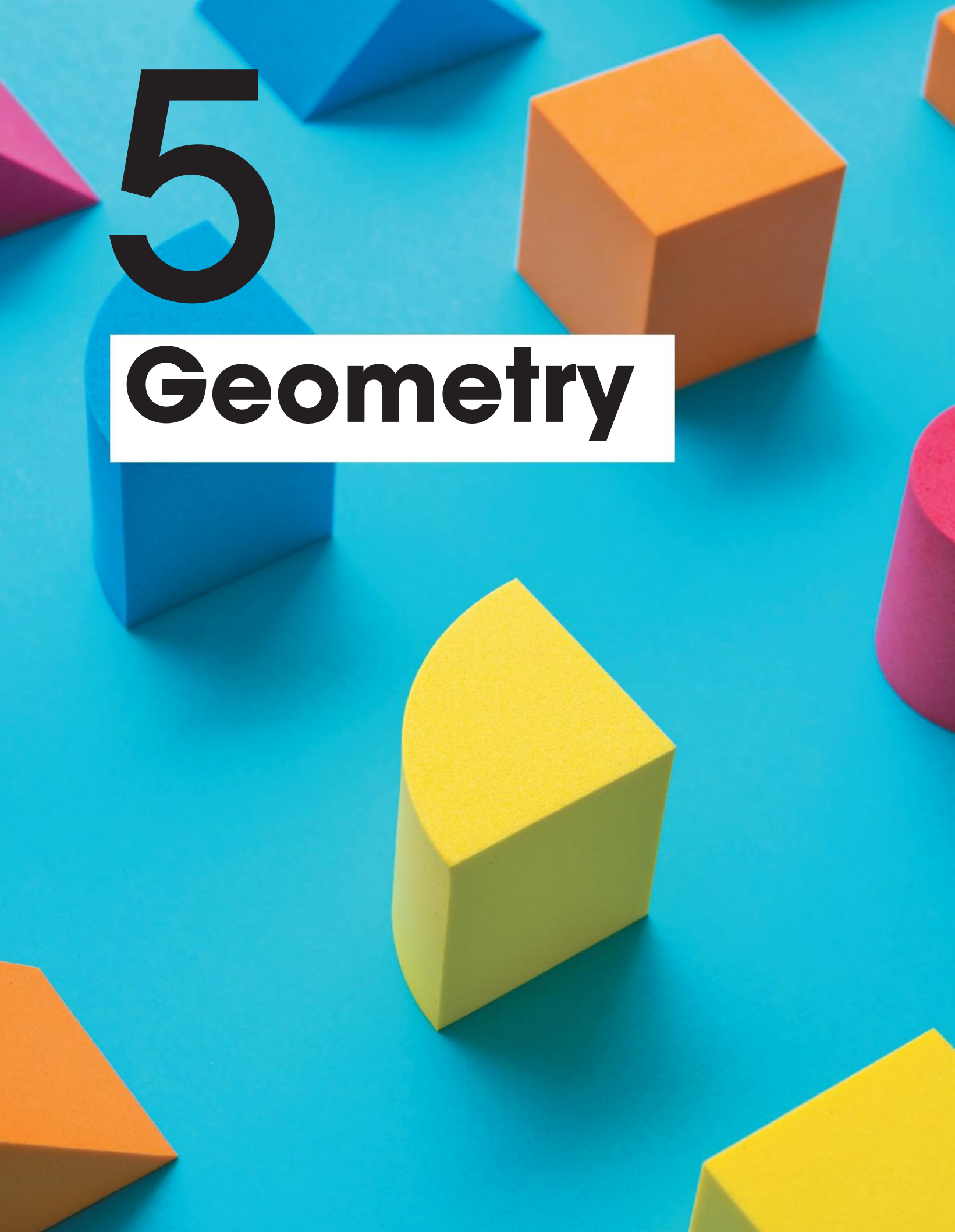
1 Use the algorithm from Example 4 to factorise $x^2 + 16x + 15$.

2 What will happen if the algorithm tries to factorise $x^2 + 7x + 17$?

3 What are the conditional actions of the algorithm in Example 4?

4 How can the algorithm from Task 1 be used to assist in factorising a quadratic expression of the form $x^2 + bx + c$?

5 **CHALLENGE:** Code the quadratic factorising algorithm into a coding language such as Snap! using parts of your coded algorithm from Task 1.



5 Geometry

Index

5A Geometry review
5B Geometric proofs
5C Congruence and similarity
5D Proofs and quadrilaterals
5E Circle geometry: angles
5F Circle geometry: chords

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Triangle properties
- ✔ Quadrilateral properties

Curriculum links

- Apply deductive reasoning to proofs involving shapes in the plane and use theorems to solve spatial problems (AC9M10SP01)

Optional content

- Relationships between angles and various lines associated with circles (radii, diameters, chords, tangents)

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Materials

- ✔ Calculator

5A Geometry review

Learning intentions

By the end of this topic you will be able to ...

✔ determine the size of unknown angles.

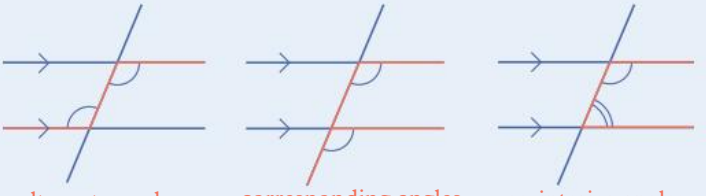
Inter-year links

Year 7 7B Angles at a point

Year 8 7A Angles

Angles

- **Complementary angles** sum to 90°.
- **Supplementary angles** sum to 180°.
- Angles at a point sum to 360°.
- **Vertically opposite angles** are equal.
- When parallel lines are crossed by a **transversal**, a number of equal and supplementary angles are formed.



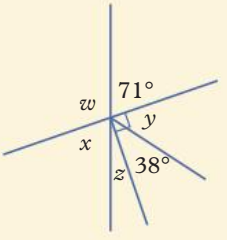
alternate angles (equal) | corresponding angles (equal) | co-interior angles (supplementary)

Polygons

- A **polygon** is a closed shape with straight sides.
- A triangle is a three-sided polygon with an interior angle sum of 180°.
- A **quadrilateral** is a four-sided polygon with an interior angle sum of 360°.
- The interior angle sum of any polygon is $(n - 2) \times 180°$, where n is the number of sides of the polygon.
- The **exterior angle** sum of any polygon is 360°.

Example 5A.1 Finding unknown angles using angle properties

Find the size of the labelled unknown angles in this diagram.



THINK

1 Angle w is supplementary to the 71° angle.

2 Angle x is vertically opposite the 71° angle.

3 Angle y is complementary to the 38° angle.

4 Angle z is supplementary to the angles 71°, y and 38°.

WRITE

$w = 180° - 71°$
$= 109°$

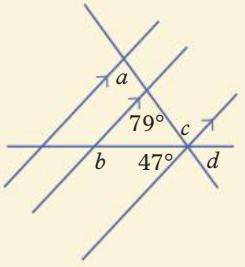
$x = 71°$

$y = 90° - 38°$
$= 52°$

$z = 180° - 71° - 52° - 38°$
$= 19°$

Example 5A.2 Finding sizes of angles using angle relationships with parallel lines

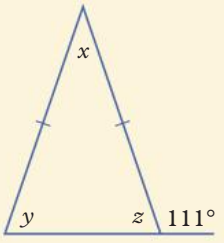
Find the values of the pronumerals in this diagram.



THINK	WRITE
1 Angle a is a corresponding angle to the 79° angle. Corresponding angles on parallel lines are equal.	$a = 79°$
2 Angle b is co-interior with the 47° angle. Co-interior angles on parallel lines are supplementary.	$b = 180° - 47°$ $= 133°$
3 Angle c is alternate to the 79° angle. Alternate angles on parallel lines are equal.	$c = 79°$
4 Angle d is supplementary to angle c and the unlabelled angle beside it, which is vertically opposite and, therefore, equal to 47°.	$d = 180° - 79° - 47°$ $= 54°$

Example 5A.3 Using triangle properties to find the sizes of angles

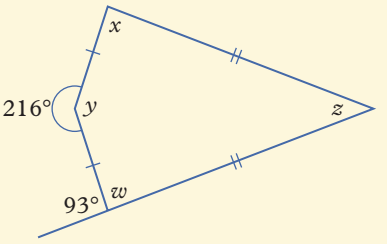
Find the value of each pronumeral in this diagram.



THINK	WRITE
1 Angle z is supplementary to the 111° angle.	$z = 180° - 111°$ $= 69°$
2 Angle y is equal to angle z because the triangle is isosceles. Base angles of an isosceles triangle are equal.	$y = z = 69°$
3 Angle x is supplementary to angles y and z because there are 180° in a triangle.	$x = 180° - 69° - 69°$ $= 42°$

Example 5A.4 Using quadrilateral properties to find sizes of angles

Find the value of each pronumeral in this diagram of a kite.



THINK	WRITE
1 Angle w is supplementary to 93°.	$w = 180° - 93°$ $= 87°$
2 Angle x is equal to angle w because the opposite angles formed from the unequal sides of a kite are equal.	$x = w = 87°$
3 Angle y and the angle labelled 216° meet at a point, therefore they add to 360°.	$y = 360° - 216°$ $= 144°$
4 Angles w, x, y and z add to 360° because the interior angle sum of a quadrilateral is 360°.	$z = 360° - 144° - 87° - 87°$ $= 42°$

Helpful hints

✔ When calculating unknown angles, you might not be given the required information to determine all the unknown angles (at first). Initially, you may need to use the angle sizes you can calculate to help you find the angle sizes you are unable to determine.

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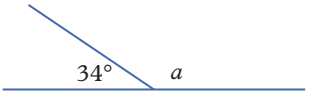
Exercise 5A Geometry review

△ 1–8, 10(a–c), 11 ■ 1–7, 9, 10(b, d, e), 11, 12(a, b), 13(b, d), 14(a–e) ⬠ 3, 5, 6, 9, 10(d–f), 11, 12(c), 13, 14

UNDERSTANDING AND FLUENCY

5A.1 **1** Find the sizes of the labelled unknown angles in these diagrams.

a



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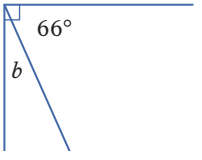

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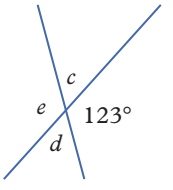

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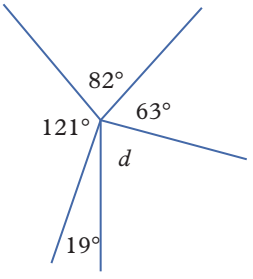

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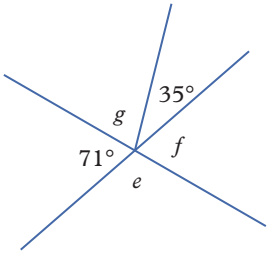

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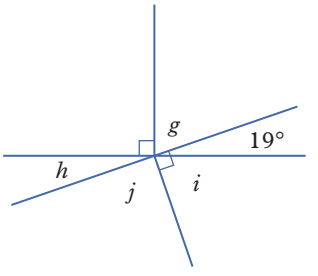
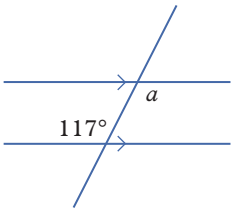


5A.2 **2** Find the value of each pronumeral in these diagrams.

a



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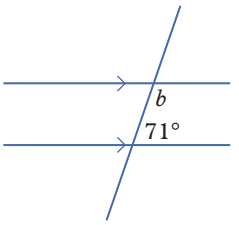

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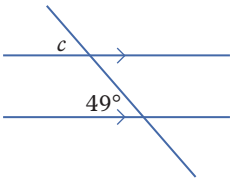

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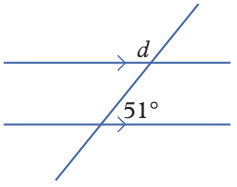

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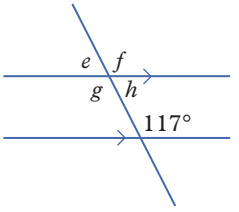

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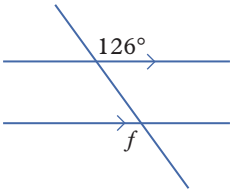
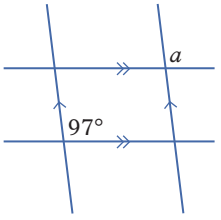


3 Find the value of each pronumeral in these diagrams.

a



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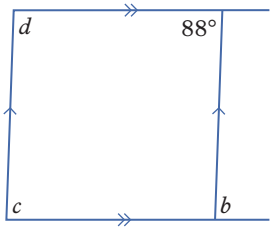

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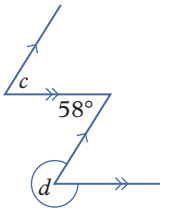

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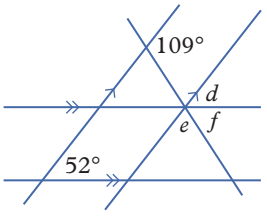

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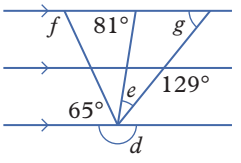

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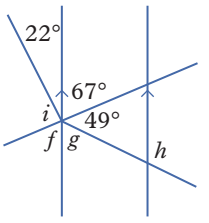
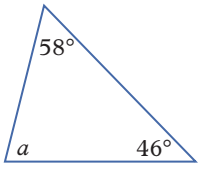


5A.3 **4** Find the value of each pronumeral in these diagrams.

a



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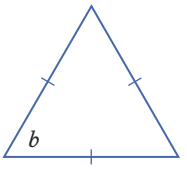

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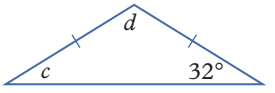

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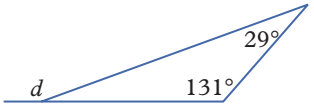

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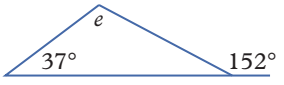

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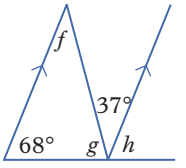
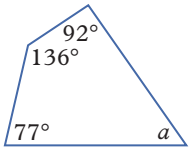


5A.4 **5** Find the value of each pronumeral in these diagrams.

a



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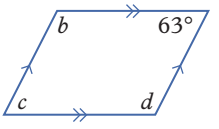

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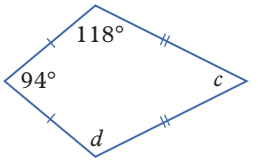

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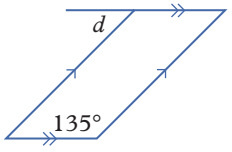

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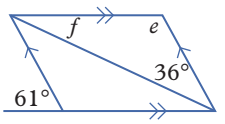

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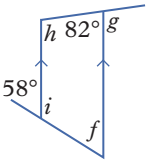


6 Use the expression $(n - 2) \times 180°$ to find the internal angle sum of a polygon with:

a 20 sides **b** 50 sides **c** 100 sides.

7 Explain why you can use the formula $x = \frac{(n-2) \times 180°}{n}$ to find the size of an individual internal angle in any regular polygon.

8 Find the size of an individual internal angle for a:

a regular heptagon **b** regular nonagon **c** regular hexagon.

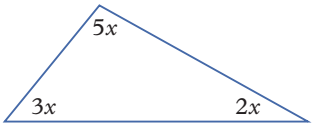
Round your answers to two decimal places.

9 Find the number of sides of a regular polygon that has each of its internal angles equal to:

a 160° **b** 144° **c** 170° **d** 179.64°

10 Find the value of x in each of these diagrams. Round the answer to two decimal places when necessary.

a



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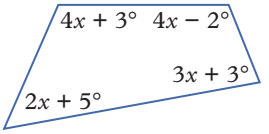

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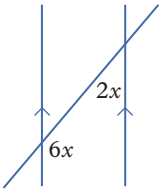

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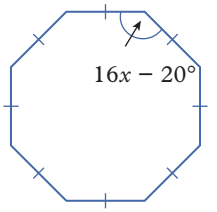

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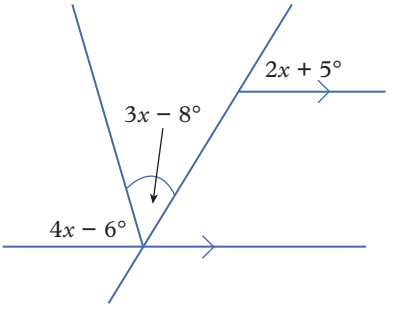

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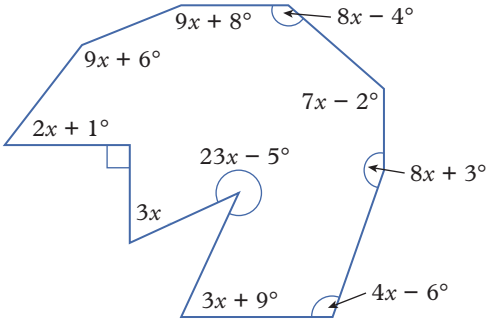


11 Decide whether each of these statements is true or false. If false, write a correct version of the statement.

a A kite has a pair of equal opposite angles.

b Angles at a point add to 180°.

c An isosceles triangle has two pairs of equal angles.

d Corresponding angles on parallel lines are supplementary.

e A square is a rhombus.

f A parallelogram is a rectangle.

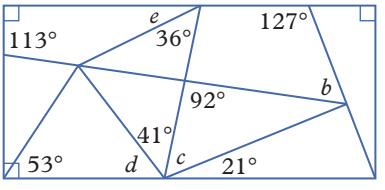
g Complementary angles add to 180°.

h A triangle that has one 60° angle must be equilateral.

i A quadrilateral can have a maximum of one concave angle.

12 Find the value of each pronumeral in these diagrams.

a



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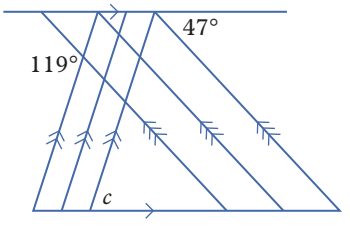

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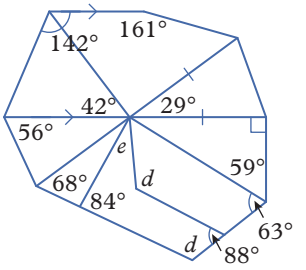
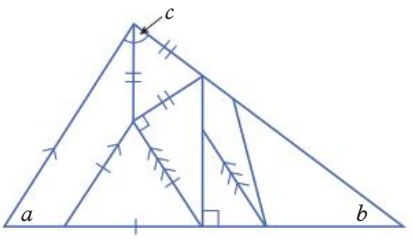


13 Find the value of each pronumeral in these diagrams.

a



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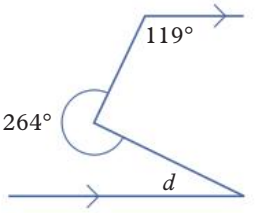

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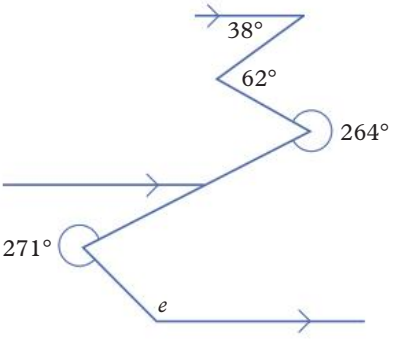

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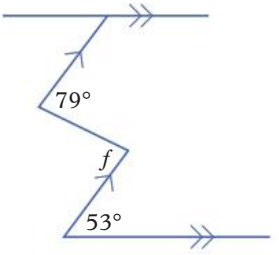

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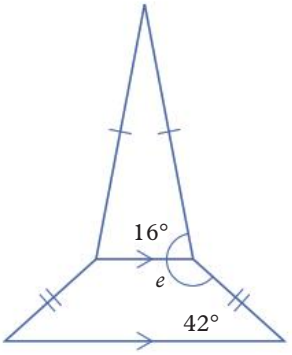

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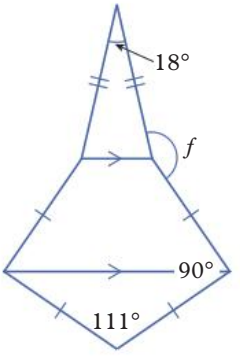


14 Explementary angles are a pair of angles that form a revolution; that is, they add to 360°.
Determine the size of each of the following angles:

a the explement of 30°
b the explement of 175°
c the explement of 286°
d the explement of the supplement of 45°
e the explement of the complement of 81°
f the complement of the explement of 324°
g the supplement of the explement of 267°
h the complement of the supplement of the explement of 200°
i the supplement of the complement of the explement of 299°
j the explement of the angle alternate to 64° on parallel lines
k the explement of the angle co-interior to 145° on parallel lines
l the explement of the exterior angle of a regular hexagon.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Angles at a point

Interactive skillsheet
Angles and parallel lines

Topic quiz
5A

5B Geometric proofs

Learning intentions

By the end of this topic you will be able to ...

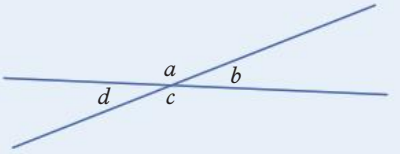
- ✔ understand what constitutes a mathematical proof
- ✔ create a mathematical proof consisting of a sequence of logically connected steps.

Inter-year links

Year 7 7C Angles and parallel lines

Year 8 7A Angles

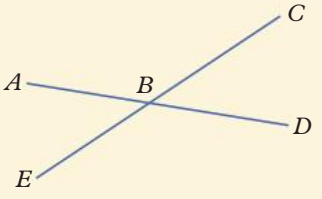
Mathematical proofs



- In mathematics, a practical demonstration can be used to show something is true. For example, we can show that vertically opposite angles are equal by cutting out one of the angles and placing it on top of the other angle.
- On the other hand, a mathematical **proof** explains why something is true through a series of logically connected statements. Mathematical proofs are constructed to show that a theory is true in all cases.
- Mathematical proofs begin with self-evident or given statements. For example, you can use the fact that there are 180° on a straight line as a given statement to prove that vertically opposite angles are equal (as shown in Example 5B.1).
- The following list of given statements can be used to create the proofs in this section:
  - → Angles at a right angle add to 90°.
  - → Angles on a straight line add to 180°.
  - → Angles at a point add to 360°.
  - → If point B lies in the interior of $\angle AOC$, then $\angle AOB + \angle BOC = \angle AOC$ (angle addition postulate).
  - → The interior angle sum of a triangle is 180° (as shown in Exercise 5B, question 4).
  - → If $a = b$ and $a = c$, then $b = c$ (transitive law).
  - → The same number can be added or subtracted from both sides of an equation to create an equivalent equation (addition and subtraction properties of equality).
  - → Any other information as indicated by the question (e.g. angles and parallel lines properties).
- The symbol $\therefore$ is shorthand for 'therefore' and the symbol $\because$ is shorthand for 'because'.
- The letters 'QED' or the 'tombstone', ■ or □, are often written at the end of a proof. The acronym QED stands for the Latin phrase 'quod erat demonstrandum', which means 'which was to be demonstrated'.
- One way to structure mathematical proofs is to lay them out in two columns, with mathematical statements given in the left-hand column, and reasons why each statement is true given in the right-hand column.

Example 5B.1 Proving that vertically opposite angles are equal

For the diagram below, prove that $\angle ABC = \angle DBE$.



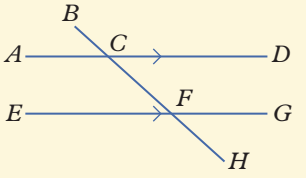
THINK

1. Set up a two-column table with 'Statements' as the heading of the left-hand column and 'Reasons' as the heading of the right-hand column.
2. Use the fact that angles on a straight line add to 180° to help you write two equations, making sure to include a common angle in both equations (for example, $\angle CBD$).
3. The right-hand sides of the equations from step **2** are the same, so the left-hand sides of the equations must also be equal (according to the transitive law). Express this in a new equation.
4. Simplify the new equation by subtracting $\angle CBD$ from both sides of the equation.
5. Include a symbol (□ or ■) or write 'QED' to show that your proof is complete.

WRITE

Statements	Reasons
$\angle ABC + \angle CBD = 180°$	Angles on a straight line add to 180°
$\angle DBE + \angle CBD = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle CBD = \angle DBE + \angle CBD$	Transitive law
$\angle ABC = \angle DBE$ ■	Subtraction property of equality

Example 5B.2 Proving that alternate angles on parallel lines are equal



For the diagram on the right, prove that $\angle ACF = \angle CFG$, given that $\angle BCD = \angle CFG$.

THINK

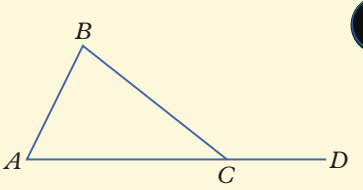
1. Set up a two-column table for the proof and write the given information in the first row.
2. Use the fact that angles on a straight line add to 180° to write two equations, making sure to include a common angle in both equations (for example, $\angle DCF$).
3. Equate the left-hand sides of the equations (because the right-hand sides are equal).
4. Simplify the equation by subtracting $\angle DCF$ from both sides of the equation.
5. Because $\angle BCD = \angle CFG$, substitute $\angle CFG$ for $\angle BCD$ in the equation (using the transitive law).

WRITE

Statements	Reasons
$\angle BCD = \angle CFG$	Given
$\angle ACF + \angle DCF = 180°$	Angles on a straight line add to 180°
$\angle BCD + \angle DCF = 180°$	Angles on a straight line add to 180°
$\angle ACF + \angle DCF = \angle BCD + \angle DCF$	Transitive law
$\angle ACF = \angle BCD$	Subtraction property of equality
$\therefore \angle ACF = \angle CFG$ ■	Transitive law

Example 5B.3 Proving the exterior angle theorem

From the diagram on the right, prove that $\angle BCD = \angle ABC + \angle CAB$.



THINK

1. Set up a two-column table for the proof.
2. Use the fact that angles on a straight line sum to 180° to write an equation linking the exterior angle of the triangle to the adjacent interior angle of the triangle.
3. Write an equation linking the interior angles of the triangle.
4. Equate the left-hand sides of the two equations (because the right-hand sides are equal).
5. Simplify by subtracting $\angle BCA$ from both sides.

WRITE

Statements	Reasons
$\angle BCD + \angle BCA = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle CAB + \angle BCA = 180°$	Interior angle sum of a triangle is 180°
$\angle BCD + \angle BCA = \angle ABC + \angle CAB + \angle BCA$	Transitive law
$\angle BCD = \angle ABC + \angle CAB$ ■	Subtraction property of equality

Helpful hints

✔ Be patient when constructing mathematical proofs. Start with the information that you are given and build on that information. You may not see the whole proof at first but, by taking it one step at a time, you will work to the required conclusion.

ANS p594

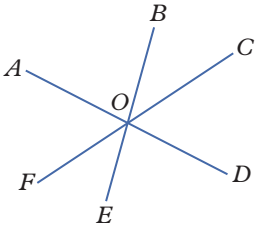
Exercise 5B Geometric proofs

1–6, 10 | 2, 4–8, 10 | 4–10

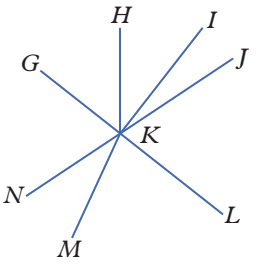
5B.1 **1** For the diagram on the right, prove that $\angle EOF = \angle BOC$. Complete the proof twice, starting with each of the following opening statements:

a $\angle EOF + \angle FOB = 180°$

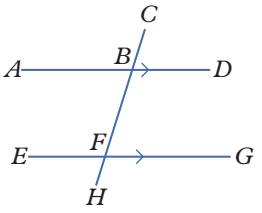
b $\angle EOF + \angle AOF + \angle AOB = 180°$.



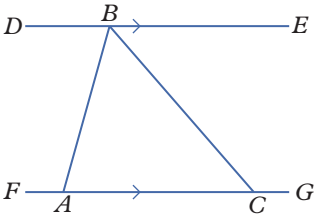
2 For the diagram below, prove that $\angle GKJ = \angle NKL$.



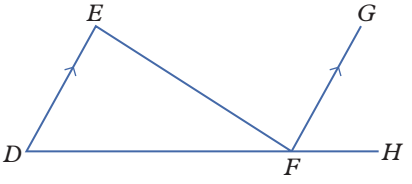
5B.2 **3** For the diagram below, prove that $\angle DBF$ is supplementary to $\angle BFG$, given that $\angle ABC = \angle EFB$.



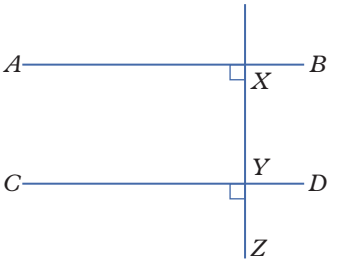
5B.3 **4** Use your knowledge of angles and parallel lines, as well as the diagram below, to prove that the internal angle sum of a triangle is 180°.



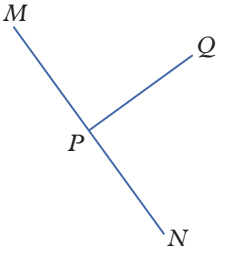
5 For the diagram below, use the fact that DE and FG are parallel to prove that $\angle EFH = \angle DEF + \angle FDE$.



6 Given that X lies on AB and Y lies on CD, prove that AB is parallel to CD.
Note: 'If co-interior angles are supplementary, they lie on parallel lines' can be used as a reason.



7 Given that $\angle MPQ$ and $\angle QPN$ are supplementary, prove that MN is a straight line.
Note: 'Supplementary angles sum to 180°' can be used as a reason.



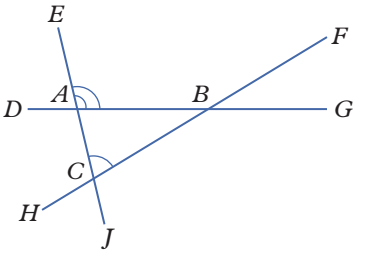
PROBLEM SOLVING AND REASONING

8 There are different forms of mathematical proofs. The proofs you have seen in this section so far are called 'direct proofs', where a statement is established from known facts. Another form of proof is 'proof by contradiction', where we assume that what we want to prove is not true, and then we demonstrate that this assumption is false.

We can use proof by contradiction to prove that, if corresponding angles are equal, the lines being considered are parallel. Start with the statement, 'If two corresponding angles are equal, the lines being considered are not parallel'.

a Why does the lines not being parallel mean that the lines would intersect?

b Consider the diagram on the right, showing intersecting lines DG and HF, cut by the transversal EJ.



Name the two angles in the diagram that you are assuming to be equal.

c The other assumption that you must make is that the length $AC > 0$. Why do you need to make this assumption?

d If $\angle EAB = \angle ACB = x$, show that $\angle BAC = 180° - x$.

e If $\angle ABC$ can be represented by y, write a statement adding together all the angles in $\triangle ABC$.

f Show that simplifying this statement leaves you with $y = 0°$.

g How does this imply that AC does equal 0?

h Explain how this is 'proof by contradiction' that, if corresponding angles are equal, the lines must be parallel.

i Write the proof in full.

CHALLENGE

9 Another form of proof is 'proof by contrapositive'. The contrapositive of the statement 'if X, then Y' is 'if not Y, then not X'. These two statements are logically equivalent. That is, if you are able to prove the contrapositive, the original statement is also proved to be true.

a Write the contrapositive of the statement, 'When two lines are cut by a transversal, if the corresponding angles are equal then the lines are parallel'.

b In the diagram from question **8**, let EJ be the transversal. Prove the contrapositive of the statement from part **a** using the diagram and the fact that AB is not parallel to CB ($AB \nparallel CB$).

Note: The following additional reasons can be used:

- Three points that do not form a line form a triangle.
- The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.
- Angles cannot be equal to 0°.

10 It is important not to make errors when completing proofs because, if you are not careful, you may think you have proven something that is actually false. The following example appears to 'prove' that $2 = 1$. Find the error.

$a = b$, where a and b are not equal to zero.

So: $a^2 = ab$

and: $a^2 - b^2 = ab - b^2$

Factorising gives: $(a + b)(a - b) = b(a - b)$

Dividing by $(a - b)$ gives: $a + b = b$

Remember that $a = b$.

So: $b + b = b$

$2b = b$

Dividing by b gives: $2 = 1$

Check your Student obook pro for these digital resources and more: pro

Topic quiz
5B

5C Congruence and similarity

Learning intentions

By the end of this topic you will be able to ...

- ✔ find unknown side lengths and angles in congruent and similar triangles
- ✔ identify whether a pair of triangles are congruent or similar
- ✔ prove that a pair of triangles are congruent or similar.

Inter-year links

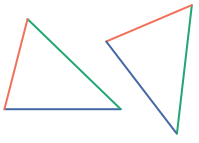
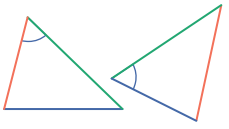
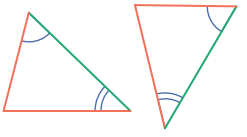
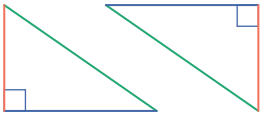
Year 7	7D Classifying triangles
Year 8	7G Congruent triangles
Year 9	6C Dilations and similar figures

Congruent figures

- **Congruent** figures are identical in shape and size but can be in any position or orientation.
- Two figures are congruent if their corresponding sides are the same length and their corresponding angles are the same size.
- The symbol for congruence is ≅ ('is congruent to').

Congruent triangles

- If two triangles meet any of the four conditions shown in the table below, they are congruent.

SSS (side–side–side)	**SAS (side–angle–side)**	**AAS (angle–angle–side)**	**RHS (right angle–hypotenuse–side)**
When two triangles have all their corresponding sides the same length (SSS), the corresponding angles are also the same size. This means that the triangles are congruent. 	Two of the side lengths are equal and the included angle (between those sides) is equal. 	Two of the angles and a corresponding pair of side lengths are equal. 	If the triangles are right-angled, the hypotenuses and another pair of side lengths are equal. 

- If a pair of triangles fails any of the four conditions for congruence shown above, they are not congruent.
- If $\triangle ABC \cong \triangle DEF$, then vertex A corresponds to vertex D, vertex B corresponds to vertex E, and vertex C corresponds to vertex F.

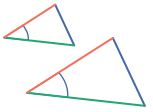
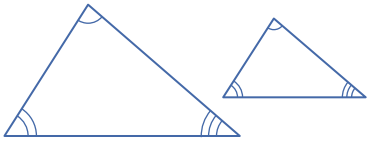
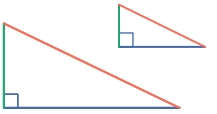
Similar figures

- **Similar** figures are identical in shape but can be different in size.
- For two figures to be similar, all angles must be equal in size and the lengths of all corresponding sides must be in the same ratio.
- If two figures are similar, an unknown side length or angle can be found if the scale factor is known.
- The symbol for similarity is ~ ('is similar to').
- The scale factor can be calculated using the formula:

$$\text{scale factor} = \frac{\text{image length}}{\text{original length}}$$

Similar triangles

- For two triangles to be similar, they must meet one of four conditions of similarity.

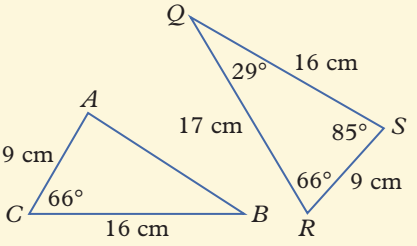
SSS (side–side–side)	**SAS (side–angle–side)**	**AAA (angle–angle–angle)**	**RHS (right angle–hypotenuse–side)**
The lengths of all three pairs of corresponding sides are in the same ratio.	The lengths of two pairs of corresponding sides are in the same ratio and the included angles are equal. 	Each of the three angles in one triangle is equal to an angle in the other triangle. 	In two right-angled triangles, the lengths of their hypotenuses are in the same ratio as the lengths of another pair of corresponding sides. 

Proving triangles are congruent or similar

- In addition to the given statements in 5B, the following statements can be used to create the proofs in this section:
  - → Vertically opposite angles are equal.
  - → Alternate angles on parallel lines are equal.
  - → Corresponding angles on parallel lines are equal.
  - → Co-interior angles on parallel lines are supplementary.
- Properties of different types of triangles can also be used.

Example 5C.1 Identifying whether two triangles are congruent

Decide whether these triangles are congruent, giving a reason for your answer.



THINK

1 Look at the given information to determine which congruence condition to use. We are given the sizes of two sides and an included angle for $\triangle ABC$. This fits the SAS condition for congruence.

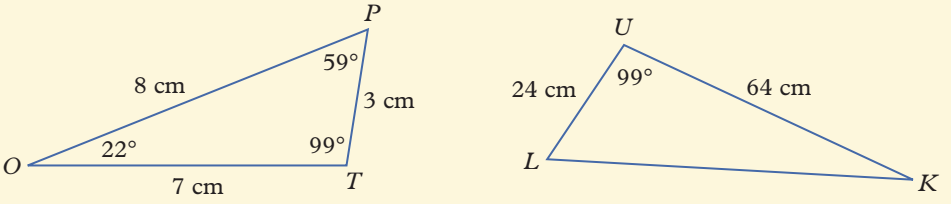
WRITE

$\triangle ABC$ has two sides and an included angle, which fits the SAS condition for congruence.

2 Identify the corresponding sides in the second triangle.	RS corresponds to AC. BC corresponds to QS.
3 Compare the sizes of the included angles between the matching sides in each triangle.	$\angle ACB = 66° \neq \angle RSQ$.
4 Write a conclusion.	The two triangles are not congruent because they fail the SAS condition for congruence.

Example 5C.2 Identifying whether two triangles are similar

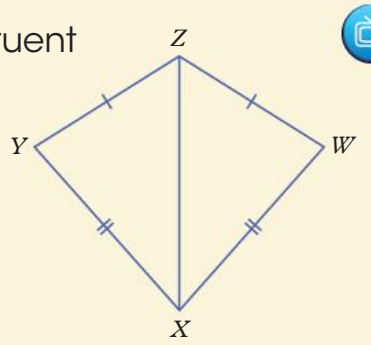
Decide whether these triangles are similar, giving a reason for your answer.



THINK	WRITE
1 Look at the given information to determine which similarity condition to use. $\triangle LUK$ has two sides and an included angle, which fits the SAS condition for similarity.	$\triangle LUK$ has two sides and an included angle, which fits the SAS condition for similarity.
2 Use the given angle in $\triangle LUK$ to find the corresponding sides.	$\angle PTO = \angle LUK = 99°$ PT corresponds to LU. TO corresponds to UK. OP corresponds to KL.
3 Determine the scale factor used for one of the pairs of matching sides (LU and PT).	scale factor $= \dfrac{\text{image length}}{\text{original length}}$ $= \dfrac{LU}{PT}$ $= \dfrac{24}{3}$ $= 8$
4 Find the scale factor used for the other pair of matching sides (TO and UK).	scale factor $= \dfrac{\text{image length}}{\text{original length}}$ $= \dfrac{UK}{TO}$ $= \dfrac{64}{7}$ ≈ 9.14
5 Write a conclusion.	The two triangles are not similar because these two scale factors are not the same, meaning their corresponding side lengths are not in the same ratio. They fail the SAS condition for similarity.

Example 5C.3 Proving that two triangles are congruent

For the diagram on the right, prove that $\triangle XYZ \cong \triangle XWZ$ using a congruence condition.



THINK

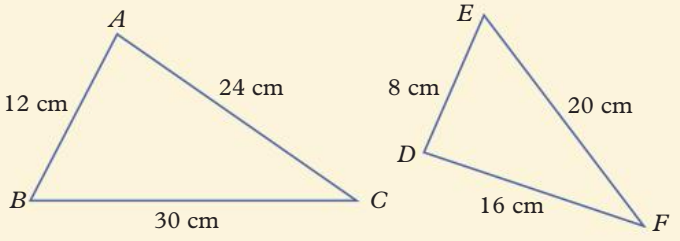
1. Set up a two-column table for your proof.
2. List the given information.
3. Determine whether the given information meets any of the conditions for congruence.
4. Complete the proof.

WRITE

Statements	Reasons
$XY = XW$	Given
$ZY = ZW$	Given
XZ is common to both triangles	Common side
$\therefore \triangle XYZ \cong \triangle XWZ$ ■	SSS (congruent)

Example 5C.4 Proving that two triangles are similar

Prove that $\triangle ABC$ is similar to $\triangle DEF$ using one of the conditions for similarity.



THINK

1. Identify the corresponding side for the two triangles. For example, AB is the shortest side in $\triangle ABC$ and DE is the shortest side in $\triangle DEF$.
2. Calculate the scale factor for each pair of corresponding sides.
3. Determine whether the given information meets any of the conditions for similarity.
4. Complete the proof.

WRITE

Statements	Reasons
$\frac{DE}{AB} = \frac{8}{12} = \frac{2}{3}$	Scale factor between corresponding sides
$\frac{EF}{BC} = \frac{20}{30} = \frac{2}{3}$	Scale factor between corresponding sides
$\frac{DF}{AC} = \frac{16}{24} = \frac{2}{3}$	Scale factor between corresponding sides
$\therefore \triangle ABC \sim \triangle DEF$ ■	SSS (similar)

Helpful hints

- ✔ If two triangles meet one of the conditions for congruence, they will meet all the conditions for congruence. Likewise, if two triangles fail one of the conditions for congruence, they will fail all the conditions for congruence.
- ✔ When writing a congruence or similarity statement, the corresponding vertices in each triangle must be written in the same order.

ANS p595

Exercise 5C Congruence and similarity

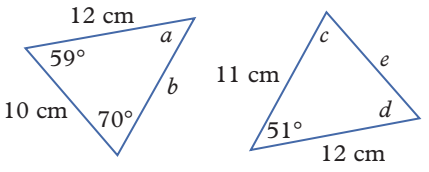
1–5, 6(a, c, d), 7, 8(a), 9(a), 10(a, b), 12, 15

1(c, d), 2(d–f), 3, 4, 5(d–f), 6, 8, 9(b), 10(c,d), 11, 13, 16, 17, 18(a)

3, 4, 6(d–f), 7–9, 11, 14, 16, 18(b), 19, 20

1 Find the unknown side lengths and angles in each of these pairs of congruent triangles.

a



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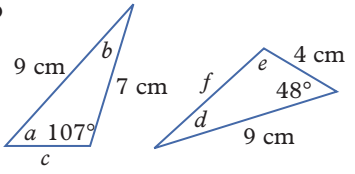
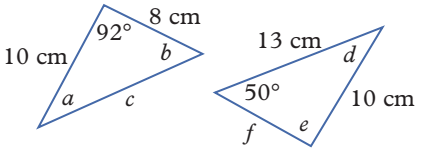


c



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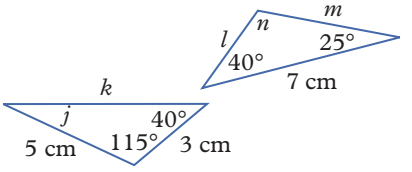
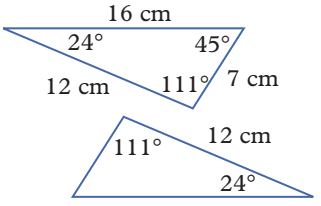


5C.1 **2** Decide whether the triangles in each of these pairs are congruent, giving a reason for your answer each time.

a



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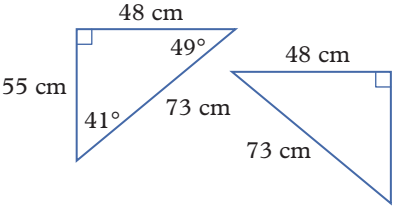

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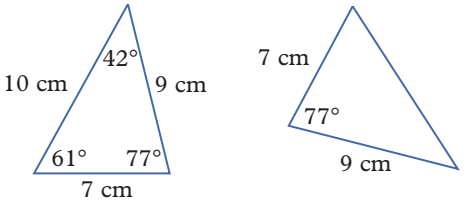

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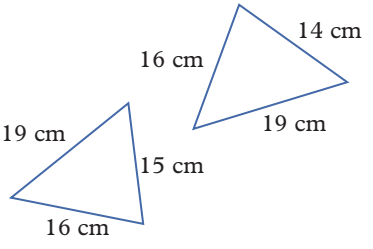

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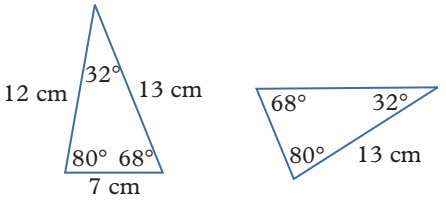

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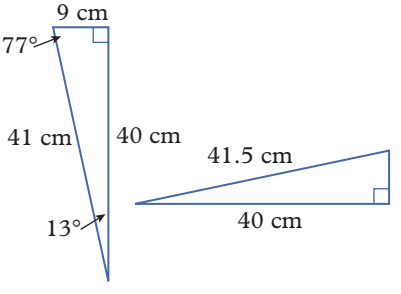
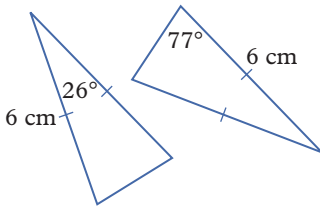


UNDERSTANDING AND FLUENCY

3 Use your understanding of triangle properties to decide whether the triangles in each of these pairs are congruent.

a



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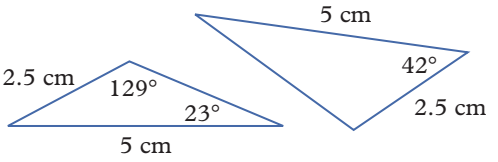

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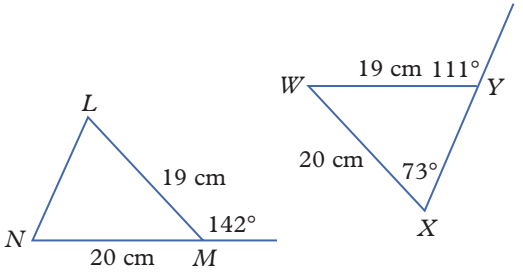

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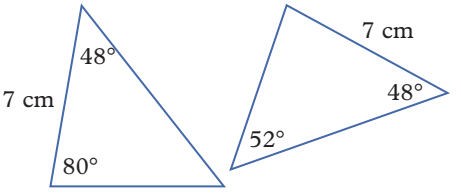
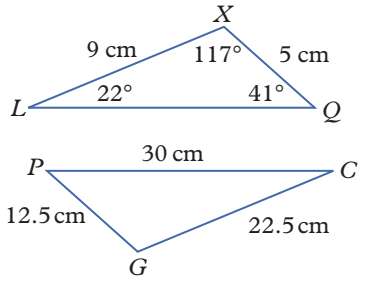


4 Find the scale factor between the triangles in each of these pairs of similar triangles. Assume that the first triangle in each pair is the original.

a



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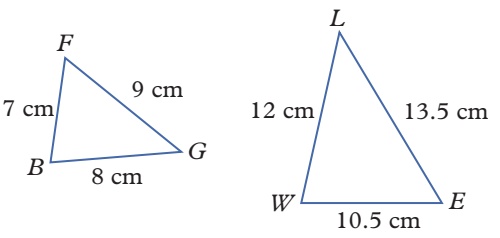

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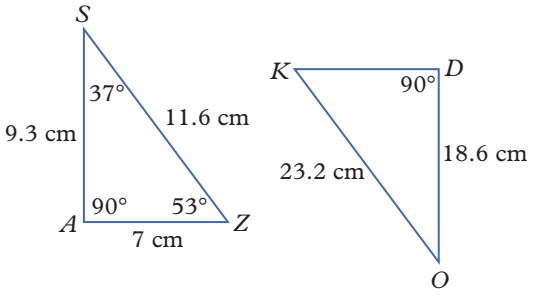

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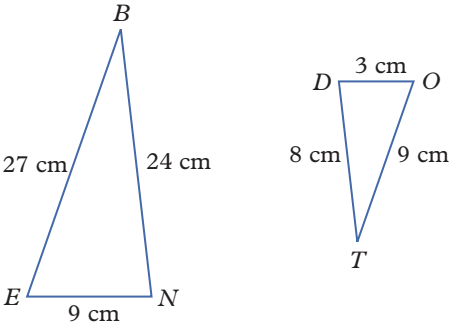
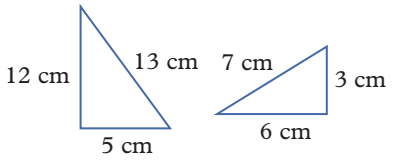


5C.2 **5** Decide whether the triangles in each of these pairs are similar, giving reasons for your answers.

a



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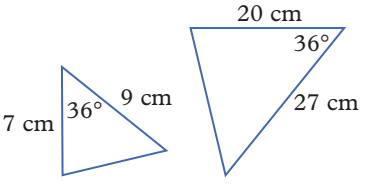

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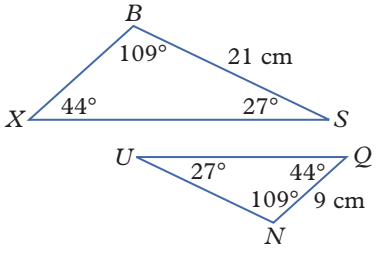

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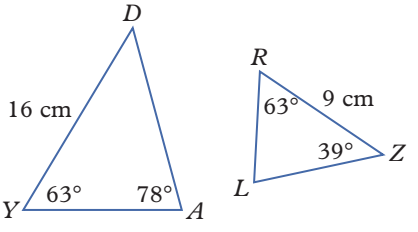

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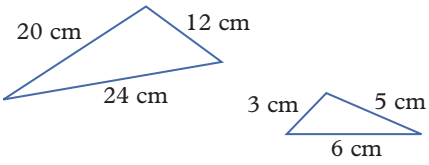

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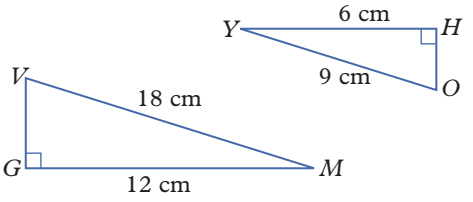
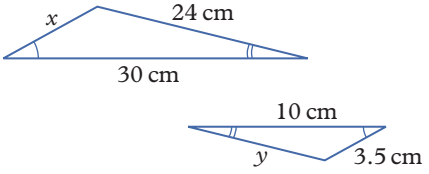


6 Find the unknown side lengths if the triangles in each pair below are similar.

a



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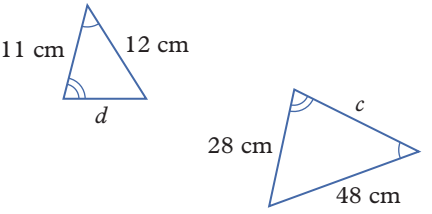

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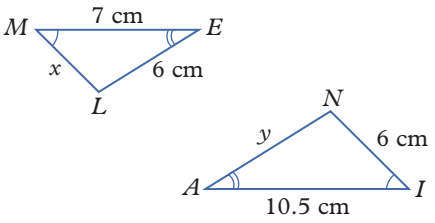

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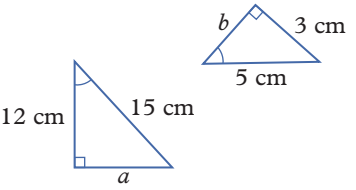

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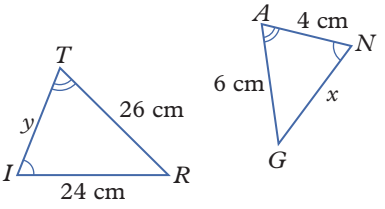

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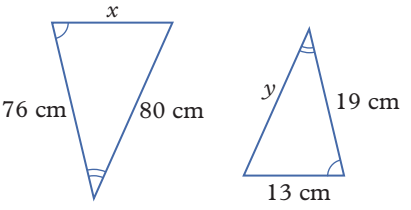
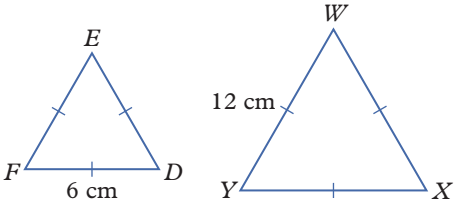


7 Use your understanding of triangle properties to decide whether the triangles in each of these pairs are similar.

a



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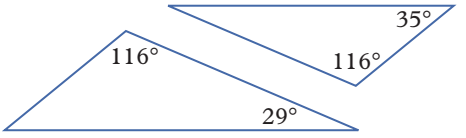

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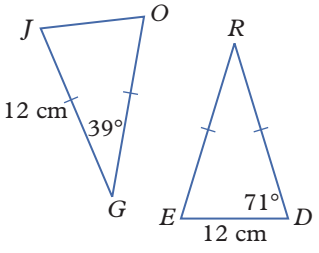

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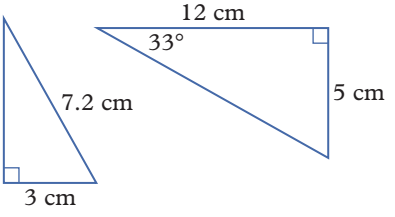
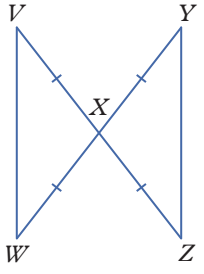
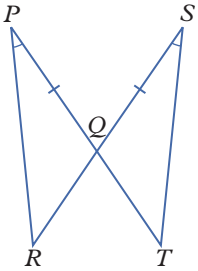


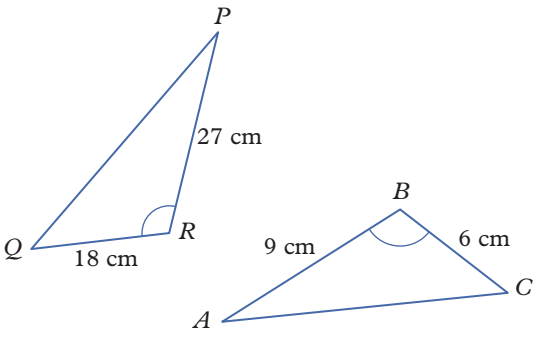
5C.3 **8** **a** For the diagram below, prove that $\triangle VWX \cong \triangle YZX$.



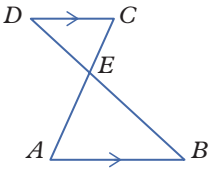
b For the diagram below, given that $\angle RPQ = \angle TSQ$, prove that $\triangle PQR \cong \triangle SQT$.



5C.4 **9** **a** Given that $\angle ABC = \angle PRQ$, prove that $\triangle ABC \sim \triangle PRQ$.



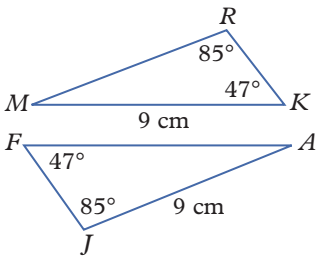
b Given that $AB \parallel CD$, prove that $\triangle ABE \sim \triangle CDE$.



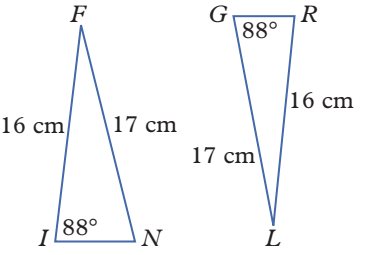
PROBLEM SOLVING AND REASONING

10 Lachlan said that the information given for the pairs of triangles below proves that they are congruent triangles. Explain why he is wrong in each case.

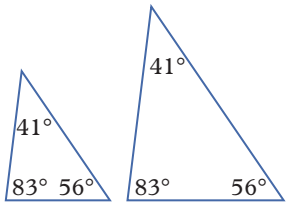
a



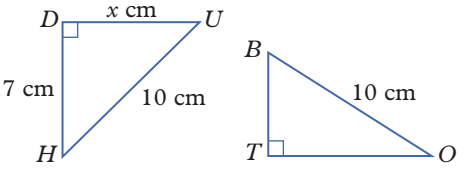
b



c



d



11 Decide whether each of these statements is true or false, giving reasons for your answers.

a If two triangles meet the condition AAA, they are not congruent.

b If two triangles have two corresponding sides of the same length and one corresponding angle of the same size, the triangles are congruent.

c A pair of triangles can meet one congruence condition but fail another.

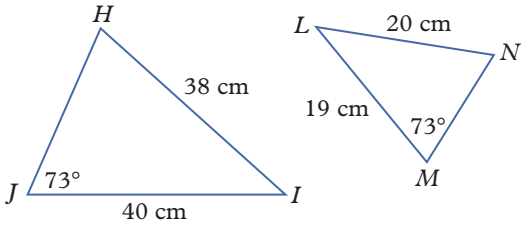
d If $\triangle ABC$ is congruent to $\triangle DEF$, and $\triangle DEF$ is congruent to $\triangle GHI$, then $\triangle GHI$ is congruent to $\triangle ABC$.

e If two quadrilaterals have all matching sides equal in length, then they are congruent.

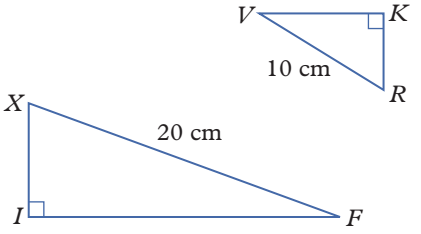
f All equilateral triangles are congruent.

12 Jessica said that the triangles in each pair below could be proven to be similar. Explain why she is wrong in each case.

a

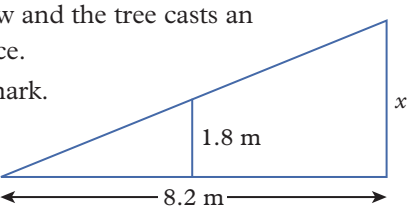


b



13 A 1 m ruler is placed upright next to a tree. If the ruler casts a 2.3 m shadow and the tree casts an 11.2 m shadow, how tall is the tree? Round your answer to one decimal place.

14 A ski jump is 8.2 m long. It has 1.8 m tall vertical supports at the halfway mark. How tall is the ski jump at its highest point?



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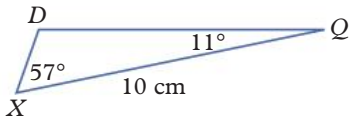
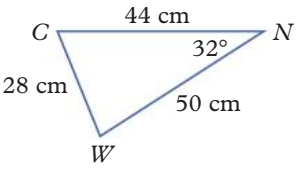
15 Select two pairs of similar triangles from the options below. Provide reasons for your selection.

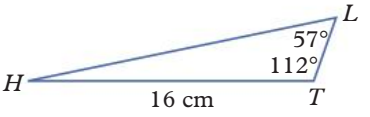
A



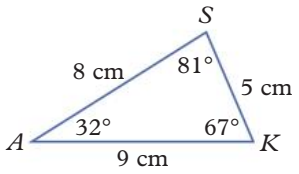
B



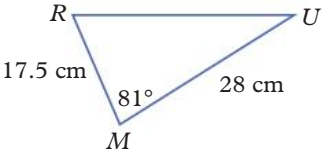
C



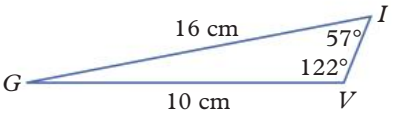
D



E



F



16 Decide whether each of these statements is true or false. Give reasons for your answers.

a If two triangles meet the condition AAA, they are not necessarily similar.

b If $\triangle ABC$ is similar to $\triangle DEF$, and $\triangle DEF$ is similar to $\triangle GHI$, then $\triangle GHI$ is similar to $\triangle ABC$.

c All isosceles triangles are similar.

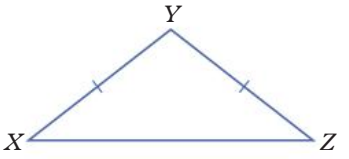
d All squares are similar.

e If two triangles fail the condition SAS, they are not necessarily similar.

f If two quadrilaterals have all sides in the same ratio, they are not necessarily similar.

17 If a triangle has two sides that are equal in length (meaning it is isosceles), then the angles opposite those sides are equal.

You can prove this fact using congruent triangles. To help you, consider $\triangle XYZ$ on the right.



a Draw a line from Y to the midpoint of XZ, and name this point W.

b Name the two triangles you have now formed.

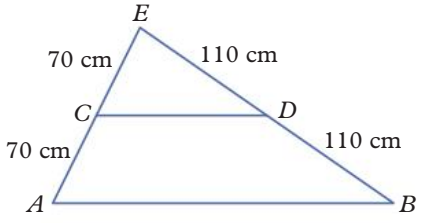
c Which pairs of sides do you know are equal in these two triangles?

d Name the side common to both triangles.

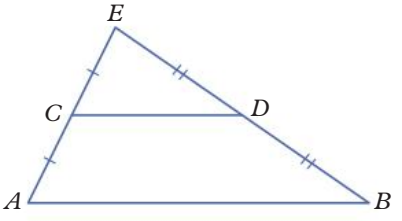
e Use a congruence condition to prove that $\triangle WXY \cong \triangle WZY$.

f Explain why you now know that $\angle WXY = \angle WZY$.

18 a Prove that $\triangle ABE \sim \triangle CDE$.

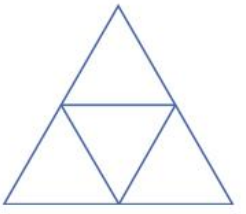


b Prove that $\triangle ABE \sim \triangle CDE$.



19 A flagpole has a guide wire attached at its midpoint and tethered to the ground 2.7 m away. A metre ruler, standing vertically, touches the guide wire when placed 75 cm away from where it is tethered to the ground. How tall is the flagpole?

20 Prove that drawing a triangle whose vertices are the midpoints of the sides of an equilateral triangle splits that equilateral triangle into four smaller congruent equilateral triangles. Hint: You will need to make use of your knowledge of isosceles triangles.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Congruent triangles

Interactive skillsheet
Similar triangles

Topic quiz
5C

5D Proofs and quadrilaterals

Learning intentions

By the end of this topic you will be able to ...

✔ prove quadrilateral properties.

Inter-year links

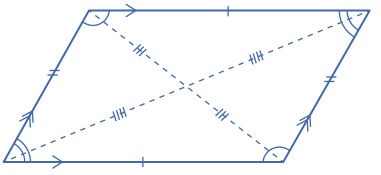
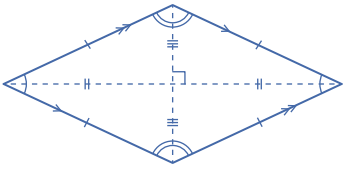
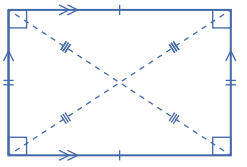
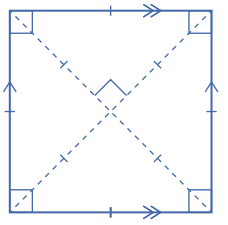
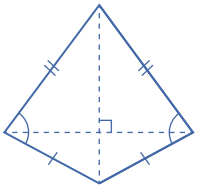
Year 7 7E Classifying quadrilaterals

Year 8 7J Quadrilaterals

Geometry terminology

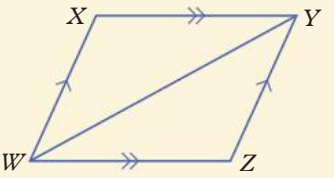
- 'Bisect' means to cut in half.
- The symbol for 'is parallel to' is ∥.
- The symbol for 'is perpendicular to' is ⊥.

Quadrilateral properties

Shape	Properties	Example
Parallelogram A quadrilateral with two pairs of parallel sides.	Opposite sides are equal in length. Opposite angles are equal. Diagonals bisect each other.	
Rhombus A quadrilateral where all sides are equal in length.	All parallelogram properties. Diagonals are perpendicular. Diagonals bisect the interior angles.	
Rectangle A quadrilateral where all interior angles are right angles.	All parallelogram properties. Diagonals are equal in length.	
Square A quadrilateral which is both a rectangle and a rhombus.	All parallelogram properties. All rhombus properties. All rectangle properties.	
Kite A quadrilateral with two pairs of adjacent sides which are equal in length.	One pair of opposite angles are equal. Diagonals are perpendicular. One diagonal bisects the other diagonal. One diagonal bisects a pair of interior angles.	

Example 5D.1 Proving opposite sides of a parallelogram are equal

Prove that the opposite sides of a parallelogram are equal in length.



THINK

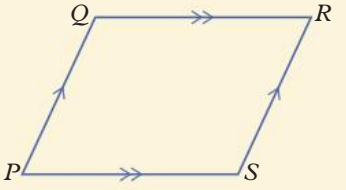
1. Set up a two-column table for your proof.
2. Use the fact that the diagonal of the parallelogram is common to both $\triangle WXY$ and $\triangle YZW$.
3. Use your knowledge of angles and parallel lines to identify equal angles.
4. Use a congruence condition to show that the triangles are congruent.
5. Show that the opposite sides of the parallelogram are equal in length (as corresponding sides of the congruent triangles).
6. Complete the proof with a closing statement.

WRITE

Statements	Reasons
YW is common to both $\triangle WXY$ and $\triangle YZW$	Common side
$\angle XYW = \angle ZWY$	Alternate angles on parallel lines are equal
$\angle YWX = \angle WYZ$	Alternate angles on parallel lines are equal
$\therefore \triangle WXY \cong \triangle YZW$	Meets the AAS condition of congruence
$WX = YZ$	Corresponding sides of congruent triangles are equal
$WZ = YX$	Corresponding sides of congruent triangles are equal
$\therefore$ Opposite sides of a parallelogram are equal in length ■	

Example 5D.2 Proving opposite angles in a parallelogram are equal

Prove that the opposite angles of a parallelogram are equal in size.



THINK

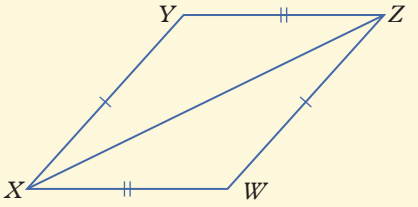
1. Use your understanding of angles and parallel lines to write two equations about co-interior angles, making sure the same angle appears in both equations.
2. Equate the left-hand sides of the equations you wrote (because the right-hand sides are equal).
3. Simplify by subtracting the same angle from both sides.
4. Repeat steps **1–3** for the other pair of co-interior angles.
5. Complete the proof with a closing statement.

WRITE

Statements	Reasons
$\angle QPS + \angle PSR = 180°$	Co-interior angles on parallel lines are equal
$\angle QPS + \angle PQR = 180°$	Co-interior angles on parallel lines are equal
$\angle QPS + \angle PSR = \angle QPS + \angle PQR$	Transitive law
$\angle PSR = \angle PQR$	Subtraction property of equality
$\angle PSR + \angle QRS = 180°$	Co-interior angles on parallel lines are equal
$\angle PSR + \angle QPS = 180°$	Co-interior angles on parallel lines are equal
$\angle PSR + \angle QRS = \angle PSR + \angle QPS$	Transitive law
$\angle QRS = \angle QPS$	Subtraction property of equality
$\therefore$ Both pairs of opposite angles of the parallelogram are equal in size ■	

Example 5D.3 Proving a quadrilateral is a parallelogram

Prove that $WXYZ$ shown on the right is a parallelogram.



THINK

1. Show that the two triangles formed by the diagonal are congruent.
2. Identify the corresponding angles in these congruent triangles.
3. If alternate angles are equal, then the lines must be parallel. (Note: || means 'is parallel to'.)
4. Use the definition of a parallelogram to prove that $WXYZ$ is a parallelogram.

WRITE

Statements	Reasons
$XY = ZW$	Given
$YZ = WX$	Given
XZ is common to both $\triangle XWZ$ and $\triangle ZYX$	Common side
$\therefore \triangle XWZ \cong \triangle ZYX$	Meets the SSS condition of congruence
$\angle WXZ = \angle YZX$	Corresponding angles of congruent triangles are equal
$\angle XZW = \angle ZXY$	Corresponding angles of congruent triangles are equal
$YZ \parallel XW$	$\angle WXZ$ and $\angle YZX$ are alternate angles. If alternate angles are equal, they lie on parallel lines
$XY \parallel WZ$	$\angle XZW$ and $\angle ZXY$ are alternate angles. If alternate angles are equal, they lie on parallel lines
$\therefore WXYZ$ is a parallelogram ■	If a quadrilateral has opposite sides that are parallel then it is a parallelogram

Helpful hints

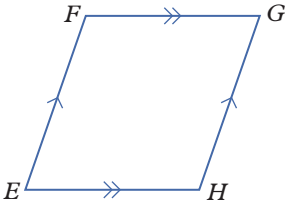
- ✔ If you are unsure where to begin when constructing a mathematical proof, start by writing all the information you are given (using mathematical terminology), as well as what you are trying to prove.
- ✔ Quadrilaterals, and other polygons, are labelled in a cyclic manner around the shape in order. That is, pairs of adjacent letters name the edges of the quadrilateral.

ANS p597

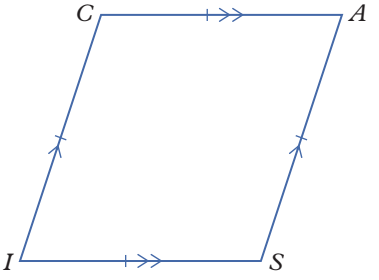
Exercise 5D Proofs and quadrilaterals

△ 1–4, 6, 8, 9 □ 1–5, 7, 10, 11, 13 ⬠ 3, 4, 7, 9, 11, 12, 14–16

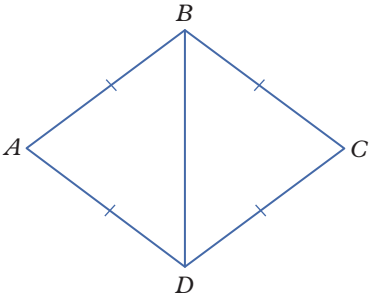
5D.1 **1** If $FG = EF$, prove that the shape on the right is a rhombus by demonstrating that all four side lengths are equal.



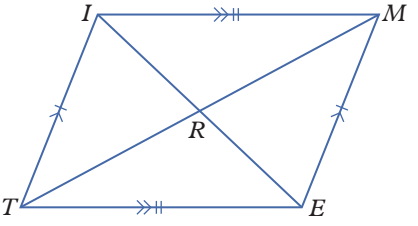
5D.2 **2** Prove that the opposite angles of the rhombus $ISAC$ on the right are equal.



5D.3 **3** Prove that the opposite sides of the rhombus $ABCD$ with four equal sides are parallel.



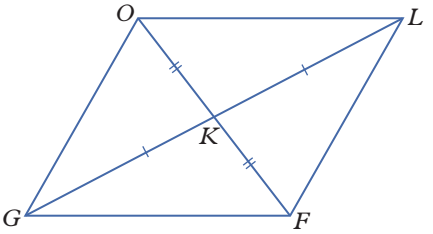
4 Consider the parallelogram on the right.



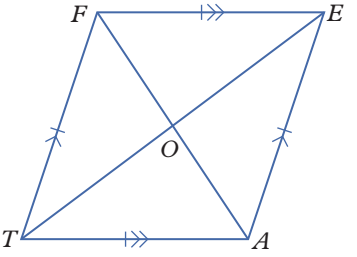
- **a** What angle is equal to $\angle IMR$?
- **b** What angle is equal to $\angle MIR$?
- **c** Prove that $\triangle MIR \cong \triangle TER$.
- **d** Copy the figure and mark the corresponding equal sides of the congruent triangles.
- **e** Explain how this shows that the diagonals of parallelogram $TIME$ bisect each other.

UNDERSTANDING AND FLUENCY

5 Prove that the shape below is a parallelogram.



6 Consider the rhombus on the right.



a Follow similar steps to those given in question **4** to prove that the diagonals of the rhombus bisect each other.

b Show that the four triangles formed by the diagonals are congruent to one another.

c Redraw the rhombus with markings showing the corresponding equal sides.

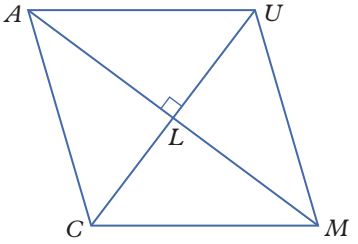
d Explain why $\angle FOE = \angle AOE$.

e Use your answer from part **d** to help you prove that the diagonals of *FEAT* bisect each other at right angles.

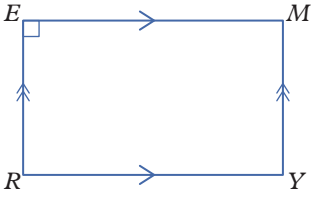
7 Prove that the shape on the right is a rhombus if:

a $AL = ML$ and $CL = UL$

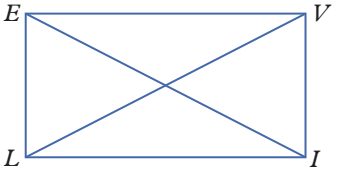
b $AU \parallel CM$ and $AC \parallel UM$.



8 Prove that parallelogram *REMY* is a rectangle if $\angle REM = 90°$.



9 Consider figure *LEVI* below, where $VE = IL$, $VE \parallel IL$ and $\angle EVI = 90°$.



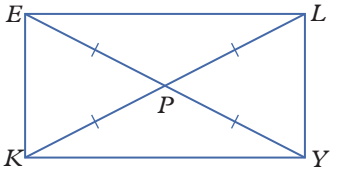
a Prove that $\angle VIL$, $\angle ILE$ and $\angle LEV$ are all 90°.

b Prove that $EL = IV$ and $EL \parallel IV$.

c Use Pythagoras' theorem to show that $LV = EI$.

d Prove that $\triangle LEV \cong \triangle ELI$.

10 Prove that the shape on the right is a rectangle by following these steps:

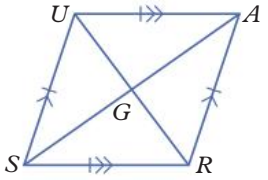


a Use what you know about isosceles triangles and about equal alternate angles to first prove that *ELKY* is a parallelogram.

b Use $\angle LYE + \angle YEL + \angle ELK + \angle KLY$ to prove that $\angle ELY = 90°$.

c Prove that all interior angles are 90° and therefore that *ELKY* is a rectangle.

11 Consider the rhombus below.



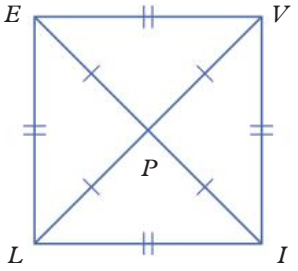
a Prove that $UG = RG$ and that $AG = SG$ (in other words, that the diagonals bisect each other).

b Prove that $\triangle UAG \cong \triangle RAG$.

c Prove that $\angle UAS = \angle RAS$.

d Explain why this method shows that the diagonals of a rhombus bisect its interior angles.

12 For the diagram below, prove that $\angle EVL = 45°$.



13 a Use mathematical reasoning with a square and a rhombus to show that having all corresponding sides the same length is not enough to prove congruence for shapes other than triangles.

b Use mathematical reasoning with a parallelogram and a rhombus to show that meeting the SAS condition of congruence does not prove congruence for shapes other than triangles.

c Use mathematical reasoning with a square and a rectangle to show that meeting the AAS condition of congruence does not prove congruence for shapes other than triangles.

14 a Use mathematical reasoning with a rectangle and a parallelogram to show that having all corresponding sides in the same ratio is not enough to prove similarity for shapes other than triangles.

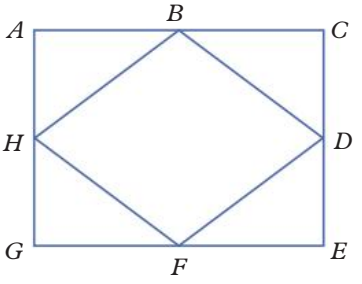
b Use mathematical reasoning with a parallelogram and a rhombus to show that having all corresponding angles the same size is not enough to prove similarity for shapes other than triangles.

15 Use your understanding of the relevant geometric properties to draw each shape described below and show that the triangles within them are congruent.

a A kite is split into two triangles by a line drawn along its line of symmetry.

b A regular hexagon is split into six triangles by drawing three diagonal lines joining opposite vertices.

16 *ACEG* is a rectangle. If the vertices of *HBDF* touch the midpoints of each of the rectangle's sides, prove that *HBDF* is a rhombus and that the opposite angles of *HBDF* are equal.



PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Investigation
Finding the shortest path around a rectangle

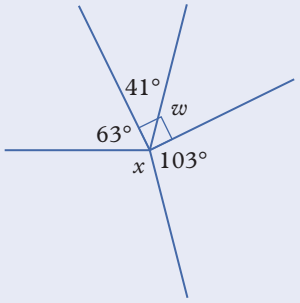
Topic quiz
5D

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

5A **1** Determine the values of the pronumerals.

a



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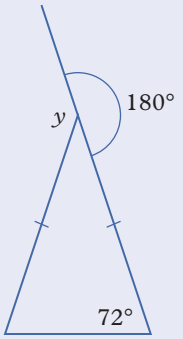
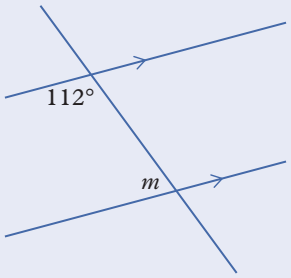


5A **2** Determine the values of the pronumerals.

a



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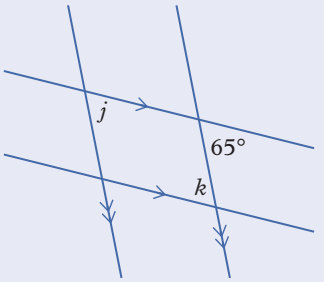
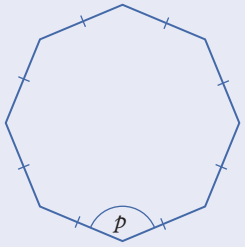


5A **3** Determine the values of the pronumerals.

a



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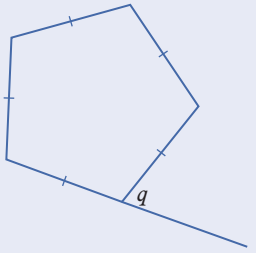
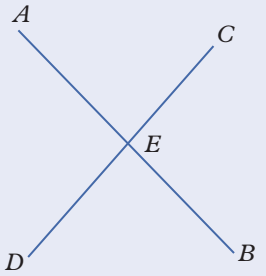
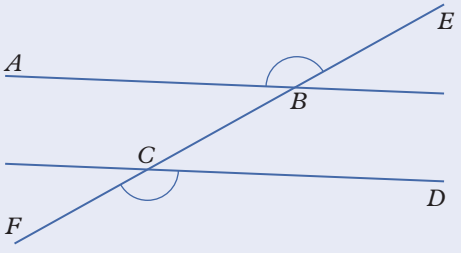


5B **4** Given that AB and CD in the diagram on the right are line segments, prove that $\angle AEC = \angle DEB$.

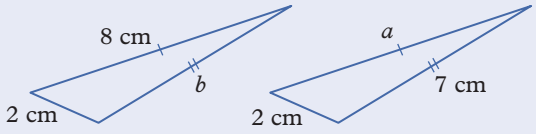


5B **5** Given that $\angle ABE = \angle DCF$, prove that $AB \parallel CD$.



5C **6** Determine the values of the pronumerals.

a



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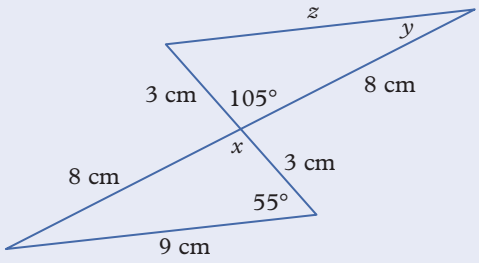
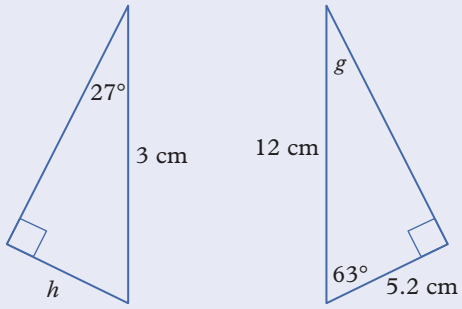
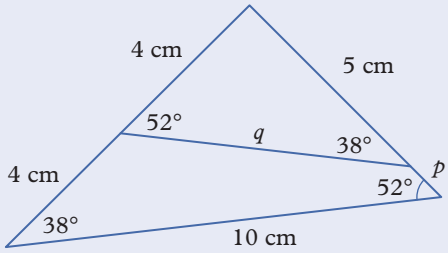


5C **7** Determine the values of the pronumerals.

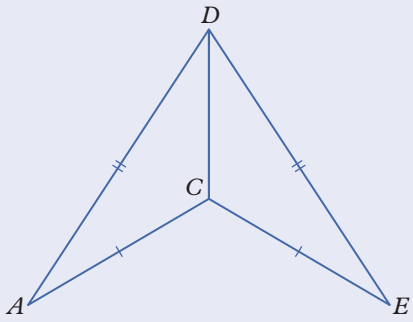
a



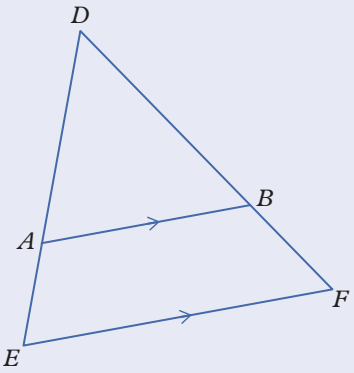
b



5C **8** **a** Prove that $\triangle ADC \cong \triangle EDC$ and, therefore, that $\angle ACD = \angle ECD$.

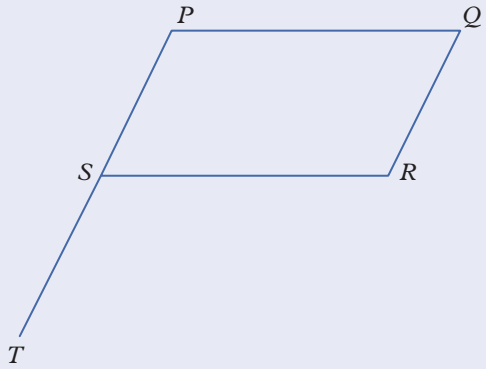


b Prove that $\triangle ADB \sim \triangle EDF$.



5D **9** Given that $AB \parallel DC$, $BC \perp CD$ and $AD \perp CD$, prove that $ABCD$ on the right is a rectangle and, therefore, that $AC = BD$.

5D **10** Given that $\angle SPQ = \angle RST = \angle SRQ$, prove that $PQRS$ is a parallelogram.



5E Circle geometry: angles

Learning intentions

By the end of this topic you will be able to ...

✔ understand and apply circle theorems 1–5.

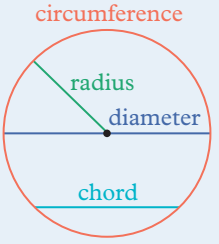
Inter-year links

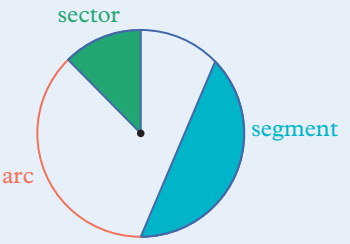
Year 7 7D Classifying triangles

Year 8 7B Triangles

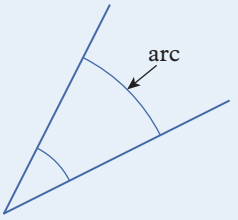
Circle terminology

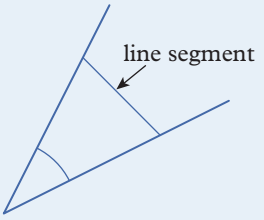
- **Arc**: A part of the **circumference** of a circle.
  A minor arc measures less than a semi-circle (180°). A major arc measures more than a semi-circle.
- **Chord**: A line segment connecting two points on a circle's circumference.
- **Sector**: A portion of a circle enclosed by two radii and an arc on the circumference.
- **Segment**: A region of a circle enclosed by a chord and an arc on the circumference.



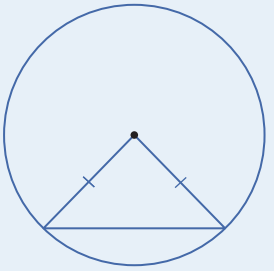


- A subtended angle is an angle whose two rays pass through the endpoints of an arc, line segment or curve. When an arc subtends an angle, the lines drawn from the arc's endpoints form that angle at the point where they meet.





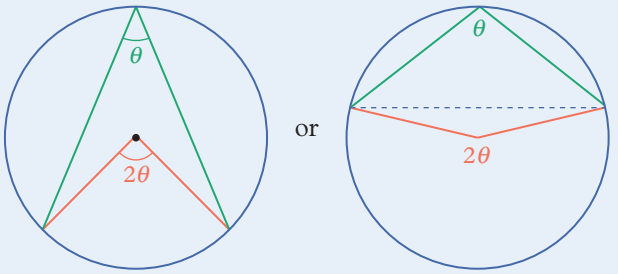
- A cyclic quadrilateral is a quadrilateral whose vertices all lie on the circumference of a single circle.
- An isosceles triangle is formed between two radii of a circle and a chord joining the points where they meet the circumference.



Circle theorems 1–5

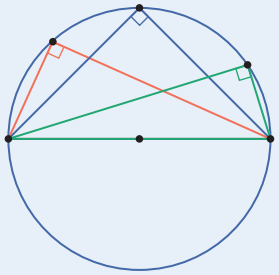
- **Circle theorem 1:** The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended by the same arc at the circumference.
  This is also known as the 'angle at the centre' theorem.

Circle theorem 1



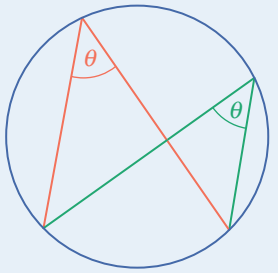
- **Circle theorem 2:** Any angle subtended at the circumference of a circle by the ends of the diameter of the circle is a right angle.
  This is also known as the 'angle in a semi-circle' theorem, or Thales' theorem.

Circle theorem 2



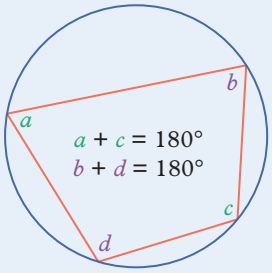
- **Circle theorem 3:** All angles subtended at the circumference by the same arc are equal in size.
  This is also known as the 'angles in the same segment' theorem.

Circle theorem 3



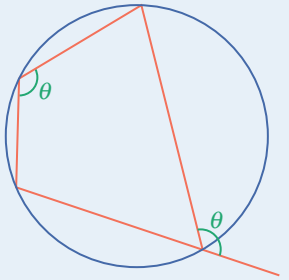
- **Circle theorem 4:** The opposite angles of a cyclic quadrilateral are supplementary.
  This is also known as the 'cyclic quadrilateral' theorem.

Circle theorem 4



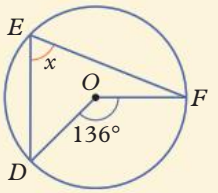
- **Circle theorem 5:** Any exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.
  This is an extension of the 'cyclic quadrilateral' theorem.

Circle theorem 5



Example 5E.1 Using circle theorem 1

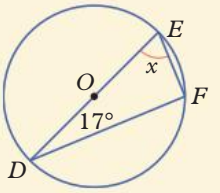
Find the value of x in the circle on the right.



THINK	WRITE
1 The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended at the circumference. So $\angle DOF = 2\angle DEF$.	$136° = 2x$
2 Write your answer.	$x = 68°$

Example 5E.2 Using circle theorem 2

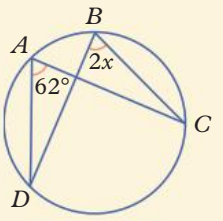
Find the value of x in the circle on the right.



THINK	WRITE
1 Any angle subtended at the circumference by the diameter of a circle is a right angle. So $\angle DFE$ is a right angle.	$\angle DFE = 90°$
2 Use the angle sum of a triangle to help you write an equation to find the value of x.	$x + 17° + 90° = 180°$
	$x = 180° - 17° - 90°$
3 Solve the equation and write your answer.	$x = 73°$

Example 5E.3 Using circle theorem 3

Find the value of x in the circle on the right.



THINK	WRITE
1 All angles subtended by the same arc at the circumference are equal in size, therefore $\angle DBC$ and $\angle DAC$ are equal in size.	$\angle DBC = \angle DAC$
2 Substitute values and solve for x.	$2x = 62°$
	$x = 31°$

Helpful hints

✔ If you can't remember the circle theorems at first, carefully inspecting the diagrams can be helpful. For example, the given diagram might help you remember the circle theorem or theorems you need to use!

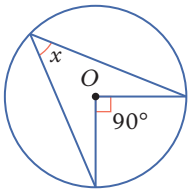
ANS p602

Exercise 5E Circle geometry: angles

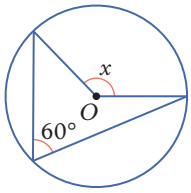
1–4, 5(a–c), 6–8 | 1–5, 9, 10, 12(a) | 4, 5, 11, 12

5E.1 **1** Find the value of x in each of these circles.

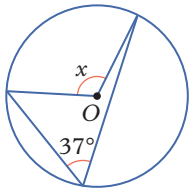
a



b

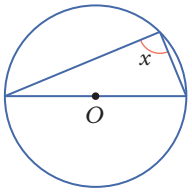


c



5E.2 **2** Find the value of x in each of these circles.

a



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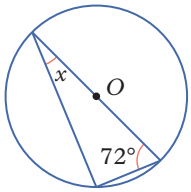

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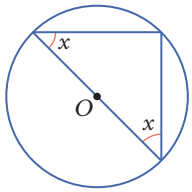
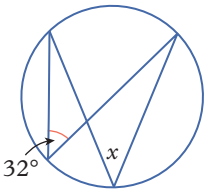


5E.3 **3** Find the value of each pronumeral in these circles.

a



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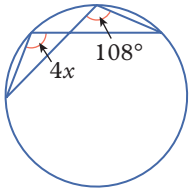

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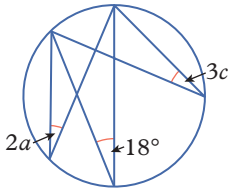
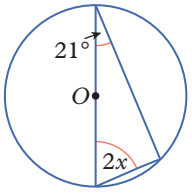
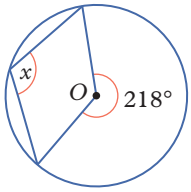


4 Find the value of each pronumeral in these circles (**a–f**), and state which circle theorem is used.

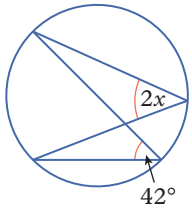
a



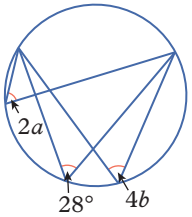
b



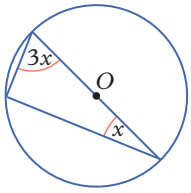
c



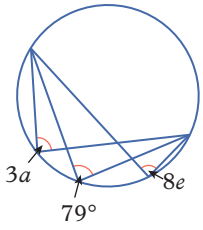
d



e

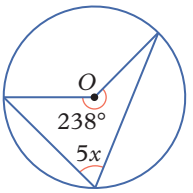


f



5 Use your understanding of triangle and circle properties and theorems to find the value of x in each of these circles.

a



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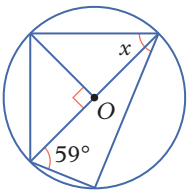

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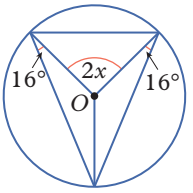

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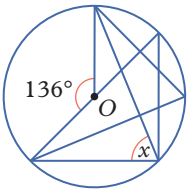

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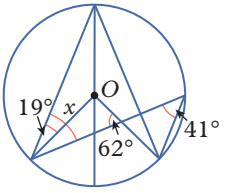

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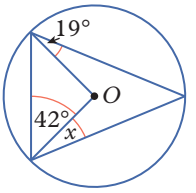
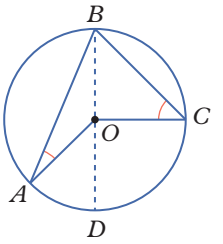


6 This question proves circle theorem 1. Consider the diagram on the right.



a How do you know that $AO = BO = CO$?

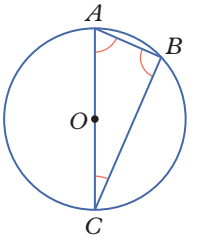
b How do you know that $\angle ABO = \angle OAB$?

c Use the theorem for the exterior angle of a triangle to show that $\angle AOD = 2\angle ABO$.

d Use your answers for parts **b** and **c** to show that $\angle DOC = 2\angle OBC$.

e Explain how this shows that $\angle AOC = 2\angle ABC$.

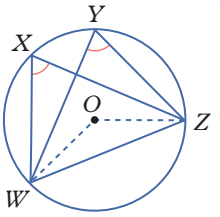
7 This question proves circle theorem 2. Consider the diagram on the right.



a What is the angle at the centre of the circle?

b Use circle theorem 1 to show that $\angle ABC = 90°$.

8 This question proves circle theorem 3.
Consider the diagram on the right.

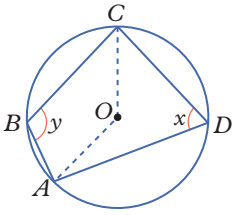


a Use circle theorem 1 to explain why $\angle WOZ = 2\angle WXZ$.

b Similarly, show that $\angle WOZ = 2\angle WYZ$.

c What can you say about $\angle WYZ$ and $\angle WXZ$?

9 This question proves circle theorem 4. Consider the diagram on the right, showing a cyclic quadrilateral.



a Explain why the obtuse angle $\angle AOC = 2x$ and the reflex angle $\angle AOC = 2y$.

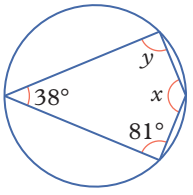
b Use what you know about angles at a point to explain why reflex $\angle AOC = 360° - 2x$.

c Use your answers from parts **a** and **b** to show that $y = 180° - x$.

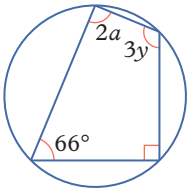
d Copy the diagram, replacing radii OC and OA with OB and OD, then use this to show that $\angle BAD = 180° - \angle BCD$.

e Find the value of each pronumeral in the circles below.

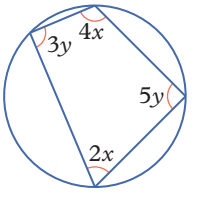
i



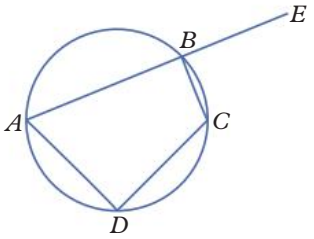
ii



iii



10 This question proves circle theorem 5. Consider the diagram below, showing another cyclic quadrilateral.



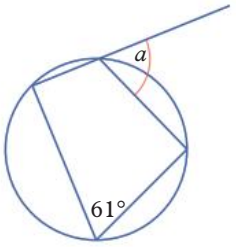
a Use circle theorem 4 to explain why $\angle ADC = 180° - \angle ABC$.

b Why does $\angle EBC = 180° - \angle ABC$?

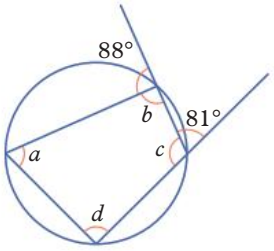
c Explain why this shows that $\angle ADC = \angle EBC$.

d Find the value of each pronumeral in the diagrams below.

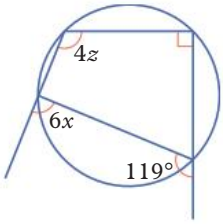
i



ii



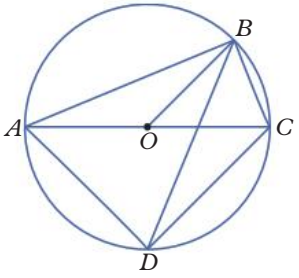
iii



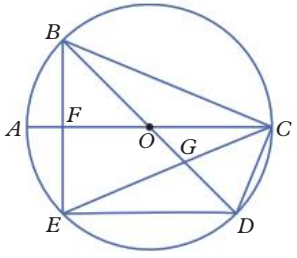
11 Write a complete proof for each of the circle theorems 1–5.

12 Use the given information to answer these questions.

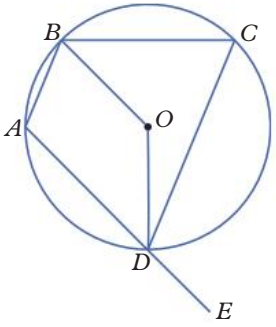
a Find $\angle DBC$, given that $\angle ADB = 68°$, $OA = OB = OC$ and $\angle BCD = 92°$.



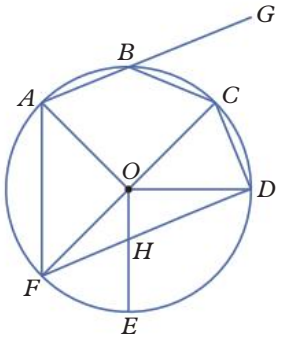
b Find $\angle DEG$, given that $BE \perp AC$, $\angle OCG = 19°$ and $\angle FOG = 142°$.



c Find $\angle ODC$, given that $\angle BAD = 96°$ and $\angle CBO = 46°$.



d Find $\angle OAB$, given that $\angle GBC = 46°$, $\angle BCD = 138°$ and $\angle AFD = 70°$.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Circle geometry: angles

Investigation
Clock angles

Topic quiz
5E

5F Circle geometry: chords

Learning intentions

By the end of this topic you will be able to ...

✔ understand and apply circle theorems 6–9.

Inter-year links

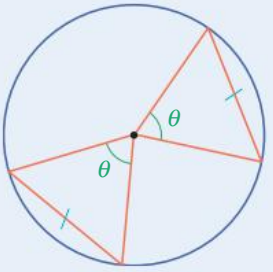
Year 7 7D Classifying triangles

Year 8 7B Triangles

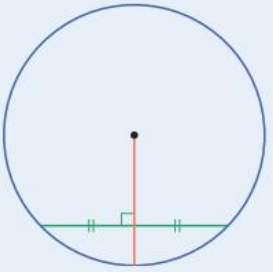
Circle theorems 6–9

- **Circle theorem 6:** Chords of equal length subtend equal angles at the centre of the circle. This is also known as the 'equal chords, equal angles' theorem.
- **Circle theorem 7:** If a radius and a chord intersect perpendicularly, then the radius bisects the chord.
- **Circle theorem 8:** Chords that are equal in length are **equidistant** (at equal distances) from the centre of the circle.
- **Circle theorem 9:** When two chords intersect inside a circle, this divides each chord into two line segments so that the product of the lengths of the line segments for both chords is the same.

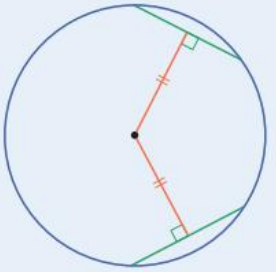
Circle theorem 6



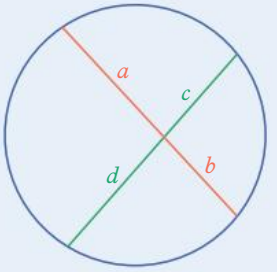
Circle theorem 7



Circle theorem 8



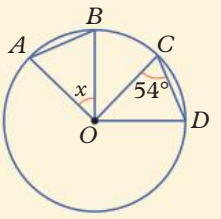
Circle theorem 9



$ab = cd$

Example 5F.1 Using circle theorem 6

Given $AB = CD$, find the value of x in the circle on the right.



THINK

1 Since OC and OD are radii, they are of equal length. This means that $\triangle COD$ is an isosceles triangle.

2 Find the two unknown interior angles of $\triangle COD$.

3 Chords that are equal in length subtend equal angles at the centre of the circle. So $\angle AOB = \angle COD$.

4 Write your answer.

WRITE

$OC = OD$

$\triangle COD$ is isosceles, so $\angle OCD = \angle ODC = 54°$.

$\angle COD = 180° - 54° - 54°$

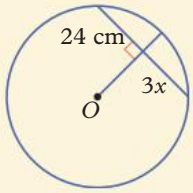
$= 72°$

$\angle AOB = \angle COD = 72°$

$x = 72°$

Example 5F.2 Using circle theorem 7

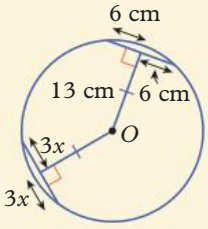
Find the value of x in the circle on the right.



THINK	WRITE
1 If a radius and a chord intersect perpendicularly, then the radius bisects the chord. So $3x$ is equal to 24 cm.	$3x = 24$ cm
2 Solve for x.	$x = 8$ cm

Example 5F.3 Using circle theorem 8

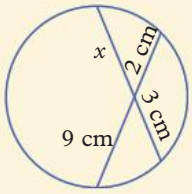
Find the value of x in the circle on the right.



THINK	WRITE
1 Chords that are equal in length are equidistant from the centre of the circle. So the two chords must be equal in length.	$3x + 3x = 6 + 6$
2 Solve for x, including units in your answer.	$6x = 12$ $x = 2$ cm

Example 5F.4 Using circle theorem 9

Find the value of x in the circle on the right.



THINK	WRITE
1 When two chords intersect, the products of the line segments they are divided into are equal. Write the product of the lengths of the line segments for each chord.	Chord 1: $x \times 3$ cm Chord 2: 2 cm $\times$ 9 cm $x \times 3 = 2 \times 9$
2 Simplify the equation.	$3x = 18$
3 Solve for x, including units in your answer.	$x = 6$ cm

Helpful hints

- ✔ All radii of the same circle are equal in size.
- ✔ When chords intersect, each of the equal products results from multiplying the lengths of the two parts of the same line segment.

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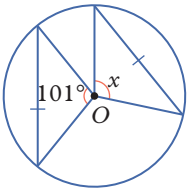
Exercise 5F Circle geometry: chords

▲ 1–9, 15 ■ 1–4(b, c), 5–7, 10–12, 16, 17 ⬟ 5–7, 13–18

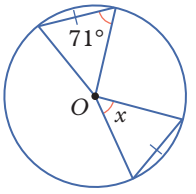
UNDERSTANDING AND FLUENCY

5F.1 **1** Find the value of x in each of these circles.

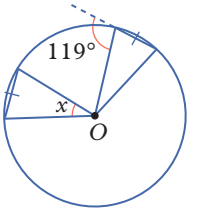
a



b

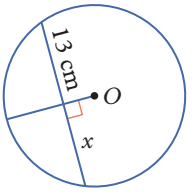


c



5F.2 **2** Find the value of x in each of these circles.

a



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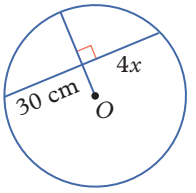

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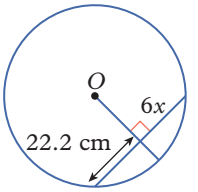
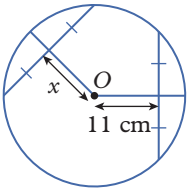


5F.3 **3** Find the value of each pronumeral in these circles.

a



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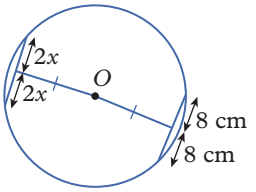

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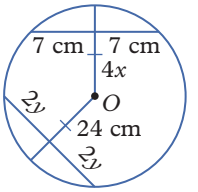
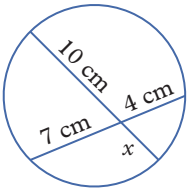
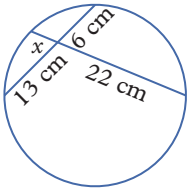


5F.4 **4** Find the value of x in each of these circles, correct to one decimal place.

a



b



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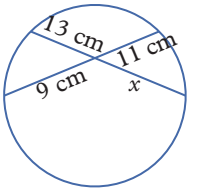
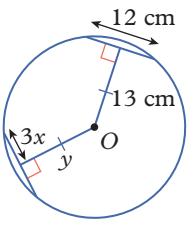


5 Find the value of each pronumeral in these circles. Round your answer to two decimal places.

a



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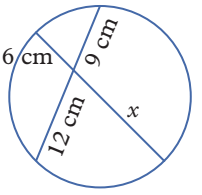

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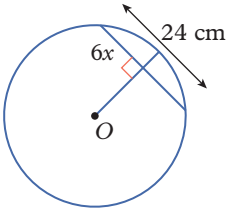
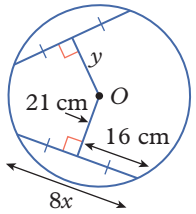


d



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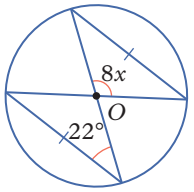

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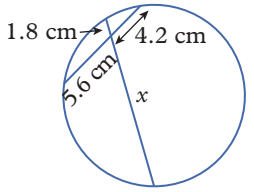

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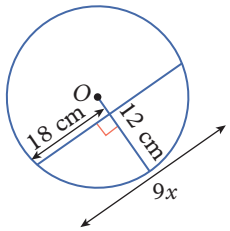

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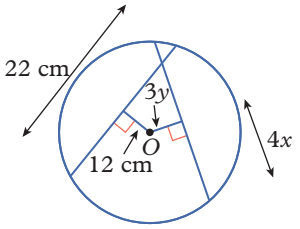

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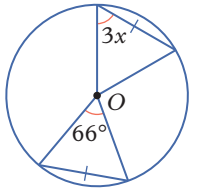
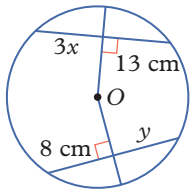


6 Use your understanding of circle properties and theorems, and any other necessary geometric properties, to find the value of each pronumeral in these circles. Round your answer to two decimal places.

a



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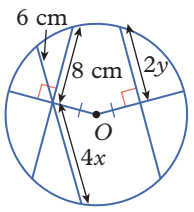

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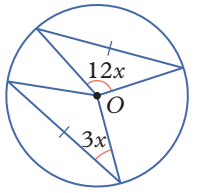

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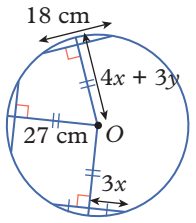

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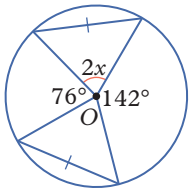

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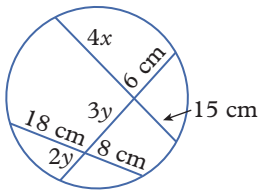
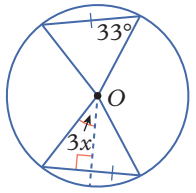


7 Find the value of each pronumeral in these circles. Round your answer to two decimal places.

a



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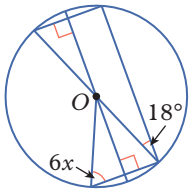

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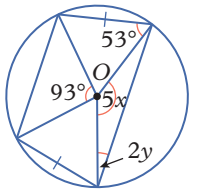

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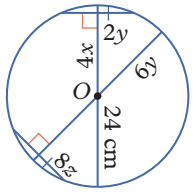

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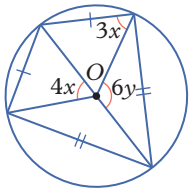

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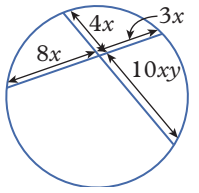
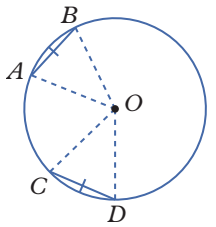


8 This question proves circle theorem 6.
Consider the diagram on the right, in which $AB = CD$.



a Given that OA, OB, OC and OD are radii, what can you say about them?

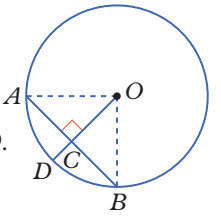
b What type of triangles does this mean $\triangle OAB$ and $\triangle OCD$ are?

c Explain why $\triangle OAB \cong \triangle OCD$.

d Explain why $\angle COD = \angle AOB$.

PROBLEM SOLVING AND REASONING

9 This question proves circle theorem 7. Consider the diagram on the right.



a Explain why $OA = OB$.

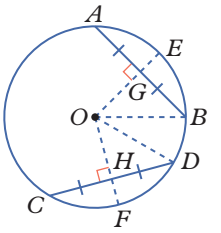
b Explain why $\angle OCA = \angle OCB$.

c Use one of the conditions for the congruence of triangles to explain why $\triangle ACO \cong \triangle BCO$.

d Explain why $AC = CB$.

10 This question proves circle theorem 8.

Consider the diagram on the right, in which $AB = CD$.



a Explain why $OE = OB = OD = OF$.

b Use circle theorem 7 to explain why $AG = GB$ and $CH = HD$.

c Likewise, use circle theorem 7 and the fact that $AB = CD$ to explain why $GB = HD$.

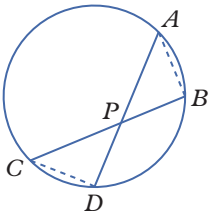
d Use one of the conditions for the congruence of triangles to explain why $\triangle OHD \cong \triangle OGB$.

e Explain why $OH = OG$.

f Use your understanding of Pythagoras' theorem to explain why the shortest distance from the centre of a circle to a chord is perpendicular to that chord.

11 This question proves circle theorem 9.

Consider the diagram on the right.



a Explain why $\angle CPD = \angle APB$.

b Use circle theorem 3 to explain why $\angle DCB = \angle BAD$.

c Explain why $\angle PDC = \angle PBA$.

d Explain how you can tell that $\triangle CPD$ is similar to $\triangle APB$.

e Explain why you can state that $\frac{AP}{BP} = \frac{CP}{DP}$ and, hence, that $AP \times DP = CP \times BP$.

12 Prove that, if a cyclic quadrilateral is a parallelogram, then it must be a rectangle.

13 Draw a circle with two chords. Show that constructing perpendicular bisectors of these two chords locates the centre of the circle.

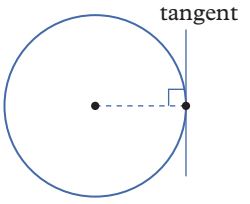
14 Write a complete proof for each of the circle theorems 6–9.

CHALLENGE

15 A tangent is a line that touches the circumference of a circle at one point only.

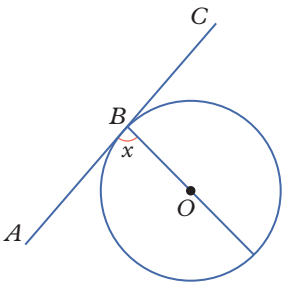
Circle theorem 10 states that, if a tangent to a circle meets a radius of that circle, the tangent and the radius are perpendicular to each other.

Circle theorem 10



Use circle theorem 10 to find the value of x in each diagram below.

a



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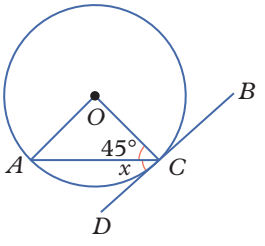

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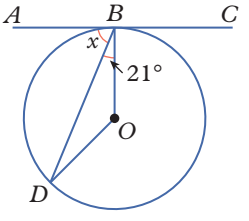
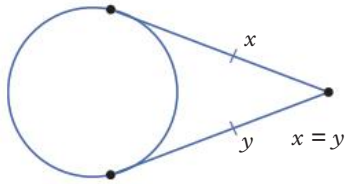


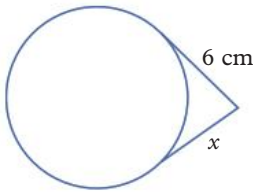
16 Circle theorem 11 states that if two tangents intersect outside a circle, the distances along the tangents from their intersection to the circumference of the circle are equal.

Use circle theorem 11 to find the value of each pronumeral in the diagrams below.

Circle theorem 11



a



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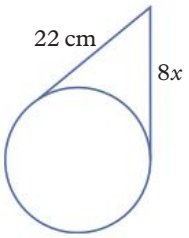

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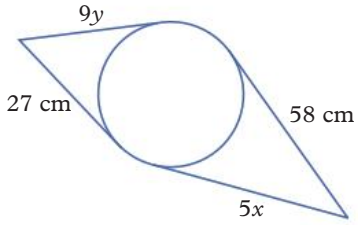
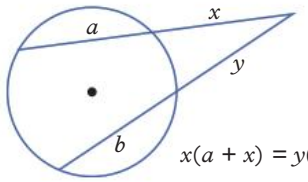


17 A secant is a line that cuts a circle twice.

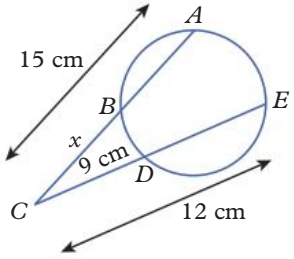
Circle theorem 12 states that if two secants intersect outside a circle, the products of one secant's whole length by its length external to the circle is equal to the product of the other secant's whole length by its length external to the circle.

Use circle theorem 12 to find the value of x in each diagram below.

Circle theorem 12



a



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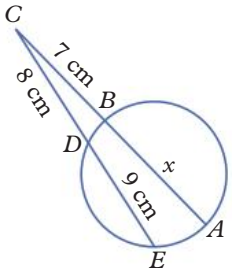

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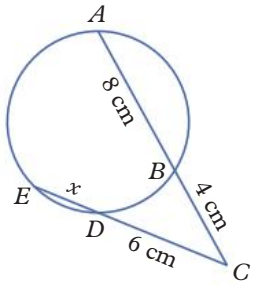
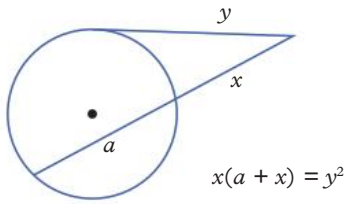


18 Circle theorem 13 states that, if a tangent and a secant intersect outside a circle, the product of the entire secant length by the external secant length is equal to the square of the tangent length.

Use circle theorem 13 to find the value of x in each diagram below.

Circle theorem 13



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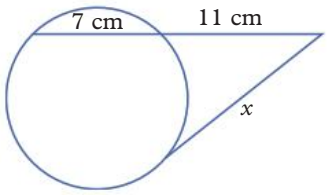

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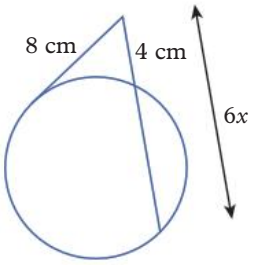

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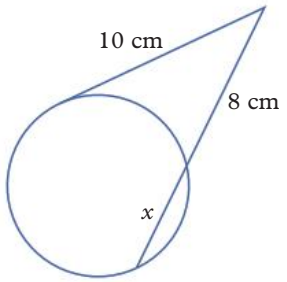


Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Circle geometry: chords

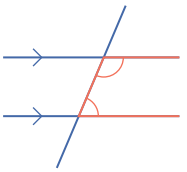
Worksheet
Tangents and secants

Topic quiz
5F

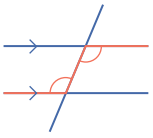
Chapter summary

Angles

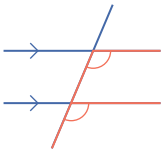
- Complementary angles add to 90°.
- Supplementary angles add to 180°.
- Angles at a point add to 360°.
- Vertically opposite angles are equal.



co-interior angles (supplementary)



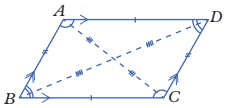
alternate angles (equal)



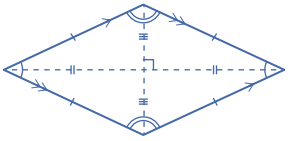
corresponding angles (equal)

Properties of quadrilaterals

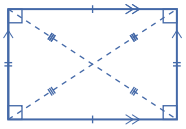
Parallelogram: Two pairs of parallel sides.



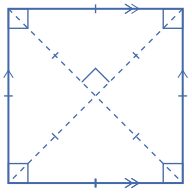
Rhombus: All sides are equal in length.



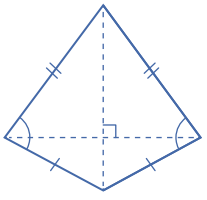
Rectangle: All angles are right angles.



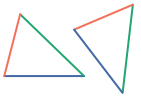
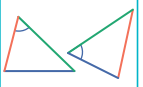
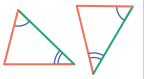
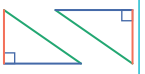
Square: Both a rectangle and a rhombus.



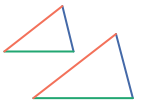
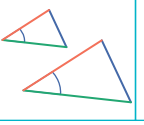
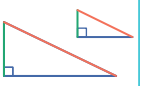
Kite: Two pairs of adjacent sides which are equal in length.



Congruent triangles

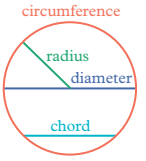
SSS (side–side–side)	SAS (side–angle–side)	AAS (angle–angle–side)	RHS (right angle–hypotenuse–side)
			

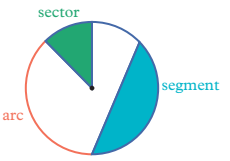
Similar triangles

SSS (side–side–side)	SAS (side–angle–side)	AAA (angle–angle–angle)	RHS (right angle–hypotenuse–side)
			

Terminology

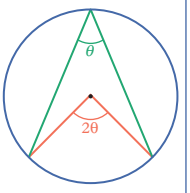
- 'Bisect' means to cut in half.
- The symbol for 'is parallel to' is ||.
- The symbol for 'is perpendicular to' is ⊥.



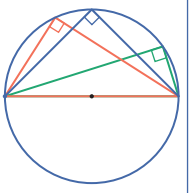


Circle theorems

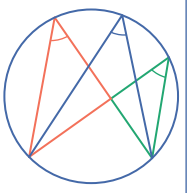
Circle theorem 1



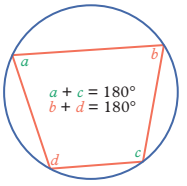
Circle theorem 2



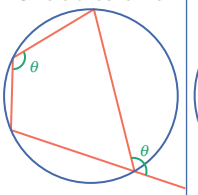
Circle theorem 3



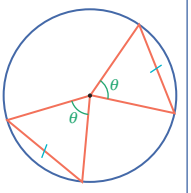
Circle theorem 4



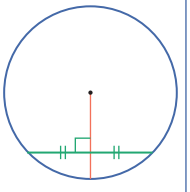
Circle theorem 5



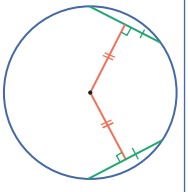
Circle theorem 6



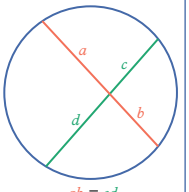
Circle theorem 7



Circle theorem 8



Circle theorem 9



$ab = cd$

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

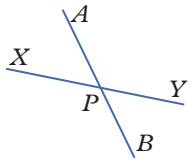
Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

5A **1** Which of these angles is supplementary to 25°?

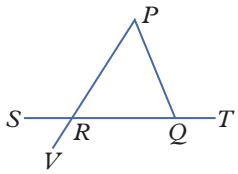
A 75° **B** 25° **C** 155° **D** 65° **E** 335°

5B **2** Straight lines AB and XY intersect at P. Given that $\angle APY = 2\angle APX$, which of the following statements is true?



A $\angle BPX = 2\angle APY$
B $\angle XPB = 2\angle APX$
C $\angle YPB = 2\angle APX$
D $\angle APX = 2\angle BPY$
E $\angle APX = 2\angle APY$

5B **3** Which of the following statements is true for the diagram below?



A $\angle PQT = \angle QRV$
B $\angle RPQ = \angle RQP$
C $\angle PRS = \angle PQT$
D $\angle TQP = \angle PRQ + \angle RQP$
E $\angle RPQ + \angle PQR = \angle SRP$

5C **4** Which of these is *not* a test for congruent triangles?

A AAA **B** SSS **C** SAS **D** AAS **E** RHS

5C **5** Which of these is *not* a test for similar triangles?

A AAA **B** SSS **C** SAS **D** AAS **E** RHS

5D **6** Which of the following reasons, by itself, cannot be used to prove that a quadrilateral is a rhombus?

A All sides are equal.
B Opposite angles are equal.
C Diagonals bisect each other at right angles.
D Diagonals bisect the interior angles.
E Parallel opposite sides and adjacent sides are equal.

5E **7** For a particular circle, the angle subtended by an arc at the centre of the circle and the angle subtended by the same arc at the circumference are complementary angles. Which of these values are possible for the two angles?

A 60° and 30° **B** 45° and 45° **C** 90° and 45° **D** 120° and 60° **E** 120° and 240°

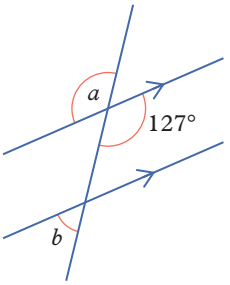
5F **8** A circle contains two chords of the same length. Which of these statements is *not* true?

A The arcs on which the two chords stand are the same length.
B The chords are the same perpendicular distance from the centre.
C The chords cannot intersect.
D The chords subtend equal angles at the centre of the circle.
E None of the above.

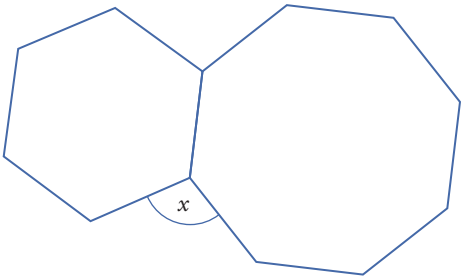
Short answer

5A **1** Determine the values of the pronumerals in these diagrams.

a



b



Note: Both polygons are regular.

5B **2** $\triangle ABC$ has an internal line drawn from vertex B, meeting the opposite side AC at point D. The following fact is known about this triangle:

$\angle ABC$ is a right angle.

a Draw a sketch of the triangle.

b Use only the given information to prove that $\angle ADB = \angle DBC + \angle BCD$.

5C **3** Rectangle $ABCD$ has side lengths of 4 cm and 3 cm. Its diagonals intersect at point P.

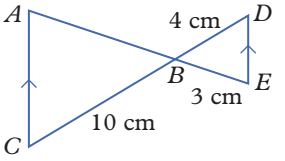
a Draw a diagram to represent this rectangle.

b Explain why $\angle ABP$ is equal to $\angle CDP$, and $\angle PAB$ is equal to $\angle PCD$.

c Explain why $\triangle PAB \cong \triangle PCD$.

d If $\angle BDC = 37°$, find the size of $\angle CBD$.

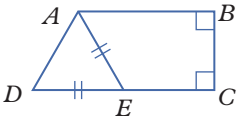
5C **4** **a** For the diagram on the right, provide a reason why $\triangle ABC$ is similar to $\triangle EBD$.



b If $\triangle EBD$ is the reduced image of $\triangle ABC$, what is the scale factor?

c What is the length of AB?

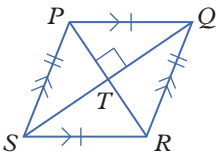
5D **5** Refer to the diagram on the right and write a proof to show that:



a $\angle AEC = 2\angle ADE$

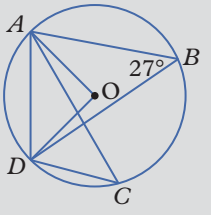
b $2\angle ADE + \angle BAE = 180°$

5D **6** Use the information from the diagram below to prove that figure $PQRS$ is a rhombus.

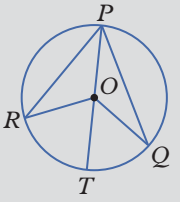


5E **7** Determine the size of these angles in the diagram on the right.

a $\angle AOD$ **b** $\angle ACD$ **c** $\angle OAD$

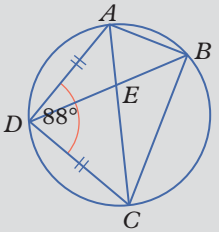


5E **8** For the diagram on the right, prove that $2\angle RPO + 2\angle QPO + \angle POR + \angle POQ = 360°$.

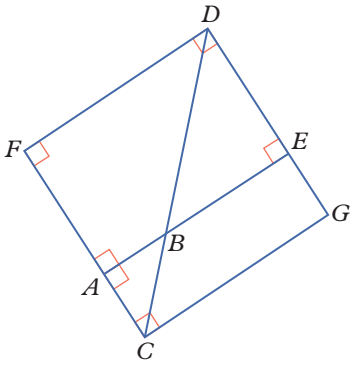


5F **9** **a** Determine the size of $\angle DBC$.

b Prove that $\angle DBC = \angle ABD$.



Analysis



1 Consider the quadrilateral $FDGC$. Any facts proven in parts **a–c** below can be used as given facts in parts **d–f**.

a Prove that $\triangle CFD \cong \triangle CGD$.

b Prove that $\triangle CAB \sim \triangle CFD$.

c Prove that $\triangle CAB \sim \triangle DEB$.

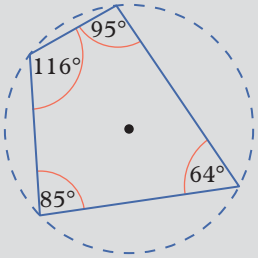
d If $AC = DE$, prove that B is the midpoint of CD.

e If $\frac{FC}{AC} = 3$, prove that $\frac{DE}{AC} = 2$.

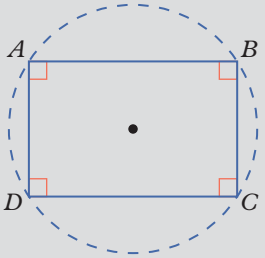
f If $\angle DBE = 45°$, prove that $FDGC$ is a square.

2 All of the edges of a cyclic quadrilateral are chords of the same circle. We can consider these chords and the chords that are the diagonals of the cyclic quadrilateral in terms of the circle theorems.

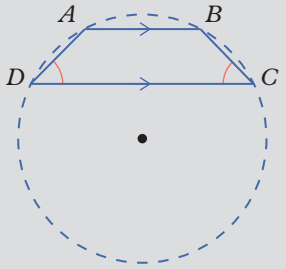
a Explain why this quadrilateral is cyclic.



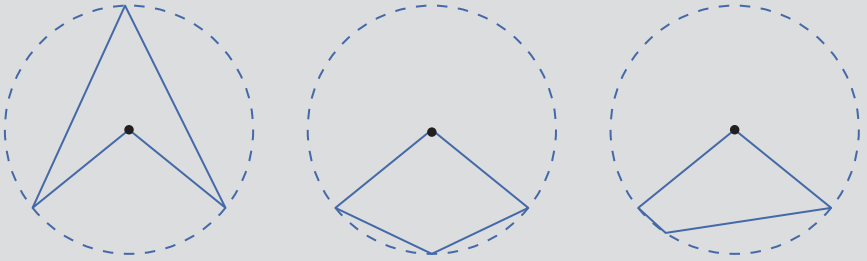
b Prove that all rectangles are cyclic, using $ABCD$ shown below.



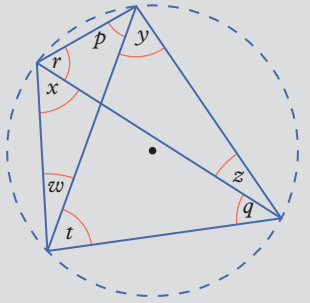
c An isosceles trapezium has equal base angles. Prove that all isosceles trapezia are cyclic, using $ABCD$ shown below.



d Explain why quadrilaterals formed using circle theorem 1 (the 'angle at the centre' theorem) are not cyclic.

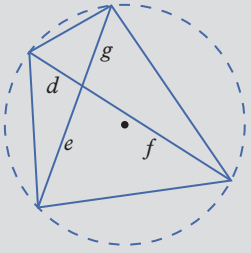


e The diagonals have been added to the cyclic quadrilateral in part **a**. Explain why $p = q$, $r = t$, $x = y$ and $w = z$.

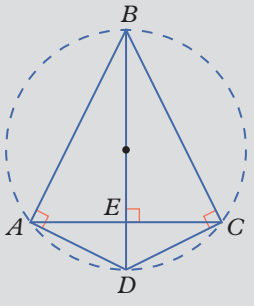


f If $x = 54°$, determine the values of p, q, r, t, w, y and z.

g Considering only the diagonals of the cyclic quadrilateral shown below, what equation can be written in terms of d, e, f and g?



h Prove that all kites with a pair of right angles, as shown in $ABCD$ below, are cyclic.



Semester 1 review

Short answer

1 Simplify each of the following. Write your answers using positive exponents.

a $2x + x^2 + 2 - 5x + 8 - 3x^2 - 5 - 6x$

b $3xy - 4yx^2 + 11xy^2 + 9yx - 5y^2x + 4x^2y + 10xy$

c $\dfrac{x^6y^{-5} \times x^{-4}y^{-11}}{x^{-3}y^3 \times xy^{-2}}$

d $(2x^{-3}y^5)^{-2} \times \left(\dfrac{y^5}{4x^3}\right)^2$

e $\dfrac{12^0x^5y^{-4}}{4^{-2}(xy)^{-4}} \div \dfrac{6^2x^{-5}y}{12^2(x^{-1}y)^3}$

f $\dfrac{3x+2}{5} - \dfrac{8x-4}{3}$

2 Expand and simplify each of the following.

a $3(2x + 5y)$

b $4x^2y(9xy^4 - 8x^3y)$

c $\dfrac{-5x}{4}(3x + 6y - 8)$

d $(2x - 7y)(8a - 7b)$

e $(x - 4)(x + 7) + (x + 4)(x + 3)$

f $(5x + 2)(3x - 4) + 3(2x + 5)$

g $(10x + 3)^2$

h $(x - 5)(3a + 8b + 2c)$

3 Factorise each of the following expressions.

a $16xy^2 - 24x^2y^2 + 40xy$

b $20xy - 5x + 16y - 4$

c $25x^2 - 49$

d $(x - 3)^2 - 9$

e $x^2 - 10x + 25$

f $x^2 + 4x + 3$

g $x^2 + 7x - 44$

h $3x^2 - 36x - 135$

4 Simplify each of the following.

a $\dfrac{6x+3}{2} \times \dfrac{2x}{2x+1}$

b $\dfrac{(x+10)(x-4)}{(x-4)^2} \times \dfrac{(x+10)(x-3)}{(x-2)(x-3)}$

c $\dfrac{x^2+12x+35}{x^2+7x+10} \times \dfrac{x^2-4}{4x+28}$

d $\dfrac{x^2-5x+6}{x^2+6x-16} \div \dfrac{x^2-9}{x^2-6x+8}$

5 Complete the square to write each of the following expressions in turning point form.

a $x^2 + 10x$

b $x^2 - 6x + 12$

c $x^2 - 5x + 12$

d $2x^2 - 28x - 18$

6 **a** Factorise $2x^2 - x - 15$.

b Factorise $6x^2 - 7x - 20$.

c **i** Complete the square for $4x^2 - 3x - 7$.

ii Using your answer from part **i**, factorise $4x^2 - 3x - 7$.

7 Determine whether each of the following numbers is rational or irrational.

a $\dfrac{3}{7}$

b $-0.229\,515$

c $24\sqrt{2}$

d $3\sqrt{-8}$

e $\sqrt{16} - 2\sqrt{2}$

f $\sqrt{\dfrac{9}{25}}$

8 Evaluate and/or simplify each of the following.

a $\sqrt{180}$

b $\sqrt{2} + \sqrt{72} + \sqrt{27} + \sqrt{50} + \sqrt{3} + \sqrt{12}$

c $\sqrt{75} \times \sqrt{6}$

d $\dfrac{\sqrt{120}}{\sqrt{5}}$

e $3\sqrt{6}(4\sqrt{6} + 9\sqrt{2})$

f $(4\sqrt{3} - 2\sqrt{5})(2\sqrt{3} + 3\sqrt{5})$

9 Write each of the following with a rational denominator.

a $\dfrac{5}{\sqrt{3}}$

b $\dfrac{4+\sqrt{2}}{\sqrt{6}}$

c $\dfrac{3}{5-3\sqrt{2}}$

d $\dfrac{3+4\sqrt{5}}{2\sqrt{5}+4}$

10 Simplify each of the following.

a $a^{\frac{5}{4}} \times a^{\frac{2}{3}}$ **b** $\left(b^{\frac{2}{5}}\right)^6$ **c** $\frac{c^{\frac{5}{4}}}{c^{\frac{5}{6}}}$

d $\frac{d^{\frac{5}{4}} \times d^{\frac{2}{3}}}{d^{\frac{5}{6}}}$ **e** $4\left(e^{\frac{2}{3}} f^{\frac{1}{2}}\right)^6$ **f** $5x^{\frac{1}{2}}y \times 3x^2 y^{\frac{1}{4}}$

11 Evaluate each of the following.

a $8^{\frac{2}{3}}$ **b** $81^{\frac{1}{4}}$ **c** $1000^{\frac{4}{3}}$ **d** $81^{\frac{1}{2}}$ **e** $100^{\frac{3}{2}}$ **f** $128^{\frac{2}{7}}$

12 For each of these, write an equivalent equation in the form specified.

a $3^4 = 81$ (in logarithmic form) **b** $10^4 = 10\,000$ (in logarithmic form)

c $6^2 = 36$ (in logarithmic form) **d** $\log_2(16) = 4$ (in exponential form)

e $\log_3(9) = 2$ (in exponential form) **f** $\log_1(1) = 0$ (in exponential form)

13 Evaluate each of the following.

a $\log_8(64)$ **b** $\log_2(128)$ **c** $\log_5\left(\frac{1}{5}\right)$ **d** $\log_{10}\left(\frac{1}{10\,000}\right)$

e $\log_4(0.25)$ **f** $\log_{\frac{1}{2}}\left(\frac{1}{4}\right)$ **g** $\log_3\left(\frac{1}{9}\right)$ **h** $\log_2\left(\frac{1}{8}\right)$

14 Calculate each of the following.

a $\log_4(16) + \log_2(16)$ **b** $2\log_3(27) - \log_{27}(27)$

c $\log_5\left(\frac{1}{5}\right) + \log_5(5) + \log_5(1)$ **d** $\log_1(1) + \log_2(2) - \log_3\left(\frac{1}{3}\right)$

15 Solve each of the logarithmic equations for x.

a $\log_x\left(\frac{1}{9}\right) = 2$ **b** $\log_x(32) = 2$

c $\log_5(x) = -2$ **d** $\log_3(3x) - \log_3(3) = 1$

16 Use the appropriate logarithm law to calculate each of the following.

a $\log_3(8) - \log_3(72)$ **b** $\log_3(30) - \log_3(10)$

c $\log_2(4^3)$ **d** $\log_{10}(50) + \log_{10}(2)$

17 Solve each of the logarithmic equations for x.

a $\log_3(3x) - \log_3(2) = 2$ **b** $\log_2(x) + \log_2(5) = \log_2(60)$

c $\log_2(x - 10) - \log_2(10) = 10$ **d** $\log_5(x) - \log_5(2) = 2$

18 Solve each of these equations

a $4^x = 4^{-2}$ **b** $2^x = \frac{1}{8}$

c $3^{x-2} = 9$ **d** $8^{x-2} = 2^6$

e $5^x = 17$ **f** $3^{2x} = 11$

g $(3^x - 27)(2^x - 4) = 0$ **h** $(4 \times 7^x)(3^x - \frac{1}{9}) = 0$

19 Solve each of the following equations.

a $\frac{1 - 2x}{3} = 5$ **b** $4 + 3(2x - 5) = 11$

c $2(x - 5) - 7(x - 4) = -9$ **d** $10(3x - 4) = 5(3 - 7x)$

e $\frac{4x + 5}{5} = \frac{6x - 5}{3}$ **f** $\frac{x + 1}{2} + \frac{4 - x}{6} = \frac{x + 4}{4}$

g $\frac{9}{3x - 1} = -5$ **h** $\frac{12}{x - 5} = \frac{9}{x + 7}$

20 Rearrange each of the following equations to make x the subject.

a $y = mx + c$ **b** $ax + by = c$ **c** $rx + t = px - q$

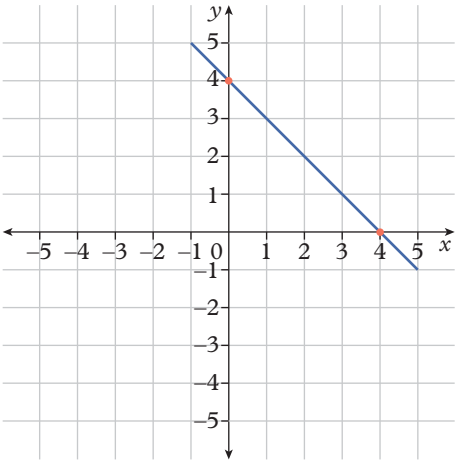
d $A = \frac{1}{2}(x + y)z$ **e** $A = \pi x^2$ **f** $V = \frac{4}{3}\pi x^3$

21 Sketch the graph of the following equations. Write the coordinates of any axis intercepts.

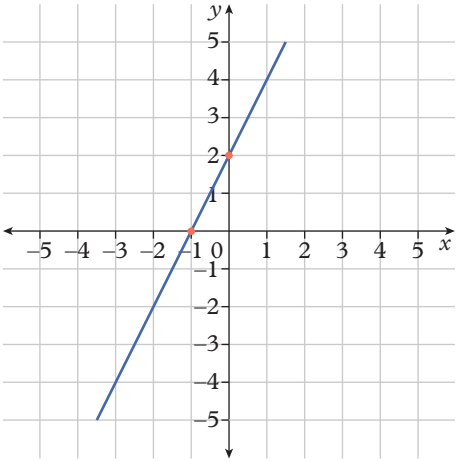
a $y = 2x - 10$ **b** $3x - 6y = 20$ **c** $x = 12$ **d** $y = 3$

22 Determine the equation of each of these linear graphs.

a



b



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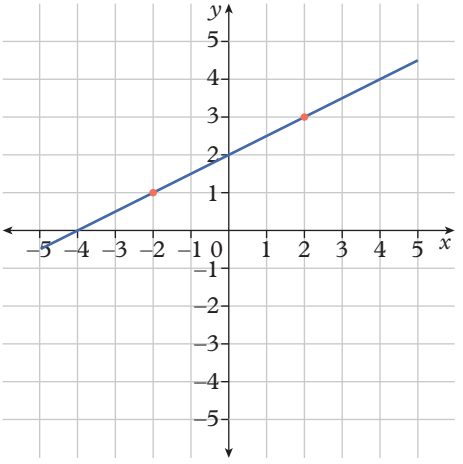
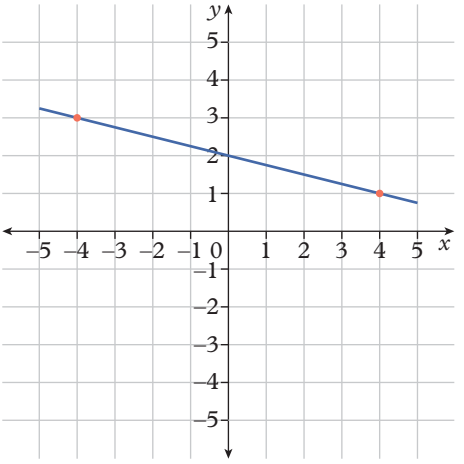
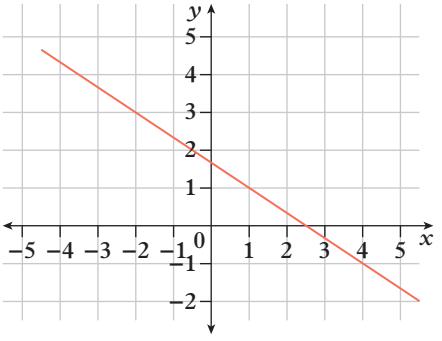


d



23 From the given information, determine the equation of each of the lines **a–c** below. Write each equation in the form $y = mx + c$.

a



b The gradient of the line is 4 and the line passes through the point (–2, 5).

c The line passes through the points (3, –4) and (8, 12).

24 Solve the following linear inequalities.

a $2 > 3x + 1$

b $23 < -3x + 2$

c $2 + x \leq 14 - 3x$

d $2x \leq 14 - 5x$

e $\frac{x + 6}{12} > \frac{5}{6}$

f $\frac{x + 6}{3} < \frac{3x - 2}{6}$

25 Shade the regions indicated by the following inequalities on the Cartesian plane. Make sure to show if the boundary line is included in each region.

a $y < 2x + 1$

b $y + \frac{x}{3} + 1 \geq 0$

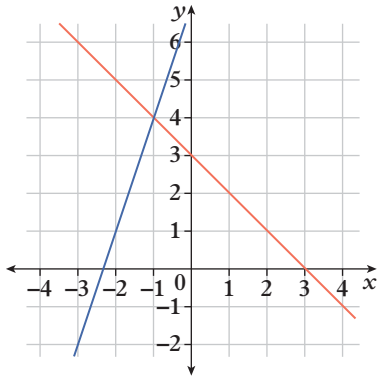
c $\frac{x}{3} \leq 2y$

d $4y + 7x < 11$

26 Use the graphs provided to solve each of the following pairs of simultaneous equations.

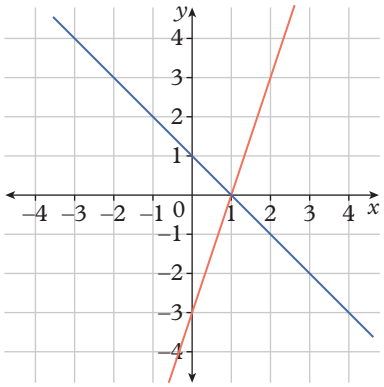
a $y = 3x + 7$

$x + y = 3$



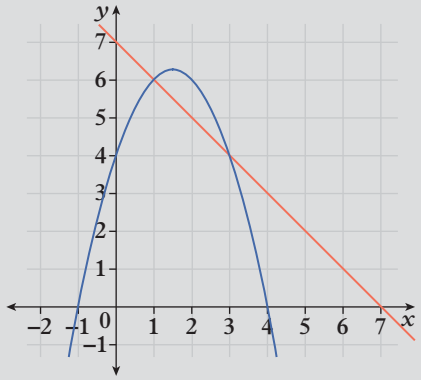
b $y = -3 + 3x$

$y = 1 - x$



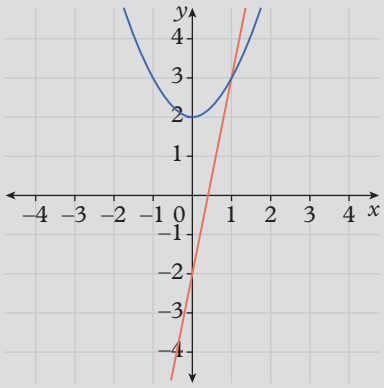
c $y = -x^2 + 3x + 4$

$y = -x + 7$



d $y = 2 + x^2$

$y = 5x - 2$



27 Solve the following pairs of equations.

a $y = 3x - 4$

$y = -7x + 16$

b $y = 2x - 5$

$4x - 5y = 12$

c $2x + 9y = 11$

$2x - 9y = 15$

d $3x - 15y = 10$

$-6x + 5y = 10$

28 Solve each of the following quadratic equations.

a $(2x - 1)(3x + 5) = 0$

b $4x^2 = 100$

c $(x - 5)^2 - 81 = 0$

d $5x^2 = 20x - 20$

e $x^2 - 10 = 0$

f $-3x^2 + 24 = 0$

29 Solve each of the following.

a $x^2 - 10x + 24 = 0$

b $5x^2 = 10x + 40$

c $x^2 + 5x - 10 = 0$

d $\frac{x-3}{5} = \frac{4}{x-2}$

e $x^2 - 14x = 2$

f $x^2 - x - 1 = 0$

30 Determine the number of x–intercepts the graphs of each of the following quadratic relationships will have.

a $y = 3(x - 4)(x + 6)$ **b** $y = (x - 4)^2$

c $y = -x^2 - 5x + 6$ **d** $y = x^2 + 1$

e $y = (x - 2)^2 - 1$ **f** $y = 4x^2 - 3x + 1$

g $y = 9x^2 + 12x + 4$ **h** $y = 2x^2 - 6x + 1$

31 Determine the coordinates of the x- and y- intercepts, where they exist, for the graphs of the equations in question **30**.

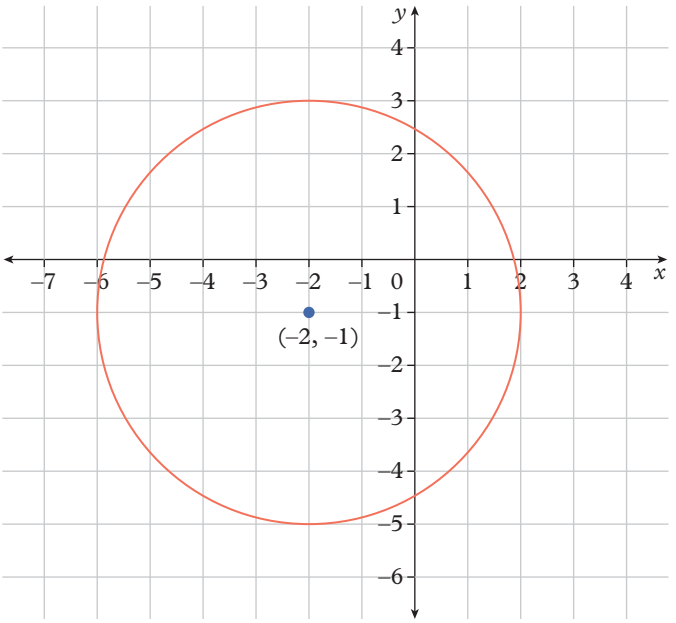
32 Graph each of the following quadratic relationships.

a $y = -x^2 + 2$ **b** $y = x^2 - 4x + 4$

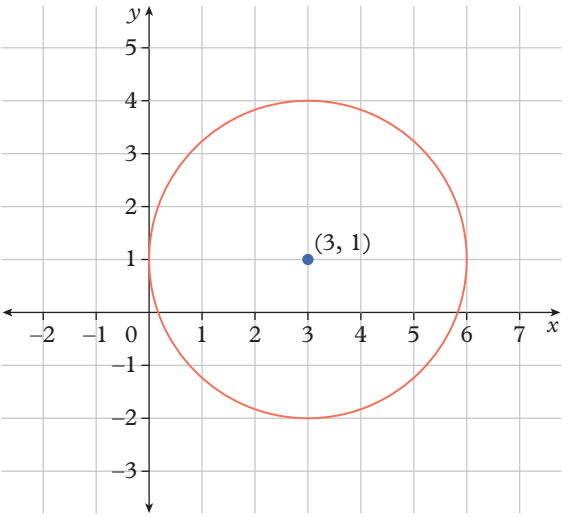
c $y = -x^2 - 4x + 1$ **d** $y = (x + 3)^2 - 2$

33 Write the equation for the following graphs.

a



b



34 Write the equation of the new graphs after performing the following transformations.

a The graph of $x^2 + y^2 = 1$ is dilated in all directions by a factor of 3, then translated 2 units right and 1 unit down.

b The graph of $(x - 1)^2 + y^2 = 0.25$ is dilated in all directions by a factor of 2, then translated 2 units right and 1 unit down.

c The graph of $y = 2^x$ is reflected in the x-axis, then translated 3 units up.

d The graph of $y = 3^{-x}$ is reflected in the y-axis, then dilated by a factor of 2 in the y-direction.

35 Calculate the average rate of change of the following relationships from $x = 3$ to $x = 4$.

a $y = 2x^2$ **b** $y = 1 - x^2$

c $y = 10^x$ **d** $y = \frac{1}{2^x}$

36 Sketch the graph of each of the following equations. Write the coordinates of any axis intercepts and turning points and the equations of any asymptotes.

a $y = 2x$ **b** $y = 2x^2$ **c** $y = (x + 2)^2$

d $x^2 + y^2 = 2^2$ **e** $y = 2^x$

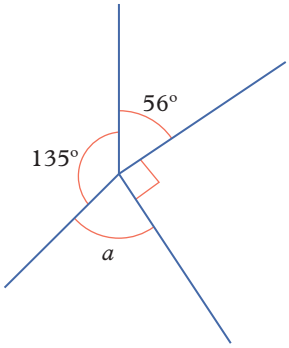
37 Describe a sequence of transformations that could have been applied to make the transformed equation from the original each time.

	Original	Transformed
a	$y = x^2$	$y = -2x^2 + 8$
b	$y = x$	$y = \frac{3}{8}(x + 2)$
c	$y = 2^x$	$y = \frac{1}{2} \times 2^x - 4$

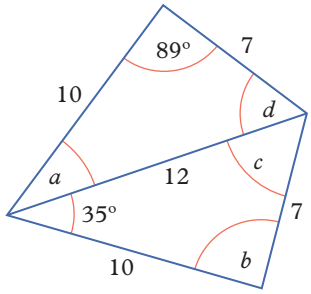
38 Sketch the graphs of the transformed equations from question **37**.

39 Determine the value of each of the pronumerals in these diagrams. Write your answers correct to one decimal place where necessary.

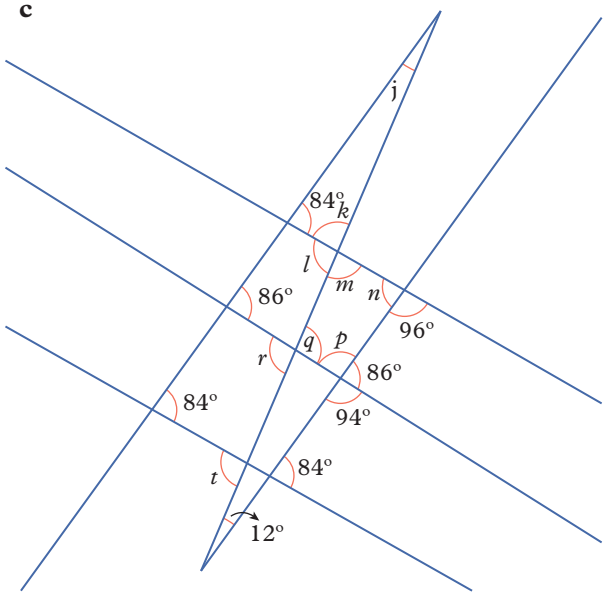
a



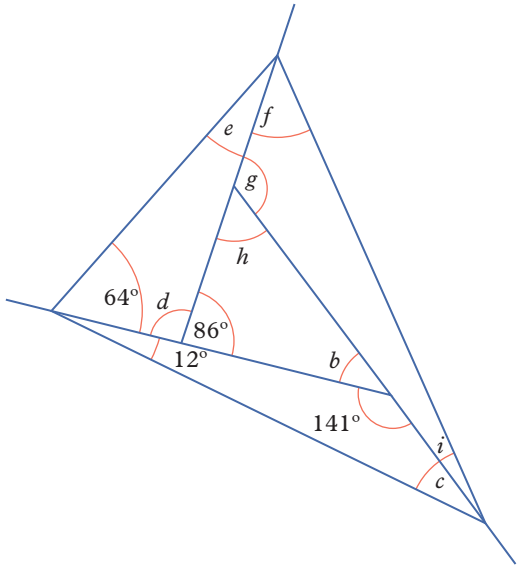
b



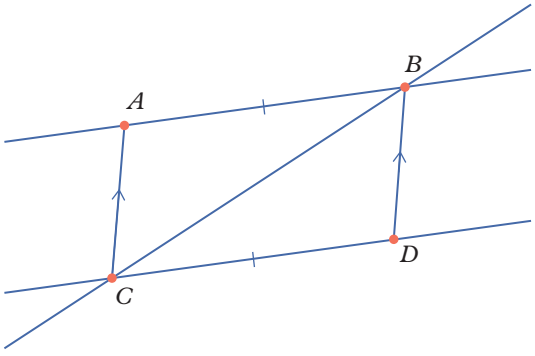
c



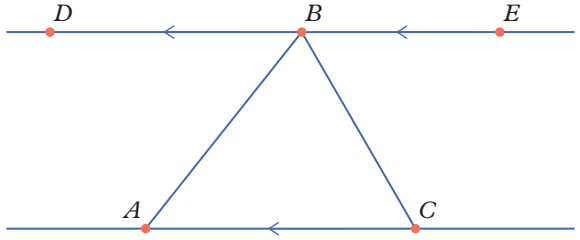
d



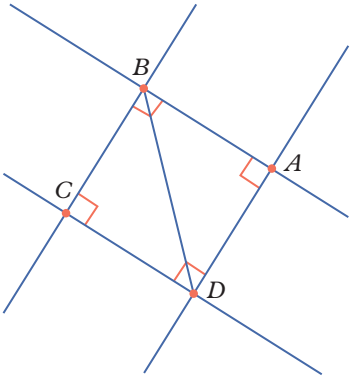
40 a Prove that $AB \parallel CD$ in this diagram.



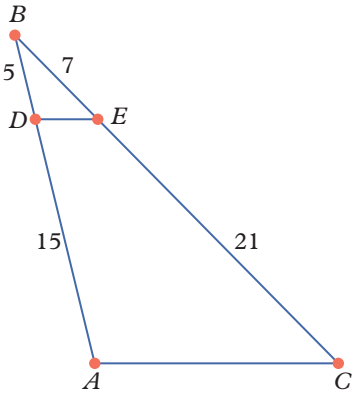
b Prove that $\angle BAC + \angle ABC + \angle BCA = 180°$.



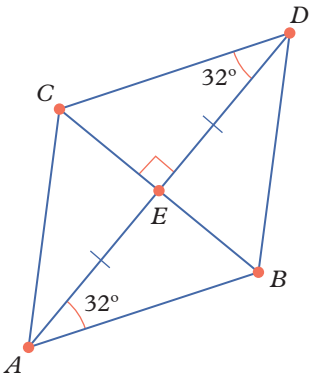
41 a Prove that $\triangle ABD \cong \triangle CDB$ in this diagram.



b Prove that $\triangle ABC$ is similar to $\triangle DBE$ in this diagram.

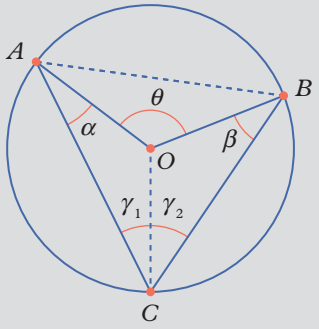


c Given that E is the intersection point of CB and AD, prove that $ABDC$ is a rhombus.



42 Use circle theorem 2 to prove that any quadrilateral whose diagonals are equal and bisect each other is a rectangle.

43 In this diagram, O is the centre of the circle. Let $\gamma = \gamma_1 + \gamma_2$ and let the reflex angle $\angle AOB = \varphi$. Without using any circle theorems, prove that $\theta = 2\gamma$.



Analysis

1 An investment of $25 000 increased in value over n years in a saving account. The value of the investment at the end of the period, A, can be calculated using the formula $A = 25\,000 \times 1.05^n$.

a Explain why $A = 25\,000$ when $n = 0$.

b Find the value of A for:

i $n = 1$

ii $n = 2$

iii $n = 3$

c Use your answer for part **b** to calculate the average rate of change between $n = 0$ and $n = 1$, $n = 1$ and $n = 2$, and $n = 2$ and $n = 3$.

d Does the value of the investment grow faster or slower over time?

e The investment increased in value to over $30 000 over a period of time, use the formula to find n. Round your answer to the nearest integer.

f How many years would it take for the value of the investment to double?

2 A new school is being designed. There will be four new buildings, A, B, C and D, arranged in the shape of a convex quadrilateral. There will be external paths from A to B (AB), B to C (BC), C to D (CD) and D to A (DA) as well as internal paths from A to C (AC) and B to D (BD).

a If the internal paths are to intersect perpendicularly, what are the possible quadrilateral layouts the school could choose?

b If the paths AB and BC are to be equal in length and the paths CD and DA are to be equal in length but the internal paths are not to be equal in length, which quadrilateral could the school choose?

c Prove that one pair of opposite angles in the quadrilateral in part **b** are equal.

The school decides to have a circular athletics track that passes the four buildings as closely as possible.

d What extra feature does the quadrilateral chosen by the school need to have? How are opposite pairs of internal angles in such quadrilaterals related?

e Use an appropriate circle theorem to prove the relationship between pairs of opposite angles in such quadrilaterals in part **d**.

f Prove that if the quadrilateral from part **b** has the feature from part **d**, then the quadrilateral is made up of two congruent right-angled triangles.

3 At a local park, the height of a hill is measured from one side to the other, where height is measured from the park's entrance near the base of the hill. The height is then modelled by an equation which gives the height, h metres, at a horizontal distance, d metres, from the park's entrance: $h = -0.05d^2 + d$, where $h \geq 0$.

a Determine the width of the hill.

b Determine the maximum height of the hill and how far, horizontally, it is from the entrance.

c Sketch the graph of the equation for an appropriate domain.

d Determine the horizontal distance to the points on the hill that are 4 m above the ground level measured at the entrance. Write your answers:

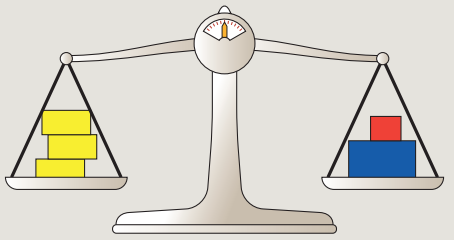
i as exact values

ii correct to two decimal places.

EXPLORATIONS 1

1 Balancing bricks

Indy has a collection of coloured plastic bricks and a balance scale. All bricks of the same colour weigh the same, but bricks of different colours weigh different amounts.



a Indy discovers the following about red, blue, yellow and green bricks:

- 1 blue brick weighs the same as 1 yellow and 1 green brick
- 3 yellow bricks weigh the same as 1 blue and 1 red brick
- 1 blue brick weighs more than 1 green and 3 red bricks
- 14 red bricks weigh more than 1 blue brick
- at least one combination of the other bricks weighs the same as 1 yellow brick.

Find all such combinations.

b Indy discovers the following about black, white, orange and purple bricks:

- 1 orange brick weighs the same as 1 white and 1 black brick
- 3 white bricks weigh the same as 1 orange and 1 purple brick
- 2 white bricks weigh less than 5 purple bricks
- **only one** combination of the other bricks weighs the same as 1 black brick. Find that combination.

2 Times tables

In each of the following partially completed multiplication tables, the letters A, B, C, … represent different whole numbers, subject to some extra conditions. Find the value of each letter and complete the table.

a Letters A to J represent the numbers 1 to 10, in some order. The sum of numbers A to E is equal to the sum of numbers F to I.

×	A	B	C	D	E
F	18				
G				20	
H					
I					24
J			10		

b Letters A to H represent the even numbers from 2 to 16, in some order. The sum of E, G and H is twice the sum of A, B, C and D.

×	A	B	C	D
E		40		
F			24	
G				112
H				

c Letters A to I represent the numbers 2 to 10, in some order. The sum of the squares of A, B and C is equal to the sum of the squares of D, E, G, H and I.

×	A	B	C	D	E
F			30		
G		28			
H				27	
I					

3 How irrational!

a Rationalise the denominator to simplify $\frac{1}{\sqrt{2}+\sqrt{3}}$.

b Let a and b be positive integers and consider the fraction $\frac{1}{\sqrt{a}+\sqrt{b}}$.

i Find, in terms of a and b, an alternative expression for this fraction in which the denominator is an integer.

ii Does your answer always make sense? If not, describe when it fails and find the correct answer in this case.

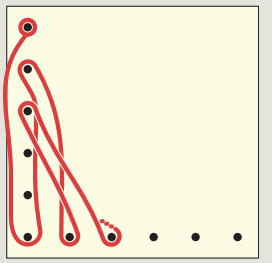
c Rationalise the denominator twice to simplify $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

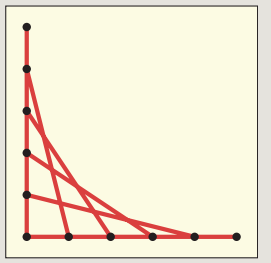
d Let a, b and c be positive integers with $c = a + b$ and consider the fraction $\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$.

i Find, in terms of a, b and c, an alternative expression for this fraction in which the denominator is an integer.

ii Use your rule to simplify $\frac{1}{2\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

4 String art





If you have access to a pegboard and string, try the following hands-on activity first. Alternatively, you can use free online geometry software, such as GeoGebra *or* Desmos. *For best results, use as many pegs (or virtual pegs) as possible.*

On a pegboard, vertical and horizontal rows of equally spaced pegs are arranged in an L-shape. A piece of string is looped from the first (top) peg of the vertical row to the first (left-most) peg of the horizontal row, then to the second vertical peg, second horizontal peg, third vertical peg and so on – see the first diagram. When the string is pulled tight, it forms the design in the second diagram. The larger the number of pegs used, the smoother the curve formed by the boundary of the intertwined string. What is this curve?

To model this mathematically, it is best to rotate the board 45° anti-clockwise.

a Follow these instructions to make your own 'string art' on a sheet of grid paper:

- Draw x- and y-axes with the origin in the middle of the page.
- Plot the lines $y = x - 10$ and $y = -x - 10$, extending them to the edge of the page. (There is nothing special about the number 10. Later we will replace it with an arbitrary number, d.)
- Pick any point on one of the lines (for example, (13, 3) on $y = x - 10$). Find the point on the other line whose y-coordinate is the negative of your first point's y-coordinate (for example, $(-7, -3)$ on $y = -x - 10$). Draw a line through your pair of points, extending it to the edge of the page.
- Repeat the previous step several times. As more and more lines are added, a familiar shape should emerge!

Let's prove that the curve produced by the lines in part **a** is the parabola $y = \frac{1}{40}x^2$. To do so, it is enough to show that each such line is a tangent to this parabola. To warm up, we start with a concrete example with two given points.

b **i** Find, in gradient–intercept form, the equation of the line through (13, 3) and $(-7, -3)$.

ii Verify that this line intersects the parabola $y = \frac{1}{40}x^2$ exactly once.

Now try the same strategy in more generality.

c Let $P(k + 10, k)$ be a point on the line $y = x - 10$, where k is a fixed but arbitrary number.

i Find the point Q on the line $y = -x - 10$ whose y-coordinate is $-k$.

ii Find, in gradient–intercept form, the equation of the line through P and Q.

iii Verify that this line intersects the parabola $y = \frac{1}{40}x^2$ exactly once.

Finally, let's see how the parabola changes when the original 'lines of pegs' are changed.

d For some arbitrary fixed number $d \neq 0$, consider the lines $y = x - d$ and $y = -x - d$ with points $P(k + d, k)$ and $Q(k - d, -k)$, respectively. There is some parabola of the form $y = ax^2$ which, for all values of k, intersects the line through P and Q exactly once. Verify this claim by finding an expression for a in terms of d.

5 Fifth degree

a Calculate the first five fifth powers: 1^5, 2^5, 3^5, 4^5 and 5^5. What do you notice about their last digits? Does this always happen?

b Fully factorise the expression $n^5 - n$. Hence show that $n^5 - n$ is divisible by 30 for all integers n. (Hint: if n, $n - 1$ and $n + 1$ are not divisible by 5, then n is of the form $5k \pm 2$ for some integer k.)

c How does the result of part **b** answer the final question of part **a**?

d Show that $n^5 - n$ is divisible by 240 for all odd integers n.

e For what values of n is $n^5 - n$ divisible by 9?

f Deduce from part **b** that every integer has the same final digit as its ninth power. For what other powers is this true?

Explorations inspired by the Australian Maths Trust's competitions and programs: www.amt.edu.au

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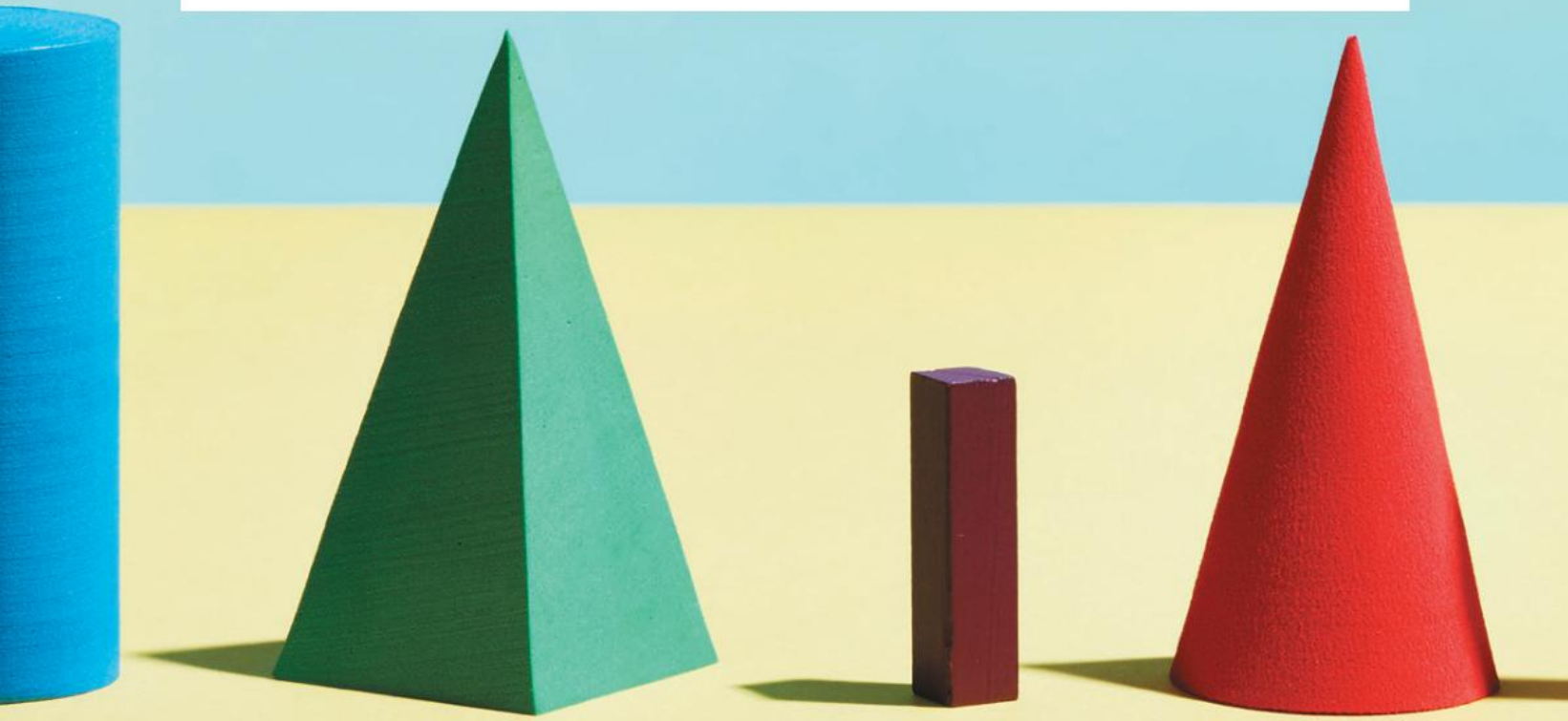
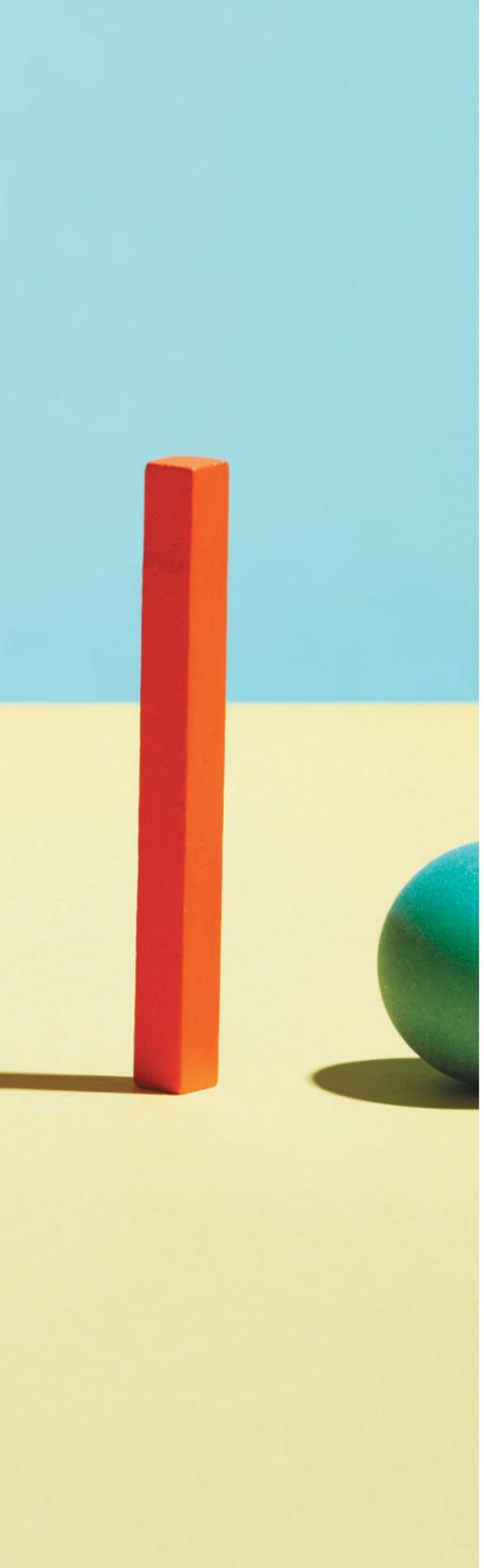
6 Measurement





Index

6A Area review
6B Surface area
6C Volume
6D Measurement errors and accuracy
6E Proportion and scaling
6F Logarithmic scales

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Converting between units of length
- ✔ Identifying shapes within composite shapes
- ✔ Area of a triangle
- ✔ Area of quadrilaterals
- ✔ Area of a circle

Curriculum links

- Solve problems involving the surface area and volume of composite objects using appropriate units [AC9M10M01]
- Interpret and use logarithmic scales in applied contexts involving small and large quantities and change [AC9M10M02]
- Identify the impact of measurement errors on the accuracy of results in practical contexts [AC9M10M04]
- Use mathematical modelling to solve practical problems involving proportion and scaling of objects; formulate problems and interpret solutions in terms of the situation; evaluate and modify models as necessary, and report assumptions, methods and findings [AC9M10M05]

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Materials

- ✔ Calculator

6A Area review

Learning intentions

By the end of this section you will be able to ...

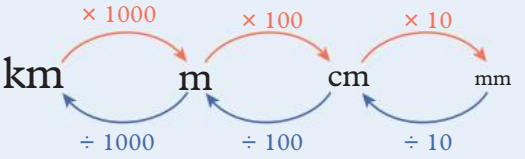
- ✔ convert units of lengths and area
- ✔ calculate the area of composite shapes
- ✔ calculate the area of sectors.

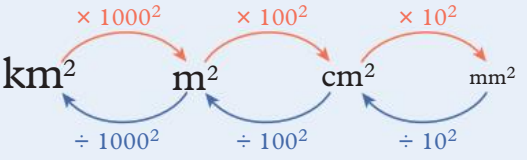
Inter-year links

Support	Understanding area
Year 7	9E Area of a triangle
Year 8	8D Area of a circle
Year 9	5A Area of composite shapes

Units of measurement

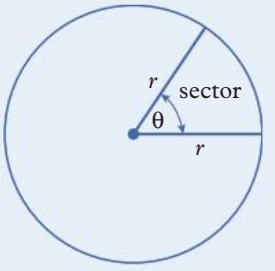
- To convert between units of length, multiply or divide by the appropriate **conversion factor**.
- A unit of **area** can be converted to another unit of area by multiplying or dividing by the appropriate conversion factor.
- 1 ha (hectare) = 10 000 m^2





Composite shapes

- To find the area of a **composite shape**:
  1. split the shape into individual parts
  2. calculate any missing dimensions
  3. calculate the areas of the individual parts
  4. add or subtract the areas to find the total area.
- The area of the sector of a circle is $\frac{\theta}{360°} \times \pi r^2$, where θ is the angle, in degrees, between the two radii forming the sector.



Example 6A.1 Converting units of length and area

Convert each length or area measurement below to the unit shown in brackets.

a 120 mm (m) **b** 1.2 km^2 (m^2) **c** 320 mm^2 (cm^2)

THINK

a Identify the correct conversion factor. To convert from millimetres (mm) to metres (m), first divide by 10 (to convert to centimetres), then divide by 100 (to convert to metres).

b Identify the correct conversion factor. To convert from square kilometres (km^2) to square metres (m^2) multiply by 1000^2.

c Identify the correct conversion factor. To convert from square millimetres (mm^2) to square centimetres (cm^2), divide by 10^2.

WRITE

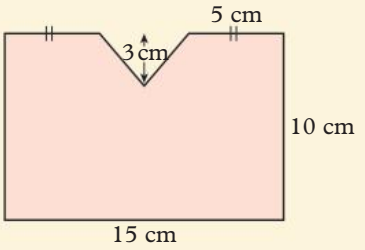
a 120 mm = (120 ÷ 10 ÷ 100) m
= 0.12 m

b 1.2 km^2 = (1.2 × 1000^2) m^2
= 1 200 000 m^2

c 320 mm^2 = (320 ÷ 10^2) cm^2
= 3.2 cm^2

Example 6A.2 Calculating the area of a composite shape

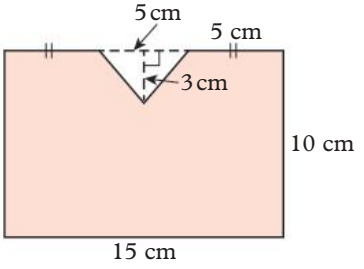
Calculate the area of this composite shape.



THINK

1. Redraw the shape with dashed lines, splitting it into individual shapes for which you can calculate the areas.
2. Label any unlabelled dimensions for each individual shape. The total length of the rectangle is 15 cm and the length either side of the triangle is 5 cm. The cut-out triangle must have a base of 5 cm (because $15 - 2 \times 5 = 5$).
3. Calculate the area of each individual shape.
4. Calculate the required area by subtracting the area of the triangle from the area of the rectangle.

WRITE



rectangle: $A = lw$

$= 10 \times 15$

$= 150 \text{ cm}^2$

triangle: $A = \frac{1}{2}bh$

$= \frac{1}{2} \times 5 \times 3$

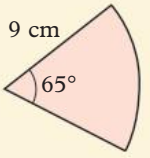
$= 7.5 \text{ cm}^2$

total area $= 150 - 7.5$

$= 142.5 \text{ cm}^2$

Example 6A.3 Calculating the area of a sector

Calculate the area of this sector correct to one decimal place.



THINK

1. Identify the values of θ and r.
2. Substitute the values into the formula for the area of a sector.
3. Calculate the area and round to one decimal place, including the appropriate unit.

WRITE

$\theta = 65°$ $\quad r = 9 \text{ cm}$

$A = \frac{\theta}{360°} \times \pi r^2$

$= \frac{65°}{360°} \times \pi \times 9^2$

$= 45.945...$

$\approx 45.9 \text{ cm}^2$

Helpful hints

- ✔ There can be more than one way to split a composite shape into individual component shapes. The way you split up the shape should not change the total area.
- ✔ When being asked to round to a given number of decimal places, the rounding should only take place at the final stage of your work and not in any intermediate steps. For example, if you are calculating the area of a composite shape, use the full numbers on your calculator when finding the areas of the individual shapes and only round after adding or subtracting.

ANS p613

Exercise 6A Area review

1–3, 4(a, c, e), 5, 6, 7(a, c, d), 8, 9 | 1, 3(c, e, f), 4(b, d, f), 5, 7(b, d, f), 9, 10, 12, 14 | 1, 3(c, f), 4(d, f), 5, 7(d, f), 11–15

UNDERSTANDING AND FLUENCY

For questions involving π, use the π key on your calculator.

6A.1 **1** Convert each length or area measurement below to the unit shown in brackets.

a 7 m (cm) **b** 4500 m (km) **c** 8.4 km^2 (m^2)

d 29 mm^2 (cm^2) **e** 24 000 m^2 (ha) **f** 0.88 m (mm)

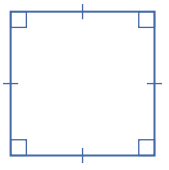
g 34 200 000 cm^2 (km^2) **h** 10.22 km (cm) **i** 3 km^2 (ha)

2 Match each shape below with a formula from this list that could be used for finding its area:

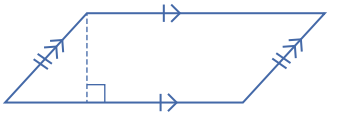
a $A = \pi r^2$ **b** $A = l^2$ **c** $A = \frac{1}{2}(a + b)h$

d $A = bh$ **e** $A = \frac{xy}{2}$

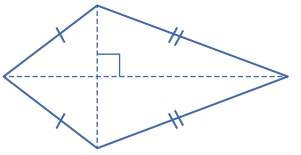
A



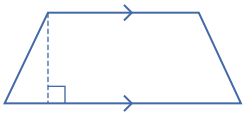
B



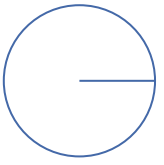
C



D

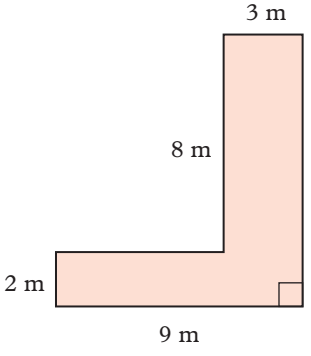


E

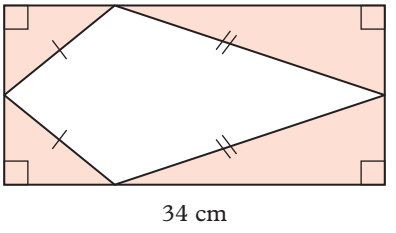


6A.2 **3** Calculate the shaded area of each of these composite shapes. Give your answers to one decimal place where necessary.

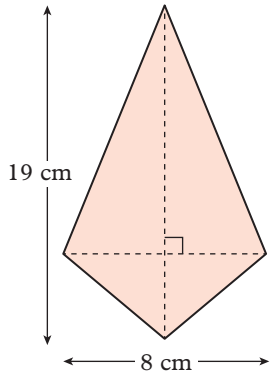
a



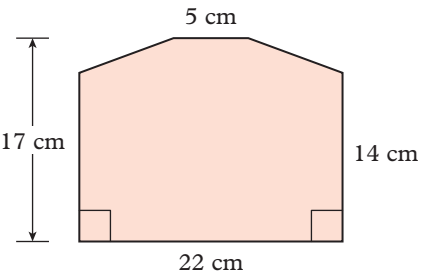
b



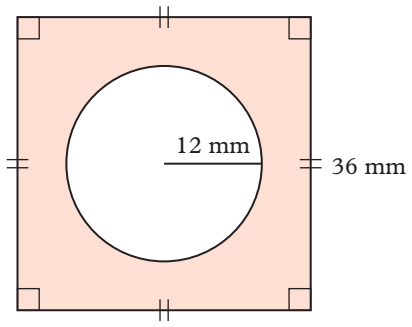
c



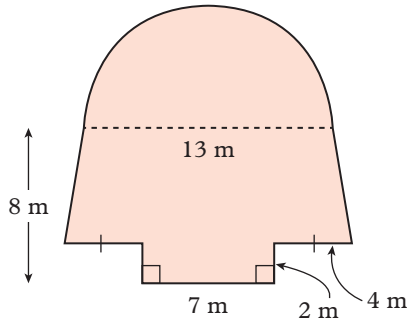
d



e

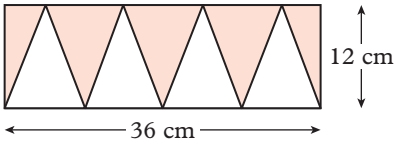


f

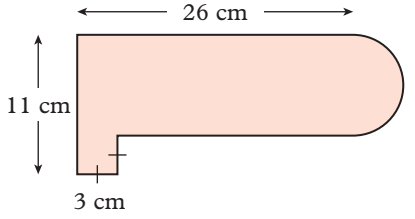


4 Calculate the shaded area of each of these composite shapes. Give your answers in the units given in the brackets, rounded to one decimal place where necessary.

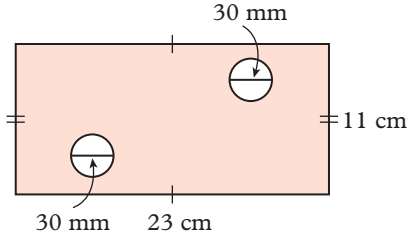
a (cm^2)



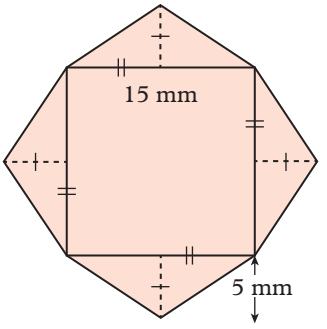
b (cm^2)



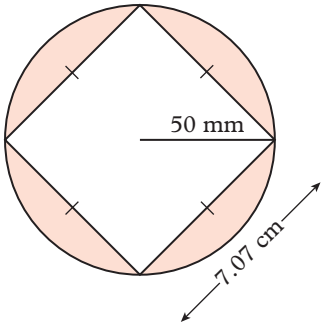
c (cm^2)



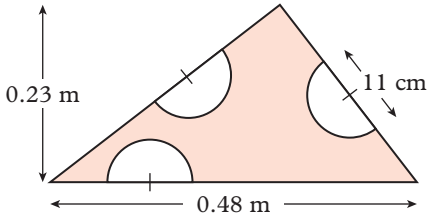
d (mm^2)



e (cm^2)

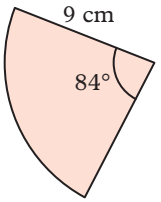


f (cm^2)

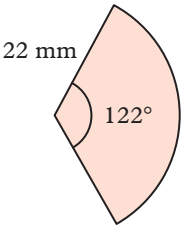


6A.3 **5** Calculate the area of each of these sectors correct to one decimal place.

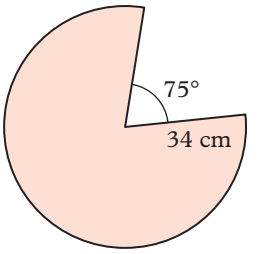
a



b



c



6 The front view of this house is made of a number of basic shapes.

a Identify each shape.

b Calculate the area of each shape you identified in part **a**.

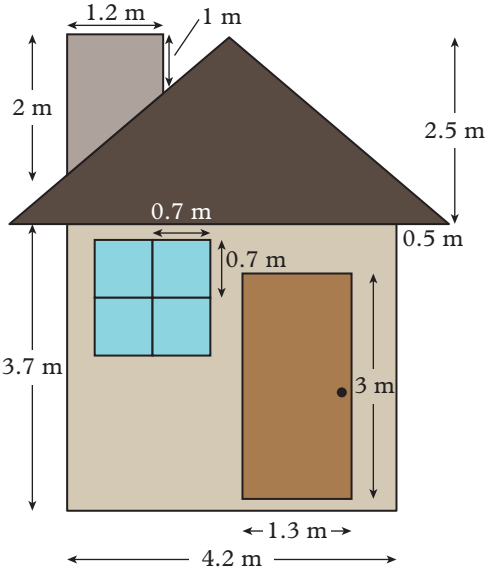
c There are four window panes of equal size. Explain how you can calculate the area of all four window panes if you know the area of only one window pane.

The owners wish to paint the front of the house (including the door, roof and chimney).

d Using your knowledge of composite shapes, explain how to calculate the total area that needs to be painted. (Remember that the windows should not be painted.)

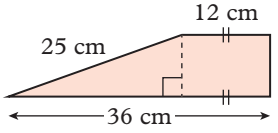
e Use your answers for parts **c** and **d** to help you find the area of the house that will be painted.

f If 1 L of paint covers 15 m^2, how many litres of paint is needed? Give your answer to the nearest half litre.

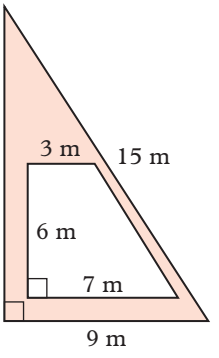


7 Use your knowledge of Pythagoras' theorem to find any unknown lengths before calculating the shaded areas. Give your answers to one decimal place where necessary.
Hint: Check that all measurements are written in the same units.

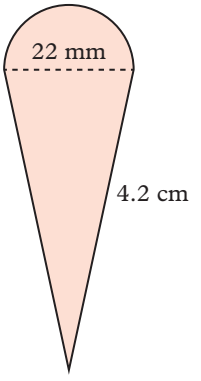
a



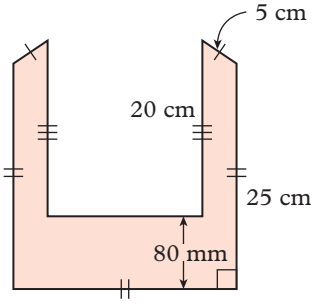
b



c



d



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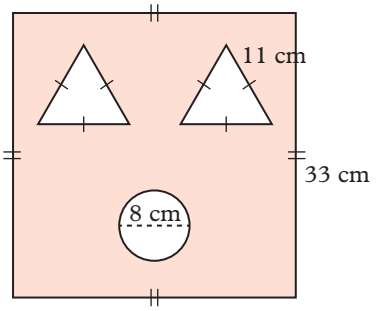

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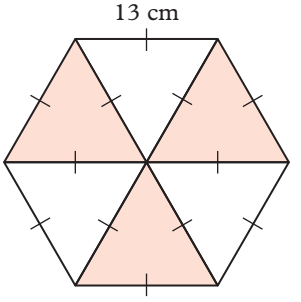


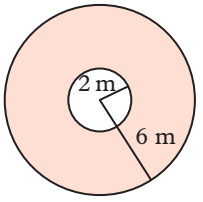
8 A farm is in the shape of a trapezium. The length of the northern boundary of the farm is 265 m and the length of the southern boundary is 180 m. The perpendicular distance from the northern boundary to the southern boundary is 90 m.

a Draw a diagram of the farm and find its area.

b Write your answer to part **a** to the nearest hectare.

9 An annulus is the region between two circles of different sizes which share the same centre.

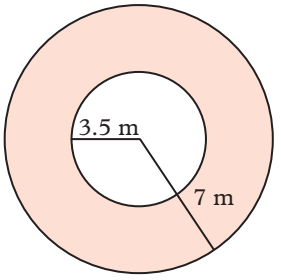
a Explain how you can find the area of the annulus below.



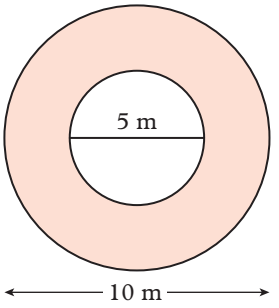
b Find the area of the annulus, correct to one decimal place.

c Find the area of the following annuli, correct to one decimal place.

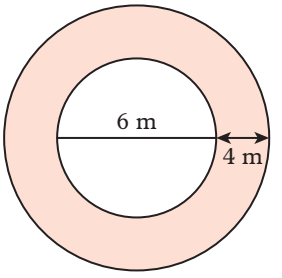
i



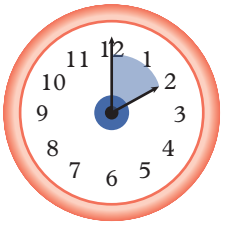
ii



iii



10 A clock has a minute hand that is 7 cm long and an hour hand that is 5 cm long.



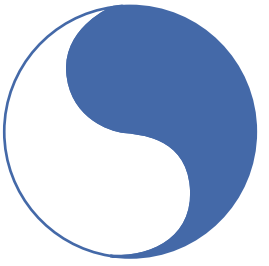
a What is the difference in cm^2, correct to one decimal place, between the areas covered by the hands when they each complete a full revolution?

b The time shown on the clock face on the right is 2 o'clock. Calculate the angle between the minute hand and the hour hand.

c Find the difference in the areas that would be covered by the hands if they both moved from the position labeled 12 to the position labeled 2 on the clock face. Give your answer correct to one decimal place.

d Repeat part **c** if the hands both move from the position labeled 12 to the position labeled 7 on the clock face.

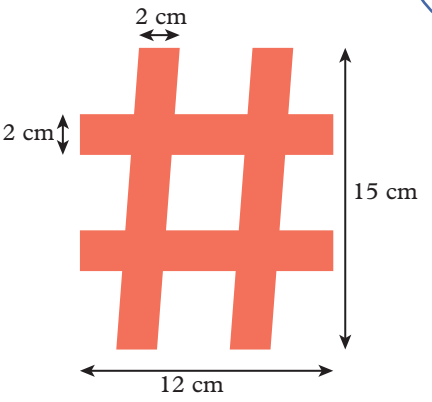
11 A design based on the ancient yin-yang symbol is shown on the right.



a If the radius of the full circle is 24 cm, find the area shaded blue correct to one decimal place.

b Find the unshaded area without performing any further calculations.

12 The area of a parallelogram is $A = bh$, where b is the base length and h is the perpendicular height.
Find the total shaded area of this hash symbol.



CHALLENGE

13 Consider the target on the right. The radius of the inner circle is 2 cm and the edge of each ring is 2 cm further from the centre of the inner circle.

a Find the shaded area, giving your answer as an exact value.

b Find the unshaded area, giving your answer as an exact value.

c Find the ratio of the shaded area to the unshaded area.

14 A regular hexagon can divided into six congruent equilateral triangles. Give your answers to the following questions as exact values.

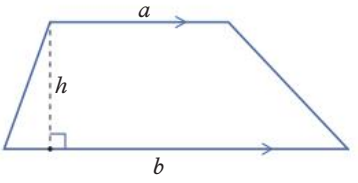
a Use Pythagoras' theorem to find the height of an equilateral triangle with side lengths 10 cm.

b Find the area of an equilateral triangle with side lengths 10 cm.

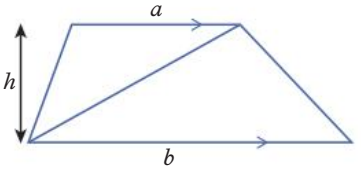
c Find the area of a regular hexagon with side lengths 10 cm.

d Find the area of a regular hexagon with side lengths of a.

15 A trapezium is a quadrilateral with a pair of parallel sides. The area of a trapezium is $A = \frac{h}{2}(a + b)$.

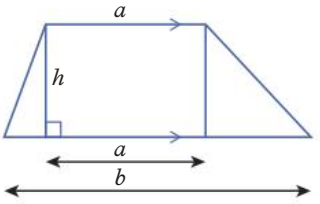


a Derive the trapezium area formula by considering a trapezium as the composite of two triangles.

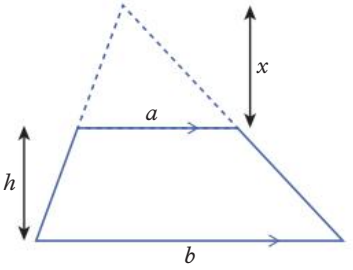


b Derive the trapezium area formula by considering a trapezium as the composite of one rectangle and two triangles.

Hint: You can combine the two triangles into one triangle, since they have the same height.



c Derive the trapezium area formula by considering a trapezium as a truncated triangle.
Hint: Use properties of similar triangles to eliminate x from your formula.



Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Converting between units of area
- Interactive skillsheet: Area of composite shapes
- Topic quiz: 6A

6B Surface area

Learning intentions

By the end of this section you will be able to ...

- ✔ calculate the surface area of prisms and cylinders
- ✔ calculate the surface area of composite solids.

Inter-year links

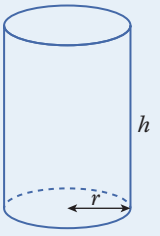
Year 7	9F Surface area
Year 8	8D Area of a circle
Year 9	5B Surface area

Surface area

- The **total surface area** (TSA) of a 3D object is the total area of its outer surface. This is the sum of the areas of the faces (or surfaces) of that object.
- **Prisms** are 3D objects that have a uniform **cross-section** that is a polygon.
  - → All faces of a prism are flat.
  - → Cubes, rectangular prisms and triangular prisms are all common examples of prisms.
- **Cylinders** are 3D objects with a uniform circular cross-section and a curved face. The surface area of the curved face of a cylinder is equal to the circumference ($2\pi r$) of either end multiplied by the height (h):

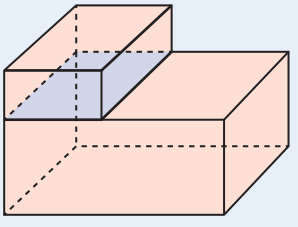
  Area of curved face of a cylinder = $2\pi rh$
- The surface area of a cylinder (the area of the two circular ends and the curved surface) can be found using the formula:

  TSA = $2\pi r^2 + 2\pi rh$



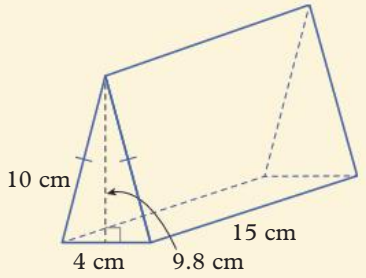
Surface area of composite solids

- A composite solid consists of two or more simple solids.
- To calculate the total surface area of a composite solid, identify all faces of the composite solid and add the area of each face to find the total surface area.
- The area where two solids join should not be included in the calculation of total surface area.



Example 6B.1 Calculating the surface area of a prism

Calculate the surface area of this triangular prism.



THINK

1 Identify the number of faces and the shapes of the faces of the triangular prism.

2 Calculate the area of each face. The triangular faces are isosceles triangles, meaning the two rectangles sharing common sides with them are identical.

3 Add the areas and include the appropriate unit.

WRITE

The prism has five faces: two identical rectangular faces, two identical triangular faces and a rectangular base.

rectangular faces: $A = 10 \times 15$
$= 150 \text{ cm}^2$ each

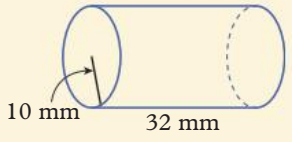
rectangular base: $A = 4 \times 15$
$= 60 \text{ cm}^2$

triangular faces: $A = \frac{1}{2} \times 4 \times 9.8$
$= 19.6 \text{ cm}^2$ each

$\text{TSA} = 2 \times 150 + 60 + 2 \times 19.6$
$= 399.2 \text{ cm}^2$

Example 6B.2 Calculating the surface area of a cylinder

Calculate the surface area of this cylinder, correct to one decimal place.



THINK

1 Identify the values of r and h for this cylinder.

2 Substitute the values you identified into the formula for the surface area of a cylinder and simplify.

3 Calculate the result using the π key on your calculator. Write the answer correct to one decimal place and include the appropriate unit.

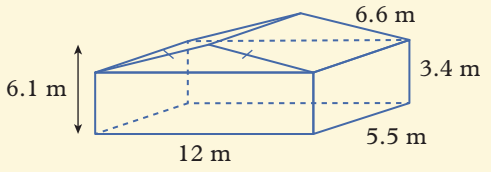
WRITE

$r = 10 \text{ mm}$ $h = 32 \text{ mm}$

$\text{TSA} = 2\pi r^2 + 2\pi rh$
$= 2 \times \pi \times 10^2 + 2 \times \pi \times 10 \times 32$
$= 200\pi + 640\pi$
$= 840\pi$
$= 2638.937...$
$\approx 2638.9 \text{ mm}^2$

Example 6B.3 Calculating the surface area of a composite solid

Calculate the surface area of this composite object.



