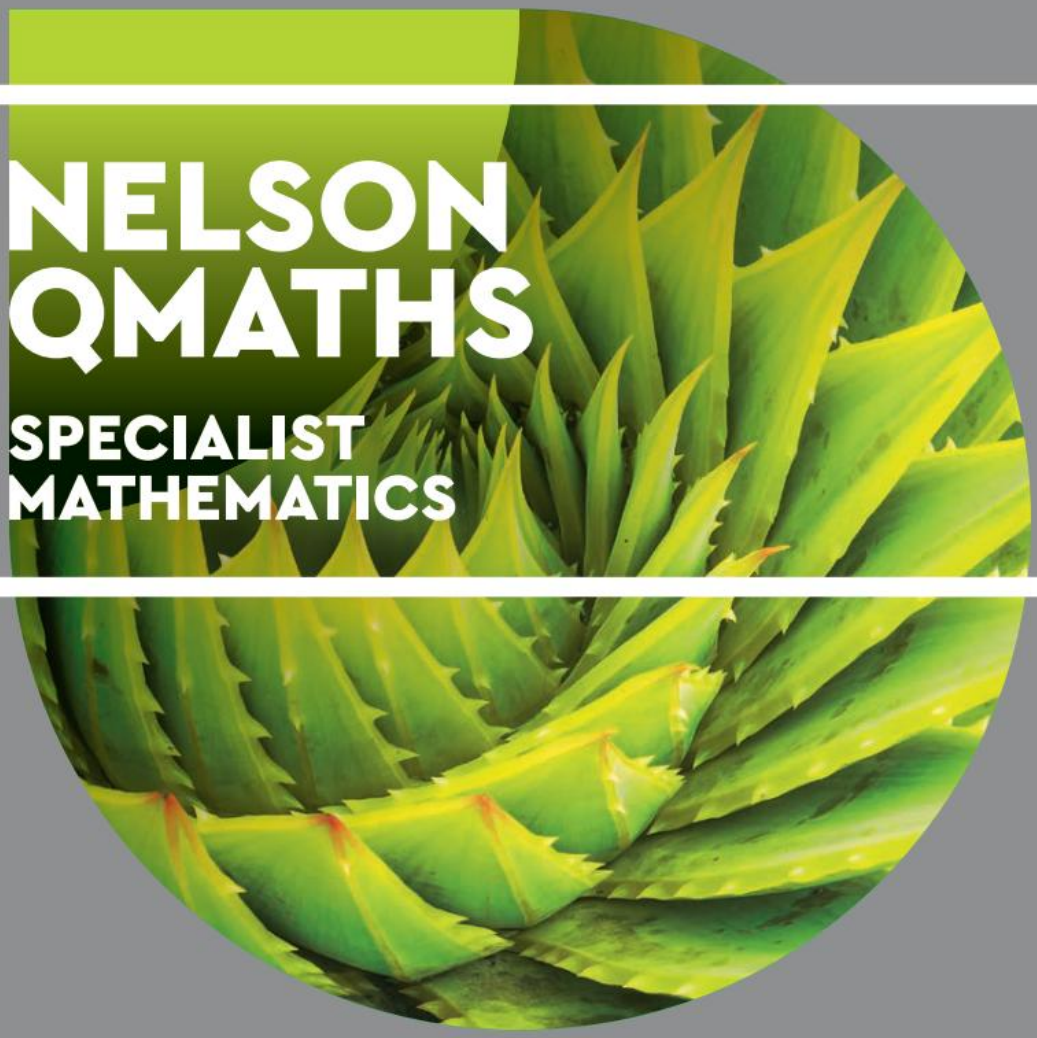


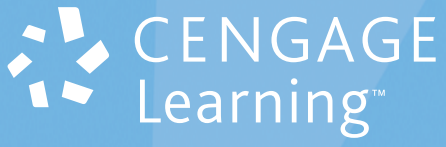
NELSON QMATHS

SPECIALIST MATHEMATICS

YEAR 11

Ross Brodie
Regina Edwards
Stephen Swift

4TH EDITION



COPYRIGHT NOTICE

Copyright in this work is owned by Cengage Learning Australia ("the work"). A condition of purchase of this electronic version of the work is that you agree to respect the copyright in the work, abide by the Copyright Act 1968 and specifically agree not to transfer, sell, assign, misuse, copy or transmit an electronic or other version of the work to any third party.

Please note: This product is accompanied by a licence (single user, network or adoption) governing the terms and conditions of its use.

This is a legal agreement between the you, (the "Customer") and Cengage Learning Australia Pty Limited (ABN 14 058 280 149) (the "Licensor") which provides the terms and conditions of this non-exclusive licence and the limited warranty for the Product. Use of the Product indicates an acknowledgement that the Customer has read and agreed to be bound by the terms and conditions of this Agreement. If you do not agree to these terms and conditions, return the Product to the place of purchase within 15 days of the date of purchase (with proof of purchase) for a full refund

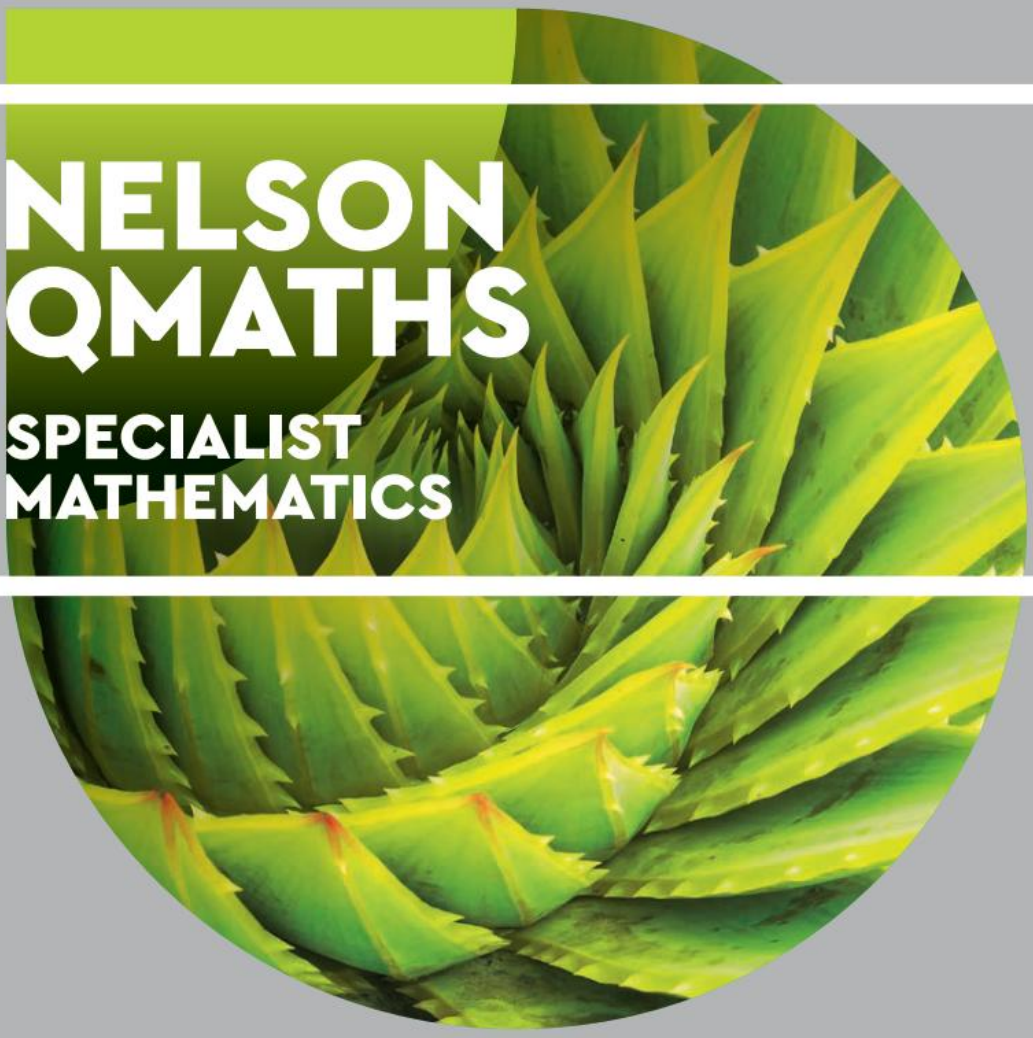
1. **Licence Grant**
   You do not receive title to the Product. Copyright in the Product (which includes all images, photographs, video, animations, audio, music and text incorporated in the Product, including all of the accompanying printed material) is owned by the Licensor and/or its suppliers and is protected by Australian copyright laws. The Licensor grants you a non-exclusive licence to use the Product subject to the restrictions and terms set out in this Agreement.

2. **A Licence allows you to:**
   Use the Product on your computer. The Customer represents that they shall in no way place the Product in the public domain or in any way compromise our copyright in the Material. You agree to take reasonable steps to protect our copyright.

3. **You may not:**
   Alter, modify, translate, reverse engineer, decompile, or adapt the software or create derivative works based on the Product. Make further copies by any means technological, electronic, digital whatsoever without the written permission of the Licensor. Rent or transfer all or any part of your rights under this Agreement. Remove or alter any copyright or other proprietary notice or label attached to the software.

4. **Termination**
   Any failure to comply with the terms and conditions of this agreement will result in the automatic termination of this licence. Upon termination of this licence for any reason, the Customer must destroy or return to the Licensor all copies of the software and accompanying documentation.

5. **Warranties**
   To the extent permitted by law, the Licensor's liability for any breach of the warranty or any term implied by law into this licence is limited to the lowest cost of replacing the goods, acquiring equivalent goods or having the goods repaired.



NELSON QMATHS

SPECIALIST MATHEMATICS

YEAR 11

Ross Brodie
Regina Edwards
Stephen Swift

4TH EDITION

NELSON
A Cengage Company

Nelson QMaths 11 Specialist Mathematics
4th Edition
Ross Brodie
Regina Edwards
Stephen Swift

Publishers: Robert Yen, Alan Stewart
Project editor: Alan Stewart
Editor: Anna Pang
Cover design: Chris Starr (MakeWork)
Text design: Sarah Anderson
Art direction: Aisling Gallagher and Dannielle Maccarone
Permissions researcher: Debbie Gallagher
Production controller: Christine Fotis
Typeset by: Cenveo Publisher Services

Any URLs contained in this publication were checked for currency during the production process. Note, however, that the publisher cannot vouch for the ongoing currency of URLs.

Acknowledgements
Specialist Mathematics 2019 v1.0 General Senior Syllabus (c) Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019.

For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

© 2018 Cengage Learning Australia Pty Limited

Copyright Notice
This Work is copyright. No part of this Work may be reproduced, stored in a retrieval system, or transmitted in any form or by any means without prior written permission of the Publisher. Except as permitted under the *Copyright Act 1968,* for example any fair dealing for the purposes of private study, research, criticism or review, subject to certain limitations. These limitations include: Restricting the copying to a maximum of one chapter or 10% of this book, whichever is greater; providing an appropriate notice and warning with the copies of the Work disseminated; taking all reasonable steps to limit access to these copies to people authorised to receive these copies; ensuring you hold the appropriate Licences issued by the Copyright Agency Limited ("CAL"), supply a remuneration notice to CAL and pay any required fees. For details of CAL licences and remuneration notices please contact CAL at Level 15, 233 Castlereagh Street, Sydney NSW 2000,
Tel: (02) 9394 7600, Fax: (02) 9394 7601
Email: info@copyright.com.au
Website: www.copyright.com.au

For product information and technology assistance,
in Australia call **1300 790 853**;
in New Zealand call **0800 449 725**

For permission to use material from this text or product, please email **aust.permissions@cengage.com**

National Library of Australia Cataloguing-in-Publication Data
Brodie, Ross, author.

Nelson Qmaths 11: specialist mathematics / Ross Brodie, Regina Edwards, Stephen Swift.

9780170412995 (paperback)
For secondary students.

Mathematics--Study and teaching (Secondary)
Mathematics--Textbooks.

Other Creators/Contributors:
Edwards, Regina, author.
Swift, Stephen (Mathematics teacher), author.

Cengage Learning Australia
Level 7, 80 Dorcas Street
South Melbourne, Victoria Australia 3205

Cengage Learning New Zealand
Unit 4B Rosedale Office Park
331 Rosedale Road, Albany, North Shore 0632, NZ

For learning solutions, visit **cengage.com.au**

Printed in China by China Translation & Printing Services.

CONTENTS

PREFACE vi
SYLLABUS REFERENCE GRID vii
ABOUT THIS BOOK viii
LIST OF SYMBOLS AND ABBREVIATIONS xii
LIST OF MATHEMATICAL VERBS xiv

1

VECTORS **2**

1.01 Vectors 4
1.02 Polar and Cartesian forms of vectors 9
1.03 Scalar multiples of vectors 18
1.04 Addition of vectors 21
1.05 Subtraction of vectors 29
1.06 Unit vectors 34
1.07 Vector components 38
1.08 Resolution of vectors 42
1.09 Applications of vectors 49
Chapter summary 56
Chapter review 58

2

METHODS OF PROOF **62**

2.01 Mathematical proof 64
2.02 Counterexamples 68
2.03 Converse 69
2.04 Equivalence and negation 72
2.05 Contrapositive 78
2.06 Contradiction 81
2.07 Euclidean geometry 84
Chapter summary 92
Chapter review 95

3

PERMUTATIONS **98**

3.01 Multiplication principle 100
3.02 Addition principle 104
3.03 The inclusion-exclusion principle 110
3.04 Permutations and factorial notation 115
3.05 Permutations with restrictions 121
3.06 Applications of permutations 126
Chapter summary 132
Chapter review 133

PRACTICE EXAMINATION 1 136

4

VECTOR ALGEBRA **140**

4.01 Geometric use of vectors 142
4.02 The scalar product 150
4.03 Parallel and perpendicular vectors 158
4.04 Projections of vectors 163
4.05 Applications of vectors 168
Chapter summary 175
Chapter review 177

ISBN 9780170412995

5

CIRCLE PROPERTIES AND VECTOR PROOFS 180

5.01 Angles at the centre of a circle 182
5.02 Angles at the circumference of a circle 186
5.03 Semicircle angle 189
5.04 Cyclic quadrilaterals 192
5.05 Chords and radii 195
5.06 Tangents 200
5.07 Secants 203
5.08 Mixed problems in circle geometry 207
5.09 Vector proofs of parallelogram properties 211
5.10 Other vector proofs in geometry 214
Chapter summary 218
Chapter review 221

6

COMBINATIONS 224

6.01 Review of multiplication principle and permutations 226
6.02 Combinations 229
6.03 Combinations with restrictions 232
6.04 Applications of permutations and combinations 237
6.05 Pascal's triangle and identities 242
6.06 The pigeon-hole principle 245
Chapter summary 249
Chapter review 250

PRACTICE EXAMINATION 2 253

7

MATRICES 258

7.01 Matrix addition and subtraction 260
7.02 Matrix multiplication 269
7.03 Matrix identities and inverses 279
7.04 Matrix equations 286
Chapter summary 291
Chapter review 293

8

COMPLEX NUMBERS 296

8.01 Complex numbers 298
8.02 Addition and subtraction of complex numbers 302
8.03 Multiplication of complex numbers 305
8.04 Complex conjugates and division of complex numbers 308
8.05 The Argand plane 313
8.06 Complex numbers as vectors 316
8.07 Operations in the complex plane 319
Chapter summary 324
Chapter review 326

ISBN 9780170412995

9

TRIGONOMETRIC FUNCTIONS 328

9.01 General solution of trigonometric equations 330
9.02 General trigonometric graphs 335
9.03 The absolute value function 348
9.04 Reciprocal functions 358
9.05 Rational functions 362
9.06 The reciprocal trigonometric functions 368
9.07 Pythagorean trigonometric identities ... 378
9.08 Angle sum and difference identities 382
9.09 Double angle identities 387
Chapter summary 392
Chapter review 394

PRACTICE EXAMINATION 3 398

10

TRANSFORMATIONS 402

10.01 Translations and vectors 404
10.02 Rotations 407
10.03 Dilations 414
10.04 Reflections 419
10.05 General linear and inverse transformations 425
10.06 Composition of transformations 430
10.07 Geometric results of transformations ... 435
Chapter summary 440
Chapter review 442

11

COMPLEX ARITHMETIC AND EQUATIONS 444

11.01 Modulus and argument 446
11.02 Polar, trigonometric and Cartesian form 450
11.03 Operations in trigonometric form 454
11.04 General solution of quadratic equations 462
11.05 Factorisation of quadratic expressions 467
Chapter summary 470
Chapter review 472

12

TRIGONOMETRIC IDENTITIES AND APPLICATIONS 474

12.01 Products of sine and cosine 476
12.02 Sums of sine and cosine 479
12.03 $a \sin(x) + b \cos(x)$ 482
12.04 Graphs of $a \sin(x) + b \cos(x)$ 487
12.05 Equations of the form $a \sin(x) + b \cos(x) = c$ 491
12.06 Multiple angle trigonometric identities 494
12.07 Periodic motion 498
12.08 Modelling with trigonometric functions 504
Chapter summary 512
Chapter review 514

PRACTICE EXAMINATION 4 516

ANSWERS 520
GLOSSARY/INDEX 596

ISBN 9780170412995

PREFACE

Nelson QMaths, Queensland's longest-running senior mathematics series, has been rewritten for the new syllabuses and assessment procedures for implementation from 2019. Based on the Australian Curriculum, four new senior mathematics courses have been introduced into Queensland schools.

- Essential Mathematics
- General Mathematics
- Mathematical Methods
- Specialist Mathematics

With the introduction of external QCE examinations, *Nelson QMaths* will have a renewed focus on assessment, and will include features such as topic tests, practice examinations, *ExamView* question banks and chapter quizzes, video tutorials and worked solutions to all exercise questions. In this book, teachers will find familiar features such as clear worked examples, graded exercises, strong syllabus coverage, Investigations, Technology, chapter summaries, chapter reviews and a glossary/index. We wish all teachers and students using this book every success in embracing the new mathematics courses.

ABOUT THE AUTHORS

Ross Brodie has taught mathematics and been Head of Mathematics at a variety of regional and urban schools in Queensland. He is an experienced teacher and author, having co-written *Nelson QMaths 7–10*, *New QMaths 11–12* and *Nelson Senior Maths 11–12* (Australian Curriculum).

Regina Edwards teaches mathematics and science at Cleveland District State High School where she coordinates International Baccalaureate High Level Mathematics and Physics. She was a district panellist for Mathematics C and Physics, and has worked in the Information Technology industry as an analyst.

Stephen Swift has taught Mathematics, Science and Computing in urban and regional schools, and was Head of Mathematics at Wellington Point State High School. He is the lead author of *Nelson QMaths 7–10*, *New QMaths 11–12* and *Nelson Senior Maths 11–12* (Australian Curriculum).

CONTRIBUTING AUTHORS

Margaret Denham wrote the topic tests.
Margaret Denham and **George Dimitriadis** wrote the worked solutions to all exercise sets.
Roger Walter wrote the *ExamView* questions.

ISBN 9780170412995

SYLLABUS REFERENCE GRID

Topics and subtopics	*Nelson QMaths 11 Specialist Mathematics chapter*
UNIT 1: COMBINATORICS, VECTORS AND PROOFS	
Combinatorics	
The inclusion-exclusion principle	3 Permutations
Permutations and combinations	3 Permutations 6 Combinations
The pigeon-hole principle	6 Combinations
Vectors in the plane	
Representing vectors in the plane	1 Vectors
Algebra of vectors in the plane	4 Vector algebra
Introduction to proof	
The nature of proof	2 Methods of proof
Rational and irrational numbers	2 Methods of proof
Circle properties and their proofs	5 Circle properties and vector proofs
Geometric proofs using vectors	5 Circle properties and vector proofs
UNIT 2: COMPLEX NUMBERS, TRIGONOMETRY, FUNCTIONS AND MATRICES	
Complex numbers 1	
Complex numbers	8 Complex numbers
The complex plane	8 Complex numbers
Complex arithmetic using polar form	11 Complex arithmetic and equations
Roots of equations	11 Complex arithmetic and equations
Trigonometry and functions	
The basic trigonometric functions	9 Trigonometric functions
Sketching graphs	9 Trigonometric functions
The reciprocal trigonometric functions, secant, cosecant and cotangent	9 Trigonometric functions
Trigonometric identities	9 Trigonometric functions 12 Trigonometric identities and applications
Applications of trigonometric functions to model periodic phenomena	12 Trigonometric identities and applications
Matrices	
Matrix arithmetic	7 Matrices
Transformations in the plane	10 Transformations

ISBN 9780170412995

ABOUT THIS BOOK

AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of syllabus subject matter (© Queensland Curriculum and Assessment Authority 2017)

MATRICES

10.

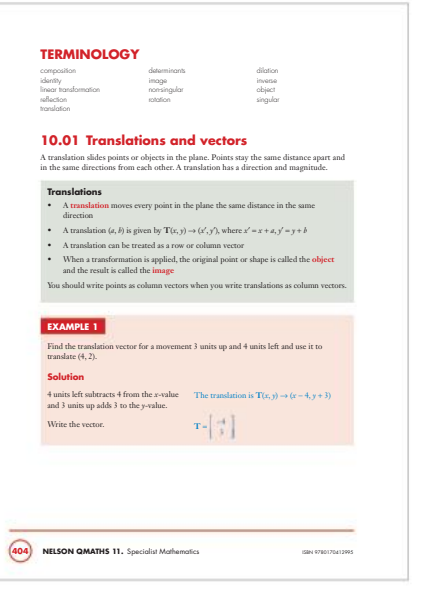
TRANSFORMATIONS

Transformations change vectors, points or objects in the plane or space to other vectors, points or objects. Translations, rotations, reflections, stretches and shears are all transformations. Transformations are used in computer graphics and animation. They have important applications in optics, atomic and nuclear theory and other branches of science. In this chapter you will study the application of matrices to transformations.

10.01 Translations and vectors
10.02 Rotations
10.03 Dilations
10.04 Reflections
10.05 General linear and inverse transformations
10.06 Composition of transformations
10.07* Geometric results of transformations
Chapter summary
Chapter review

Transformations in the plane
- understand translations and their representation as column vectors
- define and use basic linear transformations: dilations of the form $(x, y) \to (\lambda x, \lambda y)$, rotations about the origin and reflection in a line that passes through the origin, and the representations of these transformations by 2×2 matrices
- apply these transformations to points in the plane and geometric objects
- define and use composition of linear transformations and the corresponding matrix products
- define and use inverses of linear transformations and the relationship with the matrix inverse
- examine the relationship between the determinant and the effect of a linear transformation on area
- establish geometric results by matrix multiplications

- **Terminology** previews the key words and phrases from within the chapter.


TERMINOLOGY

composition, identity, linear transformation, reflection, translation, determinants, image, nonsingular, rotation, dilation, inverse, object, singular

10.01 Translations and vectors

A translation slides points or objects in the plane. Points stay the same distance apart and in the same directions from each other. A translation has a direction and magnitude.

Translations
- A translation moves every point in the plane the same distance in the same direction
- A translation (a, b) is given by $\mathbf{T}(x, y) \to (x', y')$, where $x' = x + a, y' = y + b$
- A translation can be treated as a row or column vector
- When a transformation is applied, the original point or shape is called the object and the result is called the image

You should write points as column vectors when you write translations as column vectors.

EXAMPLE 1

Find the translation vector for a movement 3 units up and 4 units left and use it to translate $(4, 2)$.

Solution

4 units left subtracts 4 from the x-value and 3 units up adds 3 to the y-value.

The translation is $\mathbf{T}(x, y) \to (x - 4, y + 3)$

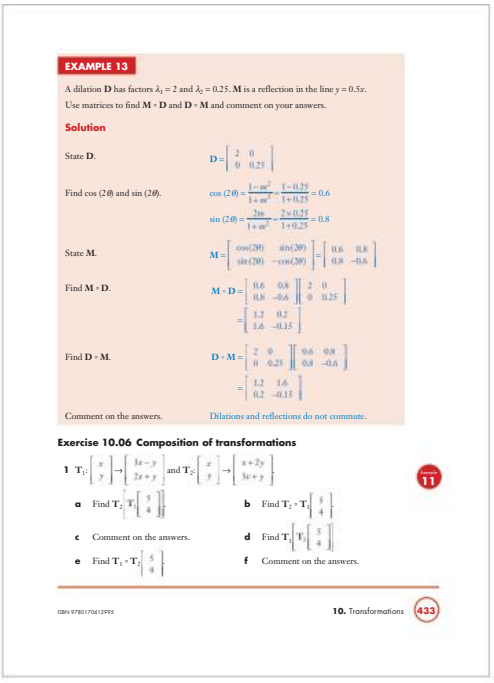
Write the vector.

$\mathbf{T} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

404 NELSON QMATHS 11. Specialist Mathematics ISBN 9780170412995

IN EACH CHAPTER

- Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand-side.
- Important facts and formulas are highlighted in shaded boxes.
- Important words and phrases are printed in red and listed in the glossary at the back of the book.
- Graded exercises, including **Problem solving** questions, are linked to worked examples and include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website.
- **Investigations** explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.
- **Technology** promotes ICT in the classroom, using spreadsheets, graphics calculators and the Internet.
- Instructions are provided for the TI-84 Plus CE, Casio fx-CG20AU and TI-Nspire CX calculators.



EXAMPLE 13

A dilation **D** has factors $\lambda_1 = 2$ and $\lambda_2 = 0.25$. **M** is a reflection in the line $y = 0.5x$.
Use matrices to find **M** ∘ **D** and **D** ∘ **M** and comment on your answers.

Solution

State **D**. $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix}$

Find cos (2θ) and sin (2θ). $\cos(2\theta) = \frac{1-m^2}{1+m^2} = \frac{1-0.25}{1+0.25} = 0.6$, $\sin(2\theta) = \frac{2m}{1+m^2} = \frac{2\times 0.25}{1+0.25} = 0.8$

State **M**. $\mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$

Find **M** ∘ **D**. $\mathbf{M} \circ \mathbf{D} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix} = \begin{bmatrix} 1.2 & 0.2 \\ 1.6 & -0.15 \end{bmatrix}$

Find **D** ∘ **M**. $\mathbf{D} \circ \mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} = \begin{bmatrix} 1.2 & 1.6 \\ 0.2 & -0.15 \end{bmatrix}$

Comment on the answers. Dilations and reflections do not commute.

Exercise 10.06 Composition of transformations

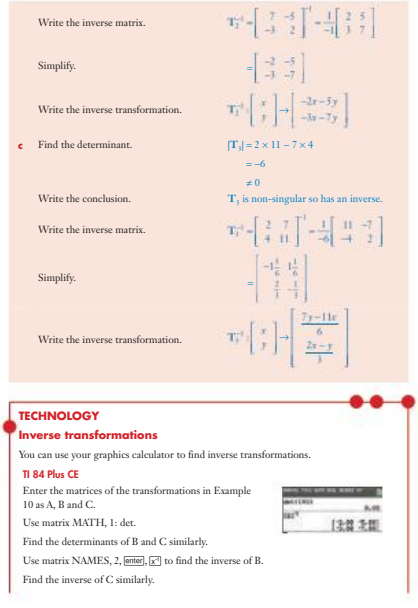
1 $T_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x-y \\ 2x+y \end{bmatrix}$ and $T_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+2y \\ 3x+y \end{bmatrix}$

a Find $T_2\left[T_1\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right]$ b Find $T_2 \circ T_1\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

c Comment on the answers. d Find $T_1\left[T_2\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right]$

e Find $T_1 \circ T_2\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ f Comment on the answers.

ISBN 9780170412995 10. Transformations 433



Write the inverse matrix. $\mathbf{T}_2^{-1} = \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

Simplify. $= \begin{bmatrix} -2 & -5 \\ -3 & -7 \end{bmatrix}$

Write the inverse transformation. $T_2^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -2x-5y \\ -3x-7y \end{bmatrix}$

c Find the determinant. $|\mathbf{T}_1| = 2 \times 11 - 7 \times 4$
$= -6$
$\neq 0$

Write the conclusion. $\mathbf{T}_1$ is non-singular so has an inverse.

Write the inverse matrix. $\mathbf{T}_1^{-1} = \begin{bmatrix} 2 & 7 \\ 4 & 11 \end{bmatrix}^{-1} = \frac{1}{-6}\begin{bmatrix} 11 & -7 \\ -4 & 2 \end{bmatrix}$

Simplify. $= \begin{bmatrix} -1\frac{5}{6} & 1\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

Write the inverse transformation. $T_1^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{7y-11x}{6} \\ \frac{2x-y}{3} \end{bmatrix}$

TECHNOLOGY

Inverse transformations

You can use your graphics calculator to find inverse transformations.

TI 84 Plus CE

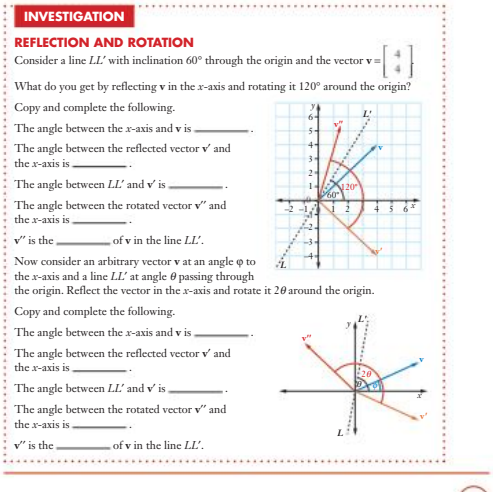
Enter the matrices of the transformations in Example 10 as A, B and C.

Use matrix MATH, 1: det.

Find the determinants of B and C similarly.

Use matrix NAMES, 2, enter, x^{-1} to find the inverse of B.

Find the inverse of C similarly.



INVESTIGATION

REFLECTION AND ROTATION

Consider a line LL' with inclination 60° through the origin and the vector $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

What do you get by reflecting **v** in the x-axis and rotating it 120° around the origin?

Copy and complete the following.

The angle between the x-axis and **v** is _____.

The angle between the reflected vector **v′** and the x-axis is _____.

The angle between LL' and **v′** is _____.

The angle between the rotated vector **v″** and the x-axis is _____.

v″ is the _____ of **v** in the line LL'.

Now consider an arbitrary vector **v** at an angle φ to the x-axis and a line LL' at angle θ passing through the origin. Reflect the vector in the x-axis and rotate it 2θ around the origin.

Copy and complete the following.

The angle between the x-axis and **v** is _____.

The angle between the reflected vector **v′** and the x-axis is _____.

The angle between LL' and **v′** is _____.

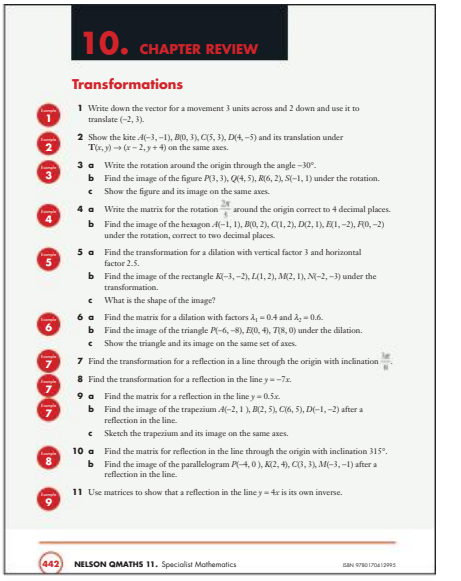
The angle between the rotated vector **v″** and the x-axis is _____.

v″ is the _____ of **v** in the line LL'.

ISBN 9780170412995 10. Transformations 419

AT THE END OF EACH CHAPTER

- **Chapter summary** summarises the key concepts, skills and formulas of the chapter.
- **Chapter review** contains revision exercises and are linked to worked examples.
- **Practice examinations** after every 3 chapters revise the skills and knowledge of those chapters.



10. CHAPTER REVIEW

Transformations

1 Write down the vector for a movement 3 units across and 2 down and use it to translate (−2, 3).

2 Show the kite $A(-3, -1)$, $B(0, 3)$, $C(5, 3)$, $D(4, -5)$ and its translation under $T(x, y) \rightarrow (x - 2, y + 4)$ on the same axes.

3 a Write the rotation around the origin through the angle −30°.
b Find the image of the figure $P(3, 3)$, $Q(4, 5)$, $R(6, 2)$, $S(-1, 1)$ under the rotation.
c Show the figure and its image on the same axes.

4 a Write the matrix for the rotation $\begin{bmatrix} 2\sqrt{5} & \sqrt{5} \\ -\sqrt{5} & 2\sqrt{5} \end{bmatrix}$ around the origin correct to 4 decimal places.
b Find the image of the hexagon $A(-1, 1)$, $B(0, 2)$, $C(1, 2)$, $D(2, 1)$, $E(1, -2)$, $F(0, -2)$ under the rotation, correct to two decimal places.

5 a Find the transformation for a dilation with vertical factor 3 and horizontal factor 2.5.
b Find the image of the rectangle $K(-3, -2)$, $L(1, 2)$, $M(2, 1)$, $N(-2, -3)$ under the transformation.
c What is the shape of the image?

6 a Find the matrix for a dilation with factors $k_1 = 0.4$ and $k_2 = 0.6$.
b Find the image of the triangle $P(-6, -8)$, $E(0, 4)$, $T(8, 0)$ under the dilation.
c Show the triangle and its image on the same set of axes.

7 Find the transformation for a reflection in a line through the origin with inclination $\frac{\pi}{6}$.

8 Find the transformation for a reflection in the line $y = -7x$.

9 a Find the matrix for a reflection in the line $y = 0.5x$.
b Find the image of the trapezium $A(-2, 1)$, $B(2, 5)$, $C(6, 5)$, $D(-1, -2)$ after a reflection in the line.
c Sketch the trapezium and its image on the same axes.

10 a Find the matrix for reflection in the line through the origin with inclination 315°.
b Find the image of the parallelogram $P(-4, 0)$, $K(2, 4)$, $C(3, 3)$, $M(-3, -1)$ after a reflection in the line.

11 Use matrices to show that a reflection in the line $y = 4x$ is its own inverse.

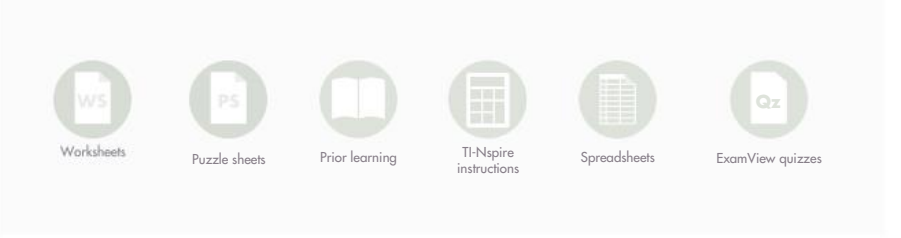
442 NELSON QMATHS 11. Specialist Mathematics ISBN 9780170412995

AT THE END OF THE BOOK

- **Glossary and index** includes a comprehensive dictionary of course terminology.
- **Answers** (with worked solutions on the teacher website).

NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website, **www.nelsonnet.com.au**. These include:



- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Prior learning** exercises that revise prerequisite skills for the chapter
- **Graphics calculator instructions**: for TI-Nspire CX calculators
- **Spreadsheets:** *Excel* files
- **ExamView quizzes:** interactive and self-marking.

NELSONNET TEACHER WEBSITE

The *NelsonNet* teacher website, also at **www.nelsonnet.com.au**, contains:

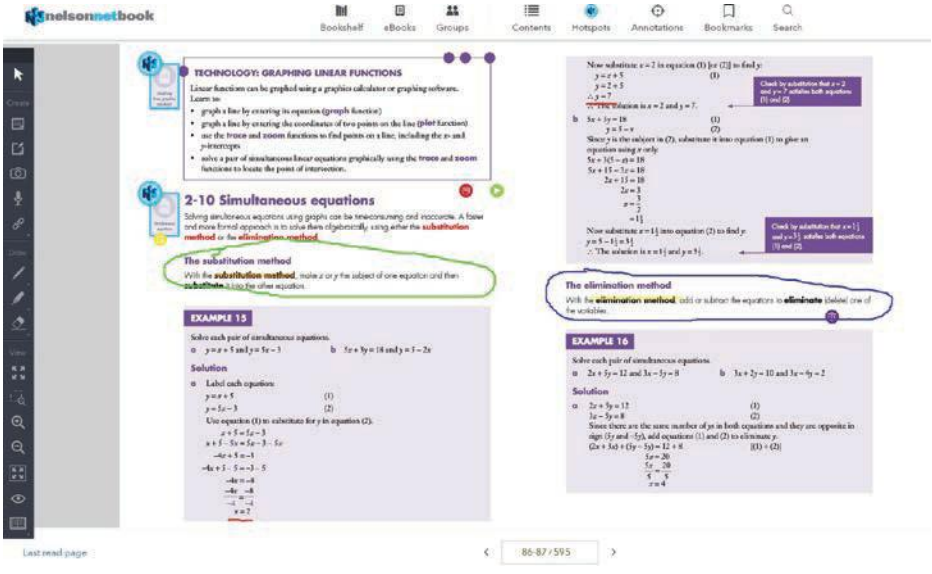
- **Teaching plan**, in Microsoft Word and PDF formats
- **Topic tests**, in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- **ExamView** exam-writing software and question banks
- **Resource Finder:** search engine for *NelsonNet* resources.

Note: Complimentary access to these resources is only available to teachers who use this book as a core educational resource in their classroom. Contact your Cengage Education Consultant for information about access codes and conditions.

NELSONNETBOOK

NelsonNetBook is the web-based interactive version of this book found on *NelsonNet*.

- To each page of **NelsonNetBook** you can add notes, voice and sound bites, highlighting, weblinks and bookmarks.
- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students.



ISBN 9780170412995

LIST OF SYMBOLS AND ABBREVIATIONS

$=$	is equal to
$\equiv$	is identically equal to, is congruent to
$\neq$	is not equal to
$\approx$	is approximately equal to
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\|\|\|$	is similar to
$\|\|$	is parallel to
$\in$	is a member of
$\cup$	union
$\cap$	intersection
$\pm$	plus or minus
π	pi ≈ 3.14159
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\notin$	is not a member of
Σ	the sum of
…	and so on
{}, $\varnothing$	the null set
U, $\mathbb{E}$	the universal set
$\boldsymbol{C}$	the complex numbers
$\boldsymbol{R}$	the real numbers
$\boldsymbol{R}^+$	the positive real numbers
$\boldsymbol{R}\setminus 0$	the real numbers except 0
$\boldsymbol{Z}, \boldsymbol{J}$	the integers
$\boldsymbol{Z}^+$	the positive integers
$\boldsymbol{N}$	the natural numbers
$^\circ$	degree
∞	infinity
$\Rightarrow$	implies
$\Leftrightarrow$, iff	if and only if
Δ	the discriminant
RTP	required to prove
QED	quod erat demonstrandum, statement that a proof is finished
$\sqrt{\ }$	square root, radical sign
$\overline{A}, A'$	not A, the complement of A
$\|A\|, n(A)$	the number of elements of set A
LHS	left-hand side
RHS	right-hand side
$\|a\|$	absolute value of a
m	gradient of a straight line
c	y-intercept of a straight line
(x, y)	Cartesian coordinates, ordered pair
${}^nP_r$	the number of permutations of r objects from n objects
${}^nC_r, \binom{n}{r}$	the number of combinations of r objects from n objects
$n!$	factorial n
$\log_{10}$	common logarithm
ln	natural logarithm
$\log_a b$	the logarithm of b to the base a
cos	cosine ratio
sin	sine ratio
tan	tangent ratio
cot	cotangent ratio
sec	secant ratio
cosec	cosecant ratio
$\sin^{-1}(a)$	inverse sine, the angle A whose sine is a
$\cos^{-1}(a)$	inverse cosine, the angle A whose cosine is a
$\tan^{-1}(a)$	inverse tangent, the angle A whose tangent is a
$\forall$	for all
$\exists$	there exists
$\mathbf{p}, \underset{\sim}{p}$	the vector $\mathbf{p}$
$\|\mathbf{p}\|, p$	the magnitude (norm) of the vector $\mathbf{p}$
$\mathbf{AB}$	displacement vector from point A to point B
OA, $\mathbf{a}$, $\overrightarrow{OA}$	$\overrightarrow{OA}$ the position vector of the point A
$(r, \theta), \begin{pmatrix} r \\ \theta \end{pmatrix}$	polar form of the vector with magnitude r and direction θ
$(x, y), \begin{pmatrix} x \\ y \end{pmatrix}$	component (Cartesian) form of vector
$-\mathbf{a}$	additive inverse of the vector $\mathbf{a}$
$\hat{\mathbf{p}}$	unit vector in the direction of the vector $\mathbf{p}$
$\mathbf{i}, \mathbf{j}$	unit vectors in the x and y directions
$\Delta\mathbf{v}$	change in the vector $\mathbf{v}$
$\mathbf{a} \cdot \mathbf{b}$	the scalar product (dot product) of the vectors $\mathbf{a}$ and $\mathbf{b}$
$\angle$	angle
$\triangle$	triangle
SSS	side, side, side test of triangle congruency
SAS	side, angle, side test of triangle congruency
AAS	angle, angle, side test of triangle congruency
RHS	right angle, hypotenuse, side test of triangle congruency

ISBN 9780170412995

$\mathbf{A}$, $\underset{\sim}{\mathrm{A}}$	the matrix $\mathbf{A}$
$\mathrm{A} = (a_{ij})_{2\times 3} = \begin{pmatrix} 3 & -2 & 5 \\ -4 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 3 & -2 & 5 \\ -4 & 1 & 0 \end{pmatrix} =$ the 2×3 matrix A with elements a_{ij}
$-\mathbf{A}$	the additive inverse of the matrix $\mathbf{A}$
$\mathbf{I}_n$	the $n \times n$ identity matrix
δ_{ij}	the Kronecker delta function
$\mathbf{A}^{-1}$	the multiplicative inverse of the (square) matrix $\mathbf{A}$
$\det \mathbf{A}$, $\lvert\mathbf{A}\rvert$, A	the determinant of the matrix $\mathbf{A}$
$\mathbf{T}: (x, y) \to (x', y')$	the transformation that changes the point (x, y) to (x', y')
$\mathbf{D}$	a dilation
λ_1, λ_2	the dilation factors in the x and y directions for a dilation
$\mathbf{R}_\alpha$	a rotation around the origin through and angle α
$\mathbf{M}$	a reflection across a line through the origin
$\mathbf{S} \circ \mathbf{T}$	the composition of the transformations $\mathbf{S}$ and $\mathbf{T}$
$\mathbf{T}^{-1}$	the inverse of the transformation $\mathbf{T}$
i	the imaginary number $\left(\sqrt{-1}\right)$
$\mathrm{Re}(z)$	the real part of the complex number z
$\mathrm{Im}(z)$	the imaginary part of the complex number z
$\overline{z}$	the complex conjugate of z
$Arg(z)$	the argument of z
$\lvert z\rvert$, $\mathrm{mod}(z)$	the modulus of the complex number z

NELSON QMATHS 11–12 SERIES

ISBN 9780170412995

LIST OF MATHEMATICAL VERBS

A glossary of 'doing words' commonly found in mathematics problems

analyse: study and state in detail the relationship of parts of a situation

apply: use knowledge or a procedure in a given situation

calculate: find a numerical value

comment: express an opinion or judgement about a statement or calculation

communicate: transmit information to others

compare: state similarities and differences and their significance

consider: take into account

construct: draw an accurate diagram or logically arrange items or ideas

convert: change from one form to another

define: give the meaning of or identify in exact terms

demonstrate: show to be correct

describe: state the features of a situation, object, pattern, event, etc

determine: find the answer or make a decision using evidence

discuss: give reasons based on evidence for and against a position or proposal

establish: introduce and develop a result

evaluate: find the value of or state the application, strengths and limitations of a solution

examine: state the details and assumptions of a situation

expand: remove brackets or change a product to an extended sum of terms

explain: state the meaning in logical detail

explore: examine or state the details and assumptions of a situation

factorise: convert to factors or change a sum of terms to a product of factors

formulate: give mathematical expression to an idea or situation

hence find/prove: find an answer or prove a result using previous answers or information supplied

identify: state the type, name or distinguishing feature of an item or situation

interpret: state a conclusion or trend from given information

investigate: establish facts, trends or conclusions from collected information

ISBN 9780170412995

justify: give reasons or evidence for an answer or conclusion

modify: change to accommodate different information

obtain: find an answer or conclusion

prove: use logical steps to establish the truth of

recall: remember (and state)

recognise: use knowledge to identify features of a situation

rationalise: make rational, remove surds

show that: in questions where the answer is given, to use mathematical reasoning to prove that the answer is true

simplify: reduce the size of numbers in a fraction, or reduce the size of an algebraic expression

sketch: draw a diagram that shows the general shape and includes relevant features

solve: find the answer or explanation for a problem, particularly the values of variables

substitute: replace a variable by a number to calculate an answer

translate (to mathematical form): express a situation as mathematical relationships

verify: check a solution or result, usually referring to the given situation

write/state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

ISBN 9780170412995

1.

VECTORS

Vectors are used to model important aspects of the physical world, to predict what will happen under different circumstances and to calculate actions to achieve desired outcomes. Physics relies heavily on vectors to model a wide range of phenomena, including forces, velocity, electromagnetic fields and applications, and astronomy. Engineers use vectors to work out how to construct cars, bridges, buildings and other objects in a safe and durable way.

1.01 Vectors
1.02 Polar and Cartesian forms of vectors
1.03 Scalar multiples of vectors
1.04 Addition of vectors
1.05 Subtraction of vectors
1.06 Unit vectors
1.07 Vector components
1.08 Resolution of vectors
1.09 Applications of vectors
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Representing vectors in the plane by directed line segments

- examine examples of vectors
- understand the difference between a scalar and a vector
- define and use the magnitude and direction of a vector
- understand and use vector equality
- understand and use both the Cartesian and polar form of a vector
- represent a scalar multiple of a vector
- use the triangle rule to find the sum and difference of two vectors

Algebra of vectors in the plane

- use ordered pair notation and column vector notation to represent a vector
- understand and use vector notation: $\overrightarrow{AB}$, $\underset{\sim}{c}$, $\mathbf{d}$, unit vector notation $\hat{n}$
- convert between Cartesian form and polar form
- determine a vector between two points
- define and use unit vectors and the perpendicular unit vectors $\hat{i}$ and $\hat{j}$
- express a vector in component form using the unit vectors $\hat{i}$ and $\hat{j}$
- examine and use addition and subtraction of vectors in component form
- define and use multiplication by a scalar of a vector in component form

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

Shutterstock/sandra zuerlein

TERMINOLOGY

air speed
direction
head
initial point
norm
position vector
scalar
speed through the water
triangle rule
vector
component form
displacement
head-to-tail
linear combination
parallelogram rule
resolve
scalar multiple
tail
true course
directed line segment
ground speed
heading
magnitude
polar form
resolution
speed and course made good
terminal point
unit vector

Representing vectors

1.01 Vectors

Many quantities, such as mass, distance, volume and speed, are specified by a single magnitude. Quantities of this type are called **scalar** quantities. Other quantities, such as displacement, velocity and acceleration, need to have both **magnitude** and **direction** to complete their specification. These are known as **vector** quantities.

The most important objects in geometry are points and lines. Points are well specified by Cartesian coordinates. You already know that a line may be specified by a linear equation. You can use the concept of a geometric vector to specify a line of a particular length. This allows you to represent quantities that have both size and direction.

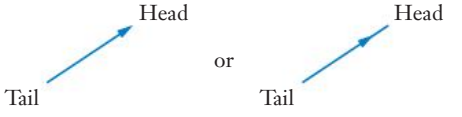
A **geometric vector** is a quantity with both magnitude (size) and direction.

A vector is shown using a lower-case bold letter such as **a**.
It can also be written as $\vec{a}$ or $\tilde{a}$ or $\underset{\sim}{a}$.

The magnitude of a vector **p** is written as $|\mathbf{p}|$ or p. The magnitude of a vector is also called the **norm** of the vector.

A vector may be shown as a **directed line segment**. The length of the segment represents the magnitude of the vector and an arrow is used to show its direction.

When drawing a vector, the arrowhead is usually placed at one end (or near the end) of the line segment. This end is called the **head** of the vector and the other end is called the **tail** of the vector.



ISBN 9780170412995

EXAMPLE 1

Draw a vector to represent each displacement on the same set of axes.

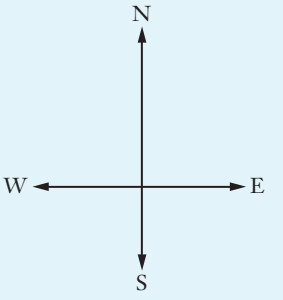
a 4 m north **b** 8 m east **c** 4 m south **d** 12 m north-west

Solution

Draw north–south and east–west lines to act as a reference when drawing the vectors.

You need to choose a suitable scale.

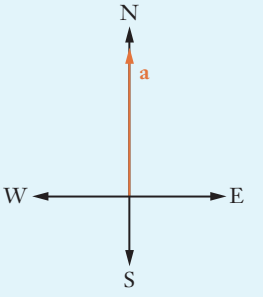
A scale of 1 cm = 2 m will do.



a Let this vector be **a**.

Using the scale, **a** will be 2 cm on the diagram.

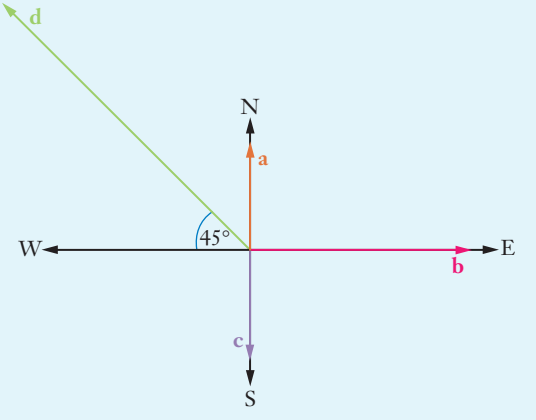
Draw **a** as a 2 cm arrow in the direction of north.



b **b** is shown by a 4 cm arrow in the direction of east.

c **c** is shown by a 2 cm arrow in the direction of south.

d **d** is shown by a 6 cm arrow in the direction of north-west.



ISBN 9780170412995

Other vector quantities such as velocity, acceleration and force can also be represented as line segments.

EXAMPLE 2

Draw a vector to represent each velocity on the same set of axes.

a **p** = 10 m/s in the direction N30°E

b **q** = 6 m/s in the direction S70°W

c **r** = 8 m/s in the direction 170°T

Solution

Use a scale of 1 cm = 2 m/s.

a Using the scale, **p** will be 5 cm long.

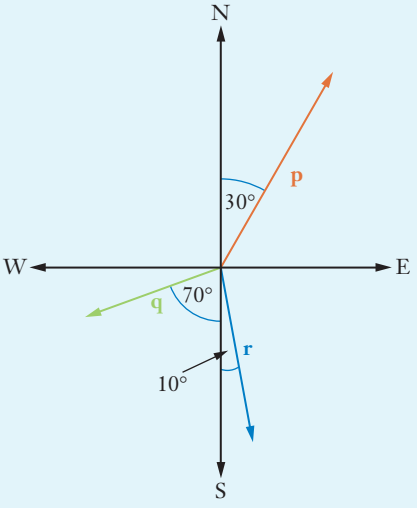
The direction of **p** is 30° east of north.

b Using the scale, **q** will be 3 cm long.

The direction of **q** is 70° west of south.

c Using the scale, **r** will be 4 cm long.

The direction of **r** is 10° east of south.



EXAMPLE 3

A car moves from a starting point and travels 5 km north. It then changes direction and travels for 4 km in the direction N60°E.

a Use a diagram to show the displacements as vectors.

b Find the bearing of the final position of the car from its starting point.

ISBN 9780170412995

Solution

a Use a scale of 1 cm = 1 km.

Let **m** be the first displacement.

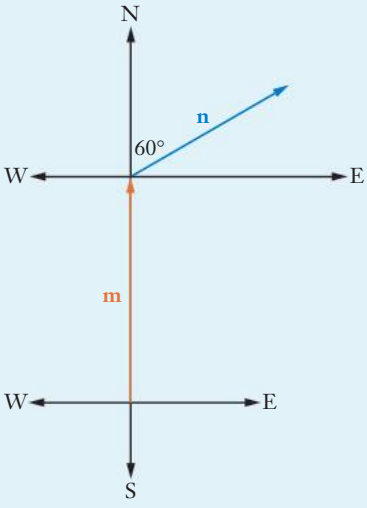
Using the scale, **m** will be 5 cm long.

The direction of **m** is north.

Let **n** be the second displacement.

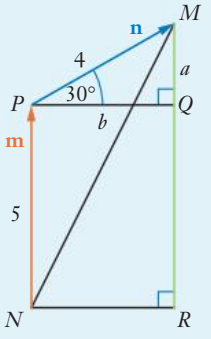
Using the scale, **n** will be 4 cm long.

The direction of **n** is 60° east of north.



b Re-draw the diagram and label it.

Using the information in the question, $\angle MPQ = 30°$.



Use the sine ratio to calculate a.

$$\sin(30°) = \frac{a}{4}$$
$$a = 4\sin(30°)$$
$$= 2$$

Use the cosine ratio to calculate b.

$$\cos(30°) = \frac{b}{4}$$
$$b = 4\cos(30°)$$
$$= 3.4641\ldots$$

You need to calculate $\angle MNP$.

Calculate $\angle MNR$ first.

$$\tan(\angle MNR) = \frac{MR}{NR}$$

ISBN 9780170412995

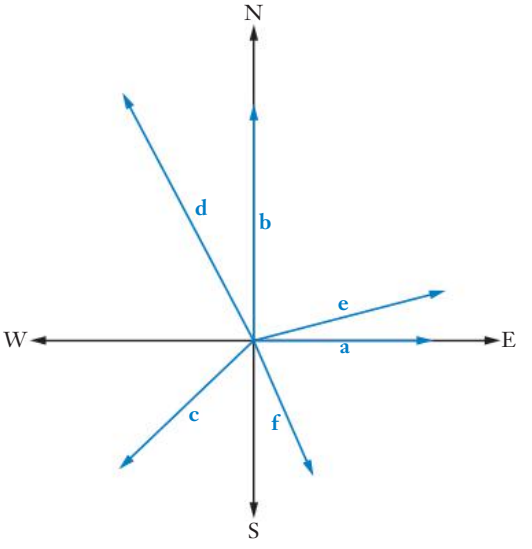
Substitute for known values.	$= \frac{2+5}{3.4641...}$ $= 2.0207...$
Use $\tan^{-1}$ to calculate $\angle MNR$.	$\tan^{-1}(2.0207...) = 63.67...°$
Calculate $\angle MNP$.	$\angle MNP + 63.67...° = 90°$
Evaluate.	$\angle MNP = 26.32...°$
Round off and state the result.	The bearing of the final position of the car from its starting point is about 026°T.

Exercise 1.01 Vectors

1 Which of the following are vector quantities?

a speed	**b** displacement	**c** length
d acceleration	**e** temperature	**f** velocity
g force	**h** volume	**i** height

2 The diagram below shows displacement vectors drawn using a scale of 1 cm = 4 km. Use direct measurement to fully describe each vector giving its magnitude and direction.



ISBN 9780170412995

3 Draw a vector to represent each vector on the same set of axes.

Example 1

a 5 m west **b** 4 m south **c** 7 m east

d 10 m south-west **e** 9 m north-east **f** 6 m south-east

4 Draw a vector to represent each force on the same set of axes.

Example 2

a **m** = 200 N in the direction S40°W **b** **r** = 160 N in the direction N20°E

c **u** = 120 N in the direction 280°T **d** **p** = 210 N in the direction N40°W

e **g** = 90 N in the direction 080°T **f** **d** = 170 N in the direction S50°E

5 Draw a vector to represent each acceleration on the same set of axes.

a 10 m s^{-2} east **b** 6 m s^{-2} north-east **c** 8 m s^{-2} 230°T

d 18 m s^{-2} N70°W **e** 12 m s^{-2} S30°E **f** 16 m s^{-2} 340°T

6 An object moves from a starting point and travels 6 km south-west. It then changes direction and travels north for 9 km.

Example 3

a Use a diagram to show the two displacements as vectors.

b Find the bearing of the final position of the object from its starting point.

Problem solving

7 A displacement vector has its tail at the origin and its head 20 m east and 15 m north of the origin. Calculate the norm of the displacement vector.

8 A cyclist sets off from a starting point and travels 8 km in the direction N30°W. The cyclist then changes direction and travels 12 km in the direction N70°E.

a Find the bearing of the final position of the cyclist from the starting point.

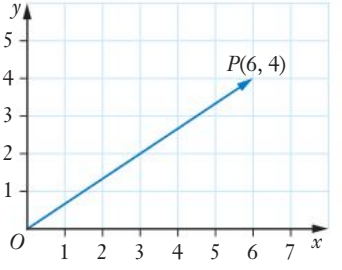
b How far is the cyclist from the starting point?

1.02 Polar and Cartesian forms of vectors

WS Displacement vectors

WS Component and polar forms of vectors

A vector that is drawn with its tail at an origin and its head at a fixed point is called a **position vector**. The diagram on the right shows a position vector to the point (6, 4).



This position vector is written as **OP** or $\overrightarrow{OP}$. For a position vector, the origin (tail) is known as the **initial point** and the head is known as the **terminal point**.

For **OP**, $O(0, 0)$ is the initial point and $P(6, 4)$ is the terminal point.

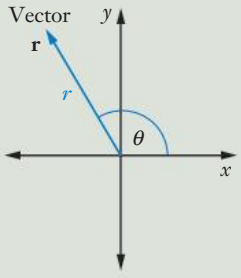
A position vector always has $O(0, 0)$ as its origin. This means you can write a position vector as the coordinates of its terminal point, so **OP** = (6, 4).

Another way of representing a two-dimensional vector is to write the magnitude and direction as an ordered pair. This is called the **polar form** of a vector.

ISBN 9780170412995

Polar form

Two-dimensional vectors are represented in **polar form** by an ordered pair. The first number gives the magnitude of the vector and the second number gives the direction of the vector. The direction is given by the anticlockwise angle from the positive direction of the x-axis.



Thus a vector **r** may be written as

the ordered pair (r, θ) or the column vector $\begin{bmatrix} r \\ \theta \end{bmatrix}$.

EXAMPLE 4

A vector **v** of magnitude 3 is in the direction of 60° to the x-axis.

a Write the polar form of the vector.

b Sketch possible representations of the vector.

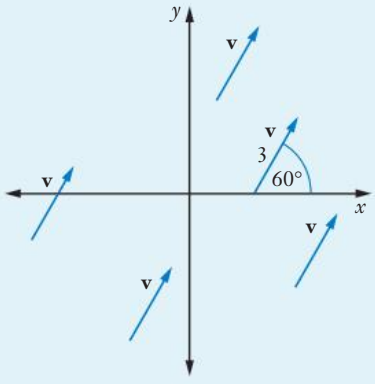
Solution

a Write as an ordered pair. $\mathbf{v} = (3, 60°)$

Write as a column vector. $\mathbf{v} = \begin{bmatrix} 3 \\ 60° \end{bmatrix}$

b The diagram on the right shows some possible representations of **v**. The vector is at an angle of 60° to the x-axis and has a magnitude of 3, as indicated in the diagram.

v may be drawn in different positions, but its magnitude and direction remain the same no matter where it is drawn in the plane.



One of the most important vector quantities in scientific applications is **displacement**. The displacement from one position to another is the *change of position*. When you look at displacement, it doesn't matter how the change has occurred. It is a vector because a change of position has both magnitude and direction.

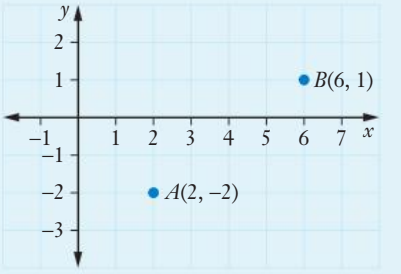
ISBN 9780170412995

EXAMPLE 5

Find the displacement **d** from $A(2, -2)$ to $B(6, 1)$.

Solution

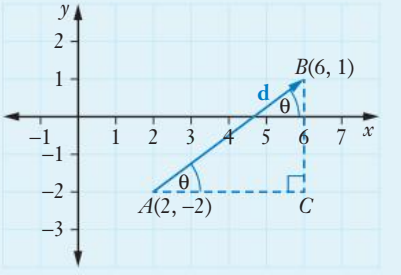
Draw a set of axes and plot $A(2, -2)$ and $B(6, 1)$.



The displacement from A to B can be shown as an arrow from A to B.

Label the arrow **d**.

You need to calculate the distance from A to B to find the magnitude of **d** and the angle θ to find the direction.



Use the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Substitute known values.

$$d = \sqrt{(6-2)^2 + [1-(-2)]^2}$$

Evaluate.

$$= \sqrt{16+9}$$

$$= \sqrt{25}$$

$$= 5$$

Calculate θ using ΔABC and the tan ratio.

$$\tan(\theta) = \frac{BC}{AC}$$

Substitute known values.

$$= \frac{3}{4}$$

Evaluate.

$$= 0.75$$

Use $\tan^{-1}$ to calculate θ.

$$\tan^{-1}(0.75) = 36.8698\ldots°$$

ISBN 9780170412995

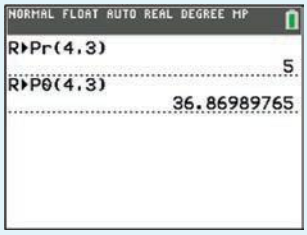
Round the angle and write **d** in polar form. $\mathbf{d} \approx (5, 36.9°)$ or $\mathbf{d} \approx \begin{bmatrix} 5 \\ 36.9° \end{bmatrix}$

You can also use a graphics calculator to find **d**.

TI-Nspire CX
Chapter 1

TI-84 Plus CE

Make sure that your calculator is set to degrees.

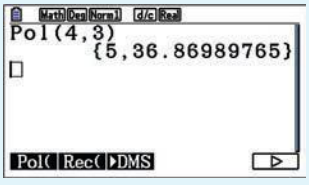


Press 2nd apps and select 5: R▶Pr. Enter the difference in the x-coordinates and the difference in the y-coordinates separated by a comma (,) and press enter. The value of r for (r, θ) is displayed.

Press 2nd apps and select 6: R▶Pθ. Enter the difference in the x-coordinates and the difference in y-coordinates separated by a comma (,) and press enter. The value of θ for (r, θ) is displayed.

Casio fx-CG20AU

Make sure that your calculator is set to degrees.



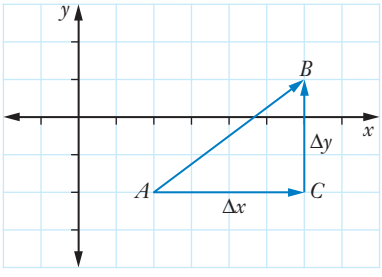
Open a Run-Matrix screen and press OPTN then select F6, F5 [**ANGLE**]. Select F6 and then F1 [**Pol(**].

Enter the difference between the x-coordinates and the difference in the y-coordinates separated by a comma (,) and then close the brackets.

Press EXE.

The polar form of **d** is then displayed.

In considering displacement, you are not concerned with how the change of position occurs.



In Example 5, the displacement could have occurred in many different ways. It could even have occurred by a movement from A to C and then from C to B, as shown in the diagram on the right. This would be an x-change of 4 and a y-change of 3.

You could show the displacement **AB** as the x-change and the y-change. This gives the ordered pair (4, 3). This can be done for any vector and is called the component form of the vector. 4 is the x-component and 3 is the y-component of the **displacement vector AB**.

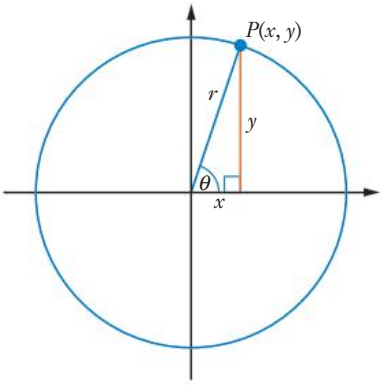
ISBN 9780170412995

Component form of a vector

A two-dimensional vector is represented in **component form** (Cartesian form) by an ordered pair, matrix or row matrix. The first number is the x-component and the second is the y-component of the vector. The x-component is in the x-direction and the y-component is in the y-direction. They form a right-angled triangle with the vector as the hypotenuse.

Thus a vector $\mathbf{p}$ may be written as the row vector (x, y) or the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

The diagram on the right shows a point $P(x, y)$ on a circle of radius r. OP makes an angle of θ with the x-axis.



It can be shown that for any point $P(x, y)$, $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ can be defined in terms of the coordinates P.

$$\sin(\theta) = \frac{y}{r}, \cos(\theta) = \frac{x}{r} \text{ and } \tan(\theta) = \frac{y}{x}.$$

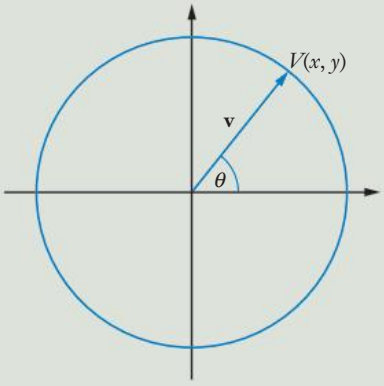
$$\text{Also, } r = \sqrt{x^2 + y^2}.$$

These definitions can be applied to vectors.

For any vector $\mathbf{v}$ with polar form $\mathbf{v} = (v, \theta)$ and component form $\mathbf{v} = (x, y)$:

$|\mathbf{v}| = v = \sqrt{x^2 + y^2}$ and $x = v\cos(\theta)$ and $y = v\sin(\theta)$

So $\mathbf{v} = (v, \theta) = (v\cos(\theta), v\sin(\theta))$



It is convenient to be able to express a vector in both component and polar form. Trigonometry of right-angled triangles is used to convert vectors between these representations.

ISBN 9780170412995

EXAMPLE 6

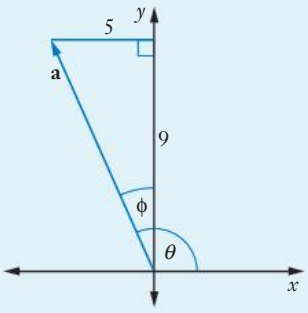
a Change the vector $\mathbf{a} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}$ to polar form.

b If **d** is a vector of magnitude 6 in the direction 245°, write **d** in component form.

Solution

a Draw **a** on the Cartesian plane using the x- and y-components. Draw a right-angled triangle to show the relationship between the two forms of the vector.

To obtain the correct directions, care must be taken to allow for the signs of the components.



You need to calculate the magnitude (a) and direction (θ) of **a**.

Use Pythagoras' theorem to find a.

$a^2 = 5^2 + 9^2$

$= 106$

$a = 10.2956\ldots$

Use the diagram to find the tan ratio.

$\tan(\phi) = \frac{5}{9} = 0.555\ldots$

Use $\tan^{-1}$ to find ϕ.

$\phi = \tan^{-1}(0.555\ldots) = 29.0546\ldots°$

Use the diagram to find θ.

$\theta = 90° + 29.0546\ldots°$

$= 119.0546\ldots°$

Round off and write **a** in polar form.

$\mathbf{a} \approx (10.30, 119.1°)$

b Write **d** in polar form.

$\mathbf{d} = (6, 245°)$

Use the definition.

$= (6\cos(245°), 6\sin(245°))$

Evaluate.

$= (-2.5357\ldots, -5.4378\ldots)$

Round off and state the result

$\mathbf{d} \approx (-2.54, -5.44)$

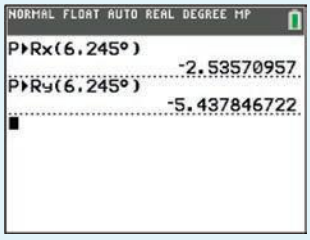
ISBN 9780170412995

You can also use a graphics calculator to find the component form of **d**.

TI-84 Plus CE

Press 2nd apps and select 7: P▶Rx. Enter 6 , 245 and then close the brackets and press enter.
The x-coordinate is displayed.

Press 2nd apps and select 8: P▶Ry. Enter 6 , 245 and then close the brackets and press enter.
The y-coordinate is displayed.



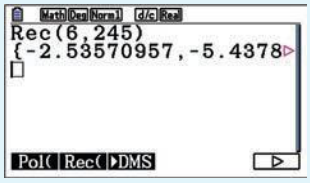
Casio fx-CG20AU

Open a Run-Matrix screen and press OPTN then select F6, F5 [**ANGLE**]. Select F6 and then F2 [**Rec(**].

Enter the norm of **d** and the angle **d** makes with the x-axis separated by a comma and then close the brackets.

Press EXE.

The Cartesian form of **d** is then displayed.



Since a displacement vector may show a change from one position to another position, it is also symbolised by writing the first and second positions in order. The displacement from A to B is shown as **AB** or $\overrightarrow{AB}$ or $\overline{AB}$ or $\widetilde{AB}$.

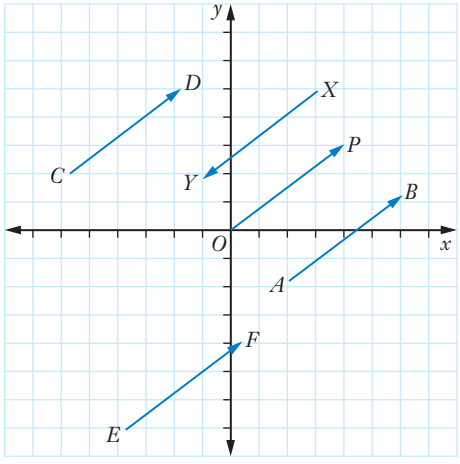
Even though a displacement from A to B can be written using A and B, it is still the same as another vector of the same magnitude and direction in another position.

In the diagram on the right, **AB**, **CD**, **EF** and **OP** represent the same vector. We could write

AB = **CD** = **EF** = **OP**

Because **XY** has the same magnitude as **OP**, but runs in the opposite direction, we can say that

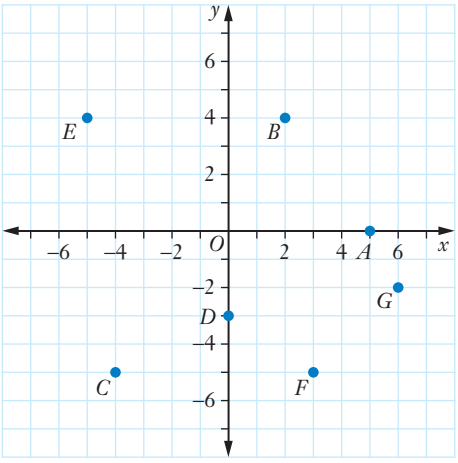
XY = –**OP** = –**AB** = –**CD** = –**EF**



ISBN 9780170412995

Exercise 1.02 Polar and Cartesian forms of vectors

1 Use the diagram to write each position vector as an ordered pair (a, b).



a OA
b OB
c OC
d OD
e OE
f OF
g OG

Example 4

2 Sketch each vector.

a (6, –2) **b** (1, 3) **c** (4, 120°) **d** (5, 210°) **e** (–2, –4)
f (–5, 2) **g** (3, –40°) **h** (5, 300°) **i** (4, 2) **j** (4, 45°)

3 Calculate the norm of the position vector to each point, correct to two decimal places.

a (2, 4) **b** (–1, 3) **c** (4, –7) **d** (6, 1)
e (–8, 3) **f** (5, –2) **g** (–4, –6) **h** (–3, 12)

4 Calculate the angle that the position vector to each point in question **3** parts **a** to **h** makes with the positive direction of the x-axis, correct to 1 decimal place.

5 Convert each vector to polar form.

a (3, –4) **b** (–5, 5) **c** (–7, –10) **d** (4, 12) **e** (–10, 16)
f (8, 15) **g** (12, 5) **h** (–9, –12) **i** (–7, –12) **j** (8, –8)

Example 5

6 Find in polar form the displacement **AB** from:

a $A(1, 3)$ to $B(6, 8)$ **b** $A(3, -4)$ to $B(7, 2)$
c $A(-1, -3)$ to $B(4, 6)$ **d** $A(5, 1)$ to $B(-3, 4)$
e $A(6, 8)$ to $B(-2, -1)$ **f** $A(5, -7)$ to $B(-2, 3)$

ISBN 9780170412995

Example 6

7 Convert each vector to component form.

a (5, 30°) **b** (10, 300°) **c** (24, 90°) **d** (16, 135°)

e (28, –120°) **f** (70, 270°) **g** (35, 0°) **h** (22, 180°)

i (4, 200°) **j** (8, –60°)

8 Write each displacement in component form and polar form.

a **AB**

b **CB**

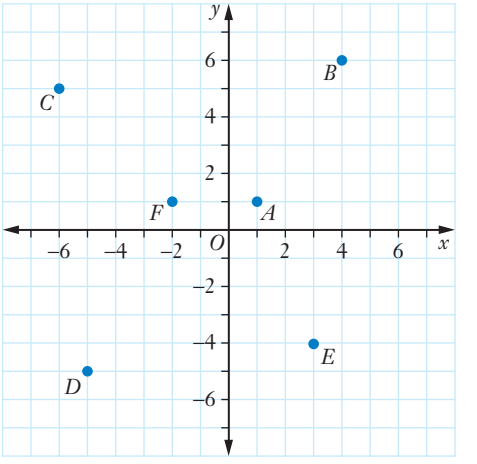
c **DE**

d **AD**

e **EC**

f **DB**

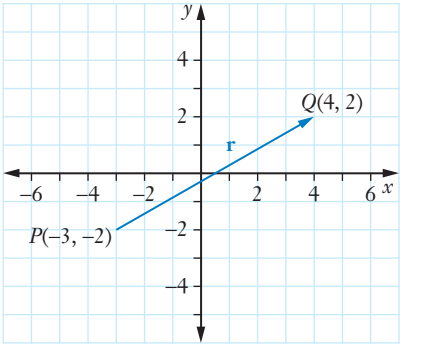
g **BF**



Problem solving

9 The directed line segment shown in the diagram represents the vector **r**.

Find a and b if $\mathbf{r} = (a, b)$.



10 The vector **p** has a magnitude of 6 and is in the direction N 40° W.

a Describe the vector **–p** in words.

b Sketch **p** and **–p** on the same set of axes so that their tails (initial points) coincide.

c On another set of axes draw **p** and **–p** so that the head (terminal point) of **p** coincides with the tail of **–p**.

d If **p** is a displacement vector, describe the displacement vector that results from a displacement of **p** followed by a displacement of **–p**.

ISBN 9780170412995

Scalar multiplication

1.03 Scalar multiples of vectors

If you wanted to make a new vector twice as big as a given vector, then you wouldn't change the direction. You would just double the magnitude.

For a vector in polar form such as $\mathbf{a} = (3, 60°)$, $2\mathbf{a} = (2 \times 3, 60°) = (6, 60°)$

For a vector in Cartesian form such as $\mathbf{b} = (4, 5)$, $2\mathbf{b} = (2 \times 4, 2 \times 5) = (8, 10)$.

Multiplying a vector by a scalar quantity is called **scalar multiplication**.

Multiplication by a scalar

For a vector $\mathbf{a} = (a_1, a_2)$ and a constant c, the **scalar multiple** $c\mathbf{a}$ is defined as:

$c\mathbf{a} = (ca_1, ca_2)$.

If the vector is in polar form, i.e., $\mathbf{a} = (r, \theta)$, the scalar multiple $c\mathbf{a}$ is defined as:

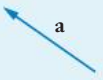
(cr, θ) for $c \geq 0$, or $(|c|r, 180° + \theta)$ for $c < 0$.

(If $c < 0$, 180° is added to reverse the direction of the scalar multiple.)

The product $(-1)\mathbf{a}$ is normally written as $-\mathbf{a}$.

EXAMPLE 7

The vector **a** is shown on the right.



Draw vectors representing:

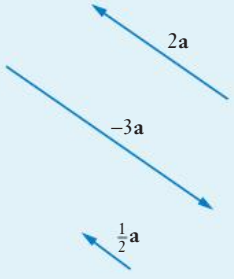
a $2\mathbf{a}$ **b** $-3\mathbf{a}$ **c** $\frac{1}{2}\mathbf{a}$

Solution

a $2\mathbf{a}$ will be in the same direction but will be twice as long as **a**.

b $-3\mathbf{a}$ will be in the opposite direction and will be 3 times as long as **a**.

c $\frac{1}{2}\mathbf{a}$ will be in the same direction but will be half as long as **a**.



ISBN 9780170412995

EXAMPLE 8

a If $\mathbf{f} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$, find $3\mathbf{f}$.

b If $\mathbf{m} = (7, 251°)$, find $-5\mathbf{m}$.

Solution

a Write the product. $3\mathbf{f} = 3\begin{bmatrix} -6 \\ 3 \end{bmatrix}$

Multiply the components of $\mathbf{f}$ by 3. $= \begin{bmatrix} 3\times(-6) \\ 3\times 3 \end{bmatrix}$

Evaluate and state the result. $3\mathbf{f} = \begin{bmatrix} -18 \\ 9 \end{bmatrix}$

b Here the scalar, $c < 0$. $c = -5$

Use the rule for scalar product. $c\mathbf{m} = (|c|m, 180° + \theta)$

Here $\mathbf{m} = (7, 251°)$ and $|c| = 5$. $-5\mathbf{m} = (5 \times 7, 180° + 251°)$

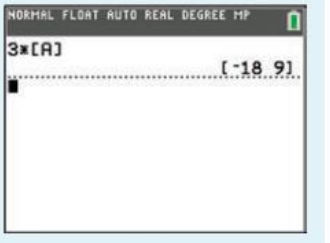
Evaluate. $= (35, 431°)$

Convert the angle so it is in the domain $0 \le \theta < 360°$ and state the result. $-5\mathbf{m} = (35, 71°)$

You can do scalar multiplication with a graphics calculator.

TI-84 Plus CE

You can't enter vectors with the TI-84 Plus CE, so enter the vector as a matrix.



Press 2nd x^{-1} (Matrix) and select EDIT, then select 1: [A] (matrix A) and press enter. Set the matrix to 1 row and 2 columns (1 × 2) and then press enter. Press (-) 6 enter and then 3 enter. Vector A is now defined as $\mathbf{A} = (-6, 3)$. Press 2nd mode to quit the screen. Enter 3 × 2nd x^{-1} (Matrix) and select A. Press enter to display the result.

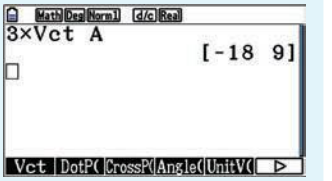
Vectors cannot be entered in polar form on the TI-84 Plus CE.

TI-Nspire CX Chapter 1

ISBN 9780170412995

Casio fx-CG20AU

Enter the vector (–6, 3) by pressing F3 [▶MAT/VCT]. Select the vector mode using [M⇔V] if necessary. Select **Vct A** and press F3 [**DIM**]. Enter 1 against **m** and 2 against **n**. Press EXE and then enter the components. Press EXIT EXIT. Now **A** = (–6, 3).

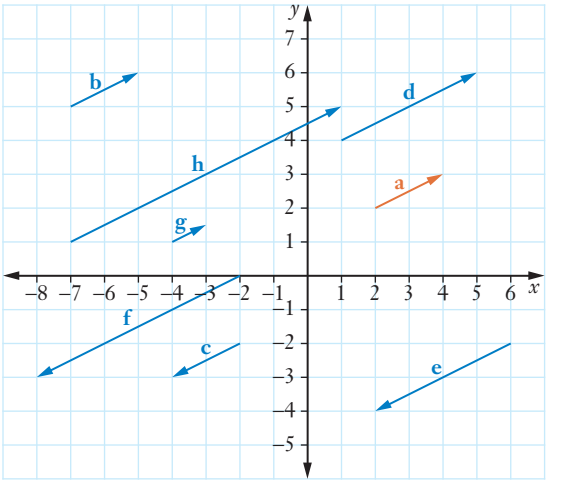


Enter 3 × OPTN F2 [**MAT/VCT**] F6 F6 F1 [**Vct**] and then press ALPHA X,θ,T [**A**]. Press EXE and the scalar product is displayed.

Vectors cannot be entered in polar form on the Casio fx-CG20AU.

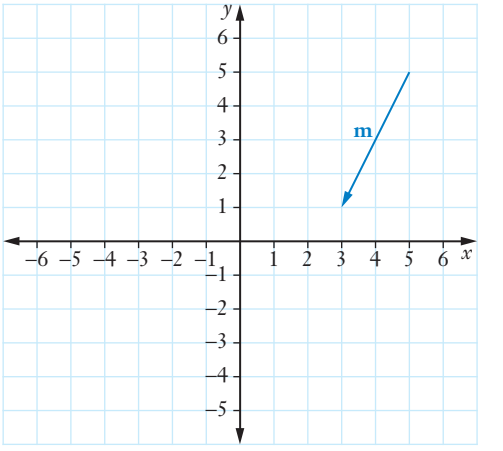
Exercise 1.03 Scalar multiples of vectors

1 Write each of the vectors **b**, **c**, **d**, …, **h** shown in the diagram below in terms of **a**.



Example 7

2 Make a copy of this diagram and draw each of the following vectors on it.



a $\frac{1}{2}$**m**
b –**m**
c –2**m**
d 3**m**
e –3**m**
f 1.5**m**
g –1.5**m**
h –4**m**

ISBN 9780170412995

3 If $\mathbf{n} = (4, -2)$, draw each vector on the same set of axes.

a $2\mathbf{n}$ **b** $\frac{1}{2}\mathbf{n}$ **c** $-\mathbf{n}$ **d** $3\mathbf{n}$

e $4\mathbf{n}$ **f** $1.5\mathbf{n}$ **g** $-2.5\mathbf{n}$ **h** $-4\mathbf{n}$

4 If $\mathbf{p} = (8, 2)$, find:

Example 8

a $3\mathbf{p}$ **b** $-\mathbf{p}$ **c** $\frac{1}{2}\mathbf{p}$ **d** $1.5\mathbf{p}$

e $-2.5\mathbf{p}$ **f** $4\mathbf{p}$ **g** $-\frac{1}{4}\mathbf{p}$ **h** $\frac{3}{4}\mathbf{p}$

5 If $\mathbf{w} = \begin{bmatrix} -12 \\ 30 \end{bmatrix}$, find:

a $2\mathbf{w}$ **b** $-3\mathbf{w}$ **c** $\frac{1}{3}\mathbf{w}$ **d** $-2.5\mathbf{w}$

e $-\frac{2}{3}\mathbf{w}$ **f** $1.5\mathbf{w}$ **g** $-\frac{1}{6}\mathbf{w}$ **h** $0.5\mathbf{w}$

6 If $\mathbf{u} = (5, 58°)$, find:

a $2\mathbf{u}$ **b** $-\mathbf{u}$ **c** $3\mathbf{u}$ **d** $\frac{1}{2}\mathbf{u}$

e $-1.5\mathbf{u}$ **f** $2.5\mathbf{u}$ **g** $-\frac{1}{5}\mathbf{u}$ **h** $-6\mathbf{u}$

7 If $\mathbf{p} = (9, 3)$, $\mathbf{q} = (-6, 4)$ and $\mathbf{r} = (4, -10)$, find:

a $4\mathbf{q}$ **b** $-2\mathbf{r}$ **c** $\frac{1}{3}\mathbf{p}$ **d** $-2.5\mathbf{r}$

e $-\frac{3}{2}\mathbf{q}$ **f** $3\mathbf{p}$ **g** $\frac{3}{4}\mathbf{r}$ **h** $-\frac{1}{4}\mathbf{q}$

8 If $\mathbf{d} = (6, 36°)$, $\mathbf{e} = (4, 124°)$ and $\mathbf{f} = (9, 218°)$, find:

a $3\mathbf{d}$ **b** $-\mathbf{f}$ **c** $2\mathbf{e}$ **d** $-\frac{1}{3}\mathbf{d}$

e $-2.5\mathbf{e}$ **f** $1.5\mathbf{d}$ **g** $-\frac{1}{3}\mathbf{f}$ **h** $-5\mathbf{e}$

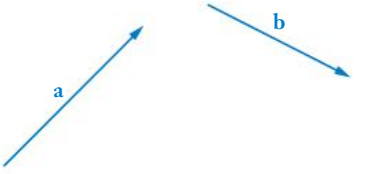
1.04 Addition of vectors

Triangle addition of vectors

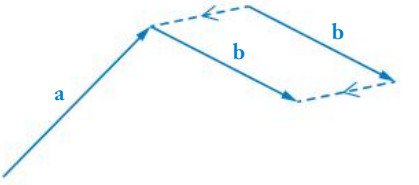
Parallelogram rule for vector addition

You have seen that you can draw a vector in any position on the plane. This means that a vector can slide (translate) from one position to another without changing its value.

Consider the two displacements **a** and **b** shown on the right.

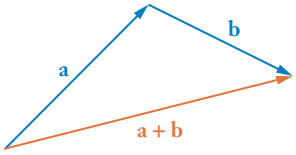


To work out the displacement that results from displacement **a** followed by displacement **b**, you can slide **b** across until its tail coincides with the head of **a**.



ISBN 9780170412995

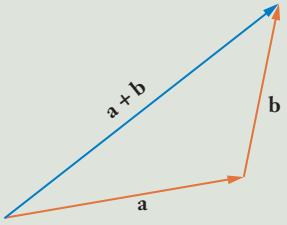
The overall result of adding the displacements **a** and **b** is the vector **a** + **b**. This vector goes from the tail of **a** to the head of **b**.



Triangle rule for vector addition

If two vectors **a** and **b** are placed head-to-tail, the sum **a** + **b**, is the third side of the triangle.

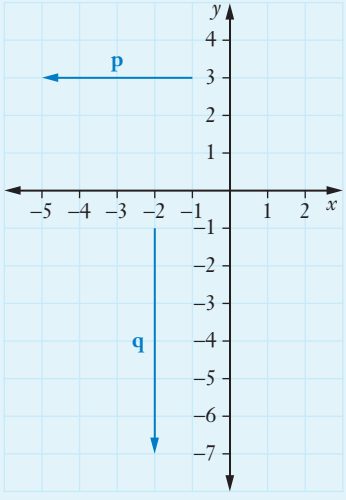
The sum **a** + **b** is also called the **resultant** (**r**) of **a** and **b**.



The triangle rule for vector addition is also called the **head-to-tail rule** because it requires that vectors be arranged head-to-tail.

EXAMPLE 9

Find the resultant (sum) of **p** and **q** in polar form.



ISBN 9780170412995

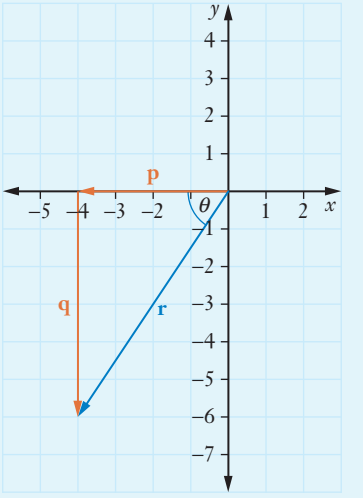
Solution

Start by re-drawing the vectors head-to-tail.

It is convenient to place the first vector with its tail at the origin.

It doesn't matter whether you start with **p** or **q**.

Draw in the resultant vector (**r**) and mark in the angle (θ) that the resultant vector makes with the x-axis.



Use Pythagoras' theorem to calculate the magnitude of **r**.

$$r = \sqrt{4^2 + 6^2}$$
$$= 2\sqrt{13}$$

Use the tan ratio to calculate θ.

$$\tan(\theta) = \frac{6}{4}$$

Use $\tan^{-1}$ to find θ.

$$\theta = \tan^{-1}\left(\frac{6}{4}\right)$$

Evaluate.

$$= 56.3099\ldots°$$

The direction of **r** is measured relative to the positive direction of the x-axis.

$$\text{Direction of } \mathbf{r} = 180° + 56.3099\ldots°$$
$$= 236.3099\ldots°$$

Round off and state the result.

$$\mathbf{r} = (2\sqrt{13}, 236.3°)$$

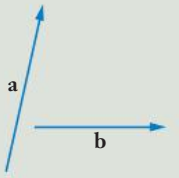
There is an alternative rule to the triangle (head-to-tail) rule for adding vectors.

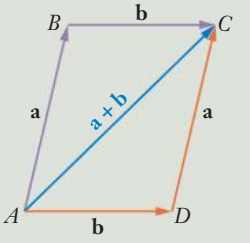
ISBN 9780170412995

Parallelogram rule for vector addition

Consider vectors **a** and **b** as shown below.

If you form the parallelogram *ABCD* using **a** and **b** as the sides, the diagonal **AC** is the resultant **a** + **b**.





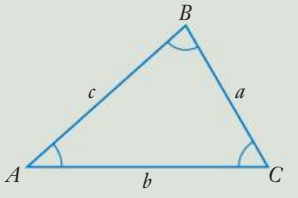
Note that the parallelogram shows that **a** + **b** = **b** + **a** so the order in which the vectors are added doesn't matter.

The application of the triangle rule and the parallelogram rule involve the use of Pythagoras' theorem and the trigonometric ratios. You may also be required to use the sine rule and the cosine rule.

Sine rule

For any $\triangle ABC$:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$



Cosine rule

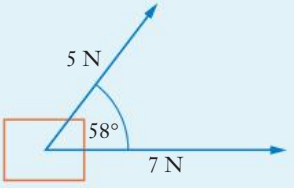
$a^2 = b^2 + c^2 - 2bc\cos(A)$

$b^2 = a^2 + c^2 - 2ac\cos(B)$

$c^2 = a^2 + b^2 - 2ab\cos(C)$

EXAMPLE 10

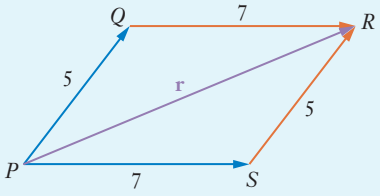
Two forces, one of 7 N and one of 5 N, act on an object. The forces are separated by an angle of 58°, as shown in the diagram. Calculate the magnitude and direction of the resultant force acting on the object.



ISBN 9780170412995

Solution

Redraw the diagram and complete the parallelogram. Label the vertices and draw in the resultant vector.



Use geometry and the given information to calculate the internal angles of the parallelogram.

$\angle P = 58°$ (Given.)

$\angle R = 58°$ (Opposite angles of a parallelogram.)

$\angle Q = 122°$ (Co-interior angles on parallel lines.)

$\angle S = 122°$ (Opposite angles of a parallelogram.)

Use the cosine rule to calculate the magnitude of **r** (r).

$PR^2 = PS^2 + SR^2 - 2 \times PS \times SR \cos(\angle S)$

Substitute for known values.

$r^2 = 7^2 + 5^2 - 2 \times 7 \times 5 \cos(122°)$

Evaluate.

$= 111.0943\ldots$

Take square roots of both sides.

$r = 10.5401\ldots$

Use the sine rule to calculate $\angle RPS$.

$$\frac{PR}{\sin(\angle RSP)} = \frac{SR}{\sin(\angle RPS)}$$

Substitute for known values.

$$\frac{10.5401...}{\sin(122°)} = \frac{5}{\sin(\angle RPS)}$$

Rearrange to isolate $\angle RPS$.

$$\sin(\angle RPS) = \frac{5 \times \sin(122°)}{10.5401...}$$

$$= 0.4022\ldots$$

Use $\sin^{-1}$ to calculate $\angle RPS$.

$\angle RPS = \sin^{-1}(0.4022\ldots)$

Evaluate.

$= 23.7217\ldots$

Round off and state the result.

The resultant force has a magnitude of about 10.5 N at an angle of 23.7° above the horizontal.

ISBN 9780170412995

EXAMPLE 11

Calculate the resultant of $\mathbf{a} = (12, 235°)$ and $\mathbf{b} = (18, 169°)$.

Solution

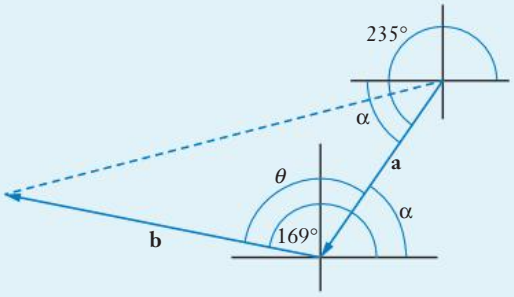
Draw a rough sketch to help with the calculations. Use either the triangle rule or the parallelogram rule.

Draw **a** at 235°.

Draw **b** in the direction of 169°, starting where **a** finishes. You should make **b** about $1\frac{1}{2}$ times the length of **a**.

You need to find the angle between **a** and **b**.

Mark in angles as required.



Use the diagram to find θ, the angle between the vectors.

$\alpha = 235° - 180° = 55°$

$\theta = 169° - \alpha$

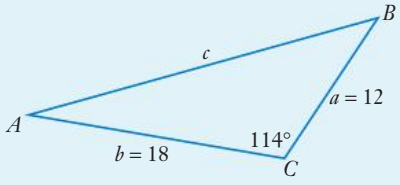
$= 169° - 55°$

$= 114°$

Represent the information in a simpler diagram.

Label the vertices and sides of the triangle.

The magnitude of the resultant is c.



Use the cosine rule to find c.

$c^2 = a^2 + b^2 - 2ab\cos(C)$

Substitute known values.

$= 12^2 + 18^2 - 2 \times 12 \times 18 \times \cos(114°)$

Evaluate.

$c^2 = 643.7102\ldots$

Take square roots of both sides.

$c = 25.3714\ldots$

Use the sine rule to find B.

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

ISBN 9780170412995

Substitute known values.	$\frac{18}{\sin(B)} = \frac{25.3714...}{\sin(114°)}$
Rearrange.	$\sin(B) = \frac{18 \times \sin(114°)}{25.3714...}$
Evaluate.	$= 0.6481\ldots$
Use $\sin^{-1}$ to calculate B.	$B = \sin^{-1}(0.6481\ldots)$ $= 40.4002\ldots°$
Refer to the first diagram to calculate the direction of the resultant.	Direction $= 235° - 40.4002\ldots°$ $= 194.5997\ldots°$
State the result.	The resultant vector $\approx (25.4, 194.6°)$.

Exercise 1.04 Addition of vectors

1 Calculate the resultant of each set of vectors. Write your answer in terms of 'left' or 'right'.

a 4 9

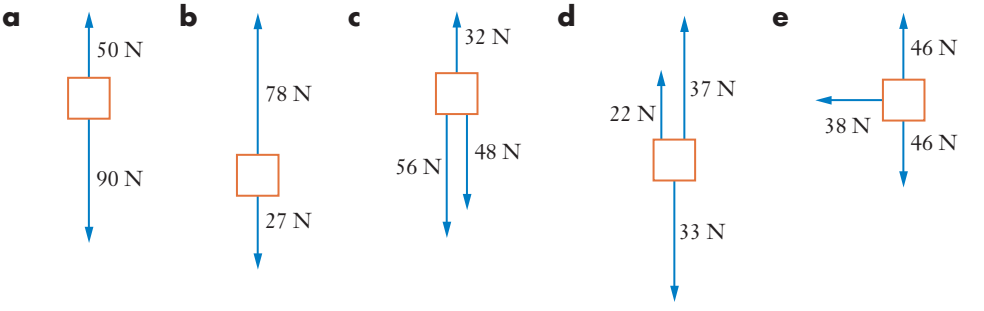
b 3 16 8

c 12 17

d 15 15

e 13 6 10

2 In each situation, a block is subjected to a number forces (in newtons). Calculate the resultant force acting on each block.



ISBN 9780170412995

Example 9

3 Find the resultant of each pair of vectors, giving your answer in polar form.

a $\mathbf{m} + \mathbf{p}$

b $\mathbf{w} + \mathbf{q}$

c $\mathbf{s} + \mathbf{p}$

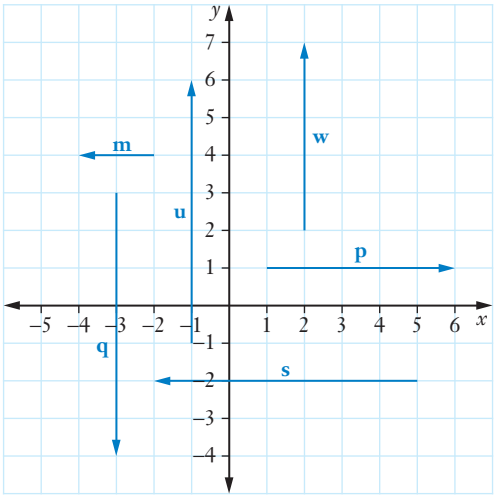
d $\mathbf{w} + \mathbf{u}$

e $\mathbf{w} + \mathbf{m}$

f $\mathbf{p} + \mathbf{u}$

g $\mathbf{s} + \mathbf{q}$

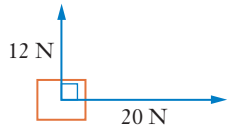
h $\mathbf{w} + \mathbf{s}$



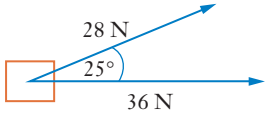
Example 10

4 Calculate the resultant force acting on each object.

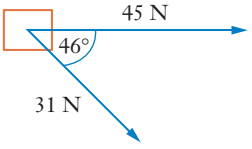
a



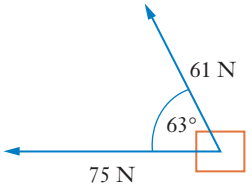
b



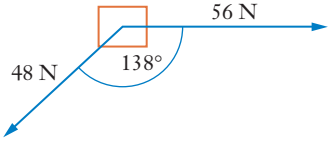
c



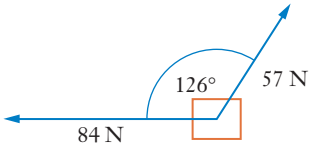
d



e



f



Example 11

5 Find the resultant of $\mathbf{p} = (6, 20°)$ and $\mathbf{q} = (9, 55°)$.

6 Calculate $(25, 120°) + (16, 80°)$.

7 Given $\mathbf{w} = (6, 220°)$, $\mathbf{x} = (10, 60°)$, $\mathbf{y} = (8, 100°)$ and $\mathbf{z} = (9, -50°)$, calculate:

a $\mathbf{w} + \mathbf{y}$ **b** $\mathbf{y} + \mathbf{z}$ **c** $\mathbf{x} + \mathbf{z}$

d $\mathbf{w} + \mathbf{z}$ **e** $\mathbf{y} + \mathbf{x}$

8 Given $\mathbf{m} = (10, 140°)$, $\mathbf{n} = (8, 70°)$, $\mathbf{p} = (11, -80°)$ and $\mathbf{q} = (12, -120°)$, calculate:

a $\mathbf{m} + \mathbf{n}$ **b** $\mathbf{p} + \mathbf{q}$ **c** $\mathbf{m} + \mathbf{p}$

d $\mathbf{q} + \mathbf{n}$ **e** $\mathbf{n} + \mathbf{p}$

ISBN 9780170412995

Problem solving

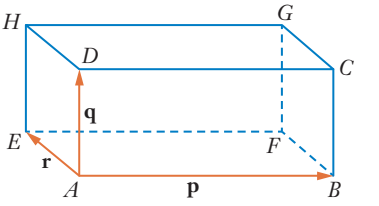
9 Vectors **a** and **b** represent two adjacent sides of a rectangle. Find a vector expression which represents the diagonals of the rectangle.

10 In the parallelogram *ABCD*, side *AD* is represented by the vector **p** and side *AB* is represented by the vector **q**. State the physical significance of the vector **p** + **q**.

11 A boat travels 5 km NE and then 7 km at a bearing of 120°. Find its distance and direction from its starting point.

Shutterstock.com/freevideophotoagency

12 For the rectangular prism *ABCDEFGH* shown, if **AB** = **p**, **AD** = **q** and **AE** = **r**, write each vector in terms of **p**, **q** and **r**.



a **AF**

b **AH**

c **BG**

d **FC**

e **GE**

1.05 Subtraction of vectors

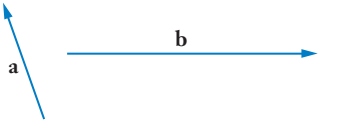
You can subtract vectors by combining the concepts of vector addition and negative vectors. Consider the vectors **a** and **b**.

You can write **a** – **b** as follows.

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

This can be shown geometrically.

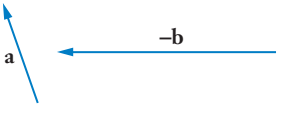
Begin with vectors **a** and **b** as shown on the right.



You can find **a** – **b** by changing the subtraction to addition:

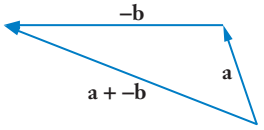
$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

Redraw **b** as –**b** by reversing its direction.



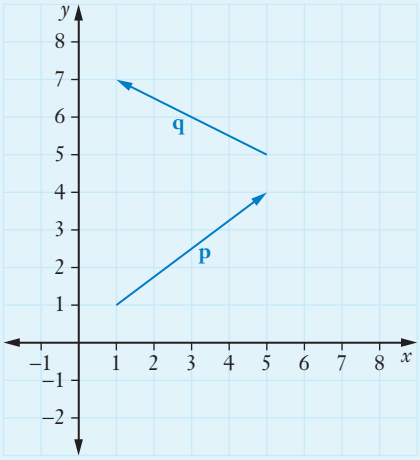
ISBN 9780170412995

The vectors **a** and **–b** may now be added by arranging the vectors head-to-tail.



EXAMPLE 12

Use the diagram to calculate **p** – **q**.



Solution

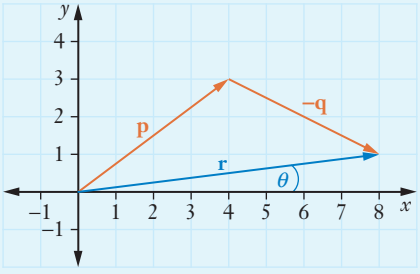
Write the subtraction as an addition.

$\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q})$

Redraw the diagram by placing **p** at the origin and reversing the direction of **q** to form **–q**. Then place **–q** head-to-tail with **p**.

Name the resultant vector **r**.

Mark in θ, the angle that **r** makes with the x-axis.



Calculate the norm of **r**.

$r = \sqrt{x^2 + y^2}$

Substitute for known values.

$= \sqrt{8^2 + 1^2}$

Evaluate.

$= \sqrt{65}$

Use the tan ratio to calculate θ.

$\tan(\theta) = \frac{1}{8} = 0.125$

ISBN 9780170412995

Use $\tan^{-1}$ to find θ.	$\tan^{-1}(0.125) = 7.1250\ldots°$
Round off and write the result.	$\mathbf{p} - \mathbf{q} \approx (\sqrt{65}, 7.1°)$

EXAMPLE 13

Given that $\mathbf{a} = (35, 130°)$ and $\mathbf{b} = (40, 50°)$, calculate $\mathbf{a} - \mathbf{b}$.

Solution

Write $\mathbf{a} - \mathbf{b}$ as an addition.

$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

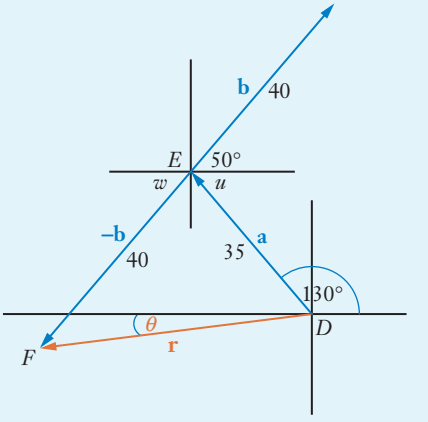
Draw a sketch showing **a** and **b**.

Draw $-\mathbf{b}$ and then place it head-to-tail with **a**.

Mark in the magnitudes and directions of **a** and **b**.

Label other angles as required.

Label the vertices of $\triangle DEF$.



Calculate required angles using the rules of geometry.

$u = 50°$ (Co-interior angles.)

$w = 50°$ (Vertically opposite angles.)

$\angle FED + 50° + 50° = 180°$ (Angles on a line.)

$\angle FED = 80°$

Use the cosine rule to find r.

$r^2 = a^2 + b^2 - 2ab\cos(\angle FED)$

Substitute known values.

$= 35^2 + 40^2 - 2 \times 35 \times 40 \times \cos(80°)$

Evaluate.

$r^2 = 2338.7851\ldots$

Take square roots of both sides.

$r = 48.3609\ldots$

Use the sine rule to find $\angle EDF$.

$$\frac{b}{\sin(\angle EDF)} = \frac{r}{\sin(\angle FED)}$$

Substitute known values.

$$\frac{40}{\sin(\angle EDF)} = \frac{48.3609...}{\sin(80°)}$$

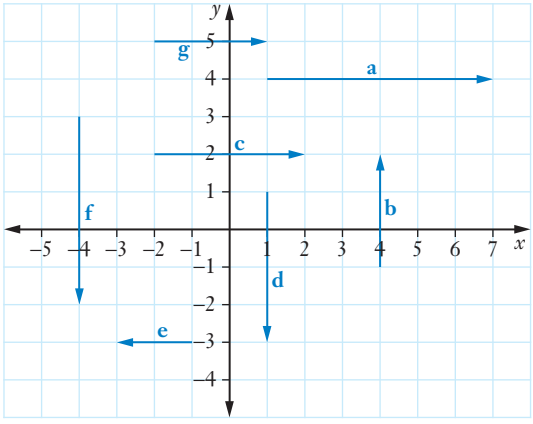
ISBN 9780170412995

Rearrange.	$\sin(\angle EDF) = \dfrac{40 \times \sin(80°)}{48.3609...}$
Evaluate.	$= 0.8145\ldots$
Use $\sin^{-1}$ to calculate $\angle EDF$.	$\angle EDF = \sin^{-1}(0.8145\ldots)$ $= 54.5426\ldots°$
Refer to the first diagram to calculate the direction of the resultant.	Direction $= 130° - 54.5426\ldots°$ $= 184.5426\ldots°$
State the result.	$\mathbf{r} = \mathbf{a} - \mathbf{b} \approx (48.4, 184.5°)$.

Exercise 1.05 Subtraction of vectors

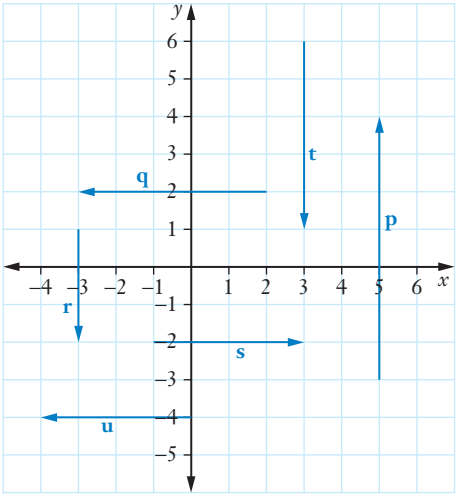
1 Find each vector in polar form.

a $\mathbf{a} - \mathbf{c}$
b $\mathbf{c} - \mathbf{e}$
c $\mathbf{d} - \mathbf{f}$
d $\mathbf{f} - \mathbf{b}$
e $\mathbf{e} - \mathbf{g}$
f $\mathbf{d} - \mathbf{b}$



2 Find each vector in polar form.

a $\mathbf{p} - \mathbf{q}$
b $\mathbf{s} - \mathbf{r}$
c $\mathbf{u} - \mathbf{t}$
d $\mathbf{q} - \mathbf{r}$
e $\mathbf{s} - \mathbf{t}$
f $\mathbf{u} - \mathbf{p}$



ISBN 9780170412995

3 Find each vector in polar form.

a a – b

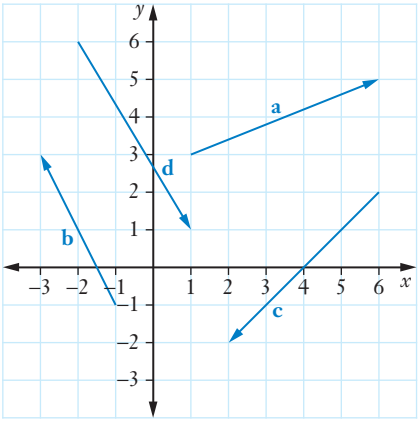
b c – b

c d – a

d c – d

e a – c

f b – a



Example 12

4 Calculate each vector in polar form.

a (7, 30°) – (9, 100°) **b** (105, 300°) – (95, 60°) **c** (6, 70°) – (6, 10°)

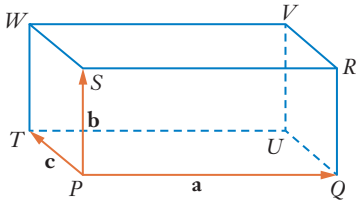
Example 13

5 Given **w** = (6, 220°), **x** = (10, 60°), **y** = (8, 100°) and **z** = (9, –50°), calculate each vector difference without changing to component form.

a w – x **b** z – y **c** x – z **d** y – w **e** w – y

Problem solving

6 A rectangular prism *PQRSTUVW* is shown. If **PQ** = **a**, **PS** = **b** and **PT** = **c**, write each vector in terms of **a**, **b** and **c**.



a **UP**

b **RP**

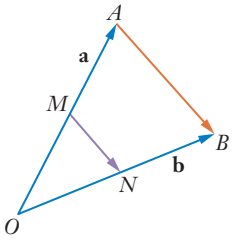
c **WP**

d **UR**

e **WU**

7 The diagram on the right shows **OA** = **a** and **OB** = **b**.

M is the midpoint of *OA* and *N* is the midpoint of *OB*.



a Write **AB** in terms of **a** and **b**.

b Show that $\mathbf{MN} = \frac{1}{2}\mathbf{AB}$.

ISBN 9780170412995

Unit vectors

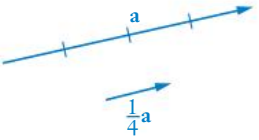
1.06 Unit vectors

So far you have seen how a vector can be written in polar form, $\mathbf{a} = (r, \theta)$, or in component form, $\mathbf{a} = (x, y)$ or $\mathbf{a} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Vectors can also be expressed in terms of **unit vectors**.

A unit vector has a magnitude of 1.

If $\mathbf{a}$ has magnitude 4, then a unit vector in the direction of $\mathbf{a}$ is $\frac{1}{4}\mathbf{a}$, as shown in the diagram.



The unit vector has a length of 1 unit and is in the same direction as $\mathbf{a}$.

Unit vectors

A unit vector has magnitude 1. The unit vector in the same direction as a given vector $\mathbf{p}$ is symbolised by placing a circumflex ($\hat{}$) over the vector symbol, for example, $\hat{\mathbf{p}}$.

A unit vector may be derived from any vector $\mathbf{p}$ by multiplying by $\frac{1}{|\mathbf{p}|}$. Thus:

$$\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}, \text{ where } |\mathbf{p}| \text{ is the norm of } \mathbf{p}.$$

The unit vectors in the directions of the x- and y-axes are commonly given the special symbols $\mathbf{i}$ and $\mathbf{j}$, so $\mathbf{i} = \hat{\mathbf{x}}$ and $\mathbf{j} = \hat{\mathbf{y}}$.

Therefore $\mathbf{i} = (1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

EXAMPLE 14

If $\mathbf{p} = (3, -4)$, find $\hat{\mathbf{p}}$.

Solution

Write the rule for the unit vector.

$$\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$$

Substitute values.

$$= \frac{1}{\sqrt{3^2 + (-4)^2}}(3, -4)$$

Evaluate.

$$= \frac{1}{5}(3, -4)$$

State the result.

$$\hat{\mathbf{p}} = \left(\frac{3}{5}, \frac{-4}{5}\right)$$

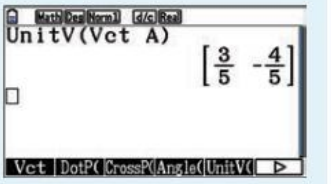
Unit vectors can be calculated on a Casio graphics calculator, but not on a TI-84 Plus CE calculator.

ISBN 9780170412995

Casio fx-CG20AU

Assign vector A as (3, –4) using the procedure outlined in Example 8.

Press OPTN F2 [MAT/VCT] F6 F6 F5 [Unit V]. Then press F1 and press ALPHA X,θ,T (A). Close the brackets and press EXE and the unit vector is displayed.



A **linear combination** of vectors is a sum of scalar multiples of the vectors. Thus, a linear combination of vectors **p** and **q** is of the form $a\mathbf{p} + b\mathbf{q}$, where a and b are real numbers.

Any vector in two dimensions may be written in the form (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$. Because **i** and **j** are the unit vectors in the direction of the x- and y-axes respectively, any two-dimensional vector can be written as a linear combination of **i** and **j**.

EXAMPLE 15

Write each vector as a linear combination of **i** and **j**.

a $\mathbf{f} = (5, -3)$ **b** $\mathbf{g} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ **c** $\mathbf{h} = (5, 300°)$

Solution

a

Write **f**.	$\mathbf{f} = (5, -3)$
Express as a sum of vectors.	$= (5, 0) + (0, -3)$
Write as multiples of unit vectors.	$= 5(1, 0) + [-3(0, 1)]$
State the result.	$\mathbf{f} = 5\mathbf{i} + (-3\mathbf{j})$

b

Write **g**.	$\mathbf{g} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$
Express as a sum of vectors in the direction of the x- and y-axes.	$= \begin{bmatrix} -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Write as multiples of unit vectors.	$= -6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
State the result.	$\mathbf{g} = -6\mathbf{i} + 0\mathbf{j}$

ISBN 9780170412995

c Write **h**. $\mathbf{h} = (5, 300°)$

Express in component form. $= (5 \cos (300°), 5 \sin (300°))$

Evaluate and round off. $\approx (2.5, -4.33)$

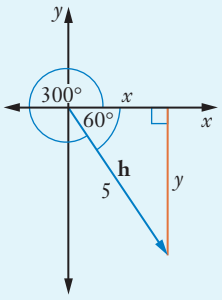
Express as a sum of vectors. $= (2.5, 0) + (0, -4.33)$

Write as multiples of unit vectors. $= 2.5(1, 0) + [-4.33(0, 1)]$

State the result. $\mathbf{h} \approx 2.5\mathbf{i} + (-4.33)\mathbf{j}$

It is also possible to write the exact value of **h**.

Draw a diagram.



Calculate the components of **h**. $x = 5 \cos (60°) = \frac{5}{2}$

$y = -5 \cos (60°) = \frac{-5\sqrt{3}}{2}$

Write as multiples of unit vectors. $\mathbf{h} = \frac{5}{2}\mathbf{i} - \frac{5\sqrt{3}}{2}\mathbf{j}$

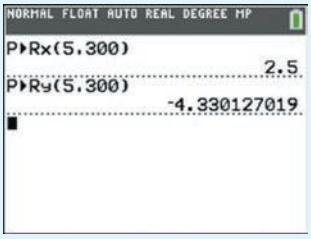
You can use a graphics calculator to express a vector in component form.

TI-Nspire CX Chapter 1

TI-84 Plus CE

For a vector in polar form e.g. (5, 300°), press 2nd apps and select 7: P▶Rx. Enter 5 , and then 300 and then close the brackets and press enter. The x-coordinate is displayed.

Press 2nd apps and select 8: P▶Ry. Enter 5 , and then 300 and then close the brackets press enter. The y-coordinate is displayed.



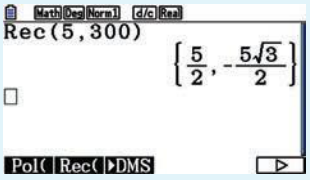
ISBN 9780170412995

Casio fx-CG20AU

Open a Run-Matrix screen and press OPTN, then select F6, F5 [**ANGLE**]. Select F6 and then F2 [**Rec(**].

Enter 5 , and then 300 and then close the brackets and press EXE.

The Cartesian form of **h** is then displayed.



Exercise 1.06 Unit vectors

1 Draw a vector to represent each expression.

a	$0\mathbf{i} + 7\mathbf{j}$	**b**	$3\mathbf{i} + 6\mathbf{j}$	**c**	$-4\mathbf{i} + 0\mathbf{j}$	**d**	$7\mathbf{i} + 0\mathbf{j}$
e	$0\mathbf{i} - 5\mathbf{j}$	**f**	$-7\mathbf{i} + 4\mathbf{j}$	**g**	$-6\mathbf{i} - 3\mathbf{j}$	**h**	$2\mathbf{i} - 7\mathbf{j}$
i	$4\mathbf{i} + \mathbf{j}$	**j**	$-4\mathbf{i} + 9\mathbf{j}$	**k**	$-6\mathbf{i} - 8\mathbf{j}$	**l**	$7\mathbf{i} - 3\mathbf{j}$
m	$4\mathbf{i} - 6\mathbf{j}$	**n**	$-\mathbf{i} - 8\mathbf{j}$				

2 If **m** is a unit vector, what is the length of $7\mathbf{m}$?

3 Calculate the unit vector for each vector as an exact value. Example 14

a	$(3, 4)$	**b**	$(-2, 8)$	**c**	$\begin{bmatrix} 5 \\ -1 \end{bmatrix}$	**d**	$(-6, -3)$
e	$(-5, 3)$	**f**	$\begin{bmatrix} -2 \\ -6 \end{bmatrix}$	**g**	$(-1, 4)$	**h**	$\begin{bmatrix} -5 \\ 12 \end{bmatrix}$

4 Calculate the unit vector for each vector as an exact value where possible.

a	$(4, 30°)$	**b**	$(7, 60°)$	**c**	$(12, 90°)$	**d**	$(6, 180°)$
e	$(11, 270°)$	**f**	$(8, 0°)$	**g**	$(4, 48°)$	**h**	$(9, 156°)$
i	$(6, 295°)$	**j**	$(13, 328°)$	**k**	$(10, -52°)$	**l**	$(15, -143°)$

5 Write each vector as a linear combination of **i** and **j**. Example 15

a	$(5, -3)$	**b**	$(-6, 4)$	**c**	$(-4, -7)$	**d**	$(6, 5)$
e	$(-3.2, -9.4)$	**f**	$\begin{bmatrix} 1 \\ -4 \end{bmatrix}$	**g**	$\begin{bmatrix} -2 \\ -8 \end{bmatrix}$	**h**	$\begin{bmatrix} -3 \\ 5 \end{bmatrix}$
i	$\begin{bmatrix} 7.59 \\ 3.68 \end{bmatrix}$	**j**	$\begin{bmatrix} 0.07 \\ 0.19 \end{bmatrix}$				

6 Write each vector as a linear combination of **i** and **j**.

a	$(7, -74°)$	**b**	$(9, 125°)$	**c**	$(11, 330°)$
d	$(16, 240°)$	**e**	$(4.8, -119°)$	**f**	$(8, 85°)$
g	$(3, 158°)$	**h**	$(10, -87°)$		
i	$(15, -255°)$	**j**	$(20, -119°)$		

ISBN 9780170412995

Problem solving

7 *OABC* is a rectangle in which the vector **OA** = 6**i** and **OC** = 4**j**. Express each vector in terms of **i** and **j**.

a CA **b** AC

8 Find a unit vector in component form that is in the opposite direction to **g** = –5**i** + 12**j**.

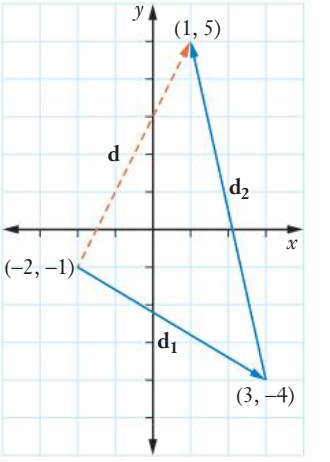
9 Given **h** = –6**i** + 8**j**, find a vector in component form that has a magnitude of 15 in the same direction as **h**.

WS
Addition of vectors using components

1.07 Vector components

You have previously seen how to add vectors in polar form. It is also possible to add vectors in component form.

Consider displacements from (–2, –1) to (3, –4) and then from (3, –4) to (1, 5), as shown in the diagram. The overall displacement is from (–2, –1) to (1, 5). The arrows in the diagram show that the total displacement is formed as the third side of the triangle made by the first two displacements, with the vectors placed head-to-tail.



In component form, the displacements are $\mathbf{d}_1 = (5, -3)$ and $\mathbf{d}_2 = (-2, 9)$. The total displacement is $\mathbf{d} = (3, 6)$.

The resultant of $\mathbf{d}_1$ and $\mathbf{d}_2$ can be found by adding the components of $\mathbf{d}_1$ and $\mathbf{d}_2$ as shown below.

$$\begin{aligned}\mathbf{d}_1 + \mathbf{d}_2 &= (5, -3) + (-2, 9)\\ &= (5 + (-2), -3 + 9)\\ &= (3, 6)\\ &= \mathbf{d}\end{aligned}$$

Addition of vectors using components

The sum of two vectors $\mathbf{a} = (x_1, y_2)$ and $\mathbf{b} = (x_2, y_2)$ is defined by:

$$\mathbf{a} + \mathbf{b} = (x_1 + x_2, y_1 + y_2)$$

In terms of the unit vectors **i** and **j**, if $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$, then:

$$\mathbf{a} + \mathbf{b} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j}$$

ISBN 9780170412995

EXAMPLE 16

Find the sum of each of the following vectors.

a $(5, -7) + (-12, 4)$ **b** $\begin{bmatrix} 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ **c** $3\mathbf{i} - 8\mathbf{j}$ and $-5\mathbf{i} + 7\mathbf{j}$

Solution

a Add the components. $(5, -7) + (-12, 4) = (5 + (-11), -7 + (-12))$

Evaluate. $= (-6, -19)$

b Add the components. $\begin{bmatrix} 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 7+5 \\ 3+(-4) \end{bmatrix}$

Evaluate. $= \begin{bmatrix} 12 \\ -1 \end{bmatrix}$

c Add the **i** and **j** parts separately. $(3\mathbf{i} - 8\mathbf{j}) + (-5\mathbf{i} + 7\mathbf{j}) = (3 + (-5))\mathbf{i} + (-8 + 7)\mathbf{j}$

Evaluate. $= -2\mathbf{i} - \mathbf{j}$

You can also add scalar multiples of vectors using the component form.

EXAMPLE 17

Given that $\mathbf{p} = (-4, 7)$ and $\mathbf{q} = (3, -5)$, calculate:

a $\mathbf{q} - \mathbf{p}$ **b** $3\mathbf{p} - 5\mathbf{q}$

Solution

a Write subtraction as an addition. $\mathbf{q} - \mathbf{p} = \mathbf{q} + (-\mathbf{p})$

Substitute for known values. $= (3, -5) + [-(-4, 7)]$

$= (3, -5) + (4, -7)$

Add the components. $= (3 + 4, -5 + (-7))$

$= (7, -12)$

ISBN 9780170412995

b Write the sum using scalar multiples. $3\mathbf{p} - 5\mathbf{q} = 3(-4, 7) - 5(3, -5)$

Evaluate. $= (-12, 21) - (15, -25)$

Write the subtraction as an addition. $= (-12, 21) + [-(15, -25)]$

Add the components. $= (-12 + (-15), 21 + 25)$

$= (-27, 46)$

You can use a graphics calculator to calculate the sums of scalar multiples of vectors.

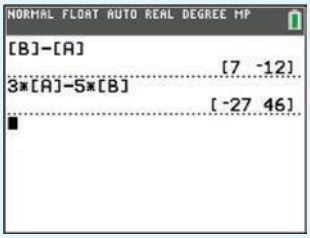
TI-Nspire CX Chapter 1

TI-84 Plus CE

Assign vector **p** as matrix A and vector **q** as matrix B as previously described in Example 8.

To find **q** – **p**, press Enter [2nd] [x^{-1}] (Matrix) and select B, then press [−] and enter [3] [×] [2nd] [x^{-1}] (Matrix) and select A. Press [enter] to display the result.

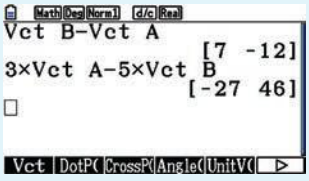
To find 3**p** – 5**q**, press [3] [×] [2nd] [x^{-1}] (Matrix) and select A, then press [−] and press [5] [×] [2nd] [x^{-1}] (Matrix) and select B. Press [enter] to display the result.



Casio fx-CG20AU

Enter the vector (–4, 7) as vector **A** and (3, –5) as vector **B**,as previously described in Example 8.

Press [OPTN], [F2] [MAT/VCT], [F6] [F6] [F1] [Vct], [ALPHA] [log] [B]. Press [−] and then press [F1] [Vct], [ALPHA] [X,θ,T] [A] and press [EXE] to display the result.



To find 3**A** – 5**B**, press [3] [×] [F1] [Vct], [ALPHA] [X,θ,T] [A] [−] [5] [×] [F1] [Vct], [ALPHA] [log] [B]. Press [EXE] to display the result.

ISBN 9780170412995

Exercise 1.07 Vector components

1 Write each vector **a**, **b**, **c**, …, **g** as a position vector. Using this information, calculate:

a $\mathbf{a} + \mathbf{b}$

b $\mathbf{c} + \mathbf{e}$

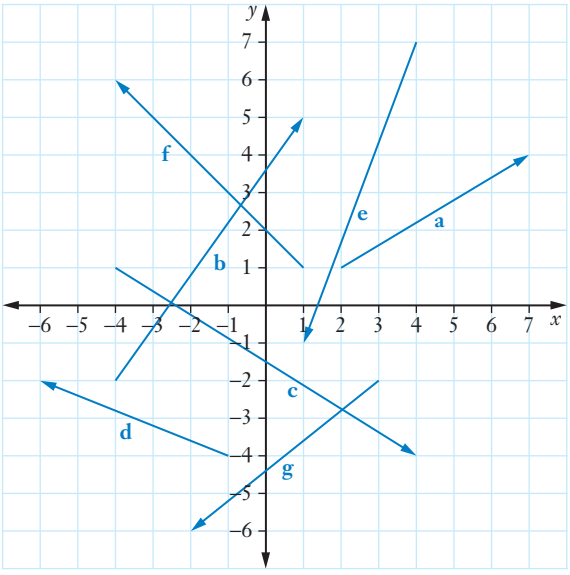
c $\mathbf{d} + \mathbf{b}$

d $\mathbf{f} + \mathbf{a}$

e $\mathbf{g} + \mathbf{f}$

f $\mathbf{b} + \mathbf{c}$

g $\mathbf{d} + \mathbf{g}$



2 Find the sum of each pair of vectors.

Example 16

a (3, 6) and (–5, 7)

b (–12, 7) and (–8, –6)

c (–11.5, –9.8) and (17.6, 13.5)

d (7.05, –4.32) and (6.11, –5.87)

e $\left(-4\frac{1}{2}, 6\frac{2}{3}\right)$ and $\left(3\frac{1}{4}, -5\frac{5}{6}\right)$

f $\left(2\frac{3}{5}, -1\frac{3}{4}\right)$ and $\left(-7\frac{7}{10}, 4\frac{1}{3}\right)$

3 Find the sum of each pair of vectors.

a $\begin{bmatrix} 9 \\ -7 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 12 \end{bmatrix}$

b $\begin{bmatrix} -4 \\ -8 \end{bmatrix}$ and $\begin{bmatrix} 11 \\ 7 \end{bmatrix}$

c $\begin{bmatrix} -8 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 13 \\ -6 \end{bmatrix}$

d $\begin{bmatrix} 3.9 \\ 5.7 \end{bmatrix}$ and $\begin{bmatrix} 7.4 \\ -2.6 \end{bmatrix}$

e $\begin{bmatrix} 7.09 \\ 5.46 \end{bmatrix}$ and $\begin{bmatrix} -4.71 \\ -3.28 \end{bmatrix}$

f $\begin{bmatrix} 2\frac{1}{3} \\ -4\frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} -3\frac{1}{6} \\ -2\frac{1}{4} \end{bmatrix}$

4 Find the sum of:

a $2\mathbf{i} - 10\mathbf{j}$ and $4\mathbf{i} + 11\mathbf{j}$

b $7\mathbf{i} + 9\mathbf{j}$ and $6\mathbf{i} - 4\mathbf{j}$

c $-8.4\mathbf{i} - 5.7\mathbf{j}$ and $6.2\mathbf{i} - 9.3\mathbf{j}$

d $-5.67\mathbf{i} + 4.92\mathbf{j}$ and $3.56\mathbf{i} - 7.84\mathbf{j}$

e $1.06\mathbf{i} - 9.22\mathbf{j}$ and $7.17\mathbf{i} - 10.09\mathbf{j}$

f $3\frac{1}{3}\mathbf{i} + 2\frac{3}{4}\mathbf{j}$ and $-5\frac{5}{9}\mathbf{i} - 6\frac{1}{3}\mathbf{j}$

5 Given $\mathbf{a} = (1, 4)$, $\mathbf{b} = (-7, 8)$, $\mathbf{c} = (2, 4)$, $\mathbf{d} = (5, -2)$ and $\mathbf{e} = (-6, 1)$, find:

Example 17

a $\mathbf{c} + \mathbf{e}$

b $\mathbf{b} + \mathbf{c} + \mathbf{d}$

c $5\mathbf{a}$

d $-2\mathbf{e}$

e $3\mathbf{a} + 2\mathbf{e}$

f $4\mathbf{b} - 2\mathbf{a}$

g $\mathbf{a} - \mathbf{c} - \mathbf{e}$

h $7\mathbf{c} - 7\mathbf{a} + \mathbf{d} + 2\mathbf{e}$

i $7\mathbf{a} + 5\mathbf{b} + \mathbf{e}$

j $2\mathbf{d} - 7\mathbf{c}$

ISBN 9780170412995

6 If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 4\mathbf{j}$ and $\mathbf{c} = -3\mathbf{i} + 2\mathbf{j}$, find each expression in $\mathbf{i}$, $\mathbf{j}$ form.

a $3\mathbf{a} - \mathbf{b}$ **b** $\mathbf{a} + 2\mathbf{b}$ **c** $3\mathbf{a} - \mathbf{c}$

d $2\mathbf{c} - 3\mathbf{b}$ **e** $\mathbf{a} + \mathbf{b} + \mathbf{c}$ **f** $2\mathbf{b} - \mathbf{c}$

g $3\mathbf{a} + 2\mathbf{b} + \mathbf{c}$ **h** $4\mathbf{c} + 2\mathbf{a}$ **i** $4\mathbf{c} - \mathbf{a}$

j $3\mathbf{c} - \mathbf{a} - \mathbf{b}$

7 Change each expression to component form, find the result, and change back to polar form.

a $(6, 20°) + (9, 55°)$ **b** $(25, 120°) + (16, 80°)$ **c** $(7, 30°) - (9, 100°)$

d $(105, 300°) - (95, 60°)$ **e** $(6, 70°) - (6, 10°)$

Problem solving

8 $\mathbf{OP} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{OQ} = 5\mathbf{i} + 3\mathbf{j}$.

a Represent **OP** and **OQ** on a diagram.

b Find **PQ** in terms of **i** and **j**.

c If M is the midpoint of **PQ**, find the vector **OM** in terms of **i** and **j**.

9 If $\mathbf{p} = (x_1, y_1)$ and $\mathbf{q} = (x_2, y_2)$ and k is a scalar, show that $k(\mathbf{p} + \mathbf{q}) = k\mathbf{p} + k\mathbf{q}$.

1.08 Resolution of vectors

While the polar form is a better description of a vector, adding vectors in component form is far easier than adding vectors in polar form.

The process of changing vectors to component form is called the **resolution** of vectors. It is possible to **resolve** a vector into components which are horizontal and vertical, east and north, or some other combination of directions. The unit vectors **i** and **j** are often used.

EXAMPLE 18

Resolve each vector into **i** and **j** components.

a $\mathbf{m} = (7, -9)$ **b** $\mathbf{p} = (5, 68°)$

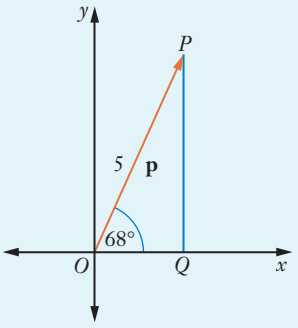
Solution

a Write **m** in terms of **i** and **j**.

$$\begin{aligned}\mathbf{m} &= (7, -9)\\ &= 7\mathbf{i} + (-9\mathbf{j})\\ &= 7\mathbf{i} - 9\mathbf{j}\end{aligned}$$

ISBN 9780170412995

b Draw a sketch.



Calculate the horizontal component. $OQ = 5 \cos(68°)$

$= 1.8730\ldots$

Calculate the vertical component. $PQ = 5 \sin(68°)$

$= 4.6359\ldots$

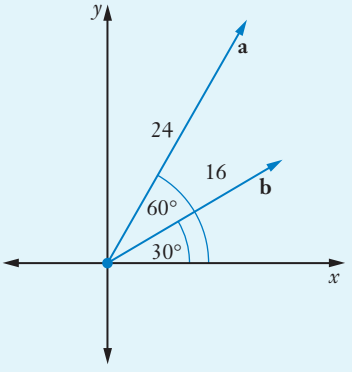
Write **p** in terms of **i** and **j**. $\mathbf{p} = 1.8730\ldots\,\mathbf{i} + 4.6359\ldots\,\mathbf{j}$

Round off and write the result. $\mathbf{p} \approx 1.87\mathbf{i} + 4.64\mathbf{j}$

You can calculate the resultant of two vectors by resolving the vectors and then adding their components.

EXAMPLE 19

Calculate the resultant, **r**, of vectors **a** = (24, 60°), and **b** = (16, 30°), as shown in the diagram.



Solution

To add the vectors, each can be resolved into its x and y-components using $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

ISBN 9780170412995

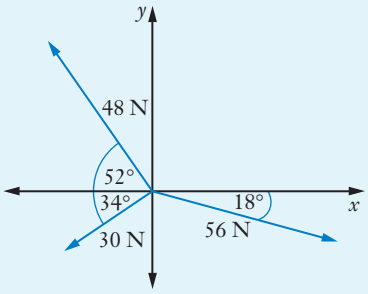
Resolve **a**.	$\mathbf{a} = (24\cos(60°), 24\sin(60°))$
Evaluate.	$= \left(24 \times \frac{1}{2}, 24 \times \frac{\sqrt{3}}{2}\right)$
	$= (12, 12\sqrt{3})$
Resolve **b**.	$\mathbf{b} = \left(16 \times \frac{\sqrt{3}}{2}, 16 \times \frac{1}{2}\right)$
Evaluate.	$= (8\sqrt{3}, 8)$
Find the resultant, **r**.	$\mathbf{r} = \mathbf{a} + \mathbf{b}$
Substitute.	$= (12, 12\sqrt{3}) + (8\sqrt{3}, 8)$
Add the x- and y-components separately.	$= (12 + 8\sqrt{3}, 12\sqrt{3} + 8)$
Calculate the magnitude of **r**.	$\lvert\mathbf{r}\rvert = \sqrt{x^2 + y^2}$
Substitute known values.	$= \sqrt{(12 + 8\sqrt{3})^2 + (12\sqrt{3} + 8)^2}$
Evaluate.	$\lvert\mathbf{r}\rvert = 38.6924\ldots$
Let **r** make an angle of θ with the x-axis. Use the tan ratio to find the direction of **r**.	$\tan(\theta) = \frac{12\sqrt{3} + 8}{12 + 8\sqrt{3}} = 1.1132\ldots$
Use $\tan^{-1}$ to find θ.	$\theta = 48.0675\ldots°$
Round off and state the result.	The resultant, $\mathbf{r} \approx (38.7, 48.1°)$.

You can use the same procedure to find the resultant of more than two vectors, as shown next page.

ISBN 9780170412995

EXAMPLE 20

Find the magnitude and direction of the resultant of the three forces shown in the diagram on the right.



Solution

Write each vector in polar form.

Let $\mathbf{a} = (48, [180 - 52]°) = (48, 128°)$

Let $\mathbf{b} = (30, [180 + 34]°) = (30, 214°)$

Let $\mathbf{c} = (56, [360 - 18]°) = (56, 342°)$

Express each force in terms of components.

$\mathbf{a} = (48 \cos(128°), 48 \sin(128°))$

$\mathbf{b} = (30 \cos(214°), 30 \sin(214°))$

$\mathbf{c} = (56 \cos(342°), 56 \sin(342°))$

Find the sum of the horizontal components.

Sum of horizontal components

$= 48 \cos(128°) + 30 \cos(214°) + 56 \cos(342°)$

$= -1.1637\ldots$

Find the sum of the vertical components.

Sum of vertical components

$= 48 \sin(128°) + 30 \sin(214°) + 56 \sin(342°)$

$= 3.7437\ldots$

Let the resultant be $\mathbf{r}$.

$\mathbf{r} = (-1.1637\ldots, 3.7437\ldots)$

Calculate the magnitude of $\mathbf{r}$.

$|\mathbf{r}| = \sqrt{x^2 + y^2}$

Substitute known values.

$= \sqrt{(-1.1637\ldots)^2 + (3.7437\ldots)^2}$

$= 3.9204\ldots$

Let $\mathbf{r}$ make an angle of θ with the x-axis. Use the tan ratio to find the direction of $\mathbf{r}$.

$$\tan(\theta) = \frac{3.7437\ldots}{-1.1637\ldots} = -3.2170\ldots$$

ISBN 9780170412995

Use $\tan^{-1}$ to find θ.	$\theta = -72.7327\ldots°$
Round off and state the direction as a positive value.	The resultant, $r \approx (3.92, 287.3°)$.

Exercise 1.08 Resolution of vectors

Example 18

1 Resolve each vector into **i** and **j** components.

a $(5, -3)$ **b** $(-6, 4)$ **c** $(-4, -7)$ **d** $(6, 5)$

e $(-3.2, -9.4)$ **f** $(2\sqrt{3}, \sqrt{5})$ **g** $\left(\frac{2}{3}, \frac{-4}{9}\right)$ **h** $(-5\sqrt{7}, 6\sqrt{2})$

2 Express each vector in terms of **i** and **j** components.

a $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ **b** $\begin{bmatrix} -2 \\ -8 \end{bmatrix}$ **c** $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ **d** $\begin{bmatrix} 7.59 \\ 3.68 \end{bmatrix}$

e $\begin{bmatrix} 0.07 \\ 0.19 \end{bmatrix}$ **f** $\begin{bmatrix} -3\sqrt{5} \\ 4\sqrt{7} \end{bmatrix}$ **g** $\begin{bmatrix} \frac{6}{11} \\ \frac{-5}{4} \end{bmatrix}$ **h** $\begin{bmatrix} -9\sqrt{11} \\ -7\sqrt{15} \end{bmatrix}$

3 Write each vector using **i** and **j** components.

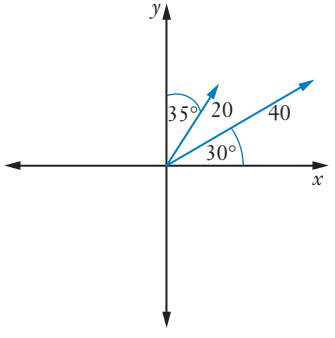
a $(7, -74°)$ **b** $(9, 125°)$ **c** $(11, 330°)$ **d** $(16, 240°)$

e $(4.8, -119°)$ **f** $(13, 53°)$ **g** $(8.7, -27°)$ **h** $(10.6, 286°)$

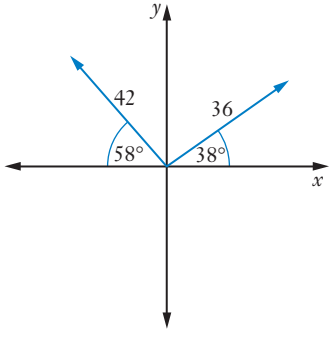
Example 19

4 Resolve each pair of vectors into components and then calculate the resultant vector in component form.

a

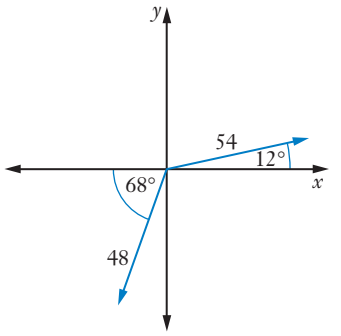


b



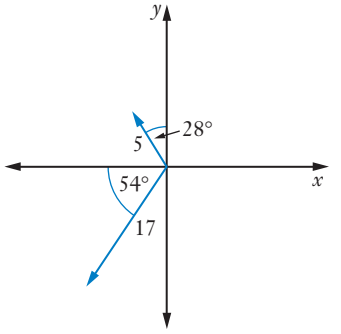
ISBN 9780170412995

c

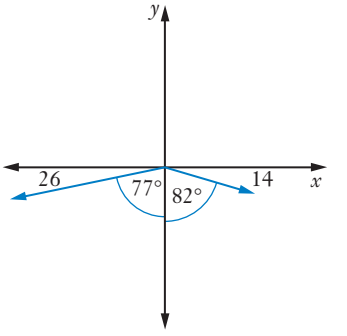


5 Calculate the magnitude and direction of the resultant of each pair of vectors.

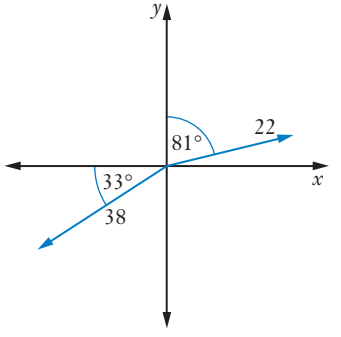
a



b



c



6 Calculate the magnitude and direction of the resultant of each pair of vectors.

a (15, 20°) and (50, 45°)

b (37, 100°) and (20, 135°)

c (60, –15°) and (75, 118°)

d (84, 230°) and (27, 320°)

e (25, –130°) and (60, 162°)

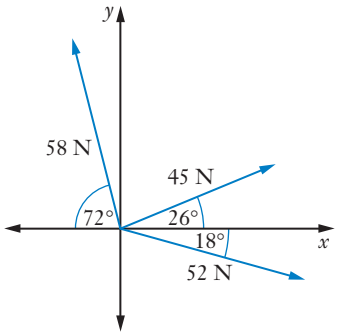
f (44, –126°) and (32, 134°)

ISBN 9780170412995

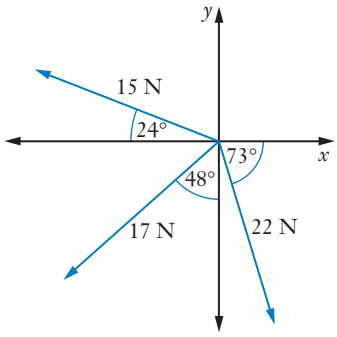
Example 20

7 For each set of forces, calculate the magnitude and direction of the resultant vector.

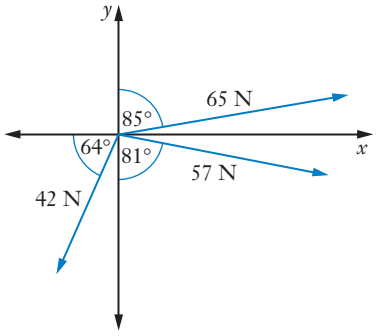
a



b



c

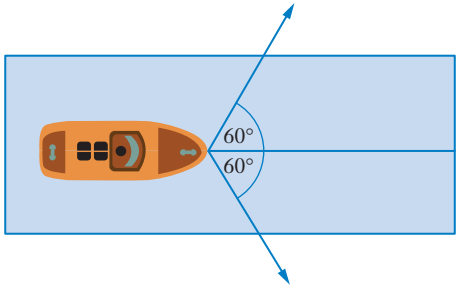


Problem solving

8 A small boat is being towed at constant speed along a canal by two people walking along the banks on opposite sides.

The people pull with equal forces of 2000 N at equal angles of 60° on ropes attached to the boat.

Calculate the resultant force exerted on the boat.



9 An object is under the influence of two separate forces. One of the forces is 52 N at N54°E and the second is 36 N at N63°W. Calculate the magnitude and direction of the minimum force required to prevent the object from moving.

10 A boat is floating on the surface of a body of water. The boat is subjected to two separate forces causing it to move. The current exerts a force of 270 N acting in the direction N6°E and the wind exerts a force of 320 N acting in the direction S42°E. In what direction does the boat move?

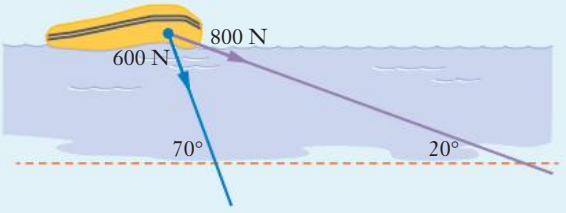
ISBN 9780170412995

1.09 Applications of vectors

Situations involving forces acting on an object are a common example of where vectors can be applied.

EXAMPLE 21

Two ropes are tied to the prow of a rubber boat and are being pulled towards the beach as shown here. The first rope is being pulled at an angle of 20° to the shoreline, with a force of 800 newtons (N).



The second rope is being pulled in the same general direction, with a force of 600 N at an angle of 70° to the shoreline.

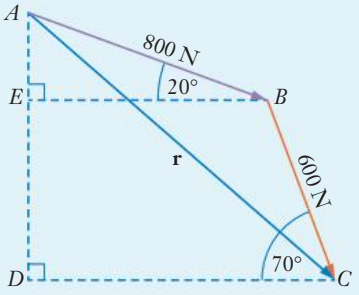
What is the total force acting on the boat and in what direction is it acting?

Solution

Draw the forces head to tail to make the triangle of forces ABC and complete the triangle with the resultant.

Since $EBCD$ is a quadrilateral,

$\angle EBC = 110°$, so $\angle ABC = 130°$.



Use the cosine rule to find r.

$$r^2 = AB^2 + BC^2 - 2 \times AB \times BC \cos(\angle ABC)$$
$$= 800^2 + 600^2 - 2 \times 800 \times 600 \cos(130°)$$

Keep the exact answer in your calculator.

$$= 1\ 617\ 0076.1\ldots$$

$$r \approx 1271.64\ldots \text{ N}$$

Use the sine rule to find $\angle BCA$.

$$\frac{AB}{\sin(\angle BCA)} = \frac{AC}{\sin(\angle ABC)}$$

Substitute for known values.

$$\frac{800}{\sin(\angle BCA)} = \frac{1271.64\ldots}{\sin(130°)}$$

Rearrange to isolate $\angle BCA$.

$$\sin(\angle BCA) = \frac{800 \times \sin(130°)}{1271.64\ldots}$$

ISBN 9780170412995

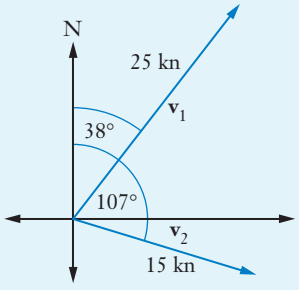
Evaluate.	$= 0.4819\ldots$
Use $\sin^{-1}$ to calculate $\angle BCA$.	$\angle BCA = \sin^{-1}(0.4819\ldots)$ $= 28.8111\ldots°$
Calculate the angle of the resultant with the shoreline.	Resultant angle $\approx 70° - 28.8111\ldots°$ $= 41.1888\ldots°$
Round off and state the answer in appropriate form.	The total force acting on the boat is about 1272 N at an angle of 41.2° to the shoreline.

As you have seen previously, if a quantity x changes from x_1 to x_2 the change is calculated using the formula $\Delta x = x_2 - x_1$. This also applies to vector quantities, but we need to use vector addition. So, for velocity, $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_2 + (-\mathbf{v}_1)$.

EXAMPLE 22

The velocity of a boat changes from 25 knots at N38°E to 15 knots at a bearing of 107° as shown in the diagram on the right.

What is the change in velocity?



Solution

Redraw the sketch.

You want to know $\Delta\mathbf{v}$.

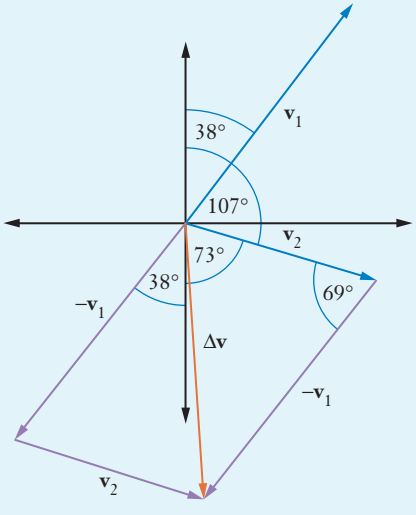
You know that $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$.

Draw $-\mathbf{v}_1$ on the diagram.

Rearrange $-\mathbf{v}_1$ and $\mathbf{v}_2$ on the diagram so that the change of velocity ($\Delta\mathbf{v}$) can be found.

Find the angle between the vectors.

The obtuse angle in the parallelogram is 111° (i.e. 38° + 73°), so the acute angle is 69°.



ISBN 9780170412995

Write the change of velocity in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$$
$$= \mathbf{v}_2 + -\mathbf{v}_1$$

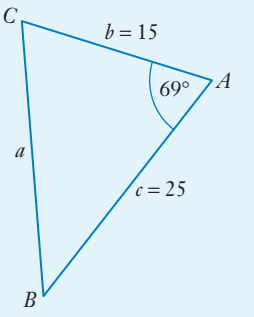
This problem could be solved by using either the component or polar form of the vectors.

The polar form will be used.

Draw a sketch to show the information of interest in a simplified form.

The resultant **a** represents the change of velocity $[\mathbf{v}_2 + (-\mathbf{v}_1)]$.

Now you can calculate the magnitude and direction of the resultant **a**.



Use the cosine rule to find $|\mathbf{a}|$ (a).

$$a^2 = b^2 + c^2 - 2bc\cos(\angle A)$$

Substitute known values.

$$= 15^2 + 25^2 - 2 \times 15 \times 25 \times \cos(69°)$$
$$= 581.2240\ldots$$

Evaluate on your calculator.

$$a = 24.1085\ldots$$

Use the sine rule to find $\angle C$.

$$\frac{c}{\sin(\angle C)} = \frac{a}{\sin(\angle A)}$$

Substitute known values.

$$\frac{25}{\sin(\angle C)} = \frac{24.1085...}{\sin(69°)}$$

Rearrange to isolate sin $(\angle C)$.

$$\sin(\angle C) = \frac{25 \times \sin(69°)}{24.1085...}$$

Evaluate.

$$= 0.9680\ldots$$

Use $\sin^{-1}$ to find $\angle C$.

$$\angle C = \sin^{-1}(0.9680\ldots)$$
$$= 75.4889\ldots°$$

Calculate the bearing of the resultant.

$$\text{Bearing} = 107° + 75.4889\ldots°$$

Evaluate.

$$= 182.4889\ldots°$$

If you refer back to the original diagram, you can see that the sketch was slightly incorrect because it shows the bearing of Δv as less than 180° (107° + 73°). So you can see that as long as the sketch is roughly correct to start with, a slight error of this nature doesn't matter.

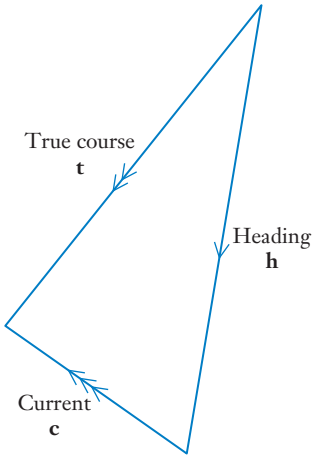
Round off and state the result.

The change in velocity is about 24.1 knots at a bearing of 182.5°.

ISBN 9780170412995

When steering a boat, the effect of the current must be taken into account. Similarly, the effect of the wind must be accounted for when determining the direction in which a plane must fly to reach its destination.

In marine navigation, the direction in which a boat is steered is called the **heading**. The **speed through the water** is the apparent speed of the boat relative to the water. The **speed and course made good** is the true speed and course of the boat when the effects of the current are taken into account.



Because a boat is carried with the current, the speed and course made good is the vector sum of the heading and speed through the water and the current. This is illustrated in the vector triangle on the right. It is standard practice to use one arrowhead for the vector showing the course steered and speed through the water (heading), two arrowheads for the vector showing the course and speed made good (true course), and three arrowheads for the current vector.

Using **h** for the heading, **c** for the current and **t** for the course made good, you get $\mathbf{t} = \mathbf{h} + \mathbf{c}$.

EXAMPLE 23

A yacht making 7 knots through the water is headed on a bearing of 285°. There is a current of 3 knots in a direction of 215°. What is the speed and course made good?

Solution

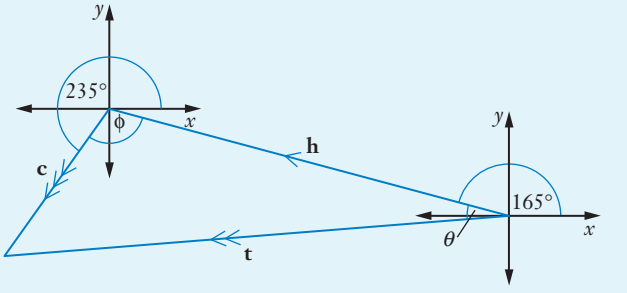
Start by making a rough sketch.

Take east as the **i** direction and north as the **j** direction.

The heading **h** is on a bearing of 285°.

This means **h** is at an angle of 165° to the **i** direction.

The current, **c**, is at an angle of 235° to the **i** direction.



First calculate the magnitude of **t**.

Write **h** in terms of **i** and **j**. $\mathbf{h} = 7\cos(165°)\mathbf{i} + 7\sin(165°)\mathbf{j}$

Evaluate. $= -6.761\ldots\mathbf{i} + 1.811\ldots\mathbf{j}$

ISBN 9780170412995

Write **c** in terms of **i** and **j**.	$\mathbf{c} = 3\cos(235°)\mathbf{i} + 3\sin(235°)\mathbf{j}$
Evaluate.	$= -1.720\ldots\mathbf{i} - 2.457\ldots\mathbf{j}$
Find **t** using vector addition.	$\mathbf{t} = \mathbf{h} + \mathbf{c}$
Substitute for **h** and **c**.	$= -6.761\ldots\mathbf{i} + 1.811\ldots\mathbf{j} + (-1.720\ldots)\mathbf{i} - 2.457\ldots\mathbf{j}$
Evaluate.	$= -8.482\ldots\mathbf{i} - 0.645\ldots\mathbf{j}$
Find the magnitude of **t**.	$\lvert\mathbf{t}\rvert = \sqrt{(-8482\ldots)^2 + (-0.645\ldots)^2}$
Evaluate.	$= 8.5067\ldots$
Now calculate the direction of **t**.	
Use the diagram to find ϕ.	$\phi = 360° - 235° - (180° - 165°)$ $= 110°$
Use the sine rule to find θ.	$\frac{t}{\sin(\phi)} = \frac{c}{\sin(\theta)}$
Substitute values.	$\frac{8.5067\ldots}{\sin(110°)} = \frac{3}{\sin(\theta)}$
Rearrange.	$\sin(\theta) = \frac{3 \times \sin(110°)}{8.5067\ldots}$
Evaluate.	$= 0.3313\ldots$
Use $\sin^{-1}$ to calculate θ.	$\theta = \sin^{-1}(0.3313\ldots) = 19.3533\ldots°$
Calculate the bearing of **t**.	Bearing of $\mathbf{t} = 360° - (165° - 90°) - 19.3533\ldots°$ $= 265.646\ldots°$
State the result.	The speed and course made good is about 8.5 knots on a bearing of 265.6°.

In air navigation, the equivalent of speed through the water is **air speed**, shown with one arrowhead. The course made good is referred to as the **true course**, and the speed made good is called the **ground speed**, shown with two arrowheads. The air current is the wind and this is shown with 3 arrowheads. In air navigation, you normally know the wind direction and speed from meteorological data, and wish to choose your heading in order to fly in a particular direction. In other words, **t** and **c** are known, and you usually wish to calculate **h**.

ISBN 9780170412995

Exercise 1.09 Applications of vectors

Example 21

1 Two winches are being used to pull out a bogged car. One winch is on the driver's side at an angle of 32° to the forward direction and the larger winch is on the passenger's side at an angle of 39° to the forward direction. The first is exerting a force of 6000 N and the other is exerting a force of 9000 N. What is the total force on the car, and in what direction does it act?

Example 22

2 A car travelling at 20 m/s south changes velocity to 18 m/s east. Find the change in velocity.

Example 23

3 Find the speed and course made good for a yacht sailing at 7 knots at 135° with a current of 2 knots at 060°.

Problem solving

4 Three horizontal forces of 300 N, 400 N and 500 N are being exerted on an object. The 300 N and 500 N forces are exerted either side of the 400 N force at angles of 30° and 50° respectively. Find the total force acting and the direction of that force.

5 The winning team of a tug-of-war competition offers to take on two opponents. The teams working together have their ropes at an angle of 70° to each other and are exerting about the same force. The winning team is exerting a force of 1200 N but is easily overcome by the other two teams. The combined force of the other teams is actually 400 N more than the winning team. What forces were exerted by the two other teams?

6 An aircraft travelling at 140 knots at a bearing of 197° changes direction to a bearing of 116° at the same speed to approach the runway from the seaward side. Find the change in velocity.

7 What is the change of velocity when a cyclist in a road race changes velocity from 15 m/s NW to 12 m/s N?

8 Find the change of velocity when an object travelling at v m/s changes direction by $\theta°$ without changing speed.

9 Find the speed and course made good for a boat sailing at 12 knots at 200° with a current of 4 knots at 340°.

10 The owner of a cruiser wants to sail on a true course of 120°. The current is very strong, being 5 knots towards the NW. The cruiser can make a speed of 15 knots through the water. At what heading should the owner steer the cruiser, and what will be the true speed?

11 A pilot wants to fly at 130 knots on a true course of 320°. The wind is a steady westerly of 15 knots. Find the heading he should take and the speed over the ground of the aircraft.

ISBN 9780170412995

12 Wind can be much stronger at high altitudes. A passenger airliner with a cruising speed of 560 km/h needs to travel 1500 km south-west at cruising height before beginning the descent for landing. At this height, the wind is an 80 km/h northerly. The plane reaches cruising height at 10:05 a.m. When will the plane commence the descent for landing?

13 A whale is moving up the east coast of Australia in a direction that is essentially at a bearing of 25°. The whale moves at a leisurely pace of about 8 knots through the water and the current up the east coast is about 2 knots northwards. What is the speed that would be observed by a whale spotter at Point Lookout?

iStock/miblue5

ISBN 9780170412995

1. CHAPTER SUMMARY

Vectors

- A **scalar** is a **magnitude** expressed by a single number, but a **vector** has both a magnitude (**norm**) and a **direction**.
- A vector is usually specified as a lower-case bold letter, such as **a**, and may be shown by a **directed line segment** whose length represents the magnitude $|\mathbf{a}| = a$. The **initial point** of a vector is called the **head** of the vector and the **terminal point** is called the **tail**.
- A **displacement** vector showing a change of position from A to B may be shown as **AB**. The **position vector** of a point P is the vector $\mathbf{p} = \mathbf{OP}$.
- The **polar form** of a two-dimensional vector $\mathbf{v} = (r, \theta)$ shows the magnitude r and the direction θ of the vector. θ is the angle in the positive direction (anticlockwise) from the x-axis. The **component form** (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$ shows the vector as components in the x- and y-directions.
- The polar and component forms of the vector **v** are related by:

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$

$$\tan(\theta) = \frac{y}{x} \text{ and } r^2 = x^2 + y^2 \text{ so } |\mathbf{v}| = \sqrt{x^2 + y^2}$$

- A **unit vector** has magnitude 1. The unit vector in the same direction as **p** is shown as $\hat{\mathbf{p}}$. $\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}$, where $|\mathbf{p}|$ is the norm of **p**.
- The unit vectors in the x- and y-directions are $\mathbf{i} = (1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Any vector can be shown as a **linear combination** of **i** and **j**, so $\mathbf{v} = r\mathbf{i} + s\mathbf{j}$ for some $r, s \in \boldsymbol{R}$.
- The **sum** (**resultant**) of two vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ is:

$$\mathbf{a} + \mathbf{b} = (x_1 + x_2, y_1 + y_2)$$

- The **scalar multiple** of a vector $\mathbf{a} = (x_1, y_1)$ and a scalar c is given by $c\mathbf{a} = (cx_1, cy_1)$.
- **Addition of vectors** is done using the triangle rule or the parallelogram rule. The triangle rule is also known as the **head-to-tail rule**.
- Subtraction of vectors is done by adding a negative vector.

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

- A vector may be **resolved** into **components** in any direction, although horizontal and vertical components are the most common.
- The change in a vector from $\mathbf{v}_1$ to $\mathbf{v}_2$ is calculated using $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_2 + (-\mathbf{v}_1)$.

ISBN 9780170412995

- The apparent velocity of a boat through water is given by its magnitude, called the **speed through the water**, and its direction, normally called its **heading**. The **speed and course made good** (**true course**) is the boat's velocity relative to the land (Earth) and is the vector sum of the **current** and the apparent velocity of the boat. That is,

$$\mathbf{t} = \mathbf{h} + \mathbf{c}$$

  where **t** = true course and speed, **h** = heading and speed through the water, and **c** is the current.

- For air navigation, speed through the air is called the **air speed** and the speed relative to the Earth is called the **ground speed**. The **true course** is the direction of flight relative to the Earth. The wind blows the aircraft so that the true course and ground speed is the vector sum of the air velocity and the wind. That is,

$$\mathbf{t} = \mathbf{h} + \mathbf{c}$$

  where **t** = true course and ground speed, **h** = heading and air speed, and **c** is the wind.

ISBN 9780170412995

1. CHAPTER REVIEW

Vectors

Examples 1,2

1 Use vectors to represent each displacement on the same set of axes.

a 7 m in the direction NW **b** 8 m in the direction S30°E

Example 3

2 An object moves from a starting point and travels 6 m west. It then changes direction and travels for 8 m in the direction S40°W.

a Use a diagram to show the displacements as vectors.

b Find the bearing of the final position of the object from its starting point.

Example 4

3 Vector **p** has a magnitude of 5 and is in the direction 120° to the x-axis.

a Write the polar form of the vector.

b Make a sketch of possible representations of the vector.

Example 5

4 Find the displacement vector for each movement.

a $D(3, -4)$ to $E(1, 2)$ **b** $T(-3, -5)$ to $S(5, 0)$

c $G(1, 1)$ to $J(-3, 5)$ **d** $R(2, 4)$ to $K(0, 8)$

e $Y(-6, 3)$ to $Z(1, -9)$

Example 5

5 Find the norm of each vector.

a $(24, 10)$ **b** $(-3, -4)$ **c** $\begin{bmatrix} -8 \\ 15 \end{bmatrix}$

d $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$ **e** $(10, 12)$

Example 6

6 Convert each vector to component form.

a $(3, 30°)$ **b** $(8, 150°)$ **c** $(4, 300°)$

d $(7, -45°)$ **e** $(9, 230°)$

Example 6

7 Convert each vector to polar form.

a $(5, 8)$ **b** $(-3, 4)$ **c** $(-6, -10)$

d $(9, -3)$ **e** $(-4, 0)$

Example 6

8 Convert each vector to polar form.

a $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} -2 \\ -6 \end{bmatrix}$ **c** $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

d $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ **e** $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$

ISBN 9780170412995

9 The vector **p** is shown on the right. Draw the vector for:

a $3\mathbf{p}$ **b** $-2\mathbf{p}$ **c** $\frac{1}{3}\mathbf{p}$

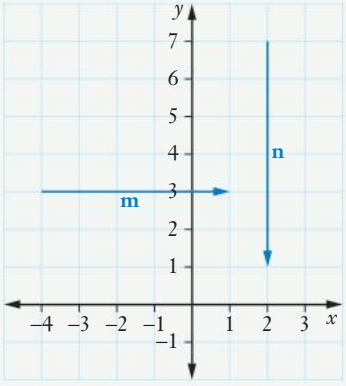
Example 7

10 If $\mathbf{c} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ and $\mathbf{u} = (8, 173°)$, calculate:

a $4\mathbf{c}$ **b** $-3\mathbf{c}$ **c** $-5\mathbf{u}$ **d** $6\mathbf{u}$

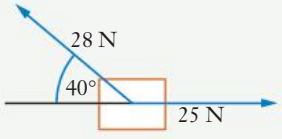
Example 8

11 Find the resultant of **m** and **n** in polar form.



Example 9

12 Two forces, one of 28 N and one of 25 N, act on an object. The 28 N force is 40° above the x-axis, as shown in the diagram. The 25 N force runs along the x-axis. Calculate the magnitude and direction of the resultant force acting on the object.



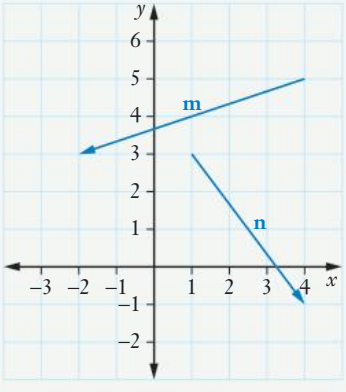
Example 10

13 Given $\mathbf{p} = (5, 85°)$, $\mathbf{q} = (7, 146°)$ and $\mathbf{r} = (6, 212°)$, find each expression without changing to component form.

a $\mathbf{p} + \mathbf{q}$ **b** $\mathbf{q} + \mathbf{r}$ **c** $\mathbf{r} + \mathbf{p}$ **d** $\mathbf{p} - \mathbf{r}$ **e** $\mathbf{r} - \mathbf{q}$

Example 11

14 Calculate $\mathbf{m} - \mathbf{n}$ in polar form.



Example 12

ISBN 9780170412995

Example 13

15 Given $\mathbf{f} = (5, 150°)$, $\mathbf{g} = (8, 60°)$ and $\mathbf{h} = (9, 225°)$, find each expression in polar form.

a $\mathbf{f} - \mathbf{g}$ **b** $\mathbf{g} - \mathbf{h}$

Example 14

16 Calculate the unit vector for each vector.

a $\mathbf{r} = (15, 20)$ **b** $\mathbf{n} = (-8, 12)$

Example 15

17 Write each vector as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

a $(1, -3)$ **b** $(4, 5)$ **c** $(-2, 5)$ **d** $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

e $\begin{bmatrix} -2 \\ -5 \end{bmatrix}$ **f** $(6, 56°)$ **g** $(8, 200°)$ **h** $(3, 30°)$

i $(10, 142°)$ **j** $(5, 120°)$

Example 16

18 Find the sum of each pair of vectors.

a $(-6, 11) + (9, -8)$ **b** $\begin{bmatrix} 6 \\ -7 \end{bmatrix} + \begin{bmatrix} 11 \\ -3 \end{bmatrix}$ **c** $10\mathbf{i} - 5\mathbf{j}$ and $-4\mathbf{i} + 11\mathbf{j}$

Example 17

19 Given $\mathbf{a} = (3, 5)$, $\mathbf{b} = (-2, -3)$ and $\mathbf{c} = (-1, 4)$, find:

a $\mathbf{a} + \mathbf{c}$ **b** $\mathbf{b} - \mathbf{a}$ **c** $3\mathbf{b} + 2\mathbf{c}$

d $5\mathbf{a} - 3\mathbf{b}$ **e** $\mathbf{a} + \mathbf{b} + \mathbf{c}$

Example 17

20 Given $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, find:

a $\mathbf{x} - \mathbf{y}$ **b** $2\mathbf{x} + 3\mathbf{y}$ **c** $\mathbf{y} + 3\mathbf{z}$

d $\mathbf{z} - 2\mathbf{x}$ **e** $\mathbf{x} + \mathbf{z}$

Example 18

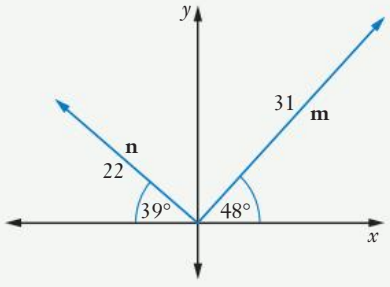
21 Resolve the each vector into $\mathbf{i}$ and $\mathbf{j}$ components.

a $(-4, 3)$ **b** $\left(\frac{1}{8}, \frac{-5}{7}\right)$ **c** $(3\sqrt{5}, \sqrt{6})$

d $(17, 38°)$ **e** $(11, 251°)$

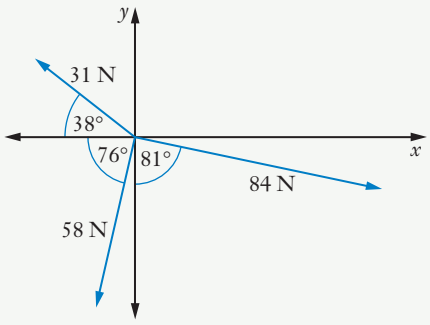
Example 19

22 Calculate the resultant of $\mathbf{m} = (24, 60°)$ and $\mathbf{n} = (16, 30°)$, giving your answer in both component and polar form.



ISBN 9780170412995

23 Calculate the magnitude and direction of the resultant of the 3 force vectors shown in the diagram.



Example 20

24 Three forces are acting on an object in a vertical plane. The first force is of 28 N and it acts vertically upwards. The second is a 45 N horizontal force and the third is a force of 36 N at an angle of 60° upward from the horizontal. Find the total force on the object.

Example 21

25 A billiard ball approaching the cushion at an angle of 60° to the cushion has been given spin that makes it bounce off at an angle of 45°. If the ball also changes speed from 0.3 m/s to 0.2 m/s, find its change of velocity.

Example 22

26 Find the speed and course made good for a yacht on a heading of 225° with a speed through the water of 12 knots, if a current is running north-west at 3 knots.

Example 23

Problem solving

27 A police officer travels 3 city blocks south and then 2 blocks east. If each block is 300 m long, find the officer's total displacement as a vector in polar form.

28 A pilot is planning to fly north-east at an airspeed of 120 knots. There is a northerly wind at 20 knots. Find the heading needed and the speed over the ground.

Practice quiz

ISBN 9780170412995

2.

METHODS OF PROOF

In Science, you use evidence to construct a theory that explains observations. The more evidence you have, the more you believe the theory. In Science though, it is always possible that some new observations will make you change the theory.

However, Mathematics is based on deductive proof: you define what is meant by something and then use logic to prove things about it. You make assumptions called axioms that apply to a particular set of proofs. In Mathematics, something that has been proven will always be true. However, you do need to take account of the axioms.

2.01 Mathematical proof
2.02 Counterexamples
2.03 Converse
2.04 Equivalence and negation
2.05 Contrapositive
2.06 Contradiction
2.07 Euclidean geometry
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

The nature of proof

- use implication, converse, equivalence, negation, contrapositive
- use proof by contradiction
- use the symbols for implication ($\Rightarrow$), equivalence ($\Leftrightarrow$), and equality ($=$)
- use the quantifiers for all and there exists
- use examples and counterexamples

Rational and irrational numbers

- prove simple results involving numbers
- express rational numbers as terminating or eventually recurring decimals and vice versa
- prove irrationality by contradiction

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

iStock_Trifonov_Evgeniy

TERMINOLOGY

acute-angled triangle
alternate angles
angle-angle (AA)
angle-angle-side (AAS)
angles at a point
co-interior angles
congruent
contradiction
contrapositive
converse
corollary
corresponding angles
counterexample
direct proof
equilateral triangle
equivalent
Euclid's postulates
if and only if (iff)
integer
irrational number
isosceles triangle
kite
lemma
member of
necessary and sufficient
negation
obtuse angled triangle
parallel
parallelogram
proof
QED
rational number
real numbers
rhombus
right angle-hypotenuse-side (RHS)
right-angled triangle
RTP
side-angle-side (SAS)
side-side-side (SSS)
similar
supplementary
theorem
trapezium
vertically opposite

2.01 Mathematical proof

When something is true in Mathematics, it means it is true in all cases. To prove something is true, you need to do more than give examples. One example just shows one case.

Consider the statement: If $x > y$, then $x + c > y + c$.

This certainly looks true, and if you try $x = 5$, $y = 3$ and $c = 2$, 0 or -3, it is correct.

But these examples do not *prove* it is true because there are other cases.

To prove it is true, you have to show that it is true for every possible case for which the statement applies.

In Mathematics, you follow the steps below.

Steps of a mathematical proof

- A precise statement of what is to be proven, usually under a heading of **Theorem**, **Lemma**, **Corollary** or **Required to Prove** (**RTP**)
- The logical steps of the **Proof** come next, usually with reasons for each step
- A final statement of what has been proven, followed by the initials **QED**

A theorem is an important result, a lemma is a small theorem used to prove a main theorem and a corollary is a simple consequence of a theorem. The letters QED stand for *quod erat demonstrandum*. This is Latin for 'that which was required to be proven' and it shows the proof is finished.

A **direct proof** is a proof that starts with the assumption and goes in easy steps to the result. It is the simplest method.

ISBN 9780170412995

EXAMPLE 1

Prove that if $x > y$, then $x + c > y + c$.

Solution

Write what has to be proven.	**RTP** If $x > y$, then $x + c > y + c$
Write the heading and start the proof with the assumption.	**Proof** Let $x > y$.
Do the steps.	Then $x - y$ is positive (otherwise $x \leq y$) Then $x - y + c - c$ is positive (adding 0) Then $x - c - y + c$ is positive (rearrangement).
Do the last step and write QED.	Thus $x - c > y - c$ (otherwise the above is not positive). QED

In Mathematics, an *elegant* proof has a small number of steps using a small amount of mathematical knowledge. Example 1 uses only addition of 0 and the meaning of greater than.

It is very clumsy to keep writing 'then', 'thus', 'so', etc. in a proof. You can use the symbol $\Rightarrow$ instead. It means 'it follows that' or 'if …. then'.

EXAMPLE 2

Prove that if $x > y$ and $c < 0$, then $cx < cy$.

Solution

Write what has to be proven.	**RTP** If $x > y$ and $c < 0$, then $cx < cy$.
Put the heading for the proof and write the starting assumption.	**Proof** Let $x > y$
Write logical steps with reasons.	$\Rightarrow x - y$ is positive (meaning of >) $\Rightarrow c(x - y)$ is negative (product of –ve and +ve) $\Rightarrow cx - cy$ is negative (expansion of brackets)
Put QED after the last step.	$\Rightarrow cx < cy$ (meaning of <) QED

ISBN 9780170412995

Not every attempt at a proof is correct. Common errors include incorrect assumptions (false premise), reversing a statement that is not reversible or applying a general truth to a case where it doesn't apply.

EXAMPLE 3

Explain why each of the following 'proofs' are wrong.

a People from England speak English

Danny speaks English

Thus Danny is from England

b Children learn to read in primary school

Kayla is an adult

Thus Kayla can read

c All boys like to wrestle

Jimmy is a boy

Thus Jimmy likes to wrestle

iStock/MarioGuti

Solution

a	What is the assumption?	From England ⇒ speaks English
	What is the problem?	The assumption is not reversible: you can speak English but not be from England.
b	What is the assumption?	Learning to read in primary school is a general statement.
	What is the problem?	It is not necessarily true, so it could be that Kayla can't read. It is a false premise.
c	What is the assumption?	All boys lie to wrestle is a general statement.
	What is the problem?	This general statement does not apply to everyone. It is a false premise.

WS
Necessary and sufficient

Some statements are reversible. In that case, you can use the symbol ⇔ to show that it works in both directions. The symbol is generally read as 'necessary and sufficient' or 'if and only if'. It just means that if the first part is true, so is the second, and vice versa.

For example, '*PQRS* is a quadrilateral' ⇔ '*PQRS* has 4 sides' works both ways.

A is sufficient for B means A ⇒ B

A is necessary for B means A ⇐ B (B ⇒ A)

ISBN 9780170412995

Exercise 2.01 Mathematical proof

Example 1

1 Prove that if $x > y$, then $x - c > y - c$.

Example 2

2 Prove that if $x > y$ and $y > z$, then $x > z$ (**transitivity**).

3 Prove that if $x > y$ and $c > 0$, then $cx > cy$.

4 Prove that if $x > y$ and $c > 0$, then $\frac{x}{c} > \frac{y}{c}$.

Example 3

5 Explain why each of the following 'proofs' are incorrect.

a Cats eat birds
People eat chickens
Chickens are birds
Therefore people are cats

b All children love chocolate
Joan is a child
Thus Joan loves chocolate

c Native Americans have always eaten a lot of rice
Sitting Bull was a Native American
Therefore Sitting Bull ate lots of rice

d The dog ate my homework is a classic lie
David said the dog ate his homework
Therefore David is lying

Problem solving

6 **a** Prove that if $x + c > y + c$, then $x > y$

b What can you conclude by combining **a** and Example 1?

7 Prove that $x > y > 1 \Rightarrow x^2 > y^2 > 1$.

8 The following is a classic 'proof' that $1 = -1$.

What is the incorrect step?

Let $x = 1$

$\Rightarrow x^2 = 1$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow (x - 1)(x + 1) = 0$

$\Rightarrow \frac{(x-1)(x+1)}{(x-1)} = \frac{0}{(x-1)}$

$\Rightarrow x + 1 = 0$

$\Rightarrow x = -1$

$\Rightarrow 1 = -1$

ISBN 9780170412995

Counterexamples

2.02 Counterexamples

When you prove a mathematical statement, you have to show it is true for all cases. It can be **disproven** by just one case where it is not true.

Counterexamples

- A **counterexample** is an example of a case where a statement is not true
- A single counterexample is all that is needed to disprove a statement

EXAMPLE 4

Find a counterexample to show that $x^2 > 1 \Rightarrow x > 1$ is not true.

Solution

The squares of negative numbers are positive.	$(-3)^2 = 9 > 1$, but $-3 < 1$
Write the conclusion.	$x^2 > 1 \Rightarrow x > 1$ is not true.

When you are looking for counterexamples, try 0, 1, negatives, large numbers, very small numbers and numbers at the boundaries of conditions.

EXAMPLE 5

In the Middle Ages, $n^2 + n + 41$ was thought to be a prime number for any positive integer n.

Prove that this is not true.

Solution

Try $n = 1$.	$n^2 + n + 41 = 1 + 1 + 41$ $= 43$, which is prime
Try $n = 41$, the other boundary.	$n^2 + n + 41 = 41^2 + 41 + 41$ $= 41(41 + 2)$ $= 41 \times 43$, which is not prime
Write the conclusion.	$n^2 + n + 41$ is not always prime because it is composite for $n = 41$.

In fact, $n^2 + n + 41$ is also composite for $n = 40$.

ISBN 9780170412995

Exercise 2.02 Counterexamples

1 Show that $x^2 > y^2 \Rightarrow x > y$ is not true.

Example 4

2 Show that $x^2 = 16 \Rightarrow x = 4$ is not true.

3 Show that $n^3 - 6n^2 + 11n - 6 = 0$ is not always true for all positive integers.

4 Show that $n^2 + n + 17$ is prime for all positive integers is not true.

Example 5

5 Show that $3n^2 + 3n + 23$ is prime for all positive integers is not true.

Problem solving

6 Decide whether $x > y \Rightarrow \frac{1}{x} < \frac{1}{y}$ is true. If it is not true:

- **a** find a counterexample
- **b** try to find a restriction on x and y to make it true.

7 Decide whether $x^2 + y^2 < (x + y)^2$ is true. If it is not true:

- **a** find a counterexample
- **b** try to find a restriction to make it true

8 Decide whether $(x - y)^2 < x^2 + y^2$ is true. If it is not true:

- **a** find a counterexample
- **b** try to find a simple change or restriction on y to make it true.

9 Decide whether $np = mp \Rightarrow n = m$ is true. If it is not true:

- **a** find a counterexample
- **b** try to find a simple change or restriction to make it true

10 Decide whether $\frac{1}{n} > \frac{1}{n+1}$ is true. If it is not true:

- **a** find a counterexample
- **b** try to find a simple change or restriction to make it true.

2.03 Converse

WS Converse

For $p \neq 0$, $mp = np \Rightarrow m = n$.

This is also true in reverse, so $m = n \Rightarrow mp = np$.

This means that you can say that for $p \neq 0$, $mp = np \Leftrightarrow m = n$.

The reverse of a conditional mathematical statement like this has a special name.

ISBN 9780170412995

Converse

- The **converse** of 'if P then Q' is 'if Q then P'
- In symbols, the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$

Writing the converse of a mathematical statement is an important skill. Writing the converse does not mean that it is true. A statement and its converse could both be true, both be false or one could be true and one false.

A statement and its converse need to be proved separately.

EXAMPLE 6

Write the converse of each statement.

a People in the army wear uniforms.

b The square of an integer is a positive integer.

c A square is a plane shape with 4 equal sides and equal angles.

Solution

a	Identify P and Q.	P means a person is in the army Q means a person wears a uniform
	Write the statement in reverse.	A person who wears a uniform is in the army.
b	Identify P and Q.	P means $x = n^2$ where n is an integer Q means $x > 0$
	Write the reverse statement.	A positive integer is the square of an integer.
c	Identify P and Q.	P means a plane shape is a square Q means a plane shape has 4 equal sides and equal angles
	Write the reverse statement.	A plane shape with 4 equal sides and equal angles is a square.

In Example 6, the statements in **a** and **b** are true but their converses are false. Both statement **c** and its converse are true. For another example, the statement $x^2 = 16 \Rightarrow x = 4$ is false, but its converse is true.

ISBN 9780170412995

EXAMPLE 7

Write the converse of each statement and decide on the truth of the statement and its converse.

a Lions hunt and eat wildebeest.

b A quadrilateral whose diagonals cross at 90° is a square.

c If x, y are integers, then xy is an integer.

Getty Images/Panoramic Images

Solution

a	Write the converse.	Wildebeest hunt and eat lions.
	Decide on their truth.	The statement is true but the converse is false.
b	Write the converse.	A square has diagonals that cross at 90°.
	Decide on their truth.	The statement is false but the converse is true.
c	Write the converse.	If xy is an integer, then x, y are integers.
	Decide on their truth.	The statement is true but the converse is false.

Exercise 2.03 Converse

Example 6

1 Write the converse of 'Sharks are dangerous animals'.

2 Write the converse of: $x + y$ an integer $\Rightarrow x, y$ integers.

3 Write the converse of: adjacent angles x, y on a straight line $\Rightarrow x + y = 180°$.

ISBN 9780170412995

Example 7

4 Write the converse of 'Crocodiles sometimes eat people' and decide on the truth of the statement and its converse.

5 Write the converse of 'A quadrilateral is a kite $\Rightarrow$ diagonals cross at 90°' and decide on the truth of the statement and its converse.

6 Write the converse of 'For integers x, y: one even and the other odd $\Rightarrow$ $x + y$ odd' and decide on the truth of the statement and its converse.

7 Write the converse of '$xy = 0 \Rightarrow x = 0$ or $y = 0$' and decide on the truth of the statement and its converse.

8 Write the converse of '$2x + 1$ is an odd integer $\Rightarrow$ x is an integer' and decide on the truth of the statement and its converse.

9 Write the converse of 'x an integer $\Rightarrow$ $2x$ is an even integer' and decide on the truth of the statement and its converse.

10 If x is an even integer, x^2 is an even integer.

a Is the statement true?

b Is the converse true?

11 If x is an integer divisible by 3, x^2 is an integer divisible by 3.

a Is the statement true?

b Is the converse true?

12 If x is an odd integer, $2x$ is an even integer.

a Is the statement true?

b Is the converse true?

2.04 Equivalence and negation

You have used the ideas of equivalence (equality) and congruence previously. Two expressions are equivalent if they have the same answer. Two geometric shapes are congruent if they are exactly the same shape and size.

Equivalence

- Two statements P and Q are **equivalent** if $P \Leftrightarrow Q$
- To prove equivalence you have to show that *both* $P \Rightarrow Q$ and its converse, $Q \Rightarrow P$ are true
- Equivalent statements are written as 'P **if and only if** Q', 'P **iff** Q' and 'P is **necessary and sufficient** for Q'

ISBN 9780170412995

EXAMPLE 8

Decide if each pair of statements are equivalent.

a x, y are rational; $x + y$ is rational

b $x^3 > 0$; $x > 0$

Solution

a

Consider the forward direction.

x, y rational

$\Rightarrow x = \frac{a}{b}, y = \frac{c}{d}$ for integers a, b, c, d

$\Rightarrow x + y = \frac{a}{b} + \frac{c}{d}$

$= \frac{ad + bc}{bd}$, which is rational as $ad + bc$, bd are integers

Consider the converse.

Consider $x = 5 + \sqrt{3}, y = 5 - \sqrt{3}$

Then $x + y$ is rational, but x, y are not rational

State the conclusion.

The converse is not true so the statements are not equivalent.

b

Consider the forward direction.

$x^3 > 0$

$\Rightarrow x \times x \times x > 0$

$\Rightarrow x > 0$ as x if x was negative, then x^3 would be negative

Consider the converse.

$x > 0$

$\Rightarrow x^2 > 0$ (product of two positives)

$\Rightarrow x^3 > 0$ (product of two positives)

State the conclusion.

The statements are equivalent as each implies the other.

Whenever you make a statement, there is a possible opposite statement. For example, the opposite of 'all 17-year-olds are in Year 11' is 'some 17-year-olds are not in Year 11'. You could even go as far as 'at least one 17-year-old is not in Year 11'. You can see this by writing the opposite by putting in the word 'not'. This gives 'not all 17-year-olds are in Year 11'.

Negation

The **negation** of a statement P is 'not P', written as $\overline{P}$ or $\neg P$.

ISBN 9780170412995

EXAMPLE 9

Write the negation of each statement.

a Antarctica is permanently covered in snow and ice.

b The inverse of a rational number is also a rational number.

c Any positive integer can be written as the sum of two different prime numbers.

Solution

a Write the statement to include the implied meanings.

The entire surface of Antarctica always has a layer of snow or ice on top of it.

Now write the negation.

There is some part of Antarctica where the surface is sometimes exposed.

b In this case, 'a rational number' really means any rational number, not just one.

There is at least one rational number whose inverse is not a rational number.

c Just one case is enough.

There is at least one positive integer that cannot be written as the sum of two different prime numbers.

iStock/PatrickPoendl

ISBN 9780170412995

INVESTIGATION

THE GOLDBACH CONJECTURE

Around 1742, Christian Goldbach thought you could write integers as a sum of small numbers of prime numbers. After corresponding with Leonhard Euler on the subject he settled on the following conjecture, but could not prove it:

'Every even number greater than 4 can be written as the sum of two prime numbers.'

This is now known as the strong Goldbach conjecture.

The following is known as the weak Goldbach conjecture:

'Every odd integer over 5 can be written as the sum of three prime numbers.'

There have been many attempted proofs of the Goldbach conjectures, but none have been accepted by the Mathematics community.

Fermat's last theorem was a conjecture for which he said he had a 'marvellous little proof'. It wasn't until 1995 that it was proven, taking 150 pages of complicated mathematics.

Investigate the Goldbach conjecture(s) and other conjectures.

Make a list of which have been proven, which disproven and which are unsolved.

Negation is also important in the definition of rational and irrational numbers.

Rational and irrational numbers

- A **rational number** is one that can be expressed as $\frac{a}{b}$, where a and b are integers
- An **irrational number** is one that is not rational
- It follows that a rational number can also be expressed as a terminating or recurring decimal and an irrational one cannot

Rational and irrational numbers

Decimal representation

Since a rational number can be written as $\frac{a}{b}$, if you divide a by b it either divides exactly to make a terminating decimal, or gives one of the remainders 1, 2, 3, 4, 5, …, $b - 1$. Since there are only a finite number of possible remainders, they must repeat, unless the division terminates. If the division terminates, so does the decimal. If it doesn't terminate, the numbers must repeat. Hence the last statement above is true. You should be able to change between the two representations of rational numbers.

ISBN 9780170412995

EXAMPLE 10

a Change $\frac{5}{16}$ and $\frac{9}{14}$ to decimal form.

b Change 0.175 and 0.318181818… to fraction form.

Solution

a Divide 5 by 16.

$$
\begin{array}{r}
0.3125 \\
16\overline{)5.00000000} \\
\underline{48} \\
20 \\
\underline{16} \\
40 \\
\underline{32} \\
80 \\
\underline{80} \\
0
\end{array}
$$

Divide 9 by 14.

$$
\begin{array}{r}
0.64285714\overline{2} \\
14\overline{)9.0000000000} \\
\underline{84} \\
60 \\
\underline{56} \\
40 \\
\underline{28} \\
120 \\
\underline{112} \\
80 \\
\underline{70} \\
100 \\
\underline{98} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40
\end{array}
$$

ISBN 9780170412995

	Write the answers.	$\frac{5}{16} = 0.3125, \frac{9}{14} = 0.\overline{6428571}$
b	Write 0.175 as x.	$x = 0.175$
	Multiply by a power of 10 to make whole numbers.	$1000x = 175$
	Write as a fraction and cancel down.	$x = \frac{175}{1000} = \frac{7}{40}$
	Write $0.3\overline{18}$ as x.	$x = 0.31818181818\ldots$
	There are two numbers recurring so multiply by 10^2.	$100x = 31.8181818181\ldots$
	Subtract to remove the recurring part.	$99x = 31.5$
	Change to only whole numbers.	$990x = 315$
	Write as a fraction and cancel down.	$x = \frac{315}{990} = \frac{63}{198} = \frac{7}{22}$
	Write the answers.	$0.175 = \frac{7}{40}, 0.3\overline{18} = \frac{7}{22}$

Exercise 2.04 Equivalence and negation

1 Decide if the following statements are equivalent.

Example 8

a $x = 10$; $5x = 50$

b $a = b$; $a^2 = b^2$

c $a^2 - 2ab + b^2 = 0$; $a = b$

d $a^3 = b^3$; $a = b$

e The opposite sides of a quadrilateral are equal; the opposite sides of a quadrilateral are parallel.

2 For positive integers, are the statements 'x is a prime number' and 'x is not a composite number' equivalent?

3 Write the negation of the following statements.

Example 9

a x is an integer

b All angles in a quadrilateral are less than 180°

c A number that can be shown on the number line is either a fraction or an integer

d A number that can be shown on the number line is either a rational or irrational number

e If $(x - y)^2 > x^2 + y^2$, then y is negative

ISBN 9780170412995

Example 10

4 Change each fraction to decimal form.

a $\frac{5}{8}$ **b** $\frac{5}{7}$ **c** $\frac{6}{13}$ **d** $\frac{17}{40}$ **e** $\frac{17}{18}$

5 Change each decimal to fraction form.

a 0.225 **b** $0.4\overline{23}$ **c** $0.\overline{514}$ **d** 0.562 **e** $0.64\overline{27}$

Problem solving

6 Are the statements 'x is a rational number' and 'x can be written as a recurring decimal' equivalent? Justify your answer.

7 Is 'x is a fraction' the negation of 'x is an integer'? Explain your answer.

8 Is being female the converse or negation of being male? Justify your answer.

9 Is being an even number the negation of being an odd number? Justify your answer.

10 Is 'If you don't work hard in Maths, you will get a poor result' equivalent to 'If you work hard in Maths, you will get a good result'? Explain your answer.

11 Are the statements 'An integer that is divisible by 2 is even' and 'An integer that is not divisible by 2 is not even' equivalent statements, converse statements or negations of each other? Justify your answer.

2.05 Contrapositive

WS Contrapositive

'Contra' means against, so **contrapositive** literally means 'against positive'. If you think of the statement 'Fish have gills' as the positive statement, then the contrapositive statement is 'Something without gills is not a fish'. You can think of the 'contra' part as saying you have to go backwards when you change to negatives.

Contrapositive

- The **contrapositive** of $P \Rightarrow Q$ is $\overline{Q} \Rightarrow \overline{P}$
- The contrapositive statements $P \Rightarrow Q$ and $\overline{Q} \Rightarrow \overline{P}$ are equivalent
- Proving $\overline{Q} \Rightarrow \overline{P}$ is the same as proving $P \Rightarrow Q$

You may find it easier to prove the contrapositive than to prove the original statement. You should always consider the contrapositive as an alternative method of proof.

ISBN 9780170412995

EXAMPLE 11

Prove that any prime number greater than 2 is odd.

Solution

Identify P and Q.	$P = n > 2$ is a prime number
	$Q = n$ is odd
Start with $\overline{Q}$.	**Proof**
	Assume $n > 2$ is not odd
	$\Rightarrow n$ is even
	$\Rightarrow n$ is divisible by 2
	$\Rightarrow n$ is not prime
Write the conclusion.	If $n > 2$ is not odd, then n is not prime
	$\Rightarrow$ If $n > 2$ is prime, then n is odd. QED

In proofs, you often use the phrases, 'for all', 'for any' or 'for all cases'. There is a symbol that makes this shorter to write. There are other symbols for the integers, rational numbers and so on.

Common symbols used in proofs

$\forall$	for all
$\boldsymbol{Z}$	the set of all integers
$\boldsymbol{Z}^+$	the set of all positive integers
$\boldsymbol{Q}$	the set of all rational numbers
$\boldsymbol{Q}^+$	the set of all positive rational numbers
$\boldsymbol{R}$	the set of all real numbers
$\boldsymbol{R}\backslash 0$	the set of all real numbers except 0
$\in$	is a member of (is one of)
$\exists$	there exists (there is)
$\parallel$	is parallel to
$\triangle$	triangle
$\angle$	angle

Quantifiers

ISBN 9780170412995

EXAMPLE 12

Prove that $\forall\ n \in Z^+$, if n^2 is divisible by 3, then n is divisible by 3.

Solution

Identify P and Q.	$P = n^2$ is divisible by 3
	$Q = n$ is divisible by 3
Start with $\overline{Q}$.	**Proof**
	Suppose n is not divisible by 3
	$\Rightarrow n = 3m + 1$ or $3m + 2$ for some integer m
Now square n.	$\Rightarrow n^2 = (3m + 1)^2$ or $(3m + 2)^2$
	$= 9m^2 + 6m + 1$ or $9m^2 + 12m + 4$
	$= 3(3m^2 + 2m) + 1$ or $3(3m^2 + 4m + 1) + 1$
	$\Rightarrow n^2$ is not divisible by 3
Complete the proof.	If n not divisible by 3, then n^2 not divisible by 3
	$\Rightarrow$ If n^2 is divisible by 3, then n is divisible by 3 QED

You can use the same method as above to show that if a square is divisible by 2, 5, 7, 11, …, then the number must be divisible by 2, 5, 7, 11, …

The contrapositive can be used in geometry. You also combine it with a counterexample to show that something is not true.

EXAMPLE 13

Show that 'If a quadrilateral has 4 equal sides, then it is a square' is false.

Solution

Identify P and Q.	P = a quadrilateral has 4 equal sides
	Q = the quadrilateral is a square
Start with $\overline{Q}$.	**Proof**
	Suppose a quadrilateral is not a square.
Find a counterexample for $\overline{P}$.	A rhombus has 4 equal sides.
Complete the proof.	It is not true that a non-square does not have 4 equal sides.
	$\Rightarrow$ 'If a quadrilateral has 4 equal sides it is a square' is false. QED

ISBN 9780170412995

Exercise 2.05 Contrapositive

Examples 11,12

Example 13

1 Prove that $\forall\ n \in Z^+$, if n^2 is even, then n is even.

2 Prove that if a quadrilateral is a rectangle, its opposite sides are parallel.

Problem solving

3 Prove that $\forall\ n \in Z^+$, if $3n + 1$ is even, then n is odd.

4 Prove that $\forall\ n, m \in Z^+$, if nm is odd, then both n and m are odd.

5 Prove that $\forall\ n, m \in Z^+$, if nm is even, then at least one of n and m are even.

6 Prove that $\forall\ n, m \in Z^+$, if $n + m$ is even, then n and m are both even or both odd.

7 Prove that $\forall\ n, m \in Z^+$, if $nm > 100$, then at least one of $n, m > 10$.

8 Prove that $\forall\ a, n, m \in Z$, if nm is not divisible by a, then n is not divisible by a.

9 Prove that $\forall\ x, y \in R$, if xy is irrational, then at least one of x, y must be irrational.

10 Prove that if x is a positive irrational number, then $\sqrt{x}$ is irrational.

2.06 Contradiction

Proof by **contradiction** is similar to contrapositive because you assume the negation of what you want to prove.

Proof by contradiction

- Assume the negation of what you want to prove
- Show that this leads to something obviously not true
- Say that this is a contradiction, so what you want must be true

Proof by contradiction is sometimes called 'reducio ad absurdum', which is Latin for 'reduction to absurdity'. It is often easier than contrapositive because you do not have to reach the negation of the assumption.

ISBN 9780170412995

EXAMPLE 14

Prove that there is no greatest integer.

Solution

	Proof
Assume the negation of what you want.	Assume there is a greatest integer, n
Find a larger one.	$\Rightarrow n + 1$ is an integer
State the contradiction.	But $n + 1 > n$
State that it is a contradiction.	This contradicts the assumption.
Finish the proof.	The assumption is not true, so there is no greatest integer. QED

Contradiction is probably the most commonly-used method of proof in higher mathematics.

EXAMPLE 15

Prove that $\sqrt{3}$ is not rational.

Solution

	Proof
Assume the negation of what you want.	Suppose $\sqrt{3}$ is rational.
Write in lowest form.	$\Rightarrow \sqrt{3} = \frac{a}{b}$ for some $a, b \in Z$ If $\frac{a}{b}$ is not in the lowest form with $a, b > 0$, cancel until it is.
Square both sides.	$\Rightarrow 3 = \frac{a^2}{b^2}$
Multiply both sides by b^2.	$\Rightarrow a^2 = 3b^2$
State the meaning.	$\Rightarrow a^2$ is divisible by 3 $\Rightarrow a$ is divisible by 3 (see Example 12, p. 80)
Write a as a multiple of 3.	$\Rightarrow a = 3c$ for some $c \in \boldsymbol{Z}$

ISBN 9780170412995

Substitute and continue.	$\Rightarrow (3c)^2 = 3b^2$
	$\Rightarrow 3b^2 = 9c^2$
	$\Rightarrow b^2 = 3c^2$
	$\Rightarrow b^2$ is divisible by 3
	$\Rightarrow b$ is divisible by 3
State the contradiction.	$\Rightarrow \frac{a}{b}$ is not in the lowest form (a and b both divide by 3)
State that it is a contradiction.	This contradicts the assumption that $\frac{a}{b} = \sqrt{3}$ is rational.
State the conclusion.	Thus, $\sqrt{3}$ is irrational. QED

The proof above uses two extra pieces of information. The first is that a fraction can always be cancelled to the lowest form. The second is that if n^2 is divisible by a prime number, then so is n. You have already seen that this is true for the prime numbers 2 and 3 in the last section. You can use the contrapositive to prove it for any prime number. It follows that the square root of any prime number will be irrational. If you wanted to show that $\sqrt{17}$ was irrational, then you could assume that 'n^2 divisible by 17' $\Rightarrow$ 'n divisible by 17' unless told to prove it first. It is also true for any integer that is not a perfect square. For example, it is true for $n = 8$ but not for $n = 9$.

Proof by contradiction is used extensively in geometry.

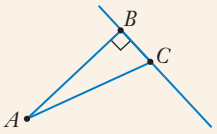
EXAMPLE 16

Prove that the perpendicular distance from a point to a line is the shortest distance to the line.

Solution

Assume the negation.	**Proof**
	Let the perpendicular distance from the point A to the line be AB.
	Assume that there is a shorter distance AC.
	$\Rightarrow AC < AB$

ISBN 9780170412995

Draw a diagram.	
Use Pythagoras' theorem.	$\Rightarrow AC^2 = AB^2 + BC^2$
State contradiction.	$\Rightarrow AC > AB$
Say it is a contradiction.	This is a contradiction.
State the conclusion.	The assumption is false, so the perpendicular distance is the shortest distance from a point to a line.

Exercise 2.06 Contradiction

Example 14

1 Prove that there is no greatest even number.

2 Prove that there is no greatest odd number.

3 Prove there are an infinite number of prime numbers. HINT: Multiply them and add 1.

Example 15

4 Prove that $\sqrt{2}$ is irrational.

5 Prove that $\sqrt{5}$ is irrational.

Example 16

6 Prove that there is at most one obtuse angle in a triangle.

Problem solving

7 Prove that if $a, b \in \mathbf{Z}$, then $a^2 - 4b \neq 2$.

8 Prove that for $a, b, c \in \mathbf{Z}^+$, $a^2 + b^2 = c^2 \Rightarrow a$ or b is even.

9 Prove that if x is irrational, then $-x$ is irrational.

10 Prove that the sum of a rational number and an irrational number is irrational.

11 Prove that the product of a rational number and an irrational number is irrational

12 Prove that the quotient of a rational number and an irrational number is irrational.

13 Prove that the opposite sides of a parallelogram are of equal length.

2.07 Euclidean geometry

The study of geometry was first set out in Euclid's *Elements*. The ancient mathematician Euclid lived from about 325 BCE to about 265 BCE in Alexandria. He systematically collected all the geometry of the time in thirteen books (chapters). They have been used ever since.

Euclid used 5 assumptions (postulates) to build Euclidean geometry.

ISBN 9780170412995

Euclid's postulates

- You can draw a straight line segment through any two points
- You can infinitely extend any line segment to make a straight line
- For any line segment you can draw a circle so that the line segment is a radius with one end at the centre
- All right angles are congruent
- If the co-interior angles of a transversal and two straight lines in the same plane do not add up to 180°, then the straight lines will meet if they are indefinitely extended on the side where the co-interior angles sum to less than 180°.

Euclid was forced to adopt the last postulate, but for a long time people thought it could be proven from the first four. It is now known to be equivalent to Pythagoras' theorem, the assumption that parallel lines are always the same distance apart and many other assumptions. Euclid defined parallel lines as lines in the same plane that never meet.

Other kinds of geometry make different assumptions about parallel lines.

The contrapositive of the 5th postulate is that the **co-interior angles of parallel lines add up to 180°**. Since it is the contrapositive, it is equivalent to the postulate.

A straight line is two right angles, so the sum of angles on a straight line must be 180°.

INVESTIGATION

EUCLID

Find out more about Euclid of Alexandria.

- Where was Alexandria?
- Why was Alexandria famous?
- How was Euclid's work preserved?
- Can you still get Euclid's Elements?

ISBN 9780170412995

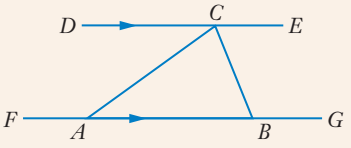
EXAMPLE 17

Use Euclid's postulates to prove that the angles in a triangle add up to 180°.

Solution

Draw a triangle ABC and a line DCE passing through C parallel to AB. Extend AB to $FABG$

Proof



State what you have to prove.

RTP

$\angle ABC + \angle ACB + \angle BAC = 180°$

Do the proof.

$\angle FAC + \angle DCA = 180°$ (co-interior)

$\angle FAC + \angle BAC = 180°$ (straight line)

$\Rightarrow \angle FAC + \angle DCA - \angle FAC - \angle BAC = 0°$

$\Rightarrow \angle DCA - \angle BAC = 0°$

$\Rightarrow \angle DCA = \angle BAC$

Similarly, $\angle ECB = \angle ABC$

Now $\angle ECB + \angle ACB + \angle DCA = 180°$

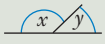
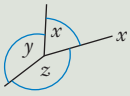
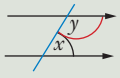
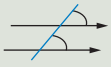
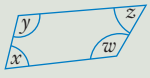
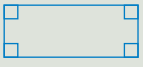
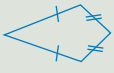
$\Rightarrow \angle ABC + \angle ACB + \angle BAC = 180°$ QED

Since the steps to show $\angle ECB = \angle ABC$ are the same as the steps for $\angle DCA = \angle BAC$, you can just say 'similarly' instead of repeating them.

The angle sum of a quadrilateral must be 360° because it is two triangles. You should remember the following from Year 10.

ISBN 9780170412995

Basic geometry and shapes

- Vertically opposite angles are equal
- Angles on a straight line add up to 180° (supplementary)  $x + y = 180°$
- Angles at a point add up to 360°  $x + y + z = 360°$
- Co-interior angles add up to 180°  $x + y = 180°$
- Alternate angles are equal
- Corresponding angles are equal 
- Angles in a triangle add up to 180° $x + y + z = 180°$
- Angles in a quadrilateral add up to 360°  $w + x + y + z = 360°$
- An equilateral triangle has all 3 sides (and angles) equal
- An isosceles triangle has 2 sides (and angles) equal
- A scalene triangle has all sides (and angles) different
- An acute-angled triangle has only acute angles (less than 90°)
- A right-angled triangle has a right angle
- An obtuse angled triangle has an obtuse angle (more than 90°)
- A trapezium has a pair of parallel sides
- A parallelogram has opposite sides parallel
- A rhombus has equal sides
- A rectangle has 4 right angles 
- A square is a regular quadrilateral; all sides and angles equal
- A kite has 2 pairs of equal adjacent sides 

ISBN 9780170412995

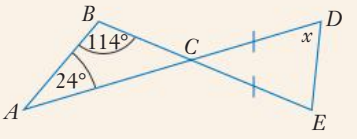
Complete the proofs

Geometrical proofs order activity

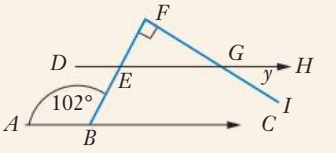
EXAMPLE 18

Find the value of each pronumeral, giving reasons for each step.

a



b



Solution

a Find the third angle in $\triangle ABC$. $\angle BCA = 42°$ (angle sum of $\triangle ABC$)

Find $\angle DCE$. $\angle DCE = 42°$ (vertically opposite)

Use the isosceles triangle. $x = 69°$ (sum of angles in isosceles triangle)

b Find the exterior angle of $\triangle EFG$. $\angle DEF = 102°$ (corresponding)

Find the adjacent angle. $\angle FEG = 78°$ (angles on a line)

Find the third angle in $\triangle EFG$ $\angle FGE = 12°$ (sum of angles in $\triangle EFG$)

Find the required angle. $y = 12°$ (vertically opposite)

Geometric proofs often used for congruent or similar triangles. You can prove that two triangles are congruent or similar using the following tests.

Congruent triangles proofs

Triangle congruency and similarity tests

Triangles are **congruent** if:

- **Side-side-side** (**SSS**); Three sides are equal
- **Side-angle-side** (**SAS**); Two sides and the included angle are equal
- **Angle-angle-side** (**AAS**); Two angles and corresponding sides are equal
- **Right angle-hypotenuse-side** (**RHS**); The hypotenuse and another side are equal in right-angled triangles

Triangles are **similar** if:

- **Angle-angle** (**AA**); Two angles are equal
- **Side-angle-side** (**SAS**); Two sides of equal angles are in proportion
- **Side-side-side** (**SSS**); Three sides are in proportion
- **Right angle-hypotenuse-side** (**RHS**); The hypotenuse and another side are in proportion in right-angled triangles

ISBN 9780170412995

You can use direct proof, counterexamples, contrapositive or contradiction in geometric proofs. You often need to prove equivalence by proving both a statement and its converse.

EXAMPLE 19

Proving properties of quadrilaterals

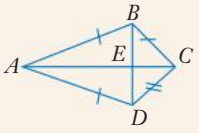
Prove that a quadrilateral is a kite if and only if one of its diagonals bisects the other at 90°.

Solution

Draw a diagram of a kite $ABCD$ with diagonals intersecting at E.

Proof

Assume that $ABCD$ is a kite. Draw the diagonals to intersect at E.



Consider $\triangle ABC$ and $\triangle ADC$.

$AB = AD$ (kite)

$BC = DC$ (kite)

$AC = AC$ (same)

$\Rightarrow \triangle ABC \equiv \triangle ADC$ (SSS)

Now consider $\triangle EBC$ and $\triangle EDC$.

$\angle BCE = \angle DCE$ ($\triangle ABC \equiv \triangle ADC$)

$BC = DC$ (kite)

$\angle CBD = \angle CDB$ (iscoceles $\triangle CBD$)

$\Rightarrow \triangle EBC \equiv \triangle EDC$

Finish the forward proof.

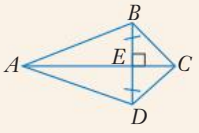
$\Rightarrow BE = ED$ and

$\angle CED = \angle CEB = 90°$ (equal angles on a line)

$\Rightarrow$ diagonal AC bisects BD at 90°.

Now start the converse. Draw a quadrilateral with one diagonal bisecting the other at 90°.

Conversely, assume that $ABCD$ is a quadrilateral and AC bisects BD at E.



ISBN 9780170412995

Consider $\triangle EBC$ and $\triangle EDC$.	$EB = ED$ (bisected)
Now consider $\triangle EBA$ and $\triangle EDA$.	$\angle CED = \angle CEB = 90°$ (given)
	$EC = EC$ (same)
	$\Rightarrow \triangle EBC \equiv \triangle EDC$ (SAS)
	Similarly, $\triangle EBA \equiv \triangle EDA$*
Use the congruent triangles.	$\Rightarrow BC = CD$ and $AB = AD$
Finish the converse proof.	$\Rightarrow ABCD$ has two pairs of equal adjacent sides
	$\Rightarrow ABCD$ is a kite
Finish the whole proof.	The proposition and its converse have been proved. QED

*The steps for $\triangle EBA \equiv \triangle EDA$ are the same as for $\triangle EBC \equiv \triangle EDC$, immediately above. You can omit the steps and write 'similarly' for such a case.

The proof is typical of many direct geometric proofs.