THINK

1 Identify the number of faces and the shapes of the faces of the solid.

2 Calculate the area of each face.

3 Add the areas and include the appropriate unit.

WRITE

The solid has nine faces: three pairs of identical rectangular faces (including the pair of sloping faces), a pair of identical triangular faces and the rectangular base.

rectangular faces (front and back): $A = 3.4 \times 12$
$= 40.8 \text{ m}^2$ each

rectangular faces (sides): $A = 3.4 \times 5.5$
$= 18.7 \text{ m}^2$ each

rectangular faces (sloping faces): $A = 6.6 \times 5.5$
$= 36.3 \text{ m}^2$ each

rectangular base: $A = 12 \times 5.5$
$= 66 \text{ m}^2$

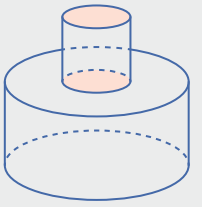
triangular faces: $A = \frac{1}{2} \times (6.1 - 3.4) \times 12$
$= \frac{1}{2} \times 2.7 \times 12$
$= 16.2 \text{ m}^2$ each

$\text{TSA} = 2 \times 40.8 + 2 \times 18.7 + 2 \times 36.3 + 66 + 2 \times 16.2$
$= 81.6 + 37.4 + 72.6 + 66 + 32.4$
$= 290 \text{ m}^2$

Helpful hints

- ✔ If you can't remember the formula for the surface area of a cylinder, remember that a cylinder consists of two circles (the ends) and a rectangle (the curved surface). Calculate the area of each individual face and add the areas together.
- ✔ In some instances, the area where the faces of solids join in a composite solid is the same as the area of the top face of the combined solid. Identifying this can help with your calculations.

  For example, in the diagram on the right the total surface area of the composite solid is the total surface area of the bottom cylinder plus just the curved surface area of the top cylinder because its exposed circular end is the same as the joined surface which is calculated as part of the surface area of the bottom cylinder.



- ✔ The surface area of a prism can also be found by using the formula:

  TSA = 2 × area of base + perimeter of base × height

ANS p614

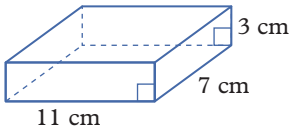
Exercise 6B Surface area

△ 1(a, b, f, g, i), 2–7, 9, 12(a) ■ 1(d, e, h, i), 2–6, 8, 9, 11, 12(a), 13(a), 14 ⬟ 1(f, h, i), 3, 6, 8, 11, 12(b), 13–16

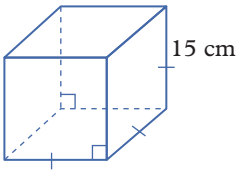
UNDERSTANDING AND FLUENCY

6B.1 **1** Calculate the surface area of each of these prisms.

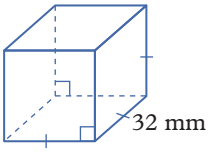
a



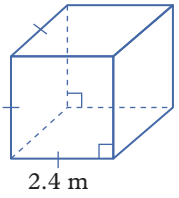
b



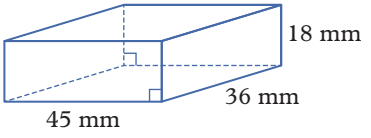
c



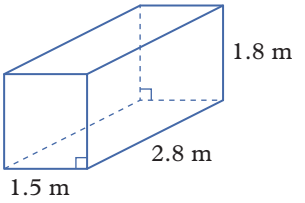
d



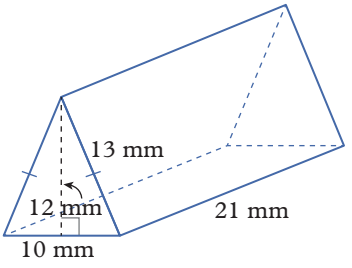
e



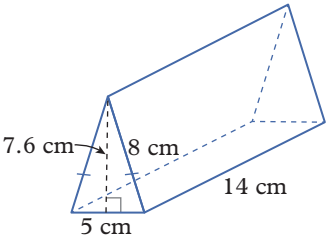
f



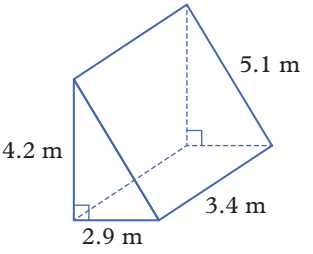
g



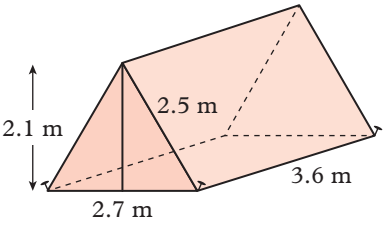
h



i

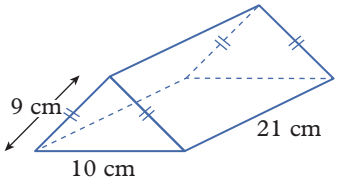


2 The tent shown below has the shape of a triangular prism. Find the area of canvas required to construct it (assuming it has no floor).

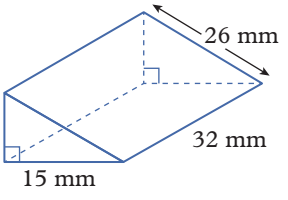


3 Calculate the surface area of each of these triangular prisms correct to one decimal place.

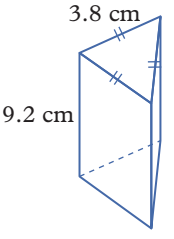
a



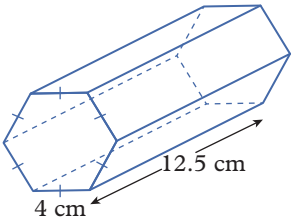
b



c

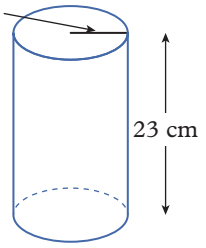


4 If the base of this hexagonal prism has an area of 41.52 cm^2, calculate its total surface area.

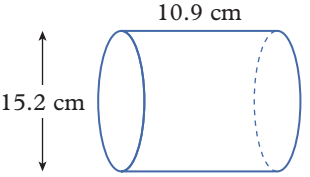


6B.2 **5** Calculate the surface area of each of these cylinders correct to one decimal place.

a



b



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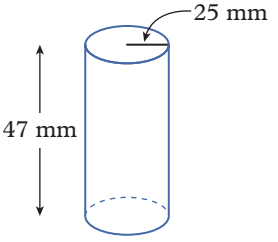
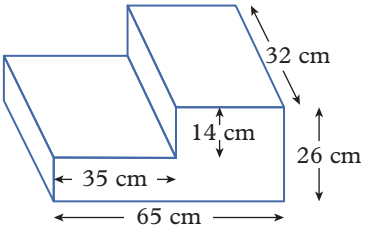
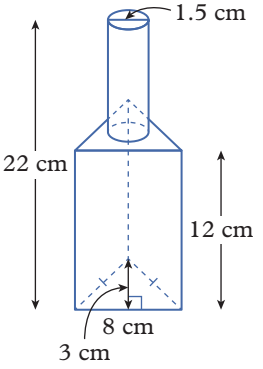


6B.3 **6** Calculate the surface area of each of these composite solids. Give your answers correct to one decimal place where appropriate.

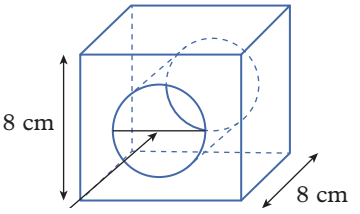
a



b



c



7 Each die in the stack on the right is a cube with side lengths of 15 mm.

a Calculate the surface area of one die.

b Use your answer for part **a** to calculate the total surface area of nine separate dice.

c When stacked, the nine dice form a rectangular prism. Draw the rectangular prism and label its dimensions.

d Calculate the total surface area of the rectangular prism.

e Find the difference between your answers to parts **b** and **d**. Explain why these answers differ. Support your explanation with calculations.

8 A cube has a total surface area of 384 cm^2.

a What is the length of each side of the cube? Explain how you found your answer.

b Use your answer for part **a** to write a formula for finding the side length of a cube when the total surface area is known.

9 An open pipe has a length of 33 cm and a diameter of 55 mm.

a Calculate the surface area of the outer curved surface of the pipe correct to the nearest cm^2.

b To find your answer for part **a**, did you need to use the entire formula for the surface area of a cylinder: TSA = $2\pi rh + 2\pi r^2$? Explain.

10 Find the surface area of the outside of an open cylinder that has a radius of 4.5 cm and a height of 9.9 cm. Give your answer correct to one decimal place.

11 A teacher wants to add two cylindrical poles to the playground. One pole is to be painted yellow and the other red. Both poles have the same diameter of 125 mm, but the heights of the poles are different.

a The yellow pole is 1.8 m tall. Calculate the area that needs to be painted yellow in cm^3 correct to one decimal place. (Assume the base of the pole does not need to be painted.)

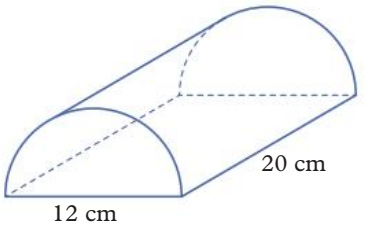
b The red pole is double the height of the yellow pole. Compared to the amount of yellow paint needed, will the teachers need to purchase double the amount of red paint in order to paint this second pole? Why or why not?

c Calculate the area of the pole that needs to be painted red correct to one decimal place. Does your result support your answer to part **b**?

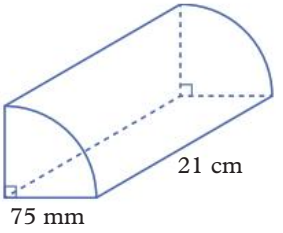
PROBLEM SOLVING AND REASONING

12 Find the total surface area of the following solids. Give your answers in cm^2 correct to one decimal place.

a

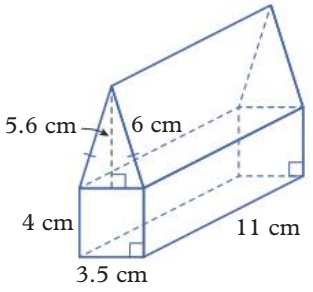


b



13 Calculate the surface area of each of these composite solids correct to one decimal place.

a



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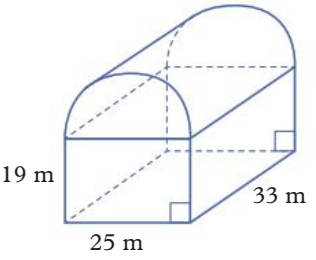
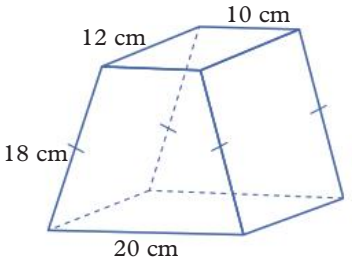


14 Calculate the surface area of this prism correct to the nearest cm^2.



CHALLENGE

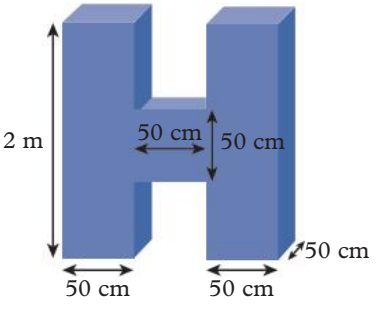
15 Seven cylinders with the same height but different diameters are stacked together as shown on the right. The top cylinder has a diameter of 1 cm and the diameter of each cylinder beneath it increases by 1 cm each time. The bottom cylinder has a diameter of 7 cm. If each cylinder is the same height, 1.5 cm, calculate the total surface area when the cylinders are sitting one on top of the other. Give your answer correct to one decimal place.

16 A company commissions an artist to make a giant logo, an H, for the foyer of their office building. The specifications are shown in the diagram on the right.

The H will be made of wood and then every face of the H will be painted, except the two rectangles that will be on the ground.

a Calculate the surface area of the H, not including the two rectangles that will be touching the ground. Give your answer in square metres.

b The H will be painted blue, as that is the company's colour. The company will use a premium blue paint which is sold in 1 L cans. Each litre of this paint is enough to paint one coat on 8 square metres. How many cans of paint must they buy if they will paint the H with a triple coat?



Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Surface area of prisms
- Interactive skillsheet: Surface area of cylinders
- Topic quiz: 6B

6C Volume

Learning intentions

By the end of this section you will be able to ...

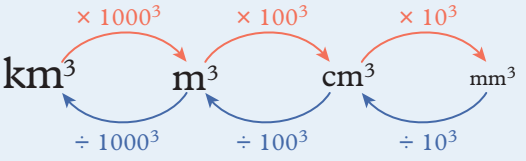
- ✔ calculate the volume of prisms and cylinders
- ✔ calculate the volume of composite solids.

Inter-year links

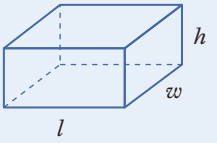
Support	Volume and capacity
Year 7	9G Volume of prisms
Year 8	8E Volume and capacity
Year 9	5E Volume of cylinders

Volume

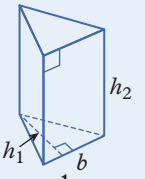
- **Volume** is measured in cubic units, such as mm^3, cm^3 and m^3.
- A unit of volume can be converted to another unit of volume by multiplying or dividing by the appropriate conversion factor.



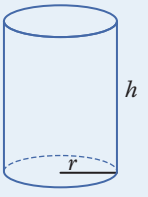
- The volume of any prism can be found using the formula $V = Ah$, where A is the area of the base and h is the height of the prism.
  - → The height is perpendicular to the base.
  - → When dealing with a triangular prism, subscripts can be used to represent the two heights, for example h_1 and h_2, as shown in the diagram of a triangular prism on the right, where h_1 is the height of the triangular base and h_2 is the height of the prism.
- The volume of a cylinder can be found by multiplying the area of the base (πr^2) by the height (h).
  $V = \pi r^2 h$



$V = lwh$



$V = \frac{1}{2}bh_1h_2$



Volume of composite solids

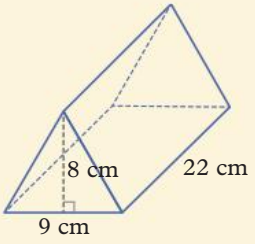
- The volume of a composite solid can be found by calculating the volumes of the individual component solids and adding them together.

Capacity

- The **capacity** of a 3D object is a measure of how much the object can hold.
  A container with an inside volume of:
  - → 1 cm^3 holds 1 mL of liquid
  - → 1000 cm^3 holds 1 L of liquid
  - → 1 m^3 holds 1 kL of liquid.

Example 6C.1 Calculating the volume of a prism

Calculate the volume of this triangular prism.



THINK

1. Identify the base of the triangular prism and the perpendicular height.
2. Calculate the area of the base.
3. Substitute the value of the area of the base and the perpendicular height into the formula for the volume of a prism.
4. Include the appropriate unit in your answer.

WRITE

The base is a triangle with base length 9 cm and height 8 cm. The height (h) of the prism is 22 cm.

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 9 \times 8$

$= 36 \text{ cm}^2$

$h = 22 \text{ cm}$

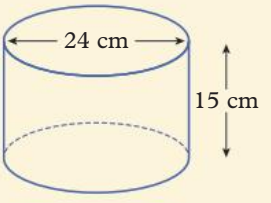
$V = bh$

$= 36 \times 22$

$= 792 \text{ cm}^3$

Example 6C.2 Calculating the volume of a cylinder

Calculate the volume of this cylinder. Give your answer to one decimal place.



THINK

1. Calculate the radius of the circular base.
2. Substitute the values of r and h into the formula to find the volume of the cylinder.
3. Calculate using the π key on your calculator, round to one decimal place, and include the appropriate unit in your answer.

WRITE

$r = D \div 2$

$= 24 \div 2$

$= 12 \text{ cm}$

$h = 15 \text{ cm}$

$V = \pi r^2 h$

$= \pi \times 12^2 \times 15$

$= 2160\pi$

$= 6785.840...$

$\approx 6785.8 \text{ cm}^3$

Helpful hints

✔ If the dimensions of a 3D object are given using different units, convert all lengths to the required unit specified in the question before you do anything else.

ANS p614

Exercise 6C Volume

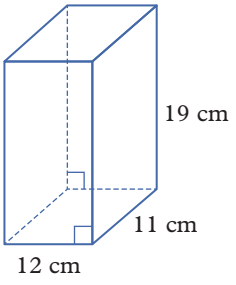
△ 1, 2(c–i), 3, 4, 5(a, c), 6(a, c, d), 7, 8, 9(b), 12, 15 ■ 1(b, d, f, h, j), 2(d, e, i–l), 3(c–e), 5(c), 6(a, d, e), 9(a, b), 10, 11, 17 ⬟ 1(g–j), 2(i–l), 6(d, f), 9(b, c), 13, 14, 16, 18, 19

1 Convert each of these volume measurements to the unit shown in brackets.

a 12.5 m^3 (cm^3)
b 240 000 mm^3 (cm^3)
c 34 200 000 cm^3 (m^3)
d 0.55 m^3 (mm^3)
e 67 200 mm^3 (m^3)
f 0.000 009 m^3 (mm^3)
g 7520 mm^3 (cm^3)
h 8.74 m^3 (cm^3)
i 142 900 cm^3 (m^3)
j 73 000 000 mm^3 (m^3)

6C.1 **2** Calculate the volume of each of these prisms.

a



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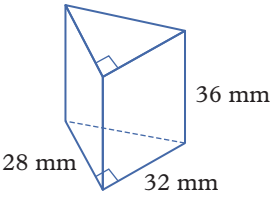

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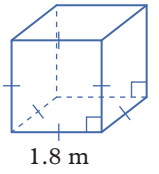

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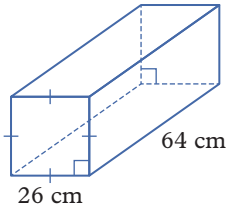

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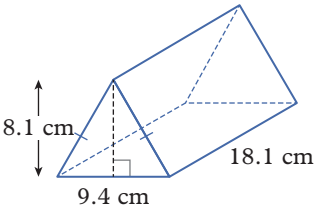

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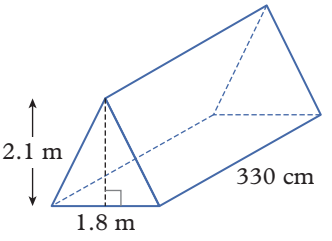

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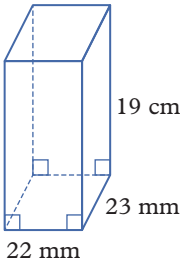

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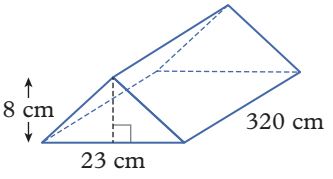

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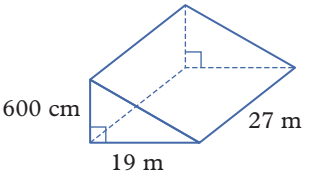

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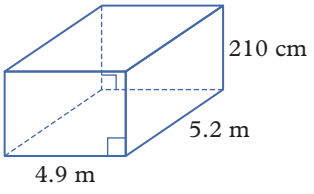

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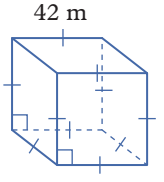

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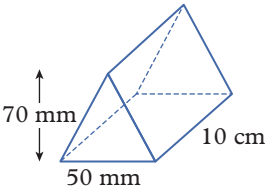
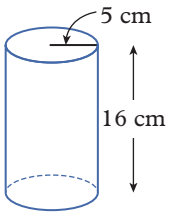


UNDERSTANDING AND FLUENCY

6C.2 **3** Calculate the volume of each of these cylinders, giving your answers correct to one decimal place. Where mixed units are provided give your answers in cm^3.

a



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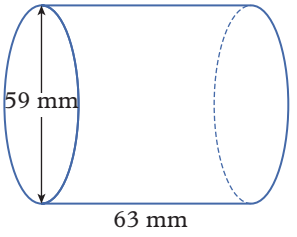

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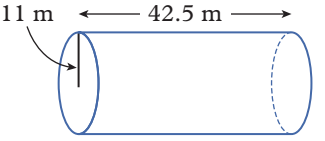

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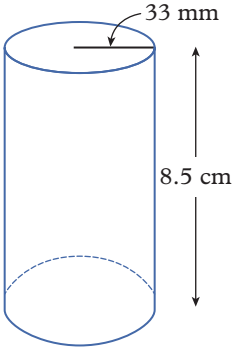

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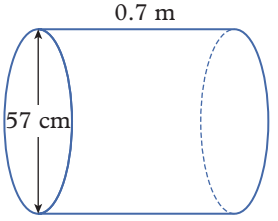

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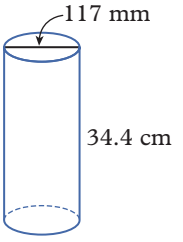


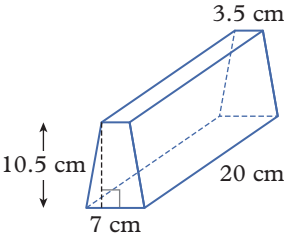
4 Paula is making a cube-shaped cushion with side lengths of 0.4 m.

a How much polyester fibre does Paula need to fill the cushion?

b Convert your answer to cubic centimetres.

5 Calculate the volume of each of these prisms.

a



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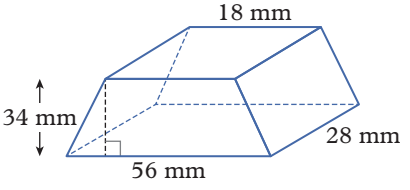

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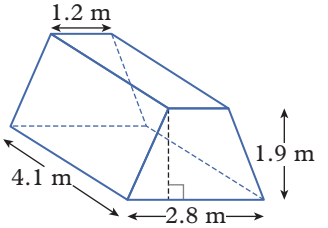
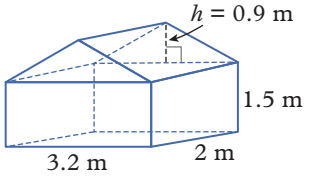


6 Calculate the volume of each of these composite solids. Where appropriate, give your answers correct to one decimal place.

Hint: Identify the individual solids first.

a



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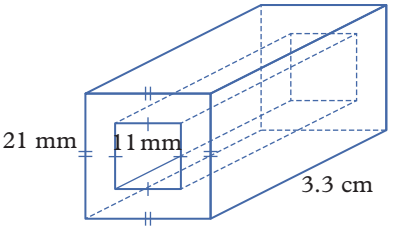

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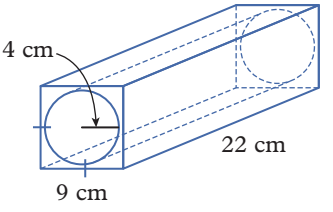

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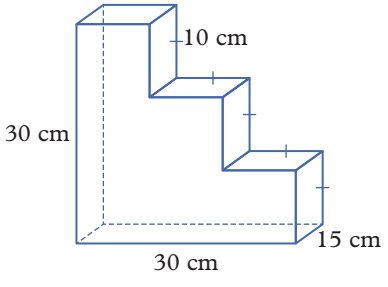

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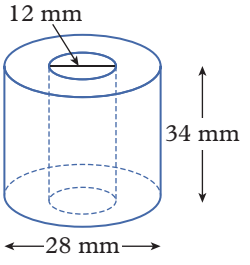

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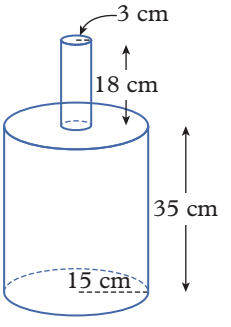


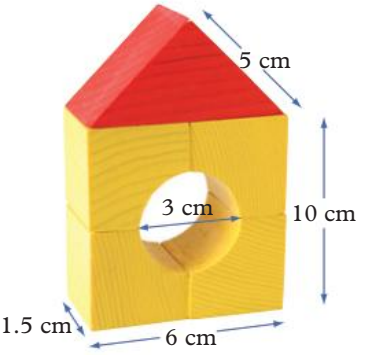
7 A round hatbox has a diameter of 23 cm and a height of 12.5 cm. Calculate its volume correct to one decimal place.

8 In the toy house on the right, constructed from different shaped blocks, three common solids can be identified.

a Identify the three solids.

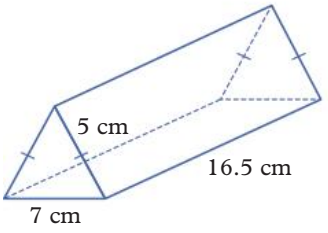
b Explain how you can calculate the volume of the toy house.

c Find the volume of wood used to make this toy house, correct to the nearest cm^3.



9 Use Pythagoras' theorem to help you calculate the volume of each of these prisms. Give your answers correct to one decimal place where appropriate.

a



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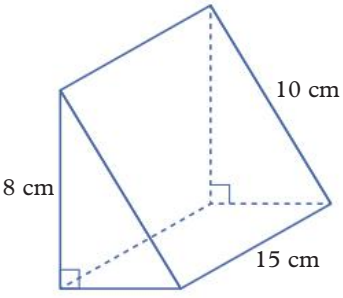

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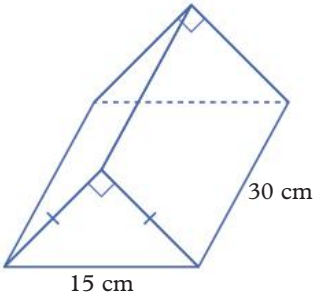
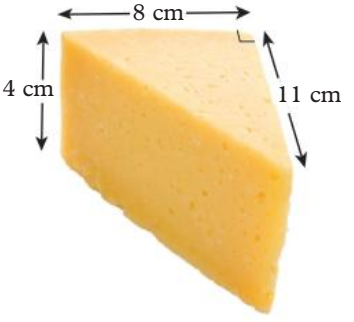


10 A piece of cheese is in the shape of a right-angled triangular prism.

a Calculate its volume.

b If 1 cm^3 of cheese has a mass of 0.001 kg, find the mass of the cheese, in kilograms, correct to four decimal places.

c The price of cheese is $21 per kilogram. How much would the cheese on the right cost? Give your answer correct to the nearest cent.



11 A rectangular fish tank is 45 cm long, 20 cm wide and 50 cm tall.

a Calculate the volume of the fish tank.

b What is the capacity of the fish tank if it is filled to the brim? Give your answer in millilitres and litres. (Remember: 1 cm^3 = 1 mL)

c The fish tank should only be filled to a level 6 cm from the top. How many litres of water will be required to fill the tank to that height?

d A rock with a volume of 360 cm^3 is placed in the fish tank. By how much will the water level rise?

12 The volume of a particular prism is 2744 cm^3. If the prism is a cube, what is its side length?

13 a If a cylinder has a volume of 1649.34 cm^3 and a radius of 5 cm, what is the height of the cylinder, to the nearest centimetre?

b If a cylinder has a volume of 207 cm^3 and a height of 5.7 cm, find the radius of its circular base, to the nearest millimetre.

14 Jack installs two cylindrical water tanks. The first tank is 2 m tall and the diameter of its base is 1.5 m. The second tank is 1.5 m tall and the diameter of its base is 2 m.

a Which tank has the larger capacity?

b What is the difference in the capacity of the two tanks? Give your answer to the nearest litre.

PROBLEM SOLVING AND REASONING

15 a A cylinder has a radius of 8 cm and a height of 15 cm. Calculate its volume.

b If the radius of the cylinder from part **a** is halved but the height remains the same:

i calculate the volume

ii describe how this volume compares to the volume of the original cylinder.

c If the radius of the cylinder from part **a** is doubled but the height remains the same:

i calculate the volume

ii describe how this volume compares to the volume of the original cylinder.

d If the radius of the cylinder from part **a** remains the same but the height is halved:

i calculate the volume

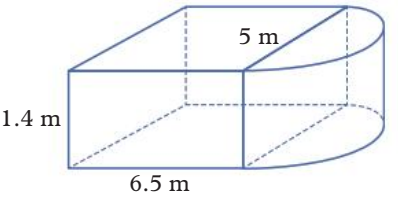
ii describe how this volume compares to the volume of the original cylinder.

e If the radius of the cylinder from part **a** remains the same but the height is doubled:

i calculate the volume

ii describe how this volume compares to the volume of the original cylinder.

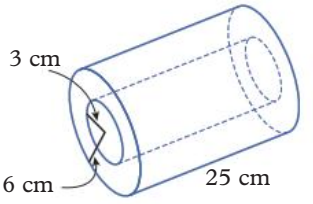
16 A swimming pool is the shape of a rectangular prism. A semi-circular spa is built onto one end of the pool, as shown in the diagram on the right.



a How much water is required if the pool and spa are filled to capacity? Give your answer to the nearest litre.

b To make the pool and spa functional, the water levels must be below the top edge. If the pool and spa are filled to 70% of their depth, how much water is required, to the nearest litre?

17 The plastic pipe on the right is hollow.



a What shape is the base of the pipe? Calculate its area correct to one decimal place.

b Use the area of the base to calculate the volume of plastic used for this pipe.

c Can you think of a different method for finding the volume? Explain your answer.

CHALLENGE

18 Nicko wishes to make an open box from a 30 cm by 30 cm piece of red cardboard. He decides to cut out a square of length 2 cm from each corner and fold up the sides to form the box. The edges of the box will be joined by sticky tape, so no tabs are needed to create the box.

a Draw a diagram showing the piece of cardboard with the corners cut out. Label the dimensions.

b Determine the length, width and height of the box once Nicko has folded up the sides.

c Calculate the volume of the box.

d Would the volume of the box increase or decrease if he cut out 3 cm square corners from the piece of cardboard before forming the rectangular prism? Explain your answer.

19 These nuts have the same dimensions but different shapes. One is circular and the other is in the shape of a regular hexagon. They both have a depth of 6.2 mm and a uniform cross-section.

a How much steel is required to manufacture each nut? Give your answers to one decimal place.

b Which nut requires more steel to make?

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Volume of prisms

Interactive skillsheet
Volume of cylinders

Topic quiz
6C

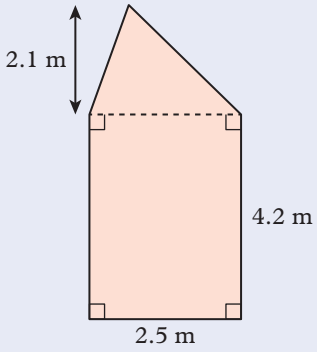
Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

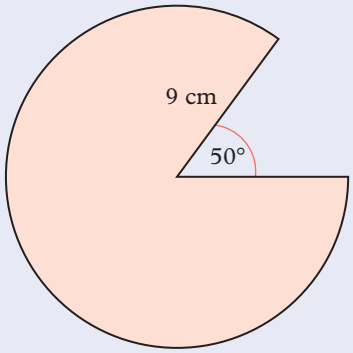
6A **1** Convert each length and area measurement to the unit shown in brackets.

a 45 mm (cm)
b 400 cm (m)
c 4.53 m (mm)
d 4000 mm^2 (cm^2)
e 5.2 km^2 (m^2)
f 68 000 cm^2 (m^2)

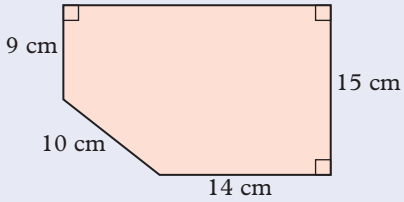
6A **2** Find the area of the following composite shape.



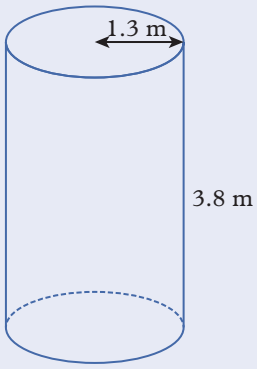
6A **3** Find the area of the following sector. Give your answer correct to one decimal place.



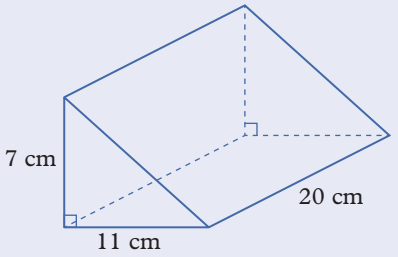
6A **4** Determine the area of the following shape.



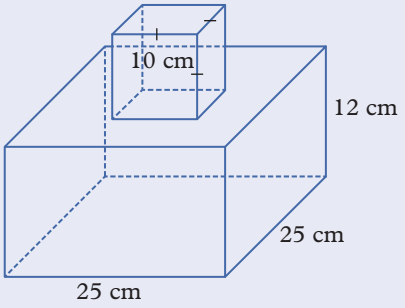
6B **5** Find the surface area of the following cylinder correct to one decimal place.



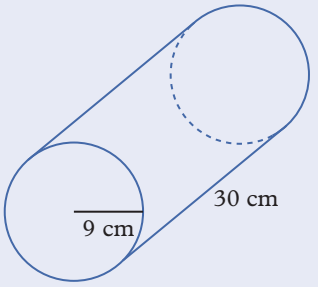
6B **6** Use Pythagoras' theorem to find the total surface area of the following prism correct to one decimal place.



6B **7** A cube with side lengths of 10 cm sits on top of a square-based rectangular prism.
Find the total surface area of the composite solid.

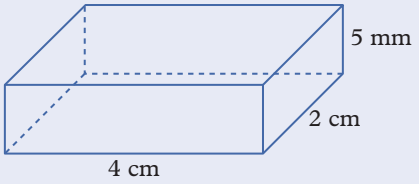


6C **8** Calculate the volume of the following cylinder correct to one decimal place.

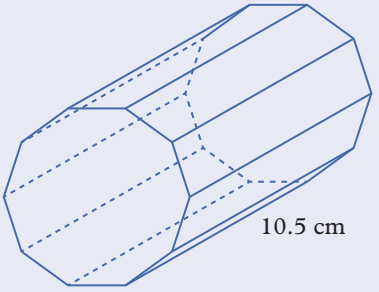


6C **9** Find the volume of this prism in:

a mm^3 **b** cm^3



6C **10** The following prism has a cross-sectional area of 123.11 cm^2. Find the volume of the prism.



6C **11** Calculate the volume of the composite solid in question **7**.

6D Measurement errors and accuracy

Learning intentions

By the end of this section, you will be able to ...

- ✔ calculate absolute, relative and percentage errors
- ✔ identify the impact of using different units of measurements
- ✔ interpret the impact of errors in context.

Inter-year links

Year 9 5F Errors

Errors

- An **error** is the difference between the exact value and the estimated value of a quantity.

  error = exact value − estimated value
- The absolute value of a number is its distance from zero. Absolute values are always positive real numbers.

  → Straight line brackets are used to indicate an absolute value. For example, $|-3.4| = 3.4$.
- An **absolute error** is the absolute value of an error.

  absolute error = |exact value − estimated value|
- A **relative error** is the ratio between the absolute error and the exact value of a quantity.

  $$\text{relative error} = \frac{\text{absolute error}}{\text{exact value}}$$
- A **percentage error** is the absolute error as a percentage of the exact value of a quantity.

  $$\text{percentage error} = \frac{\text{absolute error}}{\text{exact value}} \times 100\%$$

Errors in measurement

- All measurements are estimates of exact values. The accuracy of a measurement depends on the instrument being used.
- Exact values should be used in calculations wherever possible. If exact values cannot be used, the most accurate approximation possible should be used instead.
- Using approximated values repetitively in calculations leads to compounding errors, where the initial error is compounded in further calculations.

Determining the significance of errors

- Errors can impact the results of experiments and lead to biased findings.
- Determining whether an error is significant or not depends on the context. For example, in scientific research the tolerance for errors is low and the potential for errors to impact the result should be mentioned.
- When determining the degree (or potential degree) of an error, the relative or percentage errors are usually more relevant than the absolute error.

Example 6D.1 Errors in scientific notation

Scientific notation can be used to record quantities that are very large or very small in scientific studies. The speed of light in a vacuum is recorded as 299 792 458 m/s. A formula sheet for high school physics students states the speed of light in a vacuum is 3.0×10^8 m/s.

a Calculate the absolute error of the speed stated in the formula sheet.

b Calculate the percentage error.

c Discuss whether the error is significant or not.

THINK

a 3×10^8 m/s is the estimation of the exact value of 299 792 458 m/s. Substitute the values into the absolute error formula.

b Divide the result in part **a** by the exact value. Write the result as a percentage.

c The significance of the error depends on the context.

WRITE

a absolute error = |exact value − estimated value|
$= |299\,792\,458 - 3 \times 10^8|$
$= 207\,542$ m/s

b percentage error $= \frac{\text{absolute error}}{\text{exact value}} \times 100\%$
$= \frac{207\,542}{299\,792\,458} \times 100\%$
$\approx 0.069\%$

c The absolute error is large, but the percentage error is small. This error is acceptable for the purpose of a high school formula sheet.

Example 6D.2 Errors in measurement

Jonny uses a measuring tape to measure his height. The tape's smallest measuring units are 1 cm on one side, and 0.1 feet on the other side.

Jonny's measurement from the side of the tape using the metric system is 168 cm, and the measurement from the other side, using the imperial system, is 5.5 feet. He finds a calculator online that converts units correct to two decimal places. It converts 168 cm to 5.51 feet, and 5.5 feet to 167.64 cm.

a If Jonny's actual height is 168 cm, what is the absolute error in his measurement using the imperial system?

b If Jonny's actual height is 5.5 feet, what is the relative error in his measurement using the metric system? Give your answer to three significant figures.

THINK

a If 168 cm is his exact height, the reading of 5.5 feet is the estimated value. Find the absolute value of the difference between the exact and estimated values.

b If 5.5 feet is his exact height, the reading of 168 cm is the estimated value. Find the absolute value of the difference between the exact and estimated value, and divide by the exact value to find the relative error.

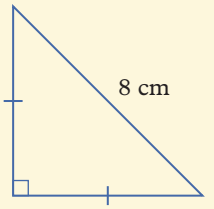
WRITE

a exact value = 168 cm
estimated value = 5.5 feet
≈ 167.64 cm
absolute error = $|168 - 167.64|$
$= 0.36$ cm

b exact value = 5.5 feet
estimated value = 168 cm
≈ 5.51 feet
absolute error = $|5.5 - 5.51|$
$= 0.01$ feet
relative error $= \frac{0.01}{5.5}$
$\approx 0.001\,82$

Example 6D.3 Compounding errors

The hypotenuse of an isosceles right-angled triangle is 8 cm.



a Find the exact length of the two shorter sides.
b Find the length of the two shorter sides, correct to one decimal place.
c Calculate the percentage error of the approximate side lengths found in part **b**.
d Find the exact area of this triangle.
e Calculate the area of this triangle using your result from part **b**.
f Calculate the percentage error of the approximate area correct to three decimal places.
g Explain why the percentage error from part **f** is larger than the percentage error from part **c**

THINK

a In an isosceles right-angled triangle, the two shorter sides are equal in length. Pythagoras' theorem can be used to find unknown side lengths in right-angled triangles.

b Convert the result in part **a** to a decimal value, correct to one decimal place.

c The exact value is the result from part **a** and the estimated value is the result from part **b**. The absolute error is the difference between the two values. Calculate the percentage error by dividing the absolute error by the exact value and multiplying by 100%.

d Both the base and height are equal to the side length found in part **a**. Substitute the values into the area formula Area $= \frac{1}{2} \times$ base $\times$ height.

e Use the approximate result found in part **b** to calculate the area.

f **1** Find the absolute error as the difference between the exact area found in part **d** and the approximate area found in part **e**.

2 Calculate the percentage error by dividing the absolute error by the exact value and multiplying by 100%.

g Look at how the two estimated values were calculated.

WRITE

a Suppose the length of its shorter side is x cm.

$x^2 + x^2 = 8^2$

$2x^2 = 64$

$x^2 = 32$

$x = \sqrt{32}$

$x = 4\sqrt{2}$

The exact length of the two shorter sides is $4\sqrt{2}$ cm.

b $4\sqrt{2} \approx 5.7$

c absolute error $= |\text{exact value} - \text{estimated value}|$

$= |4\sqrt{2} - 5.7|$

$$\text{percentage error} = \frac{\text{absolute error}}{\text{exact value}} \times 100\%$$

$$= \frac{|4\sqrt{2} - 5.7|}{4\sqrt{2}} \times 100\%$$

$\approx 0.76\%$

d $\text{Area} = \frac{1}{2} \times 4\sqrt{2} \times 4\sqrt{2}$

$= 16 \text{ cm}^2$

e $\text{Area} = \frac{1}{2} \times 5.7 \times 5.7$

$= 16.245 \text{ cm}^2$

f Absolute error $= |16 - 16.245|$

$$\text{Percentage error} = \frac{|16 - 16.245|}{16} \times 100\%$$

$= 1.531\%$

g The approximate value for the side length is used twice in the calculation for the area, so the error for this value is compounded in the area calculation, making the error in the area calculation larger than in the side length calculation.

Helpful hints

- ✔ Understand the scenario to identify the exact and estimated values before calculating errors.
- ✔ Make sure quantities are expressed in the same unit before calculating errors.
- ✔ The significance of errors needs to be interpreted depending on the context. An error considered small in daily practice could be significant in scientific research.

ANS p615

Exercise 6D Measurement errors and accuracy

1–5, 9, 12, 14 | 2, 4, 5, 7, 9, 13, 16, 17 | 4, 6, 8, 10–13, 15, 17

UNDERSTANDING AND FLUENCY

1 **a** Convert the following fractions to decimals, correct to two decimal places.

i $\frac{2}{3}$ **ii** $\frac{6}{7}$

b Determine the absolute error of each of the conversions in part **a** correct to four decimal places.

2 Two commonly used approximations of π are 3.14 and $\frac{22}{7}$. Find the percentage error of these two estimations correct to four decimal places. Which one is a more accurate approximation?

3 A bar of chocolate is labelled as 150 g on the packaging. After placing the chocolate on a scale, the actual mass is found to be 156 g.

a Calculate the relative error of the labelled mass correct to four significant figures.

b Calculate the percentage error of the labelled mass correct to four significant figures.

6D.1 **4** The radius of Earth at sea level is approximately 6378.137 km. A formula sheet for students states that the radius of Earth is 6.37×10^6 m.

a Calculate the absolute error of the radius of Earth given in the formula sheet in metres.

b Calculate the percentage error correct to four decimal places.

c Discuss whether the error is significant or not.

6D.2 **5** A carpenter measures the length of a table using a long ruler. The measurement from the side of the ruler that uses the metric system is 120 cm, and the measurement using the imperial system is 4 feet. She finds that 120 cm converts to 3.937 feet, and 4 feet converts to 121.92 cm.

a If the actual length of the table is 120 cm, what is the absolute error in her measurement using the imperial system?

b If the actual length of the table is 4 feet, what is the relative error in her measurement using the metric system? Give your answer to four significant figures.

6 A chef weighs some butter using a scale. The measurement from the scale using the metric system is 185 g, and the measurement using the imperial system is 6.5 oz. He finds that 185 g converts to 6.5257 oz, and 6.5 oz converts to 184.27 g.

a If the actual mass of the butter is 185 g, what is the percentage error in the measurement using the imperial system? Give your answer to four decimal places.

b If the actual mass of the butter is 6.5 oz, what is the percentage error in the measurement using the metric system? Give your answer to four decimal places.

6D.3 **7** The circumference of a circle is 40 cm.

a Determine the exact radius of the circle.

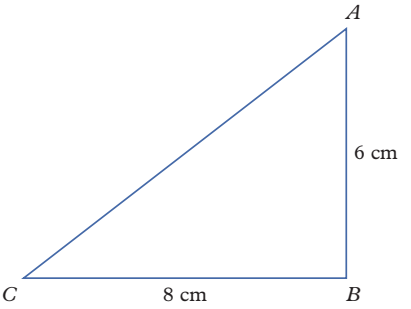
b Find the approximate radius of the circle correct to one decimal place.

c Determine the exact area of this circle.

d Calculate the area of this circle using your result from part **b**.

e Find the percentage error of the approximate area correct to two decimal places.

8 In the right-angled triangle ABC, $\angle ABC = 90°$, $AB = 6$ cm and $BC = 8$ cm.



a Using a protractor, measure the size of $\angle ACB$.

b Find the exact value of $\angle ACB$ using trigonometry.

c Find the percentage error for your measurement for $\angle ACB$. Give your answer to two decimal places.

9 Jack is visiting Canada and noticed the speed limit sign on the road has two numbers: 60 miles per hour (mph) and 100 kilometres per hour (km/h).

Jack wonders if the two speed limits are equivalent to each other and finds a unit conversion calculator online. It converts 60 mph to 96.56 km/h and 100 km/h to 62.14 mph.

a If the actual speed limit is 60 mph, what is the absolute error of the speed limit of 100 km/h?

b If the actual speed limit is 100 km/h, what is the absolute error of the speed limit of 60 mph?

c Which speed limit should Jack follow? Explain your answer.

10 A cylinder has a radius of 2 cm and a height of 4 cm.

a Calculate the area of the base of the cylinder correct to one decimal place.

b Calculate the percentage error for the base area calculated in part **a** compared to the exact base area.

c Calculate the curved surface area of a cylinder correct to one decimal place.

d Determine the percentage error for the curved surface area calculated in part **c** compared to the exact area. Give your answer to four decimal places.

e Using the result from parts **a** and **c** find the surface area of the cylinder correct to one decimal place.

f Determine the percentage error of the surface area calculated in part **e** compared to the exact surface area. Give your answer to four decimal places.

11 Sally measures the diameter of a round coaster using a ruler as 8.4 cm.

a Find the area of the coaster correct to two decimal places.

b Sally measures the coaster using a vernier caliper and it reads 8.352 cm. Determine the percentage error of the diameter measured using the ruler compared to the more accurate vernier caliper correct to four decimal places.

c What is the percentage error of the area calculated in part **a** correct to four decimal places? Explain why the percentage error is greater for the area than for the diameter.

12 Two students, Bianka and Christof, use gravimetric analysis to determine the amount of salt in a chocolate biscuit. The students work to two different levels of accuracy. The working for each student is given below.

Bianka	Christof
$\text{mass of salt} = \frac{0.064}{143.321} \times 58.442$	$\text{mass of salt} = \frac{0.1}{143.3} \times 58.4$
$= 0.026$ g	$= 0.04$ g

a Describe the level of accuracy that each student uses. Which result is more reliable?

Bianka's result matches the amount of salt given in the nutrition label on the packet of biscuits.

b Determine the percentage error in Christof's calculation correct to two decimal places.

c Explain the main cause of the error in Christof's calculation.

d Give a reason why the nutrition label is only accurate to three decimal places instead of to a higher degree of accuracy.

PROBLEM SOLVING AND REASONING

13 Pre-election polls are published with a stated margin of error. For example, before an election you may read something along the lines of 'Labor has a lead over the Coalition of 45% to 43% with a 3% margin of error.'

a Why do you think the margin of error is stated when publishing a pre-election poll?

b What could be done to decrease the margin of error in a pre-election poll?

14 James rides his bike from home to his friend Karen's house which is 5 km away. It takes James 15 minutes to complete the journey. If Karen rides from her house to James' place, it will take 25 minutes.

a Find their speeds in km/h.

b Convert the speeds found in part **a** to m/s, giving your answers correct to one decimal place.

c If James and Karen both leave their house at the same time and ride towards each other, how long will it take for them to meet using:

i the speeds calculated in part **a**

ii the speeds calculated in part **b**.

Give your answer in minutes correct to three decimal places.

d Calculate the percentage error of the time found in part **c ii**.

15 Jemima plans to make 25 gift bags for her upcoming birthday party. She can make them in 45 minutes. If her sister makes the gift bags instead, they can be made in 30 minutes.

a Find, correct to one decimal place, the number of gift bags Jemima and her sister can each make in one minute?

b If they work together, use the results from part **a** to determine how long it will approximately take to make all gift bags. Give your answer in minutes correct to one decimal place.

c If they work together, find the exact time taken to make all gift bags.

d Calculate the absolute error of your answer in part **b**.

16 A homebuyer took out a loan from the bank. The loan broker calculated the monthly repayment to be $1317.172 242 per month for 30 years. The bank system only allows transaction amounts to be accurate to the nearest cent.

a Find the absolute error for each monthly payment.

b Find the absolute error for the total loan repayment, correct to the nearest cent.

CHALLENGE

17 A supermarket sells packs of almonds priced at $19.95. The weekly promotion marks the packs at 25% off. The cashier only accepts cash payment correct to the nearest 5 cent.

a Find the money paid by a customer purchasing one pack of almonds during this promotion. Is this an overpayment or underpayment?

b Find the money paid by a customer purchasing two, three, and four packs of almonds during the promotion and calculate the error of each of these cash payments by completing the following table.

Number of packs	Exact price ($)	Payment amount ($)	Error ($)
1			
2			
3			
4			

c 52 customers visited the shop during the promotion. 24 of the customers bought one pack of almonds, 16 customers bought two packs, 7 customers bought three packs and the rest bought four packs. Determine the total error from the sale of almonds.

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Interactive skillsheet
Measurement errors

Topic quiz
6D

6E Proportion and scaling

Learning intentions

By the end of this section, you will be able to ...

- ✔ calculate length, area and volume scale factors
- ✔ use scale factors to find measurements of original and image objects.

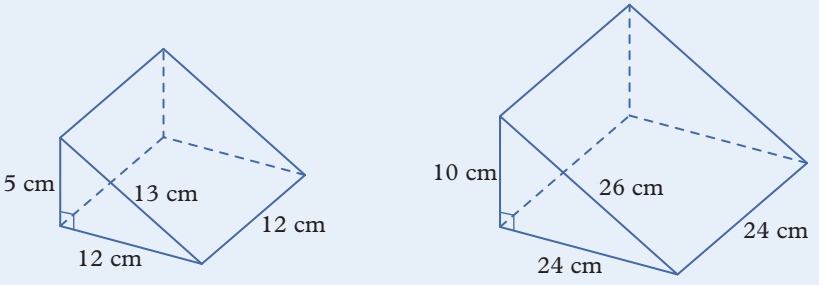
Inter-year links

Year 8 7H Dilations and similar figures

Year 9 6C Dilations and similar figures

Similar objects

- Two objects are similar if they are exactly the same shape. Similar objects do not need to be the same size.
- If two objects are similar, all side lengths will be in the same proportion. For example, these two prisms are similar as every side length in the prism on the right is twice the corresponding side length in the prism on the left.



Scale factor

- A scale factor measures the enlargement or reduction factor between similar objects.
- The **length scale factor** is the ratio between the image side length and the original side length.

$$\text{Length scale factor} = \frac{\text{image length}}{\text{original length}}$$

- The **area scale factor** is the ratio between the image area and the original area, and is equal to the square of the length scale factor between the objects.

$$\text{Area scale factor} = \frac{\text{image area}}{\text{original area}} = \left(\frac{\text{image length}}{\text{original length}}\right)^2$$

- The **volume scale factor** is the ratio between the image volume and the original volume, and is equal to the cube of the length scale factor between the objects.

$$\text{Volume scale factor} = \frac{\text{image volume}}{\text{original volume}} = \left(\frac{\text{image length}}{\text{original length}}\right)^3$$

- For example, if the length scale factor between two similar objects is 4, the area scale factor is $4^2 = 16$ and the volume scale factor is $4^3 = 64$.

Enlargement and reduction

- If the scale factor is between 0 and 1, the image is a **reduction** (smaller than the original object).
- If the scale factor is greater than 1, the image is an **enlargement** (larger than the original object).

Example 6E.1 Calculating area and volume scale factors

Two dolls from a set of Russian nesting dolls are identical in shape but differ in size. One of the dolls is twice as large as the other.

a Find the area scale factor for the surface areas of the dolls.

b Find the volume scale factor for the volume of the dolls.

c If the smaller doll has a volume of 200 cm^3, find the volume of the larger doll.

THINK

a The area scale factor is the square of the length scale factor.

b The volume scale factor is the cube of the length scale factor.

c The volume of the smaller doll mutliplied by the volume scale factor gives the volume of the larger doll.

WRITE

a Area scale factor $= 2^2$
$= 4$

b Volume scale factor $= 2^3$
$= 8$

c Volume $= 200 \times 8$
$= 1600\ cm^3$

Example 6E.2 Using scale factors to find measurements (regular objects)

A rectangular prism has length 5 cm, width 4 cm and height 3 cm. The prism is dilated by a scale factor of 2.

a Find the surface area and volume of the original rectangular prism.

b Find the surface area and volume of the new prism by first dilating the length, width and height measurements.

c Find the surface area and volume of the new prism using the area and volume scale factors.

THINK

a The surface area of a prism is the sum of areas of all six faces. The volume of a rectangular prism is the product of the length, width, and height of the prism.

WRITE

a Surface area $= 2 \times (3 \times 4 + 3 \times 5 + 4 \times 5)$
$= 94\ cm^2$
Volume $= 3 \times 4 \times 5$
$= 60\ cm^3$

b **1** Find the dilated dimensions of the prism and then find the image surface area using the dilated length, dilated width and dilated height measurements.	**b** Dilated length = 10 cm, dilated width = 8 cm, dilated height = 6 cm.
2 Find the image volume by multiplying the dilated length, width and height together.	Surface area of image $= 2 \times (6 \times 8 + 6 \times 10 + 8 \times 10)$ $= 376 \text{ cm}^2$ Volume of image $= 6 \times 8 \times 10$ $= 480 \text{ cm}^3$
c **1** Find the area scale factor as the square of the length scale factor, and then multiply the original surface area by the area scale factor.	**c** Area scale factor $= 2^2$ $= 4$ Surface area of image $= 4 \times 94$ $= 376 \text{ cm}^2$
2 Find the volume scale factor as the cube of the length scale factor and then multiply the original volume by the volume scale factor.	Volume scale factor $= 2^3$ $= 8$ Volume of image $= 8 \times 60$ $= 480 \text{ cm}^3$

Example 6E.3 Using scale factors to find measurements (irregular objects)

A gift shop in an airport sells model airplanes as souvenirs with a scale of 1 : 64 compared to the full-sized plane. The full-sized airplane is 38 m long and has a surface area of 500 m^2.

a Find the length of the model airplane in cm.

b Find the surface area of the model correct to the nearest cm^2.

THINK	WRITE
a The ratio means the full-sized plane is 64 times larger than the model. The length of the model is equal to the length of the full-sized plane divided by the scale factor.	**a** Scale factor = 64 $38 \div 64 = 0.59375 \text{ m}$ $= 0.59375 \times 100 \text{ cm}$ $= 59.375 \text{ cm}$
b The area scale factor is the square of the length scale factor. The area of the model is equal to the surface area of the full-sized plane divided by the area scale factor. Convert m^2 to cm^2 by multiplying by 100^2.	**b** Area scale factor $= 64^2$ $500 \div 64^2 = 0.12207 \text{ m}^2$ $= 0.12207 \times 100^2 \text{ cm}^2$ $\approx 1221 \text{ cm}^2$

Helpful hints

- ✔ For similar objects with length scale factor k, the area scale factor is k^2 and the volume scale factor is k^3.
- ✔ The side length, area and volume of a dilated object can be found by multiplying the original measurement by the corresponding scale factor.

ANS p615

Exercise 6E Proportion and scaling

1–4, 8, 9, 11 | 3–5, 7, 9, 12, 13, 16 | 4, 6, 7, 10, 13–16

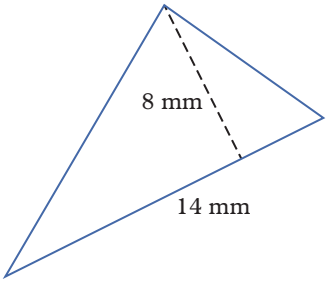
UNDERSTANDING AND FLUENCY

1 Complete the table below.

Length scale factor	Area scale factor	Volume scale factor
2		
$\frac{3}{4}$		
	16	
	$\frac{36}{25}$	
		$\frac{1}{27}$

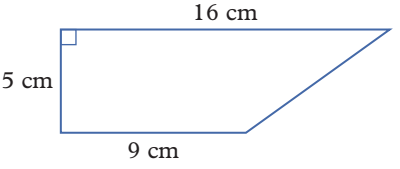
2 For each shape, state the area of the image if the original figure is dilated by the given length scale factor. Give your answers correct to the nearest whole number where necessary.

a



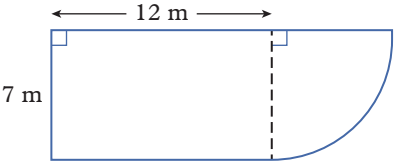
length scale factor = 2

b



length scale factor = $\frac{1}{5}$

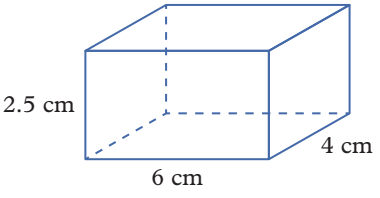
c



length scale factor = 3

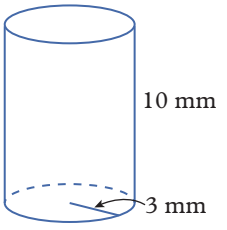
3 For each shape, state the volume of the image if the original figure is dilated by the given length scale factor.

a



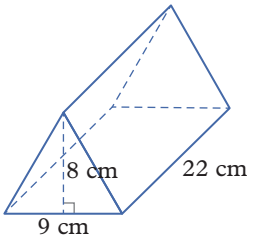
length scale factor = $\frac{1}{2}$

b



length scale factor = 5

c



length scale factor = 1.5

6E.1 **4** A set of salt and pepper shakers are similar in shape. The salt shaker is 1.5 times larger than the pepper shaker.

a Find the area scale factor for the surface areas of the shakers.

b Find the volume scale factor for the volume of the shakers.

c If the pepper shaker has a volume of 5 cm^3, find the volume of the salt shaker.

6E.2 **5** A cube of side length 5 mm is dilated by a scale factor of 3.

a Find the surface area and volume of the original cube.

b Find the surface area of the new cube.

c Find the volume of the new cube.

6 A cylinder with radius 12 cm and height 9 cm is dilated by a scale factor of $\frac{1}{3}$. Leave your answers as exact values for all parts of this question.

a Find the surface area and volume of the original cylinder.

b Find the surface area of the new cylinder.

c Find the volume of the new cylinder.

6E.3 **7** A handicraftsman created a model of the Louvre Pyramid with scale of 1 : 300.

a The Louvre Pyramid is 21.6 m high. Find the height of the model in cm.

b The surface area of the Louvre Pyramid is 1000 m^2. Find the surface area of the model to the nearest cm^2.

8 A floor plan with a scale of 1 : 50 illustrates a rectangular bedroom in a house. The dimension of the room is measured to be 8 cm by 6 cm on the floor plan. Find the area of the bedroom in m^2.

9 Lake Eacham has an area of 640 000 m^2. On one map, the area of the lake is measured to be 4 cm^2. Find the scale of this map in the form 1 : n.

10 Andrea has a set of similar rectangular containers. She tries to store 400 mL of soup in a container with dimensions 15 cm × 5 cm × 18 cm and finds that the container is too large. Find the length scale factor between this container and the smallest similar container that is sufficient to store the soup.

11 Lillian grows herbs in a pot on her balcony. The herbs thrive under her care, and she needs to find a similar pot that has twice the volume for replanting. What is the scale factor comparing the dimensions of the two pots?

12 A community recreational centre is reconstructing a rectangular swimming pool of constant depth to accommodate the growing population. The new pool will be twice as long in width and length. Due to safety restrictions, the depth of the pool will remain the same as before. Find the scale factor for the volume of the new pool.

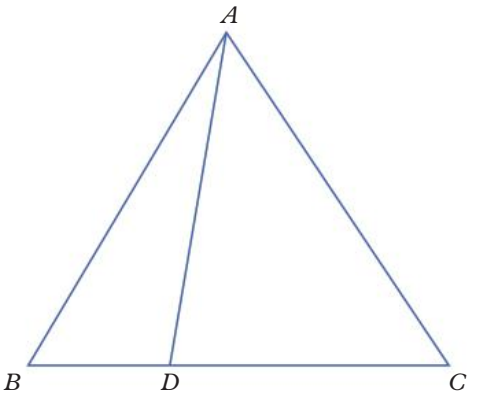
13 A student uses a 3D printer to create a solid sculpture with 25 g of filament. The student also spent 20 minutes painting its surface. This sculpture is later selected for an exhibition, and the student needs to print the same sculpture at three times the size for display.

a How much filament does the student need to print the larger statue?

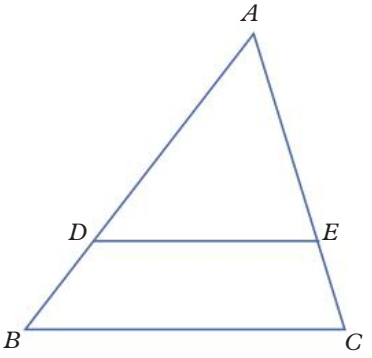
b If this student can paint at a constant rate, how long would it take to paint the surface of the larger sculpture?

PROBLEM SOLVING AND REASONING

14 In $\triangle ABC$, point D divides side BC into a ratio of 1 : 2. A student suggests that the area ratio between $\triangle ABD$ and $\triangle ADC$ is 1 : 4, calculated as the square of the length ratio. Do you agree with this statement? Explain your reasoning.



15 In $\triangle ABC$, points D and E lie on side AB and AC, respectively, and DE is parallel to BC. Point D divides side AB in a ratio of 2 : 1.



a Find the ratio between the areas of $\triangle ABC$ and $\triangle ADE$ in the form $m : n$.

b Find the ratio between the areas of $\triangle ADE$ and the quadrilateral $BCED$ in the form $m : n$.

CHALLENGE

16 A spherical balloon has a radius of 10 cm. It loses some air and deflates to half of the original size over a week. The surface area of a sphere is $SA = 4\pi r^2$, and the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. Leave your answers as exact values for all parts of this question.

a Find the surface area and the volume of the original balloon.

b Find the radius of the balloon after a week.

c Use the new radius to calculate the surface area of the balloon after a week.

d Calculate the area scale factor. Comment on your result in relation to the radius scale factor.

e Use the new radius to calculate the volume of the balloon after a week.

f Calculate the volume factor. Comment on your result in relation to the radius scale factor.

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6E

6F Logarithmic scales

Learning intentions

By the end of this section you will be able to ...

- ✔ determine the order of magnitude of base-ten numbers
- ✔ determine the position of a value on a base-ten logarithmic scale
- ✔ interpret values displayed on a logarithmic scale.

Inter-year links

Year 8 — 1A Rounding and estimating

Year 9 — 1E Scientific notation

Scientific notation and orders of magnitude

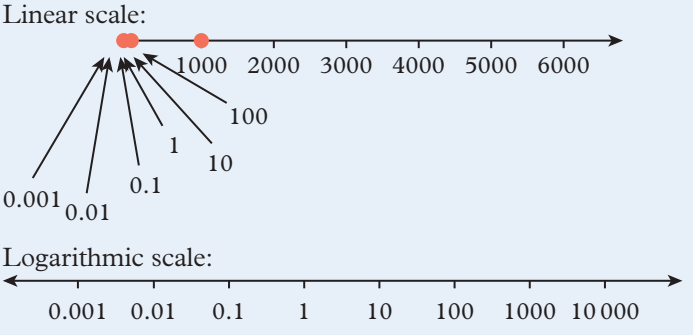
- **Base-ten** refers to the traditional number system we use, where each digit in a number has an integer value of between 0 and 9 (comprised of ten different possible digits).
- **Scientific notation** is a simple way of writing very large or very small numbers in base-ten. A number written in scientific notation is in the form $a \times 10^m$, where $1 \leq a < 10$ or $-10 < a \leq -1$, and m is an integer.
- The **order of magnitude** of a number in base-ten is the equal to the value of m when the number is written in scientific notation.

Base-ten number	0.001	0.01	0.1	1	10	100	1000
Scientific notation	1×10^{-3}	1×10^{-2}	1×10^{-1}	1×10^{0}	1×10^{1}	1×10^{2}	1×10^{3}
Order of magnitude	−3	−2	−1	0	1	2	3

- In base-ten, if a number is increased by 1 order of magnitude, it is multiplied by 10. Similarly, if a number is decreased by 1 order of magnitude, it is divided by 10.

Logarithmic scales

- **Logarithmic scales** can be used to display numerical data which covers a wide range of values of differing orders of magnitude.
- A logarithmic scale is non-linear, which means the distance between consecutive numbers on the scale is not equal. Instead, in a logarithmic scale the distance between the orders of magnitude of the numbers is equal.



- To determine the position of a value on a logarithmic scale, the logarithm of the value should be determined using a calculator. Use 1 as the reference point and move the specified number of units right or left on the scale.
  - → When working with a base-ten logarithmic scale, the base-ten logarithm of a value must be taken to determine its position on the scale. For example, to find the position of 500 on the above scale, $\log_{10}(500) = 2.70$, so 500 is located 2.70 units to the right of 1 on the scale.

- If the range of values in a numerical data set are all within one order of magnitude of each other, it is more appropriate to use a linear scale than a logarithmic scale.
- Logarithmic scales should not be used if any of the quantities do not have a real-valued logarithm. Since the logarithm of zero is undefined and the logarithm of a negative value is not a real number, they cannot be displayed on a logarithmic scale.

Example 6F.1 Determining the order of magnitude of base-ten numbers

Determine the order of magnitude of the following base-ten numbers.

a 6.29×10^8 **b** 0.001 24

THINK	WRITE
a For numbers written in scientific notation, the order of magnitude is the exponent of 10 in the scientific notation.	**a** The order of magnitude is 8.
b 1 Write the number in scientific notation.	**b** $0.00124 = 1.24 \times 10^{-3}$
2 The order of magnitude is the exponent of 10 in the scientific notation.	The order of magnitude is -3.

Example 6F.2 Determining the position of values on a logarithmic scale

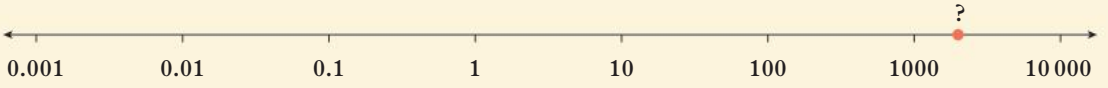
Determine the position of the following values on a base-ten logarithmic scale, relative to 1.

a 1 000 000 **b** 0.0423 (correct to two decimal places)

THINK	WRITE
a 1 Evaluate the base-ten logarithm of the number.	**a** $\log_{10}(1\,000\,000) = 6$
2 If the result if positive, the value lies this many units to the right of 1. If the result is negative, the value lies this many units to the left of 1.	1 000 000 lies 6 units to the right of 1.
b 1 Evaluate the base-ten logarithm of the number to two decimal places.	**b** $\log_{10}(0.0423) = -1.373...$ ≈ -1.37
2 If the result if positive, the value lies this many units to the right of 1. If the result is negative, the value lies this many units to the left of 1.	0.0423 lies 1.37 units to the left of 1.

Example 6F.3 Determining values on a logarithmic scale

Determine the value of the quantity which is 3.3 units to the right of 1 on a base-ten logarithmic scale. Give your answer correct to the nearest whole number.



THINK	WRITE
1 Evaluate 10 to the power of the given (logarithmic) value.	$10^{3.3} = 1995.2...$
2 Round your answer to the nearest whole number.	≈ 1995

Helpful hints

- ✔ An increase of 1 unit on a base-ten logarithmic scale corresponds to a quantity that is one order of magnitude (ten times) greater.
- ✔ A logarithmic scale is not appropriate if the range of quantities do not span multiple orders of magnitude or if any of the values do not have a real-valued logarithm.

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Exercise 6F Logarithmic scales

1–3, 4–6(a, b, e), 11, 13(a–c), 14 | 2, 3, 4–7(c, d, f), 8, 9(c–f), 11, 12, 13(d–f), 16 | 2, 3, 7, 8, 10, 11, 12(d–f), 14–18

6F.1 **1** Determine the order of magnitude of the following base-ten numbers.

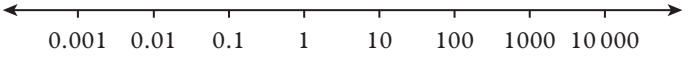
a 8.2751×10^{12} **b** 2.1582×10^{-5} **c** 6.014×10^{3}

d 190 124 098 **e** 0.000 000 281 9 **f** 5217.241

6F.2 **2** Determine the position of the following values on a base-ten logarithmic scale, relative to 1. Round your answers correct to two decimal places where necessary.

a 1000 **b** 0.000 001 **c** 10 **d** 200 **e** 0.02 **f** 0.0008

6F.3 **3** Determine the value of the quantities in the following positions on a base-ten logarithmic scale. Give your answers correct to three significant figures where necessary.



a 5 units to the right of 1. **b** 1 unit to the left of 1. **c** 3.4 units to the right of 1.

d 2.7 units to the left of 1. **e** 0.1 units to the right of 1. **f** 4.6 units to the left of 1.

4 **a** Increase 4.215×10^{3} by 1 order of magnitude.

b Increase 4.215×10^{3} by 2 orders of magnitude.

c Increase 4.215×10^{3} by 4 orders of magnitude.

d Increase 0.0729 by 1 order of magnitude.

e Increase 0.0729 by 2 orders of magnitude.

f Increase 0.0729 by 4 orders of magnitude.

5 **a** Decrease 4.215×10^{3} by 1 order of magnitude.

b Decrease 4.215×10^{3} by 2 orders of magnitude.

c Decrease 4.215×10^{3} by 5 orders of magnitude.

d Decrease 0.0729 by 1 order of magnitude.

e Decrease 0.0729 by 2 orders of magnitude.

f Decrease 0.0729 by 5 orders of magnitude.

6 For each pair of numbers, determine how many orders of magnitude higher or lower the first number is than the second.

a 4.12×10^{8} and 4.12×10^{2}

b 9.0210×10^{-4} and 9.0210×10^{-1}

c 3.14×10^{-8} and 3.14×10^{7}

d 42 000 and 420

e 0.000 88 and 0.88

f 72 400 000 and 0.000 072 4

7 What number would you need to multiply or divide by to move the following number of units on a base-ten logarithmic scale? Give your answers correct to the nearest integer where appropriate.

a 3 units to the left
b 4 units to the right
c 11 units to the right
d 6.3 units to the right
e 2.1 units to the left
f 11.1 units to the left

8 **a** Write each of the following numbers in scientific notation.

b State the order of magnitude of each number.

c State the two consecutive integers the base-ten logarithm of each number will lie between.

d Determine the base-ten logarithm of each number, correct to three decimal places.

i 7293.85
ii 105 235.79
iii 0.002 135
iv 50 123
v 61.29
vi 0.000 003 279 3

9 Complete the following sentences. Write the missing values correct to three significant figures.

a Increasing a value's base-ten logarithm by 0.5 corresponds to the value becoming _____ times greater.

b Increasing a value's base-ten logarithm by 1.5 corresponds to the value becoming _____ times greater.

c Increasing a value's base-ten logarithm by 2.5 corresponds to the value becoming _____ times greater.

d Increasing a value's base-ten logarithm by 3.5 corresponds to the value becoming _____ times greater.

e Multiplying a value by 2 increases its base-ten logarithm by _____.

f Multiplying a value by 4 increases its base-ten logarithm by _____.

g Multiplying a value by 8 increases its base-ten logarithm by _____.

h Multiplying a value by 16 increases its base-ten logarithm by _____.

10 Use the approximate logarithmic values provided to determine the positions of the following quantities on a base-ten logarithmic scale, relative to 1.

a 5.291×10^4, given that $\log_{10}(5.291) \approx 0.724$

b 5.291×10^{-7}, given that $\log_{10}(5.291) \approx 0.724$

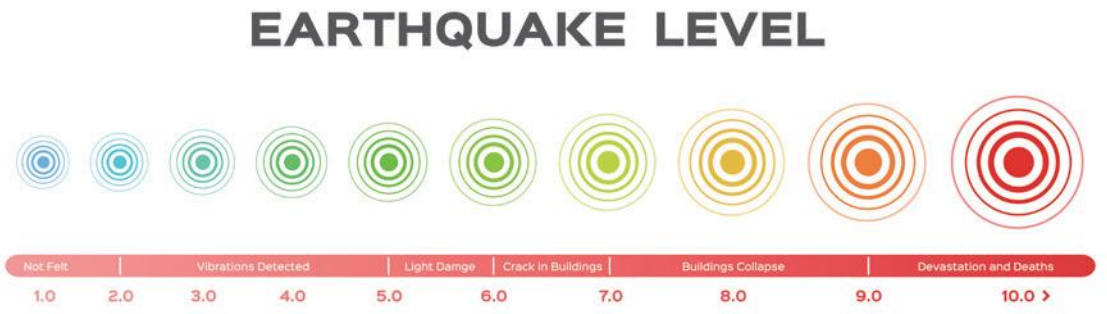
c 2.11×10^{-7}, given that $\log_{10}(2.11) \approx 0.324$

d 9.78×10^8, given that $\log_{10}(9.78) \approx 0.990$

e 7.02×10^{-6}, given that $\log_{10}(7.02) \approx 0.846$

f 1.05×10^{-1}, given that $\log_{10}(1.05) \approx 0.021$

11 The Richter scale measures the magnitude (size) of an earthquake by taking the base-ten logarithm of the amplitude of the largest seismic wave measured in micrometres (μm).



a Determine the magnitude of earthquakes with the following amplitudes of their largest seismic waves correct to one decimal place.
Hint: 1000 μm = 1 mm

i 320 000 μm
ii 1200 μm
iii 75 600 μm
iv 600 μm
v 21 460 mm
vi 146 m

b Determine the amplitudes, in micrometres, of the largest seismic waves of earthquakes with the following magnitudes correct to the nearest micrometre.

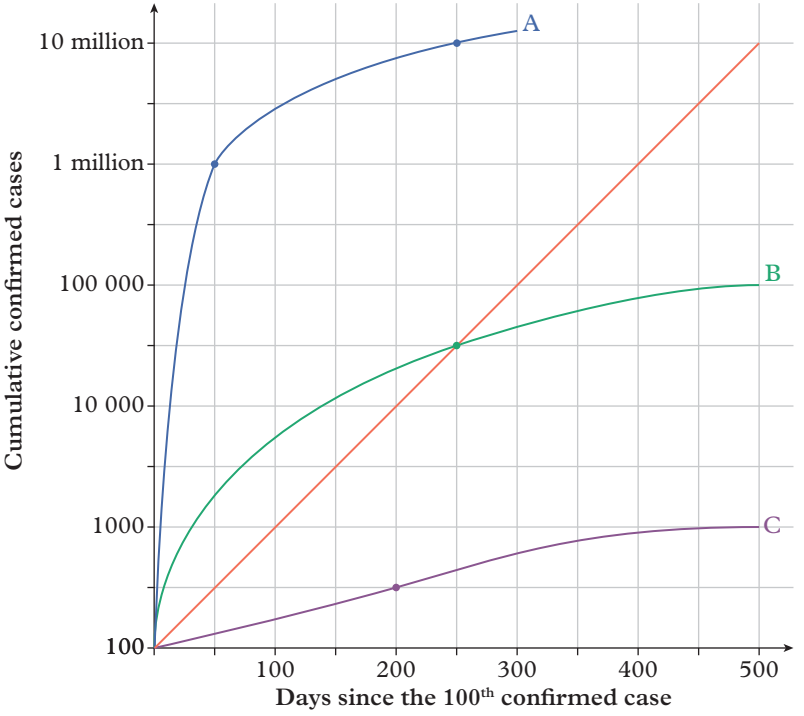
i 5.0
ii 6.0
iii 6.5
iv 1.5
v 7.8
vi 4.3

c Each increase of one unit on the Richter scale corresponds to an earthquake that is 10 times stronger. For each pair of earthquake magnitudes, determine how many times stronger the larger magnitude is than the smaller magnitude. Give your answers correct to four significant figures where necessary.

i 4.2 and 5.2 **ii** 4.2 and 4.7 **iii** 4.2 and 5.7

iv 6.1 and 7.9 **v** 1.2 and 7.3 **vi** 5.1 and 5.2

12 When change occurs at an exponential rate, the change is often displayed on a logarithmic scale. The number of people infected by a virus in three countries has been plotted against the number of days since the 100th infection on the following base-ten logarithmic scale.



a An orange line has been drawn at a 45° angle onto the graph so that it passes through the points (100, 1000), (200, 10 000), (300, 100 000). Describe what this line represents.

b Use the graph to determine:

i the number of days it took for Country A to go from its 100th infection to its one millionth infection

ii the number of days it took for Country A to go from its 1 millionth infection to its ten millionth infection

iii the number of days it took for Country B to go from its 100th infection to its hundred thousandth infection

iv correct to the nearest integer, the number of cumulative confirmed cases in Country B after 250 days

v the number of cumulative confirmed cases in Country C after 500 days.

c Describe the importance of the point where country B crosses the orange 45° line.

13 For each set of quantities, state if using a linear or base-ten logarithmic scale would be more appropriate to display the values.

a 790, 392, 597, 598, 800, 748, 828, 831, 814, 967

b 194 112, 42 461, 697 139, 17 186, 5599, 680 398, 67 184, 7065, 365 973

c 0.08, 0.8, 0.06, 0.09, 0.1, 0.05, 0.03, 0.7, 0.006

d 0.514, 0.64, 0.39, 0.421, 0.717, 0.49, 0.851, 0.874, 0.525, 0.33, 0.832

e 5.042, 0.118, 3.386, 4.4847, 0.096, 1.373, 7.717, 2.644, 5.005, 7.117, 0.476

f 0.000 173 4, 0.000 623 15, 0.000 513 54, 0.000 417 34, 0.0006, 0.000 961, 0.000 615 23, 0.000 252 31, 0.000 551 32

14 For each of the following sets of values, give a reason why a logarithmic scale would not be appropriate to display the data.

a –8.2, 62, 0.09, 1.2, –57, 0.1, –0.57, –6.23, 600, –7.92, –9.7, 4.31

b 2.4, 8.6, 6.5, 6.7, 3.2, 0, 0, 3.3, 4.4, 8.3, 7.3, 2.3

15 Logarithmic scales are also useful for plotting data in histograms that involve very small and very large values, where otherwise the smaller values would all be grouped in the same class interval. The average masses of 47 breeds of animals, in kilograms, are listed below.

1390	1100	725	475	386	190	180	154.25	117.5	100	90	64.475
60	44.984	28.25	26.625	25.5	20	16.5	13.25	9.3	8.6	7.875	6.81
6	5.765	4.25	3.1	2.25	2.06	1.25	0.425	0.397	0.336	0.15	0.29
0.2	0.186	0.173	0.115	0.1	0.07	0.06	0.055	0.05	0.028	0.018	

a Complete the grouped data tables using the average masses listed above.

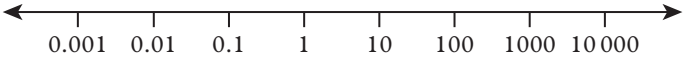
i

Class interval	Frequency
0–<200	
200–<400	
400–<600	
600–<800	
800–<1000	
1000–<1200	
1200–<1400	

ii

Class interval	Frequency
0.01–<0.1	
0.1–<1	
1–<10	
10–<100	
100–<1000	
1000–<10 000	

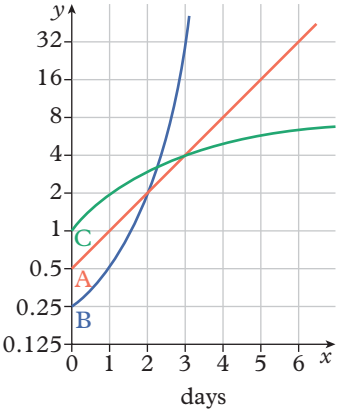
b Draw a histogram for each grouped data table in part **a**. Use a horizontal, base-ten logarithmic scale for the table in part **ii** as shown here.



c Describe the shape of the two histograms as either symmetric, positively skewed or negatively skewed.

d Explain why using a logarithmic scale helps to better understand the distribution of average masses than a linear scale.

16 A base-two logarithmic scale can be used when we want to consider the time it takes for a quantity to double in value. In a base-two logarithmic scale, the quantities double in value for each unit on the scale. Consider the following graph.



a How long does it take for A to double the first time?

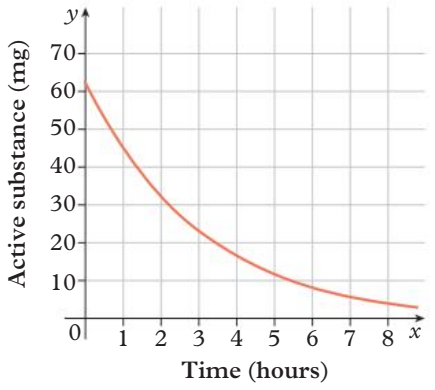
b How long does it take for the A to double the second time?

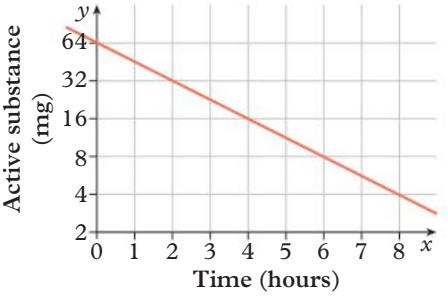
c For the data shown, is the doubling time for A constant, increasing, or decreasing?

d For the data shown, is the doubling time for B constant, increasing, or decreasing?

e For the data shown, is the doubling time for C constant, increasing, or decreasing?

17 The half-life of a drug is the time it takes for the amount of active substance of the drug in a body to reduce by 50%. The two graphs below show the amount of active amount of drug remaining using two different scales.





a What is the half-life of the drug?

b Discuss the advantages and disadvantages of using a base-two logarithmic scale instead of a linear scale to display this data.

18 Bels (B) are a unit of measurement for loud noise, named in honour of Alexander Graham Bell, the inventor of the telephone. Bels are measured on a logarithmic scale, on which an increase in 1 B is an increase of 10 times in loudness. More frequently, decibels (dB) are used where 10 dB = 1 B.

a Determine how many times louder the larger sound is in each of these pairs, correct to four significant figures.

i 15 dB and 25 dB
ii 60 dB and 70 dB
iii 50 dB and 90 dB
iv 95 dB and 145 dB
v 10 dB and 100 dB
vi 99 dB and 100 dB

b The gain, G, in decibels is 10 times the base-ten logarithm of the ratio, $\frac{P_{out}}{P_{in}}$, of the power in watts of the output power level, P_{out}, and the input power level, P_{in}.

That is, $G = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right)$.

Determine the gain, correct to the nearest decibel, given the following input and output power levels.

i output = 5 W, input = 50 W
ii output = 50 W, input = 5 W
iii output = 50 W, input = 5000 W
iv output = 2 W, input = 8 W
v output = 240 W, input = 240 W
vi output = 100 W, input = 1 W

c Determine the output power level in watts, correct to one decimal place, given the following input power levels and gains.

i input = 5 W, gain = 30 dB
ii input = 5 W, gain = 3 dB
iii input = 50 W, gain = 3 dB
iv input = 500 W, gain = 30 dB
v input = 240 W, gain = 60 dB
vi input = 240 W, gain = 20 dB

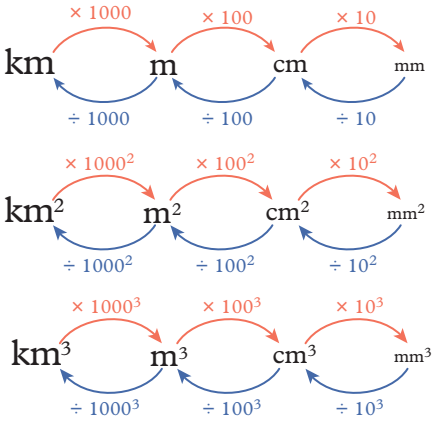
Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Logarithmic scales

Topic quiz
6F

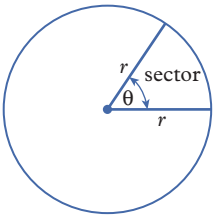
Chapter summary

Units



Area of composite shapes

1. Split shape into individual parts.
2. Calculate missing dimensions.
3. Calculate areas of individual parts.
4. Add or subtract the areas.



Area of a sector

$$\text{Area} = \frac{\theta}{360°} \times \pi r^2$$

Errors

error = exact value – estimated value

absolute error = |exact value – estimated value|

$$\text{relative error} = \frac{\text{absolute error}}{\text{exact value}}$$

$$\text{percentage error} = \frac{\text{absolute error}}{\text{exact value}} \times 100\%$$

Scale factors

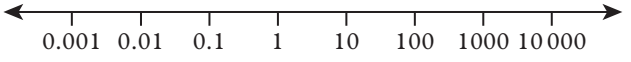
- Scale factor between 0 and 1 = reduction
- Scale factor greater than 1 = enlargement

$$\text{Length scale factor} = \frac{\text{image length}}{\text{original length}}$$

$$\text{Area scale factor} = \frac{\text{image area}}{\text{original area}} = \left(\frac{\text{image length}}{\text{original length}}\right)^2$$

$$\text{Volume scale factor} = \frac{\text{image volume}}{\text{original volume}} = \left(\frac{\text{image length}}{\text{original length}}\right)^3$$

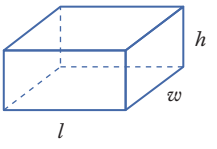
Logarithmic scales



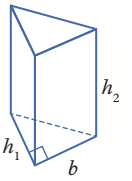
- Position of value on scale relative to 1 = logarithm of number
- Should not be used if the data set has a zero or negative values

Volume of a prism

For any prism, $V = Ah$, where A = area of base and h = height of prism.



$V = lwh$

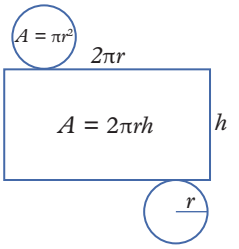


$V = bh_1h_2$

Cylinders

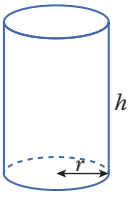
Surface area

$\text{TSA} = 2\pi r^2 + 2\pi rh$



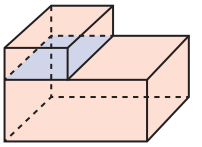
Volume

$V = \pi r^2 h$



Composite solids

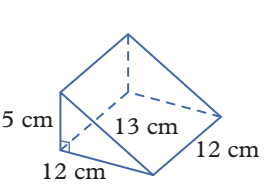
- Total surface area (TSA) of 3D object = sum of areas of the faces.

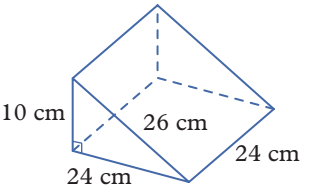


- Volume of 3D object = sum of volumes of individual composite solids

Similar objects

- Two objects with exactly the same shape.





Scientific notation & orders of magnitude

- $a \times 10^m$, where $1 \leq a < 10$ or $-10 < a \leq -1$ and m is an integer
- Order of magnitude in base-ten = m
- To increase a number by 1 order of magnitude: × 10
- To decrease a number by 1 order of magnitude: ÷ 10

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

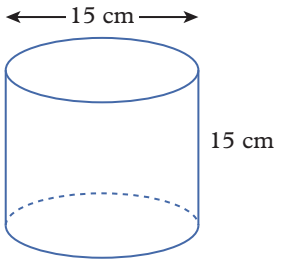
6A **1** Which of the following is equivalent to 300 cm^2?

A 3 m^2 **B** 0.3 m^2 **C** 3000 mm^2 **D** 0.03 m^2 **E** 30 mm^2

6B **2** What is the TSA of a cube with side lengths of 8 cm?

A 384 cm^2 **B** 48 cm^2 **C** 64 cm^2 **D** 512 cm^2 **E** 288 cm^2

6B **3** A closed cylinder has a base diameter of 15 cm and a height of 15 cm. Which of the following is closest to the total surface area of the cylinder?



A 225 cm^2

B 700 cm^2

C 750 cm^2

D 1060 cm^2

E 2800 cm^2

6C **4** A cylinder has a base diameter of 6 cm and a height of 20 cm. Which of the following is closest to the volume of the cylinder?

A 120 cm^3 **B** 188 cm^3 **C** 377 cm^3 **D** 565 cm^3 **E** 2262 cm^3

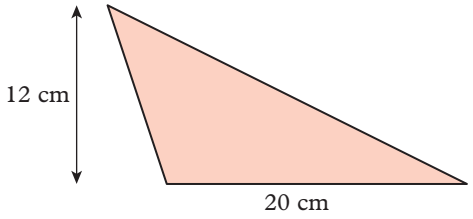
6C **5** A solid wooden cylinder has a base with radius 10 cm and is 10 cm tall. It has a cylinder of radius 2 cm drilled through its centre, from top to base. What is the volume of timber remaining?

A 640π cm^3 **B** 1000π cm^3 **C** 960π cm^3 **D** 80π cm^3 **E** 40π cm^3

6D **6** A carton of eggs is labelled as 700 g. After placing the eggs on a scale, the actual mass is found to be 684 g. Which of the following is closest to the percentage error of the labelled mass?

A 0.02% **B** 2.29% **C** 2.34% **D** 8% **E** 16%

6E **7** What would be the area of the image of this triangle if it is dilated by a factor of $\frac{1}{4}$?



A 7.5 cm^2 **B** 15 cm^2 **C** 30 cm^2 **D** 60 cm^2 **E** 120 cm^2

6E **8** A cube with a side length of 2 cm is dilated by a scale factor of 4. Which of the following is the volume of the new cube?

A 32 cm^3 **B** 96 cm^3 **C** 128 cm^3 **D** 512 cm^3 **E** 1536 cm^3

6F **9** What is the order of magnitude of 230 000?

A 1 **B** 2.3 **C** 5 **D** 6 **E** 7

6F **10** Which of the following best represents the position of 0.005 on a base-ten logarithmic scale?

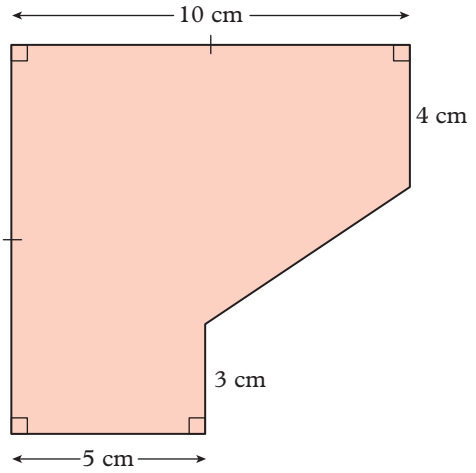
A 2 units to the left of 1 **B** 2.3 units to the left of 1
C 1.3 units to the left of 1 **D** 2 units to the right of 1
E 2.3 units to the right of 1

Short answer

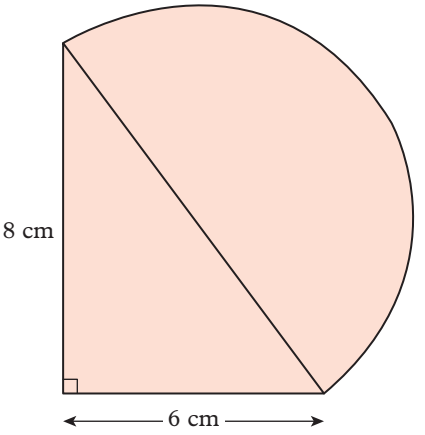
6A **1** A flat metal washer is in the form of an annulus. The radius of the inner ring of the washer is 1 cm, while the outer ring has a radius of 1.5 cm. What is the area of the annulus correct to one decimal place?

6A **2** Determine the area of each of these composite shapes. Give your answers correct to one decimal place.

a



b



6B **3** The base of a triangular prism is a right-angled triangle with a base length of 25 mm and a height of 18 mm. The height of the prism is 30 mm. What is the TSA of the prism?

6B **4** A set of three wooden cubes is made for a playground. The smallest cube has a side length of 40 cm. The next sized cube has side lengths twice as long as the smallest cube, while the side lengths of the largest cube are twice as long as those of the middle cube.

a Calculate the total surface area of each cube.

b If 1 L of paint covers an area of 15 m^2, and the paint is only available in 5 L cans, how many cans of paint would be required to paint all three cubes?

6C **5** A can of baked beans has a cylindrical shape with a base diameter of 7 cm and a height of 10 cm. What is the capacity of the can correct to the nearest millilitre?

6C **6** Calculate the volume of each cube in question **4.** Give your answers in m^3.

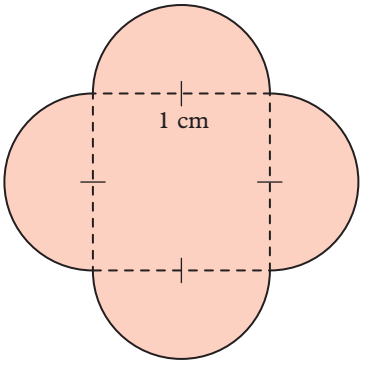
6D **7** The diameter of a circle is 14 mm.

a Find the exact area of the circle.

b Find the approximate circumference of the circle, correct to one decimal place.

c Calculate the relative error of your approximation in part **b**, correct to 3 significant figures.

6E **8** The following composite shape is dilated by a factor of 3. Calculate the area of the image of the composite shape, correct to 2 decimal places.



6E **9** A map with a scale of 1 : 500 details a park. A rectangular playground in the park has dimensions of 7 cm × 5 cm on the map. What is the area of the playground in real life? Give your answer in m^2.

6E **10 a** Increase 2.65×10^2 by 2 orders of magnitude.

b Increase 0.084 by 4 orders of magnitude.

c Decrease 7.51×10^4 by 2 orders of magnitude.

d Decrease 9091 by 5 orders of magnitude.

6F **11** Consider the following base-ten logarithmic scale.

a A quantity is placed halfway between 0.01 and 0.1. What is its value correct to 3 significant figures?

b A different quantity is placed one quarter of the way between 100 and 1000. What is its value correct to the nearest integer?

Analysis

1 Ethan bought this set of wooden blocks for his young cousins.

a Name the solids in order, from left to right.

b The cube has side lengths of 3 cm. Calculate its:

i total surface area

ii volume.

c The rectangular prism has a base the same size as the cube, and is twice the height of the cube.

i Compare the surface area of the rectangular prism with that of the cube.

ii Compare the volume of the rectangular prism with that of the cube.

d The triangular prism is the same shape as the rectangular prism cut vertically along the diagonal of its base.

i Draw a labelled diagram showing the shape of the triangular prism's base.

ii Calculate the total surface area of the triangular prism correct to one decimal place.

iii Is the TSA of the triangular prism half that of the rectangular prism that you calculated in part **c**? Explain.

iv Compare the volumes of the triangular prism and the rectangular prism.

e The base of the cylinder has a diameter the same length as the side of the cube, and the cylinder is the same height as the rectangular prism. Calculate, correct to one decimal place, the cylinder's:

i total surface area **ii** volume.

f Create two tables listing the solids in parts **b–e** in increasing order of:

i surface area **ii** volume.

2 An old treasure map details the path to a chest full of hidden treasure on a remote island. Captain Jack carefully studies the map, and measures the distance along the path on land as being 3.7 km. After eventually finding the treasure, Captain Jack measures the real distance along the path to the treasure as 3.61 km.

a Calculate:

i the absolute error of the distance on the map

ii the percentage error of the distance on the map, correct to two decimal places.

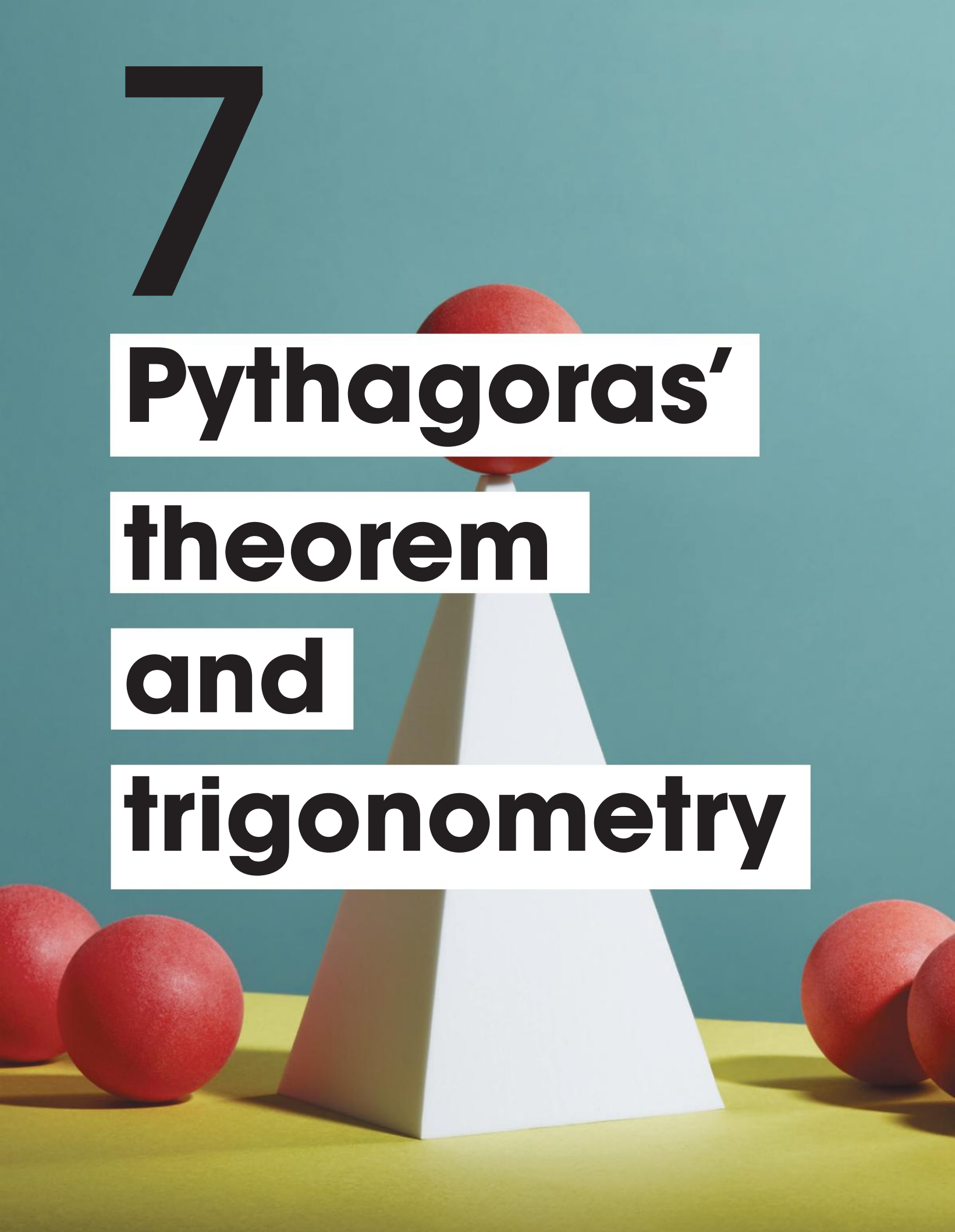
b On the map the forest covers an area of 12 cm^2. If the forest in real life covers an area of 43 200 m^2, what is the scale of the map in the form 1: n?

After collecting the treasure, Captain Jack gets it valued by an independent expert. The least valuable piece in the collection is worth $500 and the most valuable piece is worth $2.5 million.

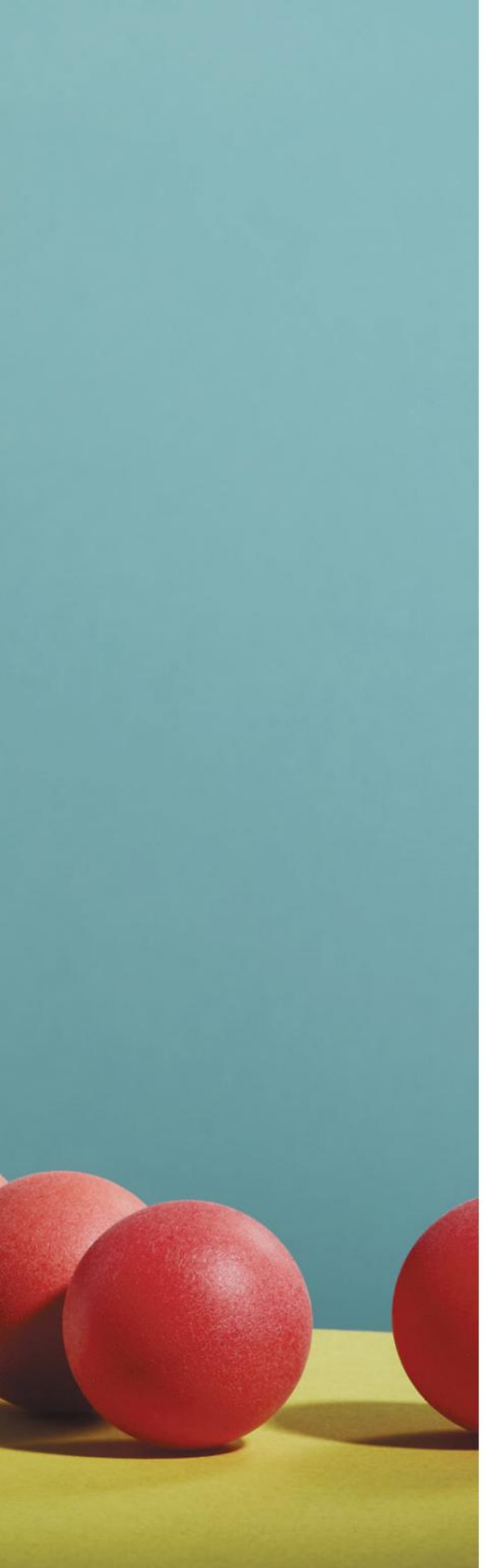
c If the values of the pieces of treasure were plotted on a base-ten logarithmic scale, in relation to $1 and correct to two decimal places, where would:

i the least valuable piece of treasure be plotted

ii the most valuable piece of treasure be plotted?



7 Pythagoras' theorem and trigonometry



Index

7A Pythagoras' theorem
7B Trigonometry
7C Applications of trigonometry
7D Three-dimensional problems
7E The unit circle
7F Exact values
7G Trigonometric graphs
7H Solving trigonometric equations

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Triangle properties
- ✔ Angles and parallel lines
- ✔ Volume of prisms
- ✔ Rounding decimals
- ✔ Simplifying surds

Curriculum links

- Solve practical problems applying Pythagoras' theorem and trigonometry of right-angled triangles, including problems involving direction and angles of elevation and depression (AC9M10M03)

Optional content

- The graphs of $y = \sin(x)$ and $y = \cos(x)$ as functions of a real variable and the solution of related equations

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Materials

- ✔ Protractor
- ✔ Calculator
- ✔ Grid paper

7A Pythagoras' theorem

Learning intentions

By the end of this topic you will be able to ...

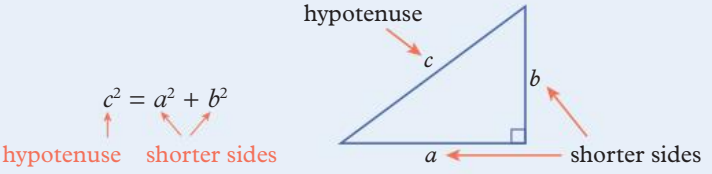
- ✔ determine whether a triangle is acute, right-angled, or obtuse given the side lengths
- ✔ use Pythagoras' theorem to determine unknown side lengths.

Inter-year links

Year 7	1G Exponents and square roots
Year 8	7C Pythagoras' theorem
Year 9	6A Pythagoras' theorem

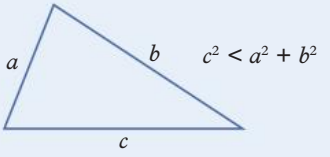
Pythagoras' theorem

- **Pythagoras' theorem** states the relationship between the three side lengths of a right-angled triangle: the sum of the squares of the shorter sides that meet at a right angle is equal to the square of the **hypotenuse**. The relationship for the triangle shown below is:

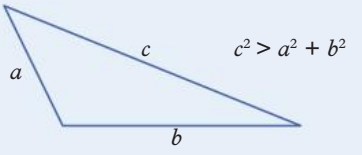


Inequalities for acute and obtuse triangles

- In an acute triangle, the square of the longest side is less than the sum of the square of the other two sides.



- In an obtuse triangle, the square of the longest side is greater than the sum of the square of the other two sides.



Example 7A.1 Determining whether a triangle is acute, right-angled or obtuse

Determine whether a triangle with side lengths of 24 cm, 20 cm and 30 cm is acute, right-angled or obtuse.

THINK

1 Identify the length of the longest side and call it c.

2 Calculate the value of c^2.

WRITE

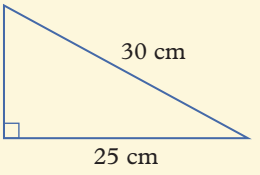
If $c = 30$ cm:

$c^2 = 30^2$
$= 900$

3 Identify the lengths of the two shorter sides and call them a and b.	If $a = 20$ cm and $b = 24$ cm:
4 Calculate the value of $a^2 + b^2$.	$a^2 + b^2 = 20^2 + 24^2$ $= 400 + 576$ $= 976$
5 Compare c^2 to $a^2 + b^2$. If these two values are equal, the triangle is right-angled. If c^2 is greater than $a^2 + b^2$, the triangle is obtuse. If c^2 is less than $a^2 + b^2$, the triangle is acute.	$c^2 < a^2 + b^2$ The triangle is acute.

Example 7A.2 Using Pythagoras' theorem to calculate an unknown side

Calculate the length of the unknown side of this triangle, give your answer as an exact value.



THINK	WRITE
1 Since the triangle is right-angled, state Pythagoras' theorem and define a, b and c. Remember that c is the hypotenuse and use b for the unknown side.	$c^2 = a^2 + b^2$, where $c = 30$ cm and $a = 25$ cm:
2 Substitute the values for a and c into the equation for Pythagoras' theorem and simplify the equation.	$30^2 = 25^2 + b^2$ $900 = 625 + b^2$
3 Solve the equation for b using inverse operations, include units in your answer.	$b^2 = 900 - 625$ $= 275$ $b = \sqrt{275}$ $= 5\sqrt{11}$ cm

Helpful hints

- ✔ Make sure you correctly identify the hypotenuse. It is always the longest side, and is opposite the right angle on the triangle. It may not be obvious, so always look at the numbers carefully.
- ✔ It doesn't matter which of the shorter sides you assign to be a or b, but the hypotenuse must always be c.
- ✔ Remember to always give your answer with correct units.
- ✔ Any set of three whole numbers that satisfies Pythagoras' theorem is called a Pythagorean triad or Pythagorean triple. For example, 3, 4, 5 and 5, 12, 13 are both Pythagorean triads.

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Exercise 7A Pythagoras' theorem

1–4, 7(a, c, f), 8, 10 | 1(a, c, e), 2(b, d, f, h), 5–9, 11, 12 | 1(a, c, e), 2(e, g, i), 3(b, d), 7(c, d, f), 11, 13

UNDERSTANDING AND FLUENCY

7A.1 **1** Determine whether each set of side lengths form an acute, right-angled or obtuse triangle.

a 1 cm, 2.4 cm, 2.6 cm

b 7.5 cm, 4.2 cm, 8.5 cm

c 5 cm, 7 cm, 6 cm

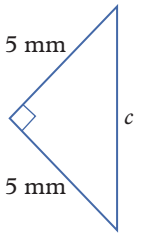
d 26 cm, 24 cm, 0.1 m

e 1 m, 96 cm, 28 cm

f 3.5 cm, 85 mm, 93 mm

7A.2 **2** Calculate the length of the unknown side in each of these triangles and give your answer as an exact value in simplest form. Leave your answer in centimetres for parts **g–i**.

a



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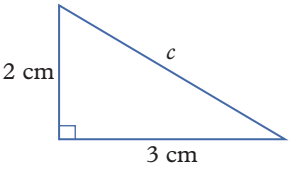

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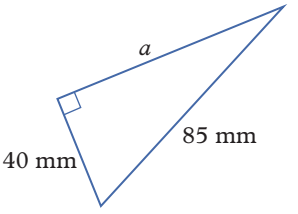

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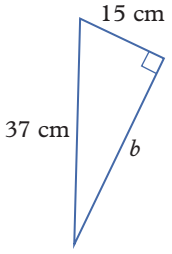

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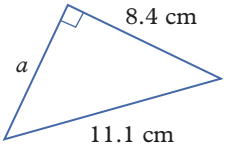

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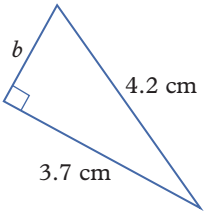
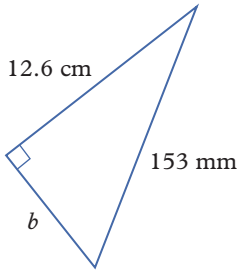


g



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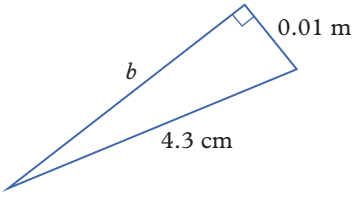
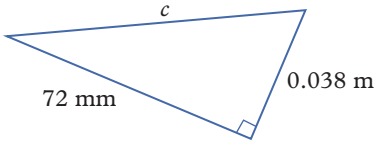


i



3 Calculate the length of the diagonal of a square with each of these side lengths. Give your answer as an exact value in simplest form.

a 2 cm **b** 30 mm **c** 13 cm **d** 35 cm

4 Calculate the length of the diagonal of a square with side lengths of x cm.

5 An equilateral triangle has side lengths of 10 cm.

a Draw a sketch of the triangle. From the top vertex, draw a line that is perpendicular to the base. Label the lengths of all sides of the figure.

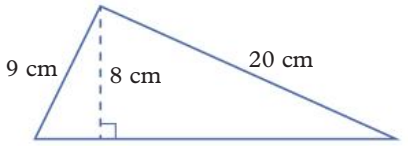
b Use Pythagoras' theorem to calculate the height of the triangle. Give your answer as an exact value.

c Calculate the area of the triangle.

6 Calculate the area of an equilateral triangle with side lengths of x cm; leave your answer as an exact value.

7 Calculate the perimeter of the following triangles by first finding the length of the unknown side, using the given height of each triangle to help you. Correct your answer to two decimal places.

a



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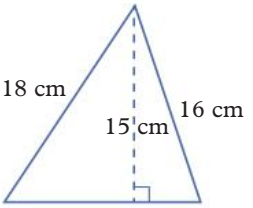

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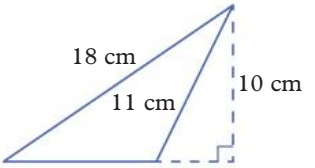

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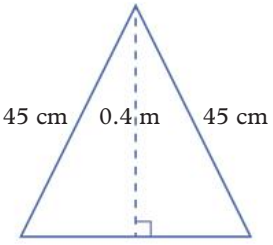
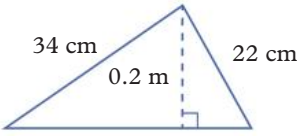


e



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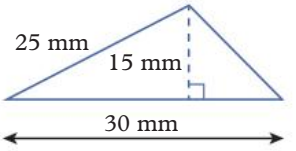


8 Determine whether the triangles in question 7 are acute, right-angled or obtuse.

9 The hypotenuse of a right-angled triangle is 25 cm long; find the length of the other two sides if:

a one of those sides is twice as long as the other

b one of those sides is three times as long as the other

c one of those sides is five times as long as the other.

10 A rectangular running field measures 50 m by 100 m. How much shorter is it to run directly from one corner to the diagonally opposite corner than it is to travel around the outside of the field?

11 The flagpole which sits on Parliament House in Canberra stands 81 m high. The Australian flag flying from the top section of the pole is said to be the largest Australian flag flying in Australia: it's about the size of the side of a double-decker bus! It measures 12.8 m wide, with a diagonal of 14.3 m.

a What is the height of the flag?

b Calculate the perimeter of the flag.

c What is the distance from the base of the flagpole to the lowest point where the flag is attached to the pole?

d Assume the flag is flying flat and directly at right angles to the pole. Calculate the distance from the base of the flagpole to the:

i lower unattached corner of the flag

ii higher unattached corner of the flag.

12 A ladder rests against a vertical wall, with the top of the ladder 7 m above the ground. When the bottom of the ladder is moved 1 m further away from the wall, the ladder is flat on the ground with one end against the wall.

a Draw a diagram to illustrate this situation.

b Calculate the length of the ladder.

13 The orthocentre of a triangle is the intersection point of the perpendicular heights of each side of the triangle. Consider $\triangle ABC$ where $A(-5, 0)$, $B(-1, 8)$ and $C(2, -1)$.

a Determine the equation of the lines of all three perpendicular heights.

b Find the intersection point of the perpendicular heights.

c Find the exact distance each vertex of $\triangle ABC$ is from the orthocentre.

Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Pythagoras' theorem
- Interactive skillsheet: Finding the length of the hypotenuse
- Interactive skillsheet: Finding the length of a shorter side
- Topic quiz: 7A

7B Trigonometry

Learning intentions

By the end of this topic you will be able to ...

✔ use trigonometry to find unknown side lengths

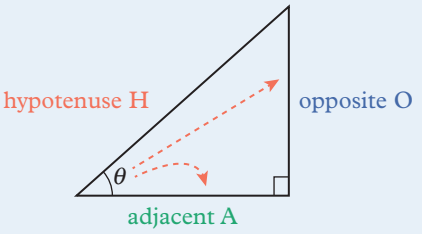
✔ use trigonometry to find unknown angles.

Inter-year links

Year 9 6D Trigonometric ratios

Finding lengths using trigonometry

- **Trigonometry** describes the relationship between the angles and side lengths of a triangle.
- The Greek letter θ (theta) is often used as a pronumeral for angles.
- The sides of a right-angled triangle can be labelled as:
  - → the hypotenuse: the longest side of the right-angled triangle opposite the right angle
  - → the opposite side: the side opposite an interior reference angle
  - → the adjacent side: the side adjacent to (or next to) an interior reference angle that is not the hypotenuse.



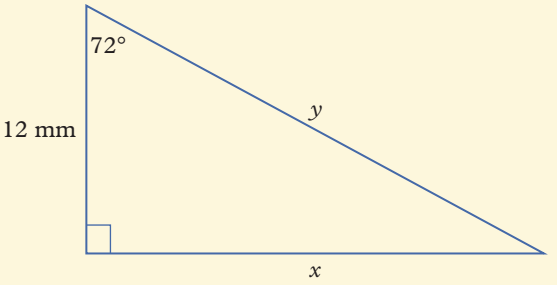
- The trigonometric ratios are known as the **sine** ratio, the **cosine** ratio and the **tangent** ratio. They are the ratios between two side lengths of a right-angled triangle with respect to a given interior reference angle.
- For right-angled triangles with a reference angle of θ:
  - → $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\text{O}}{\text{H}}$
  - → $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{\text{A}}{\text{H}}$
  - → $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\text{O}}{\text{A}}$
- A calculator can be used to determine the sine, cosine or tangent values of any angle.
- The equations for the sine, cosine and tangent ratios can be rearranged to determine the side lengths of a right-angled triangle given an angle and one side length.

Using trigonometry to determine unknown angles

- The non-right angles in a right-angled triangle can be determined using the inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) or inverse tangent ($\tan^{-1}$) of a ratio as they are the opposite operations or inverse functions of the sine, cosine and tangent functions.
  - → $\theta = \sin^{-1}\left(\frac{\text{Opposite}}{\text{Hypotenuse}}\right)$
  - → $\theta = \cos^{-1}\left(\frac{\text{Adjacent}}{\text{Hypotenuse}}\right)$
  - → $\theta = \tan^{-1}\left(\frac{\text{Opposite}}{\text{Adjacent}}\right)$
- A calculator can be used to determine the unknown angle given the sine, cosine or tangent values.

Example 7B.1 Using trigonometry to determine unknown side lengths

Find the length of each unknown side in this triangle, correct to one decimal place.



THINK	WRITE
1 Identify the hypotenuse (H), opposite side (O) and adjacent side (A) with respect to the reference angle 72°.	H = y, O = x, A = 12 mm
2 Decide which trigonometric ratio to use to find the length of x. Since this involves the sides opposite and adjacent to the reference angle (O and A), use tangent.	$\tan(\theta) = \frac{\text{O}}{\text{A}}$
3 Substitute values for θ, O and A into $\tan(\theta) = \frac{\text{O}}{\text{A}}$.	$\tan(72°) = \frac{x}{12}$
4 Solve the equation for x. Use a calculator to multiply 12 by tan(72°). Round the value you find for x to one decimal place and include units.	$x = 12 \times \tan(72°)$ $= 36.9322\ldots$ ≈ 36.9 mm
5 Decide which trigonometric ratio to use to find the length of y. Since this involves the side adjacent to the reference angle and the hypotenuse (A and H), use cosine.	$\cos(\theta) = \frac{\text{A}}{\text{H}}$
6 Substitute values for θ, A and H into $\cos(\theta) = \frac{\text{A}}{\text{H}}$.	$\cos(72°) = \frac{12}{y}$
7 Solve the equation for y. Use a calculator to divide 12 by cos(72°). Round the value you find for y to one decimal place and include units.	$y \times \cos(72°) = 12$ $y = \frac{12}{\cos(72°)}$ $= 38.8328\ldots$ ≈ 38.8 mm

Example 7B.2 Using the inverse trigonometric functions

Use a calculator to find the size of θ for these trigonometric ratios to the nearest degree.

a $\sin(\theta) = 0.439$ **b** $\tan(\theta) = \frac{17}{5}$

THINK

a **1** Rearrange to make θ the subject of the equation using the inverse sine function.

2 Use the $\sin^{-1}$ key on your calculator to find the inverse sine of 0.439.

3 Round the angle to the nearest degree.

b **1** Rearrange to make θ the subject of the equation using the inverse tangent function.

2 Use the $\tan^{-1}$ key on your calculator to find the inverse tangent of $\frac{17}{5}$.

3 Round the angle to the nearest degree.

WRITE

a $\sin(\theta) = 0.439$

$\theta = \sin^{-1}(0.439)$

$\theta = 26.0400\ldots$

$\theta \approx 26°$

b $\tan(\theta) = \frac{17}{5}$

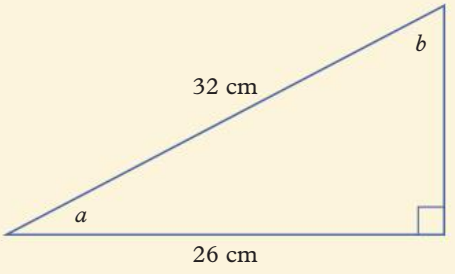
$\theta = \tan^{-1}\left(\frac{17}{5}\right)$

$\theta = 73.6104\ldots$

$\theta \approx 74°$

Example 7B.3 Using trigonometry to determine unknown angles

Find the size of each of the two unknown angles in this triangle. Give your answers correct to two decimal places.



THINK

1 Decide which angle to calculate first, and identify the known side lengths with respect to that chosen angle.

2 Decide which trigonometric ratio to use. Because the adjacent side (A) and hypotenuse (H) are involved, use cosine.

3 Substitute values for θ, A and H into $\cos(\theta) = \frac{\text{A}}{\text{H}}$.

4 Rearrange to make a the subject of the equation using the inverse cosine function.

5 Use the $\cos^{-1}$ key on your calculator to find the inverse cosine of $\frac{26}{32}$.

6 Round your answer correct to two decimal places.

7 Use the value of a and the right angle to find the value of b. Remember that the sum of the interior angles of a triangle is equal to 180°.

WRITE

$\theta = a$, H = 32 cm, A = 26 cm

$\cos(\theta) = \frac{\text{A}}{\text{H}}$

$\cos(a) = \frac{26}{32}$

$a = \cos^{-1}\left(\frac{26}{32}\right)$

$= 35.659\ldots$

$\approx 35.66°$

$b = 180° - 90° - a$

$= 90° - 35.659\ldots°$

$\approx 54.34°$

Helpful hints

- ✔ When finding an unknown side length, label the triangle to identify which side length you know (O, A or H) and which side length you want to find (O, A or H). Then choose the trigonometric ratio whose formula contains both.
- ✔ When finding an unknown angle, label the triangle to identify which two side lengths you know (O, A or H). Then choose the trigonometric ratio whose formula contains both.
- ✔ Remember that $\sin^{-1}$ does *not* mean sine to the power of –1 (the reciprocal of sine).
- ✔ Remember to write $\theta = \sin^{-1}\left(\frac{4}{5}\right)$, not $\sin^{-1}(\theta) = \frac{4}{5}$.
- ✔ It helps to draw diagrams roughly to scale. Remember, longer sides are opposite larger angles and shorter sides are opposite smaller angles.

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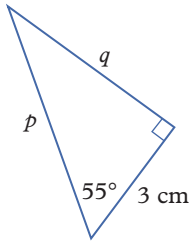
Exercise 7B Trigonometry

1–5, 7–9 | 1–4, 6, 10, 12(a, b), 14, 17–19 | 1, 3(d–f), 6, 10–13, 15–19

7B.1 **1** Find the lengths of the unknown sides in each of these triangles, correct to one decimal place.

a



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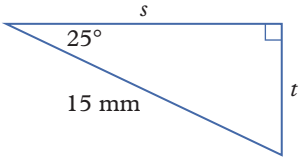

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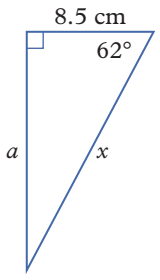

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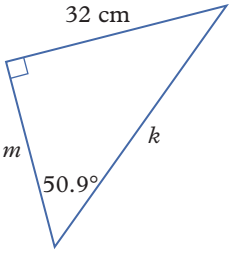

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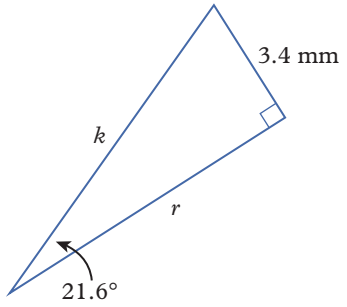

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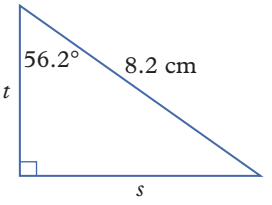


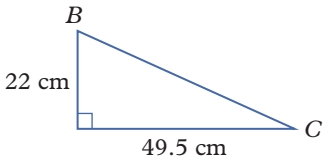
7B.2 **2** Use a calculator to find the size of θ for the trigonometric ratios below. Write your answers to the nearest degree.

a $\sin(\theta) = 0.378$ **b** $\cos(\theta) = 0.845$ **c** $\tan(\theta) = 2.376$

d $\sin(\theta) = \frac{10}{13}$ **e** $\cos(\theta) = \frac{44}{99}$ **f** $\tan(\theta) = \frac{1}{100}$

7B.3 **3** Find the size of each of the two unknown angles in each of these triangles. Give your answers to the nearest degree.

a



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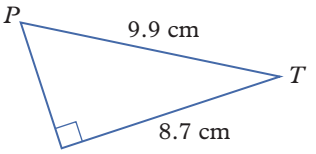

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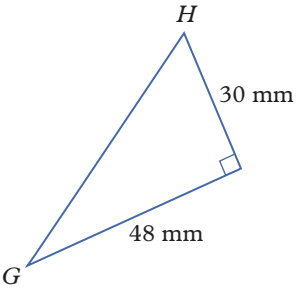

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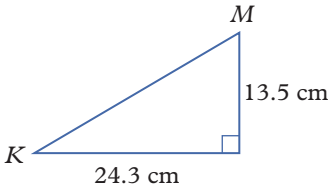

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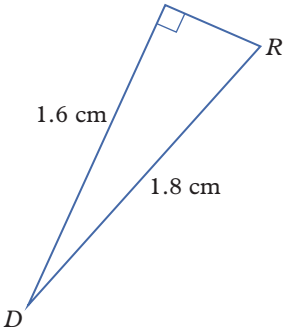

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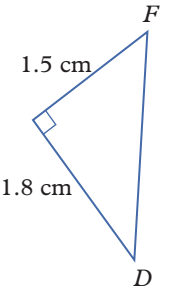
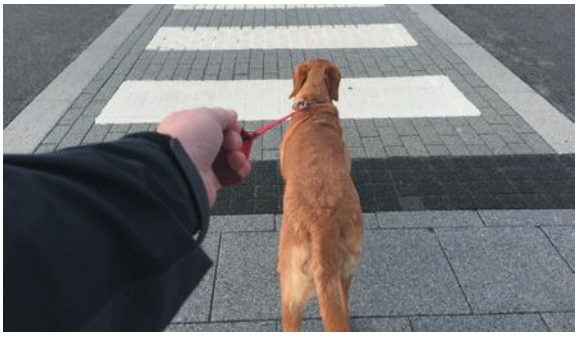


4 Shun is walking his dog on a leash. The leash is 2.5 m long, and his dog is 55 cm tall. Shun's dog is pulling him along so that the taut leash makes an angle of 85° with the vertical.



a Draw a diagram to show this information.

b How far in front of Shun is his dog, correct to the nearest centimetre?

c What is the vertical height of the end of the leash in Shun's hand to the nearest centimetre?

5 The front face of a brick has a diagonal length of 242 mm. This diagonal forms an angle of 18.3° with the longer side of the face. What are the dimensions of the face of the brick, correct to the nearest millimetre?

6 The Leaning Tower of Pisa is a freestanding bell tower next to the cathedral in the Italian city of Pisa. The top of the tower is displaced 3.9 m horizontally from where it would be if it were standing vertically. The distance from the centre of the base to the centre of the top is 56.28 cm.

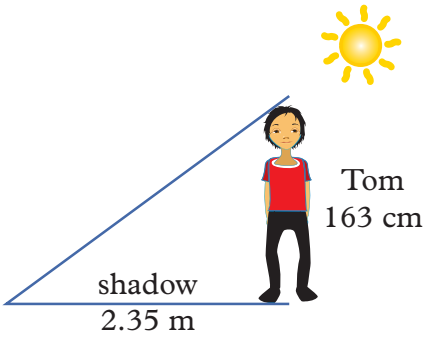
a Draw a labelled sketch of the tower using this distance and its horizontal displacement. Mark the angle of 'lean' of the tower.

b Describe which trigonometric ratio you would use to calculate the angle of the tower's lean, marked in part **b**.

c Calculate the angle of lean of the tower. Give your answer to the nearest degree.

7 Tom and his brother Sam were standing in the sunshine measuring the length of each other's shadow. They each knew their height, so they collated the measurements (see the table on the right).

	Height	Length of shadow
Tom	163 cm	2.35 m
Sam	141 cm	2.04 m



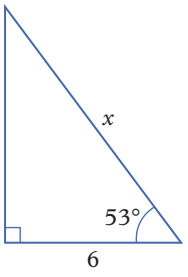
a Tom drew the diagram on the right to represent his shadow and height. Draw a similar diagram to display Sam's measurements.

b Calculate the base acute angle for the triangles in each diagram. (Remember to take into account the difference in units.) Give your answers in degrees, correct to one decimal place. What does this base angle represent?

c **i** Explain why the two triangles formed by the boys' heights and their shadows must be similar shapes.

ii Draw a diagram combining the two triangles from the diagram above and your own diagram from part **a**.

8 **a** Ji-ho has attempted to find the value of x in the following triangles. Explain the mistake Ji-ho made in all three questions.

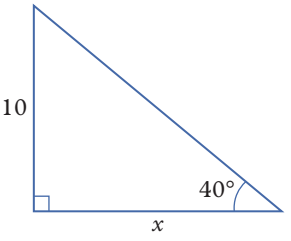


$\cos(53°) = \frac{x}{6}$

$x = 6\cos(53°)$

$x = 3.61089...$

$x \approx 3.6$

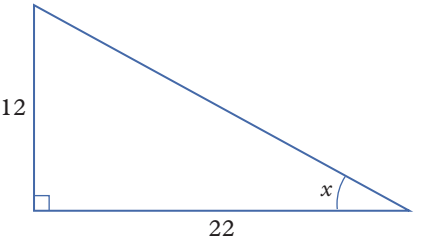


$\tan(40°) = \frac{x}{10}$

$x = 10\tan(40°)$

$x = 8.390996...$

$x \approx 8.4$



$\tan(x) = \frac{22}{12}$

$x = \tan^{-1}\left(\frac{22}{12}\right)$

$x = 61.3895...°$

$x \approx 61.4°$

b Explain how you can ensure you don't make the mistakes in part **a**.

9 Parliament House in Canberra is 300 m long and 300 m wide, being one of the largest buildings in the Southern Hemisphere.

a Draw a labelled diagram to show the shape and size of the building from above.

b Use Pythagoras' theorem to calculate the diagonal length of the building.

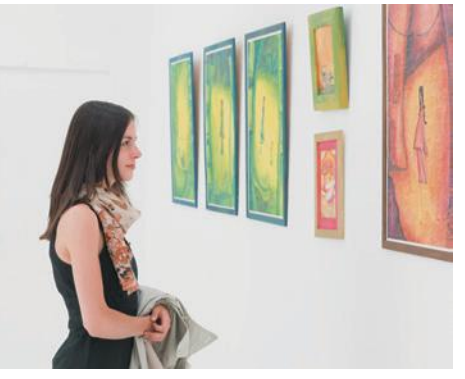
c Use trigonometry to calculate the diagonal length of the building. Hint: Identify the size of the angle you can use.

d Comment on your answers for parts **b** and **c**.

10 Sandy is standing 2 m in front of a picture hanging on a wall. The picture is 100 cm wide, and is hung so the bottom of the frame is 1.5 m from the floor. If she lowers her eyes 5.7° from the horizontal, the bottom of the frame of the painting is directly in her line of sight. If she raises her eyes 16.7° from the horizontal, the top of the frame is directly in her line of sight.

a At what height are Sandy's eyes?

b What are the dimensions of the painting?



11 Some road signs display the gradient of a road, particularly when the road is approaching a hill or mountain climb. The gradient can be written in different forms, commonly as a ratio, a percentage or a fraction. For example, 1 in 10, 1 : 10, 10% and $\frac{1}{10}$ can all be used to represent the same gradient. Patricia is a long-distance truck driver delivering goods between various locations. She generally plans her route to avoid steep hill climbs, if possible. Her next delivery takes her via a mountainous stretch, where she has a choice of three possible routes, described below.

Route 1: Over a horizontal distance of 12 km, with the rise being 6 km

Route 2: Over a horizontal distance of 18 km, with the rise being 7.5 km

Route 3: Over a horizontal distance of 15 km, with the rise being 6.5 km

a Write the gradient of each route, correct to one decimal place, as:

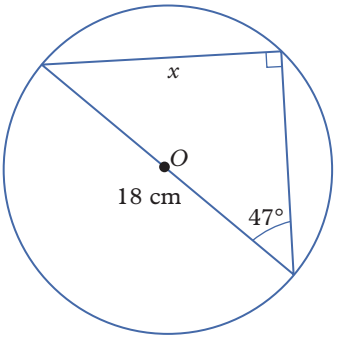
i a ratio **ii** a percentage.

b What is the angle of the rise of each road over the distances given, correct to one decimal place?

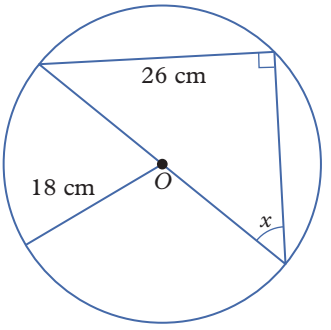
c Rank the three routes in order of increasing steepness.

12 Determine the value of x in each of the following graphs, correct your answer to two decimal places.

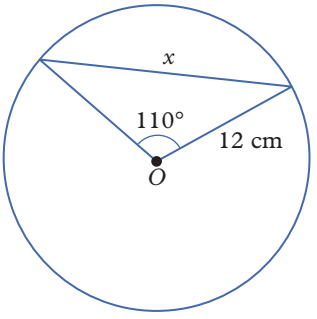
a



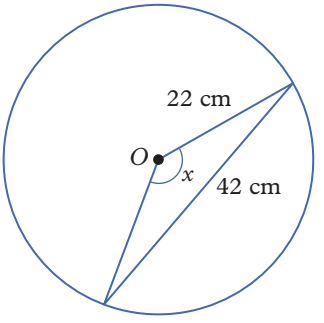
b



c



d

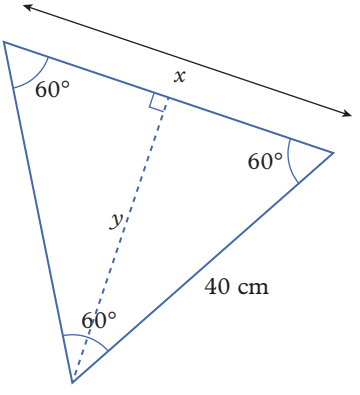


13 A hexagon is a six-sided polygon. Consider a regular hexagon with side lengths of 12 cm. Calculate the distance between its parallel sides correct to two decimal places.

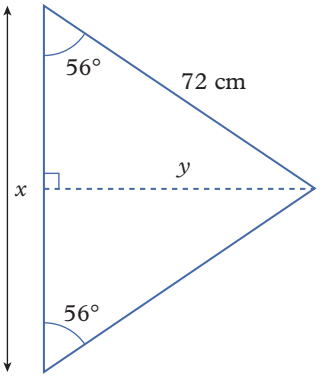
14 a Determine the value of x and y in the following triangles, correct to two decimal places where required.

b Determine the area of each triangle, correct to one decimal place.

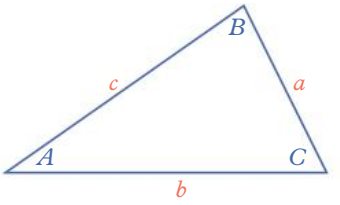
i



ii



15 a Copy the given triangle into your workbook.



b Draw the height of the triangle from vertex B to base b. Label the point where the height meets the base, D. Label the length of the height h.

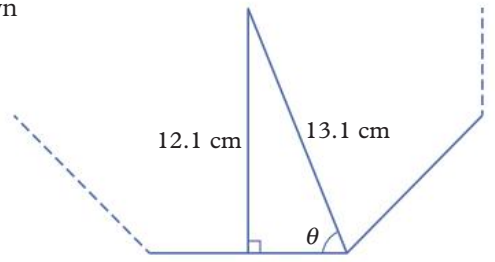
c i Use the sine ratio for angle A to write a relationship between h and c. Rearrange to make h the subject of the formula.

ii Use the sine ratio for angle C to write a relationship between h and a. Rearrange to make h the subject of the formula.

iii Equate your two answers for parts **i** and **ii** and show how this can be written as $\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$.

d Repeat the same reasoning outlined in parts **a**, **b** and **c** to find a similar relationship involving angle B. Show that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$.

16 The vertex of a regular polygon where line segments have been drawn from the polygon's centre to the vertex and the midpoint of the adjacent edge is shown to the right.

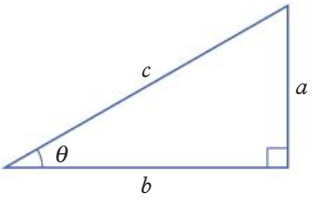


a Determine the value of θ correct to one decimal place.

b Use the angle from part **a** to determine what type of regular polygon the vertex is from.

c Determine the perimeter of the regular polygon correct to the nearest centimetre.

17 Investigate what happens to the value of the sine ratio as the angle increases from 0° to 90° in a right-angled triangle. Consider the triangle in the diagram shown on the right.



a The sine of θ is the ratio of the length of the opposite side to the length of the hypotenuse; that is, $a : c$. When θ is 0°:

i What is the length of a?

ii What does this mean for the value of the ratio $a : c$?

b Now consider the situation when θ becomes almost 90°.

i What is the value of a compared with c?

ii What is the value of the ratio $a : c$?

c Describe what happens to the value of the sine ratio as the angle increases from 0° to 90°.

d Repeat parts **a–c** for the cosine ratio. Consider the relevant line segments for that ratio.

e Once more, repeat parts **a–c** for the tangent ratio, considering the relevant line segments for that ratio.

18 From your investigations in question **17**, you can determine the minimum and maximum values for each of the three trigonometric ratios for angles in the range 0° to 90°. Copy and complete the following to summarise your findings.

The sine ratio has a minimum value of _____ and a maximum value of _____.

The cosine ratio has a minimum value of _____ and a maximum value of _____.

The tangent ratio has a minimum value of _____ and a maximum value of _____.

19 Check your answers for question **18** with a calculator. Provide some values to support your conclusions.

Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Trigonometry
- Interactive skillsheet: Finding lengths by trigonometry
- Interactive skillsheet: Finding angles by trigonometry
- CAS instructions: Trigonometry
- Topic quiz: 7B

7C Applications of trigonometry

Learning intentions

By the end of this topic you will be able to ...

- ✔ use angles of elevation and depression to calculate distances
- ✔ use compass bearings and true bearings in trigonometry problems.

Inter-year links

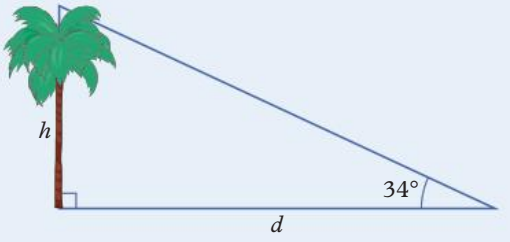
Year 7	6G Solving equations using inverse operations
Year 8	6B Solving linear equations
Year 9	6E Using trigonometry to find lengths

Angles of elevation and depression

- The **angle of elevation** is the angle measured upward from the horizontal.

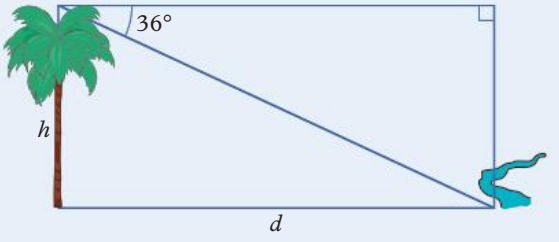
  For example, in the diagram on the right, there is a 34° angle of elevation to the top of a palm tree from a distance d along a flat beach.

  The height of the palm tree is given by $h = d\tan(34°)$.



- The **angle of depression** is the angle measured down from the horizontal.

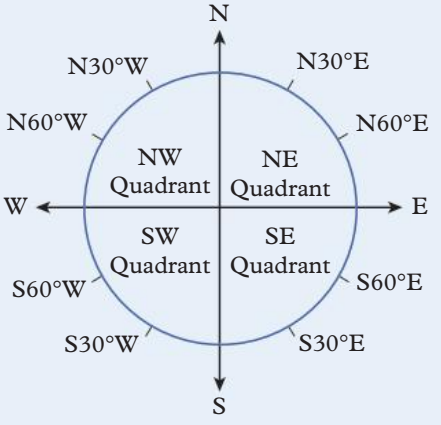
  For example, if someone climbs to the top of the same palm tree and measures a 36° angle of depression from the top of the tree to a nearby ocean inlet, then the distance from the base of the tree to the inlet would be given by $d = \frac{h}{\tan(36°)}$.



Bearings

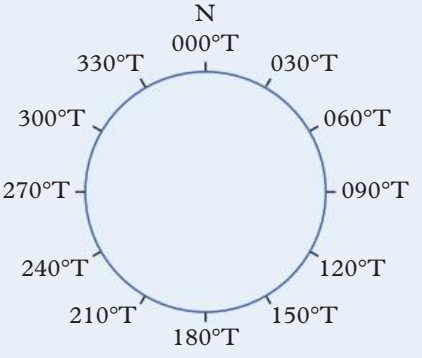
- The four cardinal directions are north, south, east and west.
- **Compass bearings** describe direction using angles between 0° and 90° within one of the four quadrants of a compass (see the diagram on the right).
  The bearings are written with a capital letter for the north–south direction, followed by the angle, followed by a capital letter for the east–west direction.

  For example, the angle 35° clockwise of north would be written as N35°E, and the opposite direction would be written as S35°W.

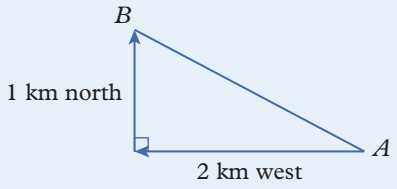


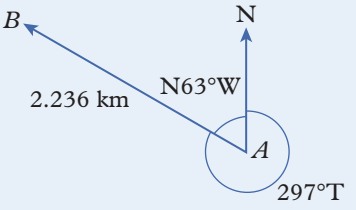
- **True bearings** describe a direction using an angle between 0° and 360° clockwise from north. They are written with three digits for the angle, followed by a capital T.

  For example, the angle 35° clockwise of north would be written as 035°T, and the opposite direction would be written as 215°T.



- It is often useful to know how far a position is in terms of distance along the cardinal directions. For example, if point B is 1 km north and 2 km west of point A, then Pythagoras' theorem can be used to find the total distance and bearings from point A to point B to be approximately 2.236 km N63°W or 297°T.





Example 7C.1 Calculating distances using angles of elevation and depression

Cape Pillar in the Tasman National Park is home to the tallest sea cliffs in the Southern Hemisphere.

a If a tourist measures her kayak to be 200 m from the base of the closest cliff face and the angle of elevation to the tip of the cliff is 56°, how tall is the cliff? Round your answer to the nearest metre.

b The next day, the same tourist hikes to the top of the cliff and spots a boat near the horizon at an angle of depression of 2°. How far away is the boat from the tourist? Round your answer to the nearest metre.

THINK

a 1 Draw a diagram to represent the situation. It does not need to be to scale.

2 Substitute the known values into the equation for the tangent ratio.

3 Solve for the height of the cliff and round to the nearest metre.

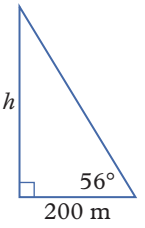
b 1 Draw a diagram to represent the situation. It does not need to be to scale. Assign the angle of depression to the corresponding angle of elevation, as they are alternate angles.

2 Substitute the known values into the equation for the sine ratio.

3 Solve for the distance from the tourist to the boat and round to the nearest metre.

WRITE

a



$\theta = 56°$, O = h, A = 200 m,

$\tan(\theta) = \frac{\text{O}}{\text{A}}$

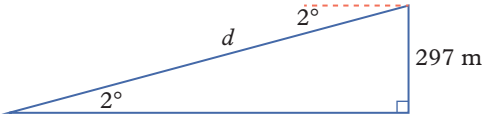
$\tan(56°) = \frac{h}{200}$

$h = 200 \tan(56°)$

$= 296.5121\ldots$

≈ 297 m

b



$\theta = 2°$, O = 297 m, H = d

$\sin(\theta) = \frac{\text{O}}{\text{H}}$

$\sin(2°) = \frac{297}{d}$

$d = \frac{297}{\sin(2°)}$

$= 8510.1514\ldots$

≈ 8510 m

Example 7C.2 Calculating distances using bearings

For ships travelling the distances and directions below:

i Draw a labelled diagram to display the distance travelled along the relevant cardinal directions after departing from the port.

ii Use trigonometry to calculate the distances travelled along the relevant cardinal directions, to the nearest kilometre.

a 180 km on a bearing of S59°E

b 85 km on a bearing of 224°T

THINK

a i Draw a diagram to represent the situation. Start with the port and draw an upwards arrow to represent north, then draw a line in the direction towards the ship, label the distance traveled on the graph. Label the distances along the east–west and north–south axes of your diagram with pronumerals x and y.

ii 1 Use the sine ratio to calculate the distance east.

2 Use the cosine ratio to calculate the distance south.

b i 1 A true bearing of between 180°T and 270°T is in the SW quadrant; calculate the angle measured clockwise from south.

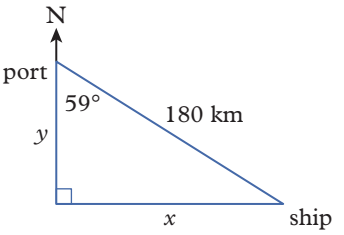
2 Draw a diagram to represent the situation. Start with the port and draw an upwards arrow to represent north, then a line in the correct direction towards the ship, label the distance travelled. Label the distance along the east–west and north–south axes of your diagram with pronumerals x and y.

ii 1 Use the sine ratio to calculate the distance west.

2 Use the cosine ratio to calculate the distance south.

WRITE

a i



ii $\sin(\theta) = \frac{O}{H}$

$\sin(59°) = \frac{x}{180}$

$x = 180 \times \sin(59°)$

≈ 154 km

The ship is 154 km east of port.

$\cos(\theta) = \frac{A}{H}$

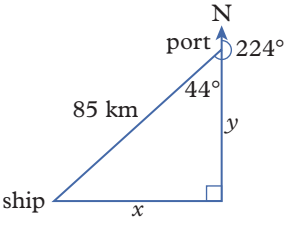
$\cos(59°) = \frac{y}{180}$

$y = 180 \times \cos(59°)$

≈ 93 km

The ship is 93 km south of port.

b i $224° - 180° = 44°$



ii $\sin(\theta) = \frac{O}{H}$

$\sin(44°) = \frac{x}{85}$

$x = 85 \times \sin(44°)$

≈ 59 km

The ship is 59 km west of port.

$\cos(\theta) = \frac{A}{H}$

$\cos(44°) = \frac{y}{180}$

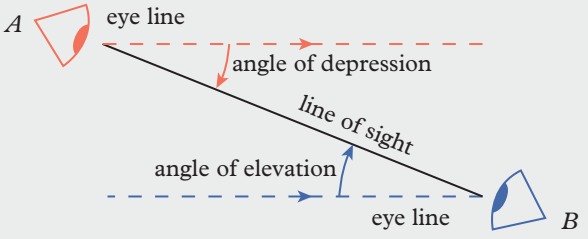
$y = 85 \times \cos(44°)$

≈ 61 km

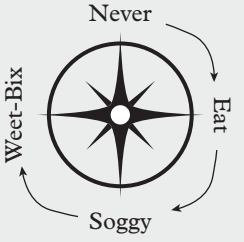
The ship is 61 km south of port.

Helpful hints

- ✔ It always helps to draw a diagram and label the sides and angles with the known and unknown values. It does not need to be exactly to scale.
- ✔ The angle of depression from point A to point B is equal to the angle of elevation from point B to point A. This is because they form alternate angles on parallel eye lines.



- ✔ One way of remembering the cardinal directions is the phrase 'Never Eat Soggy Weet-Bix'. The first letter of each word matches the cardinal directions, read clockwise from north.



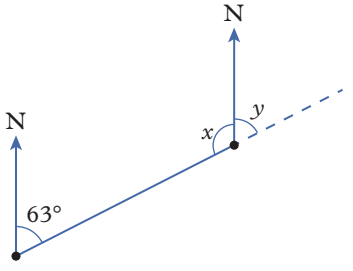
ANS p619

Exercise 7C Applications of trigonometry

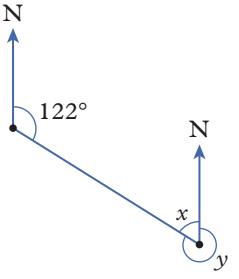
1–6, 8, 11, 12, 14, 15 | 1, 2, 5–7, 9, 10, 13, 16, 17, 19 | 1(d–f), 5, 6, 10, 14, 16–18, 20

1 Determine the value of the pronumerals in each of the following.

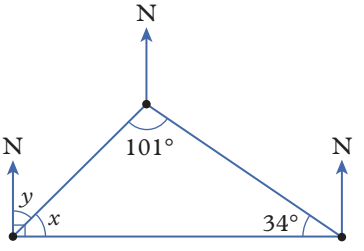
a



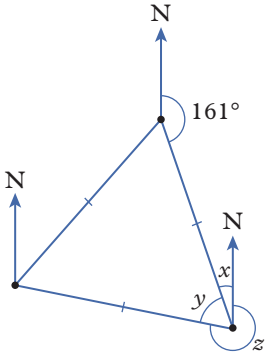
b



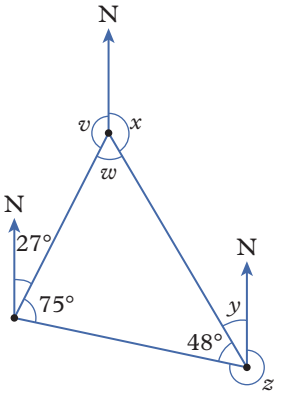
c



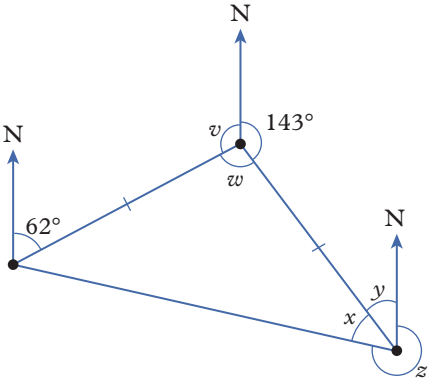
d



e



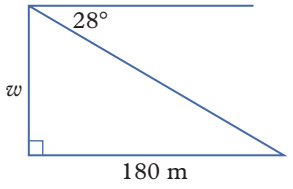
f



UNDERSTANDING AND FLUENCY

2 Use the angle of elevation or depression in each diagram to calculate the side length marked with the pronumeral. Give your answers correct to one decimal place.

a



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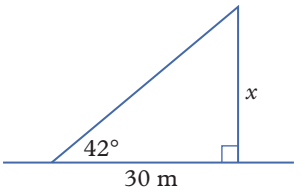

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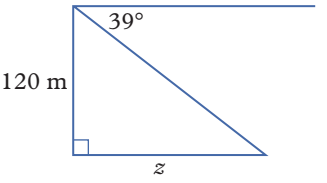

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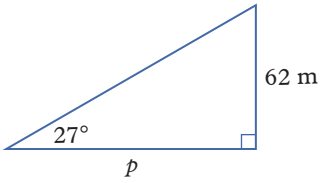

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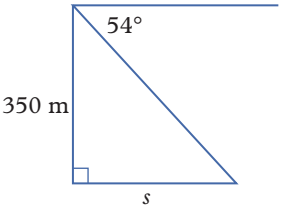

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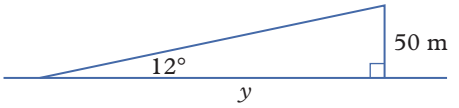


7C.1 **3** A lighthouse stands on a cliff above the sea. The angle of elevation to the top of the lighthouse from a point on level ground 100 m from its base is 14°.

a Find the height of the lighthouse, to the nearest metre.

b The angle of depression from the light at the top of the lighthouse to a ship at sea 2.5 km from the base of the cliff is 3.3°. Find the height of the light above the sea, to the nearest metre.

4 Convert the following bearings:

a from compass bearings to true bearings

i S30°E **ii** N52°W **iii** N5°E

b from true bearings to compass bearings.

i 200°T **ii** 100°T **iii** 300°T

7C.2 **5** For ships travelling each of the distances and directions below:

i draw a labelled diagram to display the distance travelled along the relevant cardinal directions after departing from the port

ii use trigonometry to calculate these distances travelled along the relevant cardinal directions, to the nearest kilometre.

a 250 km on a bearing of N30°E **b** 100 km on a bearing of 300°T

c 75 km on a bearing of S45°W **d** 210 km on a bearing of 090°T

6 If each ship in question **5** turns around and returns to its original port, write a description of the ship's path with respect to its turn-around position using:

i compass bearings and distance in kilometres

ii true bearings and distance in kilometres.

7 Stavros is descending an escalator at an angle of depression of 30°.

a If the escalator is 6 m high at the top, correct to two decimal places, what horizontal distance does it cover?

b If it takes 30 seconds for Stavros to descend the escalator, correct to one decimal place, at what speed, in m/s, is the escalator moving?

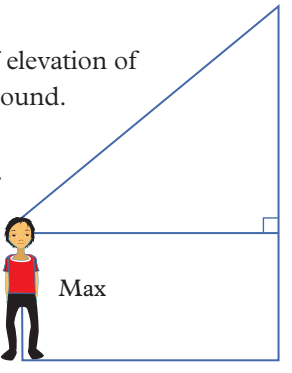
8 Max throws a ball into the air. He views the ball at its highest point to be at an angle of elevation of 40° from him and 3 m horizontally away from him. Max's eye level is 172 cm above ground.

a Copy the diagram on the right and label it with the known values.

b What height does the ball reach above the level of Max's eyes? Correct your answer to the nearest centimetre.

c What is the maximum height the ball reaches above the ground? Correct your answer to the nearest centimetre.

d Why is it important to consider the height of a person's eyes in a situation like this?



9 Can the angle of elevation or angle of depression be an obtuse angle? Explain.

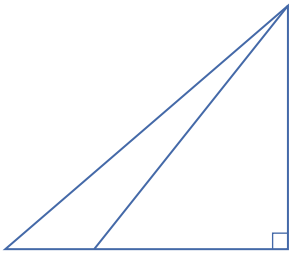
10 Toshiko is standing at the edge of a 10 m cliff and sees a boat near the base of the cliff. Toshiko measures the angle of depression to the point where the bow of the boat (the front of the boat) touches the water as 24°.

a Correct to two decimal places, how far from the base of the cliff is the boat if Toshiko's eye height is 152 cm?

Toshiko spots a shark directly below the boat and measures the angle of depression to the nose of the shark as 25°.

b If the nose of the shark is directly below where the bow of the boat touches the water, correct to two decimal places, how deep is the shark swimming in the water?

11 Ann is standing 8 m in front of a tree. She observes the angle of elevation to the top of the tree to be 52°. Beth is standing further from the tree than Ann, but in the same line. She observes the angle of elevation to be 40°. (Ignore the heights of the two girls in this problem.)



a Copy the diagram on the right and label it with the known values.

b Find the height of the tree calculated using Ann's observations, correct to one decimal place.

c Use the tree's height, together with Beth's angle of elevation, to calculate Beth's distance from the tree, correct to one decimal place.

d What is the distance between Ann and Beth? Correct your answer to one decimal place.

e Suppose Beth had been standing in the same line as Ann, but on the opposite side of the tree.

i Draw a diagram to show this situation.

ii What would have been your answer to part **d** in this case?

12 A lookout tower is built on top of a cliff. At a point 50 m from the base of the cliff, the angle of elevation to the base of the tower is 58°, while the elevation to the top of the tower is 64°.

a Draw a diagram to display this information.

b Find the height of the tower, correct to one decimal place.

13 Two bushwalkers start walking from their campsite. Roger walks due north, while Ben walks N22.5°E. They both walk 25 km.

a Display the information on a diagram.

b How far north of his starting point is Ben when he completes his walk? Correct your answer to one decimal place.

c How far east of his starting point is Ben at this stage? Correct your answer to one decimal place.

d If Ben were to walk directly to meet Roger at his finishing point, how far would he have to walk? Correct your answer to one decimal place.

14 Jake is riding his bike from home to his friend Luke's house. He rides 1.2 km directly north, then makes a right-angled turn left and continues for 3.2 km west to Luke's house.

a Draw and label a sketch of Jake's route. Mark the north direction at each significant point on his journey.

b Calculate the direct distance between Jake's house and Luke's house, correct to one decimal place.

c Use trigonometry to find the true bearing from Jake's house to Luke's house. (Remember that bearings are measured from the north in a clockwise direction.)

d What is the true bearing from Luke's house to Jake's house?

e The bearing in part **d** is known as the back-bearing of the bearing in part **c** (and vice versa). What do you notice about the relationship between these two bearings?

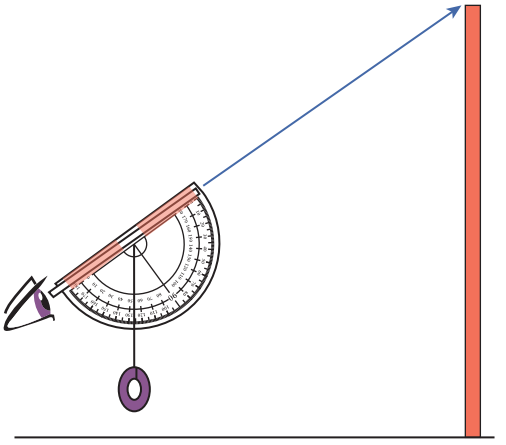
f Draw a scale drawing of Jake's route. From your drawing:

i find the distance between the two houses, correct to one decimal place.

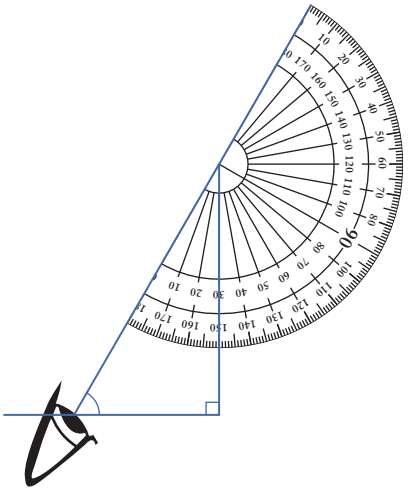
ii use a protractor to measure the bearings in both directions.

g Comment on any differences between your previous answers and those found using your scale drawing.

15 A clinometer (or inclinometer) is a tool used to measure angles of inclination. That is, angles of elevation or depression.
A makeshift clinometer can be constructed using a protractor, a straw, some string and a weight. You can then measure angles of elevation and depression by looking at the object through the straw, reading the angle the string makes with the protractor, then using geometry to determine the angle of elevation or depression.
Determine the angle of elevation or depression each of the following clinometers is measuring.



a



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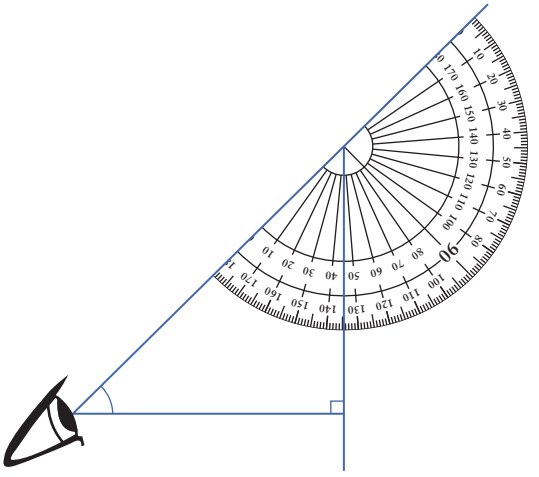

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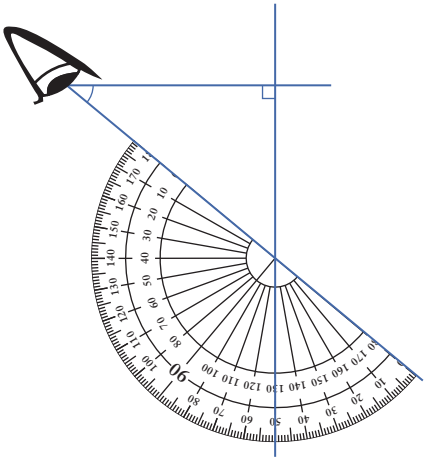

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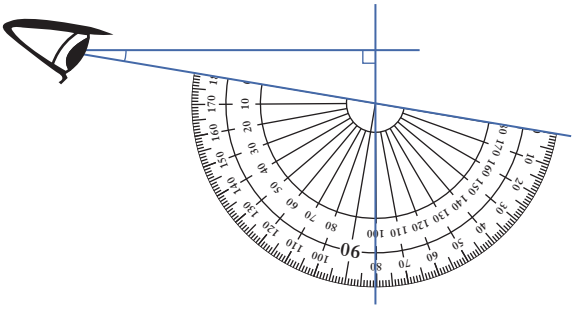


e Describe how you can use the measured angle on the clinometer to determine:

i the angle of elevation **ii** the angle of depression.

16 The members of a bushwalking group have been given instructions for their next hike.

- Start from a meeting place, labelled A, and walk 35 km at a bearing of 055°T to B.
- From B, walk 25 km at a bearing of 015°T to finish the walk at C.

a Draw a sketch of the walk. Try to make it roughly to scale because it can lead to incorrect calculations if the relative position of points is misleading.

b Correct to one decimal place, find how far north:

i B is from A **ii** C is from B **iii** C is from A.

c Correct to one decimal place, find how far east:

i B is from A **ii** C is from B **iii** C is from A.

d Consider the right-angled triangle in which AC is the hypotenuse. Calculate:

i the distance from A to C correct to one decimal place

ii the true bearing from A to C

iii the true bearing from C to A.

e Draw a scale drawing of the bushwalkers' route.

i Measure distances and bearings to obtain measured answers for part **d**.

ii Compare your answers with those you already had for part **d**.

17 A new hike for bushwalkers took them along a different route.

- From the starting point, walk 18 km at a bearing of 300°T.
- Change direction and walk 23 km at a bearing of 200°T to reach your endpoint.

a Find the direct distance from the starting point to the endpoint, correct to one decimal place.

b What is the bearing of the starting point from the endpoint?

18 Jan is standing 2 m in front of a painting on a wall. The bottom of the painting's frame is 1.4 m from the floor, and Jan's eyes are 165 cm from the floor. The painting, including the frame, has a height of 70 cm.

a Draw a diagram to show this information.

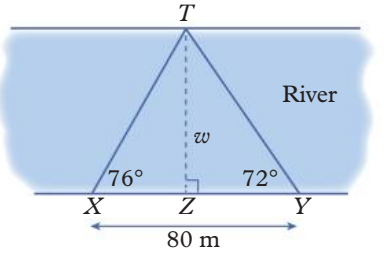
b Find the angle of elevation from Jan's eyes to the top of the painting's frame.

c What is the angle of depression from Jan's eyes to the bottom of the painting's frame?

d If Jan stood 1 m further away from the painting, through what angle would she have to sweep her eyes to look directly at the top of its frame, and then directly at the bottom of its frame?

19 Council surveyors have been set the task of measuring the width of a river along a straight stretch in preparation for building a bridge across it. They have located a tree on the opposite side of the river and have taken angle measurements from two points X and Y, which are 80 m apart, as shown in the diagram on the right.

Calculate the width of the river, correct to the nearest metre. Hint: Let the distance XZ be represented by a pronumeral.



20 A pilot was flying a routine route from A, directly east to B, a distance of 3500 km. She didn't know her compass was not operating correctly and she was actually travelling 10° south of her true direction. When she didn't arrive at her destination after travelling 3500 km, the pilot realised she was off course.

a How much further does she need to travel to fly directly to her destination from this point? Give your answer to the nearest kilometre.

b The pilot only has enough fuel for another 60 km, so she decides to land on a nearby runway 29 km away. The pilot measures the angle of depression from the plane to the runway to be 6° at an altitude of 3 km. The runway is a straight line from north to south and so the pilot must approach the runway in that direction. The bearing from the plane to the runway is 115°T. The pilot wants to travel directly east and then make a sharp right-angled turn to land the plane.

i What is the horizonal distance east before the pilot must make the turn? Give your answer to the nearest kilometre.

ii What is the horizontal distance south from the turn to the runway? Give your answer to the nearest kilometre.

c If the pilot had become aware of the error when she was halfway into the flight and adjusted her direction from that point, would she have made it to her destination?

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheets
Angles of elevation and depression

Interactive skillsheet
Compass bearings and true bearings

Investigation
How do you get to school?

Topic quiz
7C

7D Three-dimensional problems

Learning intentions

By the end of this topic you will be able to ...

✔ find lengths and angles within 3D objects.

Inter-year links

Year 8 — 8F Three-dimensional space

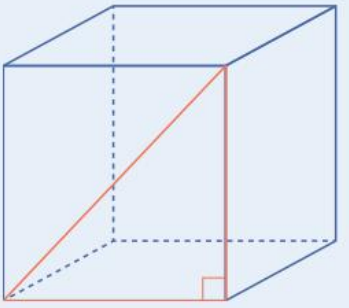
Year 9 — 6F Using trigonometry to find angles

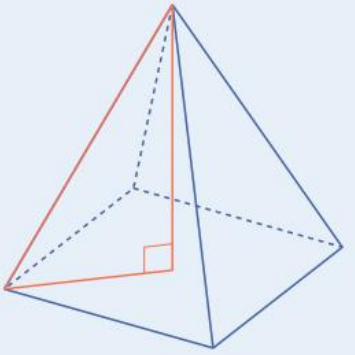
Three-dimensional problems

- Just as 2D shapes can contain many right-angled triangles, there are also many right-angled triangles to be found in 3D objects.

  Some right-angled triangles can be found on the surface of 3D objects, such as the triangle that has as its hypotenuse the diagonal joining vertex to vertex across the face of a cube.

  Other right-angled triangles can be found within a shape, such as the triangle shown in the diagram of the pyramid on the right. That triangle is formed by a slanting edge of the pyramid (the hypotenuse), a perpendicular line from the base of the square pyramid to the apex of the pyramid, and a line on the base joining the perpendicular to the slanting edge.
- Pythagoras' theorem can be used to find an unknown length, either on the surface of a 3D object or within a 3D object.
- Trigonometry can be used to find an unknown angle or length, either on the surface of a 3D object or within a 3D object.





Example 7D.1 Three-dimensional problems

A 20 cm long box has two square faces with 12 cm sides.

a Calculate the length of the diagonal across each of the square faces of the box, to the nearest centimetre.

b Find the length of the diagonal within the box, to the nearest centimetre.

c Find the angle the internal diagonal from part **b** makes with the diagonal across the square face, to the nearest degree.

THINK

a Substitute the side lengths of the square into Pythagoras' theorem to find the length of the diagonal across the square face.

WRITE

a $c^2 = a^2 + b^2$

$= 12^2 + 12^2$

$= 12^2 \times 2$

$c = 12\sqrt{2}$

≈ 17 cm

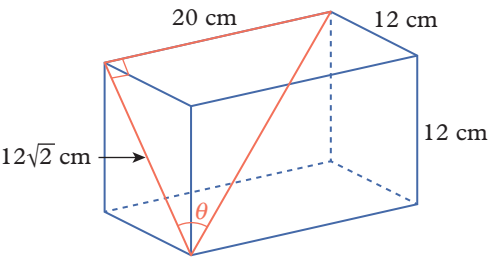
b **1** Draw a labelled diagram of the box. Identify the right-angled triangle formed by the two diagonals.

2 Substitute the side lengths of the two short sides of the right-angled triangle into Pythagoras' theorem to find the length of the diagonal within the box (the hypotenuse).

c **1** Choose a trigonometric ratio to use for calculating the angle. Note that all the sides are now known, so any trigonometric ratio can be used.

2 Solve for the angle using the inverse trigonometric function.

b



$c^2 = a^2 + b^2$

$= (12\sqrt{2})^2 + (20)^2$

$= 288 + 400$

$= 688$

$c = \sqrt{688}$

≈ 26 cm

c $\tan(\theta) = \frac{O}{A}$

$\tan(\theta) = \frac{20}{12\sqrt{2}}$

$\theta = \tan^{-1}\left(\frac{20}{12\sqrt{2}}\right)$

$\approx 50°$

Helpful hints

- ✔ It always helps to draw a diagram and label the sides and angles with the known and unknown values. It does not need to be exactly drawn to scale.
- ✔ Remember, the trick is to find right-angled triangles! This can often be done by:
  - → drawing a line across the diagonal of a rectangle
  - → drawing a line through any triangle from one vertex to the opposite side, to meet that side at a right angle.
- ✔ When being asked to round to a given number of decimal places, the rounding should only take place at the final stage of your work and not in any intermediate steps. For example, rounding the size of angles in the middle of your work can lead to considerable differences in the values of trigonometric ratios.

ANS p621

Exercise 7D Three-dimensional problems

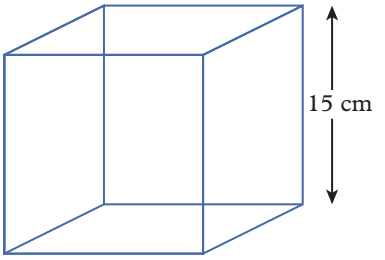
△ 1–5, 7, 8, 10, 11 ▢ 1–4, 6, 9, 10, 12, 13, 15 ⬠ 3, 6, 9, 10, 12–14, 16, 17

7D.1 **1** The side lengths of a cube are 15 cm.

a Calculate the length of the diagonal of the base of the cube correct to two decimal places.

b Find the length of the diagonal within the cube correct to two decimal places.

c Find the angle the internal diagonal makes with the base diagonal correct to two decimal places.



2 A box has a base 20 cm by 15 cm and is 10 cm tall.

a Provide working to show whether a 30 cm rod will fit in the box.

b If a rod was a snug fit, correct to two decimal places, what angle would it make with the base diagonal of the box?

3 A solid cube of side length 5 cm is sliced along its base diagonal.

a What is the shape of the exposed face? Draw a diagram with its dimensions labelled.

b Compare the size of the exposed face with the size of one of the faces of the cube.

4 A piece of cheese has the shape of a triangular prism. Its base is an isosceles triangle with side lengths of 10 cm, 10 cm and 4 cm, and the prism's height is 5 cm.

a Draw a labelled diagram of the triangular prism, showing all dimensions.

b Draw a labelled diagram of:

i the front view of the largest side

ii the top view (or base view).

c On your diagram of the front view, draw a diagonal.

i Find the length of the diagonal you drew correct to one decimal place.

ii What angle does the diagonal make with the longer side, correct to two decimal places?

iii Without using trigonometry, explain how you could find the angle the diagonal makes with the shorter side.

d Calculate, correct to one decimal place:

i the perpendicular height of the base triangle

ii the area of the base.

e The cheese is sliced along the height of the base triangle.

i Draw the shape of one of the exposed faces. Label its dimensions.

ii Find the area exposed on both sides of the slice.

iii What is the diagonal length of the exposed face, correct to the nearest centimetre?

5 Jack was given a tent for Christmas. He wants to make a cover the same size as the tent to protect it when it rains. The front of the tent forms an equilateral triangle, with a height of 1.5 m.

a Sketch the front of the tent. Label the height 1.5 m, and let the side lengths be x m. Highlight a right-angled triangle, and write the length of its base in terms of x.

b Use Pythagoras' theorem to find the value of x correct to one decimal place.

c Jack has bought a rectangular tarpaulin sheet with one side long enough to fit the length of the tent from front to back. If the other side length is 3.5 m, explain whether this is sufficient to reach the ground from one side of the tent to the other.

6 The rectangular perimeter of a football field is lined with trees. Ethan is standing 70 m in front of one of the trees. He notices a much taller tree 37° to his left.

a Draw a sketch showing the relative position of the two trees described. Label the known values.

b Find the distance along the ground from Ethan to the base of the taller tree correct to one decimal place.

c He measures the angle of elevation to the top of the taller tree as 13°. Draw a sketch showing the elevation from Ethan to the top of the tall tree. Label the known values.

d Use this diagram to find the height of the tall tree correct to one decimal place.

7 The Pyramid of Giza is the largest stone monument in the world. At one stage, it was also the tallest.

The base of the pyramid is a square with side lengths of 230 m. It stands 140 m tall.

a Draw a labelled sketch of the pyramid.

b Calculate the diagonal length of the base correct to two decimal places.

c The apex of the pyramid lies directly over the midpoint of the base diagonal. Use the vertical right-angled triangle to calculate the angle that the sloping edge makes with the base diagonal, correct to two decimal places.

d A different right-angled triangle can be used to find the angle the sloping sides make with the base by measuring from the midpoint of one of the base sides.

i Draw a labelled diagram to display the dimensions of this triangle.

ii Calculate the angle the sloping side makes with the base, correct to two decimal places.

8 Cooks sometimes find that if they leave their stirring spoon unattended, it falls into the saucepan.

This saucepan is in the shape of a cylinder open at one end, with a circumference of 60 cm and a height of 12 cm.

a Find the diameter of the saucepan correct to one decimal place.

b A stirring spoon is 20 cm in length. Explain whether it could fall below the rim of the saucepan.

9 An ice cream cone has a height of 10 cm. The circumference of the circular top of the cone is 15.5 cm.

a Find the radius of the circular top of the cone correct to one decimal place.

b Draw a labelled two-dimensional sketch of the outline of the cone. Define any ight angles within the shape.

c Find the slant length of the cone correct to one decimal place.

10 A Christmas tree in the shape of a cone is 3 m tall with a base diameter of 2 m. Sandy wants to put streamers from the top to the base of the tree, along the outer surface of the tree.

She has one 30 m long roll of streamer paper that she can cut up to do this. How many streamers the full length of the slant of the tree is she able to attach?

11 A regular tetrahedron is a three-dimensional object with four equilateral triangular faces.

a Identify the tetrahedron in the photo below and draw a sketch of it.

b Suppose that the length of each side of the tetrahedron you drew is 10 cm and its height from the base to the upper apex is 8.2 cm. Add these measurements to your sketch from part **a**.

c **i** Draw a diagram of a right-angled triangle with corresponding measurements from the tetrahedron. Label one side of the triangle with the height of the tetrahedron from part **a** and label the hypotenuse with the length of one edge of the tetrahedron.

ii Use your diagram to help you calculate the angle between the edge and the base of the tetrahedron correct to the nearest degree.

d Draw the shape of a face of the tetrahedron. Draw the perpendicular height, and label all known values.

i What is the perpendicular height of the face of the tetrahedron correct to one decimal place?

ii Draw a right-angled triangle showing the perpendicular height of the face and the height of the tetrahedron. Use your diagram to calculate the angle between the base and a face of the tetrahedron correct to the nearest degree.

12 A tower stands at point T on flat ground. From point X on the ground, the angle of elevation to the top of the tower, point A is 54°. At point Y on the ground, in the same straight line as XT, and 25 m beyond X, the angle of elevation to the top of the tower is 28°.

a Draw a labelled diagram to display the given information. Label AT, the height of the tower, as h, and the distance XT along the ground as d.

b Use a trigonometric ratio with $\triangle AXT$ to find an equation for h in terms of d.

c Use a trigonometric ratio with $\triangle AYT$ to find an equation for h in terms of d. Note that the length of YT is $(25 + d)$ m.

d Equate your two equations from parts **b** and **c**. Solve to find a value for d correct to one decimal place.

e Use the value you found for d in part **d** with $\triangle AXT$ to find the height of the tower correct to one decimal place.

13 A rectangular prism has width, w, length, l and height, h.

a Write expressions for the lengths of the diagonals along each of the faces.

b Write an expression for the length of the diagonal of the rectangular prism from one vertex to the vertex it is furthest away from.

14 A rectangular prism has a base with length, l, twice its width, w. The height of the prism, h, is three times its width.

a Write the length and height of the prism in terms of its width, w.

b Draw a labelled diagram of the prism.

c Find the length of the base diagonal in terms of w.

d Calculate the length of the diagonal within the prism in terms of w.

e Find the angle that the internal diagonal of the prism makes with the diagonal of the base. Give your answer to the nearest degree.

15 The length of the diagonal within a cube is 6.2 cm. What is the side length of the cube?

16 Consider a cube of side length x cm.

a Draw a diagram of the cube.

b An ant crawls from the top front left-hand corner to the bottom back right-hand corner along the outer surface of the cube.

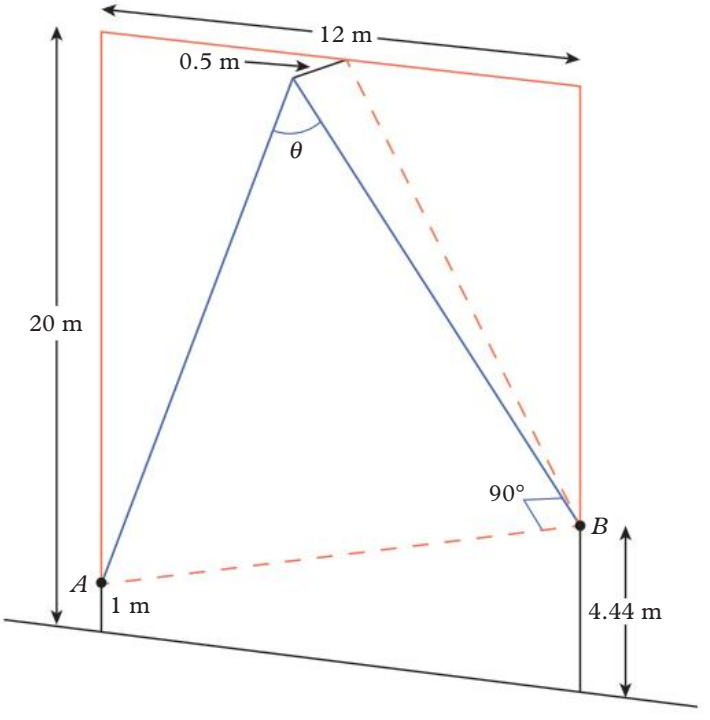
i Draw a labelled net of the cube and label the ant's starting and finishing points. Mark the shortest path the ant could take.

ii Find the length of the shortest path from your answer to part **i**. Give your answer in terms of the side length x.

17 Yolandi goes rock-climbing at an indoor rock-climbing gym. She starts by climbing a rectangular practice wall that is 12 m wide and 20 m high. A belay rope is suspended from the ceiling by a pulley and attached to Yolandi's waist to stop her from falling too far. The pulley is positioned at the centre, 0.5 m away from the wall. Yolandi starts from the left side of the wall with the rope tied to her waist at point A, 1 m above the floor. She then climbs directly to point B, 4.44 m high on the right side of the wall. She then continues directly to the top of the wall at its centre. Given that the angle the rope makes with the line from point A to point B is a right angle, correct to one decimal place, determine:

a the distance Yolandi climbs in total

b the angle, θ, that the rope turns from when Yolandi is on the left side of the wall to when Yolandi is at the right side of the wall.



CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Finding lengths within 3D objects

Interactive skillsheet
Finding angles within 3D objects

Topic quiz
7D

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

7A **1** Determine whether the following lengths can form a right-angled triangle.

a 9 cm, 40 cm, 41 cm

b 39 cm, 89 cm, 80 cm

c 137 cm, 104 cm, 89 cm

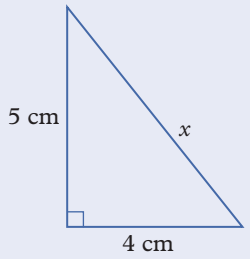
d 21 cm, 221 cm, 220 cm

7A **2** Determine the value of x in each of the following:

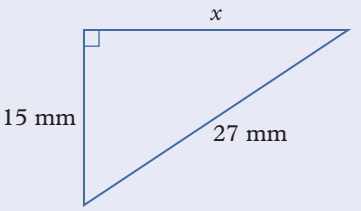
i as a simplified exact value

ii correct to two decimal places.

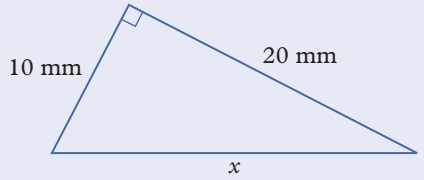
a



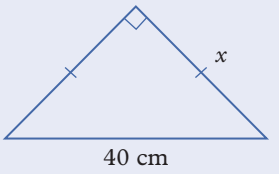
b



c

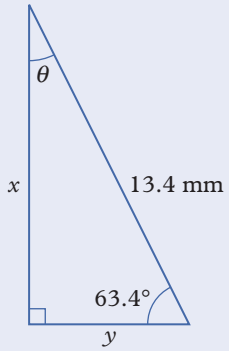


d

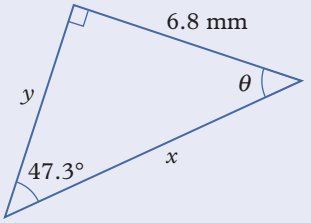


7B **3** Determine the value of each of the pronumerals correct to one decimal place.

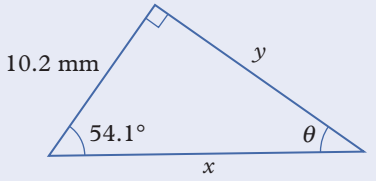
a



b

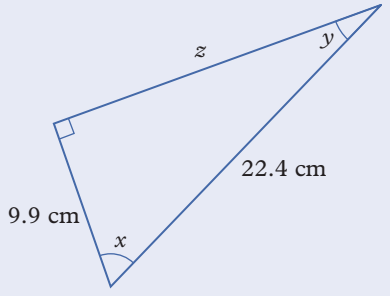


c

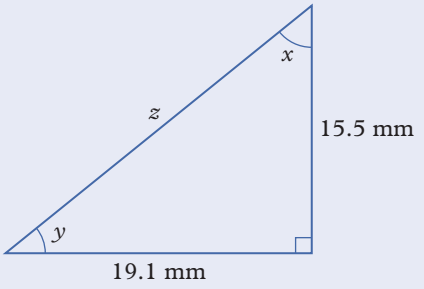


7B **4** Determine the value of each of the pronumerals correct to one decimal place.

a



b

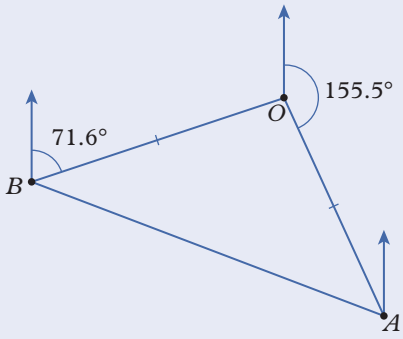


7C **5** Determine the true bearing of B from A in each of the following.

a The bearing of B from A is S85°E.

b The bearing of A from B is N35°W.

c



7C **6** Calculate each of the following correct to one decimal place.

a The angle of elevation of the top of a large tree from a point on the ground 6 m away is 78°. Determine the height of the tree.

b A building is 120 m tall. Another building is 105 m tall. Determine the angle of depression from the top of the taller building to the top of the shorter building if the tops of the buildings are 30 m apart.

c The angle of depression of an overboard passenger from the deck of a ship is 24°. A 12 m rope pulled taut just reaches from the deck of the ship to the overboard passenger in the water. Determine how far, in total, the overboard passenger needs to be pulled towards the boat through the water and then up the side of the ship to get them back onto the deck of the boat.

d A child holds the 8 m string of a kite 130 cm from the ground. If the angle of elevation of the kite is 65°, how high above the ground is the kite flying?

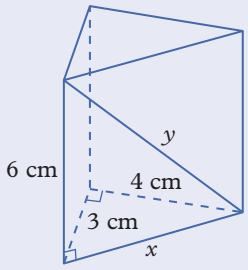
7C **7** Calculate each of the following distances correct to one decimal place and bearings correct to the nearest degree.

a A snorkeller swims to a point 230 m at a bearing of 160° T from their boat. Determine how far south and east they are from the boat.

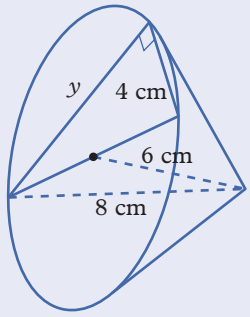
b A farmer walks 1.2 km east and then 800 m north along two sides of a paddock. Determine the true bearing of the location the farmer ended at from their starting point.

7D **8** Determine the value of each of the pronumerals correct to one decimal place.

a

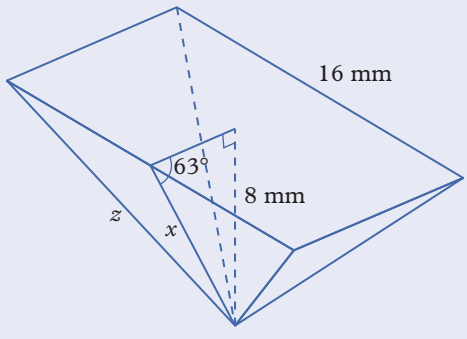


b

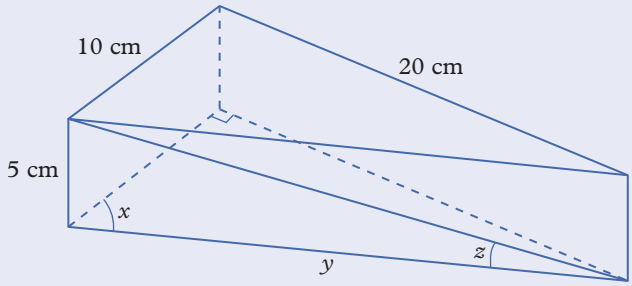


7D **9** Determine the value of each of the pronumerals correct to one decimal place.

a



b



7E The unit circle

Learning intentions

By the end of this topic you will be able to ...

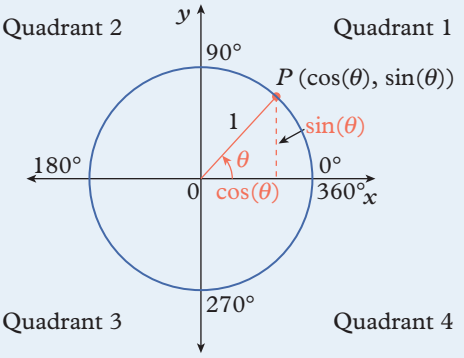
- ✔ determine the quadrant in which an angle lies
- ✔ determine the sign of a trigonometric expression given an angle
- ✔ find the reference angle in the first quadrant of any angle using symmetry properties.

Inter-year links

Year 9 6D Trigonometric ratios

The unit circle

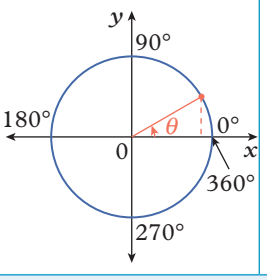
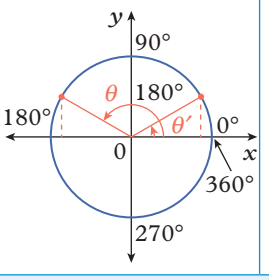
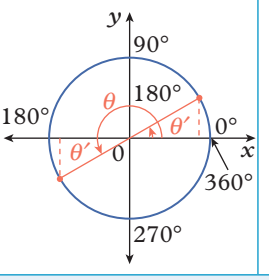
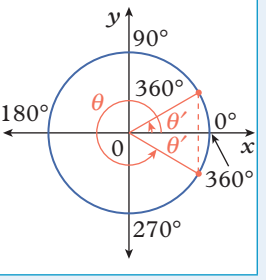
- The **unit circle** is a circle on the Cartesian plane, having its centre at the origin and a radius of 1 unit. The quadrants on the Cartesian plane are labelled in an anti-clockwise direction from the positive x-axis.



- If θ is the angle measured anti-clockwise from the positive x-axis to the radius that meets the unit circle at point P, then:
  - → The x-coordinate of the point on the unit circle is the cosine of θ.
  - → The y-coordinate of the point on the unit circle is the sine of θ.
  - → The gradient of the line from the origin to the point on the unit circle is the tangent of θ.

The reference angle

- The sine and cosine values for any angle θ can be related by symmetry to an acute reference angle θ' in quadrant 1.
- The **reference angle** is the acute angle between the x-axis and the line connecting the origin to the point on the unit circle.

	Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
Angle (θ)	$0° < \theta < 90°$	$90° < \theta < 180°$	$180° < \theta < 270°$	$270° < \theta < 360°$
Reference angle (θ')	$\theta' = \theta$ 	$\theta' = 180° - \theta$ 	$\theta' = \theta - 180°$ 	$\theta' = 360° - \theta$ 

Example 7E.1 Determining the quadrant in which an angle lies

Identify the quadrant in which each of these angles lies.

a 34° **b** 122° **c** 230° **d** 325°

THINK

a Sketch the unit circle and mark the angle 34°. Because the size of the angle is between 0° and 90°, the angle is in quadrant 1.

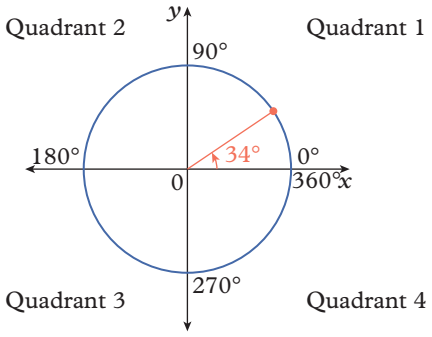
b Sketch the unit circle and mark the angle 122°. Because the size of the angle is between 90° and 180°, the angle is in quadrant 2.

c Sketch the unit circle and mark the angle 230°. Because the size of the angle is between 180° and 270°, the angle is in quadrant 3.

d Sketch the unit circle and mark the angle 325°. Because the size of the angle is between 270° and 360°, the angle is in quadrant 4.

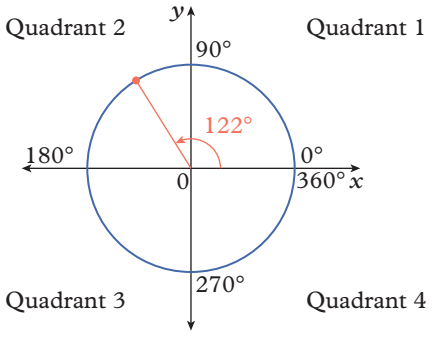
WRITE

a



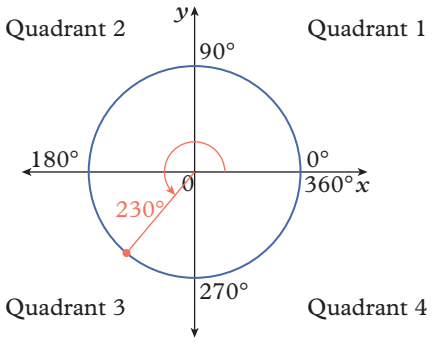
34° is in quadrant 1.

b



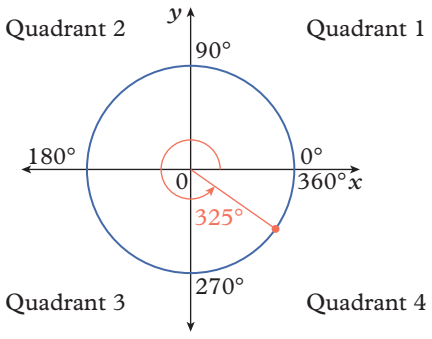
122° is in quadrant 2.

c



230° is in quadrant 3.

d



325° is in quadrant 4.

Example 7E.2 Determining the sign of a trigonometric expression

Identify the sign (positive or negative) of each of these trigonometric expressions.

a cos(55°) **b** tan(103°) **c** sin(230°) **d** tan(349°)

THINK

a Sketch the unit circle and mark the angle 55°. The angle is in quadrant 1, and the cosine value is positive in quadrant 1 because the x-coordinate of a point in quadrant 1 is positive.

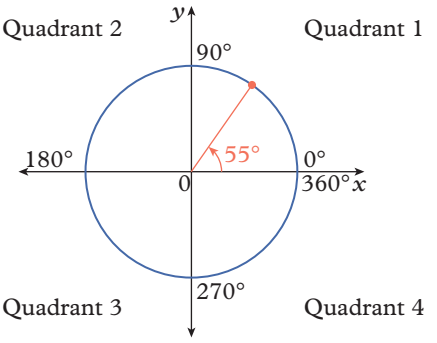
b Sketch the unit circle and mark the angle 103°. The angle is in quadrant 2, and the tangent value is negative in quadrant 2 because the gradient of the line from the origin to a point in quadrant 2 is negative.

c Sketch the unit circle and mark the angle 230°. The angle is in quadrant 3, and the sine value is negative in quadrant 3 because the y-coordinate of a point in quadrant 3 is negative.

d Sketch the unit circle and mark the angle 349°. The angle is in quadrant 4, and the tangent value is negative in quadrant 4 because the gradient of a line from the origin to a point in quadrant 4 is negative.

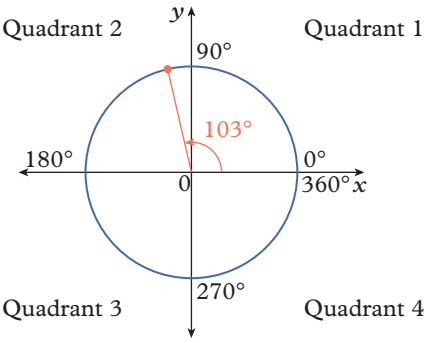
WRITE

a



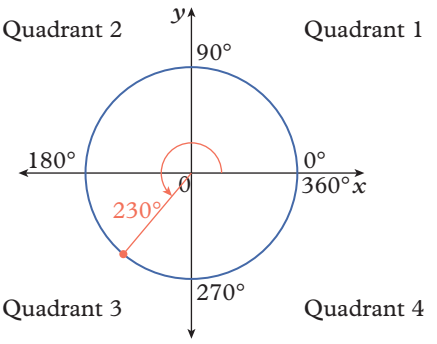
cos(55°) is positive.

b



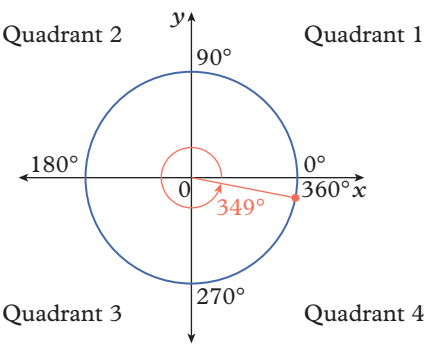
tan(103°) is negative.

c



sin(230°) is negative.

d



tan(349°) is negative.

Example 7E.3 Determining the reference angle

Write each trigonometric expression in terms of the reference angle using the correct sign.

a sin(255°) **b** cos(317°) **c** tan(145°)

THINK

a **1** Sketch the unit circle and mark the angle 255°.

2 The angle is in quadrant 3, so the reference angle is $\theta - 180°$.

3 sin(255°) is negative because the y-coordinate of a point in quadrant 3 is negative.

b **1** Sketch the unit circle and mark the angle 317°.

2 The angle is in quadrant 4, so the reference angle is $360° - \theta$.

3 cos(317°) is positive because the x-coordinate of a point in quadrant 4 is positive.

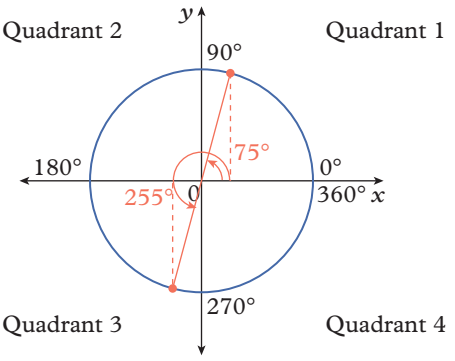
c **1** Sketch the unit circle and mark the angle 145°.

2 The angle is in quadrant 2, so the reference angle is $180° - \theta$.

3 tan(145°) is negative because the gradient of a line from the origin to any point in quadrant 2 is negative.