Exercise 2.07 Euclidean geometry

Example 17

1 Use Euclid's postulates to prove that corresponding angles on parallel lines are equal.

2 Prove the converse of 'co-interior angles on parallel lines add up to 180°'.
Hint: Use the contrapositive.

3 Prove that alternate angles on parallel lines are equal.

4 Prove the converse of question **3**.

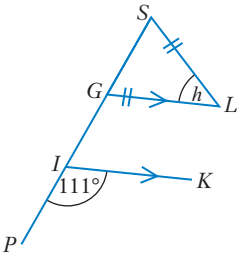
5 Prove the converse of question **1**.

6 Show that the exterior angle of a triangle is equal to the sum of the interior opposite angles.

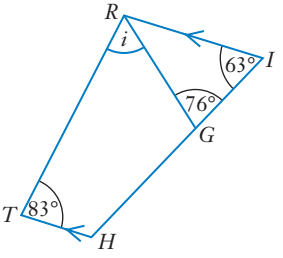
Example 18

7 Find the value of each pronumeral, giving reasons for each step.

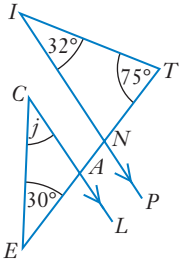
a



b



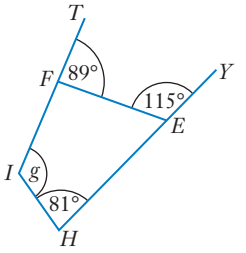
c



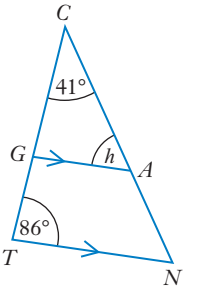
ISBN 9780170412995

8 Find the value of each pronumeral, giving reasons for each step.

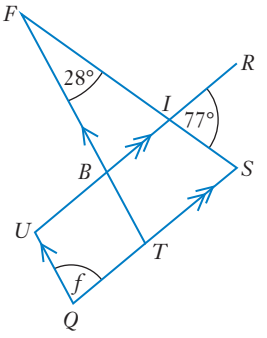
a



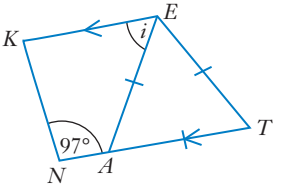
b



c



d



9 Show that a quadrilateral is a parallelogram if and only if the diagonals bisect each other.

Example 19

Problem solving

10 Show that a quadrilateral is a rhombus if and only if the diagonals bisect each other at 90°.

11 Show that a line segment between the midpoints of two sides of a triangle is parallel to and half the length of the third side.

12 State and show the converse of question **11**.

13 Show that the diagonally opposite angles of a quadrilateral are equal if and only if it is a parallelogram.

ISBN 9780170412995

2. CHAPTER SUMMARY

Methods of proof

Steps of a mathematical proof

- A precise statement of what is to be proven, usually under a heading of **Theorem**, **Lemma**, **Corollary** or **Required to Prove** (**RTP**)
- The logical steps of the **Proof** come next, usually with reasons for each step
- A final statement of what has been proven, followed by the initials **QED**

Types of proof

- A **direct proof** is a proof that starts with the assumption and goes in easy steps to the result.
- A **counterexample** is an example of a case where a statement is not true
- A single counterexample is all that is needed to disprove a statement
- The **converse** of 'if P then Q' is 'if Q then P'
- In symbols, the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$
- Two statements P and Q are **equivalent** if $P \Leftrightarrow Q$
- To prove equivalence you have to show that *both* $P \Rightarrow Q$ and its converse, $Q \Rightarrow P$ are true
- Equivalent statements are written as 'P **if and only if** Q', 'P **iff** Q' and 'P is **necessary and sufficient** for Q'

Contrapositive

- The **negation** of a statement P is 'not P', written as $\overline{P}$ or $\neg P$.
- The **contrapositive** of $P \Rightarrow Q$ is $\overline{Q} \Rightarrow \overline{P}$
- The contrapositive statements $P \Rightarrow Q$ and $\overline{Q} \Rightarrow \overline{P}$ are equivalent
- Proving $\overline{Q} \Rightarrow \overline{P}$ is the same as proving $P \Rightarrow Q$

Proof by contradiction

- Assume the negation of what you want to prove
- Show that this leads to something obviously not true
- Say that this is a contradiction, so what you want must be true

Rational and irrational numbers

- A **rational number** is one that can be expressed as $\frac{a}{b}$, where a and b are integers
- An **irrational number** is one that is not rational
- It follows that a rational number can also be expressed as a terminating or recurring decimal and an irrational one cannot

ISBN 9780170412995

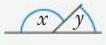
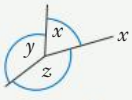
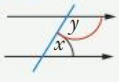
Mathematics symbols used in proofs

$\forall$	for all
$\boldsymbol{Z}$	the set of all integers
$\boldsymbol{Z}^+$	the set of all positive integers
$\boldsymbol{Q}$	the set of all rational numbers
$\boldsymbol{Q}^+$	the set of all positive rational numbers
$\boldsymbol{R}$	the set of all real numbers
$\boldsymbol{R}\backslash 0$	the set of all real numbers except 0
$\in$	is a member of (is one of)
$\exists$	there exists (there is)
$\parallel$	is parallel to
$\triangle$	triangle
$\angle$	angle

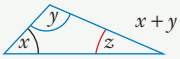
Euclid's postulates

- You can draw a straight line segment through any two points
- You can infinitely extend any line segment to make a straight line
- For any line segment you can draw a circle so that the line segment is a radius with one end at the centre
- All right angles are congruent
- If the co-interior angles of a transversal and two straight lines in the same plane do not add up to 180°, then the straight lines will meet if they are indefinitely extended on the side where the co-interior angles sum to less than 180°.
- The last postulate is equivalent to 'co-interior angles of parallel lines add up to 180°'

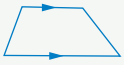
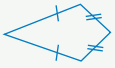
Basic geometry and shapes

- Vertically opposite angles are equal
- Angles on a straight line add up to 180° (supplementary)  $x + y = 180°$
- Angles at a point add up to 360°  $x + y + z = 360°$
- Co-interior angles add up to 180°  $x + y = 180°$
- Alternate angles are equal
- Corresponding angles are equal

ISBN 9780170412995

- Angles in a triangle add up to 180° 

$x + y + z = 180°$
- Angles in a quadrilateral add up to 360°

$w + x + y + z = 360°$
- An equilateral triangle has all 3 sides (and angles) equal
- An isosceles triangle has 2 sides (and angles) equal
- A scalene triangle has all sides (and angles) different
- An acute-angled triangle has only acute angles (less than 90°)
- A right-angled triangle has a right angle
- An obtuse-angled triangle has an obtuse angle (more than 90°)
- A trapezium has a pair of parallel sides 
- A parallelogram has opposite sides parallel
- A rhombus has equal sides
- A rectangle has 4 right angles
- A square is a regular quadrilateral; all sides and angles equal
- A kite has 2 pairs of equal adjacent sides 

Triangles are **congruent** if:

- **Side-side-side** (**SSS**); Three sides are equal
- **Side-angle-side** (**SAS**); Two sides and the included angle are equal
- **Angle-angle-side** (**AAS**); Two angles and corresponding sides are equal
- **Right angle-hypotenuse-side** (**RHS**); The hypotenuse and another side are equal in right-angled triangles

Triangles are **similar** if:

- **Angle-angle** (**AA**); Two angles are equal
- **Side-angle-side** (**SAS**); Two sides of equal angles are in proportion
- **Side-side-side** (**SSS**); Three sides are in proportion
- **Right angle-hypotenuse-side** (**RHS**); The hypotenuse and another side are in proportion in right-angled triangles

ISBN 9780170412995

2. CHAPTER REVIEW

Methods of proof

1 Prove that for positive numbers w, x, y, z: $w > x$ and $y > z \Rightarrow wy > xz$ Examples 1, 2

2 Explain why each of the following 'proofs' are wrong. Example 3

a Boys like to wrestle.
Mohamed is a boy.
Therefore Mohamed likes to wrestle.

b Working in a noisy place without ear protection causes deafness.
Tom is deaf.
Thus Tom must have worked in a noisy environment.

3 Find a counterexample for each statement. Examples 4, 5

a $x > y \Rightarrow x^2 > y^2$

b $x^2 + 24 > 10x$

4 What is the converse of each statement? Example 6

a All lions are carnivores.

b x, y integers $\Rightarrow xy$ is an integer

5 Write the converse of 'xy is a perfect square $\Rightarrow x, y$ are perfect squares' and decide on the truth of the statement and its converse. Example 7

6 Decide if each pair of statements are equivalent. Example 8

a $0 \leq x \leq 1$; $0 \leq x^n \leq 1$ for $n > 0$

b Opposite sides of a quadrilateral are equal; Opposite sides of a quadrilateral are parallel

c A quadrilateral is a kite; The diagonals of a quadrilateral cross at 90°

7 Write the negation of each of statement. Example 9

a All rational numbers can be written as recurring decimals

b Australians live in the southern hemisphere

c An acute-angled triangle has only acute angles

8 **a** Express $\frac{5}{12}$ and $\frac{9}{25}$ in decimal form. Example 10

b Express 0.745 and $0.3\overline{75}$ in fraction form.

9 Use the contrapositive to prove that if x^2 is irrational then x is irrational. Examples 11, 12

ISBN 9780170412995

Example 13

10 Prove that if a quadrilateral has an acute angle then it is not a rectangle.

Example 14

11 Prove that there is no least integer.

Example 15

12 Prove that $\sqrt{7}$ is irrational.

Example 16

13 Prove that there is at most one right angle in a triangle.

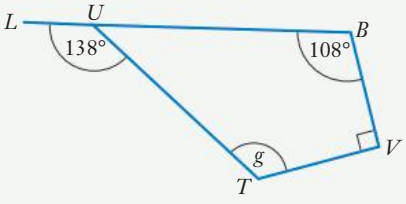
Example 17

14 Use Euclid's postulates to prove that the angle sum of a trapezium is 360°.

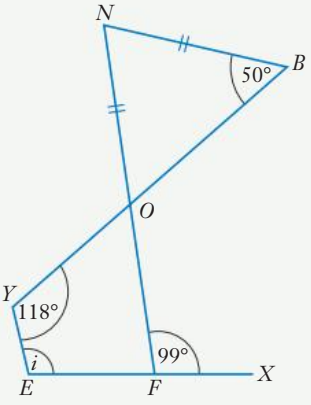
Example 18

15 Find the value of each pronumeral, giving reasons for each step.

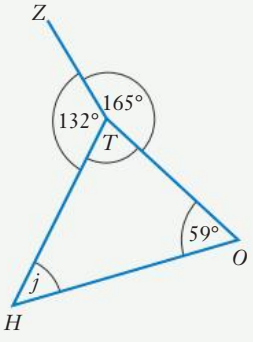
a



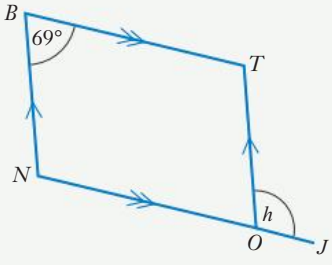
b



c



d



Example 19

16 Prove that a quadrilateral is a square if and only if it has equal diagonals that bisect each other at 90°.

ISBN 9780170412995

Problem solving

17 Prove that for any real numbers $x^2 + y^2 \geq 2xy$.

18 Decide whether $\frac{1}{n} < \frac{1}{n+m}$ is true for $m < 0$. If it is not true:

- **a** find a counterexample
- **b** try to find a simple change or restriction to make it true.

19 Are the statements '$x > 0$' and '$x < 0$' converses, negations of equivalents of each other? Justify your answer.

20 Prove that $\forall\ n \in \boldsymbol{Z}^+$, if n^2 is divisible by 6, then n is divisible by both 2 and 3.

21 Prove that if $\frac{x}{y}$ is irrational, then at least one of x, y must be irrational.

22 Prove that the difference of a rational and irrational number is irrational.

23 Show that a line segment between the points on two sides of a triangle that divide the sides in the ratio 1:4 (from the common vertex) is parallel to and $\frac{1}{5}$ the length of the third side.

ISBN 9780170412995

3.

PERMUTATIONS

The ability to count is a fundamental skill in mathematics. There is a branch of mathematics known as **combinatorics** that includes 'counting' the objects that satisfy certain conditions or criteria and deciding when the criteria can be met, for example, the number of possible orderings of a deck of 52 distinct playing cards. Combinatorics is related to many areas of study such as algebra, probability theory, geometry, computer science, design theory, game theory and statistical physics.

Methods of counting can become quite sophisticated, particularly when you are counting things that have a pattern.

3.01 Multiplication principle
3.02 Addition principle
3.03 The inclusion-exclusion principle
3.04 Permutations and factorial notation
3.05 Permutations with restrictions
3.06 Applications of permutations
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

The inclusion-exclusion principle for the union of two sets and three sets

- determine and use the formulas (including the addition principle) for finding the number of elements in the union of two and the union of three sets
- use the multiplication principle

Permutations (ordered arrangements) and combinations (unordered selections)

- solve problems involving permutations
- use factorial notation
- use the notation ${}^{n}P_{r} = \dfrac{n!}{(n-r)!}$
- solve problems involving permutations with restrictions

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

Shutterstock/Kris Schmid

TERMINOLOGY

addition principle
circular permutation
combinatorics
counting technique
enumeration
factorial notation
inclusion-exclusion principle
multiplication principle
mutually exclusive
permutation
probability
sample point
sample space
tree diagram

3.01 Multiplication principle

A **counting technique** is a method of determining how many objects are in a collection without actually listing out all the objects in the collection.

Shutterstock.com/Lotus_studio

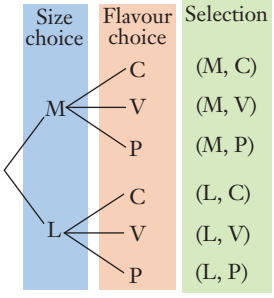
You work out how many squares are on a chess board by counting 1, 2, 3, …, 64. This is called **enumeration**.

This is a simple counting technique. What about a situation that is more complex? Consider the problem of assigning vehicle number plates that have three letters and three numbers. How many different number plates could formed? It doesn't take long to realise that there are more than 17 million different number plates that are possible.

Counting techniques are needed to be able to answer questions like the number plate problem.

Consider the following example.

An ice cream shop serves two different sizes, medium (M) and large (L), and three different flavours, chocolate (C), vanilla (V) and pistachio (P). How many different selections of size and flavour are possible?



When a customer orders an ice cream, the size is selected first and then a flavour.

A **tree diagram** can be used to show these selections more clearly.

Using the tree diagram, you can see that there are 6 different selections possible.

You can also see that there are 2 ways of selecting the size and 3 ways of selecting the flavour.

$2 \times 3 = 6$

This is an example of the counting technique called the **multiplication principle**.

ISBN 9780170412995

Multiplication principle

If one operation can be performed in n ways, followed by another operation that can be performed in m ways, then the combined operation can be performed in $n \times m$ ways.

General case

If operations $O_1, O_2, O_3, \ldots, O_n$ are performed in order, with the possible number of outcomes $n_1, n_2, n_3, \ldots, n_n$, respectively, then there are $n_1 \times n_2 \times n_3 \times \ldots \times n_n$, possible outcomes of the operations performed in the given order.

EXAMPLE 1

A restaurant offers a choice of 3 entrées, 5 main courses and 6 desserts.

a How many different ways can a diner choose a 3-course meal?

b How many different ways can an entrée and main course be selected?

c A diner selects a main course and a dessert. How many different ways can this be done?

Solution

a

There are 3 courses.	Number of choices $= n_1 \times n_2 \times n_3$
Substitute for known values.	$= 3 \times 5 \times 6$
Evaluate.	$= 90$
Write the result.	There are 90 different choices of three-course meals.

b

There are 2 operations.	Number of ways $= n_1 \times n_2$
Substitute for known values.	$= 3 \times 5$
Evaluate.	$= 15$
Write the result.	There are 15 different choices of entrée and main course.

c

There are 2 operations.	Number of ways $= n_1 \times n_2$
Substitute for known values.	$= 5 \times 6$
Evaluate.	$= 30$
Write the result.	There are 30 different choices of main course and dessert.

ISBN 9780170412995

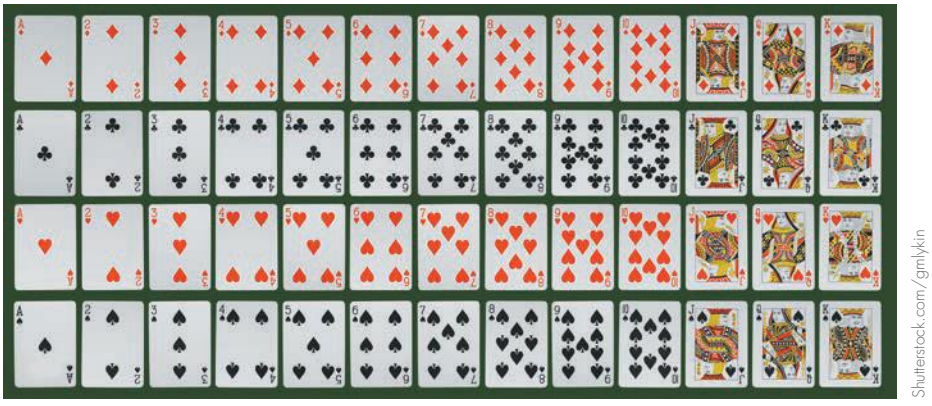
EXAMPLE 2

3 dice are rolled and the numbers on the upper faces are noted. How many different combined outcomes are possible?

Solution

There are 3 operations.	Number of ways $= n_1 \times n_2 \times n_3$
Each die has 6 numbered faces.	$= 6 \times 6 \times 6$
Evaluate.	$= 216$
Write the result.	There are 216 different combined outcomes from rolling 3 dice.

A standard deck of playing cards has 52 cards in all. There are four different suits – hearts (♥), diamonds (♦), clubs (♣) and spades (♠). Each suit has 13 different cards – ace, 1, 2, 3, …, jack, queen and king. A number of people each choose a card from a standard deck of playing cards, and each person has one less card to choose from than the person before.



Shutterstock.com/gmlykin

EXAMPLE 3

In how many ways can 5 people select a card from a deck of standard playing cards?

Solution

There are 5 operations.	Number of ways $= n_1 \times n_2 \times n_3 \times n_4 \times n_5$
There are 52 cards available for the first person, 51 for the second, 50 for the third and so on.	$= 52 \times 51 \times 50 \times 49 \times 48$

ISBN 9780170412995

Evaluate.	$= 311\,875\,200$
Write the result.	There are 311 875 200 different ways 5 people can select a card from a pack.

Exercise 3.01 Multiplication principle

Example 1

1 Ella has 3 hats (A, B and C), and 2 coats (1 and 2), in her wardrobe.

a List out the different choices of hat and coat combinations.

b Calculate the number of different choices Ella can make using the multiplication principle.

2 How many 3-letter code words are possible using the first 8 letters of the alphabet if letters can be repeated?

3 I can travel from Brisbane to Bundaberg by bus, car, rail or air. However, because of schedules, I can return only by car or air. In how many different ways can I travel from Brisbane to Bundaberg and return?

4 A new number plate made up of 2 letters followed by 2 numbers and then 2 letters is proposed by the transport authority. How many different number plates of this type are possible?

5 Matilda's wardrobe contains 3 hats, 2 coats and 4 scarves. Assuming she is comfortable wearing any combination of hat, coat and scarf, how many different combinations could she select from her closet?

6 A car manufacturer produces 4 different models. Each model can come equipped as a standard finish, an elegance finish or a sports finish. The cars are available with either manual or automatic transmission and can be painted in 7 different colours. How many different combinations of vehicles are possible?

Example 2

7 A die is rolled and a coin is tossed. How many different combined outcomes are possible?

8 If 7 coins are tossed, how many different outcomes are possible?

9 Two 8-sided die are rolled and a coin is tossed. How many different combined outcomes are possible?

10 A standard die is rolled, a card is selected from a deck of playing cards and a coin is tossed. How many different outcomes are possible?

Example 3

11 A group of 5 boys and 3 girls is to be photographed. How many ways can they be arranged in one row?

12 How many 3-letter code words are possible using the first 8 letters of the alphabet if no letter can be repeated?

ISBN 9780170412995

13 How many different selections of 4 letters are possible from the letters in the word MARINE, if:

a no repetition is allowed? **b** repetition is allowed?

14 A vehicle is spotted at the scene of a crime. The number plate has 3 letters followed by 3 numbers. Unfortunately, the CCTV image is blurred so that the second letter and last 2 digits of the number plate cannot be read. How many number plates will need to be checked by police to identify the vehicle?

15 Two people are playing a card game. The dealer gives each player one card and the players keep the card that is dealt to them.

a How many different outcomes can result?

b In how many of the possible outcomes do both players have clubs?

Problem solving

16 The Cleveland train line runs from Brisbane Central to Cleveland station. There are 23 stations at which it stops along the line. Assuming that you cannot depart and arrive at the same station, how many one-way tickets could be printed showing a point of departure and a destination?

17 **a** How many different number plates consisting of 3 letters followed by 3 digits are possible?

b Australia's population is about 25 million and there are about 736 registered vehicles per 1000 in the population. Would there be enough of these types of number plates for all the registered vehicles in Australia?

18 How many 3-letter code words are possible using the first 8 letters of the alphabet if adjacent letters cannot be alike?

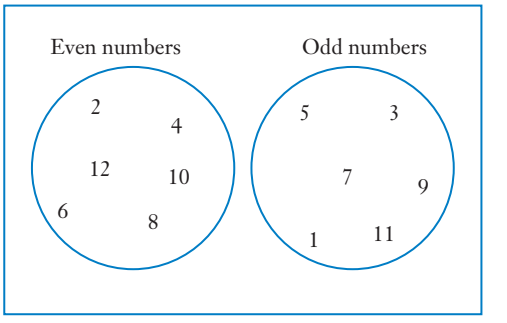
19 A large high school has 1150 students. Students are issued with an ID number. For security reasons, repetition of digits is not permitted in the ID number. How many digits should the ID number be so that there will be enough to identify each of the students at the school?

20 Most Australian banks require that customers use a 4-digit personal identification number (PIN) for accessing accounts. In Europe, some banks require a 6-digit PIN.
How many more different PINs would be available for a bank requiring a 6-digit PIN?

3.02 Addition principle

Consider the integers from 1 to 12 inclusive. The set of odd integers and the set of even integers are completely separate as they have no common elements.

The Venn diagram shows that there is no overlapping region between the sets.



ISBN 9780170412995

Completely separate events are described as **mutually exclusive**.

In more complex counting problems, you may be able to break the task into subtasks that can be counted separately. Provided that the subtasks are mutually exclusive, you can add the results.

Addition principle

If two mutually exclusive operations can be performed in n and m ways respectively, then the number of ways of performing one or the other is $n + m$.

More generally, for mutually exclusive events, the total number of choices is the sum of the choices for each of the events.

EXAMPLE 4

A bag contains coloured marbles that are numbered. There are 5 blue marbles numbered 1 to 5, 7 yellow marbles numbered 1 to 7 and 3 red marbles numbered 1 to 3. How many different selections of 1 marble can be made?

Solution

Selecting a blue marble, yellow marble and a red marble are mutually exclusive events since you can't have a marble that is more than one colour.

Use the addition principle. Number of ways $= 5 + 7 + 3$

$= 15$

State the result. There are 15 ways a marble can be selected.

In many cases, the addition principle is used with the multiplication principle.

EXAMPLE 5

A drawer contains socks that are the same shape and size. The only difference is that 3 are black and 4 are white. In how many ways can you choose a matching pair of socks?

Solution

You can either choose a black sock or a white sock first.

Use the multiplication principle to calculate the ways of choosing a black pair. Number of black pairs $= 3 \times 2$

$= 6$

Use the multiplication principle to calculate the ways of choosing a white pair. Number of white pairs $= 4 \times 3$

$= 12$

ISBN 9780170412995

Choosing a black pair and choosing a white pair.	Total number of ways = 6 + 12 = 18
Write the result.	There are 18 ways of choosing a matching pair of socks.

It is sometimes useful to use boxes to help visualise a problem. One box is used for each position to be filled. For example, if you wanted to know how many 5-digit numbers could be formed with a given set of numbers, you would draw 5 boxes joined together. You can work out the number of arrangements by considering how each box can be filled.

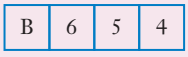
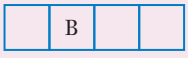
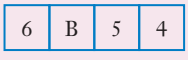
The procedure is as follows:

1. Draw a row of boxes to show the positions available.
2. Write the number of possible choices into the first box.
3. Having 'filled' one box, consider the next box, and so on, until all choices have been made.
4. The product of the number of possible choices for each box then gives the overall number of arrangements.

EXAMPLE 6

How many 4-letter 'words' can be made using the letters A, B, C, D, E, F and G without repetition, if the letter B must be used?

Solution

Draw a set of boxes for this situation. The letter B could be placed in any of the boxes.	
Let B be placed in the first box.	
There are 6 letters remaining, so the second box can be filled in 6 ways. The third box can be filled in 5 ways and the last box in 4 ways.	
Use the multiplication principle.	Number of ways with B 1st = 1 × 6 × 5 × 4 = 120
Let B be placed in the second box.	
The first box can be filled in 6 ways. The third box can be filled in 5 ways and last box in 4 ways.	

ISBN 9780170412995

Use the multiplication principle.	Number of ways with B 2nd = $6 \times 1 \times 5 \times 4$ = 120
Let B be placed in the third box.	6 | 5 | B | 4
Use the multiplication principle.	Number of ways with B 3rd = $6 \times 5 \times 1 \times 4$ = 120
Let B be placed in the fourth box.	6 | 5 | 4 | B
Use the multiplication principle.	Number of ways with B 4th = $6 \times 5 \times 1 \times 4$ = 120
Placing B first, second, third and fourth are mutually exclusive events. Use the addition principle.	Total number of ways = 120 + 120 + 120 + 120 = 480
Write the result.	The number of four-letter words containing B selected from the letters A, B, C, D, E, F and G without repetition is 480.

You could have used the multiplication principle alone to calculate the solution to Example 6 in the following way.

Total words = ways of placing B × ways of filling the other 3 letters

$= 4 \times 5 \times 4 \times 3$

$= 480.$

INVESTIGATION

ARRANGEMENTS

Work in groups of 2 or 3 for this discussion. You may find it useful to obtain some 'props' to assist in visualising the problem. For example, large dice and a supply of Post-it notes could be used. The sticky part of the notes can be cut into small pieces that just fit onto the faces of the dice, so that they can be coloured.

The problem

How many different ways can the faces of a cube can be painted using 6 colours?

This is a 3-dimensional problem and the difficulty is to determine which colourings are really different, considering that the cube can take on different orientations.

Work in your group to discuss the problem if:

- each face is to be a different colour
- faces can be coloured the same.

ISBN 9780170412995

Exercise 3.02 Addition principle

Example 4

1 Noah owns 5 different red ties, 4 different blue ties and 6 multi-coloured ties in different patterns. He selects a tie to wear to work. How many different selections can be made?

2 A restaurant offers 4 different meat dishes, 5 seafood meals, 3 vegetarian options and 2 vegan meals. How many different selections are available for diners?

3 Mia is researching mobile phones. She is only interested in looking at the phones provided by 3 different manufacturers. She is quite particular about the camera quality and internal memory. The first manufacturer has 5 different phones that are suitable, the second has 4 and the third has only one. From how many different phones may Mia make her selection?

Example 5

4 **a** How many arrangements of 5 different letters can be made using the letters of the word BEGAN?

b How many of the arrangements in part **a** begin with either A or N?

5 A code consists of 3 symbols. For each code, at least one symbol is a letter of the alphabet and the other symbols are digits (0, 1, 2, …, 9). How many possible codes are there?

Example 6

6 How many 'words' can be made from the letters of the word CATERING using different arrangements, according to each of the following conditions?

a Any 4 letters may be used, without repetition.

b Any 4 letters may be used, with repetition permitted.

c One vowel and 3 consonants must be used, without repetition.

d 2 vowels and 2 consonants must be used, without repetition.

e 3 vowels and one consonant must be used, without repetition.

f At least one vowel and at least one consonant must be used, without repetition, in 4-letter words.

g At least 2 vowels and at least 2 consonants must be used, without repetition, in 6-letter words.

7 A queue has 4 girls and 4 boys standing in a line. How many different arrangements are possible if:

a the boys and girls stand anywhere?

b boys and girls alternate?

c all the boys stand together?

d 2 particular girls wish to stand together?

e a particular boy refuses to stand next to another boy called Ethan?

ISBN 9780170412995

8 **a** How many numbers of four different digits can be formed using the digits 0, 4, 5, 9?

b How many of the numbers in part **a** end in 0?

c How many end in 4?

d How many are even?

e How many of the numbers in part **a** are odd?

f How many of the numbers in part **a** are divisible by 5?

Problem solving

9 A theme park has two major attractions: Underwater World and Future Land. People can go between Underwater World and Future Land by water ferry in 3 different ways, and by light rail in 2 ways. In how many different ways can people go:

a Between Underwater World and Future Land by either water ferry or by light rail?

b By water ferry and return by light rail?

c By one of the two forms of transport and return by the other?

10 3-pin locks have 3 'pins' inside the lock that are pushed into the correct position by the key to open the lock. The pins have different possible lengths that require different keys. The pins are normally made to be of different lengths, so that the key is not straight.

a A cheap lock has 5 possible pin lengths. How many different keys are possible?

b A better 3-pin lock has 10 possible pin lengths. How many different keys are possible?

c Good locks have 5 pins with 10 possible pin lengths. How many different keys are possible for these locks?

11 The genetic code of a DNA molecule consists of arrangements of 4 nucleotides with the bases adenine, cytosine, guanine and thymine. The bases, referred to by their initial letters A, C, G and T, are arranged in long chains but are often broken up into 3-letter 'words' for legibility. How many such words are possible?

12 Dimitri makes flutes from different materials in different sections of the flute to suit the customer's needs. The head joint may be made entirely from alloy, be silver-plated, have a silver lip-plate or be solid silver. The body may be alloy, silver-plated or solid silver, and the foot joint may be alloy, silver-plated or solid silver. Furthermore, the foot joint may be the standard C foot joint or the alternative B foot joint. How many different flutes are possible?

ISBN 9780170412995

13 In horse racing, a quadrella requires a punter to select the winners of four particular races or 'legs' of the quadrella. A punter makes the following selections for each leg of the quadrella.

1st leg	2nd leg	3rd leg	4th leg
Ripabuy	Roy's Joy	Fab	Cunning As
Run Barry Run	Did We Win	Gadzoox	Notta Brass Razoo
	Petal To The Metal	Just A Con	

How many possible quadrellas can be formed from these selections?

14 A lab rat is in an enclosure with 5 doors.

a In how many ways can the lab rat enter and leave the enclosure via a door?

b In how many ways can the lab rat enter the enclosure through one door and leave via a different door?

The inclusion-exclusion principle

3.03 The inclusion-exclusion principle

Consider the situation where 30 students are lined up waiting to enter a classroom and 14 of these are wearing hats. How many are not wearing hats?

The answer is found using subtraction: $30 - 14 = 16$ students who are not wearing hats.

This is a very simple example of the **inclusion-exclusion principle**. It does provide an opportunity to introduce some notation that will be useful.

If E = {All students waiting to enter the classroom} and H = {Students wearing caps} then

$\overline{H}$ = {Students not wearing caps}.

Using set notation, $|E| = 30$ and $|H| = 14$, so $|\overline{H}| = |E| - |H|$.

EXAMPLE 7

If $S = \{1, 2, 3, \ldots, 100\}$, how many elements are not divisible by 5?

Solution

Rather than try to count the elements of S that are *not divisible* by 5, it is easier to count the elements of S that *are divisible* by 5 and then use the inclusion-exclusion principle.

State $\lvert S\rvert$.	$\lvert S\rvert = 100$
Let F be the set of integers divisible by 5.	$F = \{5, 10, 15, \ldots, 100\}$
State $\lvert F\rvert$.	$\lvert F\rvert = 20$
Write an expression for $\lvert\overline{F}\rvert$.	$\lvert\overline{F}\rvert = \lvert S\rvert - \lvert F\rvert$

ISBN 9780170412995

Substitute for known values.	$= 100 - 20$ $= 80$
State the result.	There are 80 elements of S that are not divisible by 5.

Consider a class of 30 students in which 15 study maths, 18 study science and 6 study both maths and science. You wish to know how many students study maths or science.

Let M = {Students who study maths}

Let S = {Students who study science}

If you use the addition principle, you get:

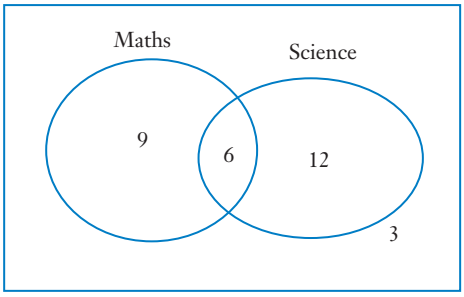
$|M| + |S| = 15 + 18 = 33$

But this is clearly too many as there are only 30 students in the class. So in this case, the number of students who study maths or science, $|M \text{ or } S| \neq |M| + |S|$. This is because M and S are not mutually exclusive.

Construct a Venn diagram to help clarify this situation.

15 students study maths and 6 study both maths and science, so 9 study maths and not science.

18 students study science and 6 study both maths and science, so 12 study science and not maths.



You can also see that 3 students study neither maths nor science.

By adding $|M|$ and $|S|$, the overlap between M and S, $(M \cap S)$ is added twice.

So $|M \text{ or } S| = |M| + |S| - |M \cap S|$

$= 15 + 18 - 6$

$= 27$

This avoids counting the students who study maths and science twice.

Inclusion-exclusion principle for 2 sets

For two sets A and B:

$$|A \text{ or } B| = |A| + |B| - |A \cap B|$$

ISBN 9780170412995

EXAMPLE 8

For the integers 1 to 50 inclusive, how many are multiples of 2 or 3?

Solution

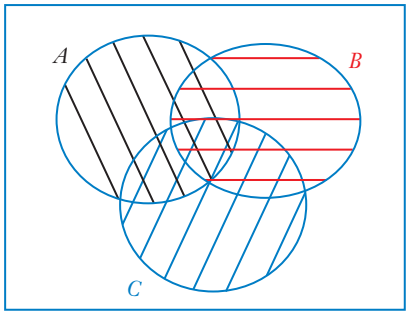
Assign the sets.	Let $A = \{x: x \text{ is a multiple of 2 and } x \leq 50\}$ Let $B = \{x: x \text{ is a multiple of 3 and } x \leq 50\}$
Find the cardinality of A and B.	$A = \{2, 4, 6, \ldots, 50\}$ so $\lvert A\rvert = 25$ $B = \{3, 6, 9, \ldots, 48\}$ so $\lvert B\rvert = 16$
Find $A \cap B$.	$A \cap B = \{6, 12, 18, \ldots, 48\}$
Find the cardinality of $A \cap B$.	$\lvert A \cap B\rvert = 8$
Apply the inclusion-exclusion principle.	$\lvert A \text{ or } B\rvert = \lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$
Substitute for known values.	$= 25 + 16 - 8$
Evaluate.	$= 33$
State the result.	There are 33 integers from 1 to 50 inclusive that are multiples of 2 or 3.

The inclusion-exclusion principle can be extended to three sets, say A, B and C.

To calculate $\lvert A \text{ or } B \text{ or } C\rvert$, you could add $\lvert A\rvert + \lvert B\rvert + \lvert C\rvert$.

However, some elements would be counted twice, or even three times.

For example, the elements of $A \cap B$ will be counted at least twice and those of $A \cap B \cap C$ are counted three times.



To correct the first issue, the elements of $A \cap B$, $B \cap C$ and $A \cap C$ need to be subtracted from the elements of $A + B + C$.

Next, you need to correct for the elements of $A \cap B \cap C$.

By subtracting the elements of $A \cap B$, $B \cap C$ and $A \cap C$, the elements of $A \cap B \cap C$ have also been subtracted three times, so these must be added back in.

This gives the formula below.

Inclusion-exclusion principle for 3 sets

For three sets A, B and C:

$$\lvert A \text{ or } B \text{ or } C\rvert = \lvert A\rvert + \lvert B\rvert + \lvert C\rvert - \lvert A \cap B\rvert - \lvert B \cap C\rvert - \lvert A \cap C\rvert + \lvert A \cap B \cap C\rvert$$

ISBN 9780170412995

EXAMPLE 9

Consider a group of 26 students in a room. 8 study maths, 11 study science and 13 study technology studies. 3 study both maths and science, 5 study both science and technology and 4 study both maths and technology. One student studies maths, science and technology.

a How many students study either maths or science or technology?

b How many students study none of the three subjects?

Solution

a

Assign sets.	M = {students who study maths} S = {students who study science} T = {students who study technology}
Use the information to assign cardinalities.	$\|M\| = 8$ $\|S\| = 11$ $\|T\| = 13$ $\|M \cap S\| = 3$ $\|S \cap T\| = 5$ $\|M \cap T\| = 4$ $\|M \cap S \cap T\| = 1$
Use the inclusion-exclusion principle.	$\|M \text{ or } S \text{ or } T\|$ $= \|M\| + \|S\| + \|T\| - \|M \cap S\| - \|S \cap T\| -$ $\|M \cap T\| + \|M \cap S \cap T\|$
Substitute for known values.	$= 8 + 11 + 13 - 3 - 5 - 4 + 1$
Evaluate.	$= 21$
State the result.	There are 21 students who study either maths or science or technology.

b There are 26 students altogether and 21 of these study either maths or science or technology.

The remainder do not study any of the subjects.	Number of students not studying these subjects $= 26 - 21$ $= 5$
State the result.	There are 5 students who study none of the three subjects.

ISBN 9780170412995

Exercise 3.03 The inclusion-exclusion principle

Example 7

1 If $M = \{1, 2, 3, \ldots, 100\}$, how many elements are:

a not divisible by 7? **b** not divisible by 6?

2 A drawer contains 34 socks of different colours. There are 6 black socks, 8 white socks, 4 red socks and the rest are multi-coloured. How many socks are not white?

3 There are 61 customers at a fast food restaurant. Of those there, 17 like pizza, 21 like burgers, 26 like fried chicken, 42 like chunky chips and 19 like wraps. How many don't like burgers?

Example 8

4 If $Q = \{1, 2, 3, \ldots, 100\}$, how many elements are not divisible by 2 or 5?

5 In a group of 50 students, 26 like chocolate ice-cream and 34 like strawberry. Each student likes at least one of the two flavours. How many like both chocolate and strawberry ice-cream?

6 At a particular high school there are 200 Year 11 students. 100 of these are studying history and 70 are studying maths. If there are 50 Year 11 students who study both history and maths, how many Year 11 students are studying neither subject?

7 The 60 customers at a takeaway food outlet are asked if they like pizza or burgers. 30 said they liked pizza, 18 said they liked burgers and 7 said they liked both. How many didn't like pizza or burgers?

8 There are 100 people at a board games convention. Of the people present, 39 played Scrabble and 30 played Monopoly. 47 played neither Scrabble nor Monopoly. How many played both Scrabble and Monopoly?

9 A class at a foreign language school has 27 students. Of those in the class, 18 are studying Japanese, 12 are studying Chinese and 4 are not studying Japanese or Chinese.

a How many are studying both languages?

b How many students are studying Japanese but not Chinese?

c Of those studying Chinese, how many are not studying Japanese?

Example 9

10 If $P = \{1, 2, 3, \ldots, 100\}$, how many elements are not divisible by 2, 3 or 5?

11 Consider the sets $P = \{1, 2, 3, 4\}$, $Q = \{1, 3, 5, 6, 7\}$ and $R = \{1, 4, 6, 8, 9\}$. Calculate:

a $|P \cap Q|$ **b** $|P \text{ or } Q|$

c $|P \cap Q \cap R|$ **d** $|P \text{ or } Q \text{ or } R|$

12 A universal set E has 3 subsets G, K and L, where $|E| = 85$, $|G| = 26$, $|K| = 32$ and $|L| = 38$. It is known that $|G \cap K| = 9$, $|K \cap L| = 11$, $|G \cap L| = 8$ and $|G \cap K \cap L| = 3$. Calculate:

a the number of elements of E that are not elements of G or K or L.

b $|G \text{ or } K|$ **c** $|K \text{ or } L|$ **d** $|\overline{L}|$

ISBN 9780170412995

13 A group of 94 people at a supermarket were interviewed and asked about their online viewing preferences. Of those interviewed, 37 said they watched Netflix, 27 said they watched Stan and 17 said they watched Foxtel. There were 5 who watched both Netflix and Stan, 7 who watched Netflix and Foxtel and 8 who watched Foxtel and Stan. Two people said they watched all three services. Calculate how many of the people interviewed:

a did not watch any of the three services

b watched only Foxtel

c watched either Netflix or Stan

d watched either Stan or Foxtel

Problem solving

14 Find the number of positive integers not exceeding 100 that are either odd or a square number.

15 Consider the universal set, E = {Integers from 1 to 50 inclusive}. Subsets of E are defined as follows:

N = {multiples of 9}

S = {square numbers}

T = {triangular numbers}

(Triangular numbers form the sequence 3, 6, 10, 15, ….)

Calculate:

a $|N \cap S|$ **b** $|N \text{ or } T|$ **c** $|\overline{S}|$ **d** $|N \cap S \cap T|$

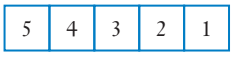
3.04 Permutations and factorial notation

You saw how the multiplication principle worked at the beginning of this chapter. Using the multiplication principle, you saw that the letters MATHS could be arranged in many ways.

If the letters MATHS are arranged without repetition, the number of different arrangements is found as shown below.

Draw up a line of 5 boxes – one for each letter in the arrangement.

The first box can be filled in 5 different ways, the second in 4 different ways, and so on.



Now use the multiplication principle.

Number of arrangements = $5 \times 4 \times 3 \times 2 \times 1 = 120$

Each of these arrangements, e.g., MATHS, THAMS, SMATH, etc. is called a **permutation**.

ISBN 9780170412995

Permutation

A **permutation** of a group of objects is an arrangement of some or all of the objects in a particular order.

It is normal to assume that permutations of objects are performed without repetition. If repetition of the same object is to be permitted, this should be stated explicitly.

Calculating the number of permutations of objects often involves extended multiplication such as $5 \times 4 \times 3 \times 2 \times 1$, as shown in the 'MATHS' example above.

This type of extended multiplication is so common that mathematicians give it a special name. $5 \times 4 \times 3 \times 2 \times 1$ is called '5 **factorial**' and is written as 5!.

Factorial notation

Factorial notation

For integers $n > 0$, n factorial (or factorial n) is symbolised $n!$ and defined as

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

You have seen that the number of permutations of 5 things is 5!. You can now use factorial notation to state the general rule for permutations of n things.

Permutation notation

The number of permutations of n different objects is given by

$$^nP_n = n!$$

There are many different ways of writing ^nP_n, Other texts may use alternative notations:

$$^nP_n = {}_nP_n = P(n, n) = P_n^n$$

EXAMPLE 10

In how many different orders can David, Hannah, Georgie, Farah, Marc and Cyndi queue for tickets to the movies?

Solution

There are six people and six positions, so there are 6! permutations.

Write using permutation notation	$^6P_6 = 6!$
Use the definition of n!.	$= 6 \times 5 \times 4 \times 3 \times 2 \times 1$
Evaluate.	$= 720$

ISBN 9780170412995

Write the result. There are 720 different orders.

You can use a graphics calculator to evaluate factorials.

TI-84 Plus CE

To calculate 6!, enter 6 and then press [math] and select PROB (Probability) and then 4: ! (Factorial). Press [enter] to display the result.

TI-Nspire CX Chapter 3

Casio fx-CG20AU

To calculate 6!, enter 6 and then press [OPTN], [F6], [F3] (PROB) and [F1] ($x!$). Press [EXE] to display the result.

As you saw in previous sections of this chapter, you are not always interested in calculating the number of arrangements of *all* objects under consideration. Sometimes, only the arrangements of some of the available objects is required.

Consider the situation where you need to calculate the number of permutations (different arrangements) of any three letters from the word FRACTION. You can use the multiplication principle. This time, 3 letters are to be chosen from a total of 8.

Draw up a line of three boxes – one for each letter in the arrangement.

The first box can be filled in 8 different ways, the second in 7 different ways and the third in 6.

Now use the multiply using the multiplication principle.

Number of arrangements $= 8 \times 7 \times 6$

You should notice that: $8 \times 7 \times 6 = \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$

Using factorial notation: $8 \times 7 \times 6 = \dfrac{8!}{5!} = \dfrac{8!}{(8-3)!}$

This result represents the number of permutations of 3 objects selected from 8 objects.

This result can be generalised to the following rule for calculating permutations.

Expression for ${}^nP_r$

The number of permutations of r objects selected from n objects is written as ${}^nP_r$.

$$ {}^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) $$

Note that the final form of the expression has r factors.

W5 Permutation calculations

ISBN 9780170412995

The definition of ${}^nP_r$ can now be used to establish a number of useful results.

Using the definition, ${}^nP_n$ must denote the ways of filling n places with n objects at our disposal.

So ${}^nP_n = n(n-1)(n-2)\cdots 3 \times 2 \times 1 = n!$

Also, if ${}^nP_r = \dfrac{n!}{(n-r)!}$

Then ${}^nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!}$

But ${}^nP_n = n!$

So $n! = \dfrac{n!}{0!}$

For this to be true, we must define 0! to be 1.

$$ {}^nP_n = n! $$
$$ 0! = 1 $$

EXAMPLE 11

a Evaluate $\dfrac{8!}{5!}$.

b Evaluate $\dfrac{(n-2)!}{(n-4)!}$.

c Find n if $7 \times {}^nP_3 = 6 \times {}^{n+1}P_3$.

Solution

a Write in expanded form. $\dfrac{8!}{5!} = \dfrac{8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1}$

Cancel. $= 8 \times 7 \times 6$

Evaluate. $= 336$

b Write in expanded form. $\dfrac{(n-2)!}{(n-4)!} = \dfrac{(n-2)(n-3)(n-4)(n-5)(n-6)\times\cdots\times3\times2\times1}{(n-4)(n-5)(n-6)\times\cdots\times3\times2\times1}$

Cancel. $= (n-2)(n-3)$

ISBN 9780170412995

c Write the equation. $7 \times {}^nP_3 = 6 \times {}^{n+1}P_3$

Apply the formula for ${}^nP_r$. $7n(n-1)(n-2) = 6(n+1)n(n-1)$

Gather terms on the LHS. $7n(n-1)(n-2) - 6(n+1)n(n-1) = 0$

Factorise. $[7(n-2) - 6(n+1)]n(n-1) = 0$

Simplify. $(7n - 14 - 6n - 6)n(n-1) = 0$
$(n-20)n(n-1) = 0$

Solve. $n = 0, 1$ or 20

$n = 0$ and $n = 1$ are not defined for ${}^nP_3$ and ${}^{n+1}P_3$.

Write the result. $n = 20$

EXAMPLE 12

A horserace has 12 runners. In how many ways can the first 3 places be filled?

Solution

There are 12 horses and 3 different places that can be filled.

Write using permutation notation. ${}^{12}P_3 = 12 \times 11 \times 10$

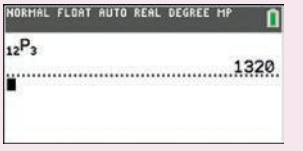
Evaluate. $= 1320$

Write the result. There are 1320 ways of filling the first three places.

You can use a graphics calculator to evaluate permutations.

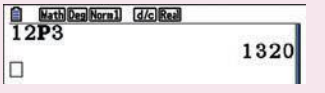
TI-84 Plus CE

To calculate ${}^{12}P_3$, type 12, press [math] and select PROB (Probability) and then 2: nPr (Permutations). Type 3 and press [enter] to display the result.



Casio fx-CG20AU

To calculate ${}^{12}P_3$, enter 12 and then press [OPTN] OPTN, [F6], [F3] (PROB) and [F2] (*nPr*). Enter 3 and press [EXE] to display the result.



ISBN 9780170412995

Exercise 3.04 Permutations and factorial notation

1 Write each factorial in expanded form.

a 4! **b** 6! **c** 5! **d** 8! **e** 7!

2 Evaluate each factorial.

a 3! **b** 4! **c** 6! **d** 8! **e** 10!

3 Evaluate each permutation.

a ${}^3P_3$ **b** ${}^5P_5$ **c** ${}^6P_6$ **d** ${}^4P_4$ **e** ${}^8P_8$

Example 10

4 Calculate each permutation.

a The number of ways you can arrange the letters from the word CODES.

b The number of different ways it is possible for 7 students to arrange themselves among 7 seats on the stage at a school assembly.

c The number of different passwords possible if it has 4 letters selected from 'ABCDEF' and no repeated letters.

d The number of different orders of finish for 8 greyhounds in a race, assuming that there are no dead heats.

e The number of 4-letter passwords that can be formed out of the letters of HOSE, where repetition of the letters is not allowed.

5 4 cars enter a carpark and 4 parking spaces are available. In how many ways can the cars occupy the parking spaces?

6 In how many ways can the letters of the word FIGURES be arranged?

Example 11

7 Evaluate each expression.

a $\frac{8!}{3!}$ **b** $\frac{9!}{4!}$ **c** $\frac{10!}{7!}$ **d** $\frac{7!}{6!}$ **e** $\frac{5!}{0!}$

8 Simplify each expression.

a $\frac{n!}{(n-4)!}$ **b** $\frac{(n+4)!}{(n+2)!}$ **c** $\frac{(n-4)!}{n!}$ **d** $\frac{(n-2)!}{(n-6)!}$ **e** $\frac{(n-3)!}{(n+3)!}$

9 Calculate each value.

a ${}^3P_2$ **b** ${}^5P_3$ **c** ${}^7P_2$ **d** ${}^9P_8$ **e** ${}^8P_4$

f ${}^5P_5$ **g** ${}^{11}P_2$ **h** ${}^{10}P_5$ **i** ${}^{85}P_3$ **j** ${}^{14}P_5$

10 Find n if:

a ${}^nP_4 = 56 \times {}^nP_2$ **b** $12 \times {}^nP_3 = {}^{3n}P_2$ **c** $8 \times {}^nP_2 = 7 \times {}^{n+1}P_2$

d $9 \times n + {}^1P_3 = 10 \times {}^nP_3$ **e** $n + {}^1P_4 = 18 \times {}^nP_2$ **f** $56 \times {}^nP_2 = {}^nP_4$

ISBN 9780170412995

Example 12

11 How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 if:

a repetition of the digits is not allowed?

b repetition of the digits is allowed?

12 How many 4-letter codes can be formed using the first 10 letters of the alphabet? (No letter can be repeated.)

13 How many numbers between 100 and 1000 use only odd digits, no digit being repeated?

14 A code is picked using 4 out of 10 different colours. How many codes are possible, if no colour can be used twice?

15 There are 14 athletes in a 1500 m race. In how many ways can the first 3 places be filled?

Problem solving

16 How many 7-digit numbers can be formed from the digits 0, 2, 4, 5, 7, 8, 9 that are divisible by 10 if no digit is repeated?

17 7 beds are arranged in a line in a long narrow room in a hostel. There are 7 backpackers who want to share the room. Among them, Lucas does not want a bed next to Riley because Riley snores. In how many ways can the hostel manager assign the beds?

3.05 Permutations with restrictions

WS Permutations with repetitions

WS Permutations

WS Circular calculations

So far in this chapter you have looked at permutations of objects that are different. But what if some of the objects are the same and cannot be distinguished from each other?

Consider the permutation of the 10 letters: A, B, C, D, E, E, E, F, G and H. Imagine that you could distinguish between the three Es by adding subscripts. Under this circumstance, there would be 10 different symbols: A, B, C, D, E_1, E_2, E_3, F, G and H. There are 10! different permutations of these symbols.

However, if the Es cannot be distinguished, some of the 10! permutations are actually the same. For example, the permutations $FABE_1CE_3DGHE_2$ and $FABE_2CE_1DGHE_3$ are actually the same, i.e., FABECEDGHE. Because there are 3 Es, the permutation FABECEDGHE will contain 3! arrangements of the Es when they are distinguished by subscripts. This will be the case for every other permutation of the 10 letters.

Because the Es are indistinguishable, the true number of permutations is $\frac{10!}{3!}$.

This can be generalised to the following formula.

ISBN 9780170412995

Permutations of objects, some alike

The number of permutations of n objects, of which a are alike, is given by $\frac{n!}{a!}$.

If there are another b, c and so on alike, the number of permutations is given by $\frac{n!}{a!b!c!\cdots}$.

EXAMPLE 13

In how many ways can the letters of the word GREENSLEEVES be arranged?

Solution

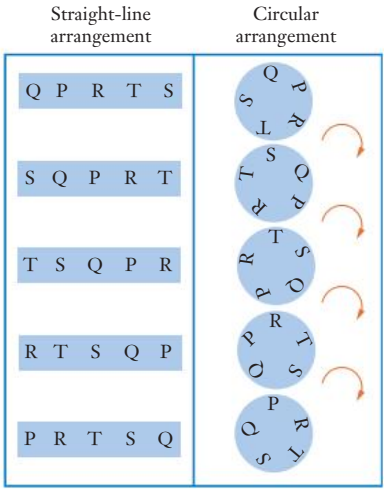
There are 12 letters altogether, but there are 5 Es and 2 Ss.

Use the formula. $\text{Number of arrangements} = \frac{12!}{5! \times 2!}$

Evaluate. $= 1\,995\,840$

State the result. There are 1 995 840 different ways to arrange the letters.

In the case of arrangements in a circle, some of the arrangements that would be different in a straight line are the same when the circle is rotated. This is shown in the diagram for a circular arrangement of the letters P, Q, R, S and T.



Notice that for a permutation with 5 objects arranged in a circle, there are 5 positions that are the same. We will refer to permutations like this one as circular permutations.

Thus, in this case, the total number of circular permutations is

$$5! \div 5 = 4!$$

This leads to the general rule for circular permutations.

Circular permutations

The number of circular permutations of n different objects is given by $(n - 1)!$.

ISBN 9780170412995

EXAMPLE 14

In a barn dance, the boys are arranged in a circle on the outside. If there are 12 boys in one circle, how many different possible arrangements are there?

Solution

In this case, $n = 12$. Use the rule for circular arrangements.

$$\text{Number of circular arrangements} = (12 - 1)!$$
$$= 11!$$

Evaluate.

$$= 39\,916\,800$$

State the result.

There are 39 916 800 arrangements.

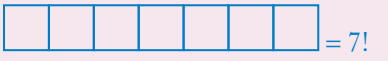
In some cases you may be interested in arrangements where there is a requirement to treat a number of objects as a block that remains together.

EXAMPLE 15

7 people, including 3 friends, are randomly assigned seats in one row at a concert. In how many ways can the 7 people be arranged if the 3 friends must be seated together?

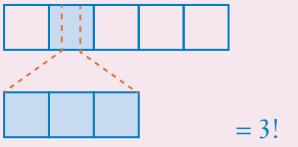
Solution

Without restrictions, there would be 7 elements to arrange.



With the restriction, we can consider the 3 friends as a single block in the arrangements with the other 4 people. There are 5 elements in this case.

Within the block of 3 friends, there are 3! arrangements.



There are 5! ways of arranging 4 people and the block of 3 friends. There are 3! ways of arranging the friends within the block of 3 friends. Use the multiplication rule.

$$5! \times 3! = 120 \times 6$$
$$= 720$$

State the result.

There are 720 arrangements of 7 people where 3 friends sit together.

ISBN 9780170412995

Exercise 3.05 Permutations with restrictions

Example 13

1 How many different arrangements of the word IMMUNE are possible?

2 How many different arrangements of the word MATHEMATICS are possible?

3 There are 7 discs that are identical except for their colour. 4 of the discs are blue and 3 are red. How many different arrangements of the 7 discs are possible?

4 How many different arrangements of the word VISIBILITY are possible?

5 There are 12 discs of equal size and shape. 3 of the discs are white, 5 are black and 4 are green. How many different arrangements of the 12 discs are possible?

6 How many different arrangements of the word SLEEPLESSNESS are possible?

Example 14

7 A merry-go-round proprietor has 10 different 'horses' to place on the outer edge of the merry-go-round. How many different placements can be made?

8 In how many ways can 4 men and 2 women be seated at a round table?

iStock/Christian Mueller

ISBN 9780170412995

9 Five people, Archie, Ben, Chloe, Delta and Ethan are to be seated around a circular table. In how many ways can they be seated if:

- **a** Archie and Ben must sit next to each other?
- **b** Archie and Ben must not sit next to each other?
- **c** Archie and Ben must be together and Chloe and Delta must be together?

10 In how many ways can you arrange 6 boys and 6 girls around a circular table if:

- **a** there are no restrictions?
- **b** boys and girls must alternate?

11 6 men and 6 women are arranged around a circular table at a dinner party. In how many ways can you arrange the people at the dinner party if:

- **a** Levi and Lucy must sit together?
- **b** Levi, Lucy and Layla must sit together?

12 Example 15

- **a** How many permutations of the letters in the word MODULES are there?
- **b** In how many of these do the letters O and U appear side by side?
- **c** In how many permutations do the letters O and U appear apart?

13 Consider the permutations of the letters of the word WANDER.

- **a** How many begin with WA?
- **b** How many do not begin with WA?

14 How many odd numbers of 4 digits can be formed with the digits 1, 4, 5, 7 if repetitions are allowed?

15 Consider the arrangements of the letters in the word EQUATION. How many different arrangements can be formed using all the letters so that vowels and consonants occur together?

Problem solving

16 How many ways are there to arrange 5 boys and 5 girls around a table such that no 2 boys are next to each other?

17 10 books are to be arranged on a shelf. 3 of the books must be together (in any order) because they form a trilogy, and another 2 must be together in the correct order (volume 1, then volume 2). In how many ways can the books be arranged?

18 The flags of a number of countries consist of 3 vertical stripes in different colours. For example, France's flag, from left-to-right, is blue, white and red. If the colours red, blue, green, white, yellow and black may be used, how many flags with 3 vertical stripes are possible, given that:

- **a** the 3 colours must be different?
- **b** the centre colour must be different from the end colour(s)?

ISBN 9780170412995

3.06 Applications of permutations

The probability of an event is calculated by dividing the number of ways it can occur by the total number of possibilities (called the **sample space**). Each element of the sample space is called a **sample point**. We assume that the sample points are all equally likely.

In formal terms, the probability of an event is defined as follows.

Probability of an event

The probability of an event that can occur in n ways is given by

$$P(\text{event}) = \frac{n(\text{event})}{n(\text{sample space})}$$

where n(sample space) is the number of occurrences in the sample space.

The closer the probability of an event is to 1, the more likely the event is to happen. The closer the probability is to 0, the less likely is the event. If P(event) = 1, then the event is a certain event and if P(event) = 0, then the event is impossible.

Counting techniques, such as the multiplication principle and the addition principle, are an essential part of calculating probability.

EXAMPLE 16

The first 8 letters of the alphabet are placed in a hat and 4 letters are drawn at random, forming a 4-letter code. Calculate the probability that the code:

a begins with the letter A

b begins with the letter A or ends with the letter H

Solution

a For any code, there are 8 ways of filling the first place, 7 ways of filling the second place, and so on.

8	7	6	5

Calculate the total number of codes (sample space).

$n(\text{all codes}) = 8 \times 7 \times 6 \times 5$
$= 1680$

If a code begins with A, there is only one way of filling the first place. Because the letter A has been used, there are 7 ways of filling the second place, and so on.

1	7	6	5

Calculate the number of codes that begin with A.

$n(\text{code beginning with A})$
$= 1 \times 7 \times 6 \times 5$
$= 210$

ISBN 9780170412995

Calculate the probability of a code beginning with A.

$$P(\text{code beginning with A}) = \frac{n(\text{code beginning with A})}{n(\text{all codes})}$$

Substitute for known values.

$$= \frac{210}{1680}$$

Evaluate.

$$= 0.125$$

State the result.

The probability that a code begins with A is 0.125 or 12.5%.

b If a code ends with H, there is only one way of filling the last place. Because the letter H has been used, there are 7 ways of filling the first place and so on.

7	6	5	1

Calculate the number of codes that end in H.

$$n(\text{code ending with H}) = 7 \times 6 \times 5 \times 1 = 210$$

Some codes begin with A *and* end in H. In these cases, there is only 1 way to fill the first place and 1 way to fill the last place. This leaves 6 ways to fill the second place and 5 ways to fill the third place.

1	6	5	1

Calculate the number of codes that begin with A *and* end in H.

$$n(\text{code beginning with A and ending with H}) = 1 \times 6 \times 5 \times 1 = 30$$

Assign symbols to the events.

Let X = code begins with A, Y = code ends with H and Z = all codes

Calculate the number of codes that begin with A or end in H, i.e., X or Y.

$$n(X \text{ or } Y) = n(X) + n(Y) - n(X \text{ and } Y) = 210 + 210 - 30 = 390$$

n(code begins with A) = 210 from part **a**.

Calculate the probability that a code begins with A or ends in H.

$$P(X \text{ or } Y) = \frac{n(X \text{ or } Y)}{n(Z)}$$

n(all codes) = 1680 from part **a**.

$$= \frac{390}{1680}$$

$$= 0.2321\ldots$$

Round off and state the result.

The probability that a code will begin with A or end with H is about 0.23 or 23%.

EXAMPLE 17

In horseracing, a quadrella is a bet on the first 4 places in the correct order. A particular race has 12 starters that are judged to be equally likely to win the race. Assuming there are no dead heats, what is the probability of successfully picking the quadrella with a single selection?

Solution

Calculate the total number of different ways the horses can finish the race.

$$n(\text{quadrellas}) = {}^{12}P_4$$
$$= 12 \times 11 \times 10 \times 9$$
$$= 11\ 880$$

Find the probability of picking the quadrella.

$$P(\text{quadrella}) = \frac{1}{11880}$$
$$= 0.000\ 084\ 17\ldots$$

Round off.

$$\approx 0.000\ 084 \text{ or } 0.0084\%$$

State the result.

The probability of picking the quadrella is about 0.0084%.

When doing calculations that involve multiple steps, it is often easier to do all the calculations at the end.

EXAMPLE 18

5 couples are seated in a row of 10 seats in the front row in a theatre. If the couples are seated randomly, what is the probability that all 5 couples will be sitting next to their partners?

Solution

Calculate the number of different arrangements without restrictions.

$$\text{Number of arrangements} = {}^{10}P_{10}$$
$$= 10!$$

Calculate the number of arrangements of the 5 couples.

$$\text{Arrangements of couples} = {}^{5}P_5$$
$$= 5!$$

Calculate the number of arrangements for each couple.

$$\text{Arrangements of a couple} = {}^{2}P_2$$
$$= 2!$$

ISBN 9780170412995

Use the multiplication principle.	Possible arrangements with couples together $= 5! \times 2! \times 2! \times 2! \times 2! \times 2!$
Simplify using $2! = 2 \times 1 = 2$.	$= 5! \times 2 \times 2 \times 2 \times 2 \times 2$
Calculate the probability.	$P(\text{couples are together}) = \frac{5! \times 2 \times 2 \times 2 \times 2 \times 2}{10!}$
Expand.	$= \frac{5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 2 \times 2 \times 2 \times 2}{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$
Cancel.	$= \frac{1}{5 \times 9 \times 1 \times 7 \times 3}$
Evaluate.	$= \frac{1}{945}$ $= 0.001\,058\ldots$
Round off.	≈ 0.001 or 0.1%
State the result.	The probability of 5 couples sitting together is about 0.1%.

Exercise 3.06 Applications of permutations

1 A hand of 5 cards is dealt from a standard deck. What is the probability that it:

Example 16

- **a** contains 4 queens?
- **b** is a flush (all cards of the same suit)?

2 There are 8 horses in a race. A trifecta is the first, second and third place finishers in that order. What is the probability of picking the trifecta if there are no dead heats?

Example 17

3 There are 10 contestants in an archery competition. 4 of the contestants are women. Prizes are awarded to the top 4 competitors. What is the probability that at least one woman finishes in the first 4 places?

Example 18

Problem solving

4 A coin is tossed 6 times. Find:

- **a** the number of ways it can fall
- **b** the number of ways you could get 2 heads, by finding the number of permutations of HHTTTT
- **c** the probability of obtaining exactly 2 heads from 6 tosses.

ISBN 9780170412995

5 Find the probability of obtaining exactly:

- **a** 3 heads from 5 tosses of a coin
- **b** 2 heads from 8 tosses of a coin
- **c** 6 heads from 8 tosses of a coin
- **d** 2 heads from 4 tosses of a coin
- **e** 10 heads from 20 tosses of a coin.

6 One of the possible arrangements of the digits 5, 7, 8 and 9 is selected randomly.

- **a** What is the probability that the selected arrangement is greater than 6000?
- **b** What is the probability that the selected arrangement is divisible by 5 and greater than 6000?

7 A three-digit number is randomly formed using the digits 2, 3, 5, 6, 7 and 9. Calculate the probability that the number is:

a less than 600 **b** even **c** odd **d** multiple of 5

8 A single arrangement of the letters of the word FASHION is selected randomly. Calculate the probability that:

- **a** the vowels are all together
- **b** the vowels are not all together
- **c** the letter N is the last letter

9 The digits 1, 2, 3, 4 and 5 are arranged randomly. What is the probability that the arrangement is a number greater than 50 000?

10 3 adults and 5 children are seated randomly in a row. What is the probability that the 3 adults are seated together and the 5 children are also seated together?

11 **a** Consider the letters of the word STUMBLE. How many different arrangements of the letters are possible?

- **b** If one of the arrangements is selected at random, what is the probability that:
  - **i** the vowels are together?
  - **ii** the last letter is not a vowel?
  - **iii** the selected arrangement will start with M and end with L?

12 The letters of the word EYELET are randomly arranged. What is the probability that the arrangement:

- **a** begins and ends with an E?
- **b** has the 3 Es together?
- **c** begins with E and ends with T?

ISBN 9780170412995

13 Six students, Ava, Ben, Charlotte, Dan, Emma and Flynn are randomly seated in a row. What is the probability that:

- **a** Ava and Ben sit beside each other?
- **b** Ava and Ben do not sit beside each other?
- **c** Dan, Emma and Flynn sit beside each other?
- **d** Ava and Flynn sit at each end of the row?

14 6 women and 6 men attend a dinner at a restaurant that has round dining tables. If the diners are seated around the table randomly, what is the probability that no 2 women sit together?

15 9 discs of equal shape and size consist of 4 blue and 5 red discs. The 9 discs are randomly arranged in a row. Calculate the probability that discs are placed so that the colours alternate.

16 A competition involves 20 competitors and has 3 prizes. What is the probability that a randomly selected competitor wins a prize:

- **a** if no competitor can win more than one prize?
- **b** if any competitor can win all the prizes?
- **c** if no competitor can win more than 2 prizes?

17 A photographer needs to photograph 10 people. The people are arranged in 2 rows of 5, one in front of the other. What is the probability that 2 particular people:

- **a** are in the front row?
- **b** are in different rows?

ISBN 9780170412995

3. CHAPTER SUMMARY

Permutations

- A **counting technique** is a method of determining how many objects are in a collection without actually listing out all the objects in the collection, such as a **tree diagram**.
- The **multiplication principle** states that if one operation can be performed in n ways, followed by another operation that can be performed in m ways, then the combined operation can be performed in $n \times m$ ways. More generally, if operations $O_1, O_2, O_3, \ldots, O_n$ are performed in order, with the possible number of outcomes $n_1, n_2, n_3, \ldots, n_n$, respectively, then there are $n_1 \times n_2 \times n_3 \times \ldots \times n_n$, possible outcomes of the operations performed in the given order.
- **Mutually exclusive** events are completely separate.
- The **addition principle** states that if two mutually exclusive operations can be performed in n and m ways respectively, then the number of ways of performing one or the other is $n + m$. More generally, for mutually exclusive events, the total number of choices is the sum of the choices for each of the events.
- **Inclusion-exclusion principle**: For 2 sets A and B: $|A \text{ or } B| = |A| + |B| - |A \cap B|$

  For 3 sets A, B and C:

$$|A \text{ or } B \text{ or } C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

- A **permutation** of a group of objects is an arrangement of some or all of the objects in a particular order.
- **Factorial notation:** For integers $n > 0$, n factorial (or factorial n) is symbolised $n!$ and defined as $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$
- The number of permutations of n different objects is given by ${}^nP_n = n!$
- The number of permutations of r objects selected from n objects is written as ${}^nP_r$.

$${}^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$$

- The number of permutations of n objects, of which a are alike, is given by $\frac{n!}{a!}$. If there are another b, c and so on alike, the number of permutations is given by $\frac{n!}{a!b!c!\cdots}$
- The number of **circular permutations** of n different objects is given by $(n-1)!$.
- The probability of an event that can occur in n ways is given by

$$P(\text{event}) = \frac{n(\text{event})}{n(\text{sample space})}$$

- $P(\text{certain event}) = 1$ and $P(\text{impossible event}) = 0$.

ISBN 9780170412995

3. CHAPTER REVIEW

Permutations

1 Max has 5 reggae CDs and 4 heavy metal CDs. Example 1-4

a How many different selections of 1 CD can be made?

b A reggae CD is chosen and then a heavy metal CD is selected. How many different ways can this happen?

c A CD rack has 3 spare slots. In how many ways can Max fill the 3 slots with CDs from the same music style?

d In how many ways can Max fill the 3 slots if any CD may be used?

2 A bag contains marbles that are the same shape and size. 4 are red and 6 are blue. In how many ways can you choose a pair of marbles that are the same colour? Example 5

3 Calculate how many 'words' can be made from the letters of the word PERMUTATION if: Example 6

a 4-letter words are to be made, with no letter repeated.

b 5-letter words are to be made, without repetition.

c 5-letter words including exactly one vowel are to be made, without repetition.

d 5-letter words including exactly 2 vowels are to be made, without repetition.

e 5-letter words including exactly 3 vowels are to be made, without repetition.

f 5-letter words including at least one vowel are to be made, without repetition.

4 If $S = \{1, 2, 3, \ldots, 100\}$, how many elements are: Example 7,8

a divisible by 7?

b not divisible by 7?

c divisible by 2 or 3?

5 Consider a group of 30 students. 15 study biology, 15 study chemistry and 13 study art. 9 study both biology and chemistry, 6 study both chemistry and art and 5 study both biology and art. 3 students study biology, chemistry and art. Example 9

a How many students study either biology or chemistry or art?

b How many students study none of the three subjects?

6 In how many ways can 7 people be arranged in: Example 10, 14

a a row? **b** a circle?

7 Calculate: Example 11

a ${}^{7}P_{3}$ **b** ${}^{12}P_{2}$ **c** ${}^{20}P_{4}$

ISBN 9780170412995

Example 11

8 **a** Evaluate $\frac{12!}{9!}$.

b Evaluate $\frac{(n-3)!}{(n-1)!}$.

c Find n if $9 \times {}^{n-1}P_3 = {}^{n}P_4$.

Example 11

9 5 people, A, B, C, D, and E are randomly seated on a bench. Calculate the probability that for a selected arrangement:

a A and B are seated together

b A and B are not seated together

c A and C are seated together and B and E are seated together also.

Example 12

10 An Olympic race has 8 finalists competing for gold, silver and bronze medals for 1st, 2nd and 3rd places. In how many different ways can the medals be won?

Example 12

11 A bookshelf has room for 5 more books. Ruchi has 8 books that may be placed on the shelf. In how many ways can the books be placed?

Example 13

12 How many different permutations can be made from the letters of the word BEGINNING?

Example 15

13 How many different 5-digit numbers can be made by arranging the digits of 37273?

Example 10

14 How many anagrams of the name QMATHS are possible?

Example 15, 16

15 A bag contains 3 black jelly beans, 5 red ones, 2 green ones and 4 yellow ones.

a In how many ways can 2 jelly beans be removed from the bag?

b What is the probability that 2 jelly beans taken at random will both be black?

c What is the probability that 2 jelly beans taken at random will be of different colours?

Example 16

16 Consider the letters in the name PATRICK.

a How many different 4-letter codes can be formed?

b What is the probability that one of these randomly selected codes:

i starts with R?

ii starts with vowels in the first 2 positions?

iii has vowels and consonants alternating?

iv contains only consonants?

Example 17

17 In a 10-horse race, what is the probability of selecting the trifecta (the correct placings of the first 3 horses)?

ISBN 9780170412995

18 3 couples decide to go to a concert together, so one person buys 6 tickets. If the tickets are for adjacent seats, find the probability that, if the tickets are given out at random, each person will be seated with his or her partner.

Example 18

Problem solving

19 A fund-raising competition at a school challenged students to match 10 baby photos to 10 teachers. It cost 20 cents to enter the competition. How much would it cost to put in every possible entry?

20 A toy train had an engine, a flat car, a box car, a passenger car, a baggage car and a little red caboose. The cars were arranged in different orders but the little caboose always came last! If the engine always came first, how many orders were possible?

21 The 25 students in a maths class decide to sit in a different arrangement every day. There are 30 desks in the room. Find the number of years it would take for the students to exhaust all the possible arrangements, assuming that there are 200 school days each year.

22 Find the probability of flipping exactly 6 heads from 10 tosses of a coin.

23 In a game of Ludo, Elena rolled a 6 on 4 successive throws of the die. What is the probability of this occurrence?

Qz Practice quiz

ISBN 9780170412995

Practice examination 1 ●○○○

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (2 marks)

Convert:

a $(5, 210°)$ to component form

b $(-4, 7)$ to polar form

Question 2 (1 marks)

Find the norm of $(2.6, 3.5)$.

Question 3 (2 marks)

Sketch each vector.

a $(3, -5)$

b $(6, 155°)$

ISBN 9780170412995

Question 4 (2 marks)

Given $\mathbf{a} = (-3, 5)$, $\mathbf{b} = (2, 7)$, $\mathbf{c} = (1, 9)$ and $\mathbf{d} = (8, -2)$, find each expression.

a $6\mathbf{d} - 4\mathbf{b}$ **b** $5\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$

Question 5 (2 marks)

Given $\mathbf{u} = (30, 36°)$, $\mathbf{v} = (25, 152°)$, $\mathbf{w} = (18, 105°)$ and $\mathbf{x} = (27, 216°)$, find each expression without changing to component form:

a $\mathbf{u} + \mathbf{v}$ **b** $\mathbf{w} - \mathbf{x}$

Question 6 (1 mark)

Find the displacement vector for the movement $A(8, -4)$ to $B(3, 4)$.

Question 7 (2 marks)

A force of 150 N is exerted at an angle of 20° downwards from the horizontal on a sledge lying on horizontal ground. Resolve the force into horizontal and vertical components.

Question 8 (2 marks)

Show that $x^2 + 8 > 6x$ is not true.

Question 9 (2 marks)

What is the converse of 'numbers divisible by 4 are even'?

Question 10 (1 mark)

What is the negation of '$a \in Z \Rightarrow \exists\, b \in Z$ such that $b - 2a = 0$?

Question 11 (1 mark)

What is the contrapositive of 'parallelograms have supplementary co-interior angles'?

Question 12 (3 marks)

Show that the sum of 2 odd integers is even.

ISBN 9780170412995

Question 13 (1 mark)

There are 5 main roads by which motorists can travel between Brisbane City and Ipswich. How many possible ways are there to make a round trip from Brisbane to Ipswich?

Question 14 (5 marks)

How many 'words' can be made from the letters of the word MATHEMATICS using different permutations, according to each of the following rules?

a 4-letter words are to be made, without repetition.

b 5-letter words are to be made, without repetition.

c 5-letter words with one or two vowels are to be made, without repetition.

d 4-letter words containing the letter M are to be made, without repetition.

e 5-letter words containing the letter M are to be made, without repetition.

Question 15 (2 marks)

How many different numbers of 4 digits can be formed with the digits 1, 4, 6, 8:

a if no digit occurs more than once in any number?

b if there are no restrictions?

Question 16 (1 mark)

A multiple-choice test consists of 10 items, each with 4 choices for the answers. Only one answer is correct in each case. In how many ways can the test be answered?

Question 17 (2 marks)

Find the speed and course made good for a yacht sailing at 10 knots at 065° with a current of 2 knots at 215°.

Question 18 (2 marks)

Find the heading required to fly a plane with an airspeed of 105 knots on a true course of 225° with a south-westerly wind of 10 knots. Determine the ground speed.

Question 19 (3 marks)

Show that if $x - y$ is irrational, then at least one of x, y is irrational.

ISBN 9780170412995

Question 20 (3 marks)

Three of the letters E, F, G, H and I are selected randomly to form a code. What is the probability that a selected code starts with a consonant and the second letter is a vowel?

Question 21 (4 marks)

A boat on a trailer is being pulled up the beach before hitching it to a car off the sand. The slope of the sand is 5° and the person at the front is pulling upwards with a force of 400 N at an angle of 15° to the horizontal. Another person is pushing horizontally at the back with a force of 350 N. What is the total force moving the boat and trailer up the beach?

Question 22 (3 marks)

Prove that $\sqrt{6}$ is irrational.

Question 23 (1 mark)

Amy and Tristan invite 6 guests to a dinner party. If the people are randomly seated around a circular table, what is the probability that Amy and Tristan sit together?

Question 24 (2 marks)

In a meeting, everyone has shaken hands with everyone else. It is found that a total of 66 handshakes were exchanged. How many people were present at the meeting?

END OF EXAMINATION

ISBN 9780170412995

4.

VECTOR ALGEBRA

You were introduced to the concept of vectors in Chapter 1 and looked at the differences between scalar and vector quantities. You have also seen that vectors can be expressed in both polar and component forms.

In this chapter, you will discover various operations on vectors and the algebraic and geometric properties of vectors.

4.01 Geometric use of vectors
4.02 The scalar product
4.03 Parallel and perpendicular vectors
4.04 Projections of vectors
4.05 Applications of vectors
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Algebra of vectors in the plane

- define and use a vector representing the midpoint of a line segment
- define and use scalar (dot) product
- apply the scalar product to vectors expressed in component form
- examine properties of parallel and perpendicular vectors and determine if two vectors are parallel or perpendicular
- define and use projections of vectors
- solve problems involving displacement, force, velocity, equilibrium, and relative velocity involving the above concepts

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

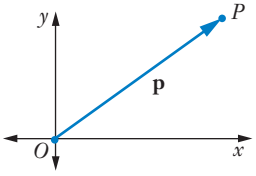
iStock/seventotheofe

TERMINOLOGY

commutative property
dot product
force
inner product
joule
midpoint theorem
newton
orthogonal
projection of (**a** on **b**)
scalar product
vector projection (of **a** on **b**)
work

4.01 Geometric use of vectors

There are many situations where vectors can be applied. For example, in geometry we are often interested in the position of one point in relation to another.



When dealing with the Cartesian plane, it is convenient to relate the position of a point to the origin O. The position of any point P with respect to the origin can be described by the position vector **OP** or **p**.

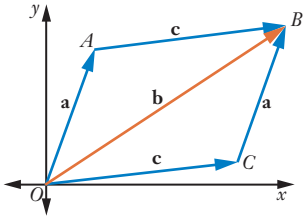
Consider the parallelogram $OABC$ shown: **a** is the position vector of A, **b** is the position vector of B and **c** is the position vector of C.

You have previously seen that vectors that have the same magnitude and direction are equal. So for parallelogram $OABC$ you know that:

$$OA = CB \text{ and } OA \parallel CB$$

and

$$OC = AB \text{ and } OC \parallel AB$$



In terms of vectors, this means that:

$$\mathbf{OA} = \mathbf{CB} = \mathbf{a}$$

and

$$\mathbf{OC} = \mathbf{AB} = \mathbf{c}$$

You have also added vectors using the parallelogram rule, so:

$$\mathbf{OA} + \mathbf{OC} = \mathbf{OB} \text{ or } \mathbf{a} + \mathbf{c} = \mathbf{b}$$

These vectors can also be added using the triangle rule with $\triangle OAB$:

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB} \text{ or } \mathbf{a} + \mathbf{c} = \mathbf{b}$$

However, the triangle rule can also be applied to $\triangle OCB$:

$$\mathbf{OC} + \mathbf{CB} = \mathbf{OB} \text{ or } \mathbf{c} + \mathbf{a} = \mathbf{b}$$

So $\mathbf{a} + \mathbf{c} = \mathbf{b} = \mathbf{c} + \mathbf{a}$

This result is important. It is known as the commutative property for vector addition.

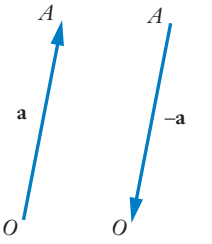
ISBN 9780170412995

Commutative property (vector addition)

For any two vectors **a** and **b**:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

You know that the direction of a vector is important.



Vectors **a** and **–a** shown here have the same magnitude, i.e., $|\mathbf{a}| = |-\mathbf{a}|$

The negative sign shows that **a** and **–a** are in opposite directions.

From the diagram, you can see that **OA** = **a** and **AO** = **–a**.

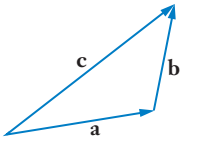
You have previously seen that the subtraction of vectors can be expressed as follows:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

Consider the vectors **a** and **b** shown on the right.

Using head-to-tail addition (or the triangle rule) you can write:

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$



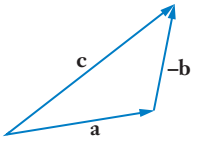
If the direction of **b** is reversed, you have:

$$\mathbf{c} + (-\mathbf{b}) = \mathbf{a}$$

or

$$\mathbf{c} - \mathbf{b} = \mathbf{a}$$

If the direction of **a** is reversed, you have:

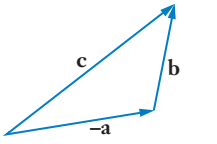


$$-\mathbf{a} + \mathbf{c} = \mathbf{b}$$

or

$$\mathbf{c} - \mathbf{a} = \mathbf{b}$$

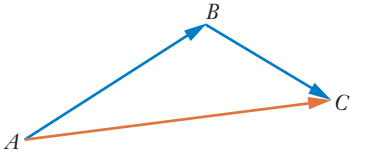
Using head-to-tail addition and the diagram on the right, it is clear that:



$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$$

But you don't need a diagram to simplify vector addition and subtraction.

Consider **PQ** + **QR**. This represents a vector from *P* to *Q* with the vector from *Q* to *R* added on. This is the same as the vector from *P* to *R*. So, **PQ** + **QR** = **PR**.



Now consider **MN** – **ON**. This can be written as the following sum:

$$\mathbf{MN} - \mathbf{ON} = \mathbf{MN} + (-\mathbf{ON})$$

$$= \mathbf{MN} + \mathbf{NO}$$

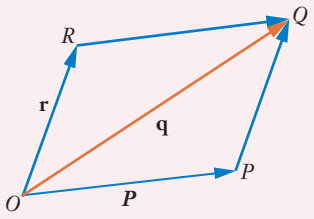
Now this can be simplified as shown previously:

$$\mathbf{MN} + \mathbf{NO} = \mathbf{MO}$$

ISBN 9780170412995

EXAMPLE 1

Parallelogram *OPQR* is shown.



If **OP** = **p**, **OQ** = **q** and **OR** = **r**, express each of the following vectors in terms of **p** and **r**.

a **OQ** **b** **QO** **c** **RP**

Solution

a	Use head-to-tail addition.	$\mathbf{OP} + \mathbf{PQ} = \mathbf{OQ}$
	Substitute for **r** and **p**.	$\mathbf{p} + \mathbf{r} = \mathbf{OQ}$
	State the result.	$\mathbf{OQ} = \mathbf{p} + \mathbf{r}$
b	Use head-to-tail addition.	$\mathbf{QP} + \mathbf{PO} = \mathbf{QO}$
	Substitute for **r** and **p**.	$-\mathbf{r} + (-\mathbf{p}) = \mathbf{QO}$
	State the result.	$\mathbf{QO} = -(\mathbf{p} + \mathbf{r})$
c	Use head-to-tail addition.	$\mathbf{RO} + \mathbf{OP} = \mathbf{RP}$
	Use the result from part **b** and substitute for **r** and **p**.	$-\mathbf{r} + \mathbf{p} = \mathbf{RP}$
	State the result.	$\mathbf{OQ} = \mathbf{p} - \mathbf{r}$

You can use vectors to determine the relationship between two lines.

ISBN 9780170412995

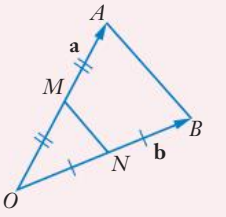
EXAMPLE 2

Consider $\triangle OAB$ as shown.

Take O as the origin and so **a** is the position vector of A and **b** is the position vector of B.

M is the midpoint of OA and N is the midpoint of OB.

What is the relationship between **MN** and **AB**?



Solution

Write **AB** as a sum of vectors.	$\mathbf{AB} = \mathbf{AO} + \mathbf{OB}$
Use the fact that $\mathbf{AO} = -\mathbf{OA} = -\mathbf{a}$. Substitute for **a** and **b**.	$= (-\mathbf{a}) + \mathbf{b}$
Simplify.	$= \mathbf{b} - \mathbf{a}$
Write **MN** as a sum of vectors.	$\mathbf{MN} = \mathbf{MO} + \mathbf{ON}$
M and N are the midpoints of OA and OB respectively.	$\mathbf{MO} = \frac{1}{2}\mathbf{AO} = -\frac{1}{2}\mathbf{a}$ $\mathbf{ON} = \frac{1}{2}\mathbf{OB} = \frac{1}{2}\mathbf{b}$
Substitute in the equation for **MN**.	$\mathbf{MN} = -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
Rearrange and simplify.	$= \frac{1}{2}(\mathbf{b} - \mathbf{a})$
Compare with **AB**.	$= \frac{1}{2}\mathbf{AB}$
State the result.	$\mathbf{MN} = \frac{1}{2}\mathbf{AB}$

The solution to Example 2 shows two things. In terms of magnitude, it shows that:

$$MN = \frac{1}{2}AB.$$

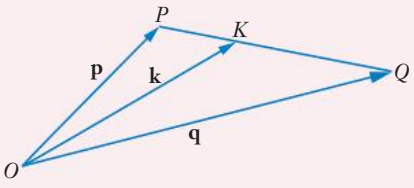
In terms of direction, it shows that MN and AB must be in the same direction, so $MN \parallel AB$.

This is called the **midpoint theorem** for a triangle. It states that if you join the midpoints of 2 sides of a triangle, then the resulting line is equal to half of the third side of the triangle, and is parallel to it.

ISBN 9780170412995

EXAMPLE 3

In the diagram, K divides PQ in the ratio 1 : 2.
Find an expression for **OK** terms of **p** and **q**.



Solution

K divides PQ in the ratio 1 : 2.

$$\frac{\mathbf{KP}}{\mathbf{QK}} = \frac{1}{2}$$

Rearrange.

$$2\mathbf{KP} = \mathbf{QK} \qquad [1]$$

Write **KP** as the sum of vectors.

$$\mathbf{KO} + \mathbf{OP} = \mathbf{KP}$$

Rearrange and use **KO** = –**OK**.

$$\mathbf{KP} = \mathbf{OP} - \mathbf{OK}$$

Substitute for **p** and **k**.

$$= \mathbf{p} - \mathbf{k}$$

Write **QK** as the sum of vectors.

$$\mathbf{QO} + \mathbf{OK} = \mathbf{QK}$$

Rearrange and use **QO** = –**OQ**.

$$\mathbf{QK} = -\mathbf{OQ} + \mathbf{k}$$

Substitute for **q** and **k**.

$$= \mathbf{k} - \mathbf{q}$$

Substitute for **KP** and **QK** into [1].

$$2(\mathbf{p} - \mathbf{k}) = \mathbf{k} - \mathbf{q}$$

Expand.

$$2\mathbf{p} - 2\mathbf{k} = \mathbf{k} - \mathbf{q}$$

Group like terms.

$$3\mathbf{k} = 2\mathbf{p} + \mathbf{q}$$

Isolate **k**.

$$\mathbf{k} = \frac{2\mathbf{p} + \mathbf{q}}{3}$$

Write the result.

$$\mathbf{OK} = \frac{2\mathbf{p} + \mathbf{q}}{3}$$

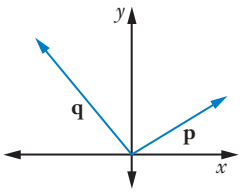
ISBN 9780170412995

Exercise 4.01 Geometric use of vectors

1 Simplify each expression.

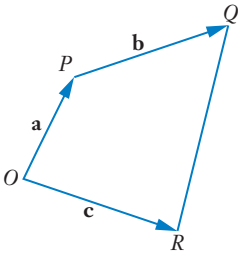
a $\mathbf{OA} + \mathbf{AB} + \mathbf{BC}$ **b** $\mathbf{AB} + \mathbf{BC} - \mathbf{EC}$ **c** $\mathbf{MN} - \mathbf{QP} + \mathbf{NP} + \mathbf{QR}$

2 Use the diagram to show that $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$.



3 Given the quadrilateral *OPQR* as shown, find expressions for each vector in terms of **a**, **b** and **c**.

Example 1

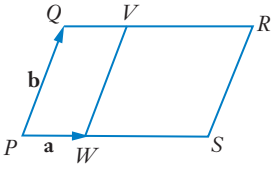


a **OQ** **b** **RP**

c **QO** **d** **QR**

e **PR** **f** **RQ**

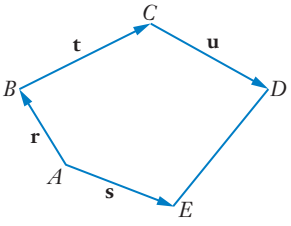
4 *PQVW* and *PQRS* are parallelograms with **PW** represented by **a** and **PQ** represented by **b**. The length of *WS* is twice the length of *PW*. Express each vector in terms of **a** and **b**.



a **WS** **b** **PS**

c **WR** **d** **RP**

5 Use the figure shown to express each of vector in terms of **r**, **s**, **t** or **u**.



a **CA** **b** **DB**

c **BE** **d** **EC**

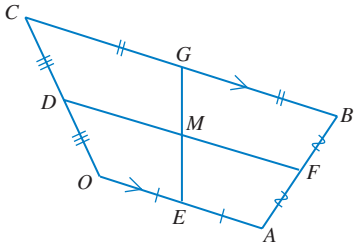
e **DE**

6 Use the diagram shown in question **3** to show that $(\mathbf{r} + \mathbf{t}) + \mathbf{u} = \mathbf{r} + (\mathbf{t} + \mathbf{u})$.

Example 2

ISBN 9780170412995

7 The upper edge of the trapezium $OABC$ shown below is twice as long as the lower. Lines have been drawn between the midpoints of the sides. Taking O as the origin, **a** and **c** are the position vectors of A and C respectively. Express each vector in terms of **a** and **c**.



a **OA**

b **CB**

c **OC**

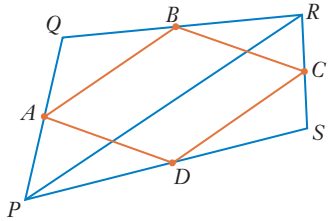
d **OB**

e The position vector of B

f **OD**

g The position vectors of D, E, F, G and M

8 $PQRS$ is a quadrilateral as shown. A, B, C and D are the midpoints of PQ, QR, RS and SP respectively.



a Use the midpoint theorem to write **AB** in terms of **PR**.

b Use the midpoint theorem to write **CD** in terms of **PR**.

c What can you conclude about **AB** and **CD**?

d Draw in QS and repeat the process for **AD** and **BC**.

e What type of quadrilateral is $ABCD$? Explain your answer.

9 In $\triangle OAB$, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. Write each vector in terms of **a** and **b**:

a **AB**

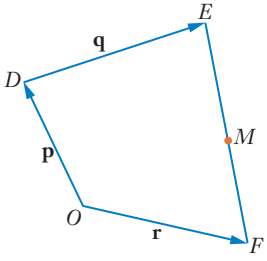
b **BA**

c **OP**, where P is the midpoint of AB

d **AP**

e **BP**

10 In the diagram, M is the midpoint of EF. Find an expression for each vector in terms of **p**, **q** and **r**.



a **OE**

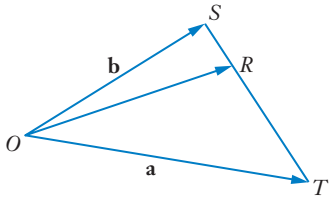
b **EF**

c **DM**

d **OM**

Example 3

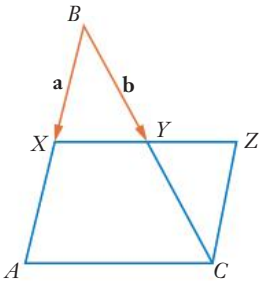
11 In the diagram, R divides ST in the ratio 1 : 3. Find an expression for **OR** terms of **a** and **b**.



12 In $\triangle OPQ$, $\mathbf{OP} = \mathbf{p}$ and $\mathbf{OQ} = \mathbf{q}$. If M divides PQ in the ratio 2 : 3, write **OM** in terms of **a** and **b**.

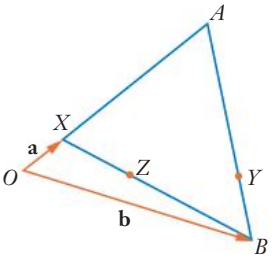
ISBN 9780170412995

13 $\triangle ABC$ is shown in the diagram. X is the midpoint of AB and Y is the midpoint of BC. XY is extended to Z so that $XY = YZ$.



- **a** Find an expression for **XY** in terms of **a** and **b**.
- **b** Find an expression for **XZ** in terms of **a** and **b**.
- **c** Find an expression for **XC** in terms of **a** and **b**.
- **d** Find an expression for **ZC** in terms of **a** and **b**.
- **e** Show that $XBZC$ is a parallelogram.

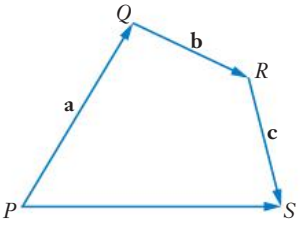
14 $\triangle OAB$ is shown in the diagram. X divides OA such that $OX : XA = 1 : 3$. Y divides AB such that $BY : YA = 1 : 2$. Z divides XB such that $XZ : ZB = 1 : 2$.



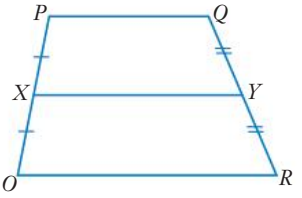
- **a** Find an expression for **OZ** in terms of **a** and **b**.
- **b** Find an expression for **OY** in terms of **a** and **b**.
- **c** Hence or otherwise show that O, Z and Y lie on the same line.

Problem solving

15 The quadrilateral $PQRS$ is shown here with **PQ** = **a**, **QR** = **b** and **RS** = **c**. Show that (**a** + **b**) + **c** = **a** + (**b** + **c**).



16 The trapezium $OPQR$ is shown in the diagram. O is the origin and so **OP** = **p** and **OR** = **r**. X and Y are the midpoints of OP and RQ respectively. Because $PQ \parallel OR$, **PQ** = k**r** for $k > 0$.



- **a** Show that **QR** = $(1 - k)$**r** − **p**.
- **b** Show that XY is parallel to OR and $\frac{1}{2}(k+1)$ times its length.

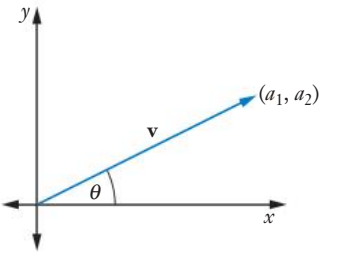
17 Point P has position vector **OP** = 3**a** − 2**b** and point Q has position vector **OQ** = **a** + **b**. Find the position vector of the point R which divides PQ in the ratio 2 : 1.

ISBN 9780170412995

Component form of the dot product

4.02 The scalar product

You have previously seen that vector, **v**, as shown in the diagram has a magnitude $|\mathbf{v}| = v$ and is in the direction of θ.



v can be written in polar form as $(|\mathbf{v}|, \theta)$ or in component form as (a_1, a_2) or $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

v can also be written in terms of the unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ as $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$.

You have also seen that a vector can be multiplied by a scalar so that for the scalar, k:

$k\mathbf{v} = (ka_1, ka_2)$ or $k\mathbf{v} = (k|\mathbf{v}|, \theta)$ for $k \geq 0$ and $k\mathbf{v} = (-k|\mathbf{v}|, 180° + \theta)$ for $k < 0$.

It is also possible to find the product of two vectors. One of the ways in which two vectors can be combined is known as the **scalar product**. The scalar product is also call the **inner product** or **dot product**. As the name suggests, it is a scalar (real number), not a vector.

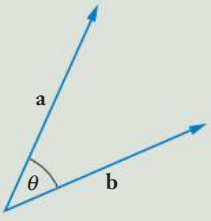
Scalar (dot) product

For vectors **a** and **b** with an angle θ between them, the scalar product is given by:

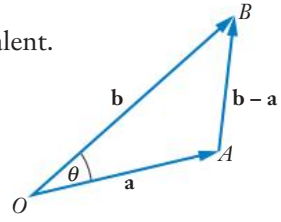
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta).$$

For vectors in component form the scalar product is given by:

$$\mathbf{a} \cdot \mathbf{b} = (a_1, a_2) \cdot (b_1, b_2) = a_1b_1 + a_2b_2$$



You can use the cosine rule to show that the polar form and component form of the scalar product of two vectors is always equivalent.



Consider $\triangle OAB$ with $\angle AOB = \theta$.

Let $\mathbf{OA} = \mathbf{a} = (a_1, a_2)$ and $\mathbf{OB} = \mathbf{b} = (b_1, b_2)$.

So, $\mathbf{AB} = \mathbf{b} - \mathbf{a} = (a_1 - b_1, a_2 - b_2)$

Using the cosine rule:

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos(\theta)$$

So $\cos(\theta) = \dfrac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b} - \mathbf{a}|^2}{2|\mathbf{a}||\mathbf{b}|}$

Write the rule for scalar product in polar form using **a** and **b**.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Substitute for $\cos\theta$:

ISBN 9780170412995

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b}-\mathbf{a}|^2}{2|\mathbf{a}||\mathbf{b}|}$$

$$= \frac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b}-\mathbf{a}|^2\right)$$

Use the rule for the norm of a vector, i.e., $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$ (or $|\mathbf{a}|^2 = a_1^2 + a_2^2$)

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 - \left[(b_1 - a_1)^2 + (b_2 - a_2)^2\right]\right)$$

$$= \frac{1}{2}\left[a_1^2 + a_2^2 + b_1^2 + b_2^2 - \left(b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2\right)\right]$$

$$= \frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 - b_1^2 + 2a_1b_1 - a_1^2 - b_2^2 + 2a_2b_2 - a_2^2\right)$$

$$= \frac{1}{2}\left(2a_1b_1 + 2a_2b_2\right)$$

$$= a_1b_1 + a_2b_2$$

So the polar form of the scalar product and the component form will always be equal.

EXAMPLE 4

Find the scalar product of:

a vectors of magnitudes 5 and 6 at an angle of 60° to each other

b $\mathbf{p} = (3, 40°)$ and $\mathbf{q} = (8, 125°)$

c $\mathbf{a} = (-1, 5)$ and $\mathbf{b} = (3, 6)$

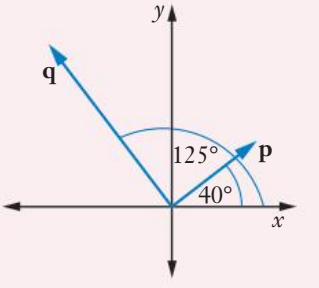
Solution

a Write the formula. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos(\theta)$

Substitute for known values. $= 5 \times 6 \times \cos(60°)$

Evaluate. $= 15$

b Draw a diagram.



Calculate the angle between **p** and **q**. $\theta = 125° - 40° = 85°$

ISBN 9780170412995

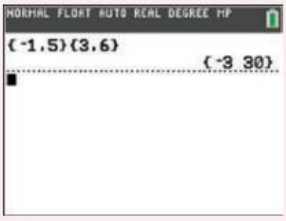
	Write the formula.	$\mathbf{p} \cdot \mathbf{q} = \lvert\mathbf{p}\rvert \lvert\mathbf{q}\rvert \cos(\theta)$
	Substitute for known values.	$= 3 \times 8 \times \cos(85°)$
	Evaluate.	$= 2.0917\ldots$
	Round off and write the result.	$\mathbf{p} \cdot \mathbf{q} \approx 2.09$
c	Write the formula.	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$
	Substitute for known values.	$= -1 \times 3 + 5 \times 6$
	Evaluate.	$= 27$

TI-Nspire CX
Chapter 4

TI-84 Plus CE

While the TI-84 Plus CE does not have the functionality to calculate the dot product of two vectors, a partial solution is possible.

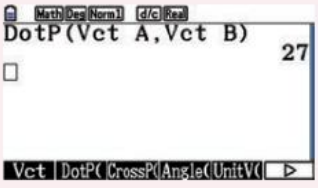
Enter the vectors with curly brackets as shown and press enter. Adding the elements of the result gives the dot product.



Casio fx-CG20AU

Enter the vectors $\mathbf{A} = (3, -2)$ and $\mathbf{B} = (3, 6)$ as previously described.

Press OPTN and then press F2 [Mat/Vct], F6, F6 and then F2 [DotP(]. Press F1 [Vct] ALPHA X,θ,T [A] , F1 [Vct] ALPHA log [B] and). Press EXE to display the result.



You can rearrange the formula for scalar products to obtain a formula that can be used to calculate the angle between two vectors.

State the formula. $\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert \lvert\mathbf{b}\rvert \cos\theta$

Make $\cos\theta$ the subject. $\cos(\theta) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$

Angle between two vectors

If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then:

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert} \quad \text{or} \quad \cos(\theta) = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + b_1^2} \times \sqrt{a_2^2 + b_2^2}}$$

ISBN 9780170412995

EXAMPLE 5

Calculate the angle between each pair of vectors.

a $\mathbf{m} = (3, -2)$ and $\mathbf{n} = (5, 8)$ **b** $\mathbf{c} = (7, 2)$ and $\mathbf{d} = (5, 123°)$

Solution

a Calculate the magnitude of each of the vectors.

$$|\mathbf{m}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$$
$$|\mathbf{n}| = \sqrt{5^2 + 8^2} = \sqrt{89}$$

Calculate the dot product.

$$\mathbf{m} \cdot \mathbf{n} = 3 \times 5 + (-2) \times 8$$
$$= 15 - 16$$
$$= -1$$

Write the rule for $\cos(\theta)$.

$$\cos(\theta) = \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{m}||\mathbf{n}|}$$

Substitute for known values.

$$= \frac{-1}{\sqrt{13} \times \sqrt{89}}$$

Use $\cos^{-1}$ to find θ.

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{13} \times \sqrt{89}}\right)$$

Evaluate.

$$= 91.7534...°$$

Round off and state the result.

The angle between $\mathbf{m}$ and $\mathbf{n}$ is about 91.8°.

b Calculate the magnitude of each of the vectors.

$$|\mathbf{c}| = \sqrt{7^2 + 2^2} = \sqrt{53}$$
$$|\mathbf{d}| = 5$$

Convert $\mathbf{d}$ to component form.

$$\mathbf{d} = (5 \cos [123°], 5 \sin [123°])$$

Calculate the dot product.

$$\mathbf{c} \cdot \mathbf{d} = 7 \times 5 \cos(123°) + 2 \times 5 \sin(123°)$$
$$= -19.0623... + 8.3867...$$
$$= -10.6756...$$

Write the rule for $\cos(\theta)$.

$$\cos(\theta) = \frac{\mathbf{c} \cdot \mathbf{d}}{|\mathbf{c}||\mathbf{d}|}$$

Substitute for known values.

$$= \frac{-10.6756...}{\sqrt{53} \times 5}$$

Use $\cos^{-1}$ to find θ.

$$\theta = \cos^{-1}\left(\frac{-10.6756...}{\sqrt{53} \times 5}\right)$$

ISBN 9780170412995

Evaluate. $= 107.0546\ldots°$

Round off and state the result. The angle between **c** and **d** is about 107.1°.

TI-84 Plus CE

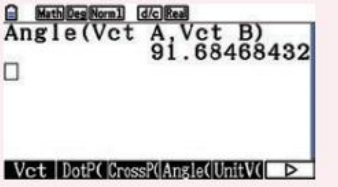
The TI-84 Plus CE does not have the functionality to calculate the angle between two vectors.

Casio fx-CG20AU

Enter the vectors **A** = (3, –2) and **B** = (5, 8) as previously described.

Press OPTN and then press F2 F6 F6 F4 [Angle(].

Press F1 (Vct) ALPHA X,θ,T (A) , F1 (Vct) ALPHA log (B) and).
Press EXE to display the result.



There are several important properties of the scalar product of two vectors.

Consider the vectors **a** = (–3, 8) and **b** = (7, 4).

$$\mathbf{a} \cdot \mathbf{b} = -3 \times 7 + 8 \times 4 = -21 + 32 = 11$$

$$\mathbf{b} \cdot \mathbf{a} = 7 \times (-3) + 4 \times 8 = -21 + 32 = 11$$

So $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

This is the commutative property of scalar product. Other properties of scalar product can be demonstrated in a similar way and are summarised below.

WS Properties of the dot product

Properties of the scalar product

1 The scalar product for vectors in polar form is:

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$, where θ is the angle between **a** and **b**.

For vectors in component form the scalar product is:

$\mathbf{a} \cdot \mathbf{b} = (a_1, a_2) \cdot (b_1, b_2) = a_1b_1 + a_2b_2$, where $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$.

2 The scalar product is a real number, *not* a vector.

3 The scalar product is commutative.

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

4 The scalar product is distributive over vector addition.

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

ISBN 9780170412995

5 $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$ for all real numbers m and vectors $\mathbf{a}$ and $\mathbf{b}$.

6 $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ (because $\cos(\theta) = 1$)

EXAMPLE 6

Prove that the scalar product is distributive over vector addition.

Solution

Write what is required to be proved (RTP).	**RTP**
	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
Commence the proof.	**Proof**
Identify the vectors.	$\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ and $\mathbf{c} = (c_1, c_2)$
Write down the LHS.	LHS $= \mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
Substitute for known values.	$= (a_1, a_2) \cdot [(b_1, b_2) + (c_1, c_2)]$
Simplify.	$= (a_1, a_2) \cdot [(b_1 + c_1), (b_2 + c_2)]$
Calculate the scalar product.	$= a_1(b_1 + c_1) + a_2(b_2 + c_2)$
Expand.	$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2$
Write the RHS.	RHS $= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
Substitute for known values.	$= (a_1, a_2) \cdot (b_1, b_2) + (a_1, a_2) \cdot (c_1, c_2)$
Calculate the scalar products.	$= a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2$
Compare the results.	LHS = RHS QED

ISBN 9780170412995

Exercise 4.02 The scalar product

Example 4

1 Find the scalar product of each pair of vectors described in the table.

	Magnitudes of vectors	Angle between vectors
a	5, 9	38°
b	3, 10	45°
c	11, 7	90°
d	15, 12	0°
e	7, 2	30°
f	6, 11	210°
g	9, 9	150°
h	8, 12	300°

2 Find each scalar product.

a $(9, 120°) \cdot (16, 48°)$ **b** $(3, 185°) \cdot (16, 200°)$

c $(19, 300°) \cdot (6, 240°)$ **d** $(3, 160°) \cdot (5, 160°)$

e $(9, -40°) \cdot (4, 280°)$ **f** $(15, 145°) \cdot (8, -15°)$

3 Find each scalar product.

a $(3, 8) \cdot (6, -4)$ **b** $(-2, 10) \cdot (5, 1)$

c $(-6, 8) \cdot (8, -6)$ **d** $(2, 6) \cdot (5, 10)$

e $(3, -4) \cdot (5, -6)$ **f** $(9, -12) \cdot (6, -3)$

g $(8, -3) \cdot (-3, -7)$ **h** $(-4, -8) \cdot (-5, 3)$

4 Find each scalar product.

a $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -9 \end{bmatrix}$ **c** $\begin{bmatrix} -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

d $\begin{bmatrix} -2 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ **e** $\begin{bmatrix} 5 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 8 \end{bmatrix}$ **f** $\begin{bmatrix} -6 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -8 \end{bmatrix}$

Example 5

5 Find the angle between each pair of vectors.

a $(5, -4)$ and $(-2, 1)$ **b** $(4, 3)$ and $(2, -3)$

c $(1, -6)$ and $(4, 3)$ **d** $(3, -4)$ and $(-2, -2)$

e $(2, \sqrt{2})$ and $(1 + \sqrt{2}, 1 - \sqrt{2})$ **f** $(5, \sqrt{3})$ and $(\sqrt{3}, 2)$

6 Find the angle between each pair of vectors.

a $(3, 4)$ and $(6, 15°)$ **b** $(7, 75°)$ and $(12, 5)$ **c** $(-2, 5)$ and $(7, 141°)$

d $(6, 118°)$ and $(-3, -1)$ **e** $(4, -3)$ and $(7, 224°)$ **f** $(10, -32°)$ and $(-1, -2)$

ISBN 9780170412995

Example 6

7 Use each pair of vectors to show that the scalar product is commutative (i.e., $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$).

a $\mathbf{a} = (4, -3)$ and $\mathbf{b} = (-7, 2)$ **b** $\mathbf{c} = (3, 8)$ and $\mathbf{d} = (9, -2)$

c $\mathbf{m} = (3, -7)$ and $\mathbf{n} = (2, 11)$ **d** $\mathbf{p} = (5, -4)$ and $\mathbf{q} = (-6, -1)$

8 Use each set of vectors to show that the scalar product is distributive over vector addition (i.e., $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$).

a $\mathbf{a} = (1, 1)$, $\mathbf{b} = (2, 1)$ and $\mathbf{c} = (3, 2)$ **b** $\mathbf{m} = (2, 1)$, $\mathbf{n} = (1, 5)$ and $\mathbf{p} = (-2, 3)$

c $\mathbf{d} = (2, -1)$, $\mathbf{e} = (3, -2)$ and $\mathbf{f} = (-5, 3)$ **d** $\mathbf{u} = (7, -2)$, $\mathbf{v} = (3, -1)$ and $\mathbf{w} = (-5, 4)$

9 In each case, use the vectors $\mathbf{a}$ and $\mathbf{b}$ and the scalar m to show that: $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$.

a $\mathbf{a} = (1, 2)$, $\mathbf{b} = (2, 1)$ and $m = 3$ **b** $\mathbf{a} = (3, 2)$, $\mathbf{b} = (1, 3)$ and $m = 5$

c $\mathbf{a} = (-2, 3)$, $\mathbf{b} = (4, -2)$ and $m = -2$ **d** $\mathbf{a} = (2, -2)$, $\mathbf{b} = (-6, 5)$ and $m = 4$

10 Use each pair of vectors to show that $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

a $\mathbf{a} = (1, 1)$ **b** $\mathbf{a} = (2, 1)$ **c** $\mathbf{a} = (-2, 3)$

d $\mathbf{a} = (-5, -3)$ **e** $\mathbf{a} = (11, -4)$

Problem solving

11 Given that $\mathbf{a} \cdot \mathbf{b} = 2$, $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 3$, calculate the value of:

a $\mathbf{a} \cdot (4\mathbf{b})$ **b** $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{b}$

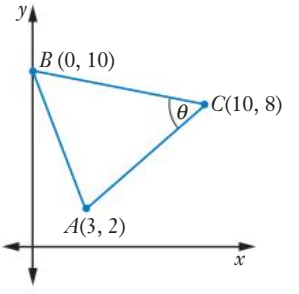
c $2\mathbf{a} \cdot (3\mathbf{a} - \mathbf{b})$ **d** $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

12 Show that $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2$.

13 Show that $(\mathbf{p} + \mathbf{q}) \cdot (\mathbf{p} + \mathbf{q}) - 2\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}|^2 + |\mathbf{q}|^2$.

14 Show that $(\mathbf{m} + \mathbf{n}) \cdot \mathbf{m} - (\mathbf{m} + \mathbf{n}) \cdot \mathbf{n} = |\mathbf{m}|^2 - |\mathbf{n}|^2$.

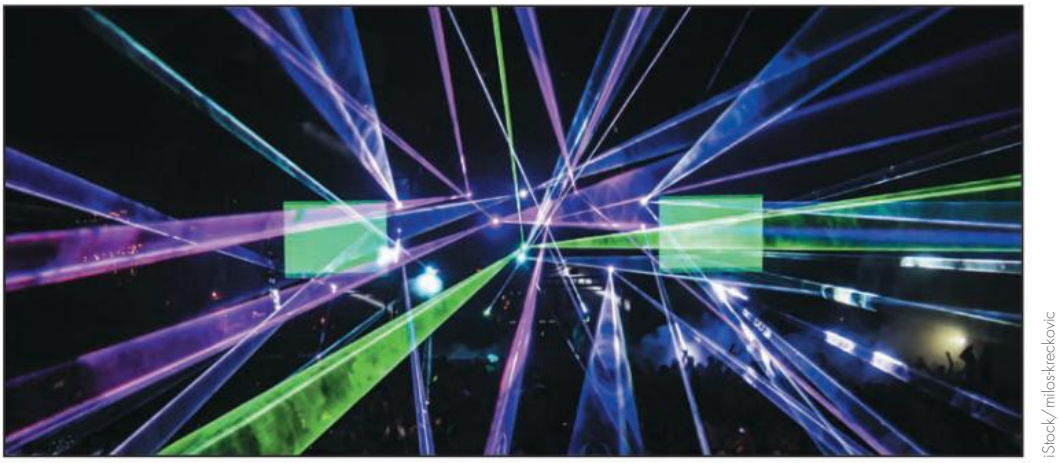
15 Calculate the value of θ in the diagram.



ISBN 9780170412995

4.03 Parallel and perpendicular vectors

Parallel and perpendicular vectors



iStock/miloskreckovic

The scalar product of vectors **a** and **b** with an angle θ between them is:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta).$$

Consider the situation where:

$$\mathbf{a} \cdot \mathbf{b} = 0$$

So $|\mathbf{a}|\,|\mathbf{b}| \cos(\theta) = 0$

This means that either $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$ or $\cos(\theta) = 0$.

If **a** and **b** are both nonzero, then the only way for $\mathbf{a} \cdot \mathbf{b} = 0$ to be true is if $\cos(\theta) = 0$, i.e. $\theta = 90°$.

This means that if **a** and **b** are both nonzero and $\mathbf{a} \cdot \mathbf{b} = 0$, then **a** and **b** must be perpendicular. Perpendicular vectors are also called **orthogonal**.

EXAMPLE 7

Show that the vectors $\mathbf{m} = (4, -3)$ and $\mathbf{n} = (6, 8)$ are perpendicular.

Solution

Calculate the scalar product.

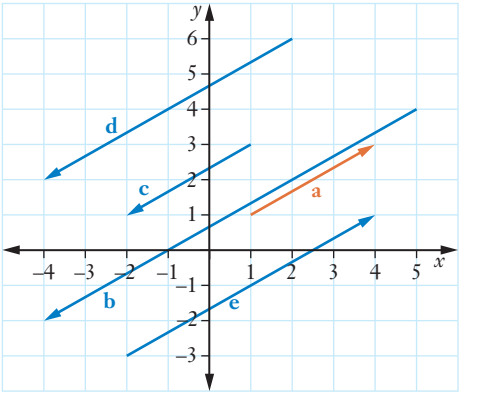
$$\begin{aligned}\mathbf{m} \cdot \mathbf{n} &= 4 \times 6 + (-3) \times 8 \\ &= 24 - 24 \\ &= 0\end{aligned}$$

State the result.

m and **n** are nonzero and $\mathbf{m} \cdot \mathbf{n} = 0$, so **m** and **n** are perpendicular.

ISBN 9780170412995

Consider the diagram on the right. You can see that **a**, **b**, **c**, **d** and **e** are all parallel.



a and **b** are in the same direction, so the angle between them is 0°.

a and **c** are in opposite directions, so the angle between them is 180°.

So $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$

$= |\mathbf{a}|\,|\mathbf{b}| \cos(0°)$

$= |\mathbf{a}|\,|\mathbf{b}| \times 1$

$= |\mathbf{a}|\,|\mathbf{b}|$

and $\mathbf{a} \cdot \mathbf{c} = |\mathbf{a}|\,|\mathbf{c}| \cos(\theta)$

$= |\mathbf{a}|\,|\mathbf{c}| \cos(180°)$

$= |\mathbf{a}|\,|\mathbf{c}| \times (-1)$

$= -|\mathbf{a}|\,|\mathbf{c}|$

Perpendicular and parallel vectors

If **a** and **b** are perpendicular, then $\mathbf{a} \cdot \mathbf{b} = 0$.

If **a** and **b** are parallel, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$ or $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$.

The unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are perpendicular and have a magnitude of 1. This makes it easy to remember the scalar products of **i** and **j**.

Scalar product of unit vectors i and j

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$$

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$$

ISBN 9780170412995

EXAMPLE 8

Determine if each pair of vectors are parallel, perpendicular or neither.

a $\mathbf{g} = (3, -2)$ and $\mathbf{h} = (-15, 10)$

b $\mathbf{u} = (1, 5)$ and $\mathbf{v} = (-3, 6)$

Solution

a Calculate the magnitudes of g and h.

$|\mathbf{g}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

$|\mathbf{h}| = \sqrt{(-15)^2 + 10^2} = \sqrt{325}$

Calculate.

$|\mathbf{g}|\,|\mathbf{h}| = \sqrt{13} \times \sqrt{325} = 65$

Calculate the dot product.

$\mathbf{g} \cdot \mathbf{h} = 3 \times (-15) + (-2) \times 10$

$= -45 - 20$

$= -65$

Compare $|\mathbf{g}|\,|\mathbf{h}|$ with $\mathbf{g} \cdot \mathbf{h}$.

$|\mathbf{g}|\,|\mathbf{h}| = -\mathbf{g} \cdot \mathbf{h}$

State the result.

g and **h** are parallel but in opposite directions.

b Calculate the magnitudes of **u** and **v**.

$|\mathbf{u}| = \sqrt{1^2 + 5^2} = \sqrt{26}$

$|\mathbf{v}| = \sqrt{(-3)^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$

Calculate.

$|\mathbf{u}|\,|\mathbf{v}| = \sqrt{26} \times 3\sqrt{5} = 3\sqrt{130}$

Calculate the dot product.

$\mathbf{u} \cdot \mathbf{v} = 1 \times (-3) + 5 \times 6$

$= -3 + 30$

$= 27$

Compare $|\mathbf{u}|\,|\mathbf{v}|$ with $\mathbf{u} \cdot \mathbf{v}$.

$\mathbf{u} \cdot \mathbf{v} \neq 0$

$|\mathbf{u}|\,|\mathbf{v}| \neq \mathbf{u} \cdot \mathbf{v}$

$|\mathbf{u}|\,|\mathbf{v}| \neq -\mathbf{u} \cdot \mathbf{v}$

State the result.

u and **v** are neither parallel nor perpendicular.

ISBN 9780170412995

Exercise 4.03 Parallel and perpendicular vectors

Example 7

1 Show that each pair of vectors are perpendicular.

a $\mathbf{d} = (3, 0)$ and $\mathbf{c} = (0, 8)$

b $\mathbf{m} = (0, -2)$ and $\mathbf{n} = (7, 0)$

c $\mathbf{a} = (4, -4)$ and $\mathbf{b} = (4, 4)$

d $\mathbf{r} = (2, 1)$ and $\mathbf{t} = (4, -8)$

e $\mathbf{g} = \left(\frac{1}{2}, \frac{1}{4}\right)$ and $\mathbf{h} = \left(\frac{1}{6}, -\frac{1}{3}\right)$

f $\mathbf{w} = \left(\frac{2}{5}, -\frac{3}{7}\right)$ and $\mathbf{v} = \left(\frac{3}{7}, \frac{2}{5}\right)$

g $\mathbf{a} = (\sqrt{6}, \sqrt{5})$ and $\mathbf{b} = (-\sqrt{5}, \sqrt{6})$

h $\mathbf{r} = (2\sqrt{3}, 5\sqrt{7})$ and $\mathbf{t} = (-5\sqrt{7}, 2\sqrt{3})$

2 Show that each pair of vectors is perpendicular.

a $\mathbf{p} = 4\mathbf{i}$ and $\mathbf{q} = 7\mathbf{j}$

b $\mathbf{u} = \mathbf{i} + 5\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$

c $\mathbf{a} = 4\mathbf{i} + 11\mathbf{j}$ and $\mathbf{b} = -11\mathbf{i} + 4\mathbf{j}$

d $\mathbf{m} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

e $\mathbf{m} = \begin{bmatrix} \frac{1}{3} \\ \frac{3}{4} \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} -12 \\ \frac{16}{3} \end{bmatrix}$

f $\mathbf{m} = \begin{bmatrix} 4\sqrt{3} \\ -\sqrt{6} \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} 3\sqrt{5} \\ 6\sqrt{10} \end{bmatrix}$

3 Use the scalar product to show that each pair of vectors are parallel.

a $\mathbf{s} = (1, 2)$ and $\mathbf{t} = (3, 6)$

b $\mathbf{g} = (-8, 7)$ and $\mathbf{h} = (4, -3.5)$

c $\mathbf{u} = (12, 8)$ and $\mathbf{v} = (18, 12)$

d $\mathbf{a} = (2.8, -4.4)$ and $\mathbf{b} = (-7, 11)$

e $\mathbf{c} = \left(-\frac{1}{3}, \frac{1}{2}\right)$ and $\mathbf{d} = \left(\frac{2}{3}, -1\right)$

f $\mathbf{p} = \left(\frac{3}{8}, \frac{2}{7}\right)$ and $\mathbf{q} = \left(\frac{1}{4}, \frac{4}{21}\right)$

g $\mathbf{r} = (3\sqrt{2}, 5\sqrt{3})$ and $\mathbf{s} = (6\sqrt{3}, 15\sqrt{2})$

h $\mathbf{w} = (-3\sqrt{6}, 2\sqrt{15})$ and $\mathbf{v} = (18\sqrt{5}, -30\sqrt{2})$

4 Use the scalar product to show that each pair of vectors are parallel.

a $\mathbf{m} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{n} = -12\mathbf{i} + 16\mathbf{j}$

b $\mathbf{p} = 10\mathbf{i} + 5\mathbf{j}$ and $\mathbf{q} = 4\mathbf{i} + 2\mathbf{j}$

c $\mathbf{a} = -12\mathbf{i} + 9\mathbf{j}$ and $\mathbf{b} = 2.4\mathbf{i} - 1.8\mathbf{j}$

d $\mathbf{m} = \begin{bmatrix} 1.4 \\ 2.6 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} 3.5 \\ 6.5 \end{bmatrix}$

e $\mathbf{c} = \begin{bmatrix} \frac{2}{9} \\ \frac{8}{11} \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} \frac{1}{6} \\ \frac{6}{11} \end{bmatrix}$

f $\mathbf{v} = \begin{bmatrix} 3\sqrt{5} \\ -6\sqrt{6} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -15\sqrt{3} \\ 18\sqrt{10} \end{bmatrix}$

ISBN 9780170412995

Example 8

5 Determine if each pair of vectors are parallel, perpendicular or neither.

a $\mathbf{a} = (3, -2)$ and $\mathbf{b} = (-6, 4)$

b $\mathbf{m} = (1, 3)$ and $\mathbf{n} = (-2, 2)$

c $\mathbf{u} = (10, -6)$ and $\mathbf{v} = (9, 15)$

d $\mathbf{p} = \left(\frac{3}{2}, -\frac{27}{4}\right)$ and $\mathbf{q} = (-2, 9)$

e $\mathbf{f} = (\sqrt{6}, 2\sqrt{3})$ and $\mathbf{h} = (3\sqrt{5}, 2\sqrt{2})$

f $\mathbf{v} = (4\sqrt{6}, 3\sqrt{10})$ and $\mathbf{w} = (12\sqrt{10}, 15\sqrt{6})$

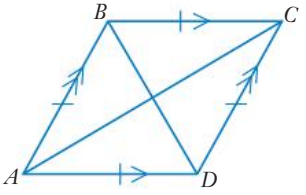
Problem solving

6 For what value of a are the vectors $\mathbf{m} = (1, a)$ and $\mathbf{n} = (2, 3)$ perpendicular?

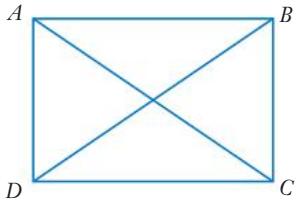
7 If $\mathbf{a} = (2, 5)$ find the vector, $\mathbf{b}$, parallel to $\mathbf{a}$ such that $\mathbf{a} \bullet \mathbf{b} = 87$.

8 Given the points $M(-2, 2)$, $N(2, 8)$ and $P(10, 20)$, show that **MN** and **NP** are parallel.

9 Use the scalar product and vectors to show that the diagonals of a rhombus are perpendicular.



10 Use the scalar product and vectors to show that the diagonals of a parallelogram $ABCD$ have equal lengths if and only if $ABCD$ is a rectangle.



ISBN 9780170412995

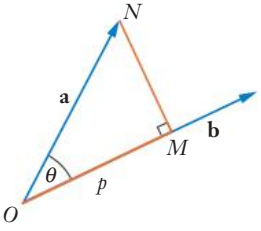
4.04 Projections of vectors

Projections

Shutterstock.com/gnomeandi

In the diagram, the angle between vectors **a** and **b** is θ.

If the line NM is drawn from **a** perpendicular to **b**, then the distance $OM = p$ is called the **scalar projection** or just the **projection** of **a** on **b**.



It is also called the projection of **a** in the direction of **b**.

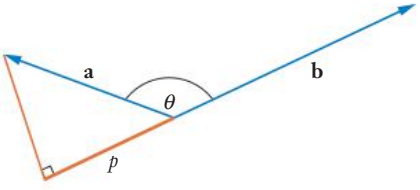
In $\triangle OMN$:

$$\cos(\theta) = \frac{p}{|\mathbf{a}|}$$

So $$p = |\mathbf{a}| \cos(\theta)$$

Or $$p = a \cos(\theta)$$

If the angle between **a** and **b** is obtuse, the projection of **a** on **b** is in the opposite direction to **b** and is negative.



Projection of a on b

Given vectors **a** and **b** with an angle of θ between them, the projection of **a** on **b** is:

$$p = |\mathbf{a}| \cos(\theta)$$

ISBN 9780170412995

EXAMPLE 9

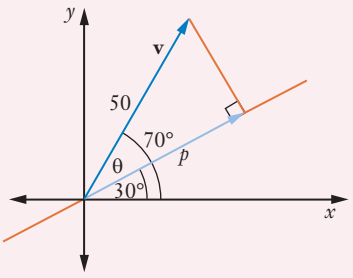
a Calculate the projection in the direction 30° of a vector of magnitude 50 in the direction 70°.

b Find the projection in the direction 46° of a vector of magnitude 20 in the direction 173°.

Solution

a Draw a diagram showing the vector with magnitude 50 in the direction 70° and direction of the projection at 30°.

Indicate the projection p and the vector $\mathbf{v}$ on the diagram.



Calculate θ.

$\theta = 70° - 30° = 40°$

Calculate the projection.

$p = 50 \cos(40°)$

Evaluate.

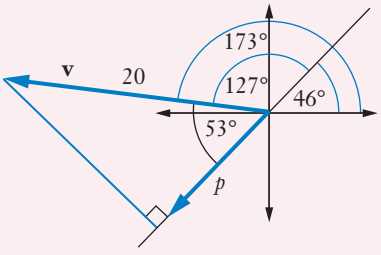
$= 38.3022\ldots$

Round off and state the result.

The projection in the direction 30° is approximately 38.3.

b Draw a diagram showing the vector with magnitude 20 in the direction 173° and the direction of the projection at 46°.

Indicate the projection p and the vector $\mathbf{v}$ on the diagram.



Calculate the projection.

$p = 20 \cos(127°)$

Evaluate.

$= -12.0363\ldots$

Round off and state the result.

The projection in the direction 46° is approximately −12.0.

ISBN 9780170412995

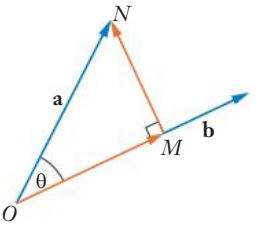
Again consider the vectors **a** and **b** as previously described.

Vector **OM** is known as the **vector projection** of **a** on **b**.

The vector projection of **a** on **b** is given by:

$$\mathbf{OM} = |\mathbf{a}| \cos(\theta)\, \hat{\mathbf{b}}$$

where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b**.



You can see from the diagram that

$$\mathbf{a} = \mathbf{OM} + \mathbf{MN}$$

OM and **MN** are perpendicular to each other and therefore are orthogonal components of **a**.

You can also use the scalar product to calculate the projection of **a** on **b**.

You have previously seen that:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$$

Or
$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

But
$$OM = |\mathbf{a}| \cos(\theta)$$

So $OM = |\mathbf{a}| \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$$= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$$

$$= \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|}$$

$$= \mathbf{a} \cdot \hat{\mathbf{b}}$$

Projection of a on b

Given vectors **a** and **b**, the projection of **a** on **b** is:

$$p = \mathbf{a} \cdot \hat{\mathbf{b}}$$

Or
$$p = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$$

The **vector projection** of **a** on **b** is $\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = |\mathbf{a}| \cos(\theta)\hat{\mathbf{b}}$, where θ is the angle between **a** and **b**.

This definition of the projection of a vector is useful when vectors are given in component form.

ISBN 9780170412995

The following example is about scalar products and projections.

EXAMPLE 10

Calculate the projection of $\mathbf{v} = (8, 3)$ on $\mathbf{w} = (5, -2)$.

Solution

Write the rule for the projection of $\mathbf{v}$ on $\mathbf{w}$.

$$\text{Projection of } \mathbf{v} \text{ on } \mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}$$

Calculate $\mathbf{v} \cdot \mathbf{w}$.

$$\mathbf{v} \cdot \mathbf{w} = 8 \times 5 + 3 \times (-2)$$
$$= 34$$

Calculate $|\mathbf{w}|$.

$$|\mathbf{w}| = \sqrt{5^2 + (-2)^2} = \sqrt{29}$$

Substitute into the rule for the projection of $\mathbf{v}$ on $\mathbf{w}$.

$$\text{Projection of } \mathbf{v} \text{ on } \mathbf{w} = \frac{34}{\sqrt{29}}$$

Evaluate.

$$= 6.3136\ldots$$

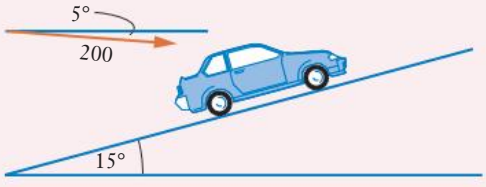
Round off and state the result.

The projection of $\mathbf{v}$ on $\mathbf{w}$ is about 6.31.

EXAMPLE 11

A car is being pushed up a road with a slope of 15° to the horizontal. The resultant force of the push is 200 N acting 5° downwards from the horizontal.

Resolve the force into components that are parallel and perpendicular to the road.

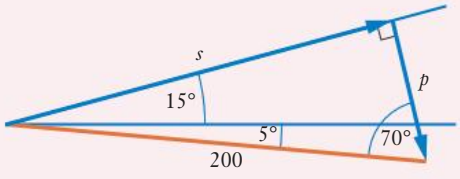


Solution

Sketch a diagram to represent the data.

The direction of the car is 15° above the horizontal. The direction of the 200 N force is 5° below the horizontal.

Let the projection of the 200 N in the direction of the car be p. Let s be the component that is orthogonal to p.



Calculate the projection p.

$$p = 200 \cos(20°)$$

Evaluate.

$$= 187.9385\ldots$$

ISBN 9780170412995

Calculate the component s.	$s = 200 \cos(70°)$ (or $200 \sin(20°)$)
Evaluate.	$= 68.4040\ldots$
Round off and state the result.	The components of the 200 N force parallel and perpendicular to the road are about 187.9 N and 68.4 N respectively.

Exercise 4.04 Projections of vectors

1 Find the projection in the direction 60° of a vector of magnitude 18 and direction 86°. Example 9

2 Find the projection in the direction 120°of a vector of magnitude 16 in the direction 82°.

3 Find the projection in the direction 206° of the vector (8, 100°).

4 Find the projection of (32, 178°) in the direction 200°.

5 What is the projection of (16, 35°) in the direction 305°?

6 What is the projection of (19, 116°) in the direction 116°?

7 In each of the following, calculate the projection of the first vector on the second. Example 10

a $\mathbf{m} = (10, 1)$ and $\mathbf{n} = (8, 3)$

b $\mathbf{c} = (5, -3)$ and $\mathbf{d} = (4, 7)$

c $\mathbf{p} = (-6, 4)$ and $\mathbf{q} = (-8, -3)$

d $\mathbf{a} = (4, 1)$ and $\mathbf{b} = (-6, 3)$

e $\mathbf{u} = (-4, 3)$ and $\mathbf{v} = (-7, -2)$

f $\mathbf{r} = (4, 7)$ and $\mathbf{s} = (6, -1)$

8 Resolve the vector (38, 90°) in the directions 16° and 106°.

9 Resolve the vector (15, 0°) in the directions 21° and 111°.

10 Resolve the vector (6, 9), given in component form, in the directions 10° and 100°.

11 Calculate the projection of $\mathbf{a} = (3, 5)$ on $\mathbf{b} = (7, 338°)$.

12 Calculate the projection of $\mathbf{m} = (8, 208°)$ on $\mathbf{n} = (9, 2)$.

13 Calculate the projection of $\mathbf{m} = (-10, -3)$ on $\mathbf{n} = (7, 163°)$.

14 A car going up a hill inclined at 7° has a weight of 9000 N acting vertically downwards. Resolve the weight into components parallel and perpendicular to the slope. Example 11

Problem solving

15 In a physics experiment, a small shot-put ball that is rolling down a plank sloped at 12° to the horizontal has a weight of 6 N. The total force acting on the ball is worked out from its acceleration to be 0.7 N acting down the slope. Resolve the weight into components parallel and perpendicular to the slope and hence find the force of friction acting back up the slope.

ISBN 9780170412995

Applications of the dot product

Resolution of forces

4.05 Applications of vectors

There are many applications of vectors, especially in the field of physics. In physics, force is calculated using the mass (m) of an object and its acceleration (a). The standard unit for mass is the kilogram (kg) and the standard unit for acceleration is m/s^2.

$$F = ma$$

The standard unit of force is the **newton** (N). 1 N is the force required to cause an object of mass 1 kg to accelerate at 1 m/s^2.

$$1 \text{ N} = 1 \text{ kg m/s}^2$$

The acceleration due to gravity is given the symbol g.

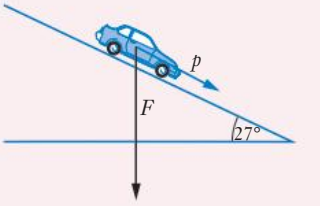
$$g \approx 9.8 \text{ m/s}^2$$

EXAMPLE 12

A car with mass 1000 kg stands on a ramp.

The angle of incline of the ramp is 27°.

Calculate the force required to prevent the car from rolling down the ramp.



Solution

The weight of the car produces a force, F, acting vertically downwards.

$F = ma$

$= 1000 \text{ kg} \times g$

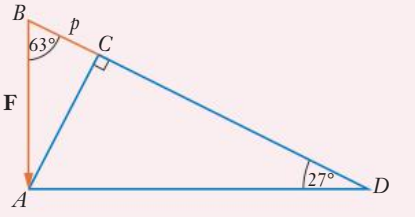
$= 1000 \text{ kg} \times 9.8 \text{ m/s}^2$

$= 9800 \text{ N}$

Write F as a vector.

$\mathbf{F} = (0, -9800)$

Draw a simplified sketch.



The component of **F** acting down the slope is the projection of **F** in the direction of p.

$p = BC = |\mathbf{F}| \cos(\theta)$

Substitute for known values.

$= 9800 \times \cos(63°)$

Evaluate.

$= 4449.1068\ldots$

ISBN 9780170412995

The force required to hold the car in place is equal and opposite to the force acting down the slope.	Required force is approximately 4449.1 N acting in the direction up the slope, i.e., (4449.1, 153°).

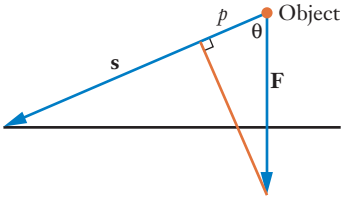
Another important quantity studied in physics is **work**. Work is a scalar quantity. In physics, work is the product of force and displacement. The unit of force is the **joule** (J) and 1 J = 1 Nm.

Only the component of force in the direction of displacement is important, so work is actually the product of the projection of the force in the direction of displacement and the displacement.

Consider the diagram on the right, which shows a force **F** acting on an object. The angle between the displacement (**s**) and the force is θ.

The projection of **F** on **s** is given by:

$$p = |\mathbf{F}| \cos(\theta)$$



The magnitude of $\mathbf{s} = |\mathbf{s}| = s$

So, work is calculated as:

$$W = p \times s$$

Or $W = |\mathbf{F}| \cos(\theta) \times |\mathbf{s}|$

$= |\mathbf{F}| \ |\mathbf{s}| \cos(\theta)$

$W = \mathbf{F} \cdot \mathbf{s}$

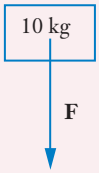
So work is the scalar product of force and displacement.

ISBN 9780170412995

EXAMPLE 13

a Calculate the work done by a horizontal force of 250 N that moves an object 10 m horizontally.

b Calculate the work done by gravity when a 10 kg object falls 5 m vertically.



Solution

a

Express the force and displacement as vectors in component form.	$\mathbf{F} = (250, 0)$ $\mathbf{s} = (10, 0)$
Calculate W using the dot product.	$W = \mathbf{F} \cdot \mathbf{s}$
Substitute.	$= (250, 0) \cdot (10, 0)$
Evaluate.	$= 250 \times 10$ $= 2500$
State the result.	Work done = 2500 J

b The force of gravity acts downwards and the displacement is also downwards.

Calculate the force due to gravity.	$\mathbf{F} = ma$
Substitute for known values.	$= 10 \times (-9.8)$ $= -98$ N
Write the force and displacement as vectors in component form.	$\mathbf{F} = (0, -98)$ $\mathbf{s} = (0, -5)$
Calculate W using the dot product.	$W = \mathbf{F} \cdot \mathbf{s}$
Substitute.	$= (0, -98) \cdot (0, -5)$
Evaluate.	$= (-98) \times (-5)$ $= 490$
State the result.	Work done = 490 J

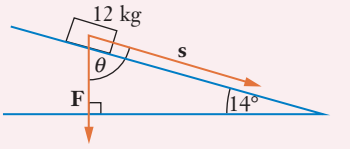
The same approach is used when the object that is subjected to a force is on an incline. In these cases, it is more convenient to use the polar form of the scalar product.

ISBN 9780170412995

EXAMPLE 14

A block weighing 12 kg is on a flat frictionless surface inclined at 14° to the horizontal. The block slides 4.7 m down the surface due to the force of gravity.

Calculate the work done.



Solution

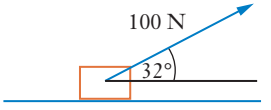
Write the magnitudes of the force and the displacement.	$\lvert\mathbf{F}\rvert = ma = 12 \times 9.8 = 117.6$ $\lvert\mathbf{s}\rvert = 4.7$
Calculate θ.	$\theta = 90° - 14° = 76°$
Write the rule for work.	$W = \lvert\mathbf{F}\rvert \lvert\mathbf{s}\rvert \cos(\theta)$
Substitute for known values.	$= 117.6 \times 4.7 \times \cos(76°)$
Evaluate.	$= 133.7150\ldots$
State the result.	The work done is approximately 133.7 J.

Exercise 4.05 Applications of vectors

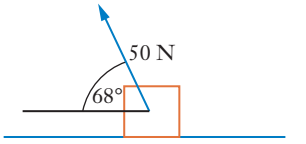
Assume that all surfaces are friction-free for this exercise.

1 For each situation, calculate the magnitude of the force acting in the direction of motion.

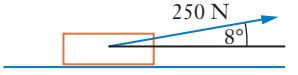
a



b



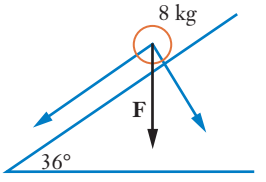
c



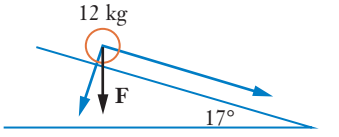
2 In each case, an object on an inclined surface is subjected to the force of gravity. Calculate the component of the force acting in the direction of the incline and the component of the force that is perpendicular to the incline.

Example 12

a

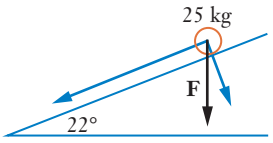


b



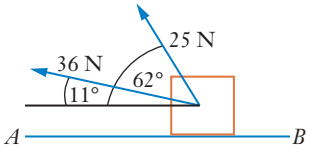
ISBN 9780170412995

c

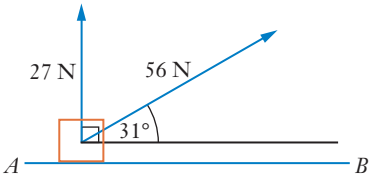


3 For each situation, calculate the magnitude of the resultant force acting in the direction of motion.

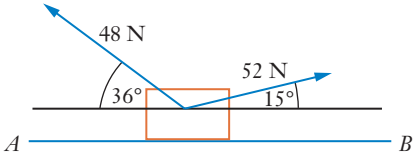
a



b



c



4 Calculate the work done by the force **F** in each situation, where the displacement is **s**.

a **F** = 36 N and **s** = 5 m

b **F** = 52 N and **s** = 3.7 m

c **F** = 105 N and **s** = 12.3 m

5 Calculate the work done in each situation, where **F** is the force acting on a body, **s** is the displacement and θ is the angle between **F** and **s**.

a **F** = 36 N, **s** = 5 m and $\theta = 18°$

b **F** = 45 N, **s** = 3.8 m and $\theta = 45°$

c **F** = 82 N, **s** = 9 m and $\theta = 36°$

d **F** = 73 N, **s** = 10.4 m and $\theta = 27°$

6 In each case, calculate the work done in moving a particle from P to Q by the force **v**.

a $P = (0, 0)$, $Q = (4, 7)$ and $\mathbf{v} = (1, 4)$

b $P = (1, 3)$, $Q = (-3, 5)$ and $\mathbf{v} = (-2, 3)$

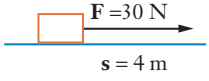
c $P = (7, 1)$, $Q = (3, -4)$ and $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}$

d $P = (-3, -7)$, $Q = (4, 9)$ and $\mathbf{v} = 5\mathbf{i} + 8\mathbf{j}$

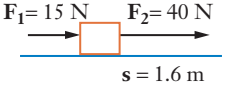
Example 13

7 Calculate the work done in each situation.

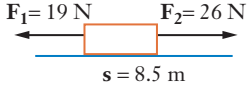
a



b



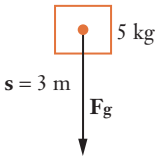
c



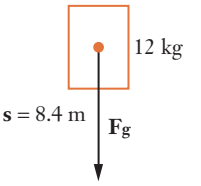
ISBN 9780170412995

8 Calculate the work done in each situation, where $\mathbf{F}_g$ is the force due to gravity.

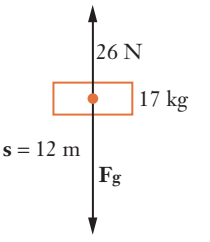
a



b



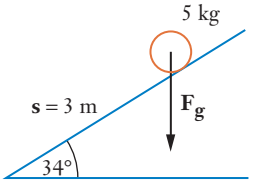
c



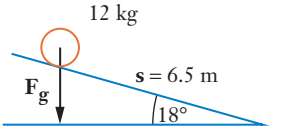
9 Calculate the work done in each situation.

Example 14

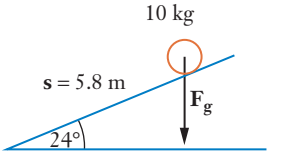
a



b

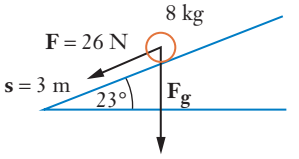


c

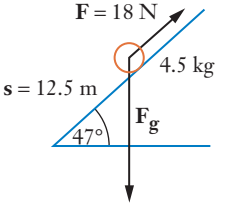


10 Calculate the work done in each situation, given that $\mathbf{F}_g$ is the force due to gravity and $\mathbf{F}_1$ and $\mathbf{F}_2$ act in the direction of the inclined plane.

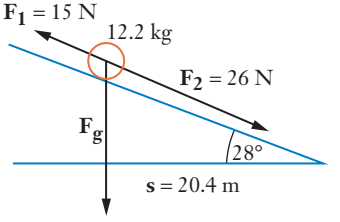
a



b



c



Problem solving

11 The acceleration of gravity is 9.81 m s^{-2} vertically downward. What is the acceleration of a toy car down a plank at 35° to the horizontal, assuming that no friction is acting?

12 A boat is being pulled along a canal by two people using ropes. One person is on the left bank pulling at an angle of 30° to the bank with a force of 400 N. The other person is on the right bank pulling at an angle of 45° to the bank. The boat is moving straight along the canal.

With what force is the second person pulling?

ISBN 9780170412995

13 An object weighing 25 kg is located 8 m up a ramp that is inclined at an angle of 18° to the horizontal. Calculate:

a the work done by the force of gravity when the object slides to the bottom of the ramp

b the minimum force (correct to 1 decimal place) acting at an angle of 60° to the ramp required to slide the object back up the ramp

c the minimum force (correct to 1 decimal place) acting at an angle of 40° to the horizontal required to slide the object back up the ramp.

14 Consider the scalar product of unit vectors in two dimensions with directions A and B. Use the scalar product to find an expression for the cosine of the angle between the vectors, $\cos(A - B)$.

15 Two forces of 30 N and 50 N act on an object. The 50 N force is at an angle of 35° to the 30 N force. The object moves 8 m at an angle of 15° to the 30 N force, away from the 50 N force.

a Find the total force acting.

b Find the work done by the:

i total force **ii** 30 N force **iii** 50 N force.

c Compare the sizes of the answers to part **b** and interpret this information mathematically.

ISBN 9780170412995

4. CHAPTER SUMMARY

Vector algebra

- The **commutative property** for vector addition states that for any two vectors **a** and **b**:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

- The **midpoint theorem** for a triangle states that the line joining the midpoints of two sides of a triangle is equal to half of the third side of the triangle, and is parallel to it.
- The **scalar** (**dot** or **inner**) **product** for vectors **a** and **b** with an angle θ between them is given by:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$$

$$\text{or } \mathbf{a} \cdot \mathbf{b} = (a_1, a_2) \cdot (b_1, b_2) = a_1b_1 + a_2b_2 \text{ (in component form)}$$

- If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ and θ is the angle between **a** and **b**, then:

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \text{ or } \cos(\theta) = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + b_1^2} \times \sqrt{a_2^2 + b_2^2}}$$

- Properties of the scalar product:

  1 The scalar product for vectors in polar form is:

  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$, where θ is the angle between **a** and **b**.

  For vectors in component form, the scalar product is:

  $\mathbf{a} \cdot \mathbf{b} = (a_1, a_2) \cdot (b_1, b_2) = a_1b_1 + a_2b_2$, where $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$.

  2 The scalar product is a real number, *not* a vector.

  3 The scalar product is commutative.

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

  4 The scalar product is distributive over vector addition.

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

  5 $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$ for all real numbers m and vectors **a** and **b**.

  6 $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ (because $\cos(\theta) = 1$)

- If **a** and **b** are perpendicular, then $\mathbf{a} \cdot \mathbf{b} = 0$.

  If **a** and **b** are parallel, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$ or $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$.

- For unit vectors **i** and **j**:

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$$

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$$

ISBN 9780170412995

- Given vectors **a** and **b** with an angle of θ between them, the **projection** of **a** on **b** is:

$$p = |\mathbf{a}| \cos(\theta) = \mathbf{a} \cdot \hat{\mathbf{b}} = \frac{\mathbf{a} \circ \mathbf{b}}{|\mathbf{b}|}$$

- The **vector projection** of **a** on **b** is given as $\frac{\mathbf{a} \circ \mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = |\mathbf{a}| \cos(\theta)\hat{\mathbf{b}}$.
- Force is the product of mass (m) and acceleration (a): $F = ma$. The unit of force is the newton (N) and 1 N = 1 kg m/s^2
- **Work** (W) is a scalar quantity. Work is the product of force and displacement and the unit of force is the joule (J), where 1 J = 1 Nm. In terms of vectors:

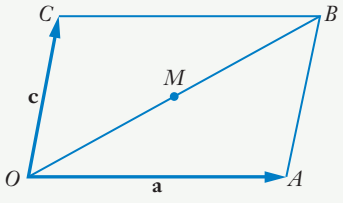
$$W = \mathbf{F} \cdot \mathbf{s}$$

ISBN 9780170412995

4. CHAPTER REVIEW

Vector algebra

1 $OABC$ is a parallelogram and M is the midpoint of OB. Taking O as the origin, the position vectors of A and C are **a** and **c** respectively. Express each vector in terms of **a** and **c**.

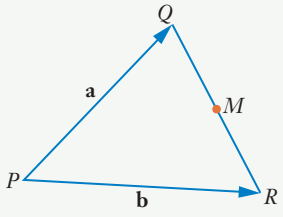


Example 1

a BC **b** AC **c** OB

d OM **e** AM **f** MC

2 $\triangle PQR$ is shown at right, where $\mathbf{PQ} = \mathbf{a}$, $\mathbf{PR} = \mathbf{b}$ and M is the midpoint of QR. Express each vector in terms of **a** and **b**.



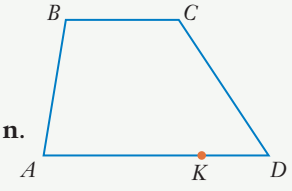
Example 2

a QR **b** QM **c** PM

3 $ABCD$ is a trapezium with BC parallel to AD and $AD = 2BC$.

K is the point on AD for which $AK : KD = 3 : 1$.

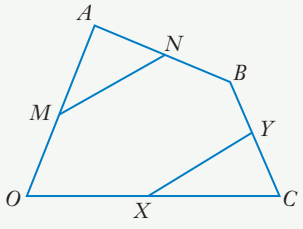
Given that $\mathbf{AB} = \mathbf{m}$ and $\mathbf{BC} = \mathbf{n}$, express **KC** in terms of **m** and **n**.



Example 3

4 Consider the quadrilateral $OABC$ shown here. O is taken to be the origin, so $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and so on.

M is the midpoint of OA, N is the midpoint of AB, X is the midpoint of OC and Y is the midpoint of BC. Use vector addition and multiplication by scalars to show that MN and XY are parallel and have the same length.



Example 3

5 Find the scalar product of:

Example 4

a vectors of magnitudes 11 and 7 at an angle of 85° to each other

b $\mathbf{c} = (7, 23°)$ and $\mathbf{d} = (13, 146°)$

c $\mathbf{u} = (-4, 3)$ and $\mathbf{v} = (5, -8)$

6 Calculate the angle between each pair of vectors.

Example 5

a $\mathbf{c} = (4, -3)$ and $\mathbf{d} = (1, 1)$

b $\mathbf{g} = (3, -8)$ and $\mathbf{h} = (3, 6)$

c $\mathbf{m} = (3, -2)$ and $\mathbf{n} = (5, 150°)$

d $\mathbf{c} = (-9, 4)$ and $\mathbf{d} = (11, 225°)$

ISBN 9780170412995

Example 6

7 Given $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$ and $\mathbf{r} = (r_1, r_2)$, prove that:

$$\mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) = \mathbf{p} \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{r}.$$

Example 6

8 Given that $\mathbf{u} \cdot \mathbf{v} = 2$, $|\mathbf{u}| = 1$ and $|\mathbf{v}| = 3$, calculate the value of:

a $\mathbf{u} \cdot 4\mathbf{v}$ **b** $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$

c $2\mathbf{u} \cdot (3\mathbf{u} - \mathbf{v})$ **d** $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

Example 7

9 Show that the vectors $\mathbf{m} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\mathbf{n} = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$ are perpendicular.

Example 8

10 Show that the vectors $\mathbf{a} = (-12, 30)$ and $\mathbf{b} = \left(\frac{1}{4}, \frac{-5}{8}\right)$ are parallel.

Example 8

11 Determine if each pair of vectors are parallel, perpendicular or neither.

a $\mathbf{c} = (-6, 15)$ and $\mathbf{d} = \left(\frac{1}{2}, \frac{-5}{4}\right)$ **b** $\mathbf{g} = (2, -5)$ and $\mathbf{h} = (3, 15)$

c $\mathbf{c} = (5, 6)$ and $\mathbf{d} = \left(\frac{3}{4}, \frac{-1}{4}\right)$ **d** $\mathbf{g} = (-2, -2)$ and $\mathbf{h} = (1, -1)$

Example 9

12 Calculate the projection in the direction 67° of a vector of magnitude 20 in the direction 32°.

Example 9

13 Find the projection in the direction 328° of a vector of magnitude 50 in the direction 38°.

Example 10

14 Calculate the projection of $\mathbf{m} = (6, 1)$ on $\mathbf{n} = (-2, 4)$.

Example 11

15 A force of 150 N acts vertically downwards on a piece of wood lying on a slope of 25°. Resolve the force into components parallel and perpendicular to the slope.

Example 12

16 A vehicle with mass 5400 kg stands on a ramp that is inclined to the horizontal at an angle of 10°. Calculate the force required to prevent the vehicle from rolling down the ramp?

Example 13

17 **a** Calculate the work done by a horizontal force of 125 N that moves an object 12 m horizontally.

b Calculate the work done by gravity when an 8 kg object falls 2.4 m vertically.

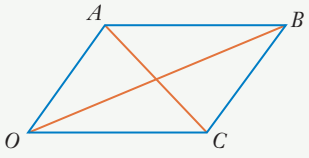
Example 14

18 A 25 kg block slides down a frictionless ramp inclined at 23° to the horizontal under the force of gravity. If the block slides down 6.3 m before reaching the bottom of the ramp, how much work is done?

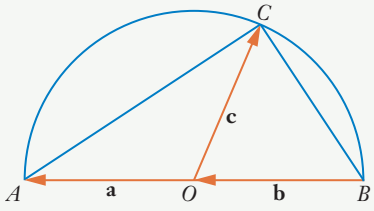
ISBN 9780170412995

Problem solving

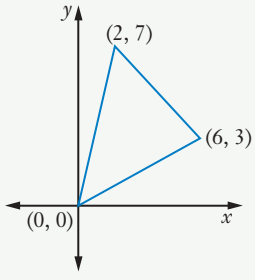
19 Consider the parallelogram shown on the right. Prove that the sum of the squares of the lengths of both diagonals of the parallelogram is equal to the sum of the squares of the lengths of all four sides.



20 Use vectors to prove that the angle subtended by a diameter of a circle is 90°, i.e. $\angle ACB = 90°$.



21 Calculate the value of the angles in the triangle shown on the right.



22 Show that $(\mathbf{v} + \mathbf{w}) \bullet \mathbf{v} - (\mathbf{v} - \mathbf{w}) \bullet \mathbf{w} = |\mathbf{v}|^2 + |\mathbf{w}|^2$

Practice quiz

ISBN 9780170412995

5.

CIRCLE PROPERTIES AND VECTOR PROOFS

You should already know many of the rules for angles in circles from Year 10. In this chapter, you will prove and extend these rules. You will also see how vectors can be used in Geometry.

It is easier to prove some geometric properties with vectors than by using other methods.

For example, connect the vertices of a triangle to points $\frac{2}{3}$ of the way along the opposite side. This makes a smaller triangle in the middle. The area of this triangle is $\frac{1}{7}$ of the area of the original triangle. This is easier to show with the dot product than with other methods.

5.01 Angles at the centre of a circle
5.02 Angles at the circumference of a circle
5.03 Semicircle angle
5.04 Cyclic quadrilaterals
5.05 Chords and radii
5.06 Tangents
5.07 Secants
5.08 Mixed problems in circle geometry
5.09 Vector proofs of parallelogram properties
5.10 Other vector proofs in geometry
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Circle properties and their proofs

- prove circle properties, such as
  - an angle in a semicircle is a right angle
  - the angle at the centre subtended by an arc of a circle is twice the angle at the circumference subtended by the same arc
  - angles at the circumference of a circle subtended by the same arc are equal
  - the opposite angles of a cyclic quadrilateral are supplementary
  - chords of equal length subtend equal angles at the centre and conversely chords subtending equal angles at the centre of a circle have the same length
  - a tangent drawn to a circle is perpendicular to the radius at the point of contact
  - the alternate segment theorem
  - when two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord and its converse
  - when a secant (meeting the circle at A and B) and a tangent (meeting the circle at T) are drawn to a circle from an external point M, the square of the length of the tangent equals the product of the lengths to the circle on the secant ($AM \times BM = TM^2$) and its converse
- solve problems finding unknown angles and lengths and prove further results using the results listed above

Geometric proofs using vectors

- prove the diagonals of a parallelogram meet at right angles if and only if it is a rhombus
- prove midpoints of the sides of a quadrilateral join to form a parallelogram
- prove the sum of the squares of the lengths of a parallelogram's diagonals is equal to the sum of the squares of the lengths of the sides
- prove an angle in a semicircle is a right angle

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

Shutterstock/B Brown

TERMINOLOGY

alternate segment
altitude
angle at the centre
angles at the circumference
centroid
chord
circumcentre
circumcircle
concyclic
cyclic quadrilateral
displacement vector
dot product
exterior angle
in-centre
in-circle
median
orthocentre
position vector
secant
segment
semicircle angle
subtend
tangent

5.01 Angles at the centre of a circle

Parts of a circle

A page of circles

Angles at the centre and circumference

In geometry, it is conventional to label the centre of the circle O.

INVESTIGATION

ANGLES AT THE CENTRE OF A CIRCLE

There is a spreadsheet called 'Angles at the centre of a circle' on NelsonNet that you can use in this investigation.

Change the position of A.

What happens to the values of $\angle ABC$ and $\angle AOC$?

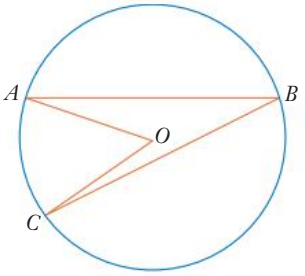
Change the position of C.

What happens to the values of $\angle ABC$ and $\angle AOC$?

Change the position of B.

What happens to the values of $\angle ABC$ and $\angle AOC$?

What do the sizes of the angles depend on?



From the investigation, it appears that the following is true.

Angle at the centre of a circle

- The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc
- Conversely, the vertex of an angle equal to half the angle at the centre of a circle subtended by the same arc lies on the circle.

ISBN 9780170412995

It is relatively easy to prove this theorem, although there are 3 cases to consider.

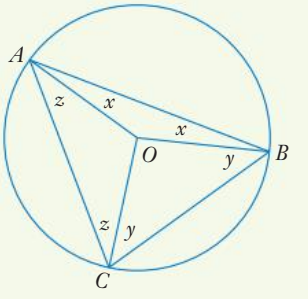
The centre of the circle can be within the angle at the circumference, on one of the arms, or outside the angle.

EXAMPLE 1

Prove the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc for the case where the centre is within the angle.

Solution

For an arc $\overset{\frown}{AC}$, construct $\angle ABC$ on a circle such that the centre lies within at the angle and A, B and C lie on the circle. Construct $\angle AOC$ and the lines AC and OB.



State what is needed.

RTP: $\angle AOC = 2\angle ABC$

Proof

Use the radii to state the isosceles triangles.

$OA = OB = OC$ (all radii)
$\Rightarrow \triangle AOC$, $\triangle BOC$ and $\triangle AOB$ are isosceles

Give the equal angles convenient names.

Let $\angle OCB = \angle OBC = y$, $\angle OAC = \angle OCA = z$ and $\angle OAB = \angle OBA = x$

Write the angle sums in the triangles with the required angles and write $\angle ABC$ in terms of the variable names.

$2x + 2y + 2z = 180°$ ($\triangle ABC$),
$2z + \angle AOC = 180°$ ($\triangle AOC$),
and $\angle ABC = x + y$

Rearrange.

$\Rightarrow \angle AOC = 180° - 2z$
$= 180° - [180° - 2(x + y)]$
$= 180° - 180° + 2(x + y)$
$= 2\angle ABC$ QED

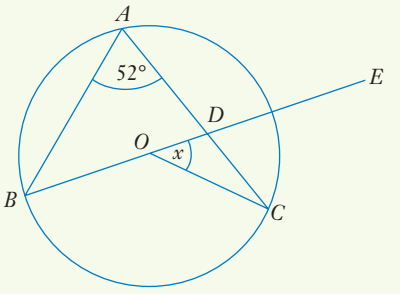
The rule can be used to find angles connected to the angle at the centre, or angles connected to the angle at the circumference. To find unknown angles, work out any angles you can.

ISBN 9780170412995

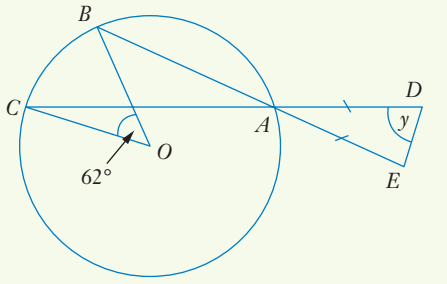
EXAMPLE 2

Find the lettered angle in these diagrams, giving reasons for each step.

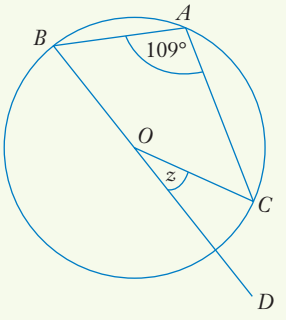
a



b



c



Solution

a You can work out the centre angle. $\angle BOC = 104°$ (angle at centre)

Now you can find x. $x = 76°$ (straight line)

b You can find another angle in the circle. $\angle BAC = 31°$ (semicircle angle)

Now you can do an angle in the triangle. $\angle DAE = 31°$ (vertically opposite)

Find the answer. $y = 74\frac{1}{2}°$ (angle sum in isosceles $\triangle ADE$)

c You can work out the centre angle. reflex $\angle BOC = 218°$ (angle at centre)

Use the straight line. $z = 218° - 180° = 38°$

ISBN 9780170412995

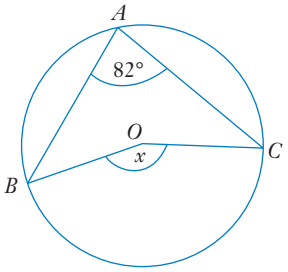
Exercise 5.01 Angles at the centre of a circle

Example 1

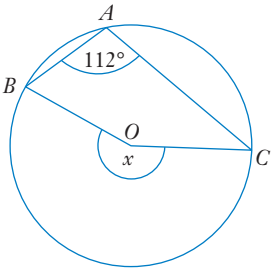
Example 2

1 Find the indicated angles in the following diagrams, giving the reason(s).

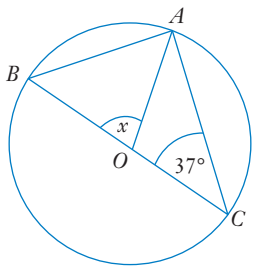
a



b

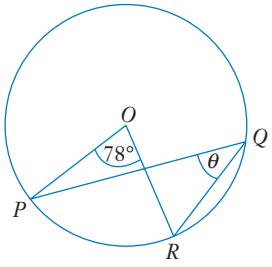


c

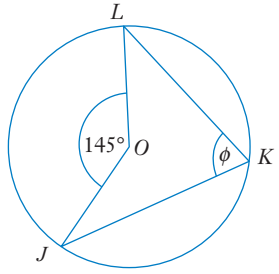


2 Find the indicated angles in the following diagrams, giving the reason(s).

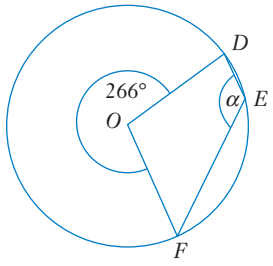
a



b

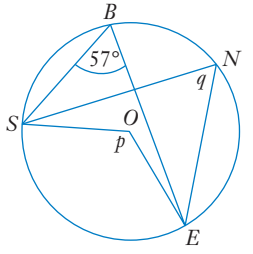


c

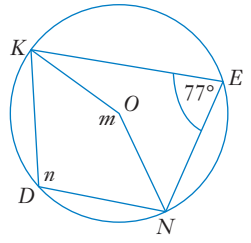


3 Find the indicated angles in the following diagrams, giving the reason(s) for each step.

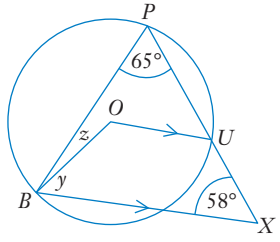
a



b

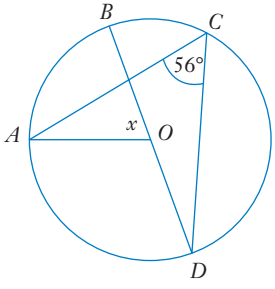


c

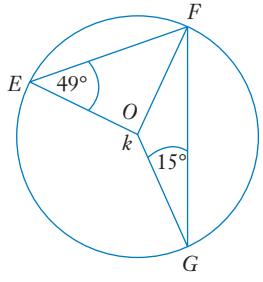


4 Find the indicated angles in the following diagrams, giving the reason(s) for each step.

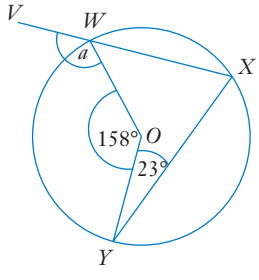
a



b



c



ISBN 9780170412995

Problem solving

5 Prove that the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc for the case where the centre is on an arm of the angle.

6 Prove that the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc for the case where the centre is outside the angle.

7 Prove that the vertex of an angle equal to half the angle at the centre of a circle subtended by the same arc lies on the circle.

8 $\angle BAD$ and $\angle BCD$ are constructed on a circle such that the centre of the circle is within the figure $ABCD$ and A, B, C and D are on the circumference. Prove that $\angle BAD$ and $\angle BCD$ are supplementary.

9 $\angle BAD$ and $\angle BCD$ are constructed on a circle such that the centre of the circle is outside the figure $ABCD$ and A, B, C and D are on the circumference. Prove that $\angle ABC$ and $\angle ADC$ are supplementary.

5.02 Angles at the circumference of a circle

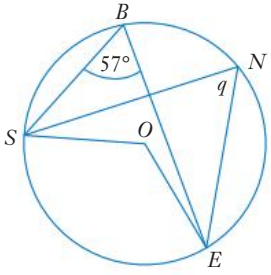
Angles at the centre and circumference

In the diagram, the angles $\angle SNE = q$ and $\angle SBE = 57°$ are both subtended by the arc $\overset{\frown}{SE}$.

Both angles are half of the angle at the centre $\angle SOE$.

$\Rightarrow q = 57°$

$\Rightarrow \angle SBE = \angle SNE$.

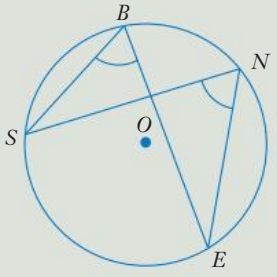


Removing the lines OS and OE does not change $\angle SBE$ or $\angle SNE$.

In the second diagram, it must still be true that $\angle SBE = \angle SNE$.

Angles at the circumference of a circle

- Angles at the circumference of a circle subtended by the same arc are equal, provided they are on the same side of the arc.
- Conversely, the vertex of an angle equal to an angle at the circumference subtended by the same arc must lie on the circle.

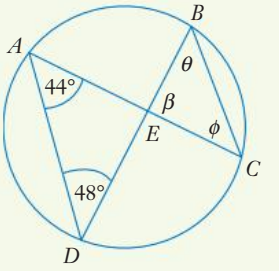


The rule for angles at the circumference can be proven by constructing the angle at the centre. Look for obvious angles you can find to calculate unknown angles.

ISBN 9780170412995

EXAMPLE 3

Find the value of each pronumeral, giving the reason(s).



Solution

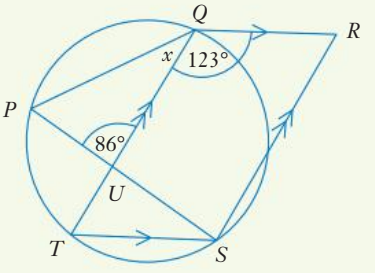
θ and 44° subtend the same side of $\overset{\frown}{CD}$.	$\theta = 44°$ (angles at circumference)
ϕ and 48° subtend the same side of $\overset{\frown}{AB}$.	$\phi = 48°$ (angles at circumference)
Use the angles in the triangle.	$\beta = 88°$ (angle sum in $\triangle BEC$)

You can often start solutions where there are known angles.

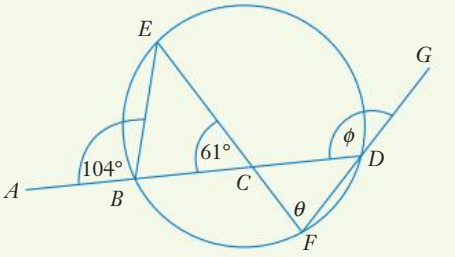
EXAMPLE 4

Find the value of each pronumeral, giving the reason(s) for each step.

a



b



Solution

a	Use the parallel lines.	$\angle UTS = 57°$ (co-interior)
	Get another angle in the triangle.	$\angle UPQ = 57°$ (angles at circumference)
	Use the angles in the triangle.	$x = 37°$ (angle sum in $\triangle UPQ$)
b	Use the 104° angle on the straight line.	$\angle EBC = 74°$ (straight line)
	Use the circle angles.	$\theta = 74°$ (angles at circumference)
	Use the 61° angle.	$\angle EBC = 74°$ (vertically opposite)
	Now you can find the other unknown.	$\phi = 74°$ (external angle of $\triangle CDF$)

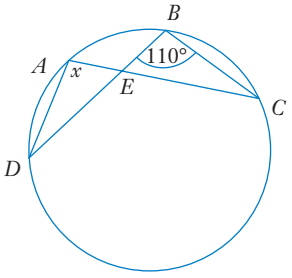
ISBN 9780170412995

Exercise 5.02 Angles at the circumference of a circle

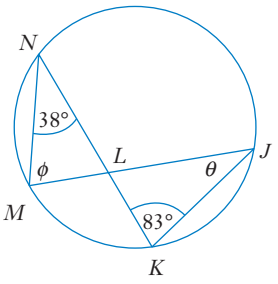
Example 3

1 Find the value of each pronumeral, giving the reason(s).

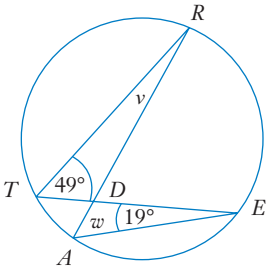
a



b

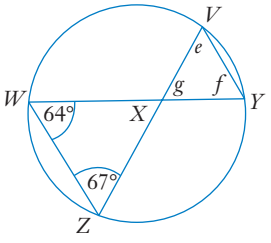


c

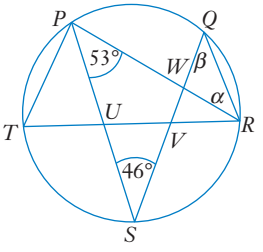


2 Find the value of each pronumeral, giving the reason(s).

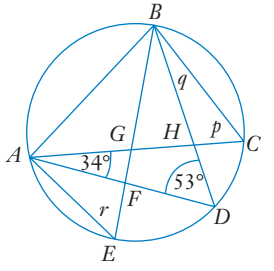
a



b



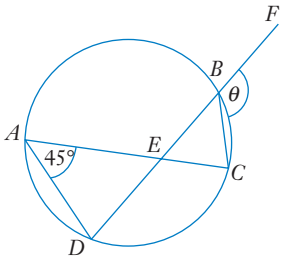
c



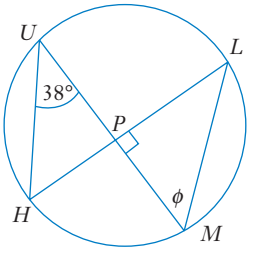
Example 4

3 Find the value of each pronumeral, giving the reason(s) for each step.

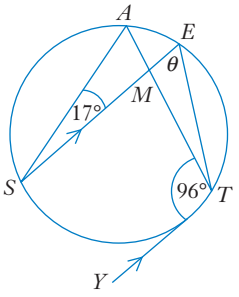
a



b



c

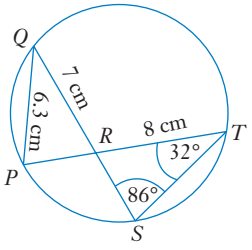


Problem solving

4 Prove that angles at the circumference of a circle subtended by the same arc and on the same side of the arc are equal.

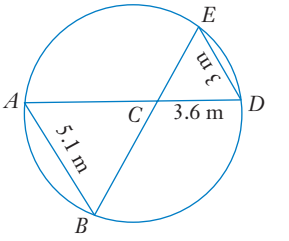
5 Prove that angles at the circumference of a circle subtended by the same arc, but on opposite sides of the arc, are supplementary.

6 Prove that $\triangle PQR \,|||\, \triangle STR$ in this diagram and hence find ST.

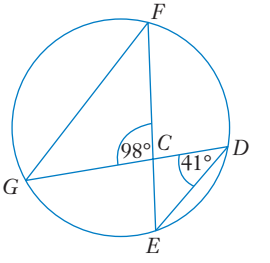


ISBN 9780170412995

7 Prove that $\Delta ABC \,|||\, \Delta DEC$ in this diagram and hence find BC.



8 Prove that $FG \parallel DE$ in this diagram.



5.03 Semicircle angle

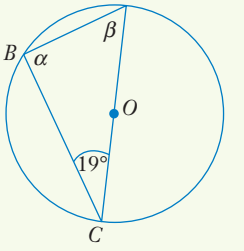
A diameter of a circle has an angle of 180° at the centre. You know that the angle at the centre is twice the angle at the circumference. This gives the semicircle rule.

Semicircle angle

- An angle at the circumference of a circle subtended by a diameter is 90°, a right angle.
- Conversely, a chord that subtends an angle of 90° at the circumference is a diameter.

EXAMPLE 5

Find the value of each pronumeral, giving the reason(s).



Solution

AC is a diameter.	$\alpha = 90°$ (semicircle angle)
Use the triangle.	$\beta = 71°$ (angle sum of $\triangle ABC$)

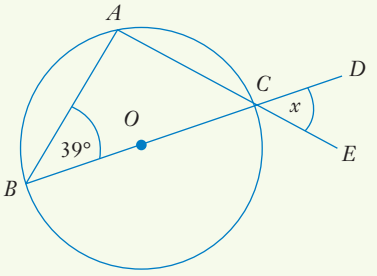
Be precise when you state a reason, such as the angle sum of a triangle or quadrilateral.

ISBN 9780170412995

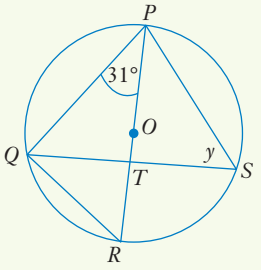
EXAMPLE 6

Find the value of each pronumeral, giving the reason(s) for each step.

a



b



Solution

a BC is a diameter. $\angle BAC = 90°$ (semicircle angle)

Use the triangle. $\angle BCA = 51°$ (angle sum of $\triangle ABC$)

Now you can find x. $x = 51°$ (vertically opposite)

b PR is a diameter. $\angle PQR = 90°$ (semicircle angle)

Use the triangle. $\angle QRP = 59°$ (angle sum of $\triangle PQR$)

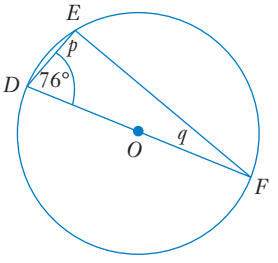
Now you can find y. $y = 59°$ (angles on the circumference)

Exercise 5.03 Semicircle angle

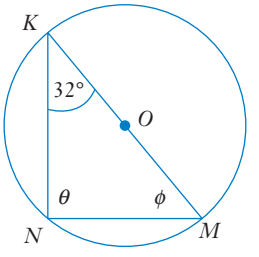
Example 5

1 Find the value of each pronumeral, giving the reason(s).

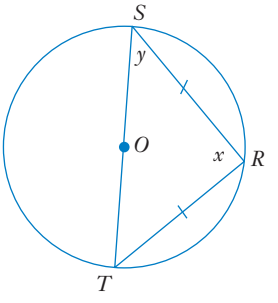
a



b

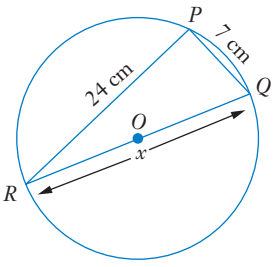


c

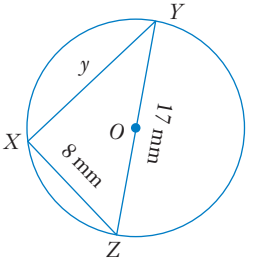


2 Find the value of each pronumeral, giving the reason(s).

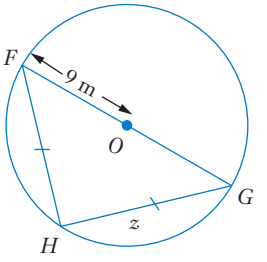
a



b



c

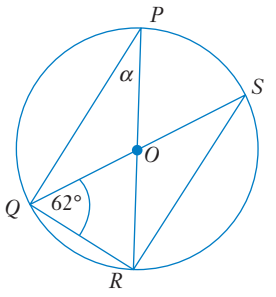


ISBN 9780170412995

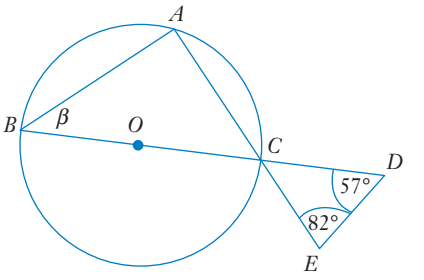
Example 6

3 Find the value of each pronumeral, giving the reason(s) for each step.

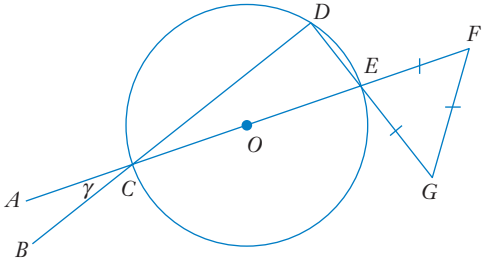
a



b

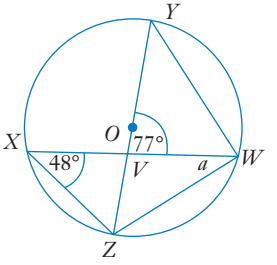


c

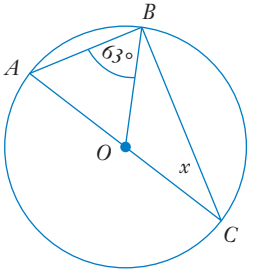


4 Find the value of each pronumeral, giving the reason(s) for each step.

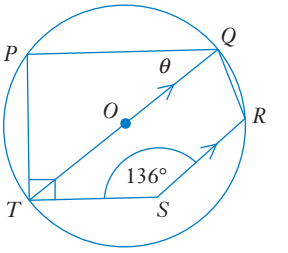
a



b



c



Problem solving

5 Prove the semicircle angle rule.

6 Prove the converse of the semicircle rule.

7 A triangle ABC has vertices on a circle and AB is a diameter. $AC = 2a$ and $BC = 4a$. Show that the radius of the circle is $\sqrt{5}a$.

8 Prove that a quadrilateral with vertices on a circle and diagonals that are diameters is a rectangle.

ISBN 9780170412995

5.04 Cyclic quadrilaterals

A circle can always be drawn through the vertices of a triangle. It is called the **circumcircle**. It's centre is the intersection of the perpendicular bisectors of the sides. Only some quadrilaterals can have a circle drawn through their vertices.

Cyclic quadrilaterals

- Four or more points are **concyclic** if they lie on the same circle
- A **cyclic quadrilateral** has all four vertices on a circle
- Opposite angles of a cyclic quadrilateral are supplementary
- Conversely, a quadrilateral with supplementary opposite angles is cyclic
- The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle
- Conversely, a quadrilateral with an exterior angle equal to the interior opposite angle is cyclic

Proof

Draw a cyclic quadrilateral $ABCD$ in a circle with centre O. Construct AO and CO.

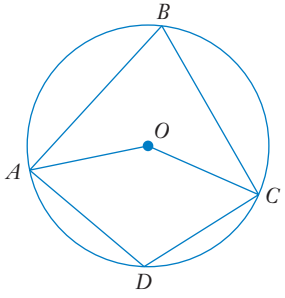
$\angle AOC = 2\angle ABC$ and reflex $\angle AOC = 2\angle ADC$

$\Rightarrow \angle AOC + \text{reflex } \angle AOC = 2\angle ABC + 2\angle ADC$

But $\angle AOC + \text{reflex } \angle AOC = 360°$

$\Rightarrow 2\angle ABC + 2\angle ADC = 360°$

$\Rightarrow \angle ABC + \angle ADC = 180°$ QED



You will prove the other results in the exercise.

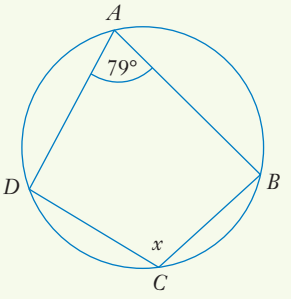
iStock/ZambeziShark

ISBN 9780170412995

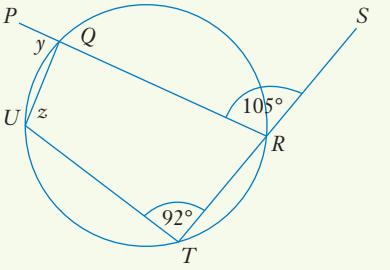
EXAMPLE 7

Find the value of each pronumeral, giving the reasons.

a



b



Solution

a *ABCD* is a cyclic quadrilateral. $x = 101°$ (opposite angles in cyclic quad. *ABCD*)

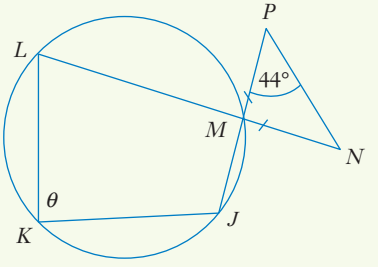
b y is an exterior angle of *QRTU*. $y = 92°$ (exterior angle in cyclic quad. *QRTU*)

105° is an exterior angle of *QRTU*. $z = 105°$ (interior opposite of cyclic *QRTU*)

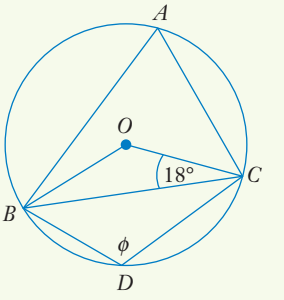
EXAMPLE 8

Find the value of each pronumeral, giving the reason(s) for each step.

a



b



Solution

a Use the equal sides. $\angle PNM = 92°$ (angle sum of isosceles $\triangle PMN$)

Now use *KLMJ*. $\theta = 92°$ (interior opposite of cyclic quad. *KLMJ*)

b Use the radii. $\angle BOC = 144°$ (angle sum of isosceles $\triangle BOC$)

Use the angle at the centre. $\angle BAC = 72°$ (angle at centre)

Now use *ABCD*. $\phi = 108°$ (opposite angles in cyclic quad. *ABCD*)

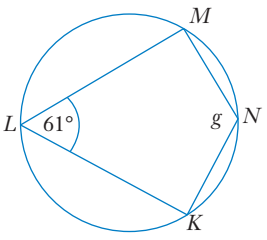
ISBN 9780170412995

Exercise 5.04 Cyclic quadrilaterals

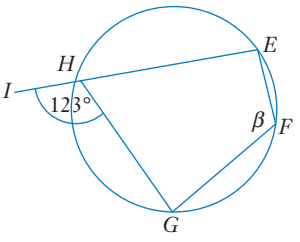
Example 7

1 Find the value of each pronumeral, giving the reason(s).

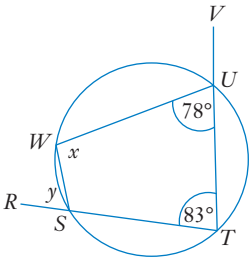
a



b

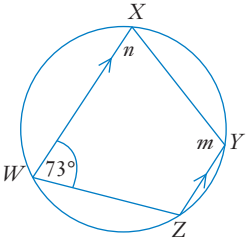


c

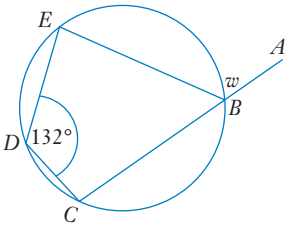


2 Find the value of each pronumeral, giving the reason(s).

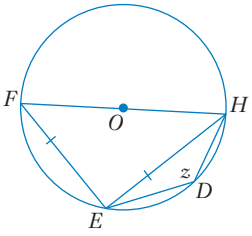
a



b



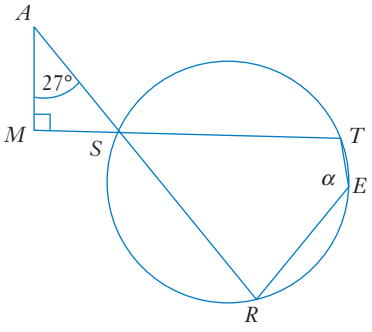
c



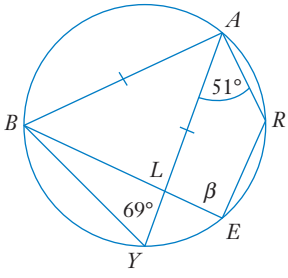
Example 8

3 Find the value of each pronumeral, giving the reason(s) for each step.

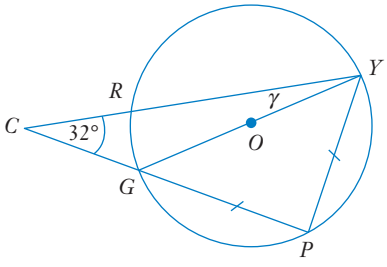
a



b



c

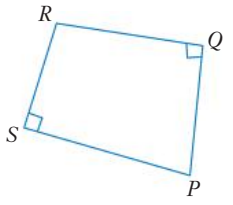


ISBN 9780170412995

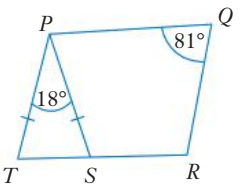
Problem solving

4 Show that, in each of the following, $PQRS$ is a cyclic quadrilateral.

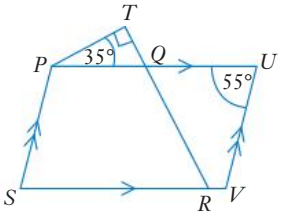
a



b



c



5 **a** Show that the intersection of the perpendicular bisectors of two sides of a triangle is equidistant from the vertices.

b Hence show that any triangle has a circumcircle centred at the intersection of the perpendicular bisectors of the sides.

6 Prove that a quadrilateral with supplementary opposite angles is cyclic.

7 Prove that the exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

8 Prove that a quadrilateral with an exterior angle equal to the interior opposite angle is cyclic.

9 Prove the converse of the angles at the circumference rule: If AB subtends equal angles at C and D, then A, B, C and D are on the circumference of a circle. HINT: Prove that $ABCD$ is a cyclic quadrilateral.

5.05 Chords and radii

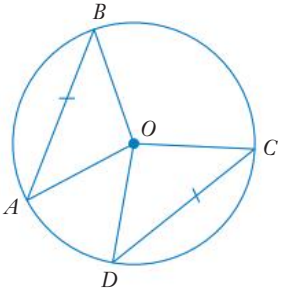
Angles in semicircles and subtended by equal chords

Consider two chords AB and CD of equal length in the same circle.

Construct the triangles $\triangle AOB$ and $\triangle COD$.

$\triangle AOB \equiv \triangle COD$ (SSS)

$\Rightarrow \angle AOB \equiv \angle COD$.



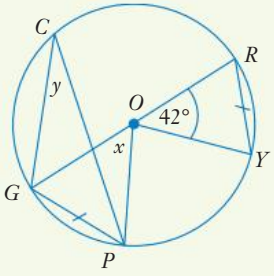
Equal chords

- Chords of equal length subtend equal angles at the centre of a circle, and conversely
- Chords subtending equal angles at the centre of a circle have the same length

ISBN 9780170412995

EXAMPLE 9

Find the value of each pronumeral, giving the reasons.

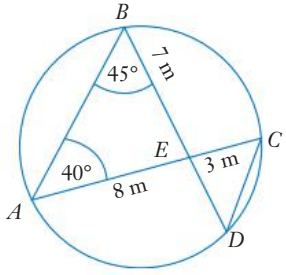


Solution

Use the equal chords. $x = 42°$ (centre angles of equal chords)

x and y are both subtended by $\overset{\frown}{GP}$. $y = 21°$ (half angle at centre)

Consider the diagram on the right. What is the length of ED?



$\angle CDE = 40°$ (angles at the circumference)

$\angle DCE = 45°$ (angles at the circumference)

$\Rightarrow \triangle ABE \,|||\, \triangle DCE$ (AA)

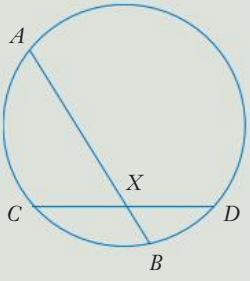
$$\Rightarrow \frac{8}{7} = \frac{ED}{3}$$

$$\Rightarrow ED = \frac{8 \times 3}{7} = 3\tfrac{3}{7} \text{ m}$$

The triangles will be similar in any similar situation, so the lengths of the intervals BE, ED, CE and AE will bear the same relationship to each other.

Intersecting chords

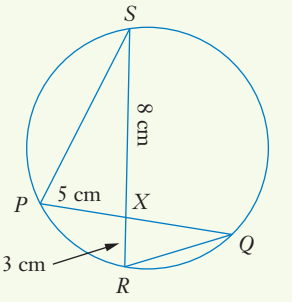
- For intersecting chords, the products of the intervals of each chord are equal: in the diagram, $AX \times XB = CX \times XD$
- Conversely, if the products of the lengths of intervals of intersecting line segments are equal, they are chords of the same circle (their ends are concyclic)



ISBN 9780170412995

EXAMPLE 10

Find the length of the chord PQ in this diagram.



Solution

PQ and RS are intersecting chords.	$PX \times QX = SX \times RX$ (intersecting chords)
Substitute values.	$5QX = 8 \times 3$
Solve for QX.	$QX = 4.8$ cm
Write the answer.	PQ is of length $5 + 4.8 = 9.8$ cm

Consider the diagram on the right.

What can you say about $\angle ACO$?

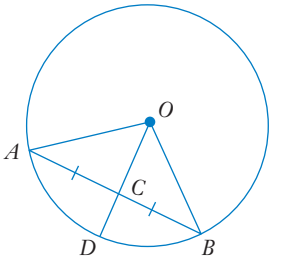
$AO = BO$ (both radii)

CO is common

$AC = CB$ (given)

$\Rightarrow \triangle AOC \equiv \triangle BOC$ (SSS)

$\Rightarrow \angle ACO = \angle BCO = 90°$



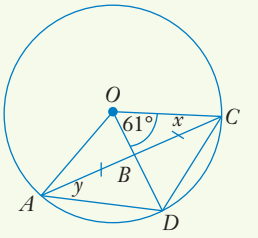
Chord bisectors

- A radius that bisects a chord is perpendicular to the chord
- Conversely, a radius that is perpendicular to a chord bisects the chord

ISBN 9780170412995

EXAMPLE 11

Find the value of each pronumeral, giving the reasons for each step.



Solution

Start with the triangle with the most information.

$\angle OBC = 90°$ (chord bisector)
$\Rightarrow x = 29°$ (angle sum of $\triangle OBC$)

Use the other half of the bisection.

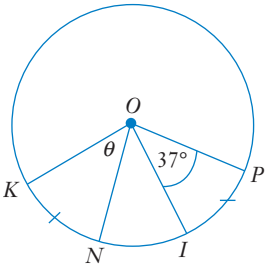
$\angle AOB = 61°$ (symmetry)
$\Rightarrow$ reflex $\angle AOC = 238°$ (angles at a point)
$\Rightarrow \angle ADC = 119°$ (centre angle)
$\Rightarrow \angle ADB = 59.5°$ (symmetry)
$\Rightarrow y = 30.5°$ (angle sum of $\triangle ABD$)

Exercise 5.05 Chords and radii

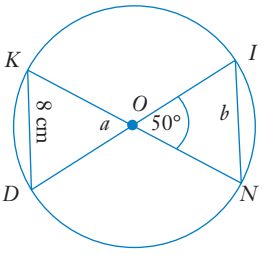
Example 9

1 Find the value of each pronumeral, giving the reasons.

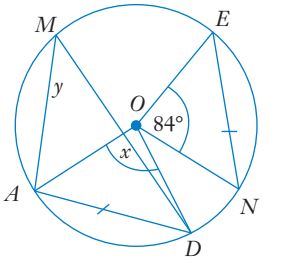
a



b



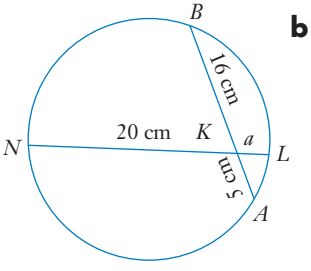
c



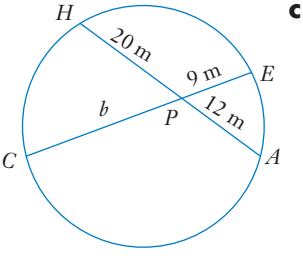
Example 10

2 Find the value of each pronumeral, giving the reasons.

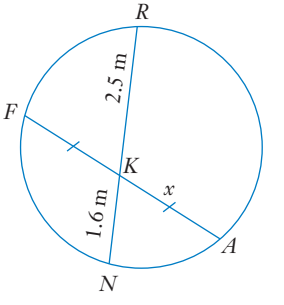
a



b



c

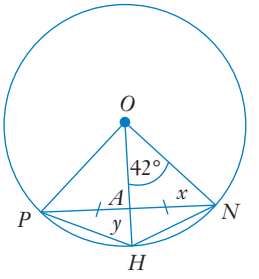


ISBN 9780170412995

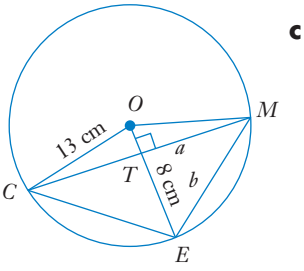
Example 11

3 Find the value of each pronumeral, giving the reasons for each step.

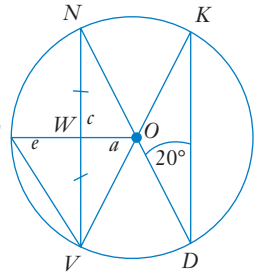
a



b

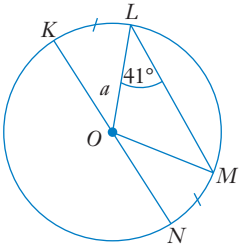


c

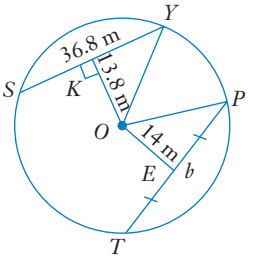


4 Find the value of each pronumeral, giving the reasons for each step.

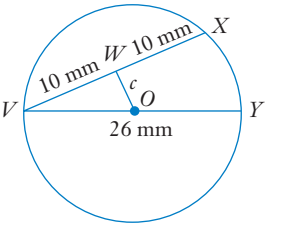
a



b



c



Problem solving

5 Prove that chords of equal length subtend equal angles at the centre of a circle.

6 Prove that chords subtending equal angles at the centre of a circle have the same length.

7 Prove that the products of the intervals of each chord of intersecting chords are equal.

8 Prove that if the products of the lengths of intervals of intersecting line segments are equal, they are chords of the same circle.

9 Prove that a radius that bisects a chord is perpendicular to the chord.

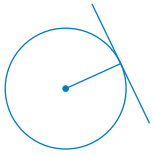
10 Prove that a radius perpendicular to a chord bisects the chord.

11 Prove that equal chords are equidistant from the centre of a circle.

ISBN 9780170412995

5.06 Tangents

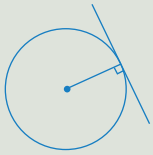
A **tangent** to a circle touches the circle at exactly one point.



From symmetry, it appears that the angles between a tangent and radius must be equal, so they should be right angles.

Tangent and radius

A tangent to a circle is perpendicular to the radius at the point of contact.

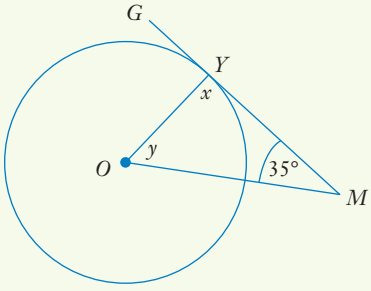


You can prove that the angle is 90° similarly to the proof that the shortest distance between a point and a line is the perpendicular distance in Example 16 of Chapter 2 on page 83, or by using the theorem.

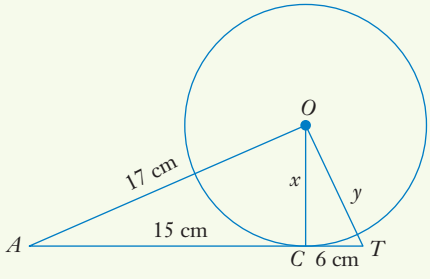
EXAMPLE 12

Find the value of each pronumeral.

a



b



Solution

a Use the tangent and radius. $x = 90°$ (tangent and radius)

Use the triangle. $y = 55°$ (angle sum of $\triangle OYM$)

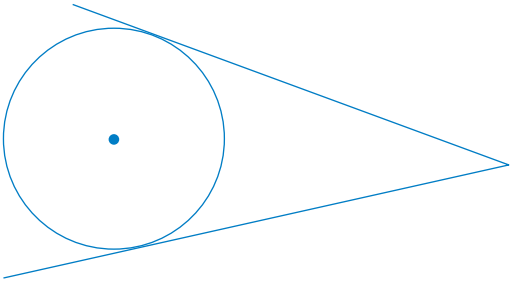
b Use the tangent and radius. $\angle ACO = 90°$ (tangent and radius)

Use $\triangle ACO$ to find x. $17^2 = 15^2 + x^2$ (Pythagoras)
$289 = 225 + x^2$
$x^2 = 64$
$x = 8$ cm

Use $\triangle TCO$ to find x. $y^2 = 8^2 + 6^2$ (Pythagoras)
$= 100$
$y = 10$ cm

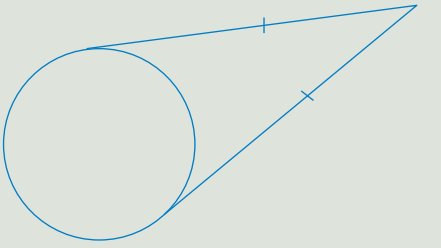
ISBN 9780170412995

You can draw two different tangents from the same point to a circle.



Tangents from a common point

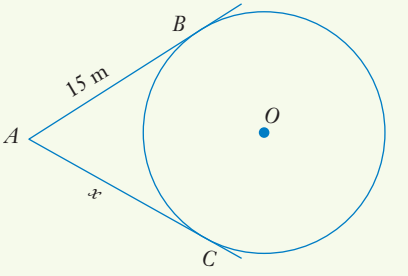
Tangents drawn from the same point to a circle are the same length.



You can prove this using congruent triangles. Draw the radii where the tangents touch and the line from the centre to the common point. Show the triangles formed are congruent, so the tangents are equal.

EXAMPLE 13

Find the value of x.



Solution

The tangents are from the same point. $x = 15$ m (tangents from a common point)

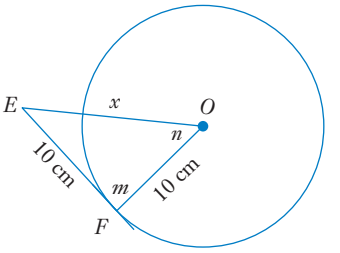
ISBN 9780170412995

Exercise 5.06 Tangents

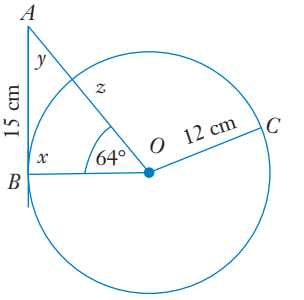
Example 12

1 Find the value of each pronumeral, giving reasons for each step.

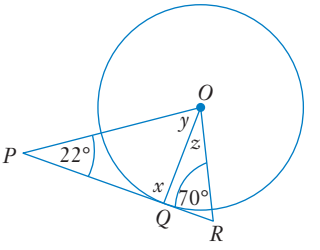
a



b



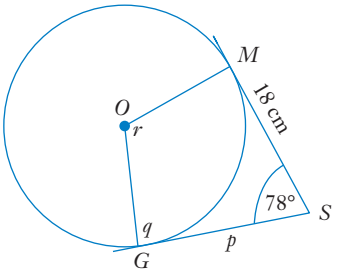
c



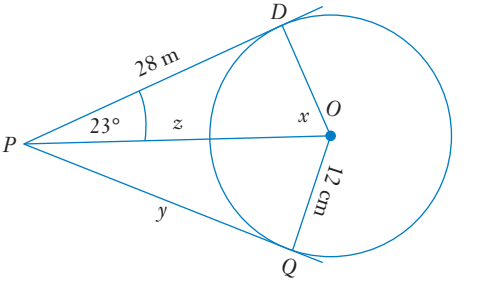
Example 13

2 Find the value of each pronumeral, giving reasons for each step.

a



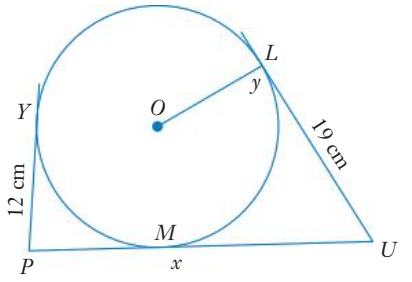
b



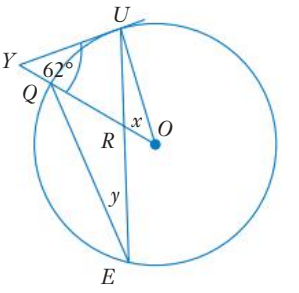
ISBN 9780170412995

3 Find the value of each pronumeral, giving reasons for each step.

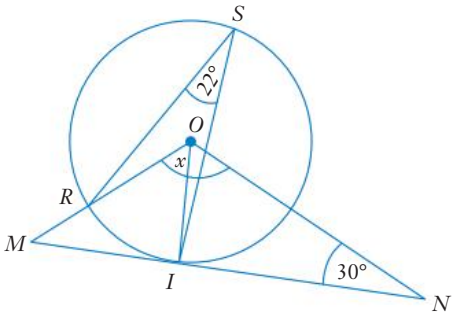
a



b



c

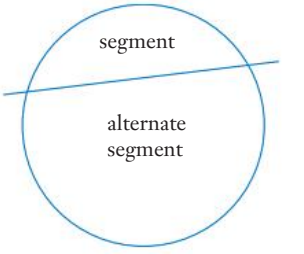


Problem solving

4 Prove that a tangent is perpendicular to the radius at the point of contact.

5 Prove that tangents drawn from the same point to a circle are the same length.

5.07 Secants



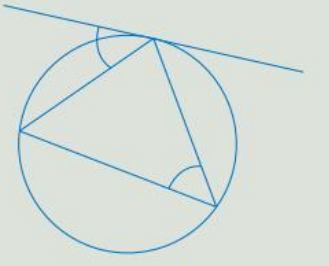
Intersecting chords, radii, tangents and secants

A **secant** is a line that intersects a circle at two points. You can also think of it as a chord that extends outside the circle. A **segment** is part of a circle cut off by a secant or chord. The remaining part of the circle is the **alternate segment**.

A chord drawn from the point that a tangent touches a circle makes an angle with the tangent. You can draw an angle at the circumference that subtends the chord. The angle between the is in the alternate segment to the angle one of the angles with the tangent. These angles are equal.

Alternate segment theorem

The angle made by a chord drawn from the point of contact with a tangent is equal to the angle in the alternate segment.

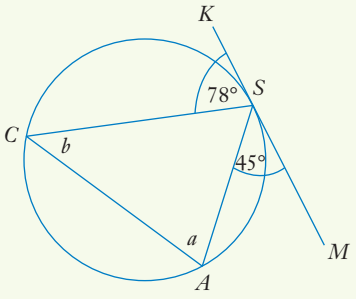


ISBN 9780170412995

You can prove the alternate segment theorem by constructing radii to the points where the chords intersect the circle. You can use the sum of the equal angles in the isosceles triangles formed to prove the theorem.

EXAMPLE 14

Find the value of each pronumeral.



Solution

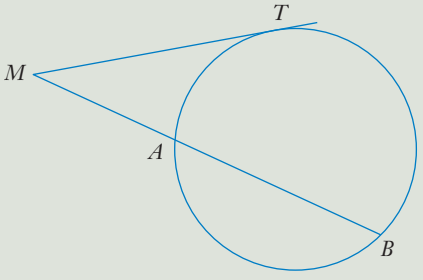
KSM is a tangent.	$a = 78°$ (alternate segment)
Use the other angle.	$b = 45°$ (alternate segment)

Secants and tangents drawn from the same point have an interesting property.

Tangent and secant

The square of the length of a tangent to a circle is the same as the product of the lengths of a secant from the same point to the circle

In the diagram, $TM^2 = AM \times BM$

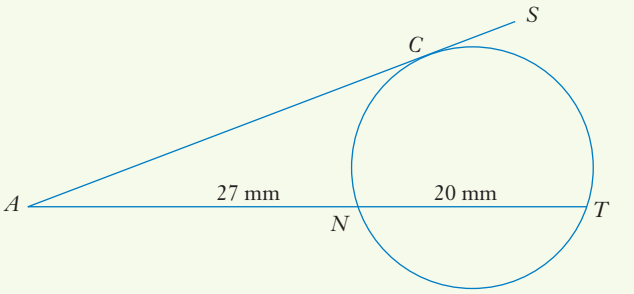


You can prove the tangent and secant theorem by constructing AT and BT. Then you can show that $\triangle MTA \parallel\!\parallel \triangle MTB$ using the alternate segment theorem. The property follows immediately from similar triangles.

ISBN 9780170412995

EXAMPLE 15

Find the length of AC in this diagram.



Solution

AC is a tangent.

$AC^2 = AT \times AN$ (tangent and secant)
$= 47 \times 27$
$= 1269$

Simplify the square root.

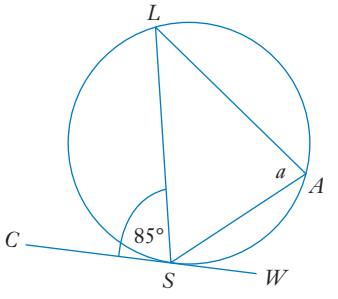
$AC = \sqrt{1269}$
$= \sqrt{9 \times 141}$
$= 3\sqrt{141}$ cm

Exercise 5.07 Secants

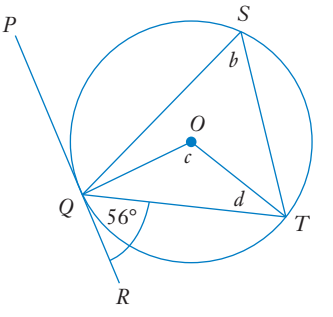
1 Find the value of each pronumeral, giving reasons.