WRITE

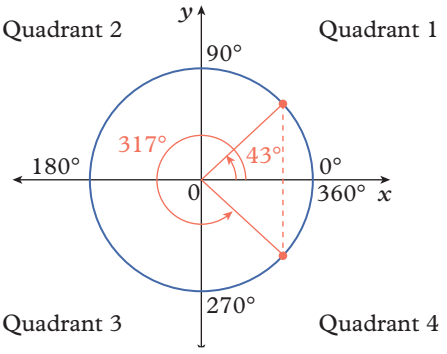
a



$255° - 180° = 75°$

$\sin(255°) = -\sin(75°)$

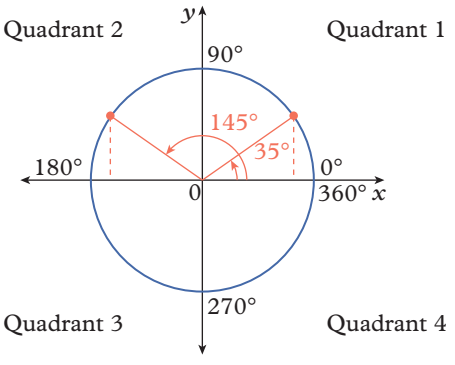
b



$360° - 317° = 43°$

$\cos(317°) = \cos(43°)$

c

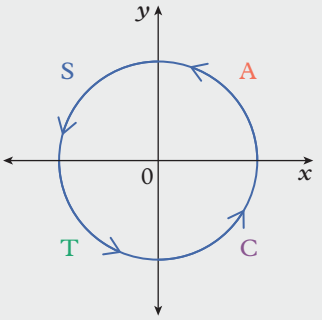


$180° - 145° = 35°$

$\tan(145°) = -\tan(35°)$

Helpful hints

- ✔ The sine, cosine and tangent values have different signs (positive or negative) depending on the quadrant in which they lie.
  - Sine is positive where the y-coordinate of the point on the unit circle is positive (quadrants 1 and 2), and negative where the y-coordinate is negative (quadrants 3 and 4).
  - Cosine is positive where the x-coordinate of the point on the unit circle is positive (quadrants 1 and 4), and negative where the x-coordinate is negative (quadrants 2 and 3).
  - Tangent is positive where the gradient from the origin to the point on the unit circle is positive (quadrants 1 and 3), and negative where the gradient is negative (quadrants 2 and 4).
- ✔ An easy way to remember the sign of the trigonometric functions in the four quadrants is with a mnemonic. Something like **A**ll **S**tations **T**o **C**entral is good:



All Stations To Central
L I A O
L N N S

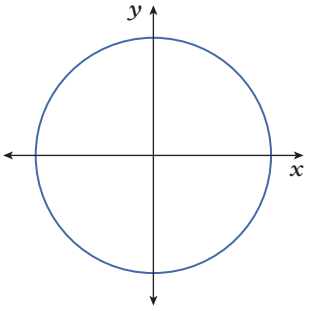
ANS p623

Exercise 7E The unit circle

1–3, 4(a–j), 5, 6, 8, 10 | 3, 4(f–o), 5, 6, 8, 9, 11, 13 | 3, 5–9, 12–14

UNDERSTANDING AND FLUENCY

1 For each of the following, shade the appropriate regions of a unit circle diagram.



- **a** Shade the regions where the sine of angles in that region is positive.
- **b** Shade the regions where the cosine of angles in that region is positive.
- **c** Shade the regions where the tangent of angles in that region is positive.
- **d** Shade the regions where the sine of angles in that region is negative.
- **e** Shade the regions where the cosine of angles in that region is negative.
- **f** Shade the regions where the tangent of angles in that region is negative.

7E.1 **2** Identify the quadrant in which each of these angles lies.

a 345° **b** 103° **c** 204°
d 265° **e** 74° **f** 139°

7E.2 **3** Identify the sign (positive or negative) of each of these trigonometric expressions.

a sin(100°) **b** cos(200°) **c** tan(140°) **d** cos(295°) **e** tan(260°)
f sin(340°) **g** cos(94°) **h** sin(130°) **i** cos(190°) **j** tan(350°)

4 Write the reference angle for each of the following.

a 140° **b** 92° **c** 200° **d** 190° **e** 302°
f 315° **g** 260° **h** 100° **i** 280° **j** 150°
k 345° **l** 170° **m** 220° **n** 255° **o** 299°

7E.3 **5** Express each of the following trigonometric expressions in terms of an appropriate reference angle. Hint: remember to consider whether the sign of each expression is positive or negative.

a sin(241°) **b** cos(304°) **c** tan(95°) **d** sin(164°) **e** cos(110°)

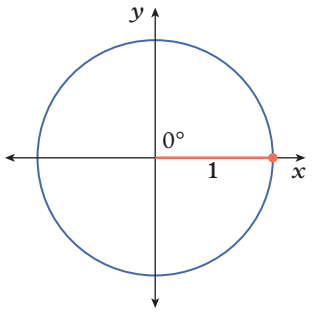
f cos(200°) **g** tan(200°) **h** tan(358°) **i** sin(285°) **j** sin(175°)

6 The unit circle can be used to evaluate the sine and cosine of angles that are multiples of 90° (including 0°) by considering the coordinates of points on the unit circle. For each of the following identify:

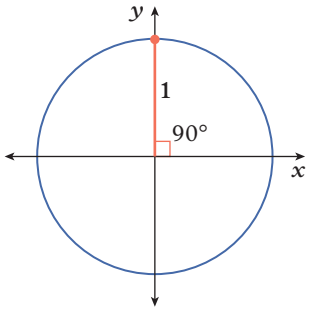
i the cosine of the marked angle (the x-coordinate of the point shown).

ii the sine of the marked angle (the y-coordinate of the point shown).

a



b



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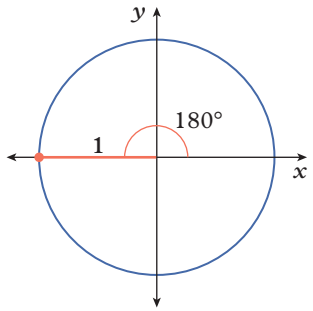

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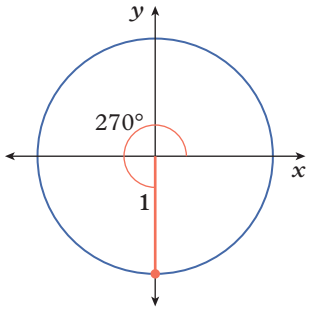
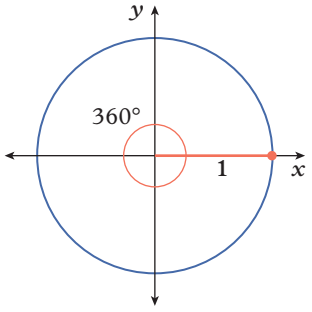


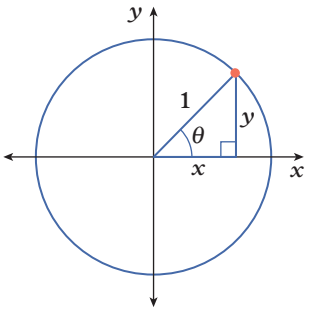
e



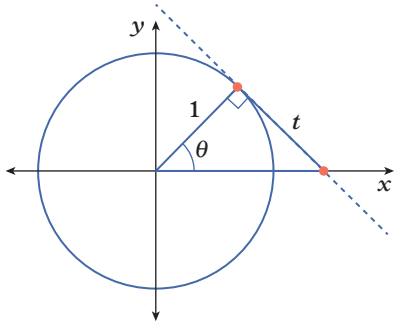
7 The tangent of an angle using the unit circle can be defined in different ways.

a State the tangent of θ in the right-angled triangles shown.

i



ii



b Explain how the definition from part **a i** can be described as 'the tangent of the angle is equal to the gradient of the line from the origin to the point on the unit circle'.

c The lengths x and y also refer to the coordinate of the point where the radius touches the circle. Write the expression of the tangent of θ from part **a i** in terms of sine and cosine.

d Use your expression from part **c** to write the following in terms of tangent.

i $\frac{\sin(20°)}{\cos(20°)}$ **ii** $\frac{\sin(152°)}{\cos(152°)}$ **iii** $\frac{\cos(218°)}{\sin(218°)}$ **iv** $\frac{\sin(347°)}{\cos(347°)}$

e The unit circle can also be used to evaluate the tangent of angles that are multiples of 90° (including 0°) by considering the gradients of a line from the origin to the point on the unit circle. Since the gradient is either horizontal or vertical at these angles, the tangent value is either zero or undefined, respectively. For each of the following state the tangent of the marked angle using the gradient of the radius shown in question **6**.

i tan(0°) **ii** tan(90°) **iii** tan(180°) **iv** tan(270°) **v** tan(360°)

8 Angles greater than 360° are coterminal with angles between 0° and 360°, as they terminate or end at the same point on the unit circle. That is, every angle greater than 360° corresponds with an angle between 0° and 360° as they point in the same direction. To determine a coterminal angle, add or subtract 360° from the given angle. Write each trigonometric expression in terms of:

i the coterminal angle between 0° and 360°

ii the reference angle using the correct sign, or its value if it is a multiple of 90°.

a $\sin(400°)$ **b** $\cos(530°)$ **c** $\tan(800°)$ **d** $\sin(630°)$

e $\cos(777°)$ **f** $\cos(1024°)$ **g** $\tan(2000°)$ **h** $\cos(900°)$

9 Using the unit circle, we can also consider negative angles as angles measured clockwise from the positive x-axis. Negative angles are also coterminal with angles between 0° and 360°. Write each trigonometric expression in terms of:

i the coterminal angle between 0° and 360°

ii the reference angle using the correct sign or its value if it is a multiple of 90°.

a $\sin(-45°)$ **b** $\cos(-225°)$ **c** $\tan(-135°)$ **d** $\sin(-315°)$

e $\cos(-508°)$ **f** $\tan(-436°)$ **g** $\cos(-630°)$ **h** $\sin(-1000°)$

10 Complete the following with different angles between 0° and 360°.

a $\sin(158°) = \sin(___°) = -\sin(___°) = -\sin(___°)$

b $\cos(213°) = \cos(___°) = -\cos(___°) = -\cos(___°)$

c $\tan(285°) = \tan(___°) = -\tan(___°) = -\tan(___°)$

11 Consider the following three equalities.

$\sin(234°) = -\sin(126°)$ $\cos(234°) = \cos(126°)$ $\tan(234°) = -\tan(126°)$

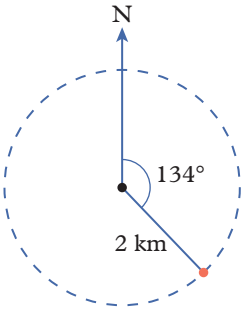
a Use a calculator to confirm that the pairs of trigonometric expressions are equal.

b Evaluate 234°+126°.

c Explain how these equalities relate to the fourth quadrant.

12 True bearings measure angles from north (the positive y-axis) clockwise, rather than from the positive x-axis, like angles anti-clockwise on the unit circle. The distance travelled along the bearing will scale the coordinates on the circle where that distance is the scale factor. If the centre point of the circle is the origin, state the coordinate of the points shown in terms of trigonometric expressions containing a reference angle.

a



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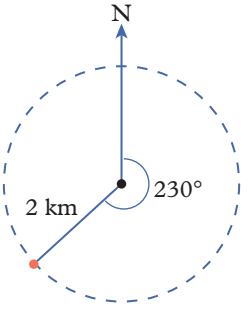

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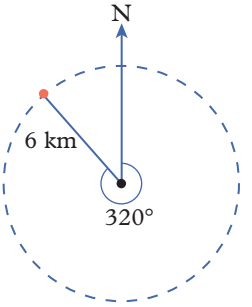

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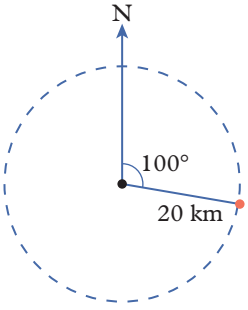

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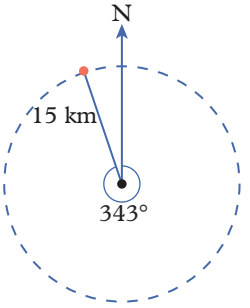
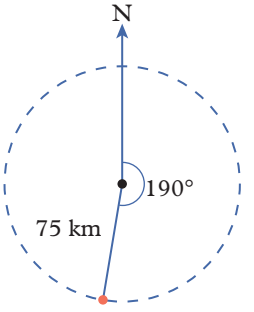
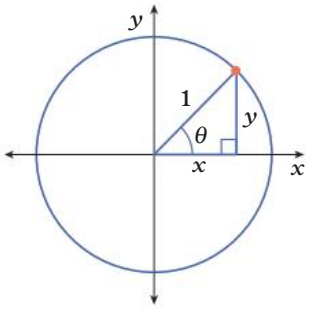


f



13 Using the diagram shown, we can write the equation $x^2 + y^2 = 1$ using Pythagoras' theorem.



a Write the equation in terms of sine and cosine of θ.

b Rearrange the equation in part **a** to make:

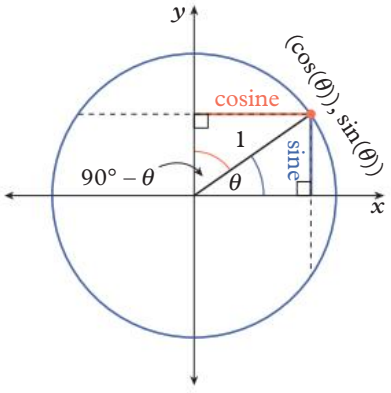
i $\sin(\theta)$ the subject **ii** $\cos(\theta)$ the subject.

c We can use the equation in part **a** (and its rearrangements in part **b**) to determine the cosine value of an angle if we know the sine of the angle and vice versa. The sign of the solution can be determined by considering which quadrant the angle lies in. Use the sine and cosine values provided to determine the value of the trigonometric expression in brackets correct to four decimal places.

i $\sin(50°) = 0.7660$ $(\cos(50°))$ **ii** $\cos(116°) = -0.4384$ $(\sin(116°))$

iii $\sin(177°) = 0.0523$ $(\cos(177°))$ **iv** $\cos(240°) = -0.5$ $(\sin(240°))$

14 The name 'cosine' comes from an abbreviation of the phrase 'the sine of the complementary angle'. That is, $\sin(90° - \theta) = \cos(\theta)$. We can use this definition to turn sine expressions into cosine expressions of the complementary angle and vice versa. Write the following sine expressions using cosine, and the following cosine expressions using sine.



a $\sin(30°)$ **b** $\sin(52°)$ **c** $\cos(24°)$ **d** $\cos(45°)$

This is true even when θ is not an acute angle. Write the following sine expressions using cosine, and cosine expressions using sine, using:

i the angle from the subtraction $90° - \theta$

ii the coterminal angle between 0° and 360° if applicable

iii the reference angle using the correct sign or its value if it is a multiple of 90° if applicable.

e $\sin(158°)$ **f** $\cos(351°)$ **g** $\sin(211°)$

h $\sin(-15°)$ **i** $\cos(482°)$ **j** $\cos(-555°)$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
The sign of trigonometric expressions

Interactive skillsheet
Reference angles

Topic quiz
7E

7F Exact values

Learning intentions

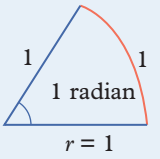
By the end of this topic you will be able to ...

- ✔ convert between degrees and radians
- ✔ determine the exact values of the sine, cosine and tangent functions
- ✔ identify points on the unit circle via angles and arc lengths in radians.

Inter-year links

Year 9 6D Trigonometric ratios

Degrees and radians



- A **radian** is a unit for measuring angles that is related to circles. Radians are the ratio between the arc length and the radius of a circle such that 1 radian is the angle subtended by the arc length that is one radius long.
- There are 2π radians in one revolution of a circle. When written in terms of π, radians describe angles as fractions of a full turn, 2π.

  For example, $\frac{2\pi}{5}$ is a fifth of a full turn and $\frac{3\pi}{4} = \frac{3(2\pi)}{8}$ is three-eighths of a full turn.
- 180° is equivalent to π radians and 1 radian is exactly $\frac{180°}{\pi}$, or approximately 57.3°. To convert from degrees to radians, we multiply by $\frac{\pi}{180°}$. To convert from radians to degrees, we multiply by $\frac{180°}{\pi}$.

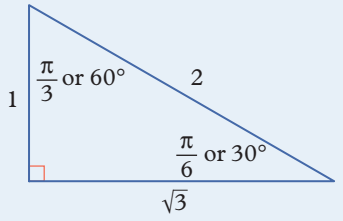
Exact values of trigonometric ratios

- The table below can be used to find exact values of the trigonometric ratios for some special angles.

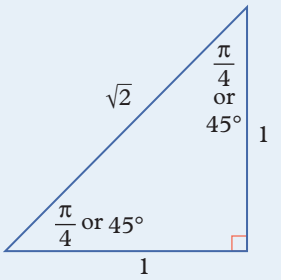
θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0° or 0	0	1	0
30° or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
45° or $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
60° or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90° or $\frac{\pi}{2}$	1	0	undefined

- There are two special triangles whose side lengths, when trigonometry is applied, provide the exact values of the trigonometric ratios for the three angles specified in the table above. We can use geometry and Pythagoras' theorem to determine all angles and lengths of the two triangles.

half-equilateral triangle



half-square triangle

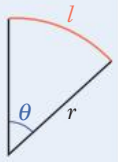


Arc length

- The circumference of the unit circle is 2π.
- The length of an arc, l, can be calculated by multiplying the angle θ (in radians) it subtends at the centre of the circle by the circle's radius, r.
- Since the unit circle has a radius of 1, any arc length on the circumference of the unit circle is equal to the radian measure of the angle θ subtending the arc.

$$l = r\theta$$

arc length | radius | angle in radians



Example 7F.1 Converting between radians and degrees

Convert each of the following to the unit indicated in the brackets.

a 110° (convert to radians)

b $\frac{4\pi}{3}$ radians (convert to degrees)

THINK

a To convert from degrees to radians, multiply by $\frac{\pi}{180°}$.

b To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.

WRITE

a $110° \times \frac{\pi}{180°} = \frac{110^{\sigma 11}}{} \times \frac{\pi}{180^{\sigma 18}}$

$= \frac{11\pi}{18}$ radians

b $\frac{4\pi}{3} \times \frac{180°}{\pi} = \frac{4\pi^{4}}{3^{1}} \times \frac{180^{\sigma 60°}}{\pi^{1}}$

$= \frac{4 \times 60°}{1}$

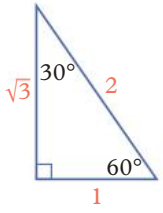
$= 240°$

Example 7F.2 Finding the exact values of a trigonometric ratio

Use the unit circle and exact values for the trigonometric ratios to evaluate: sin(150°), cos(150°), tan(150°)

THINK

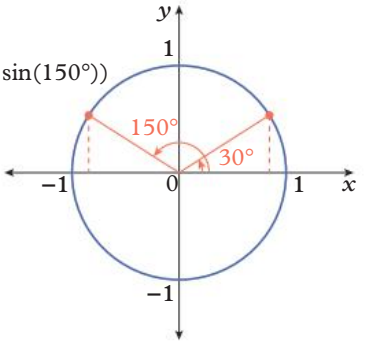
1 Sketch the unit circle and mark the angle 150°. It is in quadrant 2, so the reference angle is 180° − 150° = 30°.



2 Calculate sin(30°), cos(30°) and tan(30°) using the side lengths of the special triangle above.

3 Apply the correct signs to the trigonometric ratios for quadrant 2.

WRITE



$\sin(30°) = \frac{1}{2}$, $\cos(30°) = \frac{\sqrt{3}}{2}$, $\tan(30°) = \frac{\sqrt{3}}{3}$

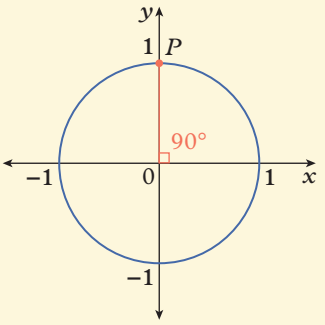
In quadrant 2, cosine and tangent are both negative, but sine is positive.

$\sin(150°) = \frac{1}{2}$, $\cos(150°) = -\frac{\sqrt{3}}{2}$, $\tan(150°) = -\frac{\sqrt{3}}{3}$

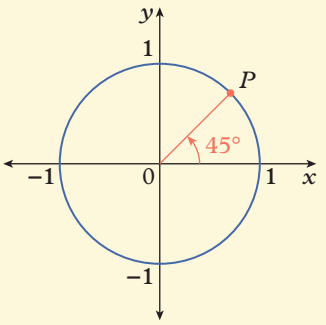
Example 7F.3 Identifying points on the unit circle

Identify the coordinates of the point P in each case below.

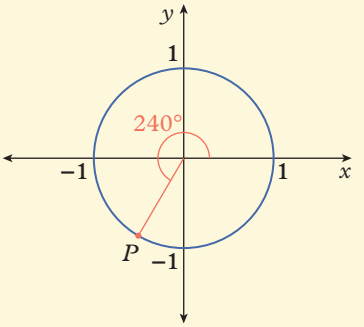
a



b



c



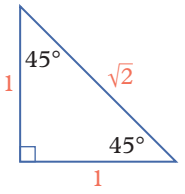
THINK

a **1** Any point on the unit circle has an x-coordinate of $\cos(\theta)$ and a y-coordinate of $\sin(\theta)$. Write P in coordinate form using $\theta = 90°$.

2 Identify the exact values of the trigonometric ratios. Remember that the unit circle has a radius of 1.

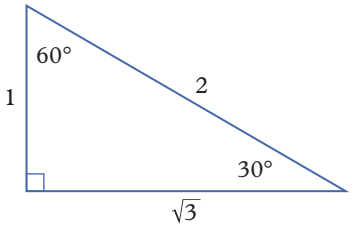
b **1** Write P in coordinate form using $\theta = 45°$.

2 Identify the exact values of the trigonometric ratios. Use the half-square triangle to determine the values.



c **1** Write P in coordinate form using $\theta = 240°$.

2 θ is in quadrant 3, so the reference angle is $240° - 180° = 60°$
Use the half-equilateral triangle to determine $\sin(60°)$ and $\cos(60°)$



Apply the correct signs to the trigonometric ratios for quadrant 3.

WRITE

a $P = (\cos(90°), \sin(90°))$

$= (0, 1)$

b $P = (\cos(45°), \sin(45°))$

$\cos(45°) = \frac{\sqrt{2}}{2}$

$\sin(45°) = \frac{\sqrt{2}}{2}$

$P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

c $P = (\cos(240°), \sin(240°))$

$\sin(60°) = \frac{\sqrt{3}}{2}, \cos(60°) = \frac{1}{2}$

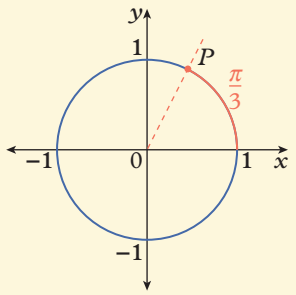
$\sin(240°) = -\frac{\sqrt{3}}{2}, \cos(240°) = -\frac{1}{2}$

$P = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

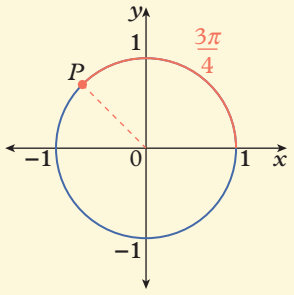
Example 7F.4 Identifying points on the unit circle from an arc length

Identify the coordinates of point P from the given arc length in each case below.

a



b



THINK

a 1 Given the arc length, determine the angle that subtends the arc in the unit circle in radians.

2 Write P in coordinate form.

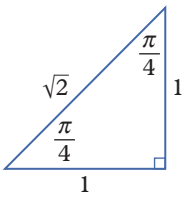
3 Identify the exact values of the trigonometric ratios. Use the half-equilateral triangle to determine the values.

b 1 Given the arc length, determine the angle that subtends the arc in the unit circle in radians.

2 Write P in coordinate form.

3 The angle is in quadrant 2, so the reference angle is $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

4 Identify the exact values of the trigonometric ratios. Use the half-square triangle to determine the values; determine the sign in quadrant 2.



WRITE

a $l = r\theta$

$\frac{\pi}{3} = 1 \times \theta$

$\theta = \frac{\pi}{3}$ radians

$P = \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

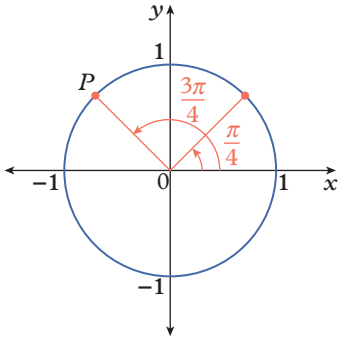
$P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b $l = r\theta$

$\frac{3\pi}{4} = 1 \times \theta$

$\theta = \frac{3\pi}{4}$ radians

$P = \left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right)$



$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$P = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Helpful hints

- ✔ When converting from radians to degrees, remember $\pi = 180°$; you substitute π with $180°$ and simplify. For example $\frac{2\pi}{5} = 2 \times \frac{180°}{5} = 72°$.
- ✔ Remember when converting from gradients to degrees, you'll need to get rid of π, so the converting factor is $\frac{180°}{\pi}$, where you divide by π. When you need to convert from degrees to gradients, you want π in your answer, so the converting factor is $\frac{\pi}{180°}$, where you multiply by π.
- ✔ It's helpful to memorize the special trigonometric ratios for 30°, 60° and 45°, and remember $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ and $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$: you will use them a lot in senior Math classes.

ANS p625

Exercise 7F Exact values

△ 1–8, 10, 11, 13 □ 2, 4–7, 9, 10, 11, 12(a–c), 13, 14, 15(a) ⬠ 5, 7, 10, 12, 14, 15(c, d), 16

UNDERSTANDING AND FLUENCY

7F.1 **1** Convert each of the following to the unit given in brackets.

a 120° (radians)
b 90° (radians)
c 135° (radians)
d 3π radians (degrees)
e $\frac{2\pi}{3}$ radians (degrees)
f $\frac{5\pi}{4}$ radians (degrees)
g 190° (radians)
h 156° (radians)
i $\frac{2\pi}{5}$ radians (degrees)
j $\frac{17\pi}{12}$ radians (degrees)
k 136° (radians)
l $\frac{143\pi}{90}$ radians (degrees)

7F.2 **2** Use the unit circle and exact values for the trigonometric ratios to evaluate the sine, cosine and tangent of each of these angles.

a 240°
b 300°
c 120°
d 315°
e 135°
f 150°

3 For each of the following angles:

i determine the reference angle in radians
ii evaluate the sine, cosine and tangent of the reference angle
iii use the unit circle to determine the sine, cosine and tangent of the original angle.

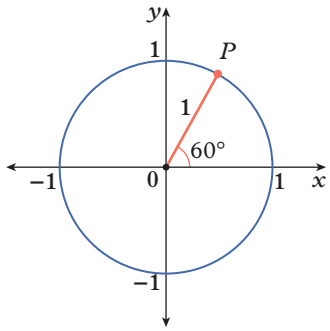
a $\frac{7\pi}{6}$ **b** $\frac{3\pi}{4}$ **c** $\frac{4\pi}{3}$

4 Use the unit circle and exact values for the trigonometric ratios to evaluate the sine, cosine and tangent of each angle.

a $\frac{\pi}{6}$
b $\frac{5\pi}{6}$
c $\frac{5\pi}{3}$
d $\frac{5\pi}{4}$
e $\frac{2\pi}{3}$
f $\frac{7\pi}{4}$

7F.3 **5** Identify the coordinates of the point P in each of these cases.

a



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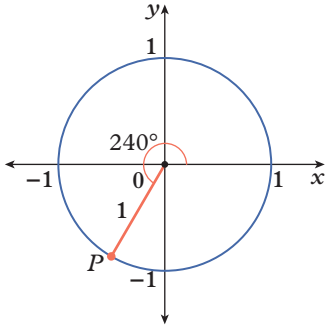

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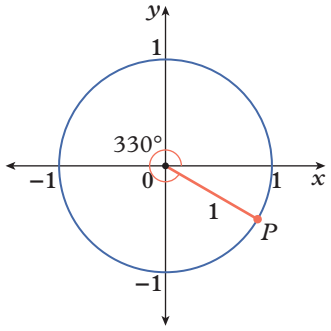

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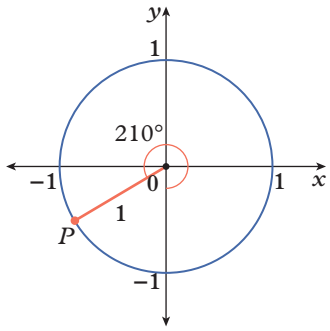

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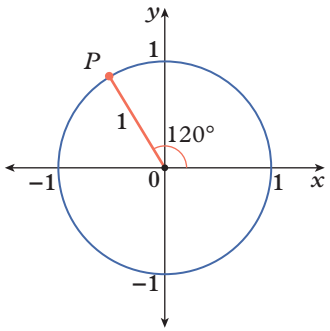

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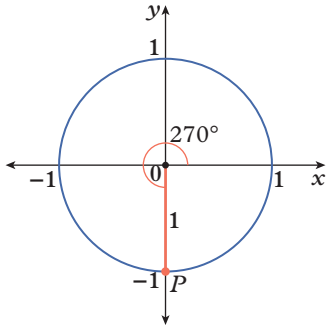


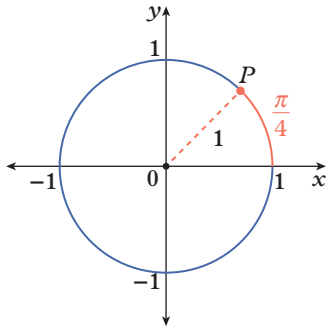
6 Give the exact coordinates of the point on the unit circle when the radial arm is rotated by each of these angles from the positive x-axis.

a 225° **b** 300 **c** 150°

d 135° **e** 315° **f** 180°

7F.4 **7** Identify the coordinates of the point P in each of these cases.

a



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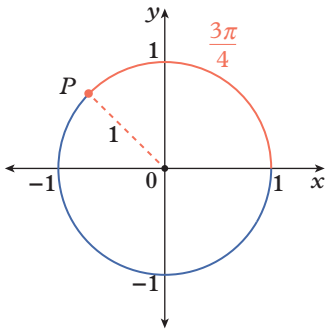

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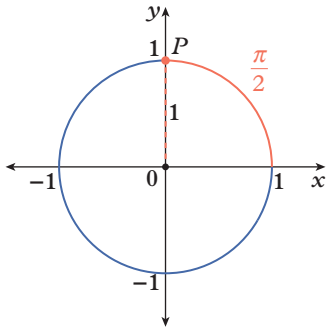

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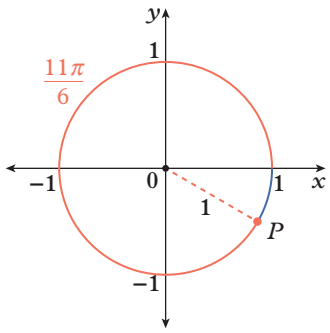

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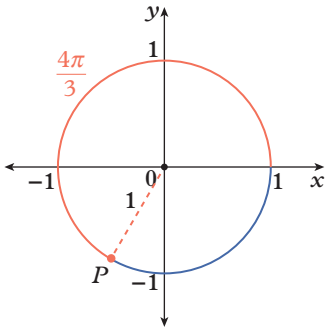
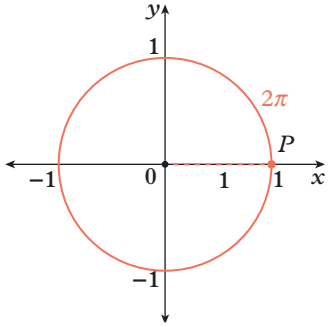


f



8 Answer each of the following in radians using your knowledge of geometry and without directly converting the angles to or from degrees.

a State the angle of:

i a revolution **ii** a straight angle **iii** a right angle.

b Write an inequality for the angle θ such that it is:

i acute **ii** obtuse **iii** reflex.

c Complete the following sentences.

i Complementary angles (two angles that form a _____) add to _____ radians.

ii Supplementary angles (two angles that form a _____) add to _____ radians.

d Determine the complement of the following angles.

i $\frac{\pi}{3}$ **ii** $\frac{3\pi}{8}$ **iii** $\frac{\pi}{4}$ **iv** $\frac{\pi}{10}$

e Determine the supplement of the following angles.

i $\frac{\pi}{3}$ **ii** $\frac{3\pi}{4}$ **iii** $\frac{\pi}{6}$ **iv** $\frac{5\pi}{12}$

f Determine the angle that would form a revolution with each of the following angles:

i $\frac{\pi}{3}$ **ii** $\frac{11\pi}{6}$ **iii** $\frac{5\pi}{4}$ **iv** $\frac{7\pi}{5}$

g State the interior angle sum of:

i a triangle **ii** a quadrilateral

iii a hexagon **iv** an n-sided polygon.

h State the size of each interior angle in:

i an equilateral triangle **ii** a square

iii a regular hexagon **iv** a regular n-sided polygon.

i Angles are said to be coterminal if the radial arm terminates or ends at the same point on the unit circle. State the coterminal angle one revolution greater than each of the following:

i $\frac{\pi}{3}$ **ii** $\frac{5\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{\pi}{2}$ **v** π

j State the coterminal angle one revolution less than each of the following:

i $\frac{\pi}{3}$ **ii** $\frac{5\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{\pi}{2}$ **v** π

9 **a** For each angle, state the fraction of a full turn that it represents. Mark each angle on a unit circle.

i $\frac{2\pi}{3}$ **ii** $\frac{\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{3\pi}{2}$

v $\frac{8\pi}{7}$ **vi** $\frac{4\pi}{5}$ **vii** π **viii** $\frac{\pi}{180}$

b For each fraction of a full turn, state the angle in radians. Mark each angle on a unit circle.

i $\frac{5}{6}$ **ii** $\frac{3}{5}$ **iii** $\frac{7}{8}$ **iv** $\frac{1}{12}$

v $\frac{6}{11}$ **vi** $\frac{9}{10}$ **vii** $\frac{8}{15}$ **viii** $\frac{5}{16}$

10 The constructions of the half-square and half-equilateral triangles are helpful to learn in case you forget any of the side lengths or angles, as you can then reconstruct the triangle from the name.

a Draw a square, mark the side lengths 1 unit, and add one of the diagonals.

b Use your knowledge of geometry and Pythagoras' theorem to label all angles in degrees and lengths with their size.

c Repeat parts **a** and **b** using radians.

d Draw an equilateral triangle with a horizontal base, mark the side lengths 2 units, and add the perpendicular height to the base such that it bisects the top angle and the base length.

e Use your knowledge of geometry and Pythagoras' theorem to label all angles in degrees and lengths with their size.

f Repeat parts **d** and **e** using radians.

11 The exact values of sine and cosine for the angles 30°, 45° and 60°, together with their multiples, can be recorded on the unit circle.

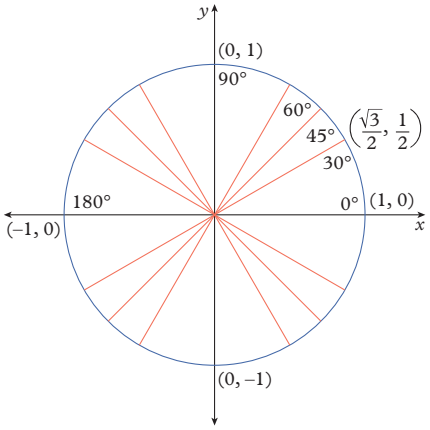
a Draw a unit circle. Mark the coordinates of the points where the circle cuts the horizontal and vertical axes. Also, mark points on the circle representing angles of 30°, 45° and 60°, together with their multiples up to 360°.

b Label each of the points shown on the unit circle with its exact cosine and sine value as coordinates.

For example, the point representing 30° would be labelled as $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
Take care to use the correct signs.

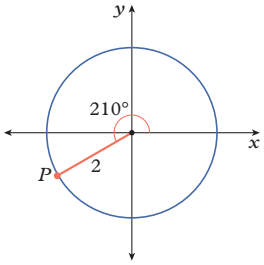
c Give the exact coordinates of each of these angles on the unit circle.

i 225° **ii** 300° **iii** 150° **iv** 135° **v** 315° **vi** 90°



12 Identify the coordinates of the point P in each case.

a



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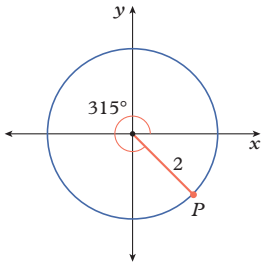
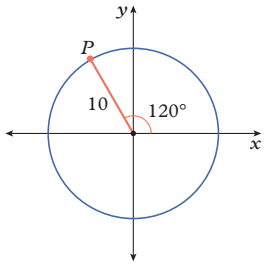


c



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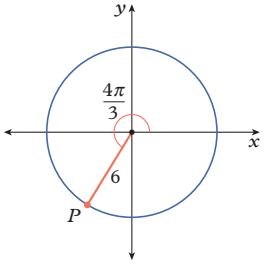

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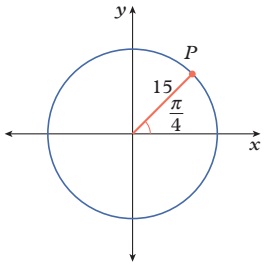

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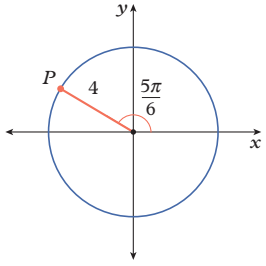


13 Complete the table.

	Radius	Angle subtending arc	Arc length
a	3 cm	$\frac{5\pi}{6}$	
b	6 cm	$\frac{2\pi}{3}$	
c		$\frac{3\pi}{4}$	
d		$\frac{5\pi}{12}$	$\frac{4\pi}{3}$ cm
e	4 cm		$\frac{9\pi}{2}$ cm
f			$\frac{2\pi}{7}$ cm

CHALLENGE

14 There is a simple method for determining the reference angle and the quadrant in which an angle given in radians lies.

a For each of the following, state the reference angle in radians.

i $\frac{2\pi}{3}$ **ii** $\frac{4\pi}{3}$ **iii** $\frac{5\pi}{3}$

iv $\frac{5\pi}{6}$ **v** $\frac{5\pi}{4}$ **vi** $\frac{7\pi}{4}$

vii $\frac{7\pi}{6}$ **viii** $\frac{11\pi}{6}$ **ix** $\frac{3\pi}{4}$

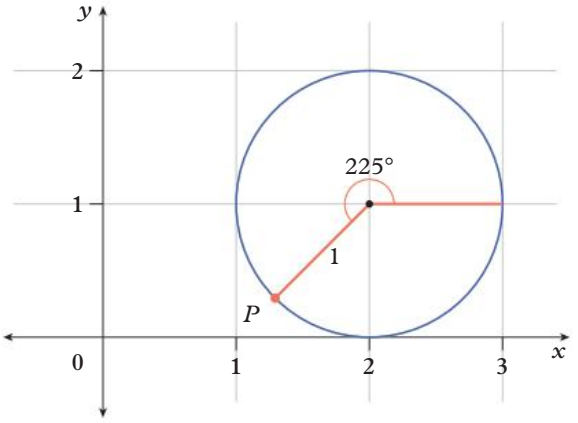
b Explain how you can determine the reference angle for an angle in radians.

c Determine the quadrant each angle in part **a** is in.

d Explain how you can determine the quadrant for an angle in radians.

15 Identify the coordinates of the point P in each case.

a



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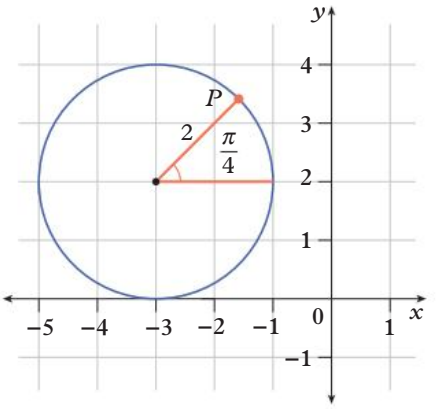

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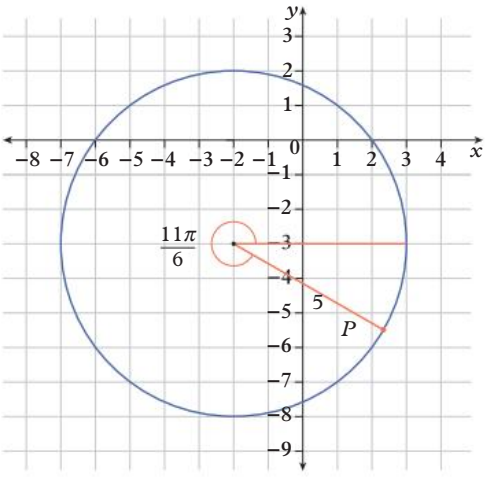

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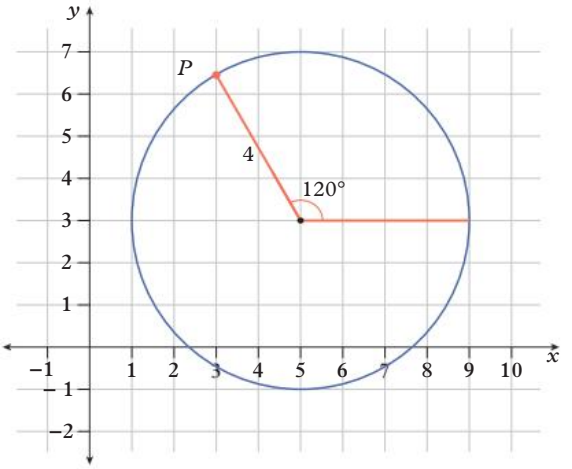
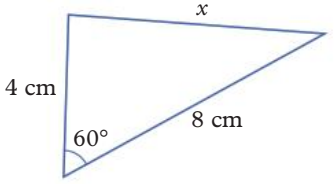


16 Determine the exact value of x for both triangles.

a



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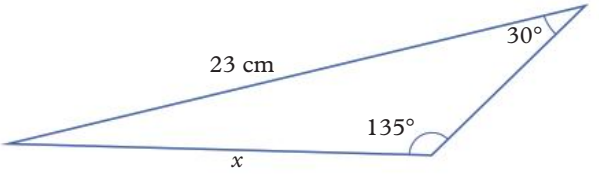


Check your Student obook pro for these digital resources and more:

- **Interactive skillsheet** — Converting between degrees and radians
- **Interactive skillsheet** — Exact values of trigonometric ratios
- **Interactive skillsheet** — Coordinates of points on the unit circle
- **Investigation** — Remembering the exact values
- **Topic quiz** — 7F

7G Trigonometric graphs

Learning intentions

By the end of this topic you will be able to ...

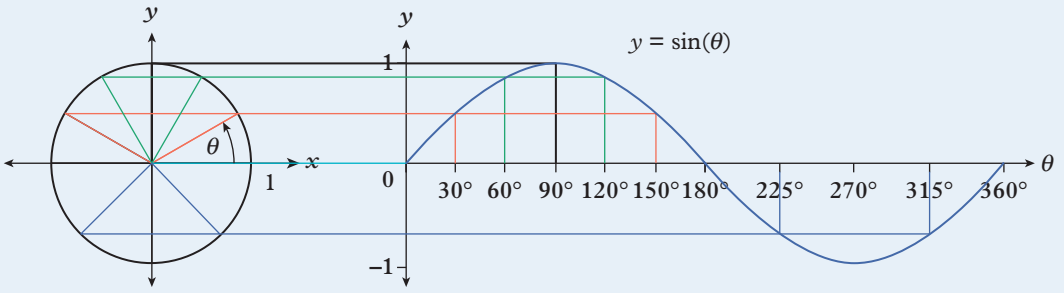
- ✔ read a point on a trigonometric graph
- ✔ determine the period and amplitude of a trigonometric function
- ✔ sketch graphs of simple trigonometric functions.

Inter-year links

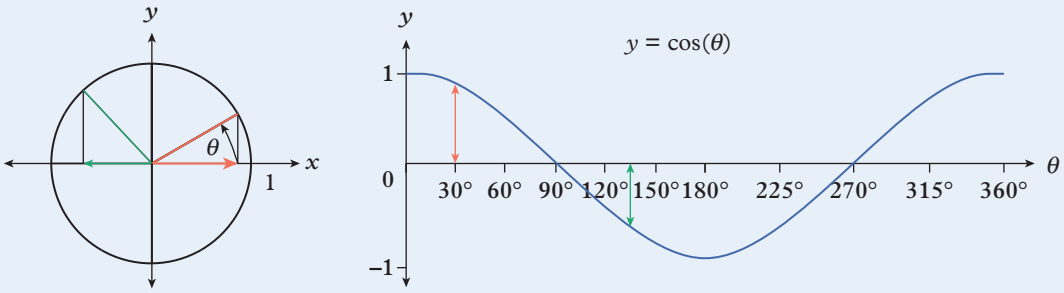
Year 9 4E Other non-linear relationships

Trigonometric functions

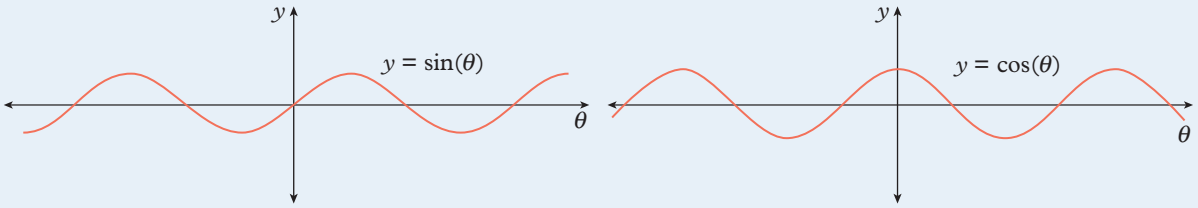
- A trigonometric function can be represented as a graph on the Cartesian plane.
- The horizontal axis represents the angle and the vertical axis represents the value for the trigonometric ratio.
- The graph of $y = \sin(\theta)$ has y values corresponding to the values of the y-coordinates on the unit circle for a given angle θ.



- The graph of $y = \cos(\theta)$ has y values corresponding to the values of the x-coordinates on the unit circle for a given angle θ.

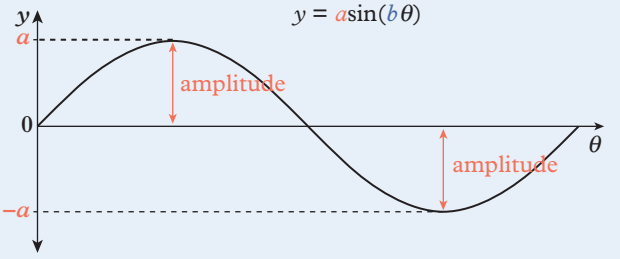


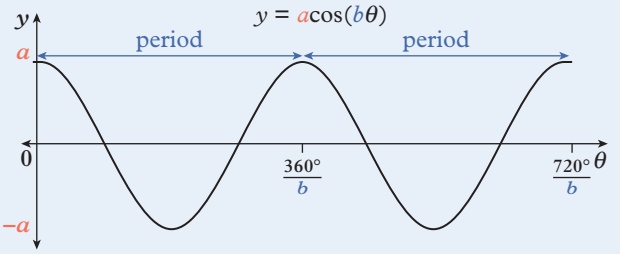
- For sine and cosine functions, the shape of the graph is called a sinusoid. They both look like waves with repeating peaks and troughs going along the horizontal axis.



Amplitude and period

- The amplitude of a trigonometric function is the distance from the centre of the curve to the maximum/minimum y value.
- For trigonometric functions of the form $y = a\sin(b\theta)$ or $y = a\cos(b\theta)$, the amplitude is equal to the magnitude of a. That means, if $a = 2$ or $a = -2$, then the amplitude is 2.
- Trigonometric functions are **periodic functions**, because their output is repeated as the angle moves around the circle.
- The **period** of a periodic function is the length of the interval along the horizontal axis that is required for the function to repeat itself.
- For trigonometric functions of the form $y = a\sin(b\theta)$ or $y = a\cos(b\theta)$, the period is given by: $\text{period} = \frac{360°}{b}$





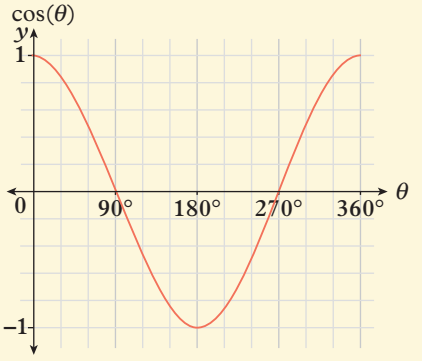
Sketching trigonometric graphs

- For trigonometric functions of the form $y = a\sin(b\theta)$ or $y = a\cos(b\theta)$:
  - → as the size of a increases, the amplitude increases, so the sinusoid is stretched on the y-axis.
  - → as b increases, the period decreases, so the sinusoid is compressed on the x-axis.
- For functions of the form $y = a\sin(b\theta)$, the y-intercept is at $y = 0$.
- For functions of the form $y = a\cos(b\theta)$, the y-intercept is at $y = a$.

Example 7G.1 Reading a trigonometric ratio from a graph

Consider the graph of the cosine function on the right. Use the graph to read the cosine value for each of these angles. Give your answers as approximate values, if necessary.

a 360° **b** 80°



THINK

a There is a maximum on the graph at $\theta = 360°$, so the vertical coordinate is exactly 1.

b Identify an approximate coordinate for the graph just to the left of (90°, 0).

WRITE

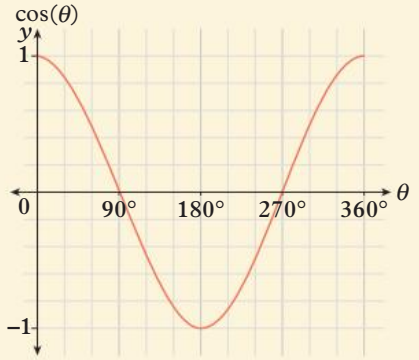
a $(360°, \cos(360°)) = (360°, 1)$
$\cos(360°) = 1$

b $(80°, \cos(80°)) \approx (80°, 0.2)$
$\cos(80°) \approx 0.2$

Example 7G.2 Reading an angle from a trigonometric graph

Consider this graph of the cosine function on the right. Use the graph to find the angles from 0° to 360° that have the following cosine values. Give your answers to the nearest 10°.

a 0.6 **b** −0.2



THINK

a **1** There are two coordinates on the graph where $\cos(\theta) = 0.6$: one near 50° and one near 310°.

2 Check the answers by comparing the reference angles and the sign of the trigonometric expression.

b **1** There are two coordinates on the graph where $\cos(\theta) = -0.2$: one near 100° and one near 260°.

2 Check the answers by comparing the reference angles and the sign of the trigonometric expression.

WRITE

a $(\theta, 0.6) \approx (50°, 0.6)$ or $(310°, 0.6)$

1st quadrant: $\theta' = \theta$
$= 50°$
4th quadrant: $\theta' = 360° - \theta$
$= 360° - 310°$
$= 50°$
Cosine is positive in the first and fourth quadrant, so $\cos(50°) = \cos(310°)$.
$\theta \approx 50°$ or $310°$

b $(\theta, 0.2) \approx (100°, -0.2)$ or $(260°, -0.2)$

2nd quadrant: $\theta' = 180° - \theta$
$= 180° - 100°$
$= 80°$
3rd quadrant: $\theta' = \theta - 180°$
$= 260° - 180°$
$= 80°$
Cosine is negative in the second and third quadrant, so $\cos(100°) = \cos(260°)$.
$\theta \approx 100°$ or $260°$

Example 7G.3 Determining the amplitude and period of a trigonometric function

Determine the amplitude and period of $y = -\frac{1}{2}\sin(6\theta)$.

THINK

1 Identify the amplitude as the magnitude of the coefficient of the sine function.

2 Write the period as 360° divided by the coefficient in front of θ and simplify.

WRITE

amplitude $= \frac{1}{2}$

period $= \frac{360°}{6}$
$= 60°$

Example 7G.4 Sketching trigonometric graphs

Sketch the graph of each of these functions over a one-period interval.

a $y = 3\sin(\theta)$ **b** $y = \cos(2\theta)$

THINK

a **1** Identify the amplitude as the magnitude of the coefficient in front of the sine function.

2 Identify the period.

3 This is a sine function, so sketch a sinusoid starting at the point (0, 0) and extending for one period.

4 Use the amplitude to label the y-axis with the maximum and minimum values. For a sine function the x-intercepts are every half period starting from 0.

b **1** Identify the amplitude as the magnitude of the coefficient in front of the cosine function.

2 Calculate the period.

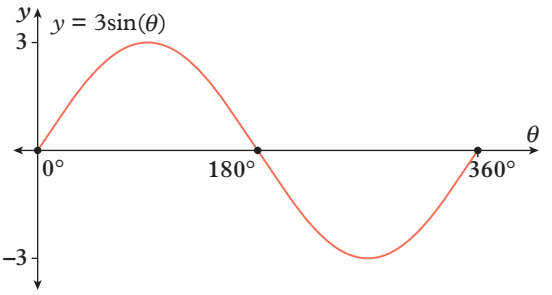
3 This is a cosine function, so sketch a sinusoid starting with a maximum at $\theta = 0$ and extending for one period.

4 Use the amplitude to label the y-axis with the maximum and minimum values. For a cosine function the x-intercepts are every half period starting from the first quarter of a period.

WRITE

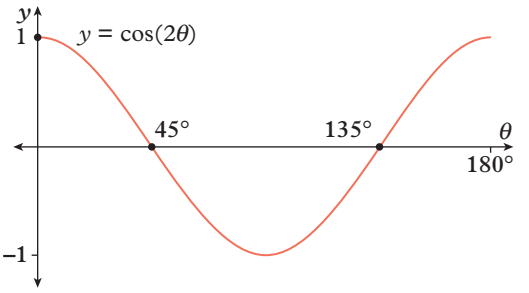
a amplitude = 3

period = 360°



b amplitude = 1

$$\text{period} = \frac{360°}{2} = 180°$$



Helpful hints

- ✔ Remember to consider all 4 quadrants of the unit circle and convert to the correct reference angle when trying to determine the angles for a given y value on a trigonometric graph. There are usually 2 angles with the same y value per period.
- ✔ Keep in mind that $\sin(30°) = \frac{1}{2}$ and $\cos(60°) = \frac{1}{2}$. That is, the graphs do not equal half the amplitude at the halfway point between the x-axis and the maximum/minimum.
- ✔ Sine and cosine graphs have the same basic shape. You only need to consider the period, amplitude and the starting point of the graph.

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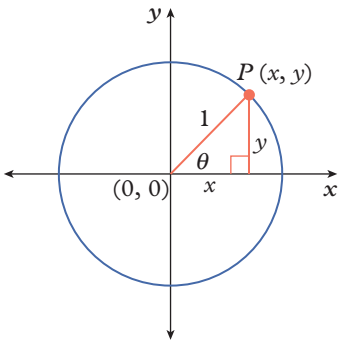
Exercise 7G Trigonometric graphs

1–8, 9(b–e), 10(b–e), 11, 14 | 3–6, 8, 9–10(c–f), 13, 15, 16 | 5, 6, 8, 9–10(d–f), 12, 13, 15–17

1 Trace the point P around the unit circle in an anti-clockwise direction from its starting point on the positive x-axis back to its starting point. As you trace, consider the size of angle θ as P moves. Also consider what happens to the x-coordinate for $P(x, y)$ as it moves along the x-axis.



a As you traced anti-clockwise around the unit circle, you will have noticed that the size of angle θ increased from 0° to 360°, and that the x-coordinate of point P decreased as it moved from 0° to 90°. Describe what happened to the x-coordinate of P when the size of the angle moved between:

i 90° and 180° **ii** 180° and 270° **iii** 270° and 360°

b The change in x value from part **a** is the change in the cosine of angle θ as it increases from 0° to 360°. Complete this table with the aid of your calculator to obtain a clearer picture of the values for cosine as the angle increases from 0° to 360°. Write your answers correct to one decimal place.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos(θ)													

c Plot the points from the table on grid or graph paper, with the size of angle θ along the horizontal axis scale and the cosine value, $\cos(\theta)$, up the vertical axis scale. Join your points with a smooth curve.

d Sketch on graph paper what would happen if you extended the angle values to 720°.

e What is the value of $\cos(\theta)$ at its first and second maximum. What is the period of the cosine graph?

f Note the difference between the maximums and the minimums of the graph. What is the amplitude of the graph?

g What is the y-intercept of the cosine graph?

2 Consider the change in the y-coordinate as point $P(x, y)$ moves and angle θ increases in size from 0° to 360°. This represents the change in the sine value of the angle.

a Repeat the procedure from question **1b**, constructing a table for the sine values, to obtain a clearer picture of the values for sine as the angle sweeps from 0° to 360°.

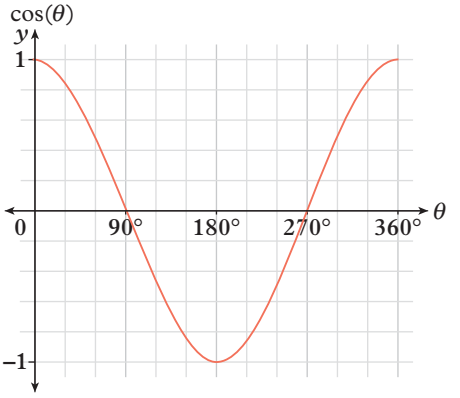
b Sketch the sine graph from 0° to 360°.

c What is the period of the graph?

d What is the amplitude of the graph?

e What is the y-intercept of the sine graph?

7G.1 **3** Consider the graph on the right of the cosine function.



a From the graph, write the cosine value for each of these angles.

i 0° **ii** 90°

iii 270° **iv** 180°

b Use the graph on the right to help you write the approximate cosine value (correct to one decimal place) for each of these angles.

i 40° **ii** 100°

iii 340° **iv** 200°

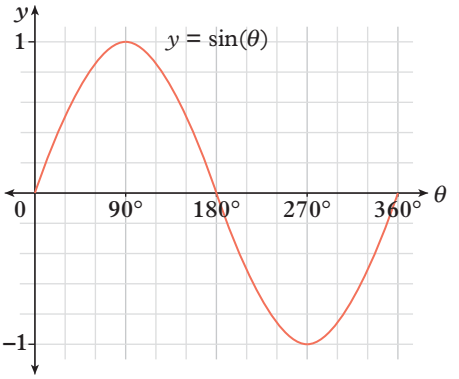
v 280° **vi** 140°

7G.2 **4** Use the graph from question **3** to help you find the angles from 0° to 360° with the following cosine values. Give your answers as approximate values, if necessary.

a 1 **b** –1 **c** 0

d 0.4 **e** –0.6 **f** 0.8

5 Consider the graph of the sine function on the right.



a From the graph, write the sine value for each of these angles.

i 360° **ii** 90°

iii 270° **iv** 180°

b Use the graph to help you write the approximate sine value (correct to one decimal place) for each of these angles.

i 40° **ii** 130°

iii 350° **iv** 190°

v 230° **vi** 140°

6 Use the graph from question **5** to help you find the angles from 0° to 360° with the following sine values. Give your answers as approximate values, if necessary.

a 1 **b** –1 **c** 0

d 0.4 **e** –0.6 **f** 0.8

7 Consider the function $y = 2\sin(3\theta)$. Calculate the y value for each of the following θ values and plot each pair of coordinates on a Cartesian plane. Draw a line to sketch the graph.

a Complete the table of values below. The first two columns have been completed for you.

b Plot each point (θ, y) on a Cartesian plane.

c Sketch the graph of $y = 2\sin(3\theta)$ for $0° \leq \theta \leq 240°$.

3θ	0°	90°	180°	270°	360°	450°	540°	630°	720°
θ	0°	30°							
$\sin(3\theta)$	0	1							
$y = 2\sin(3\theta)$	0	2							

7G.3 **8** Determine the amplitude and period in degrees for each of the following.

a $y = 3\sin(2\theta)$ **b** $y = 5\cos(3\theta)$ **c** $y = -4\sin\left(\frac{\theta}{2}\right)$ **d** $y = -6\cos\left(\frac{\theta}{6}\right)$

e $y = \frac{3}{4}\sin\left(\frac{3\theta}{4}\right)$ **f** $y = \frac{5}{3}\cos\left(\frac{3\theta}{5}\right)$ **g** $y = -\frac{7}{8}\sin\left(\frac{4\theta}{7}\right)$ **h** $y = \frac{5}{12}\cos\left(\frac{24\theta}{5}\right)$

7G.4 **9** Sketch the graphs of the following equations for $0° \leq x \leq 360°$.

a $y = 4\sin(\theta)$ **b** $y = \sin(4\theta)$ **c** $y = \sin\left(\frac{\theta}{4}\right)$

d $y = -4\sin(\theta)$ **e** $y = 5\sin(2\theta)$ **f** $y = -3\sin\left(\frac{\theta}{2}\right)$

10 Sketch the graphs of the following equations for $0° \leq x \leq 360°$.

a $y = 3\cos(\theta)$ **b** $y = \cos(3\theta)$ **c** $y = -3\cos(\theta)$

d $y = \cos\left(\frac{\theta}{3}\right)$ **e** $y = -\frac{1}{2}\cos(4\theta)$ **f** $y = 7\cos\left(\frac{\theta}{2}\right)$

11 The graphs of the sine function and the cosine function are quite similar. If you were presented with a graph of $y = \sin(\theta)$ and $y = \cos(\theta)$, and the functions weren't labelled, how could you tell which graph was which?

12 A grandfather clock has a pendulum that swings back and forth in a regular periodic motion. This motion is so regular that it has been the same method used, for hundreds of years, for keeping time. The weight of the bob on the end of the pendulum does not affect the period, but the period is affected by the length of the pendulum.

a Explain why the motion of a pendulum is periodic.

b The equation used to find the period of the motion of a pendulum is $T = \pi\sqrt{\frac{l}{g}}$, where T is the time (in seconds), l is the length of the pendulum (in metres) and g is the acceleration due to gravity (9.8 m/s^2).

Find the period of a pendulum with each of the following lengths:

i 1 m **ii** 1.5 m **iii** 2 m

c Explain how the length of a pendulum affects the period of its motion.

13 Periodic functions occur in many everyday activities. Consider this table of the sunrise and sunset times recorded for the east coast of Australia on the first day of the month throughout a year. The times are recorded as Eastern Standard Time.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sunrise (am)	4:55	5:20	5:40	5:57	6:12	6:29	6:38	6:29	6:02	5:28	4:57	4:44
Sunset (pm)	6:46	6:42	6:20	5:46	5:17	5:01	5:04	5:19	5:34	5:47	6:05	6:28

a Plot the sunrise and sunset figures from the table on separate graphs, placing the sunrise graph directly below the sunset graph. Use the horizontal axis scale for the months and the vertical axis scale for the times.

b Describe the shapes of the two curves you graphed.

c Are these two curves similar to those for the sine and cosine functions? Explain any similarities or differences.

d Are the curves periodic? Explain why or why not.

e Because of the way you positioned these two graphs, the trend in daylight hours throughout the year is visually apparent. Explain how this is so.

f If you were not told that these were figures for the east coast of Australia, how could you reason that they must be figures for somewhere in the Southern Hemisphere?

14 State the number of cycles (a repetition of a sinusoid) the graphs of each of the following equations will have in the domain $0° \leq x \leq 360°$.

a $y = \sin(3\theta)$ **b** $y = \frac{1}{2}\sin(11\theta)$ **c** $y = 5\sin\left(\frac{\theta}{2}\right)$ **d** $y = a\sin(b\theta)$

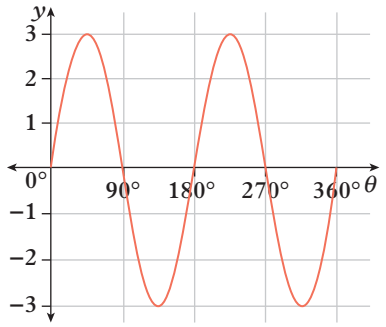
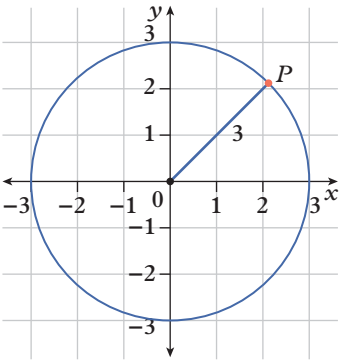
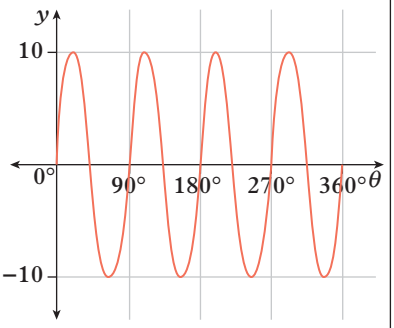
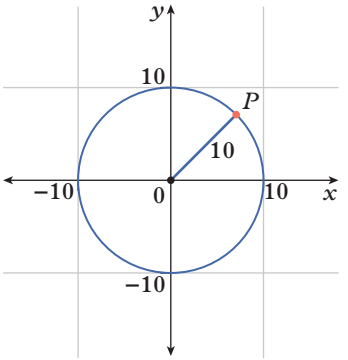
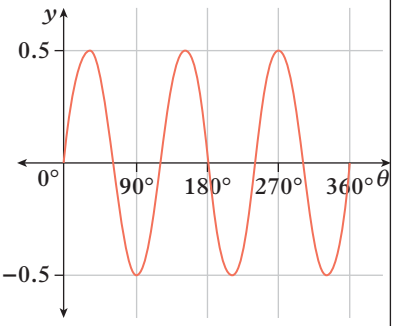
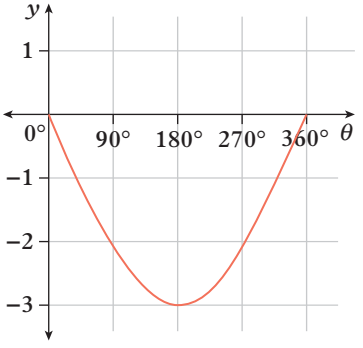
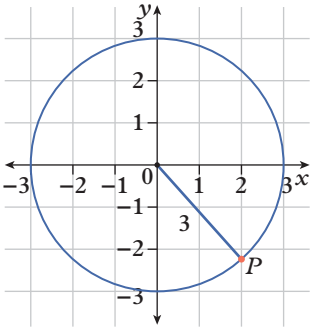
e $y = \cos(4\theta)$ **f** $y = -4\cos(90\theta)$ **g** $y = \frac{5}{2}\cos\left(\frac{3\theta}{2}\right)$ **h** $y = a\cos(b\theta)$

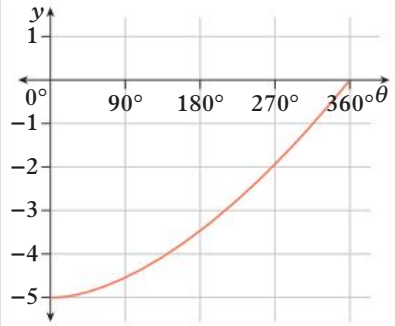
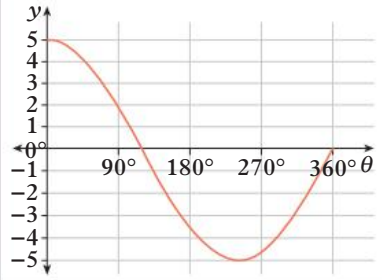
15 The graphs for sine and cosine can be considered a point moving at a constant speed anti-clockwise around a circle. The amplitude and period, then, have analogues on the circle that they are related to.

a Complete the table below.

b Determine what feature of the circle the amplitude is related to and how.

c Determine what feature of the circle the period is related to and how.

	Graph	Circle	Radius	Revolutions per 360°
			3	2
i				
ii				
iii				

	Graph	Circle	Radius	Revolutions per 360°
iv			5	
v			3	5
vi				$\frac{3}{4}$

16 The tangent ratio is also a periodic function.

a Complete a table of values, correct to one decimal place, for θ values from 0° to 360° as shown.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
tan(θ)																	

b Plot these points on a Cartesian plane and join them with a smooth curve.

c **i** Describe the shape of the graph.

ii It is different from the sine curve and the cosine curve. Why is this so?

iii At $\theta = 90°$, a vertical asymptote occurs. Explain what you understand this to mean. Where does the next asymptote occur?

iv Explain why the graph is periodic. What is its period?

v What is the amplitude of the graph?

17 Like other graphs, the graphs of trigonometric functions can be translated vertically and horizontally.

a Complete the table of values below for $y = \sin(\theta - 30°) + 4$.

$\theta - 30°$	0°	90°	180°	270°	360°
θ	30°	120°			
$\sin(\theta - 30°)$	0	1			
$y = \sin(\theta - 30°) + 4$	4	5			

b Sketch the graph of $y = \sin(\theta - 30°) + 4$ for $30° \leq \theta \leq 390°$.

c Sketch the graph of $y = \cos(\theta - 45°) - 3$ for $45° \leq \theta \leq 405°$.

Check your Student obook pro for these digital resources and more:

- Interactive skillsheet — Period of a trigonometric function
- Interactive skillsheet — Amplitude of a trigonometric function
- Interactive skillsheet — Sketching trigonometric graphs
- Topic quiz — 7G

7H Solving trigonometric equations

Learning intentions

By the end of this topic you will be able to ...

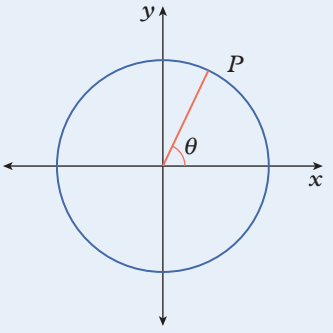
✔ solve simple trigonometric equations.

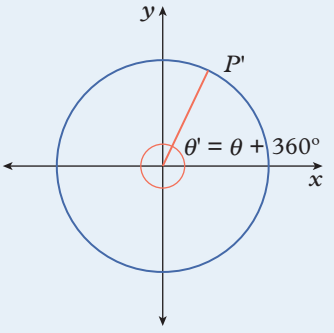
Inter-year links

Year 9 6F Using trigonometry to find angles

Trigonometric equations

- The trigonometric functions $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ are all periodic, as their values repeat with every revolution of the angle, θ.
- If θ is the angle measured anti-clockwise from the positive x-axis to the radius that meets the unit circle at point P, then:
  - → the x-coordinate of the point on the unit circle is $\cos(\theta)$
  - → the y-coordinate of the point on the unit circle is $\sin(\theta)$
  - → the gradient from the origin to the point P on the unit circle is $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.
- Adding or subtracting 360° or 2π to θ, the intersection of the new angle θ' meets the unit circle at point P' on the unit circle, then:
  - → the x-coordinate of P' on the unit circle is $\cos(\theta \pm 360°)$ or $\cos(\theta \pm 2\pi)$
  - → the y-coordinate of P' on the unit circle is $\sin(\theta \pm 360°)$ or $\sin(\theta \pm 2\pi)$
  - → the gradient from the origin to the point P' on the unit circle is $\tan(\theta \pm 360°)$ or $\tan(\theta \pm 2\pi)$.
- Because P and P' would have the same coordinates on the unit circle:
  $\sin(\theta) = \sin(\theta \pm 2\pi)$
  $\cos(\theta) = \cos(\theta \pm 2\pi)$
  $\tan(\theta) = \tan(\theta \pm 2\pi)$





- Equations involving trigonometric functions have infinitely many solutions, as trigonometric functions are periodic.
  For example, $\sin(\theta) = 0$ for $\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$ or $\theta = 0, 180°, 360°, 540°, 720°, \ldots$
- The solutions to a trigonometric equation are often restricted to a domain given by an interval, such as $0 \leq \theta \leq 2\pi$ or $0° \leq \theta \leq 360°$.
- Most calculators only give one of the solutions for $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ as the domains are restricted to only half a revolution of the unit circle.
- Once one solution for a trigonometric equation is identified, the rest can be determined using the symmetry of the unit circle within a specified domain.

Example 7H.1 Solving a trigonometric equation within a given domain

Solve the equation $\sin(\theta) = -\frac{1}{2}$, $0° \leq \theta \leq 360°$.

THINK

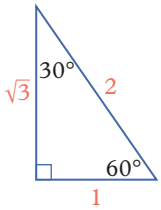
1 Determine the first solution to the equation. Use the special triangles to determine the angle in the first quadrant that has a sine of $\frac{1}{2}$.

2 Identify the quadrants in which sine is negative.

3 Write possible values for θ using symmetry properties of trigonometric ratios in the given domain.

4 Write the solution to the equation.

WRITE



$\sin(\theta) = \frac{1}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$

$= 30°$

We were given that $\sin(\theta) = -\frac{1}{2}$ and sine is negative in quadrants 3 and 4.

So: $\theta = 180° + 30°$

$= 210°$

or: $\theta = 360° - 30°$

$= 330°$

There are two solutions: $\theta = 210°$ or $\theta = 330°$

Helpful hints

✔ The trigonometric functions repeat themselves every 360° or 2π radians, because these are always coterminal angles. But don't forget that most values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ repeat themselves in another quadrant of the unit circle as well.

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Exercise 7H Solving trigonometric equations

△ 1–5, 7, 9, 11, 16(a, b), 18(a, e)

□ 1–3, 6, 8, 10(a, e, i), 11, 12, 15, 16–18(a, c, e), 19

⬠ 1, 2, 6, 10(g–i), 11–14, 17–18(a, c, e), 20, 21

7H.1

1 Solve each equation for the domain $0° \leq \theta \leq 360°$.

a $\cos(\theta) = \frac{\sqrt{3}}{2}$ **b** $\sin(\theta) = -\frac{1}{\sqrt{2}}$ **c** $\tan(\theta) = -\frac{1}{\sqrt{3}}$

d $\sin(\theta) = -\frac{\sqrt{3}}{2}$ **e** $\cos(\theta) = \frac{1}{\sqrt{2}}$ **f** $\tan(\theta) = 1$

2 Using a calculator to determine the reference angle, solve each equation for the domain $0° \leq \theta \leq 360°$ correct to the nearest degree.

a $\cos(\theta) = -0.4$ **b** $\sin(\theta) = -0.3$ **c** $\tan(\theta) = 0.9$

d $\sin(\theta) = 0.6$ **e** $\cos(\theta) = 0.6$ **f** $\tan(\theta) = -7$

UNDERSTANDING AND FLUENCY

3 Consider the equation $\sin(\theta) = 1$, $0° \le \theta \le 360°$.

a What angle has a sine value of 1? Hint: Refer to the unit circle.

b Are there any other angles within the domain with a sine value of 1? If so, which one(s)?

c What is the solution to the equation?

d Write a list of all the solutions for the domain $0° \le \theta \le 720°$.

4 Consider the equation $\cos(\theta) = -1$.

a What solution(s) can be found in the domain $0° \le \theta \le 360°$?

b What are the solutions in the domain $0° \le \theta \le 720°$?

5 Consider the equation $\sin(\theta) = 0$, $0° \le \theta \le 360°$.

a What angle has a sine value of 0? Hint: Refer to the unit circle.

b Are there any other angles within the domain with a sine value of 1? If so, which one(s)?

c What is the solution to the equation?

d Write a list of all the solutions for the domain $0° \le \theta \le 720°$.

6 Solve each equation for the domain $0° \le \theta \le 360°$.

a $\cos(\theta) = 0$ **b** $\tan(\theta) = 0$ **c** $\sin(\theta) = -1$ **d** $\cos(\theta) = 1$

PROBLEM SOLVING AND REASONING

7 Give the smallest domain within $0° \le \theta \le 360°$ for there to be two solutions for each equation.

a $\sin(\theta) = -1$ **b** $\cos(\theta) = 0$ **c** $\cos(\theta) = \frac{1}{2}$

d $\sin(\theta) = \frac{\sqrt{3}}{2}$ **e** $\sin(\theta) = \frac{1}{\sqrt{2}}$ **f** $\cos(\theta) = -\frac{\sqrt{3}}{2}$

8 Find the smallest domain for each equation to give:

i two solutions within $0° \le \theta \le 360°$

ii four solutions within $0° \le \theta \le 720°$.

a $\sin(\theta) = -\frac{1}{2}$ **b** $\cos(\theta) = -\frac{1}{2}$ **c** $\sin(\theta) = -\frac{1}{\sqrt{2}}$ **d** $\cos(\theta) = -\frac{1}{\sqrt{2}}$

9 It is possible that a trigonometric equation has no real solution. Explain why each of these equations has no solution within any real domain.

a $3 - \sin(\theta) = 1$ **b** $2\cos(\theta) = 3$ **c** $\frac{1}{4}\sin(\theta) = 1$

10 Indicate whether each equation has a real solution within the domain $0° \le \theta \le 360°$. For those without a real solution, explain why this is so. There is no need to solve the equation.

a $\tan(\theta) = 4$ **b** $1 - \sin(\theta) = 1$ **c** $2\cos(\theta) = 2$

d $\frac{1}{2}\sin(\theta) = -\frac{1}{2}$ **e** $1 + \cos(\theta) = -1$ **f** $4\sin(\theta) = -\frac{1}{4}$

g $3 - \cos(\theta) = \frac{1}{3}$ **h** $\frac{1}{2}\cos(\theta) = -\frac{1}{4}$ **i** $\frac{1}{4} + \sin(\theta) = -1$

11 Take care when you consider the domain of a trigonometric equation. Don't always assume that a trigonometric equation has two or more solutions. Consider the following equations with the domains indicated. Write the number of solutions for each. (Don't write the values of the solutions.)

a $\sin(\theta) = \frac{1}{2}$, $0° \le \theta \le 90°$ **b** $\sin(\theta) = -\frac{1}{2}$, $0° \le \theta \le 90°$

c $\cos(\theta) = -\frac{1}{2}$, $0° \le \theta \le 180°$ **d** $\tan(\theta) = 0.3$, $0° \le \theta \le 180°$

e $\sin(\theta) = -0.4$, $0° \le \theta \le 270°$ **f** $\cos(\theta) = 0.8$, $0° \le \theta \le 270°$

12 If the pronumeral inside the brackets of the trigonometric function is multiplied by a coefficient, then start by redefining the specified domain in terms of the expression inside the brackets.

For example, if the equation is $\cos(3\theta) = \frac{1}{2}$, $0° \le \theta \le 180°$

Then $\cos(3\theta) = \frac{1}{2}$, $0° \le 3\theta \le 540°$

So, solving normally, $3\theta = 60°, 300°, 420°$

Therefore, $\theta = 20°, 100°, 210°$

Solve each equation for the domain $0° \le \theta \le 180°$.

a $\sin(2\theta) = \frac{1}{\sqrt{2}}$ **b** $\cos(4\theta) = -1$ **c** $\tan(3\theta) = -1$

13 Solve each equation for the domain indicated.

a $2\sin(\theta) - 3 = \sin(\theta) - 2, 0° \leq \theta \leq 180°$

b $4\cos(\theta) + 5 = 4 + 5\cos(\theta), 0° \leq \theta \leq 180°$

c $\sqrt{2}\cos(\theta) + 1 = 0, 0° \leq \theta \leq 270°$

d $2\sin(\theta) + \sqrt{3} = 0, 0° \leq \theta \leq 270°$

e $4 - 4\tan(\theta) = 0, 90° \leq \theta \leq 270°$

f $\sqrt{48}\tan(\theta) + 2 = -3 - \sqrt{3}\tan(\theta), 90° \leq \theta \leq 270°$

14 Use the method shown in question **11** to solve each equation for the domain $0° \leq \theta \leq 1080°$.

a $\sin\left(\frac{\theta}{2}\right) = -\frac{1}{2}$ **b** $\cos\left(\frac{\theta}{3}\right) = \frac{1}{\sqrt{2}}$ **c** $\tan\left(\frac{2\theta}{5}\right) = -\sqrt{3}$

15 Explain why solving $\tan(\theta) = 1$ is different to solving $\sin(\theta) = 1$ and $\cos(\theta) = 1$.

16 Determine the sum of the solutions of the following equations for the domain $0° \leq \theta \leq 360°$.

a $\sin(\theta) = \frac{1}{2}$ **b** $\cos(\theta) = \frac{1}{2}$ **c** $\tan(\theta) = \frac{1}{\sqrt{3}}$

d $\sin(\theta) = -\frac{1}{2}$ **e** $\cos(\theta) = -\frac{1}{2}$ **f** $\tan(\theta) = -\frac{1}{\sqrt{3}}$

17 Solve each equation for the domain $-360° \leq \theta \leq 0°$.

a $\sin(\theta) = \frac{\sqrt{3}}{2}$ **b** $\cos(\theta) = -\frac{1}{\sqrt{2}}$ **c** $\tan(\theta) = \sqrt{3}$

d $\sin(\theta) = -1$ **e** $\cos(\theta) = 0$ **f** $\tan(\theta) = 0$

18 Trigonometric equations can be solved in degrees as well as radians. To determine whether you should write your answer in degrees or radians, use the same unit that the domain is written in. Solve the following equations for the domain $0 \leq \theta \leq 2\pi$.

a $\sin(\theta) = 0$ **b** $\cos(\theta) = \frac{\sqrt{3}}{2}$ **c** $\tan(\theta) = -1$

d $\sin(\theta) = \frac{1}{\sqrt{2}}$ **e** $\cos(\theta) = -1$ **f** $\tan(\theta) = \frac{1}{\sqrt{3}}$

19 The equation $\sin(\theta) = \cos(\theta)$ can be solved by using the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

a Show that $\sin(\theta) = \cos(\theta)$ is equivalent to $\tan(\theta) = 1$.

b Solve the following equations for $0° \leq \theta \leq 360°$.

i $\sin(\theta) = \cos(\theta)$

ii $\sin(\theta) = \sqrt{3}\cos(\theta)$

iii $\sqrt{3}\sin(\theta) + \cos(\theta) = 0$

iv $3\sin(\theta) - \sqrt{3}\cos(\theta) = 0$

20 Solve the following equations.

a $2\sin(2\theta) - 1 = 0, 0° \leq \theta \leq 360°$

b $4\cos\left(\frac{\theta}{3}\right) + 2 = 0, 180° \leq \theta \leq 540°$

c $10\sin(2\theta - 30°) + 5 = 0, 180° \leq \theta \leq 540°$

d $12\cos\left(\frac{\theta}{4} + \pi\right) - \sqrt{72} = 0, -5\pi \leq \theta \leq 12\pi$

21 a Solve the equation $2x^2 - 3x - 2 = 0$.

b Hence, solve the equation $2(\sin(\theta))^2 - 3\sin(\theta) - 2 = 0, 0° \leq \theta \leq 360°$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Solving trigonometric equations

CAS instructions
Solving trigonometric equations

Topic quiz
7H

OPTIONAL

PROBLEM SOLVING AND REASONING

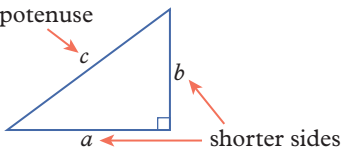
CHALLENGE

Chapter summary

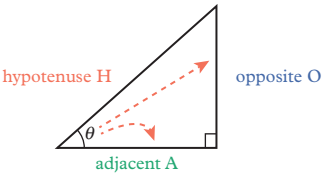
Pythagoras' theorem and trigonometric ratios

$c^2 = a^2 + b^2$

hypotenuse, shorter sides



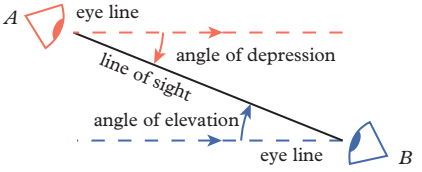
- $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\text{O}}{\text{H}}$
- $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{\text{A}}{\text{H}}$
- $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\text{O}}{\text{A}}$



Pythagoras' theorem and trigonometric ratios can be used to find an unknown length or an unknown angle, either on the surface of a 3D object or within a 3D object.

Angles of elevation and depression

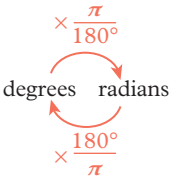
- The angle of depression from A to B is equal to the angle of elevation from B to A.



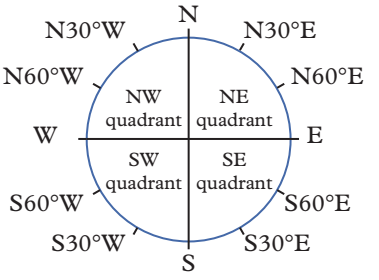
Exact values

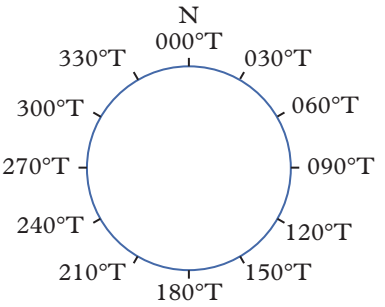
θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Radians

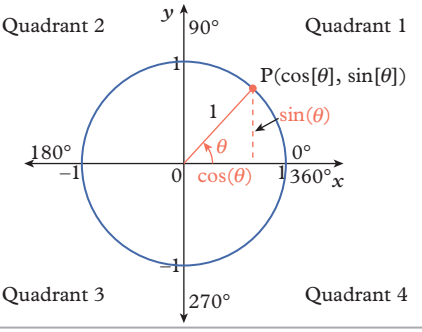


Compass bearings and true bearings

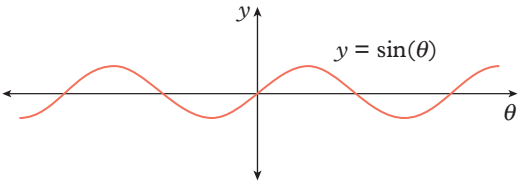


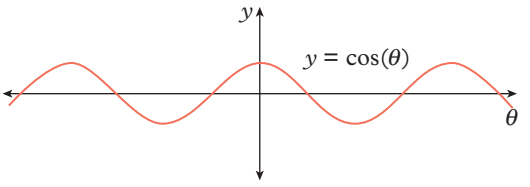


The unit circle

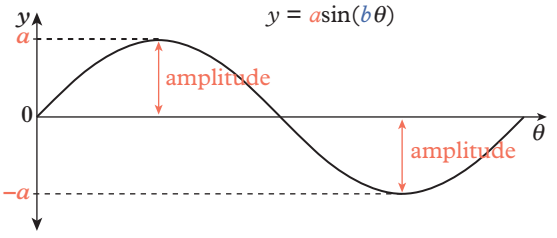


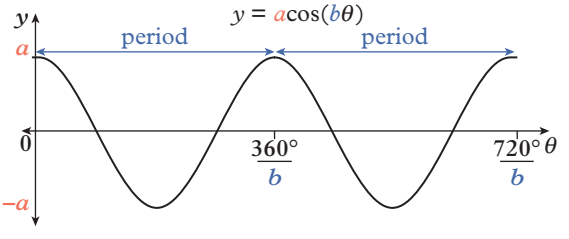
Trigonometric graphs





Amplitude and period





Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

7A **1** Which of these describes a triangle with side lengths measuring 4 cm, 5 cm and 6 cm?

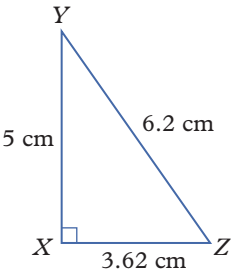
A right-angled **B** obtuse-angled **C** acute-angled **D** isosceles **E** equilateral

7A **2** The ratio $\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ of a right-angled triangle is known as which of the following?

A sine **B** cosine **C** tangent **D** π **E** none of these

7B **3** What is the size of $\angle YZX$, to the nearest degree?

A 54° **B** 55° **C** 35° **D** 36° **E** 90°



7B **4** Which of these ratios represents the sine of $\angle ZYX$?

A $\frac{5}{6.2}$ **B** $\frac{3.62}{6.2}$ **C** $\frac{3.62}{5}$ **D** $\frac{5}{3.62}$ **E** $\frac{6.2}{3.62}$

7C **5** What is the equivalent true bearing of N12°W?

A 348°T **B** 012°T **C** 192°T **D** 168°T **E** 078°T

7D **6** Which of these values is closest to the diagonal length within a cube of side length 5 cm?

A 5 cm **B** 8 cm **C** 9 cm **D** 10 cm **E** 7 cm

7E **7** Which of the following has a different acute reference angle to the others?

A 330° **B** $\frac{5\pi}{6}$ **C** 210° **D** $\frac{13\pi}{6}$ **E** 120°

7E **8** If a point P on the unit circle in quadrant 2 is given in terms of an acute reference angle, θ, then which of these coordinates for P are correct?

A $(-\cos(\theta), \sin(\theta))$ **B** $(\cos(\theta), -\sin(\theta))$ **C** $(-\cos(\theta), -\sin(\theta))$ **D** $(\cos(\theta), \sin(\theta))$ **E** $(-\sin(\theta), \cos(\theta))$

7F **9** Which of these is the exact sine value for 30°?

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{2}}$ **E** $\sqrt{3}$

7G **10** What is the period of $y = \sin(\theta)$?

A 90° **B** 180° **C** 270° **D** 360° **E** 720°

7G **11** How many solutions does $\sin(\theta) = -\frac{1}{2}$ have in the domain $0° \leq \theta \leq 720°$?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

7G **12** What is the amplitude of $y = 3\cos(4\theta)$?

A 3 **B** −3 **C** 4 **D** $\frac{1}{3}$ **E** $\frac{1}{4}$

7H **13** Which of the following is not a solution to $2\cos(2\theta) = 1$?

A 30° **B** 60° **C** 150° **D** 210° **E** 330°

Short answer

For these questions, calculate each angle correct to the nearest degree and each length correct to two decimal places (unless stated otherwise).

7A **1** An equilateral triangle has side lengths of 5 cm. Find the height of the triangle as:

a an exact value **b** an approximate value, correct to one decimal place.

7B **2** A rectangle has a diagonal length of 15.8 cm. The diagonal forms an angle of 31.3° with the longer side. What are the dimensions of the rectangle?

7C **3** A map shows a stretch of road as a horizontal distance of 18.5 km. The road rises 6.2 km from start to finish.

a Find the angle of the rise of the road, to the nearest degree.

b Find the average gradient of the road as a:

i ratio **ii** percentage.

c What is the actual distance you would travel along this stretch of road?

7C **4** Two buildings stand 20 m apart. One is taller than the other. From the top of the taller building to the base of the shorter building, the angle of depression is 75°. From the top of the shorter building to the top of the taller building, the angle of elevation is 60°. What is the height of the shorter building?

7D **5** Find the length of the longest diagonal within each 3D object:

a a cylinder with base circumference of 20 cm and height of 15 cm

b a box 10 cm long, 8 cm wide and 15 cm tall

c a cube of side length 25 cm.

7D **6** Calculate the angle each diagonal you calculated in question **5** makes with the height of the object correct to the nearest degree.

7E **7** **a** Convert the following angles to radians.

i 120° **ii** 315° **iii** 130°

b Convert the following angles to degrees.

i 3π **ii** $\frac{8\pi}{3}$ **iii** $\frac{9\pi}{8}$

7F **8** Show how the unit circle can be used to find each of these values by drawing the corresponding points on a unit circle and labelling their coordinates.

a $\sin(300°)$ **b** $\cos(225°)$ **c** $\sin(135°)$

d $\cos\left(\frac{7\pi}{4}\right)$ **e** $\sin\left(\frac{4\pi}{3}\right)$ **f** $\cos\left(\frac{11\pi}{6}\right)$

7G **9** Sketch the graph of the following functions for the domain $0° \leq \theta \leq 360°$.

a $y = -3\sin(\theta)$ **b** $y = \cos(2\theta)$ **c** $y = 4\sin\left(\frac{\theta}{2}\right)$ **d** $y = -2\cos(3\theta)$

7H **10** Solve each of these equations for the domain $0° \leq \theta \leq 360°$.

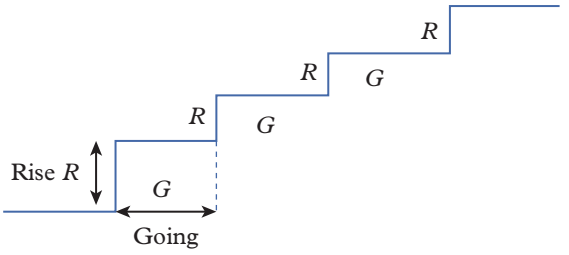
a $\cos(\theta) = -1$ **b** $\sin(\theta) = 0$ **c** $\cos(\theta) = \frac{\sqrt{3}}{2}$

d $\sin(\theta) = -\frac{1}{2}$ **e** $\cos(\theta) = -\frac{1}{\sqrt{2}}$ **f** $\sin(\theta) = -\frac{\sqrt{3}}{2}$

Analysis

1 There are building code regulations for the construction of staircases. For safety purposes, the tread cannot be too narrow, nor the staircase too steep. There are particular terms for staircases, as listed and shown in the diagram below.

- The rise (R) must have a minimum height of 115 mm and a maximum height of 190 mm.
- The going (G) must have a length within the range of 250–355 mm.
- The regulation also states that $2R + G$ must lie within the range of 550–700 mm.



a Consider a staircase with the minimum rise measurement of 115 mm and the minimum going measurement of 250 mm.

i Explain whether this staircase would be within the building regulation guidelines.

ii What is the slope of the staircase described? Give your answer to the nearest degree.

b Repeat part **a** using the maximum values allowed for the rise and going measurements.

c If the going measurement is the maximum value allowed, 355 mm, and the value of $2R + G$ falls within the required limits:

i find a range of acceptable values for the rise

ii calculate the range of acceptable slope values.

d The number of steps is counted by the number of goings from the bottom to the top, starting with a going and ending with a going. Draw a set of five stairs with a going measurement of 300 mm and a rise of 150 mm.

i What is the total horizontal distance of the five stairs?

ii Find the total vertical height of the five stairs.

iii Calculate the direct distance between the base of the first step and the total vertical height calculated in part **ii**.

iv What is the slope of the staircase? Correct your answer to the nearest degree.

e What do you consider would be ideal measurements for the going and rise of a comfortable staircase? Draw a sketch of your design, show that it falls within the regulation guidelines, and provide its slope.

f There are different building code regulations for wheelchair ramps. The slope of the ramp is of particular importance, because a wheelchair may tip backwards or run away if the ramp is too steep. The code specifies that an internal ramp should be no steeper than 1 to 14, while an external ramp should have a gentler slope of 1 to 20.

i Draw diagrams of 20 m long internal and external ramps with these slopes.

ii For each ramp you drew in part **i**, find the going and rise measurements, rounding to three decimal places. Mark these measurements on your diagrams.

2 Emma rides her bicycle at a constant speed with a tag attached to a spoke at the rim of the wheel.

Tom used a video to record the height of the tag with respect to its starting position. He also made a record of the results in the table below.

Time (s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Height (cm)	20	30	20	0	–20	–30	–20	0	20	30

a Graph the data from the table, showing time on the x-axis.

b Suggest whether the graph models a sine curve or a cosine curve.

c How long does it take for the wheel to rotate once?

d What was the starting position of the tag on the wheel? Explain how you deduced this.

e What is the radius of the wheel?

f Calculate the circumference of the wheel.

g At what speed does Emma ride? Give your answer in metres per second.

h What is the position of the tag after 10 seconds of riding at this constant speed? Explain how you arrived at this answer.

8 Networks

Index

8A Introduction to networks
8B Planar graphs
8C Platonic solids
8D Eulerian trails and circuits
8E Applications of networks

Prerequisite skills

Before starting work on this chapter, complete the Colouring in the plane investigation.

Investigation
Colouring in the plane

AC curriculum links

- Interpret networks and network diagrams used to represent relationships in practical situations and describe connectedness (AC9M10SP02)

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8A Introduction to networks

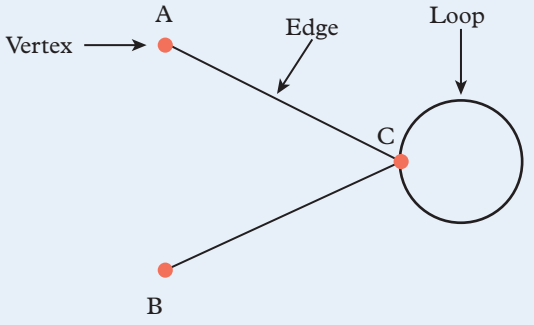
Learning intentions

By the end of this section, you will be able to ...

- ✔ draw networks to represent real-world situations
- ✔ use terminology relating to networks
- ✔ reason with a network diagram.

Networks

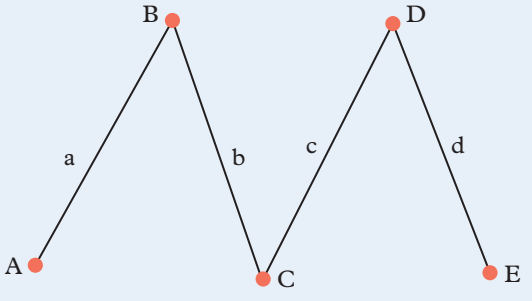
- A **network** (or **graph**) is a collection of objects and the relations between them. The objects in a network are called **vertices** and a relation between two vertices is called an **edge**.
  - → Vertices and edges often have physical meanings. For example, vertices may correspond to things such as people, cities or electronic devices. Edges may correspond to connections such as friendship, highways or power cables.
  - → For convenience, assume that all networks have at least one vertex.
- Edges connect exactly two vertices. Vertices on either side of an edge are said to be **neighbours**.
- The **degree** of a vertex is the number of edges connected to that vertex.
- A **loop** is an edge with the same vertex at both ends.
  - → A loop contributes two to the degree of that vertex.



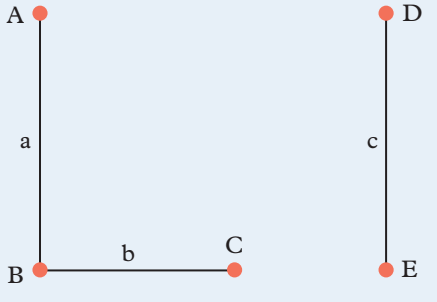
- The **handshaking lemma** (edges version) states that in every network the sum of vertex degrees is twice the total number of edges.
  - → Intuitively, both ends of an edge contribute to the overall sum of vertex degrees.

Walks

- A **walk** is a sequence of vertices and the edges between them, starting and ending with a vertex.
  - → A graph is said to be **connected** if there is a walk between every pair of vertices.



Connected network



Disconnected network

Example 8A.1 Determining the degree of vertices in a network

State the degree for each vertex in this network.

THINK

1. Count the number of edges connected to each vertex.
2. Remember that a loop adds two to the degree of a vertex.

WRITE

Vertex A has degree 2.

Vertex B has degree 4.

Example 8A.2 Drawing a network

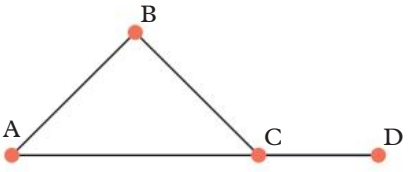
Annabel, Benjamin, Cecilia and Derek are students. Their friendships are given in the table below. Display this information in a network.

Student	Annabel	Benjamin	Cecilia	Derek
Friends	Benjamin, Cecilia	Annabel, Cecilia	Annabel, Benjamin, Derek	Cecilia

THINK

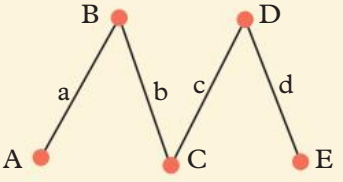
1. Draw a vertex for each student.
2. Label the vertices with the students' initials.
3. Draw an edge between students who are friends with each other.

WRITE



Example 8A.3 Finding a walk in a network

Find a walk from A to E in this network.



THINK

1. Start at vertex A.
2. Move along the edges and through the vertices of the network until you reach vertex E.
3. Write all the vertices and edges that were touched, in order, in your walk. Include the start and finish vertex in your list.

WRITE

A, a, B, b, C, c, D, d, E

Helpful hints

- ✔ Two networks may look quite different even if they contain the same information.
- ✔ When drawing a graph, make sure that the vertices and edges are legible and easy to interpret.

ANS p633

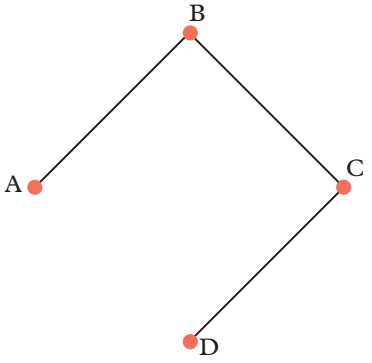
Exercise 8A Introduction to networks

1–4, 6, 7, 9, 12, 14–16 | 4–15 | 4–11, 14, 17, 18

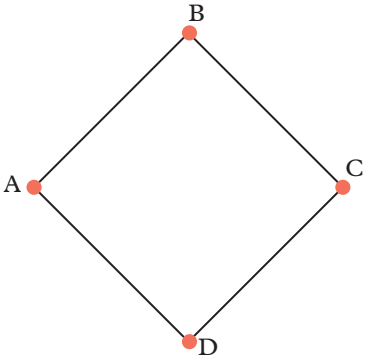
UNDERSTANDING AND FLUENCY

1 State the number of **i** vertices and **ii** edges in each of the following networks.

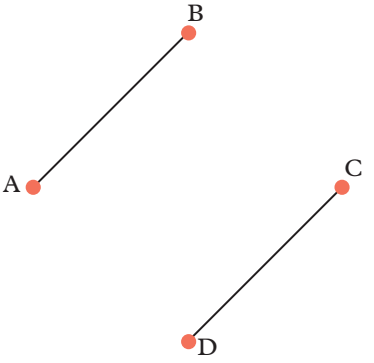
a



b



c



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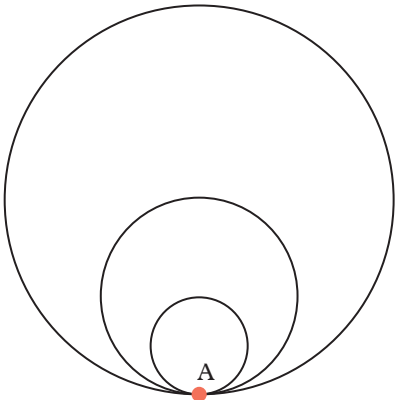
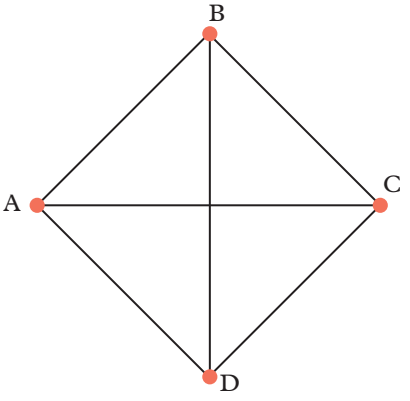
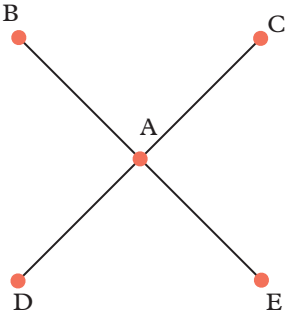


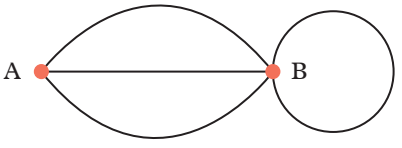
e



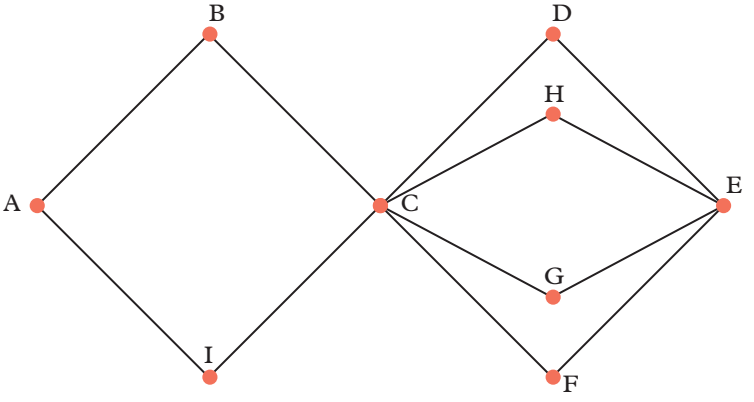
f



g



h



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8A.1 **2** Make a table of vertex degrees for each network in question **1**.

3 Which network in question **1** is disconnected?

8A.2 **4** The following table represents regional NSW airports and the destinations serviced by those airports. Display this information in a network.

Airport	Albury (ABX)	Bathurst (BHS)	Coffs Harbour (CFS)	Dubbo (DBO)	Evans Head (EVH)
Destinations	BHS, DBO	ABX, CFS, DBO	BHS, EVH	ABX, BHS	CFS

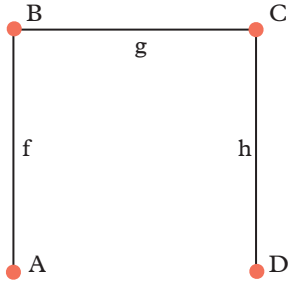
5 A house contains a printer, computer, tv and gaming console. All these devices are connected to a single router via ethernet cables.

a Draw a network diagram to represent this information. Use vertex names P, C, T, G and R for the printer, computer, tv, gaming console and router, respectively, and use edges to represent ethernet connections between the devices.

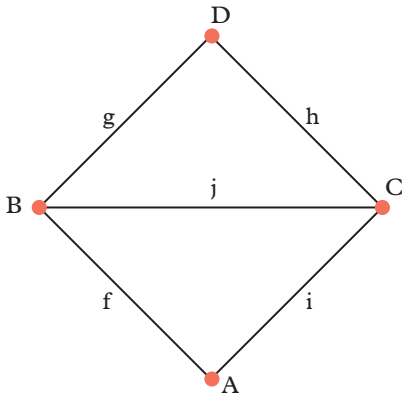
b How many ethernet connections are required at R?

8A.3 **6** Which networks contain a walk from A to D? State the vertices and edges in that walk (if it exists).

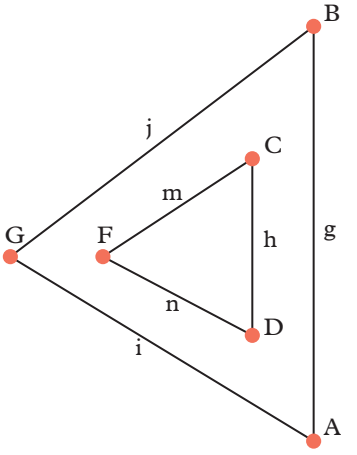
a



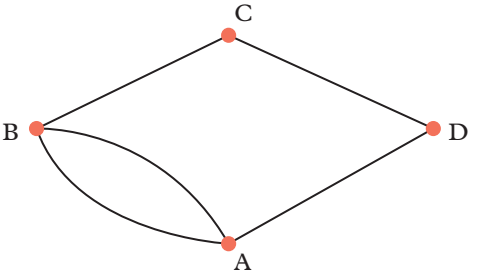
b



c



7 Consider the following network, where the vertices represent 4 people at a party and the edges represent handshakes.



a How many handshakes did each person make?

b What is the total number of handshakes?

8 What is the sum of the vertex degrees in the network from question **7**?

9 A network has five vertices and each vertex has degree 4. How many edges does it have?

10 Why is it impossible to have a graph with three vertices, where the vertex degrees are 3, 2 and 2?
Hint: If there were such a graph, how many edges must it have?

11 a Draw the network which represents the following list of people and their friends.

Person	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Friends	Vanessa, Zhang	Wasim, Uma	Vanessa, Zhang, Yvonne, Xavier	Wasim, Yvonne	Wasim, Xavier	Uma, Wasim

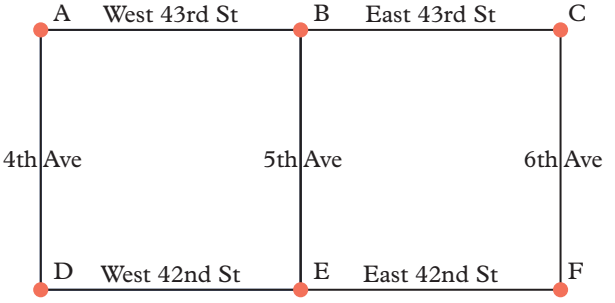
b Fill out the table of vertex degrees.

Vertex	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Degree						

c Uma is telling Vanessa a story about what happened to a friend of a friend of hers. Use your answer to part **a** to find out who the story is about.

d Yvonne is telling Wasim a story about what happened to a friend of a friend of hers. Use your answer to part **a** to find out who the story is *not* about.

12 The streets of Manhattan are divided into East and West by 5th Avenue. The following network is a map of Manhattan between 4th and 6th Avenues and 43rd and 42nd streets.



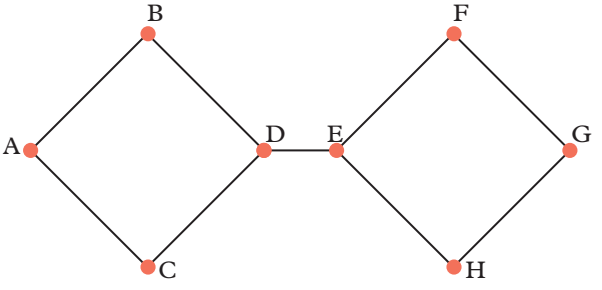
You are given some directions: Start at corner A. Follow West 43rd St to corner B. Follow 5th Avenue to corner E. Follow East 42nd St to corner F. Then follow 6th Avenue to corner C.

a Is this a valid walk?

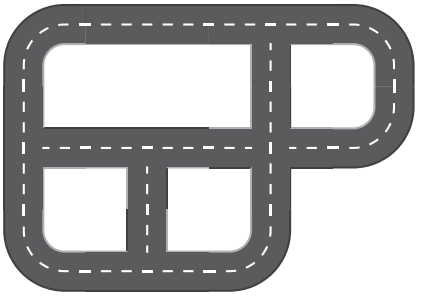
b If so, where does this walk lead to?

13 You now have a different set of directions for the network in question **12**: Start at corner A. Follow East 42nd St to corner F. Explain why this walk is not valid.

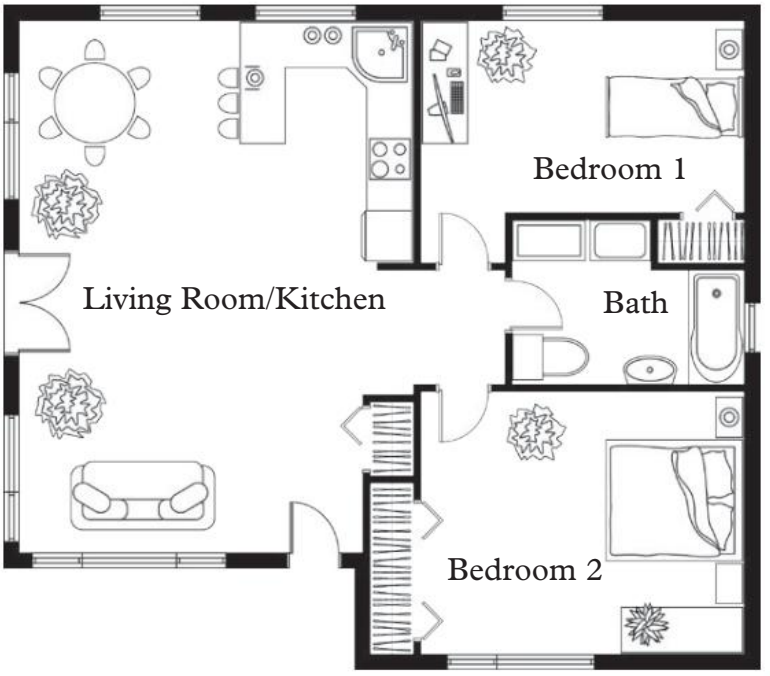
14 Disconnect this network by removing exactly one edge.



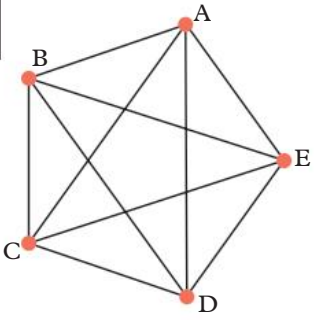
15 Redraw the roads below as a network using vertices to represent intersections, and edges to represent a span of road connecting two intersections.



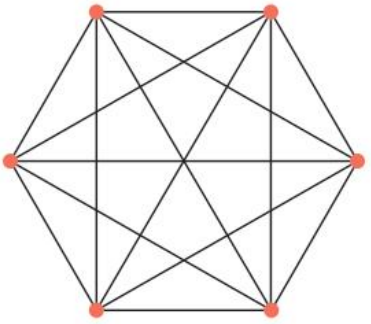
16 Draw the following house plan as a network. Use vertices to represent the Living Room/Kitchen, Bathroom, Bedroom 1 and Bedroom 2, and use edges to represent the doorways that connect the rooms.



17 Copy the network diagram on the right and use a red and blue pen to colour all the edges. Try to do this without drawing a red triangle or a blue triangle (i.e. three vertices where the edges between them are all the same colour).
Hint: First colour as many edges in red as you can without drawing a triangle, then do the same for blue.



18 a Copy the network below and colour all the edges blue or red.



b Find a red or blue triangle in your coloured network.

c Suppose you use a graph with coloured edges to represent whether dinner guests have met before or not: blue edges represent people who have met before, and red edges represent people who have not met before. If you are hosting a dinner party and want to ensure that either three people will all have met before, or three people will have never met before, what is the smallest number of people you should invite?
Hint: Compare your answers to questions **17** and **18b**.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Networks

Topic quiz
8A

8B Planar graphs

Learning intentions

By the end of this section, you will be able to ...

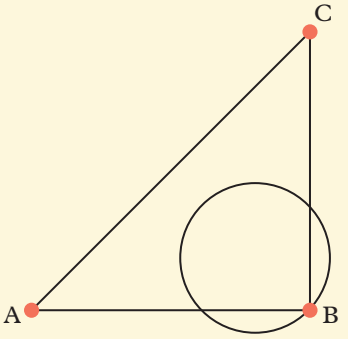
- ✔ identify planar graphs
- ✔ identify non-planar graphs
- ✔ use Euler's formula for connected planar graphs.

Planar graphs

- A **planar graph** is a graph that can be drawn on a sheet of paper without edges crossing.
  → If a graph is drawn with edges crossing but can be redrawn without edges crossing then it is a planar graph.
- A drawing of a planar graph without edges crossing divides a sheet of paper into regions, called **faces**, bordered by edges. There is always one infinite 'outside' face.
- The number of vertices (V), the number of edges (E) and the number of faces (F) in a connected planar graph are related by **Euler's formula**:
  $F + V = E + 2$.
- The **degree** of a face is the number of edges along its border.
- The handshaking lemma (faces version) states that in every planar graph the sum of face degrees is twice the number of edges.
  → Intuitively, both sides of an edge contribute to the overall sum of face degrees.
- Two edges with the same endpoints are parallel.
- A **simple graph** is a graph without loops or parallel edges.
- For a simple, connected graph with $V \geq 3$ vertices and E edges, if $E > 3V - 6$ then the graph is not planar.
  → If $E \leq 3V - 6$ then the graph may or may not be planar.

Example 8B.1 Determining whether a graph is planar

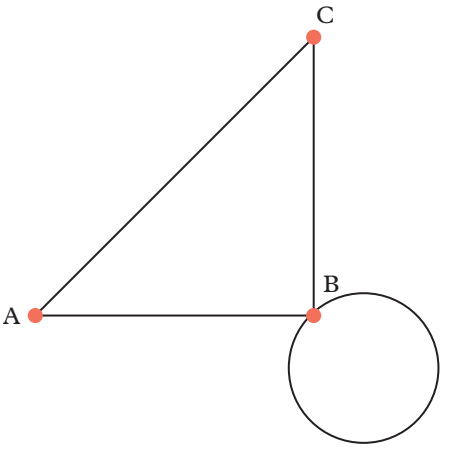
Try to redraw the following graph without edges crossing, and hence determine whether it is planar or not.



THINK

If the graph can be redrawn without edges crossing, it is a planar graph.

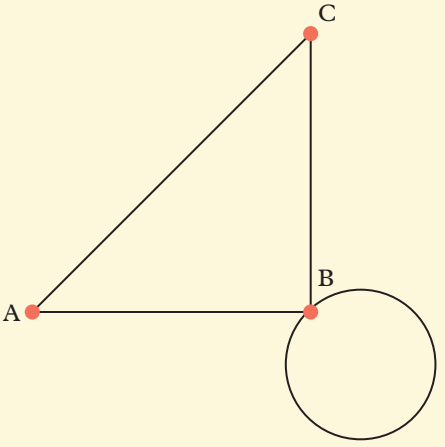
WRITE



The graph can be redrawn without edges crossing, so it is a planar graph.

Example 8B.2 Using Euler's formula to find the number of edges

Consider the connected planar graph from Example 8B.1 (reproduced here).



a Count the number of vertices in the graph.

b Count the number of faces in the graph.

c Use Euler's formula (instead of counting) to find the number of edges in the graph.

THINK

a There are three vertices: A, B and C.

b There are three faces: inside the triangle, inside the circle, and outside both shapes.

c Euler's formula states $F + V = E + 2$.

WRITE

a $V = 3$

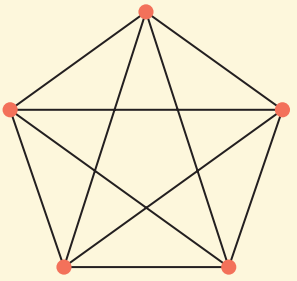
b $F = 3$

c $3 + 3 = E + 2$

$E = 4$

Example 8B.3 Using $E > 3V - 6$ to prove a graph is not planar

Consider this simple and connected graph.



Use the number of edges and vertices to prove this graph is not planar.

THINK	WRITE
1 Count the number of vertices in the graph.	$V = 5$
2 Count the number of edges in the graph.	$E = 10$
3 If the number of edges in the graph exceeds $3V - 6$, then the graph is not planar.	$3V - 6 = 9$ As $E > 3V - 6$ $(10 > 9)$, the graph is not planar.

Helpful hints

- ✔ $3V - 6$ is an upper limit on the number of edges a simple, connected, planar graph can have. If the number of edges exceeds this limit, then there must be some edges crossing and the graph cannot be planar. If the number of edges does not exceed this limit, we still don't know if the graph is planar or not.
- ✔ The handshaking lemma (faces version) is true for connected and disconnected planar graphs, whereas Euler's formula only applies to connected planar graphs.

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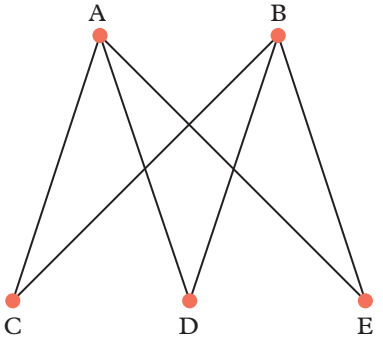
Exercise 8B Planar Graphs

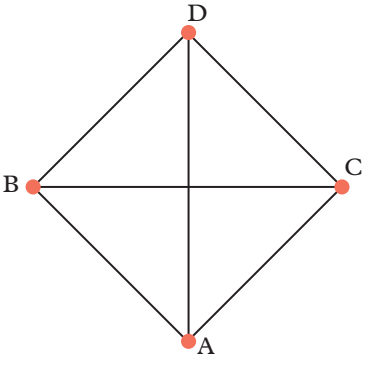
△ 1–5, 9, 11, 14, 17 □ 1–11, 13, 15, 17 ⬠ 1–8, 10–13, 15–17

8B.1 **1** Try to redraw the following graphs without edges crossing, and hence determine whether they are planar or not.

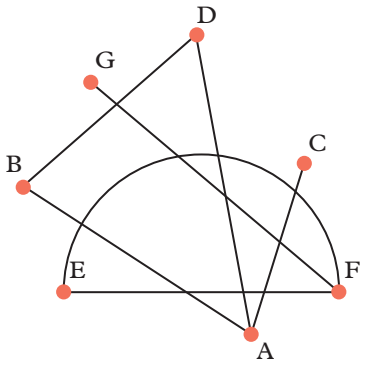
a



b



c



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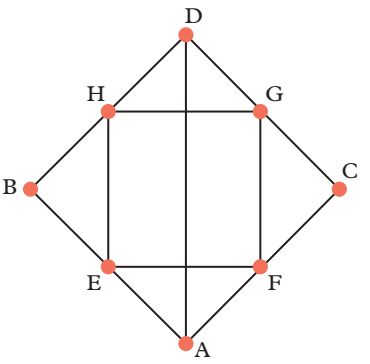
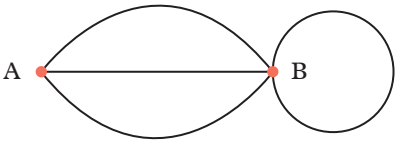


8B.2 **2** Consider the connected graph on the right.



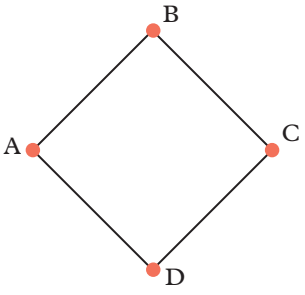
a Count the number of vertices in the graph.

b Count the number of faces in the graph.

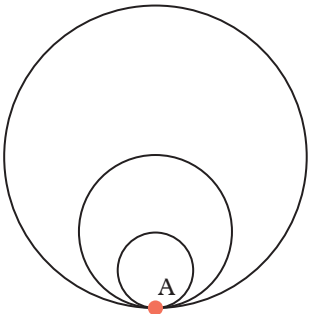
c Use Euler's formula (instead of counting) to find the number of edges in the graph.

3 Count the number of edges, faces and vertices in the graphs below and show Euler's formula is satisfied.

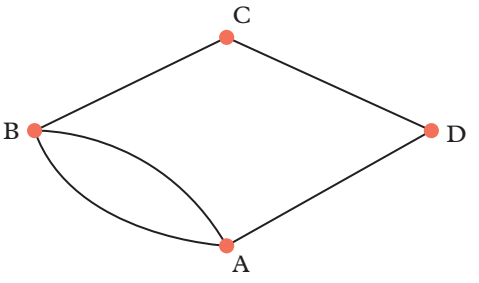
a



b



c



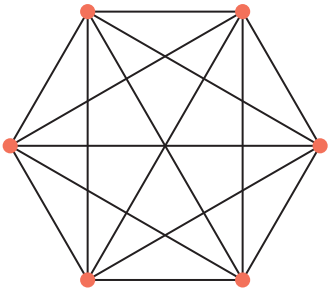
4 For the networks in question **3**, verify that the sum of face degrees is twice the number of edges.

5 Which networks from question **3** are simple?

6 A connected planar graph has 2 vertices and 4 faces. How many edges does it have?

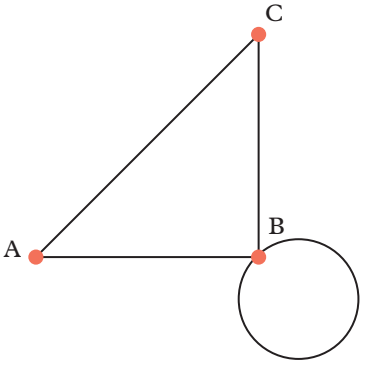
7 Why does the graph from question **1c** not satisfy Euler's formula?

8B.3 **8** Consider this simple and connected graph.



Use the number of edges and vertices to prove this graph is not planar.

9 Consider this graph from Example 8B.2 which has 4 edges.



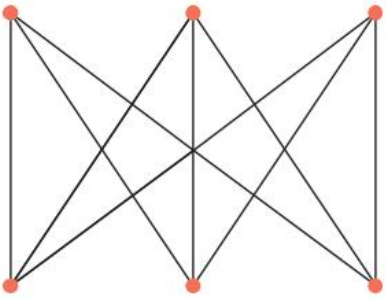
a State the degree of each face.

b Find the sum of the face degrees.

c How is the sum of the face degrees related to the number of edges?

PROBLEM SOLVING AND REASONING

10 If there are *fewer* than $3V - 6$ edges, a graph might be planar or not planar. For example, consider the following non-planar graph.



a Count the number of edges and vertices.

b Is this graph simple?

c Is this graph connected?

d Show that this non-planar graph has fewer than $3V - 6$ edges.

11 Two houses need to be connected to three underground utilities: power, water and internet. Is it possible to make the connections without utility lines crossing?

12 Repeat question **11** with three houses instead of two.

13 A simple, connected, planar graph has 3 vertices.

a What is the largest number of possible edges?

b Draw this graph.

14 The vertex degrees of a connected planar graph are 5, 3 and 2. How many faces does the graph have?
Hint: determine the number of edges and then use Euler's formula or draw the graph.

15 Can a connected, simple, planar graph have 5 faces and 4 vertices? Explain your answer.

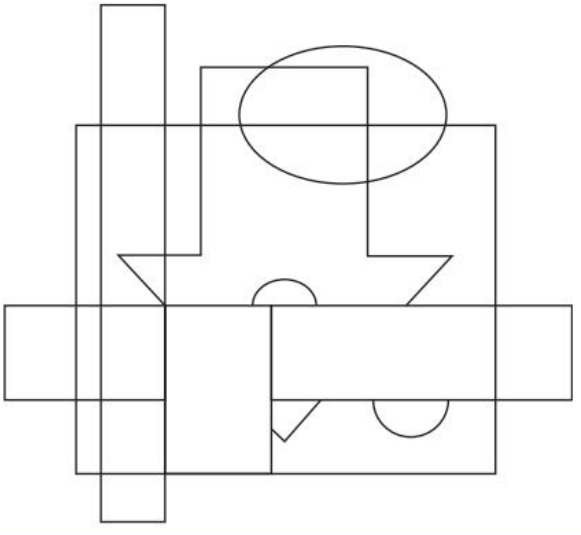
16 a What does a face of degree 1 look like?

b What does a face of degree 2 look like?

c Given that a simple graph has no loops or parallel edges, what is the smallest face degree that a simple graph can have?

CHALLENGE

17 The four-colour theorem states only four colours are needed to colour a planar graph without loops so that neighbouring region have different colours. Colour in this planar graph using only four colours. Remember to colour the outside face.



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Interactive skillsheet
Planar graphs

BLM
Four-colour theorem

Topic quiz
8B

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8C Platonic solids

Learning intentions

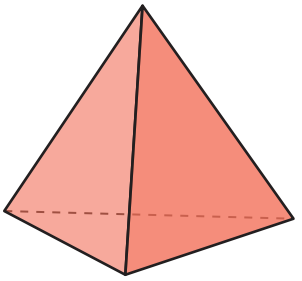
By the end of this chapter, you will be able to ...

✔ identify the five Platonic solids

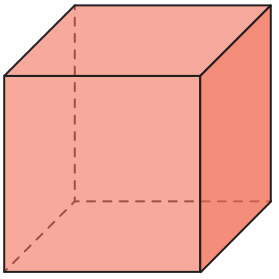
✔ think of the Platonic solids as planar graphs.

Platonic solids

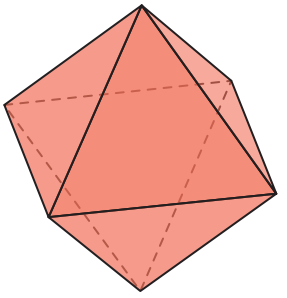
- The **Platonic solids** get their name from the famous philosopher Plato, who associated them with the four elements: fire, earth, air, water. A fifth Platonic solid was associated with the heavens and constellations. These are:



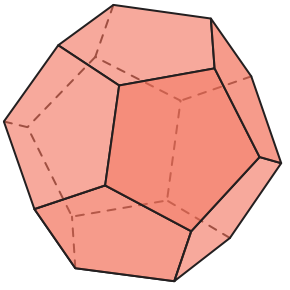
Tetrahedron (fire)



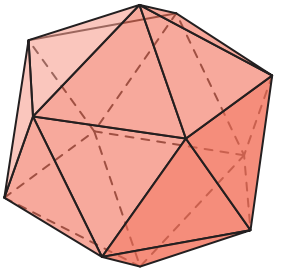
Cube (earth)



Octahedron (air)



Dodecahedron (heavens)

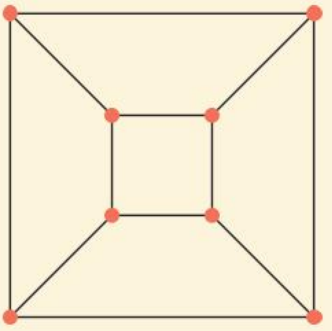


Icosahedron (water)

- A Platonic solid is a convex three-dimensional shape where each face is a polygon with n equal sides, and where m faces meet at each corner.
  → Each face is identical in shape and size.
- There are only five Platonic solids. This was postulated by Euclid in approximately 300 BCE.
- Each Platonic solid can be viewed as a planar graph.
  → Imagine a Platonic solid with vertices at each corner and edges around each face. Now puncture a hole in one of the faces and peer through that hole. You will see a simple, connected, planar graph.

Example 8C.1 Viewing Platonic solids as planar graphs

Imagine that we puncture a hole in one of the faces of a Platonic solid. Peer through that hole into the shape. Each Platonic solid will appear as a simple, connected, planar graph. We represent each side of the face with an edge, and the point at which faces meet with a vertex. If you peer inside a cube you will see something like this:



Use the handshaking lemma (faces version) to calculate the number of edges on the cube.

THINK	WRITE
1 The graph is planar, so the handshaking lemma (faces version) gives the number of edges.	
2 A cube has six faces, and each face is bordered by 4 edges and therefore has a degree of 4.	The sum of face degrees is $4 + 4 + 4 + 4 + 4 + 4 = 24$.
3 Half of the sum of face degrees is 12, which is the number of edges.	Therefore, there are $\frac{24}{2} = 12$ edges.

Example 8C.2 Using Euler's formula to find the number of vertices

Use Euler's formula to find the number of vertices in the graph from Example 8C.1

THINK	WRITE
1 From the previous example, we know the number of faces and edges.	$F = 6, E = 12$
2 The graph is connected and planar, so Euler's formula holds.	$F + V = E + 2$ $6 + V = 12 + 2$
3 Solve Euler's formula for V.	$V = 12 + 2 - 6$ $V = 8$

Helpful hints

✔ Each face of the cube is a square, so each face has degree 4. This makes it easy to calculate the sum of face degrees as 4 times the number of faces. Therefore, the handshaking lemma (faces version) can be simplified to:

$4 \times$ number of faces = sum of face degrees = $2 \times$ number of edges

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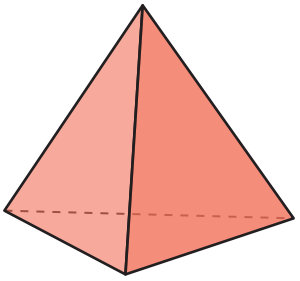
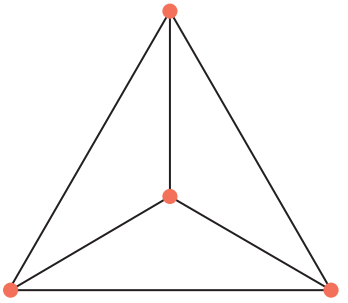
Exercise 8C Platonic solids

1–9, 13 1, 2, 5–12, 14–16 1, 3, 5, 9–12, 14, 16, 17

8C.1 **1** Peer inside the tetrahedron and you will see a simple, connected, planar graph with four faces.





Use the handshaking lemma (faces version) to calculate the number of edges on the tetrahedron.

2 Label the following statements as true or false.

a Every Platonic solid corresponds to a planar graph.

b Every planar graph corresponds to a Platonic solid.

3 **a** Which Platonic solids have triangular (3-sided) faces?

b Which have square (4-sided) faces?

c Which have pentagonal (5-sided) faces?

4 **a** Which Platonic solids have 3 faces meeting at each corner?

b Which have 4 faces meeting at each corner?

c Which have 5 faces meeting at each corner?

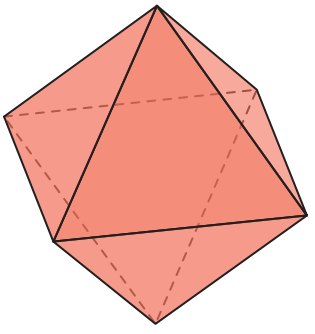
5 The game of Dungeons & Dragons has five kinds of dice known as d4, d6, d8, d12 and d20. The number after 'd' indicates the number of faces the die has. Match these dice with the Platonic solids.

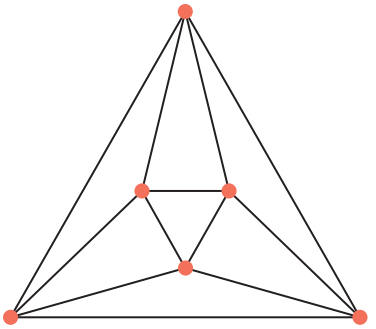
6 If the faces of a Platonic solid have degree 3, what solid could it be?

7 A connected planar graph has three faces. Each face has degree 4. How many edges does it have?

8C.2 **8** Use Euler's formula to find the number of vertices in the graph from question **7**.

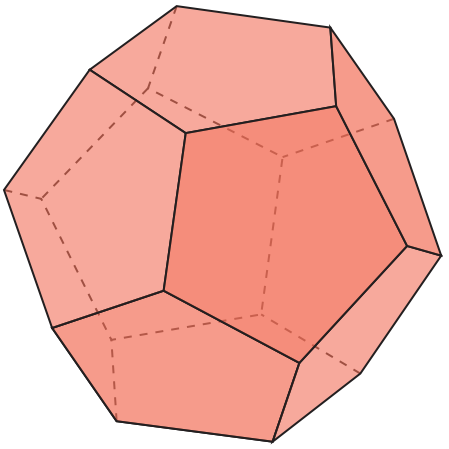
9 Repeat question **1** for the octahedron, given that it has 8 faces.

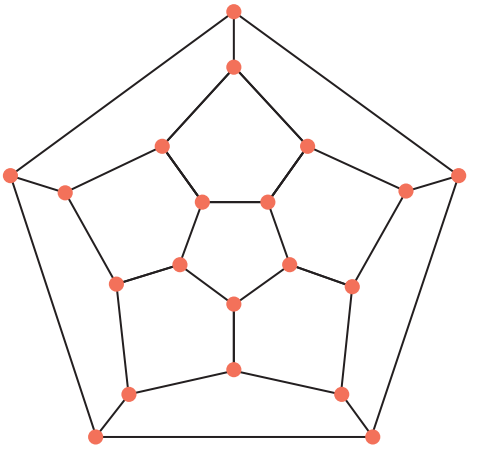




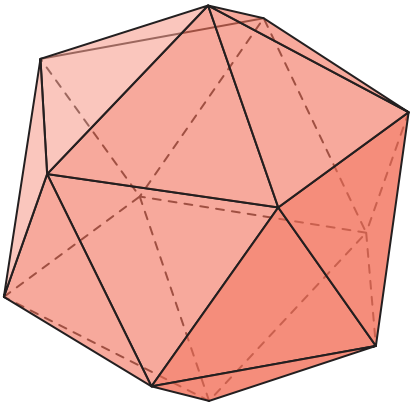
UNDERSTANDING AND FLUENCY

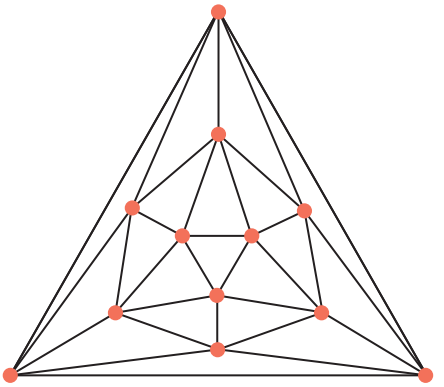
10 Repeat question **1** for the dodecahedron, given that it has 12 faces.





11 Repeat question **1** for the icosahedron, given that it has 20 faces.





12 Use Euler's formula and your answers to questions **1**, **9**, **10** and **11** to complete the following table.

Platonic solid	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Vertices		8			
Faces	4	6	8	12	20
Edges		12			

13 For the cube:

a state the sequence of vertex degrees

b state the sequence of face degrees

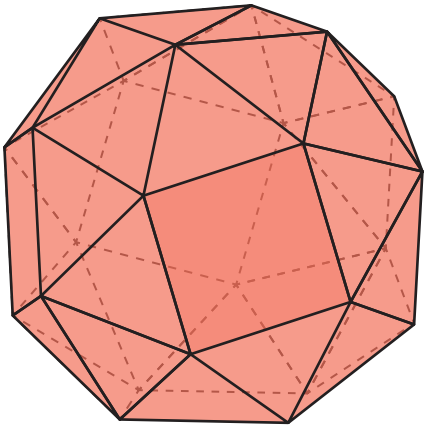
c show both sequences sum to 24.

14 One unique feature of the Platonic solids is that each face is identical. Other solids can also be created, but the faces will be different. For example, the snub cube is a non-Platonic solid with two kinds of faces. It has 6 square faces and 32 triangular faces.

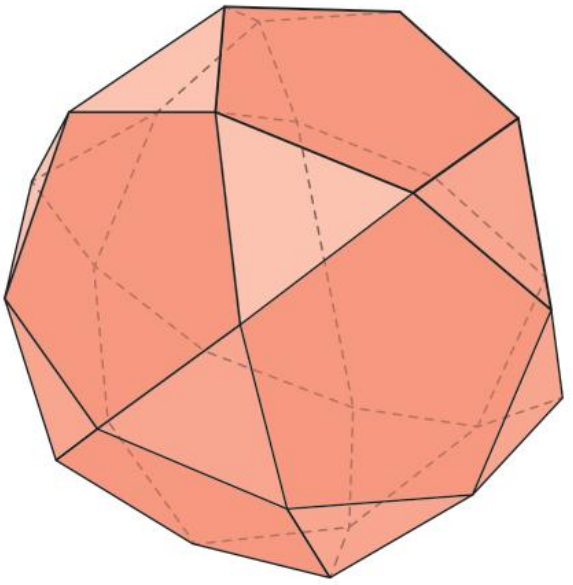
a What is the sum of the face degrees?

b How many edges does it have?

c How many vertices does it have?



15 Repeat question **14** for the icosidodecahedron, given it has 20 triangular faces and 12 pentagonal faces.



16 Each Platonic solid can be imagined as a planar graph with V vertices, E edges and F faces. Earlier in this section we used the handshaking lemma to show if each vertex of a Platonic solid has degree $m \geq 3$ then

$2 \times E$ = sum of vertex degrees = $m \times V$.

Similarly, if each face of the Platonic solid is a polygon with $n \geq 3$ equal sides then

$2 \times E$ = sum of face degrees = $n \times F$.

Putting this into Euler's formula, and rearranging, gives

$\frac{1}{E} = \frac{1}{n} + \frac{1}{m} - \frac{1}{2}$

a Fill out the missing entries in the following table with the value of $\frac{1}{n} + \frac{1}{m} - \frac{1}{2}$.

	$n = 3$	$n = 4$	$n = 5$
$m = 3$	$\frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$		
$m = 4$			
$m = 5$			$\frac{1}{5} + \frac{1}{5} - \frac{1}{2} = -\frac{1}{10}$

b Which entries in this table have a positive value for $\frac{1}{E}$? How do these entries relate to the Platonic solids?

c Is it possible to have a graph where $\frac{1}{E}$ is negative?

d Does this means Euclid was right and our five shapes are the only Platonic solids?

17 Why don't we need to consider $m \geq 6$ in the previous question?

Hint: This would force $n < 3$; compare with 8B question **16**.

Check your Student obook pro for these digital resources and more:

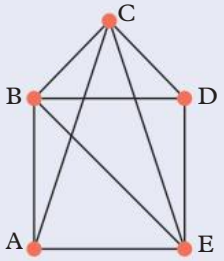
- Interactive skillsheet: Platonic solids
- Investigation: Constructing solid shapes
- Topic quiz: 8C

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

8A **1** Fill out the table of vertex degrees for the network below

Vertex	A	B	C	D	E
Degree					



8A **2** Is the network in question **1** connected?

8A **3** Explain why there is no network with a vertex degree sequence 4, 3, 2?

8B **4** Is the network in question **1** simple?

8B **5** Is the network in question **1** planar?

8B **6** Consider a simple, connected network of 6 vertices where each vertex has degree 5.

a How many edges does this graph have?

b Is this graph planar?

8B **7** Draw a network with 6 vertices, where each vertex has degree 3 and the graph is:

a planar

b not planar.

8B **8** Suppose the network in question **1** is planar. Using Euler's formula, how many faces should it have?

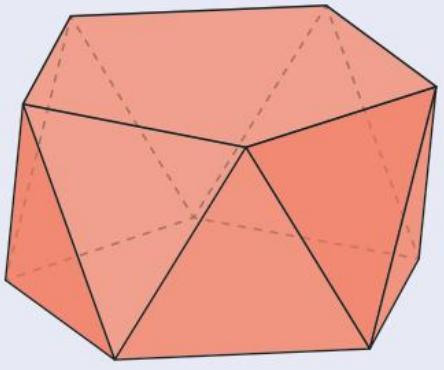
8C **9** In Dungeons & Dragons d4 represents a die with 4 sides and d20 represents a die with 20 sides. A chaotic goblin wizard taunts you and you take two d4 emotional damage.

a What is the maximum possible total of the numbers shown on two d4 dice?

b What is the smallest amount of emotional damage you can take?

c You attempt a witty retort to the chaotic goblin wizard; however goblins are thick-skinned creatures. You roll a d20 and if the die shows a score of 13 or more your retort is successful. Are you more likely to succeed or to fail?

8C **10** A pentagonal antiprism has two pentagonal faces and ten triangular faces.



a What is the sum of the face degrees?

b How many edges does it have?

c How many vertices does it have?

8D Eulerian trails and circuits

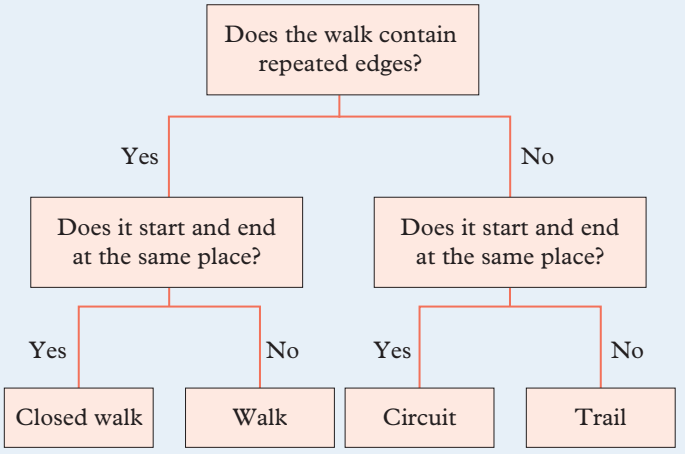
Learning intentions

By the end of this chapter, you will be able to ...

- ✔ identify trails and circuits
- ✔ know when a trail or circuit is Eulerian
- ✔ determine when Eulerian trails or circuits exist.

Eulerian trails and circuits

- Recall that a walk is a sequence of vertices and the edges between them. Some special kinds of walks have names.
  - → A **trail** is a walk without repeated edges.
  - → A **closed walk** is a walk with the same first and last vertices (i.e. ending where it started).
  - → A **circuit** is a trail with the same first and last vertices.
- **Eulerian trails** and **Eulerian circuits** are special because they contain every edge exactly once.
- There is an easy way to know when an Eulerian trial or circuit exists: just count the number of vertices with odd degree. Specifically, suppose your graph is connected and has n vertices with odd degree.
  - → If $n = 0$ then there must be an Eulerian circuit and the graph is **Eulerian**.
  - → If $n = 2$ then there must be an Eulerian trail and the graph is **semi-Eulerian**.
  - → Otherwise there is no Eulerian circuit or Eulerian trail, and the graph is not Eulerian.



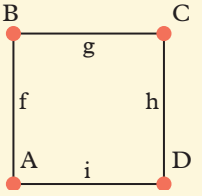
Example 8D.1 Classifying walks

For the network below, classify the following sequences as walks, trails, circuits, closed walks or invalid walks.

a A, i, D, i, A

b A, i, D, h, C

c A, i, C



THINK

a This sequence re-uses edge i, so it must be a walk. As it starts and ends with the same vertex (A), it is a closed walk.

b This sequence has no repeated edges. As it starts and ends with different vertices, it is a trail.

c Edge i does not connect vertices A and C, so this sequence is not possible.

WRITE

a This is a closed walk.

b This is a trail.

c This is an invalid walk.

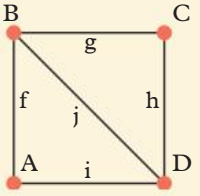
Example 8D.2 Finding an Eulerian circuit in a network

Find an Eulerian circuit, if one exists, in the network from Example 8D.1.

THINK	WRITE
1 Check the number of vertices with odd degree to see if an Eulerian circuit exists. There are zero vertices with an odd degree, so there must be an Eulerian circuit.	
2 Start at A, then move clockwise around the network until you return to A. This is a circuit because it starts and ends at the A without re-using edges. This circuit is Eulerian because it touches all edges exactly once.	An Eulerian circuit for this network is A, f, B, g, C, h, D, i, A.

Example 8D.3 Finding an Eulerian trail in a network

Find an Eulerian trail, if one exists, in the network below.



THINK	WRITE
1 Check the number of vertices with odd degree to see if an Eulerian trail exists. There are two vertices with an odd degree, so there must be an Eulerian trail.	
2 Start at B, then move clockwise around the network until you return to B. Then use the diagonal edge to finish at D. This is a trail because edges are not re-used. This trail is Eulerian because it touches all edges exactly once.	An Eulerian trail for this network is B, g, C, h, D, i, A, f, B, j, D.

Helpful hints

- ✔ A visual way to understand walks, trails and circuits is to place your finger on a vertex, then move your finger along the graph and finish at any vertex. Everything that your finger touches (i.e. both vertices and edges) is a walk. Trails and circuits are just special kinds of walks.
- ✔ If a trail or circuit contains every edge exactly once, then it said to be Eulerian.
- ✔ If a graph has exactly 2 vertices of odd degree then these are the start and endpoints of the Eulerian trail.

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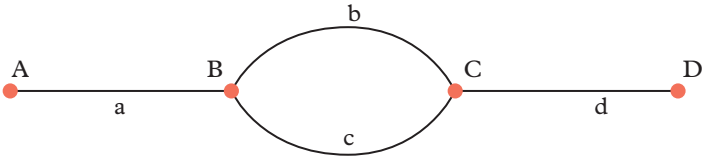
Exercise 8D Eulerian trails and circuits

1, 2, 4–11, 13, 15, 16 | 2, 4–12, 14, 15 | 2–4, 6, 7, 9–12, 14, 17

1 Consider the following network.



Remember from Section 8A that vertices are neighbours if they are connected by an edge.

a List the neighbours of B.

b Is A a neighbour of C?

8D.1 **2** For the network in question **1**, classify the following sequences as walks, trails, circuits, closed walks or invalid walks.

a A, a, B, c, C **b** A, a, B, b, D **c** A, b, B, a **d** B, c, C, b, B, c, C

3 Consider two different walks from A to D in the network from question **1**.

One walk is: A, a, B, b, C, d, D.

The other walk is: A, a, B, c, C, d, D.

a Write the walks as sequences of vertices only.

b In this example, is it a good idea to abbreviate walks to just a sequence of vertices?

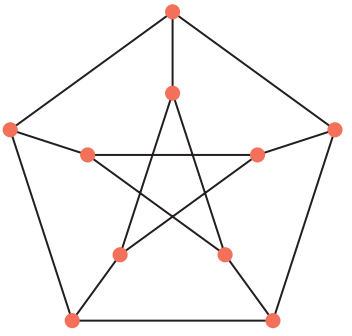
4 For the network in question **1**, consider the walk A, a, B, b, C, c, B, a, A.

What kind of walk is this?

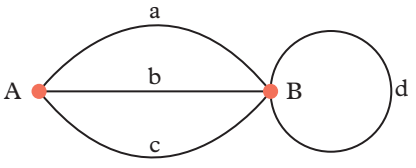
5 Find a circuit in the network from question **1**.

8D.2 **6** Find an Eulerian circuit for the network in Example 8D.1 that is different from that given in Example 8D.2.

7 Show that the following network is not Eulerian by counting the number of vertices with odd degree.

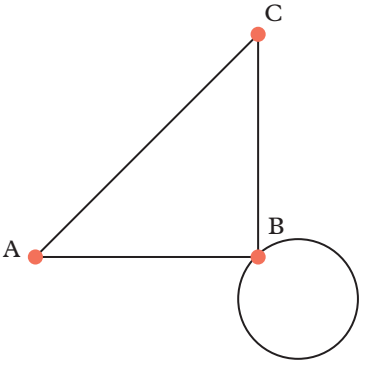


8 Show that the following network is semi-Eulerian by counting the number of vertices with odd degree.



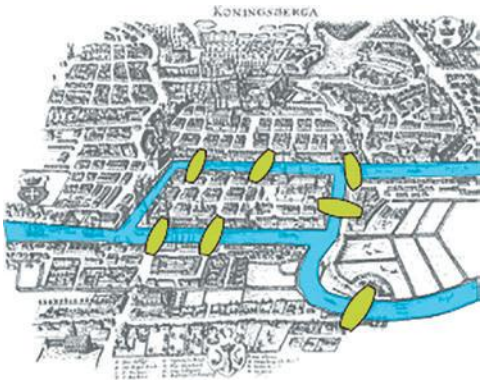
9 Find an Eulerian trail, if one exists, in the network from question **8**.

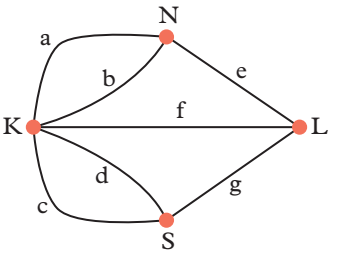
10 Show that the network on the right is Eulerian by counting the number of vertices with odd degree.



UNDERSTANDING AND FLUENCY

11 The old city of Königsberg lies on the river Pregel. The city has seven bridges, as shown to the right. Work through this problem to see if there is there a walk through Königsberg which crosses every bridge exactly once.





We represent city regions with vertices: N for the North side of the river, S for the south side, K for Kneiphof Island and L for Lomse Island. The bridges are represented by edges, and we label them a, b, c, d, e, f and g. The network above shows the city regions as vertices and the bridges between them as edges. The question can be rephrased: does this graph have an Eulerian trail?

The answer to the seven bridges of Königsberg problem is far from obvious. In 1736 Leonhard Euler said the problem has nothing to do with how the edges are connected. The only thing that matters is the number of vertices with odd degree.

a Fill out the table of vertex degrees.

Vertex	N	K	S	L
Degree				

b How many vertices have odd degree?

c Is the graph Eulerian, semi-Eulerian or not Eulerian?

d Is there a walk through Königsberg which crosses every bridge exactly once? Explain your answer.

12 The mayor of Königsberg wants to encourage tourism with a grand tour of the bridges. Tour groups must start and end outside the mayor's offices in the northern part of the city and traverse every bridge exactly once.

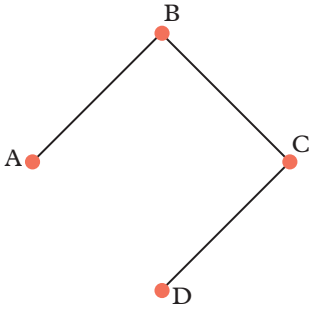
a The mayor will allow you to build one additional bridge. How can you explain to the mayor that the task is still impossible?

b The mayor really wants this to happen and has secured funding for two additional bridges, on the condition that both of them start from the northern side of the river. Is the task now possible? Explain your answer.

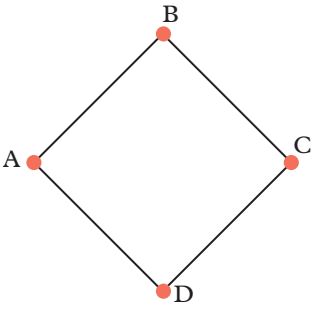
c The mayor is willing to budge a little and as well as the two additional bridges will allow the tour to finish in any location, provided it starts from the mayor's offices. Is the task now possible? Explain your answer.

13 Determine whether the following networks are Eulerian, semi-Eulerian or not Eulerian by counting the number of vertices with odd degrees.

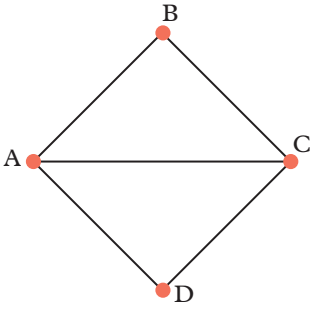
a



b



c



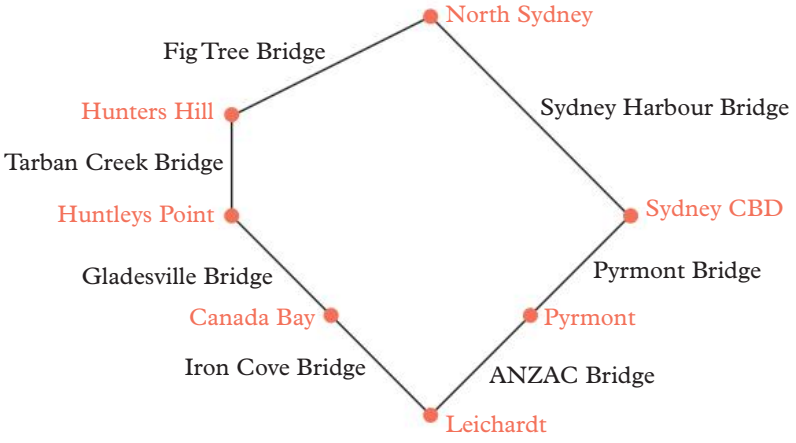
14 Which Platonic solid has an Eulerian circuit along its edges?

15 Design a garbage truck route that visits every street in these maps exactly once or explain why this task is impossible.

a

b

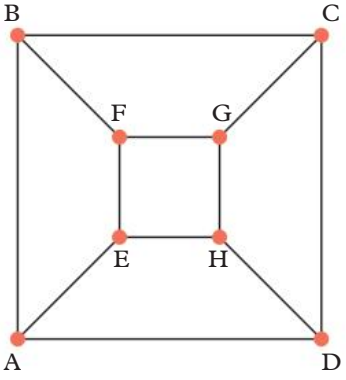
16 In 2017, the 'Seven Bridges Walk' toured some of the scenic bridges of Sydney.



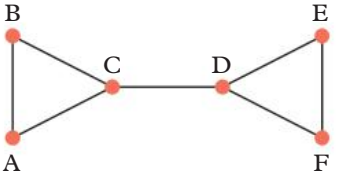
Count the vertex degrees, then state why it is possible to cross the seven bridges of Sydney without visiting the same bridge more than once.

17 A Hamiltonian circuit is a circuit that visits every vertex exactly once. Unlike Eulerian circuits, a simple way to determine if a graph has a Hamiltonian circuit has never been found.

a Find a Hamiltonian circuit for the cube.



b Is there a Hamiltonian circuit in the graph below? Explain your answer.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Eulerian trails and circuits

Topic quiz
8D

8E Applications of networks

Learning intentions

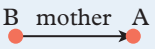
By the end of this chapter, you will be able to ...

✔ use directed edges in networks

✔ model authentic real-world situations using networks.

Directed networks

- A **directed edge** is an edge that has a direction indicated by an arrow.
  → For example, suppose person B is the mother of person A. We can represent this directional relationship with a directed edge. The arrow on the tip of the edge represents the direction of the 'mother of' relationship



- A **directed network** is a network with directed edges.

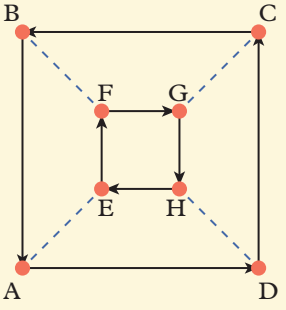
Example 8E.1 Using networks to represent kin systems

The Warlpiri are a people whose country is found in Central Australia around the Tanami Desert. The following kin system is inspired by the Warlpiri people and has been simplified for the purposes of this example. Suppose everyone belongs to one of eight kin groups A, B, C, D, E, F, G or H, and that:

- Marriage is recommended between kin groups A and E, B and F, C and G, or D and H. Recommended marriages are represented by the dashed blue edges in the network below. Here and throughout we will simplify things by assuming all children are born from recommended marriages.
- The kin group of the mother determines the group of all her children.

	Kin group							
Mother	A	B	C	D	E	F	G	H
Children	D	A	B	C	F	G	H	E

The directed edges in the network below represent the 'mother of' relation.

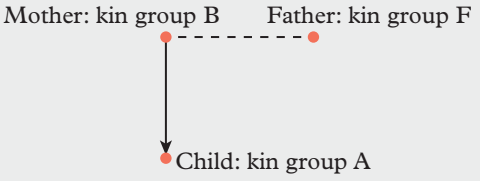


What is the kin group of the parents for children in kin group A?

THINK	WRITE
1 Follow the 'mother of' arrow backwards from A to find the kin group of their mother.	Their mother is in kin group B.
2 Once you know the kin group of their mother, follow the dashed line to find the recommended kin group of her partner.	Their father is in kin group F.

Helpful hints

✔ Draw a family tree, and then decorate that tree with kin groups.



ANS p636

Exercise 8E Applications of networks

1–3, 6–9 | 1, 3, 5, 10–13 | 1, 3–5, 10–12, 14

Use the kin diagram in Example 8E.1 to answer questions **1** to **5**.

8E.1

1 Suppose you are in kin group D.
- **a** What is the kin group of your siblings?
- **b** What is the recommended kin group of your spouse?
- **c** What is the kin group of your sister's children?
- **d** What is the kin group of your brother's children?

2 Repeat question **1**, supposing that you are in kin group H.

3 Complete this paternity table. Assume that children are only born from recommended marriages.

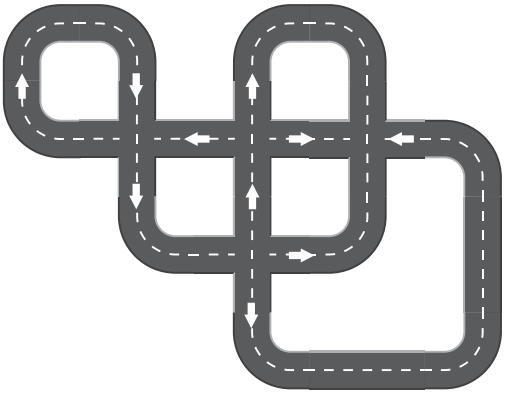
	Kin group							
Father	A	B	C	D	E	F	G	H
Children	F					A		

4 Suppose you belong to kin group A. What is the kin group of:
- **a** your mother's sister's children
- **b** your mother's brother's children
- **c** your father's sister's children
- **d** your father's brother's children?

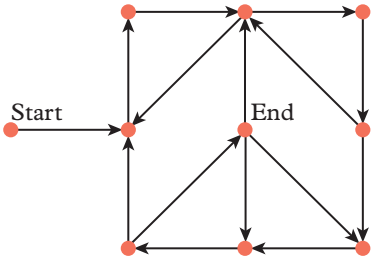
5 Complete the following table.

	Kin group							
You	A	B	C	D	E	F	G	H
Cousins	A or G						G or A	

6 Draw this system of one-way streets as a directed network using labelled vertices to represent intersections and directed edges to represent one-way streets.



7 Find your way to the end of the labyrinth. You can only travel along edges in the direction of the arrows.



a How many vertices did you pass on your way?

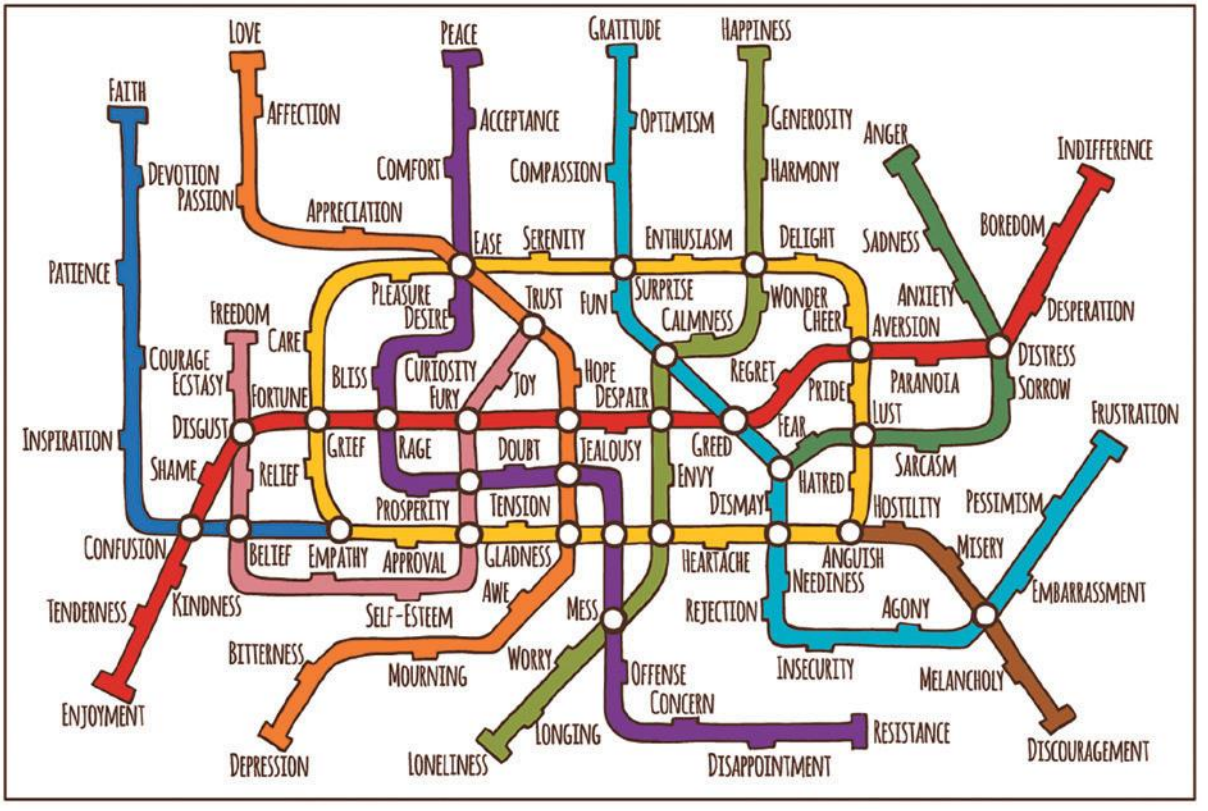
b How many edges did you travel through?

8 In the underground map of feelings, how many stops does it take to get from:

a Comfort (purple line) to Concern (purple line).

b Heartache (yellow line) to Serenity (yellow line). Is it faster to pass through Pride or Tension?

c Indifference (red line), changing at Grief, to Empathy (yellow line).



Use the kin diagram in Example 8E.1 to answer questions **9** to **12**.

9 A child is born into kin group A. What is the kin group of the:

a maternal grandmother **b** maternal grandfather

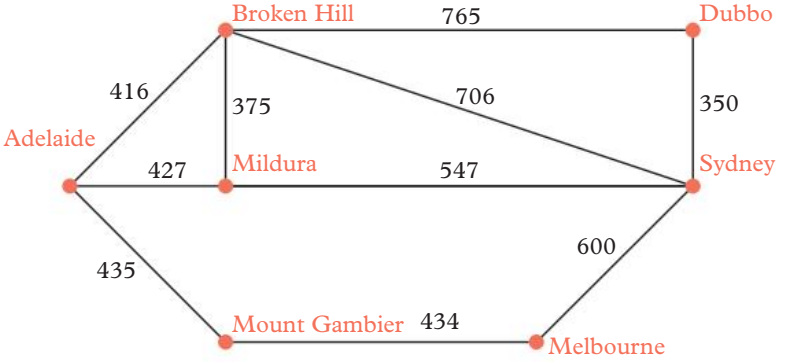
c paternal grandmother **d** paternal grandfather?

10 Are you always in the same kin group as your paternal grandfather?
Hint: Use the table from question **3**.

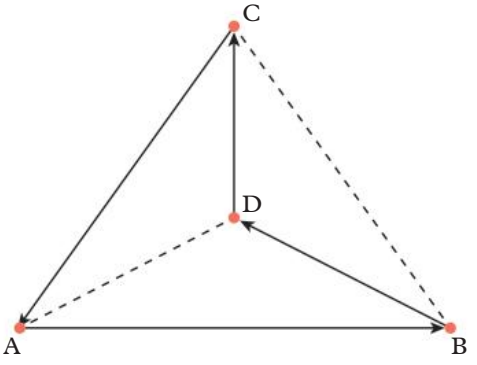
11 Are you ever recommended to marry your cousin?
Hint: Use the table from question **5**.

12 You are in kin group A and your mother re-marries. What is the kin group of your step-sisters and step-brothers?

13 Consider the network of regional flights. Each edge has a number that represents the cost (in $) of each route. What is the cheapest route from Dubbo to Mount Gambier?



14 Suppose we define a new kin system based on Example 8E.1. Instead of using a cube we use a tetrahedron with just four kin groups.



a Fill out the maternity table.

	Kin group			
Mother	A	B	C	D
Children	B			

b Fill out the paternity table.

	Kin group			
Father	A	B	C	D
Children	C			

c Can any other Platonic solids be used to define kin group relations?

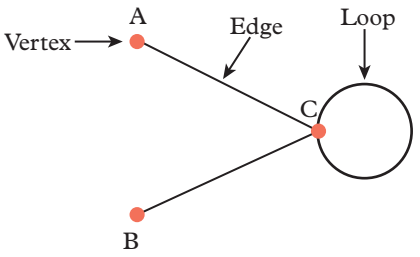
Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Directed networks

Topic quiz
8E

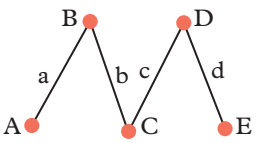
Chapter summary

Networks

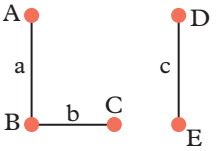


Walks

- Walk: sequence of vertices and the edges between them.
- Connected graph: a walk exists between every pair of vertices.



Connected network



Disconnected network

Planar graphs

- A graph that can be drawn without edges crossing.
- There is always one infinite 'outside' face.
- Euler's formula for connected planar graphs:
  $F + V = E + 2$

Handshaking lemma

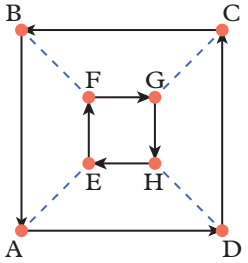
- Sum of vertex degrees = 2 × number of edges
- Sum of face degrees = 2 × number of edges

Simple graphs

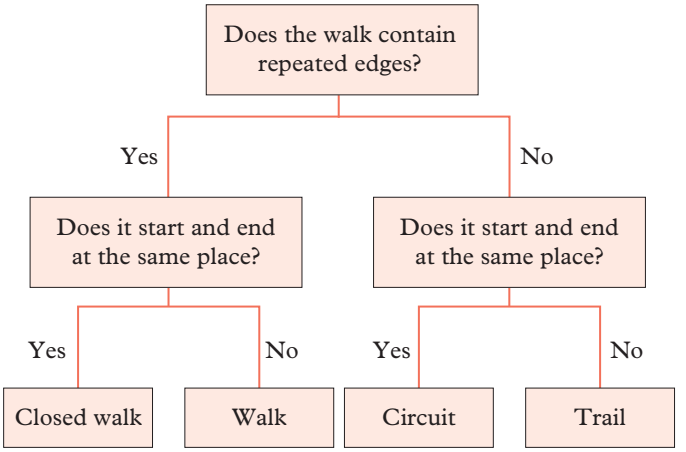
- No loops or parallel edges.
- Simple, connected graph with $V \geq 3$ vertices and E edges:
  → If $E > 3V - 6$ graph is not planar.
  → If $E \leq 3V - 6$ graph may or may not be planar.

Directed networks

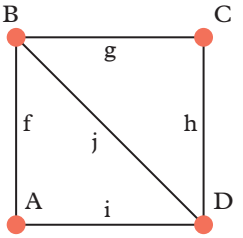
- A directed edge has a direction indicated by an arrow.
- A directed network is a network with directed edges.



Trials and circuits



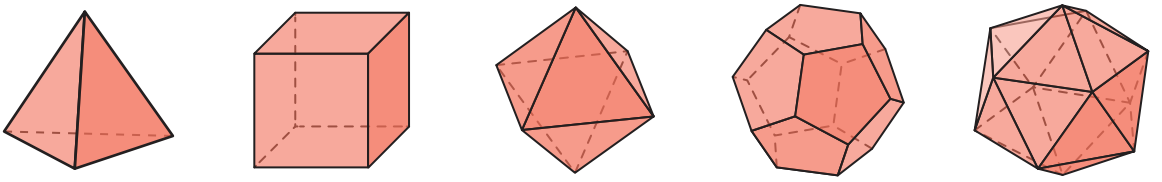
- Eulerian trails and Eulerian circuits contain every edge exactly once.
- If a connected graph has n vertices with odd degree:
  → if $n = 0$, there is an Eulerian circuit and graph is Eulerian
  → if $n = 2$, there is an Eulerian trail and graph is semi-Eulerian
  → otherwise no Eulerian circuit or trail and graph is not Eulerian.



There are two vertices of odd degree, so an Eulerian trail exists.

Platonic solids

- Convex, three-dimensional shapes where each face is a regular polygon.



Tetrahedron | Cube | Octahedron | Dodecahedron | Icosahedron

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Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

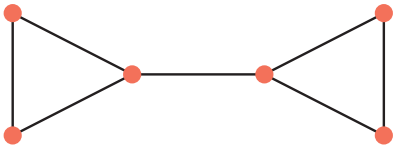
Multiple choice

8A **1** A graph has a vertex degree sequence of 3, 2, 1. How many edges does it have?

A 2 **B** 3 **C** 4 **D** 5 **E** 6

8A **2** What is the largest vertex degree in this network?

A 1
B 2
C 3
D 4
E 5

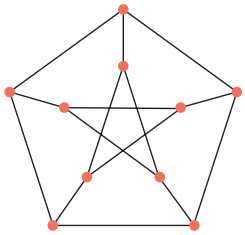


8B **3** What is the largest face degree in the network from question **2**?

A 4 **B** 5 **C** 6 **D** 7 **E** 8

8B **4** What is the largest number of edges that you can remove from this network without disconnecting the network?

A 3
B 4
C 5
D 6
E 7



8B **5** What is the smallest number of edges that must be removed from the network in question **4** to disconnect the network?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

8C **6** Which platonic solid has 12 faces?

A Tetrahedron **B** Cube **C** Octahedron **D** Dodecahedron **E** Icosahedron

8C **7** Which platonic solid has 12 vertices?

A Tetrahedron **B** Cube **C** Octahedron **D** Dodecahedron **E** Icosahedron

8C **8** Every vertex in a connected graph has an even degree. This means the graph is:

A not Eulerian **B** half-Eulerian **C** semi-Eulerian **D** Eulerian **E** Directed

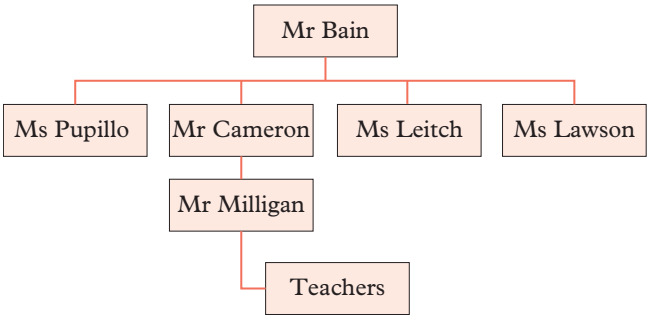
8D **9** Suppose three vertices in a graph have an odd degree, and the rest have an even degree. This graph is:

A not Eulerian **B** half-Eulerian **C** semi-Eulerian **D** Eulerian **E** Impossible

8E **10** The following organisational chart shows the hierarchy within Oban High School. A teacher reports a serious incident to Mr Milligan. How many steps will it take for this report to be handed up from Mr Milligan to Mr Bain?

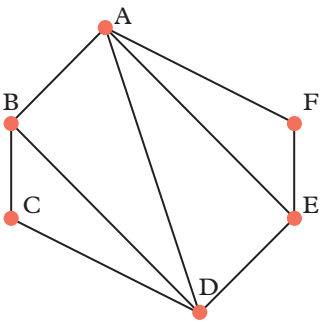
A 1
B 2
C 3
D 4
E 5

Oban High School Organisation Chart



Short answer

8A **1** Fill out the table of vertex degrees for the following network.

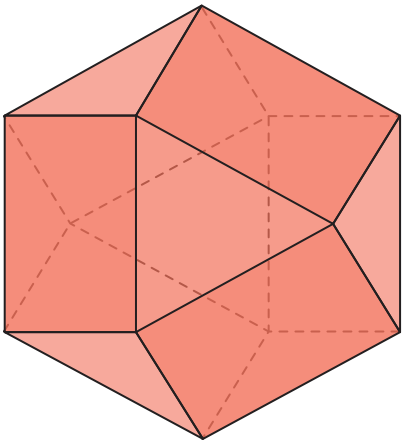


Vertex	A	B	C	D	E	F
Degree						

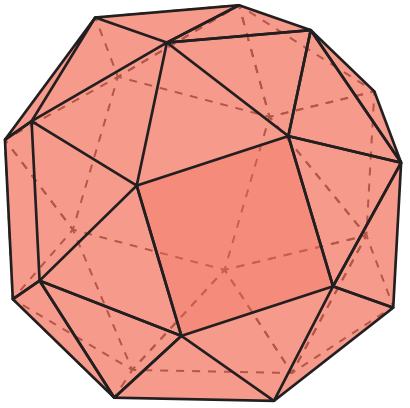
8B **2** The network from question **1** is planar. How many faces does it have?

8B **3** Use Euler's formula to find the number of edges in the network from question **1**.

8C **4** The cuboctahedron is related to the cube and octahedron. According to Archimedes, it may have been known to Plato. It has 6 square faces and 8 triangular faces. What is the sum of the face degrees?

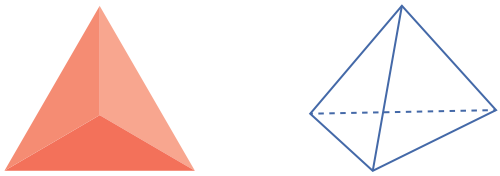
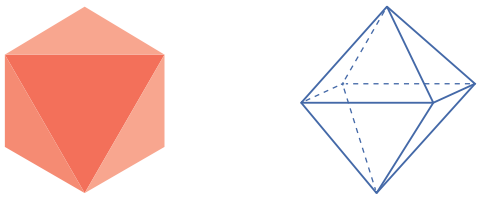
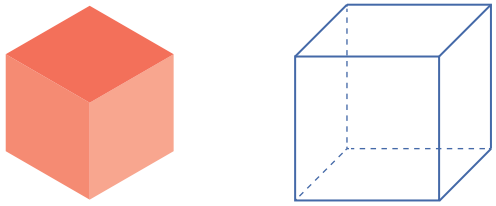
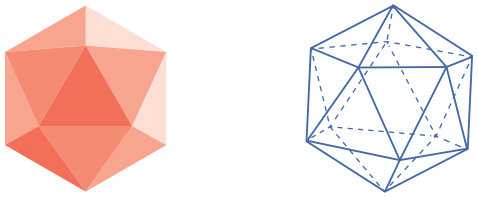
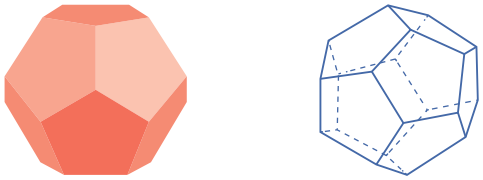


8C **5** The snub cube is a convex three-dimensional shape with 6 square faces and 32 triangular faces. Why is it not considered a Platonic solid?



8C **6** Why is it impossible for a planar graph to exactly have five faces, where each face has a degree of three?

8C **7** Fill out the table for the number of faces, vertices and edges of the Platonic solids.

Solid	Faces	Vertices	Edges
TETRAHEDRON 			
OCTAHEDRON 			
CUBE 			
ICOSAHEDRON 			
DODECAHEDRON 			

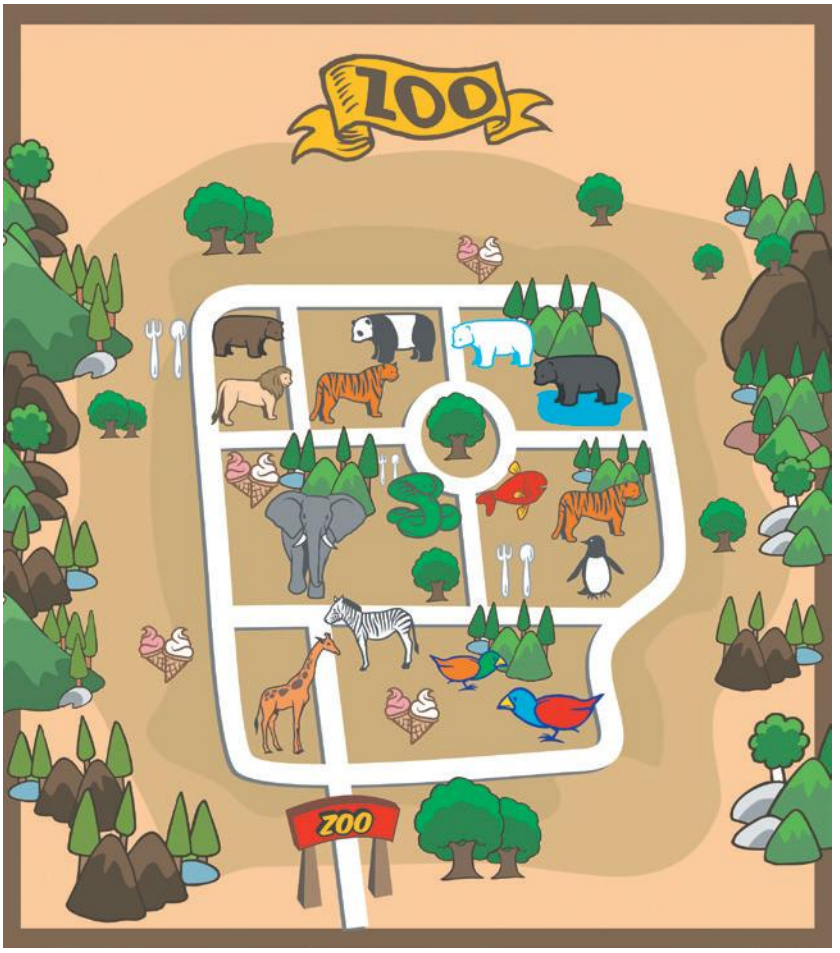
8D **8** Consider the following town map as a network of intersections (vertices) connected by roads (edges).

a How many vertices are in the network?

b What is the highest vertex degree?

c Can you design a bus route that visits every street exactly once?

8D **9** Why is it impossible to pass through every footpath in the zoo below exactly once?



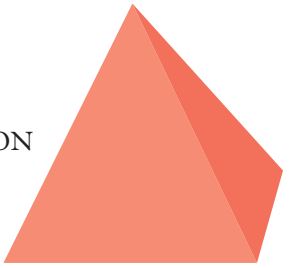
8E **10** By referring to the kin system in Example 8E.1, complete the table to show that you are always in the same kin group as your maternal great-great-grandmother (i.e. your mother's mother's mother's mother).

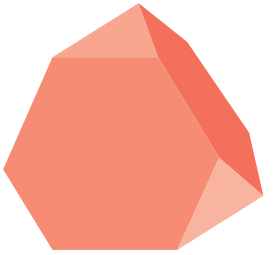
	Kin group							
You	A	B	C	D	E	F	G	H
Mother	B							
Grandmother	C							
Great-grandmother	D							
Great-great-grandmother	A							

Analysis

1 a Platonic solids can be truncated into different shapes by flattening the pointy bits. For example, the tetrahedron has faces of degree 3, but the truncated tetrahedron has faces of degree 3 or 6.

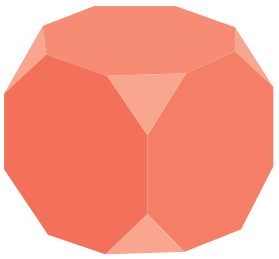
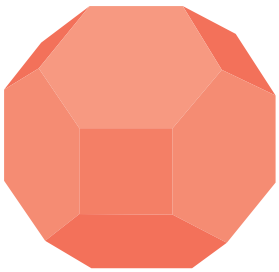
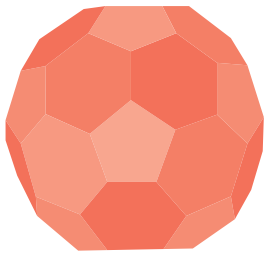
TETRAHEDRON





TRUNCATED TETRAHEDRON

Fill out the table of face degrees.

Shape	 TRUNCATED CUBE	 TRUNCATED OCTAHEDRON	 TRUNCATED ICOSAHEDRON
Face degrees			

b For the truncated tetrahedron:

i How many faces have a degree of 3?

ii How many faces have a degree of 6?

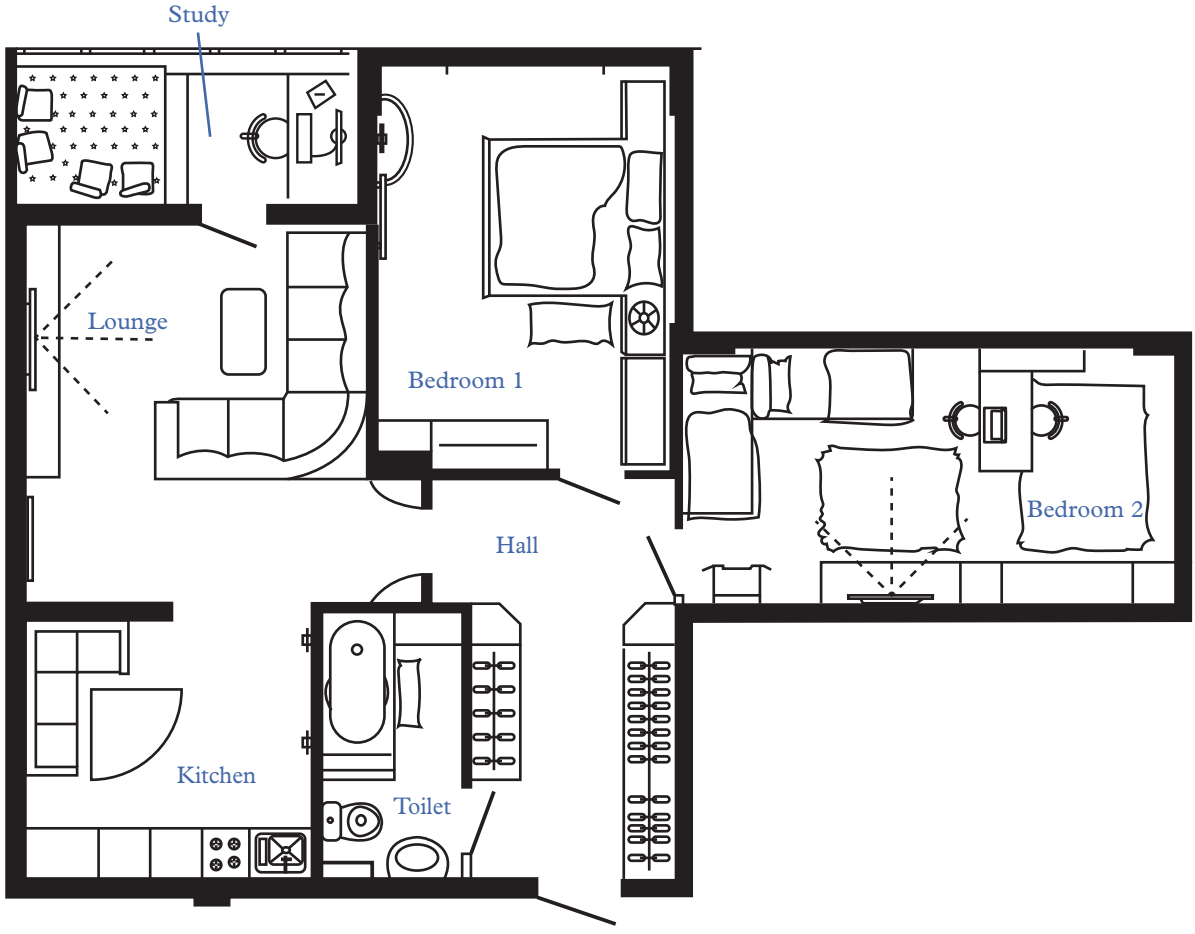
iii What is the sum of all the face degrees?

iv How many edges does it have?

v Using Euler's formula or otherwise, state the number of vertices.

c Repeat **iii**, **iv** and **v** from part **b** for the truncated cube.

2 a Draw a network which represents the following floor plan. Use vertices to represent rooms, and represent rooms connected by a door with an edge.



b Can you walk through the house passing through every doorway exactly once? Why or why not?

Algorithmic Thinking

Algorithmic thinking: Measurement and space

An algorithm is a set of precise instructions that when followed will achieve a goal or solve a problem. Once the actions in an algorithm have been identified, it may be possible to encode the algorithm in a coding language so it can be automated by a computer or machine.

The term iteration refers to repetitive actions in algorithms. Examples 1 and 2 use pseudocode and the Repeat directive to indicate iteration. The examples have also been coded using the free block-coding language Snap!

Example 1

This example uses the Repeat directive, the variables k, x, y and difference, and lists L1 and L2.

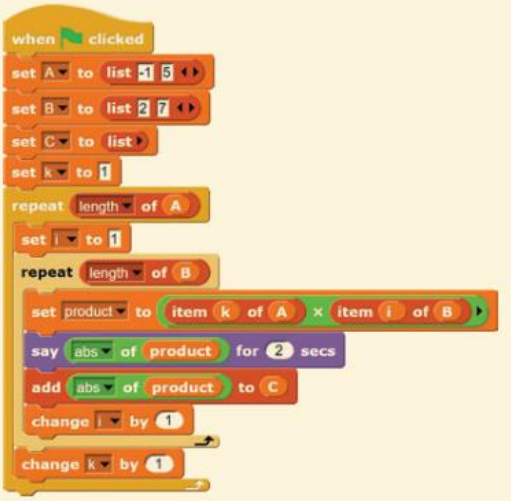
Create list of numbers L1 = {1, 3}
Create a list of numbers L2 = {4, 7}
Set the variable k to 1
Repeat (length of list L1) times:
    Set the variable x to the k^{th} element of L1
    Set the variable y to the k^{th} element of L2
    Set the variable difference to x – y
    Print (difference)
    Add 1 to k

Some algorithms contain iterations that are nested, as shown in Example 2. Nested iterations are useful for working on lists and tables of numbers.

Example 2

This example uses nested iteration, the variables i, k and product, and the lists A, B and C. When coded in Snap! the function abs is used which finds the absolute value.

Create list of numbers A = {−1, 5} (so A[1] = −1, A[2] = 5)
Create list of numbers B = {2, 7} (so B[1] = 2, B[2] = 7)
Create an empty list C
Set the variable k to 1
Repeat (length of list A) times:
    Set the variable i to 1
    Repeat (length of list B) times:
        Set product to A[k] × B[i]
        Print size of product ignoring the sign
        Add product to list C
        Add 1 to i
    Add 1 to k



Notice how indentation is used in Example 1 to group the actions in the iteration that need to be repeated, and that double indentation is used in Example 2 to show the nested iteration actions.

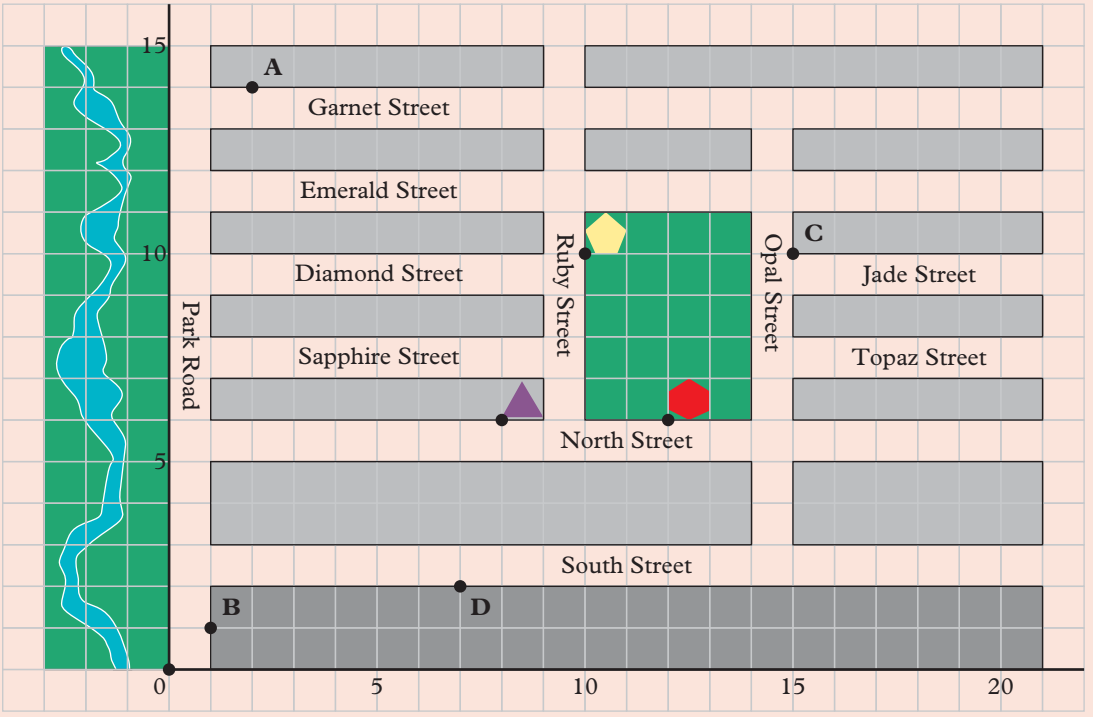
Task 1

1 What are the algorithms in Examples 1 and 2 printing out?

Task 2

The street map below shows the location of the homes of the air drone club members Anh (A), Bo (B) and Cal (C), who live near the corner of different streets in Gem City. The pentagon, hexagon and triangle are possible in-person meet-up locations or drone destinations.

When the friends want to meet up in person, they would like to pick a meeting location so that the total travel walking distance is minimised. People can only travel along the streets in Gem City; they must go around the park to keep off the grass. However, each person's drone can fly directly to the pentagon, hexagon and triangle in a straight line (as the crow flies) over the buildings and the park.



There are 3 possible meeting locations: H (Hexagon), P (Pentagon) and T (Triangle), and 3 friends: A (Anh), B (Bo) and C (Cal). A grid with regular spacing has been overlaid onto the street map.

1 Using the elements of variables, lists, iteration and nested iteration used in the algorithms in Examples 1 and 2, define an algorithm in pseudocode to work out the walking distances travelling along footpaths only for each person (A, B, C) to each location (H, P, T).

2 **CHALLENGE:** Code your path-walking algorithm into Snap! or another coding language and try it out.

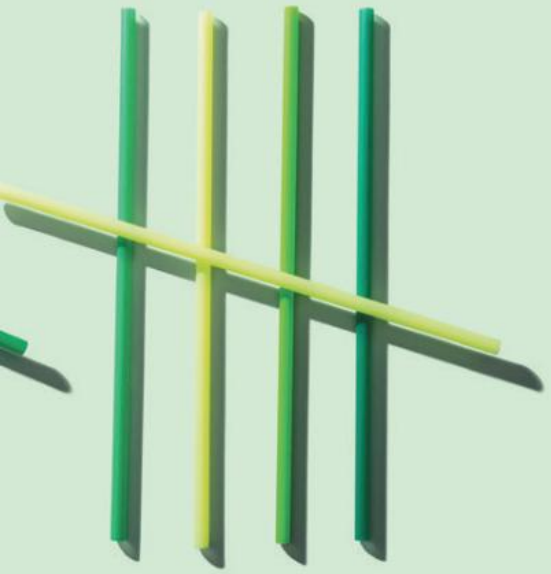
3 Modify your algorithm to also find all the shortest straight line flying distances (using Pythagoras' theorem) for each person's air drone (A, B, C) to each location (H, P, T).

4 **CHALLENGE:** Code your as-the-air-drone-flies algorithm into Snap! or another coding language and try it out.

5 When a new person at location D wants to join the air drone club, will your algorithms be affected? Explain why or why not.

9

Statistics



Index

9A Five-number summary and interquartile range
9B Boxplots
9C Distributions of data
9D The mean and standard deviation
9E Scatterplots and bivariate data
9F Analysing statistical reports

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Stem-and-leaf plots
- ✔ Classifying data
- ✔ Interpreting graphs
- ✔ Histograms
- ✔ The Cartesian plane

Curriculum links

- Analyse claims, inferences and conclusions of statistical reports in the media, including ethical considerations and identification of potential sources of bias (AC9M10ST01)
- Compare data distributions for continuous numerical variables using appropriate data displays including boxplots; discuss the shapes of these distributions ... (AC9M10ST02)
- Construct scatterplots and comment on the association between the 2 numerical variables in terms of strength, direction and linearity (AC9M10ST03)
- Plan and conduct statistical investigations of situations that involve bivariate data; evaluate and report findings with consideration of limitations of any inferences (AC9M10ST05)

Optional content

- Measures of spread, their interpretation and usefulness with respect to different data distributions

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9A Five-number summary and interquartile range

Learning intentions

By the end of this topic you will be able to ...

- ✔ find the five-number summary for a numerical data set
- ✔ calculate the interquartile range of a numerical data set.

Inter-year links

Support	Understanding data
Year 7	10A Summary statistics
Year 8	9B Summary statistics
Year 9	7D Summary statistics from tables and displays

Measures of data distribution

- **Quartiles** divide a numerical data set into four quarters.
- The **median** is the middle value of an ordered numerical data set. The median divides a data set into two halves.
  - → The median is also known as the second quartile, Q_2.
- The **lower quartile** (Q_1) is the median of the lower half of an ordered numerical data set.
- The **upper quartile** (Q_3) is the median of the upper half of an ordered numerical data set.
- When calculating the quartiles for a data set with an odd number of data points, the median is not included in either half of the data set.

Q_1 Q_2 Q_3

1 2 **3** 4 5 **6** 7 8 **9** 10 11

lower half upper half

- If there is an even number of data points, both middle values are included in the quartile calculations.

Q_1 Q_2 Q_3

1 2 3 4 5 6 7 8 9 10 11 12

lower half upper half

$$Q_1 = \frac{(3 + 4)}{2} = 3.5$$

$$Q_2 = \frac{(6 + 7)}{2} = 6.5$$

$$Q_3 = \frac{(9 + 10)}{2} = 9.5$$

- The **five-number summary** for a numerical data set consists of the minimum and maximum values, the lower and upper quartiles, and the median.
  - → The five values that make up the five-number summary are displayed in ascending order: minimum, Q_1, Q_2 (median), Q_3, maximum
- The **range** of a data set can be found by subtracting the smallest value (minimum) from the largest value (maximum).
- The **interquartile range** (IQR) represents the range of the middle 50% of a data set. It can be found by subtracting the lower quartile from the upper quartile.

$$\text{IQR} = Q_3 - Q_1$$

- An **outlier** is an extreme value that is much higher or lower than the other values in a data set.
  - → Outliers do not have an impact on the median or interquartile range.

Example 9A.1 Finding the five-number summary

Find the five-number summary for this ordered data set: 2, 2, 3, 4, 5, 6, 6, 6, 8.

THINK	WRITE
1 The data set is already ordered, so find the median by either circling the middle number or finding the average of the two middle numbers.	2 2 3 4 (5) 6 6 6 8 $Q_2 = 5$
2 Divide the data set into two equal halves, remembering not to include the median in either half.	
3 Find the upper quartile (Q_1) and the lower quartile (Q_3) by finding the median of each half of the set.	2 2 3 4 (5) 6 6 6 8 $Q_1 = 2.5$ $Q_3 = 6$
4 Identify the minimum and maximum scores from the data set.	minimum = 2, maximum = 8
5 Write your answer.	2, 2.5, 5, 6, 8

Example 9A.2 Calculating the interquartile range

Find the interquartile range for this data set:
6, 32, 7, 2, 8, 9, 9, 4, 6, 6, 21, 19, 13, 5.