Example 14

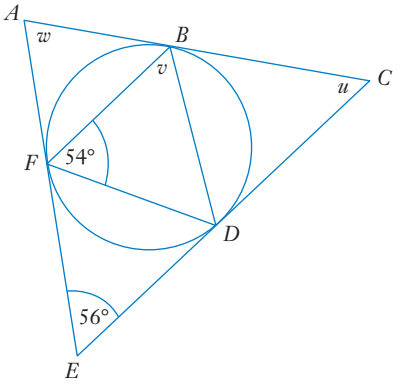
a



b



c

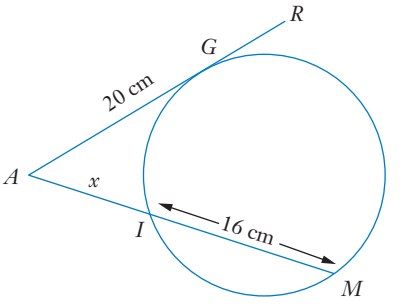


ISBN 9780170412995

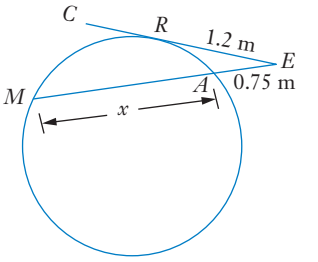
Example 15

2 Find the value of each pronumeral, giving reasons.

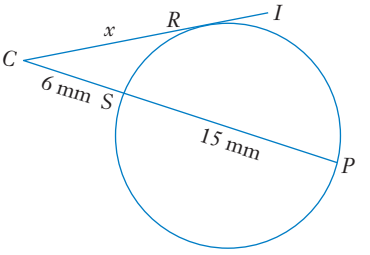
a



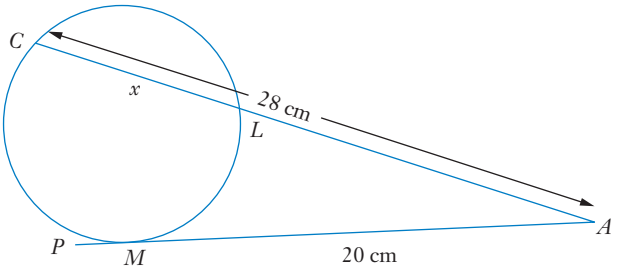
b



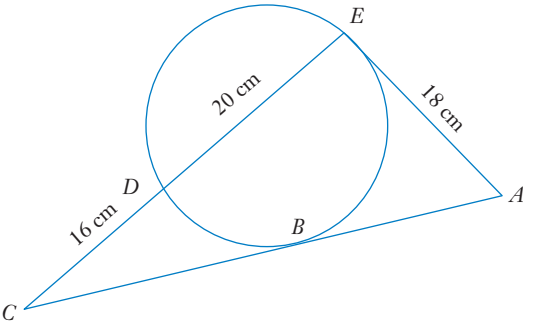
c



d



3 Find AC, giving reasons for each step.



Problem solving

4 Prove that the angle made by a chord drawn from the point of contact with a tangent is equal to the angle in the alternate segment.

5 Prove that the square of the length of a tangent to a circle is the same as the product of the lengths of a secant from the same point to the circle.

ISBN 9780170412995

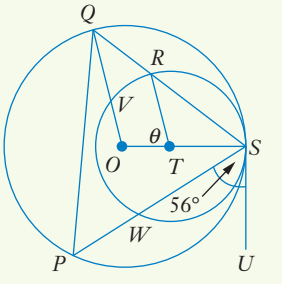
5.08 Mixed problems in circle geometry

iStock/rusm

In some applications, you will have two circles involved.

EXAMPLE 16

Find the value of each pronumeral, giving reasons for each step.



Solution

Work out as many angles as possible.

$\angle QPS = 56°$ (alternate segment)

The angle θ is in the inner circle, so try to find an angle in the inner circle.

$\Rightarrow \angle QOS = 112°$ (angle at centre of larger circle)

$\Rightarrow \angle QSO = 34°$ (angle sum of isosceles $\triangle QSO$)

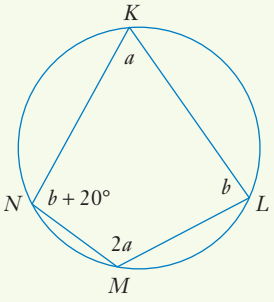
$\Rightarrow \theta = 68°$ (angle at centre of smaller circle)

In some problems, you will have to solve equations to find an unknown value or values.

ISBN 9780170412995

EXAMPLE 17

Find the value of each pronumeral, giving reasons for each step.



Solution

KLMN is a cyclic quadrilateral.

$a + 2a = 180°$ (opposite angles in cyclic quad.)
$\Rightarrow 3a = 180°$
$\Rightarrow a = 60°$

Now work out b.

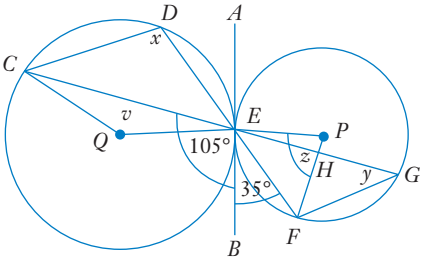
$b + b + 20° = 180°$ (opposite angles in cyclic quad.)
$\Rightarrow 2b = 160°$
$\Rightarrow b = 80°$

Exercise 5.08 Mixed problems in circle geometry

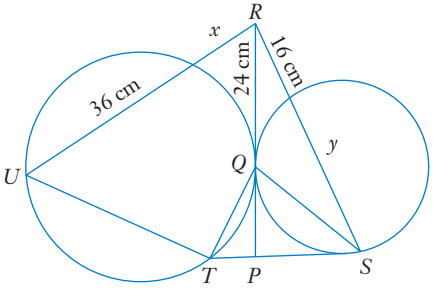
Example 16

1 Find the value of each pronumeral, giving reasons for each step. Write answers correct to 2 decimal places if necessary.

a

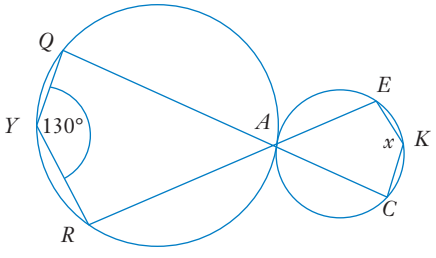


b

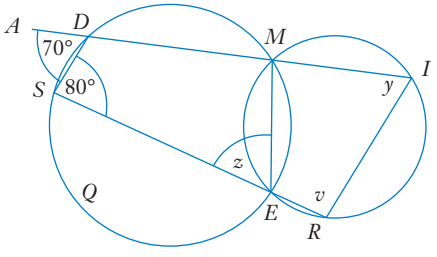


2 Find the value of each pronumeral, giving reasons for each step. Write answers correct to 2 decimal places if necessary.

a



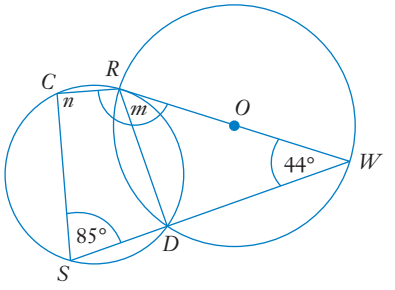
b



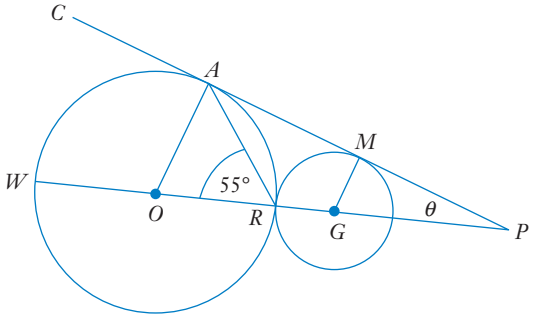
ISBN 9780170412995

3 Find the value of each pronumeral, giving reasons for each step.

a

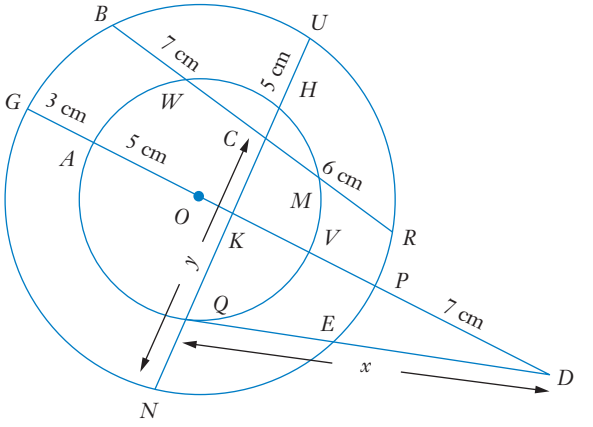


b

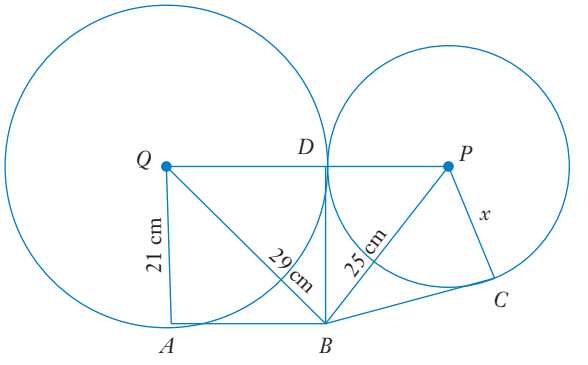


4 Find the value of each pronumeral, giving reasons for each step.

a

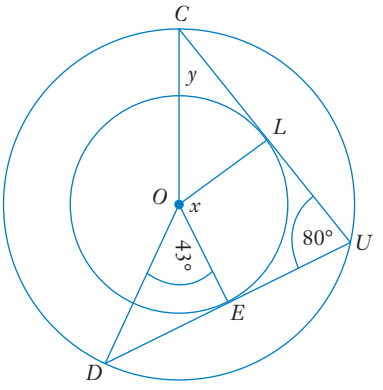


b

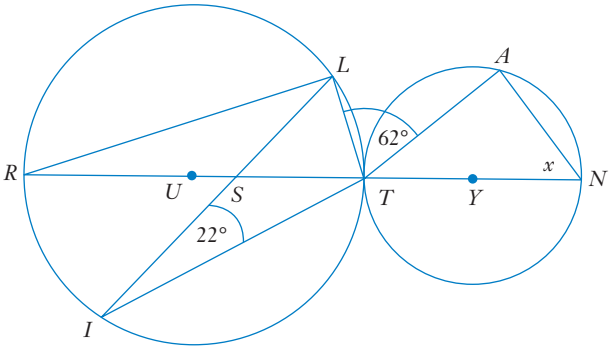


5 Find the value of each pronumeral, giving reasons for each step.

a



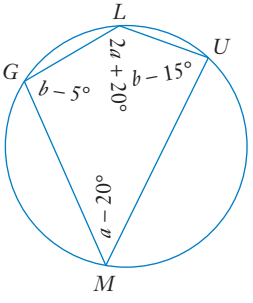
b



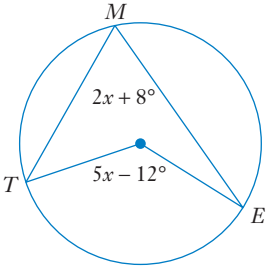
Example 17

6 Find the value of each pronumeral, giving reasons for each step. Write answers correct to 2 decimal places if necessary.

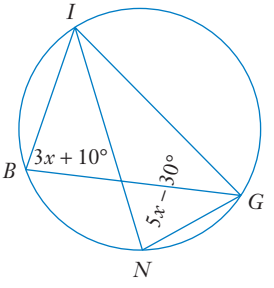
a



b

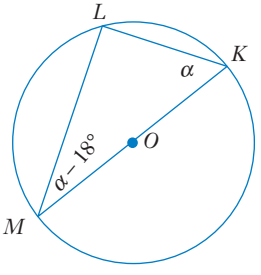


c

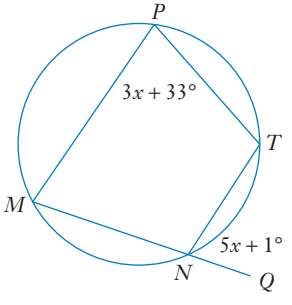


7 Find the value of each pronumeral, giving reasons for each step. Write answers correct to 2 decimal places if necessary.

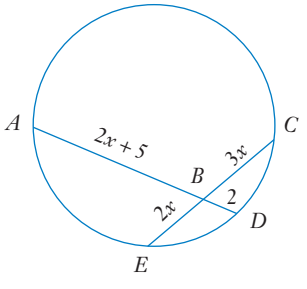
a



b

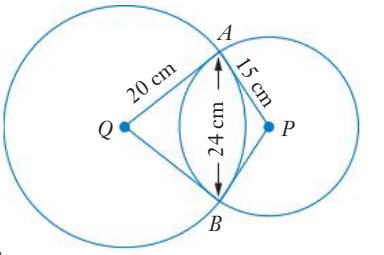


c



ISBN 9780170412995

Problem solving



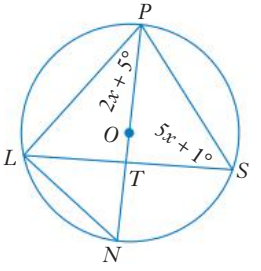
8 P and Q are the centres of these circles.

a Show that $\triangle QAP \equiv \triangle QBP$

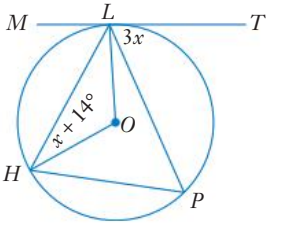
b Find the distance between the centres.

9 Find the value of each pronumeral, giving reasons for each step. Write your answers correct to 2 decimal places if necessary.

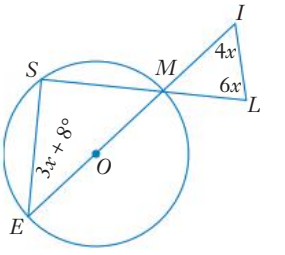
a



b



c



10 Prove that if 2 circles just touch each other, the point of contact and centres are collinear.

5.09 Vector proofs of parallelogram properties

You have already studied vectors in the plane, including position vectors and the dot product. You should recall the following.

Vector properties

- The position vector of a point $P(a, b)$ in the plane is $\mathbf{p} = \mathbf{OP} = a\mathbf{i} + b\mathbf{j}$
- The displacement vector from P to Q is given by $\mathbf{PQ} = \mathbf{q} - \mathbf{p}$
- The position vector of the midpoint of PQ is given by $\frac{1}{2}(\mathbf{p} + \mathbf{q})$
- The dot product of $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j}$ and $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j}$ is $\mathbf{p} \cdot \mathbf{q} = p_1q_1 + p_2q_2$
- The vectors for parallel lines of equal length are the same
- If $\mathbf{p} \neq \mathbf{0}$, $\mathbf{q} \neq \mathbf{0}$ and $\mathbf{p} \cdot \mathbf{q} = 0$, then $\mathbf{p}$ and $\mathbf{q}$ are perpendicular
- If $\mathbf{p} \neq \mathbf{0}$, $\mathbf{q} \neq \mathbf{0}$ and $\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}|$, then $\mathbf{p}$ and $\mathbf{q}$ are parallel
- $|\mathbf{p}|^2 = \mathbf{p} \cdot \mathbf{p}$

You can choose the origin and axis directions for a geometric proof. Try to make the situation as simple as possible without losing generality.

ISBN 9780170412995

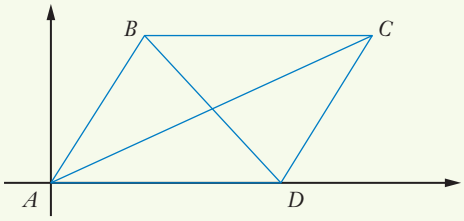
EXAMPLE 18

Use vectors to prove that the diagonals of a parallelogram bisect each other.

Solution

For the parallelogram $ABCD$, choose the origin at A and the x-axis in the direction of AD.

Sketch the diagram and construct the diagonals.



Write what you have to prove.	RTP: Midpoint of BD = midpoint of AC
Use the parallelism of AD and BC.	$AD \parallel BC \Rightarrow \mathbf{BC} = \mathbf{AD} = \mathbf{d}$
Use the parallelism of AB and DC.	$AB \parallel DC \Rightarrow \mathbf{DC} = \mathbf{AB} = \mathbf{b}$
Write $\mathbf{c}$ in terms of the other vectors.	$\mathbf{c} = \mathbf{AC} = \mathbf{AD} + \mathbf{DC} = \mathbf{d} + \mathbf{b}$
Find the midpoints of AC and BD.	Midpoint $AC = \frac{1}{2}(\mathbf{0} + \mathbf{d} + \mathbf{b}) = \frac{1}{2}(\mathbf{b} + \mathbf{d})$ Midpoint $BD = \frac{1}{2}(\mathbf{b} + \mathbf{d})$
State the conclusion.	The position vectors of the midpoints are the same, so the midpoints are the same. QED

Notice that $\mathbf{AB} + \mathbf{BC} + \mathbf{CD} + \mathbf{DA} = \mathbf{0}$. Any sum of displacements around a polygon must be $\mathbf{0}$ because you return to the same point.

You can show the converse of Example 18 by reversing the proof.

Two lengths are equal if the dot products of their displacement vectors are equal.

ISBN 9780170412995

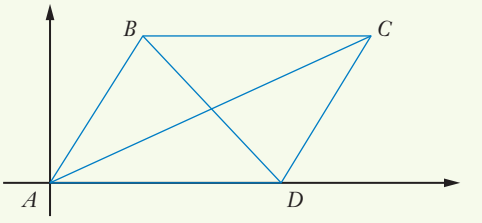
EXAMPLE 19

Prove that a parallelogram with equal diagonals is a rectangle.

Solution

For the parallelogram $ABCD$, choose the origin at A and the x-axis in the direction of AD.

Sketch the diagram and construct the diagonals.



Write what you need to prove.	RTP: AB is perpendicular to AD
Use the parallelism of AD and BC.	$AD \parallel BC \Rightarrow \mathbf{BC} = \mathbf{AD} = \mathbf{d}$
Use the parallelism of AB and DC.	$AB \parallel DC \Rightarrow \mathbf{DC} = \mathbf{AB} = \mathbf{b}$
Write $\mathbf{c}$ in terms of the other vectors.	$\mathbf{c} = \mathbf{AC} = \mathbf{AD} + \mathbf{DC} = \mathbf{d} + \mathbf{b}$
Write the vectors for the diagonals.	$\mathbf{AC} = \mathbf{d} + \mathbf{b}$ $\mathbf{BD} = \mathbf{d} - \mathbf{b}$
State the condition.	$\lvert\mathbf{AC}\rvert = \lvert\mathbf{BD}\rvert$
Change to use dot products.	$\Rightarrow \lvert\mathbf{AC}\rvert^2 = \lvert\mathbf{BD}\rvert^2$
Replace with dot products.	$\Rightarrow (\mathbf{d} + \mathbf{b}) \cdot (\mathbf{d} + \mathbf{b}) = (\mathbf{d} - \mathbf{b}) \cdot (\mathbf{d} - \mathbf{b})$
Expand products and simplify.	$\Rightarrow \mathbf{d} \cdot \mathbf{d} + 2\mathbf{b} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{b} = \mathbf{d} \cdot \mathbf{d} - 2\mathbf{b} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{b}$ $\Rightarrow 4\mathbf{b} \cdot \mathbf{d} = 0$ $\Rightarrow \mathbf{b} \cdot \mathbf{d} = 0$ $\Rightarrow \mathbf{b}$ and $\mathbf{d}$ are perpendicular
State the conclusion.	AB is perpendicular to AD. QED

You can also show the converse of Example 19 by reversing its proof.

ISBN 9780170412995

Exercise 5.09 Vector proofs of parallelogram properties

Example 18

1 Use vectors to prove that the midpoints of the sides of a quadrilateral join to form a parallelogram.

2 Use vectors to prove the converse of Example 18, that is, if the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Example 19

3 Use vectors to prove that if the diagonals of a parallelogram meet at right angles, then it is a rhombus.

4 Use vectors to prove the converse of question **3**, that the diagonals of a rhombus meet at right angles.

5 Use vectors to prove the converse of Example 19, that the diagonals of a rectangle are of equal length.

Problem solving

6 Use vectors to prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

7 Use vectors to prove that if the diagonals of a rhombus are of equal length, then it is a square.

8 Use vectors to prove that the opposite angles of a parallelogram are equal.

9 Equilateral triangles $\triangle BCE$ and $\triangle CDF$ are constructed on adjacent sides of the parallelogram $ABCD$. Use vectors to prove that $\triangle AEF$ is equilateral.

5.10 Other vector proofs in geometry

You can use vectors to prove geometric results in figures other than parallelograms. You still use the properties of vectors covered in the previous section.

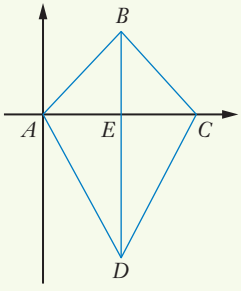
EXAMPLE 20

Prove that the diagonals of a kite intersect at right angles.

Solution

Draw a kite $ABCD$ with $|AB| = |BC|$ and $|CD| = |DA|$. Construct the diagonals to intersect at E.

Place the origin at A and the x-axis in the direction of AC.



ISBN 9780170412995

State what you need to prove.	RTP: $\mathbf{AC}$ is perpendicular to $\mathbf{BD}$
Use the equality of sides.	$\lvert\mathbf{AB}\rvert = \lvert\mathbf{BC}\rvert$ and $\lvert\mathbf{DC}\rvert = \mathbf{DA}\rvert$ $\Rightarrow \lvert\mathbf{AB}\rvert^2 = \lvert\mathbf{BC}\rvert^2$ and $\lvert\mathbf{DC}\rvert^2 = \mathbf{DA}\rvert^2$ $\Rightarrow (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = (\mathbf{c}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})$ and $(\mathbf{c}-\mathbf{d})\cdot(\mathbf{c}-\mathbf{d}) = (\mathbf{a}-\mathbf{d})\cdot(\mathbf{a}-\mathbf{d})$
Simplify using $\mathbf{a} = \mathbf{0}$.	$\Rightarrow \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} - 2\,\mathbf{c}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}$ and $\mathbf{c}\cdot\mathbf{c} - 2\,\mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{d} = \mathbf{d}\cdot\mathbf{d}$
Simplify each expression.	$\Rightarrow \mathbf{c}\cdot\mathbf{c} = 2\,\mathbf{c}\cdot\mathbf{b}$ and $\mathbf{c}\cdot\mathbf{c} = 2\,\mathbf{c}\cdot\mathbf{d}$
Write as a single equation.	$\Rightarrow 2\,\mathbf{c}\cdot\mathbf{b} = 2\,\mathbf{c}\cdot\mathbf{d}$
Look for $\mathbf{BD} = \mathbf{d} - \mathbf{b}$.	$\Rightarrow \mathbf{c}\cdot\mathbf{d} - \mathbf{c}\cdot\mathbf{b} = 0$ $\Rightarrow \mathbf{c}\cdot(\mathbf{d}-\mathbf{b}) = 0$ $\Rightarrow \mathbf{AC}\cdot\mathbf{BD} = 0$
State the conclusion.	$\Rightarrow$ $\mathbf{AC}$ is perpendicular to $\mathbf{BD}$ QED

The proof above is found by actively looking for $\mathbf{BD} = (\mathbf{d} - \mathbf{b})$ in the expressions arising from the equality of the sides.

Triangles have three different centres that are important in geometry.

Triangle centres

- The **medians** of a triangle join the vertices to the midpoints of the opposite sides
- The **centroid** is the intersection of the medians
- The **circumcentre** is the intersection of the perpendicular bisectors of each side. It is the centre of the **circumcircle** that passes through the vertices
- The **in-centre** is the intersection of the angle bisectors of each vertex. It is the centre of the **in-circle** to which each side is a tangent
- The **altitudes** of a triangle pass through the vertices and are perpendicular to the opposite sides
- The **orthocentre** is the intersection of the altitudes

ISBN 9780170412995

EXAMPLE 21

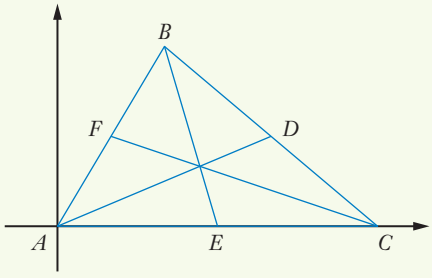
Use vectors to show that the medians of a triangle are concurrent and that the centroid divides the median in the ratio 2 : 1 from the vertex to the opposite side.

Solution

Draw a triangle $\triangle ABC$ and place the origin at A with the x-axis in the direction of AC.

Construct the medians AD, BE, CF.

Let G, H and K divide AD, BE and CF in the ratios 2 : 1 respectively.



Write what you need to prove.

RTP: AD, BE and CF are divided in the ratio 2 : 1 by a common point.

Write the position vectors of D, E and F in terms of $\mathbf{b}$ and $\mathbf{c}$.

$\mathbf{f} = \frac{1}{2}\mathbf{b}$, $\mathbf{d} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\mathbf{e} = \frac{1}{2}\mathbf{c}$

Find the position vector of the point G that divides AD in the ratio 2 : 1.

$$\begin{aligned}\mathbf{g} &= \mathbf{0} + \tfrac{2}{3}\mathbf{AD} \\ &= \tfrac{2}{3} \times \tfrac{1}{2}(\mathbf{b} + \mathbf{c}) \\ &= \tfrac{1}{3}(\mathbf{b} + \mathbf{c})\end{aligned}$$

Find the position vector of the point H that divides BE in the ratio 2 : 1.

$$\begin{aligned}\mathbf{h} &= \mathbf{b} + \tfrac{2}{3}\mathbf{BE} \\ &= \mathbf{b} + \tfrac{2}{3}(\mathbf{e} - \mathbf{b}) \\ &= \mathbf{b} + \tfrac{2}{3} \times \tfrac{1}{2}\mathbf{c} - \tfrac{2}{3}\mathbf{b} \\ &= \tfrac{1}{3}(\mathbf{b} + \mathbf{c})\end{aligned}$$

Find the position vector of the point K that divides CF in the ratio 2 : 1.

$$\begin{aligned}\mathbf{k} &= \mathbf{c} + \tfrac{2}{3}\mathbf{CF} \\ &= \mathbf{c} + \tfrac{2}{3}(\mathbf{f} - \mathbf{c}) \\ &= \mathbf{c} + \tfrac{2}{3} \times \tfrac{1}{2}\mathbf{b} - \tfrac{2}{3}\mathbf{c} \\ &= \tfrac{1}{3}(\mathbf{b} + \mathbf{c})\end{aligned}$$

Write the conclusion.

G, H and K are the same point
$\Rightarrow$ AD, BE and CF are divided in the ratio 2 : 1 by a common point. QED

ISBN 9780170412995

Exercise 5.10 Other vector proofs in geometry

Example 20,21

1 Use vectors to prove that the line between the midpoints of two sides of a triangle is parallel to and half the length of the third side.

2 Use vectors to prove that the midpoints of an isosceles triangle form another isosceles triangle.

Problem solving

3 Use vectors to prove that the line dividing two sides of a triangle in the ratio 1 : 2 from the same vertex is parallel to the third side and one-third of its length. Generalise the result.

4 Use vectors to show that one diagonal of a kite is bisected by the other.

5 Use vectors to show that if one line segment perpendicularly bisects another, the figure obtained by joining their ends is a kite.

6 Use vectors to show that the angle bisectors of a triangle are concurrent and the intersection is the centre of a circle tangent to the sides.

7 Use vectors to show that the altitudes of a triangle are concurrent (Hint: find the position vector of the intersection of two of the altitudes).

8 Use vectors to prove that the line joining the midpoint of two adjacent sides of a quadrilateral is parallel to and the same length as the line joining the midpoints of the other two sides.

9 Use vectors to prove that the line joining the point that divides the sides from one vertex of a quadrilateral in the ratio 1 : 2 is parallel to and the same length as the similar line joining the sides from the opposite vertex. Generalise the result.

10 Use vectors to show that the angle in a semicircle is a right angle.

11 The points D, E, F and G are on sides AB and AC of $\triangle ABC$ such that $AD = DE = EB$ and $AF = FG = GC$. Use vectors to prove that BF is bisected by EG.

12 Construct the X on the line segment AB and the point C such that A, B and C are equidistant from X. Use vectors to prove that $\angle ACB$ is a right angle.

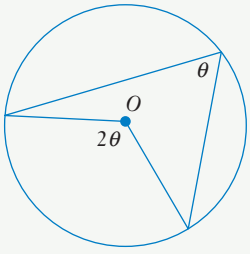
ISBN 9780170412995

5. CHAPTER SUMMARY

Circle properties and vector proofs

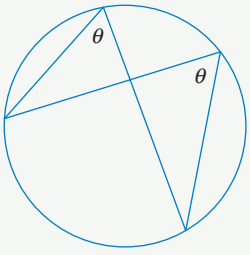
Angle at the centre of a circle

- The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc
- Conversely, the vertex of an angle equal to half the angle at the centre of a circle subtended by the same arc lies on the circle.



Angles at the circumference of a circle

- Angles at the circumference of a circle subtended by the same arc are equal, provided that they are on the same side of the arc.
- Conversely, the vertex of an angle equal to an angle at the circumference subtended by the same arc must lie on the circle.

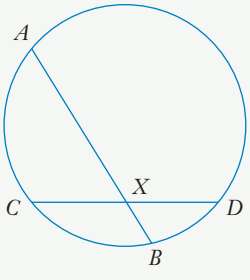


Cyclic quadrilaterals

- Four or more points are **concyclic** if they lie on the same circle
- A **cyclic quadrilateral** has all four vertices on a circle
- Opposite angles of a cyclic quadrilateral are supplementary
- Conversely, a quadrilateral with supplementary opposite angles is cyclic
- The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle
- Conversely, a quadrilateral with an exterior angle equal to the interior opposite angle is cyclic

Chords

- Chords of equal length subtend equal angles at the centre of a circle, and conversely
- Chords subtending equal angles at the centre of a circle have the same length
- For intersecting chords, the products of the intervals of each chord are equal: in the diagram, $AX \times XB = CX \times XD$

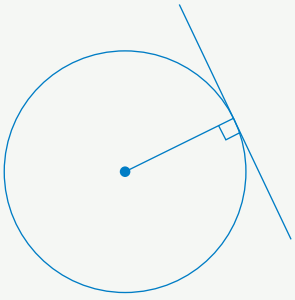


ISBN 9780170412995

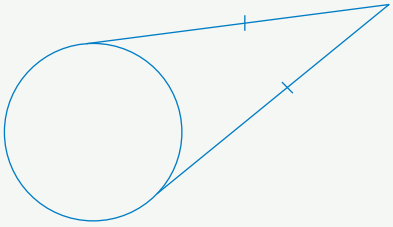
- Conversely, if the products of the lengths of the intervals of intersecting line segments are equal, they are chords of the same circle (their ends are concyclic)
- A radius that bisects a chord is perpendicular to the chord
- Conversely, a radius that is perpendicular to a chord bisects the chord

Tangents, secants and radii

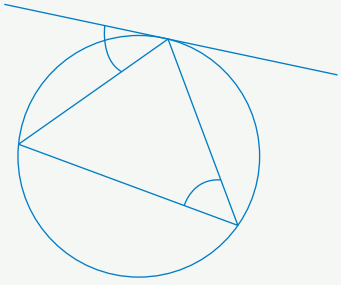
- A **tangent** to a circle touches the circle at exactly one point.



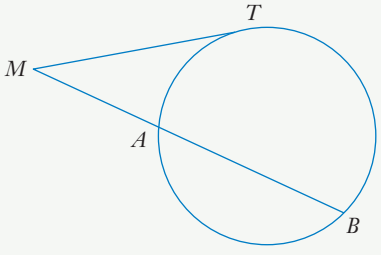
- A tangent to a circle is perpendicular to the radius at the point of contact



- Tangents drawn from the same point to a circle are the same length
- The angle made by a chord drawn from the point of contact with a tangent is equal to the angle in the **alternate segment**



- The square of the length of a tangent to a circle is the same as the product of the lengths of a secant from the same point to the circle
- In the diagram, $TM^2 = AM \times BM$



Vector properties

- The position vector of a point $P(a, b)$ in the plane is $\mathbf{p} = \mathbf{OP} = a\mathbf{i} + b\mathbf{j}$
- The displacement vector from P to Q is given by $\mathbf{PQ} = \mathbf{q} - \mathbf{p}$
- The position vector of the midpoint of PQ is given by $\frac{1}{2}(\mathbf{p} + \mathbf{q})$
- The dot product of $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j}$ and $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j}$ is $\mathbf{p} \cdot \mathbf{q} = p_1q_1 + p_2q_2$
- The vectors for parallel lines of equal length are the same
- If $\mathbf{p} \neq \mathbf{0}$, $\mathbf{q} \neq \mathbf{0}$ and $\mathbf{p} \cdot \mathbf{q} = 0$, then $\mathbf{p}$ and $\mathbf{q}$ are perpendicular
- If $\mathbf{p} \neq \mathbf{0}$, $\mathbf{q} \neq \mathbf{0}$ and $\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}|$, then $\mathbf{p}$ and $\mathbf{q}$ are parallel
- $|\mathbf{p}|^2 = \mathbf{p} \cdot \mathbf{p}$
- Any sum of displacements around a polygon must be $\mathbf{0}$ because you return to the same point: $\mathbf{AB} + \mathbf{BC} + \mathbf{CD} + \mathbf{DA} = \mathbf{0}$
- When proving geometric properties with vectors, choose the position of the origin and coordinate axes to simplify the proof

Triangle centres

- The **medians** of a triangle join the vertices to the midpoints of the opposite sides
- The **centroid** is the intersection of the medians
- The **circumcentre** is the intersection of the perpendicular bisectors of each side. It is the centre of the **circumcircle** that passes through the vertices
- The **in-centre** is the intersection of the angle bisectors of each vertex. It is the centre of the **in-circle** to which each side is a tangent
- The **altitudes** of a triangle pass through the vertices and are perpendicular to the opposite sides
- The **orthocentre** is the intersection of the altitudes

ISBN 9780170412995

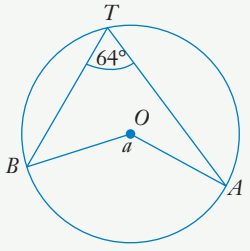
5. CHAPTER REVIEW

Circle properties and vector proofs

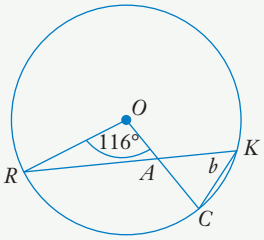
1 Find the value of each pronumeral, giving reasons for each step.

Examples 1, 2

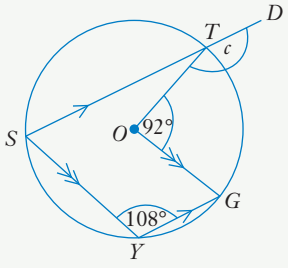
a



b



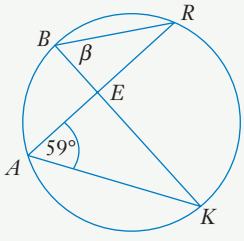
c



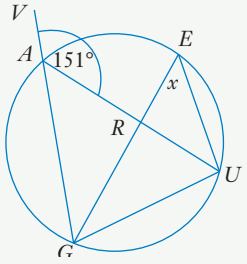
2 Find the value of each pronumeral, giving reasons for each step.

Examples 3, 4

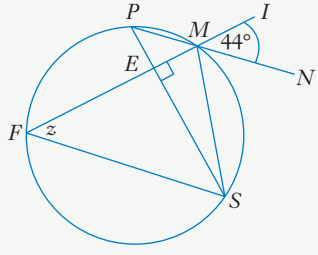
a



b



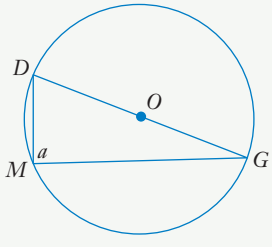
c



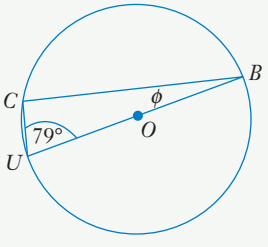
3 Find the value of each pronumeral, giving reasons for each step.

Examples 5, 6

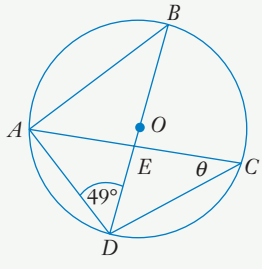
a



b



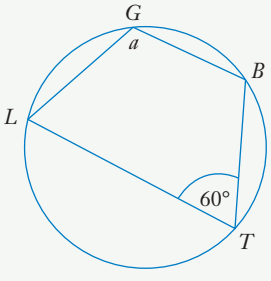
c



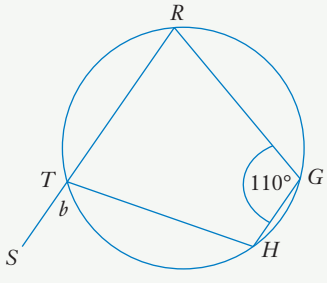
4 Find the value of each pronumeral, giving reasons for each step.

Examples 7, 8

a

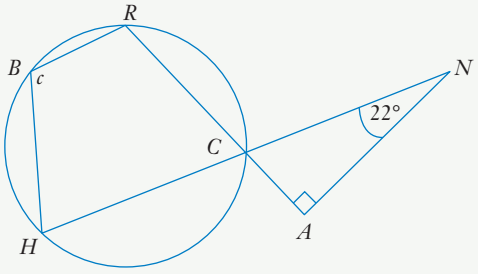


b



ISBN 9780170412995

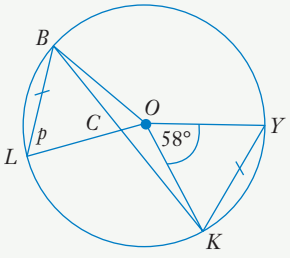
c



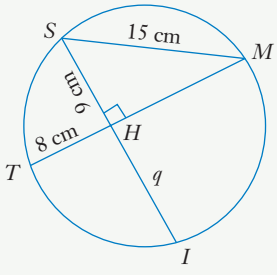
Examples 9-11

5 Find the value of each pronumeral, giving reasons for each step.

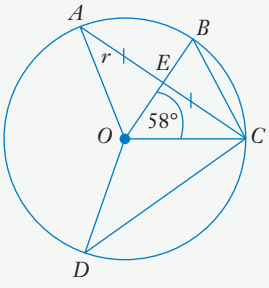
a



b



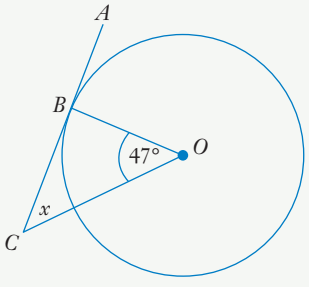
c



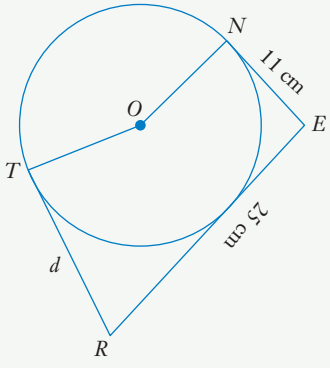
Examples 12,13

6 Find the value of each pronumeral, giving reasons for each step.

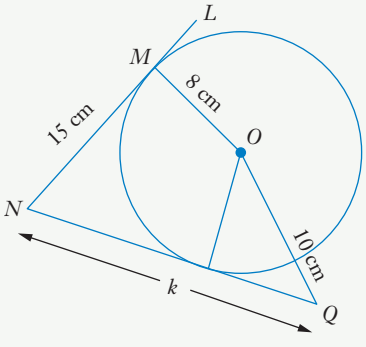
a



b



c

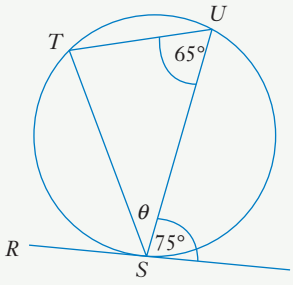


ISBN 9780170412995

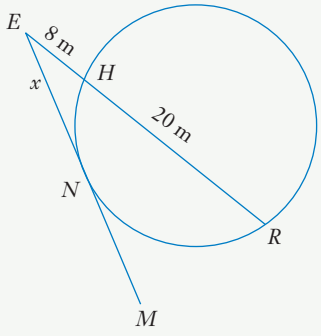
7 Find the value of each pronumeral, giving reasons for each step.

Examples 14,15

a

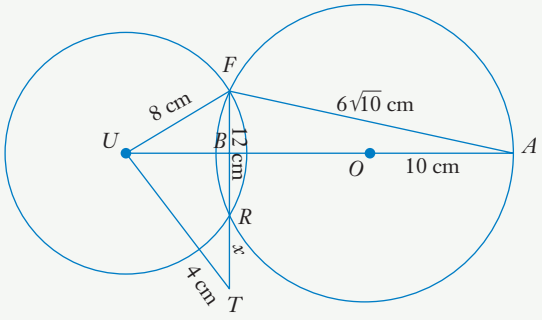


b



8 Find the distance between the centres and the value of x in the following diagram, giving reasons for each step.

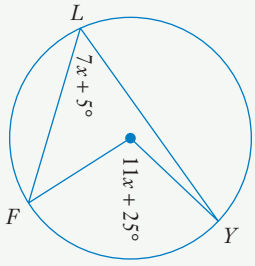
Example 16



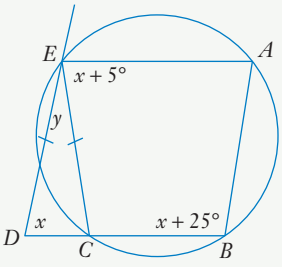
9 Find the value of each pronumeral, giving reasons for each step.

Example 17

a



b



Problem solving

10 Prove that the triangles formed by angles subtended at the circumference by the same chord on the same side of the chord are similar.

11 Prove that the perpendicular bisector of a chord passes through the centre of the circle.

12 Use vectors to prove that the lines joining one vertex of a parellelogram to the midpoints of the opposite sides trisects the other diagonal.

Qz Practice quiz

ISBN 9780170412995

COMBINATORICS

6.

COMBINATIONS

You studied permutations (arrangements) in Chapter 3, where order is important. Imagine an Olympic event. If Germany won the gold medal and Australia the silver, it is different from Australia winning the gold and Germany the silver.

In contrast, imagine the selection of 12 people from a shortlist of 20 to sit on a jury. Every possible combination is distinct, but the order in which they sit in the courtroom is not important as each have the same role.

In this chapter, you will study problems involving unordered selections or **combinations**.

6.01 Review of multiplication principle and permutations
6.02 Combinations
6.03 Combinations with restrictions
6.04 Applications of permutations and combinations
6.05 Pascal's triangle and identities
6.06 The pigeon-hole principle
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Permutations (ordered arrangements) and combinations (unordered selections)

- solve problems involving combinations
- use the notation $\binom{n}{r}$ and ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
- derive and use simple identities associated with Pascal's triangle
- solve problems involving combinations with restrictions
- apply permutations and combinations to probability problems

The pigeon-hole principle

- solve problems and prove results using the pigeon-hole principle

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

iStock / peepo

TERMINOLOGY

combination
factorial
multiplication principle
Pascal's triangle
permutation
pigeon-hole principle
probability

6.01 Review of multiplication principle and permutations

EXAMPLE 1

Evaluate:

a $\frac{7!}{4!}$ **b** $\frac{12!}{2!9!}$

Solution

a Write in expanded form. $\frac{7!}{4!} = \frac{7\times6\times5\times4!}{4!}$

Cancel. $= 7\times6\times5$

Evaluate. $= 210$

b Write in expanded form. $\frac{12!}{2!9!} = \frac{12\times11\times10\times9!}{2\times1\times9!}$

Cancel. $= \frac{12\times11\times10}{2\times1}$

Evaluate $= 660$

EXAMPLE 2

Jasmine is trying to decide between 5 jumpers, 6 pairs of jeans and 9 pairs of shoes to wear to lunch. How many different outfits could she wear?

Solution

There are 3 items of clothing.

5	6	9

Substitute for known values No. of arrangements $= 5\times6\times9$

Evaluate. $= 270$

Write the result. There are 270 different outfits Jasmine could wear.

ISBN 9780170412995

EXAMPLE 3

Ben, Albie, Vien, Connor, Paige, Chris, Yoonis and Humza are at the cinema.

a In how many ways can they sit together?

b In how many ways can they be seated if Ben and Paige want to sit together?

Solution

a There are 8 people and 8 positions, so there are 8! permutations.

Write using permutation notation. $^8P_8 = 8!$

Use the definition of $n!$. $= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

Evaluate. $= 40\,320$

Write the answer. There are 40 320 different seating arrangements.

b We can consider Ben and Paige as a single block in the arrangements with the other 6 friends. There are 7 elements in this case.

B&P	1	2	3	4	5	6

Within the block of Ben and Paige, there are 2! arrangements.

Arrangements of two people $= {}^2P_2$
$= 2!$

There are 7! ways of arranging 6 people and the block of Ben and Paige. There are 2! ways of arranging Ben and Paige. Use the multiplication rule.

$^7P_7 \times 2! = 7! \times 2!$
$= 10\,080$

Write the answer. There are 10 080 different seating arrangements.

Exercise 6.01 Review of multiplication principle and permutations

Example 1

1 Evaluate each expression.

a 0! **b** 5! **c** $\frac{6!}{2!}$ **d** $\frac{10!}{9!}$ **e** $\frac{9!}{2!5!}$

2 Evaluate each expression.

a $^4P_4$ **b** $^5P_4$ **c** $^6P_3$ **d** $^4P_0$ **e** $^7P_6$

f $^6P_4$ **g** $^7P_3$ **h** $^{10}P_6$ **i** $^5P_2$ **j** $^{11}P_{10}$

k $^9P_4$ **l** $^{12}P_2$ **m** $^3P_3$ **n** $^{90}P_3$ **o** $^{15}P_5$

ISBN 9780170412995

Example 2

3 A pizza restaurant allows patrons to choose from 4 different bread bases, 3 different meat toppings and 6 sauces to go with a standard assortment of vegetable toppings. How many different pizzas could be made with these options?

Example 3

4 How many ways can the letters of the word HARMONY be arranged so that vowels occur together and the consonants occur together?

Problem solving

5 Conventional Queensland number plates consist of 3 letters followed by 3 numbers. How many different number plates are possible with this arrangement? Suggest why some other states changed from 2 letters and 4 numbers to this arrangement.

iStock/Kokkai Ng

6 In how many ways can the letters of the word LUMBERJACK be arranged?

7 In how many ways can 5 people be arranged in a row?

8 In how many ways can the letters of the word CREAMY be arranged?

9 How many anagrams of the word SEPARATE are possible? (An anagram is a rearrangement of the letters.)

10 A school orchestra has to play 6 pieces at a concert. The conductor is trying to decide the order of the pieces. How many choices does he have?

11 In how many ways can 4 black and 5 red discs be placed in a row if they are placed alternately?

12 A combination lock is opened by turning the dial to 3 different numbers in succession. How many different combinations are possible if the numbers run from:

a 0 to 9? **b** 0 to 19? **c** 0 to 29?

13 A singer has arrangements of 20 different songs from her repertoire that are suitable for a particular backing group. Only 3 of them can be used as the opening number, and there are 4 that she would prefer to choose from for the finale. In a 6-song bracket, how many different programs are possible if the middle 4 songs are chosen from the remaining 13 arrangements?

ISBN 9780170412995

6.02 Combinations

Combination calculations

When the order in which a group of objects is selected makes no difference, combinations rather than permutations are relevant. A **combination** of a group of objects is a selection of some or all of the objects. The same combination may be written in different orders, but this does not change the combination. However, you can use your study on permutations to find the basic formula for the number of combinations.

The number of combinations of n objects, taken r at a time, is given the symbol ${}^nC_r$, which is also written as ${}_nC_r$, $\binom{n}{r}$, $C^n{}_r$ or $C(n, r)$.

$$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!} $$

Proof

The number of permutations of r objects is $r!$. The number of permutations of n objects, taken r at a time, must be the number of combinations multiplied by the number of permutations of each combination. That is,

$$ {}^nP_r = {}^nC_r \times r! $$

Thus
$$ {}^nC_r = \frac{{}^nP_r}{r!} $$

Since
$$ {}^nP_r = \frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1) $$

then
$$ {}^nC_r = \frac{n(n-1)\cdots(n-r+1)}{r!} \qquad \text{QED} $$

The result of this theorem is very important in probability and counting, and it arises in other branches of mathematics. It should be committed to memory. The calculation of the value can be simplified using some simple properties of the formula.

Formulas for combinations

1 The number of combinations of n objects, taken r at a time, is given by

$$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!} $$

Note: The last expression has r factors on the top and r factors on the bottom.

2 ${}^nC_n = {}^nC_0 = 1$

3 ${}^nC_1 = n$

4 ${}^nC_r = {}^nC_{n-r}$

ISBN 9780170412995

EXAMPLE 4

Find n if ${}^nC_2 = 10$.

Solution

Write the equation.	${}^nC_2 = 10$
Apply the formula for ${}^nC_r$.	$\frac{n(n-1)}{2!} = 10$
Simplify.	$n(n-1) = 20$
Expand and rearrange.	$n^2 - n - 20 = 0$
Factorise.	$(n-5)(n+4) = 0$
Solve.	$n = 5$ or -4
$n > 2$ in this example.	$n = 5$

EXAMPLE 5

Find n if ${}^{n+1}C_5 = {}^nC_4$.

Solution

Apply the formula.	$\frac{(n+1)!}{5!(n+1-5)!} = \frac{n!}{4!(n-4)!}$
Expand.	$\frac{(n+1)n(n-1)(n-2)(n-3)}{1\times2\times3\times4\times5} = \frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}$
Simplify.	$\frac{(n+1)}{5} = \frac{1}{1}$
Solve.	$n = 4$

ISBN 9780170412995

EXAMPLE 6

A committee of 4 people is to be selected from a group of 15. How many different committees can be selected?

Solution

We are selecting a group of 4 from 15.

$${}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1}$$

Evaluate.

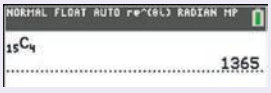
$$= 1365$$

Write the result.

There are 1365 ways of selecting a committee of four from 15 people.

TI-84 Plus CE

Type 15, press MATH, select PROB, 3: nCr, type 4 and press ENTER.



TI-Nspire CX Chapter 6

Casio fx-CG20AU

Type 15. Select **OPTN**, **PROB**, **nCr**. Type 4. Press **EXE**.

Evaluating ${}^{15}C_4$ gives 1365.

Exercise 6.02 Combinations

1 Calculate each value.

a ${}^{6}C_2$ **b** ${}^{8}C_3$ **c** ${}^{10}C_6$ **d** ${}^{12}C_4$ **e** ${}^{8}C_8$

f ${}^{25}C_1$ **g** ${}^{13}C_5$ **h** ${}^{13}C_8$ **i** ${}^{9}C_3$ **j** ${}^{12}C_2$

2 Find n if:

a ${}^{n}C_2 = 6$ **b** ${}^{(n+1)}C_3 = {}^{n}C_2$ **c** ${}^{n}C_2 = 45$

d ${}^{(n+1)}C_7 = {}^{n}C_6$ **e** ${}^{n}C_2 = 66$ **f** ${}^{n}C_5 = {}^{n}C_{11}$

Examples 4, 5

3 How many different selections of 5 cards can be made from a deck of 52 different cards?

Example 6

4 How many combinations of 4 flavours of gelato can be chosen from 20 flavours?

5 For one exam, students are given 5 possible questions before the exam, but only 2 are actually examined. How many different possible exams are there?

ISBN 9780170412995

6 10 people are introduced and shake hands with each other.

a How many handshakes are there?

b How many handshakes are there for n people?

7 How many triangles can be drawn using 5 given points on the circumference of a circle?

8 A short test has 10 questions, all of which must be answered *Yes* or *No*. In how many sets of answers can a person get half the questions wrong?

Problem solving

9 A car radio has 5 preset-buttons that can be tuned to different stations. In a city with 12 radio stations, how many different selections of those stations can be chosen for the buttons?

10 A simple Lotto game uses the numbers 1 to 40, with 6 numbers being chosen. How many different combinations are possible?

11 Use the formula of ${}^nC_r$ to prove that ${}^nC_n = {}^nC_0 = 1$.

12 Use the formula of ${}^nC_r$ to prove that ${}^nC_1 = n$.

13 Use the formula of ${}^nC_r$ to prove that ${}^nC_r = {}^nC_{n-r}$.

Combinations

6.03 Combinations with restrictions

In more complex problems you might want to count the number of combinations that include (or exclude) specific objects. This reduces the number of objects you have to select and the number of objects from which you are selecting.

EXAMPLE 7

Khouri and Astrid are 2 of 11 members of a chess squad. In how many ways can a team of 5 be chosen if:

a both Khouri and Astrid are in the team?

b Khouri is in the team but Astrid is not?

c neither Khouri nor Astrid are in the team?

Solution

a Write the rule for ${}^nC_r$.

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

If Khouri and Astrid are both in the team, then there are three places to be filled from the remaining nine members. Substitute $n = 9$ and $r = 3$.

$${}^9C_3 = \frac{9!}{3!(9-3)!}$$

ISBN 9780170412995

Simplify.

$${}^9C_3 = \frac{9\times8\times7\times6!}{3!6!}$$

$$= \frac{9\times8\times7}{3\times2\times1}$$

Evaluate.

$$= 84$$

Write the answer.

If Khouri and Astrid are both in the team, then there are 84 ways to fill the team.

b If Khouri is included but Astrid is not, then there are 4 places to be filled from the remaining 9 members. Substitute $n = 9$ and $r = 4$.

$${}^9C_4 = \frac{9!}{4!(9-4)!}$$

Simplify.

$${}^9C_4 = \frac{9\times8\times7\times6\times5!}{4!5!}$$

$$= \frac{9\times8\times7\times6}{4\times3\times2\times1}$$

Evaluate.

$$= 126$$

Write the answer.

If Khouri is included but Astrid is not, then there are 126 ways to fill the team.

c If neither Khouri nor Astrid are included, then there are 5 places to be filled from the remaining 9 members. Substitute $n = 9$ and $r = 5$.

$${}^9C_5 = \frac{9!}{5!(9-5)!}$$

$$= \frac{9\times8\times7\times6\times5!}{5!4!}$$

$$= \frac{9\times8\times7\times6}{4\times3\times2\times1}$$

$$= 126$$

Write the answer.

If neither Khouri nor Astrid is included, then there are 126 ways to fill the team.

ISBN 9780170412995

EXAMPLE 8

A straight poker hand is made up of 5 cards taken from a deck of 52 cards. How many hands contain the Ace of Hearts but none of the Jacks?

Solution

Write the rule for ${}^nC_r$.

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

The Ace of Hearts must be included, but the four Jacks must not. Four cards must be filled from 47 cards. Substitute $n = 47$ and $r = 4$.

$${}^{47}C_4 = \frac{47!}{4!(47-4)!}$$
$$= \frac{47 \times 46 \times 45 \times 44 \times 43!}{4!43!}$$
$$= \frac{47 \times 46 \times 45 \times 44}{4 \times 3 \times 2 \times 1}$$
$$= 178\ 365$$

Write the answer.

There are 178 365 hands that contain the Ace of Hearts and no Jacks.

Sometimes you need to make selections from different groups. Use the multiplication principle to find the total number of ways of making the selections.

EXAMPLE 9

From 6 women and 4 men in a singing ensemble, how many groups of 5 can be chosen:

a with no restrictions?

b containing 3 men and 2 women?

c containing at least one man?

Solution

a Write the rule for ${}^nC_r$.

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

With no restrictions, five people are selected from ten. Substitute $n = 10$ and $r = 5$.

$${}^{10}C_5 = \frac{10!}{5!(10-5)!}$$
$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!5!}$$
$$= \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$$
$$= 252$$

ISBN 9780170412995

Write the answer.

There are 252 ways to select a group of 5 people from 10.

b There are ${}^4C_3$ ways of selecting 3 men from 4 and ${}^6C_2$ ways of selecting 2 women from 6. Apply the multiplication principle.

$$\begin{aligned}{}^4C_3 \times {}^6C_2 &= \frac{4!}{3!(4-3)!} \times \frac{6!}{2!(6-2)!} \\ &= \frac{4 \times 3!}{3!1!} \times \frac{6 \times 5 \times 4!}{2!4!} \\ &= \frac{4}{1} \times \frac{6 \times 5}{2 \times 1} \\ &= 60\end{aligned}$$

Write the answer.

There are 60 ways of selecting 3 men and 2 women from the ensemble.

c The number of ways of having at least one man is the total number of ways of making a selection minus the number of ways of choosing only women.

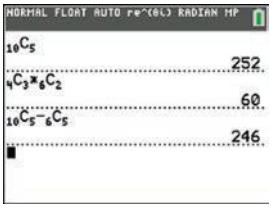
$$\begin{aligned}{}^{10}C_5 - {}^6C_5 &= 252 - \frac{6!}{5!(6-5)!} \\ &= 252 - \frac{6 \times 5!}{5!1!} \\ &= 252 - 6 \\ &= 246\end{aligned}$$

Write the answer.

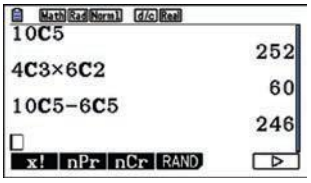
There are 246 ways of choosing a group containing at least one man.

Example 9 can also be solved using a calculator, using the methods shown on page 231.

TI-84 Plus CE



Casio fx-CG20AU



TI-Nspire CX Chapter 6

ISBN 9780170412995

Exercise 6.03 Combinations with restrictions

Example 7

1 Robert and Boris are part of a 15 player soccer squad. In how many ways can a team of 11 be chosen if:

- **a** both Robert and Boris are in the team?
- **b** Boris is in the team but Robert is not?
- **c** neither Robert nor Boris are in the team?

Example 8

2 In poker, a full house consists of 3 of one card and 2 of another, such as **KKK88**. How many possible full house hands are there?

Example 9

3 A pet shop owner wishes to display 5 animals in the shop window. There are 6 cats and 3 dogs available for display. How many combinations of 5 display animals can be chosen:

- **a** with no restrictions?
- **b** to include one dog only?
- **c** to include at least one dog?

Alamy Stock Photo/David Crossland

4 David and Elise are in a class of 20 students. How many ways can you select a group of 4 students from the class if both David and Elise are to be included?

5 How many subsets of the set {1, 2, 3,… , 10} have exactly 5 elements and contain the number 10?

6 How many ways are there of selecting 4 letters from the English alphabet if each selection contains 2 vowels?

7 A committee of 5 doctors is chosen from 4 psychiatrists, 4 surgeons and 3 neurologists. In how many ways can this be done if the committee contains:

- **a** exactly 3 psychiatrists and one surgeon?
- **b** exactly 2 psychiatrists?

Problem solving

8 A company board has 10 directors. 6 of the directors are in favour of a merger proposal while 4 are opposed to it. A subcommittee of 3 directors is to be chosen to investigate the merger proposal. How many subcommittees can be selected so that exactly 2 directors are in favour of the proposal?

ISBN 9780170412995

9 12 members of parliament are available for selection on a 5-person committee of enquiry. 7 of those available are from the Opposition while 5 are from the Government. How many such committees are possible if each committee must contain no more than 2 members from the Opposition?

10 The menu at a small Chinese restaurant has 15 dishes. You generally order the same number of dishes as there are people in your party. How many different combinations may be chosen by a party of 6 people?

11 A formal committee of 5 students is to be selected from 12 nominations. 7 girls and 5 boys have been nominated. The principal of the school has stated that there must be a balance of boys and girls on the committee. How many possible committees are there?

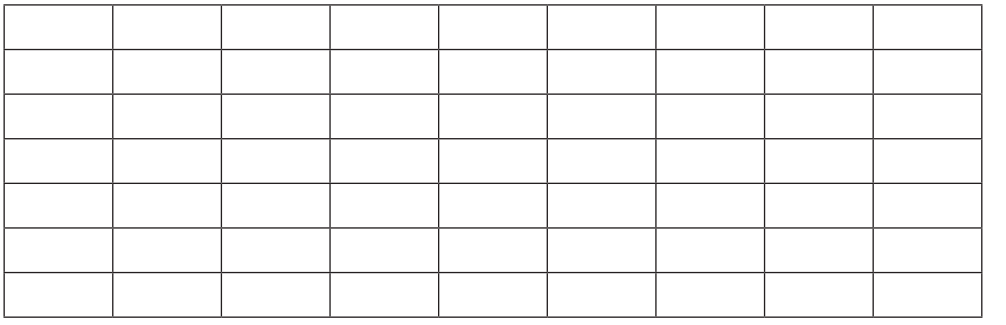
12 An amateur theatrical company is forming a new committee for next year. Since nobody really wants to do the committee work, it is decided to select the 3-member committee at random from the 20 members of the company. It is generally agreed that no committee should include both Jodie and Michael, as they do not work well together. What is the probability that they would both be selected?

13 In how many ways can a group of 6 people be divided into:

a 2 equal groups?

b 2 unequal groups, if there must be at least one person in each group?

14 The grid below contains many rectangles (more than 63). Every time 2 horizontal grid lines and 2 vertical grid lines are chosen, a different rectangle is formed.



How many rectangles does the grid contain?

ISBN 9780170412995

6.04 Applications of permutations and combinations

In real-life problems, you need to work out whether permutations, combinations or both are useful in modelling the situation. For each case, you should carefully consider whether the order of a selection is important. In some cases, you need to set up a systematic method of counting to ensure that all possibilities are included.

EXAMPLE 10

How many arrangements of the letters in the word EXHAUSTINGLY can be made that have 2 vowels and 3 consonants?

Solution

There are ${}^4C_2$ ways of selecting 2 vowels from 4. There are ${}^8C_3$ ways of choosing 3 consonants from 8. Then there are 5! ways of arranging the chosen letters.

$${}^4C_2 \times {}^8C_3 \times 5!$$

$$= \frac{4!}{2!(4-2)!} \times \frac{8!}{3!(8-3)!} \times 5!$$

$$= \frac{4 \times 3 \times 2!}{2!2!} \times \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!5!} \times 5!$$

$$= \frac{4 \times 3}{2!} \times \frac{8 \times 7 \times 6 \times 5 \times 4}{1}$$

$$= 40\,320$$

Write the answer.

There are 40 320 arrangements.

EXAMPLE 11

4 marbles are taken at random from a bag containing 5 red, 3 green and 4 yellow marbles. What is the probability that 2 red and 2 yellow marbles are taken?

Solution

Use the probability formula.

$P(\text{obtaining desired selection})$

$= P(2\text{R and } 2\text{Y})$

$$= \frac{n(2\text{R and } 2\text{Y})}{n(\text{sample space})}$$

ISBN 9780170412995

There are ${}^5C_2$ ways of choosing 2 red marbles from 5 and ${}^4C_2$ ways of choosing 2 yellow marbles from 4. There are ${}^{12}C_4$ ways of selecting 4 marbles from 12.

$$= \frac{{}^5C_2 \times {}^4C_2}{{}^{12}C_4}$$

$$= \frac{5\times4}{2\times1} \times \frac{4\times3}{2\times1} \div \frac{12\times11\times10\times9}{4\times3\times2\times1}$$

$$= \frac{4}{33}$$

Write the answer.

The probability that 2 red and 2 yellow marbles are taken is $\frac{4}{33}$.

EXAMPLE 12

In a poker hand, a flush has 5 cards from the same suit. What is the probability of being dealt a flush?

Solution

Use the probability formula.

$$P(\text{flush}) = \frac{n(\text{flushes})}{n(\text{hands})}$$

There are ${}^{13}C_5$ ways of selecting a flush in each suit. There are 4 suits so there are $4 \times {}^{13}C_5$ ways of being dealt a flush. There are ${}^{52}C_5$ ways of selecting a poker hand.

$$= \frac{4 \times {}^{13}C_5}{{}^{52}C_5}$$

$$= \frac{33}{16660}$$

$$\approx 0.001\ 981$$

Write the answer.

The probability of being dealt a flush is approximately 0.001 981.

Some of the flushes counted in Example 12 are actually straight flushes. There are 9 of these in each suit, from 2-3-4-5-6 to 10-J-Q-K-A. If you just wanted to count flushes, but not straight flushes, the number of flushes would be $4 \times {}^{13}C_5 - 36$, so the number of flushes would be 5112 instead of 5148.

$$\text{The probability (not including straight flushes)} = \frac{5112}{2\ 598\ 960}$$
$$\approx 0.001\ 967.$$

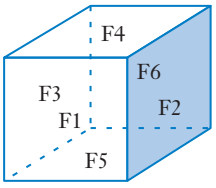
In some kinds of counting it is easy to become confused, so the visualisation and structuring of the information become an important part of counting. **Euler's formula** is a simple relation between the number of edges, faces and vertices of flat-faced solid figures. Counting is reasonably difficult for many-sided figures, so the information is changed to a two-dimensional graph, as shown in the following investigation.

ISBN 9780170412995

INVESTIGATION

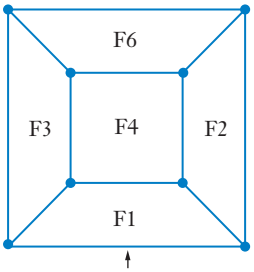
FACES OF A CUBE

The diagrams on the right show a cube and a *connected graph* of that cube. On the graph, each vertex is shown as a point on a sheet of paper, and each edge is shown as a connection between two points. Each face is shown as an enclosed area, if we agree that the outside edges also 'enclose' a face. Counting edges, faces and vertices has become easier because we have drawn the cube in two dimensions.



1 Work in groups of 2 or 3 to count the edges, faces and vertices of a cube, using the two-dimensional representation.

2 Work in your group to find the corresponding numbers for a tetrahedron, dodecahedron, truncated cube (with the corners cut off) and other three-dimensional shapes with flat faces. You can even just draw some connected graphs and calculate the number of vertices (v), edges (e) and faces (f) for the graphs. However, you should remember to count the face enclosed by the outside edges.



3 Now examine your results and calculate the value of $v + f - e$.

This is known as Euler's formula, after its discoverer, Leonhard Euler.

4 Try to prove your result by using induction on connected graphs, starting with $v = 3$.

Alamy Stock Photo/photonic 14

ISBN 9780170412995

Exercise 6.04 Applications of permutations and combinations

1 How many arrangements of the letters in the word UNCOPYRIGHTABLE can be made that have 2 consonants and 3 vowels?

Example 10

2 A committee of 5 people is to be chosen at random from a class of 20. What is the probability that Esrif and Johanna are both on the committee?

3 In the card game Pontoon, the highest hand (a pontoon) consists of an ace and a court card (Jack, Queen, King) or 10. Each player is dealt 2 cards to start with. What is the probability of getting a pontoon?

4 3 letters are chosen at random from the word MONKEY. What is the probability that:

- **a** they are all consonants?
- **b** they form a 'word' with consonant-vowel-consonant?
- **c** both vowels are chosen?

5 6 marbles are taken at random from a bag containing 6 blue, 3 yellow and 5 red marbles. What is the probability that 2 blue, 2 yellow and 2 red marbles are taken?

Example 11

6 In poker, a full house consists of 3 of one card and 2 of another, such as **KKK88**. What is the probability of being dealt a full house?

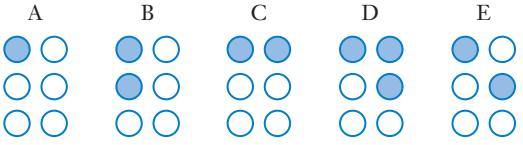
Example 12

Problem solving

7 6 letters were to be placed in addressed envelopes for posting. Unfortunately, the letters were dropped before being put in the envelopes, and they were placed in at random. What is the probability that:

- **a** exactly 4 are in the correct envelopes?
- **b** exactly 3 are in the correct envelopes?
- **c** exactly 2 are in the correct envelopes?
- **d** exactly one is in the correct envelope?
- **e** none are in the correct envelope? (*Hint:* Start with 2 letters and 2 envelopes and work upwards.)

8 In the Braille system of writing, dots are raised above the page so that they can be distinguished by touch. There are six possible dot positions. For each letter, a different combination of one to six dots is raised. The first five letters have the dots raised as shown.



- **a** In how many ways can one dot be raised?
- **b** In how many ways can 2 dots be raised?
- **c** How many combinations of raised dots are possible?

ISBN 9780170412995

9 A square has 2 diagonals that intersect in one place. Find the number of diagonals and intersections for regular:

a pentagons **b** hexagons **c** heptagons **d** octagons.

10 Some students arrive at the basketball court and decide to make up 2 teams to play a practice game. How many different pairs of teams of equal numbers can be made if there are:

a 7 students? **b** 9 students? **c** 11 students?

d 15 students? (*Note:* The odd one out must umpire!)

6.05 Pascal's triangle and identities

Pascal's triangle, named after French mathematician Blaise Pascal (1623–1662), is also known as Yang Hui's triangle in China, Khayyam's triangle in Iran and Tartaglia's triangle in Italy. The first five rows of Pascal's triangle are depicted below. Observe that:

- each row begins and ends with 1
- the middle entries are found by adding the 2 entries directly above.
- each entry is ${}^nC_r$, where n is the row number (starting from the 0th row) and r is the rth element in the row (starting from the 0th element)

Row 0					1							
Row 1				1		1						
Row 2			1		2		1					
Row3		1		3		3		1				
Row4	1		4		6		4		1			
Row5		1		5		10		10		5		1
Row6	1		6		15		20		15		16	1

					${}^0C_0$							
				${}^1C_0$		${}^1C_1$						
			${}^2C_0$		${}^2C_1$		${}^2C_2$					
		${}^3C_0$		${}^3C_1$		${}^3C_2$		${}^3C_3$				
	${}^4C_0$		${}^4C_1$		${}^4C_2$		${}^4C_3$		${}^4C_4$			
${}^5C_0$		${}^5C_1$		${}^5C_2$		${}^5C_3$		${}^5C_4$		${}^5C_5$		
${}^6C_0$		${}^6C_1$		${}^6C_2$		${}^6C_3$		${}^6C_4$		${}^6C_5$		${}^6C_6$

Pascal's triangle identity

A value in Pascal's triangle is equal to the sum of the 2 values above it.

${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, where $1 < r < n$.

Proof

Consider the subsets of $\{1, 2, 3, \ldots, n\}$. The number of subsets containing r elements is ${}^nC_r$. Each subset falls into one of two groups:

- subsets that contain n
- subsets that do not contain n

If a subset contains n, then its remaining $r - 1$ elements must be chosen from $\{1, 2, 3, \ldots, n - 1\}$. Hence this group contains ${}^{n-1}C_{r-1}$ subsets.

ISBN 9780170412995

If a subset does not contain n, then r elements still have to be chosen from $\{1, 2, 3, \ldots, n-1\}$. So the second group contains ${}^{n-1}C_r$ subsets.

Therefore, the 2 groups combined contain all ${}^nC_r$ subsets. This can be written as

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \qquad \text{QED}$$

EXAMPLE 13

a Write down the 7th row of Pascal's triangle.

b Write down the value of ${}^7C_4$.

Solution

a Each number in the 7th row is the sum of the 2 numbers directly above.

$n = 7$: 1 7 21 35 35 21 7 1

b ${}^7C_4$ is the 5th entry in the row, since the first entry is ${}^7C_0 = 1$

${}^7C_4 = 35$

EXAMPLE 14

If ${}^{20}C_3 = 1140$ and ${}^{20}C_4 = 4845$, find ${}^{21}C_4$.

Solution

Let $n = 21$ and $r = 4$ in the identity ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

Simplify and evaluate.

$$\begin{aligned} {}^{21}C_4 &= {}^{21-1}C_{4-1} + {}^{21-1}C_4 \\ &= {}^{20}C_3 + {}^{20}C_4 \\ &= 1140 + 4845 \\ &= 5985 \end{aligned}$$

Recall the identity $(a+b)^2 = a^2 + 2ab + b^2$. Notice that the coefficients of each term correspond to the 2nd row of Pascal's triangle. The powers of a decrease from 2 to 0 and the powers of b increase from 0 to 2. Pascal's triangle can be used to expand expressions with higher powers.

Consider the expansion: $(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$. The coefficients correspond to the 3rd row of Pascal's triangle. The powers of a decrease from 3 to 0. The powers of b increase from 0 to 3.

Expanding expressions in the form $(a+b)^n$ for higher powers of n using algebraic methods becomes tedious. Pascal's triangle gives the coefficients for an expanded binomial of the form $(a+b)^n$ and can be used to expand binomial expressions more efficiently.

ISBN 9780170412995

If you wanted to expand an expression with a much larger power such as $(x + 1)^{35}$, then writing down all the rows of Pascal's triangle would itself become time consuming. An alternative method is to use the **binomial theorem.**

The binomial theorem

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-2}a^2b^{n-2} + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$$

Notice that in the expansion of $(a + b)^n$, the coefficients are entries in Pascal's triangle and the powers of a start from n and descend to 0 and the powers of b do the converse.

EXAMPLE 15

a Use Pascal's triangle to expand $(x + 2)^4$.

b Find the first 4 terms in the expansion of $\left(2+\frac{x}{3}\right)^{12}$.

Solution

a Use the 4th row of Pascal's triangle to find the coefficients.

1 4 6 4 1

Decrease the powers of x from 4 to 0 and increase the powers of 2 from 0 to 4.

$$\begin{aligned}(x+2)^4 &= 1x^42^0 + 4x^32^1 + 6x^22^2 + 4x^12^3 + 1x^02^4\\ &= x^4 + 8x^3 + 24x^2 + 32x + 16\end{aligned}$$

b Write the first 4 terms of the binomial expansion of $(a + b)^n$.

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \ldots$$

Substitute the values $a = 2$, $b = \frac{x}{3}$ and $n = 12$ into the binomial theorem.

$$\left(2+\frac{x}{3}\right)^{12}$$

$$= {}^{12}C_02^{12}\left(\frac{x}{3}\right)^0 + {}^{12}C_12^{12-1}\left(\frac{x}{3}\right) + {}^{12}C_22^{12-2}\left(\frac{x}{3}\right)^2 + {}^{12}C_32^{12-3}\left(\frac{x}{3}\right)^3 + \ldots$$

Simplify.

$$= 4096 + 8192x + \frac{22528}{3}x^2 + \frac{112640}{27}x^3 + \cdots$$

ISBN 9780170412995

Exercise 6.05 Pascal's triangle and identities

Example 13

1 Write down the 7th row of Pascal's triangle to find the values of ${}^7C_3$ and ${}^7C_5$.

Example 14

2 Write down the 8th row of Pascal's triangle to find the values of ${}^8C_2$ and ${}^8C_6$.

3 Evaluate ${}^9C_2$ and ${}^9C_3$. Hence find ${}^{10}C_3$.

Example 15

4 **a** Use Pascal's triangle to expand each binomial.

i $(2+x)^3$ **ii** $(1-5x)^5$ **iii** $(a-b)^7$ **iv** $\left(1+\frac{3}{a}\right)^4$

b Use the binomial theorem to find the first 3 terms in the expansion of $\left(1-\frac{x}{2}\right)^8$.

5 Use the binomial theorem to expand $(1+2x)^3$.

6 Use the binomial theorem to expand $(1-3x)^4$.

7 Find the coefficient of x^5 in the expansion of $(1+4x)^9$.

8 In the expansion of $(1-x)^8$, find the coefficient of x^7.

9 Prove that the third number of any row in Pascal's triangle is $\frac{1}{2}n(n-1)$ when $n > 1$.

10 Prove that in Pascal's triangle, the numbers in any row are always symmetrical.

11 If ${}^nC_0 = 1$ and $n > 0$, prove that ${}^nC_{r+1} = \frac{n-r}{r+1} \times {}^nC_r$.

6.06 The pigeon-hole principle

The pigeon-hole principle is an intuitive counting principle that has many uses. Assume that there are 11 pigeons but only 10 pigeon-holes (bird houses). You could place 10 pigeons in each of the 10 pigeon-holes, but there would still be one pigeon left over. That pigeon would have to share one of the pigeon-holes with another pigeon. It follows that if there are more pigeons than pigeon-holes, then at least one pigeon-hole will house more than one pigeon.

The idea was first formalised by German mathematician Peter Dirichlet in 1834 under the name *Schubfachprinzip* ('drawer' or 'pigeon-hole principle'). More generally, the principle can be stated as follows:

Pigeon-hole principle

If m objects are distributed among n boxes where $m > n$, then at least one box must contain more than one object.

ISBN 9780170412995

Shutterstock.com/napocska

EXAMPLE 16

13 friends gather for a party. Show that at least 2 friends will have the same star sign.

Solution

Think of each friend as a pigeon and each star sign as a pigeon-hole.	There are 13 friends and 12 star signs.
If you put 13 pigeons into 12 pigeon-holes, at least one pigeon-hole will have more than one pigeon.	If you assign a star sign to 13 friends, at least 2 friends will have the same star sign.

EXAMPLE 17

In a group of 49 people, show that at least 5 will have a birthday in the same month.

Solution

Think of each person as a pigeon and each month as a pigeon-hole.	There are 49 people and 12 months.
If you try to distribute the pigeons as evenly as possible, 48 pigeons could be housed in 12 pigeon-holes with 4 pigeons in each. This leaves 1 extra pigeon to be housed so one pigeon-hole must contain 5 pigeons.	If the birthdays were spread as evenly as possible across the 12 months, there would be 4 birthdays in each of 11 months while one month would have 5 birthdays.

Note that in the example above, not all the birthdays would necessarily be evenly spread across all 12 months. All 49 birthdays may be in the same month (in an extreme case) while the more likely scenario is that there would be more (perhaps 5 or 6) in some months and

ISBN 9780170412995

fewer (say 2 or 3) in other months. The pigeon-hole principle dictates that there will be *at least one month with 5 birthdays* irrespective of the distribution.

EXAMPLE 18

Frank has a large number of red, yellow, blue and green counters in his pocket. He needs three of the same colour to demonstrate a magic trick to his friends. What is the least number he must take out to be *certain* of having three the same colour?

Solution

You can set up a pigeon-hole for each colour so you can deal counters to each pigeon-hole, avoiding repeating colours.

Colour	R	Y	B	G
4 counters selected (one of each colour)	1	1	1	1
8 counters selected (2 of each colour)	2	2	2	2

The next (9th) counter must be the same as 2 already taken out.

9 counters must be drawn to be certain of having 3 of the same colour.

Exercise 6.06 The pigeon-hole principle

1 If there are 13 people in a room, show that at least 2 people will have a birthday in the same month. Example 16

2 Show that a sentence of 27 English words will have at least 2 words that begin with the same letter.

3 In a group of 22 people, show that at least 4 were born on the same day of the week. Example 17

4 If 25 students each received a grade of A, B or C for a test, show that at least 9 of the students received the same grade.

5 Show that for any 4 natural numbers chosen at random, at least 2 will have the same remainder when divided by 3.

6 12 red and 12 black checker pieces are contained in a cloth bag. How many must you remove in the dark to be sure of getting 5 of the same colour? Example 18

7 How many playing cards need to be dealt from a standard 52-card deck to ensure you will get 2 of the same:

a colour? **b** suit?

ISBN 9780170412995

Problem solving

8 How many times would you have to roll a die to obtain the same number at least twice?

9 Two whole numbers add to 41. Show that at least one of the numbers must be greater than 20.

10 101 dots are drawn at random on a 1 m ruler. Show that at least 2 dots will be less than 1 cm apart.

11 Show that for any 10 points chosen at random inside a 3 by 3 square, at least 2 points will be no more than $\sqrt{2}$ units apart.

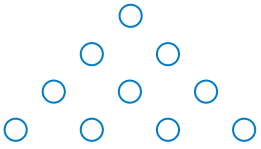
12 Show that at least one integer must be greater than or equal to the average of a group of integers.

13 7 girls and 5 boys sit evenly-spaced at a round table. Prove that some pairs of girls are sitting opposite each other.

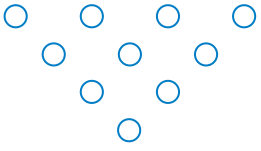
INVESTIGATION

MOVING TRIANGLES

The following triangle of counters points upwards.



1 Work in groups of 2 or 3 to show that, by moving a minimum of 3 counters, it can be made to point downwards, like this:



2 Now work in your group to show that the number of counters for triangular arrangements of 1, 2, 3, 4, 5, … rows is 1, 3, 6, 10, 15, … .

3 For each triangular arrangement, show that the *minimum* number of counters that must be moved to make it point the opposite way is 0, 1, 2, 3, 5, … .

4 Use any method to find a formula for the minimum number of moves necessary to point a triangular arrangement of n rows in the opposite direction. (*Hint*: It is not a Fibonacci sequence.)

ISBN 9780170412995

6. CHAPTER SUMMARY

Combinations

- A **combination** of a group of objects is a selection of some or all of the objects.
- The number of combinations of n things taken r at a time ${}^nC_r$, $\binom{n}{r}$, nC_r, $C(n, r)$ or C_r^n is given by

$$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!} $$

- Important results for combinations include:

  ${}^nC_n = {}^nC_0 = 1$

  ${}^nC_1 = n$

  ${}^nC_r = {}^nC_{n-r}$

- **Pascal's triangle** is an arrangement of numbers in rows that begin and end with 1. The middle numbers in each row are generated by adding the two numbers in the row directly above.

Row 0				1			
Row 1			1		1		
Row 2		1		2		1	
Row 3	1		3		3		1
Row 4	1	4		6		4	1
Row 5	1	5	10		10	5	1
Row 6	1	6	15	20	15	16	1

- Each value in Pascal's triangle is given by ${}^nC_r$ and the numbers in the nth row give the coefficients in the binomial expansion of $(a + b)^n$.
- Pascal's triangle identity

$$ {}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r, \text{ where } 1 < r < n $$

- Binomial theorem

$$ (a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \cdots + {}^nC_{n-2}a^2b^{n-2} + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n $$

- Pigeon-hole principle

  If m objects are distributed among n boxes, where $m > n$, then at least one box must contain more than one object.

ISBN 9780170412995

6. CHAPTER REVIEW

Combinations

Example 1

1 Evaluate each expression.

a 4! **b** $\frac{7!}{3!}$ **c** $\frac{8!}{2!3!}$

Example 2

2 How many combinations of 3 different-coloured beanies can be selected at random from a bag containing 12 beanies of different colours?

Example 3

3 Evaluate each expression.

a ${}^6P_3$ **b** ${}^5P_2$ **c** ${}^6P_2$ **d** ${}^7P_4$ **e** ${}^9P_5$

Example 4

4 Calculate the value of:

a ${}^{12}C_4$ **b** ${}^{16}C_3$ **c** ${}^{25}C_{20}$

Example 5

5 Find n if:

a ${}^nC_2 = 15$ **b** ${}^nC_2 = 36$.

Example 6

6 How many combinations of 4 flavours of gelato can be chosen from 20 flavours?

Examples 7–9

7 In the Canasta family of card games, a meld consists of a sequence of 3 or more cards of the same suit in a row (such as 8♥–9♥–10♥–J♥) or 3 or more cards of the same rank (such as 5♣–5♦–5♥–5♥).

a How many different melds of 3-card straights are possible?

b How many different melds of 3 cards of the same rank are possible?

Example 10

8 How many arrangements of the letters in the word AMBIDEXTROUS can be made that have 3 vowels and 4 consonants?

Examples 11, 12

9 Three letters are chosen at random from the word WONDERFUL. What is the probability that:

a they are all consonants?

b they form a 'word' with vowel-consonant-vowel?

c all 3 vowels are chosen?

Example 13

10 **a** Write down the 8th row of Pascal's triangle.

b Calculate the value of ${}^8C_4$.

Example 14

11 If ${}^{15}C_2 = 105$ and ${}^{15}C_3 = 455$, find ${}^{16}C_3$.

Example 15

12 Use Pascal's triangle to expand $(x - 1)^5$.

ISBN 9780170412995

Examples 16–17

Example 18

13 How many students must you ask to be sure of getting three whose birthdays this year are on the same day of the week?

14 A bank teller has a bag of mixed Australian coins. How many coins must she take out to be sure of obtaining 4 of the same kind?

Problem solving

15 A jury may consist of different numbers of people, depending on the seriousness of the case. However, a traditional jury has 12 people. In criminal cases, there must be a unanimous verdict, but in civil cases a majority verdict is sufficient.

- **a** In how many ways can 2 people be in disagreement with the majority in a criminal case?
- **b** In how many ways can a 6-6 result arise in a civil case?
- **c** In how many ways can a combination of jurors give a verdict in a civil case?

16 The organising committee of an international conference wants to provide translators between any pair of the official conference languages: English, French, German, Japanese and Chinese. How many translators are needed?

17 A sports roster is to be organised for 8 teams so that each team plays a home game and an away game against every other team.

- **a** How many games are needed?
- **b** A round consists of pairings of teams for one week of the competition. How many different rounds are possible, without considering the difference between home and away games?
- **c** How many rounds are possible when the difference between home and away is taken into account?

18 For one exam, students are given 5 possible questions before the exam, but only 2 are actually examined. How many different possible exams are there?

19 A young customer has 40 cents to spend on lollies. There are 12 types of 10-cent lollies available. If the customer gets different types of lollies, how many different selections can be made if:

- **a** the whole 40 cents is spent?
- **b** 30 cents is spent?
- **c** 20 cents is spent?

20 A 'scratch-it' ticket has 6 windows. To win a prize, the first 3 windows you scratch must match. If every ticket has 3 matching windows, what is the probability of winning a prize?

ISBN 9780170412995

21 A magazine wanted to publish an article about people's attitudes towards education. 6 interviews were to be published from a selection of 12 interviews with. 5 Australian-born women, 3 overseas-born women, 2 Australian-born men and 2 overseas-born men. If at least one interview from each category was to be included, how many different selections were possible?

22 Morse code (which was once an important method of communication by sound or by light) is made up of short and long units (beeps or flashes) called dots and dashes. How many different codes could be made using:

a 2 units? **b** 3 units? **c** 4 units?

d 5 units? **e** n units?

Practice quiz

23 A domino has 2 squares, each with a number of dots from 0 to 6. The smallest possible set of dominoes has one domino of each possible combination, including doubles. How many dominoes are there in the smallest possible set?

ISBN 9780170412995

Practice examination 2 ●●○○

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (3 marks)

Find the scalar product of:

a vectors of magnitudes 4 and 5 at an angle of 46° to each other

b $\mathbf{c} = (5, 46°)$ and $\mathbf{d} = (6, 87°)$

c $\mathbf{u} = (2, 5)$ and $\mathbf{v} = (3, 1)$

Question 2 (2 marks)

Calculate the angle between each pair of vectors.

a $\mathbf{m} = (7, 1)$ and $\mathbf{n} = (2, 6)$

b $\mathbf{p} = (4, 3)$ and $\mathbf{q} = (3.6, -56.3°)$

Question 3 (1 mark)

Show that the vectors $\mathbf{a} = (-5, 6)$ and $\mathbf{b} = \left(\frac{3}{4}, \frac{5}{8}\right)$ are perpendicular.

Question 4 (1 mark)

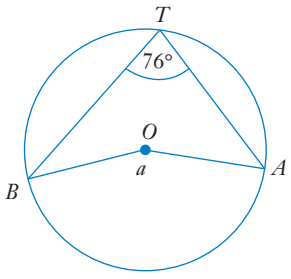
Calculate the projection of $\mathbf{a} = (5, 7)$ on $\mathbf{b} = (6, 2)$.

ISBN 9780170412995

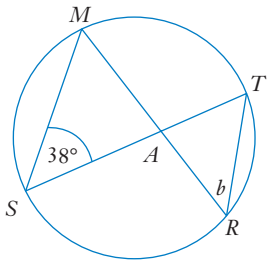
Question 5 (3 marks)

Find the value of each pronumeral, giving reasons.

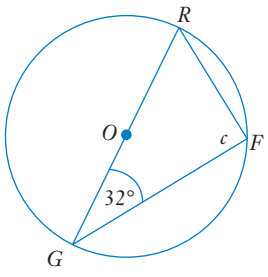
a



b



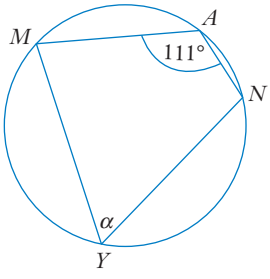
c



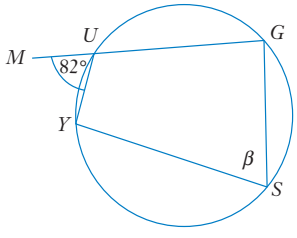
Question 6 (3 marks)

Find the value of each pronumeral, giving reasons.

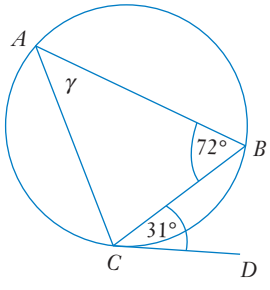
a



b



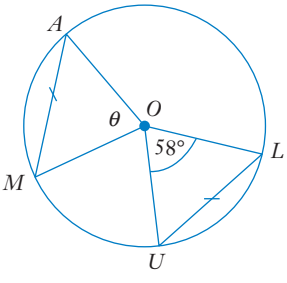
c



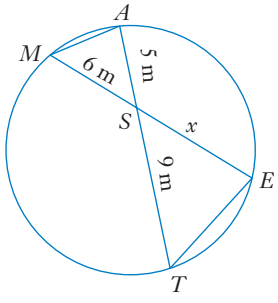
Question 7 (4 marks)

Find the value of each pronumeral, giving reasons.

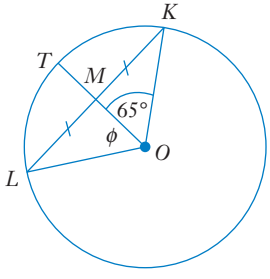
a



b



c

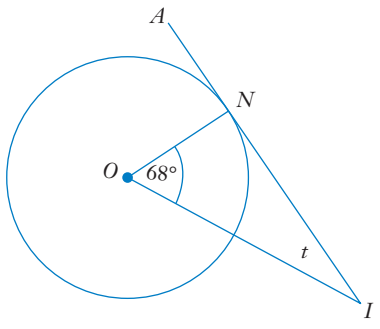


ISBN 9780170412995

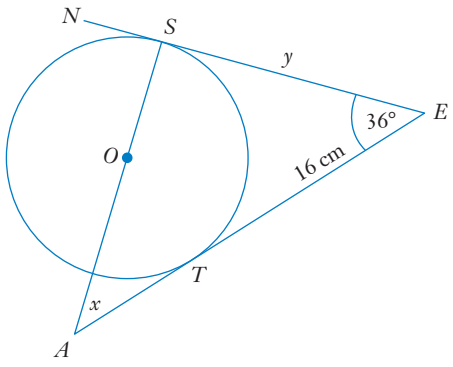
Question 8 (3 marks)

Find the value of each pronumeral, giving reasons.

a

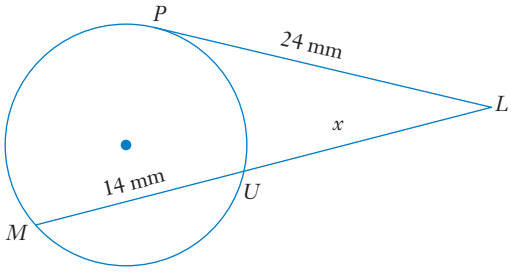


b



Question 9 (3 marks)

Find the value of x, giving reasons.



Question 10 (2 marks)

Evaluate:

a ${}^6P_2$

b ${}^9C_3$

Question 11 (2 marks)

John and Andrew are two members of a 12-player squad. How many different possible teams of 7 can be selected if:

a John is selected but Andrew is not?

b neither John nor Andrew are selected?

ISBN 9780170412995

Question 12 (3 marks)

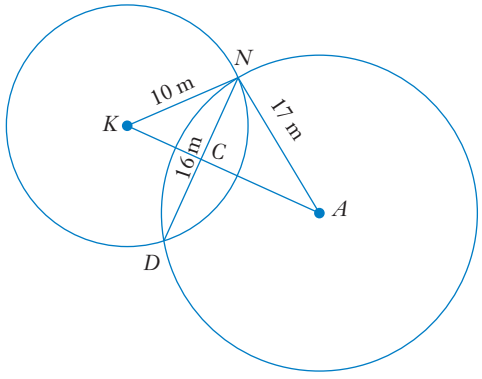
Use Pascal's triangle to calculate ${}^4C_3 \times {}^5C_4$.

Question 13 (2 marks)

If **AC** : **CB** = 2 : 3, express **c** in terms of **a** and **b**.

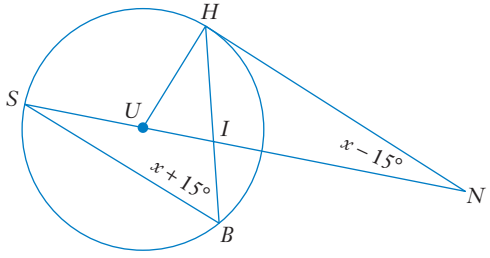
Question 14 (3 marks)

Find the distance between the centres of the circles in the diagram below, giving reasons for each step.



Question 15 (2 marks)

Find the value of x in the diagram below, giving reasons for each step.



Question 16 (2 marks)

Sandra and Susan are on a committee of 15 people. A group of 6 is chosen at random to evaluate a business proposal. What is the probability that both Sandra and Susan are selected?

ISBN 9780170412995

Question 17 (1 marks)

A drawer contains 10 black, 10 grey and 10 white socks. Without looking, how many must be taken out to be sure of picking 1 pair of black socks and 1 more pair of another colour?

Question 18 (2 marks)

Use the fact that $|\mathbf{a}|^2 = \mathbf{a} \bullet \mathbf{a}$ to prove $\mathbf{a} \bullet \mathbf{b} = \frac{1}{2}\left(|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}|^2 - |\mathbf{b}|^2\right)$.

Question 19 (5 marks)

Use vectors to prove that the opposite angles of a parallelogram are equal.

Question 20 (3 marks)

A bag contains 800 balls that are coloured either red, blue, green, yellow or black. The ratio of red to blue to green is 1 : 4 : 2. The ratio of green to yellow to black is 1 : 3 : 6. What is the smallest number of balls that must be taken at random to ensure that 100 balls of one colour are chosen?

END OF EXAMINATION

ISBN 9780170412995

7.

MATRICES

Tables of numbers are often used to show mathematical information. In many cases, the information in different tables is combined to solve problems. For example, in physics, lens systems are designed by combining many simple lenses to make zoom lenses. The properties of the simple lenses are shown by a square table of numbers. These are combined to show the properties of the system. Economic policy changes can be worked out using large tables of numbers that show the effect of changes on the economy.

7.01 Matrix addition and subtraction
7.02 Matrix multiplication
7.03 Matrix identities and inverses
7.04 Matrix equations
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Matrix arithmetic

- understand the matrix definition and notation
- define and use addition and subtraction of matrices, scalar multiplication, matrix multiplication, multiplicative identity and multiplicative inverse
- calculate the determinant and inverse of 2×2 matrices algebraically and solve matrix equations of the form $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{X}$ and $\mathbf{B}$ are column vectors
- calculate the determinant and inverse of higher order matrices and solve matrix equations using technology

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

iStock/monsitj

TERMINOLOGY

column matrix
commutative
conformable
determinant
dimension
element
identity
inverse
Kronecker delta function
matrix equation
matrix, matrices
non-singular
order
row matrix
scalar multiple
singular
size
square matrix
zero matrix

Addition and subtraction of matrices

7.01 Matrix addition and subtraction

A warehouse distributes greetings cards to shops like newsagents. Mixed packs of birthday cards, engagement cards, get-well cards, etc. are supplied in budget, standard, quality and deluxe packs of 50 cards. The packs ordered by some retailers in a small town in January are shown below using their customer codes.

Order no.	Budget	Standard	Quality	Deluxe
#0657	3	2	0	2
#0659	5	8	9	6
#0665	3	6	7	5
#0671	2	1	5	9
#0672	3	4	3	2

The rectangle of numbers is used to pack and send the orders to the town retailers.

Matrices

- A **matrix** (plural **matrices**) is a rectangle of numbers in large parentheses () or brackets [ ].
- The symbol is a bold capital letter like **A**, or handwritten with a wavy underline like $\underset{\sim}{\mathrm{A}}$.

$\mathbf{M} = \begin{bmatrix} 3 & 2 & 0.6 & \sqrt{2} \\ -4 & 5 & \frac{3}{4} & -6 \end{bmatrix}$ would be shown as $\underset{\sim}{\mathrm{M}}$

- Each number is called an **element** of the matrix.

ISBN 9780170412995

- The **size** (**dimension** or **order**) is the number of rows × number of columns, so **M** is 2×4.
- A **row matrix** has only one row.
- A **column matrix** has only one column.
- A **square matrix** has the same number of rows and columns.
- Every entry of a **zero matrix** is 0.
- The elements of a matrix are shown by the lower case letter with the row and column as subscripts: $m_{23} = \frac{3}{4}$

EXAMPLE 1

a Show the greetings card orders for January above as the matrix **J**.

b What are the dimensions of **J**?

c What are the values of j_{23}, j_{42} and j_{45}?

Solution

a Show the numbers in brackets.

$$\mathbf{J} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 5 & 8 & 9 & 6 \\ 3 & 6 & 7 & 5 \\ 2 & 1 & 5 & 9 \\ 3 & 4 & 3 & 2 \end{bmatrix}$$

b **J** has 5 rows and 4 columns. **J** is 5×4

c What is in the 2nd row, 3rd column? $j_{23} = 9$

What is in the 4th row, 2nd column? $j_{42} = 1$

What is in the 4th row, 5th column? There are only 4 columns so j_{45} does not exist.

The orders from the same retailers for February are given by $\mathbf{F} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 4 & 0 & 1 & 3 \\ 3 & 0 & 2 & 4 \\ 2 & 3 & 4 & 0 \\ 3 & 2 & 4 & 4 \end{bmatrix}$.

How would you find the total orders for January and February?

When would you say that the orders for two different months were the same?

How would you find the difference between two months' orders?

ISBN 9780170412995

Matrix addition and subtraction

- Addition of matrices is only possible for matrices of the same size
- Matrices are added by adding the corresponding elements;

$$\text{for } \mathbf{A} = (a_{ij}) \text{ and } \mathbf{B} = (b_{ij}),\ \mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$$

- Equal matrices are the same size and have equal elements;

$$\mathbf{A} = \mathbf{B} \Leftrightarrow a_{ij} = b_{ij}\ \forall\ i, j$$

- Subtraction of matrices is only possible for matrices of the same size
- Matrices are subtracted by subtracting the corresponding elements;

$$\text{for } \mathbf{A} = (a_{ij}) \text{ and } \mathbf{B} = (b_{ij}),\ \mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$$

EXAMPLE 2

$\mathbf{J} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 5 & 8 & 9 & 6 \\ 3 & 6 & 7 & 5 \\ 2 & 1 & 5 & 9 \\ 3 & 4 & 3 & 2 \end{bmatrix}$, $\mathbf{F} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 4 & 0 & 1 & 3 \\ 3 & 0 & 2 & 4 \\ 2 & 3 & 4 & 0 \\ 3 & 2 & 4 & 4 \end{bmatrix}$ and $\mathbf{M} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 5 & 8 & 9 & 6 \\ 3 & 6 & 7 & 3 \\ 2 & 1 & 5 & 9 \\ 3 & 4 & 3 & 2 \end{bmatrix}$. Find

a $\mathbf{J} + \mathbf{F}$ **b** $\mathbf{J} + \mathbf{F} + \mathbf{M}$ **c** $\mathbf{J} - \mathbf{F}$ **d** Is $\mathbf{J} = \mathbf{M}$?

Solution

a Add the corresponding elements.

$$\mathbf{J} + \mathbf{F} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 5 & 8 & 9 & 6 \\ 3 & 6 & 7 & 5 \\ 2 & 1 & 5 & 9 \\ 3 & 4 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 & 2 \\ 4 & 0 & 1 & 3 \\ 3 & 0 & 2 & 4 \\ 2 & 3 & 4 & 0 \\ 3 & 2 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 2 & 4 \\ 9 & 8 & 10 & 9 \\ 6 & 6 & 9 & 9 \\ 4 & 4 & 9 & 9 \\ 6 & 6 & 7 & 6 \end{bmatrix}$$

b Add the corresponding elements.

$$\mathbf{J} + \mathbf{F} + \mathbf{M} = \begin{bmatrix} 7 & 5 & 2 & 6 \\ 14 & 16 & 19 & 15 \\ 9 & 12 & 16 & 12 \\ 6 & 5 & 14 & 18 \\ 9 & 10 & 10 & 8 \end{bmatrix}$$

ISBN 9780170412995

c Subtract the corresponding elements.

$$\mathbf{J}-\mathbf{F}=\begin{bmatrix}3&2&0&2\\5&8&9&6\\3&6&7&5\\2&1&5&9\\3&4&3&2\end{bmatrix}-\begin{bmatrix}1&1&2&2\\4&0&1&3\\3&0&2&4\\2&3&4&0\\3&2&4&4\end{bmatrix}$$

$$=\begin{bmatrix}2&1&-2&0\\2&8&8&3\\0&6&5&1\\0&-2&1&9\\0&2&-1&-2\end{bmatrix}.$$

d Is every element the same? $j_{34} \neq m_{34}$ so $\mathbf{J} \neq \mathbf{M}$.

If you prefer, you can subtract matrices by adding the opposite.

TECHNOLOGY

Matrices and graphics calculators

You can use your graphics calculator to add and subtract **J** and **F** from Example 2.

TI-Nspire CX Chapter 7

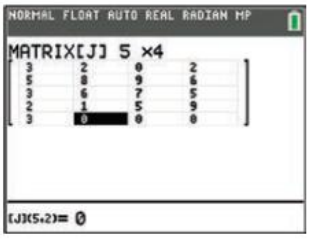
TI 84 Plus CE

Only A–J are available as matrix names.

If you want to do **M**, you have to choose a different name.

Press 2nd x⁻¹ (matrix), move the cursor across to EDIT, down to J, and press enter.

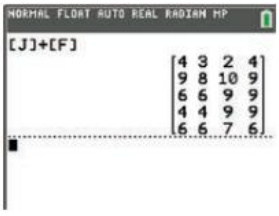
Now press 5 enter, 4 enter for the size and put in the elements of **J**.



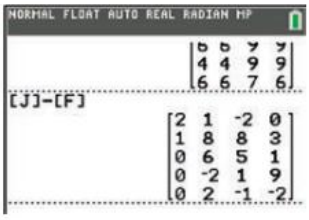
Press 2nd mode to quit and do **F**.

For **J** + **F**, press 2nd x⁻¹ and use the cursor to go down to **J** in NAMES and press enter.

Then press +, 2nd x⁻¹, go down to **F** and press enter twice.



Do **J** – **F** similarly.



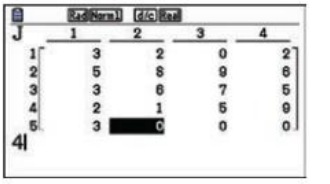
ISBN 9780170412995

Casio fx-CG20AU

A–Z are available as matrix names.

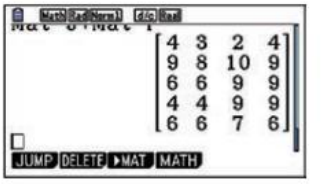
Choose 1 Run-Matrix and press F3 (▶MAT).

Use the cursor to go down to Mat J.



Press F3 again (DIM), 5 EXE and 4 EXE for the size and put in the elements of **J**.

Press EXIT, move down and do **F** in the same way.



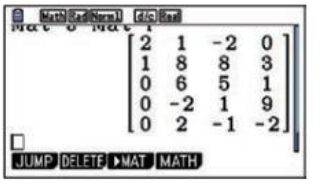
Press EXIT to leave matrix entry.

For **J** + **F**, press OPTN F2 F1 (Mat), then ALPHA J.

Press + and do Mat F as for Mat J and press EXE.

You can also use SHIFT 2 for Mat.

Do **J** – **F** similarly.



Exercise 7.01 Matrix addition and subtraction

Example 1

1 Which of the following are matrices?

a $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & & \\ & 1 & \\ & & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

d $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & -2 \\ & 1 & 3 \end{bmatrix}$ **e** $\begin{bmatrix} \sqrt{3} & \frac{5}{6} \\ -2 & 7 \end{bmatrix}$ **f** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Example 2

2 $A = \begin{bmatrix} 1 & 4 & 3 & -2 \\ 9 & 0 & 6 & -1 \\ 7 & 2 & 8 & 5 \end{bmatrix}$. If possible, write down

a a_{12} **b** a_{21} **c** a_{34} **d** a_{22}

e a_{32} **f** a_{43} **g** What is the size of **A**?

ISBN 9780170412995

3 What kind of matrix is each of the following?

a $(3, 4, 0, 2, 5)$ **b** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 2 \\ 3 \\ 0 \\ -4 \end{bmatrix}$

d $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **e** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **f** $[0\ 0\ 0\ 0\ 0\ 0]$

4 Given $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -6 & 4 \\ 8 & 10 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 5 & -7 \\ -3 & 1 \end{bmatrix}$, find:

Example 2

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{C} + \mathbf{B}$ **c** $\mathbf{B} + \mathbf{D}$ **d** $\mathbf{C} + \mathbf{D}$

e $-\mathbf{B}$ **f** $\mathbf{C} - \mathbf{A}$ **g** $\mathbf{B} - \mathbf{D}$ **h** $\mathbf{C} - \mathbf{B}$

5 Given $\mathbf{A} = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 3 & 6 \\ 5 & 0 & -1 \\ 0 & 5 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & -4 & -3 \\ 2 & -3 & 1 \\ 4 & 2 & 4 \\ -4 & -3 & -2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 5 & 8 \\ 0 & 2 & 4 \\ 0 & 6 & 2 \\ 7 & 0 & 0 \end{bmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} + \mathbf{C}$ **c** $\mathbf{C} + \mathbf{B}$ **d** $\mathbf{B} + \mathbf{A}$

e $-\mathbf{B}$ **f** $\mathbf{C} - \mathbf{A}$ **g** $\mathbf{A} - \mathbf{C}$ **h** $\mathbf{B} - \mathbf{C}$

Problem solving

6 A manufacturer builds desktop computers with a variety of central processor speeds and hard drive capacities. The result of a stock inventory is shown in the table below.