THINK	WRITE
1 Arrange the scores in order.	2 4 5 (6) 6 6 7 8 9 9 (13) 19 21 32 Q_1 Q_3
2 Divide the data set into two halves. Because there are 14 data values, each half will contain 7 values.	
3 Find the value of the lower and upper quartiles by finding the median of each half of the data set.	$Q_1 = 6$, $Q_3 = 13$
4 Subtract Q_1 from Q_3 to obtain the IQR.	IQR = $Q_3 - Q_1$ = 13 – 6 = 7

Helpful hints

- ✔ When placing a set of data in order, count to make sure the ordered set has the same amount of data values as the unordered set.
- ✔ If there are n values in an ordered data set, the median will be in the $\frac{n+1}{2}$ th position. For example, if there are 11 values, the median will be in the $\frac{11+1}{2} = 6$th position.

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Exercise 9A Five-number summary and interquartile range

1–8, 12, 13 | 2, 3, 5, 8–10, 12, 14, 16(a, b) | 3(d–f), 5(d–f), 8, 10–12, 14–16

UNDERSTANDING AND FLUENCY

1 Find the range for each of these data sets.

a 5, 9, 6, 12, 16, 3, 8, 19, 11, 8, 12

b 48, 3, 7, 3, 8, 4, 12, 19, 6, 6, 8, 19, 13

c 110, 167, 189, 102, 144, 117, 166

d 54, 46, 78, 11, 19, 22, 26, 57, 67

9A.1 **2** Find the five-number summary for each of these ordered data sets.

a 3, 3, 3, 3, 4, 5, 6, 6, 7, 8, 9

b 7, 9, 11, 12, 13, 13, 18, 19, 35, 42, 45, 46, 61, 62, 78

c 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 5, 6, 6, 6, 6, 7, 7, 7, 8, 8, 9, 9

d 2, 4, 4, 6, 7, 7, 8, 8, 9, 11, 16, 16, 18, 19, 19, 19, 21, 27, 32

3 Find the five-number summary for each of these data sets.

a 4, 8, 9, 12, 7, 6, 6, 6, 7, 4, 3, 2, 8, 9, 11

b 46, 28, 16, 23, 8, 4, 46, 49, 55, 13

c 72, 81, 16, 22, 42, 55, 67, 73, 19, 24, 16, 77, 37

d 724, 169, 843, 120, 564, 786, 341, 186

e 42, 18, 32, 18, 64, 39, 42, 18, 64, 21, 27, 34

f 8, 19, 24, 26, 17, 13, 8, 4, 13, 9, 14, 21, 17, 32, 8, 4, 3, 19

4 Given the following five-number summaries, find the interquartile range each time.

a 2, 6, 10, 16, 19 **b** 17, 22, 25, 31, 33 **c** 7, 9, 13, 19, 24 **d** 34, 40, 52, 59, 66

9A.2 **5** Find the interquartile range for each of these data sets.

a 46, 48, 32, 12, 62, 27, 19, 34, 86, 16, 16

b 8, 1, 6, 4, 23, 5, 7, 16, 3, 3, 14, 13, 7, 5

c 123, 189, 146, 179, 98, 165, 108, 142, 152

d 19, 62, 53, 14, 8, 43, 62, 9, 46, 13, 19, 20

e 5, 9, 11, 16, 13, 7, 13, 18, 21, 34, 12, 8, 7

f 125, 465, 132, 99, 164, 222, 378

PROBLEM SOLVING AND REASONING

6 The stem-and-leaf plot below shows the number of cupcakes a local bakery sold per day over the course of a month. Find the range and the interquartile range and describe the spread of the set.

Stem	Leaf
4	1 2 2 6 7 8
5	1 6 6 7 9
6	1 2 3 3 6 6 8 9
7	2 4 8 9 9
8	0 1 5 5 7
9	1 2

Key: 4 | 2 = 42

7 Consider these two data sets.

Set A: 9, 31, 46, 11, 39, 24, 3, 43, 25, 10, 19, 4, 36, 14, 16

Set B: 16, 26, 29, 18, 14, 23, 14, 57, 26, 21, 16, 18, 14, 19, 19

a Find the range for each data set. What do you notice?

b Find the interquartile range for each data set.

c Explain why the interquartile range should be used to describe the difference between the spreads of these two sets.

8 Consider this data about the number of times a year a sample of Year 10 students have dyed their hair.

Number of times hair has been dyed	0	1	2	3	4	5	6
Frequency	10	2	3	7	11	9	10

a Find the range of the data.

b Find the five-number summary.

c Find the interquartile range.

9 Explain what it means if a data set has:

a a small interquartile range

b a small interquartile range and a large range

c a large interquartile range

d an interquartile range that is very similar to its range.

10 Consider this data set, which is the income (in dollars) per week of some Year 10 students.

10 120 50 60 45 85 250 80 100 65 75 20

35 45 50 80 25 10 20 50 100 45 80

a Calculate the range and the interquartile range of this data.

b Why is the range so much larger than the interquartile range?

c If the data in a set is spread evenly, the range will be roughly twice the size of the interquartile range. Explain why.

d The presence of an outlier can affect the range. Explain why for data sets with more than 5 values an outlier would not affect the interquartile range.

11 For each data set below:

i calculate the range and interquartile range

ii describe the spread of the data.

a 11, 8, 19, 9, 11, 15, 7, 2, 6, 17, 13, 13, 12, 16, 8, 7, 16, 5, 4, 19

b 32, 68, 43, 31, 77, 44, 67, 42, 37, 55, 59, 79, 81, 83, 11, 56, 57, 61, 82, 63

c 123, 118, 131, 141, 142, 136, 128, 127, 119, 129, 138, 127, 191, 132, 134

12 Consider this back-to-back stem-and-leaf plot showing the heights of students in two classes.

a For each class, find:

i the median

ii the range

iii the interquartile range.

b Compare the heights of the students in the two classes. Which class would you say is taller? Which class would you say is more diverse?

Leaf Class A	**Stem**	**Leaf Class B**
	14	9
9	15	6 8
9 8 7 7 7 6 5 4 3 3 2 1 1	16	0 1 3 4 5 5 7 9
9 9 8 7 6 5 5 4 3 2 2 1	17	2 3 4 4 5 5 5 7 9
6 5 4 1	18	1 4 8
	19	7

Key: 1 \| 6 = 16

PROBLEM SOLVING AND REASONING

13 Consider the set of data: 80, 50, 60, 66, 74, 71, 83, 55, 70, 79.

a Determine the five-number summary for the data.

b Determine the interquartile range and range for the data.

It turns out the final value, 79, was instead meant to be 9.

c Determine the correct five-number summary for the data.

d Determine the correct interquartile range and range for the data.

e Describe the effect changing 79 to 9 had on the:

i median **ii** range **iii** interquartile range.

f Explain why outliers and other extreme values have little to no effect on the interquartile range.

14 Consider this five-number summary created from a list of 10 numbers.

10, 10, 12, 14, 14

a Determine the number of possible original data sets that would have this five-number summary.

b If the maximum was increased to 15, determine the number of possible original data sets that would have this five-number summary.

CHALLENGE

15 A k-quantile is a more general name for the points at which a data set is divided into smaller, equally sized parts. For example, the 4-quantiles are the quartiles Q_1, Q_2 and Q_3 because they divide the data set into four equally sized parts. A 5-quantile is called a quintile. Quintiles are often used in economics to describe how income is distributed across a population.

a Give one advantage of using quintiles instead of quartiles to describe the income distribution of a large population.

The term quintile is also used to describe each 20% of a data set. The following data shows the income distribution in Australia between 2013 and 2020.

Quintile	2013–2014 (%)	2015–2016 (%)	2017–2018 (%)	2019–2020 (%)
Lower (0–19%)	2.7	3.2	3.1	3.1
Second (20–39%)	8.4	8.6	8.6	8.6
Third (40–59%)	16.9	16.0	16.5	16.7
Fourth (60–79%)	25.7	25.4	25.6	25.8
Highest (80–100%)	46.4			

b Copy the above table into your workbook and fill in the missing values to complete the last row.

c Was income evenly distributed in Australia in 2019–2020? If not, what would an even distribution of income look like in the table?

The ratio of the percentage income earned by the lowest quintile compared to the highest quintile in 2013–2014 can be calculated as follows:

$46.4 : 2.7 = \frac{46.4}{2.7} : \frac{2.7}{2.7} \approx 17.185 : 1$

d Calculate the income ratios between the lowest and highest quintiles for the other years.

e Describe how the income ratios change between 2013 and 2020.

16 Determine the interquartile range for each of the following.

a $21, 4, -11, -2, -24, -18, -3, -1$

b $\frac{13}{6}, \frac{17}{3}, \frac{10}{3}, \frac{7}{3}, \frac{1}{4}, 5, \frac{17}{3}, \frac{2}{3}$

c $\frac{18}{3}, \frac{8}{19}, \frac{8}{3}, \frac{13}{10}, \frac{12}{13}, \frac{14}{13}, \frac{9}{11}, \frac{6}{7}$

d $-\frac{17}{18}, -\frac{1}{2}, -\frac{5}{13}, -\frac{11}{8}, -\frac{20}{7}, \frac{9}{10}, \frac{19}{14}, -\frac{13}{17}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Five-number summary and interquartile range

Topic quiz
9A

9B Boxplots

Learning intentions

By the end of this topic you will be able to ...

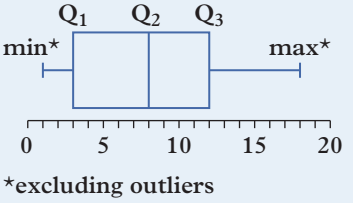
- ✔ create and interpret boxplots
- ✔ use parallel boxplots to compare data sets.

Inter-year links

Support	Understanding data
Year 7	10A Summary statistics
Year 8	9B Summary statistics
Year 9	7F Comparing data

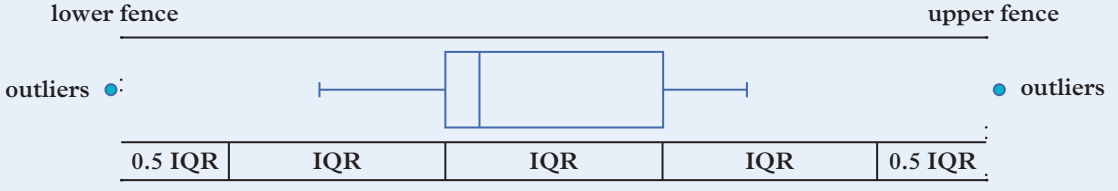
Boxplots

- A **boxplot** (or box-and-whisker plot) can be used to represent the five-number summary visually and to display the centre (median) and spread (range and interquartile range) of a numerical data set.
- A boxplot consists of a central box with horizontal lines called 'whiskers' on either side of the box.
  - → Boxplots should always be accompanied by a clear and even scale.
  - → If the minimum or maximum value is equal to the value of the lower or upper quartile, there will be no whisker on that side of the boxplot.
  - → If the median is equal to either quartile, the median line will not be visible.



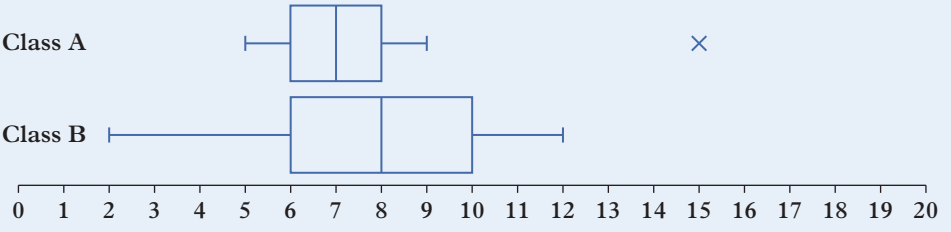
Identifying outliers

- The quartiles and the interquartile range can be used to identify outliers. The **lower fence** lies 1.5 × IQR below the lower quartile, and the **upper fence** lies 1.5 × IQR above the upper quartile. Values beyond these fences are classified as outliers.
- Outliers should be identified before boxplots are drawn. The values shown by the ends of the whiskers are the minimum and maximum values excluding any outliers. Outliers should be represented on boxplots as single points using a cross or dot lying beyond the whiskers.



Using boxplots to compare data sets

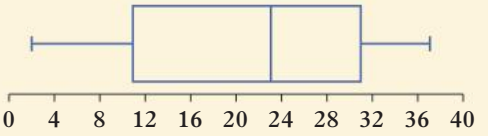
- **Parallel boxplots** can be used to compare data sets on the same scale. Place one boxplot directly above the other boxplot on the same scale to directly compare key points from each set of data.
  For example, in the parallel boxplots below, the scores of Class A are more consistent than the scores of Class B, except for the outlier in Class A.



Example 9B.1 Understanding boxplots

Use this boxplot to state the:

a five-number summary **b** interquartile range.



THINK

a The end of the left-hand whisker represents the minimum value, the left-hand side of the box represents the lower quartile, the line inside the box represents the median, the right-hand side of the box represents the upper quartile and the end of the right-hand whisker represents the maximum value.

b Subtract Q_1 from Q_3 to obtain the IQR.

WRITE

a min = 2,
$Q_1 = 11$,
$Q_2 = 23$,
$Q_3 = 31$,
max = 37

b $IQR = Q_3 - Q_1$
$= 31 - 11$
$= 20$

Example 9B.2 Creating a boxplot

Draw a boxplot to represent this data about the number of books that people own.

45	22	19	4	14	39	152	108	39	19	66	81
122	42	38	22	18	35	144	33	27	13	31	

THINK

1 Arrange the scores in order and find the value of the median, the lower quartile and the upper quartile.

2 Calculate the values of the lower fence (LF) and upper fence (UF).

3 Check for outliers by determining whether any values lie below the lower fence or above the upper fence.

4 Identify the minimum and maximum values, excluding outliers.

5 Draw an even scale that covers all values, including outliers.

6 Mark in the position of the minimum and maximum values, the quartiles, the median and any outliers. Complete your boxplot using a ruler, providing a title if appropriate.

WRITE

4, 13, 14, 18, 19, 19, 22, 22, 27, 31, 33, 35, 38, 39, 39, 42, 45, 66, 81, 108, 122, 144, 152

$Q_1 = 19$, $Q_2 = 35$, $Q_3 = 66$

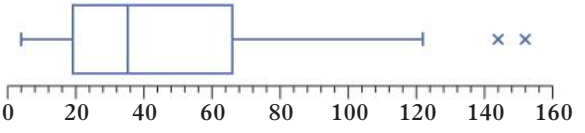
$LF = 19 - 47 \times 1.5$
$= 19 - 70.5$
$= -51.5$

$UF = 66 + 47 \times 1.5$
$= 66 + 70.5$
$= 136.5$

144 and 152 are outliers.

min = 4, max = 122

Number of books people own



Helpful hints

- ✔ The ends of the whiskers do not represent the minimum and maximum values in a data set if there are outliers. If there are outliers, the left whisker will start at the value directly after the closest outlier to the lower fence, and the right whisker will finish at the value directly before the closest outlier to the upper fence.
- ✔ Outliers lie beyond the fences. If a value is exactly equal to the lower fence or upper fence it is not an outlier.
- ✔ The height of a boxplot has no particular significance.

ANS p638

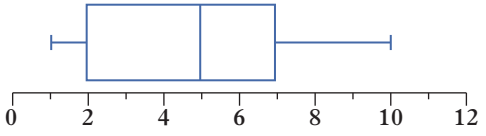
Exercise 9B Boxplots

1, 2, 3(a–d), 4–6, 8 | 1, 3(e–h), 4, 5, 7, 8, 11, 12 | 3(e–h), 4, 5, 9, 10, 12–14

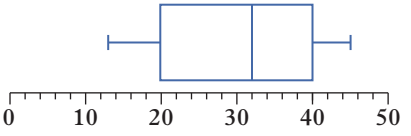
9B.1 **1** For each boxplot below, state the:

i five-number summary **ii** interquartile range.

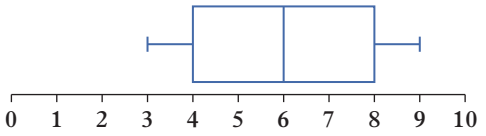
a



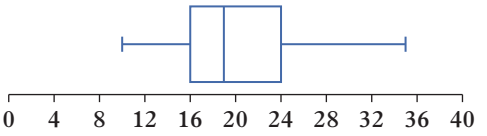
b



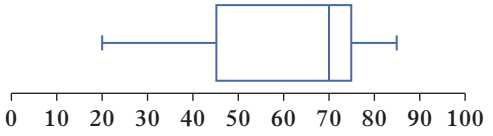
c



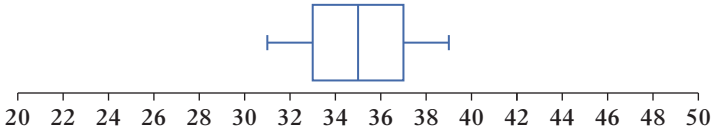
d



e



f



2 Consider this data set: 2, 6, 3, 8, 9, 10, 14, 19, 4, 8, 22, 11, 13, 17, 4, 19, 7, 7, 10.

a Arrange the data in order.

b Find the lower quartile, median and upper quartile.

c Calculate the values of the lower and upper fences.

d Determine whether there are any outliers in this data set.

e Create a boxplot for this data.

9B.2 **3** Draw a boxplot to represent each of these data sets.

a 3, 6, 2, 8, 4, 8, 10, 15, 6, 8, 11

b 11, 15, 14, 19, 20, 26, 17, 13, 6, 22

c 4, 7, 1, 8, 3, 7, 6, 6, 9, 3, 5, 1, 6, 9, 7, 5, 3

d 12, 36, 45, 92, 42, 85, 49, 66, 51, 19, 27, 78

e 16, 28, 9, 16, 13, 27, 18, 22, 33, 24, 9, 18, 8

f 5, 7, 9, 19, 6, 9, 8, 5, 5, 4, 3, 9, 9, 1, 6, 5, 4

g 11, 18, 49, 23, 24, 26, 37, 31

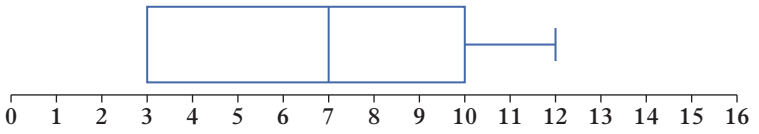
h 8, 11, 19, 21, 6, 7, 12, 18, 19, 31, 4, 6, 24, 7, 9

4 For each boxplot, state the:

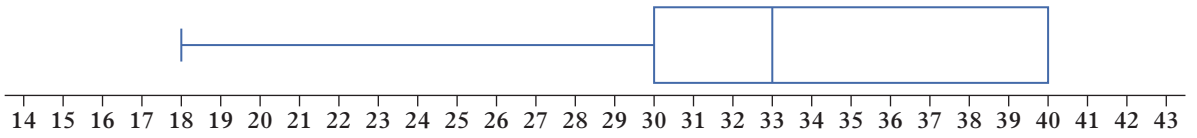
i five-number summary

ii interquartile range.

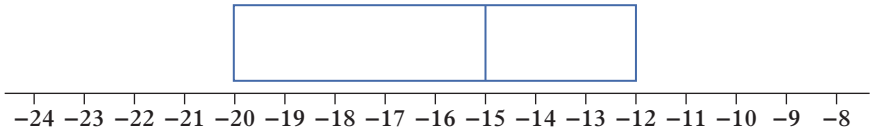
a



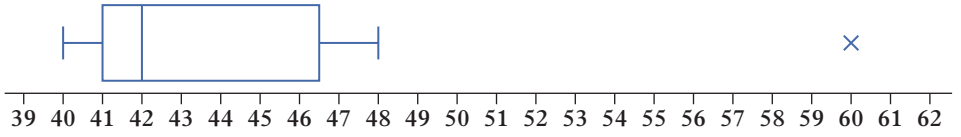
b



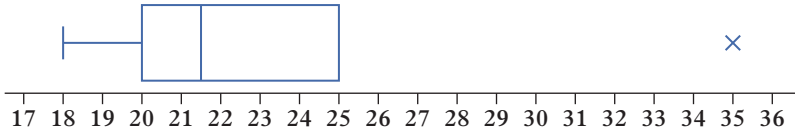
c



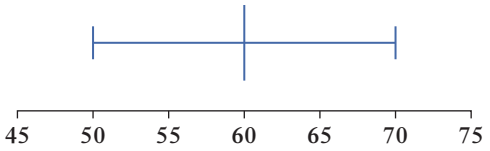
d



e

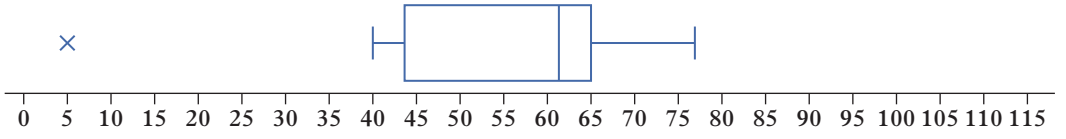


f

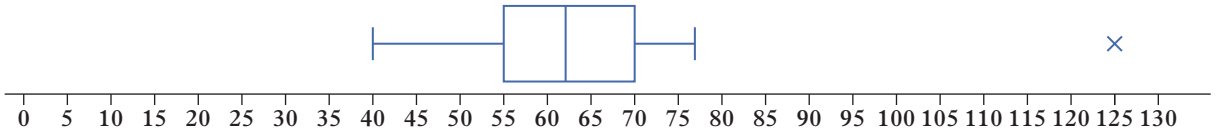


5 For each of the following boxplots, add a vertical dashed line where the upper and lower fences lie.

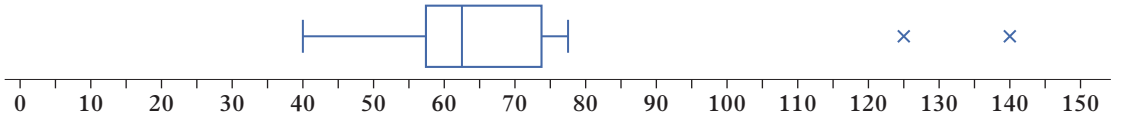
a



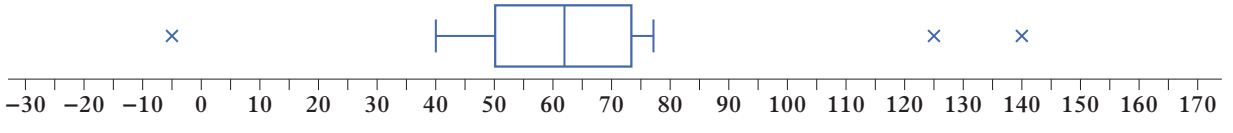
b



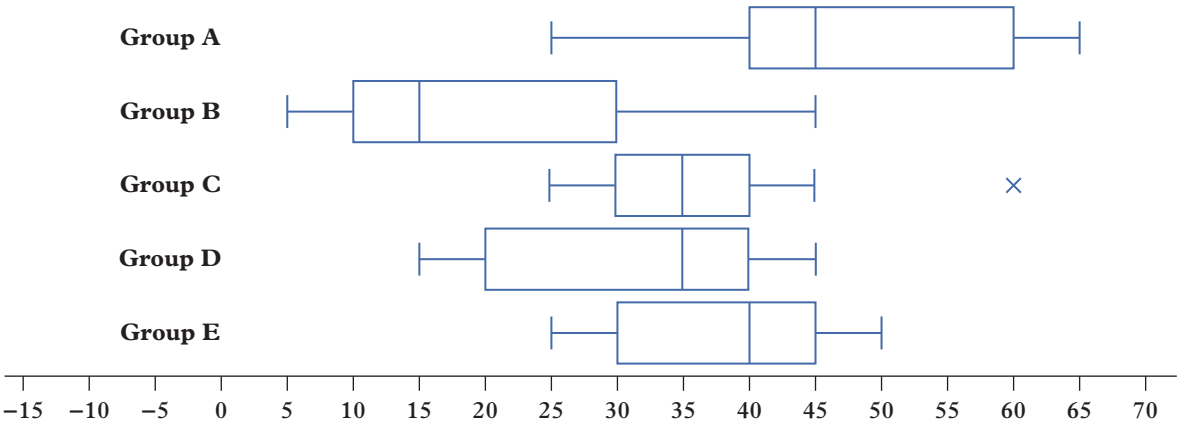
c



d



6 Consider the parallel boxplots below.



a State the group(s) with the lowest:

i minimum

ii maximum.

b At 45, state the statistic (if any) that group:

i A has

ii B has

iii C has

iv D has

v E has.

c Which group has a range equal to the interquartile range of another group?

7 This data was collected about the number of puppies in a litter:

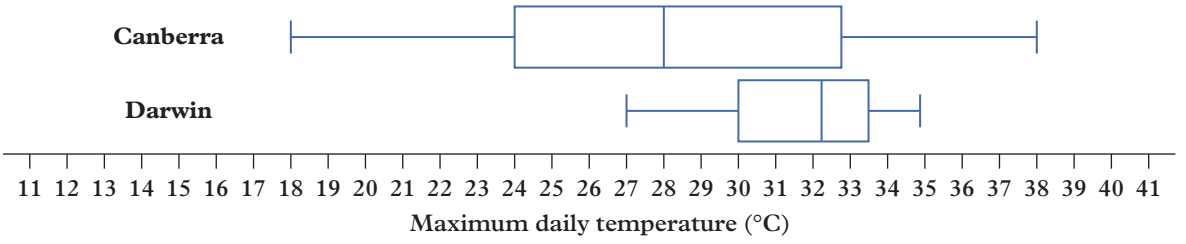
1	6	4	2	5	4	8	6	4	5	8
3	2	4	6	5	8	7	5	6	5	6
12	4	6	5	7	6	5	6	5		

a Find the lower quartile, median and upper quartile of the data set.

b Determine if there are any outliers in this data set.

c Draw a boxplot to represent the data.

8 Consider these parallel boxplots, showing the distribution of the maximum daily temperatures for a particular month in Darwin and Canberra.



a What is the centre (median) and spread (IQR) of the temperature in Darwin?

b What is the centre (median) and spread (IQR) of the temperature in Canberra?

c Write a paragraph comparing the distributions of the two sets of temperature data. Which would you say is the hotter city?

PROBLEM SOLVING AND REASONING

9 Data was collected on the ages of customers at a laser tag facility on weekdays (set A) and at weekends (set B).

Set A: 19, 18, 21, 22, 23, 18, 24, 25, 22, 21, 20, 19, 18, 18, 19, 20, 19, 20, 24, 22, 21, 23, 27, 19, 21

Set B: 21, 26, 23, 19, 18, 17, 15, 6, 10, 43, 36, 26, 23, 49, 7, 11, 14, 13, 51, 38, 32, 27, 18, 12, 19, 33

a Draw a set of parallel boxplots to represent the data.

b Describe each data set in terms of its centre (median) and spread (range).

c Compare the two sets. What differences do you notice?

10 The five-number summary for a set of data that shows the number of customers in a shoe shop at 1 pm each day for a month is 4, 6, 7, 8, 10.

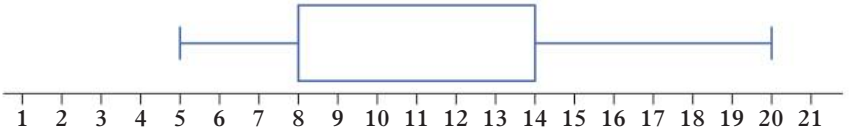
a Draw a possible boxplot for this five-number summary.

b Are other boxplots possible? Explain.

The five-number summary for another set of data is 2, 6, 7, 8, 12.

c Explain why you cannot draw the boxplot for this five-number summary with only the information provided.

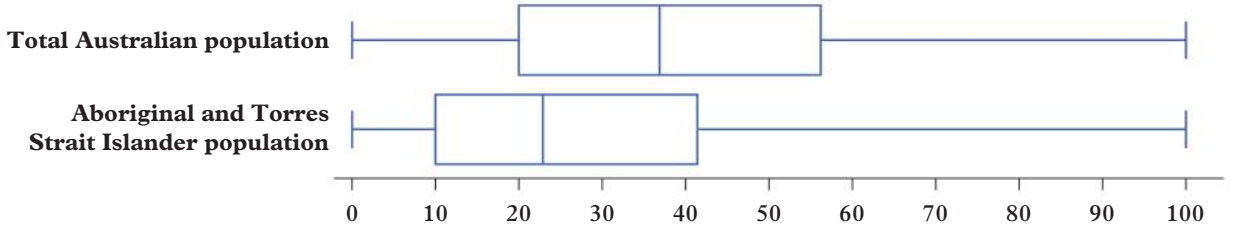
11 Consider this boxplot.



a Explain why there are two possible five-number summaries when only the boxplot (and not the original data) is provided.

b Provide the two possible five-number summaries for the data the boxplot represents.

12 The parallel boxplots below show the distribution of the ages of the total Australian population and the distribution of ages of Aboriginal and Torres Strait Islanders in 2016.



Compare the distributions of the ages of all Australians and the ages of Aboriginal and Torres Strait Islanders.

CHALLENGE

13 Draw a set of 3 parallel boxplots, A, B and C, such that:

- the maximum of A and B are equal
- 75% of the data in B lies within the same values that 25% of the data in A lies within
- 100% of the data in C, excluding an outlier, lies within the same values that the middle 50% of the data of A lies within
- the median of B is equal to the value of the outlier of C
- the medians of A and C are equal to the minimum value of B
- A is perfectly symmetric and each quartile is equally spaced, B is negatively skewed, and C is symmetric.

14 For a particular set of data, the lower fence is 15 and the upper fence is 51. Determine the:

a interquartile range

b the upper and lower quartiles.

Check your Student obook pro for these digital resources and more:

- Interactive skillsheet: Boxplots
- Investigation: How varying data values affect the interquartile range
- CAS instructions: Boxplots
- Topic quiz: 9B

9C Distributions of data

Learning intentions

By the end of this topic you will be able to ...

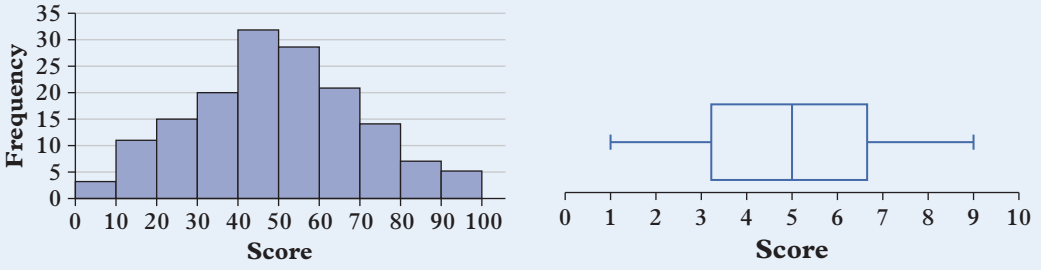
- ✔ accurately describe distributions of data
- ✔ create and interpret a cumulative frequency distribution.

Inter-year links

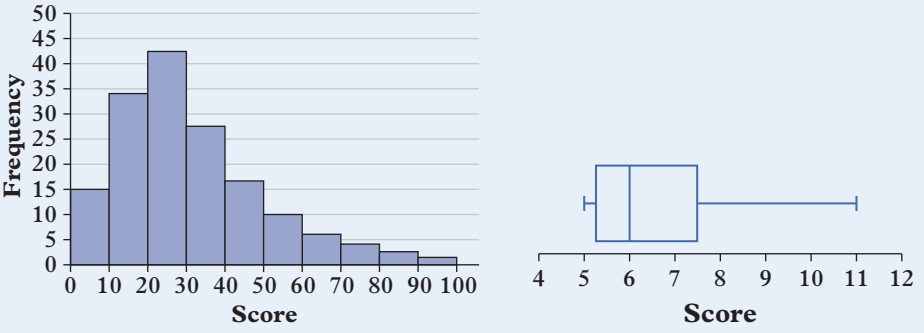
Year 8	9C Analysing data
Year 9	7E Describing data

Describing data distribution

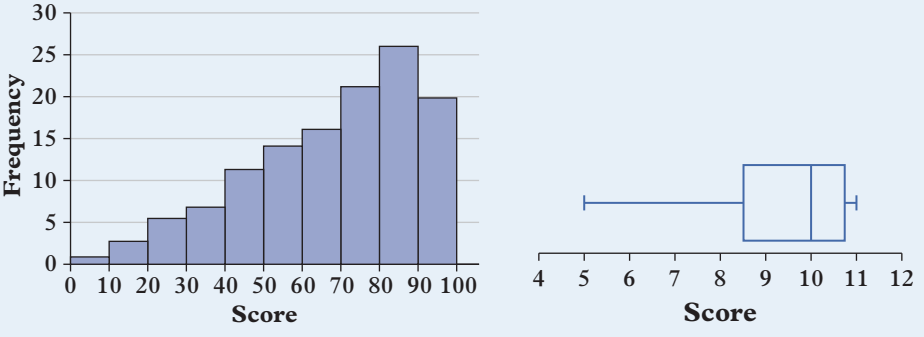
- Symmetrical or **symmetric distributions** are evenly spread on either side of a central peak. The **histogram** and boxplot below are examples of symmetric distributions.



- A **skewed distribution** has a big cluster of data points on one side of the distribution. The other side of the distribution has a tail.
  - → A **positively skewed distribution** has a cluster of values around the left-hand side of the distribution and a tail tapering to the right. The histogram and boxplot below are examples of positively skewed distributions.

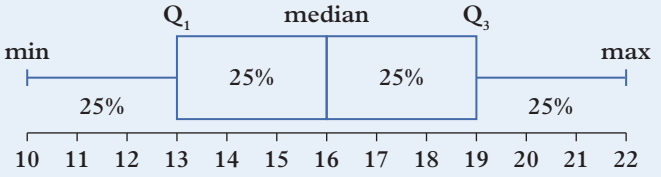


  - → A **negatively skewed distribution** has a cluster of values around the right-hand side of the distribution and a tail tapering to the left. The histogram and boxplot below are examples of negatively skewed distributions.



- The **modality** of a distribution describes the number of peaks in a data set. A distribution is **bi-modal** if there are two clear peaks in the data.

- To describe the **spread** of a data set, consider where the scores in the data set lie across the distribution. Remember that 25% of a data set lies between each pair of consecutive terms in a five-number summary, so if any two values in a five-number summary are close together, 25% of scores will be covered in a small range of data (i.e. packed closely together).
- Each section of a boxplot represents approximately 25% of a data set.

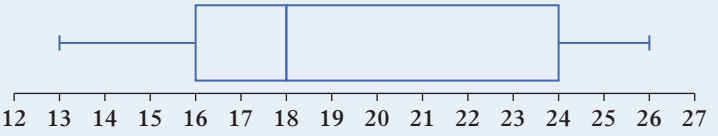


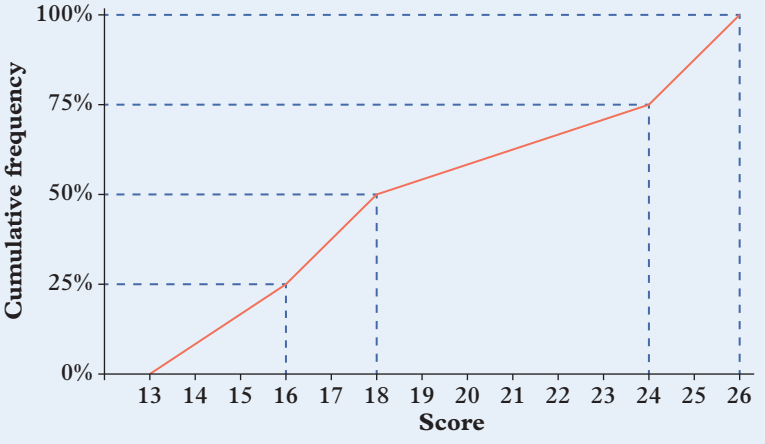
- To describe the centre of a data set, consider where the median score in the data set lies in the distribution.

Cumulative frequency distributions

- The **cumulative frequency** of a numerical data set can be calculated by adding the frequency of a particular score to the sum of the frequencies of all its preceding scores.
  → Cumulative frequencies can be expressed as raw numbers or as a percentage of the data set.
- A **cumulative frequency polygon**, also called an ogive (pronounced: 'oh-jive'), is a graph plotting the cumulative frequency of the scores in a data set (on the vertical axis) against those scores (on the horizontal axis).
  → The data displayed in the following cumulative frequency table and boxplot are plotted on the cumulative frequency polygon below.

Score	Cumulative frequency
13	0%
16	25%
18	50%
24	75%
26	100%



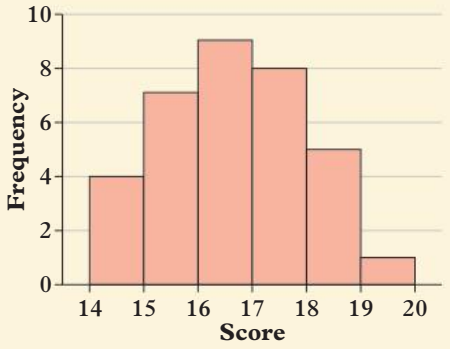


- Cumulative frequency polygons can also be used to plot grouped data. Each section of the cumulative frequency polygon should start at the lowest value of a class interval and end at the highest value of the same class interval (as shown in Example 9C.3).

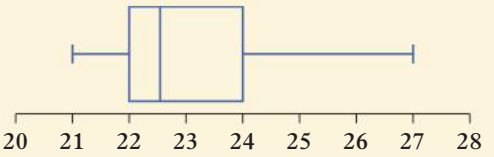
Example 9C.1 Describing a data distribution

Describe the type of distribution of data shown on each of the following.

a



b



THINK

a Look at the shape of the histogram. Is there a skew towards one side of the distribution, or is the data approximately symmetrical?

b Each section of a boxplot represents approximately 25% of the data set. Identify whether there is a cluster of points on one side of the boxplot, or whether the distribution is symmetric.

WRITE

a Symmetrical

b Positively skewed

Example 9C.2 Interpreting a cumulative frequency polygon

Use this cumulative frequency distribution to write the:

a five-number summary for the data displayed

b interquartile range for the data.

THINK

a The value at 0% represents the minimum value, the value at 25% represents the lower quartile, the value at 50% represents the median, the value at 75% represents the upper quartile and the value at 100% represents the maximum value.

b Subtract Q_1 from Q_3 to obtain the interquartile range.

WRITE

a min = 6,
$Q_1 = 8$,
$Q_2 = 13$,
$Q_3 = 20$,
max = 32

b $IQR = Q_3 - Q_1$
$= 20 - 8$
$= 12$

Example 9C.3 Creating a cumulative frequency polygon

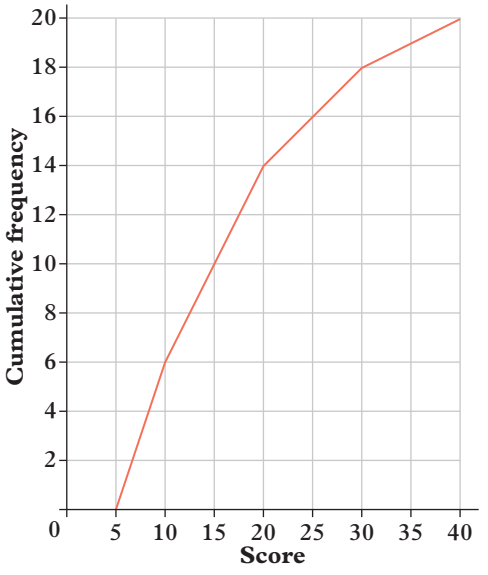
Using the data from this table, create a cumulative frequency polygon.

Score	Frequency	Cumulative frequency
5–<10	6	6
10–<15	4	10
15–<20	4	14
20–<25	2	16
25–<30	2	18
30–<35	1	19
35–<40	1	20

THINK

1. Draw and label axes to cover the full range of scores on the horizontal axis and the cumulative frequency on the vertical axis.
2. Mark the first point with a frequency of 0 at the start of the first class interval. The next point to mark will be the point at the end of the first class interval, with a cumulative frequency equal to that of the first class interval.
3. Mark each subsequent point from the table on your graph.
4. Join the points together with straight lines.

WRITE



Helpful hints

- ✔ Symmetric data does not need to be perfectly symmetric; it only needs to be approximately symmetric.
- ✔ The cumulative frequency of the final score will equal the number of pieces of data, or 100% of the data set.

ANS p640

Exercise 9C Distributions of data

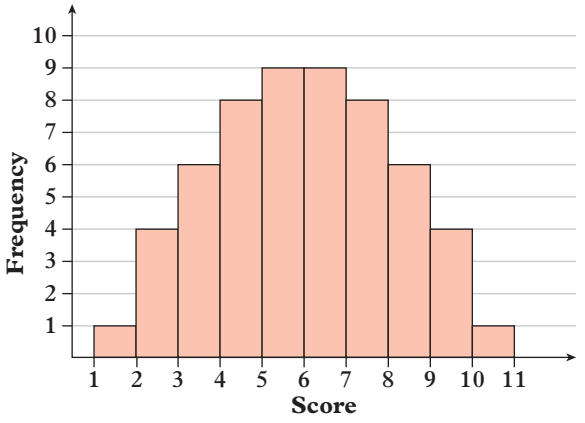
1, 2, 3(a–c), 4, 8, 10(a), 11(a)

1, 2, 3(b–d), 4, 6, 8, 10(b), 11(b), 13, 14

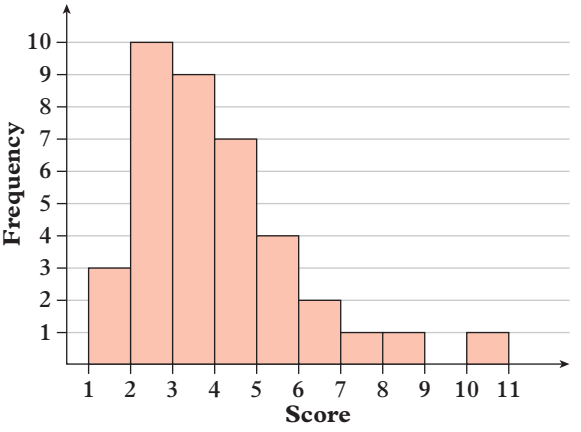
1, 2, 3(b–d), 6, 9, 10(b), 11(b), 12, 13, 15

9C.1 **1** Describe each of the following distributions of data.

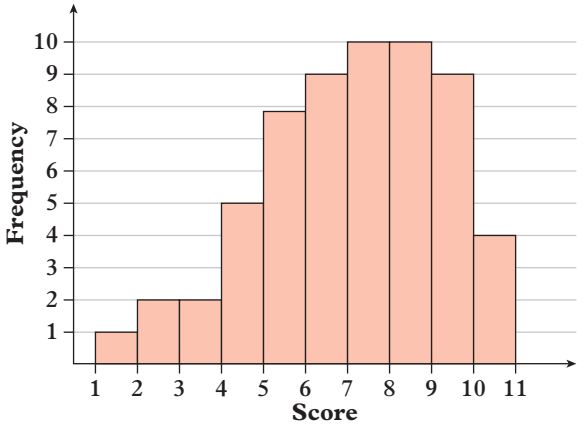
a



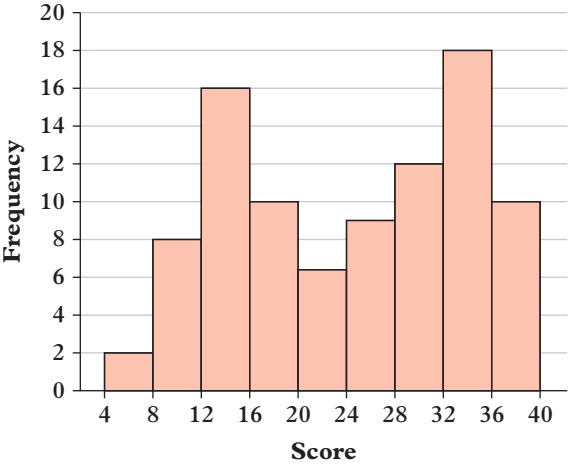
b



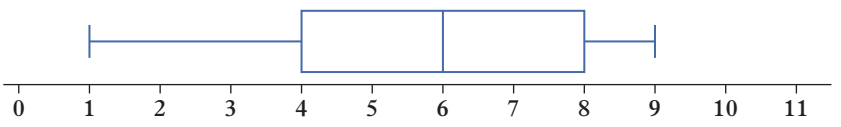
c



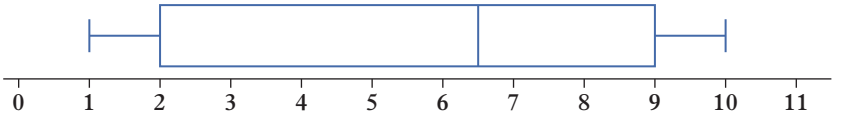
d



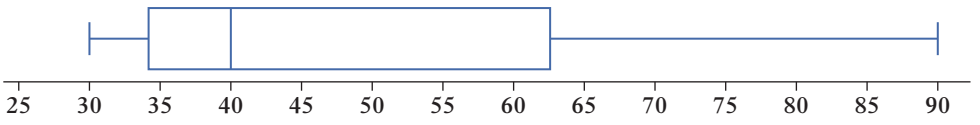
e



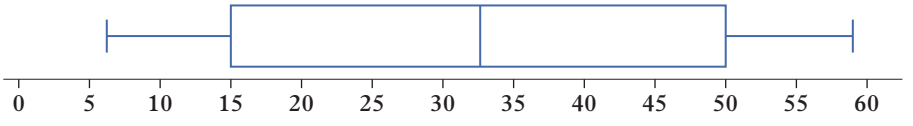
f



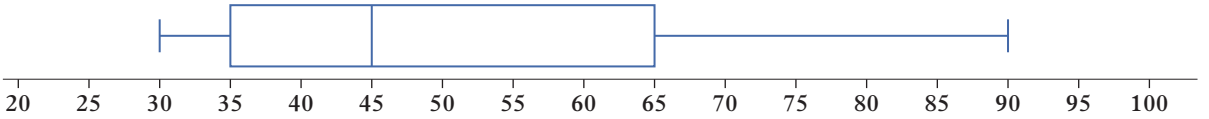
g



h



2 State the percentage of the data that lies in the following ranges of values for this boxplot.

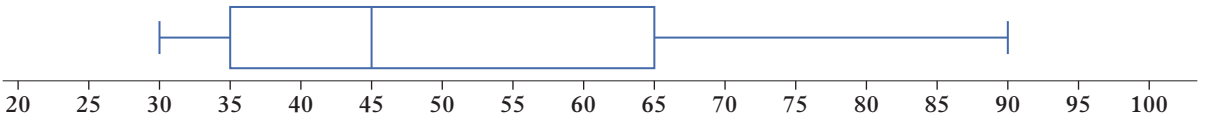


a between 35 and 45 **b** between 35 and 65 **c** between 65 and 90

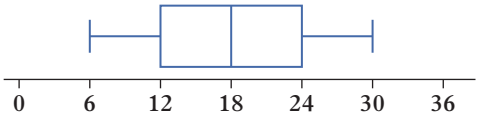
d less than 65 **e** greater than 35 **f** less than 35 or greater than 65

3 Construct a cumulative frequency polygon for the following boxplots.

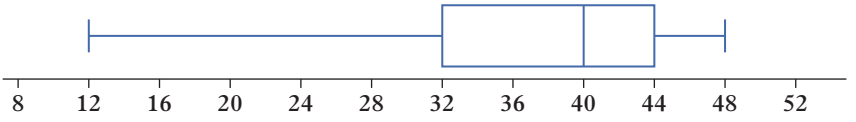
a



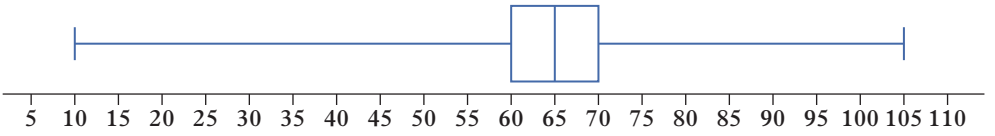
b



c



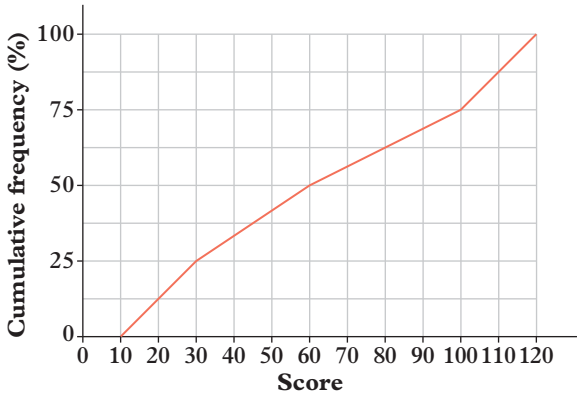
d



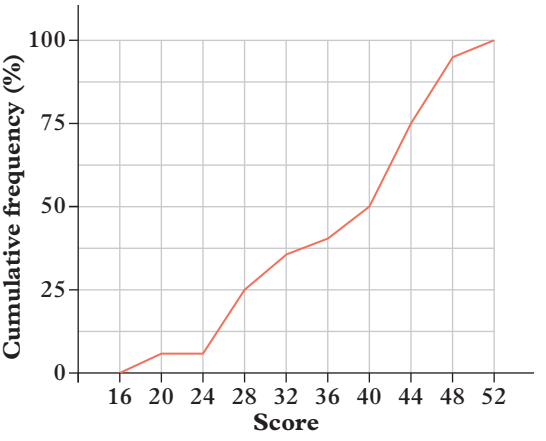
9C.2 **4** For each cumulative frequency polygon, state the:

i five-number summary **ii** interquartile range.

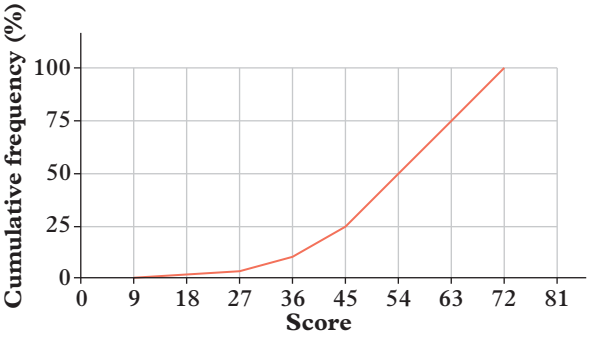
a



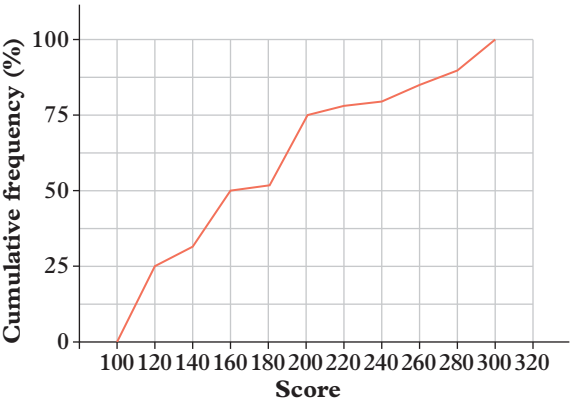
b



c



d



9C.3 **5** Create a cumulative frequency polygon using the data in the table below.

Score	Frequency	Cumulative frequency	Cumulative percentage
7	5	5	25
8	3	8	40
9	2	10	50
10	5	15	75
11	3	18	90
12	2	20	100

6 For this table:

a complete the cumulative frequency and cumulative percentage columns

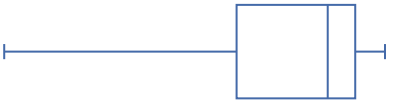
b construct a cumulative frequency polygon.

Score	Frequency	Cumulative frequency	Cumulative percentage
20–<30	10		
30–<40	40		
40–<50	50		
50–<60	20		
60–<70	10		
70–<80	20		
80–<90	10		
90–<100	40		

7 Match each of these boxplots (**a–d**) with the correct graph (**I–IV**).

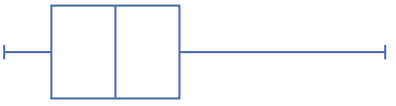
a

b

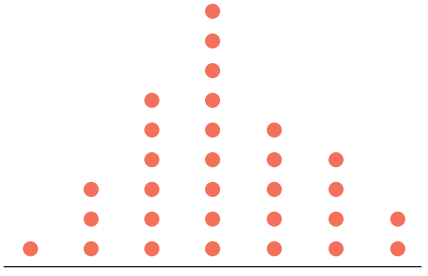


c

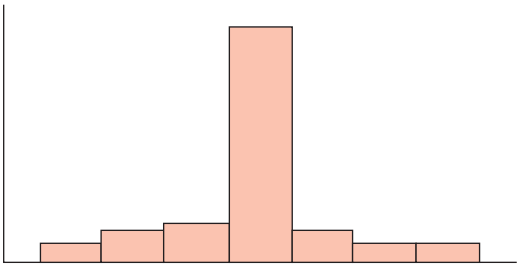
d



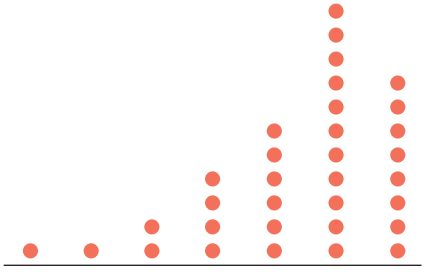
I



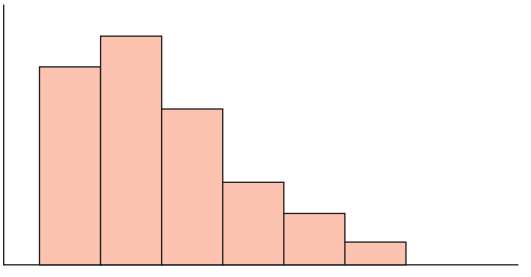
II



III



IV



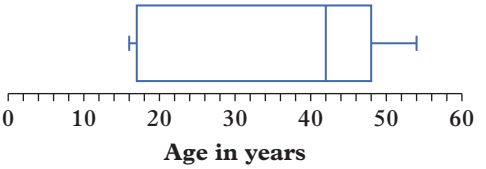
8 **a** Describe what the boxplot on the right tells you about the ages of people attending the Year 10 orientation night, by describing its:

i shape

ii spread

iii centre.

Ages of people at a Year 10 orientation night



b Can you suggest a reason for the unusual shape of the distribution?

9 A vet collected the following data about the weights (in kilograms) of dogs.

5.1	11.2	35.5	17.9	21.1	84.4	42.2	51.0	9.2	8.6	7.5	18.5	19.2	21.8	22.6
24.8	13.7	13.1	16.9	29.8	58.3	42.7	2.1	3.8	9.6	10.8	46.0	1.9	35.0	39.4

a Draw a boxplot to represent the data.

b Draw a histogram with intervals of width 10 to represent the data.

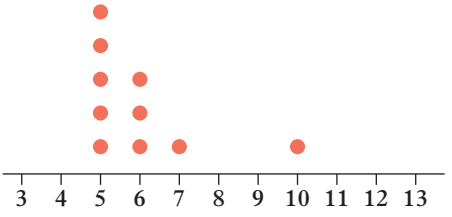
c Describe the distribution of the data.

d Which of the displays you drew in parts **a** and **b** do you think represents the data best? Explain.

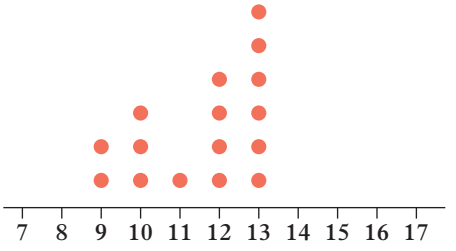
e Why would a dot plot not be as useful for looking at the distribution of the weights of these dogs?

10 Construct a cumulative frequency polygon for the each of these dot plots.

a

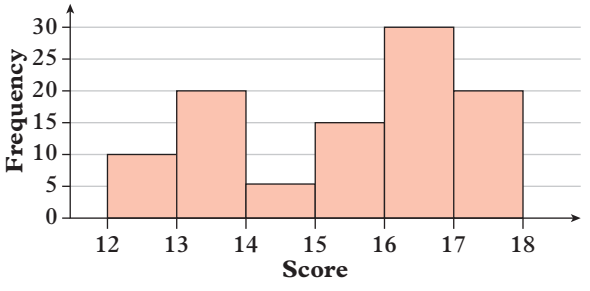


b

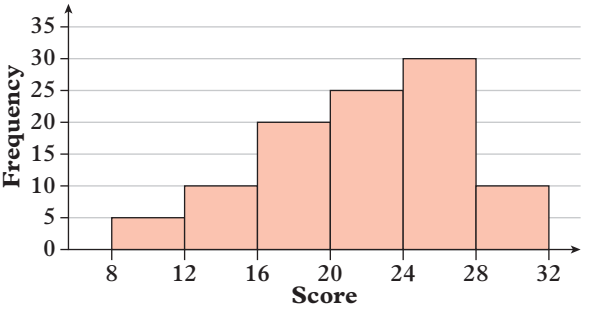


11 Construct a cumulative percentage frequency polygon for each of these histograms.

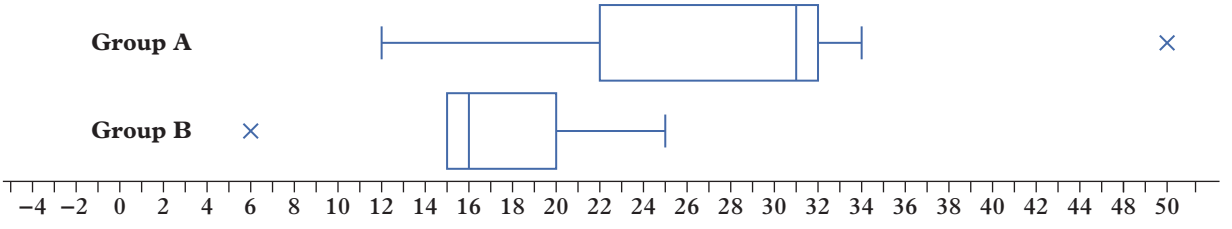
a



b



12 Consider the parallel boxplots below.



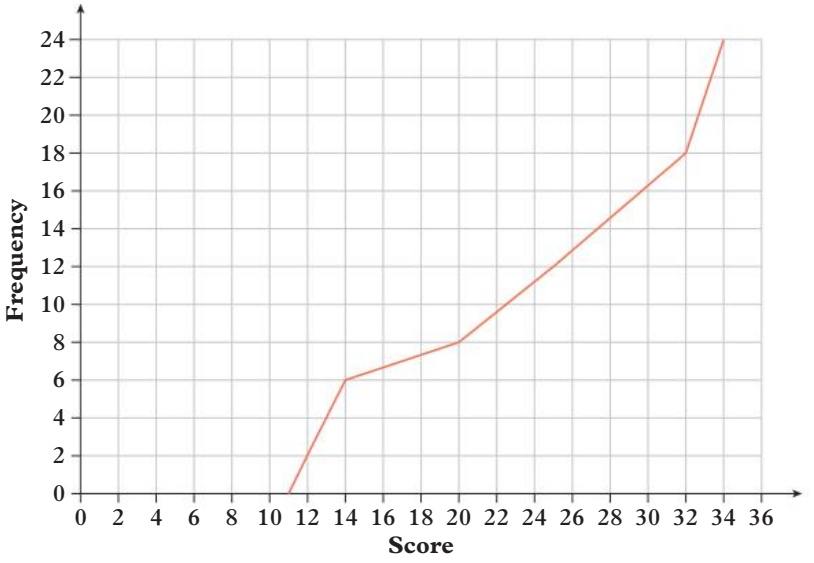
a Describe the distribution of the boxplots.

b Give three reasons why the median would be a better measure of centre than the mean to compare the boxplots.

13 Consider the cumulative frequency plot shown with a frequency scale.

a Determine the five-number summary for the data.

b Determine the interquartile range for the data.



14 Explain why no section of a cumulative frequency polygon can have a negative gradient.

15 Another way to describe a data set is to use percentiles. One way to define the the k^{th} percentile in a data set is as the data value below which k% of data values fall. For example, the median is the 50^{th} percentile as 50% of data values are below the median.

a Write the equivalent percentile for each of the following values in a data set.

i Q_1

ii Q_2

iii Q_3

b Using this definition, why doesn't it make sense to talk about the 100^{th} percentile in a data set?

c For each of the following data sets, calculate the 30^{th} and 70^{th} percentiles.

i 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

ii 10, 20, 20, 30, 30, 30, 40, 40, 40, 40, 50, 50, 50, 50, 50, 60, 60, 60, 60, 60

iii The set of all integers from 1 to 1000.

Another way to define the k^{th} percentile is as the data value at or below which k% of the data values fall.

d Recalculate the 30^{th} and 70^{th} percentiles for each of the data sets in part **c** using this definition.

e How do the the percentile values you found in part **d** differ from those you found in part **c**?

f 100 000 people chose an integer from 1 to 100 and the data is collected. The 30^{th} and 70^{th} percentiles are calculated using both definitions given above. How different or similar would you expect the values to be and why?

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Describing data

Interactive skillsheet
Cumulative frequency polygons

CAS instructions
Histograms

Topic quiz
9C

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

9A **1** Determine the five-number summary for each of the following.

a 14, 16, 50, 23, 17, 24, 32, 19, 22, 32

b 40, 35, 38, 54, 51, 37, 48, 54, 55, 39, 42, 60

c

d

Stem	Leaf
5	4 7 7
6	1 3 5 8 9
7	2 3 5 7 9 9
8	0 1 2

Key: 5 | 4 = 5.4

9A **2** Determine the interquartile range and range for the following five-number summaries:

a 14, 18, 23, 29, 32

b 58, 64, 85, 91, 93

9B **3** Construct a boxplot for each of the following five-number summaries. These boxplots will have no outliers.

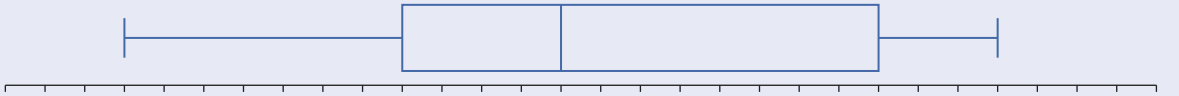
a 120, 130, 145, 150, 175

b 7, 12, 16, 22, 30

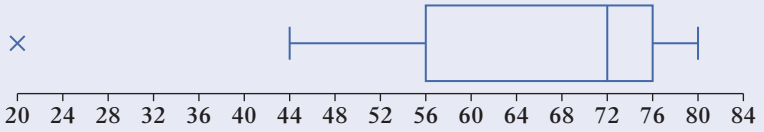
9B **4** For each of the following, state the:

i five-number summary **ii** interquartile range.

a



b

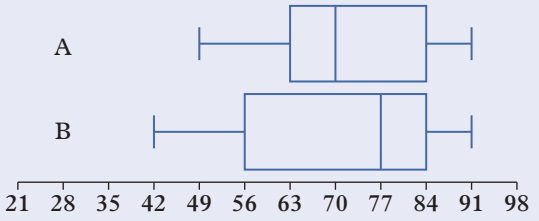


9B **5** Construct a boxplot for each set of data. Check each set for potential outliers.

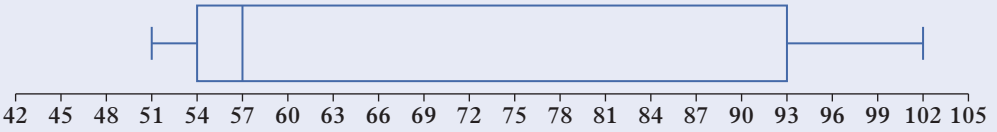
a 41, 85, 74, 67, 79, 75, 22, 84, 80, 66, 41, 79, 45, 54, 47, 84

b 56, 68, 68, 71, 56, 60, 79, 68, 76, 74, 64, 75, 62, 77, 80, 72, 76, 55, 62, 71, 72

9B **6** Consider the parallel boxplots shown below. Compare the distributions of the data in terms of the centre and spread.



9C **7** State the percentage of the data that lies in the following ranges for the boxplot shown below.



a less than 54

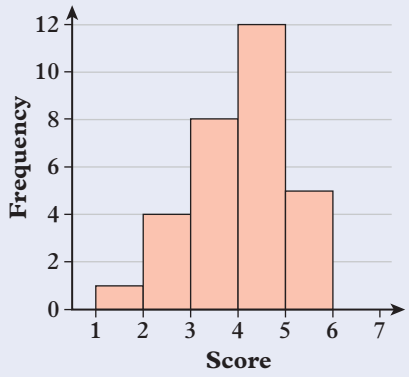
b between 54 and 93

c less than 93

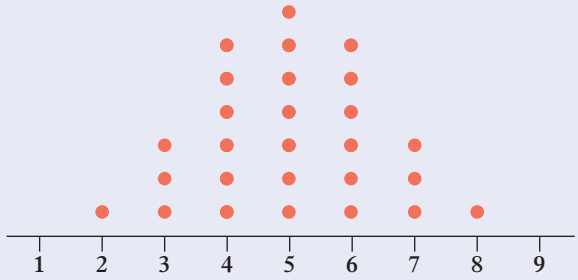
d greater than 57

9C 8 State the shape of the following distributions.

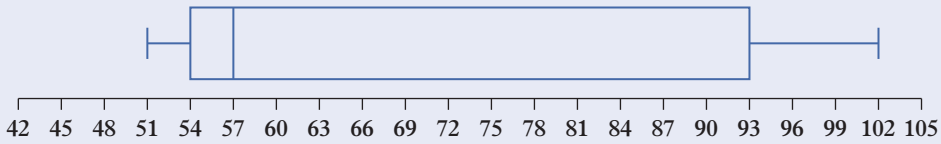
a



b



c



d

Stem	Leaf
1	2 5 8
2	0 1 4 4 5 6 9
3	1 2 3 5 7
4	2 2 9

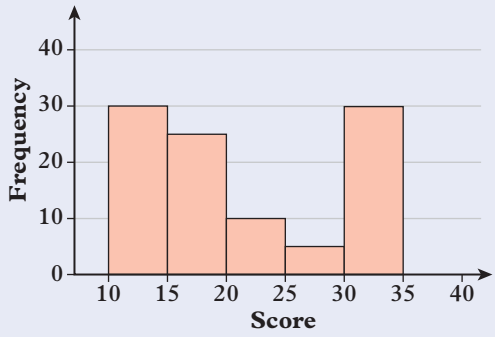
Key 1 | 2 = 12

9C 9 Construct a cumulative frequency polygon for each of the following.

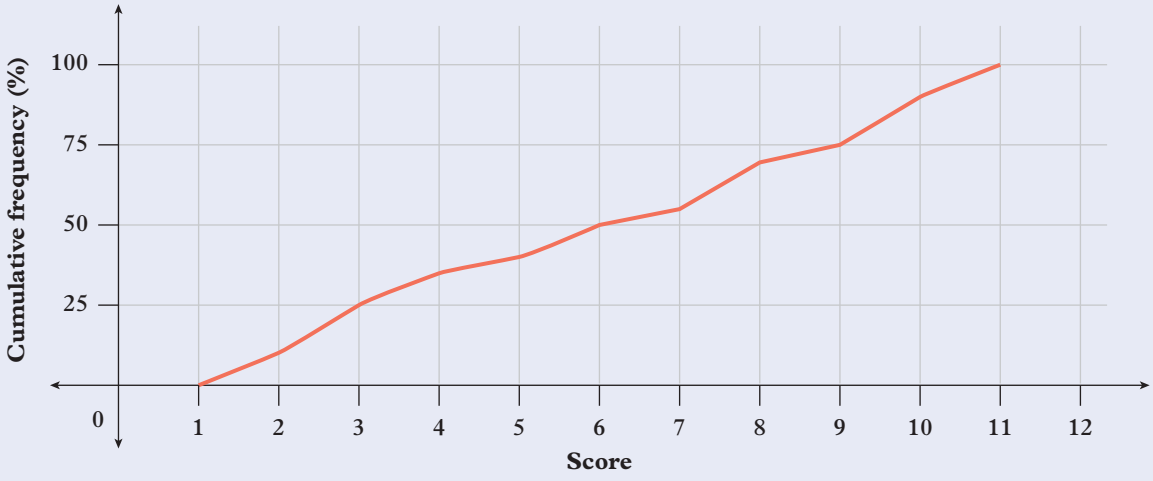
a

Score	Frequency
8	5
9	10
10	13
11	0
12	4
Total	**32**

b



9C 10 Determine the five-number summary for the cumulative frequency plot shown.



9D The mean and standard deviation

Learning intentions

By the end of this topic you will be able to ...

- ✔ calculate the standard deviation of a population and sample
- ✔ use the mean and standard deviation to compare data sets.

Inter-year links

Year 8 — 9B Summary statistics

Year 9 — 7D Summary statistics from tables and displays

Populations and samples

- When working with data, the **population** refers to all potential pieces of data under consideration.
- A **sample** is a selection of data that is part of the population.

Mean and standard deviation

- The **mean** is a numerical average of all values in a data set.
  - → The Greek letter μ (mu) is used to represent the population mean.
  - → $\bar{x}$ (x-bar) is used to represent a sample mean.
  - → The population mean and the sample mean can be calculated in the same way.
- The **standard deviation** is a measure of how much the scores in a set differ (or deviate) from the mean.
  - → A small standard deviation means that the scores in a data set are close to the mean.
  - → A large standard deviation means that the scores in a data set are spread out from the mean.
  - → The Greek letter σ (sigma) is used to represent the population standard deviation.
  - → The letter s is used to represent a sample standard deviation.
- A population's standard deviation (σ) can be calculated using the formula:

$$\sigma = \sqrt{\frac{\left(x_1 - \mu\right)^2 + \left(x_2 - \mu\right)^2 + \ldots + \left(x_n - \mu\right)^2}{n}}$$

- A sample's standard deviation (s) can be calculated using the formula:

$$s = \sqrt{\frac{\left(x_1 - \bar{x}\right)^2 + \left(x_2 - \bar{x}\right)^2 + \ldots + \left(x_n - \bar{x}\right)^2}{n - 1}}$$

  - → When calculating the standard deviation from a sample mean, the sum is divided by $n - 1$ rather than n because this provides a better estimate of the standard deviation for the population as the squared deviations from the sample mean underestimate the squared deviations from the population mean.

Using the mean and standard deviation to describe and compare data sets

- The mean describes a central point in a data set. If a data set is positively skewed the mean will be greater than the median, and if the data is negatively skewed the mean will be less than the median.
- When the data is skewed, the median represents the centre of the data better than the mean.
- The standard deviation describes the spread of a data set. When comparing data sets, a larger standard deviation will have scores that are more spread out than a smaller standard deviation.

Example 9D.1 Finding the standard deviation for a list of data

Find the standard deviation for this sample of data, correct to one decimal place:
4, 8, 12, 19, 5

THINK

1 Calculate the mean of the sample.

2 Substitute the given values into the formula for the standard deviation of a sample and round your answer to one decimal place.

WRITE

$$\bar{x} = \frac{4 + 8 + 12 + 19 + 5}{5}$$
$$= 9.6$$

$$s = \sqrt{\frac{(4 - 9.6)^2 + (8 - 9.6)^2 + (12 - 9.6)^2 + (19 - 9.6)^2 + (5 - 9.6)^2}{5 - 1}}$$
$$= \sqrt{\frac{(-5.6)^2 + (-1.6)^2 + (2.4)^2 + (9.4)^2 + (-4.6)^2}{4}}$$
$$\approx 6.1$$

Example 9D.2 Using a spreadsheet to find the standard deviation

Use Excel (or alternative spreadsheet software) to help you find the standard deviation for the data sample below. Give your answer correct to one decimal place.
10, 15, 11, 23, 18, 9, 22, 17, 14, 10

THINK

1 Enter each value from the data sample into a separate cell in column A, in a vertical list.

2 Use the formula =STDEV.S(A1:A10) to calculate the standard deviation of cells A1 to A10.

3 Round the standard deviation to one decimal place.

WRITE

	A
1	10
2	15
3	11
4	23
5	18
6	9
7	22
8	17
9	14
10	10
11	=STDEV.S(A1:A10)

	A
1	10
2	15
3	11
4	23
5	18
6	9
7	22
8	17
9	14
10	10
11	5.043147166

$s \approx 5.0$

Note: The formula for finding the standard deviation might vary if you are using a different spreadsheet program.

Helpful hints

- ✔ The formula for the population standard deviation using Excel is =STDEV.P(:), where the range of cells containing the data is placed within the brackets. The cell names of the first and last cells from the listed data are separated by the colon.
- ✔ Most calculators have a function for calculating the standard deviation of a list of data.

ANS p643

Exercise 9D The mean and standard deviation

1, 2, 3(a, d–f), 4(a, b, d, f), 5, 7, 8, 10, 14, 15 | 1, 2, 3–4(b, e, f), 5–7, 9, 11, 13, 15, 16 | 1, 2, 5, 9, 11, 12, 15–18

UNDERSTANDING AND FLUENCY

9D.1 **1** Using a calculator, find the standard deviation for each of these sample data sets. Give your answers to one decimal place.

a 6, 3, 8, 65, 4 **b** 8, 11, 15, 3, 9, 5 **c** 7, 7, 10, 12, 17, 15

d 13, 29, 21, 22, 22, 11, 19 **e** 140, 156, 120, 99, 187, 147 **f** 159, 166, 122, 171, 136, 129

2 Using a calculator, find the standard deviation for each of these population data sets. Give your answers to one decimal place.

a 2, 2, 2, 2, 2 **b** 2, 2, 2, 8, 8, 8 **c** 20, 60, 70, 80, 100

d −11, −9, −17, −23, −15 **e** 2.2, 5.0, −0.2, −0.8, 5.0, 3.2 **f** 1, 2, 4, 16, 32, 62

9D.2 **3** Calculate the standard deviation for these samples of data using technology. Give your answers to one decimal place.

a 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 **b** 39, 24, 34, 34 30, 32, 38, 36, 36, 32

c 10, 100, 1000, 10 000, 100 000, 1 000 000 **d** 20, 200, 2000, 20 000, 200 000, 2 000 000

e −1 000 000, −100 000, −10 000, −1000, −100, −10 **f** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{9}{10}, \frac{10}{11}$

4 Calculate the standard deviation for these populations of data using technology. Give your answers to one decimal place.

a 89, 104, 97, 114, 97, 90, 107, 82, 109, 103

b 1776.6, 2521.4, 2110.1, 606.5, 926.6, 2395.4, 1572.5, 12.03, 2496.3, 123.7

c 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

d 81, 82, 83, 84, 85, 86, 87, 88, 89, 90

e 5, 6, 7, 8, 9, 10, 11, 12, 13, 14

f 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256

5 For each data set below:

i decide whether the data set is from a sample or a population.

ii calculate the appropriate standard deviation.

a This data was collected about the ages of students attending a school formal by surveying everybody who attended.

16	17	18	16	17	16	15	14	16	17	18	16	17	17
16	17	16	17	15	16	17	16	17	16	16	16	15	17
18	18	17	16	17	16	17	16	17	16	16	16	17	17

b This data was collected about the number of hours 20 students in a Year 10 class (of 30 students) spent per week using social networks.

5	10	7	1	19	17	16
7	10	11	8	9	15	13
14	6	21	18	14	13	

c This data was collected about the weights of newborn babies in South Australia:

3.2	4.1	2.9	3.4	3.5	3.4	3.1
3.8	4.0	3.3	3.6	2.9	3.1	3.4
3.5	4.2	2.6	3.0	3.6	3.4	4.8
2.1	3.5	3.3	3.6			

6 Consider this data set, which shows the ages of every player in two cricket teams.

16	16	15	17	15	16	14	15	16
16	17	16	15	17	16	15	16	16
17	16	17	15	16	16	17	16	

a Find the standard deviation for the data set. Give your answers to two decimal places.

b Write a sentence about the spread of ages of players in the cricket teams.

7 Consider this data, about the number of text messages sent by a sample of Year 10 students per day.

1	5	11	17	21	9	8	25	26	19	5	2	8
9	13	34	16	29	31	26	19	18	15	20	10	

a Find the mean and standard deviation for the data. Give your answers to two decimal places.

b Write a sentence describing the centre and spread of the data set.

8 The following is a list of the number of people at a shoe sale each hour over a 24-hour period:

46	26	35	49	58	13
27	48	46	39	42	48
46	51	35	29	47	41
42	42	49	45	46	51

a Find the mean for this data. Give your answers to one decimal place.

b Find the standard deviation. Give your answers to two decimal places.

c Comment on the centre and spread of the data set.

9 Data was collected about the number of minutes a sample of passengers were kept waiting for their trains on two different train lines over the course of a day.

Train line A: 4, 2, 3, 5, 6, 7, 4, 4, 6, 5, 6, 7, 9, 4, 5, 6, 6, 7, 6, 7, 9, 4, 6, 6, 1

Train line B: 2, 9, 15, 3, 4, 3, 11, 1, 1, 6, 2, 3, 4, 2, 3, 19, 2, 3, 2, 2, 2, 3, 1, 4

a For each train line, find the mean waiting time. Give your answers to one decimal place.

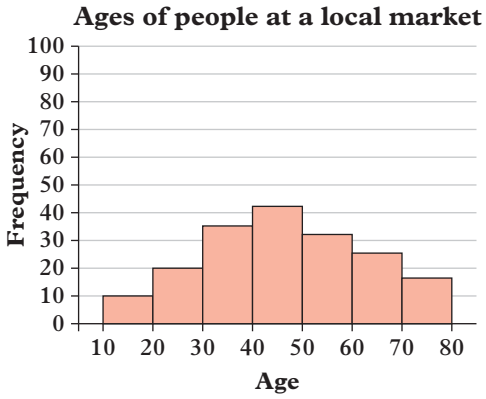
b Find the standard deviation for the data from each train line. Give your answers to two decimal places.

c Write a sentence comparing the centre and spread of the two data sets.

d On which train line would you say that the trains run late more significantly?

10 The two histograms below show data collected about the ages of people attending a local market on two different days. The data sets have the same mean and the same number of scores, and have been shown on axes with the same scale for ease of comparison.

Which data set has a small standard deviation and which has a large standard deviation? Explain how you know.



11 Consider this data collected about the price (in dollars) of a particular brand of electronic tablet bought from all stores where it is available in a city.

499 519 499 475 469 479 509 549 529 525
475 499 499 199 489 479 499 475 515 499

a Find the standard deviation for this data set.

b You may notice that one of the prices is distinctly different from the rest (an outlier). Provide a possible reason for this.

c Recalculate the standard deviation for the data set excluding the outlier.

d What do you notice about your answers for parts **a** and **c**?

e What measure of spread might you use instead of standard deviation when there is an outlier in the data set?

12 The magnitude or absolute value of x, written $|x|$, is its distance from 0 on a number line, ignoring direction. For example, $|5| = |-5| = 5$.

In a data set with n data points the mean absolute deviation is found using the formula:

$$\text{mean absolute deviation} = \frac{|x_1 - \bar{x}| + \ldots + |x_n - \bar{x}|}{n}.$$

a Calculate the mean absolute deviation and standard deviation for each of the following population data sets.

i 1, 2, 2, 3, 4, 4, 4, 5, 6, 10 **ii** 5, 5, 5, 7, 8, 10, 15, 21

b What do you notice when comparing the mean absolute deviations and standard deviations you calculated in part **a**?

13 Another measure of spread is the median absolute deviation. In a data set with n data points and median $\tilde{x}$, the median absolute deviation is the median of the data set

$$|x_1 - \tilde{x}|, \ldots, |x_n - \tilde{x}|.$$

For each of the data sets in question 12:

a find the median $\tilde{x}$

b calculate the median absolute deviation.

14 a When is the median a better measure of the centre of a distribution of data than the mean? Explain.

b When is the interquartile range a better measure of the spread of a distribution of data than the standard deviation? Explain.

15 Compare the following pairs of symmetric sample data in terms of their mean and standard deviation. Use Excel (or alternative spreadsheet software) to help you calculate the statistics.

a **A:** 20, 24, 28, 28, 30, 30, 31, 32, 35, 40
B: 24, 25, 25, 25, 26, 27, 29, 31, 34, 36, 36, 36, 37

b **A:** 75, 81, 83, 84, 88, 92, 95, 101, 105, 109, 113
B: 64, 70, 71, 71, 72, 74, 80, 81, 83, 84, 86, 90

16 We can compare individual data values by determining the number of standard deviations, s, the value, x, is above or below the mean, $\bar{x}$. That is $z = \frac{x - \bar{x}}{s}$. These values are called standard scores (or z-scores).

For example, if Julia scored 15 on a test where the mean score was 20 and the standard deviation was 2, then Julia's standard score is $z = \frac{15 - 20}{2} = -2.5$, meaning Julia's score is 2.5 standard deviations below the mean score.

i Calculate the standard score for each pair of students' results, correct to two significant figures.

ii Compare the scores of each pair of students from different classes using their standard scores.

	Class A			Class B		
	Mean	**SD**	**Student A**	**Mean**	**SD**	**Student B**
a	70	10	85	55	16	85
b	70	20	85	70	15	85
c	50	25	30	75	5	50
d	50	8	26	50	16	2

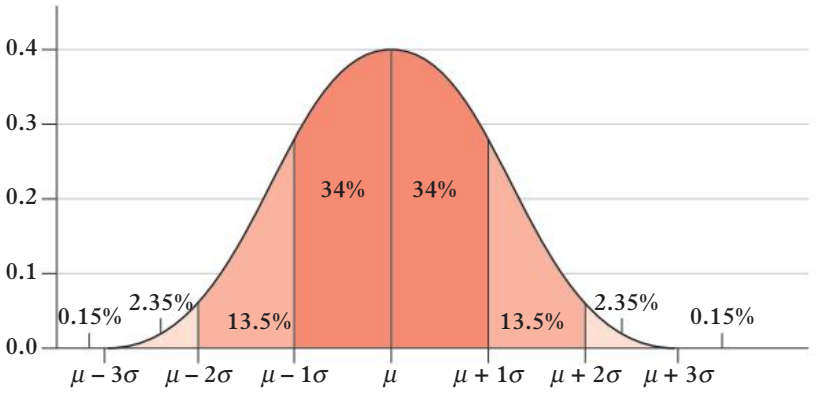
17 A population that has a symmetric distribution where the majority of the data is centred around the mean in a rough bell shape is said to be normally distributed (also known as forming a bell curve). Standard deviation plays a useful role in making predictions about populations that are normally distributed. A normal distribution will have nearly all values (99.7%) within three standard deviations (also known as three sigma) of the mean. Consider the graph below with approximate percentages of the amount of data between different standard deviations using the 68–95–99.7% rule.

a Show that 99.7% of values lie within three sigma (three standard deviations) of the mean (μ).

b What percentage of values lie within:

i one sigma of the mean

ii two sigma of the mean?



A typical example of this kind of data distribution can be seen in the data for IQ (intelligence quotient). IQ scores are a ranking system where the mean (μ) IQ score is set to 100 and the standard deviation is set to 15.

c Redraw the graph above so that the scale on the x-axis uses these values (that is, $\mu = 100$, $\mu + 1\sigma = 115$, etc.).

d What is the IQ range of 99.7% of the population?

e What IQ would you need to have in order to be in the top 0.15% of the population?

f What IQ would you need to have in order to be in the bottom 2.5% of the population?

g What percentage of the population has an IQ of 130 or greater?

h What percentage of the population has an IQ of 85 or less?

i A group of highly intelligent people had their IQs tested and the distribution was found to be normal, with a mean in the middle of these scores. If 95% of this group had an IQ between 120 and 152, find the mean and standard deviation of the group.

18 If we were to take different samples from a population, the sample mean in each sample would vary in value but would approximate the mean of the population. Regardless of the shape of the distribution of the original data, the distribution of the sample means would be approximately normally distributed with a mean of $\mu_{\bar{x}}$ and a standard deviation of $s_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ where n is the size of each sample. A particular population has a mean of 40 and a standard deviation of 9.

a Find the mean and standard deviation of the distribution of sample means when taking samples of 25.

Use the diagram in question **17** to answer the following.

b Determine the probability, as a percentage, that a randomly selected sample will be:

i between 38.2 and 41.8

ii greater than 38.2

iii less than 34.6

iv greater than 43.6 or less than 36.4.

c Determine the value(s) of the sample mean such that the probability of selecting a sample mean:

i less than this sample mean is 84%

ii greater than this sample mean is 50%

iii between these two sample means is 95%

iv between these two sample means is 15.85%.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Standard deviation

CAS instructions
Standard deviation

Topic quiz
9D

9E Scatterplots and bivariate data

Learning intentions

By the end of this topic you will be able to ...

- ✔ interpret and draw scatterplots
- ✔ determine the strength and direction of a linear relationship from a scatterplot
- ✔ draw a line of best fit by eye.

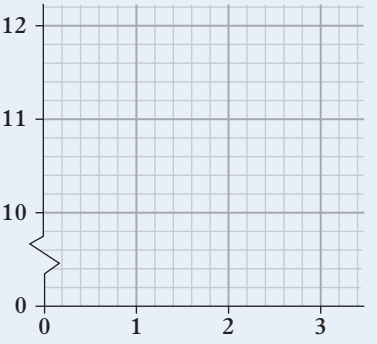
Inter-year links
Year 9 7B Classifying and displaying data

Bivariate data

- **Bivariate data** contains two variables, where each data value in one variable is paired with a data value in the other variable. The purpose of gathering bivariate data is to investigate the relationship between the two variables.
- When investigating numerical bivariate data, we usually look at how one variable (the **independent** or **explanatory variable**) affects another variable (the **dependent** or **response variable**).

Scatterplots

- A **scatterplot** can be used to display numerical bivariate data.
  - → When plotting numerical bivariate data on a scatterplot, the independent variable is plotted on the horizontal axis and the dependent variable is plotted on the vertical axis.
- Scatterplots are usually drawn focusing on the area of the axes between which the data points lie. A break in the scale on the axes, indicated by a zigzag, can be used to indicate that the focus is on the axes a distance away from the origin.

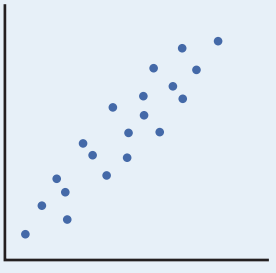


Direction

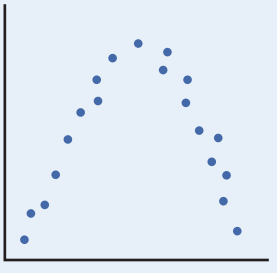
- If there is a **positive relationship** between two numerical variables, as one variable increases in value the other variable also increases in value.
- If there is a **negative relationship** between two numerical variables, as one variable increases in value the other variable decreases in value.

Linearity

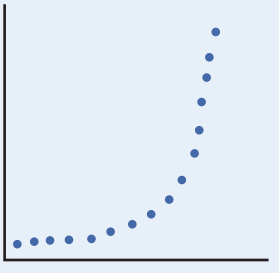
- When considering bivariate data, we need to determine whether the pattern of the data is either linear or non-linear. For a linear relationship the points can be approximated by a straight line.



Linear



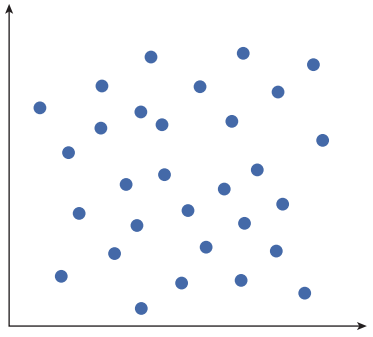
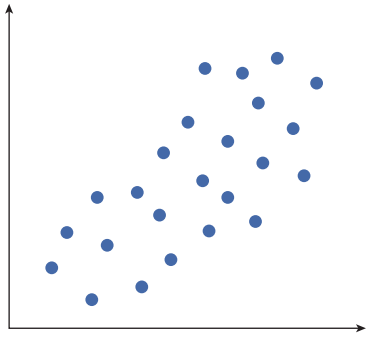
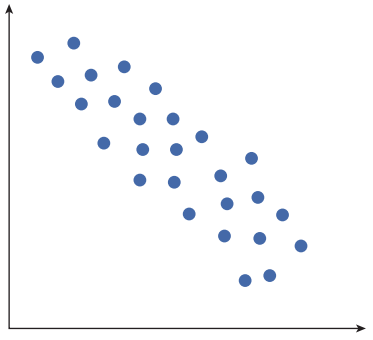
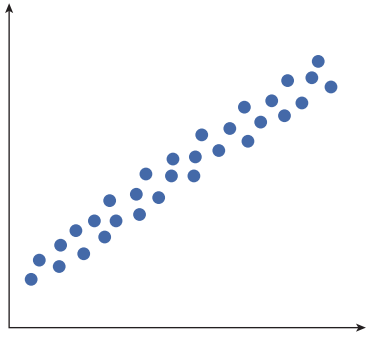
Non-linear



Non-linear

Strength

- The **correlation** between two numerical variables is a measure of the strength of the linear relationship between those variables. If the relationship is strong, the points will lie close to a straight line.

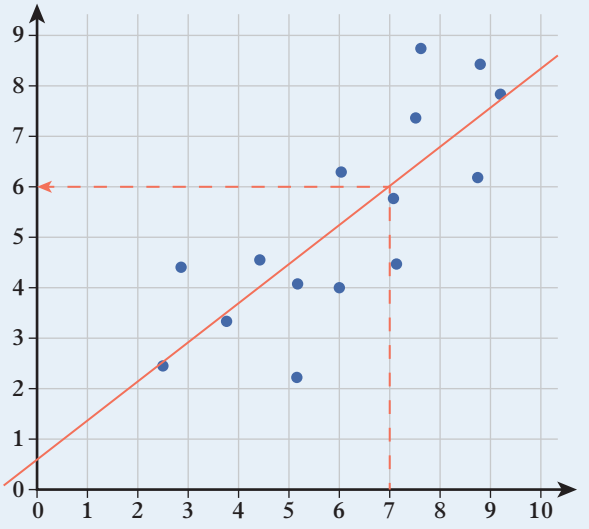
Sample scatterplot	Description	Strength of relationship	Direction of relationship
	no obvious pattern in the data	no correlation	not applicable
	a weak linear pattern in the data	weak	positive
	a clear linear pattern in the data	moderate	negative
	a strong linear pattern in the data	strong	positive

Lines of best fit by eye

- A **line of best fit** is a line that best represents the relationship between the two variables on a scatterplot.
- To draw a line of best fit by eye, draw a straight line on your scatterplot so that it lies as close to as many points as possible.
  - → There should be approximately the same number of points on each side of the line of best fit.

Using a line of best fit to make predictions

- A line of best fit can help predict the value of one variable when the other variable is known.
  - → As the strength of the correlation between two plotted variables increases, the predictions made using a line of best fit become more reliable.
  - → Predictions made within the original range of the data plotted are more reliable than predictions made outside the original range of data.
- To make a prediction using a line of best fit, locate the value on the axis you are predicting for, and draw a straight line (horizontal or vertical) to the line of best fit. From this point, draw another straight line (horizontal or vertical) to the other axis to get your predicted value.

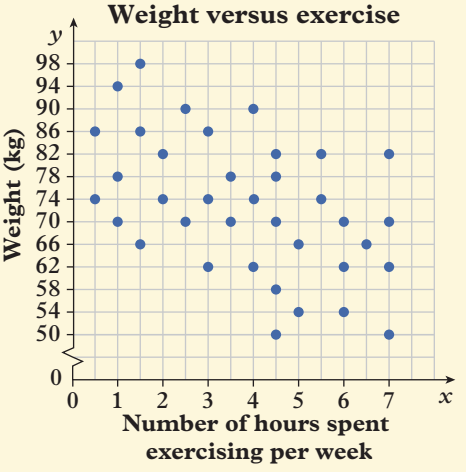


Example 9E.1 Describing the relationship in a scatterplot

Describe the relationship shown in this scatterplot:

a in terms of its strength and direction

b in the context of the data it represents.



THINK

a There is only a weak linear correlation in the data but, as the independent variable increases, the dependent variable does decrease.

b Describe the relationship shown in the context of what it tells you about the data.

WRITE

a The scatterplot shows a weak, negative linear relationship between the two variables.

b As the number of hours spent exercising increases, weight generally decreases.

Example 9E.2 Drawing a scatterplot

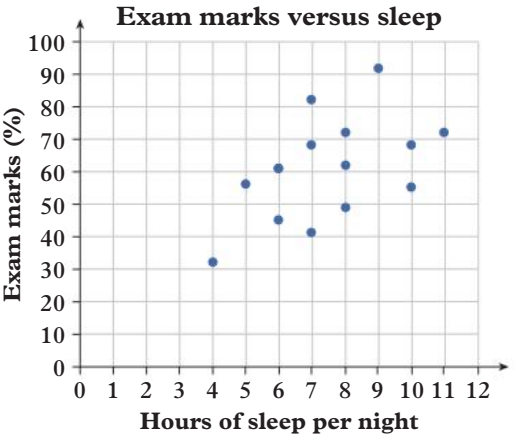
Draw a scatterplot for this data set.

Hours of sleep per night	8	6	7	9	10	11	8	7	6	5	4	8	7	6	10
Exam marks	72	61	82	92	68	72	49	68	61	56	32	62	41	45	55

THINK

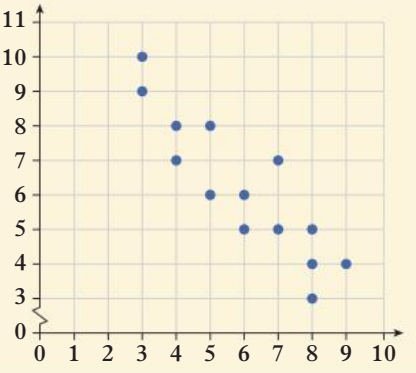
1. Determine which of the two variables is the independent variable and which variable is the dependent variable. A change in the number of hours of sleep per night could affect the exam marks, so the number of hours of sleep per night is the independent variable.
2. Use the horizontal axis for the independent variable (hours of sleep per night) and the vertical axis for the dependent variable (exam mark).
3. Identify the smallest and largest value for each variable and mark scales on the axes to cover these values. (Show breaks in the axes if the values on either axis do not begin at zero.)
4. Create a scatterplot by plotting each pair of variables from the table on the axes.

WRITE



Example 9E.3 Drawing a line of best fit by eye

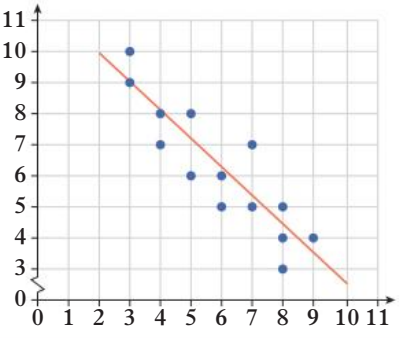
Draw a line of best fit by eye for the data shown on this scatterplot.



THINK

1. Look at the direction of the plotting of the points. There is a clear downward trend, so the line of best fit will need to be drawn in this direction.
2. Draw a straight line on the scatterplot in the direction of the data, with an approximately equal number of points below and above the line.

WRITE



Helpful hints

- ✔ When creating a scatterplot by hand, use graph paper to ensure the scatterplots you create are appropriately scaled and the points are accurately plotted.
- ✔ A line of best fit may, but does not need to, pass through any of the points on the scatterplot.
- ✔ A line of best fit drawn by eye is not expected to be perfect. As long as it looks like a good fit for the data it will be fine.

ANS p644

Exercise 9E Scatterplots and bivariate data

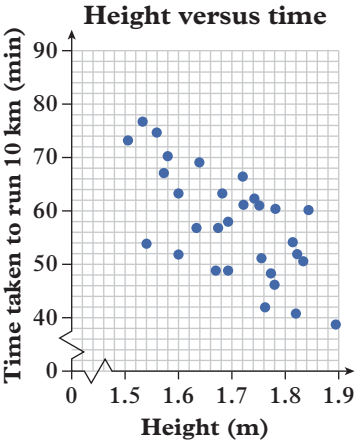
△ 1–3, 5–8 □ 1–6, 9, 13, 14 ⬠ 1–3, 7–15

UNDERSTANDING AND FLUENCY

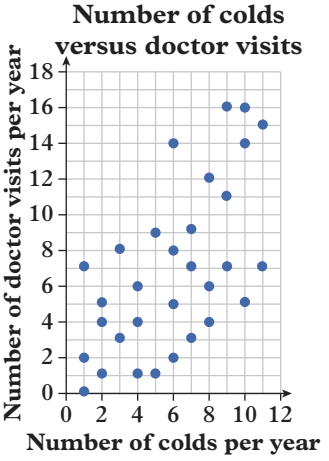
9E.1 **1** Describe the relationship shown in each of the scatterplots below:

i in terms of strength and direction **ii** in the context of what it shows about the data it represents.

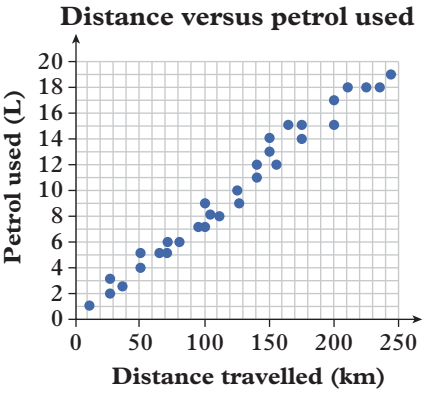
a



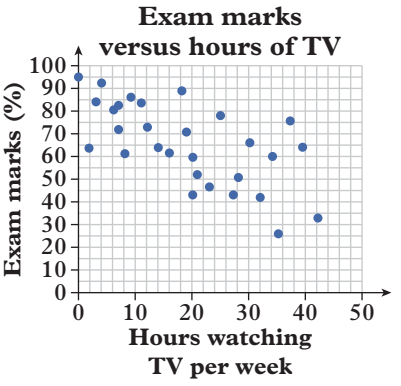
b



c



d



9E.2 **2** Draw a scatterplot for each of these data sets.

a

Height (cm)	150	181	166	168	175	178	172	155	161	168	179	184	188	159	167
Shoe size	4	9	9	7	9	8	7	4	5	7	7	10	12	7	6

b

Temperature (°C)	33	15	16	25	27	36	41	11	8	32	20	25	31	18	28
Number of pies sold	5	25	33	16	19	2	3	42	35	12	21	11	10	51	15

c

Number of customers	45	56	16	32	78	98	26	66	45	76	86	42	59	31	67
Daily profit ($100s)	12	16	8	19	21	48	7	32	22	29	36	13	18	16	12

3 Describe the relationship shown in each scatterplot you drew for question **2** in the context of what it tells you about the bivariate data it represents.

9E.3 **4** For each of these data sets, construct a scatterplot and draw a line of best fit by eye.

a

x	3	2	7	5	6	3	1	4	10	9	7	4	8	10	1	5	7	3	8	6
y	5	4	7	6	8	2	2	3	10	9	9	8	6	8	5	4	4	6	10	5

b

x	4	7	1	2	10	5	8	6	2	9	4	3	10	7	1	9	8	5
y	5	4	9	9	2	6	4	5	7	4	8	8	1	6	8	1	5	7

c

x	6	8	3	9	11	14	5	1	15	13	8	5	11	12	7	4
y	5	10	5	6	11	9	3	2	11	13	8	7	8	10	3	1

d

x	4	9	13	15	7	6	18	11	13	7	9	10	18	12	16	17
y	42	29	15	6	38	36	1	13	18	40	20	12	4	16	9	7

5 Kieran and Marie want to investigate the relationship between the number of different types of animals in zoos and the number of visitors that the zoos receive per day.

a For this investigation, predict the correlation you expect to see between the two variables in terms of linearity, strength and direction.

b Kieran and Marie collected the data in the table below. Draw a scatterplot to represent this data.

Number of different animal types	9	26	11	37	19	42	15	14	25	32	47	34	31	37	22	18
Number of visitors per day	42	261	150	541	247	387	226	291	317	401	761	396	309	311	177	151

c Describe the trend shown by the scatterplot you drew for part **b**. Compare the trend to the prediction you made in your answer to part **a**.

6 When examining the relationship between two variables in real life, we think about how one variable changes depending on how the other changes. Height and weight are a good example of this.

a Would you say that the height of a person depends on how much he or she weighs? Or does the weight of a person depend on how tall the person is?

Another way to look at this would be to think about what effect changes in one variable would have.

b Would you say that, as a person gets taller, this generally makes them heavier? Or, as a person gets heavier, this generally makes them taller?

c If we consider height and weight, which is:

i the dependent variable **ii** the independent variable?

7 For each of the pairs of variables **a–d** below, determine:

i if there might be a linear relationship between them

ii which is the independent variable and which is the dependent variable, if a linear correlation does exist between them

iii the strength and direction of any linear correlation that exists.

a time (years) and height of a tree (metres)

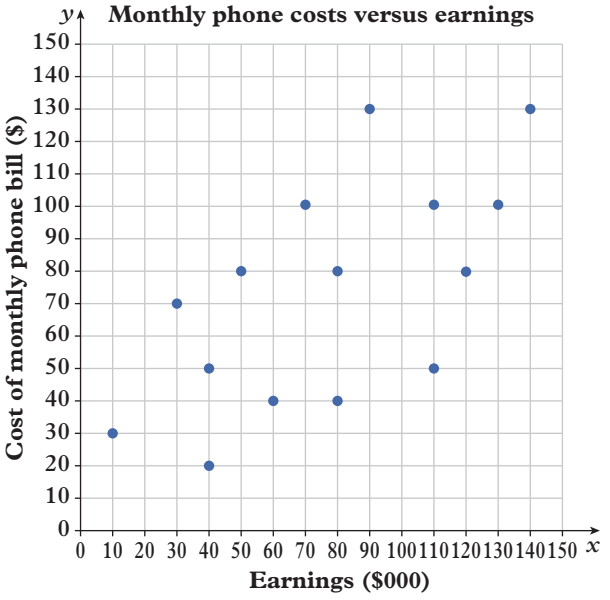
b value of a car (dollars) and age (years)

c weight of a person (kilograms) and number of languages he or she speaks

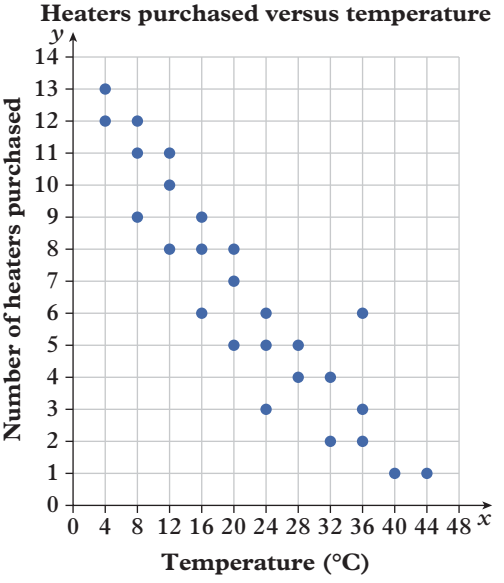
d shoe size and height of a person (metres)

8 Consider the two scatterplots below.

Scatterplot A



Scatterplot B



a For each of the scatterplots, describe the strength and direction of the linear relationship shown in the context of the data it shows.

b If the temperature on a day was 26°C, how many heaters do you predict would be bought?

c If somebody had earnings of $100 000, what would you predict their monthly phone bill to be?

d How confident are you of your predictions in parts **b** and **c**? Explain why you would not be as confident predicting a value from scatterplot A as you would be predicting a value from scatterplot B.

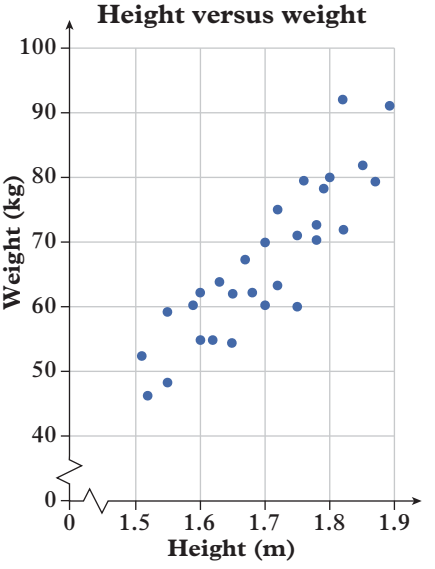
9 A strong positive linear correlation was found between the number of hours studied and the result achieved on a test.

a Does this mean that if you study for many hours you will get high marks? Explain.

b To help prevent assumptions like the one made in part **a**, there is a saying: 'correlation does not mean causation'. What does this mean?

c Explain why the saying quoted in part **b** is useful to remember, particularly when dealing with strong correlations.

10 Consider the scatterplot on the right, showing the relationship between the heights and weights of the students in a Year 10 class.



a Describe the trend shown.

b Draw a line of best fit by eye for this scatterplot.

c Use your line of best fit to predict the weight of a person who is 185 cm tall.

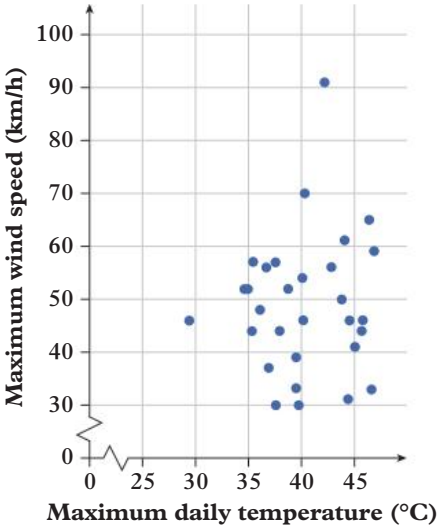
d How confident are you of your prediction in part **c**? Explain your answer.

11 Can a scatterplot be drawn with a negative section of the x- and/or y-axis? Explain.

12 Explain why it is important to plot a scatterplot with the original data rather than with grouped data.

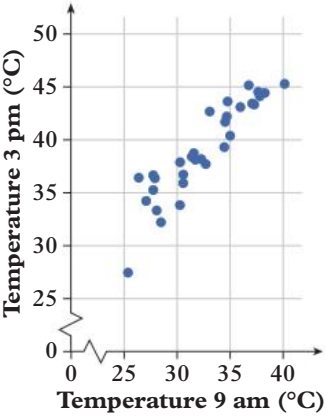
13 For each of the following, state whether there is a correlation between the two variables and, if there is, describe it in terms of linearity, strength and direction.

a The scatterplot below plots the maximum wind speed, in kilometres per hour, against the maximum temperature, in °C, for each day in January 2022 in Birdsville, Queensland.



(Source: BOM)

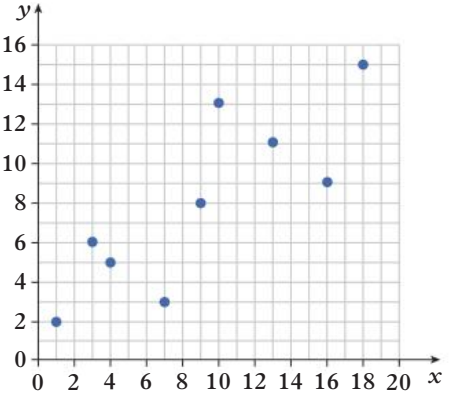
b The scatterplot below plots the temperature at 3 pm, in °C, against the temperature at 9 am, in °C, for each day in January 2022 in Birdsville, Queensland.



(Source: BOM)

14 Look up the weather data for the previous month in your local area. Determine any correlations between the numerical variables given by constructing a variety of scatterplots with the data.

15 A common method for finding a line of best fit is called median–median regression or three-median regression.



a Copy the scatterplot on the right and divide it vertically into three sections so that each section has the same number of points.

b For the median–median regression method, you need to find the median of each section. This is obtained by finding the median x-coordinate and the median y-coordinate and marking these points on the graph. Show that the points (3, 4), (9, 8) and (16, 11) are the medians of each section.

c Draw in the line of best fit by placing your ruler on the lower median and upper median, then moving it one-third of the way towards the middle median before you draw the line.

d Find the equation of the line of best fit you drew in part **c**.

e Explain why the three-median regression line is best used when a scatterplot has one or more outliers.

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Scatterplots

CAS instructions
Scatterplots and lines of best fit

Topic quiz
9E

9F Analysing statistical reports

Learning intentions

By the end of this topic you will be able to ...

- ✔ analyse statistical reports
- ✔ identify potential sources of bias
- ✔ identify where graphs or claims may be misleading.

Inter-year links

Year 8	9A Collecting data and sampling methods
Year 9	7A Surveys and sampling methods

Analysing statistical reports

- When analysing a statistical report, you need to consider the source and size of the sample and decide if the sample appropriately represents the population.
- A **biased sample** is a sample that is not reflective of the population. To avoid bias, samples should be selected using a random sampling method.

Misleading graphs and claims

- Reports may contain graphs or data displays that are misleading. Look at the scales used on the axes of graphs to be certain they are evenly spaced with consistent numbering.
  - → Look at the value the vertical axis starts at – if it does not start at 0, vertical differences will be enlarged and the differences in heights will be exaggerated.
  - → Where only 'averages' are provided, the distribution of the data is not known, so the average may not be very informative.
- Graphs and axes on graphs should be properly labelled. By not labelling the graph correctly, incorrect assumptions can be made about the data being presented.
- Look at the range of data being presented and consider whether any data has been purposefully omitted. For example, if you are looking at temperature data over a year, all months should be considered, as temperature varies due to the seasons in a year.
- Consider why the data has been presented as it has, and what may have been omitted. For example, if only summary statistics are provided, this may not provide adequate information about the distribution of the data.
- Be prepared to question the claims made in reports. Some reports are written motivated by an agenda that can result in an unwarranted claim being pushed to further that agenda.

Ethical considerations

- **Ethics** is the system of moral principles that distinguish between right and wrong behaviour.
- Ethics need to be considered throughout the statistical investigation process, from data collection through to data interpretation and reporting.
- The following ethical considerations should be made when analysing statistical reports:
  - → Was the data collected in a fair manner with the aim to find the truth as opposed to validating pre-existing beliefs?
  - → Has any of the data been manipulated or hidden?
  - → Has the data been presented in a balanced way, showing both positive and negative results?
  - → Could reporting the data cause harm to any individuals or groups of people?

Example 9F.1 Reviewing a published report

A new toll tunnel has just been opened to allow cars to travel from one side of the river to the other. The alternative route is via a no-toll bridge. Shortly after the tunnel opened, a newspaper reported:

'The tunnel is now so popular that the probability of a car using the tunnel rather than the bridge is 85%'.

Public perception was that this is not the case. Some residents decided to do a car count over a long time and found 967 cars from a total of 2472 went over the bridge. Comment on the newspaper report.

THINK

1 From the collected data, calculate the probability of a car using the tunnel.

2 Compare the collected data with the reported data, and comment on the comparison.

WRITE

Using the collected data:

$$\Pr(\text{car uses bridge}) = \frac{967}{2472} \times 100\%$$
$$\approx 39\%$$

So: $\Pr(\text{car uses tunnel}) \approx 61\%$

The collected data shows that 61% of cars use the tunnel, while the newspaper reports this figure to be 85%. It seems likely that the newspaper report is exaggerated as this gap is unlikely to occur from natural variation.

Example 9F.2 Identifying misleading features in graphs

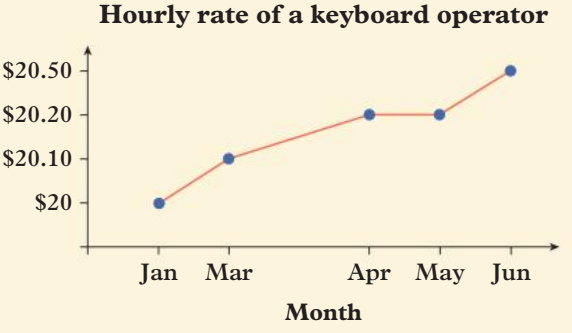
This graph displays the hourly pay rate for a keyboard operator employed by a large company. The time period covered is the first 6 months of the year.

a This graph is misleading. Give reasons why.

b The company manager claims that the hourly rate of the keyboard operator has increased dramatically over the last 6 months. The graph appears to indicate this.

i Is this true?

ii How does the graph seem to support the manager's interpretation of the data?



THINK

a Look for anything that might make the graph misleading.

b i Look at the values on the *y*-axis.

ii Consider the scale on the *y*-axis.

WRITE

a This graph is misleading for the following reasons:

- The scale on the *x*-axis is not uniform.
- The month of February is missing.
- The scale on the *y*-axis does not start at zero, and no break is indicated.
- The scale on the *y*-axis is not uniform.

b i The keyboard operator's hourly pay rate has only increased from \$20 to \$20.50 over the 6-month period, an increase of 50c per hour. This is not a 'dramatic' increase.

ii Because the scale on the *y*-axis does not start at zero, the range of \$20 to \$20.50 has been stretched out to appear larger than it would on a complete scale. This gives the impression that the increase is greater than it really is.

Helpful hints

✔ When reviewing graphs, look at where the numbering on the axes starts. Heights of columns in a column graph may not be proportional to their frequency unless the scale is accurate.

ANS p646

Exercise 9F Analysing statistical reports

△ 1, 2, 4, 6, 7 □ 1, 3, 5, 6, 8, 9, 12 ⬠ 2, 3, 5, 6, 8, 10–12

UNDERSTANDING AND FLUENCY

9F.1 **1** Critically analyse each of these reports.

a A survey conducted at a gym revealed that 90% of the gym's clients felt they were getting enough exercise each week. It was later quoted that: 'Australians are getting fitter, as 90% of them say they exercise regularly each week'.

b Four teenage boys were still left on a bus when it reached the end of the line. A reporter asked them if they had part-time jobs. Two of them said they did. It was later reported that, '50% of teenagers today have part-time jobs'.

c A survey showed that four out of five children believe they do their fair share of chores at home. A report later appeared in a newspaper with the heading, 'Families lucky: 80% of children do fair share of chores'.

2 Critically analyse each of the media reports below. Suggest what research would need to have been done to make each claim valid.

a In a study of women, having coffee with lunch seemed to reduce the risk of diabetes more than having coffee at other times of day.

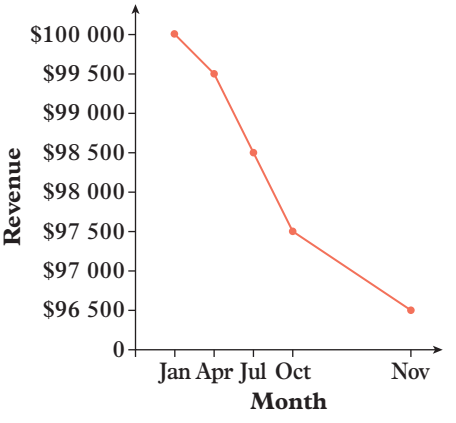
b Studies suggest that physically active people have a reduced risk of Alzheimer's disease in later life.

c Studies suggest that happiness wards off heart disease.

3 A television channel invites its viewers to vote on a topic that has just been discussed on air. Later in the program, the results of the votes are revealed, such as, '98% of viewers agree that …' Discuss whether the reported figures in this type of survey are valid.

PROBLEM SOLVING AND REASONING

9F.2 **4** The graph on the right displays the revenue a company has made during a year.



a This graph is misleading. Give reasons why.

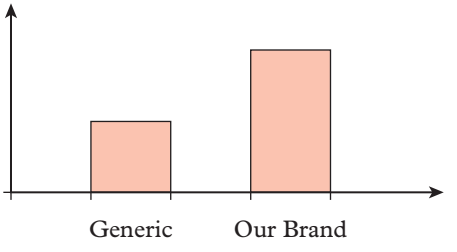
b The chief finance officer claims that the company's revenue has declined significantly this year, but declined at a slower rate at the end of the year. The graph appears to indicate this.

i Is this true?

ii How was this effect achieved?

iii By what percentage has the revenue declined from January to November?

5 The graph on the right displays the average time a brand of paracetamol takes before relief compared to a generic competitor.



a This graph is misleading. Give reasons why.

b The company claims their brand of paracetamol is twice as effective as their generic competitor's. The graph appears to indicate this.

i Is this true?

ii How was this effect achieved?

6 Fatima's employer constructed the graph on the right.

a What is the graph supposed to represent?

b Explain how the graph misrepresents the data.

c Redraw the graph as best you can so that it is fair.

7 Some advertisements claim that eating a good breakfast leads to success at school. Consider an alternative argument:

'Students who don't eat breakfast are more likely to be absent from school or experience lethargy.'

Does this statement mean the same thing as eating a good breakfast leading to success at school?

Why do the advertisements use one statement rather than the other?

8 We are often unaware of companies sponsoring studies.

Consider this headline: 'A hotdog a day increases risk of cancer by 21%.'

Whether this statement is true or not is open to debate. However, if the study was later revealed to have been conducted by an animal-rights group whose purpose is to convince people to be vegan, would this affect the credibility of the study? Suggest how this type of reporting of information from a biased source could be monitored.

9 A car manufacturer gets their new model tested by an independent company prior to release. The testing reveals that while the car is comfortable and good value for money, its fuel efficiency is significantly less than competitor models. How do you think the car company should use these test results when marketing their new model? Give reasons with your answer.

10 The CEO of a company has constructed this table showing the composition of her company.

	Total employees		Full-time		Part-time	
	25 and over	Under 25	25 and over	Under 25	25 and over	Under 25
Managers	18	4	14	4	4	0
Non-managers	42	61	40	31	2	30

She made these claims about her employees.

a 82% of 25-and-over employees are managers.

b 6% of under-25s in the company are managers.

c 29% of employees work part-time.

d 49% of under-25 non-managers work full-time.

Discuss whether each statement is correct. Provide mathematical evidence to support your answers. If a particular claim is incorrect, supply the correct figure.

11 Confirmation bias is the tendency to favour information that confirms already existing beliefs. Confirmation bias can occur throughout the statistical investigation process, from the collection of data through to the reporting of data.

Ellie is running to be the class president at her school. She conducts a survey of students in her school asking for their opinions on a range of topics and reports the results as favourable to her campaign.

a Discuss where confirmation bias may have occurred in either the collection or reporting of Ellie's data.

b Give another example of where confirmation bias may affect a statistical report. Explain how the confirmation bias may affect either the data being collected or how the results are reported.

12 Choose a topic and conduct a survey among the members of your class. Report on the results of your survey, justifying any claims with mathematical evidence.

Check your Student obook pro for these digital resources and more:

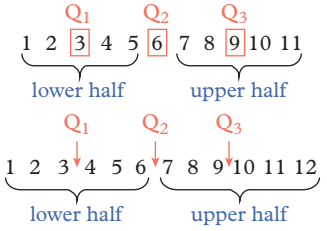
Investigation
Conducting your own statistical report

Topic quiz
9F

Chapter summary

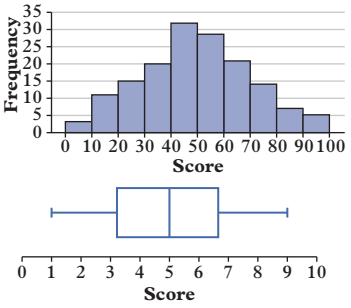
Five number summary

min, Q_1, Q_2 (median), Q_3, max

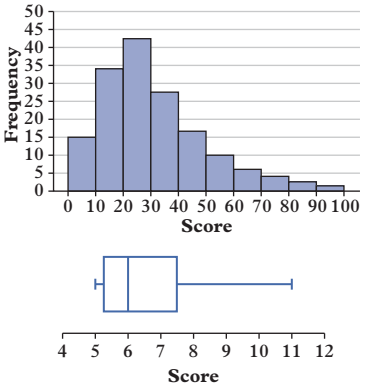


Distributions

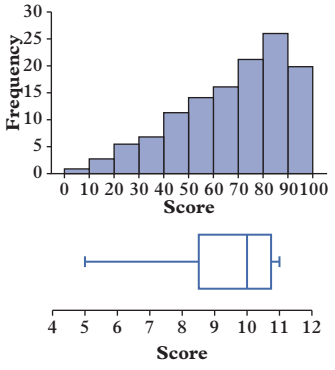
- Symmetric



- Positively skewed

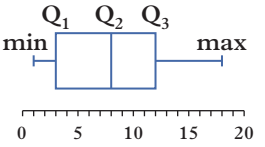


- Negatively skewed



Boxplots

- Lower fence: 1.5 × IQR below lower quartile
- Upper fence: 1.5 × IQR above upper quartile
- Outliers: values that lie beyond the fences



Standard deviation

Population's standard deviation:

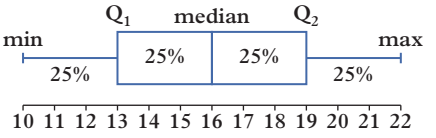
$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \ldots + (x_n - \mu)^2}{n}}$$

Sample's standard deviation:

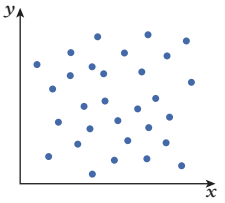
$$s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n - 1}}$$

Cumulative frequency

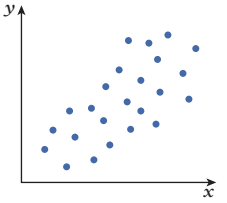
- Cumulative frequency: sum of all frequencies up to and including the given value



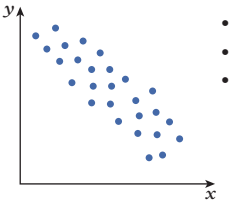
Scatterplots



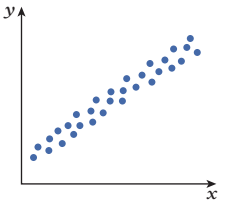
- No obvious pattern
- No correlation



- Weak linear pattern
- Weak correlation
- Positive direction



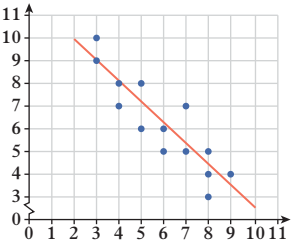
- Clear linear pattern
- Moderate correlation
- Negative direction



- Strong linear pattern
- Strong correlation
- Positive direction

Line of best fit

- A straight line that lies as close to as many points as possible
- Approximately the same number of points on each side of the line
- Can be used to make predictions



Analysing statistical reports

- Graph axes should be evenly spaced with consistent numbering
- Graphs should be properly labelled
- Check that no data has been purposefully omitted
- Question the claims made in reports
- Ask whether the data is presented in a fair manner

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

9A **1** A data set has 14 values. Which of the following is correct?

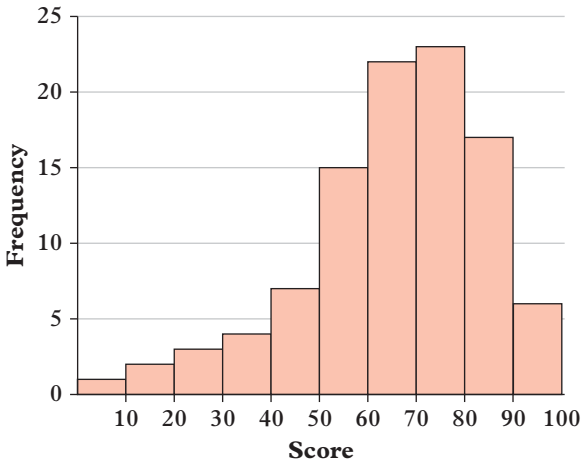
A The median and both upper and lower quartiles are values in the set.
B The median is a value in the set, but the upper and lower quartiles are not.
C The upper and lower quartiles are values in the set, but the median is not.
D None of the median, upper or lower quartiles are values in the set.
E The median and the upper quartile are values in the set, but the lower quartile is not.

9B **2** In a boxplot, what does the 'box' part represent?

A mean **B** interquartile range **C** range **D** median **E** standard deviation

9C **3** Which option best describes the histogram on the right?

A positively skewed
B symmetrical
C perfectly symmetrical
D negatively skewed
E bi-modal



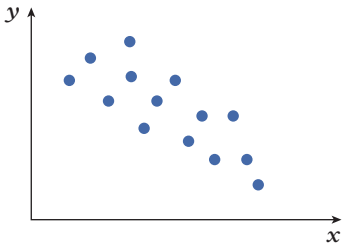
9C **4** The length from the lower quartile to the upper quartile in a boxplot represents what percentage of the scores of the data set?

A 25%
B 50%
C 75%
D 90%
E 100%

9D **5** What does the standard deviation in a set of data indicate?

A how large the mean is
B how the scores are spread out from the mean
C how many scores there are in the data set
D where the median lies in the set
E where the mean lies in the set

Questions **6–7** refer to the scatterplot below, which shows a linear relationship between two variables, x and y.



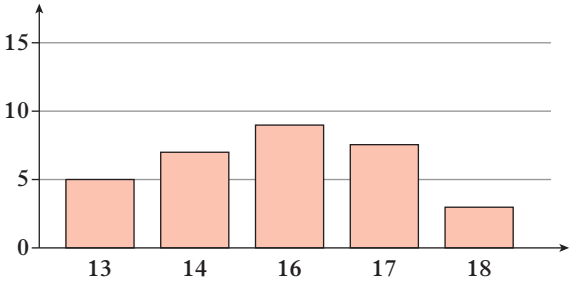
9E **6** How could you describe the linear relationship between the two variables?

A strong **B** weak **C** moderate
D no correlation **E** impossible to determine

9E **7** Which of these terms also describes the linear relationship between the two variables displayed on the graph?

A positive **B** negative **C** horizontal **D** vertical **E** random

9F **8** This graph displays the number of secondary students of different ages who joined the school band.



Why is the graph misleading?

A The x-axis doesn't start at 0.

B The y-axis doesn't start at 0.

C Not all values are present on the x-axis.

D The scale on the y-axis is inconsistent.

E The maximum value on the y-axis is too high.

Short answer

9A **1** For the data in the stem-and-leaf plot on the right showing the number of ice creams sold per day during the month of January, find:

a the range **b** the median

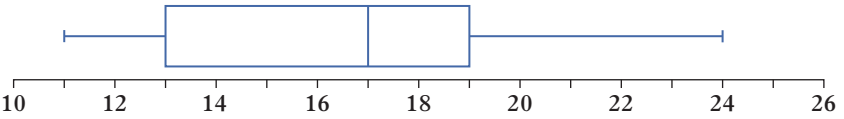
c the five-number summary **d** the interquartile range.

Stem	Leaf
4	0 1 3 5 7 9
5	2 5 5 7 8
6	2 3 3 5 7 7 9 9
7	3 5 6 7 8
8	2 4 6 7 8
9	3 6

Key: 4 \| 0 = 40

9B **2** Draw a boxplot for the data from question **1** and comment on its shape.

9B **3** State the five-number summary and interquartile range of the data presented in the following boxplot.



9C **4** **a** Create a grouped frequency table using class intervals of 10 with cumulative frequency and cumulative percentage columns for the data in question **1**. Round percentages correct to the nearest integer.

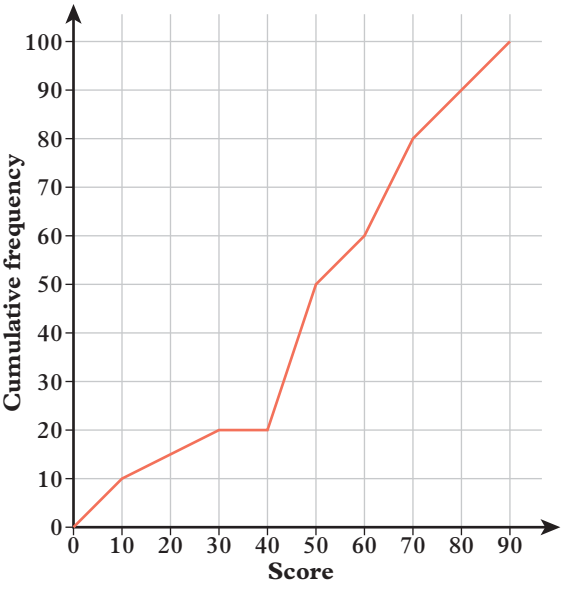
b Use the table in part **a** to construct a cumulative percentage frequency polygon.

9C **5** State the percentage of the data that lies in the following ranges of values for the cumulative frequency polygon on the right.

a between 50 and 60

b less than 30

c greater than 70



9D **6** Below are the results Lily and Cassy obtained for five tests.

Lily: 72, 77, 67, 69, 75

Cassy: 47, 92, 87, 77, 52

a For each student, calculate:

i the mean result

ii the standard deviation (to two decimal places).

b Which of these two students performed more consistently?

9F **7** This table shows the number of children enrolled at a day-care centre over the 12 years since its opening.

Years since opening	1	2	3	4	5	6	7	8	9	10	11	12
Number of children	25	27	26	33	32	35	37	39	40	39	40	45

a Create a scatterplot for the data.

b Describe the relationship between the two variables.

c Draw a line of best fit by eye on the scatterplot you have created.

d Use your line of best fit from part **c** to predict the number of children the day-care centre could expect to have 20 years after opening.

8 A company conducts an annual staff survey for all its staff, asking for feedback in several different areas. A selection of the aggregated results of the survey are shown below.

- Direction of the company: 8.0/10
- Management performance: 8.5/10
- Pay and benefits: 3.9/10
- Workload: 4.4/10
- Company culture: 6.5/10

At the all-staff meeting the HR Director highlights the strong results in the survey but doesn't mention the lower-scoring categories. Is this an ethical issue? Give reasons why or why not.

Analysis

Usain Bolt of Jamaica became a triple Olympic champion in the 100 m sprint at the Rio 2016 Olympic Games. A record of all the winning times for this race has been kept since the early 1900s. The Olympic Games usually take place every four years but were not held during the years of World War I or II, so there are some gaps in the record.

Consider this table showing winning times for the 100 m sprint since 1952.

Year	1952	1956	1960	1964	1968	1972	1976	1980	1984
Time (s)	10.79	10.62	10.32	10.06	9.95	10.14	10.06	10.25	9.99

Year	1988	1992	1996	2000	2004	2008	2012	2016
Time (s)	9.92	9.96	9.84	9.87	9.85	9.69	9.63	9.81

a From the raw data, describe the winning times over the period shown.

b From the data in the table, calculate (to two decimal places) the:

i mean and median

ii range and interquartile range

iii five-number summary.

c Find the standard deviation of the data set.

d **i** Draw a boxplot to represent the data from the table.

ii Comment on the shape of the boxplot.

iii Explain how you think the shape of the boxplot will change as future winning times are added to the data.

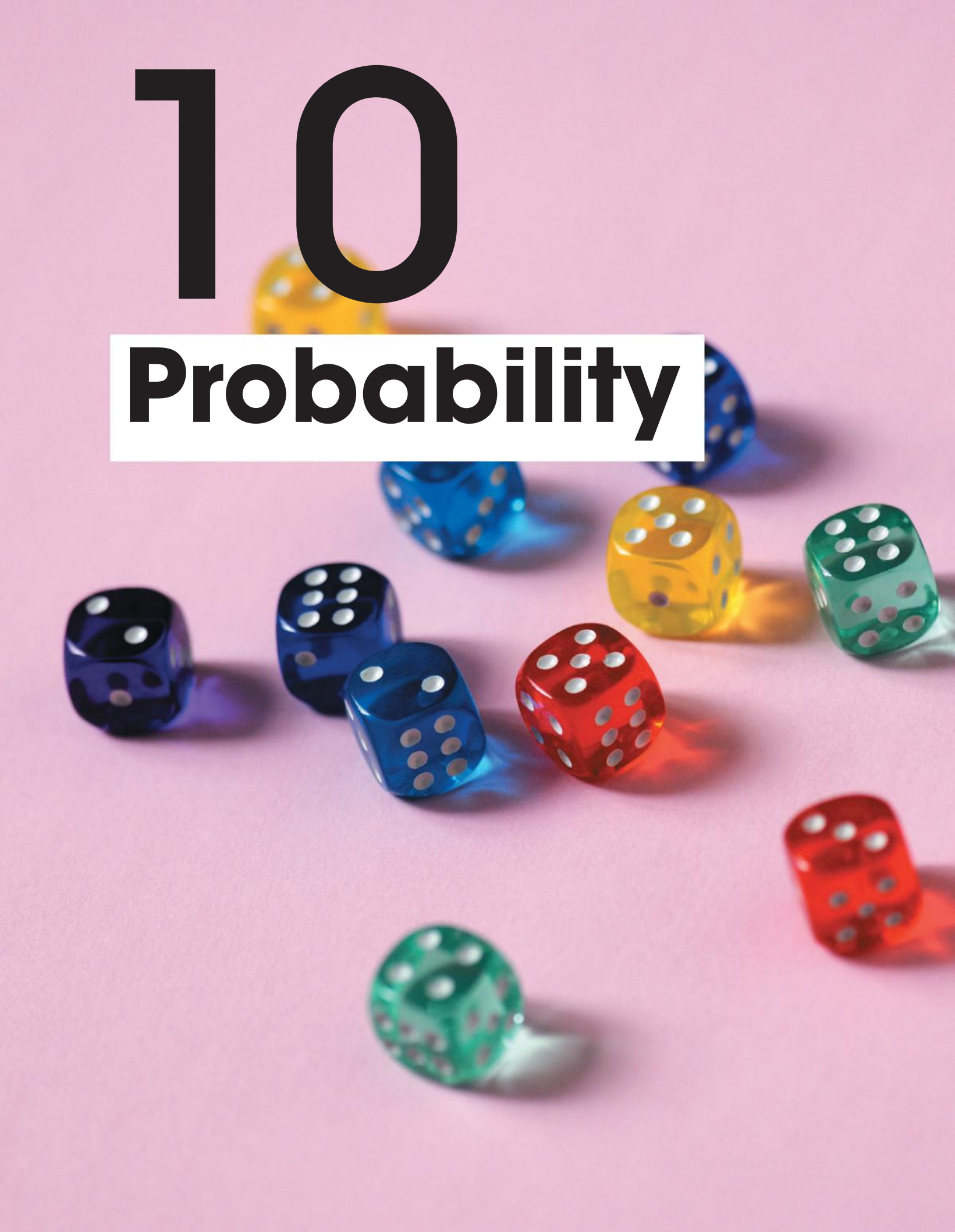
e Draw a scatterplot of the data.

f Draw a line of best fit by eye on your scatterplot.

g In the 1960s, speculation centred on the question, 'When will the 10 second barrier be broken for the 100 m sprint?' Today, the talk centres around the question, 'When will the 9.5 second barrier be broken?' Use your line of best fit from part **f** to make a prediction for when this might occur. (Remember that the Olympic Games are only held every four years.)

10

Probability



Index

10A Theoretical probability
10B Experiments with and without replacement
10C Two-way tables and Venn diagrams
10D Permutations
10E Combinations
10F Conditional probability
10G Independence

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Calculating theoretical probability
- ✔ Fractions, decimals and percentages
- ✔ Adding and subtracting fractions
- ✔ Multiplying and dividing fractions

Curriculum links

- Use the language of 'if .... then', 'given', 'of', 'knowing that' to describe and interpret situations involving conditional probability (AC9M10P01)
- Design and conduct repeated chance experiments and simulations using digital tools to model conditional probability and interpret results (AC9M10P02)
- Construct two-way tables and discuss possible relationship between categorical variables (AC9M10ST04)

Optional content

- Counting principles, and factorial notation as a representation that provides efficient counting in multiplicative contexts, including calculations of probabilities

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Materials

- ✔ Calculator

10A Theoretical probability

Learning intentions

By the end of this section you will be able to ...

- ✔ calculate the probability of experiments with and without equally likely outcomes
- ✔ calculate theoretical probabilities using tree diagrams and arrays.

Inter-year links

Year 7	10E Theoretical probability
Year 8	9D Theoretical probability
Year 9	8A Two-step chance experiments

Theoretical probability

- The **sample space** of an **experiment** is a list of all the different possible **outcomes** of that experiment.
- The different outcomes of an experiment may not be equally likely to occur.
- An **event** is a set of one or more outcomes from the sample space of an experiment.
- The probabilities of all events within the sample space sum to 1.
- If all the outcomes of an experiment are equally likely to occur, the probability of an event occurring can be calculated using the formula:

$$\Pr(\text{event}) = \frac{\text{number of outcomes in event}}{\text{number of outcomes in the sample space}}$$

- The **complement** of an event A is the event that A does not occur.
  - → The complement of A is written as A′, and is sometimes referred to as 'not A'.
  - → The sum of the values of the probabilities of complementary events is 1.

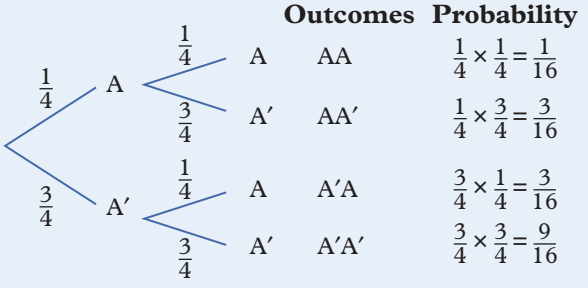
$$\Pr(A) + \Pr(A') = 1$$

Mutually exclusive events

- If two events are **mutually exclusive** then they cannot both occur at the same time. For example, when rolling a standard six-sided die, the events 'rolling an odd number' and 'rolling an even number' are mutually exclusive.

Tree diagrams

- **Tree diagrams** can be used to display the outcomes of multi-step experiments.
  - → The possible outcomes for each step of the experiment are represented by the branches.
  - → The branches might not all have the same probability.
  - → The final outcomes are listed at the ends of the branches.
  - → The probability of the final outcomes can be calculated by multiplying together the individual probabilities along the branches.



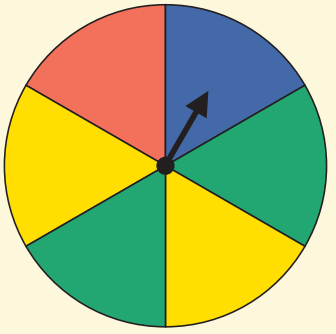
Arrays

- The sample space of a two-step experiment can be displayed in an **array** or table.
- Arrays can be used to calculate the probability of an experiment. To do this, all the cells in the array must be equally likely to occur. Divide the number of cells that satisfy the given event by the total number of cells in the array.
- The table below shows the outcomes for the two-step experiment of flipping a coin and then rolling a fair six-sided die.
  - → The outcomes of the first step are the headings of the rows and the outcomes of the second step are the headings of the columns. Reading across the columns and then down the rows to each cell will help you list all the possible combinations of outcomes.

		Rolling a fair six-sided die					
		1	**2**	**3**	**4**	**5**	**6**
Flipping a coin	**H**	(H,1)	(H,2)	(H,3)	(H,4)	(H,5)	(H,6)
	T	(T,1)	(T,2)	(T,3)	(T,4)	(T,5)	(T,6)

Example 10A.1 Identifying the sample space and calculating theoretical probability

This spinner divided into equally sized sectors (two yellow, two green, one red, one blue) is spun and the colour that the pointer lands on is recorded.



a List the sample space for this experiment.

b Calculate the probability of the spinner landing on a green sector.

THINK

a List all of the possible outcomes for this experiment inside curly brackets (also called braces).

b **1** The spinner has six equally sized sections, each of which has an equally likely chance of the pointer landing on it.

2 Divide the number of equally sized sections that are green by the total number of equally sized sections and simplify.

WRITE

a {red, blue, green, yellow}

b $\Pr(\text{green}) = \frac{2}{6}$

$= \frac{1}{3}$

Example 10A.2 Calculating the probability of multi-step experiments with equally likely outcomes

A coin is flipped twice. Calculate the probability that the outcome will be:

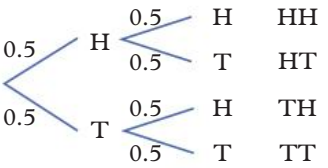
a exactly one head **b** no heads **c** at least one head.

THINK

1. Draw a tree diagram to represent all the different outcomes of this experiment.
2. Write the individual probabilities on the branches of the tree diagram. For each flip of the coin there are two different possible outcomes which are equally likely to occur, therefore they each have a $\frac{1}{2}$ or 0.5 chance of occurring.
3. The probabilities of the final outcomes can be determined by multiplying the probabilities along the related branches together.
4. Identify the required final outcomes.
5. All final outcomes have the same probability of occurring, so the required probability can be determined by multiplying the number of final outcomes by their probability.

WRITE

Outcomes



All final outcomes have a probability of: $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

a Favourable outcomes are HT and TH.

$$\text{Pr(exactly one head)} = 2 \times \frac{1}{4}$$
$$= \frac{2}{4}$$
$$= \frac{1}{2}$$

b The favourable outcome is TT.

$$\text{Pr(no heads)} = \frac{1}{4}$$

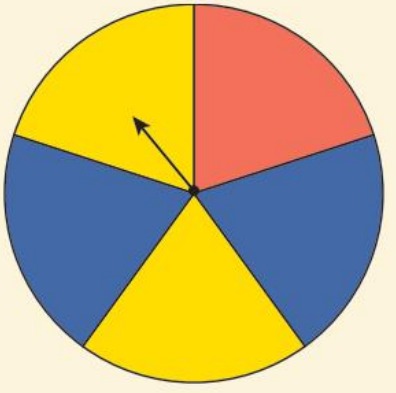
c Favourable outcomes are HH, HT and TH.

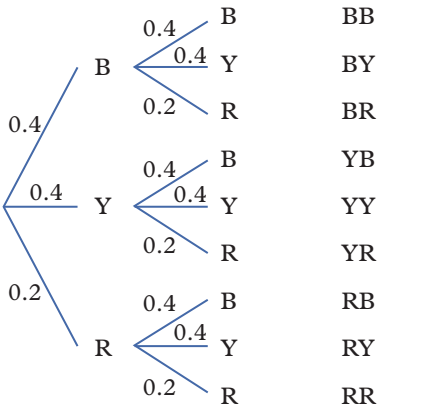
$$\text{Pr(at least one head)} = 3 \times \frac{1}{4}$$
$$= \frac{3}{4}$$

Note: The probability of this event could also be found by subtracting the probability of the complementary event ('no heads') from 1.

Example 10A.3 Calculating the probability of multi-step experiments without equally likely outcomes

This spinner divided into equally sized sectors (two blue, two yellow, one red) is spun twice, with the colour the pointer lands on being recorded both times. Calculate the probability of landing on a blue sector both times.



THINK	WRITE
1 Calculate the probabilities of landing on blue. There are 5 equally sized sectors, so the probability of landing on a blue sector is $\frac{2}{5}$, and so on.	$\Pr(\text{blue}) = \Pr(\text{B}) = \frac{2}{5} = 0.4$ $\Pr(\text{yellow}) = \Pr(\text{Y}) = \frac{2}{5} = 0.4$ $\Pr(\text{red}) = \Pr(\text{R}) = \frac{1}{5} = 0.2$
2 Draw a tree diagram. Mark the individual probabilities on the branches and show all outcomes.	
3 Identify the outcome with two blue sectors. **4** Calculate the probability of this outcome by multiplying the probabilities on each branch together.	$\Pr(\text{two blues}) = 0.4 \times 0.4 = 0.16$

Helpful hints

- ✔ The word 'event' has a specific meaning in probability that is separate to its meaning in everyday English. Read the definition carefully and make sure you do not get confused.
- ✔ Remember that you can only use the theoretical probability formula, $\Pr(\text{event}) = \frac{\text{number of outcomes in event}}{\text{number of outcomes in the sample space}}$, if the outcomes you are dealing with are equally likely to occur.
- ✔ Addition of decimals is often easier than addition of fractions, and multiplication of fractions is often easier than multiplication of decimals. Therefore, depending on the context, it may be useful to write your probabilities as decimals or fractions.
  For example, evaluating $0.25 \times 0.7 \times 0.8$ is more difficult than $\frac{1}{4} \times \frac{7}{10} \times \frac{4}{5}$ whereas $0.15 + 0.125$ is easier to evaluate than $\frac{3}{20} + \frac{1}{8}$.
- ✔ Be careful not to represent decimals incorrectly. For example, $\frac{1}{3}$ is not equal to 0.33.

ANS p648

Exercise 10A Theoretical probability

1–8, 10, 12, 13, 17 | 1, 2, 6, 7, 9, 10, 14, 17, 18 | 1, 2, 8, 10, 11, 14, 15–17, 19, 20

UNDERSTANDING AND FLUENCY

10A.1 **1** List the sample space for each of these experiments.

a rolling an eight-sided die (with sides numbered 1–8) and recording the uppermost number

b randomly selecting a letter from the word AUSTRALIA

c spinning a spinner with two red, three white and four blue sections (all sections being equal in size) and recording the colour

2 State whether the outcomes in the sample spaces for the experiments in question **1** are equally likely to occur or not.

3 A letter is selected at random from the word MATHEMATICS.

a List the sample space for this experiment.

b Calculate the probability of selecting an M.

10A.2 **4** A multiple-choice question consists of two parts. Each part has an equally likely chance of being true or false. What is the probability that:

a the answers are both true

b the answers are both false

c one answer is true, and one answer is false?

5 In a room there are two lights, each with its own on/off switch.

a Draw a tree diagram to show all the possible combinations of on or off the two lights could be.

b If it is equally likely that each light is either on or off, what is the probability that the lights will:

i both be on

ii both be off

iii not be in the same mode?

10A.3 **6** Two fair six-sided dice are rolled. Each time a 6 is rolled on either of the dice, it is recorded.

a If A represents the event 'rolling a 6', what does A′ represent?

b Draw a tree diagram showing the probabilities of the different outcomes of the experiment. Write the probabilities for each step on the branches.

c Use your tree diagram to find the probability of rolling:

i a double 6

ii at least one 6.

7 The experiment in question **6** is repeated. This time, all the numbers rolled on both dice are recorded.

a Explain how this experiment differs from the experiment in question **6**.

b How many outcomes are possible in this case?

c Draw an array to represent the experiment, and use it to determine the probability that the two numbers rolled are:

i the same

ii both prime numbers.

8 Three eight-sided dice (numbered 1–8) are thrown and the number rolled with each die is noted. It is recorded whether the number is even or odd.

Use a tree diagram to find the probability of rolling:

a no odd numbers

b one or two odd numbers

c more odd numbers than even numbers.

9 'Unders and overs' is a game that is played by rolling two six-sided dice. The numbers rolled are added together to give a total. Players predict whether the total will be under 7, equal to 7 or over 7.

a Draw an array to show all the possible outcomes when rolling two dice.

b Colour your array from part **a** to show which outcomes are:

i under 7 **ii** equal to 7 **iii** greater than 7.

c Calculate the probability of the prediction being correct if you predicted:

i under 7 **ii** equal to 7 **iii** greater than 7.

d Describe what your strategy would be if you played this game.

10 A ten-sided die (numbered 1–10) is rolled and the number facing up is recorded. Let A represent the event 'rolling an even number', B represent the event 'rolling a multiple of 4', and C represent the event 'rolling a prime number'.

a List the outcomes in each event: A, B and C.

b Which two events are mutually exclusive?

c Find each of the following:

i Pr(A) **ii** Pr(B) **iii** Pr(C).

11 When rolling two unbiased dice of the same kind, the probability of rolling a double depends on the number of sides on each die.

a Consider rolling a pair of standard six-sided dice.

i How many outcomes are possible when the two dice are rolled?

ii How many of these outcomes are doubles? List them.

iii What is the probability of rolling a double?

b Consider rolling a pair of eight-sided dice. Repeat the questions from part **a**.

c Write a general statement for the probability of rolling a double using a pair of dice with n sides.

12 A marble collection is made up of clear and opaque marbles. There are 15 clear marbles, and the probability of choosing an opaque marble from the collection is $\frac{2}{5}$. How many opaque marbles are in the collection?

13 Two fair six-sided dice are thrown and the product of the two numbers rolled is recorded.

a What is the probability that this number is:

i odd **ii** prime **iii** a square number?

b From the three given in part **a**, what is:

i the most likely outcome **ii** the least likely outcome?

14 A game involves a regular six-sided die and a fair coin. The die is rolled, and the number recorded. The coin is then flipped. If the coin lands on heads, then the die score is doubled, whereas if the coin lands on tails, then the die score remains the same.

a List the sample space of possible scores for this game.

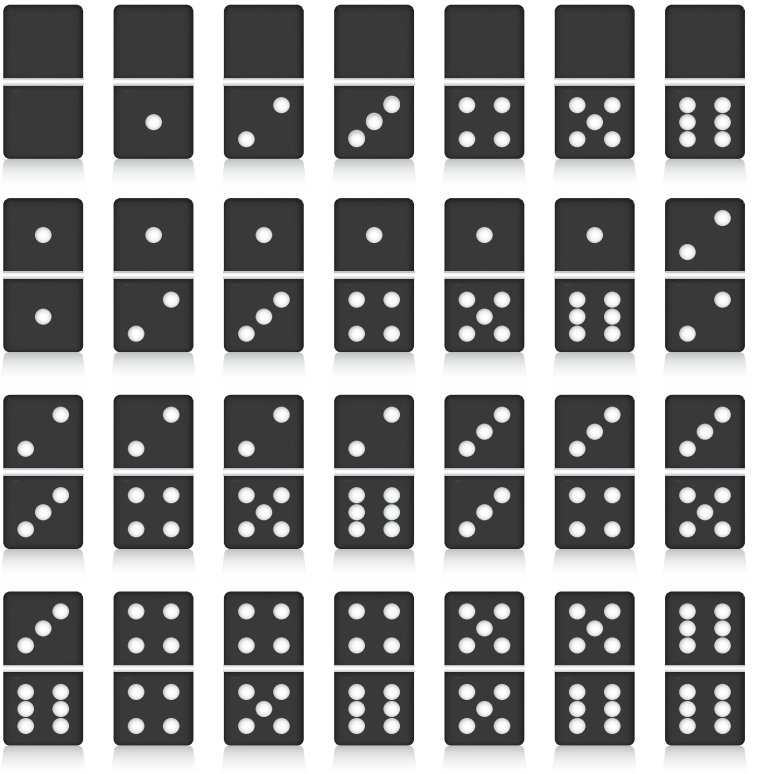
b Explain why each outcome in the sample space is not equally likely to occur.

c Find the probability that a player achieves a score of:

i 10 **ii** 4 **iii** at least 8.

15 Two standard six-sided dice are rolled and the two numbers facing up are recorded. What is the probability that the sum of the numbers is greater than the product of the numbers?

16 A set of dominoes contains tiles which have been divided in two and which represent numbers with dots (similar to dice). A blank space on a tile represents 0.



a How many tiles are in the complete set of dominoes?

b Explain the order of the tiles in the photo.

c For each of the values (0, 1, 2, 3, 4, 5 and 6), count the number of tiles on which the value appears. If the value appears on both halves of a tile, count the tile once. You should get the same number of tiles for each of the values from 0 to 6. What is this number?

d One tile is chosen at random. What is the probability that it shows:

i double 6

ii the same number twice

iii two different numbers

iv one even number and one odd number

v at least one prime number

vi two numbers, one of which is a multiple of the other (not including zero as a multiple)?

17 A coin is biased so that the chance of it showing a tail when flipped is 0.7.

a Draw a tree diagram to show the outcomes and their probabilities if the coin is flipped three times.

b From your tree diagram, what is:

i the most likely outcome

ii the least likely outcome?

c What is the probability of flipping:

i only one head

ii only one tail

iii more heads than tails?

18 Snakes and ladders is played on a board with numbered grid squares. Ladders connect certain pairs of squares together, and snakes connect other pairs of squares together. The aim of the game is to progress from square 1 to square 100 by rolling a fair six-sided die and moving the number of squares rolled. You must move up a ladder or down a snake if you land on the squares that show the base of a ladder or the head of a snake.

Imagine a grid on which the first ladder base is on square 7.

a What is the minimum number of die rolls which would take you to square 7?

b What is the maximum number of die rolls which could take you to square 7?

c Consider the minimum number of rolls required.

i List the die rolls which would take you to square 7.

ii How many different options are there?

19 A bag contains Christmas decorations of four different colours: red, green, gold and silver. There are four silver decorations and eight gold decorations. One decoration is drawn at random from the bag. The probability it is silver or gold is 0.6. If there are three green decorations, how many red decorations are in the bag?

20 Consider the target on the right.

a Discuss whether the chance of landing on a red section is the same as that of landing on a white section.

b The diameter of the innermost red circle (the bullseye) is 2 cm, and the widths of each of the subsequent rings is 2 cm. What is the radius of each of the rings?

c Calculate the area of each of the rings in terms of π.

d Assuming the arrow is equally likely to hit any point within the target, calculate the probability of the arrow landing:

i in the bullseye

ii in the outermost red ring.

Round your answers to three decimal points.

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Complementary events

Interactive skillsheet
Arrays

Investigation
Sicherman dice

Topic quiz
10A

Interactive skillsheet
Tree diagrams

10B Experiments with and without replacement

Learning intentions

By the end of this section you will be able to ...

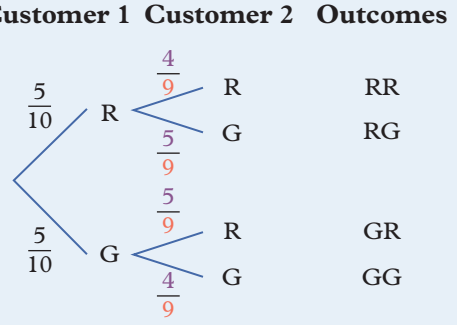
✔ calculate probabilities of multi-step experiments with and without replacement.

Inter-year links

Year 7	10E Theoretical probability
Year 8	9D Theoretical probability
Year 9	8B Experiments with replacement

Experiments with and without replacement

- When multi-step experiments involve items being selected from a group, the item can either be replaced or not replaced after each selection.
- If the items in a multi-step experiment involving selection are replaced, the probabilities for each step of the experiment remain the same.
- If the items in a multi-step experiment involving selection are not replaced, the probabilities for each step of the experiment change.
- The probability of the second step depends on the item selected at the first step.
- The number of items to select from decreases by 1 after each selection.



Depending on the first selection, the number of successful outcomes may also be reduced. The number of outcomes in the sample space is reduced for the second selection.

- Arrays can be used to show the reduced sample space for experiments without replacement. For example, if two letters are picked at random from the word 'SUN' without replacement, the sample space would be:

		Second letter		
		S	**U**	**N**
First letter	**S**	–	(S, U)	(S, N)
	U	(U, S)	–	(U, N)
	N	(N, S)	(N, U)	–

Example 10B.1 Calculating probability for experiments with replacement

A barrel contains coloured balls: two red, one blue and three green. One ball is drawn from the barrel, its colour is noted, then it is replaced. A second ball is then chosen and its colour is also noted.

a Draw a tree diagram to represent the situation.

b Determine the probability that the two balls selected are the same colour.

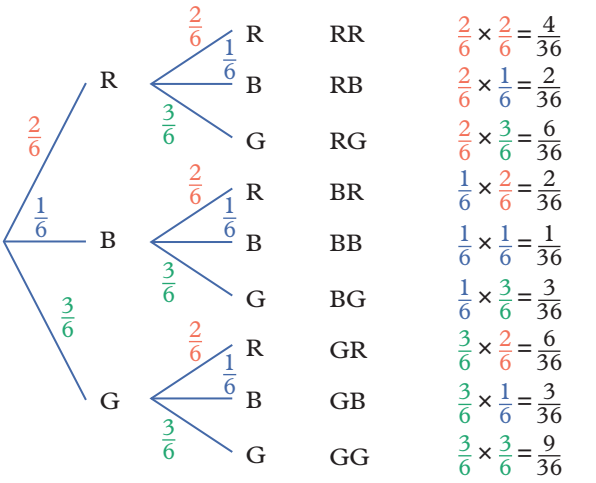
c Determine the probability that a red and green ball are selected in any order.

THINK

a Draw a tree diagram, writing all possible outcomes along with their probabilities. Because the selected ball is replaced after the first selection, the probabilities for the second selection remain the same.

WRITE

a



Outcomes	Probability
RR	$\frac{2}{6} \times \frac{2}{6} = \frac{4}{36}$
RB	$\frac{2}{6} \times \frac{1}{6} = \frac{2}{36}$
RG	$\frac{2}{6} \times \frac{3}{6} = \frac{6}{36}$
BR	$\frac{1}{6} \times \frac{2}{6} = \frac{2}{36}$
BB	$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
BG	$\frac{1}{6} \times \frac{3}{6} = \frac{3}{36}$
GR	$\frac{3}{6} \times \frac{2}{6} = \frac{6}{36}$
GB	$\frac{3}{6} \times \frac{1}{6} = \frac{3}{36}$
GG	$\frac{3}{6} \times \frac{3}{6} = \frac{9}{36}$

b On your tree diagram, identify the final outcomes in which the two balls are the same colour. Add the probabilities for those outcomes to find the required probability, and then simplify.

b Favourable outcomes are RR, BB and GG.

$$\Pr(\text{RR}) = \frac{4}{36}$$
$$\Pr(\text{BB}) = \frac{1}{36}$$
$$\Pr(\text{GG}) = \frac{9}{36}$$
$$\Pr(\text{same colour}) = \frac{4}{36} + \frac{1}{36} + \frac{9}{36}$$
$$= \frac{14}{36}$$
$$= \frac{7}{18}$$

c Identify the final outcomes in which a red and a green ball are selected. Add the probabilities for these outcomes to find the required probability, and then simplify.

c Favourable outcomes are RG and GR.

$$\Pr(\text{RG}) = \frac{6}{36}$$
$$\Pr(\text{GR}) = \frac{6}{36}$$
$$\Pr(\text{red and green}) = \frac{6}{36} + \frac{6}{36}$$
$$= \frac{12}{36}$$
$$= \frac{1}{3}$$

Example 10B.2 Calculating probability for experiments without replacement

Repeat the experiment from Example 10B.1, but this time assume the first ball is not replaced before the second ball is chosen.

THINK

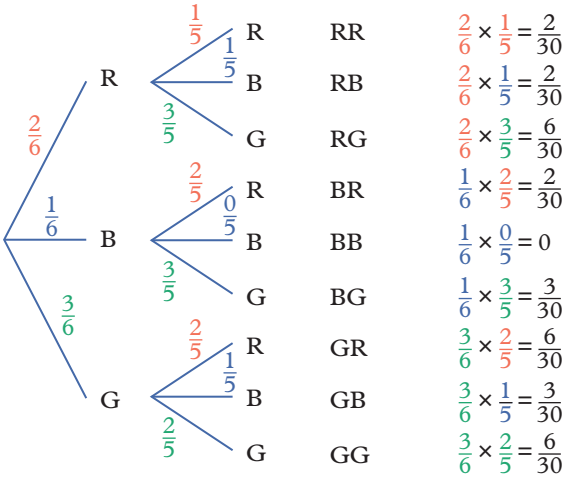
a Draw a tree diagram. The number of balls from which the second ball is chosen decreases by 1, so the possible outcomes at the second step are different from the possible outcomes for the first step.

b On your tree diagram, identify the final outcomes in which the two colours are the same. Add the probabilities for those outcomes to find the required probability and simplify. (Note the outcome BB is not possible, because there is only one blue ball.)

c Identify the final outcomes in which a red and a green ball are selected. Add the probabilities for these outcomes to find the required probability, and then simplify.

WRITE

a



Outcomes	Probability
RR	$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$
RB	$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$
RG	$\frac{2}{6} \times \frac{3}{5} = \frac{6}{30}$
BR	$\frac{1}{6} \times \frac{2}{5} = \frac{2}{30}$
BB	$\frac{1}{6} \times \frac{0}{5} = 0$
BG	$\frac{1}{6} \times \frac{3}{5} = \frac{3}{30}$
GR	$\frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$
GB	$\frac{3}{6} \times \frac{1}{5} = \frac{3}{30}$
GG	$\frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$

b Favourable outcomes are RR and GG.

$$\Pr(\text{RR}) = \frac{2}{30}$$
$$\Pr(\text{GG}) = \frac{6}{30}$$
$$\Pr(\text{same colour}) = \frac{2}{30} + \frac{6}{30}$$
$$= \frac{8}{30}$$
$$= \frac{4}{15}$$

c Favourable outcomes are RG and GR.

$$\Pr(\text{RG}) = \frac{6}{30}$$
$$\Pr(\text{GR}) = \frac{6}{30}$$
$$\Pr(\text{red and green}) = \frac{6}{30} + \frac{6}{30}$$
$$= \frac{12}{30}$$
$$= \frac{2}{5}$$

Helpful hints

- ✔ The sum of the probabilities at each step of a multi-step experiment is always equal to 1.
- ✔ For multi-step experiments, it is often helpful to leave fractions unsimplified until the final answer. In Example 10B.1, the probability that the balls are the same colour was calculated as $\frac{4}{36} + \frac{1}{36} + \frac{9}{36}$, which is easy to add. However, if each fraction were simplified then the denominators would all be different, which would make addition more difficult.

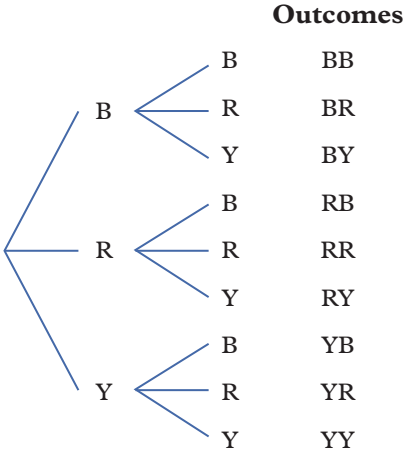
ANS p650

Exercise 10B Experiments with and without replacement

1–3, 5–7, 10, 12 | 3, 4, 8, 9, 11, 13, 14, 16, 17 | 4, 7, 9, 11, 13–18

1 A bag contains 22 coloured marbles: 6 blue, 8 red and 8 yellow. One marble is drawn from the bag, its colour is noted, then it is replaced. A second marble is then chosen and its colour is also noted.

a Complete the tree diagram representing this situation by adding probabilities to the branches.



b Determine the probability that the two chosen marbles are the same colour.

10B.1 **2** A standard deck of playing cards consists of four suits (hearts, diamonds, clubs and spades) with 13 cards in each suit (52 cards in total). One card is chosen from a deck and its suit is noted. It is replaced, then a second card is chosen and the suit noted.

a Draw a tree diagram to represent this experiment.

b Determine the probability that two hearts are selected.

c Determine the probability that a club and a diamond are selected in any order.

d Determine the probability that two cards of the same suit are drawn.

e Determine the probability that two cards of different suits are drawn.

3 John removes the loose coins from his pockets each night and throws them into a tin. There are 75 silver coins and 25 gold coins in the tin. He picks out a coin from the tin and notes its colour. After replacing it in the tin, he repeats this process another two times.

a Determine the probability of John drawing out:

i a silver coin

ii a gold coin.

b Draw a tree diagram to display this experiment. Write the individual probabilities on the branches of the tree diagram.

c Use your tree diagram to calculate the probability of John drawing out:

i all silver coins

ii no silver coins

iii at least one coin of each colour.

4 Consider the digits 1, 2, 3, 4 and 5. You are asked to write a two-digit number using these digits. The digits may be used more than once.

a Draw a tree diagram to represent this situation and use it to help you write all possible outcomes.

b How many outcomes are there?

c How many of these outcomes have two digits which are the same?

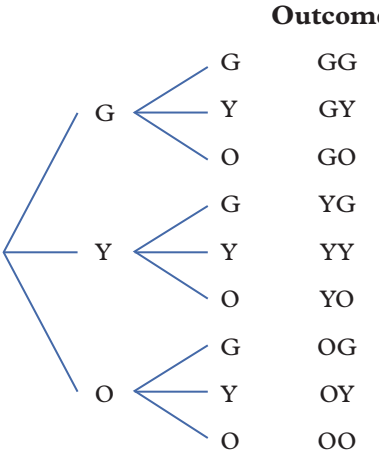
d Calculate the probability that the number has:

i two different digits **ii** two digits the same

iii the first digit less than the second.

5 A bag contains 16 coloured marbles: 6 green, 5 yellow and 5 orange. One marble is drawn from the bag, its colour is noted and it is not replaced. A second marble is then chosen and its colour is also noted.

a Complete the tree diagram representing this situation by adding probabilities to the branches.



b Determine the probability that the two chosen marbles are the same colour.

10B.2 **6** A bag contains 8 cubes: two black, three white, and three grey. One cube is drawn and its colour is noted. A second cube is drawn without the first cube being replaced and its colour is noted.

a Draw a tree diagram to represent the situation.

b Determine the probability that the two cubes selected are the same colour.

c Determine the probability that a black and a white cube are selected in any order.

7 The letters of the word PROBABILITY are written on separate pieces of paper and placed in a box. One letter is drawn from the box and it is recorded whether the letter is a consonant or a vowel. A second letter is drawn without replacing the first letter. Determine the probability that:

a the letters are both vowels

b one letter is a vowel and the other is a consonant.

8 From a set of all 26 letters in the alphabet, two letters are drawn:

a with replacement **b** without replacement.

For each of these situations, what is the probability that the letters are both vowels or both consonants?

9 Consider again the situation described in question **4** but, this time, you are only able to form a two-digit number without repeated digits.

a Draw a tree diagram to represent this situation and use it to help you write all possible outcomes.

b How many outcomes are there?

c Calculate the probability that the number has:

i two digits the same

ii two different digits

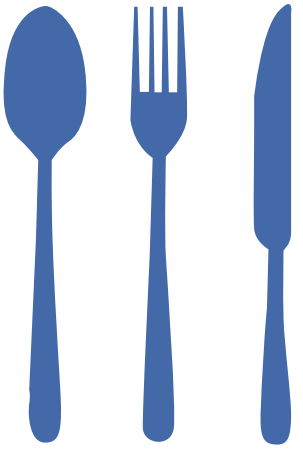
iii the first digit smaller than the second.

10 Sam has a small bag of lollies. The bag contains 10 red, 8 blue, 6 orange, 4 yellow and 12 green lollies.

a Sam prefers the red lollies. What is the probability that, if he chooses one lolly at random, it will be red?

b Sam eats all the red lollies. He then chooses another lolly at random, eats it, then chooses another one and eats it. What is the probability that the two lollies Sam ate were the same colour?

11 You are setting the table for you and your friend to have dinner. You each need a knife, fork and spoon. The cutlery drawer in the kitchen contains 15 knives, 10 forks and 15 spoons.



Without looking in the drawer, you choose three pieces of cutlery.

a What is the chance that your first choice was a knife?

b If your first choice is a knife, what is the probability that your next choice will be a fork, and the next choice a spoon?

c What is the probability of choosing a knife, a fork and a spoon in any order?

d Assume that you have already chosen a knife, fork and spoon for yourself. What is the probability that you will now choose these three items for your friend on your next selection?

12 Consider a pencil case containing 30 pencils: 10 black, 2 red, 4 blue, 3 green and 11 yellow.

Two pencils are taken out one after the other. The chance that the first pencil chosen is black (and is not replaced), then the next one is red is calculated as:

$\Pr(\text{black then red}) = \Pr(\text{black}) \times \Pr(\text{red})$

$$= \frac{10}{30} \times \frac{2}{29}$$
$$= \frac{2}{87}$$

a Use this information to help you determine the probability that:

i the first pencil chosen is black, then the next one is not red

ii the first pencil chosen is black, then the next one is red or blue

iii the first pencil chosen is black, then the next one is red, blue or green

iv both pencils are black

v both pencils are red

vi the first pencil chosen is red, then the next one is blue.

b Investigate to see whether undertaking the two events in the reverse order affects the probability by determining the probability that:

i the first pencil chosen is red, then the next one is black

ii the first pencil chosen is blue, then the next one is red.

c Comment on your answers to part **b**.

13 A bag contains 5 balls: three white and two black. Three balls are drawn, and each ball is placed back in the bag immediately after it has been drawn.

a Find the probability that all three balls drawn are black.

b Find the probability that all three balls drawn are white.

PROBLEM SOLVING AND REASONING

14 As with question **13**, a bag contains 5 balls: three white and two black. This time, three balls are drawn without replacement.

a Explain why at least one white ball will be drawn.

b Find the probability that all three balls drawn are white.

15 Five balls labelled with numbers 1–5 are placed in a bag. Two balls are selected, at random, one at a time.

a Write the sample space for the sum of the two numbers if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

b Find the probability that the sum of the two balls chosen is 4 if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

c Find the probability that the sum of the two balls chosen is 6 if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

16 It's not always necessary to draw tree diagrams to determine the number of outcomes of an experiment.

First, consider forming a two-digit number from the digits 1 to 5 with replacement (as was described in question **4**). There are five digits to choose from when writing the first digit. You can use all the digits again for the second digit, so there are five from which to choose again. This means that there should be 5×5 possible outcomes.

Next, consider forming a two-digit number from the digits 1 to 5 *without* replacement. There are five digits to choose from when writing the first digit. But, since you are not able to repeat any of the digits, there are only four digits from which to choose the second digit. This means that there should be 5×4 possible outcomes.

a Use this method to determine the number of possible outcomes when forming a three-digit number from the digits 1 to 5:

i with replacement **ii** without replacement.

b Repeat part **a** using the digits 1 to 9 to form the three-digit number.

CHALLENGE

17 A restaurant offers a menu of six entrées, six main meals and six desserts.

a How many different combinations of courses does the restaurant offer?

b During the evening, the kitchen runs out of one of the entrées and two of the desserts. What is the decrease in the number of options available?

18 a Suppose that the number plates for cars contain six digits and digits can be repeated. Remember that a number plate can start with the digit 0.

i How many different digits can be used in each place of the six-digit number?

ii How many different number plates could be produced using this system?

b A new system was introduced whereby each number plate contained three digits followed by three letters.

i How many number plates would be possible using this system?

ii Explain why the system with a mixture of digits and letters for number plates is better than the system with six digits.

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Investigation
Replacement and sample sizes

Investigation
Birthday surprises

Topic quiz
10B

Interactive skillsheet
Experiments without replacement

10C Two-way tables and Venn diagrams

Learning intentions

By the end of this section you will be able to ...

- ✔ understand and use set notation
- ✔ convert two-way tables to Venn diagrams and vice versa
- ✔ use the addition rule for probability.

Inter-year links

Year 7	10E Theoretical probability
Year 8	9F Venn diagrams
Year 9	8F Venn diagrams

Set notation

- A **set** is a collection of distinct objects.
- An **element** is a member of a set. The number of elements in set A is written as $n(A)$.
- The **universal set** contains all the elements under consideration and is represented by ξ (the Greek letter xi, pronounced 'zai').
- The **intersection** of two sets, A and B, is the set of elements in both A and B, and is denoted by $A \cap B$.
- The **union** of two sets, A and B, includes all elements from both sets and is denoted by $A \cup B$.
- The complement of set A is written as A'. This includes everything from the universal set that is not in set A (not A).
- A **subset** is a smaller set within a set.
  - → If set A is a subset of set B, this can be written as $A \subset B$.
  - → If set A is not a subset of set B, this can be written as $A \not\subset B$.
- The null set, or the empty set, is the set with no elements; that is, $\{\}$. The symbol for the null set is $\emptyset$.
- If two sets, A and B, are mutually exclusive, their intersection has no elements: $A \cap B = \emptyset$.

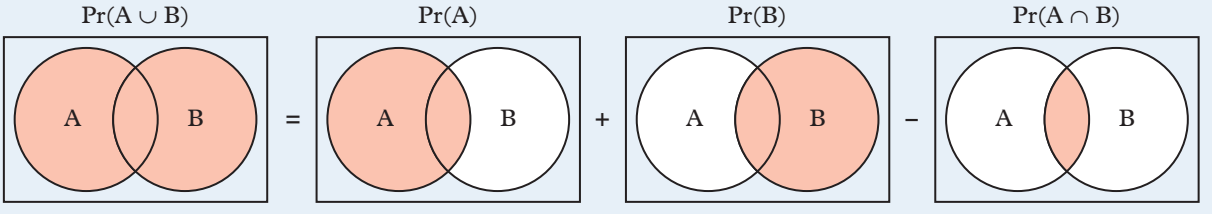
Venn diagrams

- Each circle in a **Venn diagram** represents a different set of data, and the rectangle represents the entire data set (the universal set, ξ).
- Sometimes Venn diagrams may be drawn with the circles being completely separated, which should be interpreted as meaning that the events are mutually exclusive.

The addition rule for probability

- When two sets (A and B) overlap, the following formula can be used:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$



Two-way tables

- A **two-way table** is a means of displaying the relationship between two sets of data. The sets are represented by the rows or columns of the two-way table.

	School A	School B	Total
Musical instrument	10	4	14
No musical instrument	3	13	16
Total	13	17	30

Example 10C.1 Converting a two-way table to a Venn diagram

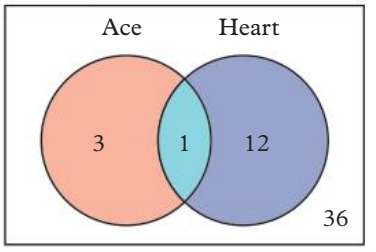
This two-way table details information about the cards found in a standard deck of playing cards. Convert the two-way table into a Venn diagram.

	Ace (A)	Not ace (A′)	Total
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

THINK

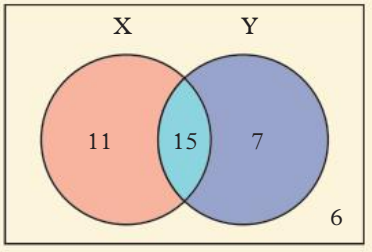
1. Draw two circles on your Venn diagram to represent the two sets. Label each circle with one of the possible options from each set.
2. Fill in the sections of the Venn diagram with the information from the two-way table.
3. Check that the total of the numbers in the Venn diagram equals the total in the bottom right-hand corner of the two-way table.

WRITE



Example 10C.2 Converting a Venn diagram to a two-way table

Convert this Venn diagram into a two-way table.



THINK

1. Create a table with columns and rows for the sets in the Venn diagram and their complements.
2. Add a row and a column for the totals.
3. Fill in the inner cells of the two-way table with the information from the Venn diagram.
4. Calculate the values in the total row and column, and check the bottom right-hand corner equals the total of the values in the Venn diagram.

WRITE

	X	X′	Total
Y	15	7	22
Y′	11	6	17
Total	26	13	39

Example 10C.3 Using the addition rule for probability

Given that $\Pr(A) = 0.35$, $\Pr(B) = 0.75$ and $\Pr(A \cap B) = 0.2$, find $\Pr(A \cup B)$.

THINK

1 Substitute the given values into the addition rule for probability.

2 Simplify and write your answer.

WRITE

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= 0.35 + 0.75 - 0.2\\ &= 0.9\end{aligned}$$

Helpful hints

- ✔ Converting between two-way tables and Venn diagrams can help you to visualise problems in a different way.
- ✔ The addition rule for probability can be rearranged to find the values of $\Pr(A)$, $\Pr(B)$ or $\Pr(A \cap B)$ as necessary.
- ✔ Two-way tables provide information about the totals of each category that isn't displayed in a Venn diagram.

ANS p651

Exercise 10C Two-way tables and Venn diagrams

△ 1–11, 14 □ 2, 4, 5, 7, 9–12, 16, 18 ⬠ 2, 7, 9–13, 15, 17, 19, 20

1 Consider the whole numbers from 1 to 12, inclusive. Then the universal set, ξ, is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Let A be the set of odd numbers and B be the set of prime numbers.

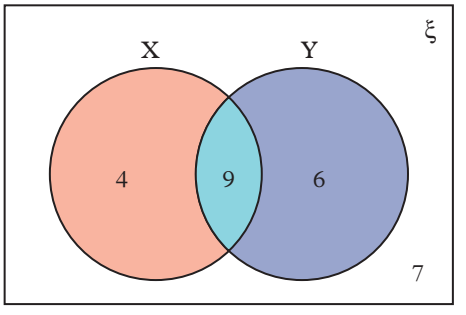
a Write the list of elements in the set:

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$

b One number from the universal set is selected at random. Find:

i $\Pr(A)$ **ii** $\Pr(B')$ **iii** $\Pr(A \cap B)$

2 Consider this Venn diagram.



a Find:

i $n(X)$ **ii** $n(X \cup Y)$

iii $n(X')$ **iv** $n(X \cap Y)$

v $n(\xi)$ **vi** $n(X' \cap Y)$

b Calculate each of these probabilities.

i $\Pr(X)$ **ii** $\Pr(X')$

iii $\Pr(X \cap Y)$ **iv** $\Pr((X \cup Y)')$

v $\Pr(X' \cap Y')$ **vi** $\Pr(X \cap Y')$

vii $\Pr(X \cup Y')$ **viii** $\Pr(Y \cup Y')$

3 Explain how you could complete all the entries in this two-way table correctly. Then complete the table.

	Primary school	High school	Total
Catch public transport	22		40
Are driven	37	11	
Walk/ride			
Total		38	102

UNDERSTANDING AND FLUENCY

4 A survey of 50 people was conducted to collect data relating to the types of movies those people like. The movie categories were comedy, drama and animation. The results were as follows.

- 26 like comedy
- 24 like drama
- 33 like animation
- 13 like comedy and drama
- 17 like drama and animation
- 14 like comedy and animation
- 8 like all three types of movies.

Draw a Venn diagram to display this data.

5 Use your Venn diagram from question **4** to help you find the probability that a person chosen:

a likes comedy

b likes comedy, but doesn't like drama

c likes drama, but doesn't like comedy

d likes comedy or drama, but doesn't like animation

e likes any of these movie forms.

10C.1 **6** The following two-way table details information about the sports that 140 students play. Convert the two-way table into a Venn diagram.

	Netball (N)	Not netball (N′)	Total
Field hockey (F)	22	10	32
Not field hockey (F′)	45	63	108
Total	67	73	140

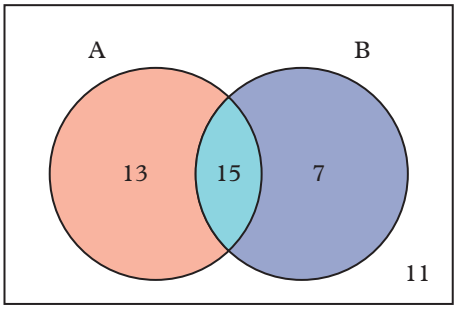
7 200 Year 10 students were asked if they have subscriptions to Netflix and Spotify. The results are partially displayed in the two-way table below.

	Netflix (N)	Not Netflix (N′)	Total
Spotify (S)	85	62	147
Not Spotify (S′)			53
Total	126		200

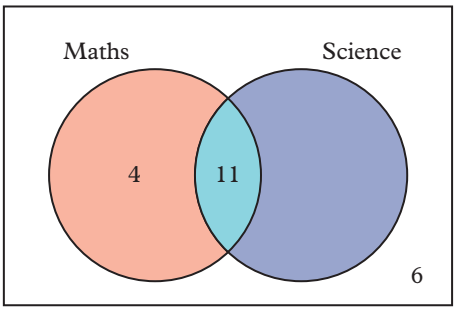
a Fill in the missing numbers in the two-way table.

b Convert the completed two-way table into a Venn diagram.

10C.2 **8** Convert the following Venn diagram into a two-way table.



9 A class of 28 students were each asked if they enjoyed studying mathematics and science. The results are partially displayed in the Venn diagram on the right.



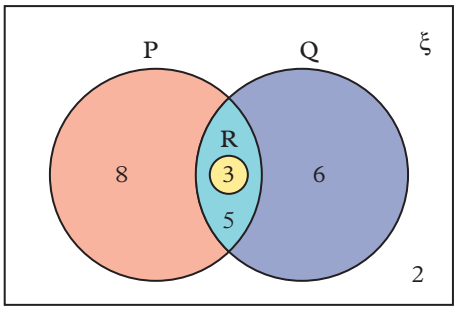
a The number of students who enjoy studying science, but not maths, is missing. What is this number?

b Convert the completed Venn diagram into a two-way table.

10C.3 **10** Given that $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.2$, find $\Pr(A \cup B)$.

11 Find the probability of A, given that $\Pr(B) = \frac{1}{3}$, $\Pr(A \cup B) = \frac{4}{9}$ and $\Pr(A \cap B) = \frac{1}{6}$.

12 Consider the Venn diagram on the right.



a Decide whether each of the following statements is true for the Venn diagram and explain your reasoning.

i $R \subset P$ **ii** $R \subset Q$

iii $R \not\subset (P \cap Q)$ **iv** $R \subset (P \cup Q)$

v $R \not\subset \xi$ **vi** $R \subset Q \subset \xi$

b Use the Venn diagram to help you calculate each of these probabilities.

i $\Pr(P \cap R)$ **ii** $\Pr(P \cap Q')$

iii $\Pr(P' \cap Q')$ **iv** $\Pr(P \cap Q \cap R)$

13 One hundred people took part in a survey to determine the readership of three monthly magazines (A, B and C). The results were as follows:

- 59 read A
- 68 read B
- 59 read C
- 42 read A and B
- 45 read B and C
- 39 read A and C
- 27 read all three magazines.

a Draw a Venn diagram to show the results of the survey.

b Find the number of people who:

i read magazines A or B

ii read magazine C but not magazine B

iii read exactly two of the magazines

iv read two magazines at most

v do not read magazine A

vi read only one magazine

vii do not read B or C

viii do not read three magazines.

14 As with question **1**, let the universal set, ξ, be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, let A be the set of odd numbers, and let B be the set of prime numbers.

a Write the list of elements in the set:

i A'

ii $A' \cap B$

iii $(A \cap B)'$, the complement of the set $A \cap B$.

b One number from the universal set is selected at random. Find:

i $\Pr(A' \cap B)$ **ii** $\Pr(A' \cap B')$ **iii** $\Pr(A' \cup B')$.

PROBLEM SOLVING AND REASONING

15 In the theory for this section, Venn diagrams were used in the proof of the addition rule. Use Venn diagrams to help you prove that $\Pr((A \cap B)') = \Pr(A' \cup B')$.

16 For two events, A and B, it is known that $\Pr(A) = 0.75$, $\Pr(B) = 0.3$ and $\Pr(A' \cap B) = 0.1$.

a Find $\Pr(A')$.

b Use the addition rule to find $\Pr(A' \cup B)$.

17 For two events, A and B, it is known that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$ and $\Pr(A \cap B) = 0.2$. These values have been entered into the following two-way probability table.

	A	**A′**	**Total**
B			0.6
B′	0.2		
Total	0.3		1

a Complete the missing values in the two-way probability table.

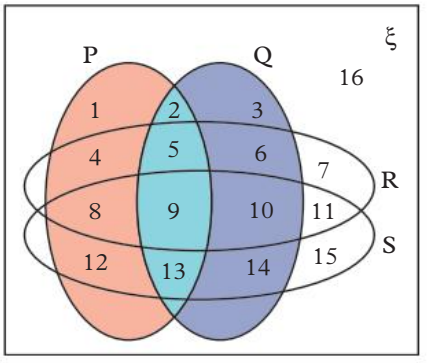
b Use the table from part **a** to help you find:

i $\Pr(A' \cap B')$ **ii** $\Pr(A' \cap B)$ **iii** $\Pr(A' \cup B')$.

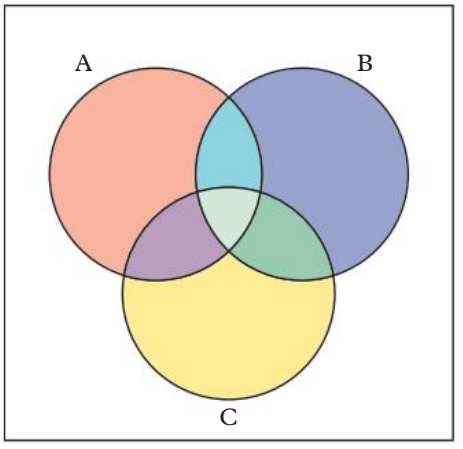
CHALLENGE

18 In a room there are 60 children, each of whom is either left-handed or right-handed. There are 30 right-handed boys, and 11 left-handed children. There are 25 girls among the 60 children. Complete a two-way table to find how many left-handed girls are in the room.

19 This Venn diagram shows the relationship between four sets P, Q, R and S. Determine the sum of the elements within the region $(P \cap Q) \cup (R \cap S')$.



20 Use the following Venn diagram to find an additional rule formula for $\Pr(A \cup B \cup C)$. That is, express $\Pr(A \cup B \cup C)$ in terms of the probabilities of A, B, C and their intersections.



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Interactive skillsheet
Set notation

Interactive skillsheet
The addition rule for probability

Topic quiz
10C

Interactive skillsheet
Venn diagrams

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

10A **1** List the sample space for each of the following experiments.

a A fair coin is tossed.

b A standard six-sided die is rolled.

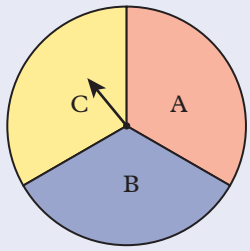
c A card is picked from a standard 52-card deck and its suit noted.

d A letter is selected at random from the word LOLLIPOP.

10A **2** A letter from the word CHANCE is selected at random. Find the probability that the selected letter is

a 'H' **b** 'C' **c** a vowel **d** not a vowel.

10A **3** The spinner on the right is spun twice.



a Draw a tree diagram to represent the possible outcomes of the two spins.

b Find the probability the spinner lands on 'A' both times.

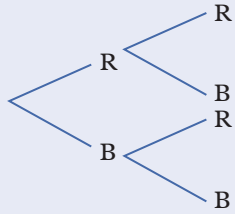
10A **4** A biased coin, which has a 60% chance of landing on heads, is tossed twice.

a Draw a tree diagram to represent this situation.

b Find the probability the coin lands on:

i heads both times **ii** heads once and tails once.

10B **5** A bag contains 3 red balls and 4 black balls. A ball is randomly selected from this bag and its colour is noted. The ball is then replaced before another ball is randomly selected from the bag.
The tree diagram on the right shows the possible outcomes of the experiment.



a Complete the tree diagram by writing the probabilities on the branches.

b Find the probability that both balls selected are red.

10B **6** Two balls are selected at random from a bag containing 2 black balls and 3 white balls. Find the probability that both balls selected are black if:

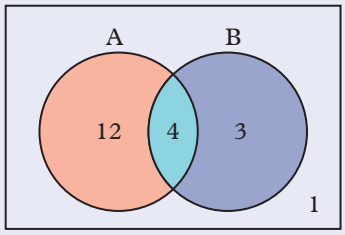
a the first ball is replaced before the second ball is selected

b the first ball is not replaced before the second ball is selected.

10B **7** A card is randomly selected from a standard 52-card deck, and its suit is noted. The card is then replaced, and another card is drawn at random. Find the probability that the two cards are:

a both hearts **b** the same suit **c** different suits.

10C **8** Consider the Venn diagram on the right.



Find:

a Pr(A)

b Pr(B)

c $\Pr(A \cap B)$

d $\Pr(A' \cap B)$.

10C **9** Consider the incomplete two-way table on the right.

	M	M′	Total
N	6	11	
N′		15	
Total			40

a Fill in the missing numbers in the two-way table.

b Convert the two-way table to a Venn diagram.

10C **10** The following two-way table shows data gathered from a class of 24 students about which brand of phone they have and which operating system they use on their home computer.

	Apple	Samsung	Other	Total
Macintosh	8	1	0	9
Windows	6	4	5	15
Total	14	5	5	24

One of the 24 students is selected at random. Find the probability that this selected student:

a has an Apple phone

b has Windows on their home computer

c has an Apple phone and Macintosh on their home computer

d has Windows on their home computer but not a Samsung phone.

10D Permutations

Learning intentions

By the end of this section you will be able to ...

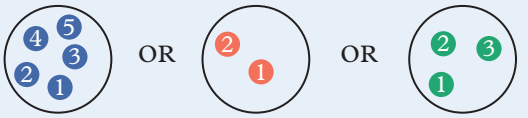
- ✔ understand and apply the addition and multiplication principles to counting problems
- ✔ evaluate factorials and apply factorial notation to counting problems involving permutations
- ✔ apply counting principles and permutations to simplify probability calculations.

Inter-year links

Year 7	10E Theoretical probability
Year 8	9D Theoretical probability
Year 9	8A Two-step chance experiments

Counting principles

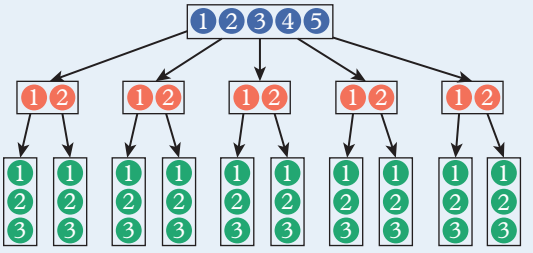
- If there are m elements in set A and n elements in set B, then there are $m + n$ elements to choose from when selecting one element from A or B. This is called the **addition principle**.



1 2 3 4 5 1 2 1 2 3

5 + 2 + 3 = 10 possibilities

- When choosing one element from set A and one element from set B, there are $m \times n$ ways to select the two objects. This is called the **multiplication principle**.
- The addition and multiplication principles can be used together. For example, when choosing an apple or a banana and a cupcake or a muesli bar, there are $(1 + 1) \times (1 + 1) = 2 \times 2 = 4$ choices in total.



5 × 2 × 3 = 30 possibilities

Permutations

- A **permutation** of a set is an ordering of the elements in the set. The number of permutations of a set depends on the number of elements in the set

ABC BCA CAB
ACB BAC CBA

3! = 3 × 2 × 1 = 6

- The number of permutations of a set with n elements is:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1.$$

This value, $n!$, is read as 'n factorial' and is found using the multiplication principle and removing one choice after each selection.

- When ordering r elements from a set with n elements, where $r \leq n$, the number of permutations is:

$$^nP_r = \frac{n!}{(n-r)!}.$$

This number, ^nP_r, is read as 'n permute r', and is found by cancelling the last $(n - r)$ factors of $n!$, as only the first r factors are used when ordering r objects.

AB BC CA
AC BA CB

$$^3P_2 = \frac{3!}{(3-2)!} = \frac{1}{6} = 6$$

Example 10D.1 Using the addition and multiplication principles

a How many choices of dish do you have if a menu has 5 chicken dishes, 4 beef dishes, 3 pork dishes, 2 seafood dishes and 3 vegetarian dishes?

b At a restaurant, a banquet consists of a soup, an entrée, a main and a dessert. If the menu lists 2 soups, 3 entrées, 5 mains and 2 desserts, how many different possible banquets are there?

THINK	WRITE
a Determine if each group of options is sequential (multiplication principle) or simultaneously available (addition principle). The options are simultaneously available. Apply the appropriate principle to determine the number of choices.	**a** $5 + 4 + 3 + 2 + 3 = 17$ choices
b Determine if each group of options is sequential (multiplication principle) or simultaneously available (addition principle). The options are sequential. Apply the appropriate principle to determine the number of choices.	**b** $2 \times 3 \times 5 \times 2 = 60$ choices

Example 10D.2 Finding the number of permutations

a In how many ways can you arrange five toys in a row?

b In how many ways can you arrange seven playing cards in a row?

THINK	WRITE
a Evaluate the appropriate factorial.	**a** $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways
b Evaluate the appropriate factorial.	**b** $7! = 7 \times 6 \times 5! = 7 \times 6 \times 120 = 5040$ ways

Example 10D.3 Permutations of a selection

In how many ways can three out of a total of eight toys be arranged in a row?

THINK	WRITE
1 Write down the factorial quotient to evaluate.	${}^8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!}$
2 Write the factorials in the numerator and denominators as their products. Divide out the common factors.	$= \frac{8 \times 7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
3 Evaluate the remaining product.	$= 8 \times 7 \times 6$
	$= 336$ ways

Example 10D.4 Finding probabilities using permutations

What is the probability that an arrangement of four of the letters of the word COUNT begins with two vowels?

THINK	WRITE
1 Use the appropriate factorial quotient to find the number of ways to arrange four of the letters of the word COUNT.	${}^5P_4 = \frac{5!}{(5-4)!} = \frac{5!}{1!} = 120$
2 Draw four lines to represent the positions of letters to arrange. If the first two letters are vowels, there are two options for the first letter and one for the second letter. The third and fourth letters can be any of the three unused letters. There are three options for the third letter and two for the fourth.	$\underline{2}\ \underline{1}\ \underline{3}\ \underline{2}$
3 Multiply the numbers together to find the number of ways to arrange four letters of the word COUNT where the first two letters are vowels.	$2 \times 1 \times 3 \times 2 = 12$
4 Divide the number arrangements beginning with two vowels by the total number of arrangements of four letters to find the desired probability.	$\Pr(\text{begins with two vowels}) = \frac{12}{120}$
5 Simplify the fraction when possible.	$= \frac{1}{10}$

Helpful hints

- ✔ Contexts that involve the addition principle usually can be described in terms of 'or' as in 'this or that', while contexts that involve the multiplication principle usually can be described in terms of 'and' as in 'this and that'.
- ✔ Some calculators have a function for factorials and permutations that can be used to quickly evaluate both.
- ✔ The addition and multiplication principles and permutations can be used to find the number of favourable outcomes in an event and the number of outcomes in the sample space when finding the probability of an event.

$$\Pr(\text{event}) = \frac{\text{number of favourable outcomes in event}}{\text{number of outcomes in the sample space}}$$

- ✔ The technique shown in Example 10D.4 of drawing lines to represent the number of positions and then filling in the number of possibilities at each position is a useful way to solve many problems involving permutations.

ANS p654 Exercise 10D Permutations

1–8, 10, 12(a,b)

1(c–f), 3(c–f), 4(a,b), 5(c–f), 6–8, 9(a,b), 10–12

1(d,e), 3(e,f), 5(e,f), 6,7,9, 10–15

10D.1 **1** **a** How many choices for a dish do you have if a menu has 7 chicken dishes, 9 beef dishes, 5 pork dishes, 6 seafood dishes and 8 vegetarian dishes?

b How many choices of yoghurt brand do you have if a grocery store has 5 brands of vanilla yoghurt, 4 brands of strawberry yoghurt and 8 other brands of yoghurts?

c How many choices of game to buy do you have if there are 8 RPGs, 9 Metroidvania games, 5 party games, 4 online multiplayer games and 3 racing games?

d At a restaurant a banquet consists of a soup, an entrée, a main and a dessert. If the menu lists 4 soups, 6 entrées, 3 mains and 3 desserts, how many choices of banquet are there?

e At the corner store, there are 8 different yoghurts, 7 different drinks and 10 different snacks on offer. In how many ways could you buy one of each item?

f There is a sale on online and you have decided to buy one of 8 RPGs, one of 9 Metroidvania games, one of 5 party games, one of 4 online multiplayer games and one of 3 racing games. In how many ways could you buy one of each type of game?

2 Evaluate the following factorials.

a 3! **b** 4! **c** 6! **d** 10! **e** 15! **f** 20!

10D.2 **3** **a** In how many ways could you arrange nine toys in a row?

b In how many ways could you arrange a hand of five playing cards in a row?

c Ten people compete in a 100-metre race. In how many ways can they place from first to tenth?

d In an octave there are twelve notes, each a semitone apart. In how many ways can they be played in a row?

e In a monster capture game, you have a party of six different monsters. In how many ways can they be ordered in your line-up?

f In how many orders could a playlist of twenty songs be played on shuffle?

4 Evaluate the following.

a ${}^{6}P_{3}$ **b** ${}^{10}P_{2}$ **c** ${}^{10}P_{5}$ **d** ${}^{7}P_{5}$ **e** ${}^{6}P_{5}$ **f** ${}^{30}P_{2}$

10D.3 **5** **a** In how many ways could you arrange four out of a total of nine toys in a row?

b In how many ways could you arrange three out of a total of five playing cards in a row?

c Ten people compete in a 100-metre race. In how many ways can the gold, silver and bronze medals be given to the first three people to complete the race?

d In an octave there are twelve notes, each a semitone apart. In how many ways can eight of them be played in a row?

e In a monster capture game, you have a party of six different monsters. In how many ways can four of them be ordered in your line-up?

f In how many ways could eight out of twenty songs in a playlist be played on shuffle?

6 A collection of symbols is made from the 26 uppercase and 26 lowercase English letters, the 24 uppercase and 24 lowercase Greek letters and the 10 Hindu–Arabic numerals. Determine the probability that:

a a randomly selected symbol is an English letter

b a randomly selected symbol is an uppercase letter

c a randomly selected symbol is a numeral or an English letter

d a randomly selected symbol is not a numeral

e a randomly selected pair of symbols is an English letter or a Greek letter in either order if each symbol is replaced before the next one is selected

f a randomly selected pair of symbols consists of two English letters if the first symbol is not replaced before the second symbol is selected.