	Processor (GHz)								
Hard drive (GB)	**1.86**	**2.0**	**3.8**	**2.33**	**2.67**	**2.8**	**3.0**	**3.2**	**3.5**
160	3	2	4	1	5	0	3	6	1
250	1	8	1	7	1	8	0	5	8
320	4	0	5	6	4	5	9	1	7
500	7	9	2	0	1	1	10	9	1

a Write the information in the table as a 4×9 matrix $\mathbf{M}$.

b Write the value of m_{32}.

c Write the value of m_{21}.

d Write the value of m_{27}.

e What is the size of $\mathbf{M}$?

ISBN 9780170412995

7 A builder is comparing the prices of timber from four different hardware suppliers, including the cost of delivery to the site. The information is shown in the following table. *Note:* All prices listed are per metre or per square metre (sheet).

Product	Jones & Co.	Brown Bros	Pine Hut	Allgoods
Hardwood framing	$4.20	$4.30	Cannot supply	$4.40
Chipboard flooring	$9.10	$8.80	$8.70	$9.00
Structural plywood	$6.20	$6.40	$6.50	$6.10
Pine chamfer	$3.80	$3.90	$3.60	$3.70

a At present, the table cannot be changed to a matrix. Why not?

b Change the table so that it can be converted to a square matrix.

c Write the matrix **T**.

d Write the value of t_{24}.

8 On a particular trading day, the exchange rates for Australian dollars (AUD), United States dollars (USD), Great Britain pounds (GBP) and Japanese yen (JPY) were as follows.

1 AUD = 0.7324 USD, 0.5874 GBP, 86.24 JPY

1 USD = 1.3654 AUD, 0.8020 GBP, 117.75 JPY

1 GBP = 1.7024 AUD, 1.2469 USD, 146.82 JPY

1 JPY = 0.011 59 AUD, 0.008 93 USD, 0.006 811 GBP

a What is the equivalent of 1 AUD in USD?

b Put the information in a 4×4 matrix **E** in the order AUD, USD, GPB and JPY.

c What is the value of e_{23}?

d What is the value of e_{32}?

e In this matrix, what is the relationship between e_{ij} and e_{ji}, where $i \neq j$?

9 A manufacturer of marbelite bathroomware uses the following quantities of materials (in kilograms) to make handbasins, shower bases, baths and spas.

Material	Handbasin	Shower base	Bath	Spa
Plaster filler	4	9	20	40
Epoxy resin	1	2	4	9
Catalyst	0.2	0.3	0.5	1

a Write the information as a matrix **B** with the materials in rows.

b What is the value of b_{23}?

c What is the value of b_{32}?

d What does the value of b_{22} represent?

ISBN 9780170412995

10 An ecologist interested in the predation of species in a freshwater pool studied the diets of water beetles, mosquito larvae, fingerlings and adult fish. She found that mosquito larvae and fingerlings both ate microscopic algae and microscopic animals in the water, but that fingerlings also ate approximately 3 mosquito larvae each day. Water beetles ate about 5 mosquito larvae and 2 fingerlings each day, and adult fish ate about 6 fingerlings and 4 water beetles each day.

a Write the information as the 4×4 matrix **P**, showing the predation of species on each other, with the animals in the order: mosquito larvae, fingerlings, water beetles and adult fish.

b What is the significance of the row of zeros for mosquito larvae?

c What is the meaning of p_{33}?

11 Construct the square matrix **Q** of order 3 with elements given by $q_{ij} = i + j$.

12 Construct the row matrix **P** of order 6 with elements given by $p_i = 4 - i$.

13 Find x in each of the following:

a $\begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix} + \begin{bmatrix} 3 & x \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & x \end{bmatrix}$

b $\begin{bmatrix} 4 & 3 & 4 \\ -2 & 0 & x \end{bmatrix} + \begin{bmatrix} -1 & x^2 & -4 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

c $\begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -x^3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix}$

14 Find **A** if $\begin{bmatrix} -1 & 2 \\ -4 & 7 \end{bmatrix} + \mathbf{A} = \begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}$

15 Find **M** if $\mathbf{M} + \begin{bmatrix} 5 & -2 \\ 10 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix}$

16 A bank has three branches in Cairns: City, Earlville and Stratford. There are four types of accounts: Low Fee, Business, Savings and Student/Pensioner. The numbers of each type of account at the beginning of a year were as shown in the table below.

Branch	Low Fee	Business	Savings	Student/Pensioner
City	300	400	200	150
Earlville	400	220	300	200
Stratford	250	150	400	150

ISBN 9780170412995

The numbers of closures of accounts in that year were:

Branch	Low Fee	Business	Savings	Student/Pensioner
City	30	70	20	20
Earlville	20	30	40	30
Stratford	40	10	50	10

The numbers of new accounts opened in that year were:

Branch	Low Fee	Business	Savings	Student/Pensioner
City	50	100	10	15
Earlville	15	50	30	10
Stratford	30	25	20	40

Use 3×4 matrices to show:

a the number of accounts at the beginning of the year

b the number of closures in the year

c the number of accounts opened in the year

d an evaluated matrix expression for the number of accounts at the end of the year.

17 The results of the first five rounds, middle five rounds and last five rounds of a netball competition are shown by the following three matrices. Each row represents the results of a team. The first column is the number of games won, the second is drawn games and the third is games lost.

$$\mathbf{F} = \begin{bmatrix} 4 & 0 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 4 \\ 3 & 0 & 2 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$$

a Write a matrix expression for the total results.

b Evaluate your expression to obtain the final results.

c Which team (1st to 6th) is the top team?

ISBN 9780170412995

7.02 Matrix multiplication

Multiplying matrices

Matrix multiplication

Budget, standard, quality and deluxe packs of 50 cards cost \$55, \$95, \$125 and \$200 respectively from the distributor. You can write the charges as $\mathbf{C} = \begin{bmatrix} 55 \\ 95 \\ 125 \\ 200 \end{bmatrix}$.

The January orders from retailers in a small town were given as below.

	Budget	Standard	Quality	Deluxe
#0657	3	2	0	2
#0659	5	8	9	6
#0665	3	6	7	5
#0671	2	1	5	9
#0672	3	4	3	2

The matrix for the orders was $\mathbf{J} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 5 & 8 & 9 & 6 \\ 3 & 6 & 7 & 5 \\ 2 & 1 & 5 & 9 \\ 3 & 4 & 3 & 2 \end{bmatrix}$. The cost of the January order for one retailer is given by

#0657: $3 \times \$55 + 2 \times \$95 + 0 \times \$125 + 2 \times \$200 = \$755$

For the other retailers, the costs are:

#0659: $5 \times \$55 + 8 \times \$95 + 9 \times \$125 + 6 \times \$200 = \$3360$

#0665: $3 \times \$55 + 6 \times \$95 + 7 \times \$125 + 5 \times \$200 = \$2610$

#0671: $2 \times \$55 + 1 \times \$95 + 5 \times \$125 + 9 \times \$200 = \$2630$

#0672: $3 \times \$55 + 4 \times \$95 + 3 \times \$125 + 2 \times \$200 = \$1320$

For each retailer, you multiply the numbers in the row by the numbers in the column and add the answers. If you bought 5 apples for \$1.30 each, they would cost $5 \times \$1.30 = \$6.50$.

The calculation for the matrices is shown as **matrix multiplication** like this:

$$\begin{bmatrix} 3 & 2 & 0 & 2 \\ 5 & 8 & 9 & 6 \\ 3 & 6 & 7 & 5 \\ 2 & 1 & 5 & 9 \\ 3 & 4 & 3 & 2 \end{bmatrix} \times \begin{bmatrix} 55 \\ 95 \\ 125 \\ 200 \end{bmatrix} = \begin{bmatrix} 755 \\ 3360 \\ 2610 \\ 2630 \\ 1320 \end{bmatrix}$$

The number of columns in the first matrix must be the same as the number of rows in the second.

ISBN 9780170412995

Matrix multiplication

- Matrices **A** and **B** are **conformable** if **A** has m columns and **B** has m rows
- The **product** of an $n \times m$ matrix **A** and an $m \times p$ matrix **B** is an $n \times p$ matrix **C**
- The elements c_{ij} of **C** are the sums of the products of the ith row of **A** and the jth column of **B**: $c_{ij} = \sum a_{ik} \times b_{kj}$.

EXAMPLE 3

$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \\ -2 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 9 & 4 & 1 & 0 \end{bmatrix}$. Find $\mathbf{D} = \mathbf{AB}$.

Solution

$d_{11} = 1 \times 2 + 3 \times 9 = 29,$

$d_{12} = 1 \times 1 + 3 \times 4 = 13,$

$d_{13} = 1 \times (-3) + 3 \times 0 = -3,$ etc.

$$\mathbf{D} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \\ -2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & -3 & 4 \\ 9 & 4 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 13 & 0 & 4 \\ 53 & 24 & -7 & 16 \\ 14 & 6 & 8 & -8 \end{bmatrix}$$

The product **AB** is done in Example 3, but you cannot do **BA** because the matrices are not conformable in this order.

You can use your graphics calculator for matrix multiplication.

EXAMPLE 4

$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -2 & 4 & 5 \\ 1 & 0 & -3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 4 & 4 \\ 4 & -1 & -2 \end{bmatrix}$. Find

a $\mathbf{C} = \mathbf{AB}$ **b** $\mathbf{D} = \mathbf{BA}$

c Comment on your answers.

ISBN 9780170412995

Solution

a Do row × column and add.

1st row × 1st column gives c_{11}.

1st row × 2nd column gives c_{12}.

1st row × 3rd column gives c_{13}.

2nd row × 1st column gives c_{21}.

2nd row × 2nd column gives c_{22}.

2nd row × 3rd column gives c_{23}.

3rd row × 1st column gives c_{31}.

3rd row × 2nd column gives c_{32}.

3rd row × 3rd column gives c_{33}.

$$\mathbf{C} = \begin{bmatrix} 2 & -3 & 1 \\ -2 & 4 & 5 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 0 & 4 & 4 \\ 4 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+4 & 4-12-1 & 4-12-2 \\ 2+0+20 & -4+16-5 & -4+16-10 \\ -1+0-12 & 2+0+3 & 2+0+6 \end{bmatrix}$$

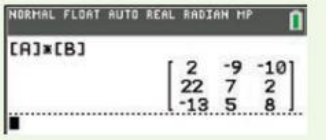
$$= \begin{bmatrix} 2 & -9 & -10 \\ 22 & 7 & 2 \\ -13 & 5 & 8 \end{bmatrix}$$

TI 84 Plus CE

Enter the matrices as shown on page 263. Use (-) for negatives.

Press 2nd x⁻¹ and press enter for **A**.

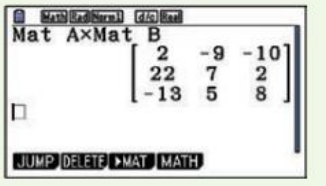
Then press ×, 2nd x⁻¹, go down to **B** and press enter twice.



Casio fx-CG20AU

Use Run-Matrix and enter the matrices as shown on page 264. Use (−) for negatives.

Press SHIFT 2 ALPHA A for Mat A and then complete the product with × SHIFT 2 ALPHA B EXE.



TI-Nspire CX Chapter 7

ISBN 9780170412995

b Do row × column and add.

1st row × 1st column gives d_{11}.

1st row × 2nd column gives d_{12}.

1st row × 3rd column gives d_{13}.

2nd row × 1st column gives d_{21}.

2nd row × 2nd column gives d_{22}.

2nd row × 3rd column gives d_{23}.

3rd row × 1st column gives d_{31}.

3rd row × 2nd column gives d_{32}.

3rd row × 3rd column gives a_{33}.

$$\mathbf{D} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 4 & 4 \\ 4 & -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -3 & 1 \\ -2 & 4 & 5 \\ 1 & 0 & -3 \end{bmatrix}$$

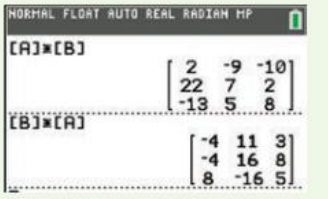
$$= \begin{bmatrix} -1 & 2 & 2 \\ 0 & 4 & 4 \\ 4 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 11 & 3 \\ -4 & 16 & 8 \\ 8 & -16 & 5 \end{bmatrix}$$

TI 84 Plus CE

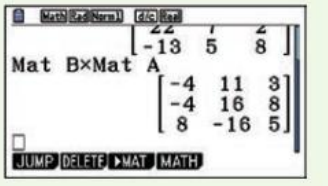
TI-Nspire CX Chapter 7

Do **BA** similarly to **AB** in part **a**.



Casio fx-CG20AU

Do **BA** similarly to **AB** in part **a**.



c Compare **C** and **D**.

AB and **BA** are both defined, but the answers are quite different, so this matrix multiplication is **not commutative**.

ISBN 9780170412995

INVESTIGATION

MATRIX MULTIPLICATION ORDER

For $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -3 & 4 \end{bmatrix}$, calculate the product **AB**.

Can you calculate the product **BA**?

Now for $\mathbf{D} = \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 0 & 3 \\ -2 & 2 \end{bmatrix}$, calculate the product **DE**.

Now calculate the product **ED**.

What can you say about the order of a matrix product?

Can you find two matrices that have products that are 2×2 in one order and 3×3 in the reverse order?

The greetings card distributor you previously looked at had a charges matrix $\mathbf{C} = \begin{bmatrix} 55 \\ 95 \\ 125 \\ 200 \end{bmatrix}$ for the prices of budget, standard, quality and deluxe packs of 50 greetings cards. The new prices after an increase of 20% would be $1.2 \times \$55$, $1.2 \times \$95$, $1.2 \times \$125$ and $1.2 \times \$200$. Every element of **C** would be multiplied by 1.2.

Multiplication by a scalar (scalar multiples)

- The product of a real number c and a matrix **A** is $c\mathbf{A} = (ca_{ij})$. Every element of the matrix is multiplied by the scalar c.
- $c\mathbf{A}$ is called a scalar multiple of **A**.

EXAMPLE 5

$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0.5 \\ 3 & 4 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -3 & 5 \\ 4 & -2 & 1 \end{bmatrix}$. Find each expression.

a $3\mathbf{A}$ **b** $\frac{1}{4}\mathbf{B}$ **c** $2\mathbf{A} - 3\mathbf{B}$

ISBN 9780170412995

Solution

a Multiply every element by 3.

$$3\mathbf{A} = 3\begin{bmatrix} 2 & -1 & 0.5 \\ 3 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -3 & 1.5 \\ 9 & 12 & 12 \end{bmatrix}$$

b Multiply every element by $\frac{1}{4}$.

$$\frac{1}{4}\mathbf{B} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{4} & 1\frac{1}{4} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

c Change $2\mathbf{A} - 3\mathbf{B}$ to $2\mathbf{A} + (-3)\mathbf{B}$ to avoid sign errors.

Now multiply and add.

$$2\mathbf{A} - 3\mathbf{B} = 2\begin{bmatrix} 2 & -1 & 0.5 \\ 3 & 4 & 4 \end{bmatrix} + (-3)\begin{bmatrix} -2 & -3 & 5 \\ 4 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -2 & 1 \\ 6 & 8 & 8 \end{bmatrix} + \begin{bmatrix} 6 & 9 & -15 \\ -12 & 6 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 10 & 7 & -14 \\ -6 & 14 & 5 \end{bmatrix}$$

Exercise 7.02 Matrix multiplication

Your teacher might want you to do some of this exercise using technology.

Examples 3, 4

1 Find each product.

a $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$ **c** $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$

d $\begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & -3 \end{bmatrix}$ **e** $\begin{bmatrix} 5 & 1 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ -2 & 4 \end{bmatrix}$ **f** $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$

g $\begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & -3 \end{bmatrix}$ **h** $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ **i** $\begin{bmatrix} 5 & -1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$

j $\begin{bmatrix} \frac{1}{2} & 3 \\ 2 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 0 & 1 \end{bmatrix}$ **k** $\begin{bmatrix} \frac{1}{3} & 1 \\ 1 & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 1 & 3 \end{bmatrix}$ **l** $\begin{bmatrix} 6 & 12 \\ 6 & 18 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} \end{bmatrix}$

2 Find each product.

a $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 0 & 4 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 5 & -1 \\ -2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

ISBN 9780170412995

c $\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 5 & 0 \\ 0 & 3 & 2 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ -1 & 2 \\ 4 & -3 \end{bmatrix}$

e $\begin{bmatrix} -2 & 4 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -5 & 1 & -4 \\ 2 & 3 & -1 \end{bmatrix}$ **f** $\begin{bmatrix} 5 & 1 & 3 \\ 2 & -3 & -2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 4 \end{bmatrix}$

g $\begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} 5 & -1 & 4 & 2 \\ 3 & 1 & 0 & -6 \\ 4 & 3 & 1 & 3 \end{bmatrix}$ **h** $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 2 \\ 4 & -1 & -6 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 3 & 0 \\ 2 & -1 \end{bmatrix}$

3 Calculate $\begin{bmatrix} 3 & 6 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 10 & 5 \end{bmatrix}$.

What is unusual about these products?

4 Calculate $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 11 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 3 \\ 1 & 11 & 1 \\ 3 & 1 & 3 \end{bmatrix}$.

What is unusual about these products?

5 Work out $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$. Is this what you expected?

6 Explain why the product $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 0 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ does not exist.

Does the order make any difference?

7 Given $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -6 & 4 \\ 8 & -10 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 5 & -7 \\ -3 & 1 \end{bmatrix}$, find:

Example 5

a $\mathbf{A} + \mathbf{A} + \mathbf{A}$ **b** $3\mathbf{A}$ **c** $5\mathbf{B}$

d $4\mathbf{C}$ **e** $\frac{1}{2}\mathbf{C}$ **f** $\frac{1}{3}\mathbf{B}$

g $0.4\mathbf{B}$ **h** $5\mathbf{A} - \mathbf{B}$ **i** $3\mathbf{B} - 2\mathbf{A}$

j $\mathbf{C} + 2\mathbf{D} - \mathbf{A}$ **k** $3\mathbf{B} + 5\mathbf{A} - \mathbf{C}$ **l** $2\mathbf{C} - (-4\mathbf{A}) + 2\mathbf{B}$

ISBN 9780170412995

8 Given $\mathbf{X} = \begin{bmatrix} 3 & 0 & -1 & 2 \\ 5 & -4 & 3 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 1 & 4 & 3 & -5 \\ -3 & 2 & -1 & -6 \\ 0 & 5 & -2 & 3 \end{bmatrix}$, and $\mathbf{Z} = \begin{bmatrix} 6 & -5 & 4 & -3 \\ -1 & 8 & -9 & 2 \\ 4 & 7 & -2 & 5 \end{bmatrix}$, find:

a $5\mathbf{X}$ **b** $-3\mathbf{Z}$ **c** $4\mathbf{Y}$ **d** $-5\mathbf{X}$

e $\frac{1}{2}\mathbf{Z}$ **f** $0.1\mathbf{y}$ **g** $\frac{1}{3}\mathbf{Z}$ **h** $4\mathbf{X} - 3\mathbf{Z}$

i $4\mathbf{Z} - 5\mathbf{X}$ **j** $2\mathbf{X} + 5\mathbf{Z} - \mathbf{Y}$ **k** $2\mathbf{Y} - 3\mathbf{X} - 6\mathbf{Z}$ **l** $3\mathbf{Z} - 6\mathbf{Y} - 5\mathbf{X}$

Problem solving

9 Find pairs of non-zero 2×2 matrices whose products are zero matrices.

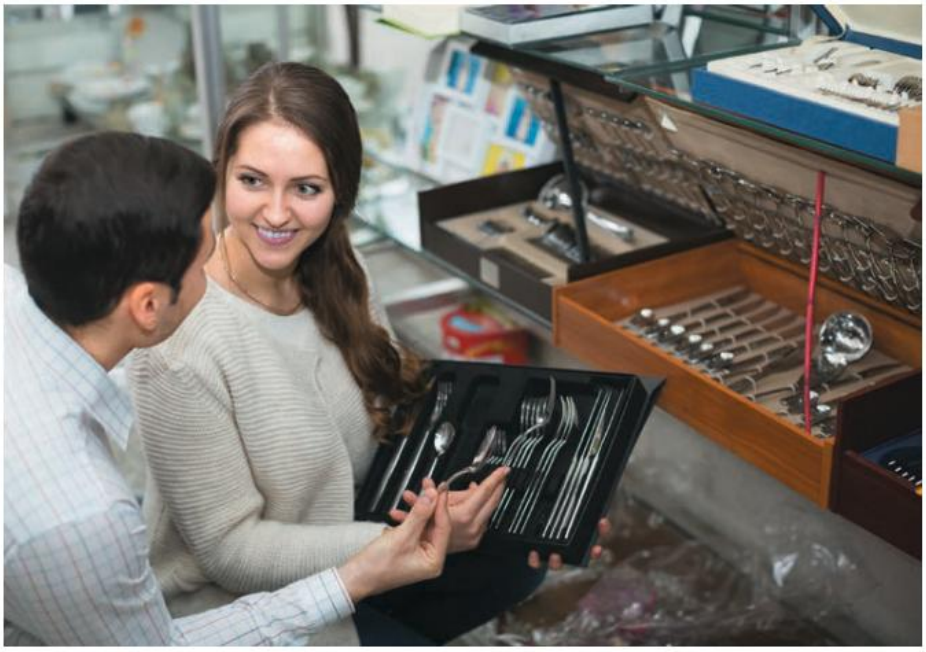
10 A kitchenware chain has 5 stores ordering stock from a central warehouse. The orders for cutlery sets for a catalogue special are shown in the table.

Setting	Brisbane	Logan	Chermside	Cairns	Gold Coast
6 place	12	15	10	12	16
8 place	10	8	10	12	6
10 place	8	4	10	4	12

a Write the orders as a 5×3 matrix.

b The prices of the sets are \$117.80, \$155 and \$190 respectively. Write the prices as a column matrix.

c Multiply the matrices to obtain the invoice amounts for each store.



Shutterstock.com/Iakov Filimonov

ISBN 9780170412995

11 An international airline sells holiday packages directly and through travel agents. These are priced according to the principal destination, number of nights accommodation and number of stopovers. Travel agents add their own commission onto the total package prices, while the airline's sales staff work on a commission of 35% if bought directly. The available packages are:

British Explorer—includes 2 stopovers and 14 days accommodation

European Wanderer—includes 4 stopovers and 21 days accommodation

East Coast US Explorer—includes 2 stopovers and 14 days accommodation

West Coast US Explorer—includes 3 stopovers and 14 days accommodation

East Coast US Wanderer—includes 5 stopovers and 28 days accommodation

West Coast US Wanderer—includes 4 stopovers and 21 days accommodation

The prices are shown in the table:

Principal destination	Accommodation	Stopovers
London $3000 (British/European)	$240/day	Singapore, New Delhi, Rome, Frankfurt, Athens, Paris $400 each
Los Angeles $2800 (West Coast US)	$220/day	Honolulu, San Francisco, Las Vegas, Seattle, Vancouver $440 each
New York $3200 (East Coast US)	$210/day	Honolulu, Los Angeles, Chicago, St Louis, Kansas City, Toronto, Quebec $500 each

a Calculate the cost of each tour and write it as a column matrix.

b Use a scalar multiple to find the commission for each tour charged by the airline's sales staff.

c Use matrix addition or another scalar multiple to find the cost of each tour if bought directly from the airline.

d 3 travel agents forward the following bookings to the airline.

Agent	BE	EW	ECE	WCE	ECW	WCW
A	2	1	0	3	2	1
B	4	2	2	4	1	2
C	3	1	5	3	3	2

Write the bookings as a 3×6 matrix.

e Use matrix multiplication to find the amounts that each agent will pay the airline.

12 The results of the first 5 rounds, middle 5 rounds and last 5 rounds of a netball competition are shown by the following 3 matrices. Each row represents the results of a team. The first column is the number of games won, the second is games drawn and the third is games lost.

$$\mathbf{F} = \begin{bmatrix} 4 & 0 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 4 \\ 3 & 0 & 2 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$$

Team scores are calculated by allotting 3 points for a win, 1 for a draw and 0 for a loss.

a Write the points as a column matrix **P**.

b Use matrix multiplication to find the team points for each group of 5 rounds.

Getty Images/Bradley Kanaris

13 The wholesale prices (in $) of some caulking compounds are shown by this matrix.

$$\mathbf{W} = \begin{bmatrix} 2.40 & 4.90 & 8.40 \\ 4.20 & 7.90 & 11.80 \\ 3.80 & 8.40 & 11.00 \\ 5.90 & 11.80 & 14.40 \end{bmatrix}$$

The first column represents the prices of 100 mL tubes, the second the prices of 250 mL tubes and the third the prices of 400 mL mastic gun cartridges. The first row is for *bathroom sealant*, the second for *gutter and downpipe sealant*, the third for *masonry sealant* and the fourth for *flexible gap filler*.

a The retailer applies a markup of 70%. Calculate the matrix for the retail prices.

b The GST is 10%. Find the matrix for the prices actually paid by the customer.

ISBN 9780170412995

7.03 Matrix identities and inverses

INVESTIGATION

SPECIAL MATRICES

Inverse and determinant of a matrix

1 Find the following products:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & -3 & 0 \\ 3 & 4 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -4 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2 What can you conclude about the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

3 Find the following products:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \end{bmatrix}, \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ and } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4 What can you conclude about the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

5 Try multiplying by some other matrices with 0s everywhere except the diagonal from the top left to the bottom right, which has 1s.

6 Write a general conclusion.

ISBN 9780170412995

Identities

- The **Kronecker delta function** is $\delta_{ij} = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases}$
- The **multiplicative identity** for $n \times n$ matrices is $\mathbf{I}_n = (\delta_{ij})$, where $i, j \in [1, n]$

From the investigation you know that $\mathbf{I}_n$ is the identity matrix. $\mathbf{I}_n$ is the left identity for $n \times m$ matrices and the right identity for $m \times n$ matrices.

EXAMPLE 6

Find $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 5 & 0.6 \end{bmatrix}$.

Solution

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity for 2×2 matrices. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 5 & 0.6 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 5 & 0.6 \end{bmatrix}$

The product of an identity matrix with any other conformable matrix is that matrix.

INVESTIGATION

INVERSE MATRICES

1 Find the following products:

$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 5 & -11 & 2 \\ -1 & 3 & -1 \\ -1 & 2 & 0 \end{bmatrix},$

$\begin{bmatrix} 4 & 5 & 2 & 1 \\ -1 & 2 & 4 & -1 \\ 2 & 2 & 0 & 1 \\ 3 & 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -3 & -3 & -1 \\ -5 & 4 & 3 & 6 \\ 4 & -3 & -2 & -5 \\ 2 & -2 & 1 & -4 \end{bmatrix}$

2 How are these like the products 0.4×2.5 and 0.625×1.6?

3 Find the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

ISBN 9780170412995

Inverse matrices

- If it exists, the **inverse** of a square matrix $\mathbf{A}$ is the matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$
- A matrix with an inverse is called **non-singular**
- A matrix with no inverse is called **singular**
- The 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular if and only if $ad - bc \neq 0$
- The **determinant** of the 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A = \det(\mathbf{A}) = |\mathbf{A}| = ad - bc$.
  It is also written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.
- The inverse of the 2×2 matrix $\mathbf{A}$ is given by $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

EXAMPLE 7

Find the inverse for each matrix that is non-singular.

a $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 8 & 3 \end{bmatrix}$ **b** $\mathbf{B} = \begin{bmatrix} 6 & 2 \\ -9 & -3 \end{bmatrix}$ **c** $\mathbf{C} = \begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}$

Solution

a Find the determinant.

$$|\mathbf{A}| = ad - bc$$
$$= 18 - 16 = 2$$

Find the inverse.

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 3 & -2 \\ -8 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 1.5 & -1 \\ -4 & 3 \end{bmatrix}$$

b Find the determinant.

$$|\mathbf{B}| = ad - bc$$
$$= -18 - (-18) = 0$$

State the result.

$|\mathbf{B}| = 0$, so the matrix is singular.

ISBN 9780170412995

c Find the determinant.

$$|\mathbf{C}| = ad - bc$$
$$= 5 - 0 = 5$$

Find the inverse.

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.2 & -0.8 \\ 0 & 1 \end{bmatrix}$$

The determinants and inverse of 3×3 and higher order matrices are much more difficult to find than for 2×2 matrices. The determinant of a 10×10 matrix needs about 3 million calculations and a 20×20 matrix about 2×10^{18} calculations. Even if a desktop computer had enough memory, it would take nearly 20 years to calculate the determinant of a 20×20 matrix exactly. Applications use much larger matrices, so they use approximations.

EXAMPLE 8

Use a graphics calculator to find the inverse of any matrix that is non-singular.

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 3 & 1 & -2 \\ 3 & -5 & -3 & 4 \\ 3 & 10 & 1 & -7 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 0 \\ 3 & 4 & 3 \end{bmatrix}, \mathbf{Z} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

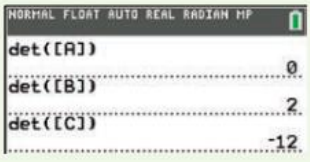
Solution

TI-Nspire CX Chapter 7

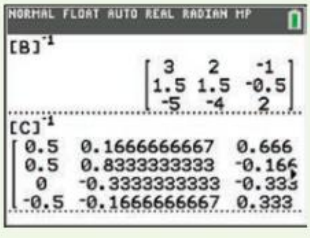
TI 84 Plus CE

Enter the matrices as shown on page 263. Remember to use (-) for negatives. You need to use, say, **A**, **B** and **C** instead of **X**, **Y** and **Z**.

Find the determinants using 2nd x^{-1} ▶ 1 (MATH det) and then use 2nd x^{-1} for each matrix.



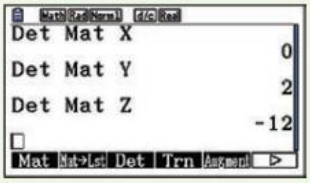
Find the inverses of **Y** and **Z** using the x^{-1} key. Use the cursor keys to look at the hidden parts of the answer(s).



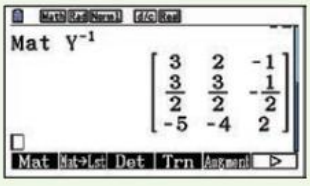
ISBN 9780170412995

Casio fx-CG20AU

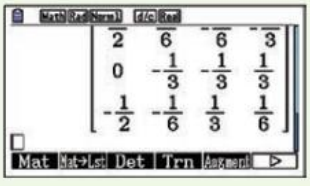
Use Run-Matrix and enter the matrices as shown on page 264. Use (−) for negatives.



Find the determinants using OPTN F3 F2 F3 ALPHA.



Find the inverses of **Y** and **Z** using SHIFT) to get x^{-1}. Use the cursor keys to look at the hidden parts of the answer(s).



Both calculators

State the results.

X is singular. The inverses of **Y** and **Z** are

$$\mathbf{Y}^{-1} = \begin{bmatrix} 3 & 2 & -1 \\ 1.5 & 1.5 & -0.5 \\ -5 & -4 & 2 \end{bmatrix} \text{ and}$$

$$\mathbf{Z}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ \frac{1}{2} & \frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

ISBN 9780170412995

Exercise 7.03 Matrix identities and inverses

Example 6

1 Find each product.

a $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -4 & 0 & 6 \\ 2 & 4 & -3 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 1 & -3 \\ -5 & 0 & 2 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 6 & -3 \\ 0 & 7 \end{bmatrix}$ **f** $\begin{bmatrix} 3 & -3 & -6 \\ 0 & 4 & 6 \\ -2 & 1 & 6 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -30 & 5 & 20 \\ 5 & 15 & 2 & 0 \\ 7 & 2 & -11 & 0 \\ 0 & -6 & 6 & 5 \end{bmatrix}$ **h** $\begin{bmatrix} 4 & 4 & 2 & 6 \\ -4 & -3 & 0 & 0 \\ 6 & 0 & 7 & 5 \\ -5 & 7 & 8 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Example 7

2 Find the determinant of each matrix and state whether it is singular or non-singular.

a $\begin{bmatrix} 2 & 4 \\ -3 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & -3 \\ 1 & -2 \end{bmatrix}$ **c** $\begin{bmatrix} 4 & -2 \\ 3 & 3 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$

e $\begin{bmatrix} -2 & -3 \\ 1 & 3 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ **g** $\begin{bmatrix} -3 & -2 \\ -4 & 4 \end{bmatrix}$ **h** $\begin{bmatrix} -4 & 3 \\ 5 & -2 \end{bmatrix}$

3 Find the inverse of any matrix that is non-singular.

a $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} -4 & 2 \\ 7 & 5 \end{bmatrix}$

e $\begin{bmatrix} -\frac{1}{4} & 0 \\ \frac{1}{16} & -\frac{1}{8} \end{bmatrix}$ **f** $\begin{bmatrix} 2 & -2 \\ 8 & -6 \end{bmatrix}$ **g** $\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$ **h** $\begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$

ISBN 9780170412995

4 Use a graphics calculator to find the determinant of each matrix and state whether it is singular or non-singular.

Example 8

a $\begin{bmatrix} -3 & -3 & -5 \\ -6 & -6 & -8 \\ 3 & 4 & 8 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 5 & 6 \\ 2 & 7 & 3 \\ -2 & 4 & -6 \end{bmatrix}$ **c** $\begin{bmatrix} -4 & 1 & -3 & 3 \\ -2 & -1 & 2 & -3 \\ -4 & -2 & 1 & -4 \\ -2 & -1 & 0 & -1 \end{bmatrix}$

d $\begin{bmatrix} 1 & -3 & -1 & 0 \\ 3 & -3 & 2 & 0 \\ -1 & 0 & 1 & -3 \\ -2 & 1 & -1 & -2 \end{bmatrix}$ **e** $\begin{bmatrix} 2 & 2 & 1 & 2 & 2 \\ 1 & 3 & 3 & 0 & 3 \\ -2 & 3 & 0 & 0 & 4 \\ -2 & 2 & -1 & 3 & 3 \\ 1 & 1 & -2 & 3 & 1 \end{bmatrix}$ **f** $\begin{bmatrix} 0 & -2 & 1 & -3 \\ 0 & 1 & -2 & 2 \\ 2 & 0 & -2 & 2 \\ 0 & -1 & -1 & -1 \end{bmatrix}$

g $\begin{bmatrix} 5 & -2 & 2 \\ -5 & 3 & -2 \\ -3 & 2 & 1 \end{bmatrix}$ **h** $\begin{bmatrix} -3 & 2 & -1 \\ 2 & 1 & -3 \\ -1 & 2 & -3 \end{bmatrix}$

5 Use a graphics calculator to find the inverse of each matrix that is non-singular.

a $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \\ 3 & 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & -1 & 3 \\ 0 & -2 & 3 \\ 3 & 0 & 4 \end{bmatrix}$

d $\begin{bmatrix} 4 & -8 & -14 & 3 \\ 2 & -5 & -9 & 2 \\ 2 & -1 & -2 & 1 \\ 3 & -2 & -4 & 2 \end{bmatrix}$ **e** $\begin{bmatrix} 37 & -4 & 36 & -26 \\ -8 & 2 & -8 & 5 \\ 2 & -2 & 2 & -1 \\ 8 & -4 & 8 & -5 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & -1 & -2 & 3 & 3 \\ 2 & 0 & 3 & 1 & 0 \\ -2 & 2 & -1 & -1 & 0 \\ 1 & 5 & 5 & 0 & 1 \\ 4 & 0 & 3 & 1 & 0 \end{bmatrix}$

g $\begin{bmatrix} -1 & 1 & -1 \\ 5 & -5 & 2 \\ 0 & 3 & 5 \end{bmatrix}$ **h** $\begin{bmatrix} 8 & -24 & 2 & -1 \\ -14 & 42 & -2 & 3 \\ 12 & -36 & 1 & -3 \\ -12 & 39 & -2 & 4 \end{bmatrix}$

Problem solving

6 Show that the identity for 2×2 matrices is unique. That is, there is no other matrix that is an identity. Hint: Assume there is another and use contradiction.

7 Show that if it exists, the inverse of a 2×2 matrix is unique. Hint: Assume there is another and use contradiction.

8 Find the square of $\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 \\ -4 & -3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. What does this mean about $\mathbf{A}$? Find another matrix (not $\mathbf{I}$) with the same property.

ISBN 9780170412995

7.04 Matrix equations

You usually solve $3x = 12$ by dividing by 3 to give $x = 4$. What if you couldn't divide?

You could multiply by $\frac{1}{3}$ to get $3^{-1} \times 3x = \frac{1}{3} \times 12$ and get the same answer.

What about $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \end{bmatrix}$? It is like $3x = 12$, but you cannot divide matrices. You can multiply by $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ instead to get

$$\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 12 \\ -4 \end{bmatrix}.$$

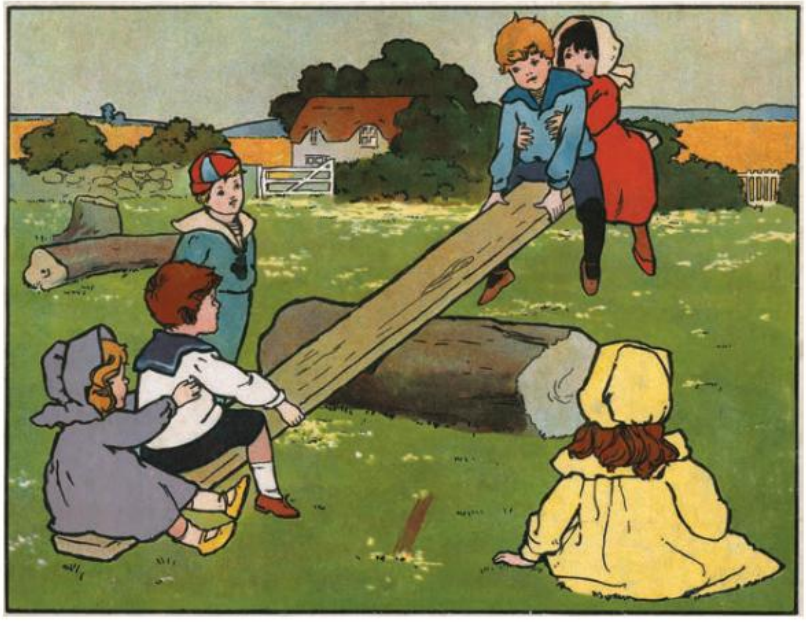
This simplifies to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 68 \\ -96 \end{bmatrix}$, which gives $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 68 \\ -96 \end{bmatrix}$.

You can write the matrix equation as $\mathbf{AX} = \mathbf{B}$ instead of using the actual matrices.

Matrix equations

- The matrix equation $\mathbf{AX} = \mathbf{B}$ has a solution if and only if $\mathbf{A}$ is non-singular
- The solution is given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

You would have to do $\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$ for $\mathbf{XA} = \mathbf{B}$ because matrix multiplication is not commutative. This makes balancing matrix equations more difficult than ordinary equations.



Alamy Stock Photo/Old Paper Studios

ISBN 9780170412995

EXAMPLE 9

If possible, solve the following for $\mathbf{X}$.

a $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & -3 \\ -8 & 6 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ **c** $\mathbf{X}\begin{bmatrix} 2 & 3 \\ -5 & -6 \end{bmatrix} = [6 \ -9]$

Solution

a Find the determinant of $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

$\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} = 18 - 20 = -2$

Find the inverse of $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2\frac{1}{2} \\ 2 & -1\frac{1}{2} \end{bmatrix}$$

Solve the equation.

$$\begin{bmatrix} -3 & 2\frac{1}{2} \\ 2 & -1\frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -3 & 2\frac{1}{2} \\ 2 & -1\frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 19 \\ -12 \end{bmatrix}$$

b Find the determinant of $\begin{bmatrix} 4 & -3 \\ -8 & 6 \end{bmatrix}$.

$\begin{vmatrix} 4 & -3 \\ -8 & 6 \end{vmatrix} = 24 - 24 = 0$

State conclusion.

There is no solution because $\begin{bmatrix} 4 & -3 \\ -8 & 6 \end{bmatrix}$ is singular.

c Find the determinant of $\begin{bmatrix} 2 & 3 \\ -5 & -6 \end{bmatrix}$.

$\begin{vmatrix} 2 & 3 \\ -5 & -6 \end{vmatrix} = -12 + 15 = 3$

Find the inverse of $\begin{bmatrix} 2 & 3 \\ -5 & -6 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 3 \\ -5 & -6 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} -6 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Solve the equation.

$$\mathbf{X}\begin{bmatrix} 2 & 3 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 1\frac{2}{3} & \frac{2}{3} \end{bmatrix} = [6 \ -9]\begin{bmatrix} -2 & -1 \\ 1\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$\mathbf{X} = [-27 \quad -12]$$

You should use your graphics calculator for matrix equations of higher order.

ISBN 9780170412995

EXAMPLE 10

Use your graphics calculator to solve for $\mathbf{X}$.

a $\begin{bmatrix} 3 & -2 & 2 \\ 0 & -1 & -2 \\ 5 & -1 & 4 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 6 \\ -4 \\ 9 \end{bmatrix}$

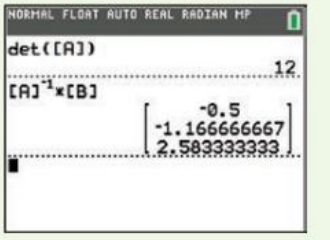
b $\begin{bmatrix} 2 & 0 & 4 & -2 \\ -2 & 3 & -2 & 2 \\ 4 & 0 & 1 & 3 \\ 3 & -2 & -2 & 2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 21 \\ -14 \\ 7 \\ 28 \end{bmatrix}$

TI-Nspire CX
Chapter 7

Solution

a TI 84 Plus CE

Enter the matrices $\mathbf{A} = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -1 & -2 \\ 5 & -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 6 \\ -4 \\ 9 \end{bmatrix}$.



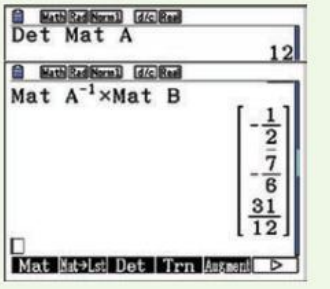
Remember to use (-) for negatives.

Check that $\mathbf{A}$ is non-singular and find $\mathbf{X}$ using $\mathbf{A}^{-1}\mathbf{B}$.

Casio fx-CG20AU

Use Run-Matrix.

Enter the matrices $\mathbf{A} = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -1 & -2 \\ 5 & -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 6 \\ -4 \\ 9 \end{bmatrix}$.



Remember to use (−) for negatives.

Check that $\mathbf{A}$ is non-singular and find $\mathbf{X}$ using $\mathbf{A}^{-1}\mathbf{B}$.

Both calculators

Write the answer.

$$\mathbf{X} = \begin{bmatrix} -\frac{1}{2} \\ -1\frac{1}{6} \\ 2\frac{7}{12} \end{bmatrix}$$

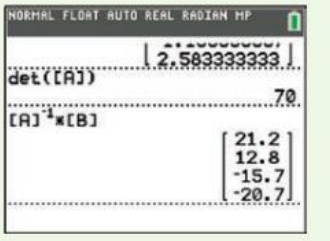
ISBN 9780170412995

b TI 84 Plus CE

Enter the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 4 & -2 \\ -2 & 3 & -2 & 2 \\ 4 & 0 & 1 & 3 \\ 3 & -2 & -2 & 2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 21 \\ -14 \\ 7 \\ 28 \end{bmatrix}.$$

Remember to use (-) for negatives.



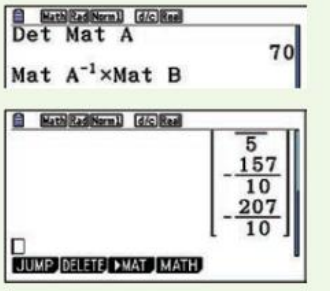
Check that **A** is non-singular and find **X** using $\mathbf{A}^{-1}\mathbf{B}$.

Casio fx-CG20AU

Enter the matrices $\mathbf{A} =$

$$\begin{bmatrix} 2 & 0 & 4 & -2 \\ -2 & 3 & -2 & 2 \\ 4 & 0 & 1 & 3 \\ 3 & -2 & -2 & 2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 21 \\ -14 \\ 7 \\ 28 \end{bmatrix}.$$

Remember to use (-) for negatives.



Check that **A** is non-singular and find **X** using $\mathbf{A}^{-1}\mathbf{B}$. Use the cursor keys to see the hidden parts of the answer.

Both calculators

Write the answer.

$$\mathbf{X} = \begin{bmatrix} 21.2 \\ 12.8 \\ -15.7 \\ -20.7 \end{bmatrix}$$

ISBN 9780170412995

Exercise 7.04 Matrix equations

Example 9

1 If possible, solve the following for $\mathbf{X}$.

a $\begin{bmatrix} 6 & 5 \\ -5 & -4 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 2 \\ -9 & -6 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

d $\mathbf{X}\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} = [10 \quad 25]$ **e** $\begin{bmatrix} -3 & 4 \\ 6 & -9 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 12 \\ -21 \end{bmatrix}$

Example 10

2 Where possible, use your graphics calculator to solve for $\mathbf{X}$.

a $\mathbf{x}\begin{bmatrix} 1 & 3 & 3 \\ -2 & 1 & -2 \\ -1 & 1 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -4 \\ 8 \\ -2 \end{bmatrix}$

b $\begin{bmatrix} 5 & -1 & 4 \\ 2 & 0 & 1 \\ 5 & 5 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 4 \\ -16 \\ 8 \end{bmatrix}$

c $\begin{bmatrix} 6 & 5 & -2 \\ 2 & 4 & 0 \\ -2 & 1 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}$

d $\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 3 & 5 & -2 \\ 2 & 3 & 2 & 6 \\ 3 & 5 & 0 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 \\ -3 \\ 2 \\ -1 \end{bmatrix}$

e $\begin{bmatrix} -3 & 1 & -3 & 3 \\ -2 & 2 & 0 & 2 \\ 2 & 1 & 0 & 2 \\ 3 & 3 & -2 & 6 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 \\ 12 \\ -15 \\ 3 \end{bmatrix}$

Problem solving

3 Solve $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 4 & 3 \\ -1 & -4 \end{bmatrix}$.

4 Solve $\mathbf{X}\begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 3 & -2 \end{bmatrix}$.

5 Solve $\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 0 & 9 \\ 4 & 1 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

6 Solve

a $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 4 & 3 \\ -1 & -4 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

b $\begin{bmatrix} 5 & 6 \\ 3 & 3 \end{bmatrix}\mathbf{X} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \mathbf{X} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

ISBN 9780170412995

7. CHAPTER SUMMARY

Matrices

Matrix definition

- A **matrix** (plural **matrices**) is a rectangle of numbers in large parentheses () or brackets [ ]
- The symbol is a bold capital letter like **A**, or handwritten with a wavy underline like $\underset{\sim}{\mathrm{A}}$.

$\mathbf{M} = \begin{bmatrix} 3 & 2 & 0.6 & \sqrt{2} \\ -4 & 5 & \frac{3}{4} & -6 \end{bmatrix}$ would be shown as $\underset{\sim}{\mathrm{M}}$

- Each number is called an **element** of the matrix.
- The **size** (**dimension** or **order**) is the number of rows × number of columns, so **M** is 2×4
- A **row matrix** has only one row
- A **column matrix** has only one column
- A **square matrix** has the same number of rows and columns
- Every entry of a **zero matrix** is 0
- The elements of a matrix are shown by the lower case letter with the row and column as subscripts: $m_{23} = \frac{3}{4}$

Addition and subtraction

- Addition of matrices is only possible for matrices of the same size
- Matrices are added by adding the corresponding elements;

$$\text{for } \mathbf{A} = (a_{ij}) \text{ and } \mathbf{B} = (b_{ij}),\ \mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$$

- Equal matrices are the same size and have equal elements;

$\mathbf{A} = \mathbf{B} \Leftrightarrow a_{ij} = b_{ij}\ \forall\ i, j$

- Subtraction of matrices is only possible for matrices of the same size
- Matrices are subtracted by subtracting the corresponding elements;

$$\text{for } \mathbf{A} = (a_{ij}) \text{ and } \mathbf{B} = (b_{ij}),\ \mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$$

ISBN 9780170412995

Matrix multiplication

- Matrices **A** and **B** are **conformable** if **A** has m columns and **B** has m rows
- The **product** of an $n \times m$ matrix **A** and an $m \times p$ matrix **B** is an $n \times p$ matrix **C**
- The elements c_{ij} of **C** are the sums of the products of the ith row of **A** and the jth column of **B**: $c_{ij} = \sum a_{ik} \times b_{kj}$
- Matrix multiplication is **not commutative**

Multiplication by a scalar (scalar multiples)

- The product of a real number c and a matrix **A** is $c\mathbf{A} = (ca_{ij})$. Every element of the matrix is multiplied by the scalar c.
- $c\mathbf{A}$ is called a scalar multiple of **A**.

Identities and inverses

- The **Kronecker delta function** is $\delta_{ij} = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases}$
- The **multiplicative identity** for $n \times n$ matrices is $\mathbf{I}_n = (\delta_{ij})$, where $i, j \in [1, n]$
- If it exists, the **inverse** of a square matrix **A** is the matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$
- A matrix with an inverse is called **non-singular**
- A matrix with no inverse is called **singular**
- The 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular if and only if $ad - bc \neq 0$
- The **determinant** of the 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A = \det(\mathbf{A}) = |\mathbf{A}| = ad - bc$. It is also written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- The inverse of the 2×2 matrix **A** is given by $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Matrix equations

- The matrix equation $\mathbf{AX} = \mathbf{B}$ has a solution if and only if **A** is non-singular
- The solution is given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$
- Matrix equations can be used to solve simultaneous linear equations

ISBN 9780170412995

7. CHAPTER REVIEW

Matrices

1 $\mathbf{M} = \begin{bmatrix} 1 & 5 & 3 & 2 & -6 \\ -7 & 4 & -3 & 0 & -5 \\ 6 & -2 & -4 & 7 & -1 \end{bmatrix}$. Example 1

a What is the size of $\mathbf{M}$?

b If possible, write the value of m_{23}, m_{34}, m_{42}.

2 For $\mathbf{P} = \begin{bmatrix} -3 & 3 & 2 \\ 4 & 5 & 0 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -2 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 5 & -3 & -1 \\ 3 & 0 & -4 \end{bmatrix}$, find: Example 2

a $\mathbf{P} + \mathbf{R}$ **b** $\mathbf{Q} - \mathbf{R}$

3 If possible, find each product. Example 3

a $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 0 \\ -4 & 5 & 1 \end{bmatrix}$

c $\begin{bmatrix} 2 & 3 & 0 \\ -4 & 5 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix}$

4 For $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 0 & 6 \\ 4 & 3 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & -4 & 2 \\ 7 & 0 & 1 \\ -6 & 0 & -1 \end{bmatrix}$, use a graphics calculator to find: Example 4

a $\mathbf{AB}$ **b** $\mathbf{BA}$

5 $\mathbf{D} = \begin{bmatrix} 4 & 2 \\ -3 & 5 \\ 0 & -2 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} -3 & 1 \\ 4 & -1 \\ -2 & 4 \end{bmatrix}$. Find each expression. Example 5

a $4\mathbf{D}$ **b** $-\mathbf{E}$ **c** $3\mathbf{E} - 2\mathbf{D}$

6 Find $\begin{bmatrix} 3 & 4 & -4 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Example 6

7 Find the determinant of $\begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix}$. Example 7

8 If possible, find the inverse of each matrix. Example 7

a $\begin{bmatrix} -2 & 5 \\ -3 & 7 \end{bmatrix}$ **b** $\begin{bmatrix} 6 & 9 \\ -2 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & -5 \\ -3 & 10 \end{bmatrix}$

ISBN 9780170412995

Example 8

9 If possible, use a graphics calculator to find the inverse of each matrix.

a $\begin{bmatrix} 3 & 5 & 5 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & -2 & -2 & 2 & -1 \\ -1 & 4 & -2 & 2 & 2 \\ -2 & 0 & 3 & -1 & -1 \\ 0 & -2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 5 & -2 \end{bmatrix}$

Example 9

10 If possible, solve for $\mathbf{X}$.

a $\begin{bmatrix} 6 & 2 \\ 12 & 4 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ **b** $\begin{bmatrix} 6 & 2 \\ 8 & 3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ **c** $\mathbf{X}\begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix} = [5 \quad -3]$

Example 10

11 If possible, use a graphics calculator to solve:

$$\begin{bmatrix} 2 & 4 & -1 & 0 \\ 3 & 0 & 4 & -1 \\ 5 & 4 & 4 & 0 \\ 2 & 5 & -2 & -2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 \\ 10 \\ -15 \\ 20 \end{bmatrix}$$

Problem solving

12 The main inhabitants of interest in the top storey of a rainforest are caterpillars and adult butterflies and moths, sap-sucking bugs, insect-eating beetles, wasps and their larvae, spiders and insectivorous birds. The following table shows the average daily diets of the non-herbivorous animals.

Animal	Daily diet
Beetle	1 caterpillar/butterfly/moth and 5 bugs
Wasp	1 caterpillar/butterfly/moth, 1 beetle and 3 bugs
Spider	1 caterpillar/butterfly/moth, 2 beetles and 1 wasp
Bird	3 caterpillars/butterflies/moths, 5 beetles, 1 wasp and 3 spiders

a Place the information in a 6×6 matrix $\mathbf{A}$ with each row showing the diet of each animal. Use the order caterpillar/butterfly/moth, bug, beetle, wasp, spider and bird.

b In one treetop area it is estimated that there are 25 000 caterpillars/butterflies/moths, 70 000 bugs, 4000 beetles, 500 wasps, 100 spiders and 6 birds. Write these numbers as a row matrix.

c In another treetop area, there are three times as many animals. Use scalar multiplication to find the numbers in the second area.

d Use matrix multiplication to find the number of each animal eaten in one day in each area.

ISBN 9780170412995

13 The costs of trucking small parcels between Brisbane, Mt Isa, Rockhampton, Sydney and Toowoomba are shown by the following matrix, where the rows give the costs from each place in alphabetical order:

$$\begin{bmatrix} 0 & 10.80 & 9.80 & 6.40 & 5.00 \\ 10.80 & 0 & 7.20 & 16.40 & 15.40 \\ 9.80 & 7.20 & 0 & 16.20 & 14.40 \\ 6.40 & 16.40 & 16.20 & 0 & 11.00 \\ 5.00 & 15.40 & 14.40 & 11.00 & 0 \end{bmatrix}$$

a Why does the diagonal have zeros?

b The handling costs for parcel delivery are a flat $4. Write a matrix **H** to give the handling costs and calculate matrix **T** to give the total delivery costs for small parcels.

c Medium-sized parcels cost $1\frac{1}{2}$ times as much for trucking but the handling costs are the same. Write a matrix expression for the delivery costs of medium-sized parcels.

d Large parcels cost twice as much for handling and three times as much for trucking as small parcels. Write a matrix expression for the delivery costs of large parcels.

14 A smallgoods manufacturer produces three types of salami—Italian, Hungarian and Austrian—in mild and hot versions. The prices of each type are shown in the table below.

Type of salami	Italian	Hungarian	Austrian
Mild	$12.50	$15.00	$17.50
Hot	$13.20	$16.40	$18.00

The orders from three delicatessens are as follows:

George's: 3 MI, 1 HI, 2 MH, 3 HH, 1 MA, 0 HA

Frank's: 2 MI, 2 HI, 3 MH, 1 HH, 3 MA, 2 MA

Maria's: 4 MI, 5 HI, 0 MH, 1 HH, 0 MA, 2 HA

a Write the prices as a 1×6 matrix.

b Write the orders as a 6×3 matrix.

c Use matrix multiplication to find the price of each deli's order.

15 Solve the equation $\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}\mathbf{X} - \mathbf{X} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

Practice quiz

ISBN 9780170412995

8.

COMPLEX NUMBERS

What happens when you try to solve the equation $x^2 - 2x + 5 = 0$?

You get $\sqrt{-16}$ in the answer.

There is no real number whose square is -16, so $\sqrt{-16}$ is called an **imaginary number**.

You might not be able to picture $\sqrt{-16}$, but imaginary numbers are incredibly useful in electronics, engineering, science and higher mathematics.

8.01 Complex numbers
8.02 Addition and subtraction of complex numbers
8.03 Multiplication of complex numbers
8.04 Complex conjugates and division of complex numbers
8.05 The Argand plane
8.06 Complex numbers as vectors
8.07 Operations in the complex plane
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Complex numbers

- define the imaginary number i as a root of the equation $x^2 = -1$
- use complex numbers in the form $a + bi$ where a and b are the real and imaginary parts
- determine and use complex conjugates
- perform complex-number arithmetic: addition, subtraction, multiplication and division

The complex plane (the Argand plane)

- consider complex numbers as points in a plane with real and imaginary parts as Cartesian coordinates
- examine and use addition of complex numbers as vector addition in the complex plane
- understand and use location of complex conjugates in the complex plane
- examine and use multiplication as a linear transformation in the complex plane

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

iStock/Mardolff

TERMINOLOGY

Argand diagram, Argand plane
argument
Cartesian form
complex number
complex plane
conjugate
imaginary number
imaginary part
linear transformation
modulus
real part
rectangular form
transformation

Complex numbers

8.01 Complex numbers

$\sqrt{-4}, \sqrt{-20}$ and $\sqrt{-150}$ are all **imaginary numbers**. Obviously, there are an unlimited number of imaginary numbers. It is simpler to define one simple imaginary number called i.

Imaginary numbers

- The **imaginary number** i is a solution of the equation $x^2 + 1 = 0$; $i = \sqrt{-1}$
- An **imaginary number** (or **pure imaginary number**) is a number of the form bi, where $b \in \boldsymbol{R}$
- A **complex number** is a number of the form $z = a + bi$, where $a, b \in \boldsymbol{R}$
- The set of all complex numbers is shown as $\boldsymbol{C}$

EXAMPLE 1

Solve each equation.

a $x^2 + 9 = 0$ **b** $x^2 - 2x + 5 = 0$

Solution

a

Write in the form $x^2 = \ldots$

$x^2 = -9$

Write both possible square roots.

$x = \pm\sqrt{-9}$

Simplify the square root.

$= \pm\sqrt{9 \times (-1)}$

$= \pm\sqrt{9} \times \sqrt{-1}$

$= \pm 3i$

Write the answer.

The solutions are $3i$ or $-3i$.

ISBN 9780170412995

b The equation will not factorise, so use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute values from the equation.

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 5 \times 1}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

Simplify.

$$= \frac{2 \pm \sqrt{16 \times (-1)}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= \frac{2(1+2i)}{2} \text{ or } \frac{2(1-2i)}{2}$$

$$= 1 + 2i \text{ or } 1 - 2i$$

Write the answer. The solutions are $1 + 2i$ or $1 - 2i$.

If the discriminant, $\Delta = b^2 - 4ac$, of a quadratic equation is negative, the solutions are complex.

There are other ways to show complex numbers.

Parts of complex numbers

- The format $a + bi$, where $a, b \in \boldsymbol{R}$ is called the **Cartesian form** or **rectangular form** of a complex number
- $\text{Re}(z) = a$ is the **real part** of the complex number $a + bi$
- $\text{Im}(z) = b$ is the **imaginary part** of the complex number $a + bi$

EXAMPLE 2

For the complex number $z = 5 - 3i$, write $\text{Re}(z)$ and $\text{Im}(z)$.

Solution

Write each part separately.

$\text{Re}(z) = 5$

$\text{Im}(z) = -3$

ISBN 9780170412995

You can simplify expressions involving i using $i^2 = -1$.

EXAMPLE 3

Simplify each expression.

a $4i + 7i$ **b** $2i \times -5i$ **c** $\dfrac{6ai - 4a^2i}{10ai}$

d i^6 **e** i^{31} **f** $\sqrt{-4} \times \sqrt{-9}$

Solution

a Collect like terms. $4i + 7i = 11i$

b Isolate i. $2i \times (-5i) = 2 \times (-5) \times i^2$

Substitute $i^2 = -1$ and simplify. $= -10 \times (-1)$

$= 10$

c Take out the common factor. $\dfrac{6ai - 4a^2i}{10ai} = \dfrac{2ai(3 - 2a)}{10ai}$

Cancel. $= \dfrac{3 - 2a}{5}$

d Write in terms of i^2. $i^6 = (i^2)^3$

Use $i^2 = -1$. $= (-1)^3$

Simplify. $= -1$

e Write in terms of i^2. $i^{31} = (i^2)^{15} \times i$

Use $i^2 = -1$. $= (-1)^{15} \times i$

Simplify. $= -i$

f Write each root in terms of i. $\sqrt{-4} \times \sqrt{-9} = 2i \times 3i$

Simplify. $= 6i^2$

Use $i^2 = -1$. $= -6$

Notice that powers of i are cyclical. $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, ...

In part **f** above, $\sqrt{-4}$ means the positive square root of -1, which is $\sqrt{4 \times (-1)} = 2i$.

$\sqrt{-4} \times \sqrt{-9} = \sqrt{-4 \times (-9)} = \sqrt{36} = 6$ is *wrong* because it hides the squaring of both sides of $x = 2i \times 3i$. It is like doing $(-6)^2 = 36$ and $6^2 = 36$, so $-6 = 6$.

ISBN 9780170412995

Exercise 8.01 Complex numbers

1 Solve each equation. Example 1

a $x^2 + 4 = 0$ **b** $3x^2 + 27 = 0$ **c** $x^2 + 12 = 0$ **d** $2x^2 + 14 = 0$

e $x^2 + 8 = 0$ **f** $45x^2 + 5 = 0$ **g** $72x^2 + 24 = 0$ **h** $20x^2 + 90 = 0$

2 Solve each equation.

a $x^2 - 6x + 25 = 0$ **b** $x^2 + 4x + 7 = 0$ **c** $3x^2 - 4x + 8 = 0$ **d** $g^2 + 5g + 7 = 0$

e $m^2 + 7 = m$ **f** $p^2 - 3p + 4 = 0$ **g** $3x^2 + 2x + 9 = 0$ **h** $5x - 4x^2 = 8$

3 Write Re (z) and Im (z) for each complex number. Example 2

a $3 - 2i$ **b** $7 + i$ **c** $-5 + 6i$ **d** $-4 - 7i$

e $\sqrt{5} - \pi i$ **f** $3i - 2\sqrt{6}$ **g** $4\pi + \sqrt{5}i$ **h** $2i - \pi$

4 Simplify each expression. Example 3

a $2i + 7i$ **b** $12i - 5i$ **c** $4i \times i$ **d** $2\sqrt{3}i \times 4i$

e $\dfrac{\sqrt{12}i}{10i}$ **f** $\sqrt{15}i \times \sqrt{20}i$ **g** $\dfrac{4a^3i - 12abi}{10ab^2i}$ **h** $\dfrac{\sqrt{8}bi - \sqrt{24}bci}{2bi}$

5 Simplify each expression.

a $\sqrt{-25}$ **b** $\sqrt{-5}$ **c** $\sqrt{-8}$ **d** $\sqrt{-40}$

e $\sqrt{-\dfrac{16}{25}}$ **f** $\sqrt{-\dfrac{11}{49}}$ **g** $\sqrt{-\dfrac{9}{19}}$ **h** $\sqrt{-\dfrac{27}{8}}$

6 Simplify each expression.

a i^5 **b** i^{10} **c** i^{24} **d** i^{97}

e i^{-2} **f** i^{-9} **g** i^{-10} **h** i^{-27}

7 Simplify each expression.

a $\sqrt{-16} \times \sqrt{-25}$ **b** $\sqrt{-6} \times \sqrt{-12}$

c $\sqrt{-27} \times \sqrt{-125}$ **d** $\sqrt{-8} \times \sqrt{-32}$

e $\sqrt{-3} \times \sqrt{-6} \times \sqrt{-8}$ **f** $\sqrt{-9} \times \sqrt{-4} \times \sqrt{-18}$

g $\sqrt{-10} \times \sqrt{-6} \times \sqrt{-5}$ **h** $\sqrt{-2} \times \sqrt{-18} \times \sqrt{-3} \times \sqrt{-12}$

8 Simplify each expression.

a $(2i)^2$ **b** $(3i)^4$ **c** $5i \times 2i \times i$ **d** $i^2 + 3i^3 + i^4$

e $5i^{-2} + 6i^5 - 11i^{10}$ **f** $i^7 \times 3i^{-5}$ **g** $4i^3 \times 6i^{-8}$ **h** $\dfrac{10i^{-6}}{2i^3}$

ISBN 9780170412995

Problem solving

9 State the nature of the roots of each equation.

a $4x^2 - 2x - 5 = 0$ **b** $12x + 4x^2 + 18 = 0$

c $y + 6y^2 = -9$ **d** $3v^2 - 5v + 2 = 0$

10 Find values of m such that the equation $mx^2 - 5x + 6 = 0$ has:

a real roots **b** equal roots **c** complex roots.

11 Find values of p such that the equation $4x^2 + 3px - 2p = 0$ has:

a real roots **b** equal roots **c** complex roots.

12 Find a new equation with roots that are exactly twice the roots of $3x^2 + 5x + 8 = 0$.

8.02 Addition and subtraction of complex numbers

When you add $3 + 4m$ and $2 - 3m$, you add the constant terms and add the variables. The expression $x + ym$ is equal to $2 - 3m$ when $x = 2$ and $y = -3$.

Complex numbers work the same way as these expressions.

Equality, addition and subtraction of complex numbers

- Complex numbers are equal if and only if their real parts are equal *and* their imaginary parts are equal.

$$a + bi = c + di \Leftrightarrow a = c \text{ and } b = d$$

- To add complex numbers, add the real and imaginary parts separately:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- To subtract complex numbers, subtract the real and imaginary parts separately:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

ISBN 9780170412995

EXAMPLE 4

For the complex numbers $z_1 = 3 + 7i$ and $z_2 = 12 - 8i$, find:

a $z_1 + z_2$ **b** $z_1 - z_2$

Solution

a Write the sum.

$$z_1 + z_2 = (3 + 7i) + (12 - 8i)$$

Add the real and imaginary parts separately.

$$= (3 + 12) + [7 + (-8)]i$$

$$= 15 - i$$

b Write the difference.

$$z_1 + z_2 = (3 + 7i) - (12 - 8i)$$

Subtract the real and imaginary parts separately.

$$= (3 - 12) + [7 - (-8)]i$$

$$= -9 + 15i$$

Separate the real and imaginary parts of equal complex numbers to solve problems.

EXAMPLE 5

Find x and y, given that $2x - 3i = x + 7 + yi - xi$.

Solution

Separate the real and complex parts on the RHS.

$$2x - 3i = x + 7 + yi - xi$$

$$= x + 7 + (y - x)i$$

Equate the real parts.

$$2x = x + 7$$

$$2x - x = 7$$

$$x = 7$$

Equate the imaginary parts.

$$-3 = y - x$$

Substitute $x = 7$.

$$-3 = y - 7$$

$$y = 4$$

Write the answers.

$x = 7$ and $y = 4$

ISBN 9780170412995

TECHNOLOGY

Addition and subtraction on a graphics calculator

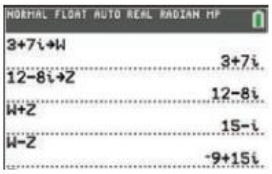
TI-Nspire CX Chapter 8

TI 84 Plus CE

Enter $z_1 = 3 + 7i$ and $z_2 = 12 - 8i$ and store in, say, w and z.

Use 2nd . for i and store the values.

Then do $w + z$ and $w - z$.



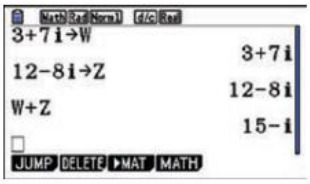
Casio fx-CG20AU

Use the Run–Matrix menu.

Enter $z_1 = 3 + 7i$ and $z_2 = 12 - 8i$ and store in, say, w and z.

Use SHIFT 0 for i and store the values.

Then do $w + z$ and $w - z$.



You should be able to recognise real, imaginary, complex, rational and irrational numbers.

Exercise 8.02 Addition and subtraction of complex numbers

Example 4

1 Simplify each expression.

a $(5 + 4i) + (2 + 7i)$ **b** $9 + (6 - 8i)$

c $(6 + 5i) - (1 + 4i)$ **d** $(8 + 3i) + (5 - 3i)$

e $[-2 + (-9i)] + (3 + 10i)$ **f** $(12 - 7i) + 11i$

g $(3 + 12i) - (6 + 8i)$ **h** $-(4 - 3i) - (1 - 6i)$

i $(13 + i) - [-(5 + 8i)]$ **j** $-(3 + 4i) + (-7 + 7i)$

2 $z_1 = 4 + 6i$ and $z_2 = 5 - 3i$. Calculate:

a $z_1 + z_2$ **b** $z_1 - 2z_2$ **c** $4z_1 - z_2$

d $-4z_1 + 3z_2$ **e** $2z_1 + 5z_2$ **f** $7z_1 - 4z_2$

Example 5

3 Find x and y, given that:

a $x + 2i = 3 + yi$ **b** $3x - 2yi = 7 + 4i$

c $2x - 3yi = \frac{1}{3}(12 - 5i)$ **d** $3x + 2yi = 15 - 12i$

e $x + (2 - 3yi) = (4 + y) - 12i$ **f** $3 + (y - 2xi) = (2 - yi) + x$

g $2x - 9i = -(4 - yi) + 2i$ **h** $(4x - i) - 5 = (x - 2yi) + 1$

ISBN 9780170412995

4 Solve for x and y.

a $(x + 2yi) + (3x + yi) = 8 - 9i$ **b** $(3x - 5yi) - (x + 2yi) = 11 + 4i$

c $-(7x + 3yi) - (2x + 3yi) = 15i$ **d** $2x + 3y - 3ix + 2y = 13 + 5i$

5 List the elements of $P = \{-7, 3i, -\sqrt{5}, 4 - 7i, 0, \sqrt{-7}, \pi, \frac{5}{8}, -\frac{2}{3} - i\}$ that are

a natural numbers **b** pure imaginary numbers

c integers **d** irrational numbers

e real numbers **f** complex numbers

g rational numbers

6 Use your graphics calculator to simplify:

a $(3 + 5i) + (-8 - 3i)$ **b** $(5 + 2i) - (3 - 4i)$ **c** $(4 - 9i) + (-3 + 3i)$

d $(-3 + 7i) + (5 - 4i)$ **e** $(-4 - 5i) - (-7 - 2i)$ **f** $(8 - 3i) - (4 + 5i)$

g $(7 - 2i) + (-3 + 6i)$ **h** $(4 - i) - (-6 - 4i)$

8.03 Multiplication of complex numbers

You multiply complex numbers by expanding brackets and then using $i^2 = -1$. In general,

$$(a + bi)(c + di) = ac + adi + bci + bdi^2$$
$$= ac + (ad + bc)i - bd.$$

Product of complex numbers

The product of $a + bi$ and $c + di$ is $(ac - bd) + (ad + bc)i$.

EXAMPLE 6

Calculate:

a $(3 + 5i)(2 + 3i)$ **b** $(2 - 5i)(3 + 4i)$ **c** $(2 + 3i)(2 - 3i)$

Solution

a Expand the brackets. $(3 + 5i)(2 + 3i) = 6 + 9i + 10i + 15i^2$

Use $i^2 = -1$. $= 6 + 19i - 15$

Simplify. $= -9 + 19i$

ISBN 9780170412995

b Expand the brackets. $(2 - 5i)(3 + 4i) = 6 + 8i - 15i - 20i^2$

Use $i^2 = -1$. $= 6 - 7i + 20$

Simplify. $= 26 - 7i$

c Expand the brackets. $(2 + 3i)(2 - 3i) = 4 - 9i^2$

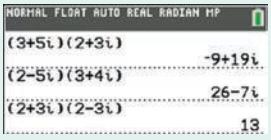
Use $i^2 = -1$. $= 4 + 9$

Simplify. $= 13$

TI-Nspire CX Chapter 8

TI 84 Plus CE

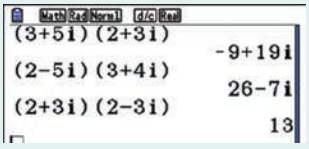
Enter the multiplications using 2nd . for i.



Casio fx-CG20AU

Use the Run–Matrix menu.

Enter the multiplications using SHIFT 0 for i.



You can simplify powers of complex numbers using the binomial expansion.

EXAMPLE 7

Calculate:

a $(1 - 2i)^3$ **b** $(3 + 2i)^5$

Solution

a Use the binomial expansion. $(1 - 2i)^3 = 1^3 + 3 \times 1^2 \times (-2i) + 3 \times 1 \times (-2i)^2 + (-2i)^3$

Simplify each term. $= 1 - 6i + 3 \times 4i^2 - 8i^3$

Use powers of i. $= 1 - 6i - 12 + 8i$

Simplify. $= -11 + 2i$

b Use the binomial expansion. $(3 + 2i)^5$

$= 3^5 + {}^5C_1 \times 3^4 \times 2i + {}^5C_2 \times 3^3 \times (2i)^2$

$+ {}^5C_3 \times 3^2 \times (2i)^3 + {}^5C_4 \times 3 \times (2i)^4 + (2i)^5$

ISBN 9780170412995

Simplify each term. $= 243 + 5 \times 81 \times 2i + 10 \times 27 \times 4i^2 + 10 \times 9 \times 8i^3 + 5 \times 3 \times 16i^5 + 32i^5$

Use powers of i. $= 243 + 810i - 1080 - 720i + 240 + 32i$

Simplify. $= -597 + 122i$

TI 84 Plus CE

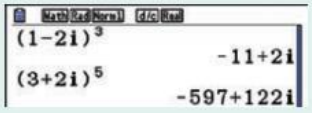
Enter the powers using 2nd . for i.

TI-Nspire CX Chapter 8

Casio fx-CG20AU

Use the Run–Matrix menu.

Enter the powers using SHIFT 0 for i.



Exercise 8.03 Multiplication of complex numbers

Your teacher might want you to use your calculator for some questions.

1 Calculate: Example 6

a $(2 + 3i)(1 + 5i)$ **b** $(1 + 2i)(8 - 3i)$ **c** $(3 - i)(7 + 2i)$

d $(4 + 3i)(2 + i)$ **e** $(3 - 2i)(1 - 5i)$ **f** $(3 + 4i)(1 + 2i)$

g $(7 - i)(2 + 3i)$ **h** $(5 + 3i)(2 - 3i)$ **i** $(1 - 4i)(4 - i)$

j $(6 + i)(2 + 3i)$

2 Find:

a $(i + i^2)(i + i^3)$ **b** $(2i^2 + 3i)(4i^2 - i)$ **c** $(1 - i)(2 + i)(3 + 2i)$

d $(1 - 2i)(1 + i)(2 - 3i)$ **e** $(2 - i)(1 - 3i)(4 - i)$ **f** $(\sqrt{3} + 2i)(\sqrt{6} - 2i)$

g $(\sqrt{5} - i)(\sqrt{15} + 2i)$ **h** $(\sqrt{8} + i)(2 - \sqrt{6}i)$ **i** $(\sqrt{27} - 2i)(\sqrt{3} - 5i)$

j $(3 + 2i)(2 + 3i)(1 - 2i)$

3 Calculate: Example 7

a $(3 - 2i)^3$ **b** $(1 + 5i)^4$ **c** $(1 + i)^8$

d $(2 - 3i)^4$ **e** $(-2 + i)^7$ **f** $(\sqrt{3} - i)^6$

g $(-2 + 3i)^5$ **h** $(2 - i)^6$ **i** $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{10}$

4 Simplify:

a $(3 - 2i)(3 + 2i)$ **b** $(1 - 5i)(1 + 5i)$ **c** $(9 + 2i)^2$

d $(2 - 3i)^2$ **e** $(-9 - 3i)(-9 + 3i)$ **f** $(3 - i)^3$

g $(-2 + 3i)^3$ **h** $(2 - i)(1 - 2i)^2$ **i** $(5 - 2i)^2(2 + 5i)$

ISBN 9780170412995

5 For the complex numbers $z_1 = 3 + 5i$ and $z_2 = 4 - 7i$, calculate:

a $z_1 z_2$ **b** $(z_1)^2$ **c** $(z_2)^3$

d $3z_1 \times 2z_2$ **e** $(z_1 + z_2)^2$ **f** $(z_1 - z_2)^2$

6 Find the real numbers a and b if:

a $a + bi = -5 + 21i$ **b** $(a + bi)(5 - 3i) = 9 - 19i$

c $(a + bi)(4 + 3i) = 2 - 3i$ **d** $(3 - i)(a + bi) = 3 + 4i$

e $(a + bi)^2 = 6 + 8i$ **f** $(a - bi)^2 = -2 - 2\sqrt{3}i$

g $(a + bi)^2 = 4 + 3i$ **h** $6 + 7i = (1 + 4i)(a + bi)$

i $(a + bi)(3 - 2i) = 2 + 3i$ **j** $(4 + i)(a + bi) = 3 - i$

Problem solving

7 Show that:

a $(1 + 5i) + (3 + i) = 2(2 + 3i)$

b $2(2 + i)^2 = (1 + i)(7 + i)$

c $(2 + 3i)(3 + 2i)$ represents a pure imaginary number

d $(1 + i)(1 - i) = (1 + i) + (1 - i)$

e $(1 - i)$ is a root of the equation $z^3 - 2z + 4 = 0$ $(z \in \boldsymbol{C})$

f $(1 + i)$ is a root of the equation $z^3 - 3z^2 + 6z = (4 + 2i)$.

8 If $z = a - ai$, prove that $z = -zi$.

Complex conjugates

Complex number operations

Complex conjugates and inverses

8.04 Complex conjugates and division of complex numbers

To divide by a decimal you change the problem to division by a whole number:

$$1.326 \div 0.03 = \frac{1.326}{0.03} = \frac{1.326}{0.03} \times \frac{100}{100} = \frac{132.6}{3} = 132.6 \div 3.$$

To divide by a fraction, you change the problem to multiplication by its reciprocal:

$$\frac{5}{8} \div \frac{3}{4} = \frac{\frac{5}{8}}{\frac{3}{4}} = \frac{\frac{5}{8}}{\frac{3}{4}} \times \frac{\frac{4}{3}}{\frac{4}{3}} = \frac{\frac{5}{8} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{5}{8} \times \frac{4}{3}}{1} = \frac{5}{8} \times \frac{4}{3}.$$

In both cases, you have multiplied by $1 = \frac{100}{100} = \frac{\frac{4}{3}}{\frac{4}{3}}$ to simplify the problem.

Can you multiply a complex number by something to make it a real number?

ISBN 9780170412995

Yes! You can multiply $5 + 3i$ by $5 - 3i$ to make it real as:

$$\begin{aligned}(5 + 3i)(5 - 3i) &= 5^2 - (3i)^2 \\ &= 25 - 9i^2 \\ &= 25 - 9 \times (-1) \\ &= 34.\end{aligned}$$

$5 - 3i$ is called the **complex conjugate** of $5 + 3i$.

Complex conjugates

- The complex **conjugate** of $a + bi$ is $a - bi$
- The complex conjugate of z is written as $\overline{z}$

EXAMPLE 8

Write the complex conjugates of the complex numbers $z = 7 + 4i$ and $w = 1 - 2i$.

Solution

For $\overline{z}$, change 4 to its negative. $\overline{z} = 7 - 4i$

For $\overline{w}$, change -2 to its negative. $\overline{w} = 1 - (-2)i$

$= 1 + 2i$

You should check the following facts about the conjugate for yourself.

Conjugate properties

- The **product** of $z = a + bi$ and its conjugate is $z\overline{z} = a^2 + b^2$
- The **sum** of $z = a + bi$ and its conjugate is $z + \overline{z} = 2a$
- The **difference** of $z = a + bi$ and its conjugate is $z - \overline{z} = 2b$

To divide by a complex number, z, multiply it by $\frac{\overline{z}}{\overline{z}}$ to change it to division by a real number.

ISBN 9780170412995

EXAMPLE 9

Calculate each quotient.

a $5i \div (2 + i)$ **b** $(6 - 5i) \div (2 - 3i)$ **c** $\dfrac{8+5i}{3+7i}$

Solution

a Change to fraction form.

$$5i \div (2 + i) = \frac{5i}{2+i}$$

Multiply by $\dfrac{2-i}{2-i} = 1$.

$$= \frac{5i}{(2+i)} \times \frac{(2-i)}{(2-i)}$$

Simplify.

$$= \frac{10i - 5i^2}{4 - i^2}$$

$$= \frac{10i + 5}{4 + 1}$$

$$= \frac{10i + 5}{5}$$

$$= 1 + 2i$$

b Change to fraction form.

$$(6 - 5i) \div (2 - 3i) = \frac{6-5i}{2-3i}$$

Multiply by $\dfrac{2+3i}{2+3i} = 1$.

$$= \frac{(6-5i)}{(2-3i)} \times \frac{(2+3i)}{(2+3i)}$$

Simplify.

$$= \frac{12 + 18i - 10i - 15i^2}{4 - 9i^2}$$

$$= \frac{12 + 18i - 10i - 15i^2}{4 - 9i^2}$$

$$= \frac{27 + 8i}{13}$$

Write in Cartesian form.

$$= 2\tfrac{1}{13} + \tfrac{8}{13}i$$

c Multiply top and bottom by the conjugate of the divisor.

$$\frac{8+5i}{3+7i} = \frac{(8+5i)}{(3+7i)} \times \frac{(3-7i)}{(3-7i)}$$

Simplify.

$$= \frac{24 - 56i + 15i - 35i^2}{9 + 49}$$

$$= \frac{24 - 56i + 15i - 35i^2}{9 + 49}$$

$$= \frac{59 - 41i}{58}$$

Write in rectangular form.

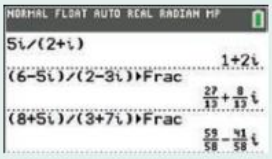
$$= \frac{59}{58} - \frac{41}{58}i$$

ISBN 9780170412995

TI 84 Plus CE

Enter the divisions using 2nd . for i.

Use math, 1: ▶Frac to express the answers as fractions.

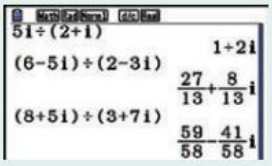


Casio fx-CG20AU

Use the Run–Matrix menu.

Enter the divisions using SHIFT 0 for i.

Press F↔D to change the answer to a fraction.



You can use $i^2 = -1$ to factorise a sum of perfect squares so that you get conjugate factors.

EXAMPLE 10

Factorise $4x^2 + 9y^2$.

Solution

Write as a difference. $4x^2 + 9y^2 = 4x^2 - 9y^2 \times (-1)$

Use $i^2 = -1$. $= 4x^2 - 9y^2 i^2$

Factorise as a difference of squares. $= (2x - 3yi)(2x + 3yi)$

Exercise 8.04 Complex conjugates and division of complex numbers

Your teacher might want you to use your calculator for some questions.

1 Write the complex conjugates of these in the form $a + bi$. Example 8

a $3 + 4i$ **b** $5 - 2i$ **c** 6 **d** $8i$

e $\frac{1}{2} + \frac{3}{4}i$ **f** $1 - i$ **g** $-7i$ **h** $-3 - 8i$

i $-(2 - 3i)$ **j** $\frac{2}{3}(3 + 4i)$ **k** $(1 - i)^2$ **l** $(1 + i)(2 - 3i)$

2 Calculate each quotient. Example 9

a $1 \div (3 + 2i)$ **b** $i \div (3 + 2i)$ **c** $1 \div (3 - 4i)$

d $\dfrac{1}{4+3i}$ **e** $\dfrac{1+2i}{3-5i}$ **f** $\dfrac{1+2i}{1-2i}$

g $(1 + i) \div (2 + 3i)$ **h** $(1 - 2i) \div (2 - 4i)$ **i** $1 \div 4i(3 - 2i)$

j $\dfrac{(1+i)(2+3i)}{1+2i}$ **k** $\dfrac{(1+2i)(2+3i)}{(4+i)(9+2i)}$ **l** $\dfrac{(3-2i)(1+4i)}{(2+i)(1-2i)}$

ISBN 9780170412995

Example 10

3 Factorise each sum.

a $a^2 + b^2$ **b** $x^2 + 16y^2$ **c** $9m^2 + 25n^2$ **d** $x^2 + 3$

e $2x^2 + 1$ **f** $3p^2 + 7q^2$ **g** $8x^2 + 27y^2$ **h** $12g^2 + 75h^2$

4 $z_1 = 2 - 3i$ and $z_2 = 5 - 6i$. Calculate:

a $z_1 z_2$ **b** $(z_1)^2$ **c** $(z_2)^3$

d $3z_1 \times 2z_2$ **e** $(z_1 + z_2)^2$ **f** $(z_1 - z_2)^2$

5 **a** $z = \dfrac{2+i}{1-i}$. Find Re (z).

b $z = \dfrac{3-2i}{4+i}$. Find Re (z).

c $z = \dfrac{1}{(1+i)(2-3i)}$. Find Im (z).

6 Simplify each expression in Cartesian form.

a $\dfrac{1}{2+i} + \dfrac{1}{3+2i}$ **b** $\dfrac{2}{2+3i} + \dfrac{3i}{2-3i}$

c $\dfrac{1-2i}{2-i} - \dfrac{2+3i}{1-2i}$ **d** $\dfrac{3-2i}{2-5i} + \dfrac{5+3i}{3-i}$

7 **a** $z = 2 - i$. Write $z + \dfrac{1}{z}$ in the form $a + bi$.

b $z = \dfrac{1}{1-i}$. Write $\dfrac{z^2+1}{z}$ in the form $a + bi$.

c $z = \dfrac{1-i}{1+2i}$. Write $\overline{z}$ in the form $a + bi$.

8 Find the value of $\dfrac{2}{1-i} + \dfrac{2}{1+i}$.

Problem solving

9 Prove that $\dfrac{2+3i}{6-4i}$ is purely imaginary.

10 Prove that $\dfrac{2+i}{1-2i} = i$.

11 Prove that $(\sqrt{2} - \sqrt{2}i)^n$ is real for $n = 4, 8, 12, 16, \ldots$

12 Prove that $(1 + \sqrt{3}i)^n$ is real for $n = 3, 6, 9, 12, \ldots$

13 Prove that $a, b \in \boldsymbol{R} \Rightarrow \dfrac{1}{a+bi} + \dfrac{1}{a-bi} \in \boldsymbol{R}$.

14 Prove that for $z_1, z_2 \in \boldsymbol{C}$, $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$, $\overline{z_1} - \overline{z_2} = \overline{z_1 - z_2}$, $\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2}$ and $\overline{z_1} \div \overline{z_2} = \overline{z_1 \div z_2}$.

ISBN 9780170412995

8.05 The Argand plane

You have been writing general complex numbers as $a + bi$. You could write them as $x + yi$ instead, where x and y are real numbers. Jean-Robert Argand observed over 200 years ago that it makes sense to represent them as points on the plane.

The Argand plane

- The **Argand plane** (or **Argand diagram**, **complex plane**) is a rectangular coordinate system with real numbers on the horizontal axis and imaginary numbers on the vertical axis
- The horizontal axis is labelled X and the vertical axis iY
- The complex number $x + yi$ is shown as the point (x, y) on the plane

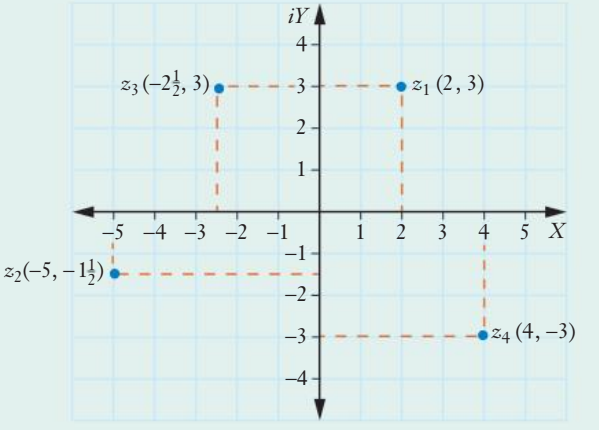
EXAMPLE 11

Show the following as points on the same Argand plane.

$z_1 = 2 + 3i,\ z_2 = -5 - 1\frac{1}{2}i, z_3 = -2\frac{1}{2} + 3i, z_4 = 4 - 3i$

Solution

Show z_1, z_2, z_3 and z_4 as the points $(2, 3), (-5, -1\frac{1}{2}), (-2\frac{1}{2}, 3)$ and $(4, -3)$ respectively.



ISBN 9780170412995

Some simple relationships between complex numbers have geometric relationships on the Argand plane.

EXAMPLE 12

$z = 3 - 4i$.

a Calculate zi and show z and zi on the same complex plane.

b What is the geometric relationship between z and zi?

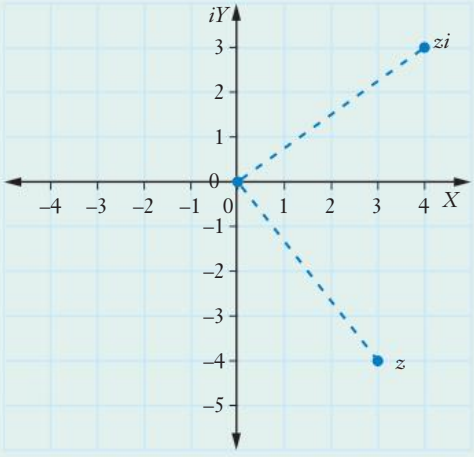
Solution

a Calculate zi.

$zi = (3 - 4i)i = 4 + 3i$

Show z and zi as the points $(3, -4)$ and $(4, 3)$ on the complex plane.

Connect to the origin with dashed lines to help find the geometric relationship.



b Check the distances from the origin, and the distances from each other.

z distance from origin $= \sqrt{3^2 + (-4)^2} = 5$

zi distance from origin $= \sqrt{4^2 + 3^2} = 5$

Distance from z to $zi = \sqrt{(4-3)^2 + [3-(-4)]^2}$

$= \sqrt{1 + 49}$

$= \sqrt{25 \times 2}$

$= 5\sqrt{2}$

State the relationship.

By the converse of Pythagoras' theorem, the triangle is right-angled at the origin so iz is rotated 90° from z.

Notice that the tangents of the angles are $-\frac{4}{3}$ and $\frac{3}{4}$, which fits with θ and $\theta + 90°$.

ISBN 9780170412995

Exercise 8.05 The Argand plane

Example 11

1 Show each complex number as a point on the same Argand plane.

a $3 + 4i$ **b** $5 - 2i$ **c** $-2 + 3i$

d 7 **e** $-6i$ **f** $-4 - 2i$

2 Show each complex number as a point on the same complex plane.

a $3 - 4i$ **b** $-2 + 3i$ **c** $4 + i$

d $-3 - 2i$ **e** $3i$ **f** -4

Example 12

3 $z = 12 + 5i$.

a Calculate zi, zi^2 and zi^3.

b Show z, zi, zi^2 and zi^3 on the same complex plane.

c What is the geometric relationship between z, zi, zi^2 and zi^3 on the complex plane?

4 $z = -8 + 6i$.

a Calculate zi, zi^2 and zi^3.

b Show z, zi, zi^2 and zi^3 on the same complex plane.

c What is the geometric relationship between z, zi, zi^2 and zi^3 on the complex plane?

5 $z = -2 - 3i$.

a Calculate zi, zi^2 and zi^3.

b Show z, zi, zi^2 and zi^3 on the same complex plane.

c What is the geometric relationship between z, zi, zi^2 and zi^3 on the complex plane?

Problem solving

6 **a** Consider $z \in \boldsymbol{C}$, zi, zi^2 and zi^3. What is the geometric relationship between z, zi, zi^2 and zi^3 on the complex plane?

b Prove the relationship.

7 What is the geometric relationship between $z \in \boldsymbol{C}$ and $\overline{z}$ on the complex plane?

8 What is the geometric relationship between $z \in \boldsymbol{C}$ and $-z$ on the complex plane?

ISBN 9780170412995

8.06 Complex numbers as vectors

You can draw a position vector from the origin to any point in the plane.

This means that you can treat complex numbers as vectors instead of points in the Argand plane.

Complex numbers as vectors

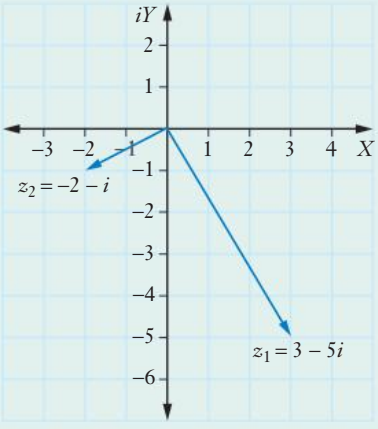
The vector for $a + bi$ is equal to the position vector from O to (a, b) in the plane.

EXAMPLE 13

Show $z_1 = 3 - 5i$ and $z_2 = -2 - i$ as vectors in the complex plane.

Solution

Show the arrows going from the origin to $(3, -5)$ and $(-2, -1)$ in the complex plane.



You know the magnitude of a vector $\mathbf{v} = (a, b)$ is given by $|\mathbf{v}| = \sqrt{a^2 + b^2}$ and its direction is given by $\tan(\theta) = \frac{b}{a}$.

When you treat complex numbers as vectors, they also have a size and direction. They are given special names to signify thay they belong to complex numbers.

Modulus and argument

- The **modulus** (size) of the complex number $z = a + bi$ is given by $\sqrt{a^2 + b^2}$
- The modulus of z is written as $|z|$ or mod (z)
- The **argument** (direction) of the complex number $z = a + bi$ is given by $\tan(\theta) = \frac{b}{a}$
- The argument of z is written as arg (z)

ISBN 9780170412995

EXAMPLE 14

Find the modulus and argument of each complex number.

a $\sqrt{3} - i$ **b** $-6 + 4i$

Solution

a Write the formula and calculate the modulus.

$$|\sqrt{3} - i| = \sqrt{(\sqrt{3})^2 + (-1)^2}$$
$$= \sqrt{4}$$
$$= 2$$

Write the formula and calculate $\tan(\theta)$.

$$\tan(\theta) = \frac{-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

Write the reference angle.

$$\text{Reference angle} = 30° \left(\text{or } \frac{\pi}{6}\right)$$

Find the angle (in the 4th quadrant).

$$\text{Arg}(\sqrt{3} - i) = -30° \left(\text{or } -\frac{\pi}{6}\right)$$

You could write the argument of $\sqrt{3} - i$ as $330°$ or $\frac{11\pi}{6}$.

b Write the formula and calculate the modulus.

$$|-6 + 4i| = \sqrt{(-6)^2 + 4^2}$$
$$= \sqrt{52}$$
$$= 2\sqrt{13}$$
$$\approx 7.21$$

Write the formula and calculate $\tan(\theta)$.

$$\tan(\theta) = \frac{4}{-6} = -\frac{2}{3}$$

Write the reference angle.

$$\text{Reference angle} = \tan^{-1}\left(\frac{2}{3}\right)$$

Find the angle (in the 2nd) quadrant.

$$\text{Arg}(-6 + 4i) = 180° - \tan^{-1}\left(\frac{2}{3}\right) \quad \left(\text{or } \pi - \tan^{-1}\left(\frac{2}{3}\right)\right)$$
$$\approx 146.3° \quad (\text{or} \approx 2.55)$$

ISBN 9780170412995

TECHNOLOGY

Modulus and argument on a graphics calculator

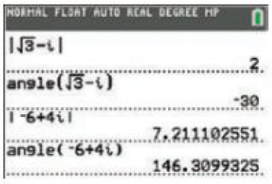
TI-Nspire CX Chapter 8

TI 84 Plus CE

Press [math] and use the cursor controls to choose CMPLX and use 5: abs (for the modulus. Use [2nd] [.] for i.

Press [mode] and use the cursor controls to change the angle measure.

Use 4: angle (in CMPLX for the argument.



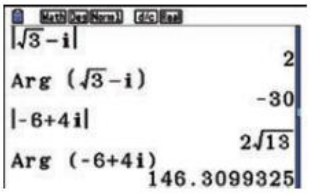
Casio fx-CG20AU

Use the Run–Matrix menu.

Press [OPTN] [F3] (COMPLEX) and [F2] (Abs) for the modulus. Use [SHIFT] [0] for i.

Press [SHIFT] [MENU] (SET UP) and use the cursor controls to change the angle measure.

Use Arg in COMPLEX for the argument.



Exercise 8.06 Complex numbers as vectors

Example 13

1 Show each complex number as a vector on the same complex plane.

a $3 - 2i$ **b** $-4 - 5i$ **c** $3 + 4i$

d $-3i$ **e** $-4 + 2i$ **f** 5

2 Show each complex number as a vector on the same Argand plane.

a $2 + i$ **b** $4 - 3i$ **c** $-5 + 3i$

d $4i$ **e** $-(4 + 2i)$ **f** $-3(1 - i)$

Example 14

3 Find the modulus and argument in degrees for each complex number, correct to one decimal place if necessary.

a $5 + 5\sqrt{3}i$ **b** $7 - 8i$ **c** $-6 - 6i$

d $-4\sqrt{2} + 4\sqrt{2}i$ **e** $7\sqrt{3} - 7i$

4 Find the modulus and argument in radians for each complex number, correct to one decimal place if necessary.

a $8 - 8i$ **b** $3 + 4i$ **c** $-5 - 12i$ **d** $-2\sqrt{3} - 2i$ **e** $-4 + 4i$

ISBN 9780170412995

Problem solving

5 Find the angle between the vectors that represent $3\sqrt{3} + 3i$ and $5 + 5\sqrt{3}i$.

6 Show that the vectors for $-4\sqrt{3} + 4i$ and $(\sqrt{2} - \sqrt{6}) + (\sqrt{2} + \sqrt{6})i$ are at an angle of 45°.

7 Show that the complex numbers $1 + 3i$, $-3 + 2i$ and $-2 - 2i$ form an isosceles triangle in the Argand plane.

8 Show that the complex numbers $-1 + 2i$, $1 - 2i$, $7 + i$ and $5 + 5i$ form a rectangle in the complex plane.

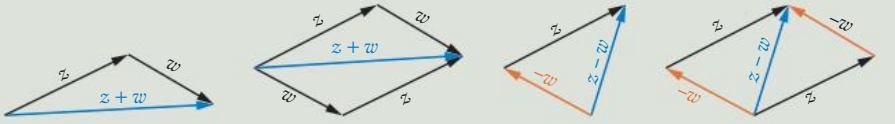
8.07 Operations in the complex plane

You can use vector triangles or parallelograms to add and subtract vectors. You can do the same with complex numbers treated as vectors.

Addition and subtraction of complex number vectors

Complex number vectors w and z can be added or subtracted as shown:

- Place the vectors head to tail
- For $z - w$, use $z + (-w)$
- The resultant vector is from the first tail to the last head



The first parallelogram shows that $z + w = w + z$.

The second parallelogram shows that $z - w = -w + z$

EXAMPLE 15

$z_1 = 3 + i$ and $z_2 = 2 - 3i$. Find each expression using vectors on an Argand diagram and state each result.

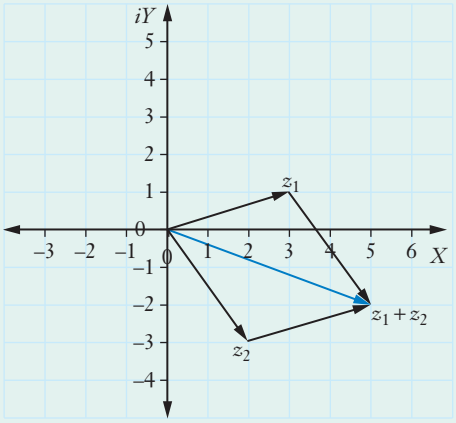
a $z_1 + z_2$ **b** $z_1 - z_2$

Solution

a Put z_1 and z_2 head to tail to find the result or complete the parallelogram.

ISBN 9780170412995

Start from the head of z_1 (3, 1) and go (2, –3). That is 2 in the real direction and –3 in the imaginary direction.



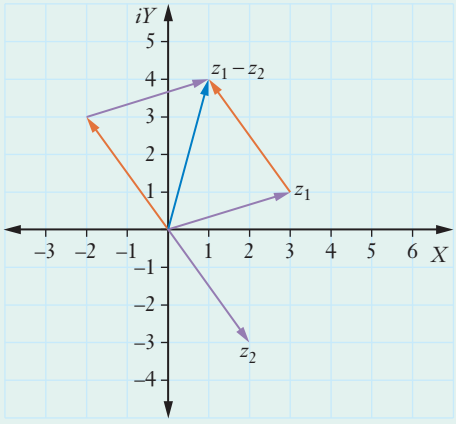
State the result.

$z_1 + z_2 = (3 + 2) + (1 - 3)i$

$= 5 - 2i$

b Put z_1 and $-z_2$ head to tail to find the result or complete the parallelogram.

Start from the head of z_1 (3, 1) and go (–2, 3). That is –2 in the real direction and 3 in the imaginary direction.



State the result.

$z_1 - z_2 = (3 - 2) + (1 + 3)i$

$= 1 + 4i$

You looked at some of the geometric relationships between numbers in the complex plane in Section 8.05.

Geometric relationships of complex numbers

- Multiplication by i rotates a complex number 90° (anticlockwise) around the origin
- Complex conjugates are reflections of each other in the X (real) axis
- Negatives are rotations of each other by 180° around the origin

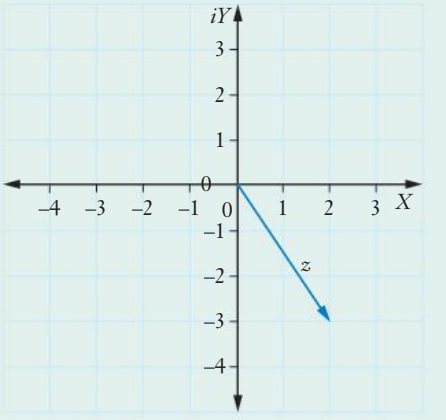
ISBN 9780170412995

EXAMPLE 16

$z = 2 - 3i$. Sketch z, zi and $\bar{z}$ as vectors without calculation.

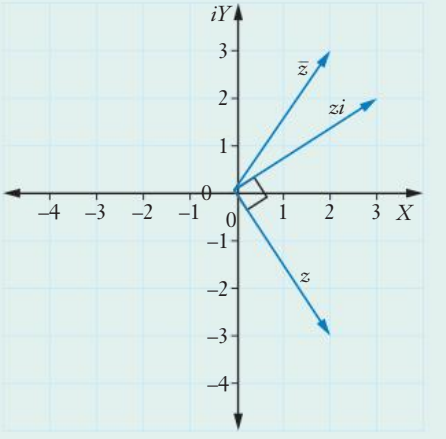
Solution

Sketch z.



Add zi by rotating z through 90°.

Add $\bar{z}$ by reflecting z in the X-axis.



Rotations, dilations, translations and reflections are all transformations of geometric figures.

A transformation changes a vector into a new vector. The transformations
$\mathbf{T}_1(x\mathbf{i} + y\mathbf{j}) = -x\mathbf{i} + y\mathbf{j}$ and $\mathbf{T}_2(x\mathbf{i} + y\mathbf{j}) = x\mathbf{i} - y\mathbf{j}$ reflect vectors in the x and y axes respectively.

ISBN 9780170412995

Linear transformations

- A **linear transformation** preserves addition and scalar multiplication
- In formal terms, a transformation $\mathbf{T}$ of vectors is linear if and only if

  $\mathbf{T}(\alpha\mathbf{v}) = \alpha\mathbf{T}(\mathbf{v})$ and $\mathbf{T}(\mathbf{v} + \mathbf{w}) = \mathbf{T}(\mathbf{v}) + \mathbf{T}(\mathbf{w})$ for any vectors $\mathbf{v}$, $\mathbf{w}$ and any $\alpha \in \boldsymbol{R}$
- The conditions for linearity can be replaced by the single condition

  $\mathbf{T}(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha\mathbf{T}(\mathbf{v}) + \beta\mathbf{T}(\mathbf{w})$ for any vectors $\mathbf{v}$, $\mathbf{w}$ and any $\alpha, \beta \in \boldsymbol{R}$

You can show that the combined condition is equivalent to the separate conditions as follows.

In the combined condition, choose $\beta = 0$ and $\mathbf{w} = \mathbf{0}$, then choose $\alpha = \beta = 1$ to get the separate conditions. To get the combined condition, just transform $\alpha\mathbf{v} + \beta\mathbf{w}$ using the separate conditions one at a time.

It is easier to prove a transformation is linear using the separate conditions.

EXAMPLE 17

Show that multiplication of complex vectors by $2 - 3i$ is a linear transformation.

Solution

State what has to be shown.

RTP: $\mathbf{T}(\alpha z) = \alpha\mathbf{T}(z)$ and $\mathbf{T}(z + w) = \mathbf{T}(z) + \mathbf{T}(w)$ for any $z, w \in \boldsymbol{C}$ and any $\alpha \in \boldsymbol{R}$.

Proof

Start the first condition.

Choose $z = a + bi$ and $\alpha \in \boldsymbol{R}$.

Put the values into the LHS.

$$\mathbf{T}(\alpha z) = \alpha z \times (2 - 3i)$$

$$= \alpha(a + bi) \times (2 - 3i)$$

Simplify the expression.

$$= (\alpha a + \alpha bi) \times (2 - 3i)$$

$$= 2\alpha a - 3\alpha ai + 2\alpha bi + 3\alpha b$$

Rearrange to get the RHS.

$$= \alpha(2a - 3ai + 2bi - 3bi^2)$$

$$= \alpha[(a + bi)(2 - 3i)]$$

$$= \alpha[z \times (2 - 3i)]$$

$$= \alpha\mathbf{T}(z)$$

Start the second condition.

Choose $z = a + bi$ and $w = c + di$.

Put the values into the LHS.

$$\mathbf{T}(z + w) = [z + w] \times (2 - 3i)$$

Simplify the expression.

$$= (a + bi + c + di) \times (2 - 3i)$$

$$= 2a + 2bi + 2c + 2di - 3ai - 3bi^2 - 3ci - 3di^2$$

ISBN 9780170412995

Rearrange to get the RHS.

$$= 2a - 3ai + 2bi - 3bi^2 + 2c - 3ci + 2di - 3\ di^2$$
$$= a(2 - 3i) + bi(2 - 3i) + c(2 - 3i) + di(2 - 3i)$$
$$= (a + bi)(2 - 3i) + (c + di)(2 - 3i)$$
$$= z \times (2 - 3i) + w \times (2 - 3i)$$
$$= \mathbf{T}(z) + \mathbf{T}(w)$$

Finish the proof. Thus $\mathbf{T}(\alpha z) = \alpha\mathbf{T}(z)$ and $\mathbf{T}(z + w) = \mathbf{T}(z) + \mathbf{T}(w)$ QED

Exercise 8.07 Operations in the complex plane

1 $z_1 = -4 + 2i$ and $z_2 = 3 + 4i$. Show each operation using vectors on an Argand diagram and state each result. Example 15

a $z_1 + z_2$ **b** $z_1 - z_2$

2 $z_1 = -2 - 3i$ and $z_2 = -1 + 2i$. Show each operation using vectors on an Argand diagram and state each result.

a $z_1 + z_2$ **b** $z_1 - z_2$

3 $z_1 = 4 - 3i$ and $z_2 = -2 - i$. Show each operation using vectors on an Argand diagram and state each result.

a $z_1 + z_2$ **b** $z_1 - z_2$

4 $z = 1 + 2i$. Sketch z, $-z$, and zi as vectors without calculation. Example 16

5 $z = -4 - i$. Sketch z, $-z$, $\overline{z}$ and zi as vectors without calculation.

6 $z = -3 + 4i$. Sketch z, zi^3 and $-\overline{z}$ as vectors without calculation.

7 $z = 3 + 2i$. Show z, $-z$, $\overline{z}$ and zi as points in the Argand plane without calculation.

8 Show that multiplication of complex vectors by $1 + 4i$ is a linear transformation. Example 17

9 Show that multiplication of complex vectors by $-3 - i$ is a linear transformation.

10 Show that multiplication of complex vectors by $-4 + 3i$ is a linear transformation.

Problem solving

11 Prove that the dual condition for a transformation to be linear is equivalent to the combined condition.

12 Prove that multiplication of complex vectors by a constant complex number is a linear transformation.

ISBN 9780170412995

8. CHAPTER SUMMARY

Complex numbers

Imaginary and complex numbers

- **The imaginary number** i is a solution of the equation $x^2 + 1 = 0$; $i = \sqrt{-1}$
- An **imaginary number** (or **pure imaginary number**) is a number of the form bi, where $b \in \boldsymbol{R}$
- A **complex number** is a number of the form $z = a + bi$, where $a, b \in \boldsymbol{R}$
- The set of all complex numbers is shown as $\boldsymbol{C}$
- The format $a + bi$, where $a, b \in \boldsymbol{R}$ is called the **Cartesian form** or **rectangular form** of a complex number
- $\text{Re}(z) = a$ is the **real part** of the complex number $a + bi$
- $\text{Im}(z) = b$ is the **imaginary part** of the complex number $a + bi$

Complex conjugates

- The complex **conjugate** of $a + bi$ is $a - bi$
- The complex conjugate of z is written as $\overline{z}$
- The **product** of $z = a + bi$ and its conjugate is $z\overline{z} = a^2 + b^2$
- The **sum** of $z = a + bi$ and its conjugate is $z + \overline{z} = 2a$
- The **difference** of $z = a + bi$ and its conjugate is $z - \overline{z} = 2b$

Operations with complex numbers

- Complex numbers are equal if and only if their real parts are equal *and* their imaginary parts are equal. $a + bi = c + di \Leftrightarrow a = c$ and $b = d$
- To add complex numbers, add the real and imaginary parts separately:

  $(a + bi) + (c + di) = (a + c) + (b + d)i$
- To subtract complex numbers, subtract the real and imaginary parts separately:

  $(a + bi) - (c + di) = (a - c) + (b - d)i$
- By expansion of brackets, the product of $a + bi$ and $c + di$ is $(ac - bd) + (ad + bc)i$
- To divide by a complex number, z, multiply it by $\frac{\overline{z}}{\overline{z}}$
- Use $i^2 = -1$ to factorise a sum of perfect squares to conjugate factors

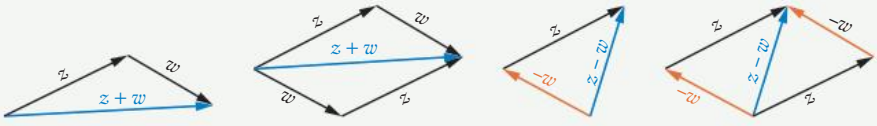
ISBN 9780170412995

The Argand plane

- The **Argand plane** (or **Argand diagram**, **complex plane**) is a rectangular coordinate system with real numbers on the horizontal axis and imaginary numbers on the vertical axis
- The horizontal axis is labelled X and the vertical axis iY
- The complex number $x + yi$ is shown as the point (x, y) on the plane
- The **complex vector** for $a + bi$ is equal to the position vector from O to (a, b) in the plane
- The **modulus** (size) of the complex number $z = a + bi$ is given by $\sqrt{a^2 + b^2}$
- The modulus of z is written as $|z|$ or mod (z)
- The **argument** (direction) of the complex number $z = a + bi$ is given by $\tan(\theta) = \frac{b}{a}$
- The argument of z is written as arg (z)

Operations in the complex plane

- Complex numbers vectors w and z can be added or subtracted by head to tail
- For $z - w$, use $z + (-w)$
- The resultant vector is from the first tail to the last head



- Multiplication by i rotates a complex number 90° (anticlockwise) around the origin
- Complex conjugates are reflections of each other in the X (real) axis
- Negatives are rotations of each other by 180° around the origin

Linear transformations

- A **linear transformation** preserves addition and scalar multiplication
- In formal terms, a transformation $\mathbf{T}$ of vectors is linear if and only if

  $\mathbf{T}(\alpha\mathbf{v}) = \alpha\mathbf{T}(\mathbf{v})$ and $\mathbf{T}(\mathbf{v} + \mathbf{w}) = \mathbf{T}(\mathbf{v}) + \mathbf{T}(\mathbf{w})$ for any vectors $\mathbf{v}$, $\mathbf{w}$ and any $\alpha \in \boldsymbol{R}$
- The conditions for linearity can be replaced by the single condition

  $\mathbf{T}(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha\mathbf{T}(\mathbf{v}) + \beta\mathbf{T}(\mathbf{w})$ for any vectors $\mathbf{v}$, $\mathbf{w}$ and any $\alpha, \beta \in \boldsymbol{R}$
- Multiplication by a complex number is a linear transformation

ISBN 9780170412995

8. CHAPTER REVIEW

Complex numbers

Example 1

1 Solve each equation.

a $x^2 + 49 = 0$ **b** $m^2 - 3m + 6 = 0$

Example 2

2 Write Re (z) and Im (z) for each complex number.

a $5i - 3$ **b** $\sqrt{5} + 2i$

Example 3

3 Simplify each expression.

a $5i \times 4i \times (-i)$ **b** $\sqrt{-49}$ **c** $\sqrt{-5} \times \sqrt{-45}$ **d** i^{15} **e** $\dfrac{\sqrt{-8}}{2i^3}$

Example 4

4 Simplify each expression.

a $(3 + 4i) - (5 - i)$ **b** $(4 - 5i) + (2 - 3i)$

c $(2 - 3i) + (2 + 3i)$ **d** $(2 + 3i) - (5 - 4i)$

Example 5

5 Solve $4 - (x - 2)i = y + 3 + 3i$ for x and y.

Example 6

6 Find each product.

a $(5 - i)(4 + 3i)$ **b** $(3 - 2i)(3 + 2i)$

Example 7

7 Calculate $(1 - 3i)^4$.

Example 6

8 Solve $3 - i + x = (5 + i)(y + i)$ for x and y.

Example 8

9 Write the complex conjugate of each complex number.

a $8 - 6i$ **b** $-2 + 3i$

Example 9

10 Simplify each quotient.

a $(15 + 5i) \div (1 + 2i)$ **b** $\dfrac{3 - 2i}{2 + 3i}$

Example 10

11 Factorise $x^2 + 16y^2$.

Example 11

12 Show $z_1 = 4 - i$ and $z_2 = -2 + 3i$ as points on the same Argand plane.

Example 12

13 For $z = 3 + 2i$, calculate zi and zi^2, show them on the same complex plane as z and state their relationships to z.

Example 13

14 Show $z_1 = -2 - 3i$ and $z_2 = 3 + 4i$ as vectors on the same complex plane.

ISBN 9780170412995

15 Find the modulus and argument of $5 + 4i$, correct to one decimal place.

Example 14

16 For the complex numbers $z_1 = -2 - i$ and $z_2 = 3 - 3i$, show each operation using vectors on an Argand diagram and state each result.

a $z_1 + z_2$ **b** $z_1 - z_2$

Example 15

17 Sketch $z = 4 - 3i$, $-z$, $\overline{z}$ and zi as vectors without calculation.

Example 16

18 Show that multiplication of complex vectors by $-2 + 4i$ is a linear transformation.

Example 17

Problem solving

19 Find values of p that make the roots of $x^2 + 3px - p = 0$

a real **b** equal **c** complex.

20 Find a new equation whose roots are the squares of the roots of $2x^2 - 5x + 1 = 0$.

21 Show that $(2 + i)$ is a root of the equation $x^3 + 21x = 8x^2 + 20$.

22 Show that $\frac{1}{2-3i} - \frac{1}{2+3i}$ is a purely imaginary number.

23 Show that 0, $2 + 3i$ and $-9 + 6i$ form a right-angled triangle in the Argand plane.

Practice quiz

ISBN 9780170412995

9.

TRIGONOMETRIC FUNCTIONS

Trigonometry are a powerful tool for indirect measurement in fields such as astronomy, surveying, navigation, construction and mapmaking. There is evidence that astronomers used trigonometric ratios more than 2000 years ago.

The most important application of trigonometric functions is the modelling of periodic phenomena. These include light, sound, electrical circuits, electronics, shock absorbers and even the way a door with a damper closes.

9.01 General solution of trigonometric equations
9.02 General trigonometric graphs
9.03 Absolute value functions
9.04 Reciprocal functions
9.05 Rational functions
9.06 The reciprocal trigonometric functions
9.07 Pythagorean trigonometric identities
9.08 Angle sum and difference identities
9.09 Double angle identities
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

The basic trigonometric functions

- find all solutions of $f(a(x-b)) = c$ where $f(\theta)$ is one of sin (θ), cos (θ) or tan (θ)
- sketch and graph functions with rules of the form $y = f(a(x-b))$ where $f(\theta)$ is one of sin (θ), cos (θ) or tan (θ)

Sketching graphs

- use and apply the notation $|x|$ for the absolute value for the real number x and the graph of $y = |x|$
- examine the relationship between the graph of $y = f(x)$ and the graphs of $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$
- sketch the graphs of simple rational functions where the numerator and denominator are polynomials of low degree

The reciprocal trigonometric functions, secant, cosecant and cotangent

- define the reciprocal trigonometric functions, sketch their graphs, and graph simple transformations of them

Trigonometric identities

- prove and apply the Pythagorean identities
- prove and apply the angle sum, difference and double angle identities for sines and cosine

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

Shutterstock/Thiff

TERMINOLOGY

absolute value
amplitude
complementary angles
double angle identities
general solution
half angle identities
horizontal asymptote
identity
mean value
modulus
period
phase shift
principal domain
Pythagorean identities
rational functions
reciprocal function
reciprocal trigonometric functions
reference angle
slant asymptote
sum and difference identities
trigonometric equation
trigonometric functions
vertical asymptote

WS Basic trigonometry

WS Degrees and radians

WS Complementary angle properties

PS Exact values 1

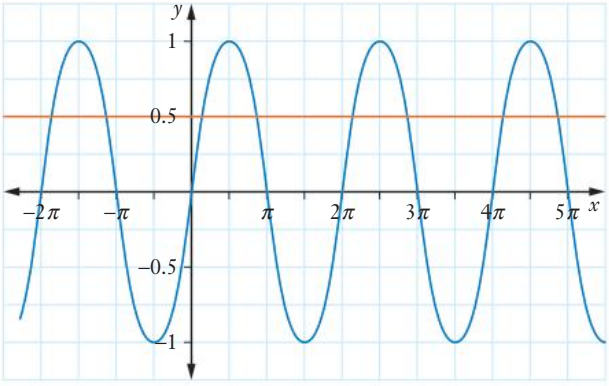
PS Exact values 2

PS Simplifying trigonometric functions

9.01 General solution of trigonometric equations

In the Mathematical Methods course, you have solved equations like $\sin(x) = \frac{1}{2}$ over limited domains such as $[0, 2\pi]$, where it has the solutions $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$ (30° or 150°).

The sketch of $y = \sin(x)$ and $y = \frac{1}{2}$ shows that there are an infinite number of solutions over the real numbers.



$\sin(x)$ has a period of 2π, so the solutions in the domain $[0, 2\pi]$ are repeated every 2π.

This means the general solutions of $\sin(x) = \frac{1}{2}$ are

$$\ldots, \frac{\pi}{6} - 6\pi, \frac{\pi}{6} - 4\pi, \frac{\pi}{6} - 2\pi, \frac{\pi}{6}, \frac{\pi}{6} + 2\pi, \frac{\pi}{6} + 4\pi, \frac{\pi}{6} + 6\pi, \ldots$$

$$\text{or} \quad \ldots, \frac{5\pi}{6} - 6\pi, \frac{5\pi}{6} - 4\pi, \frac{5\pi}{6} - 2\pi, \frac{5\pi}{6}, \frac{5\pi}{6} + 2\pi, \frac{5\pi}{6} + 4\pi, \frac{5\pi}{6} + 6\pi, \ldots$$

You can write this as $\frac{\pi}{6} + 2n\pi$ or $\frac{5\pi}{6} + 2n\pi$ for $n \in Z$.

You can see that equations involving $\cos(x)$ or $\tan(x)$ will have similar general solutions.

ISBN 9780170412995

From our work on inverse trigonometric functions, we know that:

- There is exactly one solution to equations of the form $\sin(x) = a$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
- There is exactly one solution to equations of the form $\cos(x) = a$ for $0 \leq x \leq \pi$.
- There is exactly one solution to equations of the form $\tan(x) = a$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

They are called the **principal domains** of $\sin(x)$, $\cos(x)$ and $\tan(x)$.

The $\sin^{-1}(y)$, $\tan^{-1}(y)$ and $\cos^{-1}(y)$ functions are restricted to these ranges. Your calculator only gives these values of x.

For $\tan(x) = a$, the principal domain covers a complete period. You only need to add $n\pi$ (where n is an integer) to $\tan^{-1}(a)$ to get the general solution $n\pi + \tan^{-1}(a)$.

For $\sin(x) = a$, there is an extra solution $\pi - \sin^{-1}(a)$ in the domain $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Adding $2n\pi$ gives the general solutions $2n\pi + \sin^{-1}(a)$ or $2n\pi + \pi - \sin^{-1}(a) = (2n + 1)\pi - \sin^{-1}(a)$.

Note that for even $(2n)$ multiples of π, you add $\sin^{-1}(a)$, while for odd $(2n + 1)$ multiples of π, you subtract $\sin^{-1}(a)$. These two expressions can be combined into one general solution as $n\pi + (-1)^n \sin^{-1}(a)$, where even values of n give the + expression and odd values of n give the – expression.

For $\cos(x) = a$ there is an extra solution $-\cos^{-1}(a)$ in the domain $-\pi < x < 0$. Adding $2n\pi$ gives the general solutions $2n\pi + \cos^{-1}(a)$ or $2n\pi - \cos^{-1}(a)$.

General solutions of trigonometric equations

- If $\sin(x) = a$, and $\theta = \sin^{-1}(a)$ is the solution in the domain $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then
  $x = n\pi + (-1)^n \times \theta$, where n is an integer.
- If $\cos(x) = a$, and $\theta = \cos^{-1}(a)$ is the solution in the domain $0 \leq \theta \leq \pi$, then
  $x = 2n\pi \pm \theta$, where n is an integer.
- If $\tan(x) = a$, and $\theta = \tan^{-1}(a)$ is the solution in the domain $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then
  $x = n\pi + \theta$, where n is an integer.

ISBN 9780170412995

EXAMPLE 1

Find the general solution of each equation.

a $\tan(x) = -\sqrt{3}$ **b** $\sin(x) = 0.45$

Solution

a Find the reference angle.

$x = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

$\tan(x)$ repeats every π.

The general solution is $n\pi - \frac{\pi}{3}$ for $n \in \mathbb{Z}$.

b Find the reference angle.

$x = \sin^{-1}(0.45) \approx 0.4668$

Write the general solution.

The general solution is $n\pi + (-1)^n (0.4668)$ for $n \in \mathbb{Z}$.

For equations involving $2x$, $3x$, … solve for $2x$, $3x$, … etc. and then divide by 2, 3, ...

EXAMPLE 2

Find the solutions of $2\cos(3x) + \sqrt{3} = 0$.

Solution

Rearrange to isolate $\cos(3x)$.

$\cos(3x) = -\frac{\sqrt{3}}{2}$

Find the reference angle.

$3x = \frac{5\pi}{6}$

Find the general solution and simplify.

$3x = 2n\pi \pm \frac{5\pi}{6}$

$= \frac{12n\pi \pm 5\pi}{6}$

$= \frac{(12n \pm 5)\pi}{6}$

Divide by 3.

$x = \frac{(12n \pm 5)\pi}{18}$

ISBN 9780170412995

A quadratic trigonometric equation will give two possible solutions. In some cases, one of them will be impossible.

EXAMPLE 3

Solve $2\sin^2(2B) + 7\sin(2B) - 4 = 0$.

Solution

Use $x = \sin(2B)$. $2x^2 + 7x - 4 = 0$

Solve for x. $(2x - 1)(x + 4) = 0$

$x = \frac{1}{2}$ or $x = -4$

Replace $x = \sin(2B)$. $\sin(2B) = \frac{1}{2}$ or $\sin(2B) = -4$

Discard the false solution. But $-1 \leq \sin(2B) \leq 1$, so $\sin(2B) = \frac{1}{2}$ only.

Write the solution in the principal domain. $2B = \frac{\pi}{6}$ for $-\frac{\pi}{2} \leq 2B \leq \frac{\pi}{2}$

Write the general solution and simplify. $2B = n\pi + (-1)^n \times \frac{\pi}{6} = \frac{6n\pi + (-1)^n \pi}{6}$

Divide by 2. $B = \frac{\left[6n + (-1)^n\right]\pi}{12}$

Exercise 9.01 General solution of trigonometric equations

Example 1

1 Solve each equation for $x \in \boldsymbol{R}$.

a $\sin(x) = \frac{\sqrt{3}}{2}$ **b** $\cos(x) = \frac{1}{\sqrt{2}}$ **c** $\tan(x) = \frac{1}{\sqrt{3}}$

d $\cos(x) = -\frac{\sqrt{3}}{2}$ **e** $\tan(x) = -\frac{4}{3}$ **f** $\sin(x) = \frac{1}{4}$

g $\tan(x) = \frac{5}{7}$ **h** $\cos(x) = 0.58$

2 Solve each equation.

a $2\cos(x) = -2$ **b** $\sin(x) - 1 = 0$ **c** $1 - \tan(x) = 0$

d $6\cos(x) - 3 = 3$ **e** $10\cos(x) + 5 = -5$ **f** $\cos(x) + 8 = 8$

g $3\tan(x) + 3 = 0$ **h** $2\sin(x) - 1 = 0$ **i** $6 - 4\cos(x) = 4$

j $2\cos(x) - 1 = 0$ **k** $1 + 2\sin(x) = 0$

ISBN 9780170412995

3 Solve for each equation for $x \in R$.

a $3 \sin(x) = 1$ **b** $4 \cos(x) + 1 = 0$ **c** $5 \tan(x) - 2 = 0$

d $9 \cos(x) = 5$ **e** $7 \sin(x) - 3 = 0$ **f** $3 \tan(x) + 20 = 0$

Example 2

4 Find all values of x for which:

a $\sin(2x) = 0.5$ **b** $\cos(2x) = -1$ **c** $\sin(2x) = 1$

d $\sin(2x) = -0.5$ **e** $\sin(2x) = \sqrt{3}\cos(2x)$ **f** $\sin(2x) = \cos(2x)$

5 Solve for each equation for $x \in R$.

a $\sin(2x) = 1$ **b** $\tan(3x) = 1$ **c** $\sin(2x) = -\frac{1}{4}$

d $\cos(2x) = 0.45$ **e** $\tan(4x) = 2.873$

Example 3

6 Find general solutions for each equation.

a $2\sin^2(3m) + \sin(3m) = 0$ **b** $\cos^2(5q) - \cos(5q) = 0$

c $\tan^2(4t) = \tan(4t)$ **d** $2\sin^2(6t) + 3\sin(6t) - 2 = 0$

e $2\tan^2(2t) - \tan(2t) = 0$ **f** $\cos^2(2t) - 1 = 0$

7 Solve each equation.

a $2\sin^2(3\alpha) + 3\sin(3\alpha) - 5 = 0$ **b** $2\cos^2(4\alpha) + 5\cos(4\alpha) - 3 = 0$

c $2\tan^2(5\alpha) - \tan(5\alpha) - 3 = 0$ **d** $4\sin^2(2\alpha) - \sin(2\alpha) - 3 = 0$

e $6\sin^2(3\alpha) + \sin(3\alpha) - 1 = 0$

Problem solving

8 The depth of water in metres in a deep water marina is modelled by

$$d = 2.8\cos\left(\frac{\pi t}{6}\right) + 11.6$$

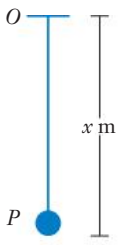
where t is the number of hours since high tide.

At what times over the next 24 hours is the depth 13 m?

iStock/JRstock

ISBN 9780170412995

9 A particle P bobs up and down on the end of an elastic string that is fixed at O. Its distance below O is given by the rule



$$x = 0.34 + 0.04 \sin(4t)$$

where x is in metres and t is the time in seconds since observation began.

a How far below O is the particle initially?

b How far below O is the particle at $t = 0.6$?

c What are the distance below O of the lowest point in the particle's path, and the value of t when the particle first reaches this point?

10 Find, correct to 2 decimal places, the two values of x closest to 0 for which

$$5 \sin\left(2x - \frac{\pi}{6}\right) - 2 = 0.$$

11 The depth of a tidal pool on a particular day could be modelled by the function

$$h(t) = 3 \sin\left(\frac{\pi t}{6}\right) + 5$$

where h is the depth in metres and t gives the hours since midnight.

a What was the depth at midnight?

b What was the depth at 6:00 a.m.?

c What was the earliest time at which the depth was 2 m?

d At what times was the pool at maximum depth?

9.02 General trigonometric graphs

Sketching periodic functions

Sketching periodic functions – amplitude and period

Sketching periodic functions – phase and vertical shift

In Mathematical Methods, you have seen how altering the values of a, b, c and d affects the graph shapes of

$$y = a \sin(bx + c) + d,$$

$$y = a \cos(bx + c) + d$$

and $y = a \tan(bx + c) + d$.

Features of sine and cosine functions

For the functions $y = a \sin\left[b\left(x + \frac{c}{b}\right)\right] + d$ and $y = a \cos\left[b\left(x + \frac{c}{b}\right)\right] + d$

- the amplitude is the magnitude of a
- the period is $\frac{2\pi}{b}$
- the phase shift is $\frac{c}{b}$ and it is the horizontal translation
- the average (mean) value is d.

ISBN 9780170412995

The average value or equilibrium value is also halfway between the maximum and minimum values of the function.

Features of tangent functions

For the function $y = a \tan\left[b\left(x + \frac{c}{b}\right)\right] + d$

- the period is $\frac{\pi}{b}$
- the phase shift is $\frac{c}{b}$ and it is the horizontal translation
- the average (mean) value is d.

It is also possible to sketch the graphs of trigonometric functions by identifying key points or by transforming the basic functions $y = \sin(x)$, $y = \cos(x)$ or $y = \tan(x)$.

Graphing trigonometric functions

Method 1: Compare the given equation with the equation of the relevant basic trigonometric function and apply the identified transformations.

Method 2: Identify the key points including:

- starting and end points
- zeros
- end of a cycle
- maxima and minima (for sine and cosine functions)
- vertical asymptotes (for a tangent function).

EXAMPLE 4

Sketch the graph of:

a $y = 4 \sin\left(2x + \frac{\pi}{6}\right) - 3$ **b** $y = 1 - 2 \cos\left(3x - \frac{\pi}{12}\right)$

Solution

a **Method 1**

Rewrite the equation. $y = 4 \sin\left[2\left(x + \frac{\pi}{12}\right)\right] - 3$

ISBN 9780170412995

First consider how the graph of $y = \sin(x)$ can be transformed into the graph of the simplified function $y = 4\sin(2x)$.

It has the basic shape of the sine function, but is compressed by a horizontal factor of 2, so the period is π instead of 2π.

It is stretched vertically by a factor of 4, so the amplitude is 4.

The graph of $y = 4\sin(2x)$ is shown in orange next page.

Now consider the translations to obtain the graph of

$y = 4\sin 2\left(x + \frac{\pi}{12}\right) - 3$

It is translated to the left by a phase shift of $\frac{\pi}{12}$ (compared with $y = 4\sin(2x)$).

It is translated 3 units downwards, so the mean value is -3.

The final graph is shown in blue next page.

Method 2

The graph $y = \sin(x)$ begins its 'first' cycle with a point at the mean value.

Find (x, y) when we get $\sin(0) = 0$ to find the start of the 'first' cycle. This is the 'first' time the graph is at the mean value.

Find x: $2x + \frac{\pi}{6} = 0 \Rightarrow x = -\frac{\pi}{12}$

Find y: $y = 4\sin(0) - 3 = -3$

The 'first' cycle starts at $\left(-\frac{\pi}{12}, -3\right)$.

Find (x, y) when we get $\sin\left(\frac{\pi}{2}\right) = 1$ to find the 'first' maximum.

Find x: $2x + \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}$

Find y: $y = 4\sin\left(\frac{\pi}{2}\right) - 3 = 1$

The 'first' maximum is at $\left(\frac{\pi}{6}, 1\right)$.

Find (x, y) so we get $\sin(\pi) = 0$. This is the 'second' time the graph is at the mean value.

Find x: $2x + \frac{\pi}{6} = \pi \Rightarrow x = \frac{5\pi}{12}$

Find y: $y = 4\sin(\pi) - 3 = -3$

The 'second' mean value point is at $\left(\frac{5\pi}{12}, -3\right)$.

Find (x, y) when we get $\sin\left(\frac{3\pi}{2}\right) = -1$ to find the 'first' minimum.

Find x: $2x + \frac{\pi}{6} = \frac{3\pi}{2} \Rightarrow x = \frac{2\pi}{3}$

Find y: $y = 4\sin\left(\frac{3\pi}{2}\right) - 3 = -7$

The 'first' minimum is at $\left(\frac{2\pi}{3}, -7\right)$.

ISBN 9780170412995

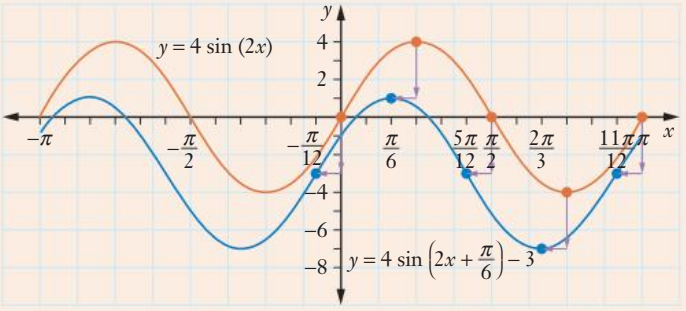
Find (x, y) so we get $\sin(2\pi) = 0$. This is the 'third' time the graph is at the mean value.

Find x: $2x + \frac{\pi}{6} = 2\pi \Rightarrow x = \frac{11\pi}{12}$

Find y: $y = 4\sin(\pi) - 3 = -3$

The 'second' mean value point is at $\left(\frac{11\pi}{12}, -3\right)$.

The five points are shown on the 'first' cycle of the graph below in blue. The cycle is replicated to extend the graph.



b **Method 1**

Rewrite the equation.

$$y = -2\cos\left[3\left(x - \frac{\pi}{36}\right)\right] + 1$$

First consider how the graph of $y = \cos(x)$ can be transformed into the graph of the simplified function $y = -2\cos(3x)$.

It has the basic shape of the cosine function, but is compressed by a horizontal factor of 3, so the period is $\frac{2\pi}{3}$ instead of 2π.

It is stretched vertically by a factor of 2, so the amplitude is 2. The graph is inverted compared with the graph of $y = \cos(x)$ because of the negative sign.

The graph of $y = -2\cos(3x)$ is shown in orange on p. 340.

Now consider the translations to obtain the graph of $y = 1 - 2\cos 3\left(x - \frac{\pi}{36}\right)$

It is translated to the left by a phase shift of $\frac{\pi}{36}$ (compared with $y = -2\cos(3x)$).

It is translated 1 unit upwards, so the mean value is 1.

The final graph is shown in blue on p. 340.

ISBN 9780170412995

Method 2

The graph is inverted compared with the graph of $y = \cos(x)$, so the 'first' cycle begins with a minimum, not a maximum.

Find (x, y) when we get $\cos(0) = 1$ to find the start of the 'first' cycle. This is the 'first' minimum.

Find x: $3x - \frac{\pi}{12} = 0 \Rightarrow x = \frac{\pi}{36}$

Find y: $y = 1 - 2\cos(0) = -1$

The 'first' cycle starts at $\left(\frac{\pi}{36}, -1\right)$.

Find (x, y) when we get $\cos\left(\frac{\pi}{2}\right) = 0$ to find the 'first' time the graph is at the mean value.

Find x: $3x - \frac{\pi}{12} = \frac{\pi}{2} \Rightarrow x = \frac{7\pi}{36}$

Find y: $y = 1 - 2\cos\left(\frac{\pi}{2}\right) = 1$

The 'first' cycle starts at $\left(\frac{7\pi}{36}, 1\right)$.

Find (x, y) so we get $\cos(\pi) = -1$. This is the 'first' maximum.

Find x: $3x - \frac{\pi}{12} = \pi \Rightarrow x = \frac{13\pi}{36}$

Find y: $y = 1 - 2\cos(\pi) = 3$

The 'first' maximum is at $\left(\frac{13\pi}{36}, 3\right)$.

Find (x, y) when we get $\cos\left(\frac{3\pi}{2}\right) = 0$ to find the 'second' time the graph is at the mean value.

Find x: $3x - \frac{\pi}{12} = \frac{3\pi}{2} \Rightarrow x = \frac{19\pi}{36}$

Find y: $y = 1 - 2\cos\left(\frac{3\pi}{2}\right) = 1$

The 'second' mean point is at $\left(\frac{19\pi}{36}, 1\right)$.

Find (x, y) so we get $\cos(2\pi) = 1$. This is the 'second' minimum.

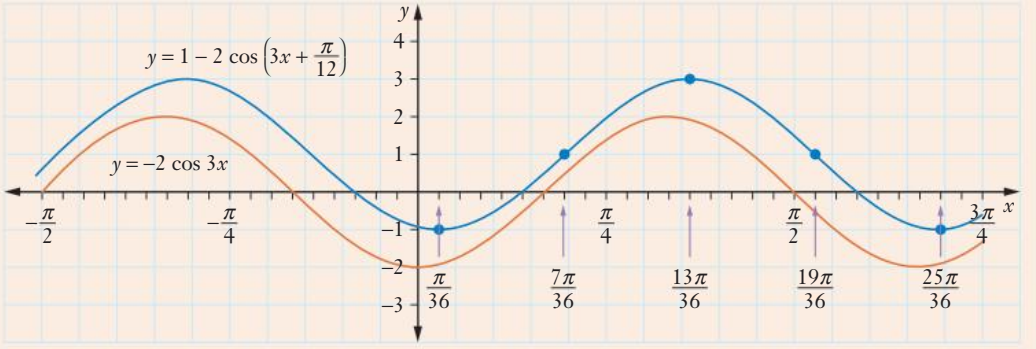
Find x: $3x - \frac{\pi}{12} = 2\pi \Rightarrow x = \frac{25\pi}{36}$

Find y: $y = 1 - 2\cos(2\pi) = -1$

The 'second' minimum is at $\left(\frac{25\pi}{36}, -1\right)$.

ISBN 9780170412995

The five points are shown on the 'first' cycle of the graph below in blue. The cycle is replicated to extend the graph.



EXAMPLE 5

Draw the graph of:

a $y = -3\tan(x) + 4$ for $-\pi \le x \le \pi$ **b** $y = 5\tan\left(3x + \frac{\pi}{4}\right)$ for one full cycle.

Solution

a State the key features of the graph.

Compared with the graph of $y = \tan(x)$, there is a vertical dilation by a factor of 3 and reflection in the x-axis.

The graph has been translated 4 units vertically upwards.

The transformations do not alter the period or asymptotes.

The x-intercept = (4, 0).

Period = π

Asymptotes occur at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

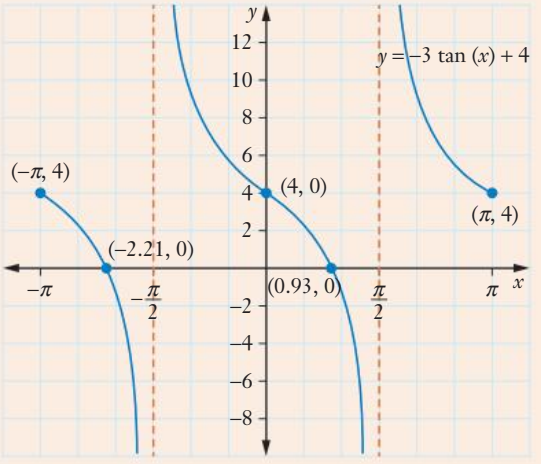
Solve $-3\tan(x) + 4 = 0$ to find the zeros.

$-3\tan(x) + 4 = 0 \Rightarrow x = \tan^{-1}\left(\frac{4}{3}\right) \approx n\pi + 0.93$

Zeros occur at (−2.21, 0) and (0.93, 0).

ISBN 9780170412995

Use the information to sketch the graph. Mark the included boundary values.



b Rewrite the equation.

$$y = 5\tan\left(3x + \frac{\pi}{4}\right)$$

$$y = 5\tan\left[3\left(x + \frac{\pi}{12}\right)\right] + 0$$

State the key features of the graph.

$a = 5$, so the graph has been dilated vertically by a factor of 5

$b = 3$, so the period is $\frac{\pi}{3}$

$d = 0$, so there is no change to the mean value.

Phase shift $= -\frac{c}{b} = -\frac{\pi}{12}$

The x-intercept $(0, 0)$ has been subjected to the phase shift.

x-intercept value $= 0 + \left(-\frac{\pi}{12}\right) = -\frac{\pi}{12}$

The x-intercept is midway between two asymptotes. The asymptotes are separated by one period, i.e., $\frac{\pi}{3}$. So the asymptotes are half a period $\left(\frac{\pi}{6}\right)$ on either side of the x-intercept.

Equation of asymptotes:

$$x = -\frac{\pi}{12} \pm \frac{\pi}{6}$$

$$x = -\frac{\pi}{12} + \frac{\pi}{6} \text{ or } x = -\frac{\pi}{12} - \frac{\pi}{6}$$

$$x = \frac{\pi}{12} \text{ or } x = -\frac{3\pi}{12} = -\frac{\pi}{4}$$

ISBN 9780170412995

To help draw the graph, calculate the coordinates of a point on either side of the x-intercept.

Let $x = -\frac{\pi}{6}$.

$$y = 5 \tan\left(-\frac{3\pi}{6} + \frac{\pi}{4}\right)$$
$$= 5 \tan\left(-\frac{\pi}{4}\right)$$
$$= -5$$

State the result.

The graph passes through $\left(-\frac{\pi}{6}, -5\right)$.

Let $x = 0$.

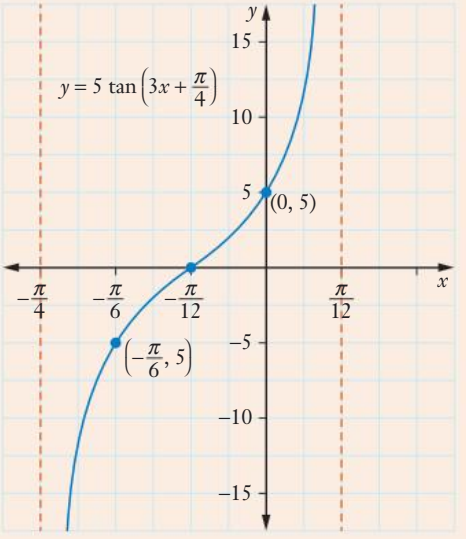
$$y = 5 \tan\left(3 \times 0 + \frac{\pi}{4}\right)$$
$$= 5 \tan\left(\frac{\pi}{4}\right)$$
$$= 5$$

State the result.

The graph passes through (0, 5).

Use the information to draw the graph.

The graph may be extended by replicating the cycle to the right or the left.



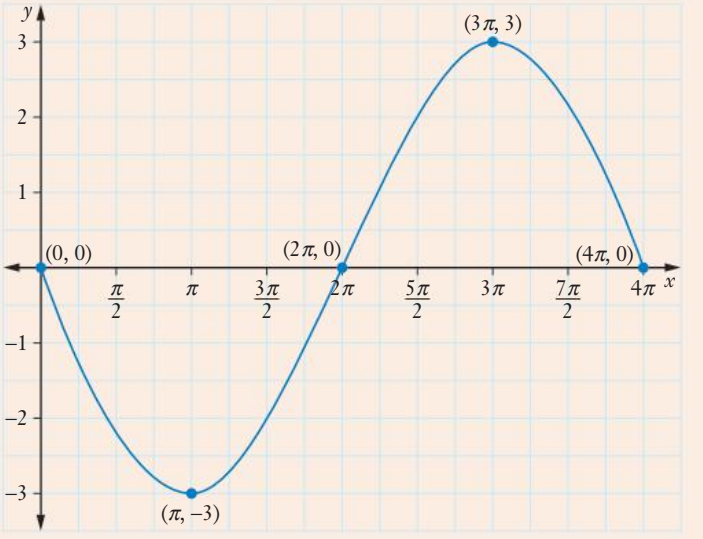
You can use your knowledge of the transformations represented by a, b, c and d with the functions $y = a \sin [b(x + c)] + d$ and $y = a \cos [b(x + c)] + d$ to form an equation for the graph of an unnamed trig function.

ISBN 9780170412995

The graph of $y = \sin\left(x + \frac{\pi}{2}\right)$ is the same as the graph of $y = \cos(x)$ because sine and cosine have a phase difference of $\frac{\pi}{2}$. This means that it is possible for the equation of the graph to be expressed in terms of either function. This is true for all sine and cosine graphs. It is usually simpler to choose the form of the trig function that does not involve a phase shift.

EXAMPLE 6

Determine 2 possible equations for the following graph.



Solution

Identify the key features of the graph.	The graph has a period of 4π, amplitude 3, and the mean position is $y = 0$.
Look at the shape of the graph, including mean values, maxima and minima.	The graph could be an inverted sine graph.
Write an equation for a sine function that does not involve a phase shift.	$y = a \sin(bx) + d$
Use the key features identified to find values for a, b and d.	$a = -3$ and $d = 0$ Period $= 4\pi$ $\frac{2\pi}{b} = 4\pi \Rightarrow b = \frac{1}{2}$

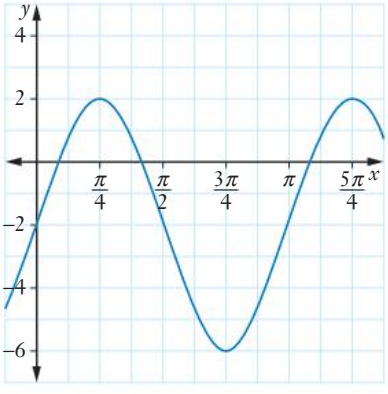
ISBN 9780170412995

Use this information to write the equation.	$y = -3 \sin\left(\frac{1}{2}x\right)$
The features of the graph suggest that it could also be a cosine function that has not been inverted, but that has been translated π units horizontally to the left.	The graph has a period of 4π, amplitude 3, and the mean position is $y = 0$.
Write an equation for a cosine function that does involve a phase shift.	$y = a \cos(bx + c) + d$
Use the key features identified to find values for a, b and d.	The amplitude is positive: $a = 3$. The period is 4π, so $b = \frac{1}{2}$. The mean value = 0, so $d = 0$.
Calculate the phase shift.	$c = \pi$
Use this information to write the equation.	$y = 3 \cos\left[\frac{1}{2}(x + \pi)\right]$
State the result.	Two possible equations for this graph are: $y = -3 \sin\left(\frac{1}{2}x\right)$ or $y = 3 \cos\left(\frac{1}{2}x - \pi\right)$.

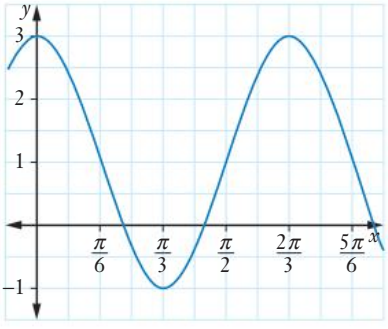
Exercise 9.02 General trigonometric graphs

1 State the amplitude and period of each graph.

a

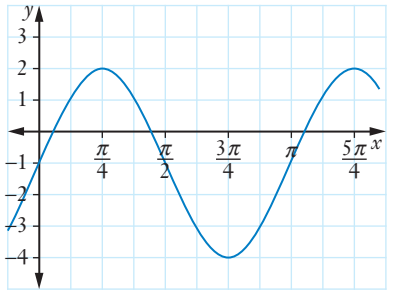


b

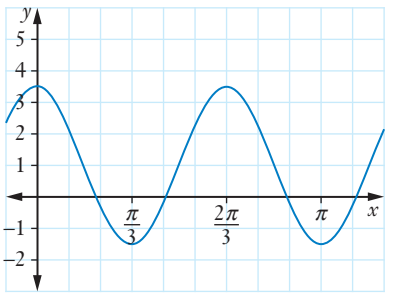


ISBN 9780170412995

c

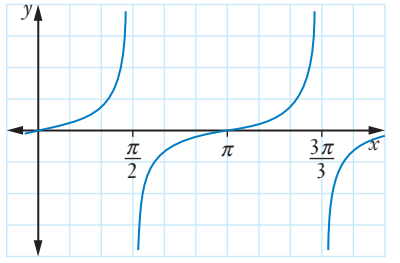


d

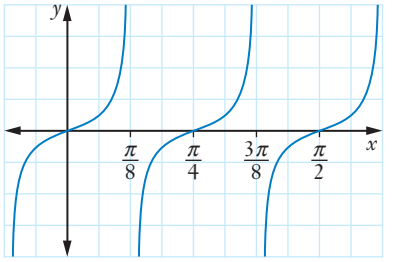


2 State the period of each graph.

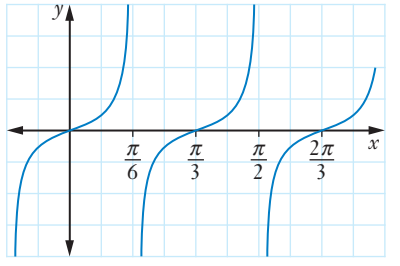
a



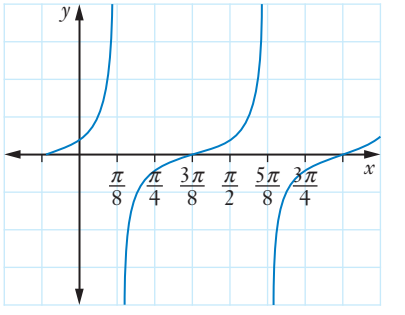
b



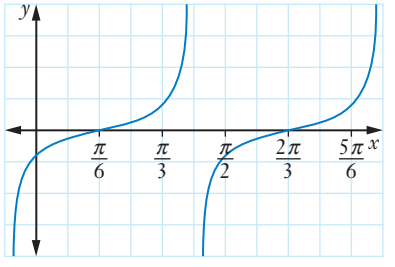
c



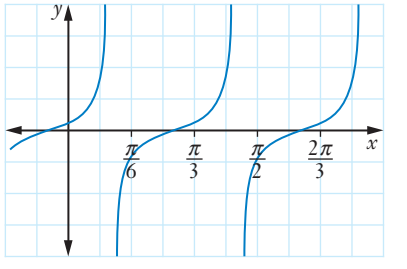
d



e



f



ISBN 9780170412995

3 State the amplitude, period and phase shift of each function.

a $y = -2\cos(3x) - 5$

b $y = 3\sin\left(\frac{1}{2}x\right) + 4$

c $y = 4\cos\left(x - \frac{\pi}{2}\right) - 2$

d $y = 5\sin\left[\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right] + 3$

e $y = -2.4\sin\left(2x - \frac{\pi}{2}\right) + 7$

f $y = 4.7\cos\left[3\left(x + \frac{\pi}{4}\right)\right] - 5$

4 State the period of each function.

a $y = \tan(2x)$

b $y = 5\tan\left(\frac{x}{4}\right) + 2$

c $y = 6\tan\left(\frac{x}{3}\right)$

d $y = -3\tan\left(\frac{3x}{2} - \frac{\pi}{3}\right)$

e $y = -4\tan(\pi x) - 3$

f $y = -2\tan\left(\frac{x}{6} + \frac{\pi}{5}\right)$

Example 4

5 Sketch the graph of each function, showing a full cycle of each. Check your answer using a graphics calculator.

a $y = 4\sin\left[2\left(x - \frac{\pi}{3}\right)\right] + 2$

b $y = 2\cos\left[3\left(x + \frac{\pi}{6}\right)\right] - 3$

c $y = 5\sin\left[4\left(x + \frac{\pi}{8}\right)\right] - 2$

d $y = 3\cos\left[2\left(x - \frac{\pi}{4}\right)\right] + 4$

e $y = 4 - 2\cos\left[4\left(x + \frac{\pi}{3}\right)\right]$

f $y = 3 - 3\sin\left[2\left(x - \frac{\pi}{4}\right)\right]$

Example 5

6 Draw the graph of each function.

a $y = \tan\left(x - \frac{\pi}{3}\right)$ for $0 \le x \le 2\pi$

b $y = \tan\left(x + \frac{\pi}{2}\right)$ for $0 \le x \le 2\pi$

c $y = -\tan\left(x + \frac{\pi}{6}\right)$ for $-\pi \le x \le \pi$

d $y = \tan\left(x - \frac{2\pi}{3}\right)$ for $-\pi \le x \le \pi$

7 Sketch the graph of each function, showing a full cycle of each. Check your answer using a graphics calculator.

a $y = 2\sin\left(2x - \frac{\pi}{3}\right) + 3$

b $y = 3\cos(4x + \pi) - 2$

c $y = 4 - 3\sin\left(6x + \frac{3\pi}{2}\right)$

d $y = 5\cos\left(2x - \frac{3\pi}{4}\right) - 3$

e $y = 3 - 5\cos\left(3x - \frac{\pi}{3}\right)$

f $y = 4 - 3\sin\left(4x + \frac{\pi}{4}\right)$

8 Draw the graph of each function.

a $y = 2\tan\left(3x - \frac{\pi}{4}\right)$

b $y = 3\tan\left(2x + \frac{\pi}{3}\right)$

c $y = -2\tan\left(4x + \frac{\pi}{3}\right) + 4$

d $y = \frac{1}{2}\tan\left(2x + \frac{\pi}{6}\right) - 3$

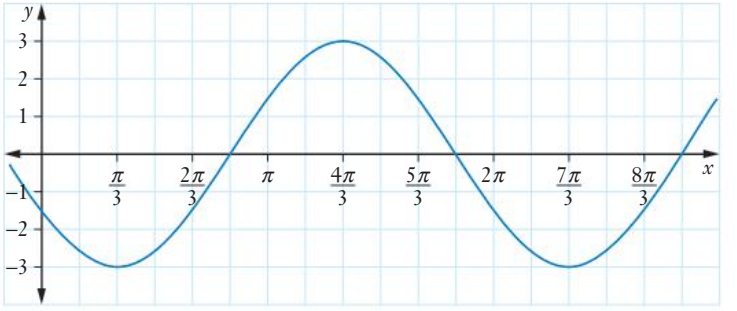
e $y = 2\tan\left(\frac{1}{4}x + \frac{\pi}{2}\right) - 5$

f $y = \frac{1}{3}\tan(2x + \pi) + 1$

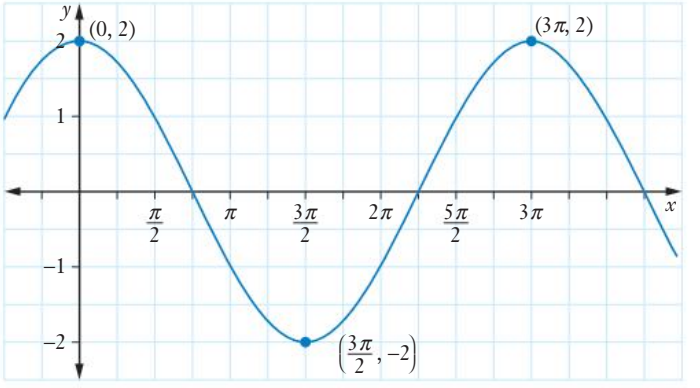
ISBN 9780170412995

9 Determine two possible equations for this graph – one using the cosine function and one using the sine function.

Example 6



10 Determine two possible equations for this graph – one using the cosine function and one using the sine function.



Problem solving

11 The depth of water, $d(t)$ m, at a pier changes according to the rule $d(t) = 5 + 2\cos\left(\frac{\pi}{6}t\right)$, where t is time in hours after high tide. On 18 March, high tide was at 7:30 a.m.

- **a** Sketch a diagram showing the depth of water for the 24 hours after 7:30 a.m. on 18 March.
- **b** When was the first low tide on that day?
- **c** When was the water at a depth of 6 m?
- **d** How long is it between low tide and high tide?

12 The sales, S in 100s of units, of a seasonal product are modelled by

$$S = 54.8 + 32.5\cos\left(\frac{\pi t}{6}\right)$$

where t is the time in months ($t = 1$ is January and $t = 12$ is December).

- **a** Draw a graph of the sales for a period of 12 months.
- **b** Use the graph to determine the months for which sales exceed 6800 units.

ISBN 9780170412995

13 In an unusual meteorological investigation, the temperature, T °C, in a town in central Queensland was found to fluctuate approximately according to the rule $T = 25 + 6 \sin(0.1\pi t)$, where t is the number of hours after 10:00 a.m.

Getty Images/Peter Harrison

a Sketch a graph of the temperature fluctuations for a sufficient number of hours to be able to determine the maximum and minimum temperatures for that day and the next night.

b Use the graph to determine the maximum and minimum temperatures.

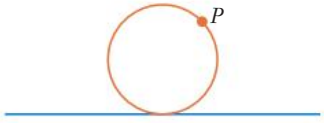
c When did they occur?

d At what time was the temperature:

i 27°C? **ii** 20°C?

e Why was this unusual?

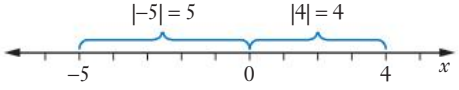
14 A large hoop is rolling along the ground. The vertical distance above the ground of a point P on the rim of the hoop is given by $y = 1.25 - 1.25 \cos(2\pi t)$, where y is in metres and t is in seconds. Find the first three times at which P is 2.00 m above the ground.



9.03 Absolute value functions

Absolute values

The **absolute value** (**modulus**) of a number $x \in \boldsymbol{R}$ is written as $|x|$. Geometrically, if x is the coordinate of a point on a real number line, then $|x|$ is the distance from the origin to x, as shown in the diagram below.



Symbolically, absolute value is defined as follows.

ISBN 9780170412995

Absolute value

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For example:

$$|7 - 11| = |-4| = 4$$

and

$$|5| \times |-9| = 5 \times 9 = 45$$

You can also use your calculator to evaluate expressions involving absolute values.

On the TI-84 Plus CE, press 2nd 0 ([catalog]) and select the absolute value option – **abs(**.

On the TI-Nspire CX, press the catalog button (📖) and select 1 in the upper menu bar. Press A (for 'absolute') and select the absolute value function – **abs(.**

On the Casio fx-CG20, from the run-matrix screen press F4 (MATH) and then press F3 (Abs) to select the absolute value option

The absolute value of x can also be written as:

$$|x| = \sqrt{x^2}$$

which works even if x is negative. For example, $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$.

Properties of absolute values

$$|-a| = |a|$$

$$|ab| = |a|\ |b|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

When solving equations involving absolute values, you need to use the definition to ensure that all solutions are found. For example:

$$|x - 3| = 7 \Rightarrow x - 3 = 7 \text{ or } -(x - 3) = 7$$

Solving each of these equations yields the solutions $x = 10$ or $x = -4$.

You should always check solutions back in the original equation.

In cases where both sides of an equation involve the absolute value of functions, when the equation $|f(x)| = |g(x)|$ is written without absolute value signs, all of the possibilities are:

1 $f(x) = g(x)$

2 $-f(x) = g(x)$

3 $f(x) = -g(x)$

4 $-f(x) = -g(x)$

ISBN 9780170412995

But equations 1 and 4 are the same and equations 2 and 3 are the same, so there are only two cases that need to be dealt with.

Equations of the form $|f(x)| = |g(x)| + a$ are more complex as you can't just 'move' the a into an absolute value expression.

EXAMPLE 7

Solve

a $|x+4| = 3x-8$ **b** $|3x-4| = |5x+6|$ **c** $|x-5| = |2x+6| - 3$

Solution

a Write the equation. $|x+4| = 3x-8$

Use the definition. $x+4 = 3x-8$ or $-(x+4) = 3x-8$

Solve the equations separately.

$12 = 2x$ or $-x-4 = 3x-8$

$x = 6$ $4 = 4x$

$x = 1$

Check the solutions.

If $x = 6$, LHS $= |6+4| = 10$

RHS $= 3 \times 6 - 8 = 10 =$ LHS

So $x = 6$ is a solution.

If $x = 1$, LHS $= |1+4| = 5$

RHS $= 3 \times 1 - 8 = -5 \neq$ LHS

So $x = 1$ is not a solution.

State the result. $x = 6$

b Write the equation. $|3x-4| = |5x+6|$

Use the definition. $3x-4 = 5x+6$ or $-(3x-4) = 5x+6$

Solve the equations separately.

$-10 = 2x$ or $-3x+4 = 5x+6$

$x = -5$ $-2 = 8x$

$x = -\frac{1}{4}$

ISBN 9780170412995

Check the solutions.

If $x = -5$, LHS $= |-15 - 4| = 19$

RHS $= |-25 + 6| = 19 =$ LHS

So $x = -5$ is a solution.

If $x = -\frac{1}{4}$, LHS $= \left|-\frac{3}{4} - 4\right| = \frac{19}{4}$

RHS $= \left|-\frac{5}{4} + 6\right| = \frac{19}{4} =$ LHS

State the result.

$x = -\frac{1}{4}$ or -5

c Write the equation.

$|x - 5| = |2x + 6| - 3$

In this case you need to deal with 4 possibilities. It helps if you also add the conditions for each absolute value expression.

$(x - 5) = (2x + 6) - 3 \quad [(x - 5) \geq 0$ and $(2x + 6) \geq 0]$

$(x - 5) = -(2x + 6) - 3 \quad [(x - 5) \geq 0$ and $(2x + 6) < 0]$

$-(x - 5) = (2x + 6) - 3 \quad [(x - 5) < 0$ and $(2x + 6) \geq 0]$

$-(x - 5) = -(2x + 6) - 3 \quad [(x - 5) < 0$ and $(2x + 6) < 0]$

Simplify each of the equations.

$x = -8 \quad [x \geq 5$ and $x \geq -3]$

$x = -\frac{4}{3} \quad [x \geq 5$ and $x < -3]$

$x = \frac{2}{3} \quad [x < 5$ and $x \geq -3]$

$x = -14 \quad [x < 5$ and $x < -3]$

Compare the solutions with the conditions for x.

$x = -8$ does not satisfy the condition $x \geq 5$ and $x \geq -3$

$x = -\frac{4}{3}$ does not satisfy the condition $x \geq 5$ and $x < -3$

$x = \frac{2}{3}$ satisfies the condition $x < 5$ and $x \geq -3$

$x = -14$ satisfies the condition $x < 5$ and $x < -3$

ISBN 9780170412995

Check the solutions in the equation.

If $x = \frac{2}{3}$, LHS $= \left|\frac{2}{3} - 5\right| = \frac{13}{3}$

RHS $= \frac{22}{3} - 3 = \frac{13}{3} =$ LHS

So $x = \frac{2}{3}$ is a solution.

If $x = -14$, LHS $= |-14 - 5| = 19$

RHS $= |-28 + 6| - 3 = 19 =$ LHS

State the result.

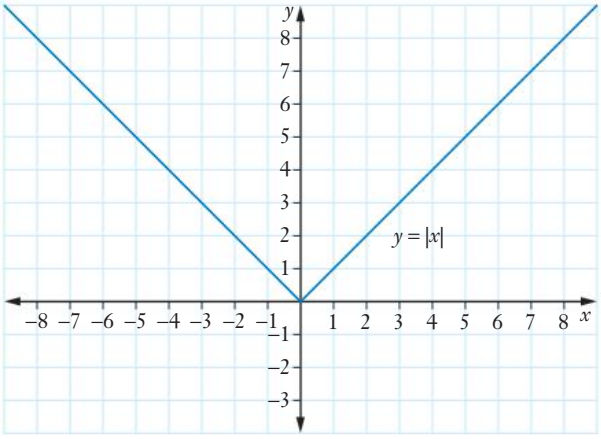
$x = -14, \frac{2}{3}$

Expressing absolute value as a function allows you to graph expressions involving absolute values. The **absolute value function** is $y = |x|$ and is defined as:

$$y = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

It is a type of **piecewise** function because it is defined in separate 'pieces'.

The graph of $y = |x|$ is shown here.



As for the trigonometric functions you dealt with in the previous sections of this chapter, the absolute value function can be written as:

$$y = a|x + c| + d$$

where the values of a, c and d will result in transforming the graph of $y = |x|$.

Features of the graph of $y = a|x + c| + d$

- a is the horizontal dilation where the scale factor is $\frac{1}{a}$. If $a > 0$, the graph is compressed and if $0 < a < 1$, the graph is dilated. If $a < 0$, the graph is reflected in the x-axis.
- c is a horizontal translation. If $c > 0$, the graph is translated to the left and if $c < 0$ the graph is translated to the right.
- d is a vertical translation. If $d > 0$, the graph is translated upwards and if $d < 0$, the graph is translated downwards.
- The graph of $y = a|x + c| + d$ has its vertex at $(-c, d)$.

ISBN 9780170412995

It is possible to draw the graphs of a range of algebraic and non-algebraic absolute value functions. When drawing a graph of an absolute value function, it is important to remember that $y = |f(x)| \geq 0$, so the graph will never be below the x-axis.

EXAMPLE 8

Draw a graph of each of the following absolute value functions and state the domain and range in each case.

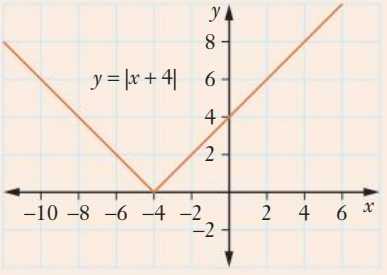
a $y = |x + 4|$ **b** $y = |x^2 - 2|$ **c** $y = |x(x - 1)(x + 1)|$

d $y = |\sin(x)|$ **e** $y = \left|\frac{1}{x}\right|$

Solution

a Comparing $y = |x + 4|$ with $y = a|x + c| + d$ shows that $c = 4$, so the graph of $y = |x|$ has been translated 4 units to the left.

Use this information to draw the graph.



State the domain and range.

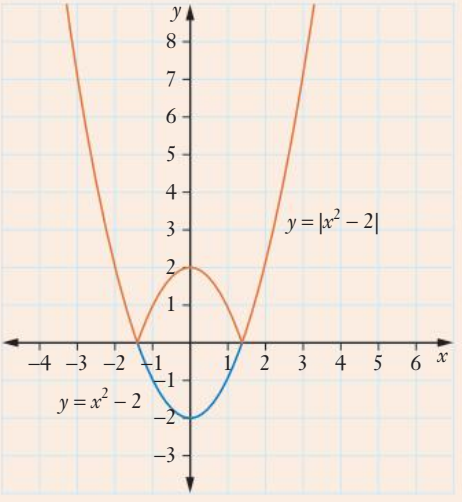
Domain: $x \in \boldsymbol{R}$, range: $y \geq 0$

b Draw the graph of $y = x^2 - 2$.

For the graph of $y = |x^2 - 2|$, $y \geq 0$.

The positive values of $y = x^2 - 2$ form part of the graph of $y = |x^2 - 2|$.

Reflect the negative y-values of $y = x^2 - 2$ in the x-axis to form the remainder of the graph of $y = |x^2 - 2|$.



State the domain.

Domain: $x \in \boldsymbol{R}$, range: $y \geq 0$

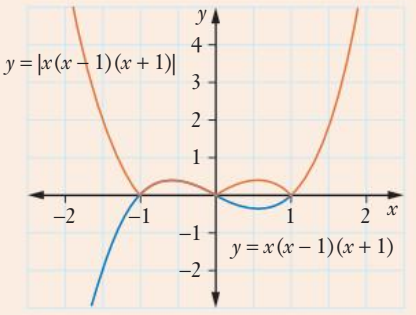
ISBN 9780170412995

c Draw the graph of $y = x(x-1)(x+1)$.

For the graph of $y = |x(x-1)(x+1)|, y \geq 0$.

The positive values of $y = x(x-1)(x+1)$ form part of the graph of $y = |x(x-1)(x+1)|$.

Reflect the negative y-values of $y = x(x-1)(x+1)$ in the x-axis to form the remainder of the graph of $y = |x(x-1)(x+1)|$.

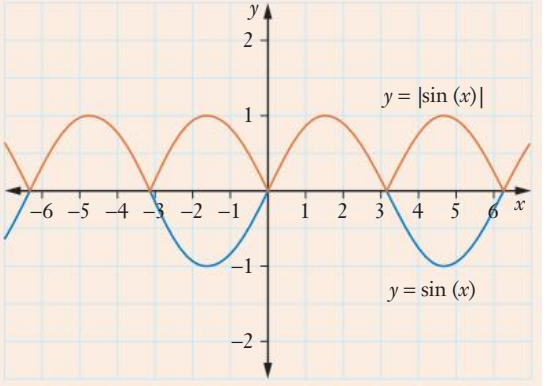


State the domain.

Domain: $x \in \boldsymbol{R}$, range: $y \geq 0$

d Draw the graph of $y = \sin(x)$.

Reflect the negative y-values of $y = \sin(x)$ in the x-axis to form the remainder of the graph of $y = |\sin(x)|$.

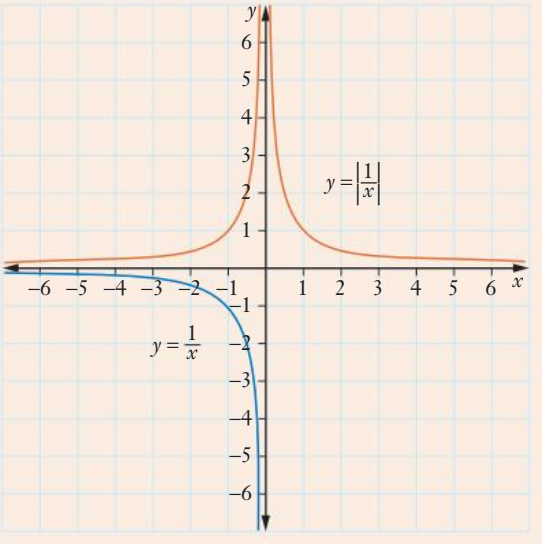


State the domain.

Domain: $x \in \boldsymbol{R}$, range: $0 \leq y \leq 1$

e Draw the graph of $y = \frac{1}{x}$.

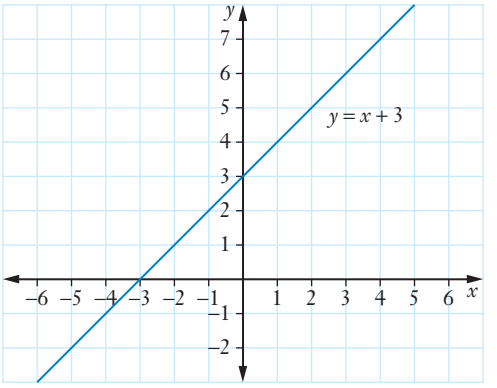
Reflect the negative y-values of $y = \frac{1}{x}$ in the x-axis to form the remainder of the graph of $y = \left|\frac{1}{x}\right|$.

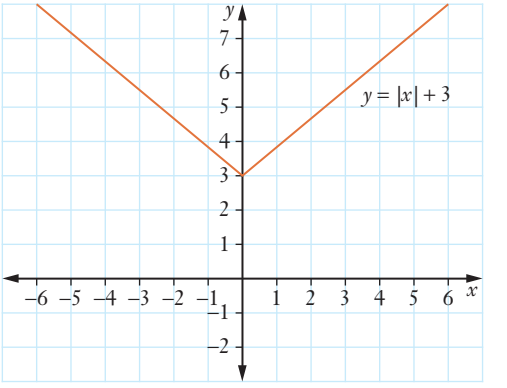


Domain: $x \in \boldsymbol{R}$, range: $y > 0$

ISBN 9780170412995

Consider the graphs of $y = x + 3$ and $y = |x| + 3$ shown below.





You can use the graph of $y = x + 3$ to draw the graph of $y = |x| + 3$.

For $x \geq 0$, the y-values of $y = x + 3$ are the same as the y-values for $y = |x| + 3$.

The value of $|x|$ is always positive, so the y-values of $y = x + 3$, where $x \geq 0$, are reflected in the y-axis to give the y-values of $y = |x| + 3$ when $x < 0$.

You can use this process to draw a range of graphs of the form $y = f(|x|)$.

EXAMPLE 9

Sketch the graph of each function.

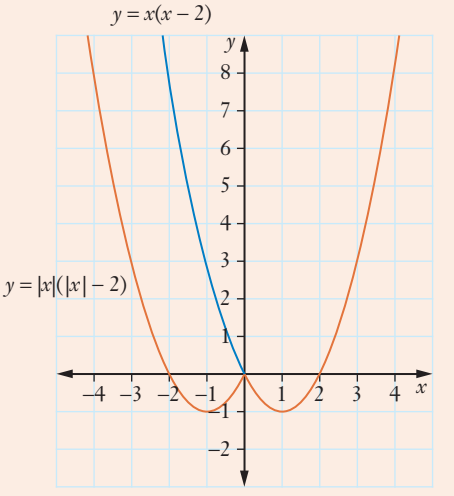
a $y = |x|(|x| - 2)$ **b** $y = e^{|x|}$

Solution

a Sketch the graph of $y = x(x - 2)$.

The y-values of $y = x(x - 2)$ for $x \geq 0$ are the y-values for $y = |x|(|x| - 2)$ for $x \geq 0$.

Reflect the y-values of $y = x(x - 2)$ for $x \geq 0$ in the y-axis to obtain the y-values of $y = |x|(|x| - 2)$ for $x < 0$.

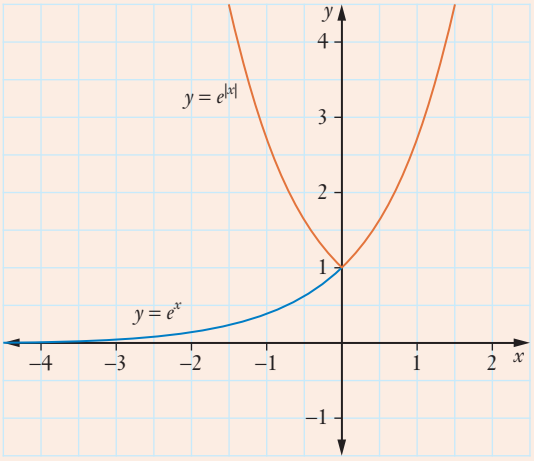


ISBN 9780170412995

b Sketch the graph of $y = e^x$.

The y-values of $y = e^x$ for $x \geq 0$ are the y-values for $y = e^{|x|}$ for $x \geq 0$.

Reflect the y-values of $y = e^x$ for $x \geq 0$ in the y-axis to obtain the y-values of $y = e^{|x|}$ for $x < 0$.



Exercise 9.03 Absolute value functions

1 Evaluate each expression.

a $|-7|$
b $|-6+4|$
c $|-5|-8|-2|-9$
d $|5\times(-4)|$
e $-|5|^2$
f $|-6\times 2|-3|5-7|$
g $|3\times(-5)|+3|-4-6|$
h $-|4\times(-3)|-2|9-12|$
i $2|3\times(-4)|-|-6+8|$

Example 7

2 Solve:

a $|x| = 5$
b $|3x+5| = 4$
c $|x+4| = 1$
d $|7-3x| = 2$
e $|2x-1| = 3$
f $|6-5x| = 16$
g $|6+5x| = 14$
h $|9x-4| = 86$
i $|3+7x| = 73$

3 Solve:

a $3|2x-3|+2=3$
b $|3x-3|+2=1$
c $-8|-x-6|+10=2$
d $8|x+7|-3=5$
e $|6x-2|+10=44$
f $7|10x-2|-9=5$
g $-5|-3x+2|-2=-2$
h $-3|x-3|+3=6$
i $2|3x-4|=7$

4 Solve:

a $|x-2| = x+3$
b $|x+5| = -6x+3$
c $|x+3| = 2(2x-1)$
d $|3-2x| = x+1$
e $|2x-7| = 2-3x$
f $|3x+1| = 2+x$

5 Solve:

a $|x|-3 = |x+3|$
b $|3x-4| = |2x+3|$
c $|5x+3| = |2x-1|$
d $|2+3x| = |4-2x|$
e $|2x-5| = |3x+4|$
f $|4x-2| = |6x+3|$
g $|3x+2| = |2x-3|$

ISBN 9780170412995

6 Solve:

a $|6x-7| = |2x+3| + 1$ **b** $|6-2x| = |x-3| + 2$

c $|2x+1| = |3x-5| + 1$ **d** $|4x-3| = -|2x+1| + 4$

e $|2x+7| - 8 = |6-3x|$ **f** $|3x+1| - 11 = -|3-2x|$

7 Graph each function and state its domain and range.

Example 8

a $y = |x| + 5$ **b** $y = |x| - 4$ **c** $y = |x-3|$

d $y = -2|x| + 3$ **e** $y = 3|x+1|$ **f** $y = |x| - 2$

g $y = -4|x+3|$ **h** $y = -\frac{1}{3}|x| + 2$

8 Graph each function and state its domain and range.

a $y = 2|x+4| + 3$ **b** $y = \frac{1}{2}|x-5| + 4$ **c** $y = -2|x+4| - 3$

9 Graph each function and state its domain and range.

a $y = |x^2 - 4|$ **b** $y = |x(x-2)(x+2)|$ **c** $y = |\cos(x)|$

d $y = |e^x|$ **e** $y = |\log_e(x)|$ **f** $y = \left|\frac{1}{x-1}\right|$

10 Graph each function.

Example 9

a $y = 3|x| - 2$ **b** $y = |x|^2 - 4$ **c** $y = \log_e|x|$

d $y = |x|(|x| - 4)$ **e** $y = 5 - |x|^2$ **f** $y = -\frac{1}{|x|}$

Problem solving

11 Prove:

a $|-a| = |a|$ **b** $|ab| = |a||b|$.

12 A worker is instructed to place a satellite dish on the sloping roof of a warehouse. The recommended height of the satellite dish is 10 m. The warehouse is 32 m wide, and the height of the roof can be described by the function $h(x) = -\frac{3}{2}|x - 16| + 4$, where x is the distance along the width of the house. Where should the worker place the dish?

iStock/IPGGutenbergUKLtd

13 A circle with centre (3, 0) and a radius of 7 is drawn on the Cartesian plane. Use an absolute value equation and the fact that all points on a circle are the same distance from the centre to find the points where the circle crosses the x-axis.

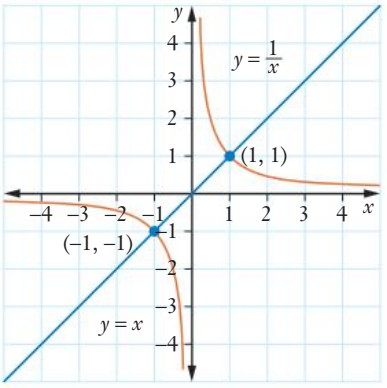
14 For which values of x is $\sqrt{(x^2-7x+12)^2} = x^2-7x+12$?

15 Solve $|x-5|^2 - 4|x-5| = 12$ for x.

16 By first sketching a graph, solve $|x^2-9| > 5$.

9.04 Reciprocal functions

For the function $y = f(x)$, the function $y = \dfrac{1}{f(x)}$ is called the **reciprocal function** of $y = f(x)$.



Consider the function $y = x$ and its reciprocal function $y = \dfrac{1}{x}$. These functions are graphed below on the same set of axes on the right.

Comparing the two graphs you can see that:

- When $y = x$ is 0, $y = \dfrac{1}{x}$ has a vertical asymptote.
- The points where $x = 1$ or -1 are common to both $y = x$ and $y = \dfrac{1}{x}$
- As $x \to \pm\infty$, $y = \dfrac{1}{x} \to 0$ and as $x \to 0$, $y = \dfrac{1}{x} \to \pm\infty$.

The points $(-1, -1)$ and $(1, 1)$ are common to both the function and the reciprocal function and are known as **invariant points**. Invariant points for any function $y = f(x)$ and its reciprocal function $y = \dfrac{1}{f(x)}$ can be found by solving the equation $f(x) = \dfrac{1}{f(x)}$.

$$f(x) = \frac{1}{f(x)} \Rightarrow (f(x))^2 = 1 \Rightarrow f(x) = \pm 1$$

For $y = x$ and its reciprocal function, $f(x) = \pm 1$ at $x = \pm 1$, so the invariant points are $(1, 1)$ and $(-1, -1)$.

ISBN 9780170412995

Properties of reciprocal functions

For any function $y = f(x)$ and its reciprocal function $y = \frac{1}{f(x)}$:

- $f(x)$ and $\frac{1}{f(x)}$ have the same sign
- $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect where $f(x) = \pm 1$. The points of intersection are invariant points.
- Any zero of $y = f(x)$ corresponds to a vertical asymptote of $y = \frac{1}{f(x)}$.
- As $f(x)$ increases, $\frac{1}{f(x)}$ decreases and vice versa.
- As $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$ and as $f(x) \to 0$, $\frac{1}{f(x)} \to \pm\infty$.
- A local maximum for $y = f(x)$ corresponds to a local minimum for $y = \frac{1}{f(x)}$ and vice versa.

EXAMPLE 10

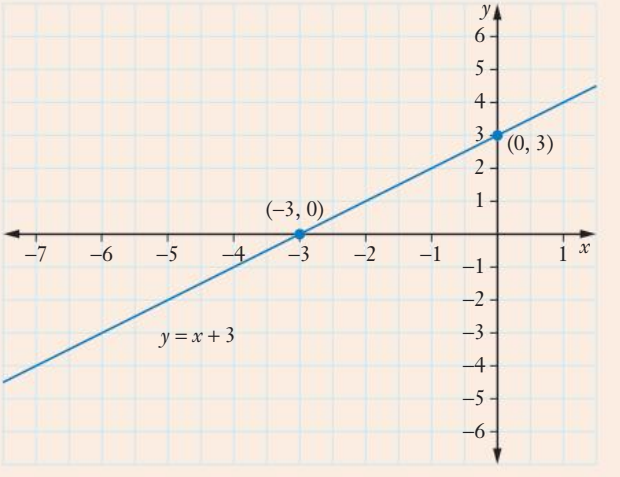
Draw the graphs of $y = x + 3$ and $y = \frac{1}{x+3}$ on the same set of axes.

Solution

Sketch $y = x + 3$.

Label the y-intercept $(0, 3)$.

Label the zero $(-3, 0)$.



ISBN 9780170412995

Now sketch the graph of $y = \frac{1}{x+3}$, which has a vertical asymptote at $x = -3$.

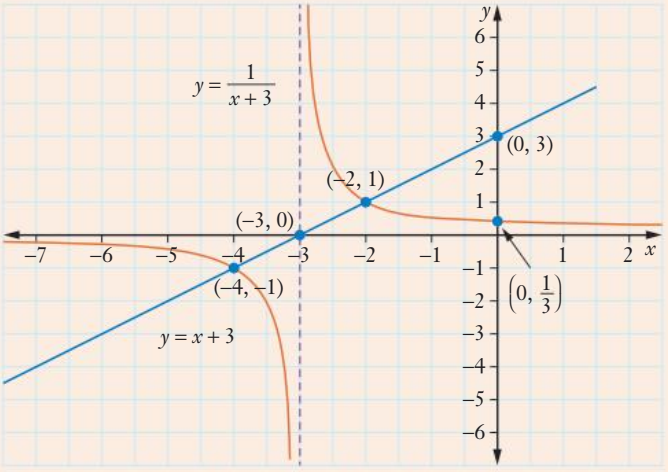
Invariant points occur when $x + 3 = \pm 1$, i.e., $x = -4$ or -2.

The invariant points are $(-4, -1)$ and $(-2, 1)$.

The y-intercept of $y = \frac{1}{x+3}$ is $\frac{1}{3}$.

As $x \to \pm\infty$, $x + 3 \to \pm\infty$, and $\frac{1}{x+3} \to 0$.

As $x \to -3$, $x + 3 \to 0$, and $\frac{1}{x+3} \to \pm\infty$.



The same procedure can be used for higher order polynomial functions.

EXAMPLE 11

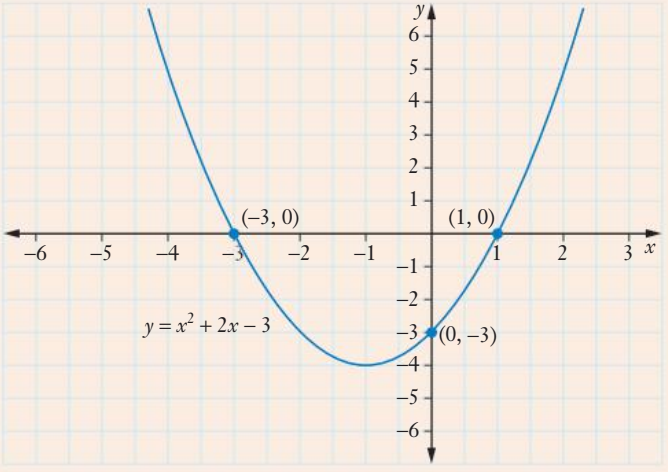
Draw the graphs of $y = x^2 + 2x + 3$ and its reciprocal function on the same set of axes.

Solution

Draw the graph of $y = x^2 + 2x + 3$.

The y-intercept = -3

The zeros are $(-3, 0)$ and $(1, 0)$.



ISBN 9780170412995

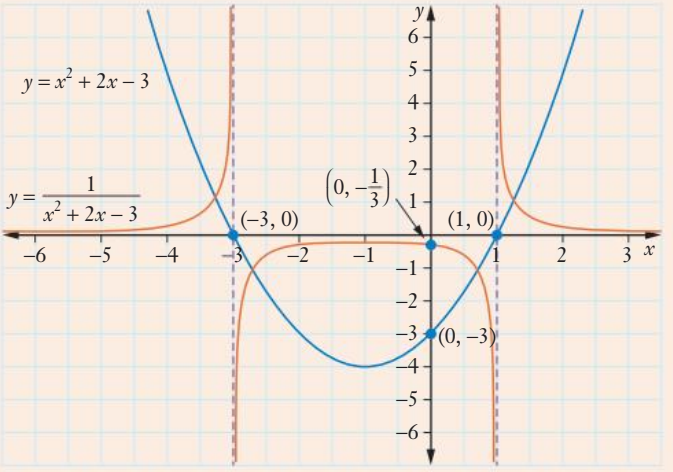
Now sketch the graph of $y = \dfrac{1}{x^2 + 2x - 3}$, which has vertical asymptotes at $x = -3$ and $x = 1$.

The y-intercept of $y = \dfrac{1}{x^2 + 2x - 3}$ is $-\dfrac{1}{3}$.

As $x^2 - 2x + 3 \to \pm\infty$, $\dfrac{1}{x^2 + 2x - 3} \to 0$.

As $x \to -3$, $\dfrac{1}{x^2 + 2x - 3} \to \pm\infty$.

As $x \to 1$, $\dfrac{1}{x^2 + 2x - 3} \to \pm\infty$.



Exercise 9.04 Reciprocal functions

1 Sketch the graph of each function and its reciprocal function on the same set of axes.

Example 10

a $y = x + 5$ **b** $y = 2x - 3$

c $y = 2 + \dfrac{1}{2}x$ **d** $y = 4 - x$

2 Sketch the graph of each function and its reciprocal function on the same set of axes.

Example 11

a $y = (x + 2)(x - 1)$ **b** $y = x^2 - 4x - 5$

c $y = 4 - 5x - x^2$ **d** $y = x(x + 3)(x - 2)$

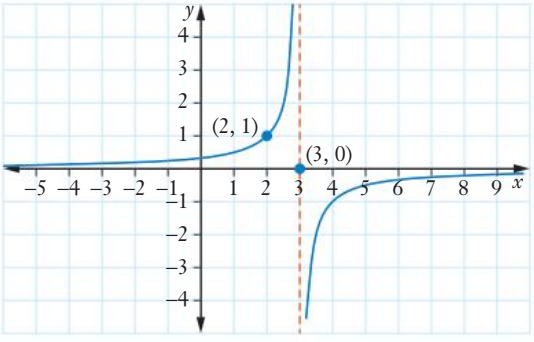
3 Sketch the graph of each function and its reciprocal function on the same set of axes.

a $y = \cos(x)$ **b** $y = e^x$

c $y = \log_e(x)$ **d** $y = -\sin(x)$

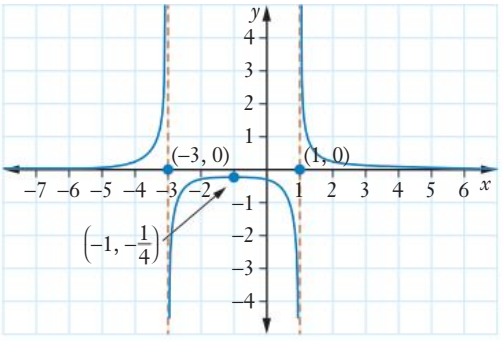
Problem solving

4 The graph of a reciprocal function $y = \dfrac{1}{f(x)}$ is shown. Use it to draw the graph of $y = f(x)$.



ISBN 9780170412995

5 The graph of a reciprocal function $y = \dfrac{1}{f(x)}$ is shown. Use it to draw the graph of $y = f(x)$.



6 Use a graphical method to solve $\dfrac{1}{x(x+1)(x-3)} < 0$.

9.05 Rational functions

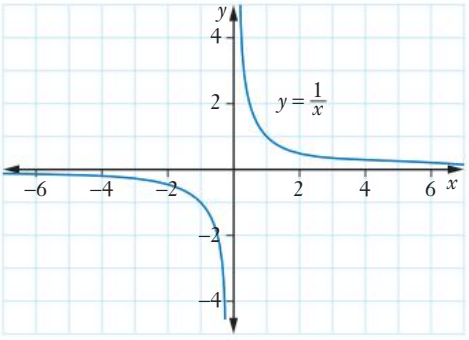
Functions where the numerator and denominator are both polynomial functions are known as **rational functions**. Examples include:

$$r(x) = \frac{x+1}{x-3}, y = \frac{x^2+2x-3}{x+2}, y = \frac{1}{x} \text{ and } y = \frac{x^3}{1-2x}$$

Rational functions

A rational function is one that can be written in the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$.

The graph of the simplest rational function $y = \dfrac{1}{x}$ shows that the y-axis is a vertical asymptote for the graph while the x-axis is a horizontal asymptote.



Being able to locate the vertical and horizontal asymptotes is an important step in drawing the graph of a rational function.

You have previously seen that for the reciprocal function $y = \dfrac{1}{f(x)}$ vertical asymptotes occur when $f(x) = 0$. This can be extended to rational functions.

ISBN 9780170412995

Vertical asymptotes of rational functions

For the rational function $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials,

if $Q(c) = 0$, then the line $x = c$ is a vertical asymptote of $f(x) = \dfrac{P(x)}{Q(x)}$.

Because $P(x)$ and $Q(x)$ are polynomials, they are continuous. This means that the rational function $f(x) = \dfrac{P(x)}{Q(x)}$ is also continuous, except when $Q(x) = 0$, i.e., at the asymptotes. This means that a rational function can change sign only at a zero or an asymptote.

The simplest types of rational functions have the form $y = \dfrac{a}{f(x)}$ (where a is a constant), so these types of rational functions are actually reciprocal functions. You can graph these types of functions by applying the techniques you used for graphing reciprocal functions.

EXAMPLE 12

Draw the graph of the function $y = \dfrac{4}{x^2 + 3x - 4}$.

Solution

First factorise the denominator.

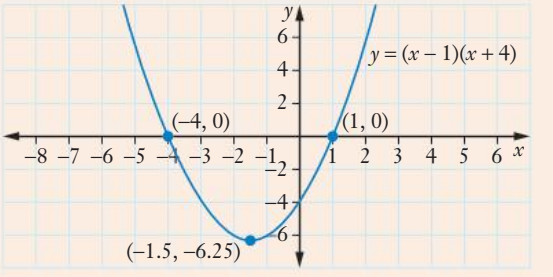
$$y = \frac{4}{x^2 + 3x - 4}$$
$$= \frac{4}{(x-1)(x+4)}$$

Graph the function $y = (x - 1)(x + 4)$.

The y-intercept $= -4$.

The zeros are $(-4, 0)$ and $(1, 0)$.

There is a local minimum at $\left(-\frac{3}{2}, -\frac{25}{4}\right)$.



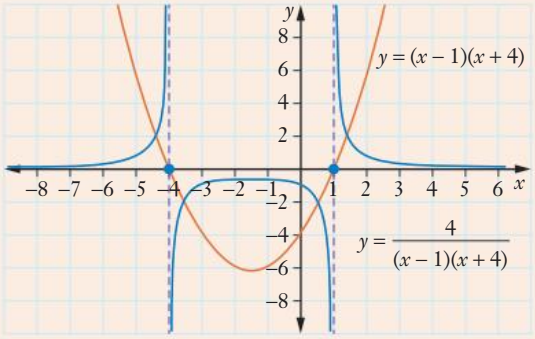
Now sketch the graph of $y = \dfrac{1}{(x-1)(x+4)}$.

There are vertical asymptotes at $x = -4$ and $x = 1$ and the x-axis is a horizontal asymptote.

y-intercept $= -\dfrac{1}{3}$.

There is a local maximum at $\left(-\dfrac{3}{2}, -\dfrac{4}{25}\right)$.

The graph of $y = \dfrac{4}{(x-1)(x+4)}$ will be the same shape as the graph of $y = \dfrac{1}{(x-1)(x+4)}$, but stretched vertically by a factor of 4. So the asymptotes remain unchanged, but the y-intercept is at $\dfrac{4}{3}$ and the maximum is at $\left(-\dfrac{3}{2}, -\dfrac{16}{25}\right)$.



A rational function can also have a horizontal asymptote.

Horizontal asymptotes of rational functions

For the rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with leading coefficients a and b respectively,

- if the degree of $P(x)$ is the same as the degree of $Q(x)$, then $y = \dfrac{a}{b}$ is the horizontal asymptote of the graph of $y = f(x)$
- if the degree of $P(x)$ is less than the degree of $Q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $y = f(x)$
- if the degree of $P(x)$ is greater than the degree of $Q(x)$, then the graph of $y = f(x)$ has no horizontal asymptotes

ISBN 9780170412995

EXAMPLE 13

Draw the graph of $y = \dfrac{x}{x^2 - 4}$.

Solution

The degree of the numerator is less than the degree of the denominator, so the horizontal asymptote is $y = 0$.

y-intercept $= 0$ (when $x = 0$)

Find the vertical asymptote.

Let $x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$

$x = \pm 2$

Vertical asymptotes at $x = 2$ and $x = -2$.

Zero occurs when $y = 0$, i.e., $x = 0$.
The zero is at $(0, 0)$.

Examine the sign of the function near the zero and asymptotes.

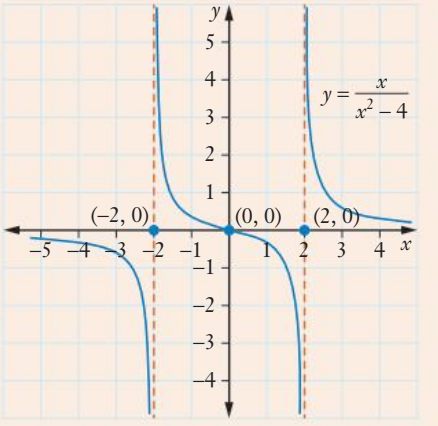
When $x < -2, y < 0$

When $-2 < x < 0, y > 0$

When $0 < x < 2, y < 0$

When $x > 2, y > 0$

Use this information to sketch the graph.



It is also possible for rational functions to have **slant** or **oblique asymptotes**.

Consider the rational expression $f(x) = \dfrac{2x^2 + 5x + 5}{x + 1}$. The degree of the numerator is one more than the degree of the denominator. You can divide the numerator by the denominator using long division.

$$
\begin{array}{r}
2x + 3 \\
x + 1 \overline{)\,2x^2 + 5x + 5} \\
\underline{2x^2 + 2x} \\
3x + 5 \\
\underline{3x + 3} \\
2
\end{array}
$$

ISBN 9780170412995

So $2x^2 + 5x + 5 = (x + 1)(2x + 3) + 2$

This means $f(x) = \dfrac{2x^2 + 5x + 5}{x + 1} = 2x + 3 + \dfrac{2}{x + 1}$

As $x \to \pm\infty$, $\dfrac{2}{x + 1} \to 0$, so $f(x)$ behaves like $y = 2x + 3$ as $x \to \pm\infty$. This is the asymptote for $f(x)$. You can see that the asymptote is a straight line with a gradient of 2, so it will be at a slant to the x-axis. All rational functions where the degree of the numerator is one more than the degree of the denominator share this feature.

Slant asymptotes of rational functions

For the rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is exactly one more than the degree of $Q(x)$, the graph of $y = f(x)$ has the slant asymptote $y = L(x)$, where $L(x)$ is the quotient obtained by dividing $P(x)$ by $Q(x)$.

EXAMPLE 14

Draw the graph of $y = \dfrac{x^2}{x - 2}$.

Solution

There is a zero where $x^2 = 0$, i.e., $x = 0$. The zero is (0, 0).

There is a vertical asymptote where $x - 2 = 0$, i.e., $x = 2$. Vertical asymptote is $x = 2$.

The degree of the numerator is one more than the degree of the denominator, so there is a slant asymptote.

Dividing x^2 by $(x - 2)$ gives $x + 2 + \dfrac{4}{x + 2}$.

Slant asymptote is $y = x + 2$.

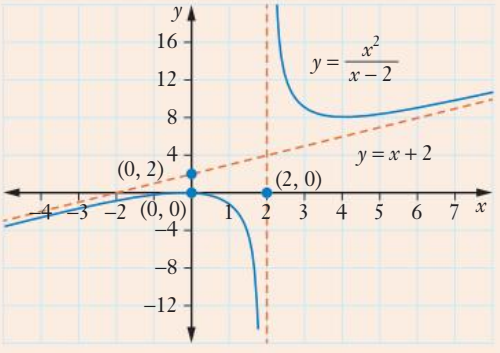
Examine the sign of the function near the zero and asymptotes.

When $x < 0$, $y < 0$

When $0 < x < 2$, $y < 0$

When $x > 2$, $y > 0$

Use this information to sketch the graph.



ISBN 9780170412995

Exercise 9.05 Rational functions

Example 12

1 Draw the graph of each function.

a $y = \dfrac{1}{x-1}$ **b** $y = \dfrac{2}{x+2}$ **c** $y = \dfrac{5}{x-3}$

d $y = \dfrac{4}{x^2-2x}$ **e** $y = \dfrac{2}{5-x^2}$ **f** $y = \dfrac{1}{x^3-3x^2}$

Example 13

2 Draw the graph of each rational function.

a $y = \dfrac{x-3}{4x+16}$ **b** $y = \dfrac{4x}{x^2-9}$ **c** $y = \dfrac{x^2-4x}{x^2-2x-3}$

d $y = \dfrac{2x^2+2x-4}{x^2+x-6}$ **e** $y = \dfrac{x^2-9x}{x^2-2x-9}$ **f** $y = \dfrac{x^2-4x+3}{-x^2+4x+5}$

Example 14

3 Draw the graph of each rational function.

a $y = \dfrac{x^2+1}{5x}$ **b** $y = \dfrac{x^2-4}{4x}$ **c** $y = \dfrac{x^2+4}{x+1}$

d $y = \dfrac{2x^2+5x-3}{2-3x}$ **e** $y = \dfrac{x^3+1}{x^2-4}$ **f** $y = \dfrac{x^3-3x+1}{x^2-1}$

Problem solving

4 Use a graphical method to solve the inequality $y = \dfrac{x^3-16x}{-4x^2+4x+24} < 0$.

5 The cost, C in dollars, to remove n% of the invasive carp from a large waterway is given by:

$C(p) = \dfrac{1870n}{100-n}$, where $0 \leq n < 100$.

Shutterstock.com/K Steve Cope

a In the context of this question, what do $C(30)$ and $C(90)$ represent?

b What does the vertical asymptote at $n = 100$ mean in this situation?

c What percentage of the carp can you remove for $24 500?

6 The cost C (in dollars) to make n electronic components is given by the expression

$$C(n) = 150n + 3000,\ n \geq 0.$$

a Find a rule for calculating the average cost $A(n)$ of producing a component.

b Find and interpret $A(2)$ and $A(90)$.

c How many components need to be produced so that the average cost per unit is $225?

d Interpret the behaviour of $A(n)$ as $n \to \infty$.

ISBN 9780170412995

9.06 The reciprocal trigonometric functions

Each trigonometric ratio sin (θ), cos (θ) and tan (θ) has a **reciprocal**.

Reciprocal trigonometric ratios

$$\text{cosec}(\theta) = \frac{1}{\sin(\theta)}, \qquad (\sin(\theta) \neq 0)$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}, \qquad (\cos(\theta) \neq 0)$$

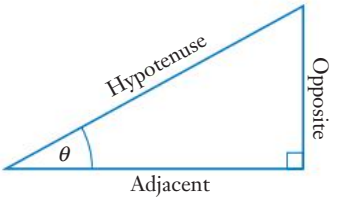
$$\cot(\theta) = \frac{1}{\tan(\theta)}, \qquad (\tan(\theta) \neq 0)$$

For right-angled triangles:

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \text{cosec}(\theta) = \frac{\text{Hypotenuse}}{\text{Opposite}}$$

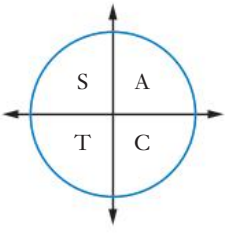
$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \sec(\theta) = \frac{\text{Hypotenuse}}{\text{Adjacent}}$$

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} \qquad \cot(\theta) = \frac{\text{Adjacent}}{\text{Opposite}}$$



Using the CAST (or ASTC) diagram, you can see that:

- cosec (θ) > 0 when sin (θ) > 0
- sec (θ) > 0 when cos (θ) > 0
- cot (θ) > 0 when tan (θ) > 0



1st quadrant: All ratios are positive (A)

2nd quadrant: Sin and cosec only are positive (S)

3rd quadrant: Tan and cot only are positive (T)

4th quadrant: Cos and sec only are positive (C)

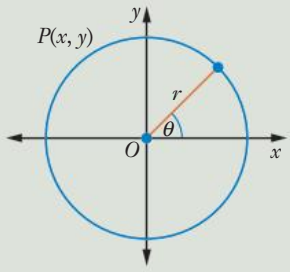
ISBN 9780170412995

Trigonometric ratios for angles of any magnitude

For any point $P(x, y)$ on a circle of radius r, and centre O with ray OP drawn at an angle θ from the x-axis:

$$\sin(\theta) = \frac{y}{r}, \cos(\theta) = \frac{x}{r} \text{ and } \tan(\theta) = \frac{y}{x}$$

$$\text{cosec}(\theta) = \frac{r}{y}, \sec(\theta) = \frac{r}{x} \text{ and } \cot(\theta) = \frac{y}{x}$$



In the case shown above, θ is the acute angle in the first quadrant and is known as the **reference angle**. The reference angle is always drawn between r and the x-axis. The definition above and the reference angle θ can now be used to define the reciprocal trigonometric ratios in quadrants 2, 3 and 4.

Reciprocal trigonometric ratios for angles of any magnitude

2nd quadrant

$$\sin(\pi - \theta) = \sin(\theta) \Leftrightarrow \text{cosec}(\pi - \theta) = \text{cosec}(\theta)$$

$$\cos(\pi - \theta) = -\cos(\theta) \Leftrightarrow \sec(\pi - \theta) = -\sec(\theta)$$

$$\tan(\pi - \theta) = -\tan(\theta) \Leftrightarrow \cot(\pi - \theta) = -\cot(\theta)$$

3rd quadrant

$$\sin(\pi + \theta) = -\sin(\theta) \Leftrightarrow \text{cosec}(\pi + \theta) = -\text{cosec}(\theta)$$

$$\cos(\pi + \theta) = -\cos(\theta) \Leftrightarrow \sec(\pi + \theta) = -\sec(\theta)$$

$$\tan(\pi + \theta) = \tan(\theta) \Leftrightarrow \cot(\pi + \theta) = \cot(\theta)$$

4th quadrant

$$\sin(-\theta) = -\sin(\theta) \Leftrightarrow \text{cosec}(-\theta) = -\text{cosec}(\theta)$$

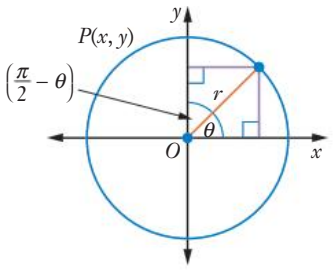
$$\cos(-\theta) = \cos(\theta) \Leftrightarrow \sec(-\theta) = \sec(\theta)$$

$$\tan(-\theta) = -\tan(\theta) \Leftrightarrow \cot(-\theta) = -\cot(\theta)$$

ISBN 9780170412995

Trigonometric ratios for complementary angles can be defined in the same way. As shown in the diagram,

θ and $\left(\frac{\pi}{2} - \theta\right)$ are complementary angles.



$$\sin(\theta) = \frac{y}{r} = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(\theta) = \frac{x}{r} = \sin\left(\frac{\pi}{2} - \theta\right)$$

These definitions can be extended to cover all quadrants and reciprocal ratios.

EXAMPLE 15

a If $\sec(x) = \frac{5}{4}$ and $0 \leq x \leq 90°$, find x (correct to 1 decimal place).

b Find the exact value of cosec (225°).

c If $\tan(x) = 3$ and $\pi < x < \frac{3\pi}{2}$, find the exact value of sec (x).

Solution

a Use $\sec^{-1}$ to find a solution. $x = \sec^{-1}\left(\frac{5}{4}\right)$

Write the solution for $0 \leq x \leq 90°$. $= 36.8698\ldots$

Round off and state the result. $x \approx 36.9°$

b Express cosec (225°) in terms of sin (225°). $\text{cosec}(225°) = \frac{1}{\sin(225°)}$

Use the angles of any magnitude definitions to simplify sin (225°). $= \frac{1}{\sin(180° + 45°)}$

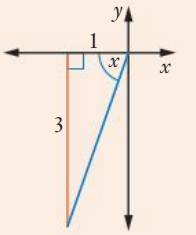
$= -\frac{1}{\sin(45°)}$

Substitute the exact value for sin (45°). $= -\frac{1}{\frac{1}{\sqrt{2}}}$

Simplify the ratio. $= -\sqrt{2}$

ISBN 9780170412995

c Draw a right-angled triangle in the 3rd quadrant, showing $\tan(x) = 3 = \frac{3}{1}$.



Use Pythagoras' theorem to calculate the hypotenuse.

$\text{Hypotenuse} = \sqrt{3^2 + 1^2}$
$= \sqrt{10}$

Use the definition of sec (x).

$\sec(x) = \frac{\text{Hypotenuse}}{\text{Adjacent}}$

Substitute for known values.

$= \frac{\sqrt{10}}{1} = \sqrt{10}$

Use CAST to determine the sign and state the result.

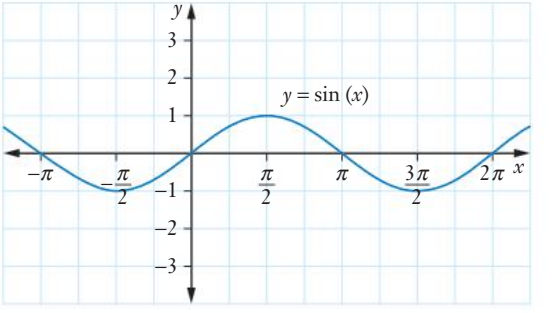
$\sec(x) = -\sqrt{10}$

You have previously studied the graphs of the sine, cosine and tangent functions, and the graphs of reciprocal functions. You can combine these two areas to draw the graphs of the **reciprocal trigonometric functions**.

y = cosec (x)

The graph of $y = \sin(x)$ is shown.
You can see that:

- the graph has maxima or minima at $x = \frac{\pi}{2} + n\pi$
- $y = \sin(x) = \pm 1$ at $x = \frac{\pi}{2} + n\pi$
- zeros exist at $x = n\pi$
- the period $= 2\pi$

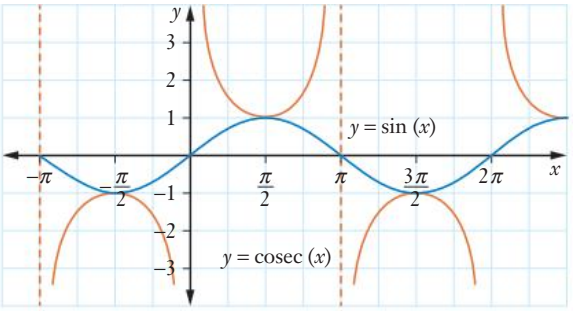


ISBN 9780170412995

Using this information, the graph of $y = \text{cosec}\,(x)$ may be drawn as shown.