10D.4 **7** **a** What is the probability that an arrangement of four of the letters of the word SAMPLE begins with two vowels?

b What is the probability that an arrangement of four of the letters of the word PIECE begins with two vowels?

c What is the probability that an arrangement of five of the letters of the word PLEASANT begins with two vowels?

8 Consider the following working out: $\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}} = 10 \times 9 \times 8 = 720$.

a Explain why we can write 10! as $10 \times 9 \times 8 \times 7!$ and how this helps to make it easier to simplify the fraction.

b Use this method to evaluate the following:

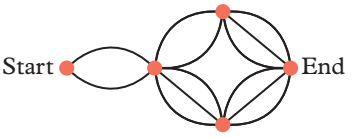
i $\frac{12!}{10!}$ **ii** $\frac{20!}{17!}$ **iii** $\frac{100!}{99!}$

iv ${}^{34}P_3$ **v** ${}^{51}P_2$ **vi** ${}^{1000}P_3$

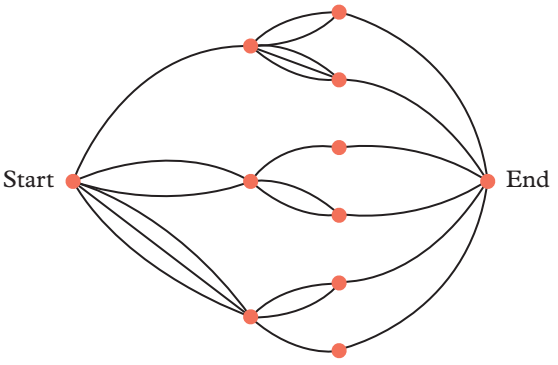
9 Determine the number of ways to get from the start to the end of each network by only moving from left to right along the edges.

a

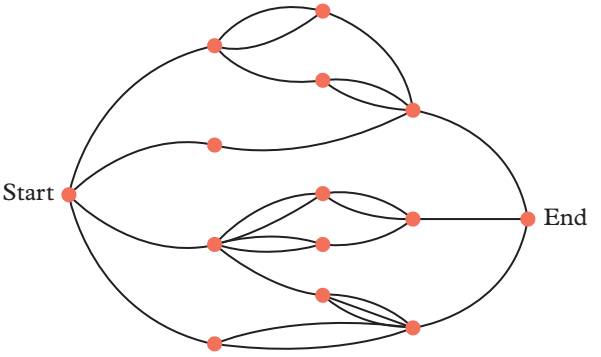
b



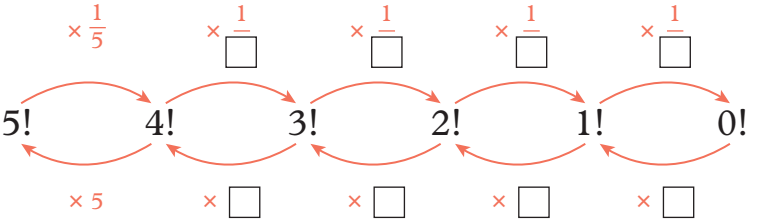
c



d



10 a Complete the following.



b Use the pattern to evaluate 0!.

c Explain why it is important that 0! is this value in the case of ${}^nP_n$.

11 The number of permutations is reduced if some of the objects being arranged are indistinguishable. For example, when considering the permutations of the letters A, A and B, we cannot distinguish between AAB and AAB as being different without some external marker that tells you which A is which. So, we need to remove the repeated arrangements from the list of all possible arrangements.

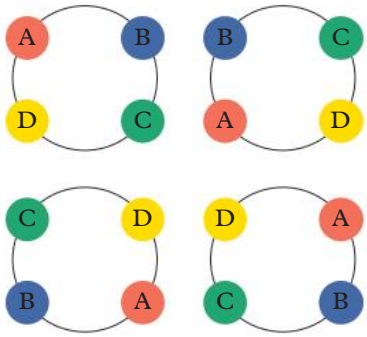
a In how many unique ways can the letters AAB be written in a row?

b In how many unique ways can the letters AABC be written in a row?

c In how many unique ways can the letters AABB be written in a row?

d In how many unique ways can the letters AABBBCCCC be written in a row?

12 The number of permutations when arranging n objects in a loop is less than the number of permutations when arranging n objects in a row as the loop does not have a distinct start and end. This means that these diagrams all represent the same arrangement.



a In how many unique ways can the letters ABCD be placed in a loop?

b In how many unique ways can the letters ABCDE be placed in a loop?

c In how many unique ways can the letters ABCDEFGHI be placed in a loop?

d In how many unique ways can n different letters be placed in a loop?

13 A group of seven friends go out for dinner.

a In how many ways could they sit around a table for dinner?

b If there are three couples that want to sit next to their partners, in how many ways could they sit around the table for dinner?

c If everyone sits down at the table randomly, what is the probability that all the couples are seated next to their partner?

The group want to take a photo together and decide to have four of them stand behind the other three.

d How many ways could they take the photo?

e If the couples want to stand next to their partners, in how many ways could they take the photo?

f If everyone takes a position randomly, what is the probability that all the couples are standing next to their partner?

14 Simplify the following where n is a natural number.

a $\dfrac{n!}{(n+1)!}$ **b** $\dfrac{(n+3)!}{(n-3)!}$ **c** $\dfrac{(n+3)!}{n!(n^2+3n+2)}$

d $\dfrac{1}{(n-1)!}-\dfrac{1}{n!}$ **e** $\dfrac{{}^{n+1}\mathrm{P}_n}{{}^{n}\mathrm{P}_{n-1}}, n \geq 1$ **f** $\dfrac{{}^{n+5}\mathrm{P}_n}{{}^{n}\mathrm{P}_{n-3}}, n \geq 3$

15 There exist variations of factorials that are products of descending sequences. An online calculator may be required to evaluate the following numbers with all of their digits.

a The double factorial, $n!!$, is the product of either all of the positive even or odd numbers less than or equal to n. For example, $9!! = 9 \times 7 \times 5 \times 3 \times 1$ and $8!! = 8 \times 6 \times 4 \times 2$. Evaluate the following.

i 9!! **ii** 8!! **iii** 15!! **iv** 16!! **v** 20!! **vi** 25!!

b The primorial, $n\#$, is the product of all the prime numbers less than or equal to n. For example, $12\# = 11 \times 7 \times 5 \times 3 \times 2$. Evaluate the following.

i 12# **ii** 15# **iii** 20# **iv** 40# **v** 71# **vi** 100#

c The superfactorial, $n\$$, is the product of the first n factorials. For example, $5\$ = 5! \times 4! \times 3! \times 2! \times 1!$. Evaluate the following.

i 2$ **ii** 3$ **iii** 4$ **iv** 5$ **v** 6$ **vi** 10$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Counting principles

Interactive skillsheet
Permutations

Topic quiz
10D

10E Combinations

Learning intentions

By the end of this section you will be able to ...

- ✔ apply factorial notation to counting problems involving combinations
- ✔ understand how to construct Pascal's triangle and apply it to evaluate combinations
- ✔ apply counting principles and combinations to simplify probability calculations.

inter-year links

Year 7	10E Theoretical probability
Year 8	9D Theoretical probability
Year 9	8C Experiments without replacement

Subsets

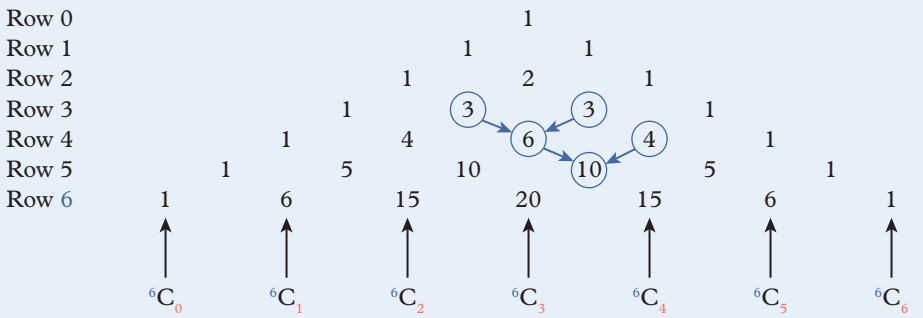
- A subset is a set contained within another set. Subsets of a set include the empty set, Ø, and the set itself.
- For example, the subsets of the set $\{a, b, c\}$ are Ø, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ and $\{a, b, c\}$.
- There are 2^n subsets of a set containing n elements.

Combinations

- A **combination** is a selection of the elements of a set where the order of the selection is not considered. Combinations are always selections made without replacement.
  For example, a hand of five playing cards from a pack of 52 cards is a combination because the order of the cards doesn't matter.
- All combinations are subsets of the set they are selected from.
- The number of combinations of r elements from a set of n elements is:

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!}.$$

  This value, ^nC_r, is read as 'n choose r' and is found by dividing ^nP_r by the number of ways to permute the r selected objects.
- The addition and multiplication principles apply when considering combinations from more than one set.
- Pascal's triangle is a triangular diagram where the entry in row n position r is equal to ^nC_r.
  - → The first row is labelled as row 0 and consists of only a 1.
  - → Numbers in subsequent rows are found by adding the two numbers diagonally above to the left and right.
  - → Blank spaces above a 1 are assumed to be 0.
  - → Entries in each row are numbered from 0 to n.
  - → The first seven rows, rows 0 to 6, are shown below.



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Example 10E.1 Counting subsets

Determine the number of different subsets there are in a set with:

a 6 elements

b 10 elements.

THINK	WRITE
a Evaluate two to the power of the number of elements in the set.	**a** $2^6 = 64$ subsets
b Evaluate two to the power of the number of elements in the set.	**b** $2^{10} = 1024$ subsets

Example 10E.2 Finding the number of combinations

In how many ways can you choose five out of a total of eight toys?

THINK	WRITE
1 Write down the factorial quotient to evaluate.	${}^8C_5 = \frac{8!}{5! \times 3!}$
2 Write the factorials in the numerator and denominators as their products. Divide out the common factors.	$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1}$
	$= 8 \times 7$
3 Evaluate the remaining product.	$= 56$ ways

Example 10E.3 Using combinations and counting principles

A grocery store has five varieties of cola and five varieties of lemonade.

a If you need three different types of each of cola and lemonade, how many different combinations of drinks could you buy?

b If you need to buy four different types of cola and two different types of lemonade, how many different combinations of drinks could you buy?

THINK	WRITE
a 1 Use the multiplication rule to multiply the combinations for each type of drink.	**a** ${}^5C_3 \times {}^5C_3$
2 Evaluate the product.	$= 10 \times 10$
	$= 100$ combinations
b 1 Use the multiplication rule to multiply the number of combinations for each type of drink.	**b** ${}^5C_4 \times {}^5C_2$
2 Evaluate the product.	$= 5 \times 10$
	$= 50$ combinations

Helpful hints

- ✔ Always check whether a scenario involves the selection being in order (permutations) or not (combinations). This may be implied rather than explicitly stated. For example, there are ${}^{25}C_2$ ways to choose two student leaders from a group of 25 students, but there are ${}^{25}P_2$ ways to choose a captain and then a vice-captain, since the order of selection matters in this case.
- ✔ Pascal's triangle can be used to quickly evaluate combinations without factorials.
- ✔ Some scientific calculators have a function which can be used to quickly evaluate ${}^{n}C_{r}$.

ANS p654

Exercise 10E Combinations

1–4, 6–8, 10 | 1(a–c), 2(c–e), 3(a,b), 4(a–c), 5,6,7(a), 8–13 | 2(c–e), 4(d–f), 5(a,f), 6(a),7(a), 8, 9, 10(a–c), 11–15

UNDERSTANDING AND FLUENCY

10E.1 **1** Determine the number of different subsets there are in a set with:

a 5 elements **b** 8 elements **c** 12 elements

d 15 elements **e** 20 elements **f** 30 elements.

10E.2 **2** **a** In how many ways could you choose four out of a total of nine toys?

b How many sets of three cards out of a hand of five playing cards are there?

c Ten people compete in a 100-metre race. How many different groups of three people can take a podium place (gold, silver and bronze)?

d In an octave there are twelve notes, each a semitone apart. How many ways are there to choose eight different notes?

e In a monster capture game, you have a party of six different monsters. How many groups of four different monsters can be chosen?

f You have a playlist of twenty songs. In how many ways could eight of the songs be heard one after the next?

3 Evaluate the following combinations.

a ${}^{6}C_3$ **b** ${}^{10}C_2$ **c** ${}^{10}C_5$ **d** ${}^{7}C_5$ **e** ${}^{6}C_5$ **f** ${}^{30}C_{28}$

8G.3 **4** **a** A grocery store has eight varieties of cola and seven varieties of lemonade. If you need four different bottles of cola and five different bottles of lemonade, how many different combinations of drinks could you buy?

b Savi is creating a playlist from two albums. She wants five tracks from the first album that has eight tracks and six tracks from the second album that has ten tracks. How many different playlists can she make if she only plays on shuffle (that is, she does not care about the order they are played in)?

c Brayden is packing clothes to take on holidays. He wants to pack three of his eight pairs of shorts, two of his five pairs of pants, and seven of his fifteen t-shirts. How many different sets of clothes could he bring on his holiday?

d Eleven players per team are needed on the field for a game of soccer. If a team wants eight of their twenty senior members on the team and three of their 10 junior members, how many different groups of members can be on the field together?

e Martine is planning to travel around Victoria and New South Wales. She has found a 'top ten places to visit' list for both states. She decides to visit six in Victoria and four in New South Wales. How many different combinations of places to visit are there?

f Xuehong is deciding which games to buy this year and has decided to buy three of eight RPGs, two of nine Metroidvania games and one of five party games. How many sets of games could Xuehong buy this year?

5 **a** In a game of five-card draw poker, you have the option to discard any number of the five cards you are dealt, including discarding none. How many different selections of cards can you discard?

b Kshipra has ten items on her wish list and recently received her paycheck. How many different selections of the items on her wish list can she choose to buy including opting to not buy any right now?

c In a deck-building card game you build your deck as you play purchasing cards using in-game resources. There are eight cards available for purchase this turn and any number of them can be purchased, or you can decline to buy any. Provided you have enough resources, how many different selections of the available cards could you choose to purchase?

d A game requires players to work as a team to remove as many of the nine jewels from a glass case as possible while not alerting the opposing players who have their eyes shut and are counting to 30. How many different selections of jewels could be stolen in a round including failing to steal any?

e There are six bags of groceries to bring in from the car. How many possible combinations of those bags could be brought in together first if at least one bag must be carried?

f Five pieces of chocolate are placed on a plate. How many ways are there to eat at least one chocolate on the plate?

6 **a** An album with ten tracks is played on shuffle. Determine the probability that tracks 4, 5, 7 and 8 are played in any order if four tracks are played in total.

b In an octave there are twelve notes, each a semitone apart. A random series of four notes in an octave are played. Determine the probability that the notes played are numbers 1, 5, 8 and 12 in any order.

7 **a** A team of four players is needed to be selected from seven seniors and four juniors. Determine the probability that two seniors and two juniors are selected.

b A bag contains five red counters and eight blue counters. Determine the probability, when six counters are selected at random, that four red and two blue counters are selected.

c A puzzle has eight '3-in-1-out' pieces, nine '2-in-2-out' pieces, and four '1-in-3-out' pieces left to go. Determine the probability that the next seven pieces to go in are four '3-in-1-out', two '2-in-2-out' and one '1-in-3-out' in any order.

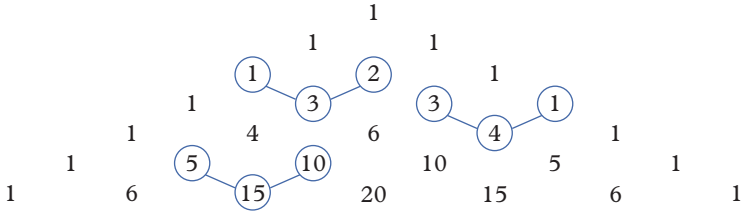
8 A netball team can have seven players on the court. Of the six seniors and eight juniors in the team, seven players are randomly selected to start the game.

a Determine the probability that all the seniors are selected.

b Determine the probability that the order the players were selected alternated between junior and senior (starting with either junior or senior).

c Suppose the size of the team can vary. Determine the probability that a team of seven is selected out of all possible selections.

9 Below are the first few rows of Pascal's triangle. The circled values are connected by addition, for example $5 + 10 = 15$.



a Write each of the circled values' sum in terms of the combinations, in the form ${}^{n}\mathrm{C}_{r}$, they represent.

b Let the bottom circled value be represented by ${}^{n}\mathrm{C}_{r}$. Write the equation of the similar type of sum to part **a** in terms of n and r.

This relationship is known as Pascal's identity, which is named after Blaise Pascal.

c Use Pascal's identity to state the value ${}^{n}\mathrm{C}_{r}$ which can be found given the following information.

i ${}^{10}\mathrm{C}_{4}$ and ${}^{10}\mathrm{C}_{5}$ **ii** ${}^{20}\mathrm{C}_{15}$ and ${}^{20}\mathrm{C}_{14}$ **iii** ${}^{16}\mathrm{C}_{14}$ and ${}^{17}\mathrm{C}_{14}$

iv ${}^{22}\mathrm{C}_{3}$ and ${}^{21}\mathrm{C}_{2}$ **v** ${}^{52}\mathrm{C}_{29}$ and ${}^{53}\mathrm{C}_{30}$ **vi** ${}^{99}\mathrm{C}_{49}$ and ${}^{99}\mathrm{C}_{48}$

PROBLEM SOLVING AND REASONING

10 Use the given values to evaluate the following.

a ${}^5C_3$ given ${}^5P_3 = 60$ and $3! = 6$

b ${}^8C_4$ given ${}^8P_4 = 1680$ and $4! = 24$

c ${}^{10}C_8$ given ${}^{10}P_8 = 1\,814\,400$ and $80! = 40\,320$

d ${}^5P_4$ given ${}^5C_4 = 5$ and $4! = 24$

e ${}^7P_5$ given ${}^7C_5 = 21$ and $5! = 120$

f ${}^{11}P_3$ given ${}^{11}C_3 = 165$ and $3! = 6$

11 a Evaluate the following sums.

i ${}^1C_0 + {}^1C_1$

ii ${}^2C_0 + {}^2C_1 + {}^2C_2$

iii ${}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3$

iv ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$

v ${}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

b Describe the pattern you see in your answers to part **a**.

c Explain why this pattern holds by viewing combinations as subsets.

12 a Write down the first and last number of every row of Pascal's triangle.

b Hence, state the values of ${}^nC_0$ and ${}^nC_n$.

c Describe the connection between the row number (starting from row 0) and the second number in from both ends of every row of Pascal's triangle.

d Hence, state the values of ${}^nC_1$ and ${}^nC_{n-1}$.

e Describe the relationship between ${}^nC_r$ and ${}^nC_{n-r}$ where $r \leq n$.

13 The general form of Pascal's identity is ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$.

a Prove Pascal's identity using the definition ${}^nC_r = \dfrac{n!}{r!(n-r)!}$. Hint: Write an expression for each term and factorise the expression on the right.

b Another way to prove Pascal's identity is to consider an element x from a set of size n.

i How many elements are there in the set not including x?

ii How many ways are there to choose r elements from the set if x is one of the r elements? Explain your answer.

iii How many ways are there to choose r elements from the set if x is not one of the r elements? Explain your answer.

iv Use your previous answers to prove Pascal's identity.

CHALLENGE

14 The n^{th} triangular number is the sum of the first n natural numbers. For example, the fifth triangular number is $1 + 2 + 3 + 4 + 5 = 15$. Use Pascal's identity to explain why ${}^{n+1}C_2$ is a triangular number for $n \geq 1$.

15 a Expand $(x + 1)^2$.

b Expand $(x + 1)^3$ by expanding $(x + 1)(x + 1)^2$.

c Expand $(x + 1)^4$ by expanding $(x + 1)(x + 1)^3$.

d Describe what the values of the coefficients are related to.

e Use the idea in part **d** to:

i expand $(x + 1)^7$

ii determine the coefficient of the x^4 term in the expansion of $(x + 1)^{20}$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Subsets

Interactive skillsheet
Combinations

Investigation
Combinations and coefficients

Topic quiz
10E

10F Conditional probability

Learning intentions

By the end of this section you will be able to ...

- ✔ understand conditional statements
- ✔ calculate conditional probability using tree diagrams
- ✔ calculate conditional probability using two-way tables and Venn diagrams.

Inter-year links

Year 8 — 9E Two-way tables

Year 9 — 8C Experiments without replacement

Conditional probability

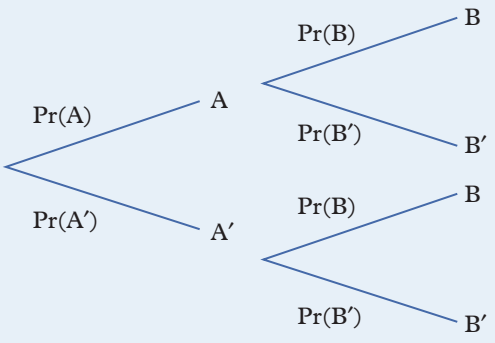
- **Conditional probability** is the probability of an event occurring, given the knowledge that another event has occurred.
  - → Conditional probability is often about a single trial of an experiment. For example, if a standard six-sided die is rolled, what is the probability the outcome is odd, given that the outcome is prime?
  - → Conditional probability restricts the sample space to a subset of the universal set. In the above example, the information 'given that the outcome is prime' restricts the sample space to $\{2, 3, 5\}$ from $\{1, 2, 3, 4, 5, 6\}$. This gives $\Pr(\text{odd given prime}) = \frac{2}{3}$, since two of the three primes are odd.
- $\Pr(A|B)$ is the probability that event A occurs given that event B has occurred. This can be read as 'the probability of A given B'.
- $\Pr(B|A)$, which represents the probability that event B occurs given that event A has occurred, is different from $\Pr(A|B)$.
- Conditional probability can be calculated by using the following formulas:

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} \quad \text{or} \quad \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

- The above formulas can be rearranged to give:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A) \quad \text{or} \quad \Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$$

  - → This explains why we multiply the probabilities on branches in tree diagrams:



$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$

$\Pr(A \cap B') = \Pr(A) \times \Pr(B'|A)$

$\Pr(A' \cap B) = \Pr(A') \times \Pr(B|A')$

$\Pr(A' \cap B') = \Pr(A') \times \Pr(B'|A')$

- Conditional probability can be identified when any of the following terms are used in a probability question: 'if ... then ...', 'given ...' or 'knowing that ...'
- When calculating conditional probability using two-way tables or Venn diagrams, the words 'of' and 'from' indicate the set from which the choice is being made. For example, 'from the group who …'

Example 10F.1 Calculating conditional probability using a tree diagram

A sandwich tray contains 30 sandwiches, 10 of which are ham (H) and 20 of which are salad (S). Jack takes two sandwiches at random, one after the other.

a Draw a tree diagram to display the possible outcomes of Jack's two choices.

b If Jack's first sandwich is ham, what is the probability that his second sandwich will also be ham?

c Knowing that Jack's first choice is a salad sandwich, what is the probability that his second one will be ham?

d What is the probability of Jack's second sandwich being salad, given that his first sandwich is ham?

THINK

a Draw a tree diagram to display all the outcomes. The probabilities for the second choice depend on the outcome of the first choice. Write the probabilities on the branches of your tree diagram.

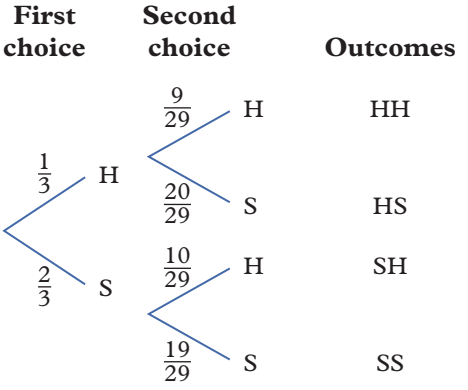
b In this case, you are looking for the probability that the second choice is H, given that the first choice is H. Follow the branches H then H on the tree diagram. The probability on the second branch is the probability of the first and second sandwiches both being ham.

c In this case, you are looking for the probability that the second choice is H, given that the first choice is S. Follow the branches S then H on the tree diagram. The probability on the second branch is the probability of the second sandwich being ham if the first sandwich was salad.

d In this case, you are looking for the probability that the second choice is S, given that the first choice is H. Follow the branches H then S on the tree diagram. The probability on the second branch is the probability of the second sandwich being salad if the first sandwich was ham.

WRITE

a H represents ham and S represents salad.



b $\Pr(H|H) = \frac{9}{29}$

c $\Pr(H|S) = \frac{10}{29}$

d $\Pr(S|H) = \frac{20}{29}$

Example 10F.2 Calculating conditional probability using a two-way table

The table below shows the results of a survey taken at a library to find whether people under 18 and people 18 and over prefer using Facebook or Twitter (or neither).

	Under 18	18 or over	Total
Facebook	20	30	50
Twitter	5	10	15
Neither	10	5	15
Total	35	45	80

Use the two-way table to find the relative frequency that a person:

a prefers Facebook, given they are 18 or over

b is 18 and over, given they prefer Facebook.

c Explain why your answers to parts **a** and **b** are not equal.

THINK

a **1** Identify the total number of people surveyed who are 18 or over by looking at the total number in the '18 or over' column.

2 Identify the number of people who prefer Facebook in the '18 or over' column.

3 Divide the number of people who are 18 or over who prefer Facebook by the total number of people who are 18 or over and give the answer as a fraction, simplifying if possible.

b **1** Identify the total number of people who prefer Facebook by looking at the Facebook row.

2 Identify the number of people who are 18 or over in the 'Facebook' row.

3 Divide the number of people who prefer Facebook who are 18 or over by the total number of people who prefer Facebook and give the answer as a fraction, simplifying if possible.

c Identify that parts **a** and **b** are asking for information about different groups of people who were surveyed. Recall that $\Pr(A|B)$ is usually not equal to $\Pr(B|A)$.

WRITE

a total number of people 18 or over surveyed = 45

number of people 18 or over who prefer Facebook = 30

$$\Pr(\text{prefer Facebook} \mid \text{18 or over}) = \frac{30}{45} = \frac{2}{3}$$

b total number of people who prefer Facebook = 50

number of the people who prefer Facebook who are 18 or over = 30

$$\Pr(\text{18 and over} \mid \text{prefer Facebook}) = \frac{30}{50} = \frac{3}{5}$$

c The selections are made from different groups of people. In **a**, the selection is made from those aged 18 or over while, in **b**, the selection is made from those who prefer Facebook.

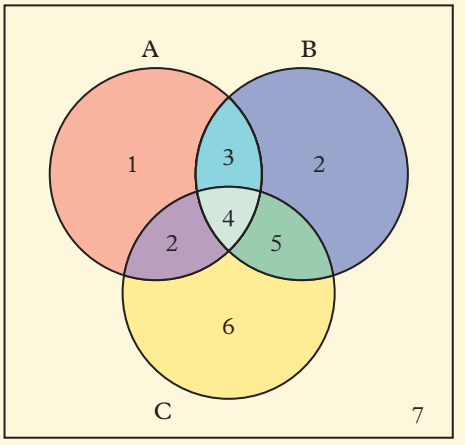
Example 10F.3 Calculating conditional probability using a Venn diagram

This Venn diagram shows the distribution of a group of students who study Art (A), Business Studies (B) and Computing (C).

Use the diagram to determine the relative frequency that a student studies:

a all three of these subjects, given that they study two of the subjects

b Art and Computing, given that they do not study Business Studies.



THINK

a **1** Identify the number of students in the 'given' set. This is the total in all the overlapping sections of the diagram.

2 Identify the number of students within this group who study all three subjects.

3 Express the two numbers you identified in parts **1** and **2** as a fraction, and simplify if possible.

b **1** Identify the number who do not study Business Studies. This is all those outside the circle B.

2 Identify the number of students within this group who study Art and Computing.

3 Express the two numbers you identified in parts **1** and **2** as a fraction, and simplify if possible.

WRITE

a number of students studying two or more subjects $= 3 + 4 + 5 + 2$

$= 14$

number of students studying all three subjects $= 4$

$\Pr(\text{all three subjects} \mid \text{two subjects}) = \frac{4}{14}$

$= \frac{2}{7}$

b number of students not studying Business Studies $= 1 + 2 + 6 + 7$

$= 16$

number of students in this group studying Art and Computing $= 2$

$\Pr(\text{Art and Computing} \mid \text{not Business Studies})$

$= \frac{2}{16}$

$= \frac{1}{8}$

Helpful hints

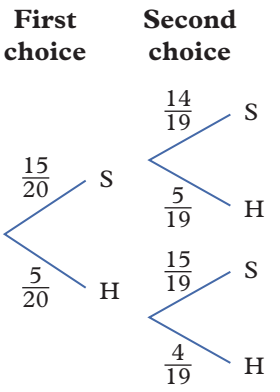
- ✔ When working with conditional probability, remember to determine the number of elements in the conditional (or given) set first.
- ✔ When calculating conditional probability, the given probability always goes on the denominator.

ANS p655

Exercise 10F Conditional probability

1, 3–8, 10, 12, 14, 15 | 2, 3, 5–7, 9–11, 13, 15, 17 | 3, 5, 6, 9, 11, 15, 16, 18, 19

1 A box of 20 chocolates contains 15 chocolates with soft centres and 5 chocolates with hard centres. Two chocolates are taken at random, one after the other. The tree diagram on the right represents the outcomes of the two choices.



a Explain the probabilities of $\frac{15}{20}$ and $\frac{5}{20}$ on the first two branches of the tree diagram.

b Explain the probabilities on the four branches for the four possible second choices.

c If the first chocolate is hard, what is the probability that the second chocolate will also be hard?

d Knowing that the first chocolate is a soft one, what is the probability the second chocolate will be hard?

e What is the probability of the second chocolate being soft, given the first chocolate is hard?

10F.1 **2** A bag contains 6 balls: two black and four white. A ball is selected at random, its colour noted, and then another ball is selected without the first ball being replaced.

a Draw a tree diagram to display this situation.

b If the first ball is white, what is the probability the second ball will also be white?

c Knowing that the first ball is black, what is the probability the second ball is white?

d What is the probability that the second ball is black, given that the first ball is white?

3 A box contains 7 balls: four green and three red. Three balls are selected without replacement.

a Draw a tree diagram to display this situation.

b If the first ball is green, what is the probability the second ball will also be green?

c Knowing that the first two balls are both green, what is the probability that the third ball will also be green?

d What is the probability that the third ball is green, given that a green ball and then a red ball were drawn in the first two selections?

10F.2 **4** Consider the two-way table from Example 10C.1 again. Use this two-way table to determine the relative frequency of a card being:

	Ace (A)	Not ace (A′)	Total
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

a not an Ace, given it is a heart

b not an Ace, given it is not a heart.

5 The two-way table on the right shows the modes of transport a group of students use to get to school. State the number of:

	Primary school	High school	Total
Catch public transport	15	27	42
Are driven	22	18	40
Walk/ride	4	14	18
Total	41	59	100

a primary school students who catch public transport to school

b high school students who are driven to school

c students who walk or ride to school

d students who are in high school.

UNDERSTANDING AND FLUENCY

6 Using the table from question **5**, calculate the relative frequency of a student:

a catching public transport to school, given they are at primary school

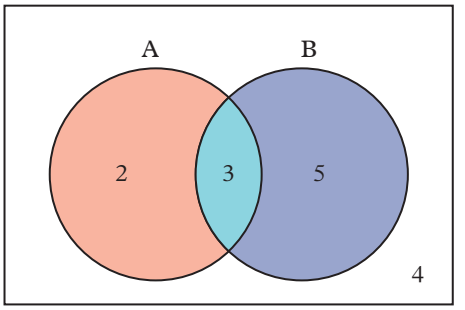
b being at high school, given they are driven to school

c walking or riding to school

d being at high school

e being driven to school, given they are at high school.

7 Consider the Venn diagram on the right.



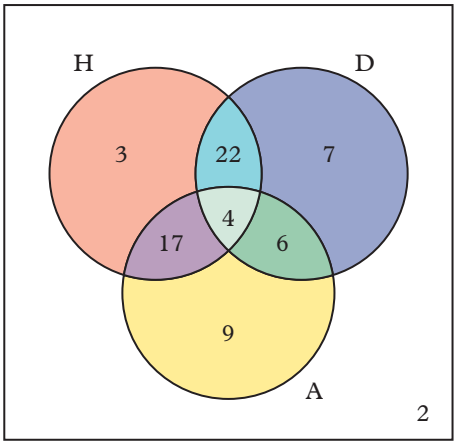
a How many elements are in set B?

b How many of the elements in set B are also in set A?

c Determine $\Pr(A|B)$.

d Using similar reasoning, determine $\Pr(B|A)$.

10F.3 **8** The following Venn diagram shows the distribution of a group of 70 students who were asked if they enjoy watching horror films (H), drama films (D) and action films (A).

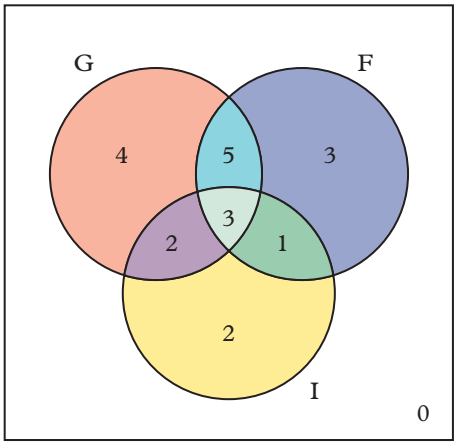


Use the Venn diagram to determine the relative frequency that a student enjoys:

a all three genres of film, given that they enjoy two of the film genres

b horror films, given that they enjoy action films.

9 Twenty members of the Swiss national soccer team were asked which languages they speak fluently, out of German (G), French (F) and Italian (I). Their responses are shown in the Venn diagram on the right.



a One of the twenty players is selected at random. Find the probability that they speak:

i German

ii German, but not French

iii German, given that they speak French

iv French, given that they speak German

v all three languages, given that they speak at least two of the languages.

b An Italian-speaking player is selected at random. Find the probability that this player can also speak French.

10 For two events, A and B, it is known that $\Pr(A|B) = 0.4$, $\Pr(A \cap B) = 0.1$ and $\Pr(A) = 0.3$.

a Find $\Pr(B|A)$. Give your answer as a fraction.

b Find $\Pr(B)$. Give your answer as a decimal.

11 A computer generates random questions for students to use for practice. From collected data, the chance that a student gets the first question correct is 0.8. If the student gets this question correct, the computer generates a harder question, with the probability of the student getting this one correct being reduced by 0.1. If the student gets the first question wrong, the computer gives the student a question of the same level of difficulty as the first question.

a Draw a tree diagram to display the possible outcomes of the two practice questions.

b Find the probability, expressed as a decimal, that the student gets the second question:

i correct, given they got the first question correct

ii correct, given they got the first question wrong

iii wrong, given they got the first question correct.

12 At a small firm with 50 employees, some work full-time and others work part-time. There are 15 employees who work from home (WFH) in total and 30 of the 34 full-time employees are hybrid workers who split their time between the office and at home..

a Draw a two-way table to record this data. Complete all the entries in the table.

b How many employees are:

i full-time employees who work from home?

ii part-time?

iii hybrid workers?

c Find the relative frequency of an employee:

i working part-time, given they are a hybrid worker

ii being a hybrid worker, given they work part-time

iii being a WFH employee given they work full-time

iv working full-time, given they are a WFH employee.

d Explain the differences in your answers for the following parts of part **c**:

i parts **i** and **ii**

ii parts **iii** and **iv**.

13 The two-way table below shows the number and types of paper clips in a pile on a desk.

	Small	**Large**	**Total**
Silver	27	35	
Coloured	18	10	
Total			

a Complete the totals for the rows and columns.

b Determine whether each of these statements is true, showing the calculations you used. (Percentages are rounded to the nearest whole number.) For those statements which are not true, show working to provide a correction.

i There is an equal number of small and large paper clips.

ii There are fewer coloured paper clips than silver ones.

iii The percentage of small paper clips which are silver is 44%.

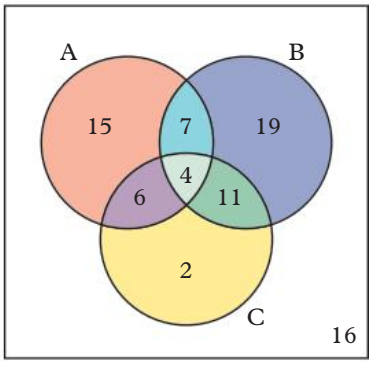
iv The percentage of coloured paper clips which are large is 22%.

v The whole pile contains 69% silver paper clips.

PROBLEM SOLVING AND REASONING

14 A jar contains black and white balls. Two balls are drawn without replacement. The probability of selecting a black ball on the first draw is 0.47, and the probability of selecting a second black ball, given that the first ball is black, is 0.22. What is the probability of selecting a white ball on the second draw, given that the first ball is a black one? Give your answer as a decimal.

15 Consider the following Venn diagram, which shows survey data gathered from 80 people.



a Find each of the following.

i $n(A)$ **ii** $n(B)$. **iii** $n(C')$

b One person from the 80 is selected at random. Find:

i $\Pr(B|A)$ **ii** $\Pr(A|B)$. **iii** $\Pr(A'|C')$

c Explain how the answers to part **a** are helpful in answering part **b**.

16 For two events, X and Y, it is known that $\Pr(X|Y) = 0.2$ and $\Pr(Y|X) = 0.5$.

a Find:

i $\Pr(X'|Y)$ **ii** $\Pr(Y'|X)$.

b If $\Pr(X \cap Y) = 0.05$, then find $\Pr(X)$ and $\Pr(Y)$.

17 There are b blue marbles and r red marbles in a bag and there are at least two marbles of each colour. Two marbles are drawn without replacement.

a Draw a tree diagram to represent this situation. Include all probabilities in terms of b and r on the branches.

b Find the probability that the second ball drawn is blue. Comment on this value.

CHALLENGE

18 If an event, A, relates to the second step of a multi-step experiment, then it may be necessary to split A into cases in order to determine its probability. That is, $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$, meaning that A can either occur with B or without B, where B relates to the first step of the experiment. Then, using the conditional probability formula, $\Pr(A) = \Pr(B) \times \Pr(A|B) + \Pr(B') \times \Pr(A|B')$.

Imagine that a soccer team has a 40% chance of winning their upcoming game if it rains and a 70% chance of winning their upcoming game if it does not rain.

a Find the probability that the team wins their upcoming game if there is a 20% chance of rain. Give your answer as a decimal.

b Find the probability that it rains on the day of their upcoming game if their chance of winning the game is 0.5. Give your answer as a fraction.

19 Baye's rule is a useful conditional probability formula which states that $\Pr(A|B) = \dfrac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$. Baye's rule can be proved using the conditional probability formulas for $\Pr(A|B)$ and $\Pr(B|A)$.

a Prove Baye's rule using $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ and $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$.

b Two boxes each contain white balls and black balls. Box X contains 7 white and 3 black, whereas Box Y contains 2 white and 3 black. One box is chosen at random and then a ball is selected from it at random. Let X be the event 'selecting box X' and W be the event 'selecting a white ball.'

i Find $\Pr(W)$. Give your answer as a decimal.

ii Use Baye's rule to find $\Pr(X|W)$, which is the probability that Box X was chosen, given that the selected ball is white. Give your answer as a fraction.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Conditional probability

Investigation
Bertrand's box

Topic quiz
10F

10G Independence

Learning intentions

By the end of this section you will be able to ...

- ✔ understand the difference between independent and dependent events
- ✔ calculate the probability of independent and dependent events.

Inter-year links

Year 9 8E Two-way tables

Independent events

- Two events are **independent** if one event occurring has no effect on the probability of the other event occurring.
- The concept of independence can be used to determine if the different steps of a multi-step chance experiment are independent of each other; for example, if the first step has an impact or not on the outcome of the second step.
  - → The outcomes for each step of a chance experiment with replacement are independent, as the probabilities for the second step are not dependent on the outcome of the first step.
  - → The outcomes for each step of a chance experiment without replacement are not independent, as the probabilities for the second step depend on the outcome of the first step.
- The concept of independence can also be used to determine if real-life events are independent or not; for example, if the weather outside has an effect on whether people take public transport on their commute to work.
- If A and B are independent events, then $\Pr(A|B) = \Pr(A)$ and $\Pr(B|A) = \Pr(B)$, since the fact that A or B occurred does not change the probability of the other event happening.
  - → Natural variation in real-life events means it can be difficult to prove independence, but you can state whether real-life events appear to be independent or not.
- If $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, then A and B are independent events.

Example 10G.1 Testing for independence

For two events, A and B, it is known that $\Pr(A) = 0.6$, $\Pr(B) = 0.75$ and $\Pr(A \cap B) = 0.5$.

a Evaluate:

i $\Pr(A|B)$ **ii** $\Pr(B|A)$.

b Are A and B independent or dependent events?

THINK

a Use the conditional probability formula to evaluate each probability.

WRITE

a i $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{0.5}{0.75} = \dfrac{2}{3}$

ii $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)} = \dfrac{0.5}{0.6} = \dfrac{5}{6}$

b Use your answers to part **a** to check if $\Pr(A|B)$ is equal to $\Pr(A)$, or if $\Pr(B|A)$ is equal to $\Pr(B)$. If yes, then the events are independent. If not, the events are dependent.

b A and B are dependent events, since $\Pr(A|B)$ is not equal to $\Pr(A)$. Equivalently, $\Pr(B|A)$ is not equal to $\Pr(B)$.

Example 10G.2 Using a two-way table to determine whether two events are independent

The two-way table below shows whether people who prefer cats or dogs have a preference for reading or exercise.

	Cats	Dogs	Total
Reading	31	18	49
Exercise	16	35	51
Total	47	53	100

Are the events 'prefers cats' and 'prefers exercise' independent?

THINK

1. Calculate the probability of a person chosen at random preferring exercise.
2. Calculate the probability of a person chosen at random preferring exercise, given that they prefer cats.
3. Compare the two probabilities. If they are similar, the two events are independent. If there is a significant difference, the two events are dependent.

WRITE

$$\Pr(\text{exercise}) = \frac{51}{100}$$
$$= 0.51$$
$$\Pr(\text{exercise}|\text{cats}) = \frac{16}{47}$$
$$\approx 0.34$$

The events 'prefers cats' and 'prefers exercise' are dependent.

Helpful hints

✔ The equation $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ is not a formula, but rather a test. This means that you can use the equation to test if A and B are independent. If you are told that A and B are independent, then you can use the equation as a formula. However, if you are not told that A and B are independent, then you can find $\Pr(A \cap B)$ using the addition rule formula or the conditional probability formulas, which are always true. These are:

→ $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$

→ $\Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$

→ $\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$

ANS p656

Exercise 10G Independence

1–7, 10, 12 | 1, 2, 4, 7–13 | 2, 4, 9–15

10G.1 **1** For two events, A and B, it is known that $\Pr(A) = 0.4$, $\Pr(B) = 0.5$ and $\Pr(A \cap B) = 0.2$.

a Evaluate:

i $\Pr(A|B)$ **ii** $\Pr(B|A)$.

b Are A and B independent or dependent events?

2 For two independent events, M and N, it is known that $\Pr(M) = \frac{2}{3}$ and $\Pr(N) = \frac{1}{4}$. Find each of the following.

a $\Pr(M|N)$ **b** $\Pr(N|M)$ **c** $\Pr(M \cap N)$

3 The two-way table on the right shows data gathered from 21 students about their preferred sport (out of cricket and football) and their preferred subject at school (out of Maths and English).

Are the events 'prefers cricket' and 'prefers Maths' independent?

	Cricket	Football	Total
Maths	6	8	14
English	3	4	7
Total	9	12	21

10G.2 **4** The two-way table below shows data gathered from 40 people about their preferred ice cream flavour (out of chocolate, strawberry, vanilla) and their preferred chocolate variety (out of milk, dark, white).

	Chocolate	Strawberry	Vanilla	Total
Milk	8	7	10	25
Dark	4	2	1	7
White	3	0	5	8
Total	15	9	16	40

Decide whether the following pairs of events are independent or dependent.

a 'Prefers vanilla' and 'prefers milk'

b 'Prefers chocolate' and 'prefers milk'

c 'Prefers strawberry' and 'prefers dark'

d 'Prefers white' and 'prefers chocolate'

5 Consider the two-way table on the right.

a Evaluate:

i $\Pr(A)$ **ii** $\Pr(A|B.)$

b Based on the results from part **a**, are A and B independent or dependent events?

	A	A′	Total
B	9	3	12
B′	6	2	8
Total	15	5	20

6 Consider the two-way table on the right.

a Evaluate:

i $\Pr(Y)$ **ii** $\Pr(Y|X)$.

b Based on the results from part **a**, are X and Y independent or dependent events?

	X	X′	Total
Y	14	6	20
Y′	4	6	10
Total	18	12	30

7 For two events, A and B, it is known that $\Pr(A) = 0.6$ and $\Pr(B) = 0.8$. If A and B are independent events, then find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

8 Alexei is told that $\Pr(A) = 0.6$, $\Pr(B) = 0.7$ and $\Pr(A \cup B) = 0.8$. He is asked to find $\Pr(A \cap B)$ and calculates the answer to be 0.42.

a What was Alexei's mistake?

b What is the actual value of $\Pr(A \cap B)$, and how must it be calculated?

9 Explain the difference between independent events and mutually exclusive events.

10 In each case, decide if A and B are independent events or dependent events.

a $\Pr(A) = 0.4$, $\Pr(B) = 0.3$, $\Pr(A \cap B) = 0.2$

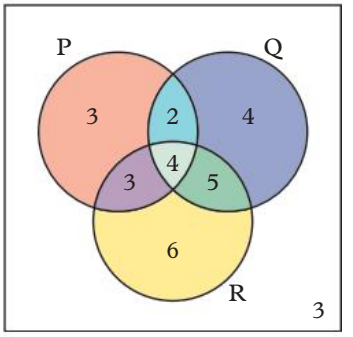
b $\Pr(A) = \frac{3}{5}$, $\Pr(B) = \frac{2}{3}$, $\Pr(A \cap B) = \frac{2}{5}$

c $\Pr(A) = 0.5$, $\Pr(B) = 0.8$, $\Pr(A \cap B') = 0.1$

d $\Pr(A) = 0.6$, $\Pr(B) = 0.5$, $\Pr(A|B) = 0.3$

11 Consider the Venn diagram on the right.

Which pairs of events are independent? Justify your answer with appropriate calculations.



12 Independent events in statistics are rare, due to random variations.

For example, outside of the Mathematics classroom we would say that someone's mode of transport to school and their favourite school subject are 'independent,' since it does not follow that knowing that a student takes a train to school would tell us anything about their favourite subject. However, it does not follow that the events 'takes train' and 'prefers Maths' would be independent events in a survey of Year 10 students.

Consider the following data which shows the results of a survey from 30 students who were asked their preferred movie genre (out of action, comedy, drama) and whether their birthday occurs on an even calendar date or an odd calendar date.

	Action	Comedy	Drama	Total
Odd	7	5	2	14
Even	5	3	2	10
Total	12	8	4	24

a One student is selected at random. Find:

i $\Pr(\text{Action})$ **ii** $\Pr(\text{Action} \mid \text{Odd})$

iii $\Pr(\text{Comedy})$ **iv** $\Pr(\text{Comedy} \mid \text{Even})$.

b Explain why 'prefers action movies' and 'born on an odd date' are independent events in this survey.

c Explain why 'born on an even date' and 'prefers comedy movies' are dependent events in this survey.

d Explain why it does not make sense to claim that being born on an even date makes someone more or less likely to prefer comedy movies.

13 'Independent events' and 'independent trials' are often confused in probability. Independent trials are separate steps of a multi-step experiment where each step does not affect the probabilities at any other step. If the trials in a multi-step experiment are independent, then this will inevitably lead to an event relating to one trial being independent of an event relating to another trial.

Consider a bag containing six balls: 4 white and 2 black. Two balls are randomly selected from the bag. Let W1 be the event 'the first ball is white' and W2 be the event 'the second ball is white.'

a Explain why W1 and W2 are independent events if the first ball is replaced before choosing the second ball.

b Find $\Pr(\text{W2} \mid \text{W1})$ if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

c Find the probability that the two balls selected are white if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

14 A and B are independent events with $\Pr(A) = \Pr(B)$. Find $\Pr(A)$ if $\Pr(A \cup B) = 0.84$.

15 a Use Venn diagrams to show to that $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$.

b Prove that if A and B are independent events, then A and B′ are also independent.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Independence

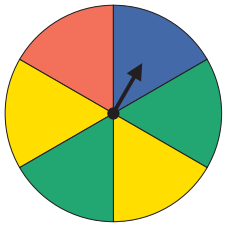
Topic quiz
10G

Chapter summary

Theoretical probability

$$\Pr(\text{event}) = \frac{\text{number of outcomes in event}}{\text{number of outcomes in the sample space}}$$

$$\Pr(\text{green}) = \frac{2}{6} = \frac{1}{3}$$



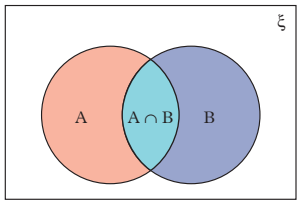
- The sum of probabilities of complementary events is 1: $\Pr(A) + \Pr(A') = 1$
- If two events are mutually exclusive then they cannot both occur at the same time.

Conditional probability

- $\Pr(A|B)$ = the probability that event A occurs given that event B has occured.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \Pr(A|B) \neq \Pr(B|A)$$

Venn diagrams



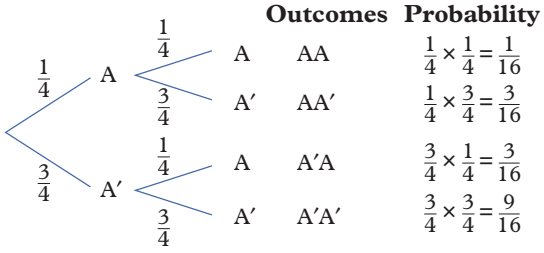
Independent events

- Two events are independent if the outcome of one event has no effect on the outcome of the other event.
- If A and B are independent events, then $\Pr(A|B) = \Pr(A)$.
- If $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, then A and B are independent events.
- The outcomes for each step of a chance experiment with replacement are independent.
- The outcomes for each step of a chance experiment without replacement are not independent.

Set notation

- A set is a collection of distinct objects.
- ξ = the universal set containing all the elements under consideration
- $A \cap B$ = the intersection of two sets, A and B
- $A \cup B$ = the union of two sets, A and B
- A' = the complement of set A, everything from ξ that is not in A
- $n(A)$ = the numbers of elements in set A
- $A \subset B$ = 'set A is a subset of set B'

Tree diagrams



Arrays

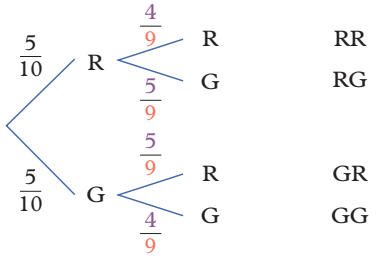
		Rolling a standard six-sided die					
		1	2	3	4	5	6
Flipping a coin	H	(H, 1)	(H, 2)	(H, 3)	(H, 4)	(H, 5)	(H, 6)
	T	(T, 1)	(T, 2)	(T, 3)	(T, 4)	(T, 5)	(T, 6)

Two-way tables

	School A	School B	Total
Musical instrument	10	4	14
No musical instrument	3	13	16
Total	13	17	30

Experiments without replacement

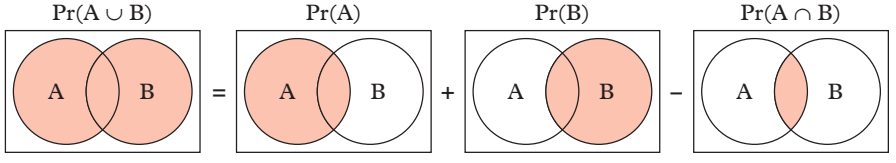
- When multi-step experiments involve items being selected from a group, if the items are not replaced after each selection, then the probabilities will change.



Permutations and combinations

- The number of ways to order n distinct objects is:
  $n! = n \times (n-1) \times \ldots \times 2 \times 1.$
- The number of ways of ordering r objects from n distinct objects is:
  $${}^nP_r = \frac{n!}{(n-r)!}$$
- The number of ways of choosing r objects from n distinct objects is:
  $${}^nC_r = \frac{n!}{r!(n-r)!}$$

Additional rule of probability



Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

10A **1** A coin is flipped three times. How many outcomes would appear on a tree diagram representing this experiment?

A 2 **B** 4 **C** 6 **D** 8 **E** 10

10A **2** A letter is randomly selected from a word. The sample space for the experiment is {A, E, L, P}. Which of the following could be the word?

A HELP **B** PLEASE **C** PLEAD **D** APPLE **E** ELLA

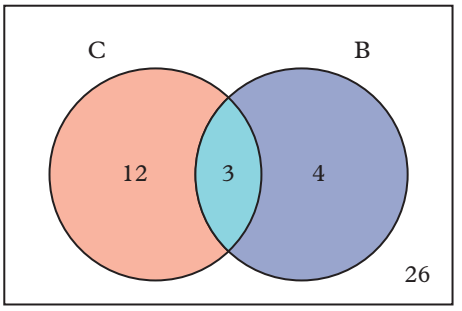
10B **3** A standard deck of cards contains 52 cards, with 13 cards of each of four suits (clubs, diamonds, hearts and spades). One card is drawn from a deck of playing cards and its suit is noted. It is replaced, then a second card is drawn. What is the probability two diamonds are drawn?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{16}$ **D** $\frac{1}{17}$ **E** $\frac{3}{52}$

10B **4** Repeat the experiment from question **3** but, this time, assume the first card is not replaced before the second card is drawn. What is the probability that two diamonds are drawn?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{16}$ **D** $\frac{1}{17}$ **E** $\frac{3}{52}$

10C **5** The following Venn diagram shows the number of students who play chess and badminton.



One of these students is selected at random. What is the probability that they play chess?

A $\frac{1}{15}$ **B** $\frac{12}{19}$ **C** $\frac{1}{3}$ **D** $\frac{4}{15}$ **E** $\frac{7}{45}$

10C **6** Based on the following incomplete two-way table, what is Pr(A)?

	A	**A′**	**Total**
B	12		24
B′		16	
Total			50

A 0.44 **B** 0.5 **C** $\frac{6}{11}$ **D** $\frac{14}{25}$ **E** 0.48

10D **7** In how many ways could three out of a total of seven playing cards be arranged in a row?

A 7! **B** 3! **C** $\frac{7!}{3!}$ **D** $\frac{7!}{4!}$ **E** 4!

10E **8** How many ways are there to choose 10 student representatives if there 120 students in the year level?

A $\frac{120!}{110!}$ **B** $\frac{120!}{110!10!}$ **C** 120! **D** $\frac{120!}{10!}$ **E** $\frac{110!}{10!}$

10F **9** Based on the Venn diagram in question **5**, what is the probability that a student plays badminton given that they also play chess?

A $\frac{1}{3}$ **B** $\frac{7}{45}$ **C** $\frac{1}{15}$ **D** $\frac{3}{7}$ **E** $\frac{1}{5}$

10F **10** For two events, A and B, it is known that $\Pr(A) = 0.45$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.85$. What is $\Pr(A|B)$?

A $\frac{1}{3}$ **B** $\frac{1}{4}$ **C** $\frac{1}{5}$ **D** $\frac{2}{3}$ **E** $\frac{2}{5}$

10G **11** If A and B are independent events with $\Pr(A) = 0.7$ and $\Pr(B) = 0.4$, then what is $\Pr(A \cup B)$?

A 0.28 **B** 1.1 **C** 0.55 **D** 0.82 **E** 0.3

10G **12** The two-way table on the right shows the data gathered from 50 repetitions of a scientific experiment.

Which pair of events is independent?

A 'Cold' and 'Success'

B 'Warm' and 'Inconclusive'

C 'Hot' and 'Failure'

D 'Hot' and 'Success'

E 'Warm' and 'Failure'

	Cold	Warm	Hot	Total
Success	12	8	0	20
Inconclusive	4	5	1	10
Failure	3	12	5	20
Total	19	25	6	50

Short answer

10A **1** A ten-sided die (with sides numbered 1–10) is rolled and the number facing up is recorded.

a List the sample space.

b If A is the event that the number facing up is odd and B is the event that the number facing up is a multiple of 3, then list the elements in the set:

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$ **v** $A \cap B'$

10A **2** A bag contains the letters of the alphabet. One letter is randomly chosen, whether it's a vowel or consonant is recorded, and then the letter is replaced. This is repeated another two times. Draw a tree diagram to find the probability of choosing:

a no vowels **b** at least one vowel

c three consonants **d** no more than two consonants.

10B **3** A three-digit number is made using the digits 0 to 9. The digits can be repeated.

a Describe any restrictions on the formation of such a three-digit number.

b How many three-digit numbers are possible?

c What is the probability the number will be 111?

d What is the probability that all the digits in the number will be the same?

10B **4** The experiment in question **3** is repeated, this time not allowing any repetition of the digits.

a Describe any restrictions on the formation of the number

b How many numbers are possible in this case?

c What is the probability the number will be 107?

d How many numbers with a value greater than 500 are possible?

10C **5** Sets A, B and C overlap, and can be described as follows:

- $n(A) = 14$
- $n(B) = 13$
- $n(C) = 15$
- $n(A \cap B) = 6$
- $n(B \cap C) = 5$
- $n(A \cap C) = 7$
- $n(A \cap B \cap C) = 4$

Draw a Venn diagram to display this information.

10C **6** This table shows the number of iPhone users in a group of students.

State the number of:

a Year 8 students who do not use an iPhone

b students who do not use an iPhone

c Year 10 students

d students who use an iPhone.

	Year 8	Year 10	Total
iPhone	20		
Other type of phone		12	
Total	35		82

10D **7** **a** What is the probability that an arrangement of six of the letters of the word PROBABILITY begins with a vowel?

b What is the probability that an arrangement of six of the letters of the word PROBABILITY begins with four vowels in any order?

c What is the probability that an arrangement of six of the letters of the word PROBABILITY begins with four vowels in alphabetical order (AIIO)?

10E **8** In a cricket match, there are eleven players on each team. The team consists of the captain plus a random combination of ten of the eight senior players and seven junior players from the cricket club.

a Determine the probability that four junior players and six senior players are selected.

b Determine the probability that all seven junior players are selected.

c What are the restrictions on the numbers of junior and senior players that could make up the team?

d What is the most likely number of senior players in the team out of all the possible team selections?

10F **9** Two pens are taken from a box containing four blue pens, five black pens and two red pens. Calculate the probability that:

a the second pen is red, given the first pen is blue

b the second pen is red, given the first pen is black

c the two pens are the same colour

d the two pens are different colours.

10F **10** A letter is chosen at random from the word CONDITIONAL. Find the probability that the chosen letter is:

a C

b N

c O, given that it is a vowel

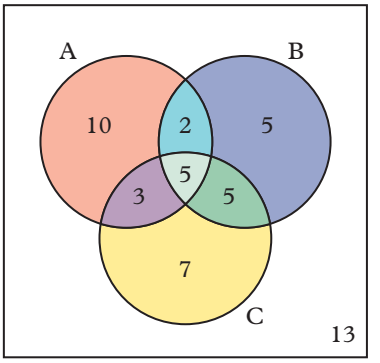
d A vowel, given that it is O

e A vowel, given that it is adjacent to an N.

10G **11** For two events, A and B, it is known that $\Pr(A) = \frac{1}{6}$ and $\Pr(A \cap B) = \frac{1}{12}$. If A and B are independent events, then find:

a $\Pr(A|B)$ **b** $\Pr(B)$ **c** $\Pr(A \cup B)$

10G **12** Consider the Venn diagram on the right.



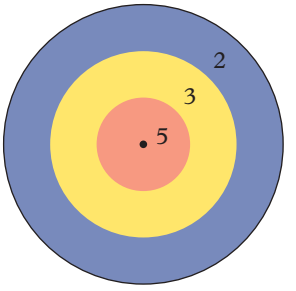
a Calculate the probability of:

i A **ii** $A \cap B$ **iii** $A \cup B$ **iv** $B' \cap C$

b Which pair of events is independent? Justify your answer.

Analysis

1 An archery target consists of an inner circle of radius 10 cm, encircled by two rings, each with width 10 cm. An arrow which lands in the outermost ring is worth 2 points, an arrow which lands in the middle ring is worth 3 points, and an arrow which lands in the inner circle is worth 5 points, as shown in the diagram.



An archer shoots two arrows, both of which hit this target.

a Find the fraction of the target area that is taken up by the region worth:

i 5 points **ii** 3 points **iii** 2 points.

b Write the sample space for the sum of the two scores from the archer's two shots.

Assume that each arrow has an equally likely chance of hitting anywhere on the target.

c Find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 5 points **iii** at least 8 points.

It is not realistic to assume that the archer has an equally likely chance to hit any point on the target. Instead, assume that, for each shot, the archer has a 20% chance of hitting the outermost ring, a 50% chance of hitting the middle ring, and a 30% chance of hitting the inner circle.

d Find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 6 points **iii** no more than 5 points.

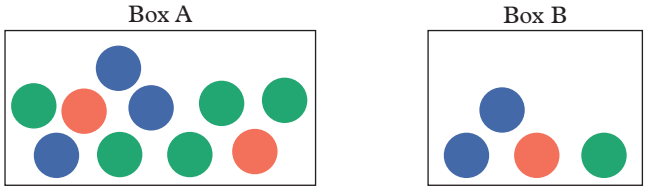
It is reasonable to think that the number of points the archer scores on the second shot is dependent on their score from the first shot. Assume that the archer learns from their first shot and will therefore score at least as many points on the second shot as they did on their first shot.

If the archer scores 2 points on their first shot, then the probabilities for their second shot remain 20%, 50%, and 30% for scoring 2, 3, or 5 points, respectively. However, if the archer scores 3 points on their first shot, then they have an even chance of scoring 3 or 5 points on their next shot. If the archer scores 5 points on their first shot, then they will score 5 points again on their second shot.

e Under these conditions, find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 8 points **iii** at least 6 points.

2 Two boxes each contain balls. Box A contains 5 green, 2 red and 3 blue. Box B contains 1 green, 1 red, and 2 blue.



a If one box is chosen at random and then one ball is chosen at random from it, then find the probability that:

i box B was chosen

ii the ball is green

iii the ball is green and was chosen from box B

iv the ball is green or box B was chosen.

b If two balls are randomly selected, one from each box, then find the probability that:

i both balls are red

ii at least one red ball is chosen

iii two red balls are chosen, given that at least one red ball was chosen.

c If one box is chosen at random and then two balls are chosen from that box, without replacement, find the probability that:

i both balls are blue

ii both balls are blue, given that box B was chosen

iii box B was chosen, given that both balls are blue.

Both boxes are emptied onto a table and the balls are jumbled. The same number of each colour ball as there were originally are randomly placed back into Boxes A and B.

d How many possible combinations of each colour ball are there that could have been placed into Box A?

e **i** Find the probability that the green balls in Box A are the same green balls which were originally in the box.

ii Find the probability that the red balls in Box A are the same red balls which were originally in the box.

iii Find the probability that the blue balls in Box A are the same blue balls which were originally in the box.

iv Find the probability that all of the balls in Box A are the same as they were originally.

f Repeat parts **d** and **e** with Box B.

g Comment on your answer to part **f**.

h Find the probability that all three sets of coloured balls in Boxes A and B are different to what they were originally. Why is this the most likely outcome?

Algorithmic Thinking

Algorithmic thinking: Statistics and probability

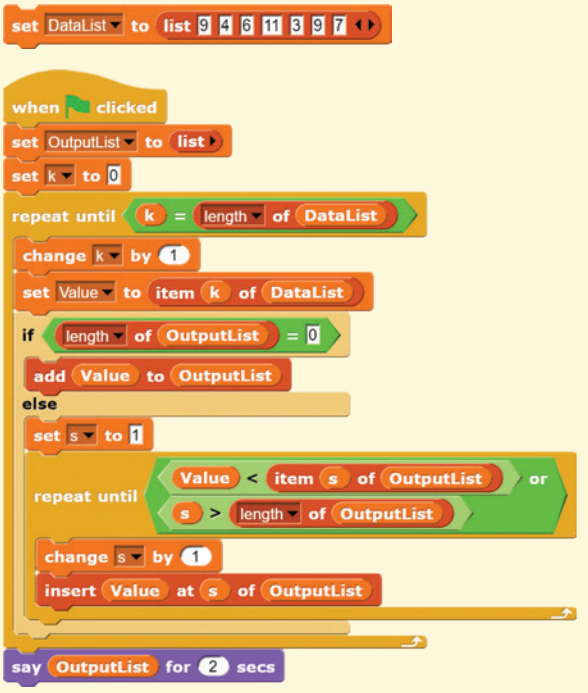
An algorithm is a set of precise instructions that when followed will achieve a goal or solve a problem. Once the actions in an algorithm have been identified, it may be possible to encode the algorithm in a coding language so it can be automated by a computer or machine.

The term iteration refers to repetitive actions, as shown in the algorithm defined using pseudocode in Example 1, which uses the Repeat until directive. When writing pseudocode, actions that need to be repeated are grouped together and indented. Some algorithms contain actions that are only done if certain conditions are true. These are called conditional actions. In pseudocode, these conditional actions can be defined using the If, Then, Else or Endif directives. Indentation is used to group the actions to be done, depending on the result of the conditional logic result (as shown in Example 1). Iteration can also be controlled by conditional logic such as Repeat until, which stops when the condition becomes true. Multiple conditions can be combined in brackets.

Example 1

This algorithm uses the Repeat until directive and conditional logic.

Input a list of numeric values called DataList
Create an empty list called OutputList
Set the variable k to 0
Repeat until k equals the length of DataList:
    Add 1 to k
    Set variable Value to the k^{th} element of DataList
    If OutputList is empty then:
        Set first element of OutputList to Value
    Else:
        Set the variable s to 1
        Repeat until (Value < s^{th} item of OutputList
                OR s > length of OutputList):
            Add 1 to s
        Insert Value into position s of OutputList
Report OutputList



Example 1 is coded in Snap! on the right using the variables k, s and Value and the lists DataList and OutputList.

Task 1

1 Where are the conditional statements and the iteration statements in Example 1?

2 What is the purpose of the conditions in each of the Repeat until directives in Example 1?

3 In what order is the algorithm in Example 1 getting each value from the list variable DataList?

4 What is the algorithm in Example 1 doing with the values in DataList to build the values of OutputList?

Note: you can check your answers by coding the algorithm example in Snap! or another coding language.

Task 2

The collection and processing of data for statistical analysis by researchers follows an algorithm with the following basic steps.

1 **Collect data:** Survey or observation

2 **Identify data:** Numeric or categorical data

3 **Sort numeric data:** Stem-and-leaf plot

4 **Analyse data:** Measures of centre and spread

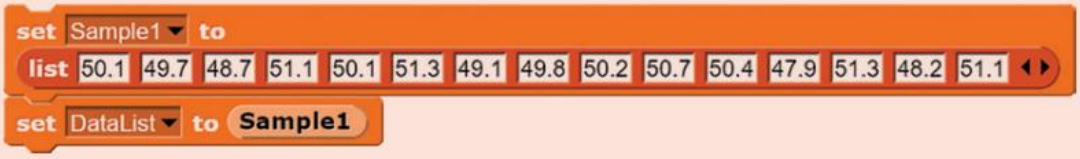
The butterfly Vanessa Atalanta, more commonly known as the red admiral butterfly, is a medium-sized butterfly with black wings, orange stripes and white spots. It has a wingspan of around 50 mm.

A butterfly researcher observes and records a sample of 15 random observations of Vanessa Atalanta butterfly wingspans from a small regional area.

The researcher records her observations of the wingspans in mm in a list variable called Sample1.

Sample1 = {50.1, 49.7, 48.7, 51.1, 50.1, 51.3, 49.1, 49.8, 50.2, 50.7, 50.4, 47.9, 51.3, 48.2, 51.1}

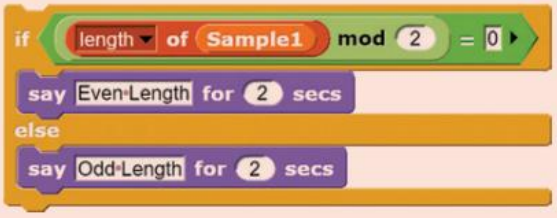
1 Use the wingspan data collected by the butterfly researcher in the algorithm from Example 1. The following code shows setting up the data in Snap!



What is the result created in OutputList?

2 Using ideas from Example 1, create an algorithm using conditional actions in pseudocode that will find the minimum, median and maximum values for the researcher's butterfly data.

3 The code below shows conditional checking for the reporting of even or odd lengths for the list variable Sample1 in the Snap! coding language using the mod function. Coding languages use the mod function to find the integer remainder after division. Try this code out in Snap! or another coding language.



4 The researcher adds an extra value to the Sample1 data to make an even number of observation values. Does your algorithm still work?

5 **CHALLENGE:** Code your algorithm to find the minimum, median and maximum values of the wingspan observations of odd or even length in Snap! or another coding language.

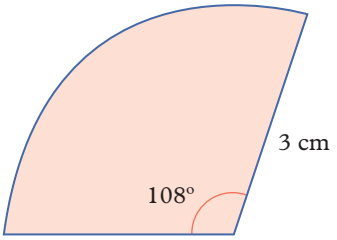
6 **CHALLENGE:** Using ideas from Example 1 and this task, modify your algorithm to find the lower quartile (Q_1) and upper quartile (Q_3) values to form the five-number summary for the observations.

Semester 2 review

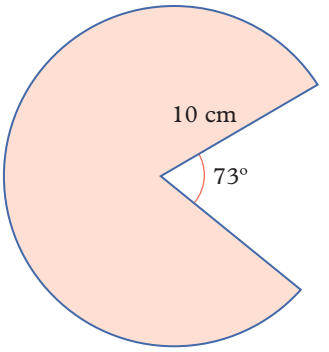
Short answer

1 Determine the area of each of the following shapes, correct to two decimal places.

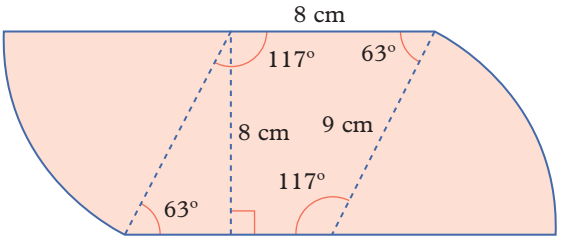
a



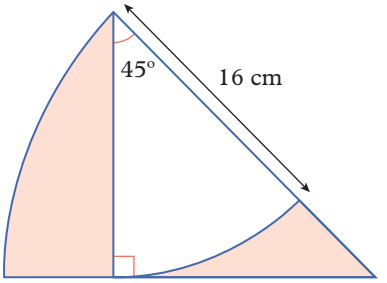
b



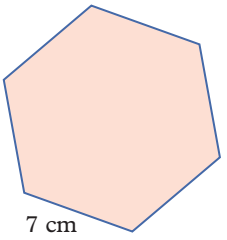
c



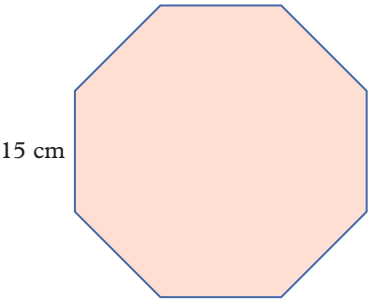
d



e



f

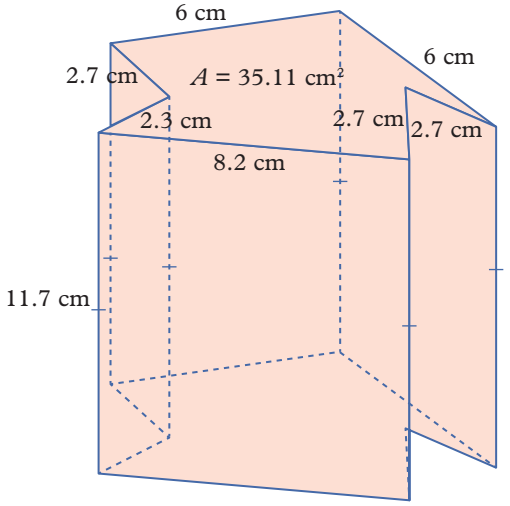


2 For each of the following objects, determine:

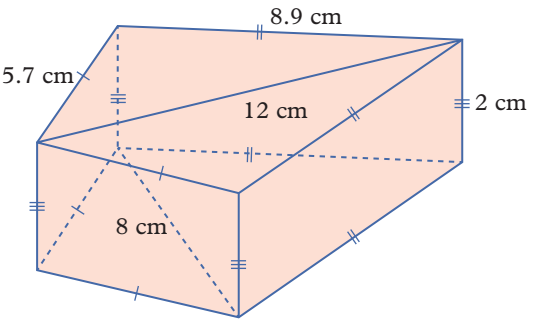
i the surface area **ii** the volume.

Write your answers correct to two decimal places where required.

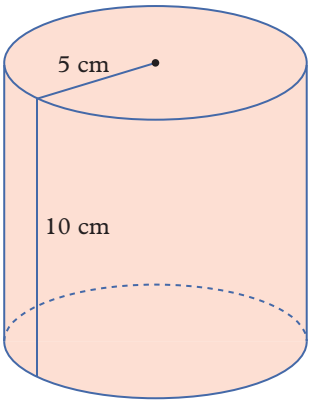
a



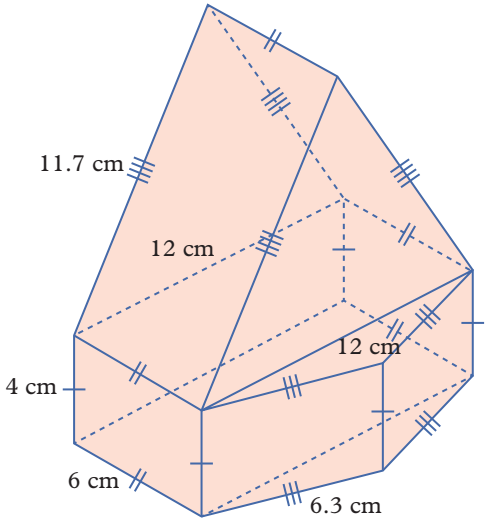
b



c



d



3 The equatorial circumference of the sun is 4 370 005.6 km. A website states the equatorial circumference of the sun as 4.4×10^6 km.

a Calculate the absolute error of this estimation.

b Calculate the percentage error.

4 Calculate the area of the image if the perimeter of the shapes in question **1** parts **e** and **f** have been dilated by the following scale factors:

i 3 **ii** $\frac{1}{2}$

5 Use your answer to question **2** and scale factor to find the surface area and volume of the solids if they are dilated by a scale factor of 1.5. Correct your answers to two decimal places.

6 For each of the following:

a write each of the numbers in scientific notation

b state the order of magnitude of each number

c determine the position of values on a base-ten logarithmic scale, relative to 1, correct to three decimal places.

i 1793.85 **ii** 5 105 232.79 **iii** 0.000 001 279 3

7 Determine if triangles with the following side lengths are acute, right-angled or obtuse.

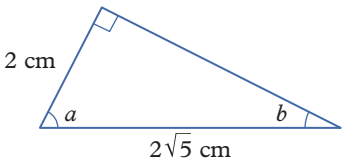
a 6 cm, 8 cm, 10 cm **b** 12 cm, 22 cm, 25 cm **c** 15 cm, 36 cm, 39 cm

8 For each of the following shapes, determine:

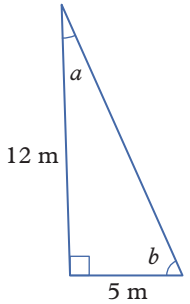
i the unknown length

ii the size of the unknown angles; correct your answers to the nearest degree

a



b



9 **a** The Forrestfield-Airport Link escalator in Perth, Western Australia, is the longest uninterrupted escalator in the Southern Hemisphere. The escalator is 15 m high and covers a horizontal distance of 35 m.

Determine the angle of elevation of the Forrestfield-Airport Link escalator correct to one decimal place.

b The Twelve Apostles are giant, natural limestone structures that are approximately 45 m tall, standing in the water off the Great Ocean Road in Victoria. If you were standing at the lookout on the adjacent 70 m high cliff, how far from the nearest of the Twelve Apostles would you be (measuring horizontally) if the angle of depression from you to the top of the nearest of the Twelve Apostles was 18°? Give your answer correct to the nearest metre.

10 a A hiker walks 2.5 km due north and then walks 1.2 km due west of that point.

i Determine the true bearing of the hiker's final position from the starting point, correct to the nearest degree.

ii Determine the true bearing of the starting point from the hiker's final position, correct to the nearest degree.

iii Determine the straight-line distance the hiker is from the starting point correct to one decimal place.

b A snorkeller swims, from a boat at point B, 35 m on a bearing of 120°T to point C, then continues for 20 m on a bearing of 200°T to point D.

i Draw a diagram showing the snorkeller's swimming path. Include all the bearings and distances provided.

ii How far south of the boat is the snorkeller at point C, correct to one decimal place?

iii How far east of the boat is the snorkeller at point C, correct to one decimal place?

iv How far south of point C is the snorkeller at point D, correct to one decimal place?

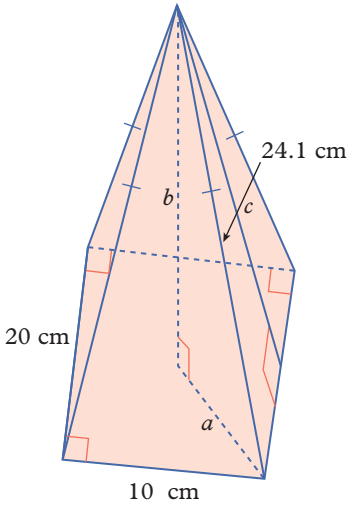
v How far west of point C is the snorkeller at point D, correct to one decimal place?

vi What is the true bearing from point D on which the snorkeller must swim in a straight line to return to the boat, correct to the nearest degree?

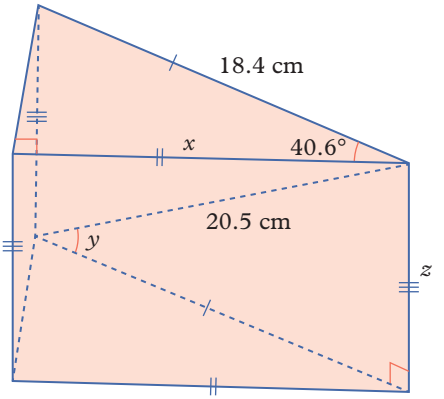
vii To return to the boat from point D, how far does the snorkeller need to swim along the bearing you wrote for part **vi**, correct to one decimal place?

11 Determine the value of the each of the pronumerals in these diagrams. Write your answers correct to two decimal places where required.

a



b



12 Convert angles expressed in degrees to radians and angles expressed in radians to degrees.

a 72° **b** $\frac{3\pi}{4}$ **c** 225° **d** $\frac{7\pi}{18}$ **e** 330° **f** $\frac{13\pi}{12}$

13 Evaluate each of the following by first writing it in terms of a positive acute angle (the reference angle). Write your answer as an exact value.

a $\sin(120°)$ **b** $\cos\left(\frac{7\pi}{6}\right)$ **c** $\tan(315°)$ **d** $\sin\left(\frac{3\pi}{4}\right)$ **e** $\cos(300°)$ **f** $\tan\left(\frac{11\pi}{6}\right)$

14 Solve each of the following equations.

a $\sin(\theta) = -\frac{1}{2}, 0° \leq \theta \leq 360°$

b $\cos(\theta) = \frac{\sqrt{2}}{2}, 0 \leq \theta \leq 2\pi$

c $\tan(\theta) = \sqrt{3}, 0° \leq \theta \leq 360°$

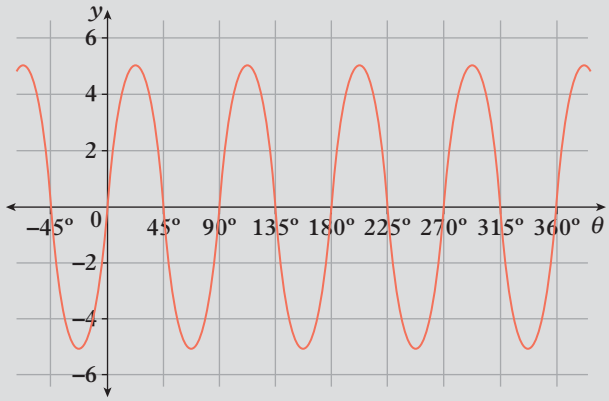
d $\sin(\theta) = -1, 0 \leq \theta \leq 2\pi$

e $\cos(\theta) = 1, 0° \leq \theta \leq 360°$

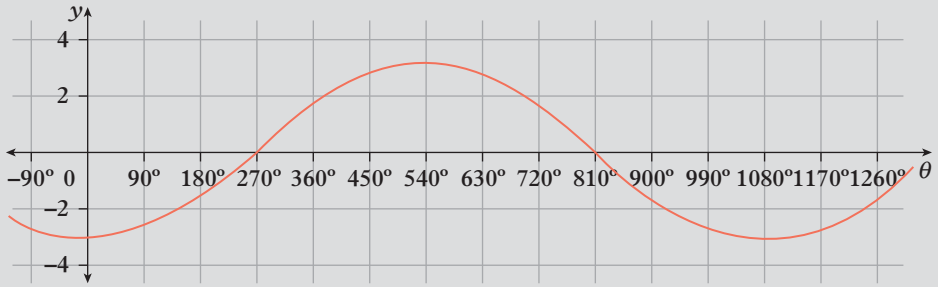
f $\tan(\theta) = -1, 0 \leq \theta \leq 2\pi$

15 State the amplitude and period of each of the following graphs.

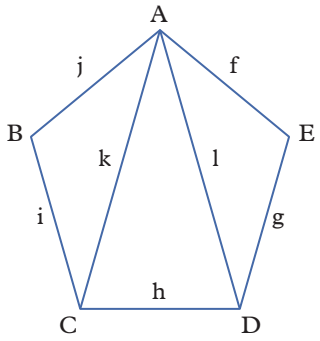
a



b



16 Fill out the table of vertex degrees for the network on the right.



Vertex	A	B	C	D	E
Degree					

17 For the network in question **16**, classify the sequences as walks, trails, circuits, closed walks or invalid walks.

a E, g, D, l, A, f, E

b A, k, C, h, D, h, C

c B, i, C, h, D, l, A

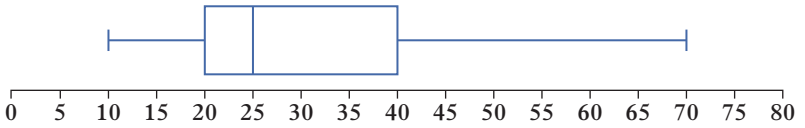
d E, g, C, k, A

18 Plan a return trip through North and South America, using every flight path on the map exactly once.

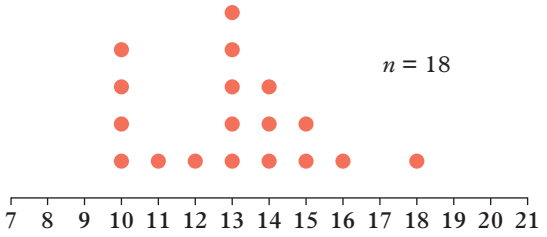
19 Determine the five-number summary and interquartile range for each of the following data sets.

a 68, 32, 62, 56, 59, 75, 38, 76, 50, 64, 48, 48, 55, 41

b



c

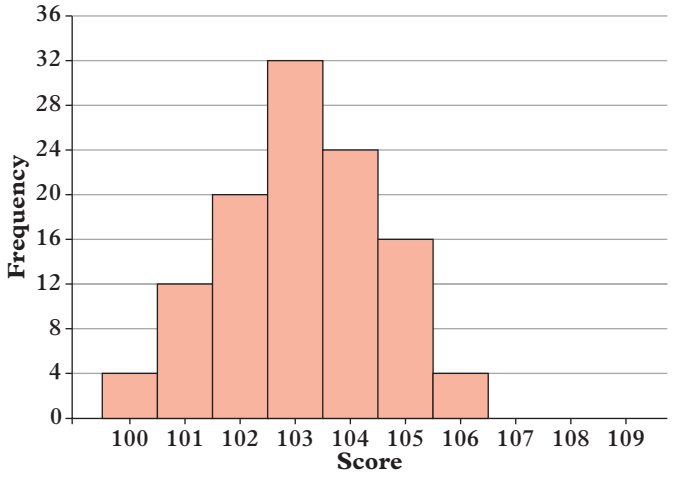


d

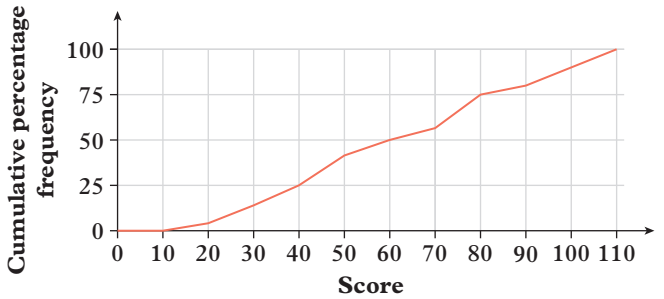
Stem	Leaf
1	2 8 9
2	0 1 3 4 7
3	1 2 2 2 4 5 6 7 8 9
4	0 0 0 1 2 3 4

Key: 1 | 2 = 1.2
$n = 25$

e



f



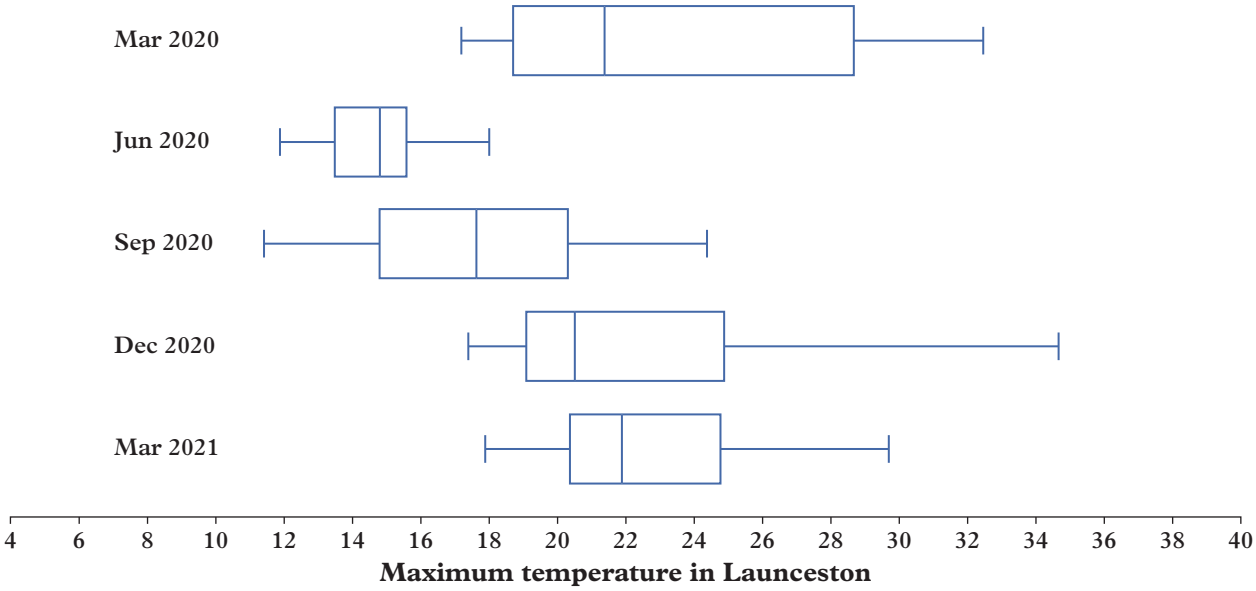
20 Construct a boxplot for each of the following sets of data. Remember to check for outliers.

a 105, 110, 123, 117, 121, 120, 125, 117, 122, 113, 118, 116, 120, 125, 122, 100, 110, 112, 115, 103

b 4, 1, 2, 7, 2, 4, 6, 21, 9, 4, 1, 1, 3, 6, 3, 2, 1, 5, 8, 4, 2, 4, 2, 5, 2

c 49, 42, 44, 45, 41, 43, 44, 47, 42, 43, 46, 58, 42, 49, 20, 45, 41, 44

21 Consider the parallel boxplots shown below and state whether each of the following statements is true or false.



a The maximum daily temperatures in June 2020 were all less than the lowest of the maximum daily temperatures in March 2021.

b The median maximum daily temperature in March 2020 and March 2021 were both approximately 22°C.

c The maximum daily temperatures in December 2020 were positively skewed.

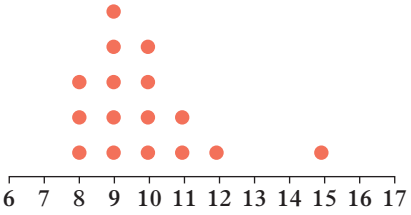
d In September 2020, approximately 50% of the days had maximum daily temperatures between 14°C and 20°C.

e In March 2020, approximately 25% of the days had maximum daily temperatures between 25°C and 30°C.

f Approximately 50% of the days in December 2020 had maximum daily temperatures greater than the median maximum daily temperature in September 2020.

22 Describe the shape of each of the following data displays.

a

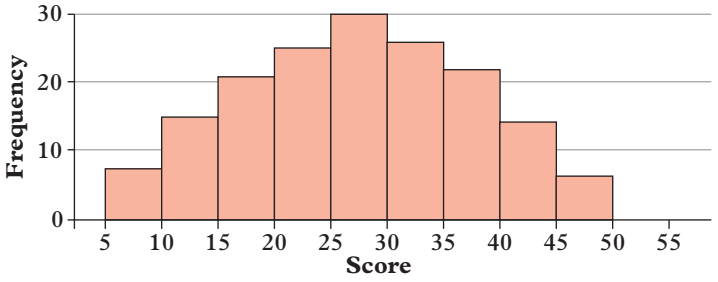


b

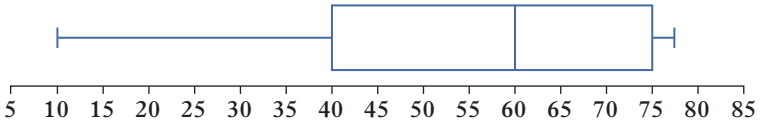
Stem	Leaf
1	9 0
2	1 2 4 6 8
3	0 1 2 3 4 5 5 6 7 8 8
4	0 0 1 1 4 7 9
5	1 2 3

Key: 1 | 2 = 1200

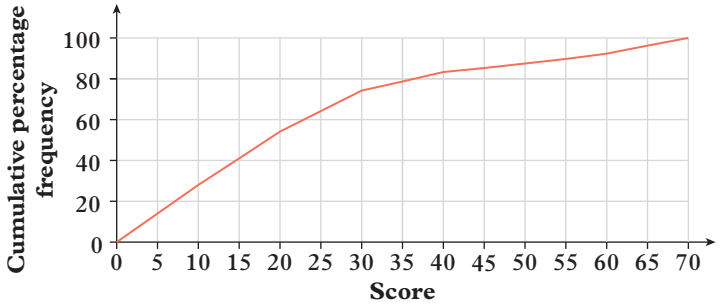
c



d

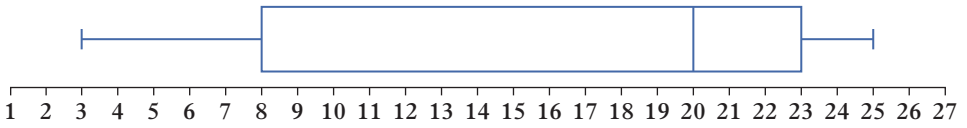


e

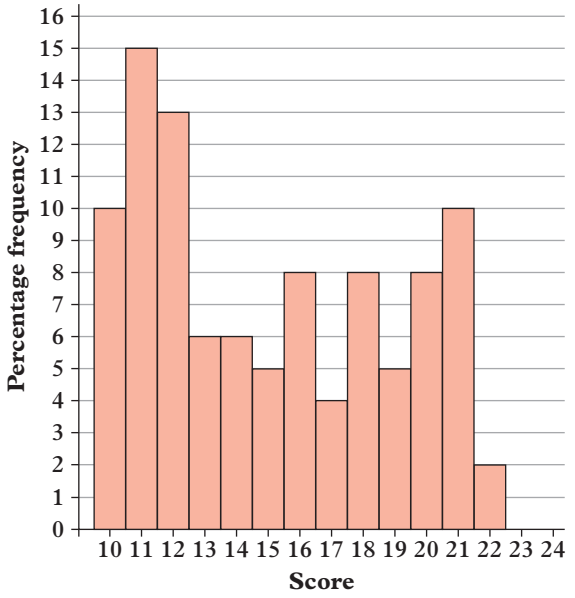


23 Construct a cumulative frequency polygon (ogive) for each of the following.

a

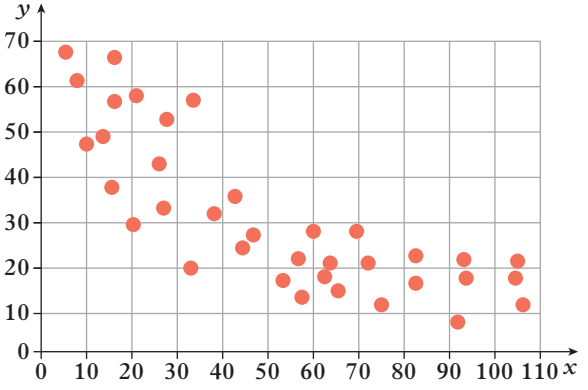


b

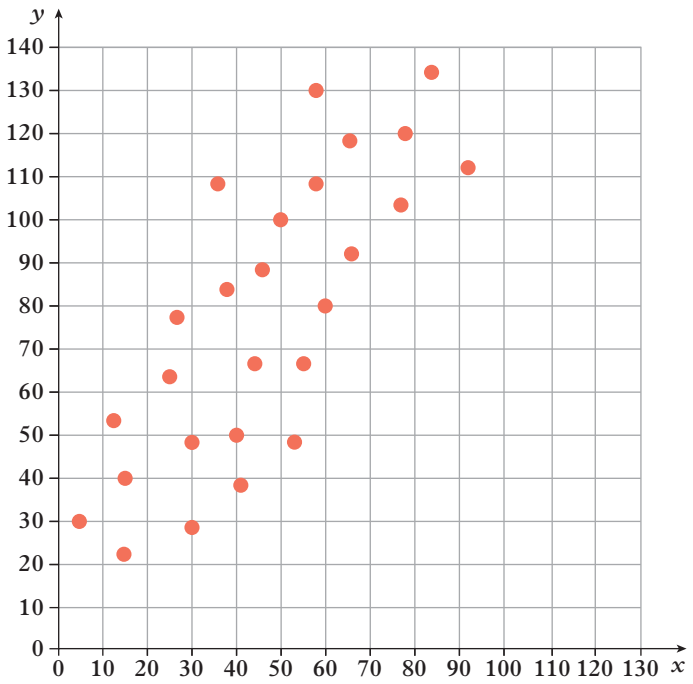


24 For each of the following scatterplots, describe the correlation between the variables in terms of strength and direction.

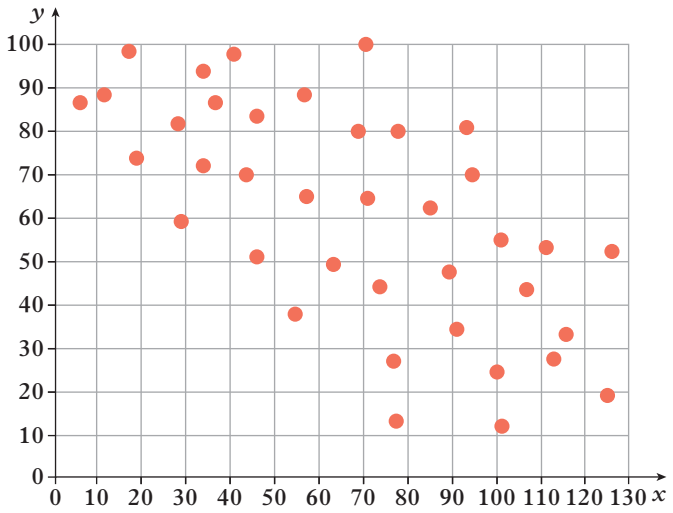
a



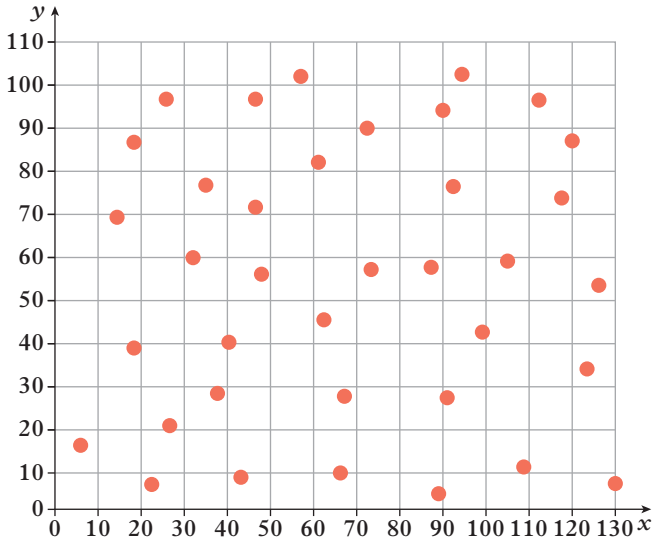
b


y
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
0 10 20 30 40 50 60 70 80 90 100 110 120 130 x

c


y
100
90
80
70
60
50
40
30
20
10
0
0 10 20 30 40 50 60 70 80 90 100 110 120 130 x

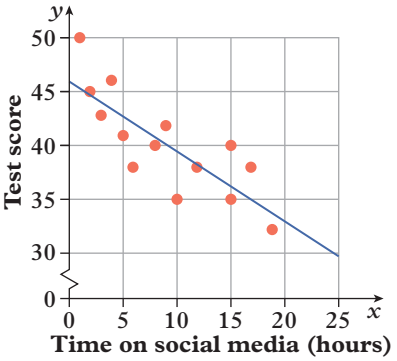
d


y
110
100
90
80
70
60
50
40
30
20
10
0
0 10 20 30 40 50 60 70 80 90 100 110 120 130 x

25 Construct a scatterplot for the following data set.

x	2	5	4	7	3	8	0	6	6	4
y	7	9	6	10	8	7	6	7	9	5

26 The test scores out of 50 of a group of students have been plotted against the time, in hours, those students spent on social media during the two days prior to the test.



a Use the line of best fit above to predict:

i a student's test score if the student spent 20 hours on social media in the two days before the test

ii a student's test score if the time the student spent on social media is 11 hours

iii the time spent on social media if the student's test score is 44

iv the time spent on social media if the student's test score is 30.

b For each answer in part **a**, state whether the prediction is reliable or not.

27 Decide whether each of these surveys will provide fair or biased results, giving reasons for your answers:

a Determine the most popular video game by asking every fifth competitor of the 128 competitors in a video game tournament (in which they are all playing the same game) what their favourite game is

b Ask a random sample of 50 students from a school with 2000 students whether a tie should be required to be worn with their school uniform, in order to determine student opinion about whether ties should be included in the school uniform policy or not

c Ask a random sample of 1000 people in Queensland (population 5 million) whether they think Queensland should adopt Daylight Saving Time

d Ask everyone in Australia what languages they speak in order to determine the most spoken language in the Southern Hemisphere.

28 Explain how each of the following headlines might be misrepresenting the given statistic.

a 'The majority of people prefer name brands to generic brands.' (51% of people surveyed said they preferred name brands to generic brands.)

b '9 out of 10 dentists recommend our toothpaste.' (10 dentists were surveyed, nine of whom are sponsored by the toothpaste company.)

29 For each of the following:

i decide whether the data collection method is fair

ii decide whether the interpretation of the data is fair

iii if appropriate, provide a suggestion to improve the survey.

a Peta wanted to know what would make more people travel on public transport. She asked every fifth person entering her local shopping centre in a five-hour period on a busy Saturday, 'What would make you more likely to take public transport?' Of the people surveyed, 60% responded that they would take public transport if it was more reliable and ran more frequently. Peta concluded that more people would take public transport if it was more reliable and ran more frequently.

b Rob wanted to know the average height of professional basketball players. He researched the heights of the players of all 30 teams in the NBA and calculated an average height of 2.01 m. Rob concluded that the average height of a basketball player is 2.01 m.

30 Calculate the mean ($\bar{x}$), sample standard deviation (s) and population standard deviation (σ) for each of the following data sets by hand. Then use a calculator to write your answers correct to two decimal places.

a 32, 36, 40, 43 **b** 20, 18, 28, 21, 24 **c** 4, 4, 4, 4, 4, 4, 4, 4, 4, 4

31 Simone was organising her book collection, which contained 65 books. She found there were 40 fiction books, and 24 books that were part of a series of which 14 were fiction.

a Construct a two-way table to represent this information.

b Convert the two-way table into a Venn diagram.

c Determine the probability of each of the following:

i Simone randomly selects a fiction book that is not part of a series.

ii Simone randomly selects a non-fiction book that is part of a series.

iii Given that Simone has selected a fiction book, it is part of a series.

iv Given that Simone has selected a book that is not part of a series, it is non-fiction.

32 A printing company is hired by two other companies to print trading cards. Company A orders twice as many cards to be printed as Company B. Due to differences in the trading card designs, there is an 8% chance that any one card for company A is misprinted compared to the 4% chance of a misprinted card for company B. A printed trading card is selected at random from a recent print run.

a Construct a tree diagram to represent the situation.

b Determine the probability of each of the following:

i The card is for company A and is misprinted.

ii The card is for company B and is not misprinted.

iii Given that the card is for company A, it is misprinted.

iv Given that the card is misprinted, it is for company A.

33 While driving, Fergus is listening to his playlist. It contains 52 tracks and Fergus knows that three songs will play at random without repeats before he reaches his destination. The playlist contains songs from four albums:

- album A has 13 tracks
- album B has 11 tracks
- album C has 12 tracks
- album D has 16 tracks.

Fergus is thinking about the probability of tracks from album A playing or not playing.

a Construct a tree diagram to represent the situation.

b Determine the probability of each of the following:

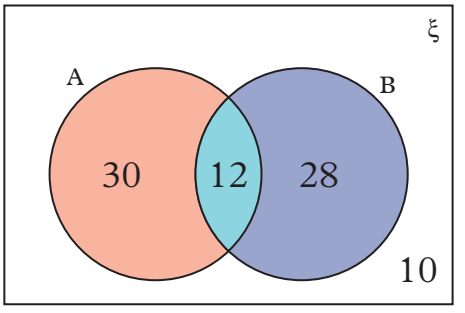
i All three songs that play are from album A.

ii Two songs that play are from album A.

iii Given that the first song that plays is from album A, the next two songs are not from album A.

iv Given that the first two songs that play are not from album A, the third song is from album A.

34 Consider the Venn diagram shown on the right. Determine:



a $\Pr(A)$

b $\Pr(B')$

c $\Pr(A \cap B)$

d $\Pr(A \cup B')$

e $\Pr(A|B)$

f $\Pr(B|A')$

35 Determine if events A and B in each of the following situations are independent or dependent.

a $\Pr(A) = 0.16$ $\Pr(B) = 0.25$ $\Pr(A \cap B) = 0.4$

b $\Pr(A) = \frac{2}{3}$ $\Pr(B) = \frac{5}{6}$ $\Pr(A \cap B) = \frac{5}{9}$

c $\Pr(A) = \frac{7}{15}$ $\Pr(A|B) = \frac{7}{15}$ $\Pr(A \cap B) = \frac{1}{3}$

d $\Pr(B) = 0.625$ $\Pr(B|A) = 0.35$ $\Pr(A \cap B) = 0.35$

36 Marion has a rock collection. She has n polished rocks and n unpolished rocks that she likes to display in different ways.

a In how many ways can Marion:

i display the rocks in a single row?

ii display the rocks in two rows, with all the polished rocks in the front row and the unpolished rocks in the back row?

iii display the rocks in two rows, with all the polished rocks in one row and the unpolished rocks in the other row?

Marion needs to nominate her two favourite rocks for a competition. Since she likes all of her rocks equally, she decides to choose her favourites randomly. Suppose all the rocks are equally likely.

b In how many ways can Marion choose two rocks to be her favourites?

c What is the probability of Marion choosing a polished rock and an unpolished rock as her favourites?

d What is the probability of Marion choosing two unpolished rocks as her favourites?

Analysis

1 Erica is planning a hike through a local national park. She plans to:

- drive to the carpark, at point P
- then walk on a bearing of 036.9°T for 10 km towards the lower bridge, at point L
- then walk on a bearing of 108.4°T for 6.3 km towards the upper bridge, at point U
- then walk on a bearing of 233.1°T for 10 km towards the new bridge, at point N.

From the new bridge there is then a 4 km track due west back to the carpark.

a Draw a diagram to show Erica's hike plan.

b Determine the length of the planned hike.

c What is the average minimum speed Erica must walk to make it back to her car in 6 hours?

d Erica reaches the upper bridge and sees a warning sign that says it is a 50 m drop to the ground below. From the upper bridge, Erica can see a kangaroo on the ground at an angle of depression of 12°. If Erica's eye height is 172 cm, determine the horizontal distance to the kangaroo, correct to the nearest metre.

e The park has multiple tracks that can be taken from the carpark. The tracks vary in difficulty. Hikers often mark which way they went on a signpost and this has allowed the park rangers to determine the following:

- From point P, 20% of hikers walk to point R and 80% walk to point L.
- From point L, 5% of hikers walk to point U and 95% walk to point A.
- From point R, 10% of hikers walk to point U and 90% walk to point A.
- They all then follow tracks that lead to point N.

Determine the probability that:

i a randomly selected hiker took the path through points P, R, U and N

ii a randomly selected hiker took the path through points P, L, A and N

iii a randomly selected hiker took the path through points P, R, A and N

iv given that they walked to point U, a randomly selected hiker took the path through points P, L, U and N.

f Draw a network that represents the paths between the points P, R, L, U, A and N.

g **i** Is the network from part **f** planar?

ii Why is there no Eulerian circuit in the network? Is there an Eulerian trail?

iii Suppose there is a path from all points to N. Is the corresponding network planar?

2 A teacher decided to retest the class two weeks after a recent test to see how much the students were able to remember. The scores the 24 students scored on the two tests are given in the table below.

The teacher hoped to be able to predict the retest scores from the test scores.

Test score (%)	95	90	90	85	85	80	80	80	75	75	75	70	70	70	65	65	65	60	60	60	55	50	40
Retest score (%)	80	85	70	50	65	65	50	90	65	60	40	80	60	75	60	65	35	35	70	50	50	30	30

a State the independent variable.

b Construct a scatterplot of the scores from the retest (vertical axis) against the scores from the original test (horizontal axis).

c Describe the correlation between the scores for the retest and the scores for the original test.

d Add a line of best fit to your scatterplot from part **b**.

e **i** If a student scored 70% for the original test, use the line of best fit to predict the student's score for the retest.

ii State whether or not each prediction is reliable.

f The teacher claims that students who scored 75% or more originally did better on the retest than students who scored less than 75%. Evaluate the teacher's claim.

3 A special Ferris wheel with enclosed, air-tight carriages has been designed so that half of the wheel is submerged underwater. Passengers board the carriage at sea level, the Ferris wheel rotates upwards to its highest point, then back to sea level, then underwater to the deepest point, then back to sea level. A full rotation of the Ferris wheel takes one hour to complete. The deepest point the Ferris wheel can reach is 20 m.

a State the amplitude and period of the Ferris wheel's height in relation to time.

b Write the equation of the height, h m, of the carriage t minutes after boarding using an appropriate function.

c Sketch the graph of the height of the carriage against the time it takes the Ferris wheel to complete one full rotation.

d Determine the height the carriage is above sea level after 20 minutes:

i as an exact value **ii** correct to two decimal places.

e Determine, as an exact value, the depth the carriage is below sea level after 35 minutes.

f Determine the time it takes after boarding for the carriage to be $10\sqrt{2}$ m below sea level.

g People on a nearby boat watch the Ferris wheel go around. Describe the vertical and horizontal location of a carriage relative to the centre of the Ferris wheel 22.5 minutes after it was boarded:

i using exact values **ii** correct to two decimal places.

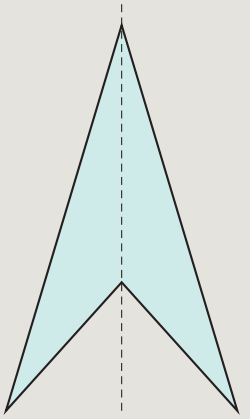
h Determine the distance the carriage has travelled during the time described in part **g**:

i as an exact value **ii** correct to two decimal places.

EXPLORATIONS 2

1 Thorn polygons

A *thorn polygon* is a shape that has at least one axis of symmetry and the maximum possible number of interior reflex angles. The symmetrical dart shown is an example of a thorn quadrilateral.



a How many reflex angles can a triangle have? What is another name for a thorn triangle?

b There are two different types of thorn pentagon: one with the reflex angles adjacent and one with them separated. Draw an example of each type.

c Explain why every thorn heptagon has at least one pair of adjacent reflex angles.

d Explain why no thorn octagon can have two obtuse angles.

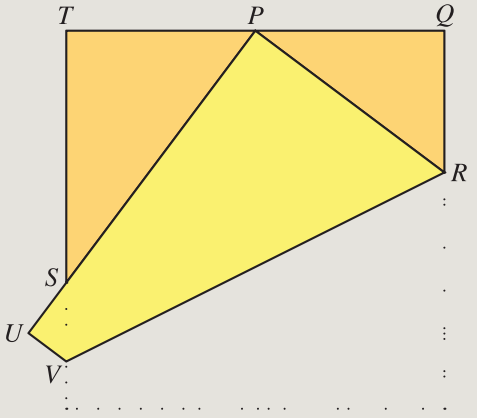
e Show that an n-sided thorn polygon has exactly $n - 3$ reflex angles. Hint: This requires two steps: 1) show that a symmetrical n-sided polygon with $n - 3$ reflex angles always exists, and 2) show that more than $n - 3$ reflex angles is impossible.

f Find the maximum number of sides in a thorn polygon in which no three consecutive angles are all reflex angles.

g Show that, for every $n \geq 3$, there exists an n-sided thorn polygon that can be bisected along its line of symmetry to form two congruent thorn polygons.

2 Know when to fold 'em

A square piece of origami paper has side length 1 unit. The bottom-right corner is folded to a point P on the top edge, as shown, creating triangles PQR, PTS and SUV.



a Suppose P is at the midpoint of the top edge of the square.

i Set up an equation to find the length of QR.

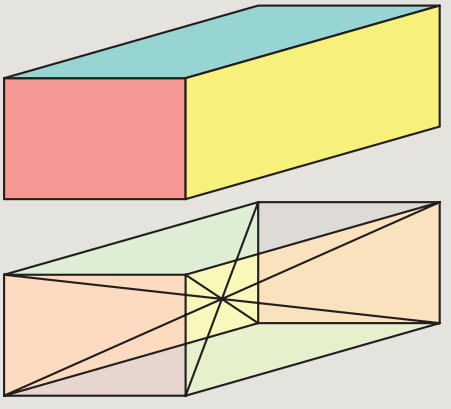
ii Use similarity to find the side lengths of triangle SUV.

iii Calculate the area of the folded yellow region of the paper.

b Find all possibilities for the length of PQ given that triangle SUV has side lengths in the ratio $7 : 24 : 25$.

c Show that, for any Pythagorean triple (a, b, c) with $a^2 + b^2 = c^2$, there are two different positions for P such that triangle SUV has sides in the ratio $a : b : c$. What is the second position for the ratio in part **a**?

3 Six pyramids



a A 4 cm × 6 cm × 12 cm rectangular prism is divided into six right rectangular-based pyramids, as shown. Each face of the prism is the base of one pyramid and the centre of the prism is the apex of each pyramid.

i Find the volume of each pyramid.

ii For each pyramid, find the sum of the lengths of its eight edges.

iii Find the surface area of each pyramid, to 2 decimal places.

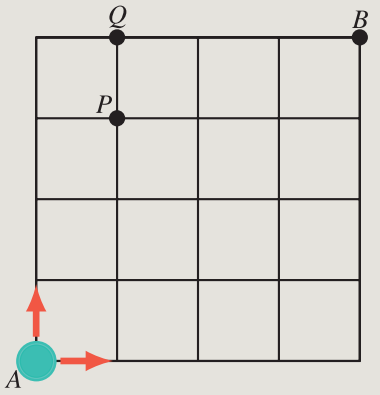
b Given an 85 cm × 132 cm × 720 cm rectangular prism, show that its six pyramids each have integer surface area, in square centimetres.

c Let (a, b, c) be any Pythagorean triple satisfying $a^2 + b^2 = c^2$ and define $u = 2a(4b^2 - c^2)$, $v = 2b(4a^2 - c^2)$ and $w = 8abc$.

Show that the six pyramids of a u cm × v cm × w cm rectangular prism each have integer surface area, in square centimetres.

4 Chance encounters

In the 4 × 4 grid shown, a counter is placed at A. Each second, the counter is moved along a grid line to an adjacent grid point, always increasing its distance from A. Where there are two possible moves, they are equally likely. After 8 seconds, the counter reaches point B.



a Explain why there is a one-in-four chance that the counter starting at A will be at P after 4 seconds.

b What is the probability that the counter starting at A will be at Q after 5 seconds?

A second counter is placed at B. Each second, it is moved along a grid line, always increasing its distance from B and with each move equally likely. The two counters start moving from A and B at the same time.

c Show on a diagram all the grid points at which the counters can meet. Explain why there are no others.

d What is the probability that the counters meet?

e The game is changed so that each counter has probability $\frac{1}{3}$ of moving vertically and probability $\frac{2}{3}$ of moving horizontally. Is it more or less likely that the counters will meet?

5 Algal bloom

Dr Al Goldblum discovered that a certain species of poisonous singled-celled algae has three different forms, called kirons, lirons and mirons. A kiron lives for one week and then changes into a liron. A liron also lives for one week and then changes into a miron. One week later, a miron divides into one kiron and one liron.

A spreadsheet is a useful way to track the numbers of kirons, lirons, mirons and total algae over time. Dr Al always takes measurements at the same time of the week: 12 pm on Wednesday.

a In one experiment, Dr Al started in week 1 with a single kiron in a flask of water.

i What will be the total number of algae in the flask in week 10?

ii In which week will the total number of algae first exceed 100?

b Show that, from week 6 onwards, the number of kirons is equal to the sum of the numbers of kirons one week earlier and five weeks earlier. Is this also true for lirons and mirons?

c In another experiment, Dr Al started in week 1 with an equal number of kirons and mirons, but no lirons. Some weeks later, there were exactly 100 algae in total. What is the smallest possible number of algae at the beginning of the experiment?

d Dr Al's goal is to develop a herbicide to kill the poisonous algae. Batch X31 destroys all lirons within one day, but it does not affect the other two forms. It remains active for several weeks.

i On Wednesday in week 11, the X31 herbicide is added to the flask that started with a single kiron in week 1. During which week will all forms of algae be completely eradicated?

ii If X31 is added to a lake infested with all three forms, what is the maximum number of days it will take to eradicate the algae?

Explorations inspired by the Australian Maths Trust's competitions and programs: www.amt.edu.au

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[STEAM project 1]

How can we use technology so that we improve the lives of people in the world's poorest nations?

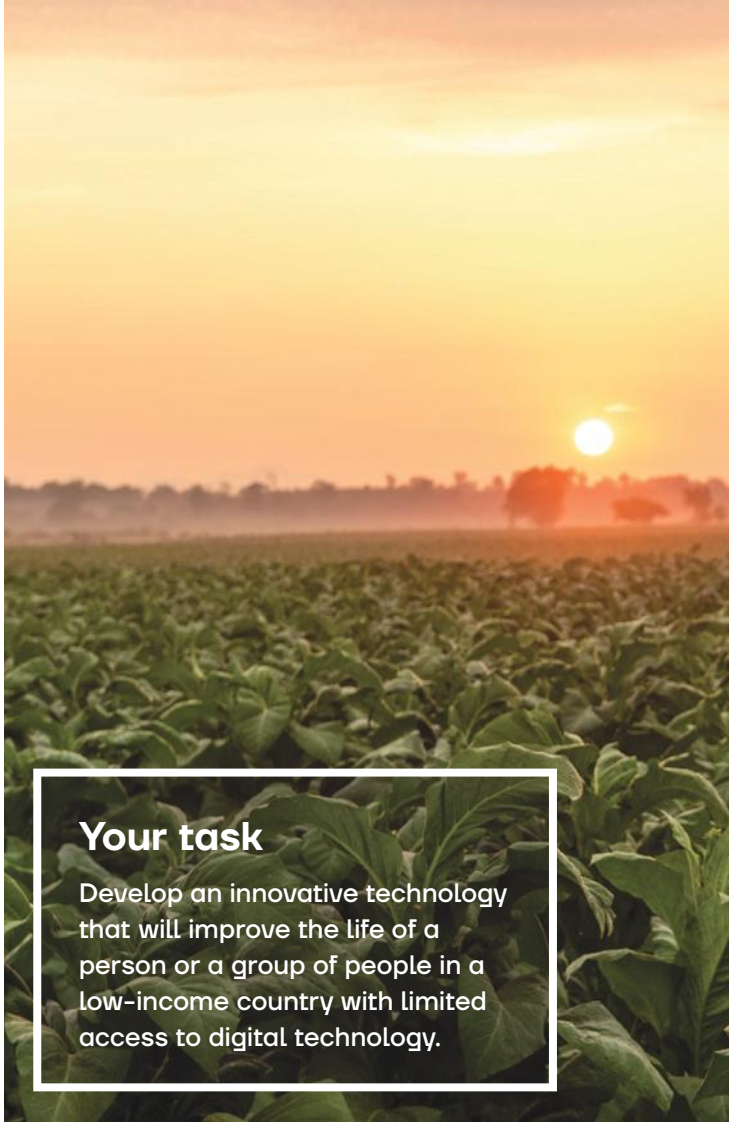
In Australia we are surrounded by technology every day. It is in the phones we use, the televisions we watch and the cars we drive. The term 'technology' is used for any machinery or equipment that applies the scientific knowledge we have discovered. Wheels and computers are both examples of technology.

In high-income countries, emergency response teams often rely on technological data supplied by electronic sensors to respond to natural disasters such as storms, fires and plagues. Drones might be used to conduct search and rescue operations. Doctors can use technology to remotely diagnose people who are sick and to perform operations that save lives.

At the end of 2017, the number of high-speed mobile subscriptions in member countries of the Organisation for Economic Co-operation and Development (OECD) reached a milestone: more subscriptions than the number of people. These mobile phones have been used to alert people to natural disasters, or to call for help in the event of floods and fires.

Technology is not just used for communication during natural disasters. It is also used to create medicine, improve farming practices and for education.

However, not everyone has access to technology.



Your task

Develop an innovative technology that will improve the life of a person or a group of people in a low-income country with limited access to digital technology.

Figure 1 Technology can be used to enhance and improve agricultural practices.

The digital divide

The term 'digital divide' is used to describe the gap between those who have access to digital technology – such as mobiles, computers and the internet – and those who do not. The Australian Bureau of Statistics has identified that almost 2.6 million Australians do not use the internet and cannot access technology in an emergency. Access is even lower in lower-income countries throughout Africa and Asia.

The OECD has identified that targeted innovation that uses technology can boost productivity, increase economic growth and help solve problems in society.

Figure 2 Technology has made attending a doctor's appointment easier and more accessible for people who may have difficulty attending in person.

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HUMANITIES

In Geography this year, you will be learning about the spatial variations in human wellbeing globally. You will need to explore a range of factors that lead to inequalities, such as social, political, economic and technological differences. In Economics and Business, you will explore variations in living standards between countries, and how living standards can be improved.

To complete this task successfully, you will need to research the initiatives of international governments and non-government organisations (NGOs) aimed at improving human wellbeing, particularly regarding health. You should consider how technology could be effectively accessed, resourced and used by a group of people to address their health concerns.

You will find more information on this in Chapter 5 'Inequalities in wellbeing' and Chapter 13 'Living standards' of *Oxford Humanities and Social Sciences 10 Australian Curriculum*.

MATHS

In Maths this year, you will extend your skills in representing, comparing and interpreting data. You will use digital technology to work with data, but also perform calculations by hand.

To complete this task successfully, you will need to find data to quantify the problem, to cost your interventions and to calculate a quantitative, evidence-based estimate of the likely benefits of your interventions. You will need to use skills in performing proportionality and other calculations with very large numbers, using scientific notation.

You will find relevant mathematical and statistical concepts in Chapter 9 'Statistics' of *Oxford Maths 10 Australian Curriculum*.

SCIENCE

In Science this year, you will learn how an understanding of evolution can contribute to the selection of desired traits (such as drought resistance) in plants and animals. You will also learn how genetic engineering can be used to develop medicines that will cure cancers and prevent disease.

To complete this task successfully, you will need to consider how the values and needs of different societies can influence the focus of scientific research. You will also need to consider the ethics of the technology that you will be offering to your selected individual or group of people.

You will find more information on this in Chapter 2 'Genetics' and Chapter 3 'Evolution' of *Oxford Science 10 Australian Curriculum*.

[STEAM project 1]

The design cycle

To successfully complete this task, you will need to complete each of the phases of the design cycle.

Discover

When designing solutions to a problem, you need to know who you are helping and what they need. The people you are helping, who will use your design, are called your end-users.

Consider the following questions to help you empathise with your end-users:

- Who am I designing for?
- What problems are they facing? Why are they facing them?
- What do they need? What do they not need?
- What does it feel like to face these problems? What words would you use to describe these feelings?

To answer these questions, you may need to investigate using different resources, or even conduct interviews or surveys.

Define

Before you start to design your innovative technology, you need to define the parameters you are working towards.

Define your version of the problem

Rewrite the problem so that you describe the group you are helping, the problem they are experiencing and the reason it is important to solve it. Use the following phrase as a guide.

'How can we help (the group) to solve (the problem) so that (the reason)?'

Determine the criteria

1. Describe the limitations in energy, communications, transport and support personnel that will need to be considered as part of the solution.
2. Describe how many copies of the solution prototype will need to be made to make a difference in the country you have chosen.
3. Identify who could pay for the construction of the solution prototypes.
4. Describe the social culture that is experienced by the individuals and groups who are affected by the problem. Why might some technologies be viewed as unwanted or even dangerous?

Ideate

Once you know who you're designing for, and you know what the criteria are, it's time to get creative!

Outline the criteria or requirements your technological design must fulfil (i.e. cost, size and weight for transportation, and cultural appropriateness).

Brainstorm at least one idea per person that fulfils the criteria.

Remember that there are no bad ideas at this stage. One silly thought could lead to a genius innovation!

Build

Each group member should select one design to draw. Label each part of the design. Include the material that will be used for its construction.

Include in the individual designs:

a a detailed diagram of the design

b a description of how it will change the life of your selected individual or group

c an outline of any similar designs that are already available to buy

d an outline of why your idea or design is better than others that are already available.

Present your design to your group.

Build the prototype

Choose one solution and build two or three prototypes. The prototype may be full size, or it may be a scale model (10 cm = 1 m).

Use the following questions as a guideline for your prototype solution.

- What materials or technology will you need to build or represent your prototype solution?
- What skills will you need to construct your prototype design? Does your group have these skills, or who can teach you those skills?
- How will you make sure your prototype design is able to be used by your selected individual or group? Will they need training?
- How will you display or describe the way the prototype design will work?

Test

Prototype 1

Use the scientific method to design an experiment that will test the effectiveness and strength of your first prototype solution. You will test the prototype more than once to compare results, so you will need to control your variables between tests.

What criteria will you use to determine the success of your prototype? Conduct your tests and record your results in an appropriate table.

Prototype 2

If your prototype will be used to help an individual, then you will need to generate a survey to test whether the prototype is appropriate for their use. (How would they use it? Would it make their work easier or harder? Would they consider buying it?)

Prototype 3

Use the information you have obtained from testing the first two versions to adapt your last prototype to be more effective and usable for the group you are helping. You may want to use the first two prototypes to demonstrate how the design has been improved over time.

Communicate

Present your design to the class as though you are trying to get your peers to invest in your designed solution.

In your presentation, you will need to:

- outline the situation of the country in which your selected individual or group lives
- outline the challenges faced by your selected individual or group
- include a working model or a detailed series of diagrams with a description of how the prototype of the solution will be used
- include a description of how you changed your design prototype as a result of testing or feedback
- include a description of how the design prototype will improve the lives of your selected individual or group.

Check your Student obook pro for the following digital resources to help you with this STEAM project:

Student booklet
This helpful booklet will guide you step-by-step through the project.

What is the design cycle?
This video will help you to better understand each phase in the design cycle.

How to manage a project
This 'how-to' video will help you to manage your time throughout the design cycle.

How to pitch your idea
This 'how-to' video will help you with the 'Communicate' phase of your project.

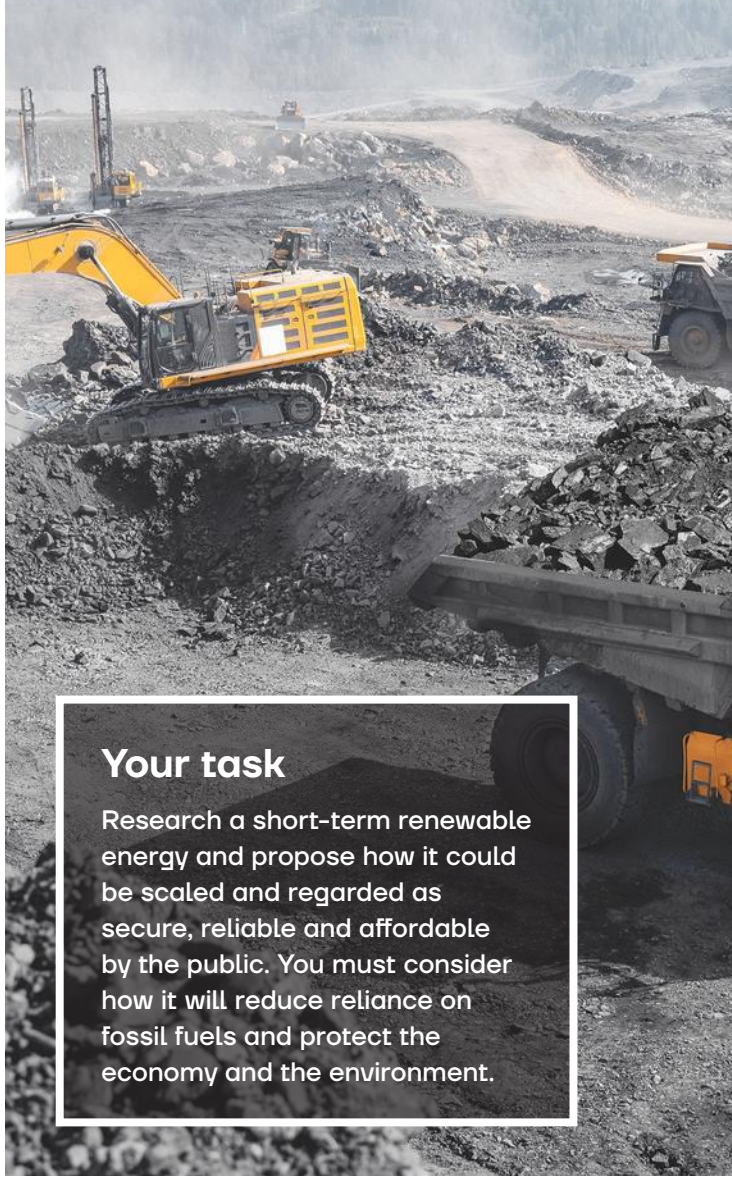
[STEAM project 2]

How can Australia reduce its reliance on fossil fuels so that we protect the environment and the economy?

Fossil fuels such as coal, oil and gas are made from fossilised, decomposed organisms that aged over millions of years in the Earth's crust. They contain carbon, which can be burned for energy. Due to the length of time it takes for these fuels to form as part of the carbon cycle, they are classified as long-term renewable sources of energy, sometimes called non-renewable.

Australia is a major user, producer and exporter of fossil fuels. Nearly 80 per cent of Australia's electricity is generated from coal and gas. Seventy-five per cent of coal mined in Australia is exported, making Australia the largest net exporter of this fuel in the world. In 2019, it was reported that Australia was the world's third-largest exporter of fossil fuels. Economically, the production and export of fossil fuels contributes hugely to Australia's GDP, and the mining industry is an important employer.

Coal emits higher amounts of CO_2 than oil or gas when used to produce energy. Measuring fossil fuel exports according to their potential to emit CO_2 makes Australia's carbon footprint per capita one of the largest in the world. This is contentious globally, particularly for nations most affected by a changing climate.



Your task

Research a short-term renewable energy and propose how it could be scaled and regarded as secure, reliable and affordable by the public. You must consider how it will reduce reliance on fossil fuels and protect the economy and the environment.

Figure 1 Most of Australia's energy is generated from coal. Almost 80 per cent of the coal produced in Australia is from open-cut mines.

Renewable alternatives

To protect both the environment and the economy, Australia needs to focus more on short-term renewable energy. When deciding on energy alternatives, it is important that energy supply be secure, reliable and affordable.

Short-term renewable energy sources include hydropower, solar power, wind power, bioenergy and ocean energy. Australia's landscape is suitable for many of these alternatives, but large investments in technology are required. As we invest more in the technology that makes short-term renewable energy possible, the better we get at making it, the more affordable it becomes and the more demand for renewable energy grows (a cyclical process).

Figure 2 A solar farm in Canberra. Solar energy is a source of renewable energy.

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HUMANITIES

In Geography this year, you will learn about environmental change and management. You will explore the environmental, technological and economic factors that have influenced the change and the consequences of human actions on the sustainability of the environment.

In Economics and Business, you will investigate how the performance of Australia's economy is measured and how Australia's economic growth has depended on natural resources. You will explore the impact that environmental policies can have on Australia's economy and living standards.

To complete this task successfully, you will need to understand how stakeholders such as governments, communities and businesses can work together to initiate environmental change and management plans that protect both the environment and the economy.

You will find more information on this in Chapter 2 'Changing and managing the environment' and Chapter 13 'Economic growth and productivity' of *Oxford Humanities and Social Sciences 10 Australian Curriculum*.

MATHS

In Maths this year, you will extend your skills in representing and interpreting data, including univariate, bivariate and multivariate data sets. This will include critical consideration of media reports that use statistics and present graphs. You will use digital technology to work with data but also perform calculations by hand.

To complete this task successfully, you will need to find data to quantify the problem, to cost your interventions and to calculate a quantitative, evidence-based estimate of the likely benefits of your interventions. You will need to have skills in performing proportionality and other calculations with very large numbers, using scientific notation.

You will find relevant mathematical and statistical concepts in Chapter 9 'Statistics' of *Oxford Maths 10 Australian Curriculum*.

SCIENCE

In Science this year, you will learn about the impacts of fossil fuel combustion reactions in the production of carbon dioxide and carbon monoxide. You will also examine how increased reliance on this form of energy has affected the way carbon cycles through Earth's spheres, and how the resulting increase in greenhouse gases (including carbon dioxide) has led to enhanced global warming, which is contributing to melting sea ice and permafrost, rising sea levels and an increased number of extreme weather events.

To complete this task successfully, you will need to consider how energy that is generated can be used efficiently.

You will find more information on this in Chapter 4 'Climate change' of *Oxford Science 10 Australian Curriculum*.

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[STEAM project 2]

The design cycle

To successfully complete this task, you will need to complete each of the phases of the design cycle.

Discover

When designing solutions to a problem, you need to know who you are helping (your end-users) and what they need.

Consider the following questions to help you empathise with your end-users:

- Who am I designing for? Will I be helping the government or members of the public?
- What problems are they facing? Why are they facing them?
- What do they need? What do they not need?

To answer these questions, you may need to investigate using different resources, or even conduct interviews or surveys.

Define

Before you start to design your solution for the potential replacement of fossil fuels, you need to define the parameters you are working towards.

Define your version of the problem

Rewrite the problem so that you describe the group you are helping, the problem they are experiencing and the reason it is important to solve it. Use the following phrase as a guide.

'How can we help (the group) to solve (the problem) so that (the reason)?'

Determine the criteria

1 What type of energy source are you trying to replace? How much of it is currently used and how is it used?

2 How will the renewable energy be used? Will it be easy for the user to access?

3 Will the renewable energy require many changes in the vehicles or equipment being used? Who will pay for this change in infrastructure? How much will it cost?

4 How long will it take to generate the resources needed to make this renewable energy resource accessible and affordable for most people?

Ideate

Once you know who you're designing for, and you know what the criteria are, it's time to get creative!

Outline the criteria or requirements your design must fulfil (i.e. type of equipment, number and amount of materials, area that needs to be covered).

Brainstorm at least one idea per person that fulfils the criteria.

Remember that there are no bad ideas at this stage. One silly thought could lead to a genius innovation!

Build

Each team member should draw one individual design. Label each part of the design. Include the material that will be used to construct a model of the design.

Include in the individual designs:

a a description of what you see as the biggest problem with the energy source that you are replacing

b a description of the renewable energy that you propose could be used instead

c a description of how this renewable energy could be scaled up so that it can be used more effectively.

Present your design to your group.

Build the prototype

As a group, choose one design and plan how to model or build it. You may need to produce two or three to-scale prototypes for your group's design. Keep each iteration so that you can show the progress of your ideas.

Use the following questions as a guideline for your prototype.

- How will you replicate or model the renewable energy source?
- How will you model how the renewable energy will be used?
- What are the limitations of the renewable energy source? Will it produce enough energy for the equipment that currently uses fossil fuels?
- Calculate the number, density or requirements of energy sources in your area. How will your model provide for these demands?

Test

Use the scientific method to design an experiment that will test the limitations of your renewable energy prototype idea. You will need to model and test more than one prototype to compare results, so you will need to consider all variables between tests.

What criteria will you use to determine the success of your renewable energy prototype?

If your prototype will be used by a particular group of individuals, then you will need to generate a survey to test whether the prototype is appropriate for their use. (How would they use the alternative energy source? Would it make their life easier or harder? Would they consider buying it? How much would they be prepared to pay to access this form of energy?)

Conduct your tests and record your results in an appropriate table.

Communicate

Present your design to the class as though you are trying to get your peers to invest in your alternative energy design.

In your presentation, you will need to:

- outline the energy needs of the selected individual or group you are supporting
- outline the energy challenges faced by your selected individual or group
- create a working model or a detailed series of diagrams, with a description of how the design prototype will be used to replace the current energy demands
- describe how you changed your design prototype as a result of testing or feedback
- describe how the renewable energy prototype will improve the life of your selected individual or group
- estimate the cost of production for each element of your energy design
- estimate the number of each element of your energy design required in your local government area
- estimate the total implementation cost to individuals, or to local, state or national government bodies
- compare how this energy system could be implemented in developed and developing countries.

Check your Student obook pro for the following digital resources to help you with this STEAM project:

Student booklet
This helpful booklet will guide you step-by-step through the project.

What is the design cycle?
This video will help you to better understand each phase in the design cycle.

How to manage your project
This 'how-to' video will help you to manage your time throughout the design cycle.

How to define a problem
This 'how-to' video will help you to narrow your ideas down and define a specific problem.

Answers

CHAPTER 1 Algebra

EX p6 1A Exponents

1 **a** a^4 **b** b^{20} **c** c^{15} **d** d
e y^{10} **f** m^{14} **g** n^{14} **h** h^7
i x^6 **j** p^{18} **k** w^3 **l** a^{10}

2 **a** 1
b **i** 1 **ii** 1 **iii** 1 **iv** 1

3 **a** 3 **b** 1 **c** 8 **d** 1
e 1 **f** 2 **g** 0 **h** 2
i 6 **j** 1 **k** 1 **l** 1

4 **a** a^5b^5 **b** k^9p^9 **c** $64c^3$ **d** $7^{13}y^{13}$
e x^4y^4 **f** c^7d^7 **g** $9w^2$ **h** 6^6g^6
i 2^8p^8

5 **a** $\frac{a^4}{b^4}$ **b** $\frac{f^7}{g^7}$ **c** $\frac{k^6}{64}$
d $\frac{5^6}{x^6}$ **e** $\frac{d^{-2}}{c^{-2}}$ **f** $\frac{m^{-5}}{3^{-5}}$

6 **a** $5x^5$ **b** $6a^7$ **c** $9x^2$ **d** $8x^{12}$
e $4a^3$ **f** $2x^5$ **g** 2 **h** $30x^{12}$
i $3a^3$

7 **a** $8y^4$ **b** $24x^9$ **c** $72k^5$ **d** $45b^{12}$
e $60a^{20}$ **f** $10c^5$ **g** $2n^2$ **h** v^6
i $25m^{13}$

8 **a** p^9 **b** $\frac{2t^{20}}{6561}$ **c** $25m^{39}n^{48}$ **d** $\frac{1}{j^6}$

9 **a** false; $x^7 \times x \times x^7 = x^{7+1+7} = x^{15}$
b false; $(5a)^3 = 5^3 \times a^3 = 125 \times a^3$
c true; $-k^0 = -(1) = -1$
d false; $a^2b^3 \times a^3b^4 = a^{2+3}b^{3+4} = a^5b^7$
e true; $x^7 \times y^4 \div y^6 = \frac{x^7y^4}{y^6} = \frac{x^7\cancel{y^4}^{1}}{\cancel{y^6}_{y^2}} = \frac{x^7}{y^2}$
f false; $\left(\frac{m}{n}\right)^5 = \frac{m^5}{n^5}$
g false; $\frac{w^6 \times w^6}{w^{12}} = \frac{w^{6+6}}{w^{12}} = \frac{w^{12}}{w^{12}} = 1$
h true; $\frac{(b^5)^4 \times b^2}{(b^3)^7} = \frac{b^{20} \times b^2}{b^{21}} = \frac{b^{22}}{b^{21}} = b$
i false; $\frac{a^8b^5}{a^4b^3} \times \frac{a^2b^2}{a^9b^4} = \frac{a^{8+2}b^{5+2}}{a^{4+9}b^{3+4}} = \frac{a^{10}b^7}{a^{13}b^7} = \frac{1}{a^3}$

10 **a** 9 **b** 0 **c** 10 **d** 4
e 7 **f** 9 **g** 6 **h** −5
i 8

11 2^5

12 3^{2x}

13 $\frac{5^{4x}}{3^{3x}}$

14 **a** $x = 4$ and $y = 8$
b $x = -2, y = 12$

15 **a** $\frac{2^{13} \times 3^{11} \times 5^9}{30^7} = \frac{2^{13} \times 3^{11} \times 5^9}{(2 \times 3 \times 5)^7}$
$= \frac{2^{13} \times 3^{11} \times 5^9}{2^7 \times 3^7 \times 5^7}$
$= 2^6 \times 3^4 \times 5^2$
$= (2^3 \times 3^2 \times 5)^2$
b $a = 5 \times 7$ or $a = 5 \times 7^3$ or $a = 5 \times 7^5$ or $a = 5 \times 7^7$ or $a = 5^3 \times 7$ or $a = 5^3 \times 7^3$ or $a = 5^3 \times 7^5$ or $a = 5^3 \times 7^7$ or $a = 5^5 \times 7$ or $a = 5^5 \times 7^3$ or $a = 5^5 \times 7^5$ or $a = 5^5 \times 7^7$
c The prime factorisation of any square number can be written as a product of squares of prime numbers.

16 50

17 **a** **i** 4 **ii** 4 **iii** 8
iv −8 **v** 100 000 **vi** −100 000
b **i** −32 **ii** 81 **iii** −125
c **i** positive, since the power is even
ii negative, since the power is odd
iii positive, since the power is even

EX p10 1B Negative exponents

1 **a** $\frac{1}{x^5}$ **b** $\frac{d^4}{c^9}$ **c** $\frac{j^5}{k^3}$ **d** $\frac{y^8}{x}$
e $\frac{2a^3}{b^6}$ **f** $\frac{3}{p^4q^5}$ **g** $\frac{w^4y^7}{x^5}$ **h** $\frac{m^5}{k^3n^8}$
i $\frac{6d^2f}{e^7}$ **j** $\frac{11}{a^3b^5c^9}$ **k** x^3 **l** b^4c^6
m $4m^3p^7$ **n** $\frac{b^2}{a^4}$ **o** $\frac{4y^5}{x^2}$ **p** $\frac{d^8e^4}{c}$

2 **a** x^3 **b** m^7 **c** k^{12} **d** y^4
e a^5 **f** d^{-15} **g** y **h** n^2

3 **a** $\frac{1}{x^4y^4}$ **b** $\frac{1}{4^6a^6}$ **c** $\frac{1}{9n}$
d $\frac{1}{5^3p^3} = \frac{1}{125p^3}$ **e** $\frac{1}{9p^2}$ **f** $\frac{1}{k^5m^5}$
g $\frac{64}{x^3}$ **h** $\frac{d}{7}$

4 **a** $14x^9$ **b** $3x^2$ **c** x^{15} **d** $20x^8$
e $4x^3$ **f** $6x^6$ **g** $\frac{3x}{7}$ **h** $\frac{3x^5}{2}$
i x^{17} **j** x^{34} **k** $2x^{23}$ **l** $5x^{35}$

5 **a** $\frac{1}{x^2}$ **b** $\frac{2}{x^5}$ **c** $42x$ **d** $\frac{20}{x^4}$
e $\frac{1}{x^2}$ **f** $2x^9$ **g** $\frac{1}{4x^9}$ **h** $\frac{3}{5x^4}$
i $\frac{1}{x^{11}}$ **j** $\frac{10}{x^6}$ **k** $\frac{3}{x^7}$ **l** $\frac{1}{x^{21}}$

6 **a** $\frac{1}{x^6}$ **b** $\frac{4}{y^6}$ **c** w^{12} **d** $4x^{14}$
e $\frac{9}{a^8}$ **f** $2m$ **g** $8h^{12}$ **h** $\frac{30}{c^9}$
i $2d^3$ **j** $\frac{2}{t^{16}}$ **k** x **l** $\frac{12}{a^8}$

7 **a** $a^{11}b^7$ **b** $2304p^5$ **c** $32y^{15}$ **d** $3m^{12}n^6$
e $\frac{d^{12}}{h^{11}}$ **f** $\frac{125w^3}{y^3}$ **g** $a^{30}b^{12}$ **h** $\frac{b^8}{a^{16}c^{28}}$
i u^3wx^{17} **j** $125c^4n$ **k** $m^{34}n$ **l** $\frac{1}{y^7}$

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8 **a** false, $\frac{a^5}{a^9} = \frac{1}{a^4}$ **b** false, $\frac{x^3}{x^6} = \frac{1}{x^3}$

c true **d** false, $\left(\frac{3a^2b^{-1}}{c^3}\right)^2 = \frac{9a^4}{b^2c^6}$

e true **f** true

g false, $\frac{3(x^{-2}y^1)^3}{x^3y^{-2}z} = \frac{3y^5}{x^9z}$ **h** true

9 **a** 3 **b** −4 **c** −2

d 3 **e** 1 **f** 5

10 **a** $x^{2a+5}y^{4b+1}$ **b** $m^{2x+4}n^3$

11 **a** $\frac{1}{25}$ **b** $\frac{1}{27}$ **c** 40

d 144 **e** $\frac{1}{8}$ **f** $\frac{13}{9}$

12 **a** x^{3a-4} **b** y^{3m+5} **c** a^{x+3} **d** b^{x+4}

13 **a** $\frac{3a^2b^2}{2c}$ **b** $\frac{3}{2b^8c^2}$ **c** $\frac{1}{y}$

14 **a** $\frac{8c^{12}}{27a^6b^9}$

15 $2c$

16 **a** **i** $\frac{1}{3}$ **ii** $\frac{1}{10}$ **iii** 4 **iv** $\frac{3}{5}$

v $-\frac{1}{3}$ **vi** $-\frac{1}{10}$ **vii** −4 **viii** $-\frac{3}{5}$

b **i** If a number is positive, then its multiplicative inverse is positive.

ii If a number is negative, then its multiplicative inverse is negative.

c $(-a)^{-1} = \frac{1}{-a} = \frac{1}{-1} \times \frac{1}{a} = -1 \times a^{-1} = -a^{-1}$

d 1 and −1

e Every number multiplied by 0 is equal to 0, so there is no number x such that $0 \times x = 1$.

EX p15 1C Simplifying

1 **a** **i** 3 **ii** Sample answer: $-4abcd$

b **i** −4 **ii** Sample answer: $2mn$

c **i** 1 **ii** Sample answer: $5xy^2$

d **i** 9 **ii** Sample answer: $-3k^2m^4p$

2 **a** −6 **b** −3 **c** 10 **d** 9

e −52 **f** −24 **g** −15 **h** −9

3 **a** $9x$ **b** $-10ab$ **c** $10y^2$

d $8m^2n$ **e** $12a + 9d$ **f** $8k + 2m$

g $2 - 3y$ **h** $7a - 4a^2$ **i** $3x^2 + 2xy$

4 **a** $11a$ **b** $-4n + 2n^2 + m - mn$

c $3x^2 - 2x + 1$ **d** $-3x^2y + 3xy - 5$

e $2p^3 + 15p^2 - 9$ **f** $-a^2 - b^2 + 3ab^2 + 9a^2b$

5 **a** $10abcd$ **b** $-21mnxy$ **c** $4k^2p$

6 **a** ac **b** $6n$ **c** $\frac{2x}{y}$

7 **a** $4x^5y^3$ **b** $6a^6b^3$ **c** $\frac{24m^7}{n^2}$

d $\frac{10x^7y^2}{z^3}$ **e** $\frac{2y^2}{x^2}$ **f** $4x$

g $\frac{36x^5z^7}{y^4}$ **h** $\frac{18y^{11}}{x}$ **i** $\frac{4a^4}{b^4}$

8 **a** 2 **b** −69 **c** 246

d −20 736 **e** −18 **f** 4

9 **a** 1550 cm^2 **b** 0.62 m^2

10 25π cm^3

11 **a** $7x + 6y$ **b** $6x + 4y - 6$ **c** $20y - 12$

12 **a** 61 cm **b** 44 cm **c** 28 cm

13 The answers are square units:

a $42ab$ **b** $4a^2$ **c** $10ab$

d $9\pi a^2$ **e** $24ab + \frac{9}{8}\pi b^2$ **f** $37ab$

14 **a** 504 m^2 **b** 64 m^2 **c** 120 m^2

d 144π m^2 **e** $288 + \frac{81\pi}{8}$ **f** 444 m^2

15 **a** $3\pi x^2$

b Any value larger than about 6.5147 mm and smaller than 7.2836 mm. Some possible values are: 6.7 mm, 7 mm and 7.1 mm.

c $6\pi x$

d 126.3 mm, 131.9 mm, 133.8 mm

16 **a** $12a$ **b** $32ab$

c $-3a + 15ab$ **d** $24xy^2 - x$

e $-6a^2 + 25a^2b^6$ **f** $16x^2y^2 + 13y^2$

17 **a** $7ab$ **b** $7abc$

c $5x^2y + 3xy$ **d** 0

e $23a^2b^2$ **f** $11y^2z^2 + 3x^2z$

18 **a** $abc - 6c$ m^3 **b** $2ab + 2ac + 2bc - 12$ m^2

EX p19 1D Expanding

1 **a** $3a + 21$ **b** $5b - 20$ **c** $12 - 2c$

d $4 + 4d$ **e** $-6e - 18$ **f** $-7f + 14$

g $-3g + 3$ **h** $-45 + 9h$ **i** $x^2 + 2x$

j $2y^2 - 8y$ **k** $-k^2 - 5k$ **l** $-3p^2 + 3p$

m $3mn + km$ **n** $12ab + 24ac$ **o** $-w^2 + 8wx$

p $-4p - 10p^2$

2 **a** $3a - 12$ **b** $-5b + 15$

c $8x + 12$ **d** $-14y - 28$

e $10a - 5a^2$ **f** $2m^3 + 6m$

g $24a^2b + 8ab$ **h** $8wx - 4w^2$

i $15k^2 - 10k^4$ **j** $12x^3y - 6x^2y^4$

k $15c^2d - 10cd^2$ **l** $-8a^3c + 6a^2bc$

3 **a** $9x + 10$ **b** $4y - 13$ **c** $mn + 3m$

d $8x^2 - 5x$ **e** $2p^3 - 3p$ **f** $x^7 - 3x^5$

4 **a** $6x + 2$ **b** $m^2 + 14m - 18$

c $6a^2 + 14a - 6$ **d** $k^5 + 7k^2 + 5$

e $3p^6 - 4p^4 + 12p$ **f** $10y^8 + y^3 + 8y$

5 **a** $ab + 2a + 4b + 8$ **b** $cd + 6c + 3d + 18$

c $mn + 5m + 7n + 35$ **d** $xy + 8x + 6y + 48$

e $kp - k + 5p - 5$ **f** $fg + 3f - 2g - 6$

g $xy - 4x - 9y + 36$ **h** $2mn + 2m + 3n + 3$

i $12xy - 28x + 15y - 35$

j $15ac - 12ad - 10bc + 8bd$

k $x^2 + 7x + 10$ **l** $y^2 + 8y - 9$

m $m^2 - 14m + 33$ **n** $p^2 - 6p + 8$

o $30 - 11k + k^2$ **p** $4x^2 + 19x + 21$

q $3a^2 - 13a - 10$ **r** $8w^2 - 23w + 14$

s $12p^2 + 52p + 35$ **t** $8x^2 + 10x - 3$

u $15y^2 - 26y + 8$

6 **a** $2a^2 + 6a - ab - 3b$

b $4t^2 - 3tu - u^2$

c $2x^2 + 11xy + 12y^2$

d $-x^2 + x + 20$

e $6x^3 + 18x^2y + xy + 3y^2$

f $ab^4 - 2a^2b^2 + b^2 - 2a$

g $ab + ac + b^2 + bc$

h $b^2 + bd + 3bc + 3cd$

i $3a - ac + 6b - 2bc$

j $3a^4b + 2a^3 + 3ab^3 + 2b^2$

k $5x^5 - 2x^4y - 15xz + 6yz$

l $c^4 - 16a^6b^2$

7 **a** **i** $x^2 - 4$ **ii** $k^2 - 49$ **iii** $m^2 - 64$
iv $w^2 - 36$ **v** $y^2 - 1$ **vi** $a^2 - b^2$

b The product of the two binomial factors produces an expression that is the difference of two squared numbers or pronumerals. The first term in each binomial factor is the same, and the second term in one factor is the negative of the second term in the other factor.

c In the expression $a^2 - b^2$, the terms a^2 and b^2 are both squares, and there is a minus sign between them indicating 'find the difference'.

d No; multiplication is commutative.

e **i** $a^2 - 9$ **ii** $x^2 - 100$ **iii** $m^2 - n^2$
iv $9 - x^2$ **v** $1 - d^2$ **vi** $4x^2 - 25$
vii $4 - 9k^2$ **viii** $16g^2 - h^2$ **ix** $25y^2 - 4w^2$

8 **a** **i** $x^2 + 6x + 9$ **ii** $y^2 + 16y + 64$
iii $a^2 + 2ab + b^2$ **iv** $k^2 - 10k + 25$
v $p^2 - 12p + 36$ **vi** $x^2 - 2xy + y^2$

b The two binomial factors are the same. The product of factors like these produces an expression made up of the squares of the two terms in the factor, and twice the product of the two terms in the factor.

c The expression $(a + b)^2$ is the square of $(a + b)$.

d **i** $a^2 + 4a + 4$ **ii** $x^2 + 10x + 25$
iii $p^2 + 8p + 16$ **iv** $y^2 + 20y + 100$
v $k^2 + 18k + 81$ **vi** $m^2 + 2mn + n^2$
vii $25 + 10x + x^2$ **viii** $1 + 4d + 4d^2$
ix $9w^2 + 42w + 49$

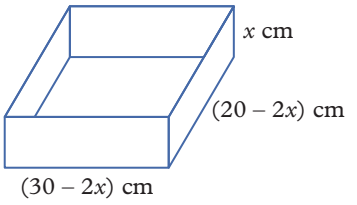
e The expression $(a - b)^2$ is the square of $(a - b)$.

f **i** $a^2 - 6a + 9$ **ii** $b^2 - 8b + 16$
iii $x^2 - 18x + 81$ **iv** $n^2 - 22n + 121$
v $4 - 4x + x^2$ **vi** $g^2 - 2gh + h^2$
vii $16w^2 - 8w + 1$ **viii** $25 - 20y + 4y^2$
ix $9m^2 - 12mp + 4p^2$

9 **a** $x^4 - 4$ **b** $x^4 - 25$ **c** $x^4 - y^4$
d $x^4 + 8x^2 + 16$ **e** $x^4 - 6x^2 + 9$ **f** $x^6 - 2x^3 + 1$

10 **a** **i** $(30 - 2x)$ cm
ii $(20 - 2x)$ cm
iii x cm

b



c **i** $x(30 - 2x)(20 - 2x) = 4x^3 - 100x^2 + 600x$
ii $30 \times 20 - 4x^2 = 600 - 4x^2$ (using the net)

d **i** 1008 cm^3 **ii** 564 cm^2

11 **a** $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

b $x^2 + 4xy + 6x + 4y^2 + 12y + 9$

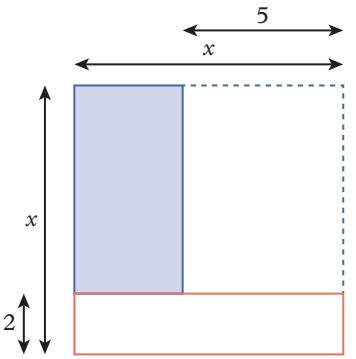
c $4a^2 - 9b^2 - c^2 + 6bc$

12 **a** $x^3 + 3x^2 + 3x + 1$ **b** $x^3 - 9x^2 + 27x - 27$

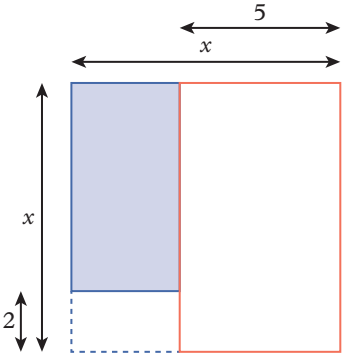
c $x^4 + 8x^3 + 24x^2 + 32x + 16$

d $x^4 - 2x^2y^2 + y^4$

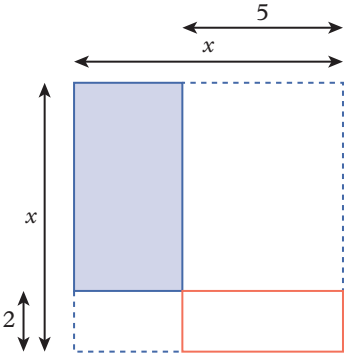
13 **a** **i**



ii



iii



b The rectangle with area 10 is subtracted as part of the $2x$ rectangle and as part of the $5x$ rectangle. Since the rectangle with area 10 is subtracted twice when it should only be subtracted once, it must be added back once.

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EX p26

1E Algebraic fractions

1 **a** $\frac{x}{2}$ **b** $\frac{1}{y}$ **c** 4 **d** $\frac{2x}{3}$
e $\frac{x+3}{5}$ **f** $2(x-5)$ **g** $x-4$ **h** $x+1$

2 **a** $\frac{x+5}{x+1}$ **b** $\frac{x-6}{x-2}$ **c** $\frac{x-1}{x+1}$ **d** $\frac{x+2}{x-8}$
e 1 **f** 1

3 **a** $\frac{2}{y}$ **b** $2b$ **c** $\frac{5}{a}$ **d** $\frac{x}{2}$
e $\frac{1}{3m}$ **f** $\frac{n+7}{2}$ **g** $3y$ **h** $\frac{x}{x+1}$
i $\frac{5(y-3)}{y-4}$ **j** $\frac{t+2}{r+2}$ **k** $\frac{x-1}{x-4}$ **l** $\frac{p-2}{3(p+2)}$

4 **a** $\frac{1}{2}$ **b** 5 **c** 1 **d** 3
e 5 **f** x

5 **a** $\frac{2a+1}{a-4}$ **b** $\frac{x+1}{x-3}$
c $\frac{2}{t}$ **d** $\frac{m+2}{(m-3)(m+7)}$
e $\frac{(x+3)(x+5)}{(x-3)}$ **f** $\frac{1}{x+5}$
g $\frac{3(2y+3)}{y+3}$ **h** $\frac{m(m+1)}{(m-3)(m-1)}$
i $\frac{c^2+6}{c+2}$ **j** $\frac{n-p}{n}$
k $\frac{y^2}{x^2(x+1)}$ **l** $\frac{10(a^2+b^2)}{a-b}$

6 **a** $\frac{2(x-5)}{x-2}$ **b** $\frac{(x+8)(x-6)}{(x-3)(x+6)}$
c $\frac{(3x+2)(5x+1)}{(x+1)(x+2)}$ **d** $\frac{x}{2(x+8)}$

7 **a** $\frac{9x}{20}$ **b** $\frac{5x}{14}$ **c** $\frac{x}{2}$ **d** $\frac{x}{40}$
e $\frac{7x}{6}$ **f** $\frac{5x}{8}$ **g** $\frac{43x}{35}$ **h** $\frac{7x^2}{18}$
i $\frac{34a}{15}$ **j** $\frac{2b^2-b}{4}$ **k** $\frac{5c+3}{12}$ **l** $\frac{7m-5}{12}$
m $\frac{3x-10}{10}$ **n** $\frac{12k+5}{20}$ **o** $\frac{29x-7}{40}$ **p** $\frac{7y}{8}$
q $\frac{-x+2}{6}$ **r** $\frac{b+10}{24}$ **s** $\frac{-5}{12}$ **t** $\frac{23x+18}{20}$

8 **a** $\frac{5x+13}{6}$ **b** $\frac{11x-8}{30}$ **c** $\frac{3x-7}{8}$ **d** $\frac{x+10}{12}$
e $\frac{x-12}{24}$ **f** $\frac{4x+1}{10}$ **g** $\frac{13x-6}{15}$ **h** $\frac{7x+17}{18}$
i $\frac{23x+5}{6}$ **j** $\frac{7x+15}{20}$ **k** $\frac{40-10x}{21}$ **l** $\frac{3x^2+1}{40}$

9 **a** $\frac{2}{x}$ **b** $\frac{4}{3a}$
c $\frac{b-5}{b^2}$ **d** $\frac{3-x^2}{3x}$
e $\frac{5m+3}{m(m+1)}$ **f** $\frac{4k^2+3}{k(k^2+1)}$
g $\frac{4y-7}{(y+2)(y-3)}$ **h** $\frac{x^2+2x+6}{x(x+3)}$
i $\frac{t^2+t+10}{(t+5)(t-1)}$ **j** $\frac{8x+5}{x(2x+1)}$
k $\frac{2x-1}{(x+2)(x+1)}$ **l** $\frac{-2h+11}{(2h-1)(h+2)}$
m $\frac{2m^2+3}{(m-3)(2m+1)}$ **n** $\frac{a^3+12a+4}{a(3a+1)}$
o $\frac{x^2-3x-6}{2(x+1)}$ **p** $\frac{2x+1}{x(x+1)}$

10 $a = x-3$

11 $b = 3x+1$

12 **a** $\frac{37x}{30}$ **b** $\frac{7x}{8}$ **c** $\frac{x}{8}$
d $\frac{3}{7}$ **e** $-\frac{8x}{21}$ **f** $\frac{7x-13}{12}$

13 **a** $\frac{(1-x^4)}{x^2}$ **b** $\frac{(x^4+2x^2+1)}{x^2}$
c $\frac{1}{x-1}$ **d** $\frac{1}{2x^2}$

14 **a** $\frac{1}{x(x+1)}$
b **i** $\frac{1}{42}$ **ii** $\frac{1}{90}$

15 **a** $\frac{x}{y(y+1)}$
b **i** $\frac{5}{42}$ **ii** $\frac{7}{110}$

16 **a** -1 **b** -4 **c** $\frac{x-3}{x+4}$

17 **a** $\frac{yz+xz+xy}{xyz}$ **b** $\frac{a^2c+ab^2+bc^2}{abc}$

18 **a** $\frac{(x+4)(5x-2)}{3x}$ m^2 **b** $\frac{2x^2+38x-12}{3x}$ m

Checkpoint

1 **a** x^9 **b** y^4 **c** $2m$
d a^{10} **e** $9x^2y^4$ **f** $\frac{8a^3}{b^6}$

2 **a** 1 **b** 2 **c** 1 **d** 3

3 **a** $\frac{1}{y^3}$ **b** $\frac{4}{x^2}$ **c** $\frac{1}{6a^7}$
d $2b^3$ **e** $\frac{1}{2a^2b^5}$ **f** $\frac{5d^4}{c^3}$

4 $\frac{12}{a^7b}$

5 **a** $6x$ **b** $8m^2$
c $7a+5b$ **d** $-3c+5d$
e $5a^2-2a$ **f** $-2x^2y+3xy+xy^2$

6 **a** $3ab$ **b** $8xy$
c $3ab^2$ **d** $-3x+12$

7 **a** $3a+21$ **b** $6y+7$
c $3x-8$ **d** $-7x+10$

8 **a** a^2+3a+2 **b** $x^2+2x-24$
c k^2-25 **d** $2x^2+x-6$

9 $x^3+2x^2y-xy-2y^2$

10 **a** $\frac{5x}{6}$ **b** $\frac{17y+2}{6}$
c $\frac{5m^2-4m+5}{20}$ **d** $\frac{11x+1}{21}$

11 **a** $\frac{a}{3}$ **b** $\frac{x+4}{5}$
c $\frac{1}{x+7}$ **d** $\frac{m-4}{m}$

12 $\frac{x^2-2x+6}{(x-2)(x+1)}$

EX p34 1F Factorising

1 **a** $5(a+7)$ **b** $6(5b+1)$ **c** $3(8+11c)$
d $4(3d+2)$ **e** $2(4e-9)$ **f** $7(5f-3)$
g $8(3+2g)$ **h** $10(1-4h)$ **i** $9(j+k)$
j $4(4w-7x)$ **k** $m(m+2)$ **l** $n(n-6)$
m $a(5+a)$ **n** $p(9-p)$ **o** $3q(q+1)$
p $2r(r-4)$

2 **a** $3(x-4)$ **b** $3(2y-5)$
c $8x(x-2)$ **d** $2k(2k+3)$
e $c(5+11c)$ **f** $4u(2-u)$
g $3m(2m-1)$ **h** $y^2(1+5y)$
i $2t(7-4t)$ **j** $3p^2(3p+1)$
k $a(4b+3)$ **l** $tu(5t-3)$
m $2mn(2m+n)$ **n** $3xy(2x+1)$
o $3c(5c+7d)$ **p** $6a^2bc(3b^2-2a^2c)$

3 **a** b, b
b $-5x, -5x, -5x$

4 **a** $-6(mn+3)$ **b** $-3b(a+c)$
c $-y(y+10)$ **d** $-9x(2x+1)$
e $-4k(1-2k)$ **f** $-10a(2a^2+3)$
g $-2xy(6x-7)$ **h** $-w^5(1-w^2)$
i $-2b^2(2b-1+3b^2)$

5 **a** $y(x-5)+2(x-5) = (x-5)\times y+(x-5)\times 2$
$= (x-5)(y+2)$
b $4a(3+k^2)-9(3+k^2) = (3+k^2)\times 4a+(3+k^2)\times(-9)$
$= (3+k^2)(4a-9)$

6 **a** $(x+3)(a+5)$ **b** $(k-2)(m+4)$
c $(y+5)(y+2)$ **d** $(x-1)(x+9)$
e $(k+6)(k-3)$ **f** $(p-9)(p-6)$
g $(4-a)(7+a)$ **h** $(2n-5)(3n+4)$
i $(y^2+2)(x+8)$ **j** $(a-4)(a^2+3)$
k $(y^2+1)(2x-5)$ **l** $(7-x^3)(5x+2)$

7 **a** $(a+4)(b+3)$ **b** $(x-7)(x+2y)$
c $(n+5)(m-6)$ **d** $(a+2)(a+5)$
e $(x-1)(x+4)$ **f** $(c+8)(c-3)$
g $(2-y)(3y+4)$ **h** $(p+2)(p^2+5)$
i $(x^2+1)(x^2+3)$ **j** $(m^2+2)(2m-3)$
k $(1+k^2)(4+k^3)$ **l** $(y-3)(x+5)$

8 **a** x^2-25
b Write each term as a square; that is, $x^2-25 = x^2-5^2$. Write the product of the sum and difference of the two values that are squared, x and 5. The factorised form is $(x+5)(x-5)$.

9 **a** $(x+2)(x-2)$ **b** $(a+6)(a-6)$
c $(10+y)(10-y)$ **d** $(8b+3)(8b-3)$
e $(5+7p)(5-7p)$ **f** $(2a+9b)(2a-9b)$
g $3(m+1)(m-1)$ **h** $2(2k+3)(2k-3)$
i $(xy+4w)(xy-4w)$ **j** $(m^3+n)(m^3-n)$
k $4(a^5+3b^3)(a^5-3b^3)$ **l** $(x^2y+w^2)(x^2y-w^2)$
m $(h+8)(h-2)$ **n** $(c-1)(c-7)$
o $4(1-x)$ **p** $(1+a+b)(1-a-b)$
q $(y-3)(7-y)$ **r** $9(2x-3)$

10 **a** $(x+\sqrt{7})(x-\sqrt{7})$ **b** $(a+\sqrt{13})(a-\sqrt{13})$
c $(\sqrt{19}+y)(\sqrt{19}-y)$ **d** $(4k+\sqrt{5})(4k-\sqrt{5})$
e $(p+8+\sqrt{2})(p+8-\sqrt{2})$
f $(m-4+\sqrt{6})(m-4-\sqrt{6})$
g $(x+7+\sqrt{3})(x+7-\sqrt{3})$
h $(y-1+\sqrt{19})(y-1-\sqrt{19})$
i $(d-17+\sqrt{17})(d-17-\sqrt{17})$

11 **a** $x^2+10x+25$
b Find the square root of the first and last terms in the expansion ($\sqrt{x^2}=x$ and $\sqrt{25}=5$). Note that the middle term is twice the product of these two values, and is positive because it is preceded by a '+' sign ($+10x$). Write the factorised form as the square of the sum of x and 5; that is, $(x+5)^2$.

12 **a** $y^2-16y+64$
b Find the square root of the first and last terms in the expansion ($\sqrt{y^2}=y$ and $\sqrt{64}=8$). Note that the middle term is twice the product of these two values, and is negative because it is preceded by a '–' sign ($-16y$). Write the factorised form as the square of the difference between y and 8; that is, $(y-8)^2$.

13 **a** $(x+6)^2$ **b** $(m+4)^2$ **c** $(p+10)^2$
d $(y-3)^2$ **e** $(x-7)^2$ **f** $(g-2)^2$
g $(1+a)^2$ **h** $(5-x)^2$ **i** $(9-b)^2$
j $(w+x)^2$ **k** $(k-m)^2$ **l** $(3x+2)^2$

14 **a** **i** $(x+10)$ cm **ii** $(2x+3)$ cm
b **i** $4x$ cm **ii** $6(x+1)$ cm

15 **a** $\frac{a}{b}$ **b** $\frac{x}{3y}$ **c** c^2-4b
d mn **e** $\frac{x-y}{2}$ **f** $\frac{3}{x}$
g $\frac{3x}{4}$ **h** $\frac{m+7}{5m}$ **i** $\frac{a+b}{a-b}$
j $\frac{x+3}{x-3}$ **k** $\frac{4a+5}{2a}$ **l** $\frac{2x+7}{7-2x}$

16 **a** $\frac{6}{x-7}$ **b** 1

17 **a** $(n+2)$ and $(n+4)$ **b** $3n+6$
c $3(n+2)$; this expression is 3 times the middle odd number.
d $425\times 3 = 1275$
e $n+(n+2)+(n+4)+(n+6)+(n+8) = 5n+20$
$= 5(n+4)$
The expression describes a value that is 5 times the middle number.
f $2675\times 5 = 13\,375$

18 **a** The sum of k odd numbers (where k is an even number)
$= k\times$ (the average of the two middle odd numbers)
or $k\times(n+k-1)$
or $k\times$ (the average of the first and last of the odd numbers)

b 10 (the average of 81 and 83) = 10 × 82 = 820
or 10(73 + 10 – 1) = 820 or 10([73 + 91] ÷ 2) = 820

19 15

20 **a** 399 **b** 896 **c** 6391 **d** 2484

21 **a** 101 **b** 50
c 50 × 101 = 5050 **d** $\frac{n}{2}(n+1)$

EX p39

1G Factorising quadratic expressions

1 **a** 7 and 1 **b** 8 and 2 **c** 9 and 5
d 10 and 6 **e** 9 and 4 **f** 8 and 3

2 **a** $(x+7)(x+1)$ **b** $(x+8)(x+2)$
c $(x+9)(x+5)$ **d** $(x+10)(x+6)$
e $(x+9)(x+4)$ **f** $(x+8)(x+3)$

3 **a** $(x+2)(x+1)$ **b** $(x+3)(x+2)$
c $(x+10)(x+1)$ **d** $(x+9)(x+3)$
e $(x+5)(x+2)$ **f** $(x+7)(x+2)$
g $(x+3)(x+3) = (x+3)^2$ **h** $(x+11)(x+1)$
i $(x+9)(x+2)$ **j** $(x+8)(x+4)$
k $(x+9)(x+7)$ **l** $(x+10)(x+4)$
m $(x+6)(x+5)$ **n** $(x+11)(x+2)$
o $(x+12)(x+3)$

4 **a** 8 and –1 **b** –6 and –1 **c** –7 and 5
d 5 and –3 **e** –4 and 2 **f** –6 and –5

5 **a** $(x+8)(x-1)$ **b** $(x-6)(x-1)$
c $(x-7)(x+5)$ **d** $(x+5)(x-3)$
e $(x-4)(x+2)$ **f** $(x-6)(x-5)$

6 **a** $(x+6)(x-2)$ **b** $(x-10)(x+2)$
c $(x-3)(x-2)$ **d** $(x+9)(x-3)$
e $(x-8)(x-5)$ **f** $(x-5)(x+4)$
g $(x+7)(x-3)$ **h** $(x-11)(x-3)$
i $(x-7)(x+4)$ **j** $(x+4)(x-3)$
k $(x-9)(x+2)$ **l** $(x-8)(x-7)$

7 **a** $2(x+8)(x+1)$ **b** $3(x+3)(x+2)$
c $7(x+4)(x-2)$ **d** $-5(x+3)(x+4)$
e $-4(x-5)(x-2)$ **f** $-(x+6)(x-1)$
g $2(x-2)(x+9)$ **h** $19(x-3)(x+1)$
i $-3(x-13)(x+2)$ **j** $-2(x-1)(x-2)$
k $\frac{1}{2}(x-1)(x+4)$ **l** $\frac{1}{3}(x+6)(x-3)$

8 **a** $x+4$ **b** $\frac{1}{x-2}$ **c** $\frac{x}{x-2}$
d $\frac{x-5}{x+3}$ **e** $\frac{2(x-5)}{x+2}$ **f** $\frac{x}{x+2}$
g $\frac{x+1}{1-x}$ **h** $\frac{2(x-3)}{x+4}$ **i** $\frac{x+2}{2(x-2)}$

9 **a** 2 **b** $\frac{x-6}{3}$ **c** $\frac{5x-2}{6}$ **d** $\frac{29x-19}{24}$

10 **a** $(x+11)(x+4)$ **b** $(x-5)(x+3)$
c $(x+2)^2$ **d** $3(x-4)(x-2)$
e $5(x+3)(x-3)$ **f** $2x(x-4)$

11 **a** $x^2 + 3^2$ is the sum of two squares.
b $(x-3)^2 + 4^2$ is the sum of two squares.
c $(x+1)^2 + 2^2$ is the sum of two squares.
d There are no numbers that add to –4 and multiply to 5.
e The expression simplifies to $x^2 + 2^2$, which is the sum of two squares.
f There are no numbers that add to 12 and multiply to 40.

12 **a** $2x^2$ square metres and $(12x + 18)$ square metres
b $(2x^2 + 12x + 18)$ square metres
c $(x + 3)$ m

13 $(x + 6)$ m

14 **a** **i** $x^2 + 2xy + y^2$ **ii** $x^2 + 5xy + 6y^2$
b **i** $(x+2)(x+8)$ **ii** $(x+2y)(x+8y)$
c **i** $(x+5)(x+2)$ **ii** $(x+5y)(x+2y)$
d **i** $(x-4y)(x-2y)$
ii $(y-7x)(y-2x)$ or $(7x-y)(2x-y)$
iii $(4x+y)(x+y)$

15 **a** $x(x+6)$ **b** not possible
c not possible **d** not possible
e not possible **f** $(x+1)(x+5)$
g not possible **h** not possible
i $(x+2)(x+4)$ **j** $(x+3)^2$

16 **a** $k = 5, 8, 9$ (positive) or $k = 0$ or
$k = -7, -16, -27, -40, \ldots$ (negative)
The negative case can be expressed as any number of the form $n(-n-6) = -n(n+6)$, where n is any positive integer.
b $k = 19, 36, 51, 64, 75, 84, 91, 96, 99, 100$ (positive) or $k = 0$ or $k = -21, -44, -69, -96, \ldots$ (negative)
The negative case can be expressed as any number of the form $-n(n + 20)$, where n is any positive integer.
The positive case can be expressed as any number of the form $n(20 - n)$, where n is positive and less than 20.

EX p45

1H Completing the square

1 **a** **i** $x^2 + 6x + 9$ **ii** $(x+3)(x+3) = (x+3)^2$
b **i** $x^2 - 18x + 81$ **ii** $(x-9)(x-9) = (x-9)^2$
c **i** $(x+4)^2$ **ii** $(x+2)^2$
iii $(x-6)^2$ **iv** $(x+8)^2$
v $(x-5)^2$ **vi** $(x-10)^2$

2 **a** 9 **b** 25 **c** 16
d 4 **e** 1 **f** 64

3 **a** $x^2 + 14x + 49$ **b** $x^2 - 6x + 9$
c $x^2 + 18x + 81$ **d** $x^2 + 20x + 100$
e $x^2 - 12x + 36$ **f** $x^2 + 22x + 121$

4 **a** $(x+7)^2$ **b** $(x-3)^2$ **c** $(x+9)^2$
d $(x+10)^2$ **e** $(x-6)^2$ **f** $(x+11)^2$

5 $(x+3)^2 - (\sqrt{2})^2 = (x+3+\sqrt{2})(x+3-\sqrt{2})$

6 **a** $(x+4+\sqrt{7})(x+4-\sqrt{7})$
b $(x-5+\sqrt{11})(x-5-\sqrt{11})$
c $(x+2+\sqrt{10})(x+2-\sqrt{10})$
d $(x+1+\sqrt{6})(x+1-\sqrt{6})$

e $(x-3+\sqrt{13})(x-3-\sqrt{13})$

f $(x-6+\sqrt{23})(x-6-\sqrt{23})$

g $(x+5+\sqrt{29})(x+5-\sqrt{29})$

h $(x-7+\sqrt{3})(x-7-\sqrt{3})$

i $(x+7)(x+1)$

j $\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)$

k $\left(x-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)\left(x-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)$

l $\left(x+\frac{11}{2}-\frac{\sqrt{85}}{2}\right)\left(x+\frac{11}{2}+\frac{\sqrt{85}}{2}\right)$

7 a $2(x+2-\sqrt{3})(x+2+\sqrt{3})$

b $-(x+3-\sqrt{11})(x+3+\sqrt{11})$

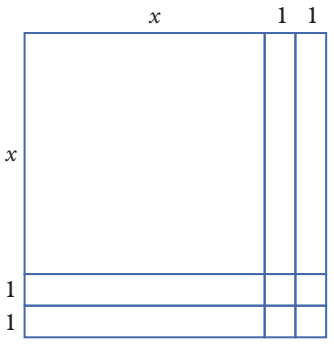
c $-5(x-1-\sqrt{2})(x-1+\sqrt{2})$

d $3\left(x-\frac{5}{2}-\frac{\sqrt{17}}{2}\right)\left(x-\frac{5}{2}+\frac{\sqrt{17}}{2}\right)$

8 a x by $y-x$ **b** x^2

c y by $y-x$ **d** y^2

9 a



b x^2+4x+4

c $(x+2)^2$

10 a $x(x+6)$

b $(x+3+\sqrt{8})(x+3-\sqrt{8})$

c $(x+3+\sqrt{7})(x+3-\sqrt{7})$

d $(x+3+\sqrt{6})(x+3-\sqrt{6})$

e $(x+3+\sqrt{5})(x+3-\sqrt{5})$

f $(x+1)(x+5)$

g $(x+3+\sqrt{3})(x+3-\sqrt{3})$

h $(x+3+\sqrt{2})(x+3-\sqrt{2})$

i $(x+2)(x+4)$

j $(x+3)^2$

11 a $x^2+7x+6=\left(x+\frac{7}{2}\right)^2-\left(\frac{5}{2}\right)^2$

$=\left(x+\frac{7}{2}+\frac{5}{2}\right)\left(x+\frac{7}{2}-\frac{5}{2}\right)$

$=(x+6)(x+1)$

b $6\times1=6$ and $6+1=7$, so the factors are $(x+6)$ and $(x+1)$.

12 a $c>16$ **b** $c>25$ **c** $-2<b<2$

13 a Can be written as $(x+3)^2+1$, which is the sum of two squares.

b Can be written as $(x-4)^2+5$, which is the sum of two squares.

c The leading coefficient is not a highest common factor.

d The leading coefficient is not a highest common factor.

e It can be written as $2(x+1)^2+2$, which is the sum of two squares.

14 a $2(x+7)(x-4)$

b not possible as it is the sum of two squares: $\left(x-\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2$

c $x(x+6)(x-3)$

d $-(x-4)(x+3)$

e $-2(x+7)(x-7)$

f not possible as it is the sum of two squares: $(2x)^2+5^2$

15 a $2\left(x+3-\sqrt{\frac{13}{2}}\right)\left(x+3+\sqrt{\frac{13}{2}}\right)$

b $2\left(x+\frac{7}{4}-\frac{\sqrt{41}}{4}\right)\left(x+\frac{7}{4}+\frac{\sqrt{41}}{4}\right)$

c $3\left(x+\frac{11}{6}-\frac{\sqrt{133}}{6}\right)\left(x+\frac{11}{6}+\frac{\sqrt{133}}{6}\right)$

16 a $\left(x+\frac{b}{2}+\sqrt{\frac{b^2}{4}-c}\right)\left(x+\frac{b}{2}-\sqrt{\frac{b^2}{4}-c}\right)$

b $\frac{b^2-4c}{4}$

EX p50 11 Factorising non-monic quadratic expressions

1 a i $ac=4, b=5$ **ii** 1, 4 **iii** $x+4x$

b i $ac=18, b=11$ **ii** 2, 9 **iii** $2x+9x$

c i $ac=16, b=17$ **ii** 1, 16 **iii** $x+16x$

d i $ac=-24, b=10$ **ii** -2, 12 **iii** $-2x+12x$

e i $ac=-18, b=-7$ **ii** -9, 2 **iii** $-9x+2x$

f i $ac=30, b=-13$ **ii** -3, -10 **iii** $-3x-10x$

2 a $(3x+2)(x+3)$ **b** $(4x+1)(x+4)$

c $(2x+1)(x+2)$ **d** $(2x-3)(3x+1)$

e $(3x-2)(x+4)$ **f** $(5x-3)(x-2)$

3 a $(2x+3)(x+2)$ **b** $(3x+2)(x+1)$

c $(2x+1)(2x+5)$ **d** $(7x+2)(x+1)$

e $(5x+4)(x+2)$ **f** $(2x+3)(x+3)$

g $(3x+2)(2x+1)$ **h** $(5x+1)(2x+3)$

i $(4x+5)(2x+1)$

4 a $(3x+1)(x-3)$ **b** $(4x-3)(x+2)$

c $(2x-5)(x-2)$ **d** $(5x+4)(x-1)$

e $(4x-1)(2x+3)$ **f** $(2x-3)(2x+5)$

g $(7x-4)(x-2)$ **h** $(4x-1)(3x+1)$

i $(3x+4)(x-5)$

5 a $2(2x+3)(x+1)$ **b** $3(4x+1)(x+2)$

c $2(5x-1)(x+4)$ **d** $(4x+5)(2x+3)$

e $3(2x+5)(x-2)$ **f** $4(3x+1)(x-1)$

g $-2(2x+1)(x+1)$ **h** $-3(4x-1)(x+2)$

i $-(3x+4)(2x+3)$

6 a $2x+3$ **b** $\frac{1}{x+3}$ **c** $2x+7$

d $\frac{2x-1}{x+5}$ **e** $\frac{5x+2}{2x-3}$ **f** -4

7 **a** $\frac{5x-3}{x-3}$ **b** $\frac{2x+7}{2x+1}$

c 1 **d** $\frac{(2x+7)(x+4)}{(2x-7)(x-4)}$

e $\frac{5x+2}{2}$ **f** $\frac{x+1}{x-3}$

8 **a** $3(x+3+\sqrt{2})(x+3-\sqrt{2})$

b $2(x-5+\sqrt{5})(x-5-\sqrt{5})$

c $5(x+1+\sqrt{7})(x+1-\sqrt{7})$

d $3(x+2+\sqrt{3})(x+2-\sqrt{3})$

e $2\left(x-3+\sqrt{\frac{21}{2}}\right)\left(x-3-\sqrt{\frac{21}{2}}\right)$

f $4\left(x+1+\frac{\sqrt{5}}{2}\right)\left(x+1-\frac{\sqrt{5}}{2}\right)$

g $6\left(x-2+\sqrt{\frac{29}{6}}\right)\left(x-2-\sqrt{\frac{29}{6}}\right)$

h $3\left(x+4+\sqrt{\frac{47}{3}}\right)\left(x+4-\sqrt{\frac{47}{3}}\right)$

i $9\left(x-1+\frac{\sqrt{7}}{3}\right)\left(x-1-\frac{\sqrt{7}}{3}\right)$

9 $\frac{-14}{3x+4}$

10 **a** $(4x-1)(x+2)$

b $-4x^2-7x+2=-(4x^2+7x-2)=-(4x-1)(x+2)$
$=(-4x+1)(x+2)$
$=(1-4x)(x+2)$

c **i** $(3-2x)(x+1)$

ii $(1-3x)(x-5)=(3x-1)(5-x)$

iii $(7-5x)(x+2)$ **iv** $(2-x)(4x+3)$

v $(1-2x)(3x+2)$ **vi** $(1-5x)(5x-1)$

11 **a** $\sqrt{9x^2-4x-5}$ metres **b** $(9x+5)(x-1)$

12 **a** $(x+4-\sqrt{11})(x+4+\sqrt{11})$

b $2\left(x+2-\sqrt{\frac{3}{2}}\right)\left(x+2+\sqrt{\frac{3}{2}}\right)$

c $3(x+1)\left(x+\frac{5}{3}\right)$ or $(x+1)(3x+5)$

13 Parts **a**, **d** and **e** cannot be factorised.

b $(5x-4)(3x+2)$

c $3(x-2+\sqrt{3})(x-2-\sqrt{3})$

f $-3(2x-9)(x-7)$

14 **a** Room A: $2x+3$ by $4x-5$ metres

Room B: $4x-5$ by $4x-5$ metres

b $(24x^2-38x+10)$ m^2

c $6x-2$ by $6x-2$ metres

d $(6x-2)^2-(4x-5)^2$ m^2

e $(10x-7)(2x+3)$ m^2

f $36-\frac{24}{x}+\frac{4}{x^2}$

g 28

15 **a** **i** $(x^2+5)(x^2+2)$

ii $(x^2+12)(x^2+2)$

iii $(x^2+3)(x^2+1)$

b **i** $(x^2+1)(x-1)(x+1)$

ii $(x^2+4)(x-2)(x+2)$

iii $(x^2+9)(x-3)(x+3)$

c **i** $(x-2)(x+2)(x-3)(x+3)$

ii $(x-1)(x+1)(x-5)(x+5)$

iii $(x-1)(x+1)\left(x-\sqrt{10}\right)\left(x+\sqrt{10}\right)$

16 **a** $a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$

b $a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)$

c b^2-4ac

CHAPTER 1 review

Multiple choice

1	D	**2**	B	**3**	E	**4**	C
5	D	**6**	B	**7**	B	**8**	A
9	E	**10**	D	**11**	E	**12**	C

Short answer

1 **a** a^7 **b** $24x^7y^5$ **c** 3 **d** $3ac^3$

2 **a** $\frac{x^3}{y^4}$ **b** $\frac{a^4}{3b^5}$ **c** $\frac{6p^4r^7}{5q^5}$

3 **a** x^8 **b** $\frac{1}{pq^2}$ **c** $\frac{1}{864a^8b^3}$

4 **a** $-6xy-3x^2+2x^2y-5xy^2$

b $-3ab^2-2a^2b^2+2a^2b$

c $-4pq-8p^2+3q^2$

5 **a** $a^2b^2c^2$ and $(2abc)^2$

b $5xyz^2$, $-2z^2xy$ and $3yz^2x$

c $2p^2q^2$, $-(pq)^2$ and $p^2\times q\times -q$

6 **a** 14 **b** 39 **c** -16

7 **a** $x^2+4x-21$ **b** x^2+4x+4

c $x^2-8x+16$ **d** $2x^2-7x-4$

e a^2-b^2 **f** $a^3+a^2b^2+ab+b^3$

8 **a** $6x^4-2x^2-20$ **b** $4x^2-9y^2$

c $9x^4-24x^2+16$

9 **a** $2x^2-5x-6$ **b** $4x-13x^2$ **c** x^2-x

10 **a** $\frac{25x}{12}$ **b** $\frac{9x-22}{20}$ **c** $\frac{14x-11}{24}$

d $\frac{31}{12x}$

11 **a** $(x-3)(x-4)$ **b** $(x+1)^2(x-1)$

c $2(8+3x)(x-2)$

12 **a** $2(3x+2y)(3x-2y)$ **b** $-3(2x-3)$

c $8x$

13 **a** $(xy+\sqrt{13})(xy-\sqrt{13})$

b $(x+1+\sqrt{3})(x+1-\sqrt{3})$

c $(\sqrt{7}+a-1)(\sqrt{7}-a+1)$

14 **a** $(x+2)(x+6)$ **b** $(x-9)(x+7)$

c $-(x-4)(x-2)$ **d** $-6(x+3)(x-2)$

15 **a** $\frac{x+4}{x+3}$ **b** $\frac{(x-3)(x+1)}{(x+4)(x+3)}$

16 **a** $(x+5+\sqrt{5})(x+5-\sqrt{5})$

b $(x-4+\sqrt{17})(x-4-\sqrt{17})$

17 **a** $(3x+2)(x+9)$ **b** $(4x-1)(2x+3)$

c $(2x-11)(x+2)$ **d** $(4x-3)(3x-2)$

18 a $2(x+3+\sqrt{11})(x+3-\sqrt{11})$

b cannot be factorised

c $3(2x-5)(x+6)$

d $-4\left(x-\frac{3}{2}+\frac{\sqrt{7}}{2}\right)\left(x-\frac{3}{2}-\frac{\sqrt{7}}{2}\right)$

Analysis

a $x^2 + 12$ is not in the correct form to apply the difference of two squares rule.

b No. Since b is an integer, the constant term would need to be b^2, but 12 is not a perfect square.

c $\left(x-\frac{b}{2}\right)^2-\frac{b^2}{4}+12$

$-\frac{b^2}{4}+12\leq 0$

$b^2 \geq 48$

Since b is an integer, its square is a square number.
The smallest square number greater than 48 is 49.

$\therefore b^2 \geq 49$

d The possible values of b are ± 7, ± 8 and ± 13.

Factor pairs of 12	b = Sum of factors	Quadratic	Factorised quadratic
1, 12	13	$x^2+13x+12$	$(x+1)(x+12)$
–1, –12	–13	$x^2-13x+12$	$(x-1)(x-12)$
2, 6	8	$x^2+8x+12$	$(x+2)(x+6)$
–2, –6	–8	$x^2-8x+12$	$(x-2)(x-6)$
3, 4	7	$x^2+7x+12$	$(x+3)(x+4)$
–3, –4	–7	$x^2-7x+12$	$(x-3)(x-4)$

e The possible values of b are ± 10, ± 11, ± 14 and ± 25.

CHAPTER 2 Real numbers

2A Rational and irrational numbers

EX p60

1 a D **b** F **c** B **d** E **e** A **f** C

2 a rational **b** rational **c** irrational
d rational **e** irrational **f** rational
g rational **h** irrational **i** rational
j rational **k** rational **l** irrational
m rational **n** irrational **o** irrational

3 **a**, **c**, **d**, **f**, **g**, **i**, **j**, **k**, **l**, **n**, **o**, **p**

4 a 4.5826 **c** 2.8284 **d** –2.4495
f 11.1803 **g** –15.5885 **i** 1.4422
j 5.1962 **k** 4.7321 **l** 1.2361
n 0.4899 **o** 0.4082 **p** –0.7071

5 $\sqrt{3}, -3\sqrt{5}, \sqrt{8}, \sqrt[4]{32}$

6 a 11 **b** 30 **c** –32 **d** –1
e 24 **f** 14 **g** 10 **h** 5
i 13 **j** –6 **k** 11 **l** 9

7 rational numbers: **a**, **b**, **c**, **f**, **g**, **j**, **k**, **l**
irrational numbers: **d**, **e**, **h**, **i**

8 **d**, **h**, **i**

9 a rational **b** irrational **c** irrational **d** rational
e irrational **f** irrational **g** irrational **h** irrational

10 Irrational, since it cannot be written exactly as a terminating or recurring decimal.

11 a 4.6856 **b** 6.7807 **c** 1.6042
d 4.9892 **e** 5.9161 **f** 54.9909
g 3.4641 **h** 2.6788 **i** 4.6291
j 0.3769 **k** 1.5157 **l** 0.1135

12 a $\sqrt{\frac{5}{8}} \approx 0.790\,569, \frac{\sqrt{5}}{\sqrt{8}} \approx 0.790\,569$

b yes

c $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

d Irrational, because it cannot be written exactly as a terminating or recurring decimal.

e i $\sqrt{\frac{32}{2}} = \sqrt{16} = 4$, rational

ii $\sqrt{\frac{15}{3}} = \sqrt{5} = 2.236\ldots$, irrational

iii $\sqrt{\frac{7}{28}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, rational

13 a i 2 **ii** 3

b 2, 3

c By choosing between which two perfect squares the number lies.

d i 3, 4 **ii** –4, –5 **iii** 11, 12
iv –6, –7 **v** 13, 14 **vi** 17, 18

14 a i 8, 9 **ii** –5, –4 **iii** 4, 5
iv 15, 18 **v** 5, 8 **vi** –15, –12

b i $\frac{3}{5}, \frac{4}{5}$ **ii** $\frac{5}{4}, \frac{5}{3}$ **iii** $\frac{13}{2}$, 7
iv $-1, -\frac{1}{2}$ **v** $-\frac{7}{2}, -\frac{10}{3}$ **vi** $-\frac{13}{3}, -\frac{22}{5}$

15 a 2, 3 **b** 1, 2 **c** 3, 4 **d** 13, 14
e 14, 21 **f** 2, 3 **g** –27, –22 **h** $6, \frac{20}{3}$

16 a $\sqrt{2^2+1^2} = \sqrt{5}$ **b** 2.24

17 a 8 cm **b i** $8\sqrt{2}$ cm **ii** 11.31 cm

18 a i $\sqrt{35}$ cm **ii** 5.92 cm **iii** 5.91 cm
b i $\sqrt{51}$ mm **ii** 7.14 mm **iii** 7.14 mm
c i $\sqrt{95}$ cm **ii** 9.75 cm **iii** 9.74 cm

19 a i 23.66 cm **ii** 23.68 cm **iii** 23.64 cm
b i 28.57 cm **ii** 28.56 cm **iii** 28.56 cm
c i 38.99 cm **ii** 39.00 cm **iii** 38.96 cm

20 The answers are only the same if the digit in the third decimal place is smaller or equal to four, that is when the rounded value is the same as the truncated value.

21 a $\frac{2}{5}$ **b** $\frac{13}{20}$ **c** $\frac{49}{100}$
d $\frac{8}{5}$ **e** $\frac{257}{100}$ **f** $\frac{851}{1000}$

22 a 1D, 2C, 3E, 4F, 5B, 6A

b To find another number with the same recurring decimal part, so that we could subtract to eliminate the decimal part.

23 a $\frac{2}{9}$ **b** $\frac{7}{9}$ **c** $\frac{7}{3}$ **d** $\frac{17}{3}$

24 a Let $x = 0.363\,636\ldots$

$100x = 36.363\,636\ldots$

$100x - x = 36.363\,636\ldots - 0.363\,636\ldots$

$99x = 36$

$x = \frac{36}{99}$

$= \frac{4}{11}$

So $0.\overline{36} = \frac{4}{11}$

b Because the recurring part of the decimal contained two digits, it was necessary to shift the decimal point two places to obtain the same recurring decimal again.

25 **a** $\frac{14}{99}$ **b** $\frac{7}{33}$ **c** $\frac{16}{33}$ **d** $\frac{172}{99}$

26 **a** $\frac{1}{3}$ **b** $\frac{8}{33}$ **c** $\frac{785}{999}$ **d** $\frac{41}{99}$

e $\frac{17}{30}$ **f** $\frac{35}{11}$ **g** $\frac{383}{90}$ **h** $\frac{3899}{1110}$

EX 2B Simplifying surds

p65

1 **a** 4 **b** 9 **c** 7 **d** 2 **e** 10 **f** 5

g 18 **h** 44 **i** 1 **j** 2

2 **a** $9\sqrt{7}$ **b** $4\sqrt{3}$ **c** $2\sqrt{6}$ **d** $5\sqrt{10}$

e $3\sqrt{2}$ **f** $6\sqrt{5}$ **g** $8\sqrt{11}$ **h** $30\sqrt{5}$

i $6\sqrt{10}$

3 **a** $2\sqrt{7}$ **b** $5\sqrt{2}$ **c** $2\sqrt{6}$ **d** $3\sqrt{3}$

e $4\sqrt{3}$ **f** $5\sqrt{3}$ **g** $3\sqrt{7}$ **h** $2\sqrt{15}$

i $2\sqrt{14}$ **j** $3\sqrt{5}$ **k** $8\sqrt{2}$ **l** $6\sqrt{3}$

m $6\sqrt{5}$ **n** $7\sqrt{3}$ **o** $9\sqrt{2}$ **p** $5\sqrt{5}$

q $15\sqrt{2}$ **r** $12\sqrt{2}$ **s** $3\sqrt{30}$ **t** $9\sqrt{7}$

4 **a** $10\sqrt{5}$ **b** $6\sqrt{2}$ **c** $12\sqrt{3}$ **d** $27\sqrt{2}$

e $16\sqrt{5}$ **f** $9\sqrt{10}$ **g** $60\sqrt{6}$ **h** $-12\sqrt{3}$

i $28\sqrt{6}$ **j** $-14\sqrt{5}$ **k** $64\sqrt{3}$ **l** $60\sqrt{7}$

m $-20\sqrt{7}$ **n** $21\sqrt{11}$ **o** $-24\sqrt{17}$ **p** $1000\sqrt{3}$

q $4\sqrt{5}$ **r** $3\sqrt{6}$ **s** $16\sqrt{62}$ **t** $3\sqrt{11}$

5 $5\sqrt{27} = 5 \times \sqrt{27}$

$= 5 \times \sqrt{9 \times 3}$

$= 5 \times \sqrt{9} \times \sqrt{3}$

$= 5 \times 3 \times \sqrt{3}$

$= 15\sqrt{3}$

6 **a** $2\sqrt{11}$ **b** $5\sqrt{5}$ **c** $2\sqrt{30}$ **d** $6\sqrt{2}$

e $5\sqrt{6}$ **f** $-6\sqrt{6}$ **g** $3\sqrt{14}$ **h** $4\sqrt{15}$

i $-9\sqrt{2}$ **j** $7\sqrt{5}$ **k** $4\sqrt{3}$ **l** $2\sqrt{3}$

m $-8\sqrt{3}$ **n** $60\sqrt{5}$ **o** $6\sqrt{21}$ **p** $9\sqrt{2}$

q $-160\sqrt{2}$ **r** -20 **s** $-10\sqrt{10}$ **t** $3\sqrt{2}$

7 **a** $4\sqrt{5}$ cm **b** $6\sqrt{5}$ cm **c** $2\sqrt{13}$ cm **d** $12\sqrt{11}$ m

8 **a** $3\sqrt{6} = \sqrt{3^2} \times \sqrt{6}$

$= \sqrt{9} \times \sqrt{6}$

$= \sqrt{9 \times 6}$

$= \sqrt{54}$

b To be simplified to a whole number coefficient it must have first been expressed as the square root of a perfect square.

c They are equal.

d Put the entire surd and the original surd into your calculator and press equal. You should get the same answer.

e $\sqrt{48}$

9 **a** $\sqrt{45}$ **b** $\sqrt{12}$ **c** $\sqrt{32}$ **d** $\sqrt{252}$

e $\sqrt{160}$ **f** $\sqrt{275}$ **g** $\sqrt{384}$ **h** $\sqrt{125}$

i $\sqrt{240}$ **j** $-\sqrt{128}$ **k** $\sqrt{1008}$ **l** $-\sqrt{75}$

10 **a** $\sqrt{a^2 b}$

c Square the whole part then multiply it by the surd part to find the number that will be under the root sign in the entire surd.

11 **a** $\sqrt{m^2 n}$ **b** $\sqrt{49a}$ **c** $\sqrt{5x^2}$ **d** $\sqrt{100xy}$

e $\sqrt{2cd^2}$ **f** $\sqrt{36abc}$ **g** $\sqrt{a^3 b^2}$ **h** $\sqrt{x^4 y^3 z^2}$

12 **a** $x\sqrt{y}$ **b** xy **c** $b\sqrt{a}$ **d** $xz\sqrt{y}$

e x^2 **f** $x^2\sqrt{y}$ **g** $x^2 y$ **h** $a^2 b\sqrt{c}$

13 **a** $2^2 \times 3$ **b** $\sqrt{2^2 \times 3} = \sqrt{2^2} \times \sqrt{3} = 2 \times \sqrt{3} = 2\sqrt{3}$

14 **a** 2×3^3 **c** $3\sqrt{6}$

15 **a** $3\sqrt{2}$ **b** $2\sqrt{7}$ **c** $4\sqrt{2}$ **d** $3\sqrt{3}$

e $2\sqrt{6}$ **f** $5\sqrt{2}$ **g** $6\sqrt{5}$ **h** $16\sqrt{2}$

i $15\sqrt{7}$ **j** $16\sqrt{10}$ **k** $28\sqrt{3}$ **l** $18\sqrt{5}$

m $6\sqrt{5}$ **n** $35\sqrt{3}$ **o** $54\sqrt{2}$ **p** $5\sqrt{5}$

q $30\sqrt{2}$ **r** $12\sqrt{2}$

16 **a** $\sqrt{18}$ m $= 3\sqrt{2}$ m **b** $\sqrt{2}$ m

c The side length of the 18 m² square is three times the size of the 2 m² square. The side length of a square divided into three smaller squares has a total area of $3 \times 3 = 9$ smaller squares, which is the specified amount.

d **i** $\sqrt{27}$ m $= 3\sqrt{3}$ m **ii** $\sqrt{48}$ m $= 4\sqrt{2}$ m

iii $\sqrt{50}$ m $= 5\sqrt{2}$ m **iv** $\sqrt{75}$ m $= 5\sqrt{3}$ m

17 **a** **i** $2\sqrt{2}$ **ii** $3\sqrt{3}$ **iii** $5\sqrt{5}$ **iv** $a\sqrt{a}$

b **i** $4\sqrt{2}$ **ii** $9\sqrt{3}$ **iii** $25\sqrt{5}$ **iv** $a^2\sqrt{a}$

c **i** $2\sqrt[3]{2}$ **ii** $3\sqrt[3]{3}$ **iii** $5\sqrt[3]{5}$ **iv** $a\sqrt[3]{a}$

d **i** $2\sqrt[4]{2}$ **ii** $3\sqrt[4]{3}$ **iii** $5\sqrt[4]{5}$ **iv** $a\sqrt[4]{a}$

18 **a** $\sqrt{5}, \sqrt{10}, \sqrt{21}, \sqrt{22}, \sqrt{27}, \sqrt{30}, \sqrt{42}, \sqrt{46}$

b $\sqrt{6}, 3, \sqrt{11}, 2\sqrt{3}, 2\sqrt{6}, 6, \sqrt{39}, 4\sqrt{3}$

c $4, 2\sqrt{5}, \sqrt{22}, 3\sqrt{3}, \sqrt{43}, 2\sqrt{11}, 3\sqrt{6}, \sqrt{83}$

d $2\sqrt{6}, 4\sqrt{5}, 3\sqrt{14}, 2\sqrt{57}, 12\sqrt{2}, 3\sqrt{43}, 20, 7\sqrt{10}$

EX 2C Multiplying and dividing surds

p70

1 **a** $\sqrt{35}$ **b** $\sqrt{33}$ **c** $-\sqrt{14}$ **d** $-\sqrt{105}$

e $\sqrt{65}$ **f** $-\sqrt{22}$ **g** $\sqrt{21}$ **h** $\sqrt{95}$

i $3\sqrt{77}$ **j** $20\sqrt{6}$ **k** $-28\sqrt{22}$ **l** $-36\sqrt{30}$

m $66\sqrt{34}$ **n** $64\sqrt{42}$ **o** $-24\sqrt{33}$ **p** $132\sqrt{15}$

2 **a** 7 **b** 3 **c** 2 **d** 6

e 5 **f** 11 **g** 10 **h** 9

3 **a** The answer is rational. **b** a

4 **a** 9 **b** -77 **c** 40 **d** 60

e $3\sqrt{10}$ **f** $30\sqrt{30}$ **g** $-32\sqrt{26}$ **h** $3\sqrt{33}$

i 84 **j** 15 **k** $-6\sqrt{26}$ **l** $\sqrt{105}$

m $72\sqrt{42}$ **n** $\frac{3\sqrt{195}}{8}$ **o** $-5\sqrt{70}$ **p** $-11\sqrt{85}$

5 **a** $\sqrt{17}$ **b** $\sqrt{3}$ **c** 1 **d** $-\sqrt{5}$

e $\sqrt{6}$ **f** $\sqrt{7}$ **g** $4\sqrt{6}$ **h** $2\sqrt{5}$

i $-3\sqrt{5}$ **j** $2\sqrt{5}$ **k** $-4\sqrt{10}$ **l** $\frac{\sqrt{5}}{2}$

6 **a** 4 **b** 2 **c** 3 **d** 2

7 **a** $-12\sqrt{5}$ **b** $5\sqrt{3}$ **c** $-3\sqrt{2}$ **d** $2\sqrt{5}$

e $4\sqrt{7}$ **f** $\frac{7\sqrt{5}}{3}$ **g** $\frac{3}{\sqrt{3}}$ **h** $\frac{\sqrt{5}}{5}$

8 $6\sqrt{3} \times 3\sqrt{2} = 6 \times \sqrt{3} \times 3 \times \sqrt{2}$
$= 6 \times 3 \times \sqrt{3} \times \sqrt{2}$
$= 18 \times \sqrt{3 \times 2}$
$= 18\sqrt{6}$

9 $\frac{24\sqrt{40}}{8\sqrt{20}} = \frac{24}{8} \times \frac{\sqrt{40}}{\sqrt{20}}$
$= 3 \times \sqrt{\frac{40}{20}}$
$= 3\sqrt{2}$

10 **a** 6 **b** $\frac{3\sqrt{5}}{4}$ **c** $\frac{11\sqrt{17}}{7\sqrt{21}}$ **d** $-\frac{4\sqrt{13}}{5\sqrt{3}}$
e $\frac{9\sqrt{5}}{4}$ **f** $-\frac{2}{3}$ **g** -48 **h** $\frac{\sqrt{15}}{16\sqrt{2}}$
i 1

11 **a** 3 **b** $\frac{9}{10}$ **c** 24 **d** $\frac{1}{3}$
e -60 **f** 2 **g** 84 **h** 576
i $\frac{24}{5}$ **j** $\frac{1}{8}$ **k** $-\frac{4}{3}$ **l** $-\frac{7}{3}$

EX p75

12 **a** $2\sqrt{10}$ **b** $\frac{\sqrt{3}}{5}$ **c** $\frac{8}{5}$
d $-\frac{11\sqrt{2}}{8}$ **e** $\frac{8\sqrt{2}}{3}$ **f** $\frac{2\sqrt{6}}{5}$
g $\frac{\sqrt{5}}{9}$ **h** $\frac{9\sqrt{7}}{5}$ **i** $-\frac{10\sqrt{11}}{9}$

13 **a** 5 **b** 8 **c** 2 **d** 7
e 9 **f** 80 **g** 54 **h** 192
i 891

14 **a** rational
b $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = \sqrt{a^2} = a$
c $\sqrt{80}, \sqrt{54}, \sqrt{192}, \sqrt{891}$
d The number beneath the square root of an entire surd is equal to the square of the surd.

15 **a** x **b** ab **c** $6y$ **d** $x + y$

16 **a** $\frac{25}{7}$ **b** $\frac{1}{5}$ **c** $\frac{3}{2}$ **d** $\frac{10}{13}$

17 **a** Method 1: $1680\sqrt{6}$ Method 2: $42\sqrt{2} \times 40\sqrt{3}$
b yes
c Use Method 2 only when the surd part can be simplified. Method 2 has the added advantage that the numbers are smaller.

18 **a** $6\sqrt{2}$ **b** $\sqrt{2}$ **c** $\frac{3\sqrt{13}}{5}$ **d** $\frac{6\sqrt{5}}{7}$
e $-7\sqrt{3}$ **f** $\frac{8\sqrt{3}}{3}$ **g** $-16\sqrt{5}$ **h** $\frac{\sqrt{6}}{5}$
i $\frac{-2\sqrt{21}}{3}$ or $\frac{-2\sqrt{7}}{\sqrt{3}}$

19 **a** $\frac{11\sqrt{2}}{8}$ **b** $8\sqrt{6}$ **c** $\frac{-7\sqrt{2}}{16}$ **d** $8\sqrt{3}$
e $-\frac{15\sqrt{3}}{2}$ **f** $\frac{6\sqrt{2}}{5}$ **g** -8 **h** $\frac{32\sqrt{6}}{11}$

20 $ab\sqrt{bc}$

21 **a** **i** 50.00 cm² **ii** 50 cm²
b **i** 137.48 cm² **ii** $30\sqrt{21}$ cm²
c **i** 19.44 cm² **ii** $3\sqrt{42}$ cm²
d **i** 169.71 mm² **ii** $120\sqrt{2}$ mm²
e **i** 6.00 cm² **ii** 6 cm²
f **i** 23.24 m² **ii** $6\sqrt{15}$ m²
g **i** 2.51 cm² **ii** $\frac{4\pi}{5}$ cm²
h **i** 37.70 cm² **ii** 12π cm²
i **i** 0.39 m² **ii** $\frac{\pi}{8}$ m²

22 The perimeter of the square is $4\sqrt{a}$ and the area is a. Equating the perimeter and area to form the equation $4\sqrt{a} = a$, which has the solutions $a = 0$, $a = 16$. Since the square roots of 0 and 16 are not surds, there is no value of a such that the perimeter and area are equal if the side length is a surd.

23 **a** $4\sqrt{5}$ m **b** $4\sqrt{17}$ m **c** $12\sqrt{2}$ m
d $8\sqrt{5}$ m **e** $2\sqrt{3}$ m **f** $\frac{10\sqrt{7}}{3}$ m

24 **a** $\sqrt{5}$ **b** $\sqrt{6}$ **c** $\sqrt{35}$
d $\sqrt{14}$ **e** $\sqrt{39}$ **f** $\sqrt{66}$

2D Adding and subtracting surds

1 **a** yes **b** no **c** no **d** yes
e no **f** no **g** yes **h** yes

2 **a** $7\sqrt{3} + \sqrt{2}$ **b** $4\sqrt{3} - \sqrt{2}$
c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$ **d** $8\sqrt{13} + 4\sqrt{39} - \sqrt{3}$
e $13\sqrt{2} - 6\sqrt{5}$ **f** $4\sqrt{6} + 4\sqrt{5} + 1$
g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$ **h** $\sqrt{21} - \sqrt{3}$
i $7\sqrt{6} + 3\sqrt{5} + 5$ **j** $-7\sqrt{15} - 3\sqrt{7}$

3 **a** $\sqrt{2}, 4\sqrt{2}, 3\sqrt{2}$ **b** none
c $3\sqrt{12}, \sqrt{3}, 2\sqrt{27}$ **d** $9\sqrt{32}, 4\sqrt{48}, \sqrt{128}, 5\sqrt{200}$

4 **a** $13\sqrt{2}$ **b** $24 - 4\sqrt{3}$ **c** $6\sqrt{6} + 3\sqrt{5}$
d $28\sqrt{7}$ **e** $59\sqrt{2}$ **f** $28 - 4\sqrt{2}$

5 **a** $-15\sqrt{3}$ **b** $-6\sqrt{5}$ **c** $-4\sqrt{7}$
d $5\sqrt{3}$ **e** $23\sqrt{2}$ **f** 0

6 **a** $4\sqrt{5}$ **b** 0
c $4\sqrt{6}$ **d** $\sqrt{2}$
e $-19\sqrt{6} + 8\sqrt{5}$ **f** $-25\sqrt{3} + 2\sqrt{2}$
g $16\sqrt{2} - 30\sqrt{3}$ **h** 0
i $6\sqrt{3} - 2\sqrt{7} + 10\sqrt{2} - 4\sqrt{5}$ **j** $3\sqrt{5} - 2\sqrt{6} - 6\sqrt{3}$

7 **a** $5\sqrt{5}$
b $42\sqrt{3}$
c $29\sqrt{2} - 80\sqrt{3}$
d $18\sqrt{6} - 5\sqrt{5} - 14\sqrt{3}$
e $66\sqrt{7} - 8\sqrt{6} + 112\sqrt{3}$
f $10\sqrt{2} - 24\sqrt{5} + 60\sqrt{6} - 12\sqrt{10}$
g $3\sqrt{3} - 25\sqrt{5} - 61$
h $88\sqrt{2} - 113\sqrt{3}$

8 **a** 0 **b** $-22\sqrt{x}$ **c** $7\sqrt{x} - 6\sqrt{3x}$
d $-5\sqrt{2xy}$ **e** $20\sqrt{6xy}$ **f** $-x\sqrt{6y}$
g $12y\sqrt{3x}$ **h** $34x\sqrt{y} - 5x\sqrt{3y}$
i $8x\sqrt{5xy} + 10\sqrt{3xy} - 5x\sqrt{5y} - 24x\sqrt{3y}$

9 **a** $12 + 6\sqrt{3}$ **b** $8\sqrt{5} - 32$
c $20 - 30\sqrt{2}$ **d** $12\sqrt{2} + 8\sqrt{3}$
e $10\sqrt{6} - 6\sqrt{5}$ **f** $-10\sqrt{42} + 15\sqrt{70} - 10\sqrt{7}$
g $45\sqrt{2} + 12\sqrt{15}$ **h** $70 - 60\sqrt{7}$
i $12\sqrt{33} - 30\sqrt{77} + 66$ **j** $3\sqrt{5} - 10 + 25\sqrt{3}$
k $-36\sqrt{2} + 6\sqrt{30} - 81\sqrt{5}$

10 a $10 + 20\sqrt{6} + 6\sqrt{3} + 36\sqrt{2}$

b $15\sqrt{10} - 24\sqrt{2} + 30\sqrt{5} - 48$

c $28\sqrt{10} + 56\sqrt{30} - 24 - 48\sqrt{3}$

d $12\sqrt{3} + 15\sqrt{2} + 12\sqrt{6} + 30$

e $24\sqrt{6} - 288 - 15\sqrt{15} + 90\sqrt{10}$

f $56\sqrt{5} - 48\sqrt{30} + 21\sqrt{10} - 36\sqrt{15}$

11 a $31 + 10\sqrt{6}$ **b** $16 - 6\sqrt{7}$

c $57 + 12\sqrt{15}$ **d** $159 - 24\sqrt{42}$

12 a $(a + b)^2 = (a + b)(a + b)$
$= a^2 + ab + ba + b^2$
$= a^2 + 2ab + b^2$

b $(a - b)^2 = (a - b)(a - b)$
$= a^2 - ab - ba + b^2$
$= a^2 - 2ab + b^2$

13 a 10 **b** 116 **c** 10 **d** –67

14 a The terms in the second bracket are exactly the same as the terms in the first bracket, but the operation sign between the terms in the second bracket is the inverse of the sign between the terms in the first bracket.

b Rational, because the surd terms add to zero each time.

c $(a + b)(a - b) = a^2 - ab + ba - b^2$
$= a^2 - b^2$
$(a - b)(a + b) = a^2 + ab - ba - b^2$
$= a^2 - b^2$

15 a $\left(32\sqrt{10} + 15\sqrt{30} - 12\sqrt{6}\right)$ m^2

b $\left(72\sqrt{5} - 40 + 10\pi\right)$ cm^2

c $\left(15 + 3\sqrt{3} + 2.5\sqrt{2} - 0.5\sqrt{6}\right)$ m^2

16 a $180\sqrt{2} - 18\sqrt{5}$ cm^2 **b** $409 + 80\sqrt{6}$ m^2

c $\left(151 - 24\sqrt{7}\right)\pi$ cm^2 **d** $\left(7 + 4\sqrt{2}\right)\pi$ cm^2

17 a $\left(10\sqrt{10} + 16\sqrt{5}\right)$ m **b** $\left(36 + 2\pi\sqrt{5}\right)$ cm

18 a $8\sqrt{6}$ **b** $3\sqrt{15}$ **c** $\frac{73\sqrt{3}}{18}$

d $2\sqrt{10} + 6\sqrt{5}$ **e** $\sqrt{11} - 3\sqrt{7}$ **f** $6\sqrt{6} - 6$

19 $25 + 3\sqrt{2} + 3\sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{10} + \sqrt{11} + \sqrt{13} + \sqrt{14} + \sqrt{15}$

20 a i add $5\sqrt{5}$ **ii** add $\sqrt{3} + 7$

iii multiply $\sqrt{2}$ **iv** multiply $2\sqrt{5}$

b i $23\sqrt{5}, 28\sqrt{5}, 33\sqrt{5}, 38\sqrt{5}, 43\sqrt{5}, 48\sqrt{5}$

ii $6\sqrt{3} + 28, 7\sqrt{3} + 35, 8\sqrt{3} + 42, 9\sqrt{3} + 49, 10\sqrt{3} + 56, 11\sqrt{3} + 63$

iii $12\sqrt{2}, 24, 24\sqrt{2}, 48, 48\sqrt{2}, 96$

iv $400\sqrt{5}, 4000, 8000\sqrt{5}, 80\,000, 160\,000\sqrt{5}, 1\,600\,000$

c i $255\sqrt{5}$ **ii** $65\sqrt{3} + 315$

iii $93\sqrt{2} + 186$ **iv** $168\,421\sqrt{5} + 1\,684\,210$

21 a i $\frac{15\sqrt{2} + 30}{8} = \frac{15\sqrt{2}}{8} + \frac{15}{4}$

ii $2 + 2\sqrt{2}$

b i $\frac{3\sqrt{5} + 4\sqrt{3} + 6\sqrt{2}}{6} = \frac{\sqrt{5}}{2} + \frac{2\sqrt{3}}{3} + \sqrt{2}$

ii $\frac{3\sqrt{5}}{2}$

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2E Rationalising the denominator

1 a i $\frac{1}{2}$ **ii** $\frac{3}{4}$ **iii** $\frac{3}{4}$ **iv** $\frac{2}{5}$

b The fraction appearing first in the multiplication remained unchanged.

c It is the same as 1, so each fraction has been multiplied by 1.

2 a i $\frac{2}{\sqrt{3}}$ **ii** $\frac{6}{3\sqrt{3}}$ **iii** $\frac{2\sqrt{3}}{3}$ **iv** $\frac{4}{2\sqrt{2}}$

b part **iii**

c Multiplication of both the numerator and the denominator by the surd in the denominator.

3 a $\frac{\sqrt{6}}{2}$ **b** $\frac{2\sqrt{7}}{7}$ **c** $\sqrt{3}$ **d** $\frac{4\sqrt{35}}{7}$

e $\frac{5\sqrt{2}}{6}$ **f** $\frac{7\sqrt{5}}{20}$ **g** $\frac{2\sqrt{15}}{5}$ **h** $-\frac{\sqrt{5}}{2}$

i $\frac{\sqrt{7}}{49}$ **j** $\frac{2\sqrt{110}}{5}$ **k** $\frac{4\sqrt{10}}{3}$ **l** $\frac{9\sqrt{30}}{20}$

4 a $-\sqrt{3}$ **b** $\frac{\sqrt{10}}{2}$ **c** $\frac{4\sqrt{14}}{21}$ **d** $\sqrt{3}$

e $\frac{6\sqrt{5}}{7}$ **f** -6 **g** $\frac{5\sqrt{x}}{x}$ **h** $\frac{\sqrt{y}}{y}$

i $\frac{2x\sqrt{2y}}{y}$ **j** $\frac{\sqrt{xy}}{xy}$ **k** $\frac{\sqrt{2xy}}{2xy}$ **l** $-\frac{\sqrt{xyz}}{2xyz}$

5 a $3 - \sqrt{2}$ **b** $2 + \sqrt{3}$ **c** $\sqrt{6} + 5$

d $-4 - 3\sqrt{5}$ **e** $-3\sqrt{5} - 2$ **f** $4\sqrt{7} - 1$

g $1 - 2\sqrt{3}$ **h** $\sqrt{2} - \sqrt{3}$ **i** $5 + \sqrt{3}$

j $5\sqrt{6} + 2$ **k** $-3\sqrt{7} - 2$ **l** $-5 + 2\sqrt{5}$

6 a $2 - \sqrt{3}$ **b** $\frac{9 + 3\sqrt{2}}{7}$ **c** $\frac{30 - 5\sqrt{5}}{31}$

d $\frac{-9 - 12\sqrt{5}}{71}$ **e** $\frac{5\sqrt{2} - 6}{7}$ **f** $\frac{12 - 2\sqrt{3}}{11}$

g $\frac{3\sqrt{5} + 3}{4}$ **h** $-6\sqrt{42} + 15\sqrt{7}$ **i** $4\sqrt{6} + 8\sqrt{2}$

j $\sqrt{15} - 2\sqrt{3}$ **k** $\frac{10\sqrt{5} - 3\sqrt{30}}{23}$ **l** $\frac{6\sqrt{10} + 12\sqrt{5}}{5}$

7 a $\frac{2\sqrt{6} - 3\sqrt{2} + 2\sqrt{3} - 3}{3}$

b $\frac{20 - 5\sqrt{2} + 8\sqrt{3} - 2\sqrt{6}}{14}$

c $\frac{3 + \sqrt{5} + 3\sqrt{3} + \sqrt{15}}{4}$

d $\frac{3\sqrt{6} - 3 + \sqrt{30} - \sqrt{5}}{5}$

e $\frac{6\sqrt{15} - 4\sqrt{3} - 6\sqrt{5} + 4}{41}$

f $\frac{10\sqrt{10} - 15\sqrt{2} - 2\sqrt{5} + 3}{11}$

8 a $\frac{x^2 + x\sqrt{y}}{x^2 - y}$ **b** $\frac{x + y\sqrt{x}}{x - y^2}$

c $\frac{x^2\sqrt{x} + xy\sqrt{x} - xy - y^2}{x^3 - y^2}$ **d** $\frac{x^2 - 2x\sqrt{y} + y}{x^2 - y}$

9 a $\frac{3\sqrt{2} + 1}{2}$ **b** $\frac{3\sqrt{2} + 1}{2}$

10 a $\frac{3\sqrt{5} + 1}{5}$ **b** $\frac{4\sqrt{3} - 1}{3}$ **c** $\frac{9\sqrt{2} + 4\sqrt{3}}{6}$

d $\frac{\sqrt{3} + 9\sqrt{2}}{12}$ **e** $\frac{9\sqrt{5} - \sqrt{6}}{9}$ **f** $\frac{12\sqrt{6} + 5\sqrt{15}}{60}$

11 a $\frac{\sqrt{5} + 1}{4} + \frac{3\sqrt{5} - 3}{4}$ **b** $\frac{2\sqrt{5} - 1}{2}$

12 **a** $2\sqrt{6}$ **b** -8

c $\frac{2\sqrt{6}+5\sqrt{3}-3}{5}$ **d** $5\sqrt{2}-4\sqrt{5}-13$

e $\frac{106\sqrt{3}-5\sqrt{6}}{238}$ **f** $\frac{16\sqrt{3}-25}{13}$

13 **a** $\frac{\sqrt{2}}{2}$ **b** $\frac{\sqrt{3}}{15}$ **c** $\frac{\sqrt{7}}{2}$

d $\frac{4\sqrt{3}}{9}$ **e** $\frac{\sqrt{6}-2}{2}$ **f** $\frac{7-3\sqrt{5}}{2}$

14 **a** 2 cm^2 **b** 1 cm^2 **c** $\frac{1}{\sqrt{2}}\text{ cm}$

d $\frac{1}{2}$ **e** $\frac{1}{2}\sqrt{2}\text{ cm}$

f The area of the shaded rectangle is given by both $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}\sqrt{2}$. Therefore, they are equal because rationalising $\frac{1}{\sqrt{2}}$ gives $\frac{1}{2}\sqrt{2}$.

g $\frac{7}{\sqrt{10}}=\frac{7}{10}\sqrt{10}$

15 **a** **i** $\sqrt{2}-\sqrt{3}, -\sqrt{2}+\sqrt{3}$

ii $\sqrt{5}+\sqrt{3}, -\sqrt{5}-\sqrt{3}$

iii $2\sqrt{5}-5\sqrt{6}, -2\sqrt{5}+5\sqrt{6}$

iv $2\sqrt{5}+3\sqrt{7}, -2\sqrt{5}-3\sqrt{7}$

b **i** -1 **ii** 1 **iii** $-2\sqrt{6}-5$

iv $2\sqrt{6}-5$ **v** 1

c **i** $\sqrt{5}+\sqrt{3}$ or $-\sqrt{5}-\sqrt{3}$

ii $2\sqrt{2}-5\sqrt{5}$ or $-2\sqrt{2}+5\sqrt{5}$

iii $\sqrt{11}+3\sqrt{2}$ or $-\sqrt{11}-3\sqrt{2}$

iv $5\sqrt{13}-4\sqrt{3}$ or $-5\sqrt{13}+4\sqrt{3}$

d **i** $\frac{\sqrt{5}+\sqrt{3}}{2}$ **ii** $\frac{5\sqrt{5}-2\sqrt{2}}{117}$

iii $\frac{\sqrt{11}+3\sqrt{2}}{7}$ **iv** $\frac{4\sqrt{3}-5\sqrt{13}}{277}$

e **i** Lucy multiplied by the conjugate that is the negative of the denominator, so it did not rationalise. The product is not the difference of two squares, it is the negative of the square of the denominator.

ii Lucy could use the conjugate of her new denominator $-2\sqrt{6}-5$ and multiply her final line by $\frac{-2\sqrt{6}-5}{-2\sqrt{6}-5}$.

f **i** $\sqrt{5}-\sqrt{3}$

ii $-4\sqrt{6}-8$

iii $\frac{\sqrt{15}-\sqrt{6}}{3}$

iv $\frac{3\sqrt{30}-10\sqrt{2}}{7}$

v $2\sqrt{5}+2\sqrt{2}$

vi $\frac{15\sqrt{6}-10\sqrt{10}-3\sqrt{3}+2\sqrt{5}}{7}$

vii -1

viii $\frac{x^2\sqrt{x}-x\sqrt{y}+xy\sqrt{x}-y\sqrt{y}}{x^3-y}$

g $2\sqrt{6}$

h $\frac{\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{3}=\frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{3}$

Checkpoint

1 **a** rational **b** rational **c** irrational

d rational **e** irrational **f** irrational

2 **a** surd **b** not surd **c** not surd

d not surd **e** surd **f** not surd

3 **a** $2\sqrt{2}$ **b** $3\sqrt{5}$ **c** $5\sqrt{7}$ **d** $10\sqrt{6}$

4 **a** $10\sqrt{3}$ **b** $21\sqrt{6}$ **c** $150\sqrt{2}$ **d** $80\sqrt{5}$

5 **a** $\sqrt{35}$ **b** $-2\sqrt{77}$ **c** $-90\sqrt{2}$ **d** $12\sqrt{14}$

6 **a** $\sqrt{6}$ **b** $-\frac{2\sqrt{17}}{3}$ **c** $-\frac{3}{2}$ **d** $\frac{5\sqrt{3}}{2}$

e $-\frac{\sqrt{30}}{3}$ **f** $\frac{6\sqrt{74}}{37}$

7 **a** 6 **b** $\frac{2\sqrt{6}}{9}$

8 **a** $12\sqrt{2}$ **b** $-6\sqrt{2}$

c $12\sqrt{3}+18\sqrt{5}$ **d** $17\sqrt{11}-2\sqrt{7}$

9 **a** $13\sqrt{2}$ **b** $2\sqrt{3}$

c $8\sqrt{6}+14\sqrt{3}$ **d** $17\sqrt{10}-17\sqrt{2}$

10 **a** $6\sqrt{3}$

b $18\sqrt{6}-18\sqrt{2}$

c $2\sqrt{7}+3\sqrt{5}-\sqrt{35}-6$

d $15\sqrt{15}+10\sqrt{10}+8\sqrt{6}+36$

e $30\sqrt{2}-54\sqrt{30}$

f $10\sqrt{154}-12\sqrt{14}+160\sqrt{11}-192$

11 **a** $\frac{5\sqrt{2}}{2}$ **b** $4\sqrt{3}$ **c** $\frac{2\sqrt{35}}{7}$ **d** $\frac{\sqrt{66}}{6}$

12 **a** $5\sqrt{2}-5$ **b** $\frac{\sqrt{3}+1}{2}$

c $\frac{4\sqrt{5}+30}{41}$ **d** $\frac{-13\sqrt{7}-35}{14}$

e $\frac{9\sqrt{5}-47}{164}$ **f** $\frac{-\sqrt{10}-8\sqrt{5}-2\sqrt{3}-8\sqrt{6}}{62}$

2F Fractional exponents

EX p86

1 **a** $x^{\frac{3}{5}}$ **b** $x^{\frac{1}{2}}$ **c** $x^{\frac{7}{8}}$ **d** $x^{\frac{17}{30}}$

e x **f** $x^{\frac{11}{12}}$ **g** $x^{\frac{1}{2}}$ **h** $x^{\frac{1}{2}}$

2 **a** $x^{\frac{4}{11}}$ **b** $x^{\frac{1}{4}}$ **c** $x^{\frac{1}{14}}$ **d** $x^{\frac{1}{20}}$

e $x^{\frac{1}{6}}$ **f** $x^{\frac{1}{10}}$ **g** $x^{\frac{7}{4}}$ **h** 1

3 **a** $x^{\frac{15}{28}}$ **b** $x^{\frac{1}{5}}$ **c** 1 **d** $x^{\frac{3}{10}}$

e $x^{\frac{1}{18}}$ **f** $x^{\frac{1}{2}}$ **g** x^{12} **h** $x^{\frac{1}{35}}$

4 **a** $10x^{\frac{7}{12}}$ **b** $30x^{\frac{31}{40}}$ **c** $-36x^{\frac{41}{63}}$ **d** $4x^{\frac{1}{12}}$

e $2x^{\frac{9}{40}}$ **f** $4x^{\frac{1}{30}}$ **g** $6x^{\frac{5}{6}}$ **h** $\frac{1}{4}x^{\frac{1}{10}}$

5 **a** $5^{\frac{3}{5}}x^{\frac{1}{5}}$ **b** $16^{\frac{1}{4}}x^{\frac{1}{10}}$ **c** $27^{\frac{2}{3}}x^{\frac{1}{4}}$ **d** $2x^{\frac{1}{5}}y^{\frac{3}{8}}$

e $4x^{\frac{2}{15}}y^{\frac{2}{7}}$ **f** $5x^{\frac{1}{4}}y^{\frac{3}{10}}$ **g** $5xy^{\frac{3}{10}}$ **h** $7x^{\frac{1}{3}}y^{\frac{20}{33}}$

6 **a** $28x^{\frac{7}{12}}y^{\frac{19}{20}}$ **b** $-72x^{\frac{3}{2}}y$ **c** $45x^{\frac{19}{24}}y^{\frac{7}{5}}$

d $5x^{\frac{2}{5}}y^{\frac{1}{6}}$ **e** $2x^{\frac{7}{40}}y^{\frac{4}{33}}$ **f** $x^{\frac{1}{14}}y^{\frac{1}{35}}$

7 **a** $a^m \times a^n = a^{m+n}$ **b** 3

c 3 **d** The answers are the same.

e **i** 2 **ii** 5 **iii** 10 **iv** x

f The fractional power of $\frac{1}{2}$ performs the same operation as finding the square root ($\sqrt{}$).

8 **a** 3

b 3

c The answers are the same.

d **i** 2 **ii** 5 **iii** 10 **iv** x

e $\sqrt[3]{}$ is a cube root and $\sqrt{}$ is a square root.

f The fractional power of $\frac{1}{3}$ performs the same operation as finding the cube root ($\sqrt[3]{}$).

9 **a** $\left(x^2\right)^{\frac{1}{3}} = x^{2\times\frac{1}{3}} = x^{\frac{2}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{1}{3}\times 2} = x^{\frac{2}{3}}$

b Since $\left(x^2\right)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2$ are both equal to $x^{\frac{2}{3}}$, we only need to show that $x^{\frac{2}{3}}$ is equal to $\sqrt[3]{x^2}$and $\left(\sqrt[3]{x}\right)^2$. So $x^{\frac{2}{3}} = x^{\frac{1}{3}\times 2} = \left(x^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{x}\right)^2$ and $x^{\frac{2}{3}} = x^{2\times\frac{1}{3}} = \left(x^2\right)^{\frac{1}{3}} = \sqrt[3]{x^2}$.

10 $x^{\frac{m}{n}} = x^{m\times\frac{1}{n}} = \left(x^m\right)^{\frac{1}{n}} = \sqrt[n]{x^m}$ and $x^{\frac{m}{n}} = x^{\frac{1}{n}\times m} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m$

11 **a** 5 **b** 6 **c** 1 **d** 12

e 10 **f** 14 **g** 2 **h** 3

i 5 **j** 1 **k** 1 **l** 2

12 **i** The answers would be the reciprocal of the original answers.

ii **a** $\frac{1}{5}$ **b** $\frac{1}{6}$ **c** 1 **d** $\frac{1}{12}$

e $\frac{1}{10}$ **f** $\frac{1}{14}$ **g** $\frac{1}{2}$ **h** $\frac{1}{3}$

i $\frac{1}{5}$ **j** 1 **k** 1 **l** $\frac{1}{2}$

iii Parts **g**, **h**, **i**, **j**, **k** and **l** can have a negative base as the root is odd (3 or 5). The cube root and 5th root of a negative number are real numbers as $(-a)^3 = -a$ so $\sqrt[3]{-a} = -a$, but the square root of a negative number is not a real number.

iv **g** $-\frac{1}{2}$ **h** $-\frac{1}{3}$ **i** $-\frac{1}{5}$

j -1 **k** -1 **l** $-\frac{1}{2}$

13 **a** $\sqrt{5}$ **b** $2\sqrt{2}$ **c** $2\sqrt{6}$ **d** $6\sqrt{3}$

e $2\sqrt{38}$ **f** $3\sqrt[3]{2}$ **g** $4\sqrt[3]{2}$ **h** $8\sqrt{5}$

i $9\sqrt{3}$ **j** $12\sqrt{3}$ **k** $5\sqrt{5}$ **l** $6\sqrt{6}$

14 **a** $x^{\frac{3}{2}}$ **b** $6^{\frac{1}{2}}$ **c** $x^{\frac{3}{2}}y^{\frac{5}{2}}$

d $3^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{3}{2}}$ **e** $4x^{-\frac{1}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{4}{3}}y^{\frac{2}{3}}$

15 **a** $108x^{\frac{9}{20}}y^{\frac{13}{15}}$ **b** $\frac{1}{5}x^{\frac{2}{9}}y^{\frac{7}{20}}$ **c** $2x^{\frac{1}{5}}y^{\frac{1}{4}}$

d $6x^4y^2$ **e** $x^{\frac{3}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{1}{2}}y^{\frac{5}{6}}$

g $96^{\frac{1}{2}}x^2y^{\frac{1}{2}}z^4$ **h** $x^4y^{\frac{2}{9}}z^{\frac{2}{3}}$

16 **a** $4x\sqrt{y}$ **b** $\frac{2y^3\sqrt{3y}}{x\sqrt{x}}$ **c** $xy\sqrt{3x}$

d $3\sqrt[3]{xy^2}$ **e** $2\sqrt[4]{x^3y^2z}$ **f** $\frac{7y^2\sqrt{7y}}{\sqrt{x}}$

17 **a** $18x^{\frac{5}{12}}y^{\frac{7}{15}}$ **b** $\frac{72y^{\frac{1}{12}}}{x^{\frac{5}{12}}}$ **c** $\frac{x^{\frac{2}{3}}}{y^{\frac{1}{2}}}$

d $\frac{1}{4x^{\frac{1}{15}}y^{\frac{7}{20}}}$ **e** $\frac{1}{x^{\frac{1}{12}}y^{\frac{3}{4}}}$ **f** $\frac{16}{x^{\frac{8}{5}}y^5}$

g $10x^{\frac{7}{30}}$ **h** $\frac{y^{\frac{13}{30}}}{3^{\frac{1}{2}}x^{\frac{1}{24}}}$

18 **a** $\frac{3^{\frac{2}{3}}}{9}$ or $3^{-\frac{4}{3}}$ **b** $x = -\frac{2}{3}$

19 **a** $x = -\frac{5}{2}$ **b** $x = 11$ **c** $x = -\frac{1}{2}$

20 **a** Both denominators specify how many repeated parts the base is broken into. When multiplying, the parts are added; when raising to the power, the parts are multiplied. Both numerators describe the number of those repeated parts to operate together.

b **i** 3 **ii** 27 **iii** 243

iv 823 543 **v** 59 049 **vi** 8

c **i** $81^{\frac{1}{4}} = \sqrt[4]{81}$ **ii** $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3$

iii $81^{\frac{5}{4}} = \left(\sqrt[4]{81}\right)^5$ **iv** $343^{\frac{7}{3}} = \left(\sqrt[3]{343}\right)^7$

v $81^{\frac{5}{2}} = \left(\sqrt[2]{81}\right)^5 = \left(\sqrt{81}\right)^5$ **vi** $256^{\frac{3}{8}} = \left(\sqrt[8]{256}\right)^3$

d The denominator of the exponent and the root of a surd both specify how many repeated parts the base is broken into. The numerator of the exponent and the index of a surd describe the number of those repeated factors of the base to multiply together.

21 **a** $\sqrt{ab} = (ab)^{\frac{1}{2}}$
$= a^{\frac{1}{2}} \times b^{\frac{1}{2}}$
$= \sqrt{a} \times \sqrt{b}$

b $\sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{2}}$
$= \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$
$= \frac{\sqrt{a}}{\sqrt{b}}$

22 **a** $\sqrt{10^5}$ **b** $\sqrt[4]{2^{17}}$ **c** $\sqrt[25]{3^4}$ **d** $\sqrt[200]{5^{1111}}$

2G Logarithms

EX p91

1 **a** $\log_3(27) = 3$ **b** $\log_4(4) = 1$ **c** $\log_5(625) = 4$

d $\log_{10}(1000) = 3$ **e** $\log_6(36) = 2$ **f** $\log_9(729) = 3$

g $\log_3(81) = 4$ **h** $\log_2(128) = 7$

2 **a** $2^5 = 32$ **b** $3^2 = 9$ **c** $5^2 = 25$

d $7^2 = 49$ **e** $4^2 = 16$ **f** $2^{10} = 1024$

g $6^3 = 216$ **h** $10^2 = 100$

3 **a** 4 **b** 4 **c** 3 **d** 3 **e** 2 **f** 3

g 2 **h** 5 **i** 4 **j** 4 **k** 2 **l** 4

m 3 **n** 3 **o** 2 **p** 5

4 **a** **i** $\frac{1}{6}$ **ii** $\log_6\left(\frac{1}{6}\right) = -1$

b **i** $\frac{1}{4}$ **ii** $\log_4\left(\frac{1}{4}\right) = -1$

c **i** $\frac{1}{4}$ **ii** $\log_2\left(\frac{1}{4}\right) = -2$

d **i** $\frac{1}{32}$ **ii** $\log_2\left(\frac{1}{32}\right) = -5$

e **i** $\frac{1}{25}$ **ii** $\log_5\left(\frac{1}{25}\right) = -2$

f **i** $\frac{1}{10}$ **ii** $\log_{10}\left(\frac{1}{10}\right) = -1$

g **i** $\frac{1}{1000}$ **ii** $\log_{10}\left(\frac{1}{1000}\right) = -3$

h **i** $\frac{1}{27}$ **ii** $\log_3\left(\frac{1}{27}\right) = -3$

i **i** 2 **ii** $\log_{\frac{1}{2}}(2) = -1$

j **i** 9 **ii** $\log_{\frac{1}{3}}(9) = -2$

k **i** 125 **ii** $\log_{\frac{1}{5}}(125) = -3$

l **i** 100 **ii** $\log_{0.01}(100) = -1$

5 **a** -3 **b** -3 **c** -2 **d** -1

e -4 **f** -6 **g** -3 **h** -2

6 **a** **i** 16 **ii** $\frac{1}{16} = 0.0625$

b no real number

c **i** no real number **ii** no real number

iii no real number **iv** no real number

v no real number **vi** no real number

d There is no real logarithm of a negative number to any positive integer base.

7 **a** no

b **i** no **ii** no **iii** no

iv no **v** no

c There is no power of base 1 that gives an answer other than 1, so there is no logarithm of any other number to base 1.

8 **a** Raise 2 to the power of 7. Then, take the logarithm with base 2 of 2^7. The result is 7.

b **i** 7 **ii** 5 **iii** 9

iv 140 **v** −6 **vi** 123

c The logarithm with base a is the exponent to raise a to get a^x. x is the exponent to raise a to get a^x. Therefore, $\log_a(a^x) = x$.

9 **a** Take the logarithm to base 2 of 7. Then, raise 2 to the power of that. The result is 7.

b **i** 7 **ii** 5 **iii** 11

iv 531 **v** 8 **vi** 147

c The logarithm to base a is the exponent, y, to raise a to get x. We raise a to that exponent, y. Therefore, $a^{\log_a(x)} = x$.

10 **a** $\log_x(64) = 2$

$x^{\log_x(64)} = x^2$

$64 = x^2$

$x = 8$

b 8, as $x > 0$ so $x = -8$, while a solution to $x^2 = 64$ is not a solution for $\log_x(64) = 2$.

c $x^2 = 8^2$ **d** It is the same.

11 **a** 2 **b** 5 **c** 4 **d** 2

e 2 **f** 3 **g** 4 **h** 2

12 **a** $\log_4(x) = 3$

$4^{\log_4(x)} = 4^3$

$x = 4^3$

b 64

c It can be easily solved by multiplying the base by itself the number of times indicated by the exponent.

13 **a** 27 **b** 25 **c** 6 **d** 10 000

e 32 **f** 64 **g** $\frac{1}{27}$ **h** $\frac{1}{4}$

14 **a** $(a^2)^{\frac{1}{2}} = a^{2\times\frac{1}{2}} = a^1 = a$

$(a^{\frac{1}{2}})^2 = a^{\frac{1}{2}\times 2} = a^1 = a$

b $12^2 = 144$

$12 = 144^{\frac{1}{2}}$

$\log_{144}(12) = \log_{144}(144^{\frac{1}{2}})$

$\log_{144}(12) = \frac{1}{2}$

c $\frac{1}{x}$

15 **a** $\frac{1}{3}$ **b** $\frac{1}{6}$ **c** $\frac{1}{4}$ **d** $\frac{1}{8}$

16 **a** 1 **b** 1 **c** 1 **d** 1

17 **a** $\frac{1}{2}$ **b** 2 **c** $-\frac{3}{2}$ **d** $\frac{7}{3}$

e $\frac{3}{2}$ **f** $\frac{3}{8}$ **g** 2 **h** $-\frac{1}{3}$

EX p96

2H Logarithm laws

1 **a** 1 **b** 1 **c** 1 **d** 4 **e** 3 **f** 0

g 0 **h** 0

2 **a** $\log_{10}(3) + \log_{10}(5)$ **b** $\log_2(6) + \log_2(3)$

c $\log_5(2.5) + \log_5(10)$ **d** $\log_2(x) + \log_2(y)$

3 **a** $\log_3(12)$ **b** $\log_{10}(15)$ **c** $\log_2(50)$

d $\log_4(72)$ **e** $\log_a(15)$ **f** $\log_2(xy)$

4 **a** $\log_{10}(5) - \log_{10}(7)$ **b** $\log_2(5) - \log_2(3)$

c $\log_5(11) - \log_5(20)$ **d** $\log_2(a) - \log_2(b)$

5 **a** $\log_3\left(\frac{12}{5}\right)$ **b** $\log_{10}\left(\frac{50}{15}\right)$ **c** $\log_2\left(\frac{25}{10}\right)$

d $\log_4\left(\frac{36}{9}\right)$ **e** $\log_a\left(\frac{3}{15}\right)$ **f** $\log_2\left(\frac{x}{y}\right)$

6 **a** 4 **b** 2 **c** 5 **d** 4 **e** 2 **f** 3

g 2 **h** 2 **i** 9 **j** 2 **k** 2 **l** 4

7 **a** 2 **b** 2 **c** 3

8 **a** $2 \times \log_3(7)$

b $9 \times \log_5(2)$

c $3 \times \log_4(8)$

d $4 \times \log_9(11)$

9 **a** 5 **b** 2 **c** 5

d 8 **e** 9 **f** 12

g 10 **h** 2

10 **a** −2 **b** −1 **c** −2

d −14 **e** −30 **f** −21

g $-\frac{1}{3}$ **h** −5

11 **a** 1 **b** 4 **c** 4

d $\frac{1}{2}$ **e** 1 **f** 1

12 **a** 3 **b** 2 **c** 2

d −1 **e** −2 **f** 3

13 **a** 3 **b** 2 **c** 6 **d** 4

e 1 **f** 7 **g** 5 **h** 3

i 3 **j** 3

14 **a** 3 **b** 3 **c** 9 **d** 4 **e** 216 **f** 64

g −3 **h** 36 **i** 1 **j** $\frac{1}{2}$ **k** $\frac{1}{2}$ **l** 4

15 **a** 0.5

b $\log_{10}(9) = \log_{10}(3^2) = 2\log_{10}(3)$

c $\frac{\log_{10}(3)}{\log_{10}(9)} = \frac{\log_{10}(3)}{2\log_{10}(3)} = \frac{1}{2} = 0.5$

16 Hailey, because $\frac{\log_3(8)}{\log_3(2)} = \frac{3\log_3(2)}{\log_3(2)} = 3$

17 **a** 4 **b** $\frac{3}{2}$ **c** 3

d $\frac{3}{4}$ **e** $\frac{2}{3}$ **f** 3

18 a i $\log_5(2x) + \log_5(3) - \log_5(12) = 0$

$\log_5(6x) - \log_5(12) = 0$

$\log_5\left(\frac{6x}{12}\right) = 0$

$\log_5\left(\frac{x}{2}\right) = 0$

ii $5^0 = \frac{x}{2}, \frac{x}{2} = 1, x = 2$

b i $\log_5(2x) + \log_5(3) = \log_5(6x) = \log_5(12)$

ii $6x = 12, x = 2$

19 a 3 **b** 4

c 20 **d** $\frac{2}{3}$

e $\frac{1}{14}$ **f** $\frac{25\,000}{3}$ or $8333\frac{1}{3}$

20 $x = 4$ or $x = 5$

21 $\log_{10}(2^{2x+1}) = \log_{10}(4)$

$(2x + 1)\log_{10}(2) = \log_{10}(2^2)$

$(2x + 1)\log_{10}(2) = 2\log_{10}(2)$

$2x + 1 = 2$

$2x = 1$

$x = \frac{1}{2}$

22 $y = 77$

2I Exponential equations

EX p100

1 a $x = 9$ **b** $x = 6$ **c** $x = -3$ **d** $x = 1$

e $x = 8$ **f** $x = 4$ **g** $x = 7$ **h** $x = -10$

i $x = 3$ **j** $x = 5$ **k** $x = 6$ **l** $x = 0$

2 a $x = 3$ **b** $x = 2$ **c** $x = 5$ **d** $x = 4$

e $x = 7$ **f** $x = 3$ **g** $x = 2$ **h** $x = -1$

3 a $x = \log_2(11)$ **b** $x = \log_5(92)$

c $x = \log_{92}(5)$ **d** $x = -\log_{14}(4)$

e $x = -\log_3(2)$ **f** $x = \log_7\left(\frac{53}{2}\right)$

g $x = \frac{1}{2}\log_5(17)$ **h** $x = \frac{1}{3}\log_{10}(75)$

i $x = \log_8(33)$ **j** $x = \frac{1}{6}\log_{11}(55)$

k $x = \log_{17}(12)$ **l** $x = -\frac{1}{4}\log_{12}\left(\frac{46}{9}\right)$

4 a $x = \log_8(10) + 1$ **b** $x = \log_3(12) - 1$

c $x = \log_5(2) - 3$ **d** $x = \log_2(6) + 5$

e $x = -\log_2(48) + 3$ **f** $x = -\log_9(20) + 2$

g $x = -\log_6(4) + 3$ **h** $x = -\log_2(30) + 6$

i $x = \frac{1}{2}\log_4(24) - \frac{3}{2}$ **j** $x = \frac{1}{4}\log_3(87) - \frac{1}{4}$

k $x = \frac{1}{5}\log_7(3) - \frac{2}{5}$ **l** $x = \frac{1}{4}\log_{10}\left(\frac{3}{2}\right) - \frac{3}{4}$

5 a $x = 2$ **b** $x = 3, x = -3$

c $x = 2, x = -2$

6 a $x = 2, x = 3$ **b** $x = 3, x = 4$

c $x = 6, x = 4$ **d** $x = 0, x = 1, x = 2$

e $x = -5, x = -3$ **f** $x = 1, x = \frac{5}{3}$

7 a 2 and 3 **b** 5 and 6 **c** 1 and 2

d 3 and 4 **e** 1 and 2 **f** 2 and 3

g 1 and 2 **h** 4 and 5

8 a $\frac{9}{32} \times 64^x = 27$ **b** $x = \log_{64}(96)$

9 a The solution to the equation would be $x = \log_3(-1)$. However, the logarithm of a negative number is not a real number. Therefore, the equation has no real solution.

b The solution to the equation would be $x = \log_3(0)$. However, the logarithm of 0 is not a defined value. Therefore, the equation has no real solution.

10 a $x = 3$ **b** $x = 1$

c $x = \log_{10}(15)$ **d** $x = 1, x = 2$

e $x = \log_3(7)$

11 a 100°C **b** 61°C **c** 4.4 **d** 20°C

e decay; temperature decreases as the amount of time increases

12 a 2 **b** 82 **c** 3.14

d population growth; population increases as the amount of time increases

13 a i 8.5 billion **ii** 11.9 billion

b 2053. It will reach 15 billion after 103.96 years, or very close to the end of 2053.

c The model predicted 7.1 billion, so the 2010 figure was slightly less than the prediction; there have been changes to birth and death rates so, despite the population continuing to grow, the overall rate of population growth has decreased slightly since 1950.

14 a 45 bacteria

b i $3\log_2(10)$ hours **ii** 10 hours

c 750 000 000 bacteria

d 5025 bacteria

e

t	66	67	68	69	70
N	204	202	200	198	196

f 68 hours

g $\log_{0.99}\left(\frac{1}{2}\right) - 1 = \log_{0.99}\left(\frac{50}{99}\right)$ hours

15 a $k = 0.13$ **b** 11 minutes

c 1.5 min or 1 min 30 sec or 90 sec

d $m = 0.28$

16 $x = -4, x = -2, x = 3, x = 7$

17 a i $\frac{\log_{10}(7)}{\log_{10}(3)}$ **ii** $\frac{\log_{10}(3)}{\log_{10}(7)}$ **iii** $\frac{\log_{10}(45)}{\log_{10}(11)}$

iv $\frac{\log_{10}(12)}{\log_{10}(101)}$ **v** $\frac{\log_{10}(24)}{\log_{10}(19)}$ **vi** $\frac{\log_{10}(b)}{\log_{10}(a)}$

b i $\log_9(8)$ **ii** $\log_8(9)$ **iii** $\log_5(18)$

iv $\log_{15}(6)$ **v** $\log_2(75)$ **vi** $\log_t(v)$

c $2^x = 54$

d $x = \log_2(54)$

$2^x = 54$

$\log_{10}(2^x) = \log_{10}(54)$

$x\log_{10}(2) = \log_{10}(54)$

$x = \frac{\log_{10}(54)}{\log_{10}(2)}$

e
$$x = \log_b(a)$$
$$b^x = b^{\log_b(a)}$$
$$b^x = a$$
$$\log_{10}(b^x) = \log_{10}(a)$$
$$x\log_{10}(b) = \log_{10}(a)$$
$$x = \frac{\log_{10}(a)}{\log_{10}(b)}$$

18 **a** 1.05 **b** 2.11 **c** 13.0 **d** 0.726
e −3.18 **f** 1.03 **g** 0.377 **h** 4.87
i −0.215

CHAPTER 2 review

Multiple choice

1 A **2** D **3** C **4** D **5** B
6 B **7** E **8** A **9** B **10** E

Short answer

1 **a** rational **b** irrational **c** rational
d irrational **e** irrational **f** rational

2 **a**, **b**, **d**, **f**

3 **a** 4.899 **b** 0.059

4 **a** $0.65 = \frac{65}{100}$ which is a fraction and therefore rational.

b $0.2175 = \frac{2175}{10\,000}$ which is a fraction and therefore rational.

c $0.5\dot{2} = \frac{47}{90}$ which is a fraction and therefore rational.

d $1.\overline{23} = 1\frac{23}{99}$ which is a fraction and therefore rational.

5 **a** $4\sqrt{6}$ **b** $15\sqrt{5}$ **c** $3\sqrt{3}$

6 **a** $72\sqrt{2}$ **b** -150

7 **a** $24\sqrt{3}$ m² **b** $90\sqrt{2}$ cm²

8 **a** $\sqrt{77}$ **b** $27\sqrt{30}$

9 **a** $\sqrt{3}$ **b** $4\sqrt{3}$

10 **a** $-2\sqrt{2}$ **b** $2\sqrt{7} + 5\sqrt{5}$
c $2\sqrt{10} + 4\sqrt{5}$ **d** $3\sqrt{6} + 2 + 3\sqrt{5}$

11 **a** $2\sqrt{5} - 22\sqrt{6}$ **b** $4\sqrt{3} + 6\sqrt{2}$
c $15\sqrt{5}$ **d** $6\sqrt{2} + 24\sqrt{5} + 4\sqrt{10}$

12 **a** $6 - 8\sqrt{2} + 12\sqrt{3} - 16\sqrt{6}$ **b** $92 + 16\sqrt{15}$

13 **a** $\frac{5\sqrt{6}}{12}$ **b** $\frac{3\sqrt{3}}{2}$

c $\frac{8\sqrt{30} - 4\sqrt{15}}{21}$

d $\frac{-6\sqrt{7} - 10\sqrt{3} - 3\sqrt{42} - 15\sqrt{2}}{12}$

14 **a** **i** $x^{\frac{3}{2}}$ **ii** $\sqrt{x^3}$

b **i** $x^{\frac{5}{6}}$ **ii** $\sqrt[6]{x^5}$

c **i** $\frac{1}{9}x^{\frac{5}{12}}$ **ii** $\frac{1}{9}\sqrt[12]{x^5}$

d **i** $24^{\frac{1}{2}}x^2y^{\frac{3}{2}}$ **ii** $2x^2y\sqrt{6y}$

e **i** $3x^{\frac{1}{8}}y^{\frac{1}{5}}$ **ii** $3\sqrt[8]{x}\sqrt[5]{y}$

f **i** $\left(\frac{1}{5}\right)\left(\frac{3}{8}\right)^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{1}{2}}z^{-\frac{1}{2}}$

ii $\frac{x}{10}\sqrt{\frac{3xy}{2z}}$ or $\frac{x\sqrt{6xyz}}{20z}$ (rationalising the denominator)

15 **a** $\log_3 81 = 4$ **b** $\log_6 216 = 3$
c $\log_2 32 = 5$ **d** $\log^x w = y$

16 **a** $2^3 = 8$ **b** $4^3 = 64$
c $10^{-4} = \frac{1}{10\,000}$ **d** $x^b = z$

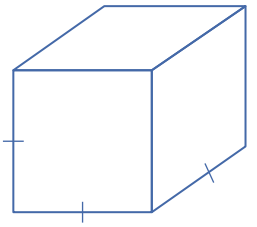
17 **a** 3 **b** 2 **c** 3 **d** 3

18 **a** 2 **b** $\frac{1}{125}$ **c** 4
d $\frac{1}{2}$ **e** $\frac{12}{5}$ **f** $\frac{625}{4}$

19 **a** $x = 0$ **b** $x = -\frac{1}{4}$
c $x = \frac{1}{3}\log_5 15$ **d** $x = \frac{1}{2}\log_6 7$

Analysis

1 **a**



$7\sqrt{10} + 11\sqrt{7}$ cm

b $(1337 + 154\sqrt{70})$ cm²

c irrational

d $(8022 + 924\sqrt{70})$ cm²

e \$20.16, correct to the nearest cent

f You need to show that a 1 L can of the protective paint contains enough paint to cover $(8022 + 924\sqrt{70})$ cm² three times, an area of $1.575\ldots \times 3 \approx 4.725\ldots < 5$; it does.

g 94.5% correct to one decimal place

h The diagonal of the base is
$$\sqrt{\left(7\sqrt{10} + 11\sqrt{7}\right)^2 + \left(7\sqrt{10} + 11\sqrt{7}\right)^2}$$
$$= \sqrt{2} \times \left(7\sqrt{10} + 11\sqrt{7}\right)$$
$$= 7\sqrt{20} + 11\sqrt{14}$$
$$= 14\sqrt{5} + 11\sqrt{14}$$
$$= 72.463...$$
72.463 cm < 80 cm, so it won't fit.

i Yes, diagonally from bottom corner to opposite top corner.

j You need to assume that the interior measurements are the same as exterior measurements.

2 **a** 10 000

b **i** $\log_{10}(2)$ **ii** 0.30

c $B = 10\,000 \times 2^t$

d 640 000 bees

e **i** $\log_2(100)$ **ii** 7

f $B = 10\,000\,000 \times 0.91^n$

g 74 years

CHAPTER 3 Linear relationships

EX p113

3A Solving linear equations

1 **a** $x = \frac{19}{3}$ **b** $x = -40$ **c** $x = \frac{2}{7}$
d $x = \frac{24}{5}$ **e** $x = -53$ **f** $x = -3$
g $x = -\frac{5}{3}$ **h** $x = -8$ **i** $x = \frac{29}{12}$
j $x = -\frac{19}{2}$ **k** $x = -\frac{43}{6}$ **l** $x = \frac{62}{5}$
m $x = -\frac{32}{9}$ **n** $x = \frac{15}{2}$ **o** $x = -\frac{7}{3}$

2 **a** $x = -8\frac{1}{2}$ **b** $x = 1.7$ **c** $x = 4$
d $x = -2.464$ **e** $x = 2.4$ **f** $x = \frac{15}{77}$
g $x = \frac{1}{2}$ **h** $x = -\frac{1}{3}$ **i** $x = -\frac{20}{21}$

3 **a** $x = -5$ **b** $x = 3$ **c** $x = -2$
d $x = 6$ **e** $x = -\frac{1}{3}$ **f** $x = 7$
g $x = 3$ **h** $x = \frac{1}{9}$ **i** $x = 6$

4 **a** $x = 2$ **b** $x = -5$ **c** $x = -6$
d $x = -4$ **e** $x = 4$ **f** $x = \frac{15}{2}$
g $x = \frac{10}{3}$ **h** $x = \frac{2}{35}$ **i** $x = 2$
j $x = -1$ **k** $x = \frac{1}{4}$ **l** $x = -\frac{12}{5}$
m $x = -\frac{11}{8}$ **n** $x = \frac{35}{2}$ **o** $x = \frac{5}{6}$

5 **a** $x = 2$ **b** $x = 5$ **c** $x = -3$
d $x = 1$ **e** $x = -7$ **f** $x = -2$
g $x = 3$ **h** $x = 4$ **i** $x = \frac{3}{7}$
j $x = -4$ **k** $x = 2$ **l** $x = -5$

6 **a** $x = 5$ **b** $x = -2$ **c** $x = 3$
d $x = -1$ **e** $x = 6$ **f** $x = 2$
g $x = -4$ **h** $x = 8$ **i** $x = 4$
j $x = -1$ **k** $x = 5$ **l** $x = 9$

7 **a** $x = -8$ **b** $x = 4$ **c** $x = -2$
d $x = 2$ **e** $x = -1$ **f** $x = 4$

8 **a** $x = -8$ **b** $x = 31$ **c** $x = 8$
d $x = 1$ **e** $x = 17$ **f** $x = 13$
g $x = -4\frac{2}{3}$ **h** $x = 6\frac{1}{7}$ **i** $x = -2\frac{1}{8}$

9 **a** $x = 3$ **b** $x = -3$ **c** $x = 12$
d $x = -\frac{3}{7}$ **e** $x = \frac{14}{15}$ **f** $x = \frac{1}{11}$

10 **a** $x = b - a$ **b** $x = k + p$ **c** $x = \frac{d}{c}$
d $x = \frac{h - g}{3}$ **e** $x = \frac{y - 5}{4}$ **f** $x = mn - a$
g $x = \frac{5 + 2w}{7}$ **h** $x = a(b - c)$ **i** $x = \frac{h}{k}$
j $x = \frac{e + f}{2}$ **k** $x = \frac{v - 2y}{2}$
l $x = p(w - q)$ **m** $x = \frac{a}{b + c}$
n $x = \frac{2ny + m}{k}$ **o** $x = \frac{5c - 3d}{8}$

11 **a** Decide that n = the number of weeks of saving. Any letter will do.
b $24n + 105 = 219$
c $n = 4.75$
d It will take Liam 5 weeks to save for his new tennis racquet.

12 **a** Let m be the cost of a makeup kit.
$4m + 29 = 387$
$m = 89.5$
Cost: \$89.50
b Let the length be l m and the width w m.
$l = w + 12$
$2l + 2w = 88$
$2(w + 12) + 2w = 88$
$4w + 24 = 88$
$w = 16$, $l = 28$
length: 28 m, width: 16 m
c Let their scores be l, s and b, respectively.
$l = s + 5$, $b = s - 3$
$l + s + b = (s + 5) + s + (s - 3)$
$3s + 2 = 29$
$s = 9$, $l = 14$, $b = 6$
Lisa: 14 goals, Sara: 9 goals, Bindi: 6 goals
d Let n = the number of people
$27n + 400 = 1800$; $n = 51.851\ldots$
51 people

13 **a** $6d + 3.8 = 4d + 9.2$, where d is the cost of a dumpling
b \$2.70

14 **a** $A = 50$ m^2 **b** $h = 7$ mm
c $a = 7$ m **d** $a = \frac{2A}{h} - b$
e **i** $a = 52$ cm **ii** $a = 7.6$ m

15 **a** $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$
$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$
$\frac{1}{c} = \frac{b}{ab} + \frac{a}{ab}$
$\frac{1}{c} = \frac{a + b}{ab}$
$c = \frac{ab}{a + b}$
b **i** 3 cm **ii** 400 ohms
c $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$
$\frac{1}{a} = \frac{1}{c} - \frac{1}{b}$
$\frac{1}{a} = \frac{b}{bc} - \frac{c}{bc}$
$\frac{1}{a} = \frac{b - c}{bc}$
$a = \frac{bc}{b - c}$
d **i** 40 cm **ii** 16 ohms

16 $5x + 20 = 3x + 50$; $x = 15$; 15, 17, 19, 21, 23

17 **a** $x = \frac{33}{5}$ **b** $x = -\frac{41}{34}$ **c** $x = \frac{33}{52}$

18 **a** $x = -6$ **b** $x = 8$ **c** $x = -1$

d $x = 13$ **e** $x = \frac{1}{8}$ **f** $x = -2\frac{7}{10}$

19 **a** $x = \frac{p - qr}{nr - m} = \frac{qr - p}{m - nr}$

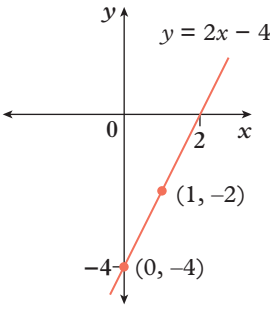
b $x = \frac{b^2 - 2aby - a^2}{a^2y - 2ab - b^2y} = \frac{a^2 + 2aby - b^2}{b^2y + 2ab - a^2y}$

c $x = -\frac{1}{y}$

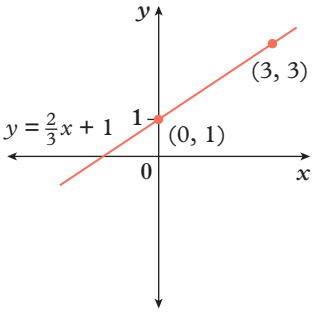
EX 3B Sketching linear graphs

p120

1 **a**



b



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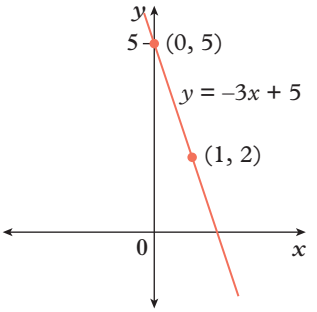

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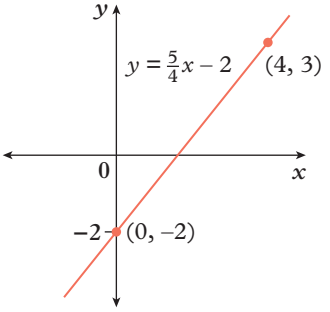

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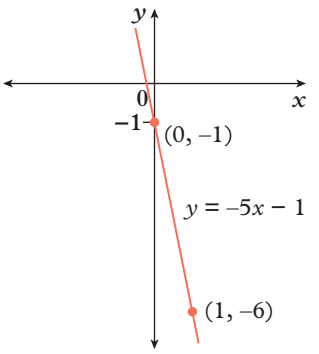

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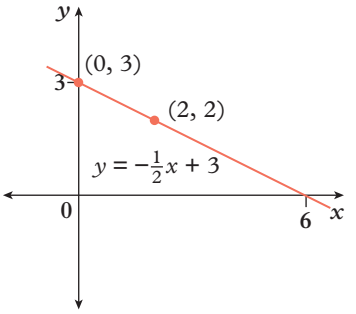

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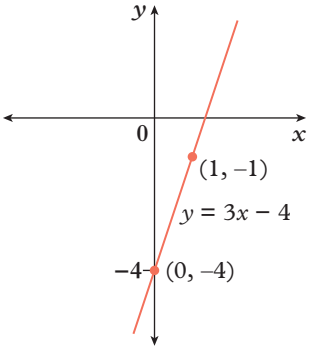

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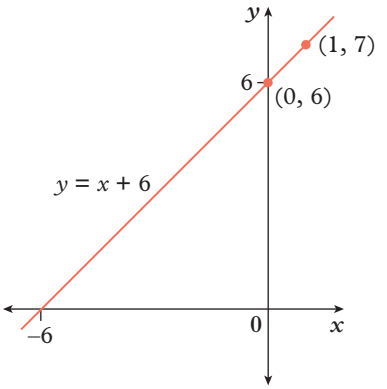

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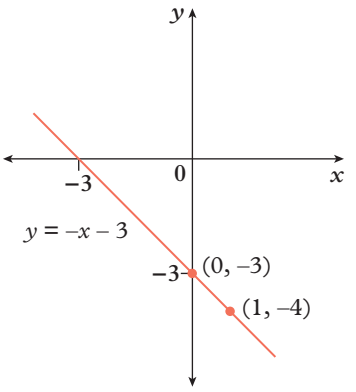
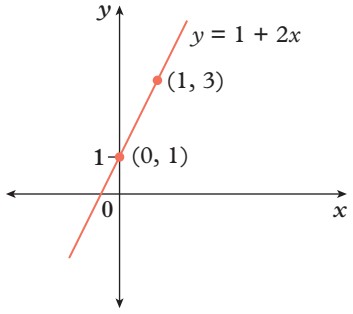
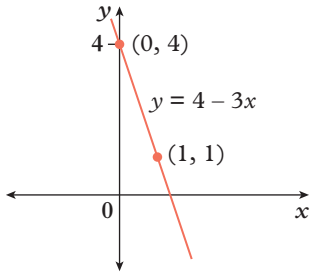


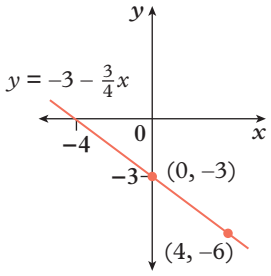
2 a i, ii, iii $y = 2x + 1$, $m = 2$, $c = 1$



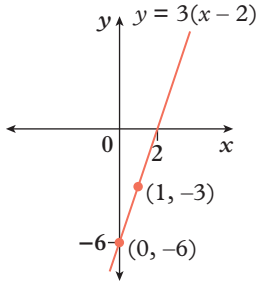
b i, ii, iii $y = -3x + 4$, $m = -3$, $c = 4$



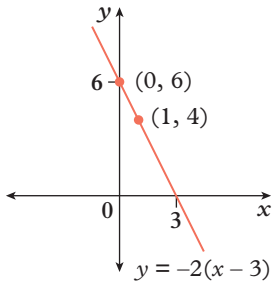
c i, ii, iii $y = -\frac{3}{4}x - 3$, $m = -\frac{3}{4}$, $c = -3$



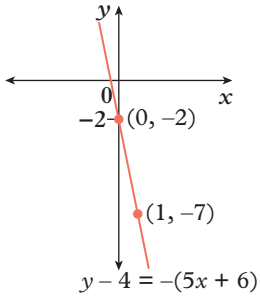
d i, ii, iii $y = 3x - 6$, $m = 3$, $c = -6$



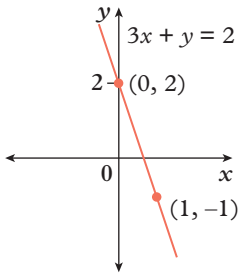
e i, ii, iii $y = -2x + 6$, $m = -2$, $c = 6$



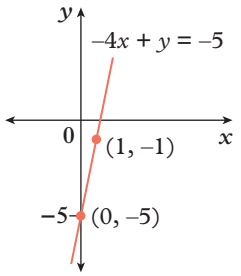
f i, ii, iii $y = -5x - 2$, $m = -5$, $c = -2$



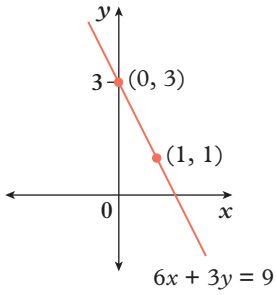
g i, ii, iii $y = -3x + 2$, $m = -3$, $c = 2$



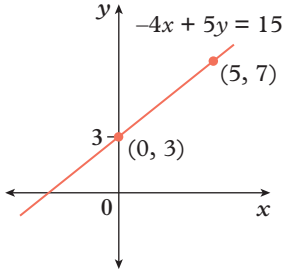
h i, ii, iii $y = 4x - 5$, $m = 4$, $c = -5$



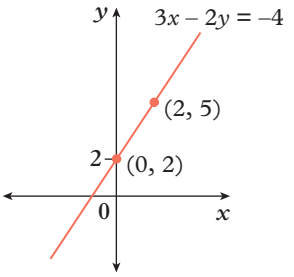
i i, ii, iii $y = -2x + 3$, $m = -2$, $c = 3$



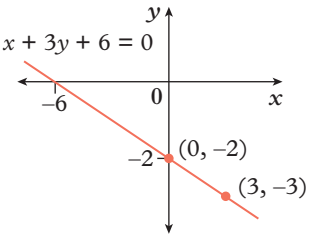
j i, ii, iii $y = \frac{4}{5}x + 3$, $m = \frac{4}{5}$, $c = 3$



k **i, ii, iii** $y = \frac{3}{2}x + 2, m = \frac{3}{2}, c = 2$

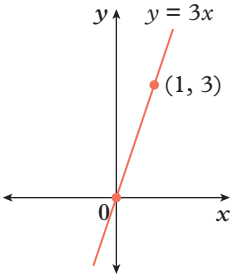


l **i, ii, iii** $y = -\frac{1}{3}x - 2, m = -\frac{1}{3}, c = -2$

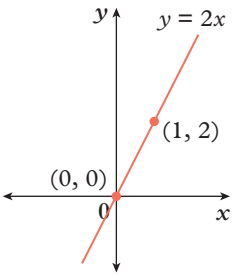


3 **a** **i** $m = 3$ **ii** $c = 0$

b



4 **a**



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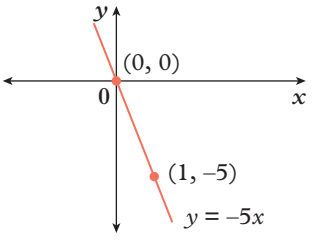

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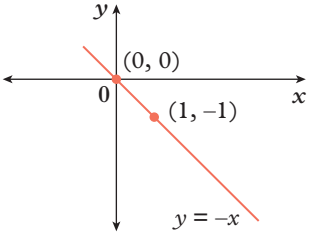

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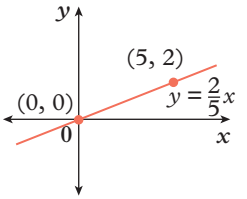
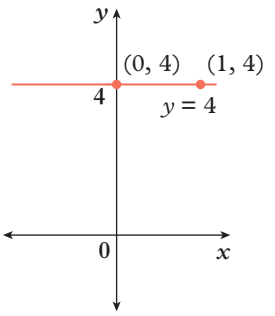
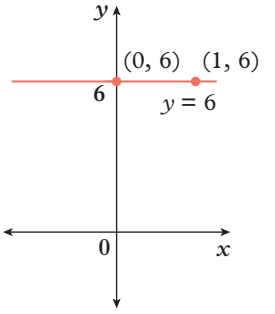


5 **a** **i** $m = 0$ **ii** $c = 4$

b



6 **a**



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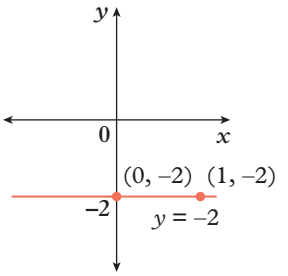

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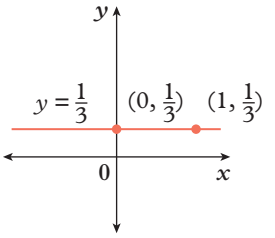

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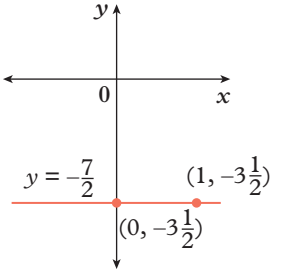
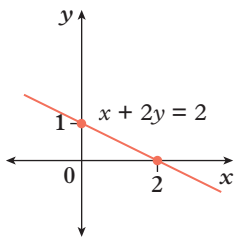
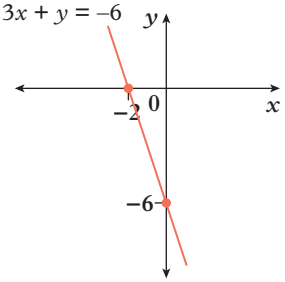


7 **a**



b $3x + y = -6$



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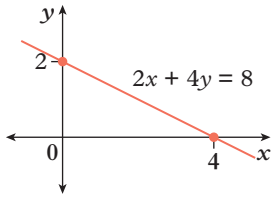

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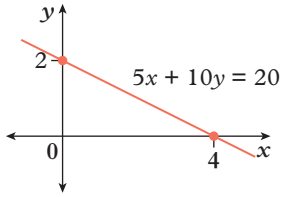

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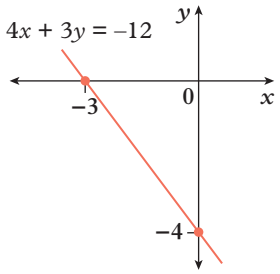

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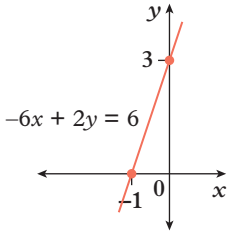

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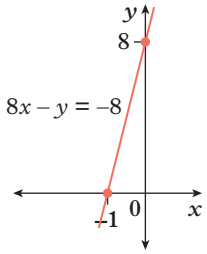

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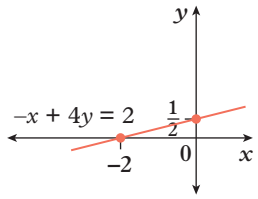

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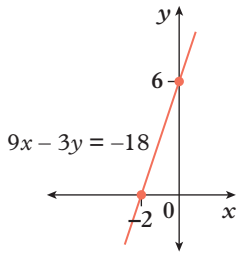

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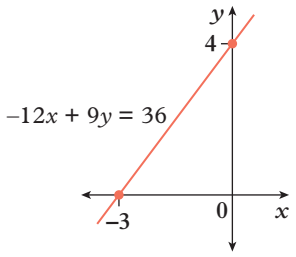

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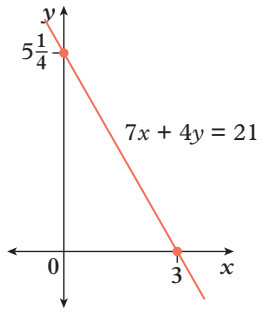

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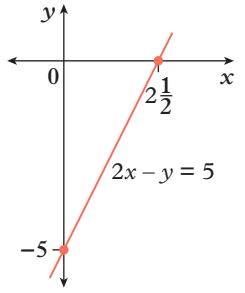
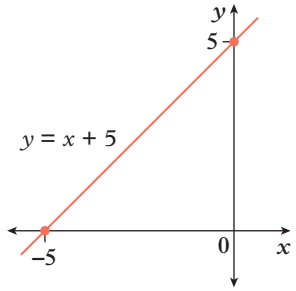
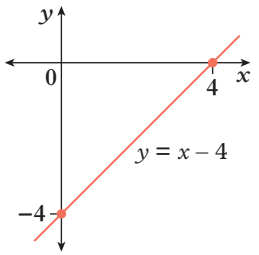


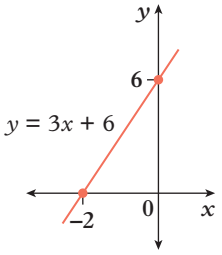
8 **a** **i, ii, iii** x-intercept: $(-5, 0)$, y-intercept: $(0, 5)$



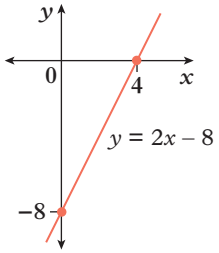
b **i, ii, iii** x-intercept: $(4, 0)$, y-intercept: $(0, -4)$



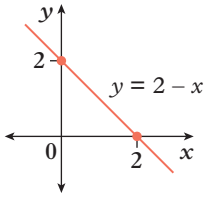
c **i, ii, iii** x-intercept: $(-2, 0)$, y-intercept: $(0, 6)$



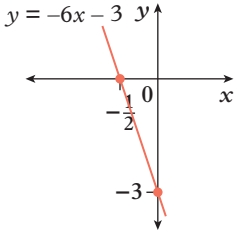
d **i, ii, iii** x-intercept: $(4, 0)$, y-intercept: $(0, -8)$



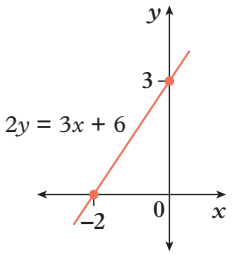
e **i, ii, iii** x-intercept: $(2, 0)$, y-intercept: $(0, 2)$



f **i, ii, iii** x-intercept: $\left(-\frac{1}{2}, 0\right)$, y-intercept: $(0, -3)$



9 **a**



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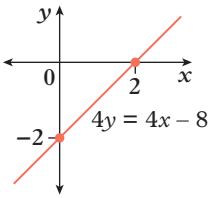

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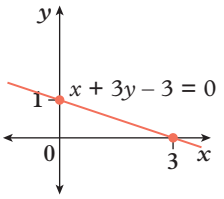

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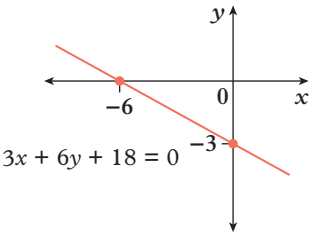

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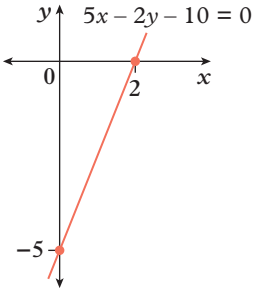
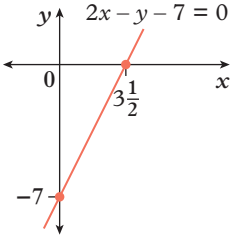


f



10 **a** **i** x-intercept: $(0, 0)$

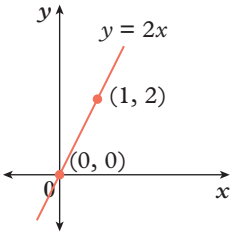
ii y-intercept: $(0, 0)$

b No; you only have one point.

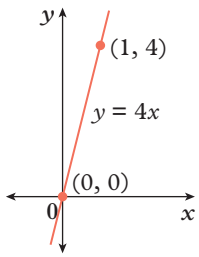
c You need the coordinates of one other point on the line.

d $y = 2$ **e** $(0, 0)$ and $(1, 2)$

f



11 a



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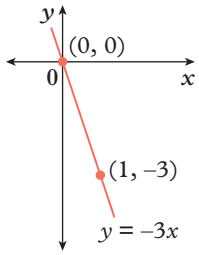

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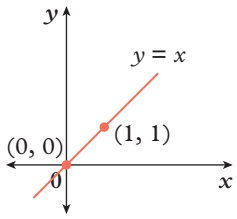

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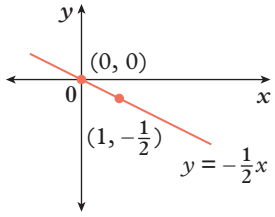
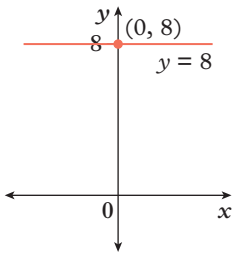


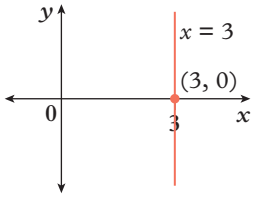
12 a i y-intercept: $(0, 8)$, no x-intercept

ii



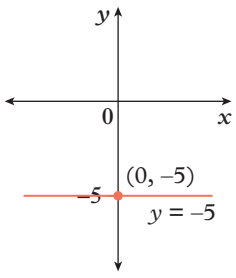
b i x-intercept: $(3, 0)$, no y-intercept

ii



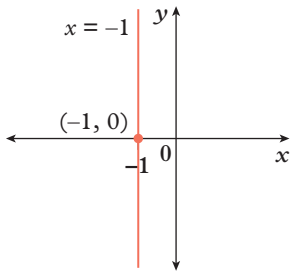
c i y-intercept: $(0, -5)$, no x-intercept

ii



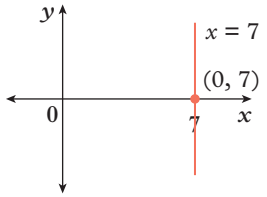
d i x-intercept: $(-1, 0)$, no y-intercept

ii



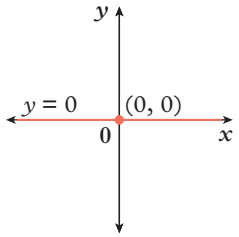
e i x-intercept: $(7, 0)$, no y-intercept

ii



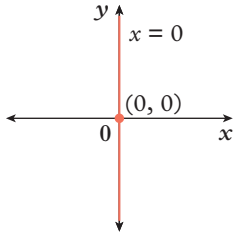
f i y-intercept: $(0, 0)$, no x-intercept

ii



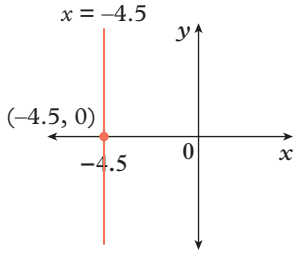
g i x-intercept: $(0, 0)$, no y-intercept

ii



h i x-intercept: $(-4.5, 0)$, no y-intercept

ii



13 A linear graph can have one or two axis intercepts (or can perhaps lie along the x-axis or y-axis).

a x-intercept: $(2, 0)$, y-intercept: $(0, -4)$, two intercepts

b no x-intercept, y-intercept: $(0, -7)$, one intercept

c x-intercept: $(3, 0)$, y-intercept: $(0, 2)$, two intercepts

d x-intercept: $(0, 0)$, y-intercept: $(0, 0)$, one intercept

e x-intercept: $(10, 0)$, y-intercept: $(0, -50)$, two intercepts

f x-intercept: $(-5, 0)$, y-intercept: $(0, 3)$, two intercepts

g x-intercept: $(28, 0)$, no y-intercept, one intercept

h x-intercept: $(6, 0)$, y-intercept: $(0, -2)$, two intercepts

i no x-intercept, y-intercept: $(0, 5.6)$, one intercept

14 a **i** $y = \frac{1}{p}(x - q)$

ii $y = \frac{1}{p}x - \frac{q}{p}$

b p is the reciprocal of the gradient $\left(\frac{\text{run}}{\text{rise}}\right)$ and q is the x-intercept.

c $x = 0y + 3$

d **i** $p = 0$ **ii** $q = 3$

e **i** $y = \frac{1}{0}(x - 3)$ and $y = \frac{1}{0}x - \frac{3}{0}$

ii The gradient of $x = 3$ is $\frac{1}{0}$, which is undefined. In general, the gradient of $x = py + q$ is $\frac{1}{p}$ which will be undefined if $p = 0$, which occurs when there is no y term.

15 a gradient–intercept method

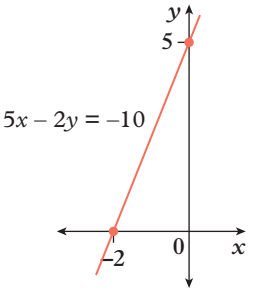
b x-and-y-intercept method

c gradient–intercept method

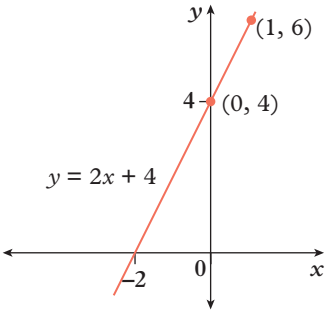
d Locate the y-intercept, and draw a line through that point parallel to the x-axis.

e Locate the x-intercept, and draw a line through that point parallel to the y-axis.

16 a



b



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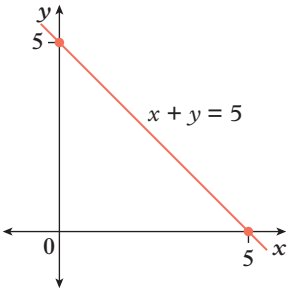

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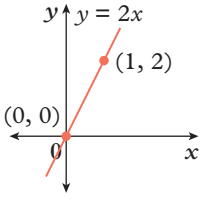

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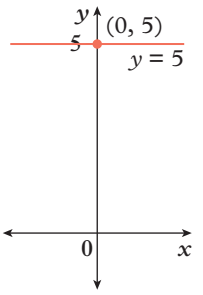

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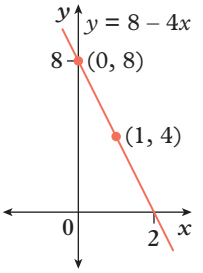

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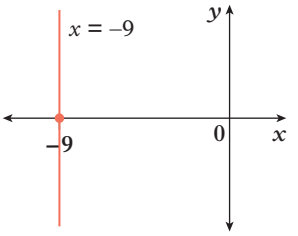

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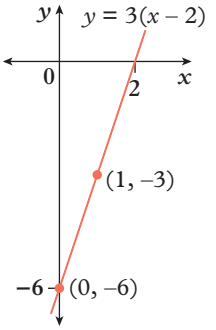

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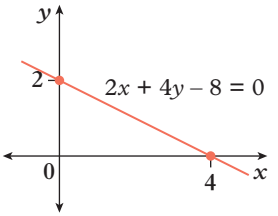

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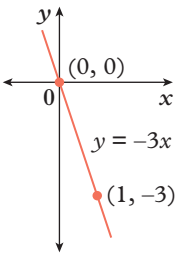

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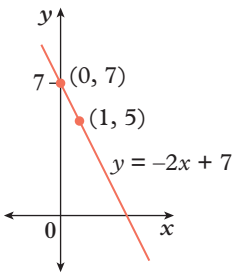
k



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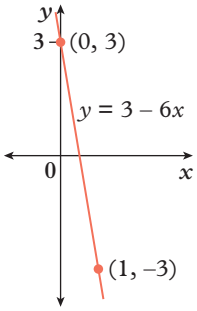
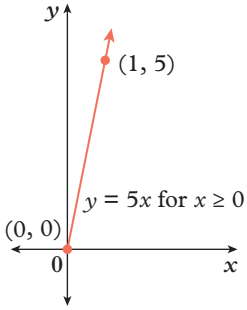


17 a



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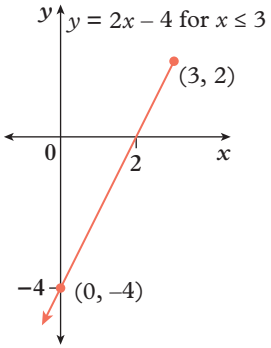

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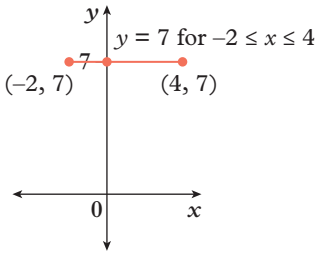

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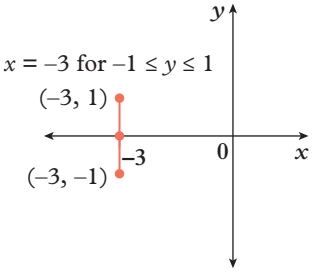

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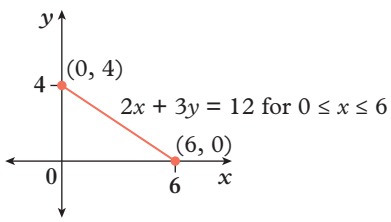

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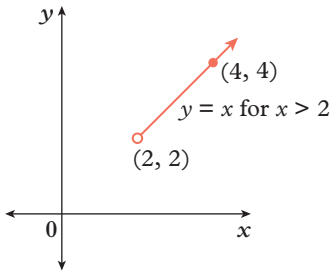

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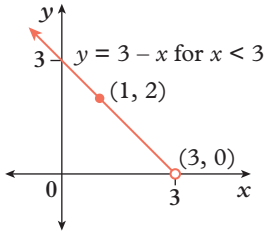

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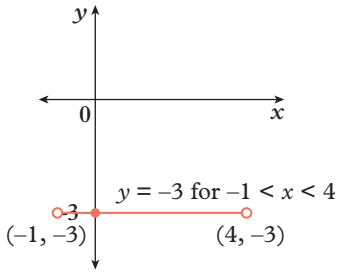

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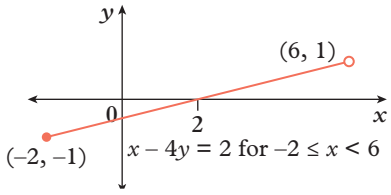

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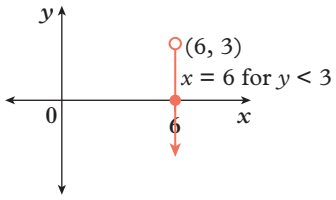
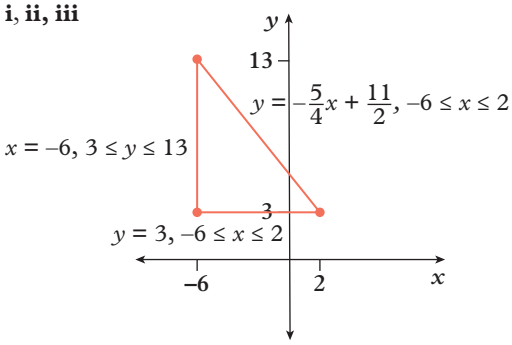


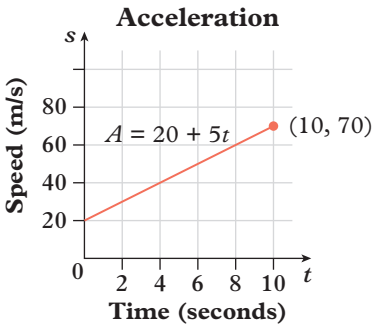
18 a i, ii, iii



b perimeter = 30.81 units

c area = 40 units^2

19 a



b the initial constant speed

c the rate of increase in speed, which is the acceleration

d **i** 35 m/s **ii** 45 m/s **iii** 57.5 m/s

e **i** 126 km/h **ii** 162 km/h **iii** 207 km/h

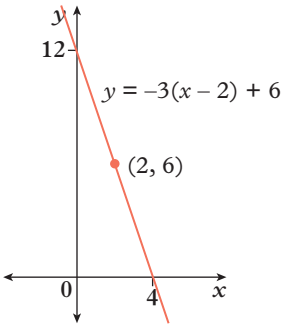
f **i** 2 s **ii** 7 s **iii** 9 s

g 8 s

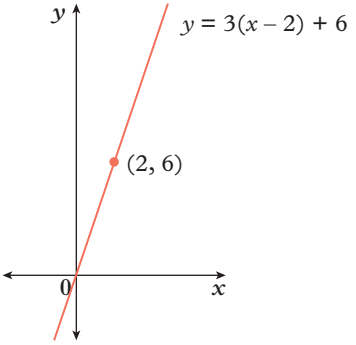
h 252 km/h reached after 10 s

20 a $y = m(x - h) + k$
$y = m(x - 0) + c$
$y = mx + c$

b **i**



ii



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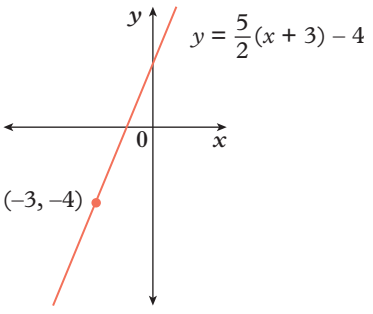

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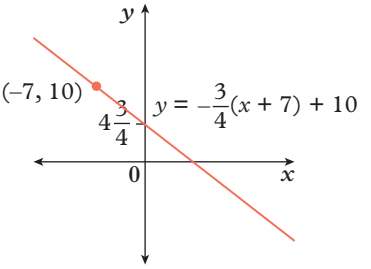


c $(0, k - mh)$ and $\left(h - \frac{k}{m}, 0\right)$

d **i** $y = m(x - b)$; $x = \frac{y}{m} + b$
This is the x-intercept version of $y = mx + c$.

ii Take the gradient, m, out as if it was a factor of the expression $mx + c$ by dividing each term by m to write the equation as $y = m\left(x + \frac{c}{m}\right)$.

iii The x-intercept is the negative of the value added to x, $\frac{c}{m}$.

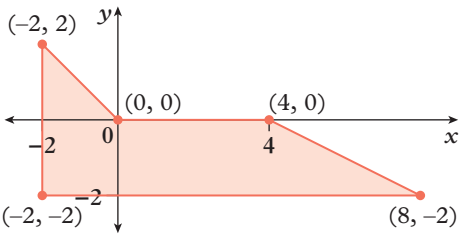
e **i** $y = 3(x - 4)$, $(4, 0)$

ii $y = 2\left(x - \frac{5}{2}\right)$, $\left(\frac{5}{2}, 0\right)$

iii $y = -3(x + 4)$, $(-4, 0)$

iv $y = 2\left(x + \frac{5}{2}\right)$, $\left(-\frac{5}{2}, 0\right)$

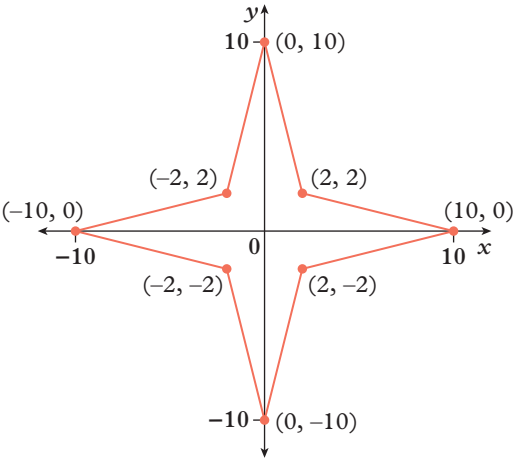
21 a



b 25.3 units

c 18 square units

22 a



b 66 units

c 80 square units

d $y = x$

e $y = -x$

23 a $\left(\frac{c}{a}, 0\right)$ and $\left(0, \frac{c}{b}\right)$

b $\frac{a}{c}x + \frac{b}{c}y = 1$

c The reciprocals of the coefficients of x and y $\left(\frac{a}{c}\text{ and }\frac{b}{c}\right)$ are the corresponding x and y values for the x- and y-intercept of the line. This is because when the equation is equal to 1, we multiply both sides of the equation by the reciprocal of the coefficient of either term when solving for the intercept.

d i $\frac{2}{5}x + \frac{3}{5}y = 1, \left(\frac{5}{2}, 0\right), \left(0, \frac{5}{3}\right)$

ii $\frac{4}{11}x - \frac{7}{11}y = 1, \left(\frac{11}{4}, 0\right), \left(0, -\frac{7}{11}\right)$

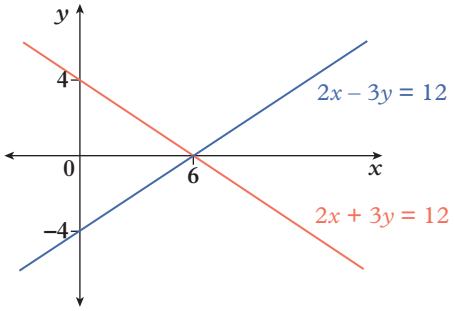
iii $\frac{1}{6}x - \frac{1}{4}y = 1$, (6, 0), (0, 4)

iv $\frac{1}{4}y - \frac{1}{18}x = 1$, (–18, 0), (0, 4)

v $-\frac{1}{4}x + \frac{1}{8}y = 1$, (–4, 0), (0, 8)

vi $2x + 5y = 1, \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{5}\right)$

24 a i



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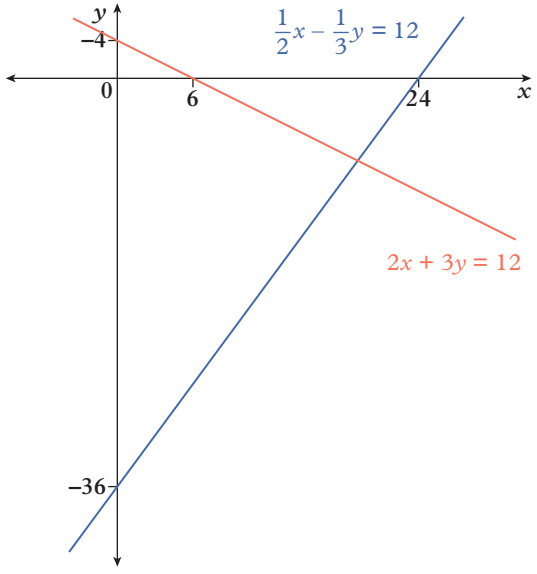

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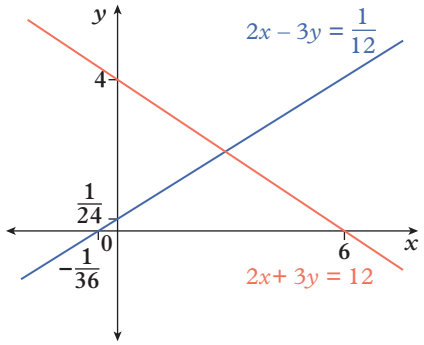

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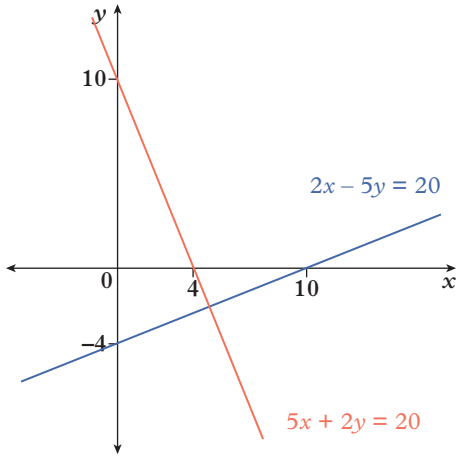
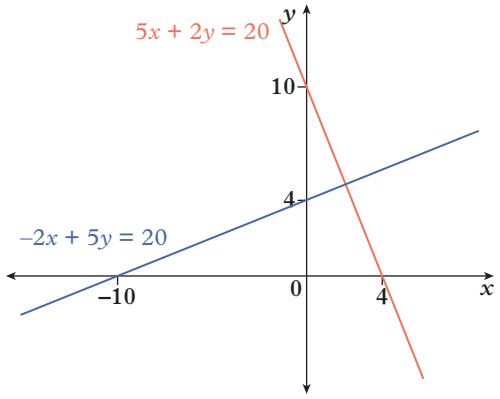
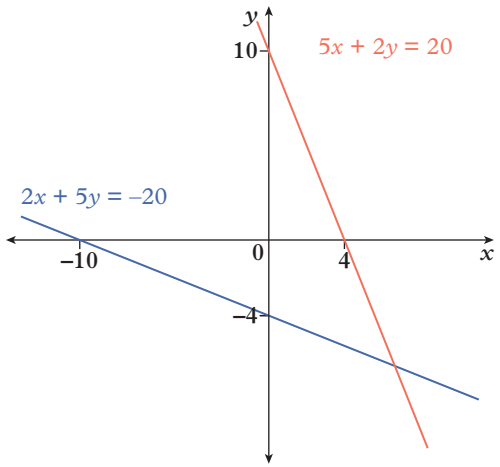


v



vi



b **ii, iv, v**

c $\frac{1}{a}x - \frac{1}{b}y = 1$ or $-\frac{1}{a}x + \frac{1}{b}y = 1$ or $bx - ay = 1$ or $-bx + ay = 1$

EX p128 3C Determining linear equations

1 **a** 4 **b** –2 **c** $\frac{1}{2}$

d –1 **e** 5 **f** –3

2 **a** **i** 4 **ii** $y = 4x + 2$

b **i** 1 **ii** $y = x - 7$

c **i** $\frac{1}{2}$ **ii** $y = \frac{1}{2}x + 5$

d **i** 10 **ii** $y = 10x + 6$

e **i** –3 **ii** $y = -3x - 5$

f **i** $-\frac{1}{4}$ **ii** $y = -\frac{1}{4}x + 4$

3 **a** $y = -2x + 4$ **b** $y = x + 3$

c $y = 2x - 6$ **d** $y = -5x + 5$

e $x = 5$ **f** $y = -2$

4 **a** $y = 3x + 1$ **b** $y = 2x + 3$

c $y = -4x - 2$ **d** $y = -x + 9$

e $y = 5x + 9$ **f** $y = -3x - 1$

g $y = \frac{6}{5}x - 14$ **h** $y = -\frac{3}{4}x + 1$

i $y = \frac{5}{2}x + \frac{3}{2}$ **j** $y = -\frac{7}{9}x - \frac{2}{3}$

5 **a** $y = 2x + 1$ **b** $y = 6x - 28$

c $y = -3x + 10$ **d** $y = -2x + 4$

e $y = x - 1$ **f** $y = -x - 6$

g $y = \frac{7}{3}x - \frac{16}{3}$ **h** $y = -\frac{7}{5}x - \frac{24}{5}$

i $y = \frac{1}{6}x + \frac{23}{6}$

6 **a** $y = 5x - 3$ **b** $y = -2x + 4$ **c** $y = \frac{4}{5}x + 2$

7 **a** $y = 3x - 11$ **b** $y = -2x + 6$ **c** $y = x + 2$

8 **a** $y = 2x - 5$ **b** $y = \frac{1}{2}x - 5$

c $y = -5x + 3$ **d** $y = -\frac{1}{5}x + 3$

e $y = 4x - 2$ **f** $y = \frac{3}{2}x + \frac{1}{2}$

g $y = 4$ **h** $y = 3x$

i $x = -3$ **j** $y = 6x - 12$

k $y = \frac{5}{12}x + 10$ **l** $y = -\frac{4}{7}x + \frac{31}{7}$

9 **a** $m = \frac{5-2}{4-(-2)}$ and $m = \frac{2-5}{-2-4}$

$m = \frac{3}{6}$ $m = \frac{-3}{-6}$

$m = \frac{1}{2}$ $m = \frac{1}{2}$

b Juan has not vertically aligned the corresponding x- and y-coordinates of the same point, so he will end up with the negative of the actual gradient, $-m = -\frac{1}{2}$.

c $m = \frac{y_1 - y_2}{x_1 - x_2}$

$= \frac{y_1 - y_2}{x_1 - x_2} \times \frac{-1}{-1}$

$= \frac{-(y_1 - y_2)}{-(x_1 - x_2)}$

$= \frac{-y_1 + y_2}{-x_1 + x_2}$

$= \frac{y_2 - y_1}{x_2 - x_1}$

10 **a** $2 = \frac{1}{2}(-2) + c$ and $5 = \frac{1}{2}(4) + c$

$2 = -1 + c$ $5 = 2 + c$

$c = 3$ $c = 3$

b Juan has substituted the x-coordinate of one of the points for the gradient and the corresponding y-coordinate for the y-intercept.

11 **a** $c = k - mh$

b $y = mx + k - mh$

c $y = mx + k - mh$

$y = mx - mh + k$

$y = m(x - h) + k$

12 **a** **i** $y = \frac{3}{4}(x - 8) + 2$

ii $y = \frac{3}{4}x - 4$

b **i** $y = -\frac{7}{3}(x - 15) - 6$

ii $y = -\frac{7}{3}x + 29$

c **i** $y = \frac{1}{4}(x + 3) + 5$ or $y = \frac{1}{4}(x - 1) + 6$

ii $y = \frac{1}{4}x + \frac{23}{4}$

d **i** $y = -\frac{8}{5}(x + 6) + 5$ or $y = -\frac{8}{5}(x + 1) - 3$

ii $y = -\frac{8}{5}x - \frac{23}{5}$

e **i** $y = -\frac{1}{2}(x + 4) + 2$ or $y = -\frac{1}{2}(x - 2) - 1$

ii $y = -\frac{1}{2}x$

f **i** $y = 0(x - 4) + 3$ or $y = 0(x + 7) + 3$

ii $y = 3$

g **i** $y = -\frac{3}{4}(x - 4) + 0$ or $y = -\frac{3}{4}(x - 0) + 3$

ii $y = -\frac{3}{4}x + 3$

h **i** $y = \frac{2}{5}(x + 5) + 0$ or $y = \frac{2}{5}(x - 0) + 2$

ii $y = \frac{2}{5}x + 2$

13 **a** **i** translation of 6 units up

ii translation of 3 units left

iii translation of 2 units left and 2 units up

b **i** $y = 2x + 6$

ii $y = 2(x + 3)$

iii $y = 2(x + 2) + 2$

c **i** translation of 1 unit left and 2 units down

ii translation of 1 unit right and 2 units up

iii translation of 1 unit right and 2 units up

iv translation of 4 units left and 6 units up

v translation of $\frac{3}{2}$ units right and $\frac{4}{3}$ units up

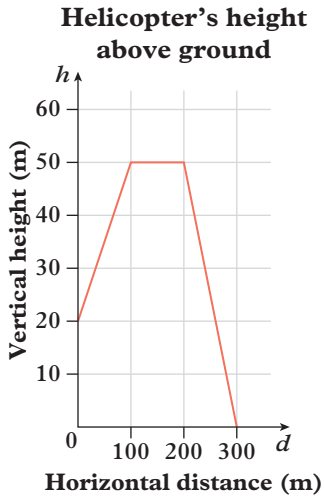
vi translation of 4 units right and 2 units up

14 **a** $y = \frac{1}{200}x + 50$

b $y = -\frac{1}{15}x + 6$

c $y = \frac{1}{5}x$

15 a



b In the order of the flight: $h = 0.3d + 20$, $h = 50$, $h = -0.5d + 150$

c **i** 29 m **ii** 44 m
iii 50 m **iv** 30 m

d $66\frac{2}{3}$ m and 220 m

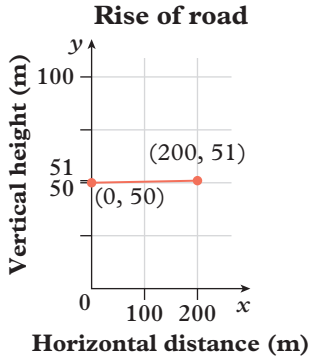
16 a $x - 2y = 4$

b $y = -23$

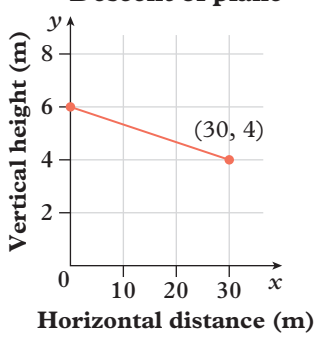
c $2x + 3y + 20 = 0$

d $12x - 17y = 0$ or $17y = 12x$

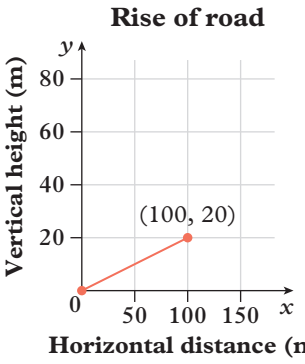
17 a $y = \frac{1}{200}x + 50$, for $0 \le x \le 200$



b $y = \frac{1}{15}x + 6$, for $0 \le x \le 30$



c $y = \frac{1}{5}x$, for $0 \le x \le 100$



18 a **i** $\frac{3}{2}x + \frac{7}{2}y = 1$ **ii** $3x + 7y = 2$

b **i** $-\frac{5}{3}x + \frac{2}{3}y = 1$ **ii** $-5x + 2y = 3$

c **i** $\frac{5}{3}x - \frac{5}{2}y = 1$ **ii** $10x - 15y = 6$

d **i** $-2x - 3y = 1$ **ii** $2x + 3y = -1$

e **i** $\frac{1}{7}x + \frac{1}{11}y = 1$ **ii** $11x + 7y = 77$

f **i** $x - y = 1$ **ii** $x - y = 1$

Checkpoint

1 a $x = 21$ **b** $x = -9$

c $x = -\frac{31}{2}$ **d** $x = -\frac{3}{7}$

2 a $x = \frac{5}{2}$ **b** $x = \frac{1}{3}$

c $x = -7$ **d** $x = \frac{41}{21}$

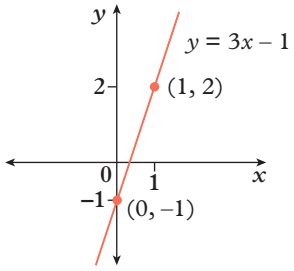
3 a $x = 19$ **b** $x = \frac{5}{7}$

c $x = \frac{1}{30}$ **d** $x = \frac{57}{10}$

4 a $x = \frac{c - by}{a}$ **b** $x = \frac{f}{dg} + \frac{e}{d} = \frac{f + eg}{dg}$

c $x = \frac{jk + mn}{m - j}$ **d** $x = \frac{rs + q}{rt + p}$

5 a



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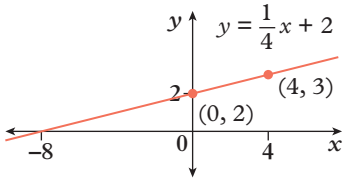

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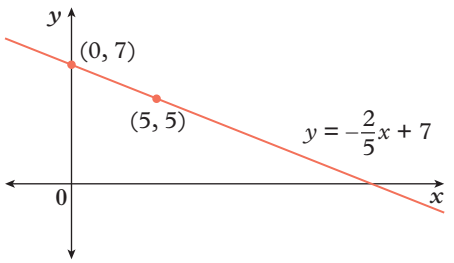

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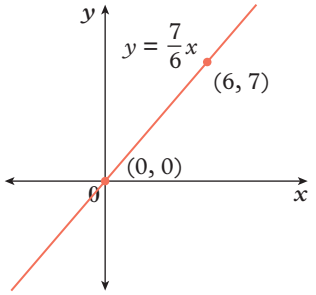
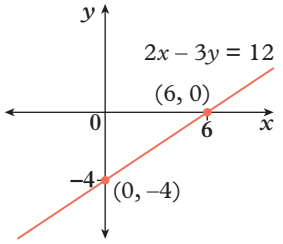


6 a



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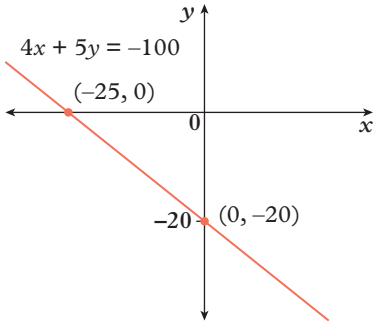

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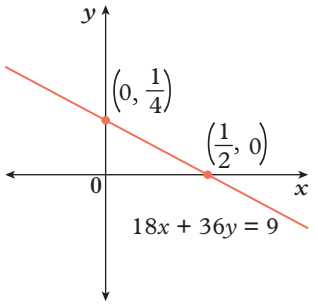

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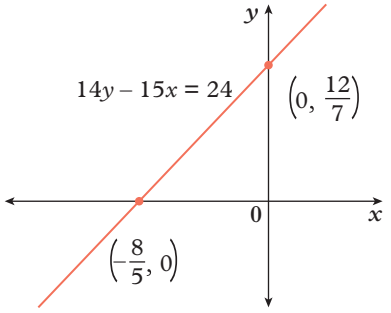
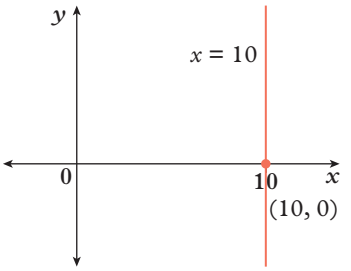


7 a



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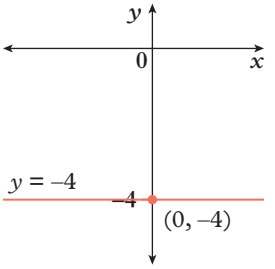

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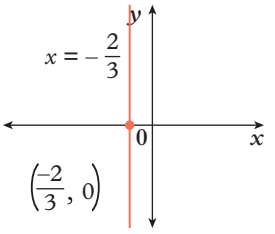
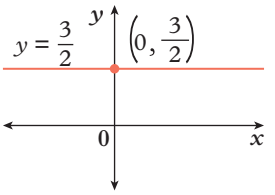


d



8 **a** $\frac{1}{2}$ **b** $\frac{1}{5}$ **c** $-\frac{9}{4}$ **d** $-\frac{7}{4}$

9 **a** $y = \frac{x}{2} + \frac{13}{2}$ **b** $y = \frac{3x}{2} + \frac{17}{2}$

c $y = -\frac{9x}{4} + \frac{13}{4}$ **d** $y = -\frac{7}{4}x + \frac{13}{4}$

10 **a** $y = 2x + 11$ **b** $y = \frac{x}{5} - \frac{3}{5}$

c $y = x + 9$ **d** $y = -2x + 3$

e $y = -\frac{x}{2} + 6$

11 **a** $y = 2x - 3$ **b** $y = -\frac{x}{2} - \frac{1}{2}$

3D Solving linear inequalities

EX p139

1 **c, d, e, h, j, l**

2 **a** $x < 4$ **b** $x \leq \frac{1}{2}$ **c** $x > 8$

d $x < -10$ **e** $x \geq -12$ **f** $x > 22$

g $x \leq 10$ **h** $x > 147$ **i** $x \leq 5$

j $x \geq \frac{500}{3}$ **k** $x \geq 25$ **l** $x > 98$

3 **a** $x > -6$ **b** $x \geq -\frac{1}{2}$ **c** $x > -10$

d $x > 10$ **e** $x \leq 6$ **f** $x < -22$

g $x \geq -9$ **h** $x < -141$ **i** $x \geq -5$

j $x \leq -\frac{10}{3}$ **k** $x \leq 10$ **l** $x < -98$

4 **a** $x > \frac{5}{8}$ **b** $x \leq -\frac{1}{3}$ **c** $x < 12$

d $x > \frac{22}{3}$ **e** $x \leq \frac{12}{5}$ **f** $x < -11$

g $x \geq \frac{2}{7}$ **h** $x < -11$ **i** $x \geq \frac{19}{26}$

j $x \leq \frac{50}{63}$ **k** $x \leq \frac{35}{17}$ **l** $x < \frac{56}{15}$

5 **a** $x < 6$ **b** $x \leq 1$ **c** $x > \frac{9}{5}$

d $x < \frac{4}{3}$ **e** $x \leq \frac{12}{5}$ **f** $x \leq 2$

6 **a** **i** no **ii** no **iii** no

b **i** no **ii** no **iii** yes

c **i** yes **ii** yes **iii** yes

d **i** yes **ii** yes **iii** yes

7 **a** **i** yes **ii** yes **iii** yes

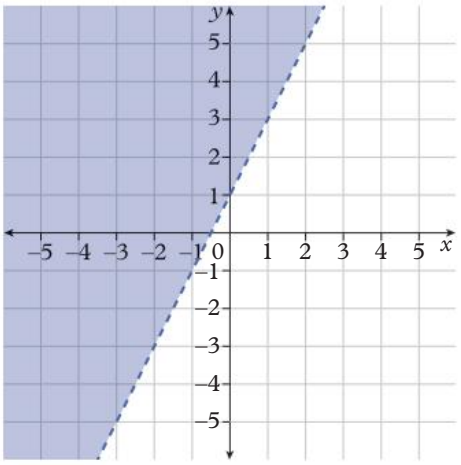
b **i** no **ii** yes **iii** yes

c **i** yes **ii** yes **iii** no

d **i** yes **ii** yes **iii** no

8 **a–A, b–D, c–C, d–B**

9 **a**



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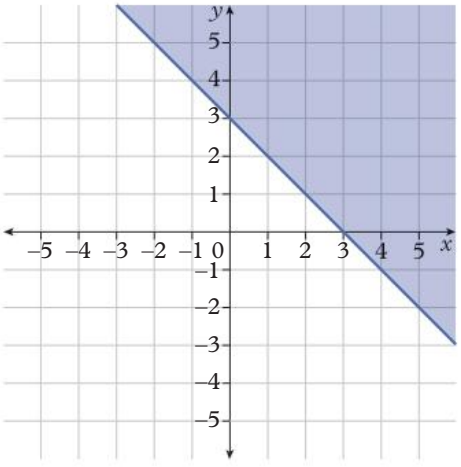

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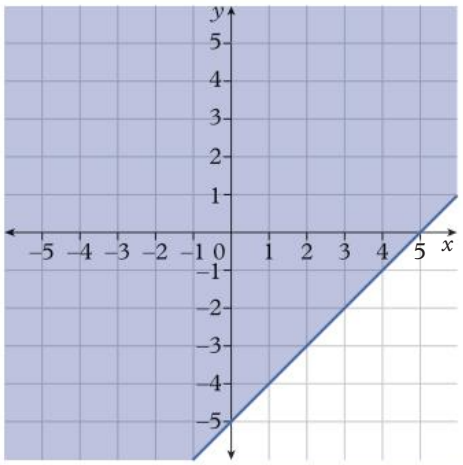

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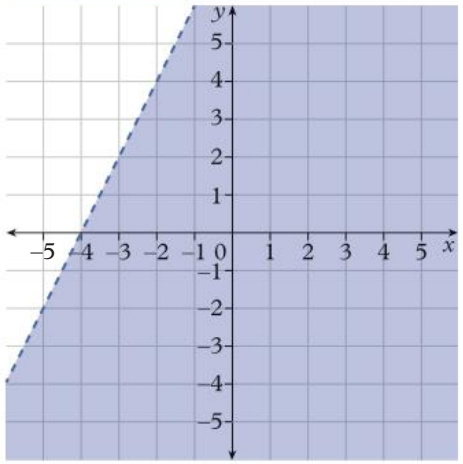
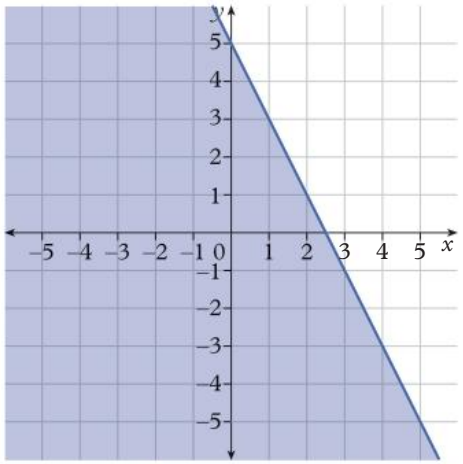


10 **a**



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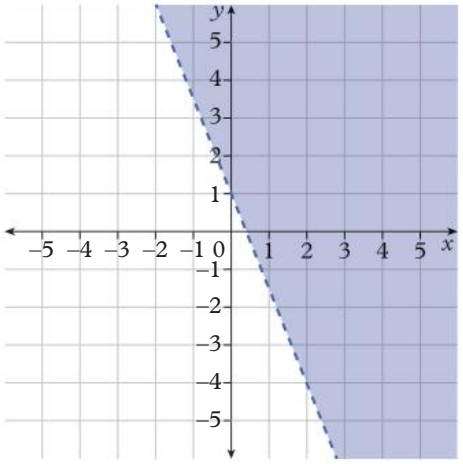
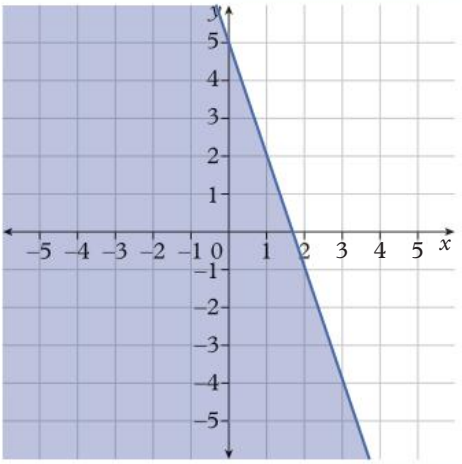


11 a



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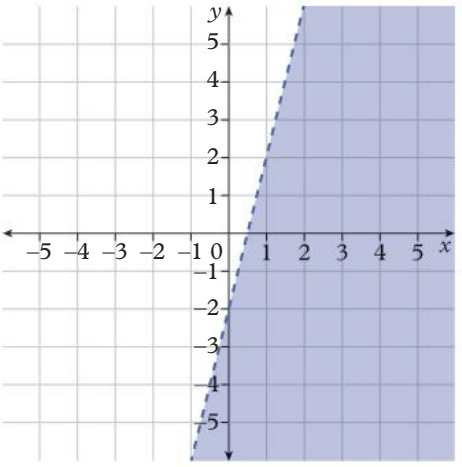

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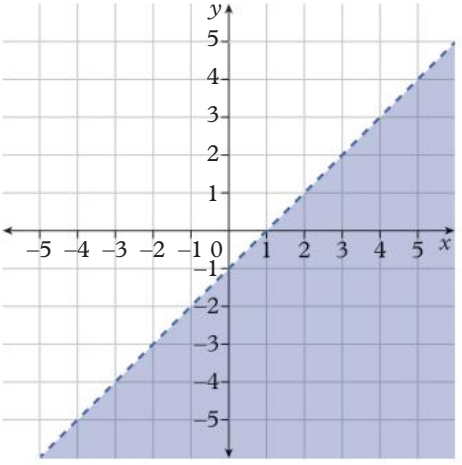

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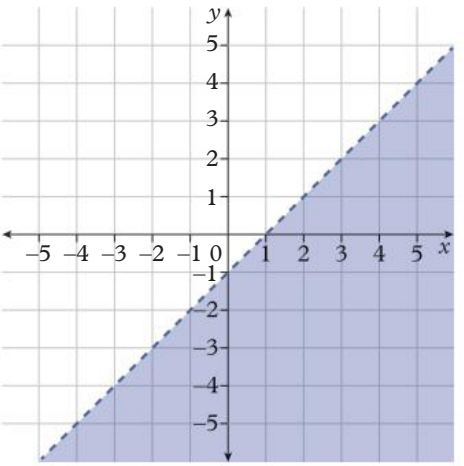

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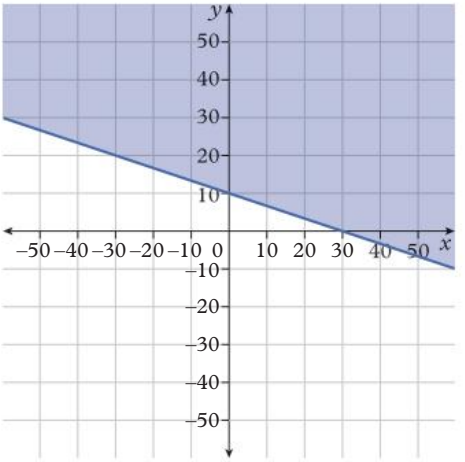

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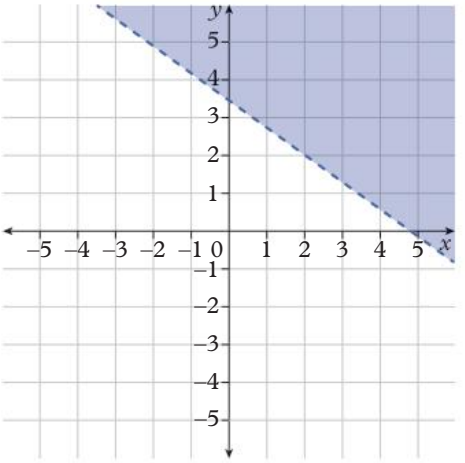


12 a $y > x + 3$ **b** $y \leq \frac{x}{2} - 1$

c $y \leq -x - 1$ **d** $y > -x + 1$

13 a $2(w + l) > 20$ **b** $22f + 6p \leq 100$

c $55s + 80b \leq 2145$ **d** $8p \geq 4 + 3g$

e $s + b + 3.75 \leq 12$

14 a $10x + 20y \leq 400$

b The right-hand side of the inequality should be $10x + 20y + 100$, because 400 is the maximum of how much Ken can spend, not the actual spending.

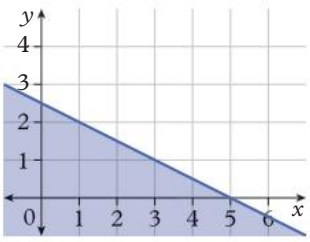
c $5x + 5y \geq 100$,

d $100

e Ken should stock up 40 product A. Because the profit for selling each product is $5 for both product A and B, as A is cheaper to acquire, Ken can make more profit by selling product A.

15 a $10x + 20y \leq 50$

b



c Because the quantity of snow globes and hoodies can't be negative.

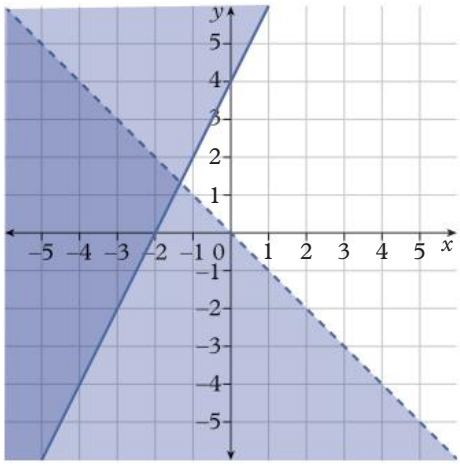
d No

e 0 snow globe, 0 hoodie
0 snow globe, 1 hoodie
0 snow globe, 2 hoodies
1 snow globe, 0 hoodie
1 snow globe, 1 hoodie
1 snow globe, 2 hoodies
2 snow globe, 0 hoodie
2 snow globe, 1 hoodie
3 snow globe, 0 hoodie
3 snow globe, 1 hoodie
4 snow globe, 0 hoodie
5 snow globe, 0 hoodie

f $x + 0.7y + 3 \leq 7$

g all combinations with three or less snow globes

16 a i



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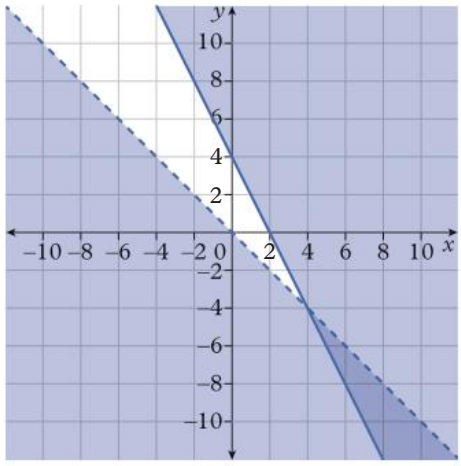

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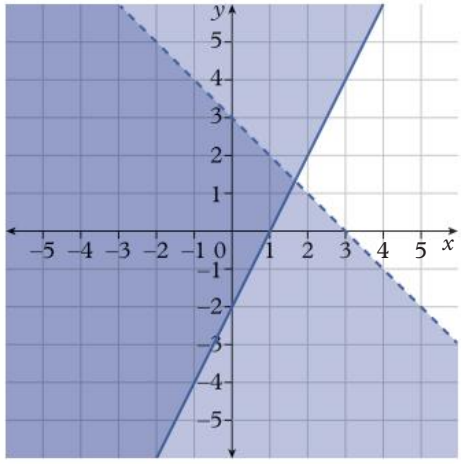

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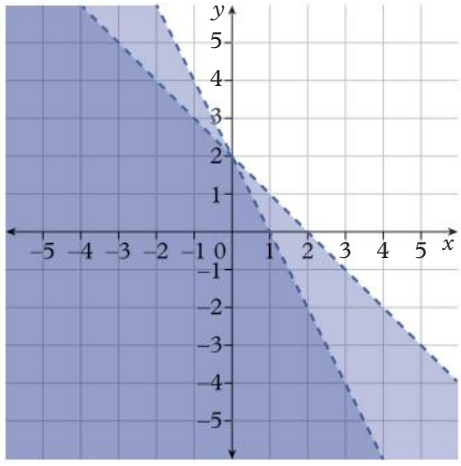


b i yes **ii** no **iii** no
iv yes **v** no **vi** no

EX p148

3E Simultaneous linear equations

1 a is a solution **b** is a solution
c is not a solution **d** is a solution
e is not a solution **f** is not a solution

2 a i (4, 3) **ii** $x = 4, y = 3$
b i (2, –4) **ii** $x = 2, y = -4$
c i (–5, –2) **ii** $x = -5, y = -2$
d i (–2, 3) **ii** $x = -2, y = 3$

3 **a** $x = 5, y = 4$ **b** $x = 2, y = 6$
c $x = 6, y = 4$ **d** $x = -2, y = 5$
e $x = -1, y = 3$ **f** $x = -3, y = 1$
g $x = 3, y = 2$ **h** $x = 4, y = -2$
i $x = -2, y = -3$ **j** $x = -5, y = 6$
k $x = 2, y = -2$ **l** $x = 3, y = 5$
m $x = -4, y = -1$ **n** $x = 1, y = -4$
o $x = -6, y = -5$ **p** $x = -1, y = 2$

4 **a** $x = 2, y = 7$ **b** $x = 4, y = 1$
c $x = -1, y = 3$ **d** $x = 6, y = 5$
e $x = -4, y = -2$ **f** $x = 2, y = 12$
g $x = 1, y = 3$ **h** $x = -3, y = 11$
i $x = 5, y = -5$ **j** $x = 5, y = 3$
k $x = -4, y = -1$ **l** $x = 5, y = 6$

5 **a** $x = 1, y = 8$ **b** $x = 6, y = 4$
c $x = -3, y = -5$ **d** $x = 5, y = -1$
e $x = -1, y = 4$ **f** $x = 4, y = -7$
g $x = 2, y = 1$ **h** $x = -2, y = 6$

6 **a** $x = 2, y = 5$
b $x = -3, y = 1$
c $x = 1, y = -5$

7 **a** $x = 5, y = 3$
b $x = -2, y = 9$
c $x = 3, y = -7$

8 **a** $x = -4, y = -2$ **b** $x = 6, y = 2$
c $x = -1, y = 2$ **e** $x = -1, y = 5$
d $x = 5, y = 3$ **f** $x = -3, y = -4$

9 **a** $x = 3, y = 2$ **b** $x = 2, y = -1$
c $x = -4, y = 7$ **d** $x = 5, y = 3$
e $x = -3, y = -4$ **f** $x = -2, y = 6$
g $x = 1, y = 4$ **h** $x = -3, y = 1$
i $x = 2, y = -1$ **j** $x = -4, y = 2$
k $x = 5, y = 5$ **l** $x = -3, y = -2$

10 **a** **i** The graphical method is most appropriate when the equations of the lines are easy to sketch accurately on the same set of axes and you have a grid on which to read the intersection point.
ii The substitution method is most appropriate when one or both of the equations have a variable as the subject of the equation.
iii The elimination method is most appropriate when the terms with the same variable in each equation have the same coefficient or the negative value of the other coefficient.

b **i** The graphical method is least appropriate when the equations of the lines are not easy to sketch accurately on the same set of axes or the intersection point cannot be read off accurately.
ii The substitution method is least appropriate when neither equation has a variable the subject of the equation.
iii The elimination method is least appropriate when the coefficients of the same variable cannot be simply multiplied by a number to make them the same or the negative value of each other.

c **i** The graphical method can be made more appropriate by using digital technology to sketch and locate the intersection point.
ii The substitution method can be made more appropriate by writing one or both equations with a variable as the subject.
iii The elimination method can be made more appropriate by multiplying the equations by a number to make the coefficients of the same variable the same or the negative value of each other.

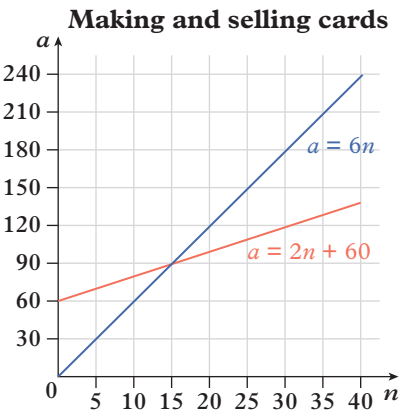
11 **a** $x = 2, y = 0$
b $x = \frac{1}{3}, y = -3$
c $x = \frac{3}{2}, y = \frac{2}{5}$ or $x = 1.5, y = 0.4$
d $x = -9, y = -7$
e $x = 4, y = 2$
f $x = -2, y = -3$
g $x = 3, y = 10$
h $x = 5, y = -2$
i $x = 2, y = -1$
j $x = -2, y = 2$
k $x = 7, y = 4$
l $x = -4, y = 1$
m $x = -0.8, y = 2.3$
n $x = -\frac{1}{4}, y = 5$ or $x = -0.25, y = 5$
o $x = 28, y = 41$

12 Both graphs have a gradient of 2. This means the lines are parallel, so there is no point of intersection.

13 **a** To make n cards, the cost of plain white cards is $2 \times n$ or $2n$. So, the amount of money spent (in $) is the cost of the plain white cards plus the start-up cost of $60.

b Each card sells for $6, so for selling n cards, the amount of money received (in $) is $6 \times n$ or $6n$.

c



d (15, 90); this shows the number of cards that need to be sold (15) so that the amount of money received is the same as the amount of money spent ($90).

e 15 cards

f 16 to 40 cards

14 a $2l + 2w = 810$ or $2(l + w) = 810$ or $l + w = 405$

b $l = 2w$

c 270 cm long and 135 cm wide

15 They are equivalent equations. When graphed, the position of the lines is identical; that is, one line lies on top of the other. There are an infinite number of points where the lines intersect so there are infinite solutions.

16 a Strategy A: Equation ③ is $10x - 6y = -2$.

① – ③: $-3x = -12$, $x = 4$

Substituting $x = 4$ into ②: $20 - 3y = -1$, $y = 7$

The solution is $x = 4$, $y = 7$.

Strategy B: Equation ④ is $-10x + 6y = 2$.

① + ④: $-3x = -12$, so $x = 4$

Substituting $x = 4$ into ②: $20 - 3y = -1$, $y = 7$

The solution is $x = 4$, $y = 7$.

Strategy C: Equation ⑤ is $35x - 30y = -70$ and equation ⑥ is $-35x - 21y = -7$.

⑤ – ⑥: $-9y = -63$, so $y = 7$

Substituting $y = 7$ into ②: $5x - 21 = -1$, so $x = 4$

The solution is $x = 4$, $y = 7$.

The strategies all provide the same solution of $x = 4$ and $y = 7$.

b Some possible answers are:

Multiply equation ① by 5 to form equation ⑦ and multiply equation ② by –7 to form equation ⑧. Add equations ⑦ and ⑧.

Graph equations ① and ② and identify the coordinates of the point of intersection.

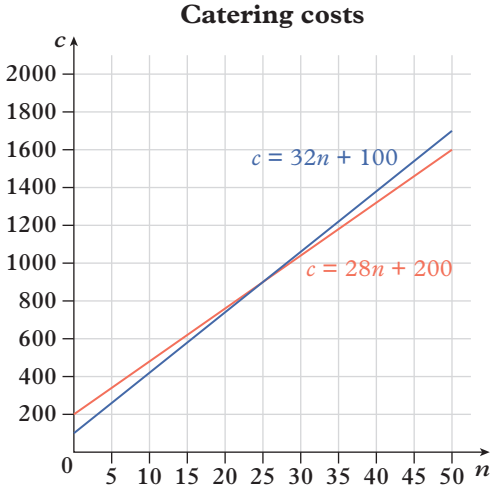
Rearrange equation ② to obtain $3y = 5x - 1$ and substitute that into equation ①. In other words, substitute $5x - 1$ for $3y$ in $7x - 2(3y) = -14$.

17 n = number of people and c = catering cost

a $c = 28n + 200$

b $c = 32n + 100$

c



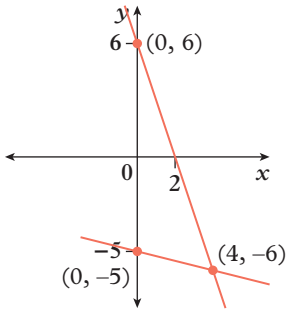
d Cool Food Club is cheaper for 18 people; at $n = 18$, the line for Cool Food Club is lower than the line for Angie's Catering. This represents a lower cost for 18 people.

e Angie's Catering

f 25 people at a cost of $900

g When catering for fewer than 25 people, Cool Food Club is cheaper. When catering for more than 25 people, Angie's Catering is cheaper. The cost is the same when catering for 25 people.

18 a



b $y = -\frac{1}{4}x - 5$, $y = -3x + 6$

c $x = 4$, $y = -6$

d If $x = 4$ and $y = -6$:

For $y = -\frac{1}{4}x - 5$, RHS $= -\frac{1}{4} \times 4 - 5 = -6 =$ LHS.

For $y = -3x + 6$, RHS $= -3 \times 4 + 6 = -6 =$ LHS.

19 a $15 + (-5) = 10$ and $15 + 5 = 20$

b $22 + (-12) = 10$ and $22 + (-2) = 20$

c Since there are three variables but only two equations, if we add, subtract or substitute, the resulting equation will still have two variables in it when we need it to have only one.

20 a

$$ax + by = c$$
$$ax + by - (ax + qy) = c - (r)$$
$$by - qy = c - r$$
$$(b - q)y = c - r$$
$$y = \frac{c - r}{b - q}$$
$$ax + b\left(\frac{c - r}{b - q}\right) = c$$
$$ax + \frac{bc - br}{b - q} = \frac{c(b - q)}{b - q}$$
$$ax = \frac{cb - cq}{b - q} - \frac{bc + br}{b - q}$$
$$ax = \frac{cb - cq - bc - br}{b - q}$$
$$ax = \frac{br - cq}{b - q}$$
$$x = \frac{br - cq}{ab - aq}$$

b $x = \frac{4 \times 7 - 9 \times 6}{3 \times 4 - 3 \times 6} = \frac{13}{3}$ and $y = \frac{9 - 7}{4 - 6} = -1$

c

$$\begin{aligned} ax + by &= c \ (\times p) \\ apx + bpy &= cp \\ px + qy &= r \ (\times a) \\ apx + aqy &= ar \\ apx + bpy - (apx + aqy) &= cp - (ar) \\ bpy - aqy &= cp - ar \\ (bp - aq)y &= cp - ar \\ y &= \frac{cp - ar}{bp - aq} \\ ax + b\left(\frac{cp - ar}{bp - aq}\right) &= c \\ ax + \frac{bcp - abr}{bp - aq} &= \frac{c(bp - aq)}{bp - aq} \\ ax &= \frac{bcp - acq}{bp - aq} - \frac{bcp - abr}{bp - aq} \\ ax &= \frac{bcp - acq - bcp + abr}{bp - aq} \\ ax &= \frac{abr - acq}{bp - aq} \\ x &= \frac{br - cq}{bp - aq} \end{aligned}$$

d $x = -13$ and $y = 12$

CHAPTER 3 review

Multiple choice

1	B	**2**	A	**3**	E	**4**	B	**5**	C
6	D	**7**	E	**8**	E	**9**	C	**10**	D
11	E	**12**	B	**13**	A				

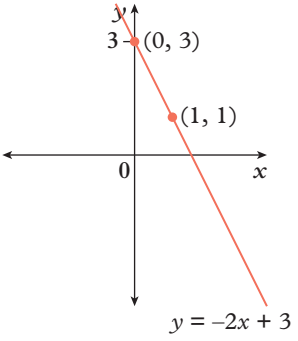
Short answer

1 a $x = -1\frac{1}{2}$ **b** $x = 9\frac{2}{3}$

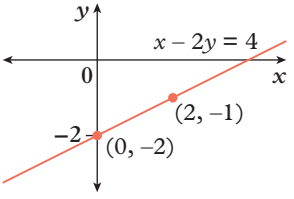
2 a $x = 9$ **b** $x = \frac{5}{13}$

3 a $b = \frac{2A}{h} - a$ **b** $h = \frac{A}{2(l + w)}$

4 a



b



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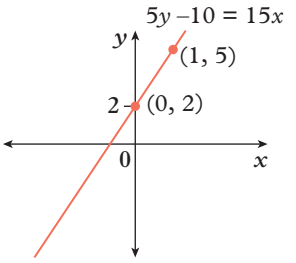
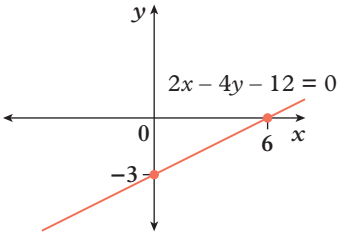


5 a



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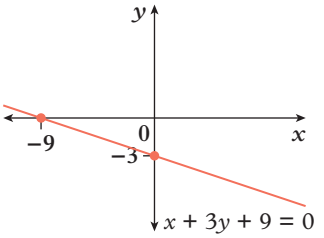

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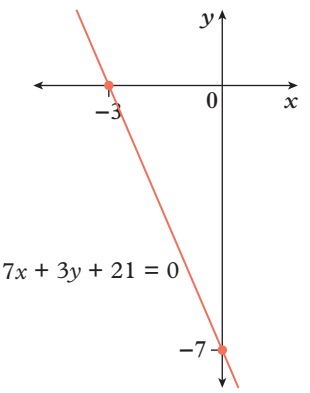


6 a $y = 2x - 4$

b $y = 6$

c $x = -3$

7 a $y = 2x + 3$

b $y = -2x + 7$

8 a $y = x + 2$

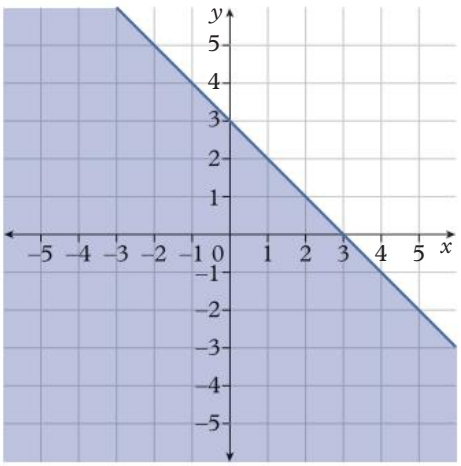
b $y = -3x + 2$

c $y = x$

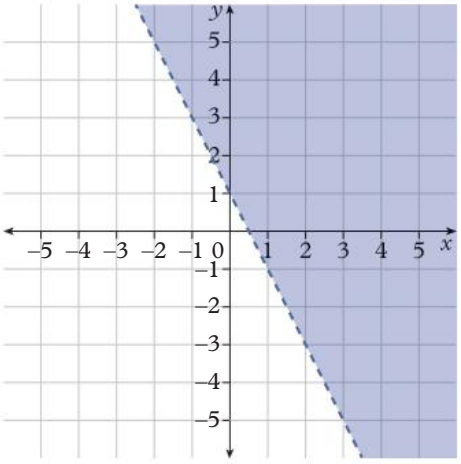
9 a $x > -1$

b $x < -2\frac{2}{3}$

10 a



b



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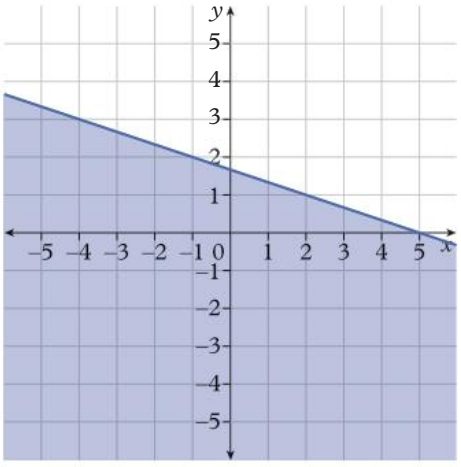
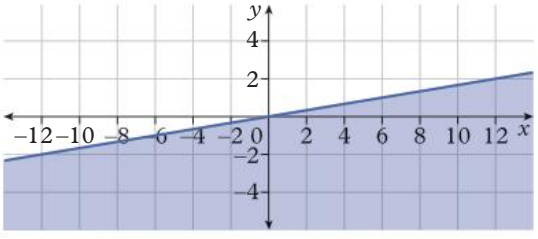


d



11 a $x = 1, y = 2$ **b** $x = 6, y = 17$

12 a $x = 1, y = -1$ **b** $x = -2, y = -5$

c $x = 2, y = 10$

13 a $x = 2, y = 4$

b $x = -2, y = 3$

c $x = 3, y = -1$

Analysis

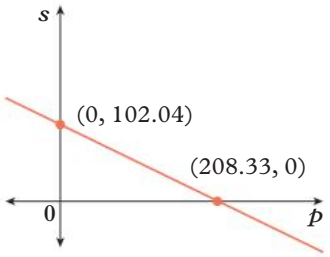
1 a $0.71s + 0.26p \leq 60$

b $p \leq 94$. Answer must be an integer.

c $s \leq 29$. Answer must be an integer.

d $P = 0.49s + 0.24p$

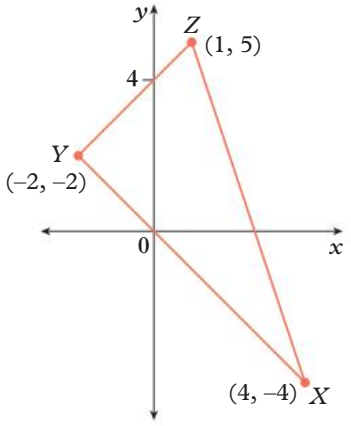
e



f 208 party pies

g 32 sausage rolls and 143 party pies

2 a



b For XY: $y = -x, -2 \leq x \leq 4$

For YZ: $y = x + 4, -2 \leq x \leq 1$

For ZX: $y = -3x + 8, 1 \leq x \leq 4$

c $x = \sqrt{18} = 3\sqrt{2}$ units

$y = \sqrt{90} = 3\sqrt{10}$ units

$z = \sqrt{72} = 6\sqrt{2}$ units

d $x^2 + z^2 = \left(\sqrt{18}\right)^2 + \left(\sqrt{72}\right)^2$

$= 18 + 72$

$= 90$

$= \left(\sqrt{90}\right)^2$

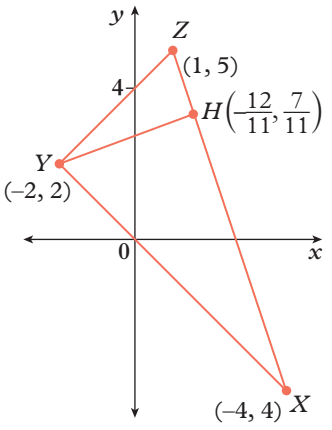
$= y^2$

e M is at the point $(2.5, 0.5)$.

f $\frac{3\sqrt{10}}{2}$

g Since $MX = MZ = \frac{\sqrt{90}}{2} = MY$, these are both isosceles triangles.

h



i $1.2\sqrt{10}$ or approximately 3.79 units

j Using YX as the base and YZ as the perpendicular height:

Area $= \frac{1}{2} \times 6\sqrt{2} \times 3\sqrt{2} = 18$ square units

Using XZ as the base and YH as the perpendicular height:

Area $= \frac{1}{2} \times 3\sqrt{10} \times 1.2\sqrt{10} = 18$ square units

k Using the two methods in **j**,

$2 \times$ Area of $XYZ = y \times YH = xz$,

so $YH = \frac{xz}{y}$

l With centre M and radius $MX = MY = MZ$, a circle can be drawn to touch the vertices of the right-angled triangle. This is one of the geometric facts which states that the angle subtended by the diameter of a semi-circle is a right angle.

CHAPTER 4 Non-linear relationships

4A Solving quadratic equations

EX p160

1 **a** $x = -2$ or $x = 4$ **b** $x = -7$ or $x = -1$

c $x = 3$ or $x = 6$ **d** $x = \pm 9$

e $x = 0$ or $x = 5$ **f** $x = 0$ or $x = \frac{1}{2}$

g $x = 0$ or $x = -1$ **h** $x = 0$ or $x = -2$

i $x = 0$ or $x = \frac{3}{4}$ **j** $x = \pm\sqrt{2}$

k $x = \pm\sqrt{5}$ **l** $x = \pm 4\sqrt{3}$

m $x = -1\frac{1}{2}$ or $x = 4$ **n** $x = \pm\frac{1}{3}$

o $x = -\frac{2}{7}$ or $x = 1\frac{1}{4}$

2 **a** $x = -3$ or $x = -4$ **b** $x = -5$ or $x = 2$

c $x = 1$ or $x = 8$ **d** $x = \pm 5$

e $x = \pm 9$ **f** $x = \pm 2$

g $x = 0$ or $x = 3$ **h** $x = 0$ or $x = -1$

i $x = 0$ or $x = -4$ **j** $x = 0$ or $x = \frac{2}{5}$

k $x = 0$ or $x = 6$ **l** $x = 0$ or $x = -10$

4 **a** Both sides of the original equation have been divided by 3, so the equation remained in balance and the values of the variables are unchanged.

b $x = -6$ or $x = 8$

5 **a** $x = -5$ or $x = 6$ **b** $x = 2$ or $x = 4$

c $x = -8$ or $x = -1$ **d** $x = 3$ or $x = -9$

e $x = \pm 4$ **f** $x = 0$ or $x = -7$

g $x = 0$ or $x = 2$ **h** $x = 0$ or $x = -1\frac{2}{3}$

i $x = 1\frac{1}{2}$ or $x = -10$

6 **a** $x = -2$ or $x = -1$ **b** $x = -5$ or $x = 3$

c $x = 1$ or $x = 4$ **d** $x = -3$ or $x = 2$

e $x = -2$ or $x = 6$ **f** $x = -8$ or $x = 7$

g $x = \pm 2$ **h** $x = \pm 3$

i $x = 0$ or $x = \frac{1}{2}$

7 **a** $x = \pm\sqrt{3}$ **b** $x = \pm\sqrt{11}$ **c** $x = \pm\sqrt{7}$

d $x = \pm 2\sqrt{2}$ **e** $x = \pm 3\sqrt{3}$ **f** $x = \pm\sqrt{2}$

g $x = \pm\sqrt{5}$ **h** $x = \pm\sqrt{6}$ **i** $x = \pm 4\sqrt{2}$

j $x = \pm\sqrt{19}$ **k** $x = \pm 2\sqrt{5}$ **l** $x = \pm 2\sqrt{13}$

8 **a** $x = -6$ or $x = 2$ **b** $x = 1$ or $x = 4$

c $x = \pm 3$ **d** $x = -5$ or $x = 8$

e $x = 0$ or $x = 9$ **f** $x = 2 \pm \sqrt{3}$

g $x = 0$ or $x = 4$ **h** $x = 2$ or $x = 9$

i $x = -3$ or $x = 6$ **j** $x = -7$ or $x = 3$

k $x = 4$ or $x = 6$ **l** $x = -3$

9 Dividing both sides of the first equation by 3 gives the equivalent equation $x - 2 = 0$ and this does not affect the solution (with $x = 2$ being a solution for both equations). The solution for the second equation is $x = 0$ or $x = 2$. But dividing both sides of the second equation by x does not produce an equivalent equation because, when $x = 0$ is a solution, we cannot divide by 0.

10 **a** two **b** one **c** two

d one **e** two **f** none

11 **a** $x = \pm 3$ **b** $x = -11$ or $x = 7$

c $x = 4$ **d** no solutions

e $x = 0$ or $x = -4$ **f** $x = 3 \pm \sqrt{10}$

12 **a** **i** 8 m **ii** 6 m

b 6 m

c 3 s

d It is not possible to have negative time values.

13 **a** 25 kg **b** 9 kg **c** 5 weeks

d No. If the amounts were the same each week, the relationship would be linear.

14 **a** $x(x + 3) = 70$ or $x^2 + 3x - 70 = 0$

b $x = -10$ or $x = 7$. The feasible solution is $x = 7$, while $x = -10$ is not feasible because it is not possible to have a negative length.

c Width 7 m, length 10 m

15 $x(x - 5) = 66$, where x is the length in centimetres. The dimensions are length 11 cm, width 6 cm.

16 **a** $x = -1 \pm \sqrt{7}$ **b** $x = 2 \pm \sqrt{2}$

c $x = -4 \pm \sqrt{5}$ **d** $x = 6 \pm 2\sqrt{2}$

e $x = -10 \pm 5\sqrt{3}$ **f** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

g $x = -3 \pm \sqrt{2}$ **h** $x = -1 \pm \sqrt{5}$

i $x = 2 \pm \sqrt{3}$ **j** $x = -4 \pm \sqrt{10}$

k $x = 3 \pm \sqrt{\frac{21}{2}}$ or $x = 3 \pm \frac{\sqrt{42}}{2}$

l $x = -\frac{1}{2} \pm \sqrt{\frac{13}{20}}$ or $x = -\frac{1}{2} \pm \frac{\sqrt{65}}{10}$

17 a $x = -5$ or $x = -\frac{3}{2}$ **b** $x = \frac{2}{5}$ or $x = 4$

c $x = -\frac{2}{3}$ or $x = \frac{1}{2}$ **d** $x = -7$ or $x = \frac{3}{4}$

e $x = -1$ or $x = \frac{1}{3}$ **f** $x = -\frac{3}{7}$ or $x = 2$

EX 4B The quadratic formula

p166

1 a none **b** two **c** two

d none **e** one **f** two

2 a i 64 **ii** two **iii** $x = -3$ or $x = 5$

b i 0 **ii** one **iii** $x = -2$

c i 196 **ii** two **iii** $x = -7$ or $x = 7$

d i 324 **ii** two **iii** $x = 0$ or $x = 6$

e i 8 **ii** two **iii** $x = 3 - \sqrt{2}$ or $x = 3 + \sqrt{2}$

f i -8 **ii** none **iii** no solutions

3 a i $x = \frac{7 \pm \sqrt{29}}{2}$ **ii** $x \approx 0.81$ or $x \approx 6.19$

b i $x = \frac{1 \pm \sqrt{13}}{6}$ **ii** $x \approx -0.43$ or $x \approx 0.77$

c i $x = \frac{-1 \pm \sqrt{33}}{4}$ **ii** $x \approx -1.69$ or $x \approx 1.19$

d i $x = \frac{-1 \pm \sqrt{109}}{6}$ **ii** $x \approx -1.91$ or $x \approx 1.57$

e i $x = \frac{-5 \pm \sqrt{97}}{12}$ **ii** $x \approx -1.24$ or $x \approx 0.40$

f i $x = 1 \pm \sqrt{2}$ **ii** $x \approx -0.41$ or $x \approx 2.41$

g i $x = -4 \pm 2\sqrt{3}$ **ii** $x \approx -7.46$ or $x \approx -0.54$

h i $x = \frac{3 \pm 2\sqrt{2}}{2}$ **ii** $x \approx 0.09$ or $x \approx 2.91$

i i $x = 2 \pm \sqrt{6}$ **ii** $x \approx -0.45$ or $x \approx 4.45$

j i $x = \frac{-5 \pm 2\sqrt{10}}{5}$ **ii** $x \approx -2.26$ or $x \approx 0.26$

k i $x = \frac{-3 \pm \sqrt{97}}{4}$ **ii** $x \approx -3.21$ or $x \approx 1.71$

l i $x = -1 \pm 2\sqrt{2}$ **ii** $x \approx -3.83$ or $x \approx 1.83$

4 a $x = \frac{-3 \pm \sqrt{13}}{2}$ **b** no solutions **c** $x = -3 \pm \sqrt{5}$

d no solutions **e** $x = \frac{-5 \pm \sqrt{17}}{4}$ **f** $x = 4$

g $x = -1 \pm \sqrt{5}$ **h** $x = \frac{7 \pm \sqrt{61}}{2}$ **i** $x = -4 \pm \sqrt{26}$

j $x = \frac{1 \pm \sqrt{21}}{2}$ **k** no solutions **l** $x = \frac{-\sqrt{5} \pm 3}{2}$

5 a $m = \frac{-3 \pm \sqrt{37}}{2}$ **b** $u = -2 \pm \sqrt{5}$

c $r = \frac{-9 \pm \sqrt{77}}{2}$ **d** $y = -1$ or $y = 3$

e $h = \frac{-7 \pm \sqrt{61}}{6}$ **f** $w = 3 \pm \sqrt{14}$

6 $x = \frac{3 \pm \sqrt{13}}{2}$

7 Step 3: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Step 4: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$

Step 5: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

Step 7: $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)^2 = 0$

Step 10: $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$

Step 11: $x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0$

Step 12: $x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$

8 a, b $x = 2$ or $x = 3$

c In cases such as this, the method described in part **a** is often found to be quicker and easier.

d If integer solutions exist, the method described in part **a** is usually quicker.

9 a \$620 **b** 59

10 length 81.4 m, width 61.4 m

11 length 369 mm, width 131 mm

12 a There is no real number that is the square root of a negative number, so we cannot obtain a result using the quadratic formula. There are no solutions.

b The square root of zero is zero, so there will be only one distinct value obtained using the quadratic formula: $x = \frac{-b \pm 0}{2a} = -\frac{b}{2a}$

c The square root of a positive number gives a positive result so there will be two distinct solutions:

$x = \frac{-b - \text{positive}}{2a}$ or $x = \frac{-b + \text{positive}}{2a}$.

13 a i $\Delta = 33$. Solutions will be irrational.

ii $\Delta = 25$. Solutions will be rational.

iii $\Delta = 9$. Solutions will be rational.

iv $\Delta = 41$. Solutions will be irrational.

v $\Delta = 49$. Solutions will be rational.

b The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
If $\sqrt{b^2 - 4ac}$ is an integer and a, b and c are integers then the solutions are rational. $\sqrt{b^2 - 4ac}$ is an integer if $b^2 - 4ac$, the discriminant, is a square number, since the square root of a square number is an integer.

14 a i $(-2, 0)$ and $(-5, 0)$ **ii** $(2, 0)$ and $(4, 0)$

iii no x-intercepts **iv** $(4, 0)$

v no x-intercepts

b The discriminant relates to the number of solutions to a quadratic when it equals zero, and therefore the number of x-intercepts of its graph.

15 a $k = 3$

b i $k < 3$ **ii** $k > 3$

16 a $k = 1$ **b** $k < 2$

c $k < -9$ **d** $k = \frac{9}{4}$

e $k < -6$ and $k > 6$ **f** $-10 < k < 10$

EX 4C Graphing parabolas using intercepts

p173

1 **a** **i** $(-6, 0), (0, 0)$ **ii** $(0, 0)$

b **i** $(0, 0), (10, 0)$ **ii** $(0, 0)$

c **i** $(1, 0), (5, 0)$ **ii** $(0, 5)$

d **i** $(-1, 0), (1, 0)$ **ii** $(0, -1)$

e **i** $(-6, 0), (2, 0)$ **ii** $(0, 12)$

f **i** $(-3, 0), (3, 0)$ **ii** $(0, 9)$

2 **a** upright **b** inverted **c** upright

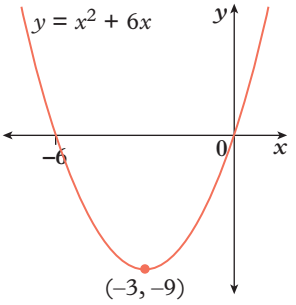
d upright **e** inverted **f** inverted

3 **a** $(-3, -9)$ **b** $(5, 25)$ **c** $(3, -4)$

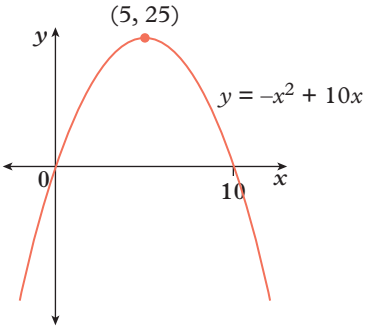
d $(0, -1)$ **e** $(-2, 16)$ **f** $(0, 9)$

4 **a** $(-1, -16)$ **b** $\left(\frac{1}{2}, 12\frac{1}{4}\right)$

5 **a**



b



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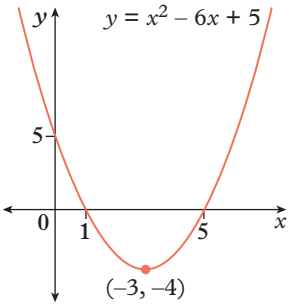

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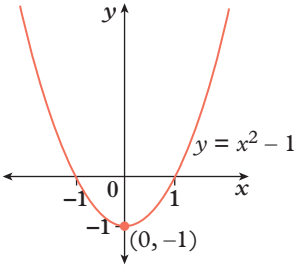

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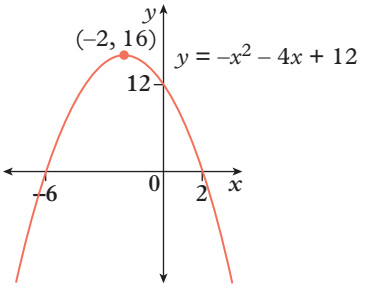
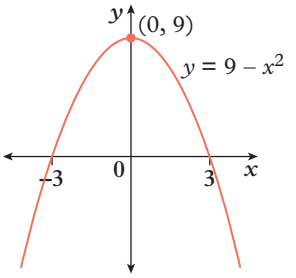
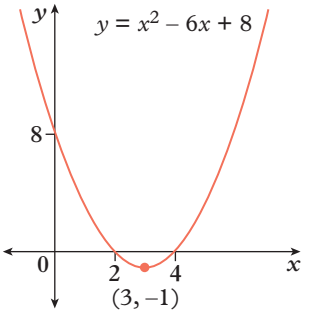


f



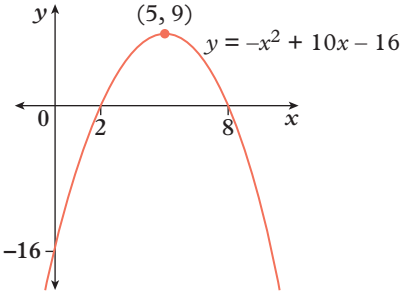
6 **a** Domain is $\mathbb{R}$.

Range is $y \geq -1$.



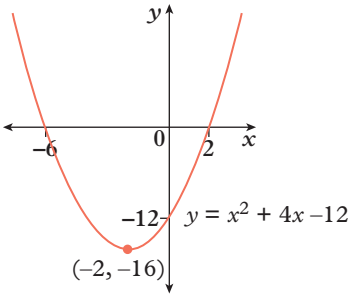
b Domain is $\mathbb{R}$.

Range is $y \leq 9$.



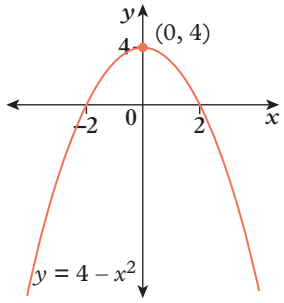
c Domain is $\mathbb{R}$.

Range is $y \geq -16$.



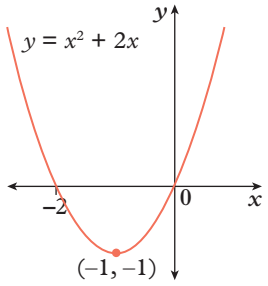
d Domain is $\mathbb{R}$.

Range is $y \leq 4$.



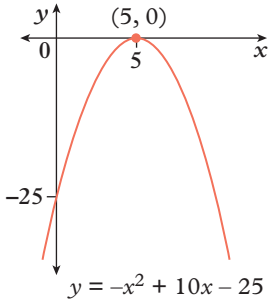
e Domain is $\mathbb{R}$.

Range is $y \geq -1$.



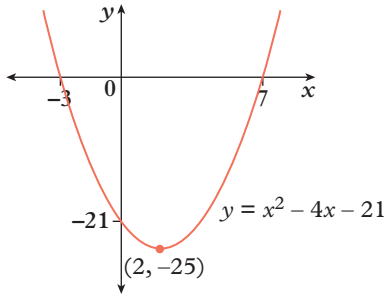
f Domain is $\mathbb{R}$.

Range is $y \leq 0$.



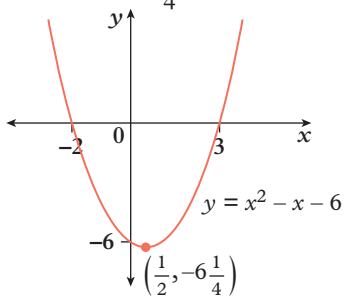
g Domain is $\mathbb{R}$.

Range is $y \geq -25$.



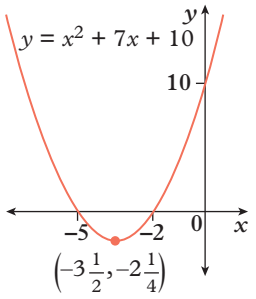
h Domain is $\mathbb{R}$.

Range is $y \geq -6\frac{1}{4}$.



i Domain is $\mathbb{R}$.

Range is $y \geq -2\frac{1}{4}$.



7 **a** **i** two **ii** one **iii** none

b **i** $x = -1$ or 2 **ii** $x = 2$ **iii** no solution

8 **a** For all these graphs, the x-intercepts are at $(-5, 0)$ and $(1, 0)$.

b The x-intercepts are the points when $y = 0$. For these equations, when $y = 0$ the equations all have the same solution for x. But, for other y values, the x values differ.

c For $y = (x + 5)(x - 1)$, the y-intercept is at $(0, -5)$.

For $y = 2(x + 5)(x - 1)$, the y-intercept is at $(0, -10)$.

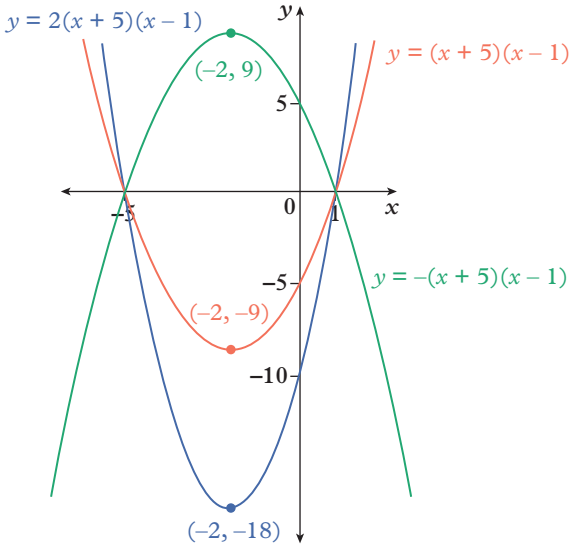
For $y = -(x + 5)(x - 1)$, the y-intercept is at $(0, 5)$.

d For $y = (x + 5)(x - 1)$, the turning point is $(-2, -9)$.

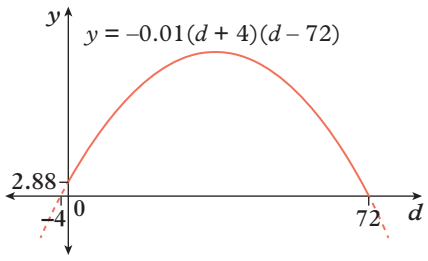
For $y = 2(x + 5)(x - 1)$, the turning point is $(-2, -18)$.

For $y = -(x + 5)(x - 1)$, the turning point is $(-2, 9)$.

e



9 **a**

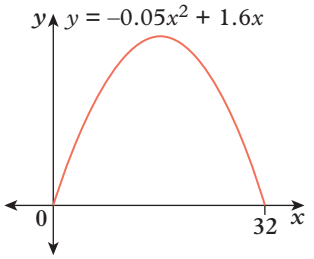


b 2.88 m

c 72 m

d 14.44 m

10 a

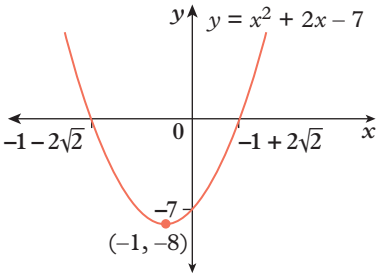


b height of 12.8 m at horizontal distance of 16 m

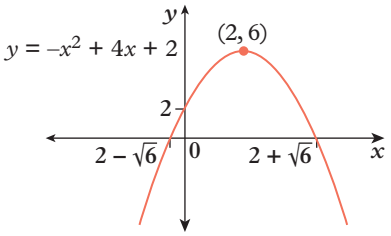
c 32 m

d 30.8 m

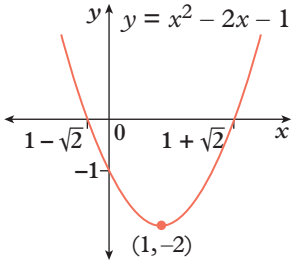
11 a



b



c



12 a $y = (x - 1)(x - 3)$

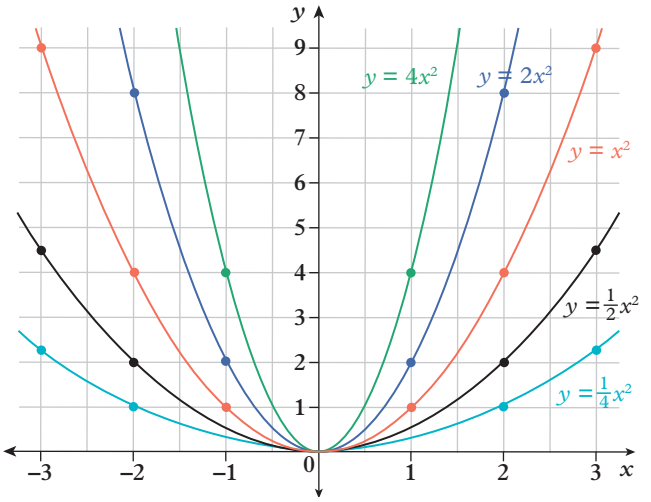
b $y = 2(x + 1)(x - 2)$

c $y = -2(x - 2)(x + 3)$

EX p179

4D Graphing parabolas using transformations

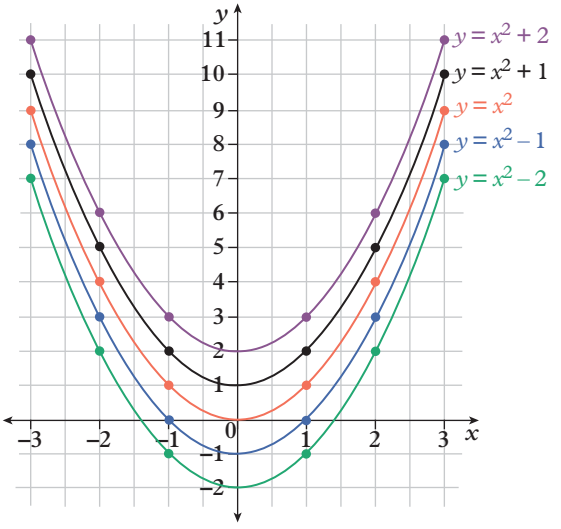
1 a i



ii All the graphs have the same minimum turning point (0, 0), the same axis of symmetry and the same x- and y-intercepts (same point as the turning point). The graphs of $y = 2x^2$ and $y = 4x^2$ are narrower than the graph of $y = x^2$; and the graphs of $y = \frac{1}{2}x^2$ and $y = \frac{1}{4}x^2$ are wider than the graph of $y = x^2$.

iii stretching; a graph is narrower when the coefficient of x^2 is greater than 1; a graph is wider when the coefficient of x^2 is between 0 and 1.

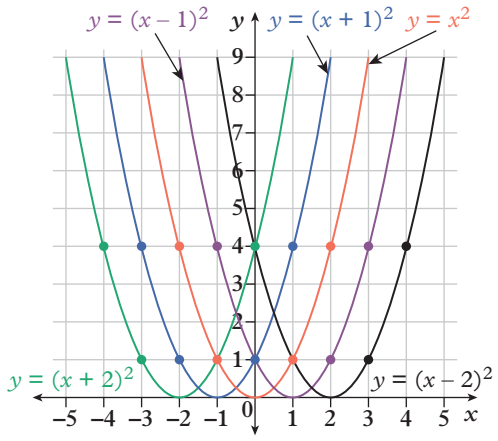
b i



ii All graphs have the same axis of symmetry and the same shape but they have different turning points.

iii vertical translation; the graph of $y = x^2$ moves up when a positive constant is added; the graph of $y = x^2$ moves down when a negative constant is added (or a positive constant is subtracted)

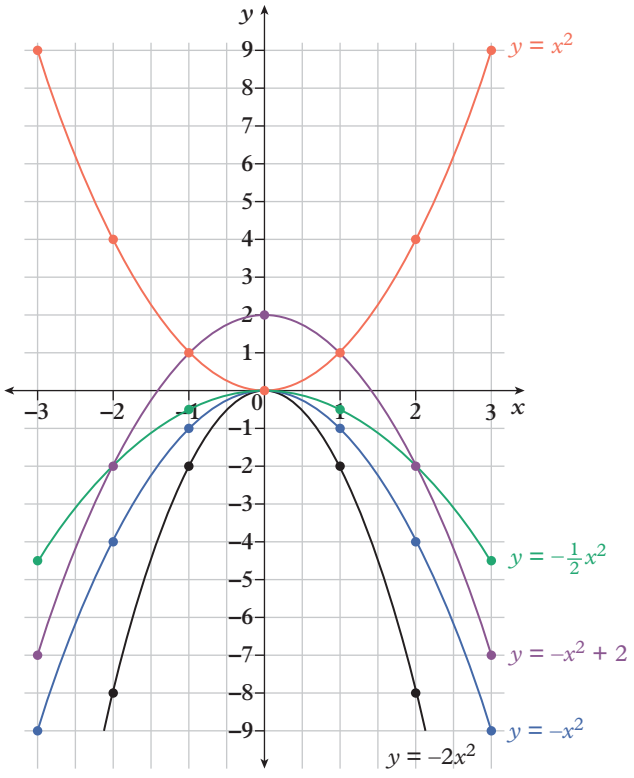
c i



ii All the graphs have the same shape but different turning points.

iii horizontal translation; the graph of $y = x^2$ moves to the right when a positive constant is subtracted from x before squaring; the graph of $y = x^2$ moves to the left when a negative constant is subtracted from x (or a positive constant is added to x) before squaring

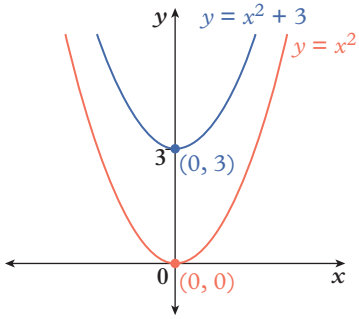
d i



ii The graph of $y = -x^2$ is a reflection in the x-axis of the graph of $y = x^2$, but it has the same shape, the same axis of symmetry and the coordinates of its turning point (0, 0) are the same as for the graph of $y = x^2$. The graph of $y = -x^2 + 2$ results from the graph of $y = x^2$ being reflected in the x-axis and moved 2 units up. The graph of $y = -2x^2$, compared to the graph of $y = x^2$, is reflected in the x-axis and is narrower. The graph of $y = -\frac{1}{2}x^2$, compared to the graph of $y = x^2$, is reflected in the x-axis and is wider.

iii All the graphs have been reflected in the x-axis when compared to the graph of $y = x^2$ because each has a negative coefficient of x^2. There is also a vertical translation for $y = -x^2 + 2$ because a constant is added to x^2; there is a dilation for $y = -2x^2$ (it is narrower than $y = x^2$ because its coefficient of x^2 is greater than 1); and there is a dilation for $y = -\frac{1}{2}x^2$ (it is wider than $y = x^2$ because its coefficient of x^2 is between 0 and 1).

2 a



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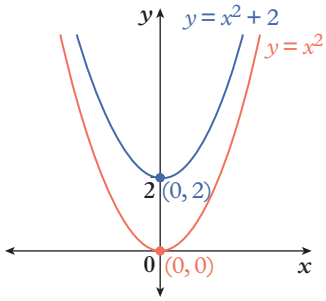

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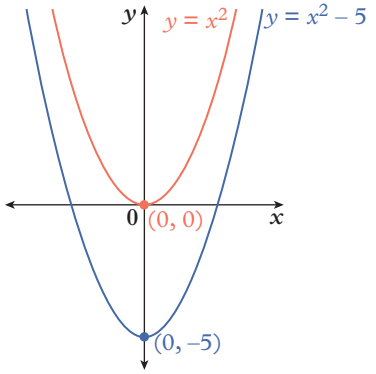

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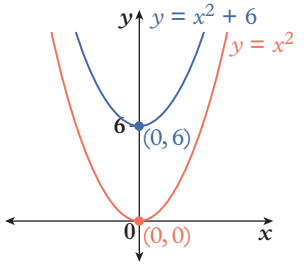

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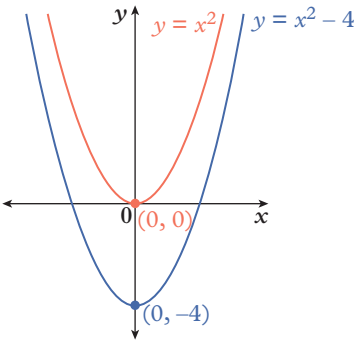

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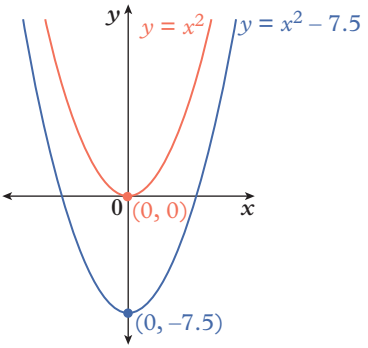
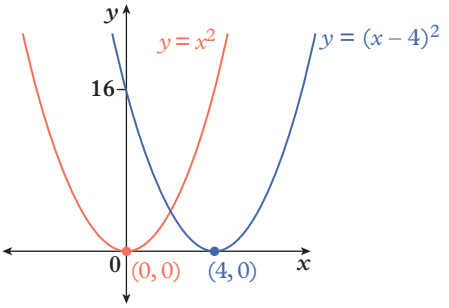


3 a



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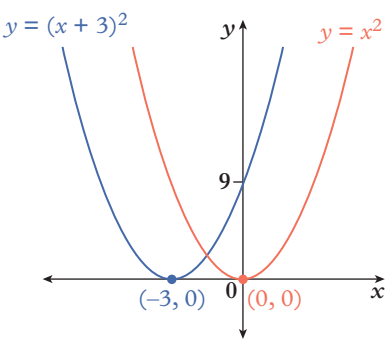

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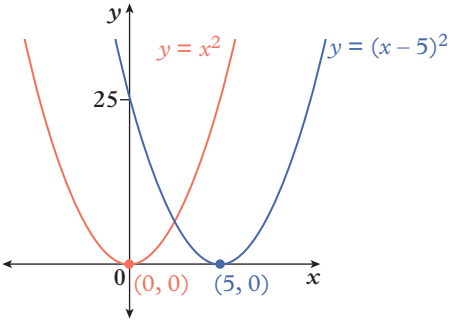

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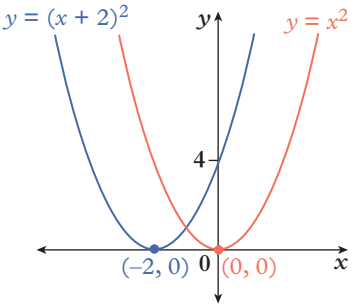

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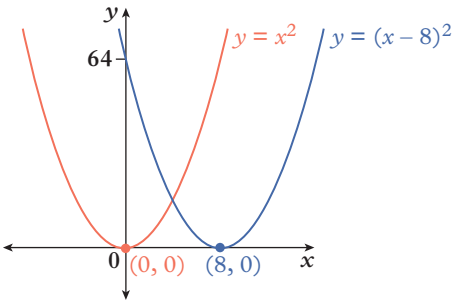

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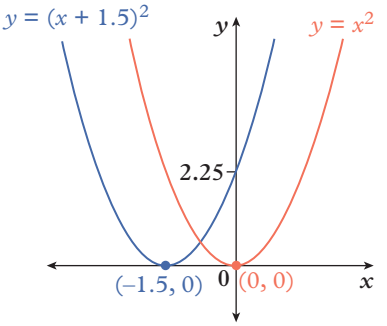
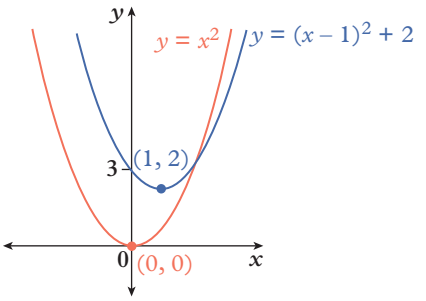


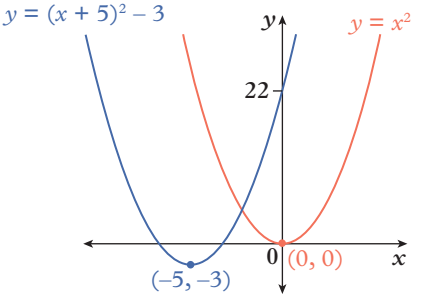
4 **a** B **b** D **c** A
d F **e** E **f** C

5 **a** $y = 6x^2$ **b** $y = -x^2$ **c** $y = (x - 5)^2$
d $y = x^2 - 2$ **e** $y = -\frac{1}{4}x^2$ **f** $y = -x^2 + 3$

6 a



b



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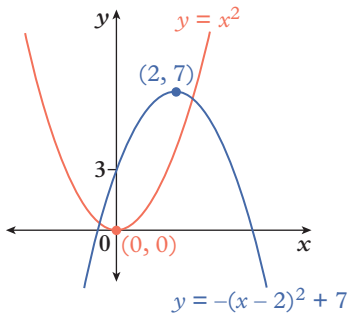

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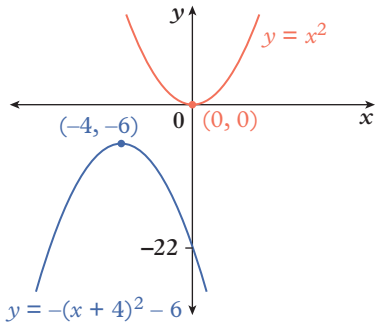

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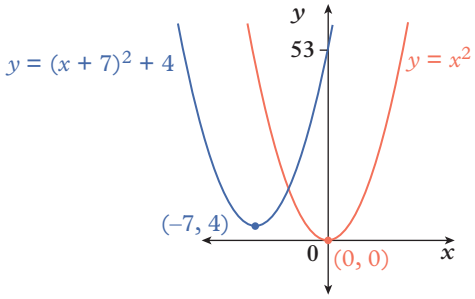

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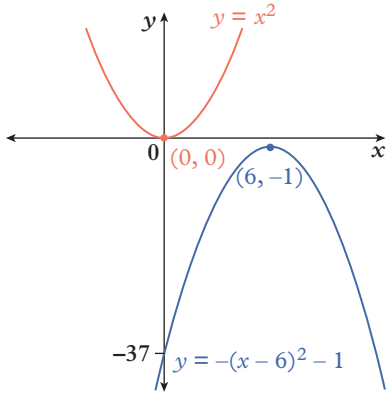


7 **a** The quadratic relationship is always positive

b $x < -5 - \sqrt{3}$ and $x > -5 + \sqrt{3}$

c $2 - \sqrt{7} < x < 2 + \sqrt{7}$

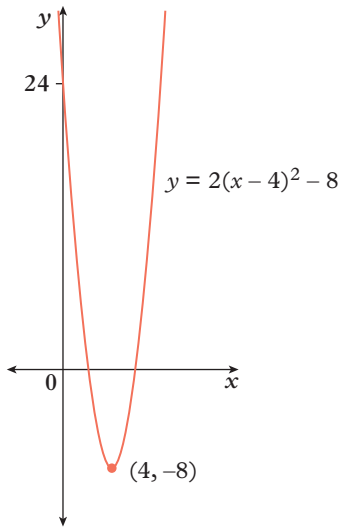
d The quadratic relationship is never positive

e The quadratic relationship is always positive

f The quadratic relationship is never positive

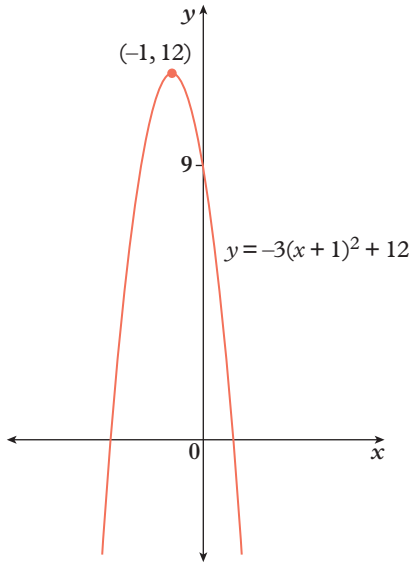
8 **a** **i** narrower **ii** upright **iii** (4, –8)

iv



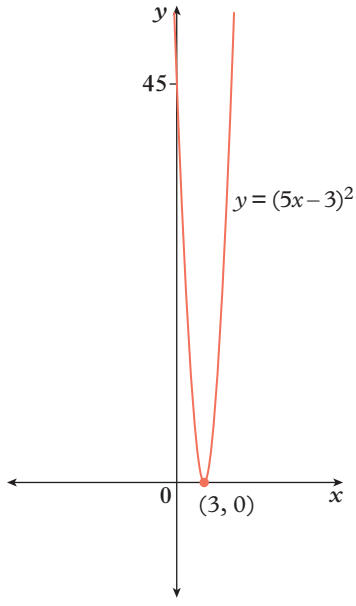
b **i** narrower **ii** inverted **iii** (–1, 12)

iv



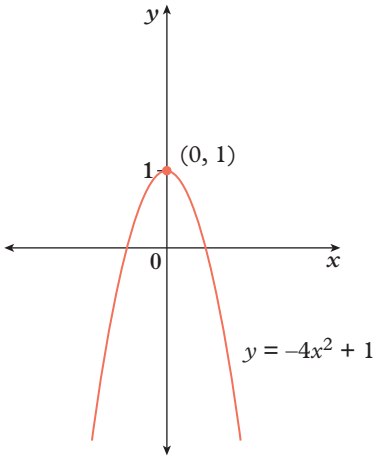
c **i** narrower **ii** upright **iii** (3, 0)

iv



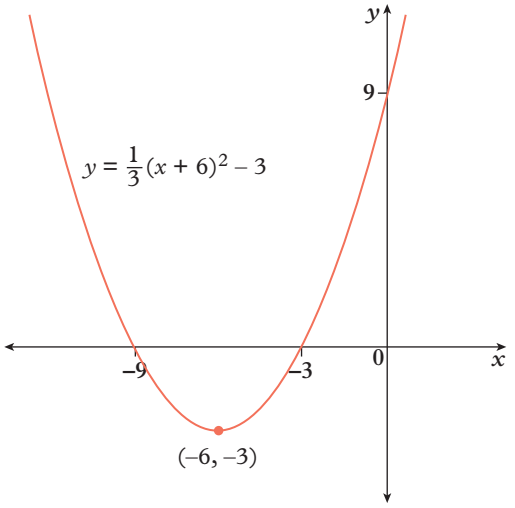
d **i** narrower **ii** inverted **iii** (0, 1)

iv



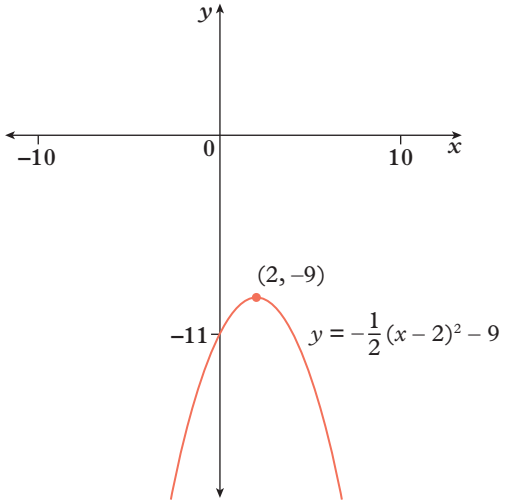
e **i** wider **ii** upright **iii** (–6, –3)

iv



f **i** wider **ii** inverted **iii** (2, –9)

iv



9 **a** $y = 2(x - 5)^2$ **b** $y = -(x + 4)^2 - 3$

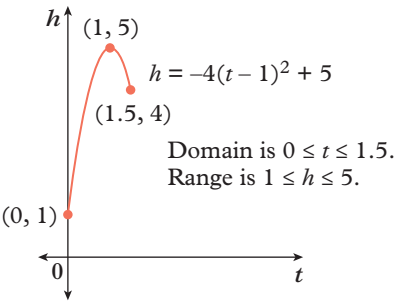
c $y = -\frac{1}{3}x^2$ **d** $y = -5x^2 + 7$

10 4 is the smallest, because the graph is upright and has a minimum turning point at (–2, 4). There is no largest value because the y values increase as the x values move away from –2.

11 –1 is the largest, because the graph is inverted and has a maximum turning point at (6, –1). There is no smallest value because the y values decrease as the x values move away from 6.

12 **a** (1, 5)

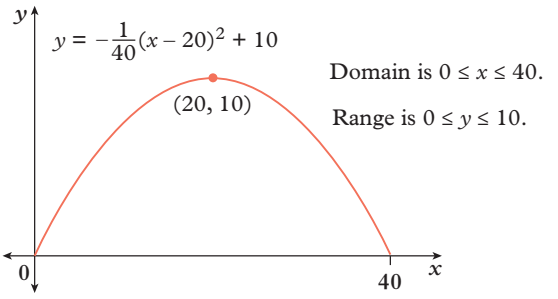
b



c **i** 1 m **ii** 5 m **iii** 4 m

d 2.1 seconds

13 **a**

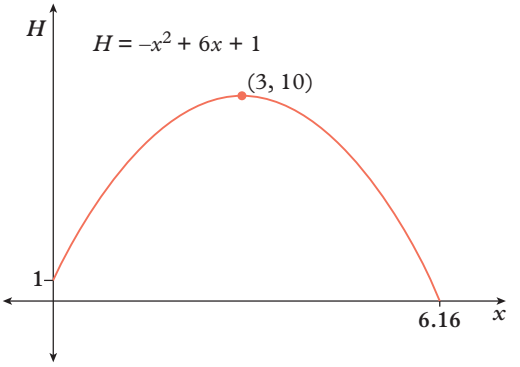


b 10 m **c** 40 m

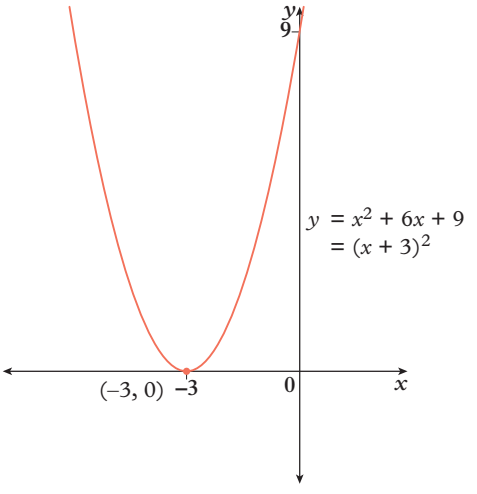
14 **a** 1 metre **b** $-(x - 3)^2 + 10$

c 10 metres **d** 6.16 metres

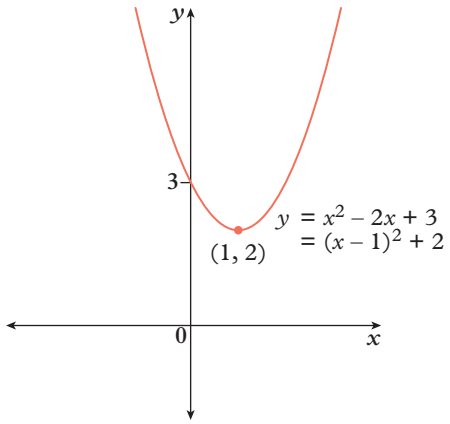
e



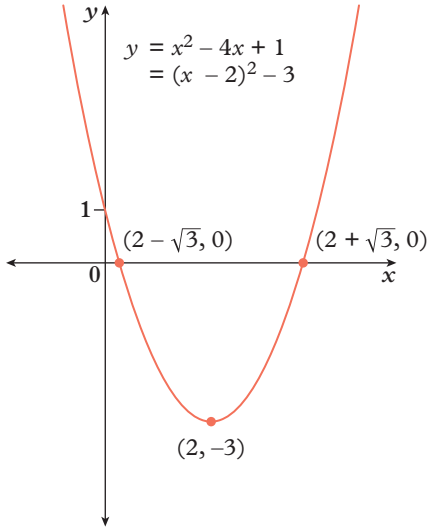
15 **a**



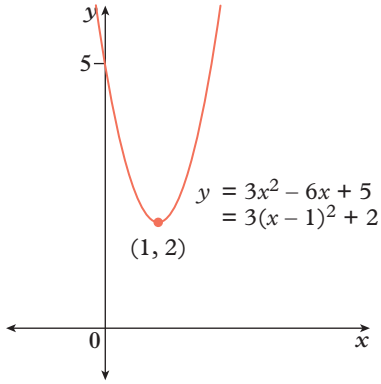
b



c



d

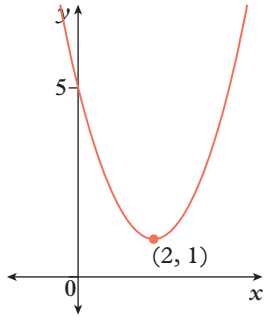


16 a $y = (x-2)^2 + 6$ **b** $y = 2(x+4)^2 - 12$

c $y = -3(x+1)^2 + 7$ **d** $y = -\frac{1}{3}(x-3)^2 - 5$

17 a
$$\begin{aligned}(2-x)^2 &= (2-x)(2-x)\\ &= 4 - 2x - 2x + x^2\\ &= x^2 - 4x + 4\\ &= (x-2)(x-2)\\ &= (x-2)^2\end{aligned}$$

b i



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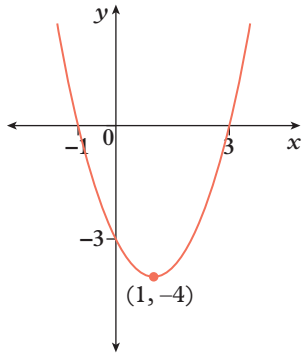

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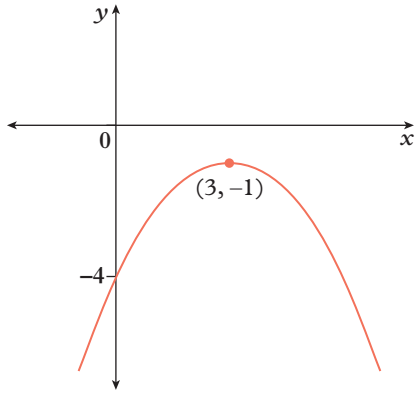

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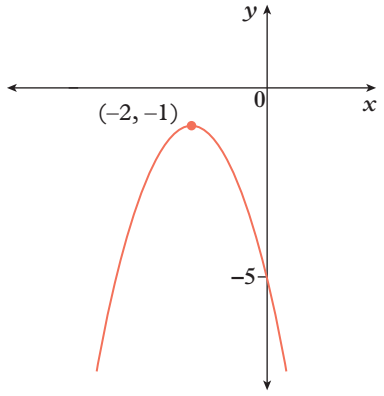


Checkpoint

1 a $x = 0$ or $x = 2$ **b** $x = -3$ or $x = 4$

c $x = -1$ or $x = 1$ **d** $x = -\frac{3}{2}$ or $x = 5$

2 a $x = 0$ or $x = -5$ **b** $x = 0$ or $x = 6$

c $x = -2$ or $x = -4$ **d** $x = 2$ or $x = 20$

e $x = -17$ or $x = 1$ **f** $x = -3$ or $x = 2$

3 **a** $x = \pm 2$ **b** $x = \pm\sqrt{7}$

c $x = \pm\sqrt{22}$ **d** $x = \pm 3$

e $x = \pm\sqrt{5}$ **f** $x = \pm\sqrt{6}$

4 **a** $\Delta = -3$ **b** $\Delta = 24$

c $\Delta = -4$ **d** $\Delta = 169$

5 **a** $\Delta = 13$. Therefore, there are two solutions.

b $\Delta = 0$. Therefore, there is one solution.

c $\Delta = 129$. Therefore, there are two solutions.

d $\Delta = -7$. Therefore, there are no solutions.

6 **a** $x = \frac{-3 \pm \sqrt{5}}{2}$ **b** $x = 1$ **c** $x = \frac{3 \pm \sqrt{13}}{2}$

d $x = \frac{-1 \pm \sqrt{41}}{10}$ **e** $x = -\frac{3}{2}$ **f** $x = -5 \pm \sqrt{31}$

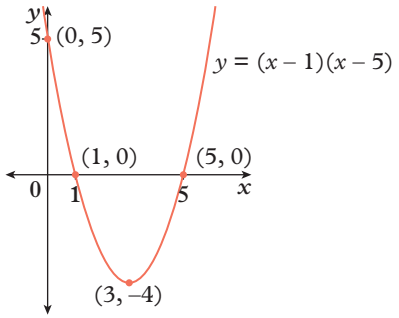
7 **a** x-intercepts: $(0, 0)$ and $(4, 0)$, y-intercept: $(0, 0)$

b x-intercepts: $(-3, 0)$ and $(1, 0)$, y-intercept: $(0, -3)$

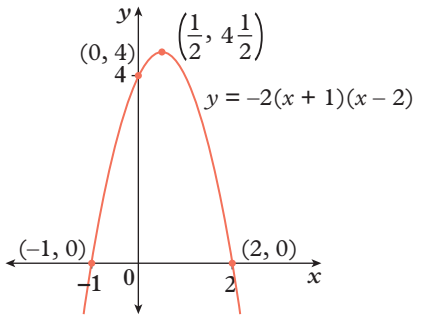
c x-intercepts: $(-1, 0)$ and $(2, 0)$, y-intercept: $(0, -2)$

d x-intercepts: $(-8, 0)$ and $(2, 0)$, y-intercept: $(0, -16)$

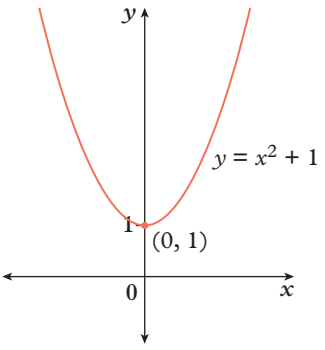
8



9



10 **a**



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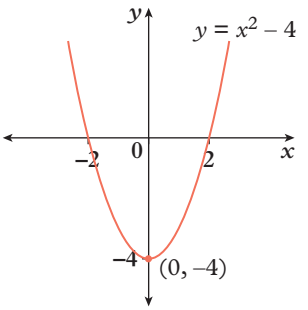

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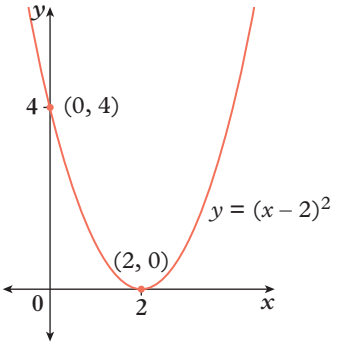

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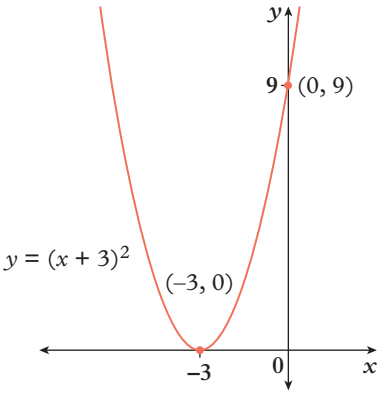
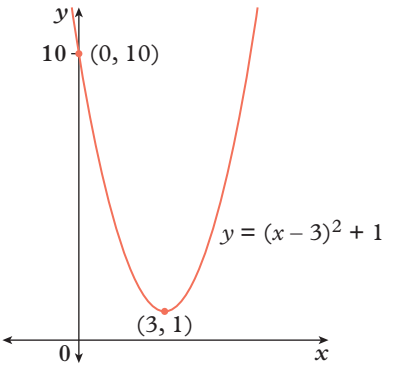


11 **a**



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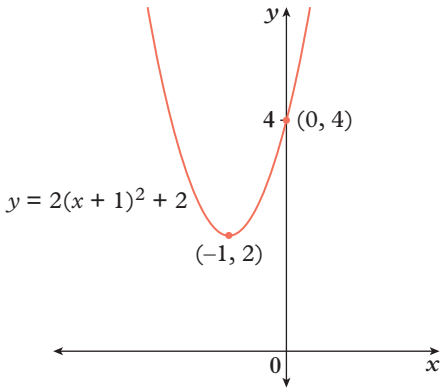

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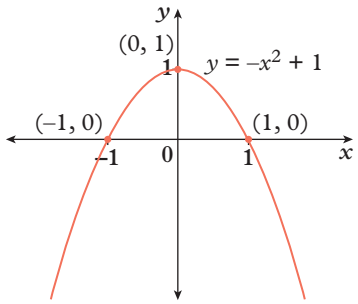

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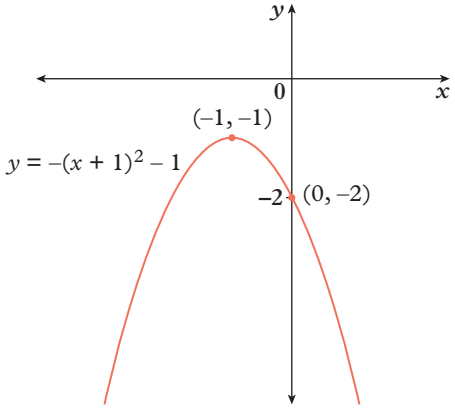

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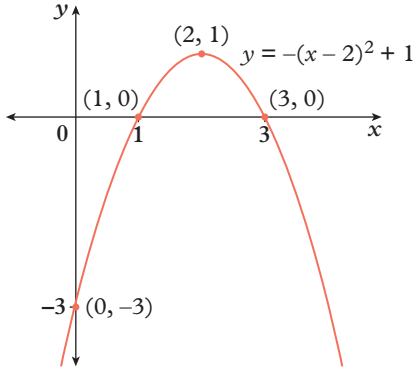
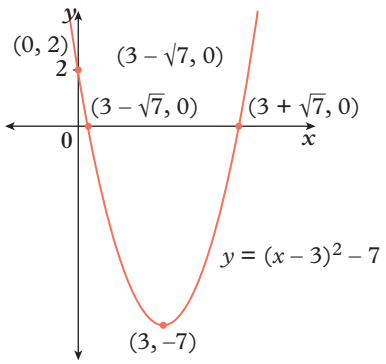


f



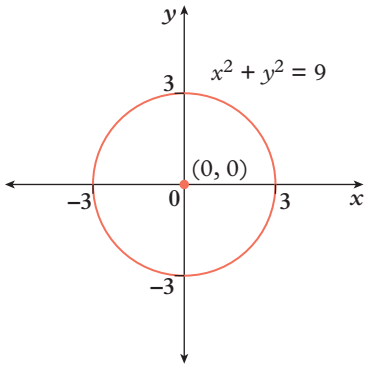
12 a $y = (x-2)^2 + 1$ **b** $y = -(x+3)^2 + 1$

c $y = 2x^2 - 3$ **d** $y = -3(x-4)^2 - 2$

EX p185

4E Graphing circles

1 a



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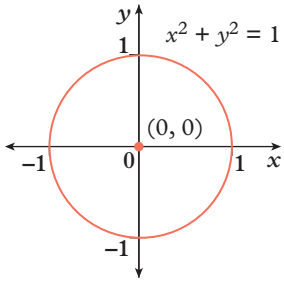

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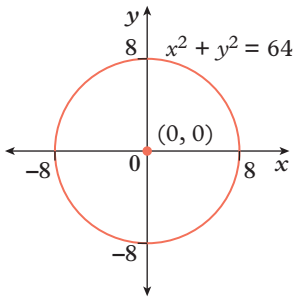

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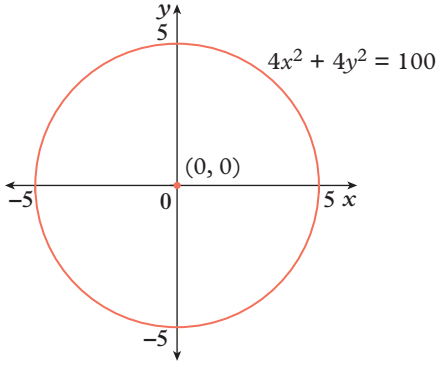
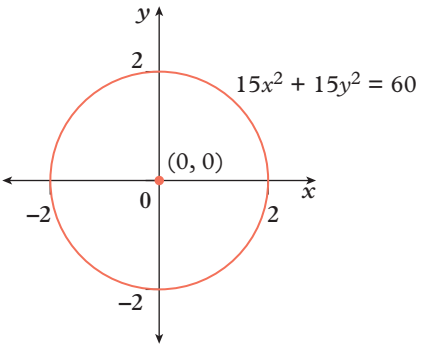


e



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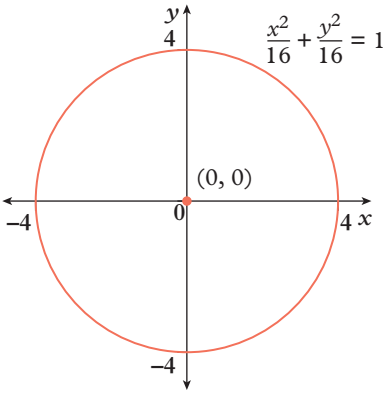
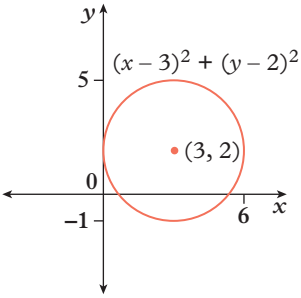


2 **a** $x^2 + y^2 = 9^2$ or $x^2 + y^2 = 81$

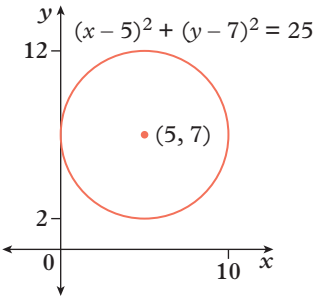
b $x^2 + y^2 = 36$

c $x^2 + y^2 = 400$

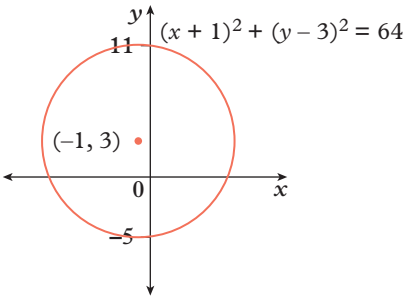
3 **a** centre (3, 2), radius 3 units



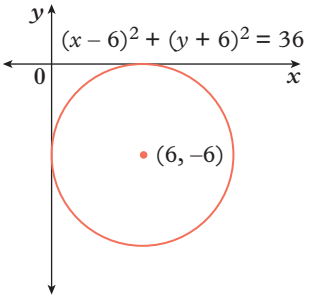
b centre (5, 7), radius 5 units



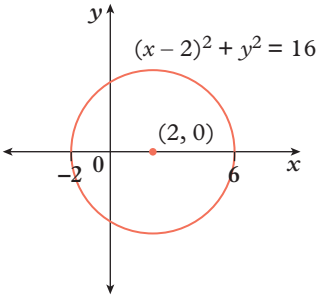
c centre (–1, 3), radius 8 units



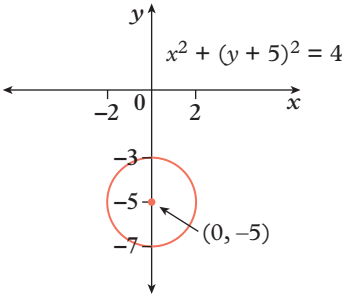
d centre (6, –6), radius 6 units



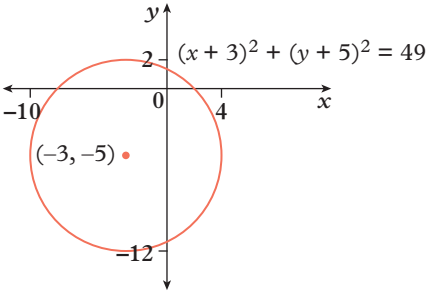
e centre (2, 0), radius 4 units



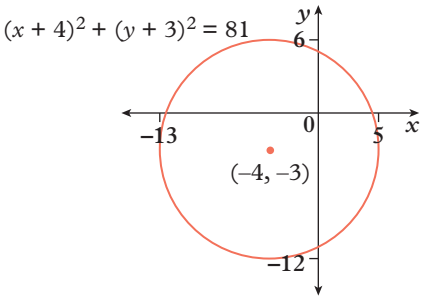
f centre (0, –5), radius 2 units



g centre (–3, –5), radius 7 units



h centre (–4, –3), radius 9 units



5 **a** $x^2 + y^2 = 144$

b $x^2 + y^2 = 225$

c $(x - 4)^2 + (y + 2)^2 = 1$

d $(x + 5)^2 + (y + 6)^2 = 49$

e $(x - 4)^2 + (y - 4)^2 = 9$

f $x^2 + (y + 5)^2 = 36$

g $(x + 1)^2 + (y - 2.5)^2 = 2.25$

h $(x - 2)^2 + (y + 3)^2 = 5$

6 **a** $(x - 7)^2 + (y - 3)^2 = 9$ **b** $(x + 1)^2 + (y + 4)^2 = 25$

c $(x + 6)^2 + (y - 5)^2 = 4$ **d** $(x - 2)^2 + (y + 8)^2 = 16$

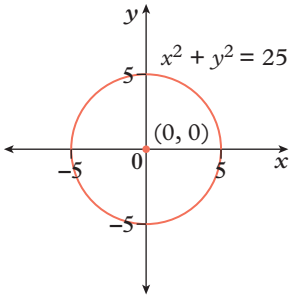
7 **a** $(x - 2)^2 + (y + 1)^2 = 9$

b A dilation in all directions by a factor of 5, then a translation of 7 units left and 4 units up

8 **a** a is the horizontal distance between the coordinates (3, 4) and (x, y). This distance is measured as $x - 3$, hence $a = x - 3$. b is the vertical distance between the coordinates (3, 4) and (x, y). This distance is measured as $y - 4$, hence $b = y - 4$.

b Pythagoras' theorem, $a^2 + b^2 = c^2$, is useful for determining the side lengths of a right-angled triangle. The right-angled triangle in the diagram can be explained by $a^2 + b^2 = 2^2$. Substituting a and b into the equation gives $(x - 3)^2 + (y - 4)^2 = 2^2$. Since every point (x, y) on the circle can be found by this process, this is the equation of the circle.

9 **a**



b $y = \pm\sqrt{25 - x^2}$

c **i** B; because $y \leq 0$ for $-5 \leq x \leq 5$

ii A; because $y \geq 0$ for $-5 \leq x \leq 5$

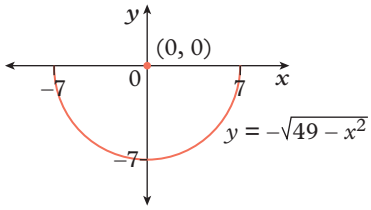
d $y = \sqrt{25 - x^2}$ gives the upper half of the circle and $y = -\sqrt{25 - x^2}$ gives the lower half of circle. Together, $y = \pm\sqrt{25 - x^2}$, or $x^2 + y^2 = 25$, gives the whole circle.

10 **a** $y = \sqrt{9 - x^2}$ **b** $y = -\sqrt{64 - x^2}$

c $y = \sqrt{1 - x^2}$

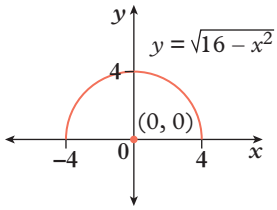
11 **a** Domain is $-7 \leq x \leq 7$.

Range is $-7 \leq y \leq 0$.



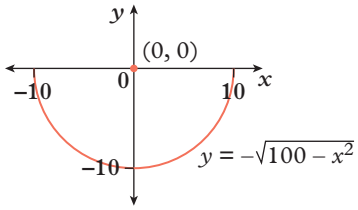
b Domain is $-4 \leq x \leq 4$.

Range is $0 \leq y \leq 4$.



c Domain is $-10 \leq x \leq 10$.

Range is $-10 \leq y \leq 0$.

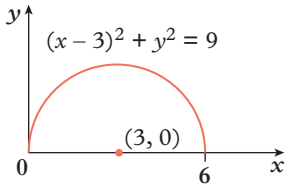


12 **a** $x = \pm\sqrt{25 - y^2}$

b For $x = \sqrt{25 - y^2}$, the graph is the right-hand half of the circle with equation $x^2 + y^2 = 25$.

For $x = -\sqrt{25 - y^2}$, the graph is the left-hand half of the circle with equation $x^2 + y^2 = 25$.

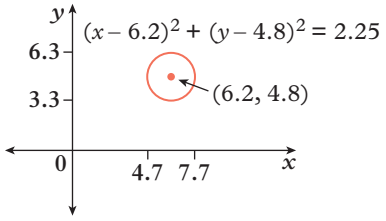
13 **a**



b 6 m **c** 3 m

d When $y = 1.5$, $x \approx 0.4$ and 5.6, so the arch is at half its maximum height at a distance of 0.4 m from either end.

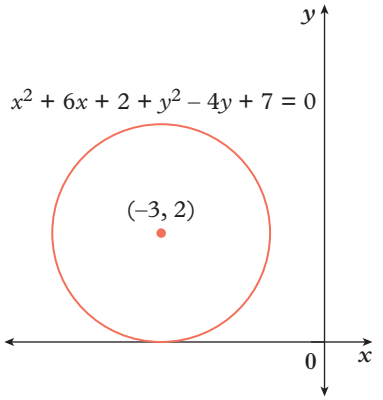
14 **a**



b 3.3 m **c** 7 m² **d** 4.7%

15 a i $(x+3)^2-7$ **ii** $(y-2)^2+3$

b



16 $(-4, 3), (-3, 4)$

17 a $(0, -3), (3, 0)$ **b** $(-3, 6), (-1, 8)$

18 $(x+4)^2 + (y-5)^2 = 36$

19 a $(1, 2)$

b Substituting in $(3, 3)$ gives $(3-1)^2 + (3-2)^2 = 5$, which is true. Therefore, since $(3, 3)$ satisfies the equation, it is a point on the circle.

c $\frac{1}{2}$

d -2

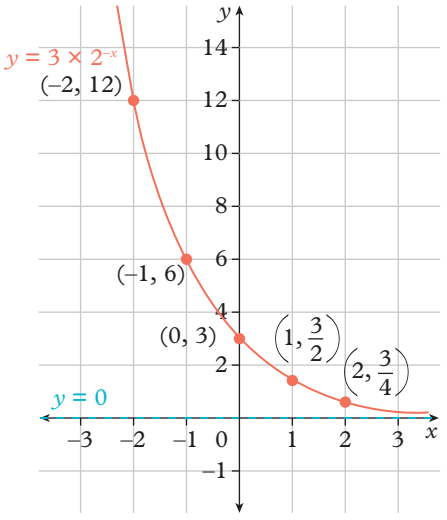
e $y = -2x + 9$

EX p195

4F Graphing exponential relationships

1 a

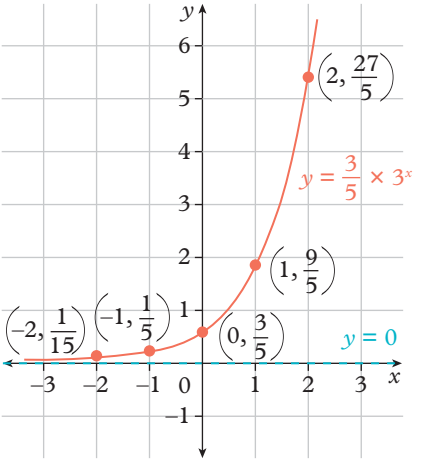
x	-2	-1	0	1	2
$y = 3 \times 2^{-x}$	12	6	3	$\frac{3}{2}$	$\frac{3}{4}$



When the x-value increases by 1, the y-value is divided by two.

b

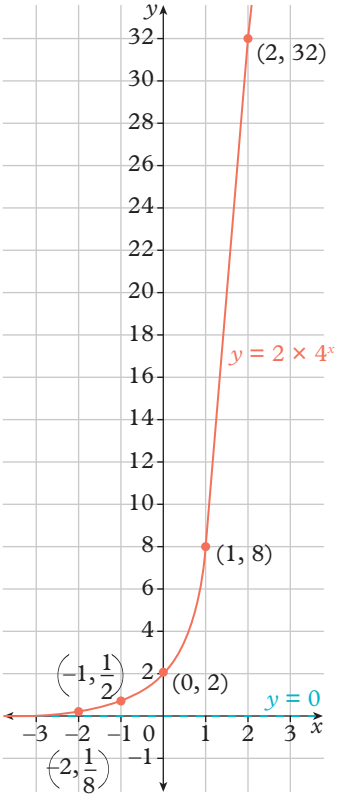
x	-2	-1	0	1	2
$y = \frac{3}{5} \times 3^x$	$\frac{1}{15}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{9}{5}$	$\frac{27}{5}$



When the x-value increases by 1, the y-value is multiplied by three.

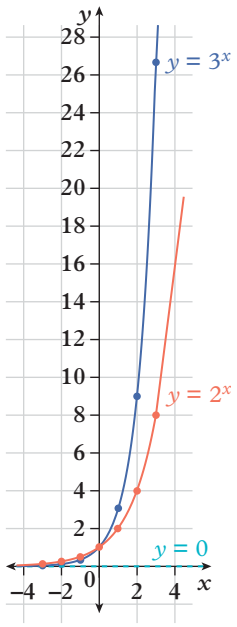
c

x	-2	-1	0	1	2
$y = 2 \times 4^x$	$\frac{1}{8}$	$\frac{1}{2}$	2	8	32



When the x-value increases by 1, the y-value is divided by four.

2 a



b The curve for $y = 3^x$ becomes steeper as the positive x values increase. That is, the y values for $y = 3^x$ are bigger than those for $y = 2^x$ when $x > 0$. When $x < 0$, the y values for $y = 3^x$ are smaller than those for $y = 2^x$.

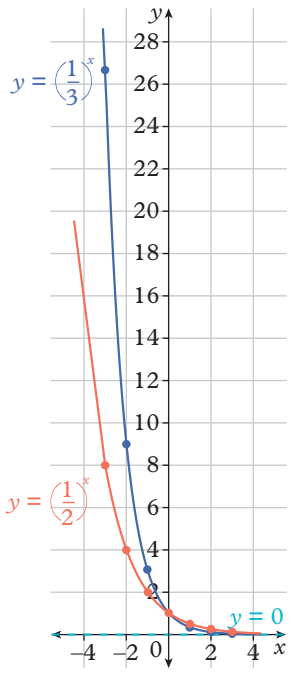
c They have the same asymptote and y-intercept. They are the same general shape.

d (0, 1) for both graphs

e $y = 0$ for both graphs

f The curve for $y = 5^x$ becomes steeper than both as the positive x values increase. That is, the y values for $y = 5^x$ are bigger than those for both $y = 2^x$ and $y = 3^x$ when $x > 0$. For $x < 0$, the y values for $y = 5^x$ are smaller than those for $y = 2^x$ and $y = 3^x$. However, the asymptote and y-intercept would be the same, and the general shape would be the same.

3 a



b The curve for $y = \left(\frac{1}{3}\right)^x$ becomes steeper as the x values decreases. That is, the y values for $y = \left(\frac{1}{3}\right)^x$ are bigger than those for $y = \left(\frac{1}{2}\right)^x$ when $x < 0$. When $x > 0$, the y values for $y = \left(\frac{1}{3}\right)^x$ are smaller than those for $y = \left(\frac{1}{2}\right)^x$.

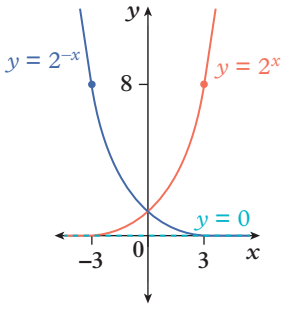
c They have the same asymptote and y-intercept. They are the same general shape.

d (0, 1) for both graphs

e $y = 0$ for both graphs

f The curve for $y = \left(\frac{1}{5}\right)^x$ becomes steeper than both as the x values decreases. That is, the y values for $y = \left(\frac{1}{5}\right)^x$ are bigger than those for both $y = \left(\frac{1}{2}\right)^x$. and $y = \left(\frac{1}{3}\right)^x$ when $x < 0$. For $x > 0$, the y values for $y = \left(\frac{1}{5}\right)^x$ are smaller than those for $y = \left(\frac{1}{2}\right)^x$. and $y = \left(\frac{1}{3}\right)^x$. However, the asymptote and y-intercept would be the same, and the general shape would be the same.

4



a $y = 2^x$ has positive y values that increase as x increases. $y = 2^{-x}$ has positive y values that decrease as x increases.

b They have the same asymptote and y-intercept, and the same general shape (reflected in the y-axis).

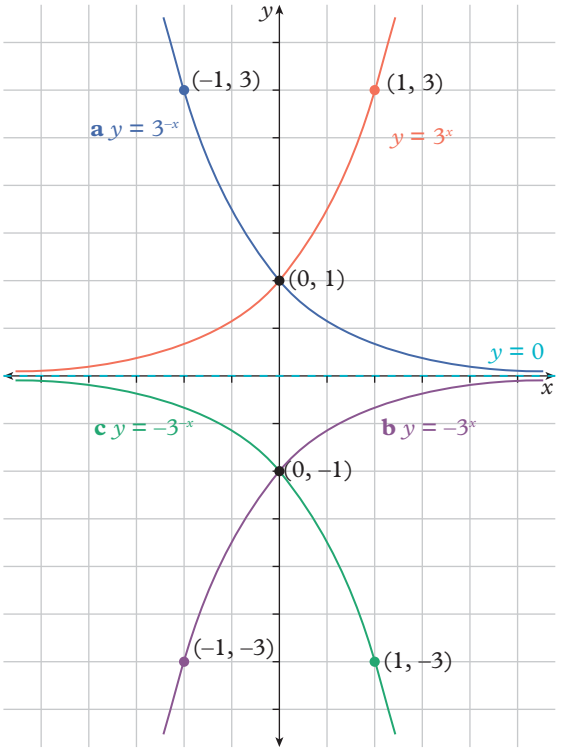
c 1 for both

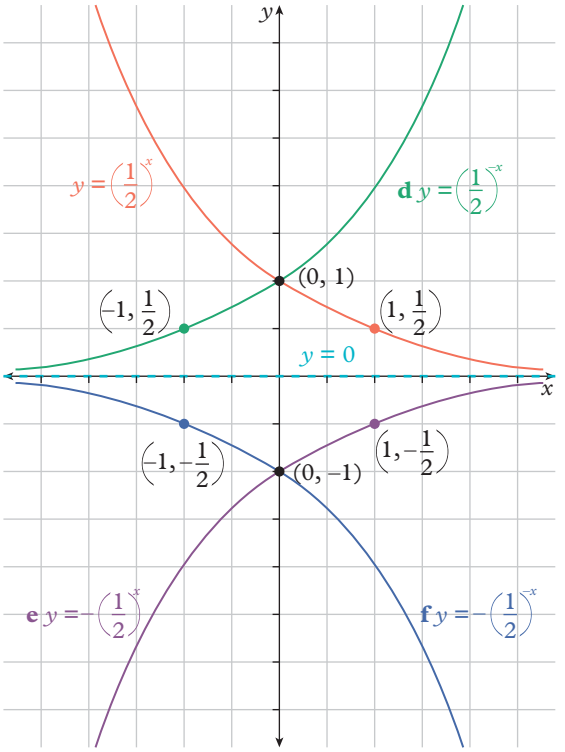
d $y = 0$ for both

e reflection in y-axis

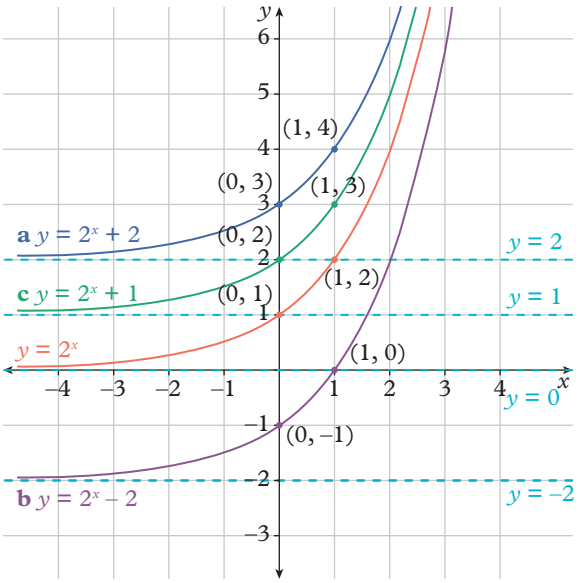
f a reflection in the x-axis and then a reflection in the y-axis

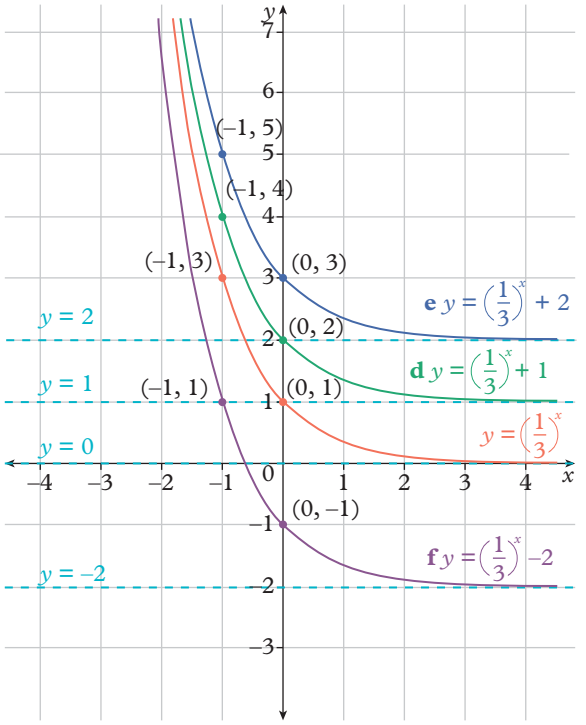
5


y
x
(−1, 3)
(1, 3)
a $y = 3^{-x}$
$y = 3^x$
(0, 1)
$y = 0$
c $y = -3^{-x}$
b $y = -3^x$
(0, −1)
(−1, −3)
(1, −3)

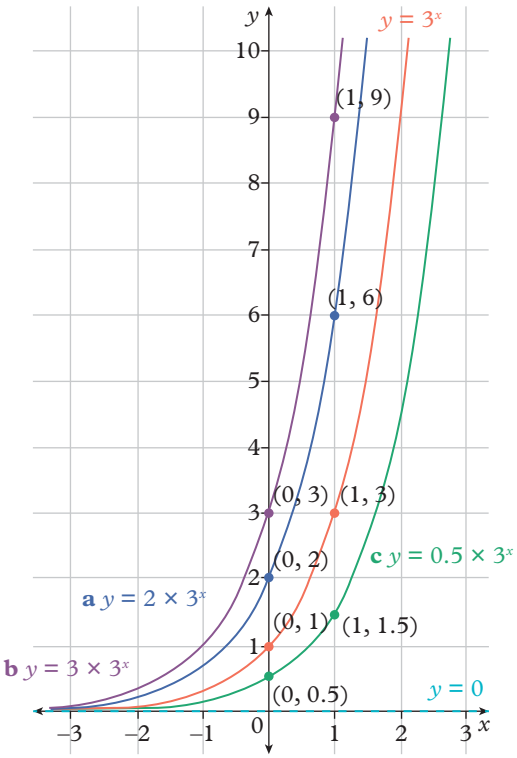

y
x
$y = \left(\frac{1}{2}\right)^x$
d $y = \left(\frac{1}{2}\right)^{-x}$
(0, 1)
$\left(-1, \frac{1}{2}\right)$
$\left(1, \frac{1}{2}\right)$
$y = 0$
$\left(-1, -\frac{1}{2}\right)$
$\left(1, -\frac{1}{2}\right)$
(0, −1)
e $y = -\left(\frac{1}{2}\right)^x$
f $y = -\left(\frac{1}{2}\right)^{-x}$

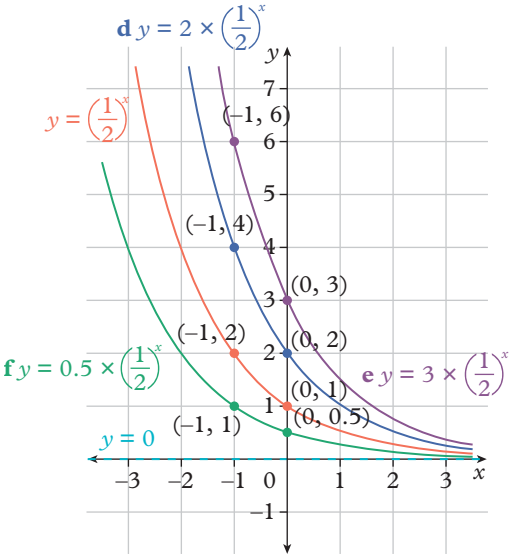
6


y
x
(1, 4)
(0, 3)
(1, 3)
a $y = 2^x + 2$
(0, 2)
$y = 2$
c $y = 2^x + 1$
(0, 1)
(1, 2)
$y = 1$
$y = 2^x$
(1, 0)
$y = 0$
(0, −1)
b $y = 2^x - 2$
$y = -2$


y
x
(−1, 5)
(−1, 4)
(−1, 3)
(0, 3)
e $y = \left(\frac{1}{3}\right)^x + 2$
$y = 2$
(0, 2)
d $y = \left(\frac{1}{3}\right)^x + 1$
$y = 1$
(−1, 1)
(0, 1)
$y = \left(\frac{1}{3}\right)^x$
$y = 0$
(0, −1)
f $y = \left(\frac{1}{3}\right)^x - 2$
$y = -2$

7





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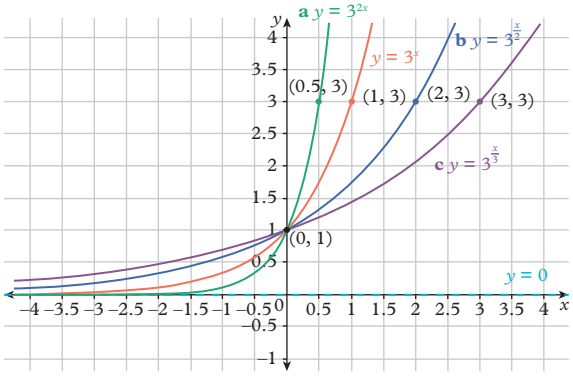
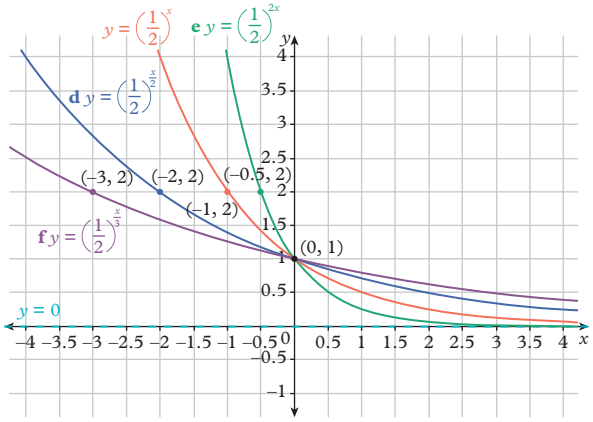




9 **a** $y = -a^x$ **b** $y = a^{-x}$ **c** $y = -a^{-x}$

10 a

x	−2	−1	0	1	2
$y = 3 \times 2^{-x}$	12	6	3	$\frac{3}{2}$	$\frac{3}{4}$
Ratio	$6 \div 12 = \frac{1}{2}$	$3 \div 6 = \frac{1}{2}$	$\frac{3}{2} \div 3 = \frac{1}{2}$	$\frac{3}{4} \div \frac{3}{2} = \frac{1}{2}$	

b

x	−2	−1	0	1	2
$y = \frac{3}{5} \times 2^x$	$\frac{1}{15}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{9}{5}$	$\frac{27}{5}$
Ratio	$\frac{1}{5} \div \frac{1}{15} = 3$	$\frac{3}{5} \div \frac{1}{5} = 3$	$\frac{9}{5} \div \frac{3}{5} = 3$	$\frac{27}{5} \div \frac{9}{5} = 3$	

c

x	−2	−1	0	1	2
$y = 2 \times 4^x$	$\frac{1}{8}$	$\frac{1}{2}$	2	8	32
Ratio	$\frac{1}{2} \div \frac{1}{8} = 4$	$2 \div \frac{1}{2} = 4$	$8 \div 2 = 4$	$32 \div 8 = 4$	

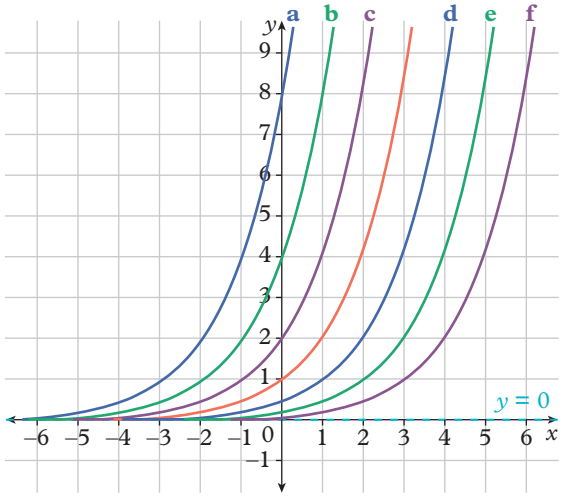
The ratio is a constant. For an exponential equation in the from $y = Ab^{nx}$, the ratio between consecutive values is a constant because $\frac{Ab^{n(x+1)}}{Ab^{nx}} = \frac{b^{nx+n}}{b^{nx}} = \frac{b^{nx} \times b^n}{b^{nx}} = b^n$

11 **a** **i** reflection in the x-axis **ii** $y = -7^x$

b **i** translation 2 units up **ii** $y = 4x + 2$

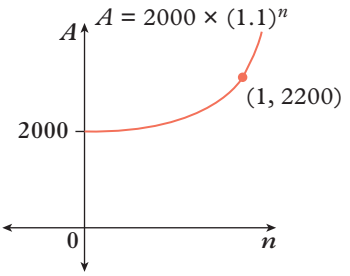
12 **a** $y = 8 \times 2^x$ **b** $y = 4 \times 2^x$ **c** $y = 2 \times 2^x$

d $y = \frac{1}{2} \times 2^x$ **e** $y = \frac{1}{4} \times 2^x$ **f** $y = \frac{1}{8} \times 2^x$



13 **a** 3 units left **b** 2 units left **c** 1 unit left
d 1 unit right **e** 2 units right **f** 3 units right

14 **a**

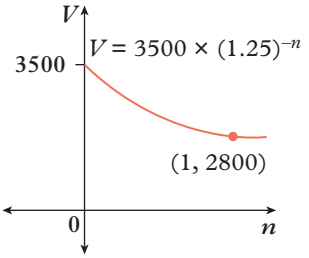


b A increases

c **i** approximately \$2900 read from the graph (the exact value is \$2928.20)

ii 8 years

15 **a** The graph will decrease as time increases.



b \$3500 **c** V decreases **d** 6 years

16 **a** $3 \times 2^2 = 12$ whereas $6^2 = 36$

b $x = 1$

c 'Equivalent to' means the expressions are exactly the same for all values of x, whereas 'equal to' means solving for a particular instance when the expressions evaluate to the same number.

d Two expressions may be equal for some values, but that does not mean they are necessarily equal for all values.

17 **a**

x	−2	−1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
2^x	0.25	0.5	1	2	4	8	16	32

Therefore, $x = 2$ and $x = 4$.

b As at $x = -1$, $x^2 > 2^x$ but at $x = 0$, $x^2 < 2^x$. This means the expressions must be equal at some value in between −1 and 0.

c $x = -0.767$

18 **a** **i** dilation in the y-direction by factor of 2, translation 4 units up

ii The equation for A is $y = 5^x$. The equation for B is $y = 2 \times 5^x + 4$.

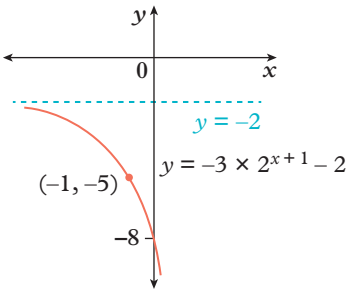
b **i** translation of 2 units right and 5 units down

ii The equation for A is $y = 4^x$. The equation for B is $y = 4^{x-2} - 5$.

c **i** dilation in the y-direction by factor of 4, reflection in x-axis and y-axis

ii The equation for A is $y = 3^x$. The equation for B is $y = -4 \times 3^{-x}$.

19 dilation in the y-direction by factor of 3, reflection in the x-axis, translation of 1 unit left and 2 units down



EX 4G Rates of change

p200

1 **b**, **d**, **f**

2 **a** 3 **b** 7 **c** 1.5
d $\frac{2}{3}$ **e** −2 **f** $\frac{1}{90}$

3 **a** −5 **b** 46.125 **c** 4.95 **d** −2

4 **a** −5.9, −5.99, −5.999, hence $m_T \approx -6$

b 34.481, 32.2408, 32.0240, hence $m_T \approx 32$

c 4.95, 2.84605, 2.322986, hence $m_T \approx 2.3$

d −2, −2.99, −2.9999, hence $m_T \approx -3$

5 **a** **i** y-intercept = 2 **ii** −1

b **i** y-intercept = 1 **ii** −3

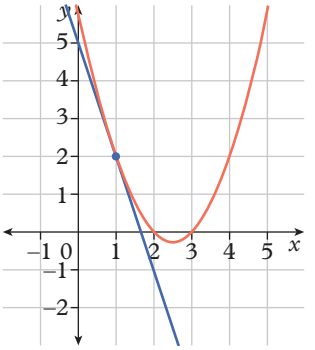
c **i** y-intercept = 1 **ii** 0

d **i** y-intercept does not exist **ii** undefined

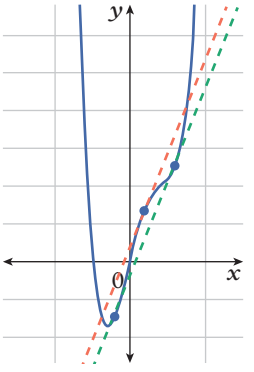
6 **a** 2 km/min **b** 1 km/min **c** 0.5 km/min
d 1 km/min **e** 1 km/min

7 **a** −2.5, −2.9, −2.99, hence $m_T \approx -3$

b



8 Not necessarily, the graph of a function could have a gradient of 5 at three different x values without being linear such as the one below.



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9 a The rate starts at zero and steadily increases (becomes more positive).

b The rate starts at zero and steadily decreases (becomes more negative).

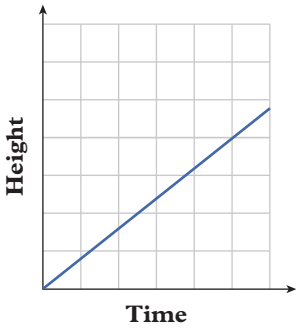
c The rate decreases (becomes less positive) until it is zero then continues decreasing (becomes more negative).

d The rate starts at nearly zero and increases at an increasing rate.

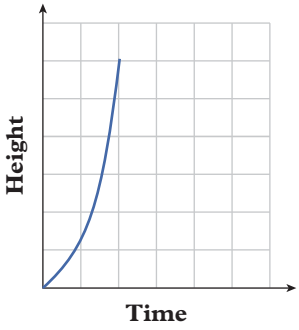
e The rate decreases (becomes less positive) until it is zero then continues decreasing (becomes more negative), then begins increasing (becomes less negative) until it is zero again then continues increasing (becomes more positive).

f The rate alternates between decreasing and increasing repeatedly.

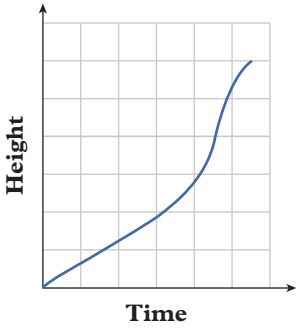
10 a



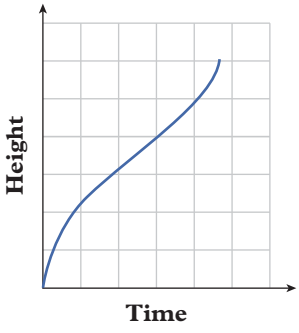
b



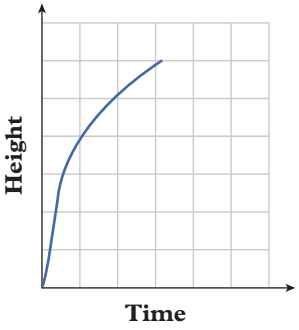
c



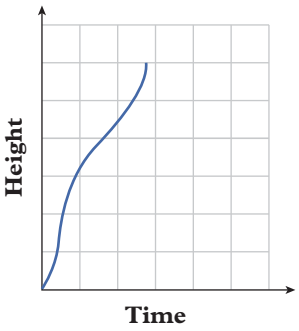
d



e



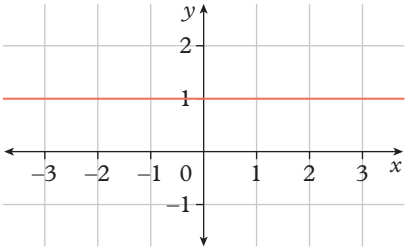
f



11 a $h + 6$ **b** $h^2 + 3h + 3$ **c** $3h + 5$

12 a 6 **b** $h^2 + 3$ **c** 5

13 a



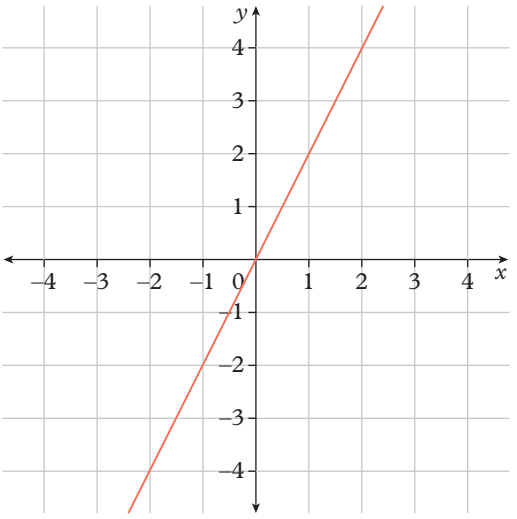
b No, the gradient of a linear relationship in the form $y = x + b$ is 1, the graph of the gradient is always a horizonal line passing through 1 on the vertical axis. The graph of the gradient is the same for different values of b. Similarly, the gradient of linear relationship in the form $y = ax + b$ is a, the graph of the gradient is always a horizonal line passing through a on the vertical axis regardless of the value of b.

c Gradient between (–2, 5) and (–1, 2) is –3
Gradient between (–1, 2) and (0, 0) is –1
Gradient between (0, 0) and (1, 2) is 1
Gradient between (1, 2) and (2, 5) is 3

d

x	–1.5	–0.5	0	0.5	1.5
Gradient	–3	–1	0	1	3

e



f C

g No, the graph of the gradient of $y = x^2 + 1$ and $y = x^2 - 1$ are the same. If a quadratic relationship $y = a(x - h)^2$ is translated k units up or down, the rates of change are still the same for a given x value, the graph of the gradient remains the same regardless the value of k.

CHAPTER 4 review

Multiple choice

1	A	**2**	C	**3**	A	**4**	C	**5**	D
6	B	**7**	A	**8**	C	**9**	C	**10**	B

Short answer

1 **a** $x = 2$ **b** $x = \frac{3}{4}$ or $x = -\frac{4}{3}$ **c** $x = \pm\sqrt{2}$

d $x = -2$ or $x = 4$ **e** $x = \pm\sqrt{\frac{7}{5}}$

2 **a** $x = -3 \pm \sqrt{5}$ **b** $x = \frac{3 \pm \sqrt{21}}{2}$

3 **a** 1 **b** 0 **c** 0 **d** 2 **e** 2 **f** 0

4 **a** **i** $x = -5$ or $x = 2$ **ii** $x \approx -5.00$ or $x \approx 2.00$

b **i** $x = \frac{3 \pm \sqrt{29}}{10}$ **ii** $x \approx -0.24$ or $x \approx 0.84$

c **i** $x = \frac{5 \pm \sqrt{17}}{2}$ **ii** $x \approx 0.44$ or $x \approx 4.56$

d **i** $x = \frac{-5 \pm \sqrt{17}}{2}$ **ii** $x \approx -4.56$ or $x \approx -0.44$

e **i** $x = \frac{1 \pm \sqrt{17}}{4}$ **ii** $x \approx -0.78$ or $x \approx 1.28$

f **i** $x = \frac{-7 \pm \sqrt{13}}{6}$ **ii** $x \approx -1.77$ or $x \approx -0.57$

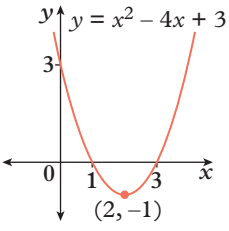
5 **a** **i** $(-5, 0)$ and $(5, 0)$ **ii** $(0, 25)$

b **i** $(-1, 0)$ and $(4, 0)$ **ii** $(0, -4)$

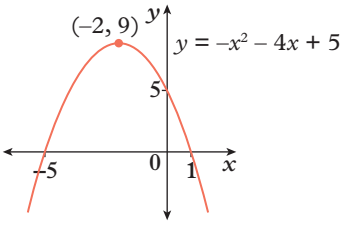
c **i** $(2, 0)$ **ii** $(0, 8)$

d **i** $(-3, 0)$ and $(2, 0)$ **ii** $(0, 6)$

6 **a**



b



7 **a** **i** 5 units right, 2 units up **ii** (5, 2)

b **i** 6 units left, 4 units down **ii** $(-6, -4)$

c **i** 1 unit right, 1 unit down **ii** $(1, -1)$

d **i** 3 units left, 5 units up **ii** $(-3, 5)$

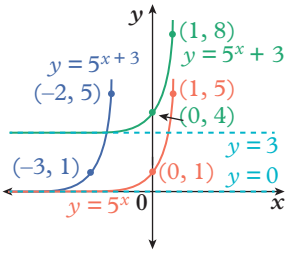
e **i** 4 units right, 2 units up **ii** (4, 2)

f **i** dilaton in the y-direction with a dilation factor of 3, then moved 7 units left, 1 unit up

ii $(-7, 3)$

8 $(x - 4)^2 + (y - 2)^2 = 4$

9 **a**, **c**



b **i** translate 3 units left **ii** translate 3 units up

10 **a** 27 **b** 3

Analysis

1 **a** **i** $a = 1, h = -4, k = -1$ **ii** $(-4, -1)$

iii $(-3, 0)$

iv $(0, -3)$ and $(0, 1)$

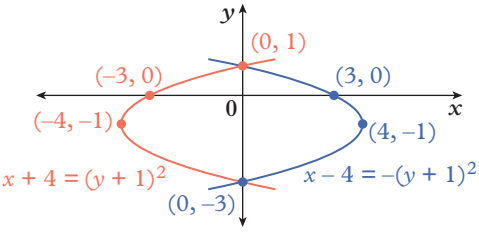
v See graph below.

vi translation of 4 units left and 1 unit down

b **i** $a = -1, h = 4, k = -1$

ii $(4, -1)$ **iii** $(3, 0)$ **iv** $(0, -3)$ and $(0, 1)$

v



vi reflection in the y-axis and translation of 4 units right and 1 unit down

c A positive value for a results in a parabola with its open end towards the right, while a negative value for a results in the open end being towards the left. The graphs from parts **a** and **b** are a reflection of each other in the y-axis.

2 **a** Using the first formula:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-3)}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{13}}{2}$$

≈ -1.3 or 2.3

Using the second formula:

$$x = \frac{2 \times -3}{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-3)}}$$
$$= \frac{-6}{1 \pm \sqrt{13}}$$
≈ -1.3 or 2.3

The two formulas give the same answer.

b Start with the positive value of the square root:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Rationalise the numerator:

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$

Multiply the terms in the numerator, leaving the denominator in factorised form.

$$= \frac{b^2 - (b^2 - 4ac)}{2a(-b - \sqrt{b^2 - 4ac})}$$

Simplify the numerator.

$$= \frac{4ac}{2a(-b - \sqrt{b^2 - 4ac})}$$

Cancel the common factor of $2a$ in the numerator and denominator.

$$= \frac{2c}{-b - \sqrt{b^2 - 4ac}}$$

A similar formula would result by finding the negative value of the square root in the original formula.

So, the two formulas are equivalent.

EX p208

Algorithmic Thinking 1 Number and algebra

Task 1

1 Example 1: 1, 2, 3, 4, 5, 6, 7, 8, 9

Example 2: 5, 10, 15, 30, 30, 60

Example 3: 2, 4

2 Sample answer:

```
For each integer A ≤ 12:
    For each integer B ≤ 12:
        If A × B equals 12:
            Print A, B
```

3 Teacher to check.

4 Teacher to check.

5 Teacher to check.

Task 2

1 $x^2 + 16x + 15 = x^2 + 1x + 15x + (15 \times 1)$
$= x(x + 1) + 15(x + 1)$
$= (x + 1)(x + 15)$

2 $x^2 + 7x + 17$ cannot be factorized with the algorithm as 17 is a prime number with only factors 1 and 17 and they do not add to 7.

3 Steps 1 and 2.

4 The algorithm from Task 1 can be used for helping with factorising the quadratic $x^2 + bx + c$ by identifying all the integer factors of c. They could then be tested using more conditional actions to determine if the factor pairs found add up to b.

```
For each integer A ≤ 12:
    For each integer B ≤ 12
        If A × B equals 12:
            If A + B = 8:
                Print A, B
```

5 Teacher to check.

CHAPTER 5 Geometry

EX p214

5A Geometry review

1 **a** $a = 146°$

b $b = 24°$

c $c = 57°$, $d = 57°$, $e = 123°$

d $d = 75°$

e $e = 109°$, $f = 71°$, $g = 74°$

f $g = 71°$, $h = 19°$, $i = 71°$, $j = 90°$

2 **a** $a = 117°$

b $b = 109°$

c $c = 49°$

d $d = 129°$

e $e = 63°$, $f = 117°$, $g = 117°$, $h = 63°$

f $f = 126°$

3 **a** $a = 97°$

b $b = 88°$, $c = 88°$, $d = 92°$

c $c = 58°$, $d = 302°$

d $d = 52°$, $e = 71°$, $f = 57°$

e $d = 180°$, $e = 30°$, $f = 115°$, $g = 51°$

f $f = 67°$, $g = 64°$, $h = 116°$, $i = 91°$

4 **a** $a = 76°$ **b** $b = 60°$

c $c = 32°$, $d = 116°$ **d** $d = 160°$

e $e = 115°$ **f** $f = 37°$, $g = 75°$, $h = 68°$

5 **a** $a = 55°$

b $b = 117°$, $c = 63°$, $d = 117°$

c $c = 30°$, $d = 118°$

d $d = 45°$

e $e = 119°$, $f = 25°$

f $f = 58°$, $g = 98°$, $h = 98°$, $i = 122°$

6 **a** 3240° **b** 8640° **c** 17 640°

7 The internal angle sum of a polygon = $(n - 2) \times 180°$

A regular polygon with n equal sides has n equal angles.

So the size of each angle = $\frac{(n - 2) \times 180°}{n}$

8 **a** 128.57 **b** 140° **c** 120°

9 **a** 18 **b** 10 **c** 36 **d** 1000

10 **a** 18° **b** 27° **c** 22.5°

d 9.69° **e** 21° **f** 17.63°

11 **a** true

b false; angles at a point add to 360°.

c false; an isosceles triangle has one pair of equal angles.

d false; corresponding angles on parallel lines are equal.

e true

f false; a rectangle is a parallelogram, but not all parallelograms are rectangles.

g false; complementary angles add to 90°.

h false; a triangle that has two 60° angles must be equilateral

i true

12 a $b = 30°$, $c = 44°$, $d = 74°$, $e = 29°$

b $c = 72°$

c $d = 109°$, $e = 8°$

13 a $a = 60°$, $b = 30°$, $c = 90°$ **b** $d = 35°$

c $e = 163°$ **d** $f = 79°$

e $e = 220°$ **f** $f = 154.5°$

14 a 330° **b** 185° **c** 74° **d** 225°

e 351° **f** 54° **g** 87° **h** 70°

i 151° **j** 296° **k** 325° **l** 300°

EX p220

5B Geometric proofs

1 a

Statements	Reasons
$\angle EOF + \angle FOB = 180°$	Angles on a straight line sum to 180°
$\angle BOC + \angle FOB = 180°$	Angles on a straight line sum to 180°
$\angle EOF + \angle FOB = \angle BOC + \angle FOB$	Transitive law
$\angle EOF = \angle BOC$ □	Subtraction property of equality

b

Statements	Reasons
$\angle EOF + \angle AOF + \angle AOB = 180°$	Angles on a straight line sum to 180°
$\angle BOC + \angle AOB + \angle AOF = 180°$	Angles on a straight line sum to 180°
$\angle EOF + \angle AOF + \angle AOB = \angle BOC + \angle AOF + \angle AOB$	Transitive law
$\angle EOF = \angle BOC$ □	Subtraction property of equality

2

Statements	Reasons
$\angle GKJ + \angle JKL = 180°$	Angles on a straight line sum to 180°
$\angle NKL + \angle JKL = 180°$	Angles on a straight line sum to 180°
$\angle GKJ + \angle JKL = \angle NKL + \angle JKL$	Transitive law
$\angle GKJ = \angle NKL$ □	Subtraction property of equality

3

Statements	Reasons
$\angle ABC = \angle EFB$	Corresponding angles on parallel lines are equal
$\angle ABC + \angle CBD = 180°$	Angles on a straight line sum to 180°
$\angle DBF + \angle CBD = 180°$	Angles on a straight line sum to 180°
$\angle ABC + \angle CBD = \angle DBF + \angle CBD$	Transitive law
$\angle ABC = \angle DBF$	Subtraction property of equality
$\angle EFB + \angle BFG = 180°$ $\angle EFB = \angle ABC = \angle DBF$	Angles on a straight line sum to 180°
$\therefore \angle DBF + \angle BFG = 180°$ □	Transitive law

4

Statements	Reasons
$\angle DBA + \angle ABC + \angle CBE = 180°$	Angles on a straight line sum to 180°
$\angle DBA = \angle BAC$	Alternate angles on parallel lines are equal
$\angle CBE = \angle BCA$	Alternate angles on parallel lines are equal
$\therefore \angle BAC + \angle ABC + \angle BCA = 180°$ □	Transitive law

5

Statements	Reasons
$\angle EFH + \angle DFE = 180°$	Angles on a straight line sum to 180°
$\angle DFE + \angle DEF + \angle FDE = 180°$	Interior angle sum of a triangle is 180°
$\angle EFH + \angle DFE = \angle DFE + \angle DEF + \angle FDE$	Transitive law
$\angle EFH = \angle DEF + \angle FDE$ □	Subtraction property of equality

6

Statements	Reasons
X lies on $\overleftrightarrow{AB}$	Given
Y lies on $\overleftrightarrow{CD}$	Given
$\angle AXY = 90°$	Given
$\angle CYZ = 90°$	Given
$\angle CYX + \angle CYZ = 180°$	Angles on a straight line are supplementary
$\angle CYX = 180° - \angle CYZ$ $= 180° - 90°$ $= 90°$	Subtraction law of equality
$\angle AXY + \angle CYX = 180°$	Addition law of equality
$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ □	If co-interior angles are supplementary, they lie on parallel lines

7

Statements	Reasons
$\angle MPQ + \angle QPN = 180°$	Supplementary angles sum to 180°
$\angle MPQ + \angle QPN = \angle MPN$	Angle addition postulate
$\angle MPN = 180°$	Transitive law
MN is a straight line □	Angles on a straight line sum to 180°

8 a Parallel lines will never meet, no matter how far they extend. Lines which are not parallel will meet at some point.

b $\angle EAB$ and $\angle ACB$

c If AC is equal to 0, the points A and C would be the same point. Any value less than zero is meaningless.

d $\angle EAB + \angle BAC = 180°$

But: $\angle EAB = x$

So: $x + \angle BAC = 180°$

That is: $\angle BAC = 180° - x$

e $(180° - x) + x + y = 180°$

f $180° + y = 180°$ simplifies to $y = 0°$

g If $\angle ABC$ is 0°, line BC would lie on top of line BA, so points A and C would lie on top of each other.

h This contradicts the assumption that $AC > 0$. In other words, the lines must be parallel.

i Assume that $\angle EAB$ and $\angle ACB$ are equal corresponding angles, and that DG is not parallel to HF.

Also assume that the length of AC is greater than zero.

In $\triangle ABC$: $\angle ABC + \angle ACB + \angle CAB = 180°$

But: $\angle CAB = 180° - \angle EAB$

So: $\angle ABC + \angle ACB + (180° - \angle EAB) = 180°$

It was assumed that $\angle EAB = \angle ACB$.

This relationship then reduces to $\angle ABC = 0°$.

This implies that $AC = 0$.This contradicts the statement that $AC > 0$. So the lines DG and HF must be parallel.

9 a If two lines cut by a transversal are not parallel, then the corresponding angles are not equal.

b

Statements	Reasons
$AB \nparallel CB$	Given
$\angle EAB$ and $\angle ACB$ are corresponding angles	Given
ABC is a triangle	Three points that do not form a line form a triangle
$\angle EAB = \angle ABC + \angle ACB$	The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles
$\angle ABC \neq 0°$	Angles cannot be equal to 0°
$\angle ABC + \angle ACB \neq 0° + \angle ACB$	Angle addition postulate
$\angle EAB \neq 0° + \angle ACB$	Transitive law
$\angle EAB \neq \angle ACB$ □	Subtraction property of equality

10 If $a = b$, then $(a - b) = 0$. Dividing by $(a - b)$ is actually dividing by 0, which gives an answer which is undefined.

EX p227

5C Congruence and similarity

1 a $a = 51°$, $b = 11$ cm, $c = 70°$, $d = 59°$, $e = 10$ cm

b $a = 48°$, $b = 25°$, $c = 4$ cm, $d = 25°$, $e = 107°$, $f = 7$ cm

c $a = 38°$, $b = 50°$, $c = 13$ cm, $d = 38°$, $e = 92°$, $f = 8$ cm

d $j = 25°$, $k = 7$ cm, $l = 3$ cm, $m = 5$ cm, $n = 115°$

2 a congruent; AAS

b congruent; RHS

c congruent; SAS

d not congruent; SSS does not apply

e not congruent; AAS does not apply

f not congruent; RHS does not apply

3 a SAS or AAS **b** SAS does not apply

c SAS **d** AAS

4 a $\frac{5}{2}$ **b** $\frac{3}{2}$ **c** 2 **d** $\frac{1}{3}$

5 a not similar; does not meet the requirements of the SSS condition for similarity

b not similar; does not meet the requirements of the SAS condition for similarity

c similar; meets the requirements of the AAA condition for similarity

d similar; meets the requirements of the AAA condition for similarity

e similar; meets the requirements of the SSS condition for similarity

f similar; meets the requirements of the RHS condition for similarity

6 a $x = 10.5$ cm, $y = 8$ cm

b $c = 44$ cm, $d = 7$ cm

c $x = 4$ cm, $y = 9$ cm

d $a = 9$ cm, $b = 4$ cm

e $x = 5\frac{7}{13}$ cm, $y = 17\frac{1}{3}$ cm

f $x = 52$ cm, $y = 20$ cm

7 a similar; passes SSS condition

b similar; meets the requirements of the AAA condition for similarity

c not similar; does not meet the requirements of the AAA condition for similarity

d not similar; does not meet the requirements of the RHS condition for similarity

8 a

Statements	Reasons
$VX = YX$	Given
$WX = ZX$	Given
$\angle VXW = \angle YXZ$	Vertically opposite angles are equal
$\therefore \triangle VWX \cong \triangle YZX$ □	SAS (congruent)

b

Statements	Reasons
$QP = QS$	Given
$\angle PQR = \angle SQT$	Vertically opposite angles are equal
$\angle RPQ = \angle TSQ$	Given
$\therefore \triangle PQR \cong \triangle SQT$ □	AAS (congruent)

9 a

Statements	Reasons
$\angle ABC = \angle PRQ$	Given
$\frac{PR}{AB} = \frac{27}{9} = 3$	Scale factor between corresponding sides
$\frac{RQ}{BC} = \frac{18}{6} = 3$	Scale factor between corresponding sides
$\therefore \triangle ABC \sim \triangle PRQ$ □	SAS (similar)

b

Statements	Reasons
$\angle AEB = \angle CED$	Vertically opposite angles are equal
$\angle EAB = \angle ECD$	Alternate angles on parallel lines are equal
$\angle EBA = \angle EDC$	Alternate angles on parallel lines are equal
$\therefore \triangle ABE \sim \triangle CDE$ □	AAA (similar)

10 a The 9 cm side length is opposite the 85° angle in one figure, and adjacent to the 85° angle in the other figure. None of the angles are equal, so the case of an isosceles triangle with two equal sides is ruled out.

b The 88° angle is opposite the 17 cm side in one figure and opposite the 16 cm side in the other figure.

c Three pairs of equal angles is not a condition for congruence. One triangle could be a dilation of the other.

d The congruence condition RHS does not apply because the only side measurement given is the hypotenuse.

11 a False; the triangles may or may not be congruent, depending on their side lengths.

b False; the corresponding angle must be between the two corresponding sides for these triangles to be congruent.

c False; if a pair of triangles meets one condition of congruence, they will meet all conditions of congruence (but, perhaps, further calculations are necessary to show the other conditions of congruence are met).

d True; figures which are congruent to each other all have the same shape and size.

e False; a quadrilateral with all sides of equal length could be a square or a rhombus. A square and a rhombus are only ever congruent if all the interior angles are right angles.

f False; all equilateral triangles are similar, but they are only congruent if their side lengths are the same.

12 a It appears at first glance that the scale factor is $\frac{1}{2}$. However, the sides compared are not corresponding sides.

b The RHS test requires a right angle, the hypotenuse and one of the shorter sides (which is not provided). This means that the pair of triangles fails the RHS test.

13 4.9 m

14 3.6 m

15 Pairs of similar triangles: **A** and **C** (AAA), **D** and **E** (SAS)

16 a False; they must be similar if their corresponding angles are equal

b True; the corresponding angles in these three triangles must be equal

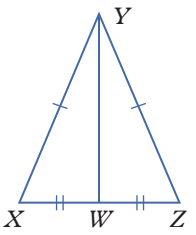
c False; the angles in isosceles triangles are not fixed

d True; all squares have all angles equal to 90° and all four lengths equal to each other, which makes them all the same shape

e False; if they fail SAS they cannot be similar

f True; consider a rhombus and a square with the same side lengths.

17 a



b $\triangle YWX$ and $\triangle YWZ$

c $YX = YZ$ and $WX = WZ$

d YW is common to both triangles

e

Statements	Reasons
$YX = YZ$	Equal sides in an isosceles triangle
$WX = WZ$	By construction
YW is common to both triangles	Common side
$\therefore \triangle WXY \cong \triangle WZY$	SSS (congruent)
$\therefore \angle WXY = \angle WZY$ □	Corresponding angles in congruent triangles

f $\angle WXY = \angle WZY$ as they are corresponding angles in congruent triangles.

18 a

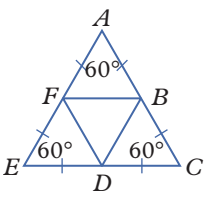
Statements	Reasons
$\angle AEB = \angle CED$	Common angle to both triangles
$AE = 70 + 70 = 140$	Substitution
$BE = 110 + 110 = 220$	Substitution
$\frac{DE}{BE} = \frac{110}{220} = \frac{1}{2}$	Scale factor between corresponding sides
$\frac{CE}{AE} = \frac{70}{140} = \frac{1}{2}$	Scale factor between corresponding sides
$\therefore \triangle ABE \sim \triangle CDE$ □	SAS (similar)

b

Statements	Reasons
$\angle AEB = \angle CED$	Common angle
$AC = CE$	Given
$AE = AC + CE$	Given
$AE = CE + CE = 2AE$	Substitution
$BD = DE$	Given
$BE = BD + DE$	Given
$BE = DE + DE = 2DE$	Substitution
$\frac{DE}{BE} = \frac{DE}{2DE} = \frac{1}{2}$	Scale factor between corresponding sides
$\frac{CE}{AE} = \frac{CE}{2CE} = \frac{1}{2}$	Scale factor between corresponding sides
$\therefore \triangle ABE \sim \triangle CDE$ □	SAS (similar)

19 7.2 m

20



Statements	Reasons
$AC = AE = CE$	Side lengths of an equilateral triangle are equal
$AB = BC = CD = DE = EF = FA$	Points B, D and F lie at the midpoints of equal side lengths
$\angle ACE = \angle CEA = EAC = 60°$	Interior angles of an equilateral triangle are 60°
$\triangle ABF \cong \triangle CDB \cong \triangle EDF$	SAS (congruent)
$FB = BD = DF$	Corresponding sides on congruent triangles
$\therefore \triangle FBD$ is equilateral	A triangle with equal side lengths is equilateral
$\angle FBD = \angle BDF = \angle DFB = 60°$	Interior angles of an equilateral triangle are 60°
$\triangle ABF$ is isosceles	A triangle with two equal side lengths is isosceles
$\angle ABF = \angle AFB$	Angles opposite equal sides of an isosceles triangle are equal
$\angle ABF = \angle AFB = 60°$	Interior angle sum of a triangle is 180°
$\therefore \triangle ABF$ is equilateral	A triangle with equal angles is equilateral
$AB = FA = FB$	Side lengths on equilateral triangles are equal
$AB = BC = CD = DE = EF = FA = FB = BD = DF$	Transitive law
$\therefore \triangle CDB$ is equilateral	A triangle with equal side lengths is equilateral
$\therefore \triangle EDF$ is equilateral	A triangle with equal side lengths is equilateral
$\therefore \triangle ABF \cong \triangle CDB \cong \triangle EDF \cong FBD$ □	SSS (congruent)

5D Proofs and quadrilaterals

EX p235

1 Join EG.

Statements	Reasons
$\angle FGE = \angle HEG$	Alternate angles on parallel lines are equal
$\angle FEG = \angle HGE$	Alternate angles on parallel lines are equal
EG is common to both $\triangle FGE$ and $\triangle HEG$	Common side
$\therefore \triangle FGE \cong \triangle HEG$	AAS (congruence)
$FG = HE$	Corresponding sides in congruent triangles are equal
$EF = GH$	Corresponding sides in congruent triangles are equal
$FG = EF$	Given
$\therefore FG = EF = HE = GH$	Transitive law
$\therefore FEGH$ is a rhombus □	A quadrilateral with four equal side lengths is a rhombus

2

Statements	Reasons
$\angle ACI + \angle CIS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ASI + \angle CIS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ACI + \angle CIS = \angle ASI + \angle CIS$	Transitive law
$\angle ACI = \angle ASI$	Subtraction property of equality
$\angle ICA + \angle CAS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ICA + \angle CIS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ICA + \angle CAS = \angle ICA + \angle CIS$	Transitive law
$\angle CAS = \angle CIS$	Subtraction property of equality
$\therefore$ the opposite angles of a rhombus are equal □	

3

Statements	Reasons
$AB = CD$	Given
$AD = CB$	Given
BD is common to both $\triangle ABD$ and $\triangle CDB$	Common side
$\triangle ABD \cong \triangle CDB$	SSS (congruence)
$\angle ABD = \angle CDB$	Corresponding angles in congruent triangles are equal
$\angle ADB = \angle CBD$	Corresponding angles in congruent triangles are equal
$AB \parallel CD$	If alternate angles are equal, they lie on parallel lines
$AD \parallel CB$	If alternate angles are equal, they lie on parallel lines
$\therefore ABCD$ is a rhombus □	A quadrilateral with all sides equal in length is a rhombus

4 **a** $\angle ETR$ **b** $\angle TER$

c

Statements	Reasons
$\angle MIR = \angle TER$	Alternate angles on parallel lines are equal
$\angle IMR = \angle ETR$	Alternate angles on parallel lines are equal
$IM = ET$	Given
$\therefore \triangle MIR \cong \triangle TER$ □	AAS (congruent)

d It follows that: $IR = ER$ and $MR = TR$.

e IR and ER together form the parallelogram's diagonal IE, while MR and TR form the diagonal TM.
The diagonals bisect each other because we know that $IR = ER$ and $MR = TR$.

5

Statements	Reasons
$OK = FK$	Given
$LK = GK$	Given
$\angle OKL = \angle FKG$	Vertically opposite angles are equal
$\therefore \triangle OKL \cong \triangle FKG$	SAS (congruence)
$\angle LOK = \angle GFK$	Corresponding angles in congruent triangles are equal
$\angle OLK = \angle FGK$	Corresponding angles in congruent triangles are equal
$OL \parallel FG$	If alternate angles are equal, they lie on parallel lines
$\angle OKG = \angle FKL$	Vertically opposite angles are equal
$\therefore \triangle OKG \cong \triangle FKL$	SAS (congruence)
$\angle GOK = \angle LFK$	Corresponding angles in congruent triangles are equal
$\angle FLK = \angle OGK$	Corresponding angles in congruent triangles are equal
$OG \parallel FL$	If alternate angles are equal, they lie on parallel lines
$\therefore GOLF$ is a parallelogram □	A quadrilateral with parallel opposite sides is a parallelogram

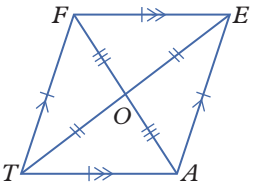
6 a

Statements	Reasons
$\angle EFO = \angle TAO$	Alternate angles on parallel lines are equal
$\angle FEO = \angle ATO$	Alternate angles on parallel lines are equal
$FE = AT$	Given
$\therefore \triangle FEO \cong \triangle ATO$	AAS (congruence)
$FO = AO$	Corresponding sides in congruent triangles are equal
$EO = TO$	Corresponding sides in congruent triangles are equal
$\therefore$ the diagonals bisect each other □	

b

Statements	Reasons
$\triangle FEO \cong \triangle ATO$	Proven in part **a**
$\angle AEO = \angle FTO$	Alternate angles on parallel lines are equal
$\angle EAO = \angle TFO$	Alternate angles on parallel lines are equal
$AE = FT$	Given
$\therefore \triangle AEO \cong \triangle FTO$	AAS (congruence)
$FE = AE$	Given
$FO = AO$	Proven in part **a**
EO is common to both $\triangle HEO$ and $\triangle AEO$	Common
$\triangle FEO \cong \triangle AEO$	SSS (congruence)
$\therefore \triangle HEO \cong \triangle AEO \cong \triangle HTO \cong \triangle ATO$ □	Transitive law

c



d $\angle FOE$ and $\angle AOE$ are corresponding angles in congruent triangles, therefore they are equal.

e

Statements	Reasons
$\angle FOE + \angle AOE = 180°$	Angles on a straight line sum to 180°
$\angle FOE = \angle AOE$	As shown in part **d**
$\angle FOE + \angle FOE = 180°$	Substitution
$\angle FOE = 90°$	Solving equation
$\angle AOE = 90°$	Transitive law
$\therefore$ the diagonals of a rhombus bisect each other at right angles □	

7 a

Statements	Reasons
$AL = ML$	Given
$\angle ALU = 90°$	Given
$\angle ALU + \angle MLU = 180°$	Angles on a straight line sum to 180°
$\angle MLU = 90°$	Substitution and solving equation
UL is common to both $\triangle AUL$ and $\triangle MUL$	Common
$\therefore \triangle AUL \cong \triangle MUL$	SAS (congruence)
$CL = UL$	Given
$\angle ALU = \angle MLC$	Vertically opposite angles are equal
ML is common to both $\triangle MUL$ and $\triangle MCL$	Common
$\therefore \triangle MUL \cong \triangle MCL$	SAS (congruence)
$\angle MLU = \angle ALC$	Vertically opposite angles are equal
$\therefore \triangle MUL \cong \triangle ACL$	SAS (congruence)
$\therefore \triangle AUL \cong \triangle MUL \cong \triangle MCL \cong \triangle ACL$	Transitive law
$AU = MU = MC = AC$	Corresponding sides in congruent triangles are equal
$\therefore AUMC$ is a rhombus □	If a quadrilateral has all sides equal in length it is a rhombus

b

Statements	Reasons
$AUMC$ is a parallelogram	If a quadrilateral has opposite sides that are parallel (given), it is a parallelogram
$AU = MC$	Opposite sides of a parallelogram are equal in length

Statements	Reasons
$\angle UAL = \angle CML$	Alternate angles on parallel lines are equal
$\angle AUL = \angle MCL$	Alternate angles on parallel lines are equal
$\therefore \triangle ALU \cong \triangle MLC$	AAS (congruence)
$AL = ML$	Corresponding sides in congruent triangles are equal
UL is common to both $\triangle ALU$ and $\triangle MLU$.	Common side
$\angle ULA = \angle ULM$	Angles on a straight line sum to 180°
$\therefore \triangle ALU \cong \triangle MLU$	SAS (congruence)
$AU = MU$	Corresponding sides in congruent triangles are equal
$AC = MU$	Opposite sides of a parallelogram are equal in length
$AU = MU = MC = AC$	Transitive law
$\therefore AUMC$ is a rhombus □	If a quadrilateral has all sides equal in length it is a rhombus

8

Statements	Reasons
$RE \parallel YM$	Opposite sides of a parallelogram are parallel
$\angle REM + \angle EMY = 180°$	Co-interior angles on parallel lines are supplementary
$\angle REM = 90°$	Given
$\angle EMY = 90°$	Substitution and solving equation
$RY \parallel EM$	Opposite sides of a parallelogram are parallel
$\angle MYR + \angle EMY = 180°$	Co-interior angles on parallel lines are supplementary
$\angle MYR = 90°$	Substitution and solving equation
$\angle MYR + \angle YRE = 180°$	Co-interior angles on parallel lines are supplementary
$\angle YRE = 90°$	Substitution and solving equation
$\therefore REMY$ is a rectangle □	A quadrilateral with four right angles is a rectangle

9 a

Statements	Reasons
$EV = IL$	Given
EI is common to both $\triangle EVI$ and $\triangle ILE$	Common side
$\angle VEI = \angle LIE$	Alternate angles on parallel lines are equal
$\therefore \triangle EVI \cong \triangle ILE$	SAS (congruence)
$\angle EVI = 90°$	Given
$\therefore \angle ILE = 90°$	Corresponding angles in congruent triangle are equal
$VE \parallel IL$	Given

Statements	Reasons
$\angle EVI + \angle VIL = 180°$	Co-interior angles on parallel lines are supplementary
$\angle VIL = 90°$	Substitution and solving equation
$\angle ILE + \angle LEV = 180°$	Co-interior angles on parallel lines are supplementary
$\angle LEV = 90°$ □	Substitution and solving equation

b

Statements	Reasons
$EL = IV$	Corresponding sides in congruent triangles are equal
$\angle LEV = \angle EVI = 90°$	Proved in part **a**
$\angle LEV + \angle EVI = 180°$	Substitution
$\therefore EL \parallel IV$ □	If co-interior angles are supplementary, they are on parallel lines

c

Statements	Reasons
$EI^2 = EL^2 + IL^2$	Pythagoras' theorem
$LV^2 = EL^2 + EV^2$	Pythagoras' theorem
$EV = IL$	Given
$LV^2 = EL^2 + IL^2$	Substitution
$LV^2 = EI^2$	Transitive law
$\therefore LV = EI$ □	Transitive law

d

Statements	Reasons
$EV = IL$	Given
$LV = EI$	Proved in part **c**
EL is common to both $\triangle LEV$ and $\triangle ELI$	Common side
$\therefore \triangle LEV \cong \triangle ELI$ □	SSS (congruence)

10 a

Statements	Reasons
$PE = PY$	Given
$PL = PK$	Given
$\angle EPL = \angle YPK$	Vertically opposite angles are equal
$\therefore \triangle ELP \cong \triangle YKP$	SAS (congruence)
$\angle PEL = \angle PYK$	Corresponding angles in congruent triangles are equal
$EL \parallel YK$	If alternate angles are equal, they lie on parallel lines
$\angle EPK = \angle YPL$	Vertically opposite angles are equal
$\therefore \triangle EPK \cong \triangle YPL$	SAS (congruence)
$\angle PEK = \angle PYL$	Corresponding angles in congruent triangles are equal
$EK \parallel YL$	If alternate angles are equal, they lie on parallel lines
$\therefore ELYK$ is a parallelogram □	If a quadrilateral has two pairs of parallel sides, it is a parallelogram

b

Statements	Reasons
$\angle LYE + \angle YEL + \angle ELK + \angle KLY = 180°$	Interior angle sum of a triangle is 180°
$\angle YEL = \angle ELK$	Angles opposite equal sides in an isosceles triangle are equal
$\angle LYE = \angle KLY$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle KLY + \angle ELK + \angle ELK + \angle KLY = 180°$	Substitution
$2\angle ELK + 2\angle KLY = 180°$	Solving equation
$\angle ELK + \angle KLY = 90°$	Solving equation
$\angle ELY = 90°$ □	Transitive law

c Since *ELYK* is a parallelogram,
$\angle ELY + \angle LYK = 180°$ (co-interior angles)
Since $\angle ELY = 90°$, $\angle LYK = 90°$.
So $\angle YKE = 90°$ and $\angle KEL = 90°$, since opposite angles in a parallelogram are equal. Since *ELYK* is a parallelogram, and all its interior angles are 90°, *ELYK* is a rectangle.

11 a

Statements	Reasons
$UA = RS$	Sides of a rhombus are equal
$\angle AUG = \angle SRG$	Alternate angles on parallel lines are equal
$\angle UAG = \angle RSG$	Alternate angles on parallel lines are equal
$\therefore \triangle GUA \cong \triangle GRS$	AAS (congruence)
$UG = RG$	Corresponding sides in congruent triangles are equal
$AG = SG$ □	Corresponding sides in congruent triangles are equal

b

Statements	Reasons
$UA = RA$	Given
$UG = RG$	Proved in part **a**
GA is common to both $\triangle UAG$ and $\triangle RAG$	Common side
$\therefore \triangle UAG \cong \triangle RAG$ □	SSS (congruence)

c

Statements	Reasons
$\angle UAG = \angle RAG$	Corresponding angles in congruent triangles are equal
$\angle UAS = \angle RAS$ □	Transitive law

d The two equal angles are the angles at vertex *A*. The angle at *A* has been bisected by the diagonal *SA* to produce them. The same argument from part **b** can be applied to the other interior triangles to show, as was shown in question **8b**, that all four interior triangles of a rhombus are congruent to each other. The same argument from part **c** can then be applied at each vertex of the rhombus to show that the diagonals of a rhombus bisect the interior angles of a rhombus.

12

Statements	Reasons
$EP = IP$	Given
$EV = IV$	Given
PV is common to both $\triangle EVP$ and $\triangle IVP$	Common side
$\therefore \triangle EVP \cong \triangle IVP$	SSS (congruence)
$\angle VPE = \angle VPI$	Corresponding angles in congruent triangles are equal
$\angle VPE + \angle VPI = 180°$	Angles on a straight line sum to 180°
$\therefore \angle VPE = \angle VPI = 90°$	Solving equation
$\angle EVP = \angle VEP$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle EVP = \angle VEP = 45°$	Angles in a triangle sum to 180° and solving equation
$\therefore \angle EVL = 45°$ □	Transitive law

13 a Consider a square and a rhombus which have equal side lengths. The interior angles of the square are all 90°, but those in the rhombus are not. This means the two figures do not have the same shape, so they are not congruent.

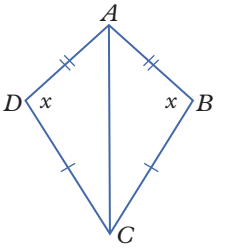
b A rhombus has four sides of equal length. If the SAS condition for congruence held for these shapes, the adjacent sides of the parallelogram would be equal in length. This would make it a rhombus.

c The AAS condition for congruence considers two angles (in this case, right angles) and one side. This could hold true for the square and rectangle for two angles and the included side, but the other sides do not need to be equal lengths unless the rectangle is also square.

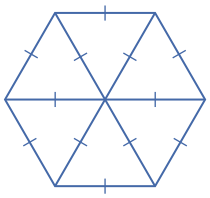
14 a A rectangle and a parallelogram can have all corresponding sides in the same ratio without being similar. They will not be the same shape unless all the interior angles of the parallelogram are right angles, which would make the shape a rectangle.

b A parallelogram and a rhombus might have their corresponding angles all the same size, but they only have the same shape if the sides of the parallelogram are all equal. This would make the parallelogram a rhombus.

15 a A kite has two pairs of adjacent sides equal, and one side in common to both triangles. The SSS condition for congruence applies.



b The six equilateral triangles formed all have the same side length. The SSS condition for congruence applies.



16

Statements	Reasons
$\angle HAB = \angle DCB = \angle DEF = \angle HGF = 90°$	Interior angles of a rectangle are 90°
$AB = CB = EF = GF$	Opposite sides of a rectangle are equal and B and F lie at the midpoints of these sides
$AH = CD = ED = GH$	Opposite sides of a rectangle are equal and H and D lie at the midpoints of these sides
$\therefore \triangle ABH \cong \triangle CBD \cong \triangle EFD \cong \triangle FGH$	SAS (congruence)
$BH = BD = FD = FH$	Corresponding sides in congruent triangles are equal
$\therefore HBDF$ is a rhombus	A quadrilateral with four equal sides is a rhombus
$\angle ABH = \angle CBD = \angle EFD = \angle GFH$	Corresponding angles in congruent triangles are equal
$\angle ABH + \angle HBD + \angle CBD = 180°$	Angles on a straight line sum to 180°
$\angle GFH + \angle HFD + \angle DFE = 180°$	Angles on a straight line sum to 180°
$\therefore \angle HBD = \angle HFD$	Substitution and transitive law
$\angle BHA = \angle BDC = \angle FDE = \angle FHG$	Corresponding angles in congruent triangles are equal
$\angle AHB + \angle BHF + \angle GHF = 180°$	Angles on a straight line sum to 180°
$\angle CDB + \angle BDF + \angle EDF = 180°$	Angles on a straight line sum to 180°
$\therefore \angle BHF = \angle BDF$ □	Substitution and transitive law

Checkpoint

1 **a** $w = 49°, x = 104°$ **b** $y = 144°$

2 **a** $m = 68°$ **b** $j = 65°, k = 65°$

3 **a** $p = 135°$ **b** $q = 72°$

4

Statements	Reasons
AB is a line segment	Given
CD is a line segment	Given
$\angle AEC + \angle CEB = 180°$	Angles on a straight line sum to 180°
$\angle CEB + \angle DEB = 180°$	Angles on a straight line sum to 180°
$\angle AEC + \angle CEB = \angle CEB + \angle DEB$	Transitive law
$\angle AEC = \angle DEB$ □	Subtraction law of equality

5

Statements	Reasons
$\angle ABE = \angle DCF$	Given
$\angle ABE = \angle CBE$	Vertically opposite angles are equal
$\angle DCF = \angle CBE$	Transitive law
$AB \parallel CD$ □	If two lines are cut by a transversal such that corresponding angles are equal, then the lines are parallel

6 **a** $a = 8, b = 7$ cm

b $x = 105°, y = 20°, z = 9$ cm

7 **a** $g = 27°, h = 1.3$ cm

b $p = 1.4$ cm, $q = 6.25$ cm

8 **a**

Statements	Reasons
$AD = DE$	Given
$AC = CE$	Given
CD is common to both $\triangle ADC$ and $\triangle EDC$	Common side
$\triangle ADC \cong \triangle EDC$	SSS (congruence)
$\angle ACD = \angle ECD$ □	Corresponding angles in congruent triangles are equal

b

Statements	Reasons
$AB \parallel EF$	Given
$\angle DAB = \angle DEF$	Corresponding angles on parallel lines are equal
$\angle DBA = \angle DFE$	Corresponding angles on parallel lines are equal
$\angle ADB = \angle EDF$	Common angle
$\therefore \triangle ADB \sim \triangle EDF$ □	AAA (similarity)

9

Statements	Reasons
$\angle ADC = 90°$	Given $AD \perp CD$
$\angle BCD = 90°$	Given $BC \perp CD$
$\angle DAB + \angle ADC = 180°$	Co-interior angles on parallel lines are supplementary
$\angle DAB = 90°$	Substituting and solving equation
$\angle ABC + \angle BCD = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ABC = 90°$	Substituting and solving equation
$\therefore ABCD$ is a rectangle	A quadrilateral with all interior angles of 90° is a rectangle
$AC = BD$ □	Diagonals of a rectangle are equal

10

Statements	Reasons
$\angle SPQ = \angle RST$	Given
$PQ \parallel SR$	If two lines are cut by a transversal such that corresponding angles are equal, then the lines are parallel
$PS \parallel QR, \angle RST = \angle SRQ$	If alternate angles are congruent, they lie on parallel lines
$PQRS$ is a parallelogram □	If a quadrilateral has two pairs of opposite sides that are parallel, it is a parallelogram

EX 5E Circle geometry: angles

p243

1 **a** 45° **b** 120° **c** 74°

2 **a** 90° **b** 18° **c** 45°

3 **a** $x = 32°$ **b** $x = 27°$

c $a = 9°$, $c = 6°$

4 In the following answers, CT stands for circle theorem.

a $x = 34.5°$, CT2 and angle sum of a triangle

b $x = 109°$, CT1

c $x = 21°$, CT3

d $a = 14°$, $b = 7°$, CT3

e $x = 22.5°$, CT2 and triangle angle sum

f $a = 26\frac{1}{3}°$, $e = 9\frac{7}{8}°$, CT3

5 **a** 12.2° **b** 76° **c** 32°

d 68° **e** 55° **f** 29°

6 **a** Each one is a radius of the circle.

b $\triangle AOB$ is isosceles.

c $\angle AOD = \angle ABO + \angle OAB = 2\angle ABO$

d $\triangle AOB$ is isosceles, so $\angle OBC = \angle OCB$.

$\angle DOC = \angle OBC + \angle OCB = 2\angle OBC$

e $\angle AOC = \angle AOD + \angle DOC = 2\angle ABO + 2\angle OBC = 2(\angle ABO + \angle OBC) = 2\angle ABC$

7 **a** 180°

b Since the angle at the centre is twice the angle at the circumference subtended by the same arc, the angle at the circumference is equal to $\frac{1}{2}$ of 180°, or 90°.

8 **a** $\angle WOZ$ is an angle at the centre of the circle, while $\angle WXZ$ is an angle at the circumference. Both are subtended by the same arc, and so circle theorem 1 applies so that $\angle WOZ = 2\angle WXZ$.

b Similarly, $\angle WOZ$ and $\angle WYZ$ are both subtended by the same arc, so circle theorem 1 applies and $\angle WOZ = 2\angle WYZ$.

c It follows that $\angle WYZ = \angle WXZ$.

9 **a** Obtuse $\angle AOC$ is an angle at the centre of the circle, subtended by the same arc as angle x at the circumference.

Similarly, the reflex angle $\angle AOC$ is an angle at the centre of the circle, subtended by the same arc as angle y at the circumference.

In both cases, circle theorem 1 applies, so that $\angle AOC = 2x$ and $\angle AOC = 2y$.

b obtuse $\angle AOC$ + reflex $\angle AOC = 360°$

That is: $2x$ + reflex $\angle AOC = 360°$

or: reflex $\angle AOC = 360° - 2x$

c Since: reflex $\angle AOC = 2y$

$2y = 360° - 2x$

Simplifying: $y = 180° - x$

d obtuse $\angle BOD = 2\angle BCD$

reflex $\angle BOD = 2\angle BAD$

obtuse $\angle BOD$ + reflex $\angle BOD = 360°$ (angles at a point)

So: $2\angle BAD + 2\angle BCD = 360°$

Rearranging and simplifying: $\angle BAD = 180° - \angle BCD$

e **i** $x = 142°$, $y = 99°$

ii $a = 45°$, $y = 38°$

iii $x = 30°$, $y = 22.5°$

10 **a** $\angle ADC + \angle ABC = 180°$ (opposite angles of a cyclic quadrilateral)

So: $\angle ADC = 180° - \angle ABC$

b supplementary angles on a straight line

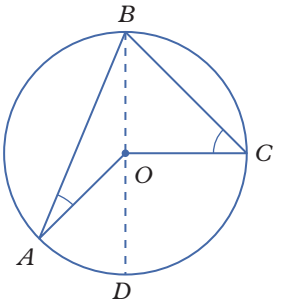
c both are equal to the same expression $(180° - \angle ABC)$

d **i** $a = 61°$

ii $a = 81°$, $b = 92°$, $c = 99°$, $d = 88°$

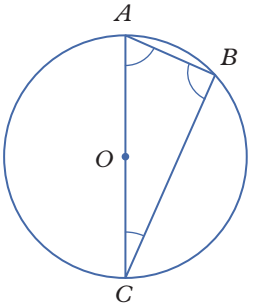
iii $x = 15°$, $z = 29.75°$

11 Circle theorem 1



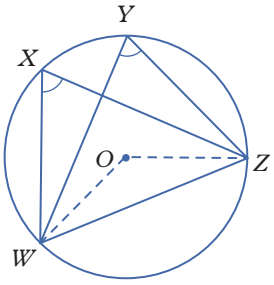
Statements	Reasons
$OA = OB = OC$	Radii of the circle
$\triangle AOB$ and $\triangle COB$ are isosceles	If two sides in a triangle are equal, the triangle is isosceles
$\angle ABO = \angle OAB$ $\angle CBO = \angle OCB$	Angles opposite equal sides in an isosceles triangle are equal
$\angle AOD = \angle ABO + \angle OAB$ $\angle DOC = \angle CBO + \angle OCB$	The exterior angle of a triangle is equal to the sum of the two opposite interior angles
$\angle AOD = 2\angle ABO$ $\angle DOC = 2\angle CBO$	Transitive law, substitution and simplifying
$\angle AOC = \angle AOD + \angle DOC$	Angle addition postulate
$\angle AOC = 2\angle ABO + 2\angle CBO$ $= 2(\angle ABO + \angle CBO)$	Transitive law and factorising
$\angle ABC = \angle ABO + \angle CBO$	Angle addition postulate
$\angle AOC = 2\angle ABC$ □	Transitive law

Circle theorem 2



Statements	Reasons
$\angle AOC = 180°$	A straight angle equals 180°
$\angle AOC = 2\angle ABC$	Circle theorem 1
$2\angle ABC = 180°$	Transitive law
$\angle ABC = 90°$ □	Solving equation

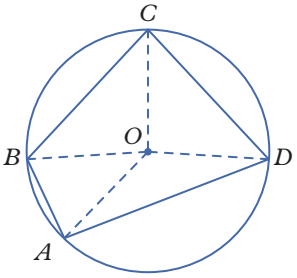
Circle theorem 3



Statements	Reasons
$\angle WOZ = 2\angle WXZ$	Circle theorem 1
$\angle WOZ = 2\angle WYZ$	Circle theorem 1
$2\angle WXZ = 2\angle WYZ$	Transitive law
$\angle WXZ = \angle WYZ$ □	Simplify equation

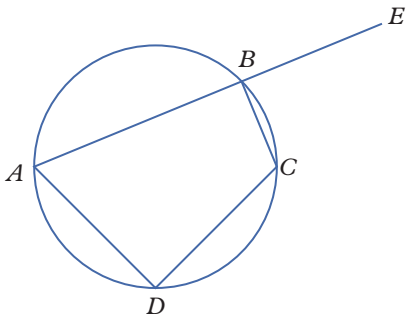
Circle theorem 4

The labels $\angle_O$ and $\angle_R$ are used to label obtuse and reflex angles, respectively.



Statements	Reasons
$\angle_R COA = 2\angle CBA$	Circle theorem 1
$\angle_O COA = 2\angle CDA$	Circle theorem 1
$\angle_R COA + \angle_O COA = 360°$	Angles at a point add to 360°
$2\angle CBA + 2\angle CDA = 360°$	Transitive law
$\angle CBA + \angle CDA = 180°$	Simplify equation
$\angle_R BOD = 2\angle BAD$	Circle theorem 1
$\angle_O BOD = 2\angle BCD$	Circle theorem 1
$\angle_R BOD + \angle_O BOD = 360°$	Angles at a point add to 360°
$2\angle BAD + 2\angle BCD = 360°$	Transitive law
$\angle BAD + \angle BCD = 180°$ □	Simplify equation

Circle theorem 5



Statements	Reasons
$\angle ABC + \angle ADC = 180°$	Circle theorem 4
$\angle ABC + \angle CBE = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle ADC = \angle ABC + \angle CBE$	Transitive law
$\angle ADC = \angle CBE$ □	Subtraction property of equality

12 a $\angle DBC = 66°$

b $\angle DEG = 19°$

c $\angle ODC = 38°$

d $\angle OAB = 62°$

EX p248

5F Circle geometry: chords

1 a 101° **b** 38° **c** 58°

2 a 13 cm **b** 7.5 cm **c** 3.7 cm

3 a $x = 11$ cm

b $x = 4$ cm

c $x = 6$ cm, $y = 3.5$ cm

4 a 2.8 cm **b** 3.5 cm **c** 7.6 cm

5 a $x = 2$ cm, $y = 13$ cm

b $x = 18$ cm

c $x = 2$ cm

d $x = 4$ cm, $y = 21$ cm

e $x = 17°$

f $x = 13.07$ cm

g $x = 4$ cm

h $x = 2.75$ cm, $y = 4$ cm

i $x = 19°$

6 a $x = 4.33$ cm, $y = 8$ cm

b $x = 2.67$ cm, $y = 4$ cm

c $x = 10°$

d $x = 3$ cm, $y = 5$ cm

e $x = 35.5°$

f $x = 2$ cm, $y = 4$ cm

7 a $x = 19°$

b $x = 12°$

c $x = 23.8°$, $y = 15.25°$

d $x = 5.66$ cm, $y = 4$ cm, $z = 1$ cm

e $x = 18°$, $y = 18°$

f x can take on any positive value, $y = 0.6$

8 **a** They are all equal in length.

b isosceles triangles

c $OA = OC$ (radii)

$OB = OD$ (radii)

$AB = CD$ (given)

So: $\triangle OAB \cong \triangle OCD$ (using SSS)

d It follows that $\angle COD = \angle AOB$, because they are corresponding angles in congruent triangles.

9 **a** They are radii of the circle.

b Since $\angle OCA$ is given as 90°, $\angle OCB$ is also equal to 90°, because they are supplementary angles on the chord AB.

c $OA = OB$ (radii)

$\angle OCA = \angle OCB = 90°$ (supplementary angles)

OC is a side common to both triangles.

So: $\triangle ACO \cong \triangle BCO$ (using RHS)

d AC and BC are corresponding sides in congruent triangles.

10 **a** They are all radii of the circle.

b Since OG is perpendicular to AB, it bisects the chord.

So: $AG = GB$.

Similarly, since OH is perpendicular to chord CD:

$CH = HD$

c If chords AB and CD are same length, and are both bisected by radii that are perpendicular to them, it follows that $AG = GB = CH = HD$.

d $OD = OB$ (radii)

$HD = GB$ (from **c**)

$\angle OHD = \angle OGB = 90°$

So: $\triangle OHD \cong \triangle OGB$ (using RHS)

e $OH = OG$ because they are corresponding sides of congruent triangles.

f In a right-angled triangle, the longest side is the hypotenuse (in this case, the radius). The other two sides are shorter in length than that. Comparing any other line segment from the centre to the chord to the line segment which is perpendicular to the chord will always result in a right-angled triangle with the perpendicular line shorter than the other line.

11 **a** They are vertically opposite angles.

b They are angles at the circumference subtended by the same arc, DB.

c They are angles at the circumference subtended by the same arc, CA.

d The two triangles have three equal corresponding angles, AAA.

e In similar triangles, the ratio of the lengths of corresponding sides is constant:

$$\frac{AP}{CP} = \frac{BP}{DP}$$

Rearranging these ratios: $\frac{AP}{BP} = \frac{CP}{DP}$

or: $AP \times DP = CP \times BP$

12 Since the quadrilateral is cyclic, opposite angles are supplementary.

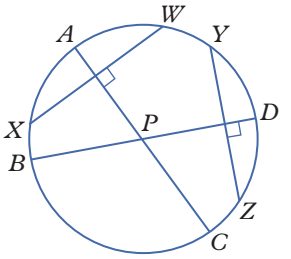
Since the quadrilateral is a parallelogram, opposite angles are equal.

If two angles are equal and supplementary, they must each be 90°.

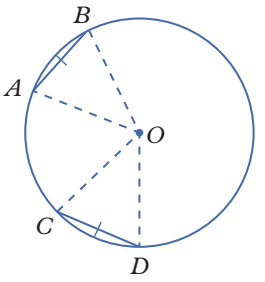
With all angles equal to 90°, the quadrilateral must be a rectangle.

13 Draw chords WX and YZ that are not parallel to each other. Let AC and BD be the perpendicular bisectors of WX and YZ, respectively. Since WX and YZ are not parallel to each other, AC and BD (or their extensions outside the circle) must intersect exactly once at a point P. Let O be the centre of the circle. We show $O = P$.

Construct the radius which intersects perpendicularly with YZ. Circle theorem 7 implies that the radius bisects YZ and so O must lie on BD. Similarly, construct the radius which intersects perpendicularly with WX. Circle theorem 7 implies that the radius bisects WX and so O must lie on AC. Since O lies on both AC and BD, it must be located at their unique intersection point, P. Hence, $O = P$. □

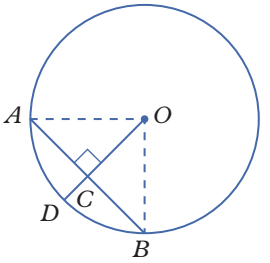


14 Circle theorem 6



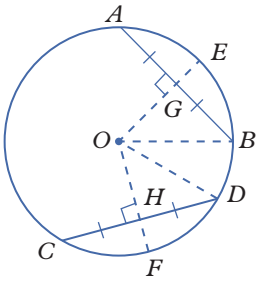
Statements	Reasons
$AB = CD$	Given
$OB = OD = OA = OC$	Radii of the circle
$\triangle AOB \cong \triangle COD$	SSS (congruence)
$\angle BOA = \angle COD$ □	Corresponding angles in congruent triangles are equal.

Circle theorem 7



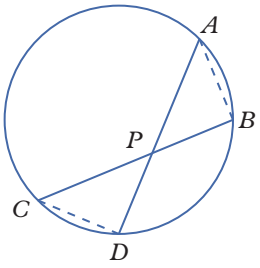
Statements	Reasons
$OA = OB$	Radii of the circle
$\angle OCA = 90°$	Given
$\angle OCB + \angle OCA = 180°$	Supplementary angles add to 180°
$\angle OCB = 90°$	Substitution and solving equation
OC is common to $\triangle ACO$ and $\triangle CBO$	Common side
$\triangle ACO \cong \triangle CBO$	RHS (congruence)
$AC = CB$ □	Corresponding sides in congruent triangles are equal.

Circle theorem 8



Statements	Reasons
$OE = OF = OB = OD$	Radii of the circle
$\angle OGB = \angle OHD = 90°$	Shortest distance
$AG = BG$ and $CH = HD$	Circle theorem 7
$AB = AG + BG = 2AG$ $CD = CH + HD = 2CH$	Given and substitution
$AB = CD$	Given
$2AG = 2CH$	Substitution
$BG = AG = CH = DH$	Simplification and transitivity
$\triangle AGO \cong \triangle DHO$	RHS (congruence)
$OG = OH$ □	Corresponding sides in congruent triangles are equal

Circle theorem 9



Statements	Reasons
$\angle APB = \angle CPD$	Vertically opposite angles are equal
$\angle BCD = \angle BAD$	Circle theorem 3
$\angle ABC = \angle ADC$	Circle theorem 3
$\triangle APB \sim \triangle UPV$	AAA (similar)
$\frac{AP}{PD} = \frac{BP}{PC}$	Scale factor between corresponding sides
$AP \times PC = BP \times PD$ □	Rearranging the equation

15 **a** 90° **b** 45° **c** 69°

16 **a** $x = 6$ cm

b $x = 2.75$ cm

c $x = 11.6$ cm, $y = 3$ cm

17 **a** 7.2 cm **b** 12.4 cm **c** 2 cm

18 **a** 14.1 cm **b** 2.7 cm **c** 4.5 cm

CHAPTER 5 review

Multiple choice

1 C **2** B **3** E **4** A

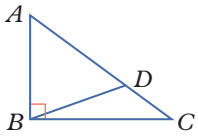
5 D **6** B **7** A **8** C

Short answer

1 **a** $a = 127°$, $b = 53°$

b $x = 105°$

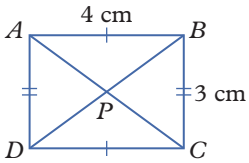
2 **a**



b

Statements	Reasons
$\angle ADB + \angle BDC = 180°$	Angles on a straight line sum to 180°
$\angle DBC + \angle BCD + \angle BDC = 180°$	Interior angle sum of a triangle is 180°
$\angle ADB + \angle BDC = \angle DBC + \angle BCD + \angle BDC$	Transitive law
$\angle ADB = \angle DBC + \angle BCD$ □	Subtraction property of equality

3 **a**



b They are alternate angles between parallel lines AB and CD.

c AAS **d** 53°

4 **a** AAA (alternate and vertically opposite angles)

b $\frac{2}{5}$

c 7.5 cm

5 **a**

Statements	Reasons
$\angle AEC = \angle ADE + \angle DAE$	Exterior angle of a triangle is equal to the sum of the non-adjacent interior angles
$\angle ADE = \angle DAE$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle AEC = \angle ADE + \angle ADE$ $= 2\angle ADE$ □	Substitution and rearrangement

b

Statements	Reasons
$\angle ABC = \angle BCE = 90°$	Given
$\angle ABC + \angle BCE = 180°$	Substitution
$\therefore \angle AEC + \angle BAE = 180°$	Interior angles of a quadrilateral sum to 360°
$\angle AEC = 2\angle ADE$	Given
$2\angle ADE + \angle BAE = 180°$ □	Substitution

6

Statements	Reasons
$PQ = RS$	Given
$\angle PQT = \angle RST$	Alternate angles on parallel lines are equal
$\angle PTQ = \angle RTS$	Vertically opposite angles are equal
$\therefore \triangle PQT \cong \triangle RST$	AAS (congruence)
$TQ = TS$	Corresponding sides in congruent triangles are equal
$TQ^2 + PT^2 = TS^2 + PT^2$	Transitive law
$PQ = PS$	Pythagoras' theorem
$QR = PS$ and $SR = PQ$	Given
$QR = SR = SP = QP$	Transitive law
$\therefore PQRS$ is a rhombus □	A quadrilateral with four equal sides is a rhombus

7 **a** 54° **b** 27° **c** 63°

8

Statements	Reasons
$\angle RPQ = \angle RPO + \angle QPO$	Angle addition postulate
$\angle ROQ = 2\angle RPQ$	Circle theorem 1
$\therefore \angle ROQ = 2\angle RPO + 2\angle QPO$	Substitution
$\angle ROQ = 360° - \angle POR - \angle POQ$	Angles at a point sum to 360°
$\therefore 2\angle RPO + 2\angle QPO = 360° - \angle POR - \angle POQ$	Transitive law
$2\angle RPO + 2\angle QPO + \angle POR + \angle POQ = 360°$ □	Rearranging

9 **a** 46°

b

Statements	Reasons
$\angle DAC = \angle DBC$	Circle theorem 6
$\angle ACD = \angle ABD$	Circle theorem 6
$\angle DAC = \angle ACD = \angle ABD$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle DBC = \angle ABD$ □	Transitive law

Analysis

1 **a**

Statements	Reasons
$\angle DGC = 90°$	Interior angles of quadrilateral $FDGC$ add to 360°
$\angle DFC = \angle DGC = 90°$	Transitive law
$FC \parallel DG$	If co-interior angles are supplementary, the lines must be parallel
$\angle FCD = \angle CDG$	Alternate angles on parallel lines are equal

Statements	Reasons
CD is common to both $\triangle CFD$ and $\triangle CGD$	Common side
$\therefore \triangle CFD \cong \triangle CGD$ □	AAS (congruence)

b

Statements	Reasons
$\angle CAB = \angle CFD = 90°$	Given
$\angle CBA = \angle CDF$	The angle sum of triangles is 180°
$\angle ACB = \angle FCD$	Common to both triangles
$\therefore \triangle CAB \sim \triangle CFD$ □	AAA (similar)

c

Statements	Reasons
$\angle CAB = \angle DEB = 90°$	Given
$\angle CBA = \angle DBE$	Vertically opposite angles are equal
$\angle ACB = \angle BDE$	The angle sum of triangles is 180°
$\therefore \triangle CAB \sim \triangle DEB$ □	AAA (similar)

d

Statements	Reasons
$AC = DE$	Given
$\angle ACB = \angle BDE$	Given
$\angle CBA = \angle DBE$	Given
$\therefore \triangle CAB \cong \triangle DEB$ □	AAS (congruence)
$CB = BD$	Corresponding sides in congruent triangles are equal
$\therefore B$ is the midpoint of CD □	Definition of midpoint (divides a line segment into two congruent pieces)

e

Statements	Reasons
$\frac{FC}{AC} = 3$	Given
$FC = FA + AC$	They make up the same line
$FC = 3AC$	Rearranging equation
$FA + AC = 3AC$	Transitive law
$FA = 2AC$	Rearranging the equation
$\angle AFD = \angle FDE = \angle DEA = \angle EAF = 90°$	Given
$\therefore AFDE$ is a rectangle	A quadrilateral with four right interior angles is a rectangle
$FA = DE$	The opposite sides of a rectangle are equal
$DE = 2AC$	Substitution
$\frac{DE}{AC} = 2$ □	Rearrangement

f

Statements	Reasons
$\angle DBE = 45°$	Given
$\angle DBE + \angle BDE + \angle DEB = 180°$	The interior angle sum of a triangle is 180°
$\angle BDE = 45°$	Substitution and solving equation
$\angle CDG = \angle BDE$	Given

Statements	Reasons
$\angle FCD = \angle CDG$	Alternate angles on parallel lines are equal
$\angle DCG = 45°$	The interior angle sum of $\triangle CGD$ is 180°
$\triangle CGD$ is isosceles with $CG = DG$	If two angles in a triangle are equal, then the sides opposite the equal angles will be equal
$DG = CG$	Equal sides of an isosceles triangle
$CG = FD$	Opposite sides of rectangle $FDGC$
$CG = FD$	Opposite sides of rectangle $FDGC$
$FC = FD = DG = GC$	Sides of a triangle opposite equal angles are equal
$FDGC$ is a rectangle □	A rectangle with equal sides must be a square
$\angle CFD = \angle FDG = \angle CGD = \angle GCF = 90°$	Given
$\therefore FDGC$ is a square □	A quadrilateral with four equal side lengths and four right angles is a square

2 a The opposite angles are supplementary: $85° + 95° = 180°$ and $116° + 64° = 180°$.

b

Statements	Reasons
$ABCD$ is a rectangle	Given
$\angle ABC = \angle BCD = \angle CDA = \angle DAB = 90°$	Interior angles of a rectangle are 90°
$\angle ABC$ and $\angle CDA$ are supplementary	Supplementary angles sum to 180°
$\angle BCD$ and $\angle DAB$ are supplementary	Supplementary angles sum to 180°
$\therefore ABCD$ is a cyclic quadrilateral □	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

c

Statements	Reasons
$AB \parallel DC$	Given
$\angle CDA = \angle BCD$	Given
$\angle ABC + \angle BCD = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ABC + \angle CDA = 180°$	Substitution
$\angle DAB + \angle CDA = 180°$	Co-interior angles on parallel lines are supplementary
$\angle DAB + \angle BCD = 180°$	Substitution
$\therefore ABCD$ is a cyclic quadrilateral □	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

d Two of the edges of these quadrilaterals are radii of the circle, not chords.

e Each pair of equal pronumerals are angles in the same segment subtended by the same chord.

f $p = 41°$, $q = 41$, $r = 62°$, $t = 62°$, $w = 23°$, $y = 54°$, $z = 23°$

g $df = eg$

h

Statements	Reasons
$\angle BAD = \angle BCD = 180°$	Given
$\angle ABC + \angle ADC + \angle BAD + \angle BCD = 360°$	Angle sum of a quadrilateral = 360°
$\angle ABC + \angle ADC + 180° = 360°$	Substitution
$\angle ABC + \angle ADC = 180°$	Simplifying
$ABCD$ is a cyclic quadrilateral □	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

SEMESTER 1 review

EX p256

Short answer

1 a $-2x^2 - 9x + 5$ **b** $6xy^2 + 22xy$

c $\frac{x^4}{y^{17}}$ **d** $\frac{1}{64}$

e $64x^{11}y^2$ **f** $\frac{26 - 31x}{15}$

2 a $6x + 15y$

b $36x^3y^5 - 32x^5y^2$

c $-\frac{15x^2}{4} - \frac{15xy}{2} + 10x$

d $16ax - 14bx - 56ay + 49by$

e $2x^2 + 10x - 16$

f $15x^2 - 8x + 7$

g $100x^2 + 60x + 9$

h $3ax + 8bx + 2cx - 15a - 40b - 10c$

3 a $-8xy(3xy - 2y - 5) = 8xy(2y - 3xy + 5)$

b $(5x + 4)(4y - 1)$

c $(5x - 7)(5x + 7)$

d $x(x - 6)$

e $(x - 5)^2$

f $(x + 1)(x + 3)$

g $(x + 11)(x - 4)$

h $3(x - 15)(x + 3)$

4 a $3x$ **b** $\frac{(x + 10)^2}{(x - 4)(x - 2)}$

c $\frac{x - 2}{4}$ **d** $\frac{(x - 4)(x - 2)}{(x + 3)(x + 8)}$

5 a $(x + 5)^2 - 25$ **b** $(x - 3)^2 + 3$

c $\left(x - \frac{5}{2}\right)^2 + \frac{23}{4}$ **d** $2(x - 7)^2 - 116$

6 a $(2x + 5)(x - 3)$ **b** $(3x + 4)(2x - 5)$

c i $4\left(x - \frac{3}{8}\right)^2 - \frac{121}{16}$

ii $4(x + 1)\left(x - \frac{7}{4}\right) = (x + 1)(4x - 7)$

7 a rational **b** rational

c irrational **d** rational

e irrational **f** rational

8 a $6\sqrt{5}$ **b** $6\sqrt{3} + 12\sqrt{2}$

c $15\sqrt{2}$ **d** $2\sqrt{6}$

e $54\sqrt{3} + 72$ **f** $8\sqrt{15} - 6$

9 **a** $\frac{5\sqrt{3}}{3}$ **b** $\frac{2\sqrt{6}+\sqrt{3}}{3}$

c $\frac{9\sqrt{2}+15}{7}$ **d** $\frac{14-5\sqrt{5}}{2}$

10 **a** $a^{\frac{23}{12}}$ **b** b^4 **c** $c^{\frac{5}{12}}$

d $d^{\frac{13}{12}}$ **e** $4e^4f^3$ **f** $15x^{\frac{5}{2}}y^{\frac{5}{4}}$

11 **a** 4 **b** 3 **c** 10 000

d 9 **e** 1000 **f** 4

12 **a** $\log_3(81) = 4$ **b** $\log_{10}(10\,000) = 4$

c $\log_6(36) = 2$ **d** $2^4 = 16$

e $3^2 = 9$ **f** $1^0 = 1$

13 **a** 2 **b** 7 **c** –1 **d** –4

e –1 **f** 2 **g** –2 **h** –3

14 **a** 6 **b** 5 **c** 0 **d** 2

15 **a** $\frac{1}{3}$ **b** 5 **c** $\frac{1}{25}$ **d** 3

16 **a** –2 **b** 1 **c** 6 **d** 2

17 **a** 6 **b** 12 **c** 10 250 **d** 50

18 **a** $x = -2$ **b** $x = -3$

c $x = 4$ **d** $x = 4$

e $x = \log_5(17)$ **f** $x = \frac{\log_3(11)}{2}$

g $x = 2$ or $x = 3$ **h** –2

19 **a** $x = -7$ **b** $x = \frac{11}{3}$ **c** $x = \frac{27}{5}$

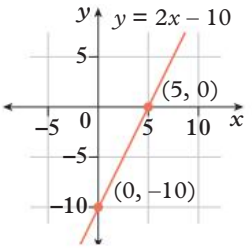
d $x = \frac{11}{13}$ **e** $x = \frac{20}{9}$ **f** $x = -2$

g $x = -\frac{4}{15}$ **h** $x = -43$

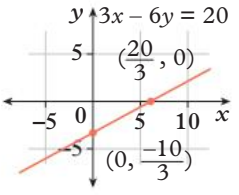
20 **a** $x = \frac{y-c}{m}$ **b** $x = \frac{c-by}{a}$ **c** $x = \frac{t+q}{p-r}$

d $x = \frac{2A}{z} - y$ **e** $x = \pm\sqrt{\frac{A}{\pi}}$ **f** $x = \sqrt[3]{\frac{3V}{4\pi}}$

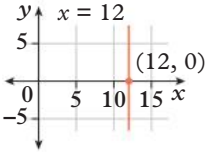
21 **a**



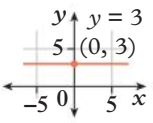
b



c



d



22 **a** $y = -x + 4$ **b** $y = 2x + 2$

c $y = \frac{x}{2} + 2$ **d** $y = -\frac{x}{4} + 2$

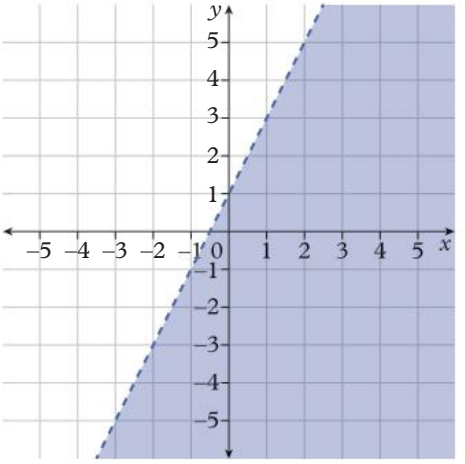
23 **a** $y = -\frac{2}{3}x + \frac{5}{3}$ **b** $y = 4x + 13$

c $y = \frac{16}{5}x - \frac{68}{5}$

24 **a** $x < \frac{1}{3}$ **b** $x < -7$ **c** $x \leq 3$

d $x \leq 2$ **e** $x > 4$ **f** $x > 14$

25 **a**



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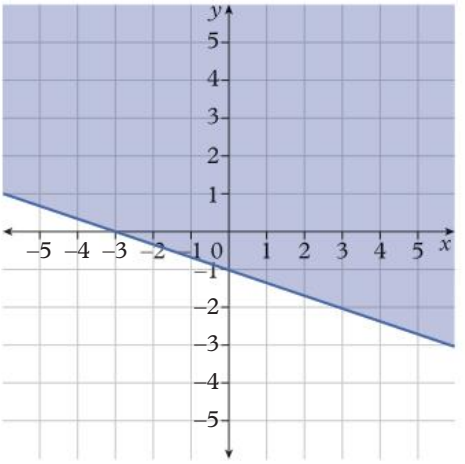

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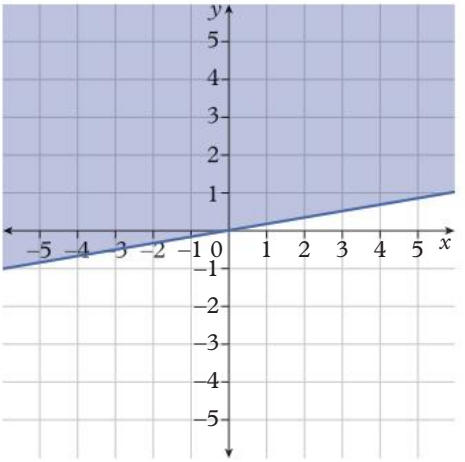
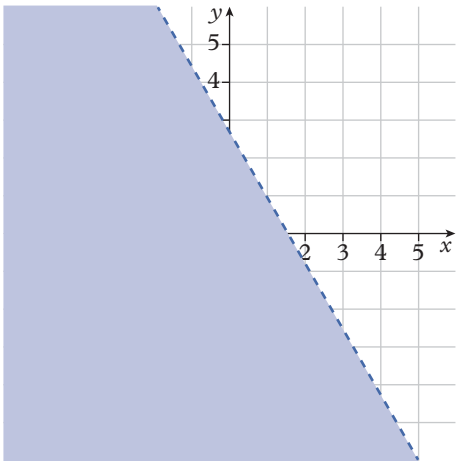


d



26 **a** $x = -1, y = 4$

b $x = 1, y = 0$

c $x = 1, y = 6$ or $x = 3, y = 4$

d $x = 1, y = 3$

27 **a** $x = 2, y = 2$

b $x = \frac{13}{6}, y = -\frac{2}{3}$

c $x = \frac{13}{2}, y = -\frac{2}{9}$

d $x = -\frac{8}{3}, y = -\frac{6}{5}$

28 **a** $x = \frac{1}{2}, x = \frac{5}{3}$ **b** $x = \pm 5$

c $x = -4, x = 14$ **d** $x = 2$

e $x = \pm\sqrt{10}$ **f** $x = \pm 2\sqrt{2}$

29 **a** $x = 4, x = 6$

b $x = -2, x = 4$

c $x = -2, x = 1$

d $x = -2, x = 7$

e $x = 7 \pm \sqrt{51}$

f $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

30 **a** 2 **b** 1 **c** 2 **d** 0

e 2 **f** 0 **g** 1 **h** 2

31 **a** x-intercepts: $(4, 0), (-6, 0)$, y-intercept: $(0, -72)$

b x-intercept: $(4, 0)$, y-intercept: $(0, 16)$

c x-intercepts: $(1, 0), (-6, 0)$, y-intercept: $(0, 6)$

d y-intercept: $(0, 1)$

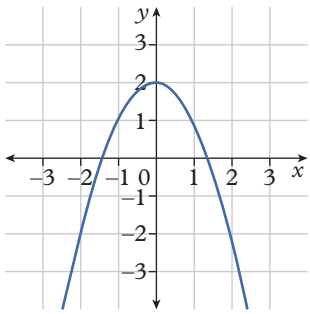
e x-intercepts: $(1, 0), (3, 0)$, y-intercept: $(0, -1)$

f y-intercept: $(0, 1)$

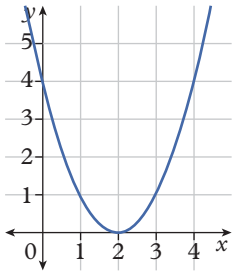
g x-intercept: $\left(-\frac{2}{3}, 0\right)$, y-intercept: $(0, 4)$

h x-intercepts: $\left(\frac{3}{2} - \frac{\sqrt{7}}{2}, 0\right), \left(\frac{3}{2} + \frac{\sqrt{7}}{2}, 0\right)$, y-intercept: $(0, 1)$

32 **a**



b



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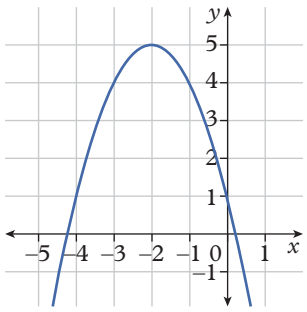

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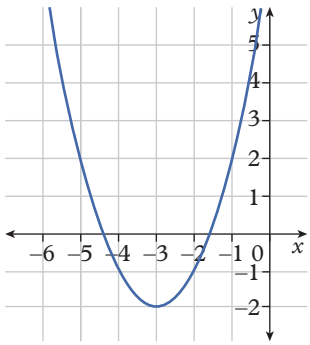


33 **a** $(x + 2)^2 + (y + 1)^2 = 16$

b $(x - 3)^2 + (y - 1)^2 = 9$

34 **a** $(x - 2)^2 + (y + 1)^2 = 9$

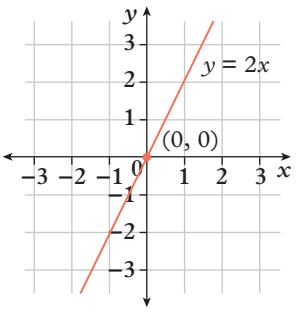
b $(x - 3)^2 + (y + 1)^2 = 1$

c $y = 3 - 2^x$

d $y = 2 \times 3^x$

35 **a** 14 **b** -7 **c** 9000 **d** $-\frac{1}{16}$

36 a



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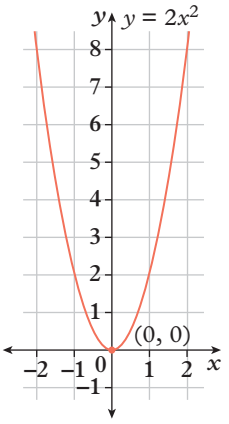

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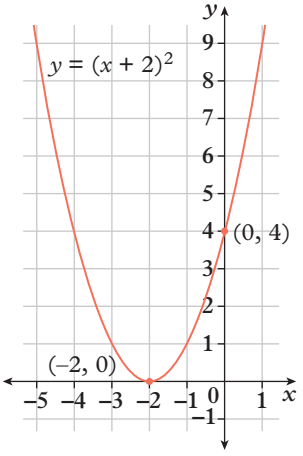

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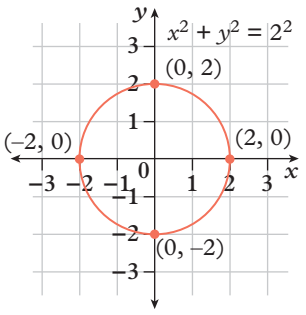

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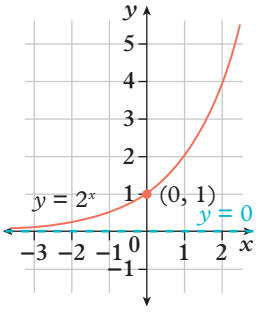
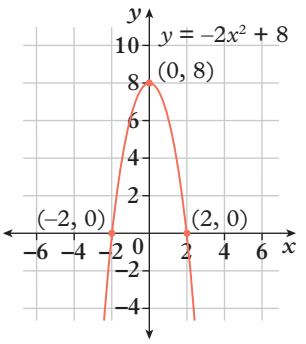


37 a stretched by a factor of 2 from the x-axis
reflected in the x-axis
translated 8 units up

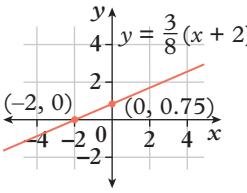
b stretched by a factor of $\frac{3}{8}$ from the x-axis
translated 2 units left

c stretched by a factor of $\frac{1}{2}$ from the x-axis, or
translated 1 unit right
translated 4 units down

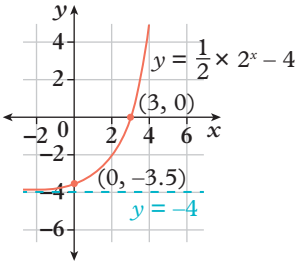
38 a



b



c



39 a $a = 79°$

b $a = 35°$, $b = 89°$, $c = 56°$, $d = 56°$

c $j = 12°$, $k = 84°$, $l = 96°$, $m = 84°$, $n = 84°$, $p = 94°$, $q = 98°$, $r = 98°$, $t = 96°$

d $b = 39°$, $c = 27°$, $d = 94°$, $e = 22°$, $f = 42°$, $g = 125°$, $h = 55°$, $i = 13°$

40 a

Statements	Reasons
$AC \parallel BD$	Given
$AC = BD$	Given
$\angle ACB \cong \angle DBC$	Alternate angles on parallel lines
$BC = BC$	Side common to $\triangle ACB$ and $\triangle DBC$
$\triangle ACB \cong \triangle DBC$	The SAS condition for congruence
$\angle ABC \cong \angle DCB$	Corresponding angles of congruent triangles, alternate angles
$\therefore AB \parallel CD$ □	Alternate angles formed by a transversal cutting AB and CD are equal.

b

Statements	Reasons
$AC \parallel DE$	Given
$\angle DBA + \angle ABC + \angle EBC = 180°$	Angles on a straight line
$\angle DBA = \angle BAC$	Alternate angles on parallel lines
$\angle EBC = \angle BCA$	Alternate angles on parallel lines
$\therefore \angle BAC + \angle ABC + \angle BCA = 180°$ □	By substituting the equal angles

41 a

Statements	Reasons
$\angle ABC = \angle BCD = \angle CDA = \angle DAB = 90°$	Given
$BC \parallel AD$	Co-interior angles are supplementary.
$\angle CBD = \angle ADB$	Alternate angles on parallel lines
$BD = BD$	Side common to both triangles
$\angle BCD = \angle BAD$	Given; both 90°
$\therefore \triangle ABD \cong \triangle CDB$ □	Meets the AAS condition for congruence

b

Statements	Reasons
$AD = 15$	Given
$DB = 5$	Given
$BE = 7$	Given
$CE = 21$	Given
$\angle ABC = \angle DBE$	Different names for the same angle
$AD + DB = AB$	Addition of line segments on the same line
$AB = 15 + 5 = 20$	Substitution of given measurements
$CE + EB = CB$	Addition of line segments on the same line
$CB = 21 + 7 = 28$	Substitution of given measurements
$\frac{AB}{DB} = \frac{20}{5} = 4$	Scale factor of corresponding sides
$\frac{CB}{EB} = \frac{28}{7} = 4$	Scale factor of corresponding sides
$\therefore \triangle ABC \sim \triangle DBE$ □	Meets the SAS condition for similarity

c

Statements	Reasons
$\angle CDA = \angle DAB = 32°$	Given
$AE = ED$	Given
$\angle CED = 90°$	Given
$\angle AEB = 90°$	Vertically opposite angles are congruent.
$\triangle AEB \cong \triangle DEC$	Meets the ASA condition for congruence
$CE = BE$	Corresponding sides of congruent triangles
$\angle BED = 90°$	Supplementary with $\angle CED$; angles on a straight line
$\angle CEA = 90°$	Vertically opposite $\angle BED$
$\triangle BED \cong \triangle CEA$	Meets the SAS condition for congruence
$\triangle AEB \cong \triangle DEB \cong \triangle DEC \cong \triangle AEC$	All have been proven to be congruent
So: $CD = DB = AB = AC$	Corresponding sides of congruent triangles are equal.
$\therefore ABDC$ is a rhombus □	A quadrilateral with four equal sides is a rhombus.

42 Label the four vertices of the quadrilateral as $ABCD$ with diagonals AC and BD. Let O be the intersection of AC and BD. Let Γ be the circle centred at O with radius OA. Since the diagonals are equal and bisect each other, $OA = OB = OC = OD$. Hence, Γ passes through all four points A, B, C and D, and AC and BD are both diameters of the circle. Circle theorem 2 implies that the four triangles $\triangle BAD$, $\triangle ADC$, $\triangle BCD$ and $\triangle ABC$ are all right-angled triangles. This means that the four internal angles of quadrilateral $ABCD$ are all right angles and so $ABCD$ is a rectangle.

43

Statements	Reasons
$\gamma = \gamma_1 + \gamma_2$	Given
$\theta + \varphi = 360°$	Angles at a point sum to 360°
$\alpha + \beta + \varphi + \gamma_1 + \gamma_2 = 360°$	Interior angles of a quadrilateral sum to 360°
$\theta + \varphi = \alpha + \beta + \varphi + \gamma_1 + \gamma_2$	Transitive law
$\theta = \alpha + \beta + \gamma_1 + \gamma_2$	Subtraction law of equality
$\theta = \alpha + \beta + \gamma$	Substitution of equal angles
$OA = OB = OC$	All radii of the circle
$\alpha = \gamma_1$	Base angles of an isosceles triangle
$\beta = \gamma_2$	Base angles of an isosceles triangle
$\theta = \gamma_1 + \gamma_2 + \gamma$	Substitution of equal angles
$\therefore \theta = 2\gamma$ □	Substitution of equal angles

Analysis

1 a When $n = 0$ the value of the investment is the same as the principle.

b **i** \$26 250 **ii** \$27 562.5 **iii** \$28 940.63

c 1250, 1312.5, 1378.13

d Faster

e 4 years

f 15 years

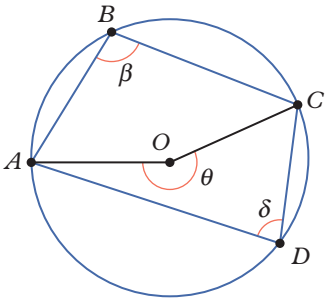
2 a Square, rhombus, kite

b Kite

c Let K be a kite $ABCD$ such that $AB = AD$ and $BC = CD$. Consider the two triangles $\triangle ABC$ and $\triangle ADC$. Since $AB = AD$, $BC = CD$ and AC is a shared edge, $\triangle ABC \cong \triangle ADC$ by the test SSS. This means that the angles $\angle ADC$ and $\angle ABC$ are equal since they are in corresponding positions in the two triangles.

d Cyclic quadrilateral. Opposite pairs of internal angles are supplementary.

e Let $ABCD$ be a cyclic quadrilateral with centre O. Let the interior angles $\angle ABC = \beta$ and $\angle ADC = \delta$ and the angle at O subtended by the arc containing B be θ, as shown. Circle theorem 1 implies that $\theta = 2\beta$. Since angles at a point sum to 360°, circle theorem 1 also implies that $360° - \theta = 2\delta$. Substituting $\theta = 2\beta$ and rearranging implies $360° = 2\delta + 2\beta$. Dividing by 2 gives $\delta + \beta = 180°$. The same logic implies that $\angle BAD + \angle BCD = 180°$. Hence, opposite pairs of angles in a cyclic quadrilateral are supplementary.

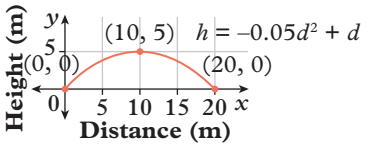


f From part **c**, $\triangle ABC \cong \triangle ADC$ and angles $\angle ADC$ and $\angle ABC$ are equal. Since $ABCD$ is a cyclic quadrilateral, opposite pairs of angles are supplementary. Since $\angle ADC$ and $\angle ABC$ are opposite angles, $\angle ADC + \angle ABC = 2\angle ABC = 2\angle ADC = 180°$. Dividing by two gives $\angle ABC = \angle ADC = 90°$. Since both $\triangle ABC$ and $\triangle ADC$ contain a right angle, they are congruent right-angled triangles that make up $ABCD$.

3 a 20 m wide

b maximum height = 5 m, 10 m away from the entrance

c



d i $10 - 2\sqrt{5}$ m or $10 + 2\sqrt{5}$ m

ii 5.53 m or 14.47 m

EX EXPLORATIONS 1

p264

1 a 4 red bricks

b 2 purple bricks

2 a

×	2	7	10	5	3
9	18	63	90	45	27
4	8	28	40	20	12
6	12	42	60	30	18
8	16	56	80	40	24
1	2	7	10	5	3

b

×	6	4	2	8
10	60	40	20	80
12	72	48	24	96
14	84	56	28	112
16	96	64	32	128

c Four tables fit the given criteria:

×	10	7	5	3	2
6	60	42	30	18	12
4	40	28	20	12	8
9	90	63	45	27	18
8	80	56	40	24	16

×	10	7	5	3	8
6	60	42	30	18	48
4	40	28	20	12	32
9	90	63	45	27	72
2	20	14	10	6	16

×	10	7	5	9	2
6	60	42	30	54	12
4	40	28	20	36	8
3	30	21	15	27	6
8	80	56	40	72	16

×	10	7	5	3	2
6	60	42	30	54	48
4	40	28	20	36	32
9	30	21	15	27	24
8	20	14	10	18	16

3 a $\sqrt{3} - \sqrt{2}$

b i $\dfrac{\sqrt{a} - \sqrt{b}}{a - b}$

ii It fails when $a = b$. Then $\dfrac{\sqrt{a}}{2a}$ is the correct simplified fraction.

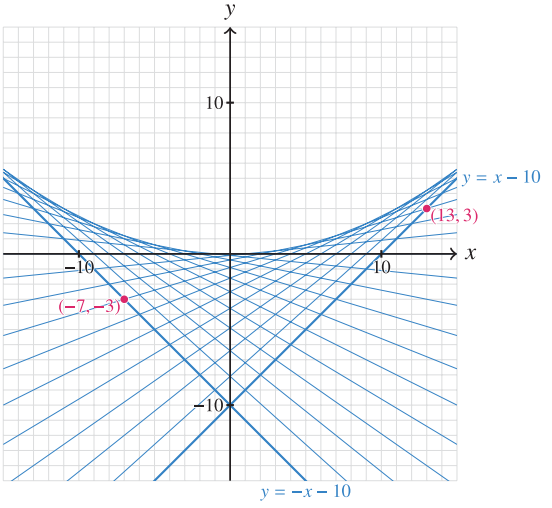
c $\dfrac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}$

d i $\dfrac{a\sqrt{b} + b\sqrt{a} - \sqrt{abc}}{2ab}$

ii It works for all a and b.

iii $\dfrac{1}{\sqrt{3} + \sqrt{5} + \sqrt{8}} = 3\dfrac{\sqrt{5} + 5\sqrt{3} - 2\sqrt{30}}{30}$

4 a



b **i** $y = \frac{3}{10}x - \frac{9}{10}$

ii Show that $\frac{1}{40}x^2 = \frac{3}{10}x - \frac{9}{10}$ simplifies to $(x-6)^2 = 0$, which has exactly one solution.

c **i** $Q(k-10, -k)$

ii $y = \frac{k}{10}x - \frac{k^2}{10}$

iii Simplify to $(x-2k)^2 = 0$ or show that the discriminant is zero.

d Show that $ax^2 = \frac{k}{d}x - \frac{k^2}{d}$ has discriminant equal to zero when $a = \frac{1}{4d}$.

5 **a** 1, 32, 243, 1024, 3125. The last digit of the power is equal to the base. Yes, this also works for other fifth powers.

b $n(n-1)(n+1)(n^2+1)$. Show that 2, 3 and 5 each divide at least one of the four factors. If n is of the form $5k \pm 2$, show that 5 divides $n^2 + 1$.

c Since $n^5 - n$ ends in 0, n^5 and n must have the same last digit.

d Explain why the second and third factors are divisible by 2 and 4, or vice versa.

e Any n that is equal to, one more than, or one less than a multiple of 9.

f $n^9 - n = (n^5 - n)(n^4 + 1)$ is a multiple of 30. It is true for all 1st, 9th, 13th, …, $(4m+1)^{th}$, … powers.

CHAPTER 6 Measurement

EX p270 6A Area review

1 **a** 700 cm **b** 4.5 km
c 8 400 000 m² **d** 0.29 cm²
e 2.4 ha **f** 880 000 mm
g 0.003 42 km² **h** 1 022 000 cm
i 300 ha

2 **a** E **b** A **c** D
d B **e** C

3 **a** 42 m² **b** 272 cm²
c 76 cm² **d** 348.5 cm²
e 843.6 mm² **f** 164.4 cm²

4 **a** 216 cm² **b** 242.1 cm²
c 238.9 cm² **d** 375 mm²
e 28.6 cm² **f** 409.5 cm²

5 **a** 59.4 cm²
b 515.3 mm²
c 2875.1 cm²

6 **a, b** chimney: trapezium, 1.8 m²
roof: triangle, 6.5 m²
front wall: rectangle, 15.54 m²
front door: rectangle, 3.9 m²
windows: square, 0.49 m² each window

c multiply by 4

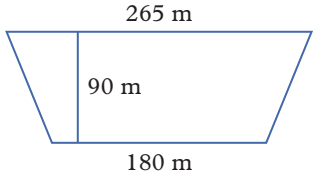
d area to be painted = area of chimney + area of roof + area of front wall − area of windows

e 21.88 m²

f 1.5 L

7 **a** 168 cm² **b** 24 m²
c 6.4 cm² **d** 348 cm²
e 933.9 cm² **f** 219.5 cm²

8 **a** 20 025 m²



b 2 ha

9 **a** Subtract the area of the inner circle from the area of the outer circle.

b 100.5 m²

c **i** 115.5 m² **ii** 58.9 m² **iii** 125.7 m²

10 **a** 75.4 cm² **b** 60°
c 12.6 cm² **d** 44.0 cm²

11 **a** 904.8 cm²

b The area of the white section is the same as that of the blue section: 904.8 cm².

12 92 cm²

13 **a** 60π cm² **b** 40π cm² **c** 3 : 2

14 **a** $5\sqrt{3}$ cm **b** $25\sqrt{3}$ cm²
c $150\sqrt{3}$ cm² **d** $\frac{3\sqrt{3}}{2}a^2$

15 **a** $A = \frac{1}{2}ah + \frac{1}{2}bh$
$= \frac{h}{2}(a+b)$

b $A = ah + \frac{1}{2}(b-a)(h)$
$= \frac{ah}{2} + \frac{bh}{2}$
$= \frac{h}{2}(a+b)$

c The two triangles are similar, and so $\frac{x}{x+h} = \frac{a}{b}$.
Solving for x gives:
$xb = a(x+h)$
$xb = ax + ah$
$xb - ax = ah$
$x(b-a) = ah$
$x = \frac{ah}{b-a}$
Therefore: $A = \frac{1}{2}b(h+x) - \frac{1}{2}ax$
$= \frac{bh}{2} + \frac{bx}{2} - \frac{ax}{2}$
$= \frac{bh}{2} + \frac{x}{2}(b-a)$
$= \frac{bh}{2} + \frac{ah}{2(b-a)}(b-a)$
$= \frac{bh}{2} + \frac{ah}{2}$
$= \frac{h}{2}(a+b)$

EX p278 6B Surface area

1 **a** 262 cm^2 **b** 1350 cm^2 **c** 6144 mm^2
d 34.56 m^2 **e** 6156 mm^2 **f** 23.88 m^2
g 876 mm^2 **h** 332 cm^2 **i** 53.66 m^2

2 23.67 m^2

3 **a** 662.8 cm^2 **b** 2310.1 mm^2 **c** 117.4 cm^2

4 383.04 cm^2

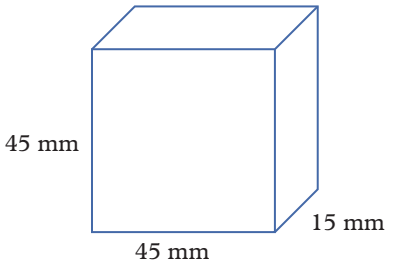
5 **a** 1809.6 cm^2 **b** 883.4 cm^2 **c** 11 309.7 mm^2

6 **a** 8224 cm^2 **b** 287.1 cm^2 **c** 470.4 cm^2

7 **a** 1350 mm^2

b 12 150 mm^2

c



d 6750 mm^2

e 5400 mm^2; 24 surfaces of the dice are hidden within the structure ($24 \times 15^2 = 5400$).

8 **a** 8 cm; divide total surface area by 6 to find the area of one face, then find the square root of the area answer to find the length of one side.

b $l = \sqrt{\frac{TSA}{6}}$

9 **a** 570.2 cm^2

b No. The pipe is open, so there is no top or bottom. This means $2\pi r^2$ is removed from formula.

10 279.9 cm^2

11 **a** 7191.3 cm^2

b No. The curved surface area will be double, but the one circular end which needs painting has the same area for both poles.

c 14 259.9 cm^2

$2 \times 7191.3 = 14\,382.6$ cm^2

This is 122.7 cm^2 larger than 14 259.9 cm^2.

This difference is the area of one end which is not doubled.

This supports the answer to part **b**.

12 **a** 730.1 cm^2 **b** 650.8 cm^2

13 **a** 306.1 m^2 **b** 4815.8 m^2

14 1311 cm^2

15 208.9 cm^2

16 **a** 9 m^2 **b** 4 cans

EX p283 6C Volume

1 **a** 12 500 000 cm^3 **b** 240 cm^3
c 34.2 m^3 **d** 550 000 000 mm^3
e 0.000 067 2 m^3 **f** 9000 mm^3
g 7.52 cm^3 **h** 8 740 000 cm^3
i 0.1429 m^3 **j** 0.073 m^3

2 **a** 2508 cm^3 **b** 16 128 mm^3
c 5.8 m^3 **d** 43 264 cm^3
e 689.1 cm^3 **f** 6.2 m^3
g 96 140 mm^3 or 96.1 cm^3 **h** 29 440 cm^3
i 1539 m^3 **j** 53.5 cm^3
k 74 088 m^3 **l** 175 cm^3

3 **a** 1256.6 cm^3 **b** 172 240.2 mm^3
c 16 155.6 m^3 **d** 290.8 cm^3
e 178 623.1 cm^3 or 0.2 m^3 **f** 3698.5 cm^3

4 **a** 0.064 m^3 **b** 64 000 cm^3

5 **a** 1102.5 cm^3 **b** 35 224 mm^3 **c** 15.58 m^3

6 **a** 12.48 m^3 **b** 10 560 mm^3 **c** 676.2 cm^3
d 9000 cm^3 **e** 17 090.3 mm^3 **f** 25 249.0 cm^3

7 5193.4 cm^3

8 **a** rectangular prism, triangular prism, cylinder

b Subtract the volume of the cylinder from the sum of the volumes of the rectangular and triangular prisms.

c 97 cm^3

9 **a** 206.2 cm^3 **b** 360 cm^3 **c** 1687.5 cm^3

10 **a** 176 cm^3 **b** 0.176 kg **c** \$3.70

11 **a** 45 000 cm^3 **b** 45 000 mL, 45 L
c 39.6 L **d** 0.4 cm

12 14 cm

13 **a** 21 cm **b** 34 mm

14 **a** second tank **b** 1178 L

15 **a** 3015.9 cm^3

b **i** 754 cm^3 **ii** $\frac{1}{4}$ the volume

c **i** 12 063.7 cm^3 **ii** 4 times the volume

d **i** 1508 cm^3 **ii** $\frac{1}{2}$ the volume

e **i** 6031.9 cm^3 **ii** 2 times the volume

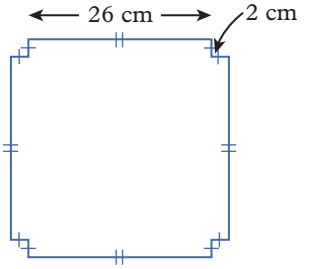
16 **a** 59 244 litres **b** 41 471 litres

17 **a** annulus, area = 84.8 cm^2

b 2121 cm^3

c Subtract the volume of the inner cylinder from the volume of the outer cylinder.

18 **a**



b length = 26 cm, width = 26 cm, height = 2 cm

c 1352 cm^3

d The box would be 24 cm × 24 cm × 3 cm, giving a volume of 1728 cm^3. The volume of the box would increase.

19 **a** circular nut: 107.71 mm^3
hexagonal nut: 144.68 mm^3

b hexagonal nut

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Checkpoint

1 **a** 4.5 cm **b** 4 m **c** 4530 mm
d 40 cm² **e** 5 200 000 m² **f** 6.8 m²

2 13.125 m²

3 219.1 cm²

4 306 cm²

5 41.7 m²

6 697.8 cm²

7 2850 cm²

8 7634.1 cm³

9 **a** 4000 mm³ **b** 4 cm³

10 1292.655 cm³

11 8500 cm³

EX p292 6D Measurement errors and accuracy

1 **a** **i** 0.67 **ii** 0.86
b **i** 0.0033 **ii** 0.0029

2 0.0507% and 0.0402%. $\frac{22}{7}$ is a more accurate approximation.

3 **a** 0.038 46 **b** 3.846%

4 **a** 8137 m **b** 0.1276%
c The error seems large in magnitude, but the percentage error is not large for high school study purpose and the radius of the Earth is not a constant value.

5 **a** 1.92 cm **b** 0.015 75

6 **a** 0.3946% **b** 0.3954%

7 **a** $\frac{20}{\pi}$ cm **b** 6.4 cm **c** $\frac{400}{\pi}$ cm²
d 128.68 cm² **e** 1.06%

8 **a** 37° **b** $\tan^{-1}\left(\frac{3}{4}\right)$ **c** 0.35%

9 **a** 2.14 mph **b** 3.44 km/h
c It's safer to follow the 60 mph limit. This could be an underestimate of the actual speed rather than overestimate.

10 **a** 12.6 cm² **b** 0.2676% **c** 50.3 cm²
d 0.068 67% **e** 75.5 cm²
f 0.1350%

11 **a** 55.42 cm² **b** 0.5747%
c 1.1569%. The error in the measurement of the diameter is compounded when calculating the area of the coaster.

12 **a** Bianka uses values rounded to three decimal places while Christof uses values rounded to one decimal place. Bianka's working is more reliable as her measurements are more accurate.
b 53.85%
c The use of 0.1 instead of 0.065 in Christof's calculation is the main cause of the error.
d Sample answer: Further decimal places would represent less than 0.001 g, which is a level of detail not required for a nutrition label.

13 **a** Sample answer: A margin of error is stated so there is an understanding that the poll is unlikely to be 100% accurate.
b Sample error: The margin of error could be decreased by increasing the sample size or by ensuring that the sample accurately reflects the voting population.

14 **a** James: 20 km/h; Karen: 12 km/h
b James: 5.6 m/s; Karen: 3.3 m/s
c **i** 9.375 minutes
ii 562 seconds = 9.363 minutes
d 0.128%

15 **a** Jemima: 0.6; Sister: 0.8 **b** 17.9 minutes
c 18 minutes **d** 0.1 minutes

16 **a** \$0.002 242 **b** \$0.81

17 **a** \$14.95; under payment
b

Number of packs	Exact price (\$)	Payment amount (\$)	Error (\$)
1	14.9625	14.95	–0.0125
2	29.925	29.95	–0.025
3	44.8875	44.90	0.0125
4	59.85	59.85	0

c −\$0.6125

EX p298 6E Proportion and scaling

1

Length scale factor	Area scale factor	Volume scale factor
2	4	8
$\frac{3}{4}$	$\frac{9}{16}$	$\frac{27}{64}$
4	16	64
$\frac{6}{5}$	$\frac{36}{25}$	$\frac{216}{125}$
$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

2 **a** 224 mm² **b** 2.5 cm² **c** 1102 m²

3 **a** 7.5 cm³ **b** 35 342 mm³ **c** 2673 cm³

4 **a** 2.25 **b** 3.375 **c** 16.875 cm³

5 **a** SA = 150 mm², V = 125 mm³
b 1350 mm²
c 3375 mm³

6 **a** SA = 504π cm², V = 1296π cm³
b 56π cm²
c 48π cm³

7 **a** 7.2 cm **b** 111 cm²

8 12 m²

9 1 : 40 000

10 1.5

11 $2^{\frac{2}{3}}$

12 4

13 **a** 675 g **b** 180 min

14 The area scale factor is the square of the length scale factor only for similar objects. $\triangle ABD$ and $\triangle ADC$ are not similar. The bases have ratio 1 : 2, but the heights of the triangles are the same. So their area ratio is 1 : 2.

15 **a** 9 : 4 **b** 4 : 5

16 **a** SA = 400π cm^2, Volume = $\frac{4000}{3}\pi$ cm^3

b 5 cm

c 100π cm^2

d $\frac{1}{4}$. The area scale factor is the square of the radius scale factor.

e $\frac{500}{3}\pi$ cm^3

f $\frac{1}{8}$. The area scale factor is the cube of the radius scale factor.

EX p303 6F Logarithmic scales

1 **a** 12 **b** −5 **c** 3

d 8 **e** −7 **f** 3

2 **a** 3 units to the right of 1

b 5 units to the left of 1

c 1 unit to the right of 1

d 2.30 units to the right of 1

e 1.70 units to the left of 1

f 3.10 units to the left of 1.

3 **a** 100 000 **b** 0.1 **c** 2510

d 0.002 00 **e** 1.26 **f** 0.000 025 1

4 **a** 4.215×10^4 **b** 4.215×10^5

c 4.215×10^7 **d** 0.729

e 7.29 **f** 729

5 **a** 4.215×10^2 **b** 4.215×10^1

c 4.215×10^{-2} **d** 0.007 29

e 0.000 729 **f** 0.000 000 729

6 **a** 6 orders higher **b** 3 orders lower

c 15 orders lower **d** 2 orders higher

e 3 orders lower **f** 12 orders higher

7 **a** divide by 1000

b multiply by 10 000

c multiply by 100 000 000 000 (or 10^{11})

d multiply by 1 995 262

e divide by 126

f divide by 125 892 541 179 (or $10^{11.1}$)

8 **a** **i** $7.293\,85 \times 10^3$ **ii** $1.052\,357\,9 \times 10^5$

iii 2.135×10^{-3} **iv** 5.0123×10^4

v 6.1290×10^1 **vi** 3.2793×10^{-6}

b **i** 3 **ii** 5 **iii** −3

iv 4 **v** 1 **vi** −6

c **i** 3 and 4 **ii** 5 and 6 **iii** −3 and −2

iv 4 and 5 **v** 1 and 2 **vi** −6 and −5

d **i** 3.863 **ii** 5.022 **iii** −2.671

iv 4.700 **v** 1.787 **vi** −5.484

9 **a** Increasing a value by 0.5 orders of magnitude corresponds to the value becoming 3.16 times greater.

b Increasing a value by 1.5 orders of magnitude corresponds to the value becoming 31.6 times greater.

c Increasing a value by 2.5 orders of magnitude corresponds to the value becoming 316 times greater.

d Increasing a value by 3.5 orders of magnitude corresponds to the value becoming 3160 times greater.

e Multiplying a value by 2 increases its order of magnitude by 0.301.

f Multiplying a value by 4 increases its order of magnitude by 0.602.

g Multiplying a value by 8 increases its order of magnitude by 0.903.

h Multiplying a value by 16 increases its order of magnitude by 1.20.

10 **a** 4.724 units to the right of 1

b 6.276 units to the left of 1

c 6.676 unit to the left of 1

d 8.990 units to the right of 1

e 5.154 units to the left of 1

f 0.979 units to the left of 1

11 **a** **i** 5.5 **ii** 3.1 **iii** 4.9

iv 2.8 **v** 7.3 **vi** 8.2

b **i** 3 162 278 μm **ii** 1 000 000 μm

iii 3 162 000 μm **iv** 32 μm

v 63 095 734 μm **vi** 19 953 μm

c **i** 10 **ii** 3.162 **iii** 31.62

iv 63.10 **v** 1 259 000 **vi** 1.259

12 **a** The number of cases becomes 10 times greater every 100 days.

b **i** 50 days **ii** 200 days **iii** 500 days

iv 31 623 cases **v** 1000 cases

c The growth in cases has slowed down below the point where the cumulative number of cases in country B is the same as if the number of cases had been increasing at a rate of 10 times the amount of cases every 100 days.

13 **a** linear **b** logarithmic

c logarithmic **d** linear

e logarithmic **f** linear

14 **a** The logarithm of a negative number is not a real number.

b The logarithm of zero is undefined.

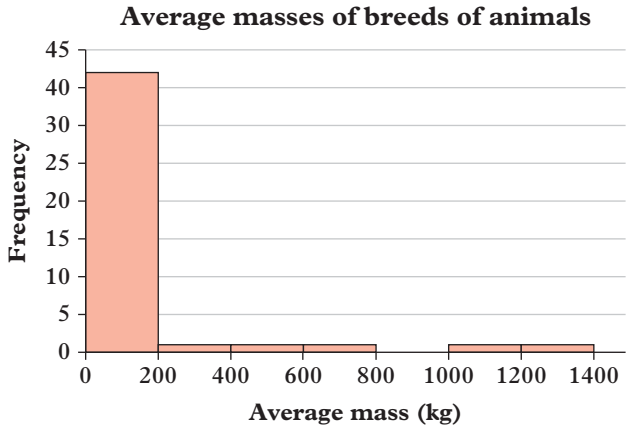
15 **a** **i**

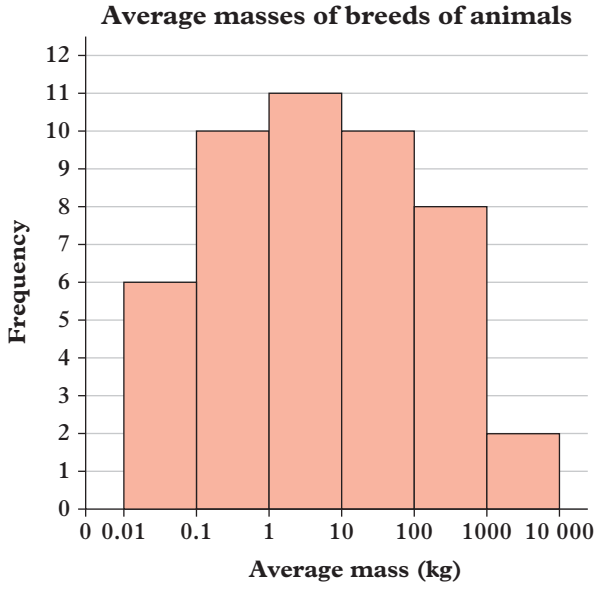
Class interval	Frequency
0–<200	42
200–<400	1
400–<600	1
600–<800	1
800–<1000	0
1000–<1200	1
1200–<1400	1

ii

Class interval	Frequency
0.01–<0.1	6
0.1–<1	10
1–<10	11
10–<100	10
100–<1000	8
1000–<10 000	2

b





c Linear scale: positively skewed
Logarithmic scale: approximately symmetric

d The logarithmic scale spreads out the average masses in the 0–<200 column (42 out of 47 masses) into five different columns and groups the data in the remaining columns (with only 5 masses) into two columns.

16 a 1 day **b** 1 day **c** constant
d decreasing **e** increasing

17 a 2 hours

b With the base-two logarithmic scale you can more easily identify the amount of the active substance remaining after a high number of hours, however it is more difficult to identify the amount of active substance remaining for periods between each half-life interval (every two hours).

18 a i 10 **ii** 10 **iii** 10 000
iv 100 000 **v** 1 000 000 000 **vi** 1.259
b i −10 dB **ii** 10 dB **iii** −20 dB
iv −6 dB **v** 0 dB **vi** −20 dB
c i 5000 W **ii** 10.0 W
iii 99.8 W **iv** 500 000 W
v 240 000 000 W
vi 24 000 W

CHAPTER 6 review

Multiple choice

1 D **2** A **3** D **4** D **5** C
6 C **7** A **8** D **9** C **10** B

Short answer

1 3.9 cm^2

2 a 77.5 cm^2 **b** 63.3 cm^2

3 2664.2 mm^2

4 a 9600 cm^2, 38 400 cm^2, 153 600 cm^2
b 1 can

5 385 mL

6 0.064 m^3, 0.512 m^3, 4.096 m^3

7 a 49π mm^2 **b** 153.9 mm^2 **c** 0.000 25

8 23.14 cm^2

9 875 m^2

10 a 265 **b** 840 **c** 0.0751
d 0.090 91

11 a 0.0316 **b** 178

Analysis

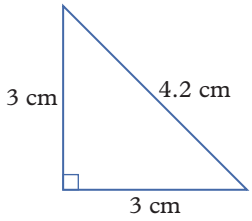
1 a cube, rectangular prism, square pyramid, triangular prism, cone, cylinder, sphere

b i 54 cm^2 **ii** 27 cm^3

c i The rectangular prism has a surface area of 90 cm^2, but the surface area of the cube is 54 cm^2. The rectangular prism is $1\frac{2}{3}$ times as large.

ii The volume of the rectangular prism is 54 cm^3. Its volume is twice the volume of the cube.

d i



ii 70.5 cm^2

iii No; the total surface area of the triangular prism is more than half the TSA of the rectangular prism because two new faces have been created by cutting the prism in half.

iv The volume of the triangular prism is half that of the rectangular prism, because it is the same size as the rectangular prism cut in half.

e **i** 70.7 cm² **ii** 42.4 cm³

f **i**

Solid	Surface area
Cube	54 cm²
Triangular prism	70.5 cm²
Cylinder	70.7 cm²
Rectangular prism	90 cm²

ii

Solid	Volume
Cube and triangular prism	27 cm³
Cylinder	42.4 cm³
Rectangular prism	54 cm³

2 **a** **i** 0.09 km **ii** 2.49%

b 1 : 6000

c **i** 2.70 units to the right

ii 6.40 units to the right

CHAPTER 7 Pythagoras' theorem and trigonometry

EX p316 7A Pythagoras' theorem

1 **a** right-angled **b** acute-angled

c acute-angled **d** right-angled

e right-angled **f** obtuse-angled

2 **a** $5\sqrt{2}$ mm **b** $\sqrt{13}$ cm **c** 75 mm

d $2\sqrt{286}$ cm **e** $\sqrt{52.65}$ cm **f** $\sqrt{3.95}$ cm

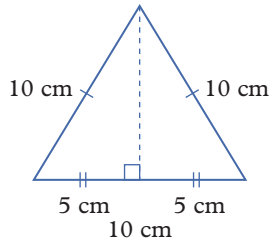
g $\sqrt{75.33}$ cm **h** $\sqrt{17.49}$ cm **i** $\sqrt{66.28}$ cm

3 **a** $2\sqrt{2}$ cm **b** $30\sqrt{2}$ mm **c** $13\sqrt{2}$ cm

d $35\sqrt{2}$ cm

4 $x\sqrt{2}$ cm

5 **a**



b **i** $5\sqrt{3}$ cm **ii** 8.66 cm

c $25\sqrt{3}$ mm

6 $\frac{(x^2\sqrt{3})}{4}$

7 **a** 22.45 cm **b** 15.52 cm **c** 10.38 cm

d 41.23 cm **e** 36.66 cm **f** 18.03 cm

8 **a** acute **b** acute **c** obtuse

d acute **e** acute **f** acute

9 **a** 11.2 cm, 22.4 cm

b 7.9 cm, 23.7 cm

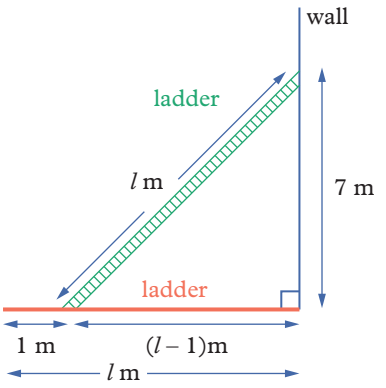
c 4.9 cm, 24.5 cm

10 38 m

11 **a** 6.4 m **b** 38.4 m **c** 74.6 m

d **i** 75.7 m **ii** 82 m

12 **a**



b 25 m

13 **a** $y = -\frac{1}{2}x, y = \frac{1}{3}x + \frac{5}{3}, y = 7x + 15$

b (–2, 1)

c $2\sqrt{5}$ units, $\sqrt{10}$, units, $5\sqrt{2}$ units

EX p321 7B Trigonometry

1 **a** $p = 5.2$ cm, $q = 4.3$ cm

b $t = 6.3$ mm, $s = 13.6$ mm

c $x = 15.1$ cm, $a = 16.0$ cm

d $k = 41.2$ mm, $m = 26.0$ mm

e $r = 8.6$ mm, $k = 9.2$ mm

f $s = 4.6$ cm, $t = 6.8$ cm

2 **a** 22° **b** 32° **c** 67°

d 50° **e** 64° **f** 1°

3 **a** $B = 66°, C = 24°$

b $T = 29°, P = 61°$

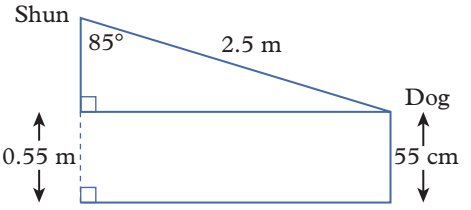
c $G = 32°, H = 58°$

d $K = 29°, M = 61°$

e $D = 27°, R = 63°$

f $F = 40°, D = 50°$

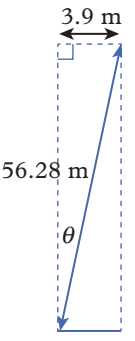
4 **a**



b 249 cm **c** 77 cm

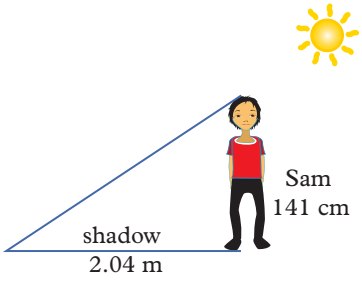
5 $l = 230$ mm, $w = 76$ mm

6 **a**



b $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$ **c** 4°

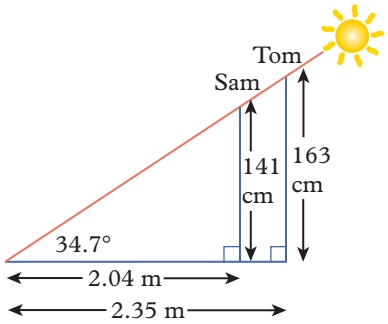
7 a



b Tom 34.7°, Sam 34.7°. The base angle represents the angle of elevation of the sun from the ground at the given time.

c i If the two triangles have the same base angle and a right angle then all three corresponding angles are equal, meaning the triangles are similar.

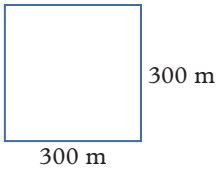
ii



8 a Ji-ho has written the trigonometric ratios in the wrong order, the numerator and denominator should be reversed

b When writing trigonometric ratios, make sure to follow the SOH CAH TOA rule

9 a



b 424.3 m

c 424.3 m

d The answers are identical.

10 a 1.7 m **b** $l = 100$ cm, $w = 80$ cm

11 a For Route 1 **i** 1 : 2 **ii** 50%

For Route 2 **i** 5 : 12 **ii** 41.7%

For Route 3 **i** 13 : 30 **ii** 43.3%

b Route 1, 26.6°; Route 2, 22.6°; Route 3, 23.4°

c Route 2, Route 3, Route 1

12 a $x = 13.16$ cm **b** $x = 46.24°$

c $x = 19.66$ cm **d** $x = 145.32°$

13 20.78 cm

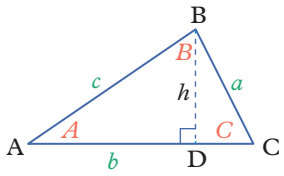
14 a i $x = 40$ cm, $y = 34.64$ cm

ii $x = 80.52$ cm, $y = 59.69$ cm

b i 692.8 cm^2

ii 2403.3 cm^2

15 a, b



c i $h = c\sin(A)$ **ii** $h = a\sin(C)$

iii $a\sin(C) = c\sin(A)$

$$a = \frac{c\sin(A)}{\sin(C)}$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

d $a\sin(B) = b\sin(A)$

$$a = \frac{b\sin(A)}{\sin(B)}$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

So, with what was found in part **c**, we know that:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

16 a 67.5° **b** octagon **c** 80 cm

17 a i 0 **ii** It is 0 : 1.

b i a is equal in length to c. **ii** 1

c The sine increases from 0 : 1 at 0° to 1 : 1 at 90°.

d The cosine ratio has a value of 1 : 1 at 0°, becoming 0 : 1 at 90°.

e The tangent is 0 : 1 at 0°, becoming infinitely large as the angle approaches 90°.

18 The sine ratio has a minimum value of 0 and a maximum value of 1.

The cosine ratio has a minimum value of 0 and a maximum value of 1.

The tangent ratio has a minimum value of 0 and a maximum value of ∞.

EX p329

7C Applications of trigonometry

1 a $x = 117°, y = 63°$ **b** $x = 58°, y = 302°$

c $x = 45°, y = 45°$ **d** $x = 19°, y = 60°, z = 281°$

e $v = 153°, w = 57°, x = 150°, y = 30°, z = 282°$

f $v = 118°, w = 99°, x = 40.5°, y = 37°, z = 282.5°$

2 a $w = 95.7$ m **b** $x = 27.0$ m **c** $z = 148.2$ m

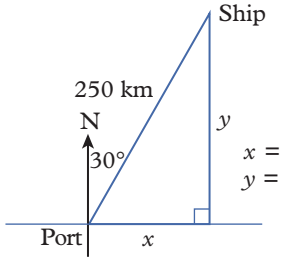
d $p = 121.7$ m **e** $s = 254.3$ m **f** $y = 235.2$ m

3 a 25 m **b** 144 m

4 a i 150°T **ii** 308°T **iii** 005°T

b i S20°W **ii** S80°E **iii** N60°W

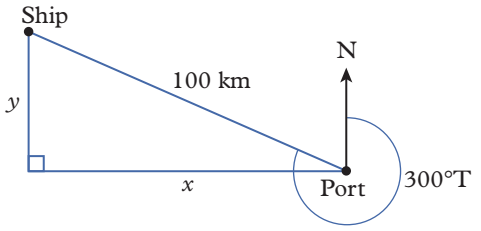
5 a i



x = distance travelled east
y = distance travelled north

ii 125 km east, 217 km north

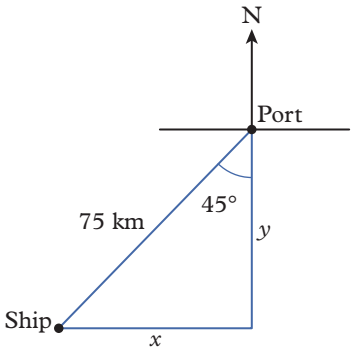
b i



x = distance travelled west
y = distance travelled north

ii 87 km west, 50 km north

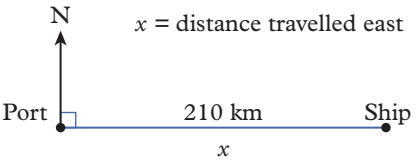
c i



x = distance travelled west
y = distance travelled south

ii 53 km west, 53 km south

d i



ii 210 km east, 0 km north

6 a i 250 km on a bearing of S30°W

ii 250 km on a bearing of 210°T

b i 100 km on a bearing of S60°E

ii 100 km on a bearing of 120°T

c i 75 km on a bearing of N45°E

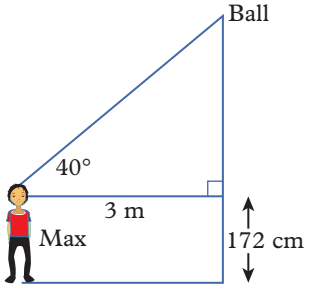
ii 75 km on a bearing of 045°T

d i 210 km on a bearing west

ii 210 km on a bearing of 270°T

7 a 10.39 m **b** 0.4 m/s

8 a



b 2.52 m (252 cm)

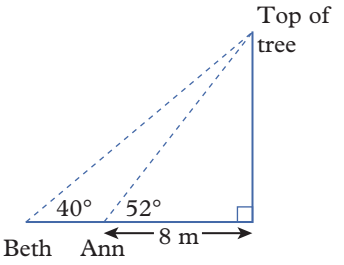
c 4.24 m (424 cm)

d The height of the ball is not significantly larger than the height of the eye level. If Max was measuring the height of a cliff, the height of the eye level may not be as important. If we do not consider the height of a person's eye in this situation, the error would be too big.

9 The angle of elevation and the angle of depression cannot be obtuse, because they are always measured from the line of sight pointing towards the other point. An obtuse angle would point backwards and therefore be measured facing the opposite direction.

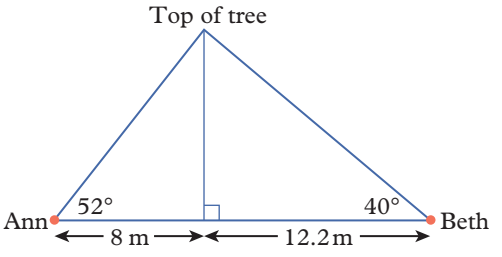
10 a 25.87 m **b** 1.17 m

11 a



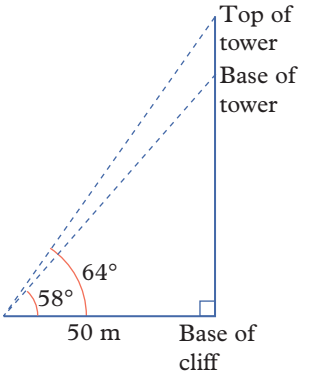
b 10.2 m **c** 12.2 m **d** 4.2 m

e i



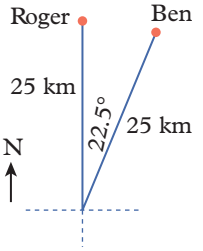
ii 20.2 m

12 a



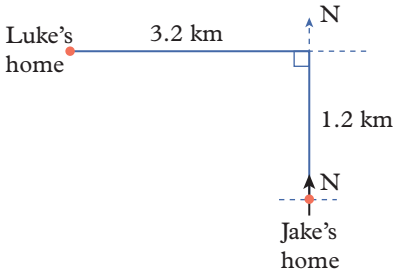
b 22.5 m

13 a



b 23.1 km **c** 9.6 km **d** 9.8 km

14 a



b 3.4 km **c** 291°T **d** 111°T

e The bearing and its back-bearing differ by 180°.

f **i** 3.4 km

ii The bearing from Jake's house to Luke's house is 291°T.

The bearing from Luke's house to Jake's house is 111°T.

g The answers are the same.

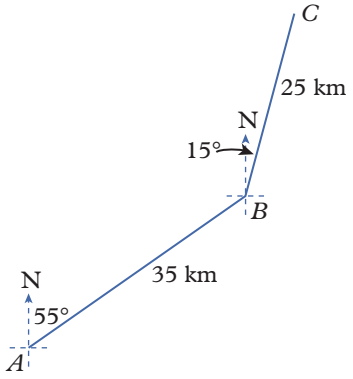
15 a 60° **b** 45°

c 40° **d** 10°

e **i** The angles in a triangle add to 180°; subtracting the acute measured angle and the right angle from 180° gives the angle of elevation. Alternatively, the acute measured angle and the angle of elevation are complementary as they are the non-right angles in a right-angled triangle.

ii The acute measured angle is vertically opposite the other non-right angle in the right-angled triangle. Alternatively, the obtuse measured angle forms a straight angle with the other non-right angle, so they are supplementary and must add to 180°. Then find the angle of depression by subtracting the acute measured angle and the right angle from 180° (or by finding the complement of the acute measured angle).

16 a



b **i** 20.1 km **ii** 24.1 km **iii** 44.2 km

c **i** 28.7 km **ii** 6.5 km **iii** 35.2 km

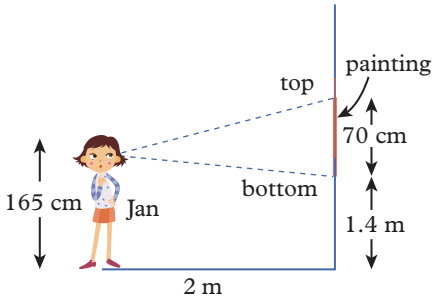
d **i** 56.5 km **ii** 039°T **iii** 219°T

e **i** 56 km, bearing of C from A is 039°T, bearing of A from C is 219°T

ii The answers are almost the same.

17 a 26.6 km **b** 062°T

18 a



b 12.7° **c** 7.1° **d** 13.3°

19 139 m

20 a 610 km

b **i** 26 km **ii** 12 km

c Yes, total distance would be 3552 km, total fuel was enough to fly 3560 km.

7D Three-dimensional problems

EX p335

1 a 21.21 cm **b** 25.98 cm **c** 35.26°

2 a diagonal of base (d):

$d = \sqrt{20^2 + 15^2}$

$d = \sqrt{625}$

$d = 25$ cm

diagonal of the box (e):

$e = \sqrt{10^2 + 25^2}$

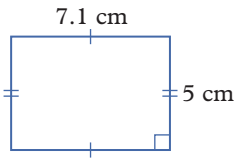
$e = \sqrt{725}$

$e = 26.93$ cm

The length of the rod is greater than the length of the diagonal of the box, so a 30 cm rod will not fit into the box.

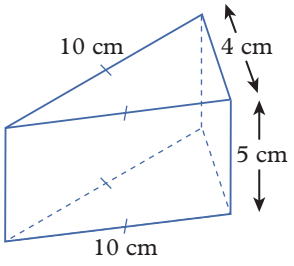
b 21.80°

3 a rectangle:

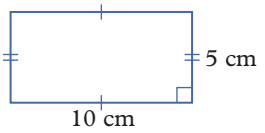


b Area of exposed face = 35.5 cm^2; area of cube face = 25 cm^2. So the area of the exposed face is 1.4 times greater than the area of one of the cube's faces.

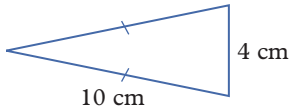
4 a



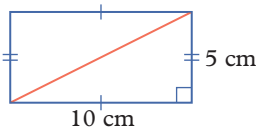
b **i**



ii



c



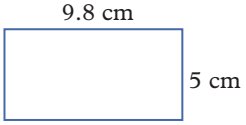
i 11.2 cm

ii 26.57°

iii The angles are complementary.
$90° - 26.57° = 63.43°$

d **i** 9.8 cm **ii** 19.6 cm^2

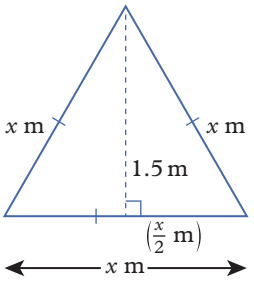
e **i**



ii Cut side: $9.8 \times 5 = 49$ cm^2
Other side: $10 \times 5 = 50$ cm^2

iii 11 cm

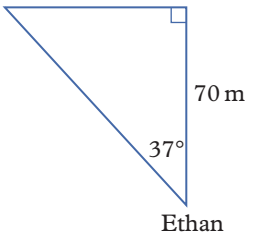
5 **a**



b 1.73 m

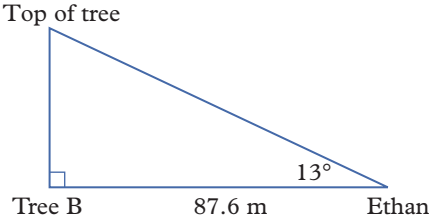
c Jack requires $2 \times 1.73 = 3.46$ m of material to cover the tent. $3.46 < 3.5$, there is sufficient material.

6 **a** Tree B Tree A



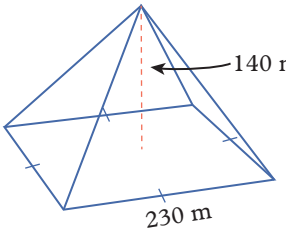
b 87.6 m

c Top of tree



d 20.2 m

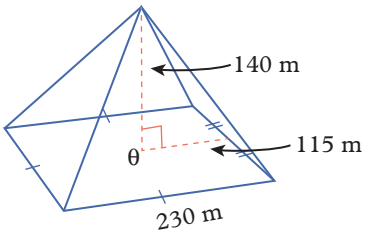
7 **a**



b 325.27 m

c 40.72°

d **i**



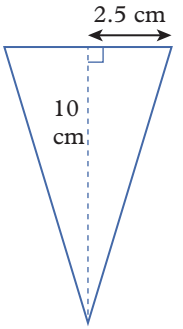
ii 50.60°

8 **a** 19.1 cm

b Internal diagonal = 22.6 cm. The spoon could fall in, because the internal diagonal of the saucepan is longer than the spoon.

9 **a** 2.5 cm

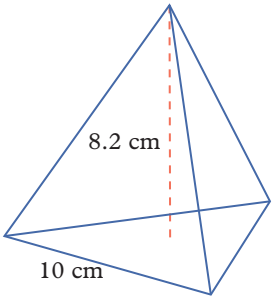
b



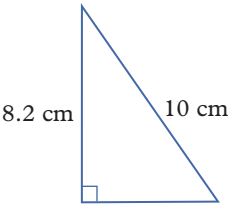
c 10.3 cm

10 9 streamers equalling the full slant length of the tree

11 **a, b**

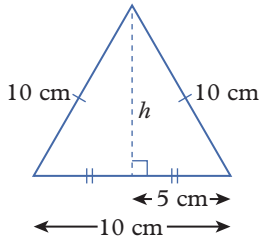


c **i**



ii 55°

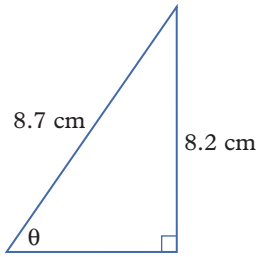
d



h = height of face

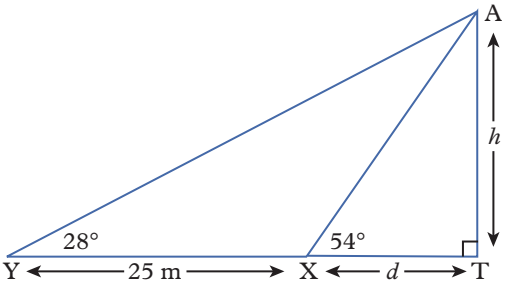
i 8.7 cm

ii



The angle (θ) between the base and a face of the tetrahedron is 71°.

12 a



b $h = d \times \tan(54°)$

c $h = (25 + d) \times \tan(28°)$

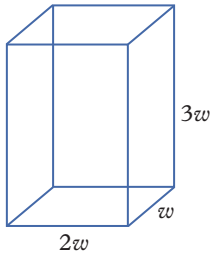
d 15.7 m **e** 21.7 m

13 a $\sqrt{w^2 + l^2}$, $\sqrt{l^2 + h^2}$, $\sqrt{w^2 + h^2}$

b $\sqrt{w^2 + l^2 + h^2}$

14 a $l = 2w$; $h = 3w$

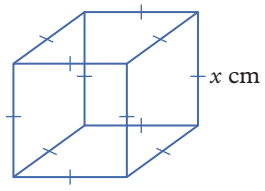
b



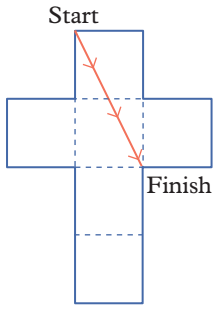
c $\sqrt{5}w$ **d** $\sqrt{14}w$ **e** 53°

15 3.6 cm

16 a



b i



ii $\sqrt{5}x$ cm

17 a 29.2 m **b** 36.8°

Checkpoint

1 a yes **b** yes **c** no **d** yes

2 a i $\sqrt{41}$ cm **ii** 6.40 cm

b i $6\sqrt{14}$ cm **ii** 22.45 cm

c i $10\sqrt{5}$ cm **ii** 22.36 cm

d i $20\sqrt{2}$ cm **ii** 28.28 cm

3 a $x = 12.0$ cm, $y = 6.0$ cm, $\theta = 26.6°$

b $x = 9.3$ cm, $y = 6.3$ cm, $\theta = 42.7°$

c $x = 17.4$ cm, $y = 14.1$ cm, $\theta = 35.9°$

4 a $x = 63.8°$, $y = 26.2°$, $z = 20.1$ cm

b $x = 50.9°$, $y = 39.1°$, $z = 24.6$ cm

5 a 095°T **b** 145°T **c** 294°T

6 a 28.2 m **b** 26.6° **c** 15.8 m

d 8.6 m

7 a 78.7 m east and 216.1 m south **b** 056°

8 a $x = 5$ cm, $y = 7.8$ cm

b $y = 9.8$ cm

9 a $x = 9.0$ cm, $z = 12.0$ cm

b $x = 63.4°$, $y = 22.4$ cm, $z = 12.6°$

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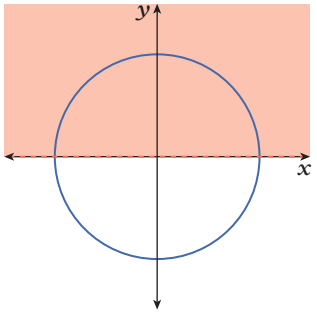
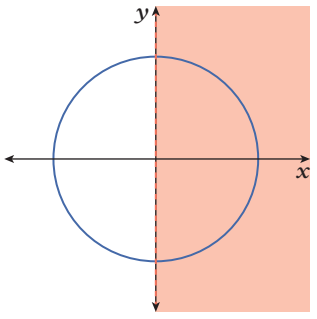
7E The unit circle

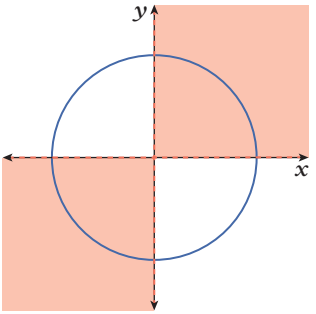
1 a



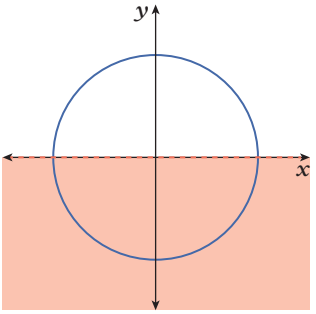
b



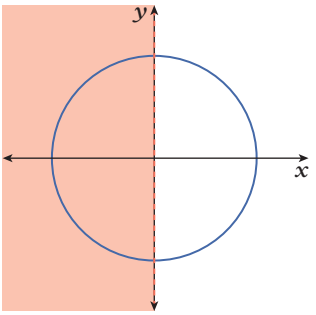
c



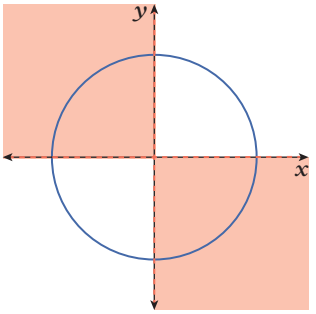
d



e



f



2 **a** quadrant 4 **b** quadrant 2 **c** quadrant 3
d quadrant 3 **e** quadrant 1 **f** quadrant 2

3 **a** positive **b** negative **c** negative
d positive **e** positive **f** negative
g negative **h** positive **i** negative
j negative

4 **a** 40° **b** 88° **c** 20°
d 10° **e** 58° **f** 45°
g 80° **h** 80° **i** 80°
j 30° **k** 15° **l** 10°
m 40° **n** 75° **o** 61°

5 **a** $-\sin(61°)$ **b** $\cos(56°)$ **c** $-\tan(85°)$
d $\sin(16°)$ **e** $-\cos(70°)$ **f** $-\cos(20°)$
g $\tan(20°)$ **h** $-\tan(2°)$ **i** $-\sin(75°)$
j $\sin(5°)$

6 **a** **i** $1 = \cos(0°)$ **ii** $0 = \sin(0°)$
b **i** $0 = \cos(90°)$ **ii** $1 = \sin(90°)$
c **i** $-1 = \cos(180°)$ **ii** $0 = \sin(180°)$
d **i** $0 = \cos(270°)$ **ii** $-1 = \sin(270°)$
e **i** $1 = \cos(360°)$ **ii** $0 = \sin(360°)$

7 **a** **i** $\tan(\theta) = \frac{y}{x}$ **ii** $\tan(\theta) = t$

b The gradient of the radius is its rise over its run, $\frac{y}{x}$. The tangent of θ is also equal to this ratio; therefore, the tangent of the angle is equal to the gradient of the radius.

c $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

d **i** $\tan(20°)$ **ii** $\tan(152°)$ **iii** $\frac{1}{\tan(218°)}$
iv $\tan(347°)$

e **i** 0 **ii** undefined **iii** 0
iv undefined **v** 0

8 **a** **i** $\sin(40°)$ **ii** $\sin(40°)$
b **i** $\cos(170°)$ **ii** $-\cos(10°)$
c **i** $\tan(80°)$ **ii** $\tan(80°)$
d **i** $\sin(270°)$ **ii** -1
e **i** $\cos(57°)$ **ii** $\cos(57°)$
f **i** $\cos(304°)$ **ii** $\cos(56°)$
g **i** $\tan(200°)$ **ii** $\tan(20°)$
h **i** $\cos(180°)$ **ii** -1

9 **a** **i** $\sin(315°)$ **ii** $-\sin(45°)$
b **i** $\cos(135°)$ **ii** $-\cos(45°)$
c **i** $\tan(225°)$ **ii** $\tan(45°)$
d **i** $\sin(45°)$ **ii** $\sin(45°)$
e **i** $\cos(212°)$ **ii** $-\cos(32°)$
f **i** $\tan(284°)$ **ii** $-\tan(76°)$
g **i** $\cos(90°)$ **ii** 0
h **i** $\sin(80°)$ **ii** $\sin(80°)$

10 **a** $\sin(158°) = \sin(22°) = -\sin(202°) = -\sin(338°)$
b $\cos(213°) = \cos(147°) = -\cos(33°) = -\cos(327°)$
c $\tan(285°) = \tan(105°) = -\tan(75°) = -\tan(255°)$

11 **a** $-0.8090... = -0.8090...$, $-0.587\,785... = -0.587\,785...$, $1.376\,38... = 1.376\,38...$

b 360°

c The reference angle, when the angle is in the fourth quadrant, is calculated using $360° - \theta°$. The same has been applied to 234° as $360° - 234° = 126°$. The RHS of each equation uses the sign of the trigonometric function in the fourth quadrant to ensure the expressions are equal.

12 **a** $(2\sin(46°), -2\cos(46°))$
b $(-2\sin(50°), -2\cos(50°))$
c $(-6\sin(40°), 6\cos(40°))$
d $(20\sin(80°), -20\cos(80°))$
e $(-15\sin(17°), 15\cos(17°))$
f $(-75\sin(10°), -75\cos(10°))$

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13 a $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

b i $\sin(\theta) = \pm\sqrt{1 - (\cos(\theta))^2}$

ii $\cos(\theta) = \pm\sqrt{1 - (\sin(\theta))^2}$

c i $\cos(50°) = 0.6428$

ii $\sin(116°) = 0.8988$

iii $\cos(177°) = -0.9986$

iv $\sin(240°) = -0.8660$

14 a $\cos(60°)$ **b** $\cos(38°)$

c $\sin(66°)$ **d** $\sin(45°)$

e i $\cos(-68°)$ **ii** $\cos(292°)$ **iii** $\cos(68°)$

f i $\sin(-261°)$ **ii** $\sin(99°)$ **iii** $\sin(81°)$

g i $\cos(-121°)$ **ii** $\cos(239°)$ **iii** $-\cos(59°)$

h i $\cos(105°)$ **ii** $\cos(105°)$ **iii** $-\cos(75°)$

i i $\sin(-392°)$ **ii** $\sin(328°)$ **iii** $-\sin(32°)$

j i $\sin(645°)$ **ii** $\sin(285°)$ **iii** $-\sin(75°)$

EX 7F Exact values

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1 a $\frac{2\pi}{3}$ **b** $\frac{\pi}{2}$ **c** $\frac{3\pi}{4}$

d 270° **e** 120° **f** 225°

g $\frac{19\pi}{18}$ **h** $\frac{13\pi}{15}$ **i** 72°

j 255° **k** $\frac{34\pi}{45}$ **l** 286°

2

	θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
a	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
b	300°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
c	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
d	315°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
e	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
f	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$

3

	θ	Reference angle θ'	sin (θ')	cos (θ')	tan (θ')	sin (θ)	cos (θ)	tan (θ)
a i	$\frac{7\pi}{6}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
ii								
iii								
b i	$\frac{3\pi}{4}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
ii								
iii								
c i	$\frac{4\pi}{3}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
ii								
iii								

4

	θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
a	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
b	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
c	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
d	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
e	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
f	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1

5 a $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **b** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **c** $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

d $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **e** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **f** $(0, -1)$

6 a $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **b** $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **c** $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

d $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **e** $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **f** $(-1, 0)$

7 a $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **b** $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **c** $(0, 1)$

d $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **e** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **f** $(1, 0)$

8 a i 2π **ii** π **iii** $\frac{\pi}{2}$

b i $0 < \theta < \frac{\pi}{2}$ **ii** $\frac{\pi}{2} < \theta < \pi$ **iii** $\pi < \theta < 2\pi$

c i Complementary angles (two angles that form a right angle) add to $\frac{\pi}{2}$ radians

ii Supplementary angles (two angles that form a straight angle) add to π radians

d i $\frac{\pi}{6}$ **ii** $\frac{\pi}{8}$ **iii** $\frac{\pi}{4}$ **iv** $\frac{2\pi}{5}$

e i $\frac{2\pi}{3}$ **ii** $\frac{\pi}{4}$ **iii** $\frac{5\pi}{6}$ **iv** $\frac{7\pi}{12}$

f i $\frac{5\pi}{3}$ **ii** $\frac{\pi}{6}$ **iii** $\frac{3\pi}{4}$ **iv** $\frac{3\pi}{5}$

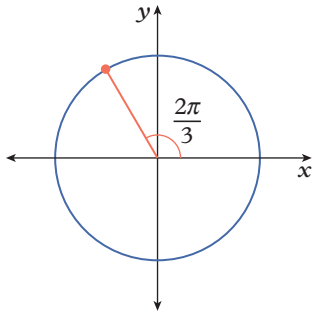
g i π **ii** 2π **iii** 4π **iv** $(n-2)\pi$

h i $\frac{\pi}{3}$ **ii** $\frac{\pi}{2}$ **iii** $\frac{2\pi}{3}$ **iv** $\frac{(n-2)\pi}{n}$

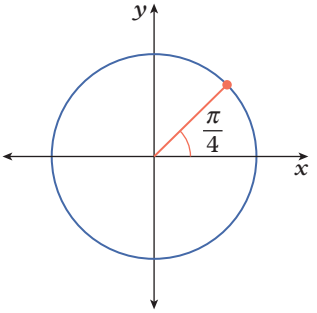
i i $\frac{7\pi}{3}$ **ii** $\frac{13\pi}{4}$ **iii** $\frac{23\pi}{6}$ **iv** $\frac{5\pi}{2}$ **v** 3π

j i $-\frac{5\pi}{3}$ **ii** $-\frac{3\pi}{4}$ **iii** $-\frac{\pi}{6}$ **iv** $-\frac{3\pi}{2}$ **v** $-\pi$

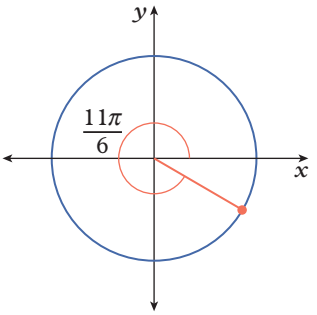
9 a i $\frac{1}{3}$



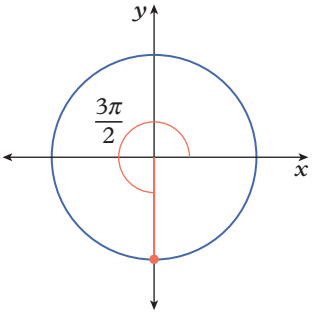
ii $\frac{1}{8}$



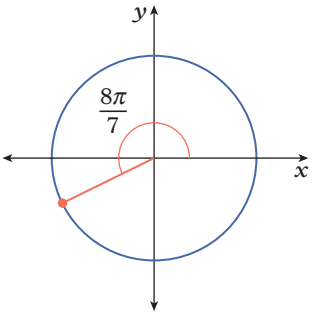
iii $\frac{11}{12}$



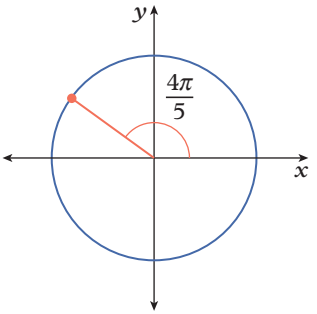
iv $\frac{3}{4}$



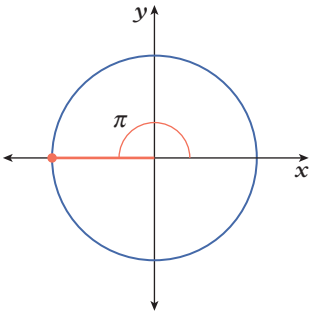
v $\frac{4}{7}$



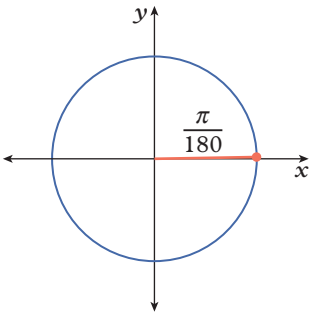
vi $\frac{2}{5}$



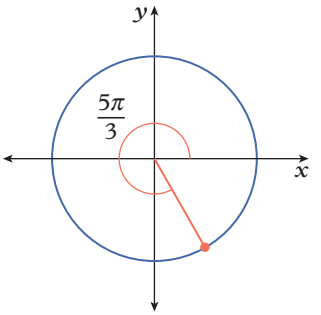
vii $\frac{1}{2}$



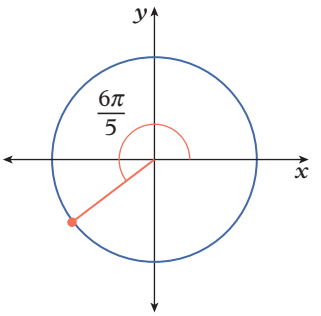
viii $\frac{1}{360}$



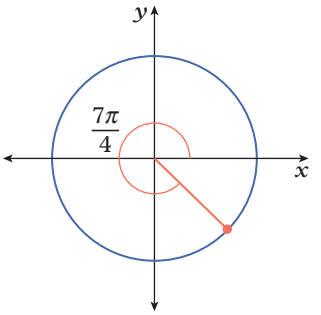
b i $\frac{5\pi}{3}$



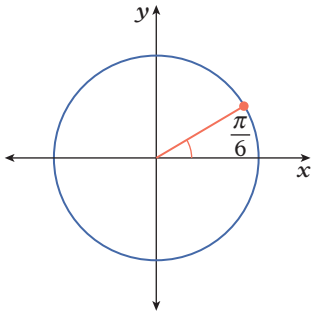
ii $\frac{6\pi}{5}$



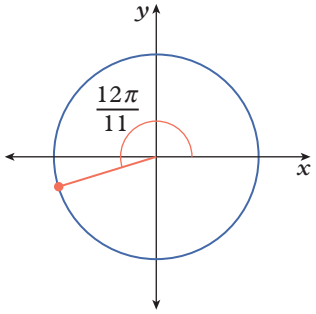
iii $\frac{7\pi}{4}$



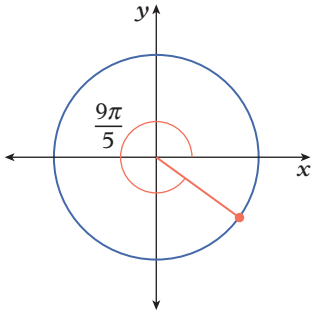
iv $\frac{\pi}{6}$



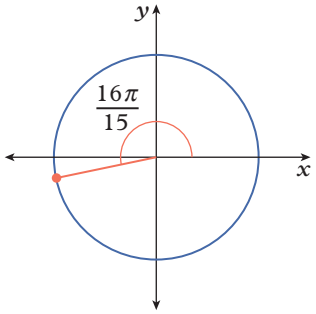
v $\frac{12\pi}{11}$



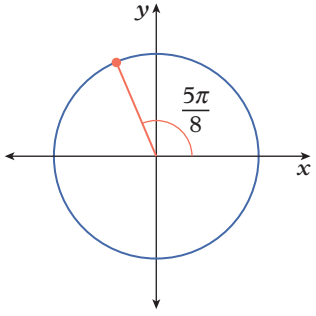
vi $\frac{9\pi}{5}$



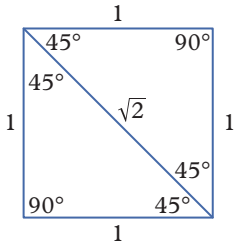
vii $\frac{16\pi}{15}$



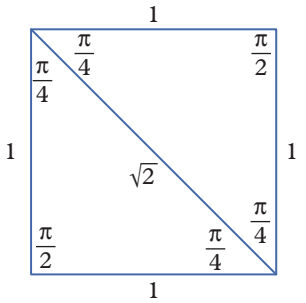
viii $\frac{5\pi}{8}$



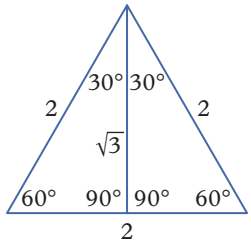
10 a, b



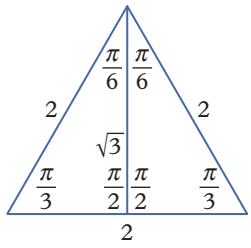
c



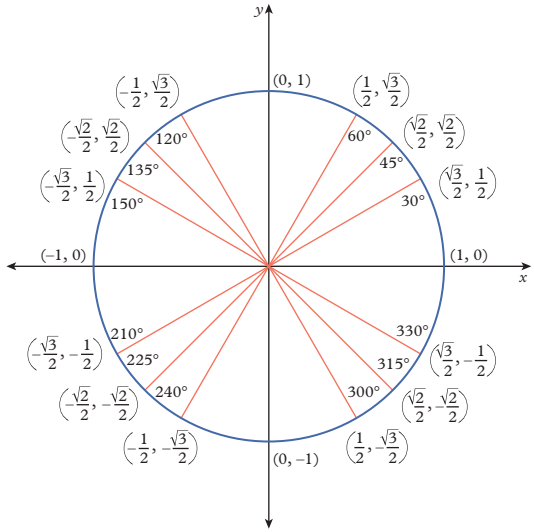
d, e



f



11 a, b



c **i** $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **ii** $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

iii $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **iv** $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

v $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **vi** $(0, 1)$

12 **a** $(-\sqrt{3}, -1)$ **b** $(\sqrt{2}, -\sqrt{2})$ **c** $(-5, 5\sqrt{3})$

d $(-3, -3\sqrt{3})$ **e** $\left(\frac{15\sqrt{2}}{\sqrt{2}}, \frac{15\sqrt{2}}{\sqrt{2}}\right)$ **f** $(-2\sqrt{3}, 2)$

13

	Radius	Angle subtending arc	Arc length
a	3 cm	$\frac{5\pi}{6}$	$\frac{5\pi}{2}$ cm
b	6 cm	$\frac{2\pi}{3}$	4π cm
c	7 cm	$\frac{3\pi}{4}$	$\frac{21\pi}{4}$ cm
d	3.2 cm	$\frac{5\pi}{12}$	$\frac{4\pi}{3}$ cm
e	4 cm	$\frac{9\pi}{8}$	$\frac{9\pi}{2}$ cm
f	2.5 cm	$\frac{4\pi}{35}$	$\frac{2\pi}{7}$ cm

14 **a** **i** $\frac{\pi}{3}$ **ii** $\frac{\pi}{3}$ **iii** $\frac{\pi}{3}$ **iv** $\frac{\pi}{6}$ **v** $\frac{\pi}{4}$
vi $\frac{\pi}{4}$ **vii** $\frac{\pi}{6}$ **viii** $\frac{\pi}{6}$ **ix** $\frac{\pi}{4}$

b The denominator of the angle and the reference angle are the same.

c **i** Q2 **ii** Q3 **iii** Q4 **iv** Q2 **v** Q3
vi Q4 **vii** Q3 **viii** Q4 **ix** Q2

d If the coefficient of π is less than one-half, the angle is in quadrant 1. If it is between one-half and one, the angle is in quadrant 2. If it is between one and three-halves, the angle is in quadrant 3. If it is between three-halves and two, the angle is in quadrant 4. If it is not between zero and two, then add or subtract two from the coefficient to determine a coterminal angle where the coefficient is between zero and two.

15 **a** $\left(2 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$ **b** $(\sqrt{2} - 3, \sqrt{2} + 2)$

c $\left(\frac{5\sqrt{3}}{2} - 2, -\frac{11}{2}\right)$ **d** $(3, 2\sqrt{3} + 3)$

16 **a** $x = 4\sqrt{3}$ cm **b** $x = \frac{23\sqrt{2}}{2}$ cm

EX p363

7G Trigonometric graphs

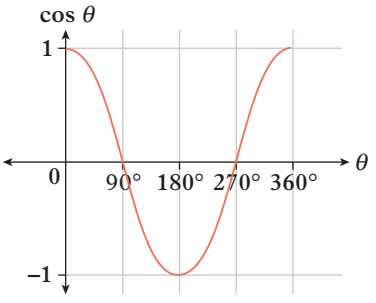
1 **a** **i** decreased and approached –1
ii increased and approached 0
iii decreased and approached 1

b

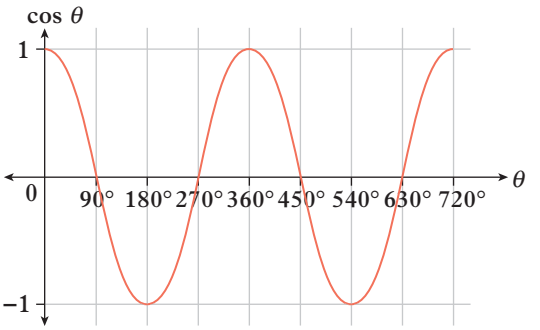
θ	0°	30°	60°	90°	120°	150°	180°
cos(θ)	1	0.9	0.5	0	–0.5	–0.9	–1

θ	210°	240°	270°	300°	330°	360°
cos(θ)	–0.9	–0.5	0	0.5	0.9	1

c



d



e 0°, 360°; the period of the cosine graph is 360°

f amplitude = 1

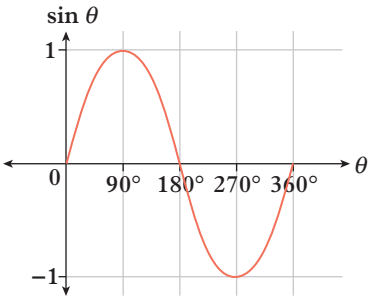
g (0, 1)

2 **a**

θ	0°	30°	60°	90°	120°	150°
sin (θ)	0	0.5	0.9	1	0.9	0.5

θ	180°	210°	240°	270°	300°	330°	360°
sin (θ)	0	–0.5	–0.9	–1	–0.9	–0.5	0

b



c 360° **d** 1 **e** (0, 0)

3 **a** **i** 1 **ii** 0 **iii** 0
iv –1

b **i** 0.8 **ii** –0.2 **iii** 0.9
iv –0.9 **v** 0.2 **vi** –0.8

4 **a** 0°, 360° **b** 180° **c** 90°, 270°
d 70°, 290° **e** 130°, 230° **f** 40°, 320°

5 **a** **i** 0 **ii** 1 **iii** –1
iv 0

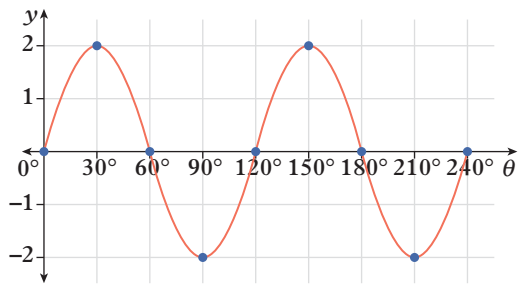
b **i** 0.6 **ii** 0.8 **iii** –0.2
iv –0.2 **v** –0.8 **vi** 0.6

6 **a** 90° **b** 270° **c** 0°, 180°, 360°
d 20°, 160° **e** 220°, 320° **f** 50°, 130°

7 **a**

3θ	0°	90°	180°	270°	360°	450°	540°	630°	720°
θ	0°	30°	60°	90°	120°	150°	180°	210°	240°
sin(3θ)	0	1	0	–1	0	1	0	–1	0
y = 2sin(3θ)	0	2	0	–2	0	2	0	–2	0

b, c



8 **a** amplitude = 3, period = 180°

b amplitude = 5, period = 120°

c amplitude = 4, period = 720°

d amplitude = 6, period = 2160°

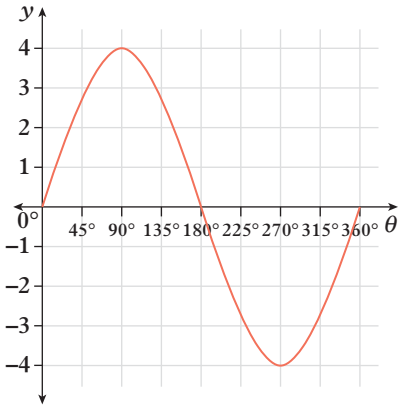
e amplitude = $\frac{3}{4}$, period = 480°

f amplitude = $\frac{5}{3}$, period = 600°

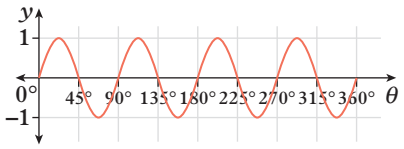
g amplitude = $\frac{7}{8}$, period = 630°

h amplitude = $\frac{5}{12}$, period = 75°

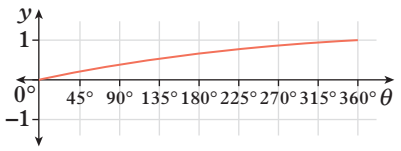
9 **a**



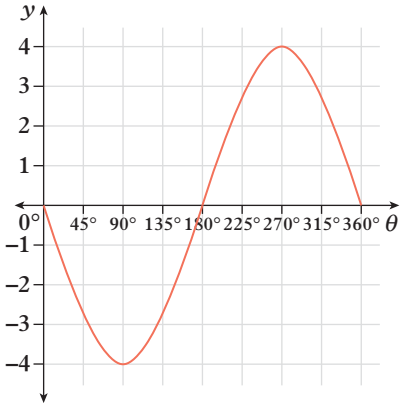
b



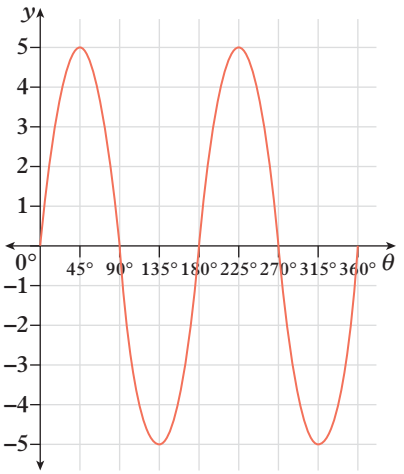
c



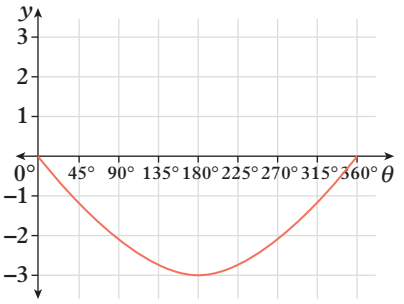
d



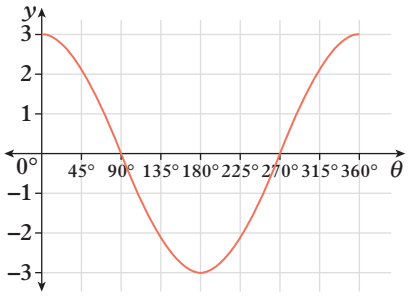
e



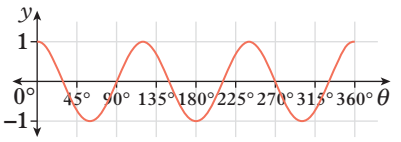
f



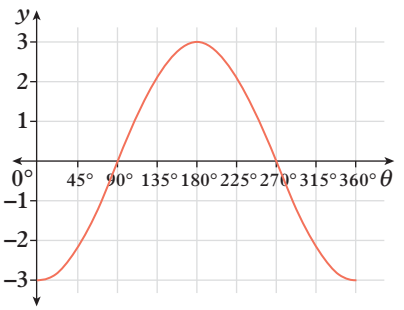
10 **a**



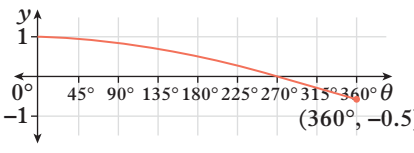
b



c



d



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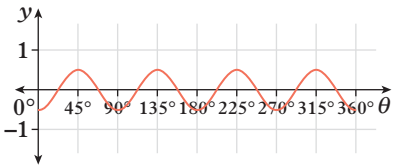

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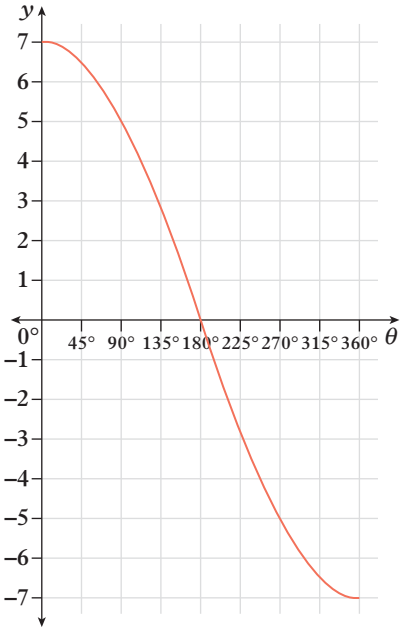


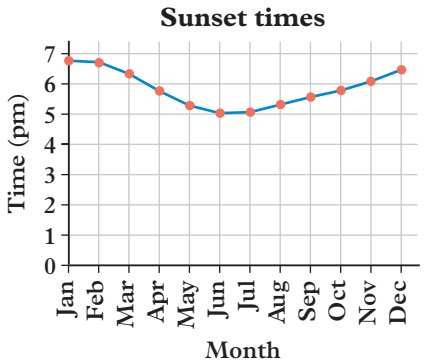
11 The sine graph starts at (0, 0) and the cosine graph starts at (0, 1).

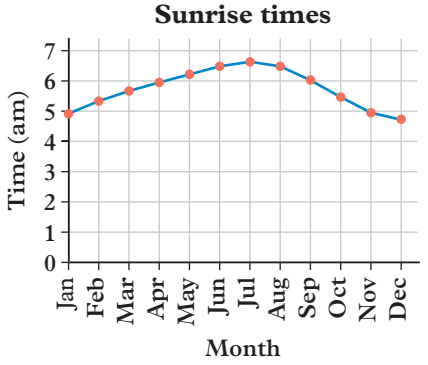
12 a A pendulum swings back and forth the same distance. Its motion is repetitive and regular, which means it is periodic.

b **i** 1.0 s **ii** 1.2 s **iii** 1.4 s

c The longer the pendulum, the longer the period of its motion.

13 a





b Answers may vary. Comments could include: The graphs are approximately cyclic or periodic. The graphs have peaks and troughs.

c Similarities to sine and cosine graphs: The graphs have a repeating pattern. The graphs are cyclic or periodic. The graphs have peaks and troughs.

Differences from sine and cosine graphs: The graphs do not start at 1 or 0. The maximum point is not 1 and the minimum point is not −1. The graphs are completely above the x-axis. The shape of the graphs resembles a cosine graph more than a sine graph. The graphs are not perfect curves.

d Yes, the values on the graphs would repeat at regular intervals. In this case the interval is 1 year. But the repeated values would be approximately the same, not exactly the same.

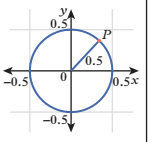
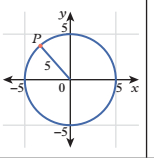
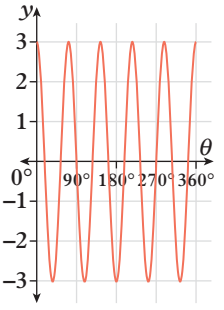
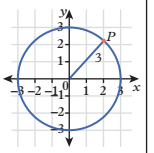
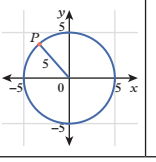
e Daylight hours are the hours which lie between the two graphs. These hours increase in summer and decrease in winter.

f Winter has the shortest days. These graphs indicate the least amount of daylight during the middle of the year. This is when winter occurs in the Southern Hemisphere.

14 a 3 **b** 11 **c** $\frac{1}{2}$ **d** b

e 4 **f** 90 **g** $\frac{3}{2}$ **h** b

15 a

	Graph	Circle	Radius	Revolution per 360°
i			10	4
ii			0.5	3
iii			3	$\frac{1}{2}$
iv			5	$\frac{1}{4}$
v			3	5
vi			5	$\frac{3}{4}$

b The amplitude is related to the length of the radius. The amplitude is equal to the length of the radius.

c The period is related to the speed the point travels around the circle. The period is the time it takes for the point to complete a revolution of the circle travelling at its constant speed. That is,

$\text{period(time)} = \dfrac{\text{distance}(360°)}{\text{speed}}$.

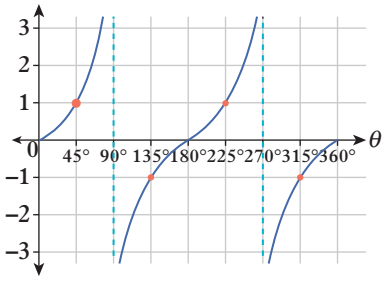
16 a

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
tan(θ)	0	0.6	1	1.7	*	−1.7	−1	−0.6	0

θ	210°	225°	240°	270°	300°	315°	330°	360°
tan(θ)	0.6	1	1.7	*	−1.7	−1	−0.6	0

*undefined

b tan θ



c i The graph is a curving line which extends from $-\infty$ to $+\infty$ with a point of inflection on the x-axis.

ii The graph line is not continuous; its cycle repeats after 180° (not 360°); it has values for which tan θ is undefined; the values approach ∞ and $-\infty$.

iii At $\theta = 90°$ the value of tan θ is undefined so a vertical asymptote is shown at this point. An asymptote is a line which the graph approaches but never reaches. The next asymptote is at $\theta = 270°$.

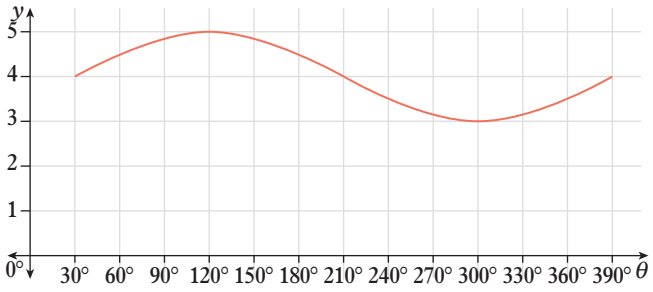
iv The graph repeats after a certain period. Period = 180°.

v The graph has no amplitude. The tan θ values approach ∞ and $-\infty$.

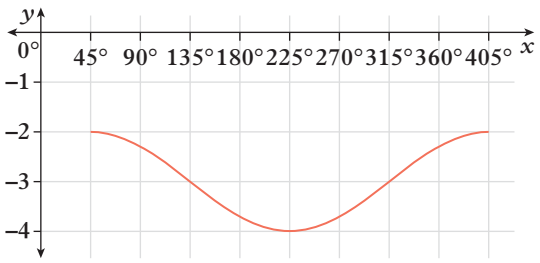
17 a

$\theta - 30$	0°	90°	180°	270°	360°
θ	30°	120°	210°	300°	390°
sin($\theta - 30°$)	0	1	0	−1	0
y = sin($\theta - 30°$) + 4	4	5	4	3	4

b



c



7H Solving trigonometric equations

EX p369

1 a $\theta = 30°, 330°$ **b** $\theta = 225°, 315°$
c $\theta = 150°, 330°$ **d** $\theta = 240°, 300°$
e $\theta = 45°, 315°$ **f** $\theta = 45°, 225°$

2 a $\theta = 114°, 246°$ **b** $\theta = 197°, 343°$
c $\theta = 42°, 222°$ **d** $\theta = 37°, 143°$
e $\theta = 53°, 307°$ **f** $\theta = 98°, 278°$

3 a 90° **b** no
c $\theta = 90°$ **d** $\theta = 90°, 450°$

4 a $\theta = 180°$ **b** $\theta = 180°, 540°$

5 a 0°, 360° **b** 180°, 360°
c $\theta = 0°, 180°, 360°$ **d** $\theta = 0°, 180°, 360°, 540°, 720°$

6 a $\theta = 90°, 270°$ **b** $\theta = 0°, 180°, 360°$
c $\theta = 270°$ **d** $\theta = 0°, 360°$

7 a no domain within the stated range
b $90° \leq \theta \leq 270°$ **c** $60° \leq \theta \leq 300°$
d $60° \leq \theta \leq 120°$ **e** $45° \leq \theta \leq 135°$
f $150° \leq \theta \leq 210°$

8 a i $210° \leq \theta \leq 330°$ **ii** $210° \leq \theta \leq 690°$
b i $120° \leq \theta \leq 240°$ **ii** $120° \leq \theta \leq 600°$
c i $225° \leq \theta \leq 315°$ **ii** $225° \leq \theta \leq 675°$
d i $135° \leq \theta \leq 225°$ **ii** $135° \leq \theta \leq 585°$

9 a–c When each of these equations is simplified the sine or cosine value is greater than 1. Sine and cosine can only take values in the range −1 to 1.

10 a yes **b** yes **c** yes **d** yes
e No, when the equation is simplified the cosine value is less than −1.
f yes
g No, when the equation is simplified the cosine value is greater than 1.
h yes
i No, when the expression is simplified the sine value is less than −1.

11 a 1 **b** 0 **c** 1
d 1 **e** 1 **f** 1

12 a $\theta = 22.5°, 67.5°$ **b** $\theta = 45°, 135°$
c $\theta = 45°, 105°, 165°$

13 a $\theta = 90°$ **b** $\theta = 0°$ **c** $\theta = 135°, 225°$
d $\theta = 240°$ **e** $\theta = 225°$ **f** $\theta = 150°$

14 **a** $\theta = 420°, 660°$

b $\theta = 135°, 945°$

c $\theta = 300°, 750°$

15 Solving $\sin(\theta) = 1$ and $\cos(\theta) = 1$ involves solutions that are multiples of 90°, whereas solving $\tan(\theta) = 1$ involves 45°. That is, the sine and cosine equations will only have one solution for $0° \leq \theta \leq 180°$, while the tangent equation will have two.

16 **a** 180° **b** 360° **c** 240°

d 540° **e** 360° **f** 480°

17 **a** $\theta = -300°, -240°$ **b** $\theta = -225°, -135°$

c $\theta = -300°, -120°$ **d** $\theta = -90°$

e $\theta = -180°$ **f** $\theta = 0°, -180°, -360°$

18 **a** $x = 0, \pi, 2\pi$ **b** $x = \frac{\pi}{6}, \frac{11\pi}{6}$ **c** $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

d $x = \frac{\pi}{4}, \frac{3\pi}{4}$ **e** $x = \pi$ **f** $x = \frac{\pi}{6}, \frac{7\pi}{6}$

19 **a** $\frac{\sin(\theta)}{\cos(\theta)} = \frac{\cos(\theta)}{\cos(\theta)}$

$\tan(\theta) = 1$

b **i** $\theta = 45°, 225°$ **ii** $\theta = 60°, 240°$

iii $\theta = 150°, 330°$ **iv** $\theta = 30°, 210°$

20 **a** $\theta = 15°, 75°, 195°, 255°$

b $\theta = 360°$

c $\theta = -5\pi, -3\pi, 3\pi, 5\pi, 11\pi$

d $\theta = -5\pi, -3\pi, 3\pi, 5\pi, 11\pi$

21 **a** $x = -\frac{1}{2}, 2$ **b** $\theta = 210°, 330°$

CHAPTER 7 review

Multiple choice

1	C	**2**	B	**3**	A	**4**	B	**5**	A
6	C	**7**	E	**8**	A	**9**	A	**10**	D
11	E	**12**	A	**13**	B				

Short answer

1 **a** $\frac{5\sqrt{3}}{2}$ cm **b** 4.3 cm

2 length = 13.50 cm, width = 8.21 cm

3 **a** 19°

b **i** $\frac{6.2}{18.5} = \frac{62}{185}$ or 62 : 185 **ii** 33.51%

c 19.51 km

4 40 m

5 **a** 16.30 cm **b** 19.72 cm **c** 43.30 cm

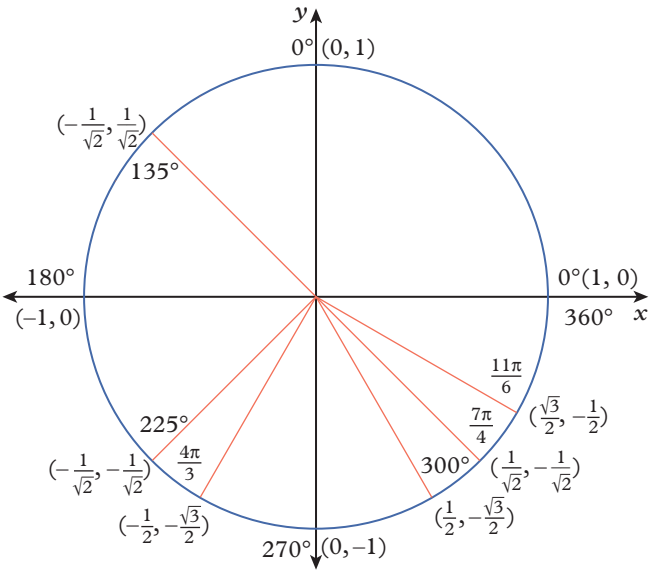
6 **a** 23° **b** 40° **c** 55°

d 15.20 m

7 **a** **i** $\frac{2\pi}{3}$ **ii** $\frac{7\pi}{4}$ **iii** $\frac{13\pi}{18}$

b **i** 540° **ii** 480° **iii** 202.5°

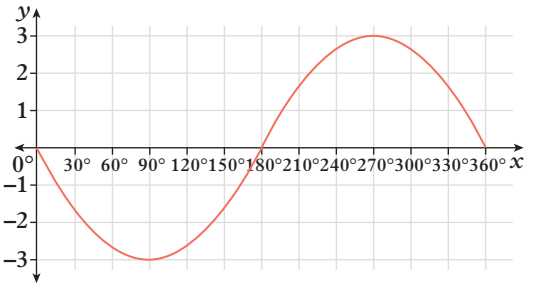
8



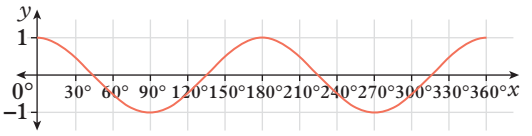
a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{1}{\sqrt{2}}$

d $\frac{1}{\sqrt{2}}$ **e** $-\frac{\sqrt{3}}{2}$ **f** $\frac{\sqrt{3}}{2}$

9 **a**

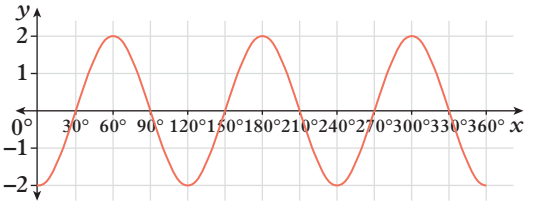


b



c

d



10 **a** $\theta = 180°$ **b** $\theta = 0°, 180°, 360°$

c $\theta = 30°, 330°$ **d** $\theta = 210°, 330°$

e $\theta = 135°, 225°$ **f** $\theta = 240°, 300°$

Analysis

1 a i $2R + G = 480$ mm; this is below the required range, so the staircase does not fall within building regulation guidelines.

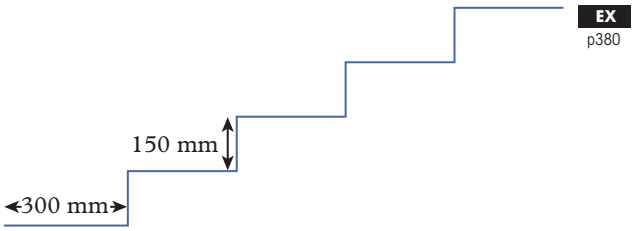
ii 25°

b i $2R + G = 735$ mm; this is above the required range, so the staircase does not fall within building regulation guidelines.

ii 28°

c i 115 mm to 172.5 mm **ii** 18° to 26°

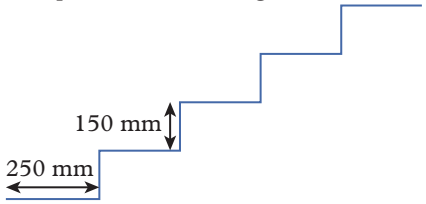
d



i 1500 mm **ii** 600 mm

iii 1615.5 mm **iv** 27°

e One possible solution is given.



115 mm ≤ 150 mm ≤ 190 mm

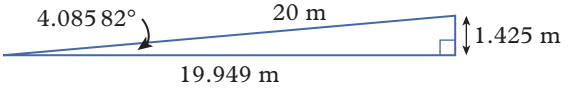
250 mm ≤ 250 mm ≤ 355 mm

$2R + G = 2 \times 150 + 250 = 300 + 250 = 550$ mm

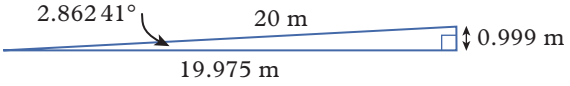
550 mm ≤ 550 mm ≤ 700 mm

f i

Internal ramp 1:14



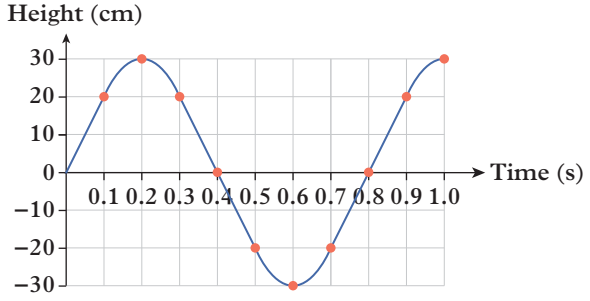
External ramp 1:20



ii Internal: rise = 1.425 m, going = 19.949 m

External: rise = 0.999 m, going = 19.975 m

2 a



b sine curve

c 0.8 seconds

d Starting position was 0 cm. It takes 0.1 seconds for the tag to go from 20 cm to 0 cm in height.

e 30 cm **f** 188.5 cm **g** 2.4 m/s

h After 10 seconds, the wheel will complete 12.5 rotations. If the tag's starting position was at 0 cm, then the tag will be at the same height of 0 cm but on the opposite side of the wheel.

CHAPTER 8 Networks

8A Introduction to networks

EX p380

1 a i 4 **ii** 3

b i 4 **ii** 4

c i 4 **ii** 2

d i 1 **ii** 3

e i 4 **ii** 6

f i 5 **ii** 4

g i 2 **ii** 4

h i 9 **ii** 12

2 a

Vertex	A	B	C	D
Degree	1	2	2	1

b

Vertex	A	B	C	D
Degree	2	2	2	2

c

Vertex	A	B	C	D
Degree	1	1	1	1

d

Vertex	A
Degree	6

e

Vertex	A	B	C	D
Degree	3	3	3	3

f

Vertex	A	B	C	D	E
Degree	4	1	1	1	1

g

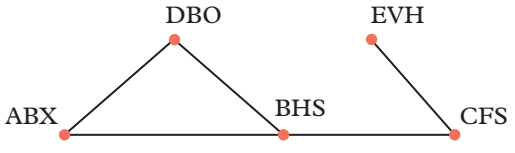
Vertex	A	B
Degree	3	5

h

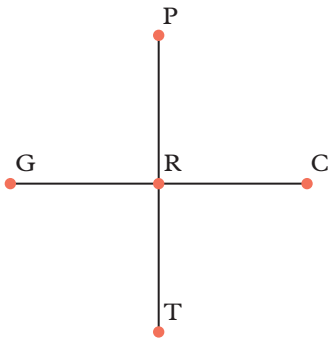
Vertex	A	B	C	D	E	F	G	H	I
Degree	2	2	6	2	4	2	2	2	2

3 c

4



5 a



b 4

6 a Sample answer: A, f, B, g, C, h, D (other walks from A to D exist)

b Sample answer: A, f, B, g, D (other walks from A to D exist)

c There is no walk from A to D.

7 a

Person	A	B	C	D
Number of hands shook	3	3	2	2

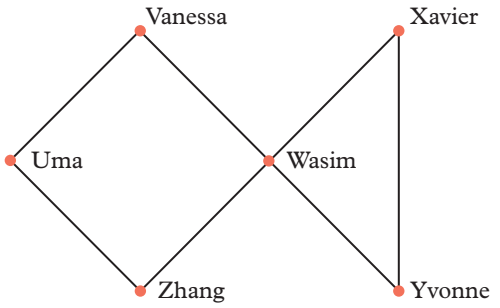
b 5 (the number of edges)

8 10 (twice the number of edges)

9 $4 \times 5 \div 2 = 10$

10 Such a graph would have $\frac{3+2+2}{2} = 3.5$ edges, which is impossible.

11 a



b

Vertex	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Degree	2	2	4	2	2	2

c Wasim or Uma

d Uma

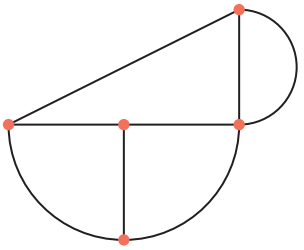
12 a Yes

b Corner C

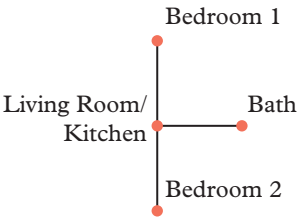
13 There are missing directions. East 42nd St does not connect to corner A.

14 Remove the edge between D and E.

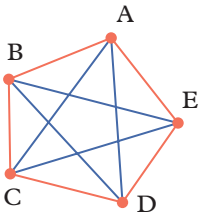
15



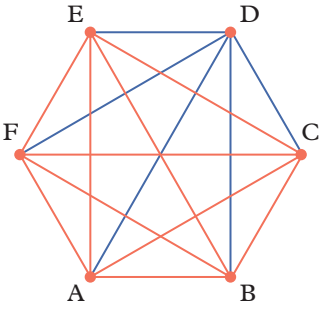
16



17 Sample answer:



18 a Sample answer:



b Sample answer: ACE.

c Question **17** shows you need to invite more than 5 people, since it contains no red or blue triangle. No matter how you answer question **18a** there is always a red or blue triangle, so you need 6 dinner party guests.

8B Planar graphs

EX p386

1 a planar **b** planar

c planar **d** planar

2 a 2 **b** 4

c $E = F + V - 2$
$= 4 + 2 - 2$
$= 4$

3 a $V = 4, F = 2, E = 4; 4 + 2 = 4 + 2$

b $V = 1, F = 4, E = 3; 1 + 4 = 3 + 2$

c $V = 4, F = 3, E = 5; 4 + 3 = 5 + 2$

4 a The face inside the square is bordered by 4 edges so has degree 4. Similarly the outside face has degree 4. Hence the sum of face degrees is $4 + 4 = 8 = 2E$.

b The inner circle and outer face have degree 1. The other two faces have degree 2. The sum of face degrees is $1 + 1 + 2 + 2 = 6 = 2E$.

c The face between A and B has degree 2. The other two faces have degree 4. The sum of face degrees is $4 + 4 + 2 = 10 = 2E$.

5 Graph **a** is simple.

6 4

7 Because it is not connected

8 A planar graph with 6 vertices has at most $3V - 6 = 12$ edges. This graph has 15 edges, so it cannot be planar.

9 **a** triangular face: 3, circular face: 1, outside face: 4

b $3 + 1 + 4 = 8$

c The sum of face degrees is twice the number of edges ($E = 4$).

10 **a** $E = 9$, $V = 6$

b yes (no loops or parallel edges)

c yes

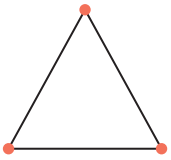
d $E = 9$ is less than $3V - 6 = 12$

11 Yes (see **8B** question **1a**)

12 No (see **8B** question **10**)

13 **a** If the graph is planar, then it has at most $3V - 6 = 3$ edges.

b



14 4

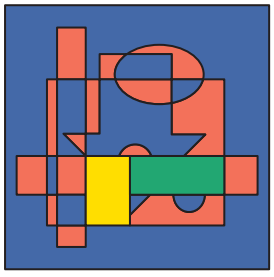
15 No. If such a graph did exist, then $E = F + V - 2 = 5 + 4 - 2 = 7$. This exceeds the limit of $3V - 6 = 6$ for planar graphs.

16 **a** a loop

b a pair of parallel edges

c 3

17 Sample answer:



EX p391 8C Platonic solids

1 6 edges

2 **a** true

b false

3 **a** tetrahedron, octahedron, icosahedron

b cube

c dodecahedron

4 **a** tetrahedron, cube, dodecahedron

b octahedron

c icosahedron

5 d4: tetrahedron, d6: cube, d8: octahedron, d12: dodecahedron, d20: icosahedron

6 tetrahedron, octahedron or icosahedron

7 6

8 5

9 12 edges

10 30 edges

11 30 edges

12

Platonic solid	Tetra-hedron	Cube	Octa-hedron	Dodeca-hedron	Icosahe-dron
Vertices	4	8	6	20	12
Faces	4	6	8	12	20
Edges	6	12	12	30	30

13 **a** 3, 3, 3, 3, 3, 3, 3, 3

b 4, 4, 4, 4, 4, 4

c $8 \times 3 = 6 \times 4 = 24$

14 **a** 120 **b** 60 **c** 24

15 **a** 120 **b** 60 **c** 30

16 **a**

	$n = 3$	$n = 4$	$n = 5$
$m = 3$	$\frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$	$\frac{1}{4} + \frac{1}{3} - \frac{1}{2} = \frac{1}{12}$	$\frac{1}{5} + \frac{1}{3} - \frac{1}{2} = \frac{1}{30}$
$m = 4$	$\frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$	$\frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0$	$\frac{1}{5} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{20}$
$m = 5$	$\frac{1}{3} + \frac{1}{5} - \frac{1}{2} = \frac{1}{30}$	$\frac{1}{4} + \frac{1}{5} - \frac{1}{2} = -\frac{1}{20}$	$\frac{1}{5} + \frac{1}{5} - \frac{1}{2} = -\frac{1}{10}$

b The five entries with positive values of $\frac{1}{E}$ correspond to the five platonic solids.

c No, the number of edges must be zero or greater.

d Yes, provided this table contains all the positive entries (see question **17**).

17 Suppose we want $\frac{1}{n} + \frac{1}{m} - \frac{1}{2} > 0$. If m is large, then n must be small. Concretely, if $m \geq 6$ then $\frac{1}{n} > \frac{1}{2} - \frac{1}{m} \geq \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$, so $3 > n$. This means $n = 2$ or $n = 1$. By 8B question **16**, we have parallel edges when $n = 2$ and loops when $n = 1$. Both cases are excluded because the graph is simple.

Checkpoint

1

Vertex	A	B	C	D	E
Degree	3	4	4	3	4

2 yes

3 A network with vertex degrees 4, 3 and 2 has $\frac{(4 + 3 + 2)}{2} = 4.5$ edges, and we must have a whole number of edges.

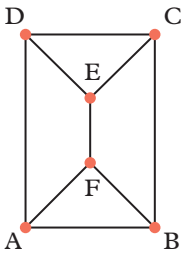
4 yes (no parallel edges or loops)

5 yes (can be redrawn without edges crossing)

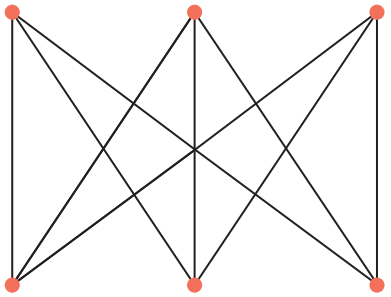
6 **a** 15

b No, a planar network with 6 vertices can have at most $3V - 6 = 12$ edges, so this network cannot be planar.

7 **a** Sample answer:



b Sample answer:



8 6

9 **a** 8 **b** 2

c More likely to fail

10 **a** $2 \times 5 + 10 \times 3 = 40$

b 20

c 10

EX p397 8D Eulerian trails and circuits

1 **a** A and C **b** No

2 **a** trail

b invalid walk

c invalid walk

d walk

3 **a** A, B, C, D and A, B, C, D

b No, because two different walks can have the same vertex sequence.

4 Closed walk

5 Sample answer: B, b, C, c, B

6 Sample answer: B, g, C, h, D, i, A, f, B

7 This network is not Eulerian because it has 10 vertices of odd degree.

8 This network is semi-Eulerian because it has 2 vertices of odd degree.

9 Sample answer: A, b, B, a, A, c, B, d, B

10 This network is Eulerian because no vertices have odd degree.

11 **a**

Vertex	N	K	S	L
Degree	3	5	3	3

b 4

c not Eulerian

d No, because the graph has too many vertices with odd degree.

12 **a** After adding one bridge, there will still be two vertices of odd degree.

b No, since the N vertex (and at least one other vertex) will still have odd degree.

c Yes, adding new bridges from N to S and N to L will make the graph semi-Eulerian, so there is a trail that crosses every bridge exactly once.

13 **a** semi-Eulerian **b** Eulerian

c semi-Eulerian

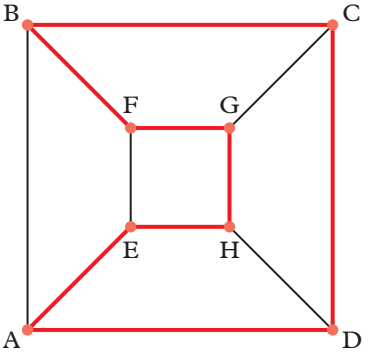
14 octahedron

15 **a** Sample answer:

b Not possible, since there are 4 intersections with odd degrees.

16 An Eulerian circuit exists because there are no vertices with odd degrees.

17 **a** Sample answer:



b There is no Hamiltonian circuit because to return to the starting vertex the central edge must be crossed twice.

EX p401 8E Applications of networks

1 **a** D **b** H **c** C **d** E

2 **a** H **b** D **c** E **d** C

3

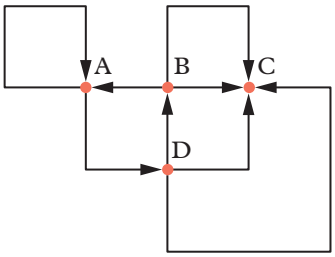
	Kin Group							
Father	A	B	C	D	E	F	G	H
Children	F	G	H	E	D	A	B	C

4 **a** A **b** G **c** G **d** A

5

	Kin group							
You	A	B	C	D	E	F	G	H
Cousins	A or G	B or H	C or E	D or F	E or C	F or D	G or A	H or B

6 Sample answer:



7 **a** 10 **b** 9

8 **a** 13

b Passing through Pride is shorter (12 stops)

c 15

9 **a** C **b** G **c** E **d** A

10 yes

11 no

12 also A

13 via Sydney and Melbourne ($1384 total)

14 **a**

	Kin group			
Mother	A	B	C	D
Children	B	D	A	C

b

	Kin group			
Father	A	B	C	D
Children	C	A	D	B

c dodecahedron

CHAPTER 8 review

Multiple choice

1 B **2** C **3** E **4** D **5** C

6 D **7** E **8** D **9** E **10** B

Short answer

1

Vertex	A	B	C	D	E	F
Degree	4	3	2	4	3	2

2 5

3 $E = F + V - 2 = 6 + 5 - 2 = 9$

4 48 ($6 \times 4 + 8 \times 3$)

5 All faces must be the same for a shape to be a platonic solid.

6 Because the number of edges would be $5 \times 3 \div 2 = 7.5$.

7

Solid	**Faces**	**Vertices**	**Edges**
Tetrahedron	4	4	6
Octahedron	8	6	12
Cube	6	8	12
Icosahedron	20	12	30
Dodecahedron	12	20	30

8 **a** 3

b 4

c Yes, because there are exactly 2 vertices with odd degrees.

9 Because there are 13 vertices of odd degree.

10

	Kin Group							
You	A	B	C	D	E	F	G	H
Mother	B	C	D	A	H	E	F	G
Grandmother	C	D	A	B	G	H	E	F
Great-grandmother	D	A	B	C	F	G	H	E
Great-great-grandmother	A	B	C	D	E	F	G	H

Analysis

1 **a**

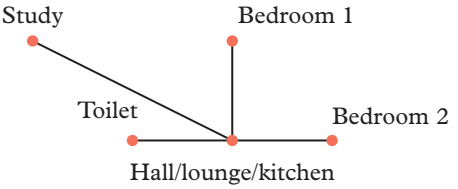
Shape	**Truncated cube**	**Truncated octahedron**	**Truncated icosahedron**
Face degrees	3 or 8	4 or 6	5 or 6

b **i** 4 **ii** 4 **iii** 36

iv 18 **v** 12

c **iii** 72 **iv** 36 **v** 24

2 **a** Sample answer (note: there is no door between the hall, lounge and kitchen):



b No, because there are 4 vertices with odd degree.

EX p411

Algorithmic Thinking 2 Measurement and space

Task 1

1 Example 1 is printing −3, −4

Example 2 is printing 2, 7, 10, 35

Task 2

1 Sample answer:

Create list of meeting spots' *x*-coordinates:
MX = {12,10,8} (so MX[1]=12, MX[2]=10, MX[3]=8)

Create list of meeting spots' *y*-coordinates:
MY = {6,10,6} (so MY[1]=6, MY[2]=10, MY[3]=6)

Create list of peoples' home *x*-coordinates:
PX = {2,1,15} (so PX[1]= 2, PX[2]=1, PX[3]=15)

Create list of peoples' home *y*-coordinates:
PY = {14,1,10} (so PY[1]=14, PY[2]=1, PY[3]=10)

Create an empty list DList

Set the variable k to 1

Repeat (length of list MX) times

  Set the variable i to 1

  Set difference to 0

  Repeat (length of list PX) times

    Set difference to difference + absolute value of (MX[k] - PX[i])

Set difference to difference + absolute value of (MY[k] - PY[i])
Add 1 to i
Add difference to DList
Add 1 to k
Print DList
(Note: DList[1] = walking distance to H,
DList[2] = walking distance to P,
DList[3] = walking distance to T)

2 Teacher to check.

3 Sample answer:
Create list of meeting spots' *x*-coordinates:
$M_X = \{12,10,8\}$ (so $M_X[1]=12$, $M_X[2]=10$, $M_X[3]=8$)
Create list of meeting spots' *y*-coordinates:
$M_Y = \{6,10,6\}$ (so $M_Y[1]=6$, $M_Y[2]=10$, $M_Y[3]=6$)
Create list of peoples' home *x*-coordinates:
$P_X = \{2,1,15\}$ (so $P_X[1]= 2$, $P_X[2]=1$, $P_X[3]=15$)
Create list of peoples' home *y*-coordinates:
$P_Y = \{14,1,10\}$ (so $P_Y[1]=14$, $P_Y[2]=1$, $P_Y[3]=10$)
Create an empty list SList
Set the variable k to 1
Repeat (length of list M_X) times
Set the variable i to 1
Repeat (length of list P_X) times
Set distance to $\sqrt{(M_x[k] - P_x[i])^2 + (M_y[k] - P_y[i])^2}$
Add 1 to i
Add distance to SList
Add 1 to k
Print SList
(Note: SList[1]=straight line distance first home to H,
SList[2]=straight line distance first home to P,
SList[3]=straight line distance first home to T,
SList[4]=straight line distance second home to H,
SList[5]=straight line distance second home to P,
SList[6]=straight line distance second home to T,
SList[7]=straight line distance third home to H,
SList[8]=straight line distance third home to P,
SList[9]=straight line distance third home to T)

4 Teacher to check.

5 Sample answer: Adding person D will change the walking algorithm since more information will be needed by the algorithm to know the layout of the street corners that people need to walk around. It does not affect the as-the-drone-flies algorithm that uses Pythagoras' theorem as the drones can fly over the houses.

CHAPTER 9 Statistics

EX p416

9A Five-number summary and interquartile range

1 **a** 16 **b** 45 **c** 87 **d** 67

2 **a** 3, 3, 5, 7, 9 **b** 7, 12, 19, 46, 78
c 1, 3, 5, 7, 9 **d** 2, 7, 11, 19, 32

3 **a** 2, 4, 7, 9, 12
b 4, 13, 25.5, 46, 55
c 16, 20.5, 42, 72.5, 81
d 120, 177.5, 452.5, 755, 843
e 18, 19.5, 33, 42, 64
f 3, 8, 13.5, 19, 32

4 **a** 10 **b** 9 **c** 10 **d** 19

5 **a** 32 **b** 9 **c** 56.5 **d** 36
e 9.5 **f** 253

6 Range = 51; IQR = 23. The difference between the least number of cupcakes sold in the month and the most number of cupcakes sold in the month is 51. On half the days, the number of cupcakes sold falls within a range of 23 (from 56 to 79 cupcakes sold).

7 **a** Both sets have a range of 43.
b IQR of set A = 26, IQR of set B = 10
c because they have the same range

8 **a** 6
b minimum = 0, $Q_1 = 2$, Q_2 (median) = 4, $Q_3 = 5$, maximum = 6
c 3

9 **a** If the IQR is small, 50% of scores are grouped closely around the median.
b If the IQR is small and the range is large, 50% of the scores are grouped closely around the median, while the 25% of scores lying below the lower quartile and the 25% of scores above the upper quartile are spread out.
c If the IQR is large, 50% of the scores are spread widely around the median.
d If the IQR is similar to the range, the range of the 25% of scores lying below the lower quartile plus the range of the 25% of scores above the upper quartile is the same as the range of the middle 50% of scores.

10 **a** range = \$240, IQR = \$45
b because there is an outlier
c The IQR represents the middle 50% of scores; those scores between Q_1 and Q_3. If the spread is even, the remaining 50% of scores will have a spread of the same size as the IQR.
d The IQR indicates a region within which the middle 50% of scores lie and the outlier will fall outside this range.

11 **a** **i** range = 17, IQR = 8.5
ii The data are spread fairly uniformly throughout the range.
b **i** range = 72, IQR = 30
ii The scores are packed more closely in the IQR. There appears to be an outlier of 11 in the lower quarter of the scores.
c **i** range = 73, IQR = 11
ii Scores are packed more closely in the IQR. There appears to be an outlier of 191 in the upper quarter of the scores.

12 a class A **i** 171.5 **ii** 27 **iii** 12
class B **i** 172.5 **ii** 48 **iii** 12.5

b The median and IQR of heights in the two classes is quite similar, with class B having a slightly larger median; however, the range of heights in class B is greater. Class B has the tallest and shortest members of the group, indicating it is a more diverse group.

13 a 50, 60, 70.5, 79, 83 **b** IQR = 19, range = 33

c 9, 55, 68, 74, 83 **d** IQR = 19, range = 74

e i The median decreased slightly from 70.5 to 68.

ii The range increased significantly from 33 to 74.

iii The interquartile range remained at 19.

f Since outliers and extreme values can only be at the ends of a list of data, they are not close enough to the position of the quartiles to impact the interquartile range.

14 a 13 **b** 26

15 a Using quintiles instead of quartiles provides a more detailed breakdown of the income distribution and gives a clearer view of the disparities between low and high income earners.

b

Quintile	2015–2016 (%)	2017–2018 (%)	2019–2020 (%)
Lower (0–19%)	3.2	3.1	3.1
Second (20–39%)	8.6	8.6	8.6
Third (40–59%)	16.0	16.5	16.7
Fourth (60–79%)	25.4	25.6	25.8
Highest (80–100%)	46.8	46.2	45.8

c No, an even distribution of income would have 20 as every entry in the table, as each 20% of the population would earn 20% of income.

d 14.625 : 1, 14.903 : 1, 14.774 : 1

e There is a decrease after 2013–2014 and then the ratio remains relatively steady at approximately 14.75 : 1.

16 a 16 **b** $\frac{47}{12}$ **c** $\frac{337}{352}$ **d** $\frac{13\,267}{9360}$

EX p421

9B Boxplots

1 a i 1, 2, 5, 7, 10 **ii** IQR = 5

b i 13, 20, 32, 40, 45 **ii** IQR = 20

c i 3, 4, 6, 8, 9 **ii** IQR = 4

d i 10, 16, 19, 24, 35 **ii** IQR = 8

e i 20, 45, 70, 75, 85 **ii** IQR = 30

f i 31, 33, 35, 37, 39 **ii** IQR = 4

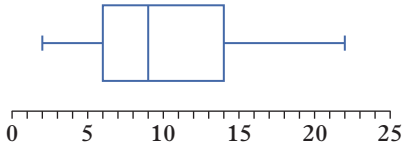
2 a 2, 3, 4, 4, 6, 7, 7, 8, 8, 9, 10, 10, 11, 13, 14, 17, 19, 19, 22

b $Q_1 = 6$, median = 9, $Q_3 = 14$

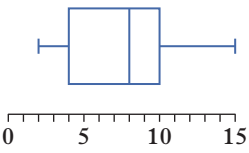
c lower fence = –6, upper fence = 26

d no outliers

e



3 a



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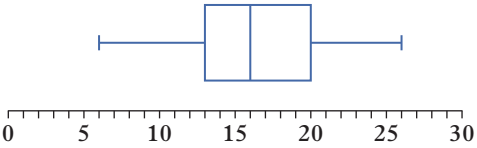

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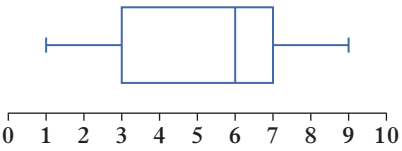

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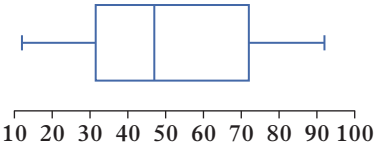

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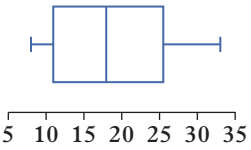

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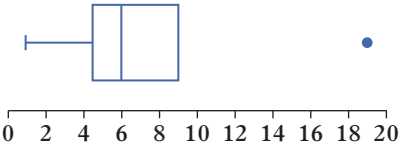

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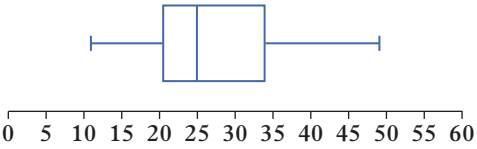

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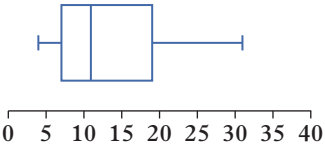


4 a i 3, 3, 7, 10, 12 **ii** 7

b i 18, 30, 33, 40, 40 **ii** 10

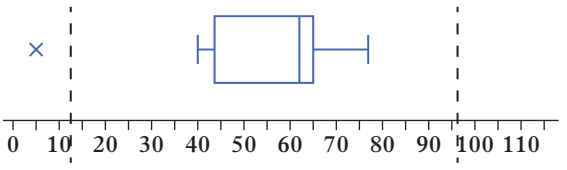
c i –20, –20, –15, –12, –12 **ii** 8

d i 40, 41, 42, 46.5, 60 **ii** 5.5

e i 18, 20, 21.5, 25, 35 **ii** 5

f i 50, 60, 60, 60, 70 **ii** 0

5 a



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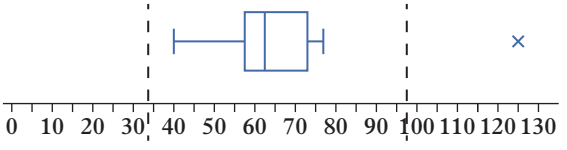

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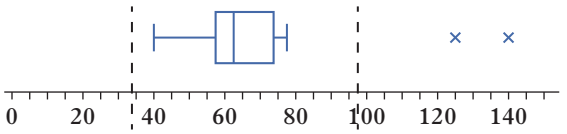

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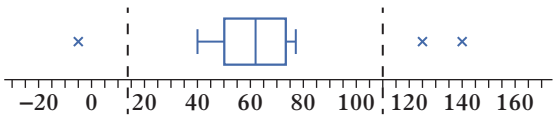


6 a i B **ii** B, D

b i median **ii** maximum **iii** none

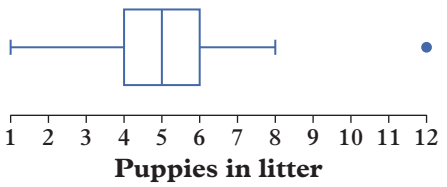
iv maximum **v** upper quartile

c Group C

7 a $Q_1 = 4$, median = 5, $Q_3 = 6$

b 12 is an outlier.

c

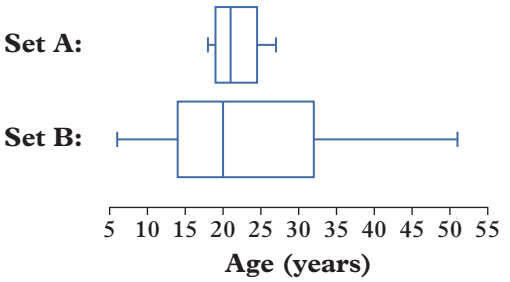


8 a median ≈ 32.2 °C, IQR ≈ 3.5 °C

b median ≈ 28 °C, IQR ≈ 8.5 °C

c Darwin's median maximum daily temperature during this month was approximately 32.2 °C, which was hotter than Canberra's median maximum daily temperature, 28 °C. Darwin's maximum daily temperature was less variable with an interquartile range of approximately 3.5 °C compared to Canberra's larger interquartile range of approximately 8.5 °C. While Canberra had the largest maximum daily temperature for the month, Darwin is, on average, hotter than Canberra.

9 a

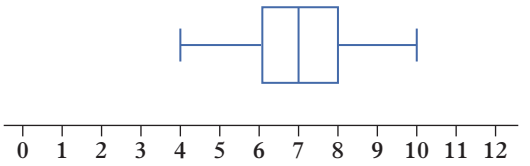


b Set A: The centre (median) = 21, the spread is 18–27 years, positively skewed. The range of ages is small, indicating that, during the week, the centre is used mainly by young people.

Set B: The centre (median) = 20, the spread is 6–51 years old, positively skewed. The range of ages is much greater during the weekend, indicating that families most probably visit then.

c The range of ages is much greater in set B than in set A, although the median ages are quite similar. More people in a younger and older age range visit during the weekend than attend on weekdays.

10 a



b No, as the minimum value (4) is greater than the lower fence (3) and the maximum value (10) is less than the upper fence (11), so the data contains no outliers.

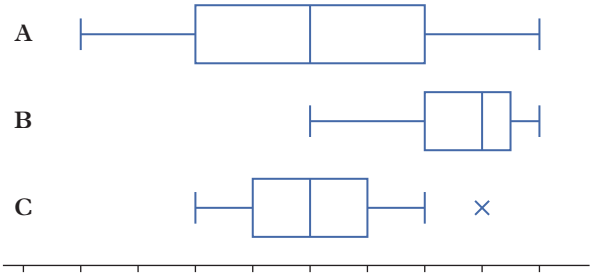
c The minimum value (2) is less than the lower fence (3) and the maximum value (12) is greater than the upper fence (11) so they are both outliers. Therefore, we would need to know at least the next value above the minimum and less than the maximum value (provided they are not also outliers) to know the location of the ends of the whiskers of the boxplot.

11 a Since the median line is not visible, it is equal to either the upper quartile or lower quartile so that it overlaps the line for either quartile.

b 5, 8, 8, 14, 20 or 5, 8, 14, 14, 20

12 The median age of all Australians is approximately 37 years, which is greater than the median age of Aboriginal and Torres Strait Islanders (approximately 23 years). The interquartile range of the ages of all Australians is approximately 37 years, which is greater than the interquartile range of the ages of Aboriginal and Torres Strait Islander peoples (approximately 31 years).

13



14 a IQR = 9 **b** $Q_1 = 28.5$, $Q_3 = 37.5$

EX p429

9C Distributions of data

1 a symmetric

b positively skewed

c negatively skewed

d bi-modal

e bi-modal

f negatively skewed

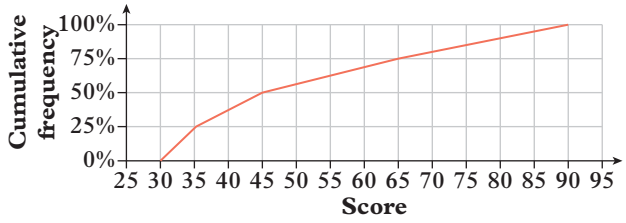
g positively skewed

h symmetric

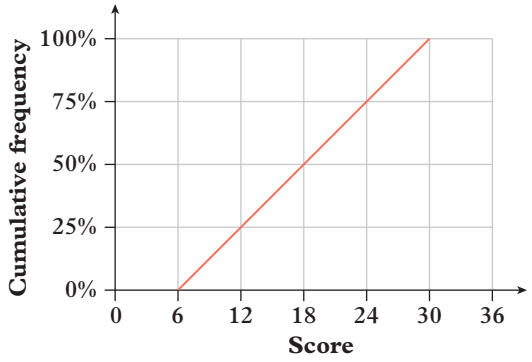
2 a 25% **b** 50% **c** 25%

d 75% **e** 75% **f** 50%

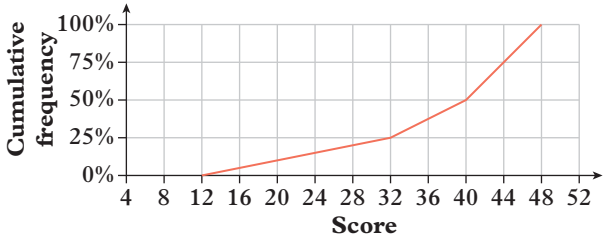
3 a



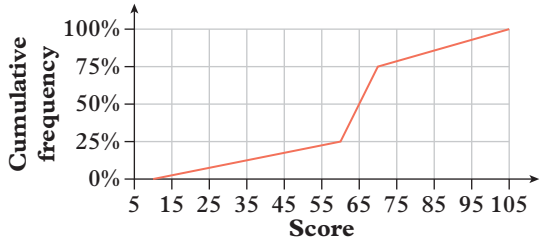
b



c



d



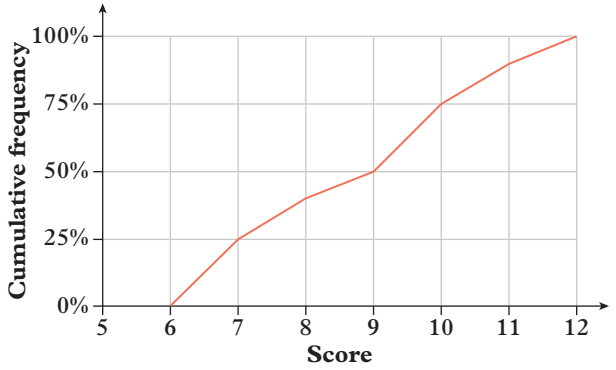
4 a i 10, 30, 60, 100, 120 **ii** 70

b i 16, 28, 40, 44, 52 **ii** 16

c i 9, 45, 54, 63, 72 **ii** 18

d i 100, 120, 160, 200, 300 **ii** 80

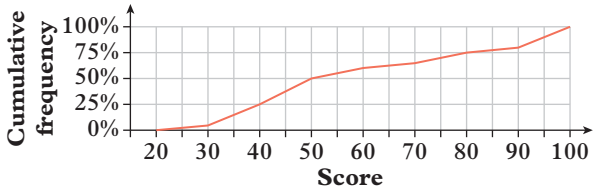
5



6 a

Class interval	Frequency	Cumulative frequency	Cumulative percentage
20–<30	10	10	5
30–<40	40	50	25
40–<50	50	100	50
50–<60	20	120	60
60–<70	10	130	65
70–<80	20	150	75
80–<90	10	160	80
90–<100	40	200	100

b



7 **A** and **iii**; **B** and **i**; **C** and **ii**; **D** and **iv**

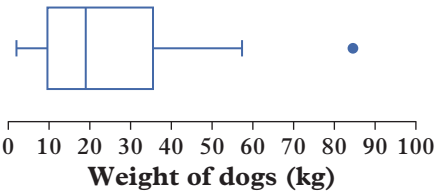
8 a i Boxplot appears negatively skewed.

ii Scores in the first quartile are very closely packed, making the lower whisker very short. In the second quartile, scores are spread far apart, making the lower half of the box long. Scores in the third and fourth quartiles appear spread out to about the same degree: less closely packed than those in the bottom whisker, but more densely packed than those in the lower half of the box.

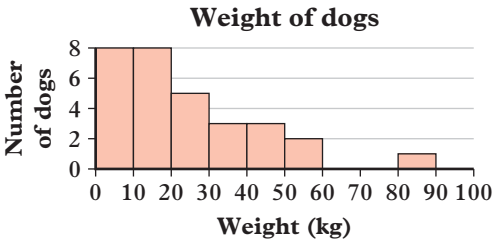
iii The median score is close to the values of data at the top end of the distribution.

b There may be multiple scores of the same value in the lower whisker.

9 a



b

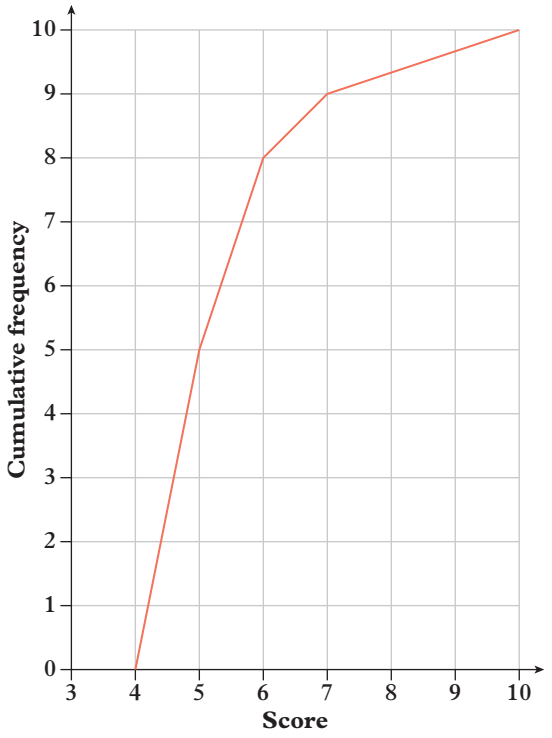


c positively skewed with an outlier

d The histogram shows more detail than the boxplot. The boxplot shows the positive skew and the median, but not the fine detail.

e To one decimal place, the weights are not repeated and so the dot plot would be a long row of scattered dots, which would be difficult to interpret.

10 a



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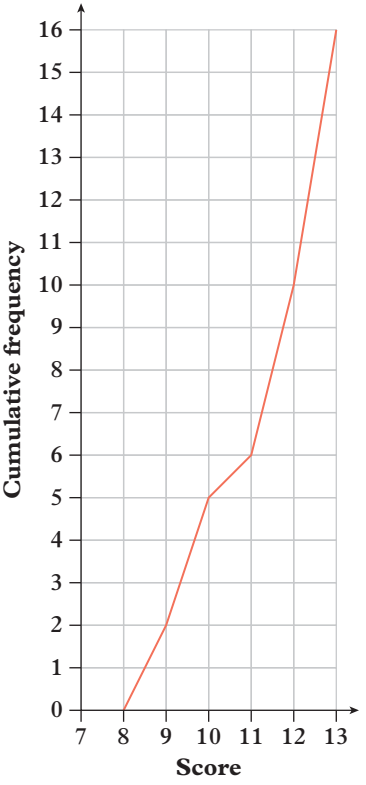
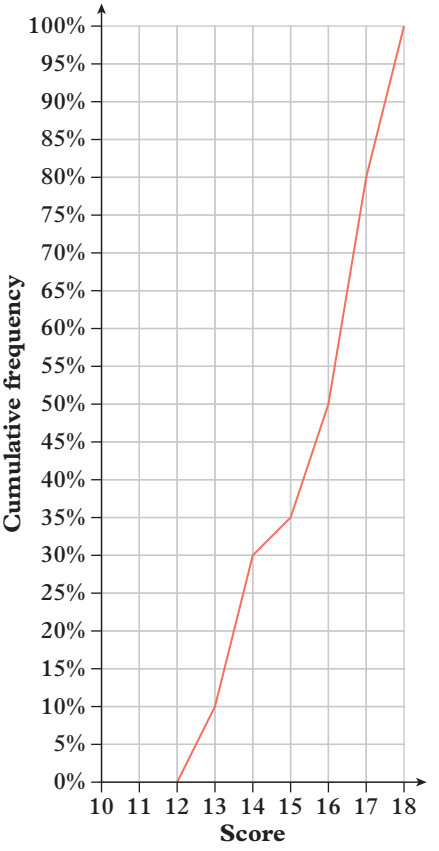


11 a



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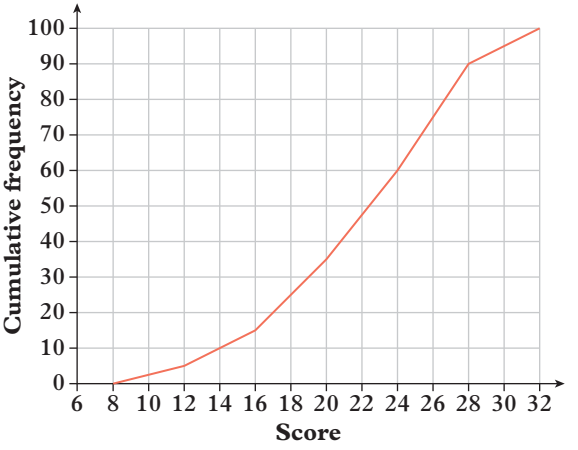


12 a Group A: negatively skewed; Group B: positively skewed

b The mean is affected by both the skew and the outliers the data contains, whereas the median is not affected by these. The mean is also not able to be calculated directly from a boxplot, whereas the median can be read off easily.

13 a 11, 14, 25, 32, 34

b 18

14 The cumulative frequency always adds to the previous amount, therefore the cumulative frequency cannot decrease. This means the gradient cannot be negative.

15 a **i** 25th **ii** 50th **iii** 75th

b The 100th percentile would be the value in the data set below which 100% of the data values fall. There is no such value as it would be included in the data set.

c **i** 4 and 8 **ii** 40 and 50 **iii** 301 and 701

d **i** 3 and 7 **ii** 30 and 50 **iii** 300 and 700

e In **i** and **iii**, the values differ by 1. In **ii**, the 30th percentile values are different by one value in the ordered data set and the 70th percentiles are the same.

f The data values would either be the same or very close as we would expect each number from 1 to 100 to appear in the data set many times.

Checkpoint

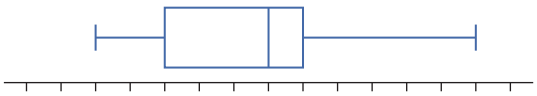
1 **a** 14, 17, 22.5, 32, 50 **b** 35, 38.5, 45, 54, 60

c 1, 4, 6, 8, 10 **d** 5.4, 6.2, 7.2, 7.9, 8.2

2 **a** IQR = 11, range = 18

b IQR = 27, range = 35

3 **a**



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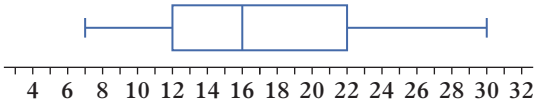
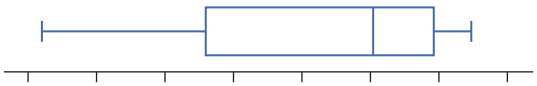


4 **a** **i** 16, 23, 27, 35, 38 **ii** 12

b **i** 20, 56, 72, 76, 80 **ii** 20

EX p438

5 **a**



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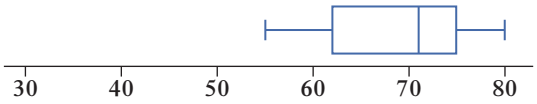


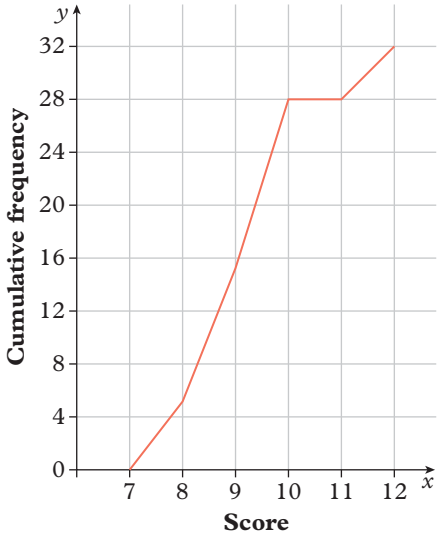
6 The median of group A is 70, which is less than the median of group B, 77. The interquartile range of group of A is 21, which is less than the interquartile range of group B, 28. Therefore, group A is, on average, less than group B and less variable.

7 **a** 25% **b** 50% **c** 75% **d** 50%

8 **a** negatively skewed **b** symmetric

c positively skewed **d** positively skewed

9 **a**



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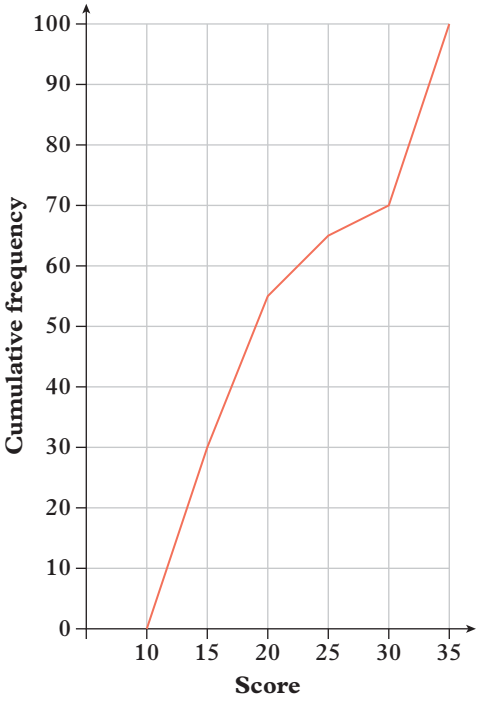


10 1, 3, 6, 9, 11

9D The mean and standard deviation

1 **a** 26.8 **b** 4.3 **c** 4.1

d 6.1 **e** 30.3 **f** 20.7

2 **a** 0.0 **b** 3.0 **c** 26.5

d 4.9 **e** 2.3 **f** 21.8

3 **a** 3.0 **b** 4.4 **c** 401 083.6

d 802 167.1 **e** 401 083.6 **f** 0.1

4 **a** 9.5 **b** 922.3 **c** 2.9

d 2.9 **e** 2.9 **f** 73.3

5 **a** **i** sample **ii** 0.85

b **i** population **ii** 5.20

c **i** sample **ii** 0.54

6 **a** 0.78

b The spread of ages of players in the cricket teams are tightly packed around the mean value, with all bar one player being between 15 and 17 years of age.

7 **a** mean = 15.88, standard deviation = 8.99

b The data are grouped pretty evenly with a large standard deviation and the mean being very close to halfway between the minimum and maximum.

8 **a** 41.7 **b** 9.18

c The spread of data is 13 to 58, with the mean at about 42. The mean is towards the higher end of the data, indicating that the data has a negative skew. The standard deviation of scores from the mean is about 10.

9 **a** the mean for train line A = 5.4

the mean for train line B = 4.5

b the standard deviation for train line A = 1.89

the standard deviation for train line B = 4.57

c The mean for train line A is higher than that for train line B, indicating trains are generally later on line A. Standard deviation for train line A is lower than that for train line B, indicating scores are bunched closely around the mean, so that line A is consistently late. Line B has four outliers of 9, 11, 15 and 19 minutes, which have increased the mean and the standard deviation. Without these outliers, the standard deviation decreases to 1.23 and mean for train line B decreases to 2.65.

d The outliers for train line B might have been caused by unusual circumstances. If these outliers are disregarded, it could be said that the train on line A runs late more often. If the outliers are not disregarded, the train on Line B, on average, runs earlier but is very unpredictable.

10 The first histogram has a larger standard deviation than the second, because the scores are more spread out.

11 a 68.51

b The shop may be a discount shop or might be having a sale; the tablet might be a discontinued old model.

c 20.98

d Standard deviation is much lower with the outlier excluded.

e interquartile range

12 a i $\sigma \approx 2.43$, mean absolute deviation = 1.74

ii $\sigma \approx 5.38$, mean absolute deviation = 4.375

b mean absolute deviation $\leq \sigma$

13 a i $\tilde{x} = 4$ **ii** $\tilde{x} = 7.5$

b i 1.5 **ii** 2.5

14 a The median is a better measure of the centre than the mean when the data has outliers or is skewed as both of these affect the value of the mean.

b The interquartile range is a better measure of the spread than the standard deviation when the data has outliers or is skewed as both of these affect the value of the mean and hence the standard deviation.

15 a Group A has a mean of 29.8, which is less than group B's mean of 30.1. Group A has a standard deviation of 5.5, which is greater than group B's standard deviation of 5.1. Therefore, on average, group A is less than and more variable than group B.

b Group A has a mean of 93.3, which is greater than group B's mean of 77.2. Group A has a standard deviation of 12.4, which is greater than group B's standard deviation of 7.9. Therefore, on average, group A is greater than and more variable than group B.

16 a i $z_A = 1.5$, $z_B = 1.875$

ii Student B scored 1.875 standard deviations above their class mean, which is greater than the 1.5 standard deviations Student A scored above their class mean. This means both students achieved better results than most of their class, with Student B performing better than Student A.

b i $z_A = 0.75$, $z_B = 1$

ii Student B scored 1 standard deviation above their class mean, which is greater than the 0.75 standard deviations Student A scored above their class mean. This means both students achieved better results than their class average, with Student B performing better than Student A.

c i $z_A = -0.8$, $z_B = -5$

ii Student B scored 5 standard deviations below their class mean, which is less than the 0.8 standard deviations Student A scored below their class mean. This means both students achieved worse results than their class average, with Student B performing worse than Student A.

d i $z_A = -3$, $z_B = -3$

ii Student B scored 3 standard deviations below their class mean, which is the same as the 3 standard deviations Student A scored below their class mean. This means both students achieved significantly worse results than their class average.

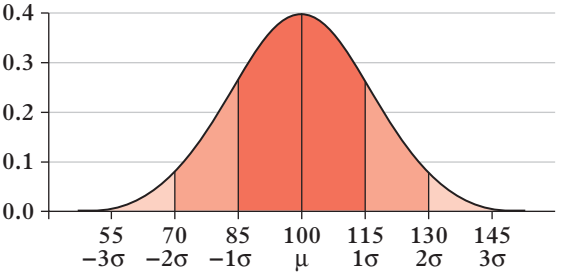
17 a Adding the percentages shown in the given graph:

2.12% + 13.6% + 34.13% + 34.13% + 13.6% + 2.12% = 99.7%

This covers the region of three sigma either side of the mean.

b i 68% **ii** 95%

c



d 55 to 145

e higher than 145

f less than 70

g 2.5%

h 16%

i mean = 136, standard deviation = 8

18 a $\mu_{\bar{x}} = 40$, $s_{\bar{x}} = \frac{9}{5} = 1.8$

b i 68% **ii** 84% **iii** 0.15%

iv 5%

c i 41.8 **ii** 40 **iii** 36.4 and 43.6

iv 36.4 and 38.2 or 41.8 and 43.6

EX p446

9E Scatterplots and bivariate data

1 a i a moderate to weak negative trend

ii With increasing height, the time taken to run 10 km decreases moderately.

b i weak positive trend

ii The number of visits to the doctor per year increases slightly as the number of colds per year increases.

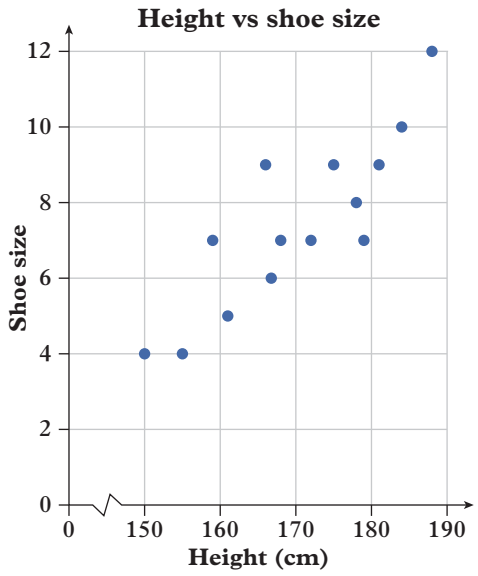
c **i** strong positive trend

ii The amount of petrol used increases greatly with an increase in the distance travelled.

d **i** moderate negative trend

ii Exam marks decrease moderately with an increase in the hours of television watched per week.

2 **a**



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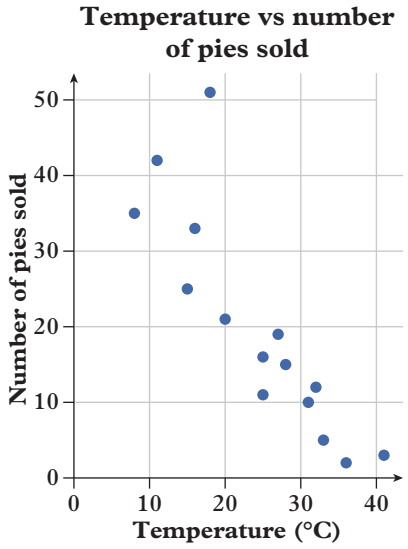

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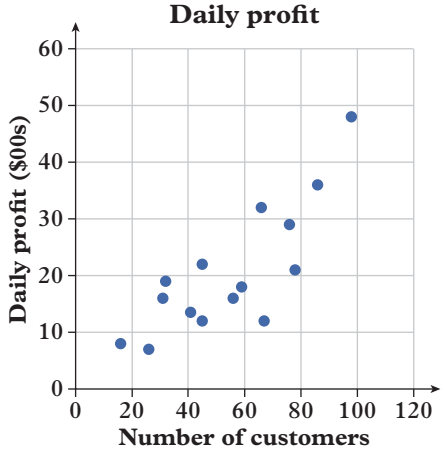
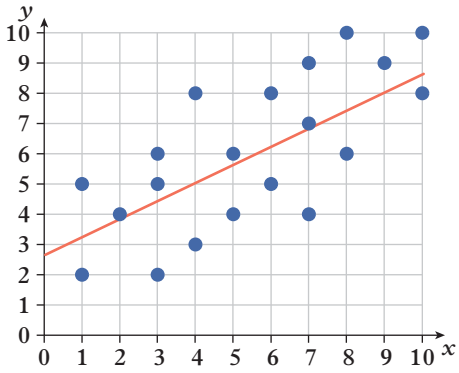


3 **a** There is a moderate increase in shoe size with an increase in people's height.

b There is a strong negative correlation, so the number of pies sold decreases greatly as temperature increases.

c There is a moderate increase in daily profit as the number of customers increases.

4 **a**



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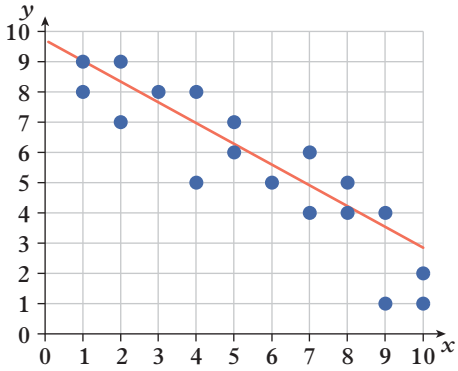
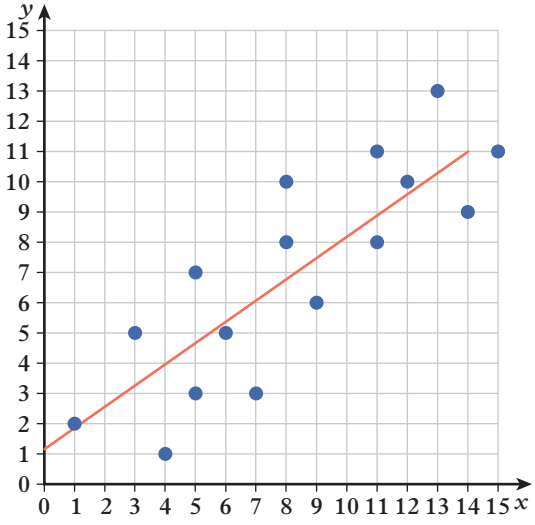
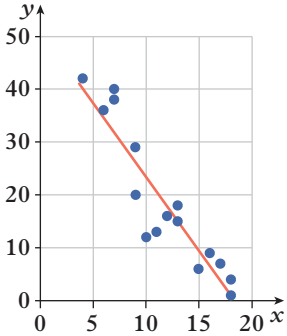


c

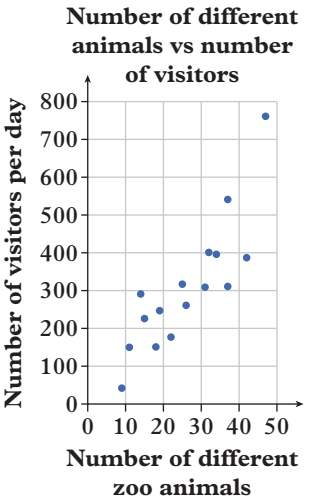


d



5 **a** moderate positive linear correlation

b



c There is a moderately strong linear increase in the number of visitors per day as the number of different zoo animals increases.

6 **a** The weight depends on height.

b As a person gets taller, he/she generally becomes heavier.

c **i** weight **ii** height

7 **a** **i** Up to a certain time (the age of the tree) there is a linear relationship.

ii The independent variable is time, the dependent variable is height.

iii The linear correlation is strong and positive in the early years of the tree's life.

b **i** Up to a certain age of car there is a linear relationship between the value of a car and its age. If the car becomes an antique, or a collectable item, this relationship changes.

ii The independent variable is the age of the car; the dependent variable is the value of the car.

iii The linear correlation is strong and negative, with the car losing value as it ages.

c There is no linear relationship between the weight of a person and the number of languages spoken.

d **i** There is a linear relationship between a person's shoe size and height.

ii The independent variable is height; the dependent variable is shoe size.

iii The linear correlation is strong and positive. Generally, as a person grows and gets taller, his/her shoe size increases.

8 **a** As temperature increases, the number of heaters purchased decreases strongly. There is a strong negative linear correlation between the temperature and the purchase of heaters, and there is a weak positive linear correlation between someone's earnings and their monthly phone bill. As the earnings increase, the phone bill does also tend to increase.

b about 4

c about $90

d The prediction from scatterplot A is not very reliable because there is not a strong relationship between the variables. For scatterplot B, the relationship is much stronger, so a prediction would be more reliable.

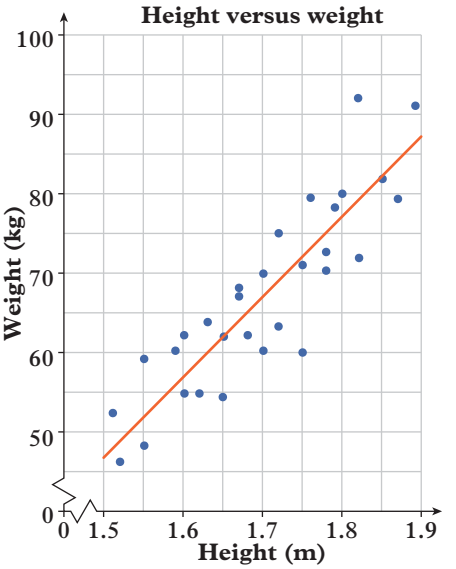
9 **a** The strong positive linear correlation between the hours studied and the marks achieved supports the view that more study time contributes to achieving a higher percentage result on a test. However, extra study time does not always guarantee a higher test score; the correlation is for a sample and for an individual the result can vary. Also, there may have been another underlying reason that explains the high marks and the extra time studying may have just been a coincidence.

b A high correlation does not imply that one variable 'caused' another to happen. There could be an underlying reason that is hidden and not known, and the correlation seen may actually just be a coincidence or due to choosing a particular sample.

c It is commonly thought if the relationship between two variables is strong, one must have a strong influence on the other. Sometimes this is true, but the particular circumstances need to be examined to determine this.

10 **a** The trend is linear, moderately strong and positive. As height increases, weight also increases.

b



c 82 kg (answers may vary)

d Quite confident because there is a strong correlation between height and weight.

11 While the previous examples have used only positive values for the dependent and independent variables, it is possible to plot data that uses negative values, such as temperature, and end up with a negative section on either axis.

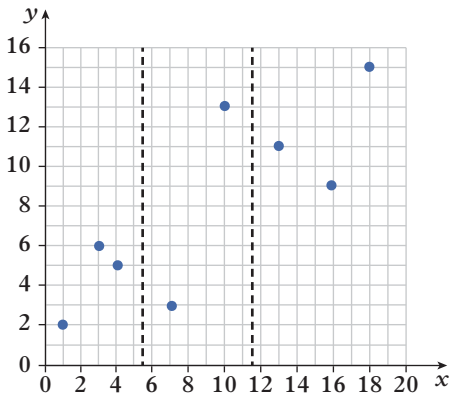
12 To plot data values in a scatterplot you need to know the exact values of each variable. When dealing with grouped data, the exact values are no longer known, so you would not be able to accurately plot each point.

13 a no correlation

b moderate, positive linear correlation

14 Answers will vary.

15 a

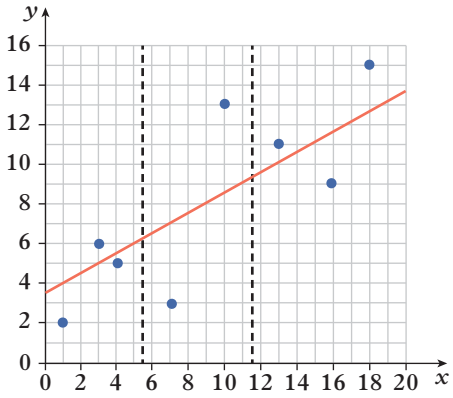


b For first section: median x-coordinate = 3, median y-coordinate = 5

For second section: median x-coordinate = 9, median y-coordinate = 8

For third section: median x-coordinate = 16, median y-coordinate = 11

c



d $y = 0.5x + 3.5$

e The effect of extreme values (outliers) is minimised by finding and using the median value of that particular third.

EX p452

9F Analysing statistical reports

1 a This survey has used results from one gym to represent the entire population of Australia, meaning it is extremely biased.

b Two boys out of four does not represent a sufficient sample on which to base the claim. Information such as socio-economic backgrounds, ages, suburbs are not considered.

c If only five children were surveyed, the sample is not large enough to represent the larger population claimed. If the reference is to four-fifths of a larger population, more details are needed about what is considered a 'fair share' of chores and of whom that population was made up. Also, the children stated they 'believe they do' their fair share, but the report simply says they 'do' their fair share.

2 a Sample answer: They have provided no evidence to support this claim and no sample size is given. They should be able to answer the following questions: Did any of the women surveyed have diabetes, and if so, what type and to what degree? Were women who had coffee at other times of day surveyed too? Is there a biological link between coffee and diabetes?

b Sample answer: No evidence has been provided to support this claim and no sample size is given. They should be able to answer the following questions: How was the survey conducted? If the information was collected from a sample containing some people with Alzheimer's disease, is the information reliable? What timeline was followed? Was it conducted over many years and were the results monitored? Is there a biological link between being physically active and Alzheimer's disease?

c Sample answer: In order to determine the accuracy of data, information is required about the source of the survey results, who conducted the survey, how it was conducted, the sample size surveyed and the population from which the sample was chosen.

3 Sample answer: Not necessarily. The sample only includes people who watch the show and from those people it only includes those viewers who decide to call.

4 a The scale on the x-axis is not uniform, and the scale on the y-axis does not start at zero, with no break indicated.

b i The revenue declined from \$100 000 to \$96 500, a loss of \$3500 which is not a significant decline. The decrease from October to November is a decrease of \$1000 over the course of a month. This is about 30% of the total loss over eleven months, which is not a slower rate.

ii Since the scale on the y-axis does not start at zero, the range of \$20 to \$20.50 can be stretched out. This gives the impression that the increase is greater than it really is.

iii 3.5%

5 a This graph is misleading as the scale on the y-axis is not labelled so we do not know if it starts at zero or not or if the scale on the y-axis is uniform or not.

b i The graph is said to display the time taken before relief, therefore the company's brand takes longer before relief and is therefore less effective.

ii By not labelling the y-axis, the company can make the graph look like a bigger bar represents a more effective product as most people assume taller or bigger is better. The bar appears twice as large but with the lack of scaling we cannot verify if the value it actually represents is twice as large.

6 a Fatima's hourly rate of pay over 60 months of employment

b The scale on the x-axis is not uniform and the scale on the y-axis doesn't start at the origin.

c

7 No. Success in school depends on more variables than just whether or not a student eats breakfast. Those who use the first statement may be trying to sell breakfast foods.

8 The source of information, the sample used to obtain the results and the source of funding for such surveys should be stated with any reported results.

9 Sample answer: The company should report both the negative and positive outcomes of the independent testing, particularly because fuel efficiency is something potential buyers would be interested in. However, they do not need to compare the fuel efficiency to their competitors' models.

10 a incorrect, $\frac{18}{60} \times 100\% = 30\%$

b correct, $\frac{4}{65} \times 100\% = 6\%$

c correct, $\frac{36}{125} \times 100\% = 29\%$

d incorrect (but close to correct), $\frac{31}{61} \times 100\% = 51\%$

11 a Sample answer: Ellie may have asked leading questions in her survey, which would make it more likely for people to answer in the way that she wanted.

b Teacher to check.

12 Teacher to check.

CHAPTER 9 review

Multiple choice

1 C **2** B **3** D **4** B **5** B

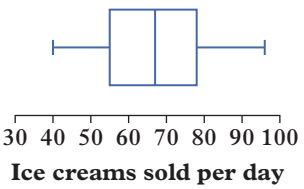
6 C **7** B **8** C

Short answer

1 a 56 **b** 67 **c** 23

d 40, 55, 67, 78, 96

2



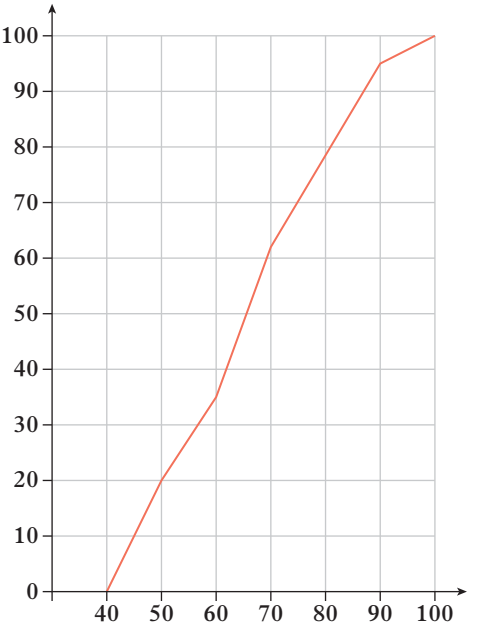
The boxplot shows that 50% of the time between 55 and 78 ice creams are sold each day and 67 ice creams are sold half of the time. The data are pretty evenly spread.

3 five-number summary: 11, 13, 17, 19, 24
interquartile range: 6

4 a

Class interval	Frequency	Cumulative frequency	Cumulative percentage
40–<50	6	6	19
50–<60	5	11	35
60–<70	8	19	61
70–<80	5	24	77
80–<90	5	29	94
90–<100	2	31	100
Total	**31**	**31**	**100**

b



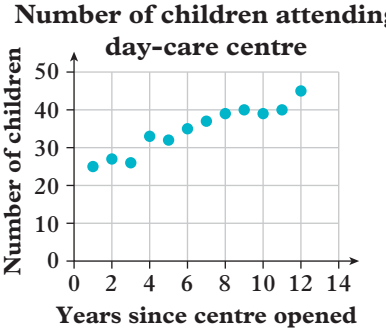
5 a 10% **b** 20% **c** 20%

6 a Lily: **i** 72 **ii** 3.69

Cassy: **i** 71 **ii** 18.28

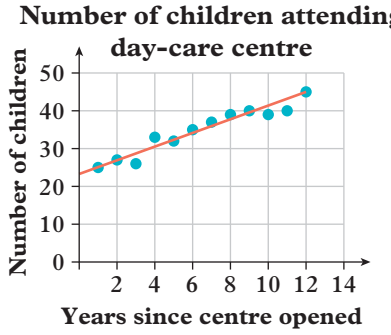
b The lower the standard deviation, the closer the scores lie to the mean. A low standard deviation indicates consistency, so Lily has performed more consistently than Cassy. This is also obvious from the raw scores, where it can be seen that Lily's scores are grouped closer together than Cassy's.

7 a



b The number of children in the day care centre increased the more years the centre was open.

c



d 60 (answers may vary)

8 Sample answer: This is an ethical issue as the company should present all the results, positive and negative, to their employees rather than just presenting the strong results.

Analysis

a Over the years, the time for winning the event has generally decreased, although there has not always been a decrease from one Olympic year to the next.

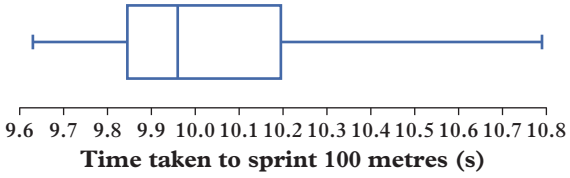
b i mean = 10.04, median = 9.96

ii range = 1.16, IQR = 0.35

iii 9.63, 9.845, 9.96, 10.195, 10.79

c 0.298

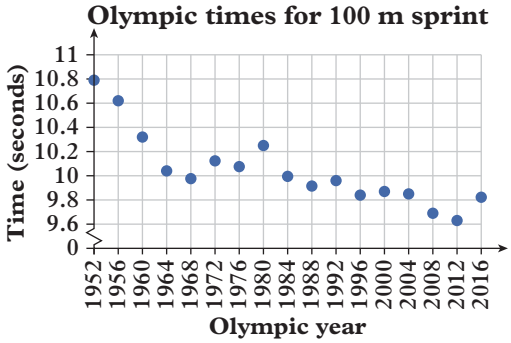
d i



ii Times are positively skewed, being packed more tightly below the median than above the median. This indicates it is becoming increasingly difficult to lower the time for the race.

iii The lower whisker will move to the left, and the lower half of the boxplot will move left and shrink in size as the times become more densely packed in that region of the plot.

e



f Teacher to check.

g 2040 (answers may vary)

CHAPTER 10 Probability

EX p464

10A Theoretical probability

1 a {1, 2, 3, 4, 5, 6, 7, 8} **b** {A, U, S, T, R, L, I}

c {red, white, blue}

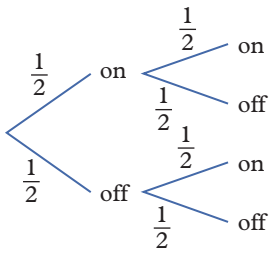
2 a equally likely **b** not equally likely

c not equally likely

3 a {M, A, T, H, E, I, C, S} **b** $\frac{2}{11}$

4 a 0.25 **b** 0.25 **c** 0.5

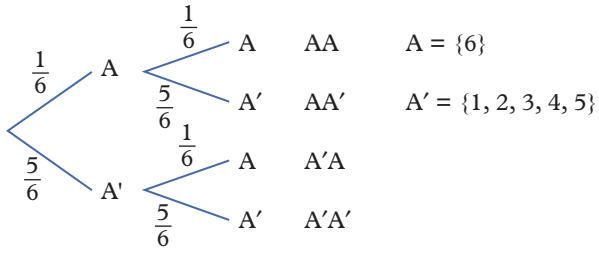
5 a



b i 0.25 **ii** 0.25 **iii** 0.5

6 a (not a 6)

b



c i $\frac{1}{36}$ **ii** $\frac{11}{36}$

7 a Each outcome now has an equal chance of occurring.

b 36

c i $\frac{1}{6}$ **ii** $\frac{1}{4}$

		2nd roll					
		1	2	3	4	5	6
1st roll	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

8 a 0.125 **b** 0.75 **c** 0.5

9 a, b i Cells coloured red

ii Cells coloured orange

iii Cells coloured green

		2nd roll					
		1	2	3	4	5	6
1st roll	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

c i There are 15 favourable outcomes out of 36 possible outcomes; $\frac{15}{36} = \frac{5}{12}$.

ii There are 6 favourable outcomes out of 36 possible outcomes; $\frac{6}{36} = \frac{1}{6}$.

iii There are 15 favourable outcomes out of 36 possible outcomes; $\frac{15}{36} = \frac{5}{12}$.

d The most successful strategy would be to predict a number greater than (or less than) 7.

10 a A = {2, 4, 6, 8, 10}, B = {4, 8}, C = {2, 3, 5, 7}

b B and C

c **i** $\frac{1}{2}$ **ii** $\frac{1}{5}$ **iii** $\frac{2}{5}$

11 a **i** 36

ii 6 outcomes are doubles; (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

iii $\frac{1}{6}$

b **i** 64

ii 8 outcomes are doubles, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8)

iii $\frac{1}{8}$

c Pr(doubles) = $\frac{1}{n}$

12 10 marbles

13 a **i** $\frac{1}{4}$ **ii** $\frac{1}{6}$ **iii** $\frac{2}{9}$

b **i** For the outcomes from part **a**, an odd number is most likely.

ii For the outcomes from part **a**, a prime number is least likely.

14 a {1, 2, 3, 4, 5, 6, 8, 10, 12}

b Some scores can only be achieved one way, whereas others can be achieved two ways.

c **i** $\frac{1}{12}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{4}$

15 $\frac{11}{36}$

16 a 28

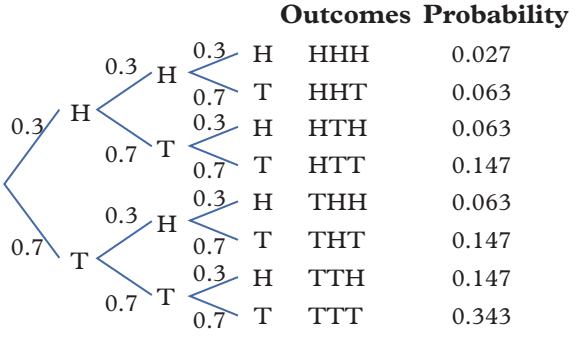
b The rows of tiles are grouped, in order from zero to six, according to the value shown on the top half of the tiles. Within each of these ordered groups, the values of the bottom half of the tiles have also been placed in order.

c 7

d **i** $\frac{1}{28}$ **ii** $\frac{1}{4}$ **iii** $\frac{3}{4}$

iv $\frac{3}{7}$ **v** $\frac{9}{14}$ **vi** $\frac{13}{28}$

17 a



b **i** TTT **ii** HHH

c **i** 0.441 **ii** 0.189 **iii** 0.216

18 a 2

b 7

c **i** (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)

ii 3

19 5 red decorations

20 a The chance of landing on a red section is not the same as the chance of landing on a white section. There are more red bands than white and the combined area of the red bands appears larger than the combined area of the white bands.

b radii of circles from inner circle out: 1 cm, 3 cm, 5 cm, 7 cm, 9 cm

c bullseye = π cm^2, inner white band = 8π cm^2, middle red band = 16π cm^2, outer white band = 24π cm^2, outermost red band = 32π cm^2

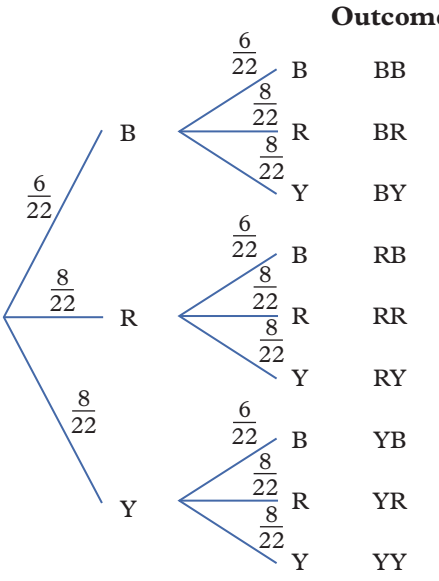
d **i** 0.012

ii 0.395

EX p471

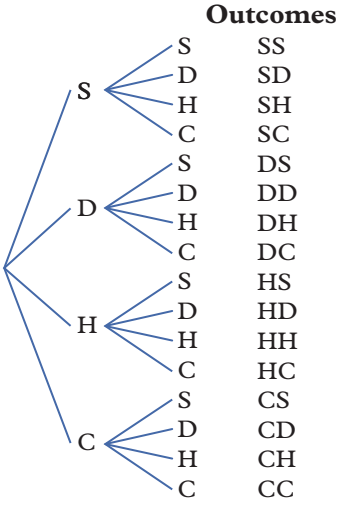
10B Experiments with and without replacement

1 a



b $\frac{41}{121}$

2 a

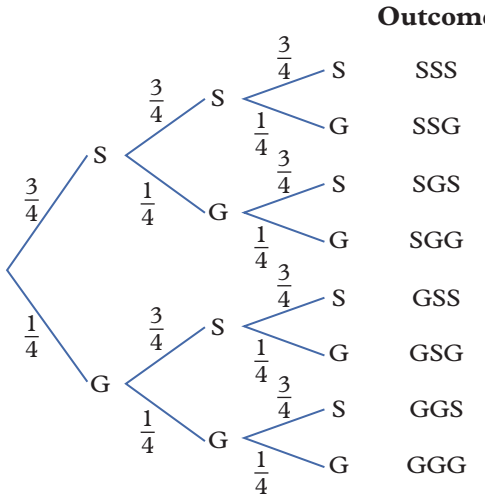


b $\frac{1}{16}$ **c** $\frac{1}{8}$

d $\frac{1}{4}$ **e** $\frac{3}{4}$

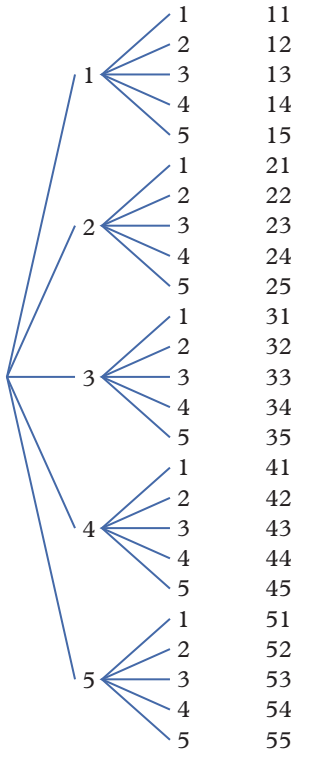
3 a i $\frac{3}{4}$ **ii** $\frac{1}{4}$

b



c i $\frac{27}{64}$ **ii** $\frac{1}{64}$ **iii** $\frac{9}{16}$

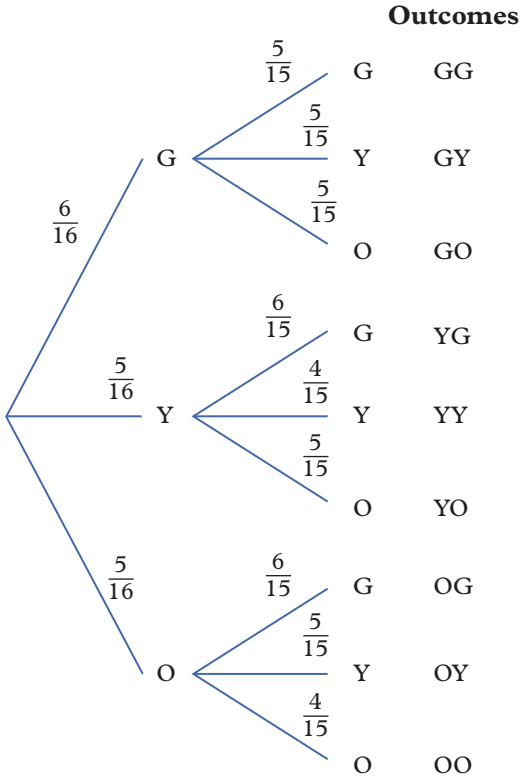
4 a



b 25 **c** 5

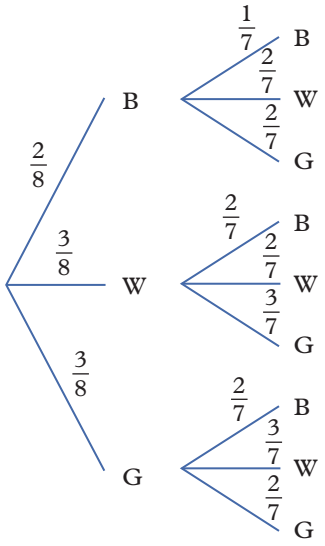
d i $\frac{4}{5}$ **ii** $\frac{1}{5}$ **iii** $\frac{2}{5}$

5 a



b $\frac{7}{24}$

6 a



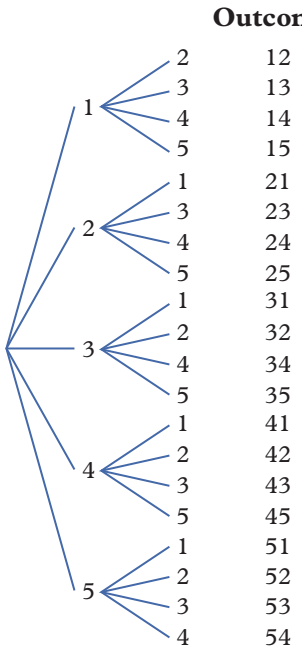
b $\frac{1}{4}$

c $\frac{5}{28}$

7 a $\frac{6}{55}$ **b** $\frac{28}{55}$

8 a $\frac{233}{338}$ **b** $\frac{44}{65}$

9 a



b 20

c **i** 0 **ii** 1 **iii** $\frac{1}{2}$

10 a $\frac{1}{4}$ **b** $\frac{23}{87}$

11 a $\frac{3}{8}$ **b** $\frac{25}{247}$ **c** $\frac{225}{988}$ **d** $\frac{42}{185}$

12 a **i** $\frac{9}{29}$ **ii** $\frac{2}{29}$ **iii** $\frac{3}{29}$

iv $\frac{3}{29}$ **v** $\frac{1}{435}$ **vi** $\frac{4}{435}$

b **i** $\frac{2}{87}$ **ii** $\frac{4}{435}$

c Undertaking the events in reverse order does not affect the overall probability of the events occurring.

13 a $\frac{8}{125}$ **b** $\frac{27}{125}$

14 a There are only two non-white balls in total.

b $\frac{1}{10}$

15 a **i** {2, 3, 4, 5, 6, 7, 8, 9, 10}

ii {3, 4, 5, 6, 7, 8, 9}

b **i** $\frac{3}{25}$ **ii** $\frac{1}{10}$

c **i** $\frac{1}{5}$ **ii** $\frac{1}{5}$

16 a **i** 125 **ii** 60

b **i** 729 **ii** 504

17 a 216 **b** 96

18 a **i** 10

ii 1 000 000

b **i** 17 576 000

ii You can form more possibilities with this system, because the alphabet has 26 letters, enabling more combinations than the 10 digits.

EX p477 10C Two-way tables and Venn diagrams

1 a **i** {1, 3, 5, 7, 9, 11}

ii {2, 3, 5, 7, 11}

iii {3, 5, 7, 11}

iv {1, 2, 3, 5, 7, 9, 11}

b **i** $\frac{1}{2}$ **ii** $\frac{7}{12}$ **iii** $\frac{1}{3}$

2 a **i** 13 **ii** 19 **iii** 13

iv 9 **v** 26 **vi** 6

b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{9}{26}$

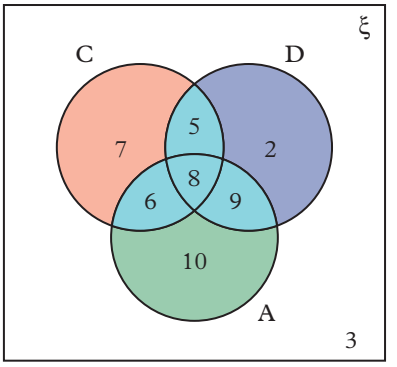
iv $\frac{7}{26}$ **v** $\frac{7}{26}$ **vi** $\frac{2}{13}$

vii $\frac{10}{13}$ **viii** 1

3 Calculate the missing entries in the rows and columns by adding or subtracting existing entries, as required.

	Primary school	High school	Total
Catch public transport	22	18	40
Are driven	37	11	48
Walk/ride	5	9	14
Total	64	38	102

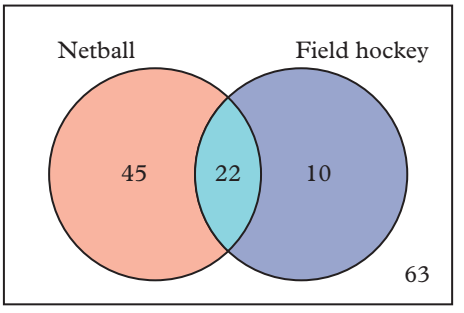
4



5 a $\frac{13}{25}$ **b** $\frac{13}{50}$ **c** $\frac{11}{50}$

d $\frac{7}{25}$ **e** $\frac{47}{50}$

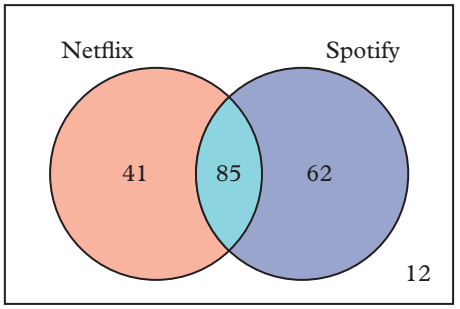
6



7 a

	Netflix (N)	Not Netflix (N′)	Total
Spotify (S)	85	62	147
Not Spotify (S′)	41	12	53
Total	126	74	200

b



8

	A	A′	Total
B	15	7	22
B′	13	11	24
Total	28	18	46

9 **a** 7

b

	Maths (M)	Not maths (M′)	Total
Science (S)	11	7	18
Not science (S′)	4	6	10
Total	15	13	28

10 0.7

11 $\frac{5}{18}$

12 **a** **i** True; set R is completely within set P so it is a subset of set P.

ii True; set R is completely within set Q so it is a subset of set Q.

iii False; set R is completely within the intersection of sets P and Q so it is a subset of the intersection of sets P and Q.

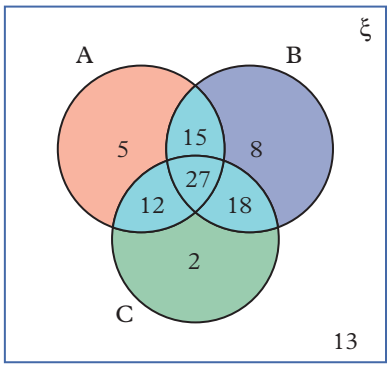
iv False; set R is completely within the union of sets P and Q so it is a subset of the union of sets P and Q.

v False; set R is within the universal set, ξ, so it is a subset of the universal set.

vi True; set R is within set Q, which is within the universal set so it is a subset of both set Q and the universal set, ξ.

b **i** $\frac{1}{8}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{12}$ **iv** $\frac{1}{8}$

13 **a**

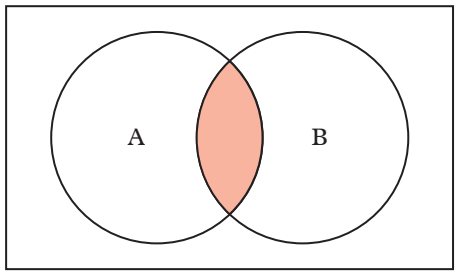


b **i** 85 **ii** 14 **iii** 45 **iv** 73 **v** 41 **vi** 15 **vii** 18 **viii** 73

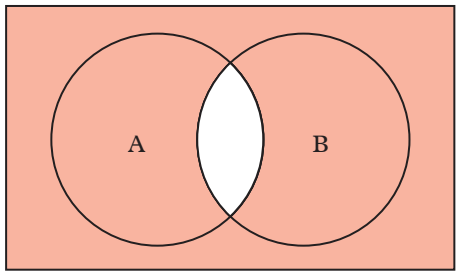
14 **a** **i** {2, 4, 6, 8, 10, 12} **ii** {2} **iii** {1, 2, 4, 6, 8, 9, 10, 12}

b **i** $\frac{1}{12}$ **ii** $\frac{5}{12}$ **iii** $\frac{2}{3}$

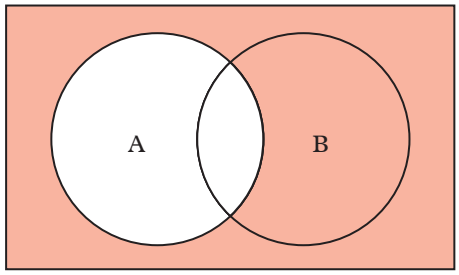
15 Pr(A ∩ B) is:

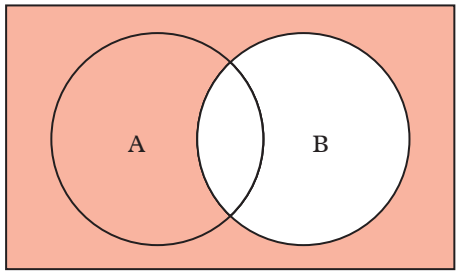


So Pr((A ∩ B)′) is:

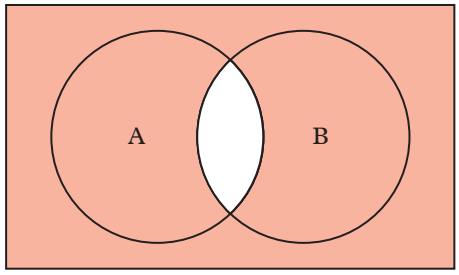


Pr(A′) and Pr(B′) are:





Hence, Pr(A′ ∪ B′) is:



16 **a** 0.25 **b** 0.45

17 **a**

	A	A′	Total
B	0.1	0.5	0.6
B′	0.2	0.2	0.4
Total	0.3	0.7	1

b **i** 0.2 **ii** 0.5 **iii** 0.9

18

	Boys	Girls	Total
Left-handed	5	6	11
Right-handed	30	19	49
Total	35	25	60

There are 6 left-handed girls.

19 $P \cap Q = \{2, 5, 9, 13\}$

$R \cap S' = \{4, 5, 6, 7\}$

$\{2, 5, 9, 13\} \cup \{4, 5, 6, 7\} = \{2, 4, 5, 6, 7, 9, 13\}$

$2 + 4 + 5 + 6 + 7 + 9 + 13 = 46$

20 $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$

Checkpoint

1 **a** {H, T}

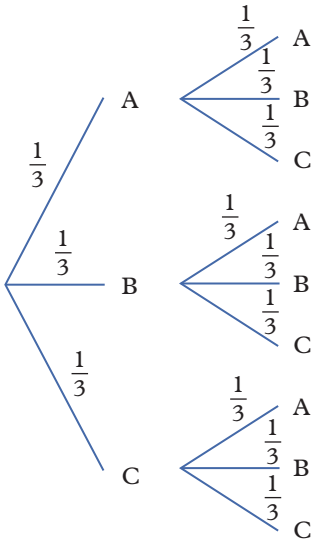
b {1, 2, 3, 4, 5, 6}

c {club, diamond, heart, spade}

d {L, O, I, P}

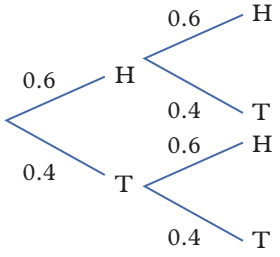
2 **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

3 **a**



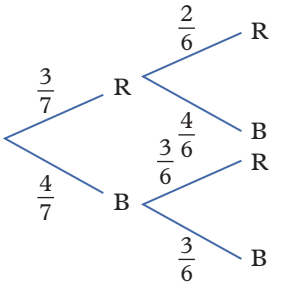
b $\frac{1}{9}$

4 **a**



b **i** 0.36 **ii** 0.48

5 **a**



b $\frac{1}{7}$

6 **a** $\frac{4}{25}$ **b** $\frac{1}{10}$

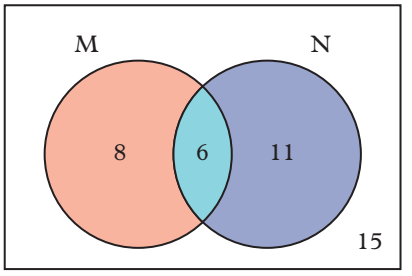
7 **a** $\frac{1}{16}$ **b** $\frac{1}{4}$ **c** $\frac{3}{4}$

8 **a** $\frac{4}{5}$ **b** $\frac{7}{20}$ **c** $\frac{1}{5}$ **d** $\frac{3}{20}$

9 **a**

	M	M′	Total
N	6	11	17
N′	8	15	23
Total	14	26	40

b



10 **a** $\frac{7}{12}$ **b** $\frac{5}{8}$ **c** $\frac{1}{3}$ **d** $\frac{11}{24}$

EX p485

10D Permutations

1 **a** 35 **b** 17 **c** 29

d 216 **e** 560 **f** 4320

2 **a** 6 **b** 24 **c** 720

d 3 628 800

e 1 307 674 368 000

f 2 432 902 008 176 640 000

3 **a** 362 880 **b** 120 **c** 3 628 800

d 479 001 600 **e** 720

f 2 432 902 008 176 640 000

4 **a** 120 **b** 90 **c** 30 240

d 2520 **e** 720 **f** 870

5 **a** 3024 **b** 60 **c** 720

d 19 958 400 **e** 360 **f** 5 079 110 400

6 **a** $\frac{26}{55}$ **b** $\frac{5}{11}$ **c** $\frac{31}{55}$

d $\frac{10}{11}$ **e** $\frac{1248}{3025}$ **f** $\frac{1326}{5995}$

7 **a** $\frac{1}{15}$ **b** $\frac{3}{10}$ **c** $\frac{3}{28}$

8 **a** $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 10 \times 9 \times 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)$

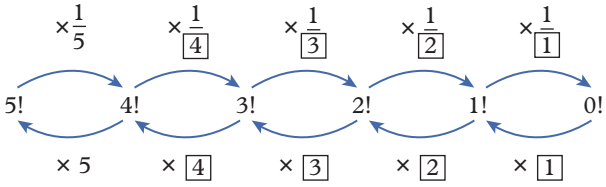
$= 10 \times 9 \times 8 \times 7!$

This shows that 7! is a common factor of the numerator and denominator that can be divided out to simplify the fraction.

b **i** 132 **ii** 6840 **iii** 100
iv 35 904 **v** 2550 **vi** 997 002 000

9 **a** 24 **b** 30 **c** 20 **d** 16

10 **a**



b $0! = 1$

c ${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$ but we know that ${}^nP_n$ means from n objects permute n of them, which is the same as $n!$ and so $n! = \frac{n!}{0!}$ which is only true if $0! = 1$.

11 **a** 3 **b** 12 **c** 6
d 1260

12 **a** 6 **b** 24 **c** 40 320
d $(n-1)!$

13 **a** 720 **b** 48 **c** $\frac{1}{15}$
d 5040 **e** 96 **f** $\frac{2}{105}$

14 **a** $\frac{1}{n+1}$
b $(n+3)(n+2)n(n+1)(n-1)(n-2)$
c $n+3$
d $\frac{n-1}{n!} = \frac{1}{n(n-2)!}$
e $n+1$
f $\frac{(n+5)(n+4)(n+3)(n+2)(n+1)}{20}$

15 **a** **i** 945
ii 384
iii 2 027 025
iv 10 321 920
v 3 715 891 200
vi 7 905 853 580 625
b **i** 2310
ii 30 030
iii 9 699 690
iv 7 420 738 134 810
v 5 579 408 030 126 698 960 967 415 390
vi 2 305 567 963 845 518 424 753 102 147 331 756 070
c **i** 2 **ii** 12 **iii** 288
iv 34 560 **v** 24 883 200
vi 6 658 606 584 104 736 522 240 000 000

EX 10E Combinations

p490

1 **a** 32 **b** 256 **c** 4096
d 32 768 **e** 1 048 576 **f** 1 073 741 824

2 **a** 126 **b** 10 **c** 120
d 495 **e** 15 **f** 125 970

3 **a** 20 **b** 45 **c** 252
d 21 **e** 6 **f** 435

4 **a** 1470 **b** 11 760 **c** 3 603 600
d 15 116 400 **e** 44 100 **f** 10 080

5 **a** 32 **b** 1024 **c** 256
d 512 **e** 63 **f** 31

6 **a** $\frac{1}{210}$ **b** $\frac{1}{495}$

7 **a** $\frac{21}{55}$ **b** $\frac{35}{429}$ **c** $\frac{28}{323}$

8 **a** $\frac{1}{429}$ **b** $\frac{8}{429}$ **c** $\frac{429}{2048}$

9 **a** ${}^2C_0 + {}^2C_1 = {}^3C_1$, ${}^3C_2 + {}^3C_3 = {}^4C_3$, ${}^5C_1 + {}^5C_2 = {}^6C_2$
b ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$
c **i** ${}^{11}C_5$ **ii** ${}^{21}C_{15}$ **iii** ${}^{16}C_{13}$
iv ${}^{21}C_3$ **v** ${}^{52}C_{30}$ **vi** ${}^{100}C_{49}$

10 **a** 10 **b** 70 **c** 45
d 120 **e** 2520 **f** 990

11 **a** **i** 2 **ii** 4 **iii** 8
iv 16 **v** 32
b They are all powers of 2
c Since a combination is really a subset, the sum of all values ${}^nC_r$ for each possible r must be equal to the number of subsets of a set of n elements, which is equal to 2^n.

12 **a** 1 and 1 **b** ${}^nC_0 = 1$ and ${}^nC_n = 1$
c The row number and the second number in from both ends are the same.
d ${}^nC_1 = n$ and ${}^nC_{n-1} = n$
e They are equal, ${}^nC_r = {}^nC_{n-r}$

13 **a**
$$\begin{aligned}
{}^{n-1}C_{r-1} + {}^{n-1}C_r &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\
&= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right) \\
&= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{n}{(r(n-r)}\right) \\
&= \frac{n(n-1)!}{r(r-1)!(n-r)(n-r-1)!} \\
&= \frac{n!}{r!\,(n-r!)} \\
&= {}^nC_r
\end{aligned}$$

b **i** $n-1$
ii ${}^{n-1}C_{r-1}$. Since x is one of the r chosen elements, there must be $r-1$ elements chosen from the remaining $n-1$ elements in the set.
iii ${}^{n-1}C_r$. There are $n-1$ elements in the set not including x from which all r elements must be chosen.
iv The element x is either one of the r chosen elements or not. The number of ways of choosing r elements from a set of size n must equal the sum of the ways to include or exclude x from the r elements, which are ${}^{n-1}C_{r-1}$ and ${}^{n-1}C_r$, respectively. Hence, ${}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r$.

14 Starting with ${}^2C_2 = 1$ and using Pascal's identity,
${}^3C_2 = {}^2C_1 + {}^2C_2 = 2 + 1 = 3$,
${}^4C_2 = {}^3C_1 + {}^3C_2 = 3 + 3 = 3 + (2 + 1) = 6$,
${}^5C_2 = {}^4C_1 + {}^4C_2 = 4 + 6 = 4 + (3 + 2 + 1) = 10$,
we can generalise to ${}^{n+1}C_2 = {}^nC_1 + {}^nC_2 = n + ((n-1) + (n-2) + \cdots 3 + 2 + 1)$.

15 a $x^2 + 2x + 1$

b $x^3 + 3x^2 + 3x + 1$

c $x^4 + 4x^3 + 6x^2 + 4x + 1$

d The coefficients of $(x + 1)^n$ are the values in the nth row of Pascal's triangle.

e i $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

ii 4845

EX 10F Conditional probability

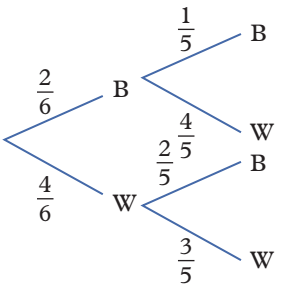
p497

1 a The probability $\frac{15}{20}$ means there are 15 soft centres out of a total of 20 chocolates. The probability $\frac{5}{20}$ means there are 5 hard centres out of the 20 chocolates.

b The branches on the tree indicate that if 1 soft centre is eaten, there are 14 soft centres and 5 hard centres left out of the remaining 19 chocolates. They also show that, if it had been 1 hard centre that had been eaten, there would have been 15 soft centres and 4 hard centres left out of the remaining 19 chocolates.

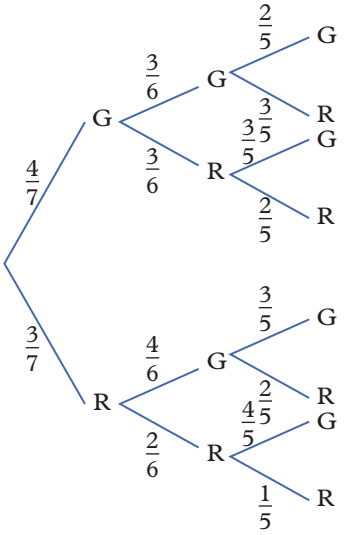
c $\frac{4}{19}$ **d** $\frac{5}{19}$ **e** $\frac{15}{19}$

2 a



b $\frac{3}{5}$ **c** $\frac{4}{5}$ **d** $\frac{2}{5}$

3 a



b $\frac{1}{2}$ **c** $\frac{2}{5}$ **d** $\frac{3}{5}$

4 a $\frac{12}{13}$ **b** $\frac{12}{13}$

5 a 15 **b** 18 **c** 18

d 59

6 a $\frac{15}{41}$ **b** $\frac{9}{20}$ **c** $\frac{9}{50}$

d $\frac{59}{100}$ **e** $\frac{18}{59}$

7 a 8

b 3 **c** $\frac{3}{8}$ **d** $\frac{3}{5}$

8 a $\frac{4}{49}$ **b** $\frac{7}{12}$

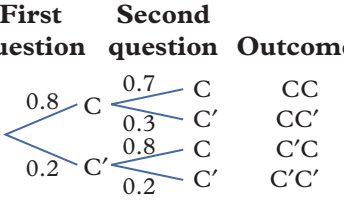
9 a i $\frac{7}{10}$ **ii** $\frac{3}{10}$ **iii** $\frac{2}{3}$

iv $\frac{4}{7}$ **v** $\frac{3}{11}$

b $\frac{1}{2}$

10 a $\frac{1}{3}$ **b** 0.25

11 a



b i 0.7 **ii** 0.8 **iii** 0.3

12 a

	Hybrid	WFH	Total
Full-time employees	30	4	34
Part-time employees	5	11	16
Total	35	15	50

b i 4 **ii** 16 **iii** 35

c i $\frac{5}{35}$ **ii** $\frac{5}{16}$

iii $\frac{2}{17}$ **iv** $\frac{4}{15}$

d i In part **i**, the selection is made from hybrid workers (total of 35), whereas in part **ii** the selection is made from part-time employees (total of 16).

ii In part **iii**, the selection is made from full time employees (total of 34), whereas in part **iv** the selection is made from WFH employees (total of 15).

13 a

	Small	Large	Total
Silver	27	35	62
Coloured	18	10	28
Total	45	45	90

b i True; there are 45 of each.

ii True; there are 28 coloured paper clips but there are 62 silver paper clips.

iii False; there are 27 small silver paper clips out of a total of 45 small paper clips. $\frac{27}{45} = 60\%$

iv False; there are 10 large coloured paper clips out of a total of 28 coloured paper clips. $\frac{10}{28} \approx 36\%$

v True; there are 62 silver paper clips out of a total of 90 paper clips. $\frac{62}{90} \approx 69\%$

14 0.78

15 a i 32 **ii** 41 **iii** 57

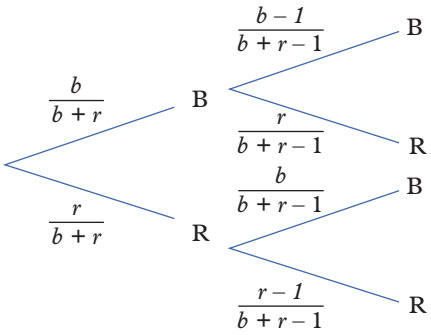
b i $\frac{11}{32}$ **ii** $\frac{11}{41}$ **iii** $\frac{35}{57}$

c They help to restrict the sample space, giving the appropriate denominator for the conditional probability.

16 a i 0.2 **ii** 0.5

b $\Pr(X) = 0.1$ and $\Pr(Y) = 0.25$

17 a



b $\frac{b}{b+r}$. This is equal to the probability that the first ball is blue. The denominator from the conditional probabilities in the second step cancels leaving this simplified fraction.

18 a 0.64 **b** $\frac{2}{3}$

19 a Start with $\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$

Multiplying through by $\Pr(A)$ gives $\Pr(A \cap B) = \Pr(B|A) \times \Pr(A)$.

Substituting this result into

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$ gives

$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$, which is Baye's rule.

b i 0.55 **ii** $\frac{7}{11}$

EX p503

10G Independence

1 a i 0.4 **ii** 0.5

b independent

2 a $\frac{2}{3}$ **b** $\frac{1}{4}$ **c** $\frac{1}{6}$

3 Yes. $\Pr(\text{cricket}) = \frac{9}{21} = \frac{3}{7}$ and $\Pr(\text{cricket}|\text{Maths}) = \frac{6}{14} = \frac{3}{7}$

4 a independent **b** dependent

c dependent **d** independent

5 a i $\frac{3}{4}$ **ii** $\frac{3}{4}$

b independent

6 a i $\frac{2}{3}$ **ii** $\frac{7}{9}$

b dependent

7 a 0.48 **b** 0.92

8 a He assumed A and B were independent events and evaluated $\Pr(A) \times \Pr(B)$.

b 0.38. It must be calculated using the addition rule: $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

9 Mutually exclusive events are events that cannot occur at the same time while independent events have no effect on each other occurring.

10 a dependent **b** independent

c independent **d** dependent

11 Q and R, P and Q

$\Pr(Q) \times \Pr(R) = \frac{9}{30} = \Pr(Q \cap R)$

$\Pr(P) \times \Pr(Q) = \frac{1}{5} = \Pr(P \cap Q)$

12 a i $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{3}$ **iv** $\frac{3}{10}$

b because $\Pr(\text{Action}|\text{Odd}) = \Pr(\text{Action})$

c because $\Pr(\text{Comedy}|\text{even}) \neq \Pr(\text{Comedy})$

d There is no logical reason that these two things should be connected. They appear dependent due to random variation.

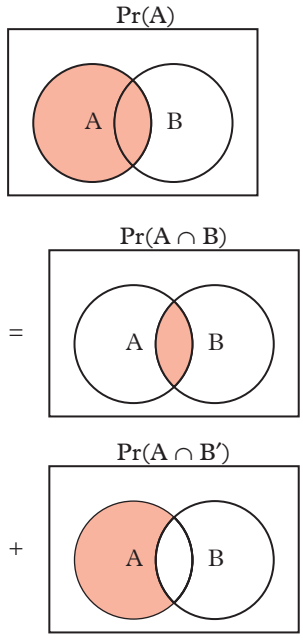
13 a The colour of the first ball selected will not affect the probability that the second ball is white.

b i $\frac{2}{3}$ **ii** $\frac{3}{5}$

c i $\frac{4}{9}$ **ii** $\frac{2}{5}$

14 $\Pr(A) = 0.6$

15 a



b $\Pr(A) \times \Pr(B') = \Pr(A) \times (1 - \Pr(B))$

$= \Pr(A) - \Pr(A) \times \Pr(B)$

$= \Pr(A) - \Pr(A \cap B)$

$= \Pr(A \cap B')$

CHAPTER 10 review

Multiple choice

1 D **2** D **3** C **4** D **5** C
6 A **7** D **8** B **9** E **10** A
11 D **12** B

Short answer

1 a {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

b i {1, 3, 5, 7, 9} **ii** {3, 6, 9}
iii {3, 9} **iv** {1, 3, 5, 6, 7, 9}
v {1, 5, 7}

2 a $\frac{9261}{17\,576}$ **b** $\frac{8315}{17\,576}$
c $\frac{9261}{17\,576}$ **d** $\frac{8315}{17\,576}$

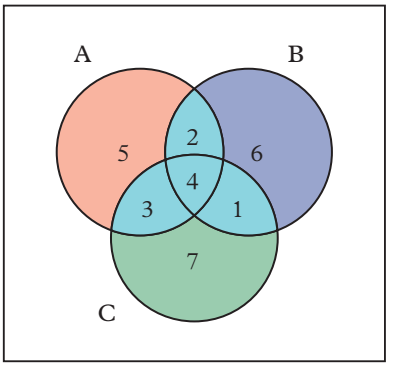
3 a You can't use zero as the first digit (the hundreds digit), because this will produce a two-digit number, not a three-digit number.

b 900
c $\frac{1}{900}$ **d** $\frac{1}{100}$

4 a You can't use zero as the first digit (the hundreds digit), because this will produce a two-digit number, not a three-digit number. There also can't be any repetitions of digits.

b 648
c $\frac{1}{648}$ **d** 360

5



6 a 15 **b** 27 **c** 47 **d** 55

7 a $\frac{4}{11}$ **b** $\frac{1}{330}$ **c** $\frac{1}{3960}$

8 a $\frac{140}{429}$ **b** $\frac{8}{429}$

c There must be between 2 and 7 junior players and 3 and 8 senior players to make up the 10 players other than the captain in the team.

d 5 seniors

9 a $\frac{1}{5}$ **b** $\frac{1}{5}$ **c** $\frac{17}{55}$ **d** $\frac{38}{35}$

10 a $\frac{1}{11}$ **b** $\frac{2}{11}$ **c** $\frac{2}{5}$
d 1 **e** $\frac{3}{4}$

11 a $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{7}{12}$

12 a i $\frac{2}{5}$ **ii** $\frac{7}{50}$
iii $\frac{16}{25}$ **iv** $\frac{1}{5}$

b A and C. $\Pr(A) = \frac{20}{50} = \frac{2}{5}$ and $\Pr(A|C) = \frac{8}{20} = \frac{2}{5}$

Analysis

1 a i $\frac{1}{9}$ **ii** $\frac{1}{3}$ **iii** $\frac{5}{9}$

b {4, 5, 6, 7, 8, 10}

c i $\frac{1}{81}$ **ii** $\frac{10}{27}$ **iii** $\frac{7}{81}$

d i 9% **ii** 25% **iii** 24%

e i 30% **ii** 25% **iii** 86%

2 a i $\frac{1}{2}$ **ii** $\frac{3}{8}$ **iii** $\frac{1}{8}$ **iv** 0

b i $\frac{1}{20}$ **ii** $\frac{2}{5}$ **iii** $\frac{1}{8}$

c i $\frac{7}{60}$ **ii** $\frac{1}{6}$ **iii** $\frac{5}{7}$

d 6 green, 3 red, 10 yellow

e i $\frac{1}{6}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{10}$ **iv** $\frac{1}{180}$

f 6 green, 3 red, 10 yellow

i $\frac{1}{6}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{10}$ **iv** $\frac{1}{180}$

g The answers for both boxes are the same because ${}^{n}C_{r} = {}^{n}C_{n-r}$. If the balls are unchanged in Box A, then the balls in Box B must also be the same, so they are equally likely.

h $\frac{1}{2}$. This is the mostly likely outcome because, for each colour, it is more likely that the set of balls making up the colour in each box will be different to what it was originally.

EX p510 Algorithmic Thinking 3 Statistics and probability

Task 1

1 The conditional statements are the If/Else statements. The iteration statements are the Repeat until statements.

2 The first Repeat until directive takes each value from the DataList. The second Repeat until directive searches for where to put the value in the OutputList so it is in order.

3 Each value is taken from the DataList in consecutive order.

4 Example 1 is sorting the data into numerical order.

Task 2

1 OutputList sorts the data in ascending order.

2 Sample answer:

```
Input SortedList
Set the value minimum to first item of SortedList
Set the value SortedListLength to length of SortedList
Set the value maximum to last item of SortedList
If the SortedListLength is even then
    Set m to SortedListLength/2
```

Set the median to the average of SortedList values at positions m and (m + 1)

Else

Set m to (SortedListLength + 1)/2

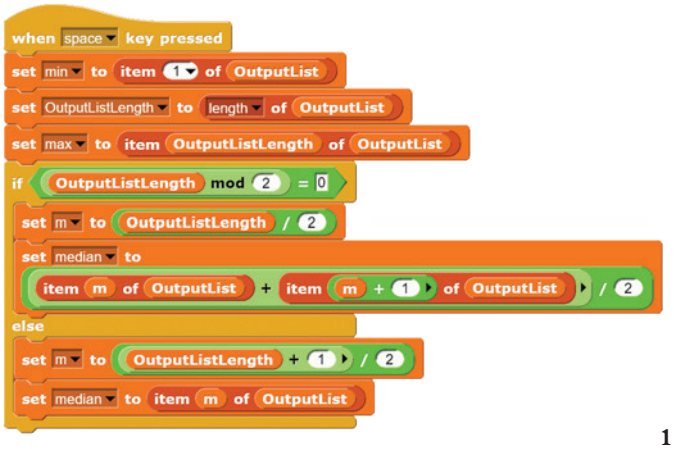
Set the median to the value at position m of SortedList

End if

3 Teacher to check.

4 Teacher to check.

5 Sample answer in Snap!:



6 Teacher to check.

EX p512

SEMESTER 2 review

Short answer

1 **a** 8.48 cm² **b** 250.45 cm²
c 153.06 cm² **d** 100.53 cm²
e 127.31 cm² **f** 1086.40 cm²

2 **a** **i** 428.24 cm² **ii** 410.79 cm³
b **i** 250.40 cm² **ii** 192.00 cm³
c **i** 471.24 cm² **ii** 785.40 cm³
d **i** 502.38 cm² **ii** 695.70 cm³

3 **a** 29 994.4 km **b** 0.68%

4 **a** **i** 1145.75 cm² **ii** 31.83 cm²
b **i** 9777.56 cm² **ii** 271.60 cm²

5 **a** **i** 963.54 cm² **ii** 1386.42 cm³
b **i** 563.40 cm² **ii** 648.00 cm³
c **i** 1060.29 cm² **ii** 2650.73 cm³
d **i** 1130.36 cm² **ii** 2347.99 cm³

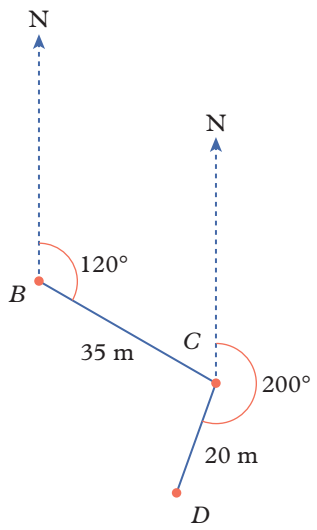
6 **a** **i** $1.793\,85 \times 10^{3}$ **ii** $5.105\,232\,79 \times 10^{6}$
iii 1.2793×10^{-6}
b **i** 3 **ii** 6 **iii** −6
c **i** 3.254 **ii** 6.708 **iii** −5.893

7 **a** right-angled triangle **b** acute triangle
c right-angled triangle

8 **a** **i** 4 cm **ii** $a = 63°, b = 27°$
b **i** 13 cm **ii** $a = 23°, b = 67°$

9 **a** 23.2° **b** 77 m

10 **a** **i** 334°T **ii** 154°T **iii** 2.8 km
b **i**



ii 17.5 m
iii 30.3 m
iv 18.8 m
v 6.8 m
vi 327°T
vii 43.2 m

11 **a** $a = 11.18$ cm, $b = 21.35$ cm, $c = 21.93$ cm
b $x = 13.97$ cm, $y = 26.16°$, $z = 9.04$ cm

12 **a** $\frac{2\pi}{5}$ **b** 135° **c** $\frac{5\pi}{4}$
d 70° **e** $\frac{11\pi}{6}$ **f** 195°

13 **a** $\sin(60°) = \frac{\sqrt{3}}{2}$
b $-\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
c $-\tan(45°) = -1$
d $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
e $\cos(60°) = \frac{1}{2}$
f $-\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

14 **a** $x = 210°, 330°$ **b** $x = \frac{\pi}{4}, \frac{7\pi}{4}$
c $x = 60°, 240°$ **d** $x = \frac{3\pi}{2}$
e $x = 0°, 360°$ **f** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

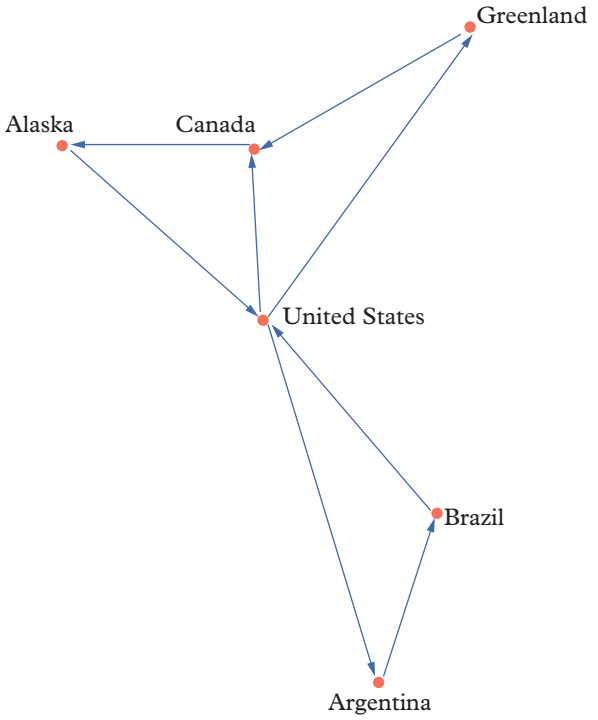
15 **a** amplitude = 5, period = 90°
b amplitude = 3, period = 1080°

16

Vertex	A	B	C	D	E
Degree	4	2	3	3	2

17 **a** closed walk **b** walk
c trail **d** invalid walk

18



19 a minimum = 32, Q_1 = 48, median = 55.5, Q_3 = 64, maximum = 76; IQR = 16

b minimum = 10, Q_1 = 20, median = 25, Q_3 = 40, maximum = 70; IQR = 20

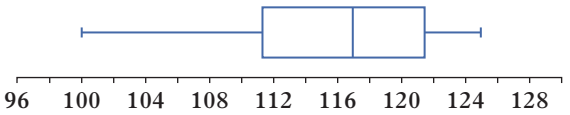
c minimum = 10, Q_1 = 11, median = 13, Q_3 = 14, maximum = 18; IQR = 3

d minimum = 1.2, Q_1 = 2.35, median = 3.4, Q_3 = 4, maximum = 4.4; IQR = 1.65

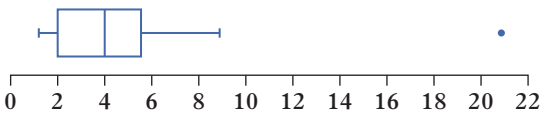
e minimum = 100, Q_1 = 102, median = 103, Q_3 = 104, maximum = 106; IQR = 2

f minimum = 10, Q_1 = 40, median = 60, Q_3 = 80, maximum = 110; IQR = 40

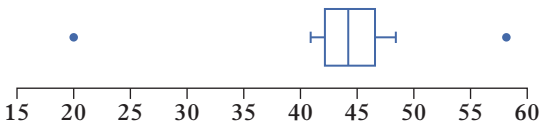
20 a



b



c



21 a false

b true

c true

d true

e false

f false

22 a positively skewed

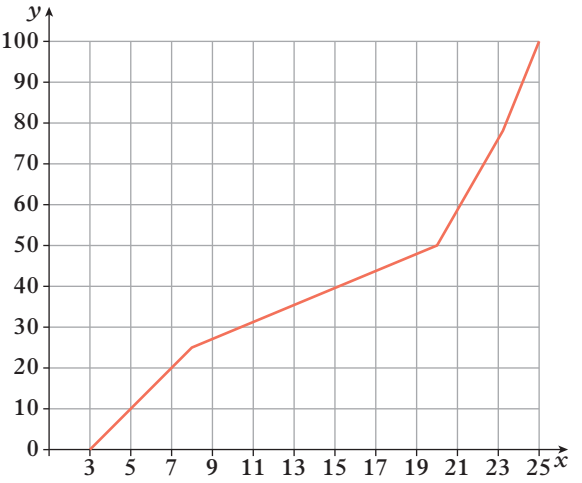
b negatively skewed

c symmetric

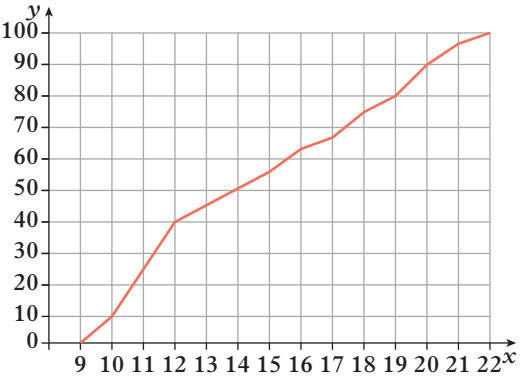
d negatively skewed

e positively skewed

23 a



b



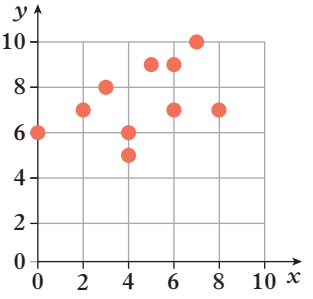
24 a moderate, negative, non-linear

b moderate, positive, linear

c weak, negative, linear

d no correlation

25



26 a i test score = 32.5

ii test score = 38

iii hours on social media = 2.5

iv hours on social media = 23

b **i** unreliable

ii reliable

iii reliable

iv unreliable

27 a Biased: the competitors at the tournament are all likely to favour the game they are playing in the tournament over other fighting games, and not enough people are being sampled.

b Fair: the sample size is large enough for the population; the students are questioned about what is being investigated.

c Biased: the sample size should be at least equal to the square root of the population so this sample size is too small (less than $\sqrt{5\,000\,000} \approx 2236$).

d Biased: the sample of all Australians does not represent all the people in the Southern Hemisphere.

28 a While 51% is the majority, it is only just over half of the responses to a two-option question and there's no indication of the reliability of the result.

b The fraction '9 out of 10' sounds like it is a simplified fraction from a large sample. However, it is only 9 from a sample of 10, and those 9 were biased because they wanted their answer to positively reflect on their sponsor.

29 a **i** unfair

ii unfair

iii Peta should either sample people from a larger population than her local area or should have made a conclusion only about people in her local area who go to the local shopping centre in a five-hour period on Saturday.

b **i** fair

ii unfair

iii Rob should have made his conclusions about basketball players in the NBA, not about all basketball players.

30 a $\bar{x} = 37.75$, $s = 4.79$, $\sigma = 4.15$

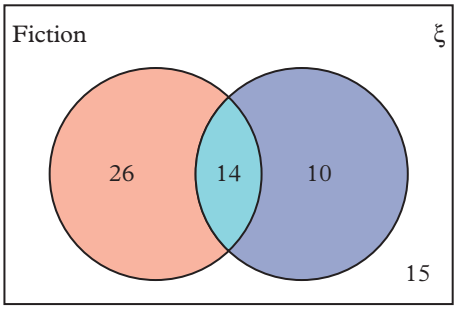
b $\bar{x} = 22.2$, $s = 3.90$, $\sigma = 3.49$

c $\bar{x} = 4$, $s = 0$, $\sigma = 0$

31 a

	Fiction	Non-Fiction	Total
In a series	14	10	24
Not in a series	26	15	41
Total	40	25	65

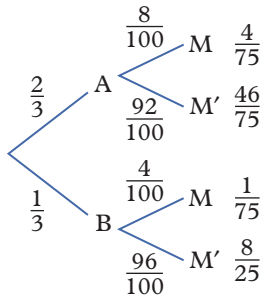
b



c **i** $\frac{2}{5}$ **ii** $\frac{2}{13}$

iii $\frac{7}{20}$ **iv** $\frac{15}{41}$

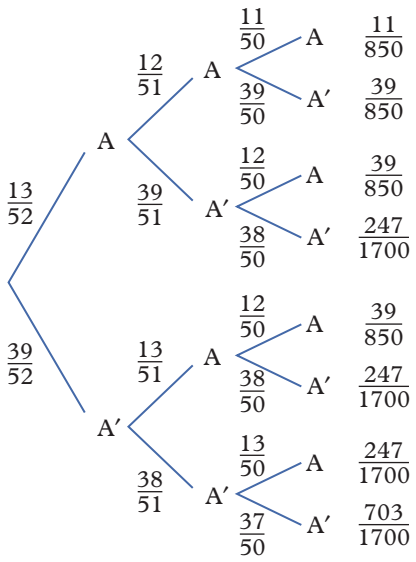
32 a



b **i** $\frac{4}{75}$ **ii** $\frac{8}{25}$

iii $\frac{2}{25}$ **iv** $\frac{4}{5}$

33 a



b **i** $\frac{11}{850}$ **ii** $\frac{117}{850}$

iii $\frac{247}{425}$ **iv** $\frac{13}{50}$

34 a $\frac{21}{40}$ **b** $\frac{1}{2}$ **c** $\frac{3}{20}$

d $\frac{13}{20}$ **e** $\frac{3}{10}$ **f** $\frac{14}{19}$

35 a dependent

b independent

c independent

d dependent

36 a **i** $(2n)!$ **ii** $(n!)^2$ **iii** $2(n!)^2$

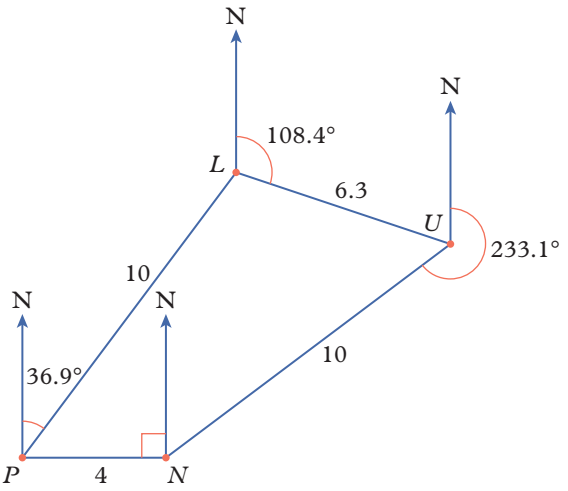
b $n(2n-1)$

c $\frac{n}{(2n-1)}$

d $\frac{(n-1)}{2(2n-1)}$

Analysis

1 a



b 30.3 km

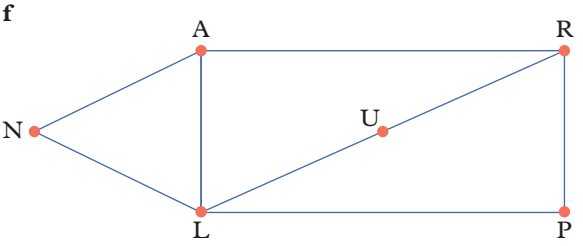
c 5.05 km/h

d 243 m

e **i** 2% **ii** 76%

iii 18% **iv** $\frac{2}{3}$

f



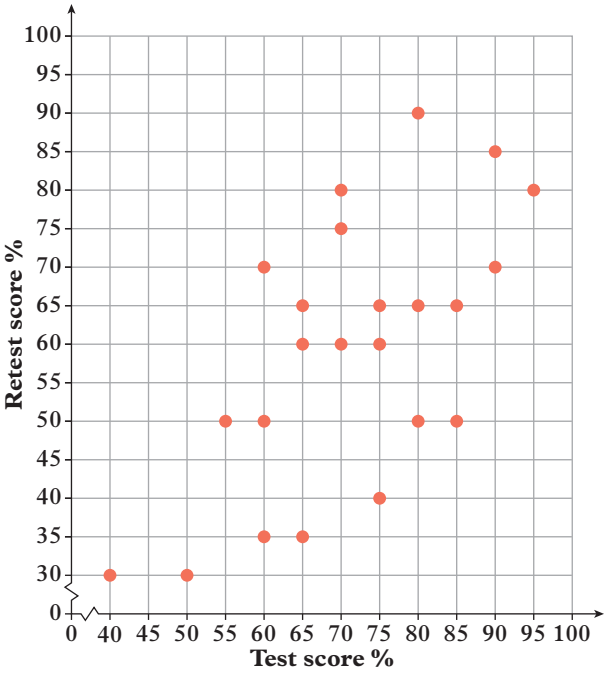
g **i** yes

ii There are two vertices of odd degree in the graph. Yes, there is an Eulerian trail.

iii no

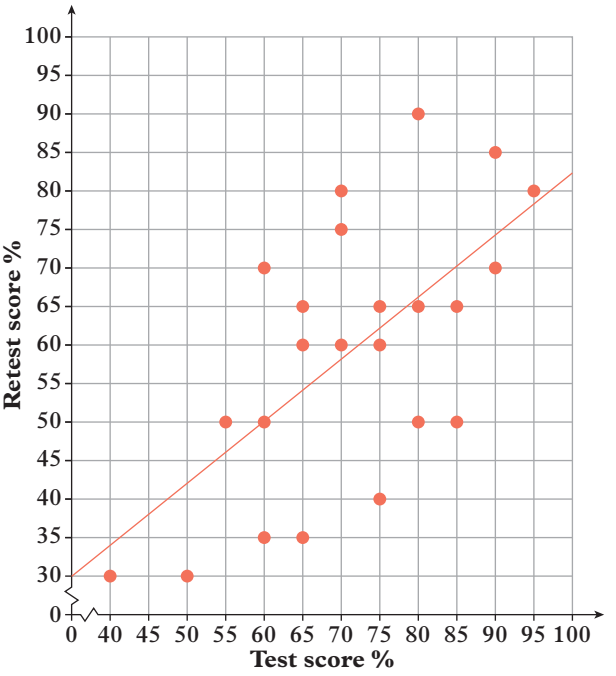
2 a test scores

b



c moderate, positive, linear

d



e **i** 60%

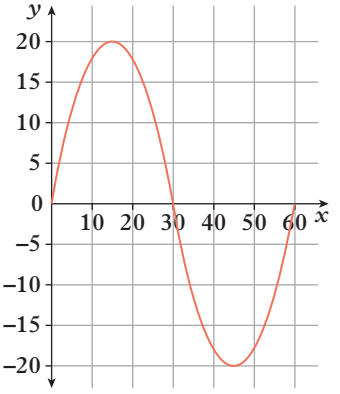
ii reliable

f There are 11 students who scored at or above 75% on the original test. Of these, 8 scored better on the retest than the median of those who scored less than 75% on the original test. The remaining 3 students of the 11 who scored at or above 75% on the original test received scores on the retest that were worse than a majority of those who scored less than 75% on the original test. Therefore, while the teacher's claim is somewhat accurate, the moderate correlation means that there are students whose results do not fit the teacher's claim.

3 a amplitude = 20 m, period = 1 hour (60 minutes)

b $h = 20 \sin\left(\frac{t\pi}{30}\right)$

c



d **i** $10\sqrt{3}$ m

ii 17.32 m

e 10 m

f 37.5 minutes and 52.5 minutes after boarding

g **i** $10\sqrt{2}$ m left and $10\sqrt{2}$ m above

ii 14.14 m left and 14.14 m above

h **i** 15π m

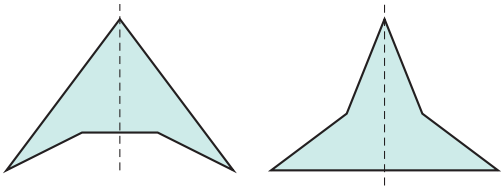
ii 47.12 m

EX EXPLORATIONS 2

p524

1 **a** None, so a thorn triangle is just an isosceles triangle.

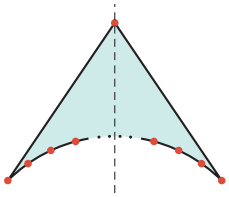
b



c Thorn heptagons have 4 reflex angles. Trying to keep them all separated would require at least 4 non-reflex angles.

d Thorn octagons have 5 reflex angles, leaving less than 180° to share among the remaining 3 angles.

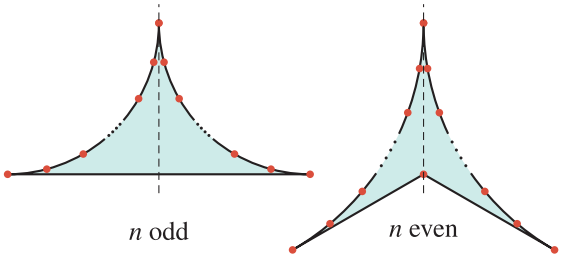
e 1) Place $n - 1$ equally spaced points along a circular arc, as shown. There are other strategies.



2) The angle sum of an n-gon is $180(n - 2)$, which is strictly less than the sum of $n - 2$ reflex angles.

f 9

g Consider equally spaced points along two identical circular arcs. Treat the odd and even cases separately.



n odd n even

2 **a** **i** $\frac{3}{8}$

ii $SU = \frac{1}{6}$, $SV = \frac{5}{24}$, $UV = \frac{1}{8}$

iii $\frac{3}{8}$

b $PQ = \frac{1}{7}$ or $\frac{3}{4}$

c $PQ = \frac{a}{b + c}$, so for 4 : 3 : 5, $PQ = \frac{1}{3}$

3 **a** **i** 48 cm^3

ii 48 cm, 60 cm, 64 cm

iii 88.78 cm^2, 118.10 cm^2, 153.21 cm^2

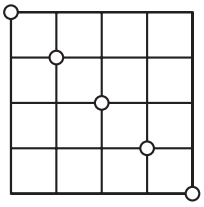
b It is enough to show the triangular faces have areas 15 555 cm^2, 23 925 cm^2 and 28 260 cm^2.

c Show that the perpendicular heights of the triangular faces are the integers c^3, $a(4b^2 + c^2)$ and $b(4a^2 + c^2)$.

4 **a** There are $2^4 = 16$ combinations of horizontal and vertical moves after 4 seconds. Four of these finish at P.

b $\frac{3}{16}$

c The counters can only meet when they are both halfway; that is, after 4 moves:



d $\frac{35}{128} \approx 0.273$

e Less likely, with probability $\frac{1120}{6561} \approx 0.171$

5 **a** **i** 9

ii week 19

b If there were k kirons, ℓ lirons and m mirons 5 weeks ago, then 1 week ago there were $\ell + m$ kirons and this week there are $k + \ell + m$ kirons. Yes, it is also true for lirons and mirons.

c 8 algae (4 kirons and 4 mirons)

d **i** week 13 **ii** 15 days

Glossary

AAA
the abbreviation for angle–angle–angle. Triangles are similar if all three angles are equal. The condition AAA does not necessarily mean the triangles are congruent

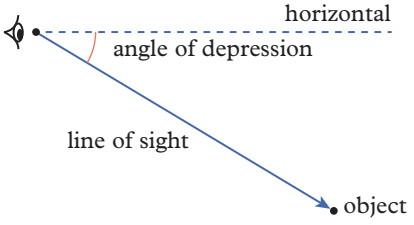
AAS
the abbreviation for angle–angle–side. Triangles are congruent if there are two pairs of corresponding equal angles and two corresponding sides are the same length

absolute error
the magnitude of an error

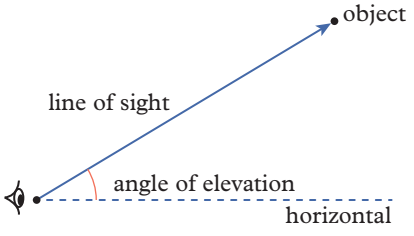
addition principle
the number of ways to choose one element from either a set containing m elements or a set containing n elements is $m + n$

algebraic fractions
fractions that contain at least one pronumeral

angle of depression
the angle between a horizontal line and an observer's line of sight to an object that is below the horizontal



angle of elevation
the angle between a horizontal line and an observer's line of sight to an object that is above the horizontal



arc
the curved edge of a sector; part of a circle's circumference; the shape usually used to mark angles on diagrams

area
the amount of space enclosed by a two-dimensional (2D) shape

area scale factor
the multiplication factor between the area of an object and an image

array
a collection of objects or numbers arranged in ordered rows and columns

asymptote
an imaginary line that a graph line approaches but never reaches

average rate of change
the gradient of the line connecting the two endpoints of an interval

axis of symmetry
an imaginary line that divides a symmetrical shape or graph so that one side is a reflection of the other

base
the number that is repeated multiplied by itself; for example, in 24 the base is 2

base-ten
the traditional number system we use which uses the digits 0 to 9

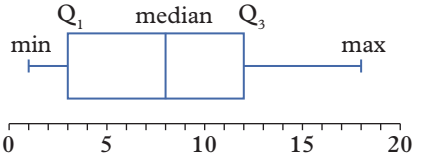
bi-modal (distribution)
a distribution of numerical data with two clear peaks

biased sample
a sample is considered biased if the method of collecting data produces a sample that is not an accurate representation of the population

binomial
an algebraic expression for the sum or the difference of two terms

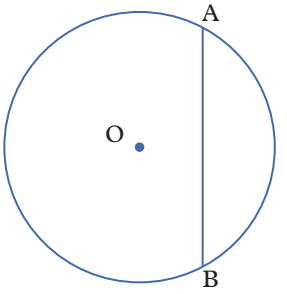
bivariate data
data showing the relationship between two variables

boxplot
a visual representation of the five-number summary of a numerical data set along a number line. The 'box' represents the middle 50% of ordered data, with vertical lines indicating the first quartile, median and third quartile. The 'whiskers' extend from the left of the box to the minimum value and from the right of the box to the maximum value
see also five-number summary



capacity
the maximum amount (usually of fluid or gas) that a three-dimensional object (container) can hold; usually measured in litres (L) or millilitres (mL)

chord
a line segment joining two points on the circumference of a circle. In the diagram below, AB is a chord



circuit
a trail with the same first and last vertices

circumference
the perimeter of a circle

closed walk
a walk with the same first and last vertices

coefficient
the number by which a pronumeral is multiplied (usually placed before the pronumeral in an algebraic expression); for example, in $3x$, the coefficient is 3

combination
a selection of the elements in a set where the order of selection does not matter. The number of combinations of size $r \leq n$ from a set containing n elements is:

$$^nC_r = \frac{n!}{r!(n-r)!}$$

compass bearing
direction describing one position from another by using the number of degrees from north or south towards east or west that the new position is from the first; for example, S43°E represents a direction that is 43° east of due south

complement
the complement of an event A is the event that A does not occur. The sum of the probabilities of an event and its complement is 1

complementary angles
angles that add to make a right angle (90°)

completing the square
the process to change a quadratic expression from the general form, $ax^2 + bx + c$, to the form $a(x - h)^2 + k$, by applying the 'perfect square' rule

composite shape
shape made up of more than one regular shape

conditional probability
the likelihood of an event or outcome occurring based on the occurrence of a previous event or outcome

congruent
two figures are congruent if they are identical in shape and size, but not necessarily in position or orientation

conjugate
the conjugate of a binomial surd expression is formed by changing the operation sign between the two terms to the opposite operation sign (+ to – and – to +); for example, the conjugate of $(3 + \sqrt{2})$ is $(3 - \sqrt{2})$. When a binomial surd expression is multiplied by its conjugate, the product is a rational number

connected (graph)
a graph where a walk exists between every pair of vertices

constant term
a term without any pronumeral component that has a fixed value in a given context

conversion factor
a multiplier for converting a quantity expressed in one set of units into another quantity expressed in a different set of units; for example, multiply by 10 to convert from cm to mm

correlation
the description of the strength (weak, moderate, strong) and direction (positive, negative) of the relationship between bivariate data displayed in a scatterplot

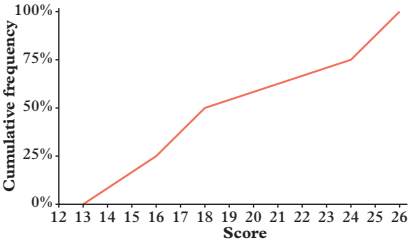
cosine (cos)
in trigonometry, the cosine of an angle in a right-angled triangle equals the ratio of the length of the adjacent side to the length of the hypotenuse.

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

cross-section
the shape of the surface revealed when we cut straight through an object. When the 'slice' is cut parallel to the base of a prism or cylinder, the cross-section has the same shape and size as that base

cumulative frequency
for any frequency distribution, the sum of all the frequencies up to the current point. Cumulative frequency can be displayed in a table or on a graph (*see* cumulative frequency polygon)

cumulative frequency polygon
a graph of a cumulative distribution, displaying data values or class intervals on the horizontal axis and the cumulative frequencies on the vertical axis



cylinder
a three-dimensional (3D) object with a uniform circular cross-section parallel to its circular base

degree (face)
the number of edges along the border of a face

degree (vertex)
the number of edges connected to a vertex

dependent variable
a variable whose value depends on the value of another variable in a relationship between two variables. The dependent variable is listed second in a table of values and in pairs of coordinates, and is shown on the vertical axis of the Cartesian plane

difference of two squares
when an expression is seen to be the difference of two perfect squares, $a^2 - b^2$, it can be factorised as: $a^2 - b^2 = (a + b)(a - b)$

dilation
a transformation that can enlarge or compress the graph of a function in the x- or y-direction

directed edge
an edge that has a direction indicated by an arrow

directed network
a network with directed edges

distributive law
the rule indicating that the result of first adding several numbers and then multiplying the sum by another number is the same as first multiplying each of the several numbers separately by that number and then adding the products: $a(b + c) = ab + ac$

domain
the complete set of possible values for the independent variable, shown on the horizontal axis of the Cartesian plane (the x- or θ-values), to produce a graph or define a rule

edge (network)
a relation between two vertices in a network

element
a member of a set; for example, 5 is an element of {1, 3, 5, 7, 9}

elimination method
the algebraic method used to solve simultaneous equations containing more than one variable by forming a new equation with all but one of the variables removed

enlargement
larger image produced after a figure has been dilated

equidistant
at equal distances from a point

error
the difference between the exact value and estimated value of a quantity

ethics
the system of moral principles that distinguishes between right and wrong behaviour

Euler's formula
a formula that states that in a connected planar graph the sum of the number of faces and vertices is equal to two more than the number of edges; $F + V = E + 2$

Eulerian
a graph is Eulerian if it contains an Eulerian circuit

Eulerian circuit
a circuit containing every edge in a graph exactly once

Eulerian trail
a trail containing every edge in a graph exactly once

event
a subset of the sample space for a random experiment. It is one of the possible outcomes or sets of outcomes for the experiment; for example, for the experiment of rolling a die, a single-outcome event might be 'rolling a 2'; and an event that includes a set of outcomes might be 'rolling a 2 or a 4'

expansion of a perfect square
a pattern observable when expanding a factorised expression in which a factor is multiplied by itself, represented as:

$(a + b)^2 = (a + b)(a + b)$
$= a^2 + 2ab + b^2$

or: $(a - b)^2 = (a - b)(a - b)$
$= a^2 - 2ab + b^2$

experiment
a repeatable procedure with a clearly defined set of possible results

explanatory variable
see independent variable

exponent
for a value expressed in exponent form (exponent notation), the exponent indicates the number of times the base is multiplied by itself as a repeated multiplication; for example, in 2^4 the exponent is 4 and $2^4 = 2 \times 2 \times 2 \times 2$

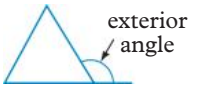
exponent form
see exponent notation

exponent laws
the set of laws that can be used to simplify calculations involving numbers in exponent form

exponent notation (exponent form)
a shorter form of writing a repeated multiplication, in which a number is written with a base and an exponent; for example, 2^4 is written in exponent notation and means $2 \times 2 \times 2 \times 2$

exponential relationship
a relationship between two variables, one of which is an exponent (power or index); for example, $y = 2^x$

exterior angle
an angle between any side of a shape and a line extended from the adjacent side



faces
the regions of a planar graph that are bordered by edges

five-number summary
five pieces of information related to a set of numerical data: minimum value, lower quartile, median, upper quartile and maximum value

gradient (m)
the measure of the steepness of a straight line; or the numerical measure of the slope of a graph

$\text{gradient} = \frac{\text{rise}}{\text{run}} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$

gradient–intercept form
the general form of the equation of a line; $y = mx + c$, where m is the gradient and c is the y value of the y-intercept of the corresponding linear graph

graph
see network

handshaking lemma
propositions that state that in every network the sum of vertex degrees is twice the total number of edges (edges version), and in every planar graph the sum of face degrees is twice the number of edges (faces version)

histogram
a graphical display of data using bars or columns of different heights, similar to a bar chart, but showing the frequency of numerical data grouped into ranges or class intervals, with the horizontal scale boundaries located at the edges of the columns, rather than at their centres

hypotenuse
the longest side in a right-angled triangle, appearing opposite the right angle

independent events
two events in a probability experiment are considered independent when the outcome of one event does not affect the outcome of the other event

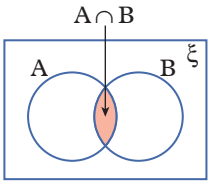
independent variable
the quantity in a relationship between variables which does not rely on any other variable for its value and is not affected by changes in another variable. The independent variable is the value listed first in a table of values and in pairs of coordinates, and the value shown along the horizontal axis on a Cartesian plane

index (*plural*: indices)
see exponent

instantaneous rate of change
the gradient at a given point

interquartile range (IQR)
a measure of spread for a set of data, calculated by subtracting the lower quartile (Q_1) from the upper quartile (Q_3); that is, IQR = $Q_3 - Q_1$. The interquartile range indicates the spread or range of the middle 50% of the data

intersection
the intersection of two sets ($A \cap B$) is the set of elements that appear in both set A and set B



irrational number
a number that cannot be written in fraction form; irrational numbers are non-terminating decimals that are not recurring

leading coefficient
the coefficient of the squared variable in a quadratic trinomial; for example, in $ax^2 + bx + c$, a is the leading coefficient

length scale factor
the multiplication factor between the length of an object and an image

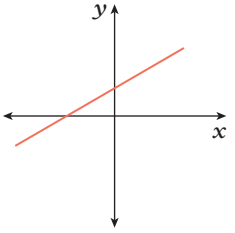
like surds
surd terms that have the same number or pronumeral/s under the root sign; for example, $2\sqrt{5}$, $-\sqrt{5}$ and $7\sqrt{5}$ are like surds

like terms
terms that contain the same pronumerals raised to the same power

line of best fit
a line that best represents the relationship between the two variables on a scatterplot. Lines of best fit are usually drawn as straight lines

linear equation
an equation that contains only a pronumeral term (or terms) for which the highest power is one; for example, $2x + 3 = 6$ is a linear equation

linear graph
the graph of a linear relationship. The graph is a straight line



linear inequality
an inequality containing pronumerals raised only to the power of 1 and no two pronumerals are multiplied together; for example, $2x + 3 > 1$

linear relationship
a relationship between two variables that produces a linear graph

logarithm (log)
the power (exponent) to which a given number (the base) must be raised in order to produce the required number; for example, in $2^3 = 8$, the logarithm is 3 and can be written as $\log_2 8 = 3$

logarithm of a power
the logarithm of a term that is expressed as a power. The related logarithm law is: $\log_a x^n = n \times \log_a x$

logarithm of a product
the logarithm of the product of two terms. The related logarithm law is: $\log_a (xy) = \log_a x + \log_a y$

logarithm of a quotient
the logarithm of the quotient of two terms. The related logarithm law is: $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

logarithmic scale
a non-linear scale where the position of a value is marked using the logarithm of the value instead of the actual value

loop
an edge with the same vertex at both ends

lower fence
the lower end point in a set of data beyond which any lower data points are considered outliers. The lower fence lies 1.5 × IQR below the lower quartile.
see also interquartile range (IQR)

lower quartile
the median of the first half (or lower half) of an ordered set of numerical data, denoted by Q_1

maximum turning point
a point at which a parabola changes direction and at which it has its maximum y value

mean
a measure of the centre of a data set, calculated by adding all of the data values together and dividing by the number of values

median
a measure of the centre of a data set (the 'middle value'), determined by placing the data in ascending order (smallest to largest), and finding the middle number of the ordered set if there is an odd number of values, and the average of the two middle numbers if there is an even number of values

minimum turning point
a point at which a parabola changes direction and at which it has its minimum y value

modality
the number of peaks in a numerical data set

monic quadratic
a quadratic in which the leading coefficient is equal to one
see also leading coefficient

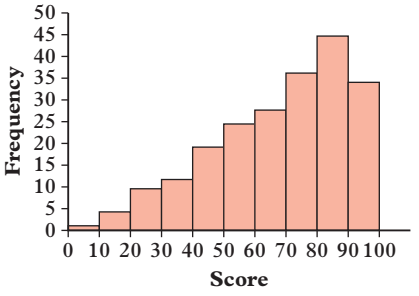
multiplication principle
the number of ways to choose one element from a set containing m elements and one element from a set containing n elements is $m \times n$

mutually exclusive events
two or more events that cannot occur at the same time

negative exponent
an exponent with a negative value, indicating repeated division

negative relationship
(between bivariate data) a relationship between two sets of numerical variables such that as one variable increases in value the other variable decreases in value

negatively skewed distribution
a distribution that is skewed away from what would be a symmetrical distribution, having a tail on the lower end of the distribution (which is the left of the distribution when graphed). The mean of this type of distribution is less than the median



neighbours
the vertices on either side of an edge in a network

network
a collection of objects and the relations between them

non-monic quadratic
a quadratic in which the leading coefficient is not equal to one
see also leading coefficient

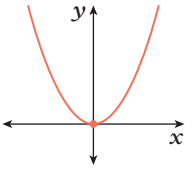
Null Factor Law
the law stating that, if the product of two factors is 0, one or both of the factors must be equal to 0

order of magnitude
the largest place value of a number in base ten

outcome
a possible result of a probability experiment

outlier
extreme piece of data that is much higher or lower than the rest of the data in a numerical data set

parabola
the approximately U-shaped graph of a quadratic relationship. A parabola can open up, down, left, right, or in any other direction, but is always symmetrical



parallel boxplots
two or more boxplots placed directly above each other on the same number line for comparison

parallel edges
two (or more) edges with the same endpoints

percentage error
the absolute error as a percentage of the exact value

period
horizontal distance covered by one full cycle of a trigonometric graph

periodic function
function or relationship that repeats itself continuously in cycles;
for example, the sine, cosine and tangent functions are periodic functions

permutation
an ordering of the elements in a set. There are $n!$ permutations of a set containing n elements. The number of permutations of $r \leq n$ elements in a set containing n elements is:

$$^nP_r = \frac{n!}{(n-r)!}$$

planar graph
a graph that can be drawn without crossing edges

Platonic solid
a convex, three-dimensional shape where each face is an identical regular polygon

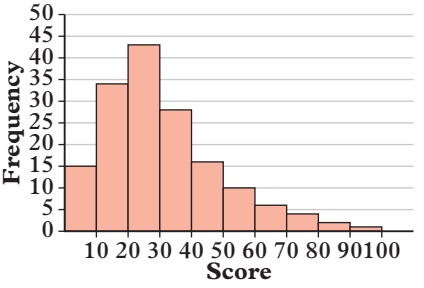
point of intersection
the point on a Cartesian plane where two graphs meet

polygon
two-dimensional (2D) shape with straight edges

population
the complete set of individuals or things about which statistical information is being collected

positive relationship
(between bivariate data) a relationship between two sets of numerical variables such that as one variable increases in value the other variable also increases in value

positively skewed distribution
a distribution that is skewed away from what would be a symmetrical distribution, having a tail on the upper end of the distribution (which is the right of the distribution when graphed). The mean of this type of distribution will be greater than the median



prism
object with two ends that are identical polygons and joined by straight edges
see also polygon

pronumeral
a letter or symbol that takes the place of a number

proof
series of logical statements showing that a statement is true in all cases

Pythagoras' theorem
the theorem stating that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the two other sides: $c^2 = a^2 + b^2$, where c is the length of the hypotenuse and a and b are lengths of the other two sides

quadratic formula
the formula used to solve equations of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$. Solutions are found using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

quadratic trinomial
any expression of the form $ax^2 + bx + c$, where a, b and c are constants

quadrilateral
a two-dimensional shape with four straight sides

quartiles
the values that divide a numerical data set into four quarters

quotient
the result of a division calculation; for example, in $28 \div 4 = 7$, the quotient is 7

radian
a unit for measurement of angles, where there are 2π radians in a revolution

range
1. the complete set of possible values for the dependent variable, shown on the vertical axis of the Cartesian plane (the y-values)
2. a measure of spread determined by finding the difference between the highest value and lowest value in a numerical data set

rational number
a number that can be written as a fraction with both the numerator and denominator being integers (whole numbers). This includes whole numbers, fractions, terminating decimals and recurring decimals

reciprocal
a value which, when multiplied by the given value, equals 1. For example, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$

reduction
smaller image produced after a figure has been dilated

reference angle
the acute angle between the x-axis and the line connecting the origin to the point on the unit circle

reflection
a transformation in which a shape or object is reflected in a mirror line to produce its exact mirror image (sometimes referred to as a 'flip')

relative error
the ratio between the absolute error and exact value of a quantity

response variable
see dependent variable

RHS
1. abbreviation for right angle–hypotenuse–side. Right-angled triangles are congruent if their hypotenuses are the same length and another pair of side lengths are equal
2. abbreviation for right angle–hypotenuse–side. Right-angled triangles are similar if their hypotenuses are in the same ratio as another pair of sides

rise
the vertical distance between two points on a line

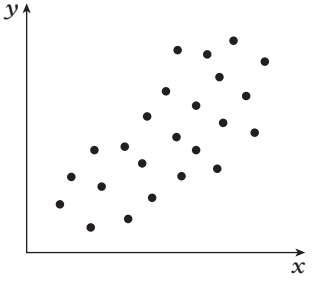
run
the horizontal distance between two points on a line

sample
a subset of a population. In statistical investigations a selection from a population is sometimes used to represent the whole population

sample space
the list of all the different outcomes possible for a probability experiment; for example, the sample space when rolling a die is {1, 2, 3, 4, 5, 6}

SAS
1. abbreviation for side–angle–side. Triangles are congruent if two corresponding side lengths are equal and the angle between them is equal
2. abbreviation for side–angle–side. Triangles are similar if two corresponding side lengths are in the same ratio and angle between them is equal

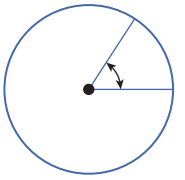
scatterplot
a graph that displays a relationship between two variables in which the points are plotted but not joined



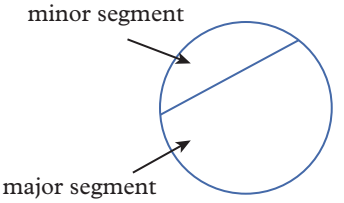
scientific notation
a value written in the form $a \times 10^m$, where $1 \le a < 10$ or $-10 < a \le -1$ and m is an integer

secant
a line that passes through at least *two* points of a non-linear graph

sector
a portion of a circle formed by two radii and part of the circumference



segment
the region enclosed in a circle by a chord and an arc
see also chord
see also arc



semi-Eulerian
a graph is semi-Eulerian if it contains an Eulerian trail

set
a collection of distinct objects or elements
see also element

similar
two figures or objects are similar if they have exactly the same shape. They do not need to be the same size

simple graph
a graph without loops or parallel edges

simultaneous linear equations
equations involving two or more unknowns that have the same values in those equations

sine (sin)
in trigonometry, the sine of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the hypotenuse

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

skewed distribution
a data set in which data points are clustered to the left or right of the distribution, unlike a symmetrical distribution

spread
a statistic describing how widely the data values in a set are distributed. Measures of spread include the range and interquartile range

SSS
1 abbreviation for side–side–side. Triangles are congruent if all three side lengths are equal
2 abbreviation for side–side–side. Triangles are similar if all three side lengths are in the same ratio

standard deviation
a measure of spread representing the difference between individual data values and the mean of a numerical data set

subject of an equation
the value that appears by itself on one side of an equation

subset
a set contained within another set

substitution
replacement of a pronumeral with a number

substitution method
the algebraic method used to solve simultaneous equations containing more than one variable, by rearranging to make one of the variables the subject of an equation equivalent to the first equation and substituting the value of that variable from that equation into the second equation, resulting in an equation that has only one variable. This equation can then be solved and its solution substituted back into the first equation so it too can be solved

supplementary angles
angles that add to make a straight angle (180°)

surd
an irrational root of an integer that cannot be simplified to remove the root; for example, $\sqrt{2}$

symmetric distribution
a similar distribution of frequencies on either side of a central peak

tangent
1 the tangent (tan) of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the adjacent side

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

2 a straight line that touches but does not cross a curve

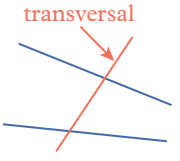
total surface area (TSA)
area of the total surface of an object. The total surface area of a prism is the sum of the areas of all the faces of that prism, commonly measured in square millimetres (mm^2), square centimetres (cm^2), or square metres (m^2)

trail
a walk without repeated edges

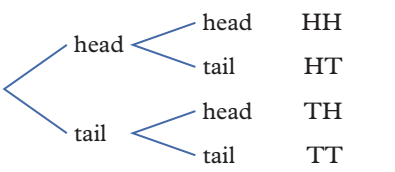
transformations
the general name for translations, reflections, rotations and dilations

translation
a transformation in which a shape or object is moved in a straight line without turning or changing size. The movement is often described as a number of units up, down, left or right

transversal
a line that crosses (intersects) two or more lines



tree diagram
a display of outcomes for a multi-step experiment; for example, a tree diagram can be used to display the possible outcomes of flipping two coins (as shown below)



trigonometry
the study of the relationships between the angles and side lengths of a triangle

trinomial
an expression containing three terms

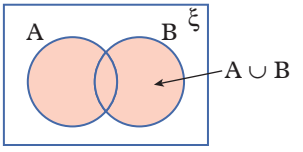
true bearing
the direction from one position to another, described by a number of degrees from north in a clockwise direction. The angle of a true bearing is written as three digits followed by T (or true); for example, 090°T is due east, 180°T is due south, and 270°T is due west.

turning point
the point on a graph where a parabola changes direction

turning point form
the general form of a quadratic equation, $y = a(x - h)^2 + k$, where a is the dilation factor, and the coordinates of the turning point are (h, k)

two-way table
a display of the outcomes of a survey or experiment that shows the relationship between two different categories

union
the union of two sets ($A \cup B$) is the set of elements that appear in A or B or both



unit circle
a circle drawn on the Cartesian plane with its centre at the origin and a radius of 1 unit. This circle is used to define trigonometric functions

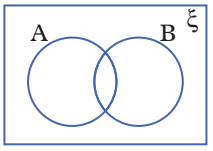
universal set
the set containing all elements. Everything inside the rectangle of a Venn diagram belongs to the universal set. The symbol for the universal set is ξ

upper fence
the upper end point in a set of data beyond which any greater data points are considered outliers. The upper fence lies 1.5 × IQR above the upper quartile *see also* interquartile range (IQR)

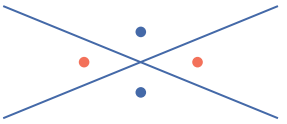
upper quartile
the median of the upper half of an ordered set of numerical data, denoted by Q_3

variable
a quantity that can have different values and can be represented with a word, symbol or pronumeral

Venn diagram
a display of the relationship between different sets of data. A rectangle represents the universal set, ξ, and each set of data within the universal set is represented by a circle



vertically opposite angles
a pair of equal angles that are not adjacent to (next to) each other, formed where two lines intersect. They are sometimes called 'X angles'. There are two pairs of vertically opposite angles in the diagram below



vertices (network)
the objects in a network, represented by single points

volume
the quantity of space that a three-dimensional (3D) object occupies

volume scale factor
the multiplication factor between the volume of an object and an image

walk
a sequence of vertices and the edges between them

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