From the graph, you can see that

- the graph has maxima or minima at $x = \frac{\pi}{2} + n\pi$
- $y = \text{cosec}\,(x) = \pm 1$ at $x = \frac{\pi}{2} + n\pi$
- vertical asymptotes occur at $x = n\pi$
- the period $= 2\pi$

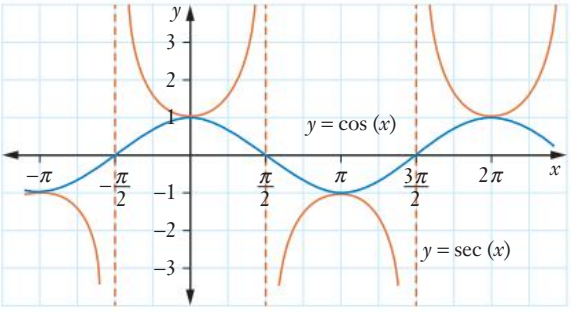


$y = \text{sec}\,(x)$

Using a similar approach, the graph of $y = \text{sec}\,(x)$ may be drawn as shown below.

From the graph, you can see that

- the maxima or minima occur at $x = n\pi$
- $y = \text{sec}\,(x) = \pm 1$ at $x = n\pi$
- vertical asymptotes occur at $x = \frac{\pi}{2} + n\pi$
- the period $= 2\pi$

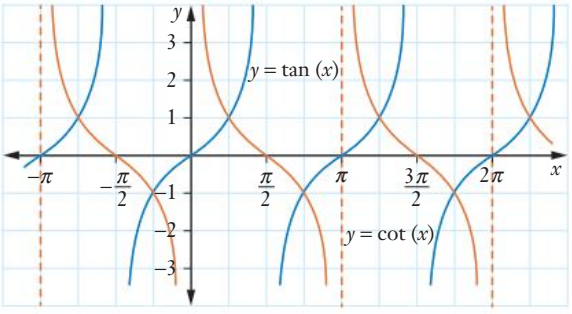


$y = \text{cot}\,(x)$

The graph of $y = \text{cot}\,(x)$ may be drawn as shown.

From the graph, you can see that

- $y = \text{cot}\,(x) = 1$ at $x = \frac{\pi}{4} + n\pi$
- $y = \text{cot}\,(x) = -1$ at $x = -\frac{\pi}{4} + n\pi$
- vertical asymptotes occur at $x = n\pi$
- the period $= \pi$



You should also refer back to earlier in this chapter on how to transform the graphs of sine, cosine and tangent functions to assist in drawing the graphs of their reciprocal trigonometric functions.

ISBN 9780170412995

EXAMPLE 16

Sketch the graph of:

a $y = 3 \operatorname{cosec}(x) + 1$ for $0 \leq x \leq 2\pi$ **b** $y = \sec(2x - \pi)$ for $-\pi \leq x \leq \pi$.

Solution

a

Compare with the graph of $y = \operatorname{cosec}(x)$.

The graph of $y = 3 \operatorname{cosec}(x) + 1$ has the same general shape as $y = \operatorname{cosec}(x)$, but it has been stretched vertically by a factor of 3 and translated up 1 unit.

Locate the vertical asymptotes.

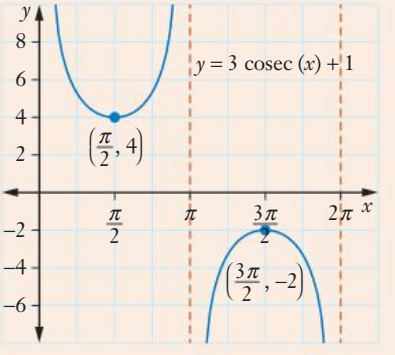
The vertical asymptotes are unchanged.
Asymptotes exist at $x = 0, \pi, 2\pi$

The position of the minima and maxima will be stretched by a factor of 3 vertically and translated up 1 unit.

$\text{Minimum} = \left(\frac{\pi}{2}, 4\right)$

$\text{Maximum} = \left(\frac{3\pi}{2}, -2\right)$.

Use this information to draw the graph.



b

Compare with the graph of $y = \sec(x)$.

The graph of $y = \sec(2x - \pi)$ is the same shape as the graph of $y = \sec(x)$, but it has been compressed by a factor of 2 horizontally, translated π units to the right and translated 1 unit up.

ISBN 9780170412995

Locate vertical asymptotes in the domain $-\pi \le x \le \pi$.

Vertical asymptotes for $y = \sec(x)$ exist at $x = \frac{\pi}{2} + n\pi$.

For $y = \sec(2x - \pi)$, asymptotes exist at:

$$2x - \pi = \frac{\pi}{2} + n\pi$$

$$2x = \frac{3\pi}{2} + n\pi$$

$$x = \frac{1}{2}\left(\frac{3\pi}{2} + n\pi\right)$$

$$x = \frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$$

Locate the y-intercept.

$y = \sec(0 - \pi) = -1$

y-intercept $= (0, -1)$

Locate maxima and minima in the domain $-\pi \le x \le \pi$.

For $y = \sec(x)$, maxima and minima occur at $x = n\pi$.
For $y = \sec(2x - \pi)$, maxima and minima occur at:

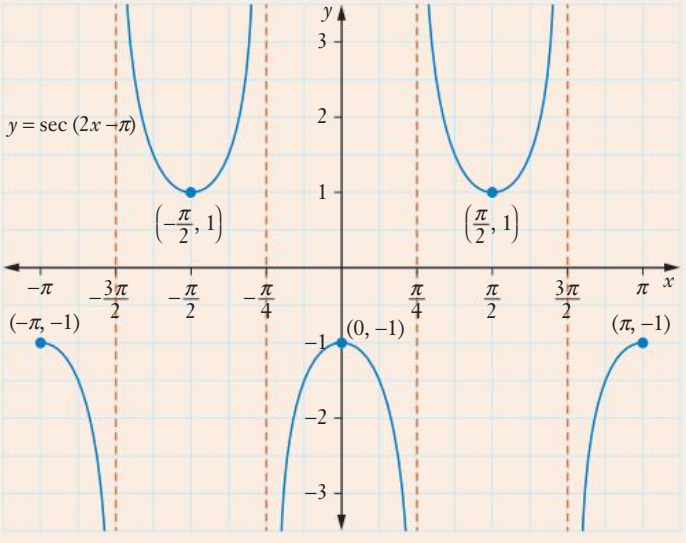
$$2x - \pi = n\pi$$

$$x = \frac{1}{2}(n\pi + \pi)$$

Maxima occur at $(-\pi, -1)$, $(0, -1)$ and $(\pi, -1)$.

Minima occur at $\left(-\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

Use this information to draw the graph.



ISBN 9780170412995

EXAMPLE 17

Sketch the graph of $y = \cot\left(x + \frac{\pi}{3}\right) + 1$ for $-\pi \leq x \leq \pi$.

Solution

Compare with the graph of $y = \cot(x)$.

The graph of $y = \cot\left(x + \frac{\pi}{3}\right) + 1$ is $y = \cot(x)$, but translated $\frac{\pi}{3}$ units to the left and 1 unit up.

The vertical asymptotes are translated $\frac{\pi}{3}$ to the left.

Vertical asymptotes exist at:

$x = 0 - \frac{\pi}{3} = -\frac{\pi}{3}$

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Find the y-intercept.

$\cot\left(0 + \frac{\pi}{3}\right) + 1 = \cot\left(\frac{\pi}{3}\right) + 1 = 1.5773\ldots$

y-intercept ≈ 1.6

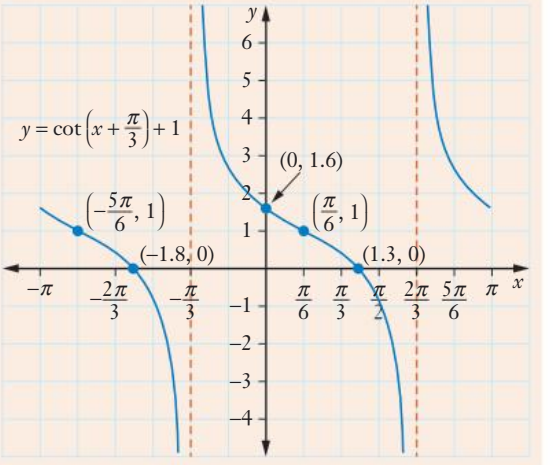
Find the zeros for $-\pi \leq x \leq \pi$.

$\cot\left(x + \frac{\pi}{3}\right) + 1 = 0$

$x \approx -1.8, 1.3$

Zeros exist at $(-1.8, 0)$ and $(1.3, 0)$.

Use this information to draw the graph.



ISBN 9780170412995

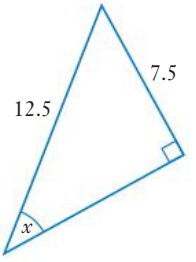
Exercise 9.06 The reciprocal trigonometric functions

Example 15

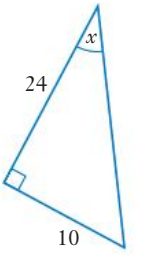
1 Copy and complete the table, using the right-angled triangles below. Give answers using exact values.

	sin (x)	cos (x)	tan (x)	cosec (x)	sec (x)	cot (x)
a						
b						
c						

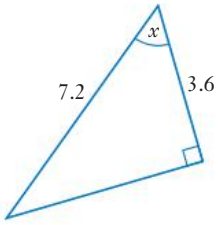
a



b



c



2 For $0° < x < 90°$, find the value of x correct to 1 decimal place.

a $\cot(x) = 0.8$ **b** $\operatorname{cosec}(x) = 2.5$ **c** $\sec(x) = 3$

d $\operatorname{cosec}(x) = \frac{5}{3}$ **e** $\sec(x) = \frac{7}{3}$ **f** $\cot(x) = \frac{1}{8}$

g $\sec(x) = \frac{9}{2}$ **h** $\cot(x) = \frac{2}{5}$ **i** $\operatorname{cosec}(x) = \frac{11}{8}$

3 Find the exact value of each expression.

a sec (45°) **b** cot (30°) **c** cosec (60°)

d cot (330°) **e** cos (–120°) **f** sec (30°)

g sec (120°) **h** cosec (420°) **i** cot (–210°)

j sec (240°) **k** cot (–315°) **l** cosec (150°)

4 If $\cos(x) = \frac{1}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find the exact value of:

a sin (x) **b** sec (x) **c** cot (x)

5 If $\sin(x) = \frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of:

a cos (x) **b** cot (x) **c** sec (x)

6 If $\tan(x) = \frac{5}{2}$ and $\pi < x < \frac{3\pi}{2}$, find the exact value of:

a cos (x) **b** cosec (x) **c** sec (x)

ISBN 9780170412995

7 If $\sec(x) = -\frac{7}{4}$ and $90° < x < 180°$, find the exact value of:

a $\cos(x)$ **b** $\sin(x)$ **c** $\cot(x)$

8 Sketch the graph of each function over the domain $0 \leq x \leq 2\pi$.

Example 16 –17

a $y = 3\operatorname{cosec}(x)$ **b** $y = \frac{1}{2}\cot(x) - 1$ **c** $y = 2\sec(x)$

d $y = 4\sec(x) + 1$ **e** $y = 5\operatorname{cosec}(x) - 2$ **f** $y = \sec\left(x + \frac{\pi}{4}\right)$

g $y = \cot\left(x - \frac{\pi}{4}\right)$ **h** $y = \operatorname{cosec}\left(x - \frac{\pi}{3}\right)$ **i** $y = \sec\left(x - \frac{\pi}{6}\right)$

9 Sketch the graph of each function over the stated domain.

a $y = \operatorname{cosec}\left(2x - \frac{\pi}{6}\right)$ for $0 \leq x \leq 2\pi$ **b** $y = \sec\left(\frac{1}{2}x + \frac{\pi}{3}\right)$ for $-2\pi \leq x \leq 2\pi$

c $y = \cot\left(3x - \frac{\pi}{4}\right)$ for $0 \leq x \leq \pi$

10 Given the function $y = \frac{3}{2}\sec\left(2x + \frac{\pi}{2}\right) + 3$ for $0 \leq x \leq \pi$,

a identify the vertical asymptotes

b find the x-intercepts

c identify any maxima and minima

d draw the graph of the function.

11 Given the function $y = -2\operatorname{cosec}\left(\frac{x}{2} - \frac{\pi}{3}\right) + 2$ for $-\pi \leq x \leq 2\pi$,

a identify any vertical asymptotes

b find the x-intercept

c find the y-intercept

d identify any maxima and minima

e draw the graph of the function.

Problem solving

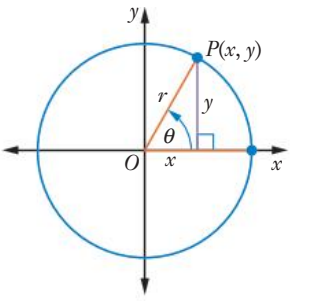
12 Simplify $\frac{\sec(x)}{\operatorname{cosec}(x)} + \frac{\sin(x)}{\cos(x)}$.

13 Simplify $\frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\operatorname{cosec}(\pi + \theta)}$.

ISBN 9780170412995

9.07 Pythagorean trigonometric identities

You have previously seen how the point $P(x, y)$ as shown on a circle can be used to define trigonometric functions.



$$\sin(\theta) = \frac{y}{r} \text{ or } y = r\sin(\theta)$$

$$\cos(\theta) = \frac{x}{r} \text{ or } x = r\cos(\theta)$$

$$\tan(\theta) = \frac{y}{x}$$

Applying Pythagoras' theorem to the same diagram gives

$$x^2 + y^2 = r^2$$

You can substitute for x and y in the equation above:

$$[r\cos(\theta)]^2 + [r\sin(\theta)]^2 = r^2$$

$$r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$$

$$\cos^2(\theta) + \sin^2(\theta) = \frac{r^2}{r^2} = 1$$

This is known as the **Pythagorean identity**.

An **identity** is true for all values of x in its implied domain. The Pythagorean identity can also be written as:

$$\sin^2(\theta) = 1 - \cos^2(\theta)$$

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

It is also possible to derive identities involving the reciprocal trigonometric functions as follows.

Write the Pythagorean identity. $\sin^2(\theta) + \cos^2(\theta) = 1$

Divide both sides by $\cos^2(\theta)$. $\dfrac{\sin^2(\theta)}{\cos^2(\theta)} + \dfrac{\cos^2(\theta)}{\cos^2(\theta)} = \dfrac{1}{\cos^2(\theta)} \quad (\cos(\theta) \neq 0)$

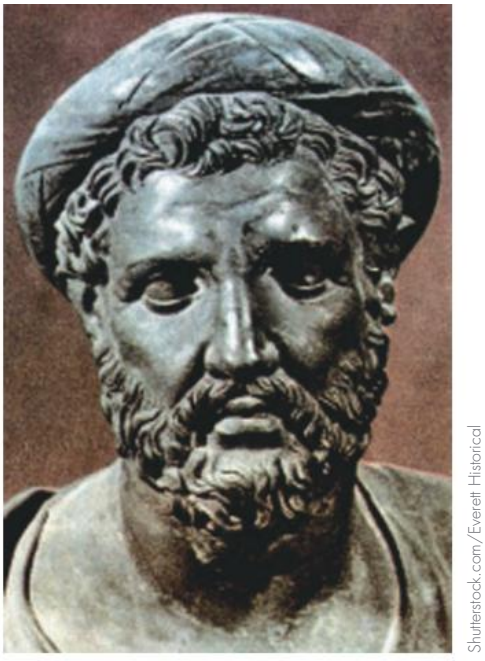
Simplify. $\tan^2(\theta) + 1 = \sec^2(\theta)$

Write the Pythagorean identity. $\sin^2(\theta) + \cos^2(\theta) = 1$

Divide both sides by $\sin^2(\theta)$. $\dfrac{\sin^2(\theta)}{\sin^2(\theta)} + \dfrac{\cos^2(\theta)}{\sin^2(\theta)} = \dfrac{1}{\sin^2(\theta)} \quad (\sin(\theta) \neq 0)$

Simplify. $1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta)$

ISBN 9780170412995



Shutterstock.com/Everett Historical

Pythagorean identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\sin^2(\theta) = 1 - \cos^2(\theta)$$
$$\cos^2(\theta) = 1 - \sin^2(\theta)$$
$$\sec^2(\theta) = \tan^2(\theta) + 1$$
$$\text{cosec}^2(\theta) = \cot^2(\theta) + 1$$

These identities can be used to simplify or evaluate trigonometric expressions.

EXAMPLE 18

Simplify the expression: $\sin^2(x)\,[1 + \tan^2(x)]$.

Solution

Write the expression. $\sin^2(x) \times [1 + \tan^2(x)]$

Use the Pythagorean identity. $= \sin^2(x) \times \sec^2(x)$

Express sec (x) in terms of cos (x). $= \sin^2(x) \times \dfrac{1}{\cos^2(x)}$

Use the definition of tan (x). $= \tan^2(x)$

ISBN 9780170412995

EXAMPLE 19

Given that $\sin(\theta) = -\frac{5}{13}$ and $\pi \leq \theta \leq \frac{3\pi}{2}$, use the Pythagorean identity to calculate $\cos(\theta)$.

Solution

Write the Pythagorean identity.	$\cos^2(\theta) = 1 - \sin^2(\theta)$
Substitute for $\sin(\theta)$.	$= 1 - \left(-\frac{5}{13}\right)^2$
Evaluate.	$= 1 - \frac{25}{169}$
	$= \frac{144}{169}$
Take the square root of each side.	$\cos(\theta) = \pm\frac{12}{13}$
$\cos(\theta)$ is negative in the 3rd quadrant.	$\cos(\theta) = -\frac{12}{13}$

The Pythagorean identities can also be used to prove other trigonometric identities.

The following hints for proving trigonometric identities may be useful.

Hints for proving trigonometric identities

1. Try starting with the most complicated-looking side of the identity.
2. Fractional expressions are generally more complicated than sums and differences. Products are the simplest.
3. If the expressions involve squares, try using the Pythagorean identities.
4. If you cannot see how to proceed, try changing all the functions to sin and cos.
5. If you are getting nowhere, try starting from the other side of the identity.

ISBN 9780170412995

EXAMPLE 20

Prove the identity $\frac{\sec(x)\sin(x)}{\tan(x)+\cot(x)} = \sin^2(x)$.

Solution

Write the LHS.

$$\text{LHS} = \frac{\sec(x)\sin(x)}{\tan(x)+\cot(x)}$$

Evaluate using sin (x) and cos (x).

$$= \frac{\frac{1}{\cos(x)}\sin(x)}{\frac{\sin(x)}{\cos(x)}+\frac{\cos(x)}{\sin(x)}}$$

Simplify.

$$= \frac{\frac{\sin(x)}{\cos(x)}}{\frac{\sin(x)^2+\cos^2(x)}{\cos(x)\sin(x)}}$$

Use the Pythagorean identity.

$$= \frac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)\sin(x)}}$$

Write as a product.

$$= \frac{\sin(x)}{\cos(x)} \times \frac{\cos(x)\sin(x)}{1}$$

Cancel.

$$= \sin^2(x)$$

Compare with the RHS.

$$= \text{RHS}$$ QED

Exercise 9.07 Pythagorean trigonometric identities

Example 18

1 Simplify each expression.

a sec (x) sin (x) **b** cos (x) tan (x) **c** $1 - \cos^2(x)$

d $\tan^2(x) - \sec^2(x)$ **e** $\sin^2(x)\cot^2(x)$ **f** $[1 - \cos(x)][1 + \cos(x)]$

2 Simplify each expression.

a $\cos(A)[\sec(A) - \cos(A)]$ **b** $\text{cosec}^2(x) - \cot^2(x)$

c $\cos^2(\theta)[\sec^2(\theta) - 1]$ **d** $\sin^4(A) - \cos^4(A)$

e $[1 - \cos(x)][1 + \sec(x)]\cos(x)$ **f** $[\sec(\theta) - 1][\sec(\theta) + 1]$

ISBN 9780170412995

3 Simplify each expression.

a $\frac{1}{\sin^2(\theta)} - \frac{1}{\tan^2(\theta)}$ **b** $\frac{\sin(x)\cos(x)}{1-\cos^2(x)}$ **c** $\frac{\cos^2(x)}{\sin(x)} + \sin(x)$

d $1 - \frac{\sin^2(\theta)}{\tan^2(\theta)}$ **e** $\frac{1}{\cos^2(x)} - \frac{1}{\cot^2(x)}$ **f** $\frac{\tan^2(x)}{\sec(x)+1} + 1$

Example 19

4 Given that $\cos(\theta) = -\frac{3}{5}$ and $\frac{\pi}{2} \leq \theta \leq \pi$, use the Pythagorean identity to calculate $\sin(\theta)$.

5 If $\tan(x) = -\frac{4}{3}$ and $\sin(x) > 0$, calculate the exact value of $\text{cosec}^2(x)$.

6 If $\tan(x) = \frac{1}{2}$ and $\pi \leq \theta \leq \frac{3\pi}{2}$, calculate the exact value of:

a $\sin(x)$ **b** $\sec(x)$ **c** $\text{cosec}(x)$

Example 20

7 Prove that $\sec(\theta) - \tan(\theta)\sin(\theta) = \frac{1}{\sec(\theta)}$.

Problem solving

8 Prove that $[1 - \sin(\theta)][1 + \sin(\theta)] = \cos^2(\theta)$.

9 Prove that $\sin^4(\theta) - \cos^4(\theta) = 1 - 2\cos^2(\theta)$.

10 Prove that $[\cot(x) + \text{cosec}(x)]^2 = \frac{1+\cos(x)}{1-\cos(x)}$.

11 Prove that $\sec(y) + \tan(y) = \frac{1+\sin(y)}{\cos(y)}$.

12 Prove that $2\,\text{cosec}^2(a) = \frac{1}{1-\cos(a)} + \frac{1}{1+\cos(a)}$.

13 Prove that $\frac{1-\tan^2(x)}{1+\tan^2(x)} = 1 - 2\sin^2(x)$.

14 Prove that $\frac{\sin(x)+\cos(x)}{\cos(x)\sin(x)} = \sin(x)[1 + \tan(x)] + \cos(x)[1 + \cot(x)]$.

15 Prove that $\frac{\sec(A)+1}{\sec(A)-1} = [\cot(A) + \text{cosec}(A)]^2$.

9.08 Angle sum and difference identities

There are some applications in mathematics where it is necessary to calculate $\cos(x - y)$, $\sin(x + y)$, etc. The rules for these calculations are called the **angle sum and difference identities**.

You could be tempted to write:

$$\cos(x - y) = \cos(x) - \cos(y)$$

ISBN 9780170412995

This can be tested using some explicit values. You know that $\cos(30°) \approx 0.866$, $\cos(60°) = 0.5$ and $\cos(90°) = 0$. From this, it is clear that

$$\cos(90° - 60°) \neq \cos(90°) - \cos(60°).$$

So for two angles x and y, the calculation of $\cos(x - y)$ is more complex than simply subtracting the cosines of the angles. In fact,

$$\cos(90° - 60°) = \cos(90°)\cos(60°) + \sin(90°)\sin(60°).$$

You can check this using exact values or by using your calculator. Generally for $\cos(x - y)$:

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y).$$

The identities for $\cos(x + y)$, $\sin(x \pm y)$ and $\tan(x \pm y)$ can all be derived from the identity for $\cos(x - y)$.

Angle sum and difference identities

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

$$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

$$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

EXAMPLE 21

a Expand $\sin(2a - b)$.

b Expand and simplify $\cos(60° + w)$.

c Simplify $\cos(m)\cos(2n) + \sin(m)\sin(2n)$.

d Simplify $\sec\left(\frac{\pi}{2} + x\right)$.

Solution

a Apply the identity for $\sin(x - y)$. $\sin(2a - b) = \sin(2a)\cos(b) - \cos(2a)\sin(b)$

b Apply the identity for $\cos(x + y)$. $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Substitute $x = 60°$ and $y = w$. $\cos(60° + w) = \cos(60°)\cos(w) - \sin(60°)\sin(w)$

ISBN 9780170412995

Use exact values. $= \frac{1}{2}\cos(w) - \frac{\sqrt{3}}{2}\sin(w)$

c Write the expression. $\cos(m)\cos(2n) + \sin(m)\sin(2n)$

Select the matching identity. $\cos(x)\cos(y) - \sin(x)\sin(y) = \cos(x+y)$

Substitute $x = m$ and $y = 2n$. $\cos(m)\cos(2n) - \sin(m)\sin(2n) = \cos(m+2n)$

d Express in terms of cos. $\sec\left(\frac{\pi}{2}+x\right) = \frac{1}{\cos\left(\frac{\pi}{2}+x\right)}$

Expand the denominator using the appropriate identity. $= \frac{1}{\cos\left(\frac{\pi}{2}\right)\cos(x) - \sin\left(\frac{\pi}{2}\right)\sin(x)}$

Simplify. $= \frac{1}{0\times\cos(x) - 1\times\sin(x)}$

$= -\text{cosec}(x)$

EXAMPLE 22

Find the exact value of tan (75°).

Solution

Express 75° in terms of 45° and 30°. $\tan(75°) = \tan(45° + 30°)$

Use the identity for tan $(x + y)$ and let $x = 45°$ and $y = 30°$. $\tan(75°) = \frac{\tan(45°) + \tan(30°)}{1 - \tan(45°)\tan(30°)}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\times\frac{1}{\sqrt{3}}} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$

Multiply the numerator and denominator by $\sqrt{3}$ and simplify. $= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

ISBN 9780170412995

Rationalise the denominator.

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{3+2\sqrt{3}+1}{3-1}$$

Simplify and state the result.

$$\tan(75°) = 2 + \sqrt{3}$$

The angle sum and difference identities can also be used to prove identities in ways similar to those previously applied.

EXAMPLE 23

Prove that $\cot(x) + \cot(y) = \dfrac{\sin(x+y)}{\sin(x)\sin(y)}$.

Solution

The right-hand side looks more complicated, so start with it.

$$\text{RHS} = \frac{\sin(x+y)}{\sin(x)\sin(y)}$$

Use the theorem for $\sin(x+y)$.

$$= \frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\sin(x)\sin(y)}$$

Express this as separate fractions.

$$= \frac{\sin(x)\cos(y)}{\sin(x)\sin(y)} + \frac{\cos(x)\sin(y)}{\sin(x)\sin(y)}$$

Simplify.

$$= \frac{\cos(y)}{\sin(y)} + \frac{\cos(x)}{\sin(x)}$$

Use the definition for cot.

$$= \cot(y) + \cot(x)$$

Compare with the LHS.

$$= \text{LHS} \qquad \text{QED}$$

Exercise 9.08 Angle sum and difference identities

Example 21

1 Expand each expression.

a $\sin(2x+y)$ **b** $\cos(4x-3y)$ **c** $\sin(3p-5q)$

d $\tan(2e-f)$ **e** $\cos(3\alpha+2\beta)$ **f** $\sin(4m+n)$

g $\tan(w+3x)$ **h** $\sin(2D-5E)$ **i** $\cos(P-4Q)$

ISBN 9780170412995

2 Expand each expression.

a $\sin(\pi - x)$ **b** $\cos(\pi + x)$ **c** $\tan\left(\frac{\pi}{2} - x\right)$

d $\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$ **e** $\cot\left(\frac{3\pi}{2} - x\right)$ **f** $\sec\left(x + \frac{\pi}{2}\right)$

g $\sin(2\pi - x)$ **h** $\cos\left(\frac{3\pi}{2} - x\right)$ **i** $\tan(x - \pi)$

3 Simplify each expression.

a $\sin(55°)\cos(37°) + \cos(55°)\sin(37°)$ **b** $\cos(68°)\cos(21°) + \sin(68°)\sin(21°)$

c $\dfrac{\tan(82°) + \tan(19°)}{1 - \tan(82°)\tan(19°)}$ **d** $\cos\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{2}\right)$

e $\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$ **f** $\dfrac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{7}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{7}\right)}$

Example 22

4 Find the exact value of each expression.

a $\tan(15°)$ **b** $\cos(75°)$ **c** $\tan\left(\frac{7\pi}{12}\right)$

d $\sin\left(\frac{5\pi}{12}\right)$ **e** $\cot\left(\frac{11\pi}{12}\right)$ **f** $\sin(105°)$

g $\cos(195°)$ **h** $\sin\left(\frac{7\pi}{12}\right)$ **i** $\sin(195°)$

Example 23

5 Use the identity for $\cos(x - y)$ and replace y with $(-y)$ to prove that:

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Problem solving

6 Solve $\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$ for $0 \leq x \leq 2\pi$.

7 Use the identity for $\cos(x - y)$ and replace x with $\left(\frac{\pi}{2} - x\right)$ to prove that:

$$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y).$$

8 Prove that $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

9 Prove $\cos(30° + x) = \dfrac{\sqrt{3}\cos(x) - \sin(x)}{2}$.

10 Prove that $\sin(x + y) + \cos(x - y) = [\sin(x) + \cos(x)][\sin(y) + \cos(y)]$.

11 Prove that $\sin(x + y)\sin(x - y) = \sin^2(x) - \sin^2(y)$.

12 Prove that $\dfrac{\cot(x) - \cot(y)}{1 - \cot(x)\cot(y)} = \dfrac{\sin(x - y)}{\cos(x + y)}$.

13 Prove that $\tan(x - y) + \tan(y - z) + \tan(z - x) = \tan(x - y)\tan(y - z)\tan(z - x)$.

ISBN 9780170412995

9.09 Double angle identities

You will recall the formula for $\sin(x+y)$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

If you replace y with x you obtain

$$\sin(x+x) = \sin(x)\cos(x) + \cos(x)\sin(x)$$

or $$\sin(2x) = 2\sin(x)\cos(x)$$

In a similar way:

$$\cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x)$$

so $$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Also $$\tan(x+x) = \frac{\tan(x)+\tan(x)}{1-\tan(x)\tan(x)}$$

so $$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

These equations are known as the **double angle identities**.

There are two other forms of $\cos(2x)$ that can be derived from the identity already established and the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$.

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= [1 - \sin^2(x)] - \sin^2(x)\\ &= 1 - 2\sin^2(x)\end{aligned}$$

and

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= \cos^2(x) - [1 - \cos^2(x)]\\ &= 2\cos^2(x) - 1\end{aligned}$$

Double angle identities

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= 1 - 2\sin^2(x)\\ &= 2\cos^2(x) - 1\end{aligned}$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

ISBN 9780170412995

EXAMPLE 24

If $\sin(x) = \frac{\sqrt{7}}{3}$ and $0 \leq x \leq \frac{\pi}{2}$, find the exact value of $\cos(2x)$.

Solution

Write the Pythagorean identity.	$\sin^2(x) + \cos^2(x) = 1$
Substitute for $\sin(x)$.	$\left(\frac{\sqrt{7}}{3}\right)^2 + \cos^2(x) = 1$
Rearrange and evaluate.	$\cos^2(x) = 1 - \frac{7}{9} = \frac{2}{9}$
Write the identity for $\cos(2x)$.	$\cos(2x) = 2\cos^2(x) - 1$
Substitute for $\cos^2(x)$.	$= 2 \times \frac{2}{9} - 1$
Evaluate and state the answer.	$\cos(2x) = -\frac{5}{9}$

By replacing $2x$ with x and x with $\left(\frac{x}{2}\right)$ it is possible to define **half angle identities**.

Half angle identities

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{x}{2}\right)$$

$$= 2\cos^2\left(\frac{x}{2}\right) - 1$$

$$\tan(x) = \frac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}$$

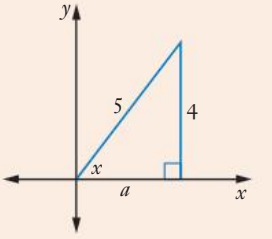
ISBN 9780170412995

EXAMPLE 25

If $\sin(x) = \frac{4}{5}$ and $0 \leq x \leq \frac{\pi}{2}$, find the exact value of $\sin\left(\frac{x}{2}\right)$.

Solution

Sketch a diagram.



Calculate a. $a^2 = \sqrt{5^2 - 4^2} = \pm 3$

x is in the 1st quadrant, so $\cos(x) > 0$. $\cos(x) = \frac{3}{5}$

Use the half angle identity for $\cos(x)$. $\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right)$

Substitute for $\cos(x)$. $\frac{3}{5} = 1 - 2\sin^2\left(\frac{x}{2}\right)$

Rearrange. $2\sin^2\left(\frac{x}{2}\right) = 1 - \frac{3}{5} = \frac{2}{5}$

Evaluate. $\sin^2\left(\frac{x}{2}\right) = \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$

Take the square root of both sides. $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{5}}$

$\sin\left(\frac{x}{2}\right) > 0$. $\sin\left(\frac{x}{2}\right) = \frac{1}{\sqrt{5}}$

Rationalise the denominator. $\sin\left(\frac{x}{2}\right) = \frac{\sqrt{5}}{5}$

ISBN 9780170412995

The multiple-angle formulas may also be used to prove some trigonometric identities.

EXAMPLE 26

Prove that $\frac{1}{\cos(\theta)+\sin(\theta)}+\frac{1}{\cos(\theta)-\sin(\theta)}=2\cos(\theta)\sec(2\theta)$.

Solution

Begin with the LHS.

$$\text{LHS}=\frac{1}{\cos(\theta)+\sin(\theta)}+\frac{1}{\cos(\theta)-\sin(\theta)}$$

Add, using the lowest common denominator.

$$=\frac{\cos(\theta)-\sin(\theta)+\sin(\theta)+\cos(\theta)}{[\cos(\theta)+\sin(\theta)][\cos(\theta)-\sin(\theta)]}$$

Simplify.

$$=\frac{2\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)}$$

Simplify the denominator using the identify for cos $(2x)$.

$$=\frac{2\cos(\theta)}{\cos(2\theta)}$$

Use the definition for cosec (x).

$$=2\cos(\theta)\sec(2\theta)$$

$$=\text{RHS}$$ QED

Exercise 9.09 Double angle identities

1 In each equation, verify the identity for the value indicated.

a $\cos(2x)=\cos^2(x)-\sin^2(x)$, $x=60°$

b $\sin(2x)=2\sin(x)\cos(x)$, $x=30°$

c $\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$, $x=\frac{\pi}{4}$

d $\tan(2x)=\frac{2}{\cot(x)-\tan(x)}$, $x=\frac{\pi}{6}$

e $\cos(2x)=1-2\sin^2(x)$, $x=\frac{\pi}{3}$

Example 24

2 Calculate exact values for sin $(2x)$ and cos $(2x)$ if:

a $\sin(x)=\frac{3}{5}$ and $\frac{\pi}{2}\le x\le\pi$

b $\cos(x)=-\frac{5}{13}$ and $\pi\le x\le\frac{3\pi}{2}$

c $\tan(x)=-\frac{5}{12}$ and $\frac{3\pi}{2}\le x\le 2\pi$

d $\tan(x)=\frac{\sqrt{3}}{3}$ and $0\le x\le\frac{\pi}{2}$

ISBN 9780170412995

3 Verify each identity.

a $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta)$ **b** $\cos(6x) = 2\cos^2(3x) - 1$

c $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ **d** $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

Example 25

4 a If $\cos(x) = \frac{12}{13}$ and $0 \le x \le \frac{\pi}{2}$, find the exact value of $\sin\left(\frac{x}{2}\right)$.

b If $\tan(\theta) = \frac{\sqrt{2}}{4}$ and $0 \le \theta \le \frac{\pi}{2}$, find the exact value of $\cos\left(\frac{\theta}{2}\right)$.

c If $\cos(x) = \frac{12}{13}$ and $0 \le x \le \frac{\pi}{2}$, find the exact value of $\tan\left(\frac{x}{2}\right)$.

d If $\tan(\theta) = \frac{4}{3}$ and $\pi \le \theta \le \frac{3\pi}{2}$, find the exact value of $\tan\left(\frac{\theta}{2}\right)$.

e If $\cos(\theta) = -\frac{15}{17}$ and $\pi \le \theta \le \frac{3\pi}{2}$, find the exact value of $\tan\left(\frac{\theta}{2}\right)$.

f If $\tan(\theta) = \frac{3}{4}$ and $0 \le \theta \le \frac{\pi}{2}$, find the exact value of $\tan\left(\frac{\theta}{2}\right)$.

Example 26

5 Show that $\sin(2x) = 2\sin(x)\cos(x)$.

Problem solving

6 Prove that $\tan(4x) - \tan(x) = \frac{\sec(4x)\sec(x)}{\operatorname{cosec}(3x)}$.

7 Show that $\cot(2x) = \frac{\cot^2(x) - 1}{2\cot(x)}$.

8 Prove that $\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$.

9 Prove that $\tan(x) = \cot(x) - 2\cot(2x)$.

ISBN 9780170412995

9. CHAPTER SUMMARY

Trigonometric functions

- The **principal domain** of $\sin(x) = -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $\cos(x) = 0 \le x \le \pi$ and $\tan(x) = -\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- The **general solutions** of trigonometric equations are such that:
  - if $\sin(x) = a$, and θ is the solution in the domain $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then
    $$x = n\pi + (-1)^n \times \theta, \text{ where } n \text{ is an integer.}$$
  - if $\cos(x) = a$, and θ is the solution in the domain $0 \le \theta \le \pi$, then
    $$x = 2n\pi \pm \theta, \text{ where } n \text{ is an integer.}$$
  - if $\tan(x) = a$, and θ is the solution in the domain $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then
    $$x = n\pi + \theta, \text{ where } n \text{ is an integer.}$$
- For the functions $y = a \sin\left[b\left(x + \frac{c}{b}\right)\right] + d$ and $y = a \cos\left[b\left(x + \frac{c}{b}\right)\right] + d$, the amplitude is the magnitude of a, the period is $\frac{2\pi}{b}$, the horizontal translation is $\frac{c}{b}$ and the average (mean) value is d.
- For the function $y = a \tan\left[b\left(x + \frac{c}{b}\right)\right] + d$, the period is $\frac{\pi}{b}$, the horizontal translation is $\frac{c}{b}$ and the average (mean) value is d.
- The **absolute value** (**modulus**) of a number $x \in \boldsymbol{R}$ is written as $|x|$ and is defined as:
$$|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$$
- The properties of absolute values include: $|-a| = |a|$, $|ab| = |a||b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- The function $y = \frac{1}{f(x)}$ is called the **reciprocal function** of $y = f(x)$.
- A **rational function** is one that can be written in the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$. If $Q(c) = 0$, then the line $x = c$ is a **vertical asymptote** of $f(x) = \frac{P(x)}{Q(x)}$.
- The **reciprocal trigonometric ratios** are defined as:

  $\text{cosec}(\theta) = \frac{1}{\sin(\theta)}$

  $\sec(\theta) = \frac{1}{\cos(\theta)}$

  $\cot(\theta) = \frac{1}{\tan(\theta)}$

ISBN 9780170412995

- An **identity** is true for all values of the variable in its implied domain.
- The **Pythagorean identities** are:

  $\sin^2(\theta) + \cos^2(\theta) = 1$

  $\sin^2(\theta) = 1 - \cos^2(\theta)$

  $\cos^2(\theta) = 1 - \sin^2(\theta)$

  $\sec^2(\theta) = \tan^2(\theta) + 1$

  $\text{cosec}^2(\theta) = \cot^2(\theta) + 1$
- The **angle sum and difference identities** are:

  $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

  $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

  $\tan(x + y) = \dfrac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$

  $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

  $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

  $\tan(x - y) = \dfrac{\tan(x) - \tan(y)}{1 - \tan(x)\tan(y)}$
- The **double angle identities** are:

  $\sin(2x) = 2\sin(x)\cos(x)$

  $\cos(2x) = \cos^2(x) - \sin^2(x)$

  $\tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)}$
- The **half angle identities** are:

  $\sin(x) = 2\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{2}\right)$

  $\cos(x) = \cos^2\left(\dfrac{x}{2}\right) - \sin^2\left(\dfrac{x}{2}\right)$

  $\tan(x) = \dfrac{2\tan\left(\dfrac{x}{2}\right)}{1 - \tan^2\left(\dfrac{x}{2}\right)}$

9. CHAPTER REVIEW

Trigonometric functions

Examples 1, 2

1 **a** Find the general solution of $\tan(x) - \sqrt{3} = 0$.

b Find the general solution for $2\cos(2x) + \sqrt{2} = 0$.

c Find the solution for $\sin\left(2x - \frac{\pi}{4}\right) + \frac{1}{2} = 0$ for $-\pi \leq x \leq \pi$.

Example 3

2 Solve:

a $4\tan^2(\theta) - 1 = 0$ for $0 \leq \theta \leq 2\pi$

b $\sin^2(x)\cos(x) = 4\cos(x)$, where $0 \leq x \leq 2\pi$

Example 4

3 Sketch the graph of each function.

a $y = -\sin\left(x - \frac{\pi}{4}\right), 0 \leq x \leq 2\pi$

b $y = 5 - 3\sin\left(2x + \frac{\pi}{6}\right), 0 \leq x \leq \pi$

c $y = 6\cos\left(3x - \frac{\pi}{4}\right) + 4, 0 \leq x \leq \pi$

Example 5

4 Sketch the graph of each function.

a $y = \tan\left(x - \frac{\pi}{3}\right)$ for $0 \leq x \leq \pi$

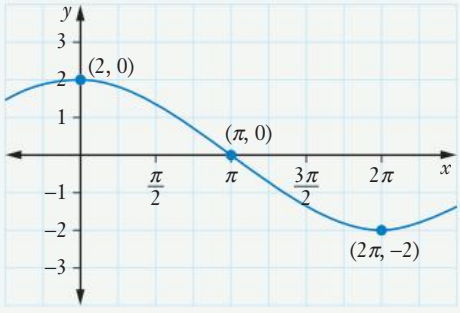
b $y = 2\tan\left[2\left(x - \frac{\pi}{4}\right)\right] + 3$ for $0 \leq x \leq \frac{\pi}{2}$

c $y = 5 - \tan\left[3\left(x + \frac{\pi}{6}\right)\right]$ for $0 \leq x \leq \frac{\pi}{3}$

ISBN 9780170412995

5 Determine 2 possible equations for this graph.

Example 6



6 Solve:

Example 7

a $|2x+3| = 5$ **b** $|x+4| = 3x-16$

c $|3x-2| = |5x+4|$ **d** $|x-4| = |2x+5| - 2$

7 Graph each absolute value function.

Examples 8, 9

a $y = |3x+6|$ **b** $y = |x-2| - 4$

c $y = |2x^2 - 2|$ **d** $y = |\cos(x) + 1|$

e $y = \left|\frac{2}{x}\right|$ **f** $y = |x|(|2x|-3)$

8 Graph each rational function.

Examples 10-14

a $y = \frac{4}{x+2}$ **b** $y = \frac{4x}{x^2-4}$

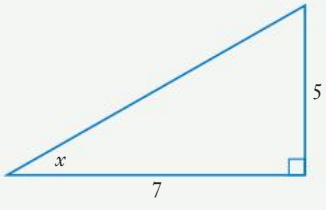
c $y = -\frac{4}{x^2-3x}$ **d** $y = \frac{x^2-4x-5}{x-3}$

e $y = \frac{2x^2+7x-4}{x^2+x-2}$ **f** $y = \frac{2x^2-3x-5}{x^2-x-6}$

9 Use the triangle to find the exact value of:

Example 15

a $\cos(x)$ **b** $\operatorname{cosec}(x)$ **c** $\cot(x)$.



10 **a** If $\operatorname{cosec}(x) = \frac{13}{7}$ and $0 \le x \le 90°$, find x correct to 1 decimal place.

Example 15

b Find the exact value of $\operatorname{cosec}(315°)$.

c If $\tan(x) = 2$ and $0 < x < \pi$, find the exact value of $\sec(x)$.

ISBN 9780170412995

Examples 16, 17

11 Sketch the graph of:

a $y = \text{cosec}\left(x + \frac{\pi}{4}\right)$ for $0 \leq x \leq 2\pi$.

b $y = \sec\left(3x + \frac{\pi}{4}\right)$ for $0 \leq x \leq \pi$.

c $y = \cot\left(2x - \frac{\pi}{4}\right)$ for $0 \leq x \leq \pi$.

d $y = \sec\left(2x - \frac{\pi}{3}\right)$ for $0 \leq x \leq \pi$.

Examples 18, 19

12 a Simplify $\frac{1}{\sin^2(x)} - \frac{1}{\tan^2(x)}$.

b Given that $\sec(\theta) = -\frac{5}{3}$ and $\pi \leq \theta \leq \frac{3\pi}{2}$, calculate $\sin(\theta)$.

Example 20

13 Prove the identity $\frac{\sec(x)\sin(x)}{\tan(x) + \cot(x)} = \sin^2(x)$.

Example 21

14 a Expand $\sin(5x - 2y)$.

b Expand $\tan(3x + y)$.

c Simplify $\cos(\pi + x)$.

d Simplify $\cot\left(\frac{3\pi}{2} - x\right)$.

Example 22

15 Find, without using a calculator:

a $\cos(75°)$ **b** $\sin\left(\frac{7\pi}{12}\right)$ **c** $\tan\left(-\frac{\pi}{12}\right)$

Example 23

16 Prove that $\cot(x) + \cot(y) = \frac{\sin(x + y)}{\sin(x)\sin(y)}$.

Examples 24, 25

17 a If $\cos(x) = \frac{3}{5}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$, find the exact value of $\cos(2x)$.

b If $\tan(x) = -\frac{5}{12}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$, find the exact value of $\cos\left(\frac{x}{2}\right)$.

Example 26

18 Prove that $\frac{1}{\cos(\theta) + \sin(\theta)} + \frac{1}{\cos(\theta) - \sin(\theta)} = 2\cos(\theta)\sec(2\theta)$.

ISBN 9780170412995

Problem solving

19 The depth of the sea near the shore changes throughout the day.
The depth of Fingal Bay measured at a particular buoy can be modelled by:

$$d = 11.5 - 7\cos\left(\frac{\pi t}{6.2}\right)$$

where d is the water depth in metres and t is the time in hours.
If $t = 0$ represents 12:00 a.m. on a particular day,

a when do the high and low tides occur?

b at what time(s) is the water depth 8.5 m?

20 Prove that $\sec^2(x) = \sec(x)\,\text{cosec}(x)\tan(x)$.

21 Prove that $\dfrac{1}{\text{cosec}(x)+1} + \dfrac{1}{\text{cosec}(x)+1} = 2\tan(x)\sec(x)[1-\sin(x)]$.

22 Show that $1 + \cos(2x) = 2\cos^2(x)$.

23 Prove that $\cot(x) + \cot(y) = \dfrac{\text{cosec}(x)\,\text{cosec}(y)}{\text{cosec}(x+y)}$.

24 Prove that $\cot(x-y) + \cot(x+y) = \dfrac{2[\cot^2(y)+1]}{\tan(x)[\cot^2(y)-\cot^2(x)]}$.

Practice quiz

Alamy Stock Photo/Suzanne Long

ISBN 9780170412995

Practice examination 3

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (3 marks)

$$\mathbf{A} = \begin{bmatrix} 5 & 6 & -4 & -6 \\ -3 & -1 & 4 & 2 \\ 0 & 3 & -2 & 1 \end{bmatrix}$$

a What is the size of $\mathbf{A}$?

b What is a_{23}?

Question 2 (3 marks)

$$\mathbf{C} = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} -3 & 1 \\ 5 & 4 \end{bmatrix}, \mathbf{E} = \begin{bmatrix} 6 & -3 \\ -3 & 5 \end{bmatrix}$$

Find:

a $\mathbf{C} + \mathbf{D}$

b $\mathbf{DE}$

c $2\mathbf{E} - 3\mathbf{C}$

ISBN 9780170412995

Question 3 (2 marks)

Solve $\begin{bmatrix} 3 & -2 \\ 4 & -4 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$.

Question 4 (1 mark)

Solve $6x^2 + 54 = 0$.

Question 5 (2 marks)

Write Re (z) and Im (z) for $z = 7i - 3$.

Question 6 (3 marks)

Simplify each expression.

a $(2 - 3i) + (5 + 6i)$

b $(-3 + i)(4 - 3i)$

c $(4 + 2i) \div (5 - 3i)$

Question 7 (1 mark)

Factorise $9x^2 + 25y^2$.

Question 8 (2 marks)

$z_1 = 3 - 2i$ and $z_2 = 2 + 5i$. Show each expression using vectors on an Argand diagram.

a $z_1 + z_2$

b $z_2 - z_1$

Question 9 (3 marks)

Find the general solution of:

a $5 \sin(x) + 2 = \sin(x)$

b $4 \cos^2(x) + 1 = 2$

c $4 \tan(x) - 7 = 3 \tan(x) - 6$

ISBN 9780170412995

Question 10 (6 marks)

Sketch the graph of each function for $0 \leq x \leq \pi$.

a $y = \cos\left(2x - \frac{\pi}{4}\right) - 1$

b $y = 5 - 4\sin\left(3x + \frac{\pi}{6}\right)$

c $y = \tan\left(2x - \frac{\pi}{3}\right)$

Question 11 (2 marks)

Solve:

a $|4x + 3| = |5 - x|$

b $|x - 5| = |2x + 6| + 1$

Question 12 (1 mark)

Given that $\tan(\theta) = -\frac{3}{4}$ and $\frac{3\pi}{2} \leq \theta \leq 2\pi$, calculate $\sin(2\theta)$.

Question 13 (1 mark)

Calculate the exact value of $\sin 165°$ without a calculator using a half-angle identity.

Question 14 (1 mark)

Write the expression $2\cos^2(0.5) - 1$ in terms of a single trigonometric function.

Question 15 (2 marks)

Three customers at a deli order the following amounts of items.

	Kalamata olives (g)	Hot Hungarian salami (g)	Artichoke hearts (g)
Customer 1	250	200	100
Customer 2	300	400	200
Customer 3	150	100	120

The olives cost \$16/kg, the salami \$18/kg and the artichoke hearts \$42/kg.

Set up a matrix for the orders and a column matrix for the prices and multiply to find the total for each customer.

ISBN 9780170412995

Question 16 (3 marks)

Show that multiplication of complex numbers by $3 - 5i$ is a linear transformation.

Question 17 (5 Marks)

Prove each identity.

a $1 + \sin(2x) = [\sin(x) + \cos(x)]^2$

b $\dfrac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta) - \cos^2(\theta)} = -\sec(2\theta)$

Question 18 (2 marks)

Solve the equation $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 6 & 1 \\ 6 & -4 & 7 \end{bmatrix} \mathbf{X} = \mathbf{X} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Question 19 (3 marks)

Find a quadratic equation whose roots are the reciprocals of the roots of $x^2 + 3x + 4 = 0$.

Question 20 (5 marks)

Prove each identity.

a $\dfrac{1 - \cos(2x)}{2} = \sin^2(x)$

b $\dfrac{[\sin(x) + \cos(x)]^2}{\sin(2x)} = \operatorname{cosec}(2x) + 1$

END OF EXAMINATION

ISBN 9780170412995

10.

TRANSFORMATIONS

Transformations change vectors, points or objects in the plane or space to other vectors, points or objects. Translations, rotations, reflections and dilations are all transformations. Transformations are used in computer graphics and animation. They have important applications in optics, atomic and nuclear theory and other branches of science. In this chapter you will study the application of matrices to transformations.

10.01 Translations and vectors
10.02 Rotations
10.03 Dilations
10.04 Reflections
10.05 General linear and inverse transformations
10.06 Composition of transformations
10.07 Geometric results of transformations
Chapter summary
Chapter review

Transformations in the plane

- understand translations and their representation as column vectors
- define and use basic linear transformations: dilations of the form $(x, y) \rightarrow (\lambda_1 x, \lambda_2 y)$, rotations about the origin and reflection in a line that passes through the origin, and the representations of these transformations by 2×2 matrices
- apply these transformations to points in the plane and geometric objects
- define and use composition of linear transformations and the corresponding matrix products
- define and use inverses of linear transformations and the relationship with the matrix inverse
- examine the relationship between the determinant and the effect of a linear transformation on area
- establish geometric results by matrix multiplications

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

Shutterstock.com/Megan Betteridge

TERMINOLOGY

composition
determinant
dilation
identity
image
inverse
linear transformation
non-singular
object
reflection
rotation
singular
translation

10.01 Translations and vectors

A translation slides points or objects in the plane. Points stay the same distance apart and in the same directions from each other. A translation has a direction and magnitude.

Translations

- A **translation** moves every point in the plane the same distance in the same direction
- A translation (a, b) is given by $\mathbf{T}(x, y) \rightarrow (x', y')$, where $x' = x + a, y' = y + b$
- A translation can be treated as a row or column vector
- When a transformation is applied, the original point or shape is called the **object** and the result is called the **image**

You should write points as column vectors when you write translations as column vectors.

EXAMPLE 1

Find the translation vector for a movement 3 units up and 4 units left and use it to translate (4, 2).

Solution

4 units left subtracts 4 from the x-value and 3 units up adds 3 to the y-value.

The translation is $\mathbf{T}(x, y) \rightarrow (x - 4, y + 3)$

Write the vector.

$$\mathbf{T} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

ISBN 9780170412995

Apply **T** to the point $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

$$\begin{bmatrix} -4 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Write the answers.

The transformation vector is $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and the new point is (0, 5).

You get the same answer whether you are adding the translation vector on the left or the right. However, it is helpful to get used to putting the transformation on the left.

INVESTIGATION

TRANSLATIONS AND LINEARITY

You saw in Chapter 8 that multiplication by complex numbers is a **linear transformation** of the complex plane. What about translation?

If you treat points as vectors, then you can add them and multiply them by real numbers.

Consider the vectors $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$ and the translation $\mathbf{T}(x, y) \to (x + 2, y - 3)$.

Work out $\mathbf{v} + \mathbf{w}$, $5\mathbf{v}$, $\mathbf{T}(\mathbf{v})$, $\mathbf{T}(\mathbf{w})$, $\mathbf{T}(\mathbf{v} + \mathbf{w})$ and $5\mathbf{T}(\mathbf{v})$.

Is it true that $\mathbf{T}(\mathbf{v} + \mathbf{w}) = \mathbf{T}(\mathbf{v}) + \mathbf{T}(\mathbf{w})$?

Is it true that $\mathbf{T}(5\mathbf{v}) = 5\mathbf{T}(\mathbf{v})$?

Is the translation $\mathbf{T}(x, y) \to (x + 2, y - 3)$ a linear transformation?

What can you conclude about translations?

EXAMPLE 2

Show the triangle $A(3, 4)$, $B(5, -2)$, $C(-1, 2)$ and its translation under $\mathbf{T}(x, y) \to (x + 3, y - 1)$ on the same axes.

Solution

Do the translations.

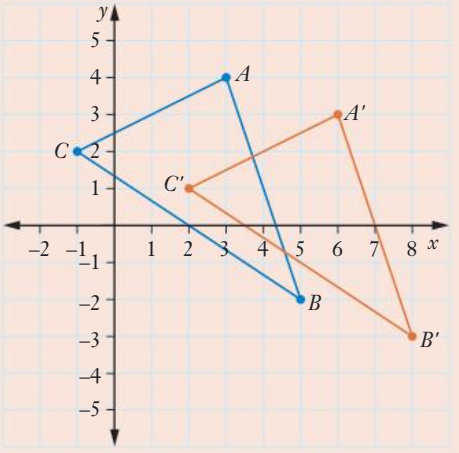
$$A\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = A'\begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$B\begin{bmatrix} 5 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = B'\begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

$$C\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = B'\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

ISBN 9780170412995

Show ABC and $A'B'C'$ in different colours on the same axes.



Exercise 10.01 Translations and vectors

Example 1

1 Write the translation vector for each translation.

a 3 units down and 5 units right
b 5 units up and 4 units right
c 4 units down and 1 unit left
d 6 units left
e 2 units up
f 6 units right and 3 units down
g 2 units left and 6 units down
h 3 units right and 7 units up

2 Find the image of each point under the translation $\mathbf{T}(x, y) \rightarrow (x + 1, y - 3)$.

a $(1, 6)$ **b** $(-1, 4)$ **c** $(3, -5)$ **d** $(-6, -4)$

3 Find the image of each point under the translation $\mathbf{T}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+4 \\ y+1 \end{bmatrix}$.

a $(-3, 7)$ **b** $(-1, 6)$ **c** $(-3, -1)$ **d** $(6, 2)$

4 Find the image of each point under the translation with vector $\begin{bmatrix} -5 \\ 6 \end{bmatrix}$.

a $(4, 7)$ **b** $(3, -5)$ **c** $(-2, -20)$ **d** $(-5, 8)$

5 Find the image of each point under the translation with vector $\begin{bmatrix} -3 \\ -6 \end{bmatrix}$.

a $(-6, -5)$ **b** $(3, 4)$ **c** $(9, -7)$ **d** $(-4, 9)$

Example 2

6 Show the triangle $P(2, 5)$, $Q(6, 4)$, $R(-2, -2)$ and its translation under $\mathbf{T}(x, y) \rightarrow (x - 2, y - 4)$ on the same axes.

7 Show the triangle $A(2, 1)$, $B(5, 3)$, $C(4, 0)$ and its translation by the vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ on the same axes.

ISBN 9780170412995

8 Show the rectangle $V(2, 4)$, $U(5, 1)$, $W(3, -1)$, $X(0, 2)$ and its translation by the vector $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$ on the same axes.

9 Show the pentagon $D(-4, -4)$, $E(-1, 2)$, $F(4, 1)$, $G(4, -2)$, $H(2, -4)$ and its translation by the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ on the same axes.

Problem solving

10 The translation $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ is applied to a point $\begin{bmatrix} x \\ y \end{bmatrix}$ in the plane followed by $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$. Find the combined transformation and generalise to the combination of any two translations.

11 Show that if two translations $\mathbf{T}_1$ and $\mathbf{T}_2$ are applied in succession, then it does not matter if the order is $\mathbf{T}_1$ then $\mathbf{T}_2$ or $\mathbf{T}_2$ then $\mathbf{T}_1$.

10.02 Rotations

Rotations

INVESTIGATION

ROTATIONS

Consider the vector $\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \end{bmatrix}$ rotated through 43° around the origin.

This is shown on the Cartesian plane below.

The vector $A'B' = OC'$, and $OA' = C'B'$, so

$OB' = OA' + OC'$

Now complete the following.

OA' is of length 5, so you can write

$OA' = \begin{bmatrix} 5\cos(43°) \\ \cdots \end{bmatrix}$

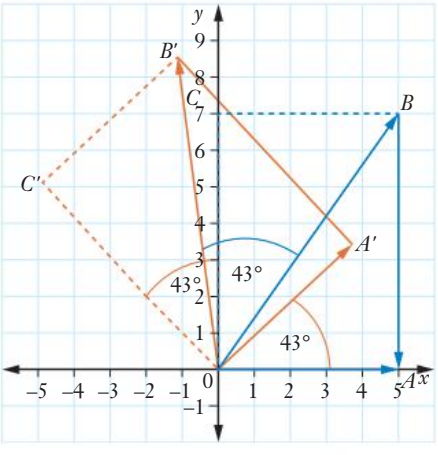
OC' is of length 7, so you can write

$OC' = \begin{bmatrix} -7___ \\ 7___ \end{bmatrix}$

You can write that:

$OB' = OA' + OC'$

$= \begin{bmatrix} 5\cos(43°) \\ \cdots \end{bmatrix} + \begin{bmatrix} -7___ \\ 7___ \end{bmatrix}$



$$= \begin{bmatrix} 5\cos(43°) - 7__ \\ +7__ \end{bmatrix}$$

$$= \begin{bmatrix} \cos(43°) & -__ \\ __ & __ \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

It is obvious that any vector will be rotated in the same way as $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ is rotated 43° around the origin, so you can generalise.

Rotation around the origin

- Rotation through the angle θ around the origin, $\mathbf{R}_\theta$, is given by

$$\mathbf{R}_\theta : \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}$$

- The matrix for a rotation is given by $\mathbf{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Shutterstock.com/Tatiana Kasyanova

ISBN 9780170412995

EXAMPLE 3

a Find the transformation for a rotation of $\frac{4\pi}{3}$ around the origin.

b Find the image of the triangle $A(1, 3)$, $B(6, 2)$, $C(4, -3)$ under the rotation.

c Sketch the triangle and its image on the same axes.

Solution

a Write the transformation.

$$\mathbf{R}_{\frac{4\pi}{3}}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x\cos\left(\frac{4\pi}{3}\right) - y\sin\left(\frac{4\pi}{3}\right) \\ \sin\left(\frac{4\pi}{3}\right)x + y\cos\left(\frac{4\pi}{3}\right) \end{bmatrix}$$

Remember sin and cos are both negative in the third quartile and substitute values.

$$\mathbf{R}_{\frac{4\pi}{3}}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x\times\left(-\frac{1}{2}\right) - y\times\left(-\frac{\sqrt{3}}{2}\right) \\ x\times\left(-\frac{\sqrt{3}}{2}\right)x + y\times\left(-\frac{1}{2}\right) \end{bmatrix}$$

Simplify.

$$\mathbf{R}_{\frac{4\pi}{3}}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{\sqrt{3}y - x}{2} \\ -\frac{\sqrt{3}x + y}{2} \end{bmatrix}$$

b Apply to the points.

$$A(1, 3) \to A'\left(\frac{3\sqrt{3}-1}{2}, -\frac{\sqrt{3}+3}{2}\right)$$

$$B(6, 2) \to B'\left(\frac{2\sqrt{3}-6}{2}, -\frac{6\sqrt{3}+2}{2}\right)$$

$$C(4, -3) \to C'\left(\frac{-3\sqrt{3}-4}{2}, -\frac{4\sqrt{3}-3}{2}\right)$$

Write the image.

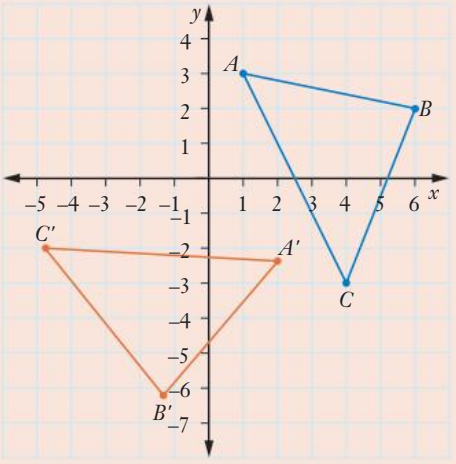
The image is $A'\left(\frac{3\sqrt{3}-1}{2}, -\frac{\sqrt{3}+3}{2}\right)$,

$B'(\sqrt{3} - 3, -3\sqrt{3} - 1)$, $C'\left(\frac{-3\sqrt{3}-4}{2}, \frac{3-4\sqrt{3}}{2}\right)$

ISBN 9780170412995

c Find the approximate image. $A'B'C' \approx (2.1, -2.4)\ (-1.3, -6.2)\ (-4.6, -2.0)$

Sketch ABC and $A'B'C'$ on the same axes.



Matrix multiplications

You can use matrix multiplication to find the image of an object under a rotation.

You should check that $\begin{bmatrix} \cos\left(\frac{4\pi}{3}\right) & -\sin\left(\frac{4\pi}{3}\right) \\ \sin\left(\frac{4\pi}{3}\right) & \cos\left(\frac{4\pi}{3}\right) \end{bmatrix} \begin{bmatrix} 1 & 6 & 4 \\ 3 & 2 & -3 \end{bmatrix}$ gives the correct answer for Example 3. The matrix for a rotation of, say, 60° around the origin is written as $\mathbf{R}_{60°}$.

EXAMPLE 4

a Write the matrix for a rotation of 30° around the origin.

b Use the matrix to find the image of the rectangle $P(-1, 2)$, $Q(3, 5)$, $R(9, -3)$, $S(5, -6)$ under the rotation.

Solution

a Write the rotation matrix.

$$\mathbf{R}_{30°} = \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix}$$

Substitute values.

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

ISBN 9780170412995

b Multiply by the matrix of points.

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -1 & 3 & 9 & 5 \\ 2 & 5 & -3 & -6 \end{bmatrix}$$

Calculate the answer.

$$= \begin{bmatrix} \frac{-\sqrt{3}-2}{2} & \frac{3\sqrt{3}-5}{2} & \frac{9\sqrt{3}+3}{2} & \frac{5\sqrt{3}+6}{2} \\ \frac{-1+2\sqrt{3}}{2} & \frac{3+5\sqrt{3}}{2} & \frac{9-3\sqrt{3}}{2} & \frac{5-6\sqrt{3}}{2} \end{bmatrix}$$

Write the image.

The image is

$$P'\left(\frac{-\sqrt{3}-2}{2}, \frac{-1+2\sqrt{3}}{2}\right), Q'\left(\frac{3\sqrt{3}-5}{2}, \frac{3+5\sqrt{3}}{2}\right),$$

$$R'\left(\frac{9\sqrt{3}+3}{2}, \frac{9-3\sqrt{3}}{2}\right), S'\left(\frac{5\sqrt{3}+6}{2}, \frac{5-6\sqrt{3}}{2}\right)$$

$\approx P'(-1.87, 2.60), Q'(0.10, 1.73), R'(9.29, -10.39),$
$S'(7.33, -9.53)$

TI 84 Plus CE

Enter the rotation matrix as, say, **A**.

Press mode to check the calculator is set to the correct angle measure, set the accuracy as desired and quit.

Press 2nd x⁻¹ (matrix), move the cursor across to EDIT, down to A, and press enter.

Now press 2 enter, 2 enter for the size and put in the values. Make sure you use (-) for negative.

Quit and do the matrix of points as, say, **B**.

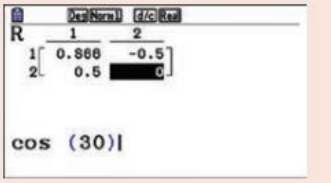
Go back to the matrices and press enter to get **A** from NAMES. Multiply by **B** from NAMES.

If you have long decimals, use the cursor keys to go up and across to get the full answers.

ISBN 9780170412995

Casio fx-CG20AU

Use the Run-Matrix menu. Use the SET UP (SHIFT MENU) to check the angle is set correctly and EXIT.



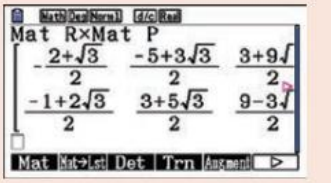
Press F3 (▶MAT).

Use the cursor to go down to Mat R.

Press F3 again (DIM), 2 EXE and 2 EXE for the size and put in the values.

Press EXIT, move down and do the matrix of points as, say, **P**. Press EXIT to leave matrix entry.

Press OPTN F2 F1 (Mat), then ALPHA R.



Press ✕ and do Mat P as for Mat R and press EXE.

You can also use SHIFT 2 for Mat.

The Casio gives exact answers where possible. Use the cursor keys to go up and across to see the complete answers. Press F↔D to get approximate answers.

A rotation around the origin has the format $\mathbf{R} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2 + b^2 = 1$. The angle of rotation is given by $\sin^{-1}(b)$. Rotations are linear transformations.

Exercise 10.02 Rotations

Your teacher will choose which problems to do using your graphics calculator.

Example 3

1 Write the transformation for a rotation around the origin of:

a $\frac{\pi}{4}$ **b** 60° **c** $-\frac{\pi}{6}$ **d** 150° **e** $\frac{5\pi}{3}$

2 Write the transformation for a rotation through each angle around the origin, correct to 2 decimal places.

a 28° **b** 3 **c** –160° **d** –2.4 **e** 38°

3 a Find the image of the triangle $A(1, 1)$, $B(4, 4)$, $C(4, -2)$ after a rotation of $\frac{\pi}{3}$ around the origin.

b Sketch the triangle and its image on the same axes.

ISBN 9780170412995

4 **a** Find the image of the rhombus $E(1, -2)$, $F(3, 3)$, $G(8, 5)$, $H(6, 0)$ after a rotation of 210° around the origin.

b Sketch the rhombus and its image on the same axes.

5 **a** Find the image of the shape $R(-1, 2)$, $S(3, 6)$, $T(4, -3)$, $U(2, 3)$ after a rotation of $-\frac{5\pi}{6}$ around the origin.

b Sketch the shape and its image on the same axes.

6 Write the matrix for each rotation around the origin.

Example 4

a $\frac{7\pi}{4}$ **b** $-60°$ **c** $\frac{3\pi}{4}$ **d** $225°$ **e** $\frac{11\pi}{6}$

7 Write the matrix for each rotation around the origin, correct to 2 decimal places.

a $-130°$ **b** 0.8 **c** 2.5 **d** $208°$ **e** -1.4

8 Use a matrix to find the image of the triangle $P(-2, 3)$, $Q(0, 2)$, $R(1, 4)$ after each rotation around the origin.

a $\frac{5\pi}{4}$ **b** $120°$ **c** 2.9 (to 2 decimal places).

9 Use a matrix to find the image of the rectangle $A(-3, 3)$, $B(0, 5)$, $C(4, -1)$, $D(1, -3)$ after each rotation around the origin.

a $240°$ **b** $-\frac{2\pi}{3}$ **c** $100°$ (to 2 decimal places).

10 Use a matrix to find the image of the parallelogram $P(-2, -3)$, $R(1, 1)$, $G(4, 2)$, $M(1, -3)$ after each rotation around the origin.

a $-\frac{7\pi}{6}$ **b** $150°$ **c** 0.4 (to 2 decimal places).

Problem solving

11 What transformation is needed to reverse the effect of a rotation through an angle θ around the origin?

12 If two rotations are performed one after the other, does it make any difference if the order is changed?

13 Show that rotation through an angle of 30° around the origin is a linear transformation.

14 Show that any rotation around the origin is a linear transformation.

ISBN 9780170412995

10.03 Dilations

iStockphoto/3D_generator

You can magnify or shrink objects or vectors in the x-direction. You multiply the x-values by the factor λ. In this case, the transformation is given by **D**: $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} \lambda x \\ y \end{bmatrix}$.

In the y-direction, you get **D**: $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ \lambda y \end{bmatrix}$.

The origin is the centre of the stretch or compression.

You can combine these **dilations** to get the general case for both directions.

Dilations

For dilations that have their centre at the origin,

- dilation by factors λ_1 and λ_2 in the x and y directions respectively is given by

  D: $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_1 x \\ \lambda_2 y \end{bmatrix}$

- the matrix for dilation is $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

In this book, all dilations are assumed to have their **centre at the origin** unless otherwise indicated.

ISBN 9780170412995

EXAMPLE 5

a Find the transformation for a vertical dilation with factor 2 and horizontal dilation with factor 1.5.

b Find the image of the triangle $A(-1, -2)$, $B(1, 2)$, $C(4, 0)$ under the transformation.

c Sketch the triangle and its image on the same axes.

Solution

a Write the general dilation.

$$\mathbf{D}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \lambda_1 x \\ \lambda_2 y \end{bmatrix}$$

Substitute values $\lambda_1 = 1.5$ and $\lambda_2 = 2$.

$$\mathbf{D}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1.5x \\ 2y \end{bmatrix}$$

b Apply to the points.

$$A\begin{bmatrix} -1 \\ -2 \end{bmatrix} \to A'\begin{bmatrix} -1.5 \\ -4 \end{bmatrix}$$

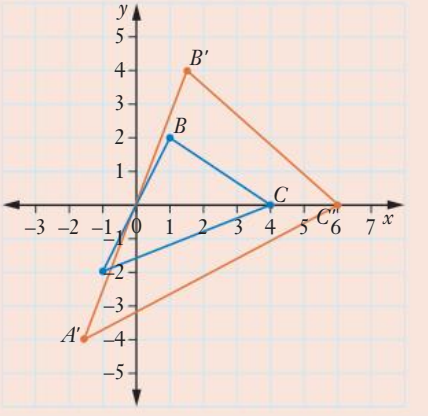
$$B\begin{bmatrix} 1 \\ 2 \end{bmatrix} \to B'\begin{bmatrix} 1.5 \\ 4 \end{bmatrix}$$

$$C\begin{bmatrix} 4 \\ 0 \end{bmatrix} \to C'\begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Write the image.

The image is $A'(-1.5, -4)$, $B'(1.5, 4)$, $C'(6, 0)$.

c Show on the same axes.



You can use matrices for dilations from the origin. Dilations are also linear transformations.

ISBN 9780170412995

EXAMPLE 6

a Write the matrix for a dilation with x-factor 2 and y-factor 0.5.

b Use the matrix to find the image of the square $P(-3, 1)$, $Q(0, 3)$, $R(2, 0)$, $S(-1, -2)$ under the transformation.

c Sketch the square and its image on the same Cartesian axes.

d What is the shape of the image?

Solution

a Write the general dilation matrix.

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Substitute values $\lambda_1 = 2$ and $\lambda_2 = 0.5$.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

b Multiply by the vectors.

$$\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

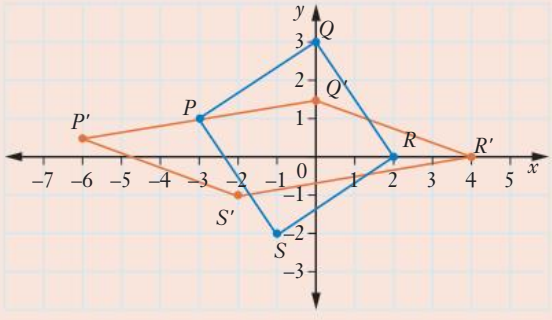
$$\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

Write the image.

The image is $P'(-6, 0.5)Q'(0, 1.5)R'(4, 0)S'(-2, -1)$.

c Draw the shapes.



d $P'Q'R'S'$ looks like a parallelogram. Check the sides.

$$m_{P'Q'} = \frac{1.5-0.5}{0-(-6)} = \frac{1}{6},\ m_{S'R'} = \frac{-2-4}{-1-0} = \frac{1}{6}$$

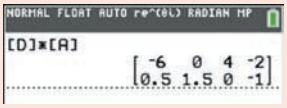
$$m_{Q'R'} = \frac{0-1.5}{4-0} = -\frac{3}{8},\ m_{P'S'} = \frac{-1-0.5}{-2-(-6)} = -\frac{3}{8}$$

ISBN 9780170412995

Write the conclusion. The opposite sides are parallel, so $P'Q'R'S'$ is a parallelogram.

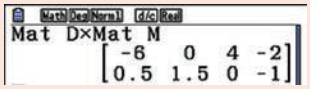
TI 84 Plus CE

Enter and do the matrix multiplication as shown on page 411. Use, say, **A** for the points matrix.



Casio fx-CG20AU

Enter and do the matrix multiplication as shown on page 412.



TI-Nspire CX Chapter 10

Exercise 10.03 Dilations

Your teacher will choose which problems to do using your graphics calculator.

Example 5

1 Write the transformation for a dilation with each pair of factors.

- **a** vertical factor 2 and horizontal factor 3
- **b** horizontal factor 0.5 and vertical factor 0.5
- **c** $\lambda_1 = 3$ and $\lambda_2 = 0.6$
- **d** x-factor 5 and y-factor 3
- **e** x-factor 1.5 and y-factor 2.5

2 **a** Find the transformation for a vertical dilation with factor 2 and a horizontal dilation with factor 1.5.

- **b** Find the image of the triangle $X(-1, 2)$, $Y(1, 2)$, $Z(4, 0)$ under the transformation.
- **c** Sketch the triangle and its image on the same axes.

3 **a** Find the transformation for a vertical dilation with factor 3 and a horizontal dilation with factor 2.

- **b** Find the image of the quadrilateral $E(1, 2)$, $F(-1, -2)$, $G(2, -3)$, $H(4, 0)$ under the transformation.
- **c** Sketch the quadrilateral and its image on the same axes.

4 **a** Find the transformation for a horizontal dilation with factor 2.5 and a vertical dilation with factor 2.5.

- **b** Find the image of the trapezium $A(-2, 1)$, $B(-1, -3)$, $C(3, -1)$, $D(6, 5)$ under the transformation.
- **c** Sketch the trapezium and its image on the same axes.

ISBN 9780170412995

Example 6

5 Write the matrix for the dilation with each factor.

a Vertical factor 4

b Horizontal factor 3

c $\lambda_1 = 4$ and $\lambda_2 = 2$

d x-factor 3 and y-factor 0.6

e x-factor 0.5 and y-factor 0.3

6 **a** Write the matrix for a dilation with x-factor 1.5 and y-factor 2.5.

b Use the matrix to find the image of the rhombus $P(-8, 3)$, $Q(0, 4)$, $R(4, -3)$, $S(-4, -4)$ under the transformation.

c Sketch the rhombus and its image on the same Cartesian axes.

d What is the shape of the image?

7 **a** Write the matrix for a dilation with x-factor 2.5 and y-factor 3.

b Use the matrix to find the image of the parallelogram $S(-3, 3)$, $T(2, 1)$, $U(2, -2)$, $V(-3, 0)$ under the transformation.

c Sketch the square and its image on the same Cartesian axes.

d What is the shape of the image?

8 **a** Write the matrix for a dilation with x-factor $\frac{1}{3}$ and y-factor $\frac{1}{2}$.

b Use the matrix to find the image of the rectangle $R(9, -6)$, $C(6, 6)$, $T(-15, 0)$, $A(-12, -12)$ under the transformation.

c Sketch the square and its image on the same Cartesian axes.

d What is the shape of the image?

Problem solving

9 What transformation is needed to reverse the effect of a dilation with factors $\lambda_1 = \frac{1}{2}$ and $\lambda_1 = 3$ centred at the origin?

10 If two dilations are performed one after the other, does it make any difference if the order is changed?

11 Show that a dilation with factors $\lambda_1 = 3$ and $\lambda_2 = 4$ is a linear transformation.

12 Show that any dilation with its centre at the origin is a linear transformation.

ISBN 9780170412995

10.04 Reflections

Reflections

Shutterstock.com/Keith Reierson

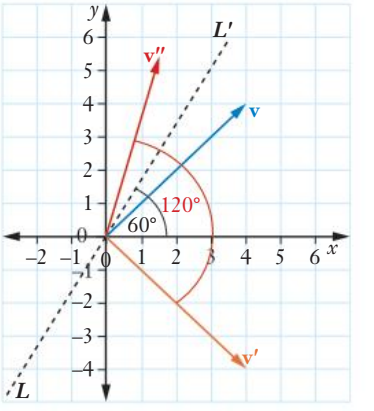
INVESTIGATION

REFLECTION AND ROTATION

Consider a line LL' with inclination 60° through the origin and the vector $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

What do you get by reflecting $\mathbf{v}$ in the x-axis and rotating it 120° around the origin?

Copy and complete the following.



The angle between the x-axis and $\mathbf{v}$ is _____ .

The angle between the reflected vector $\mathbf{v}'$ and the x-axis is _____ .

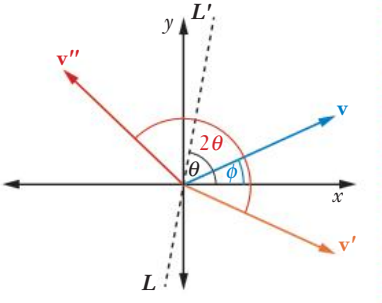
The angle between LL' and $\mathbf{v}'$ is _____ .

The angle between the rotated vector $\mathbf{v}''$ and the x-axis is _____ .

$\mathbf{v}''$ is the _____ of $\mathbf{v}$ in the line LL'.

Now consider an arbitrary vector $\mathbf{v}$ at an angle φ to the x-axis and a line LL' at angle θ passing through the origin. Reflect the vector in the x-axis and rotate it 2θ around the origin.

Copy and complete the following.



The angle between the x-axis and $\mathbf{v}$ is _____ .

The angle between the reflected vector $\mathbf{v}'$ and the x-axis is _____ .

The angle between LL' and $\mathbf{v}'$ is _____ .

The angle between the rotated vector $\mathbf{v}''$ and the x-axis is _____ .

$\mathbf{v}''$ is the _____ of $\mathbf{v}$ in the line LL'.

ISBN 9780170412995

From the investigation, you should be able to obtain the reflection of a vector in a line of slope θ through the origin. It is the same as a reflection in the x-axis, followed by rotation through an angle 2θ.

For the vector $\begin{bmatrix} x \\ y \end{bmatrix}$, the reflection in the x-axis is $\begin{bmatrix} x \\ -y \end{bmatrix}$.

Rotation through an angle of 2θ is given by $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x\cos(2\theta) - y\sin(2\theta) \\ x\sin(2\theta) + y\cos(2\theta) \end{bmatrix}$.

The reflection is given by $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ -y \end{bmatrix} \rightarrow \begin{bmatrix} x\cos(2\theta) - (-y)\sin(2\theta) \\ x\sin(2\theta) + (-y)\cos(2\theta) \end{bmatrix}$.

This simplifies to $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x\cos(2\theta) + y\sin(2\theta) \\ x\sin(2\theta) - y\cos(2\theta) \end{bmatrix}$.

Reflections

- Reflection in a line with inclination θ through the origin, **M**, is given by

  $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x\cos(2\theta) + y\sin(2\theta) \\ x\sin(2\theta) - y\cos(2\theta) \end{bmatrix}$

- The matrix for the reflection is $\mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

- Reflection in a line of slope m through the origin is given by

  $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x\left(\frac{1-m^2}{1+m^2}\right) + y\left(\frac{2m}{1+m^2}\right) \\ x\left(\frac{2m}{1+m^2}\right) - y\left(\frac{1-m^2}{1+m^2}\right) \end{bmatrix}$

- The matrix for the reflection is $\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & -\frac{1-m^2}{1+m^2} \end{bmatrix}$.

Using multi-angle formulas, you can show that for a line of slope m and inclination θ:

$$\cos(2\theta) = \frac{1-m^2}{1+m^2} \quad \text{and} \quad \sin(2\theta) = \frac{2m}{1+m^2}.$$

ISBN 9780170412995

EXAMPLE 7

a Find the transformation for a reflection in a line through the origin with inclination 30°.

b Find the transformation for the reflection in the line $y = 2x$.

c Find the reflection of the kite $A(-5, 4)$, $B(0, 4)$, $C(3, 0)$, $D(-4, -4)$ in the line $y = 2x$.

d Sketch $ABCD$ and $A'B'C'D'$ on the same coordinate system.

Solution

a Write the general reflection.

$$\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x\cos(2\theta) + y\sin(2\theta) \\ x\sin(2\theta) - y\cos(2\theta) \end{bmatrix}$$

Substitute $\theta = 30°$

$$\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x\cos(60°) + y\sin(60°) \\ x\sin(60°) - y\cos(60°) \end{bmatrix}$$

Simplify.

$$\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \dfrac{x + \sqrt{3}y}{2} \\ \dfrac{\sqrt{3}x - y}{2} \end{bmatrix}$$

b State the nature of the line.

$y = 2x$ passes through the origin.

Find the parameters.

$$\cos(2\theta) = \frac{1 - m^2}{1 + m^2} = \frac{1 - 2^2}{1 + 2^2} = -0.6$$

$$\sin(2\theta) = \frac{2m}{1 + m^2} = \frac{2 \times 2}{1 + 2^2} = 0.8$$

Write the transformation.

$$\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -0.6x + 0.8y \\ 0.8x + 0.6y \end{bmatrix}$$

c Apply to the points.

$$A\begin{bmatrix} -5 \\ 4 \end{bmatrix} \to A'\begin{bmatrix} 6.2 \\ -1.6 \end{bmatrix}$$

$$B\begin{bmatrix} 0 \\ 4 \end{bmatrix} \to B'\begin{bmatrix} 3.2 \\ 2.4 \end{bmatrix}$$

$$C\begin{bmatrix} 3 \\ 0 \end{bmatrix} \to C'\begin{bmatrix} -1.8 \\ 2.4 \end{bmatrix}$$

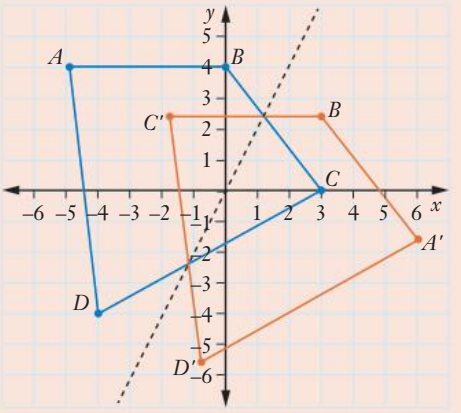
$$D\begin{bmatrix} -4 \\ -4 \end{bmatrix} \to D'\begin{bmatrix} -0.8 \\ -5.6 \end{bmatrix}$$

ISBN 9780170412995

Write the image.

The image is $A'(6.2, -1.6)$, $B'(3.2, 2.4)$, $C'(-1.8, 2.4)$, $D'(-0.8, -5.6)$.

d Sketch on the same axes.



You can use matrices for reflections.

EXAMPLE 8

a Write the matrix for a reflection in the line through the origin with inclination $\frac{2\pi}{3}$.

b Find the image of $F(2, 4)$, $G(-2, 1)$, $H(3, -1)$ under the reflection, correct to 1 decimal place.

c Sketch the object and image on the same axes.

Solution

a Write the general reflection matrix.

$$\mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Substitute $\theta = \frac{2\pi}{3}$.

$$= \begin{bmatrix} \cos\left(\frac{4\pi}{3}\right) & \sin\left(\frac{4\pi}{3}\right) \\ \sin\left(\frac{4\pi}{3}\right) & -\cos\left(\frac{4\pi}{3}\right) \end{bmatrix}$$

Simplify.

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

ISBN 9780170412995

b Multiply by points matrix.

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ 4 & 1 & -1 \end{bmatrix}$$

Simplify.

$$= \begin{bmatrix} -1-2\sqrt{3} & \frac{2-\sqrt{3}}{2} & \frac{-3+\sqrt{3}}{2} \\ -\sqrt{3}+2 & \frac{2\sqrt{3}-1}{2} & \frac{-3\sqrt{3}-1}{2} \end{bmatrix}$$

Write the image.

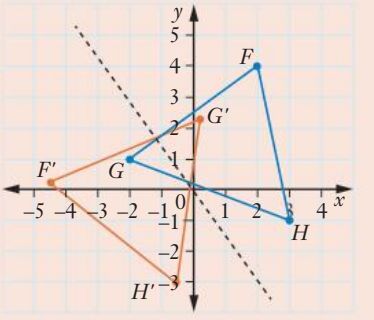
The image is $F'(-1-2\sqrt{3}, -\sqrt{3}+2)$,

$G'\left(\frac{2-\sqrt{3}}{2}, \frac{2\sqrt{3}-1}{2}\right)$, $H'\left(\frac{-3+\sqrt{3}}{2}, \frac{-3\sqrt{3}-1}{2}\right)$

Write to 1 decimal place.

$\approx F'(-4.5, 0.3)$, $G'(0.1, 2.2)$, $H'(-0.6, -3.1)$

c Sketch on the same axes.



Reflections in lines passing through the origin have the format $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$.

The angle of inclination of the line or reflection is given by $\frac{1}{2}\sin^{-1}(b)$. A reflection in a line through the origin is a linear transformation. You can use your graphics calculator for reflections just as you did for rotations.

Exercise 10.04 Reflections

Your teacher will choose which problems to do using your graphics calculator.

1 Find the transformation for a reflection:

- **a** in a line through the origin with inclination 210°
- **b** in the line $y = -2x$
- **c** in a line through the origin with inclination $\frac{3\pi}{4}$

Example 7

ISBN 9780170412995

d in the line $y = \frac{1}{2}x$

e in a line through the origin with inclination $-120°$.

f in the line $y = -x$

2 a Find the transformation for a reflection in a line through the origin with inclination $-\frac{\pi}{12}$.

b Find the reflection of the rectangle $R(1, 4)$, $E(4, 3)$, $C(2, -3)$, $D(-1, -2)$ in the line.

c Sketch $RECD$ and $R'E'C'D'$ on the same coordinate system.

3 a Find the transformation for the reflection in the line $y = -\frac{1}{2}x$.

b Find the reflection of the parallelogram $P(-2, 3)$, $L(5, 4)$, $G(4, 2)$, $M(-3, 1)$ in the line.

c Sketch $PLGM$ and $P'L'G'M'$ on the same coordinate system.

4 a Find the transformation for a reflection in a line through the origin with inclination $60°$.

b Find the reflection of the square $S(5, 4)$, $Q(1, 6)$, $R(-1, 2)$, $E(3, 0)$ in the line.

c Sketch $SQRE$ and $S'Q'R'E'$ on the same coordinate system.

5 a Find the transformation for the reflection in the line $y = x$.

b Find the reflection of the trapezium $T(-3, 2)$, $R(3, -1)$, $A(4, -3)$, $P(-4, 1)$ in the line.

c Sketch $TRAP$ and $T'R'A'P'$ on the same coordinate system.

Example 8

6 Write the matrix for a reflection in the line:

a through the origin with inclination $\frac{5\pi}{6}$

b $y = 3x$

c through the origin with inclination $-300°$

d $y = -\frac{1}{3}x$

e through the origin with inclination $\frac{\pi}{4}$

f $y = 2x$

7 a Write the matrix for a reflection in the line through the origin with inclination $120°$.

b Find the image of kite $K(3, 6)$, $I(-1, 4)$, $T(1, 0)$, $E(8, 1)$ under the reflection, correct to 1 decimal place.

c Sketch the object and its image on the same axes.

8 a Write the matrix for a reflection in the line $y = 3x$.

b Find the image of rhombus $R(2, -7)$, $M(3, 1)$, $B(-4, 5)$, $S(-5, -3)$ under the reflection.

c Sketch the object and image on the same axes.

ISBN 9780170412995

9 a Write the matrix for reflection in the line through the origin with inclination $-\frac{\pi}{3}$.

b Find the reflection of the parallelogram $G(6, -1)$, $R(-1, 1)$, $A(-4, -3)$, $M(3, -5)$ in the line.

c Sketch the object and image on the same axes.

10 a Write the matrix for reflection in the line $y = 7x$.

b Find the image of square $M(5, 5)$, $N(-2, 6)$, $P(-3, -1)$, $Q(4, -2)$ under the reflection, correct to 1 decimal place.

c Sketch the object and image on the same axes.

Problem solving

11 What transformation is needed to reverse the effect of a reflection in the line $y = -3x$?

12 If two reflections are performed one after the other, does it make any difference if the order is changed?

13 Show that reflection in the line through the origin with inclination 60° is a linear transformation.

14 Show that any reflection across any line through the origin is a linear transformation.

10.05 General linear and inverse transformations

Any linear transformation can be written as a linear combination of x and y components. This means that any linear transformation can be shown as a matrix.

Linear transformations

- Any linear transformation is of the form $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$, where $a, b, c, d \in \boldsymbol{R}$
- It follows that every linear transformation has a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Proof

Any vector in the plane can be written as a combination of the unit vectors. In particular, for any transformation $\mathbf{T}$, you can write $\mathbf{T}\mathbf{i} = a\mathbf{i} + c\mathbf{j}$ and $\mathbf{T}\mathbf{j} = b\mathbf{i} + d\mathbf{j}$.

Suppose $\mathbf{T}$ is linear

$\Rightarrow \mathbf{T}(x\mathbf{i} + y\mathbf{j}) = x\mathbf{T}\mathbf{i} + y\mathbf{T}\mathbf{j} = xa\mathbf{i} + xc\mathbf{j} + yb\mathbf{i} + yd\mathbf{j} = (ax + by)\mathbf{i} + (cx + dy)\mathbf{j}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ QED

ISBN 9780170412995

It is easy to show that a transformation of the form $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$, so the condition is both necessary and sufficient for linearity.

Notice that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} b \\ d \end{bmatrix}$, as chosen in the proof.

The 2×2 matrix identity is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A 2×2 matrix $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse if and only if the determinant $|\mathbf{M}| = ad - bc \neq 0$. In that case, the inverse is $\frac{1}{|\mathbf{M}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If follows that the same conditions apply to transformations of the plane.

Inverse transformations

- The identity transformation is $\mathbf{I}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix}$
- A transformation $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$, has an inverse if and only if the determinant of the transformation, $|\mathbf{T}| = ad - bc \neq 0$
- A transformation with no inverse is called **singular** and one with an inverse is called **non-singular**
- The **inverse** of a non-singular transformation $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ is given by

$$\mathbf{T}^{-1}: \begin{bmatrix} x \\ y \end{bmatrix} \to \frac{1}{ad-bc}\begin{bmatrix} dx-by \\ -cx+ay \end{bmatrix}$$

The inverses of rotations, dilations and reflections are those that restore the original points.

EXAMPLE 9

Use matrices to show that the inverse of a rotation through an angle of 30° is a rotation through −30°.

Solution

Write the matrix for a rotation through 30°.

$$\mathbf{R}_{30°} = \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

ISBN 9780170412995

Write the matrix for a rotation through $-30°$.

$$\mathbf{R}_{-30°} = \begin{bmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Multiply the matrices and simplify.

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4}+\frac{1}{4} & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & \frac{3}{4}+\frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Write the conclusion. $\mathbf{R}_{30°} \times \mathbf{R}_{-30°} = \mathbf{I}$, so $\mathbf{R}_{-30°}$ is the inverse of $\mathbf{R}_{30°}$. QED

EXAMPLE 10

State whether each transformation is singular or non-singular and find the inverse of any that have them.

a $\mathbf{T}_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x-6y \\ x-2y \end{bmatrix}$

b $\mathbf{T}_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 7x-5y \\ 2y-3x \end{bmatrix}$

c $\mathbf{T}_3: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x+7y \\ 4x+11y \end{bmatrix}$

Solution

a Find the determinant.

$$|\mathbf{T}_1| = 3 \times (-2) - (-6) \times 1$$
$$= -6 + 6$$
$$= 0$$

Write the conclusion. $\mathbf{T}_1$ is singular and so has no inverse.

b Find the determinant. Be careful to use $ad - bc$ correctly.

$$|\mathbf{T}_2| = 7 \times 2 - (-5) \times (-3)$$
$$= -1$$
$$\neq 0$$

Write the conclusion. $\mathbf{T}_2$ is non-singular and so has an inverse.

ISBN 9780170412995

Write the inverse matrix. $\mathbf{T}_2^{-1} = \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

Simplify. $= \begin{bmatrix} -2 & -5 \\ -3 & -7 \end{bmatrix}$

Write the inverse transformation. $\mathbf{T}_2^{-1} : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -2x - 5y \\ -3x - 7y \end{bmatrix}$

c Find the determinant. $|\mathbf{T}_3| = 2 \times 11 - 7 \times 4$

$= -6$

$\neq 0$

Write the conclusion. $\mathbf{T}_3$ is non-singular so has an inverse.

Write the inverse matrix. $\mathbf{T}_3^{-1} = \begin{bmatrix} 2 & 7 \\ 4 & 11 \end{bmatrix}^{-1} = \frac{1}{-6}\begin{bmatrix} 11 & -7 \\ -4 & 2 \end{bmatrix}$

Simplify. $= \begin{bmatrix} -1\frac{5}{6} & 1\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

Write the inverse transformation. $\mathbf{T}_3^{-1} : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{7y - 11x}{6} \\ \frac{2x - y}{3} \end{bmatrix}$

TECHNOLOGY

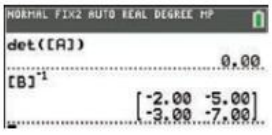
Inverse transformations

You can use your graphics calculator to find inverse transformations.

TI-Nspire CX Chapter 10

TI 84 Plus CE

Enter the matrices of the transformations in Example 10 as A, B and C.



Use matrix MATH, 1: det.

Find the determinants of B and C similarly.

Use matrix NAMES, 2, enter, x^{-1} to find the inverse of B.

Find the inverse of C similarly.

ISBN 9780170412995

Casio fx-CG20AU

Use Run-Matrix.

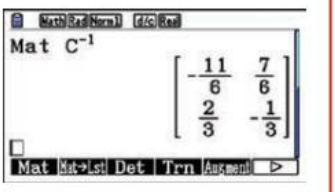
Enter the matrices of the transformations in Example 10 as A, B and C.

Use [OPTN], MAT, Det, Mat, [ALPHA], A, [EXE] to find the determinant of A.

Find the determinants of B and C similarly.

Use [OPTN], MAT, Mat, [ALPHA], C, [^], [(−)] 1, [EXE] to find the inverse of C.

You can also use [SHIFT] 2 to obtain Mat.



Exercise 10.05 General linear and inverse transformations

Your teacher will choose which problems to do using your graphics calculator.

Example 9

1 Use matrices to show that the inverse of a rotation around the origin through an angle of 120° is a rotation through −120°.

2 Use matrices to show that the inverse of a rotation around the origin through an angle of −60° is a rotation through 60°.

3 Use matrices to show that the inverse of a dilation with factors $\lambda_1 = 3$ and $\lambda_2 = \frac{1}{2}$ is a dilation with factors $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = 2$.

4 Use matrices to show that the inverse of a dilation with factors $\lambda_1 = 0.3$ and $\lambda_2 = 5$ is a dilation with factors $\lambda_1 = 3\frac{1}{3}$ and $\lambda_2 = 0.2$.

5 Use matrices to show that the reflection through a line through the origin with inclination 60° is its own inverse.

6 Use matrices to show that the reflection through the line $y = -2x$ is its own inverse.

7 Show that the inverse of the translation 2 up and 5 left is 2 down and 5 right.

8 Show that the inverse of the translation given by the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

9 What is the inverse of the rotation around the origin through the angle 80°?

10 What is the inverse of the dilation with x-factor 3 and y-factor 0.25?

11 What is the inverse of a reflection in the line $y = 0.6x$?

12 What is the inverse of the translation with vector $\begin{bmatrix} -5 \\ -3 \end{bmatrix}$?

ISBN 9780170412995

Example 10

13 State whether each transformation is singular or non-singular and find the inverses of any that have them.

a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x-2y \\ 4y-6x \end{bmatrix}$ **b** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x+y \\ 3x-y \end{bmatrix}$ **c** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x+3y \\ 3x+2y \end{bmatrix}$

d $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 5x+3y \\ 3x+2y \end{bmatrix}$ **e** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x-4y \\ 6y-3x \end{bmatrix}$ **f** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x-y \\ 6x-5y \end{bmatrix}$

Problem solving

14 Prove that a reflection in a line passing through the origin with inclination θ is its own inverse.

15 Prove that the inverse of a rotation around the origin through the angle φ is the rotation around the origin through the angle $-\varphi$.

16 Prove that the inverse of the translation $\begin{bmatrix} a \\ b \end{bmatrix}$ is the translation $\begin{bmatrix} -a \\ -b \end{bmatrix}$.

PS Linear transformations

10.06 Composition of transformations

Transformations are shown to the left of a vector, point or object. A second transformation of the new vector, point or object will also be put on the left. That means that combinations of transformations are performed right to left. This is the opposite of most combinations of operations.

Composition of transformations

- The **composition** of two transformations is the result of doing them one after the other
- The composition of $\mathbf{T}_1$ and $\mathbf{T}_2$ is shown as $\mathbf{T}_2 \circ \mathbf{T}_1$. $\mathbf{T}_1$ is done first, followed by $\mathbf{T}_2$

EXAMPLE 11

Two transformations are given by $\mathbf{T}_1$: $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x+y \\ 3y-x \end{bmatrix}$ and $\mathbf{T}_2$: $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x-3y \\ x+y \end{bmatrix}$.
Find:

a $\mathbf{T}_1\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. **b** $\mathbf{T}_2\left[\mathbf{T}_1\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right]$ **c** $\mathbf{T}_2 \circ \mathbf{T}_1$ **d** $\mathbf{T}_2 \circ \mathbf{T}_1\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

ISBN 9780170412995

Solution

a Apply $\mathbf{T}_1$.

$$\mathbf{T}_1\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}2\times2+4\\3\times4-2\end{bmatrix}$$

$$=\begin{bmatrix}8\\10\end{bmatrix}$$

b Apply $\mathbf{T}_2$ to $\mathbf{T}_1\begin{bmatrix}2\\4\end{bmatrix}$.

$$\mathbf{T}_2\left[\mathbf{T}_1\begin{bmatrix}2\\4\end{bmatrix}\right]=\mathbf{T}_2\begin{bmatrix}8\\10\end{bmatrix}$$

$$=\begin{bmatrix}8-3\times10\\8+10\end{bmatrix}$$

$$=\begin{bmatrix}-22\\18\end{bmatrix}$$

c Apply $\mathbf{T}_1$ to $\begin{bmatrix}x\\y\end{bmatrix}$.

$$\mathbf{T}_1\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2x+y\\3y-x\end{bmatrix}$$

Apply $\mathbf{T}_2$ to $\mathbf{T}_1\begin{bmatrix}x\\y\end{bmatrix}$.

$$\mathbf{T}_2\circ\mathbf{T}_1\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{T}_2\begin{bmatrix}2x+y\\3y-x\end{bmatrix}$$

$$=\begin{bmatrix}(2x+y)-3(3y-x)\\(2x+y)+(3y-x)\end{bmatrix}$$

$$=\begin{bmatrix}5x-8y\\x+4y\end{bmatrix}$$

Write the transformation.

$$\mathbf{T}_2\circ\mathbf{T}_1\begin{bmatrix}x\\y\end{bmatrix}\rightarrow\begin{bmatrix}5x-8y\\x+4y\end{bmatrix}$$

d Apply $\mathbf{T}_2\circ\mathbf{T}_1$ to $\begin{bmatrix}2\\4\end{bmatrix}$.

$$\mathbf{T}_2\circ\mathbf{T}_1\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}5\times2-8\times4\\2+4\times4\end{bmatrix}$$

$$=\begin{bmatrix}-22\\18\end{bmatrix}$$

You should think of $\mathbf{T}_2\circ\mathbf{T}_1\begin{bmatrix}2\\4\end{bmatrix}$ as $\mathbf{T}_2\left[\mathbf{T}_1\begin{bmatrix}2\\4\end{bmatrix}\right]$.

ISBN 9780170412995

EXAMPLE 12

Two transformations are given by $\mathbf{T}_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+2y \\ 3x-y \end{bmatrix}$ and $\mathbf{T}_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+y \\ x-y \end{bmatrix}$.

a Find $\mathbf{T}_2 \circ \mathbf{T}_1$.

b Find $\mathbf{T}_1 \circ \mathbf{T}_2$.

c Comment on your answers.

Solution

a Multiply the matrices in the order $\mathbf{T}_2, \mathbf{T}_1$.

$$\mathbf{T}_2 \circ \mathbf{T}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$$

Write as the transformation.

$$\mathbf{T}_2 \circ \mathbf{T}_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 4x+y \\ -2x+3y \end{bmatrix}$$

b Multiply the matrices in the order $\mathbf{T}_1, \mathbf{T}_2$.

$$\mathbf{T}_1 \circ \mathbf{T}_2 = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

Write as the transformation.

$$\mathbf{T}_1 \circ \mathbf{T}_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x-y \\ 2x+4y \end{bmatrix}$$

c Comment on the results.

$\mathbf{T}_2 \circ \mathbf{T}_1 \neq \mathbf{T}_1 \circ \mathbf{T}_2$, so the composition of transformations is not commutative.

There could be some transformations that do commute. This example shows it is not generally true, so you say they do not commute.

You can use multiplication of matrices for composition of transformations. You can also use your graphics calculator to do the multiplications.

ISBN 9780170412995

EXAMPLE 13

A dilation **D** has factors $\lambda_1 = 2$ and $\lambda_2 = 0.25$. **M** is a reflection in the line $y = 0.5x$.

Use matrices to find **M** ∘ **D** and **D** ∘ **M** and comment on your answers.

Solution

State **D**.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix}$$

Find cos (2θ) and sin (2θ).

$$\cos(2\theta) = \frac{1-m^2}{1+m^2} = \frac{1-0.25}{1+0.25} = 0.6$$

$$\sin(2\theta) = \frac{2m}{1+m^2} = \frac{2\times 0.25}{1+0.25} = 0.8$$

State **M**.

$$\mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

Find **M** ∘ **D**.

$$\mathbf{M} \circ \mathbf{D} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix} = \begin{bmatrix} 1.2 & 0.2 \\ 1.6 & -0.15 \end{bmatrix}$$

Find **D** ∘ **M**.

$$\mathbf{D} \circ \mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} = \begin{bmatrix} 1.2 & 1.6 \\ 0.2 & -0.15 \end{bmatrix}$$

Comment on the answers.

Dilations and reflections do not commute.

Exercise 10.06 Composition of transformations

Example 11

1 $\mathbf{T}_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x-y \\ 2x+y \end{bmatrix}$ and $\mathbf{T}_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+2y \\ 3x+y \end{bmatrix}$.

a Find $\mathbf{T}_2\left[\mathbf{T}_1\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right]$.

b Find $\mathbf{T}_2 \circ \mathbf{T}_1\begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

c Comment on the answers.

d Find $\mathbf{T}_1\left[\mathbf{T}_2\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right]$

e Find $\mathbf{T}_1 \circ \mathbf{T}_2\begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

f Comment on the answers.

ISBN 9780170412995

2 $T_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x+3y \\ -3x+2y \end{bmatrix}$ and $T_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -2x-4y \\ x-2y \end{bmatrix}$.

a Find $T_2\left[T_1\begin{bmatrix} -3 \\ -1 \end{bmatrix}\right]$ and $T_2 \circ T_1\begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

b Find $T_1\left[T_2\begin{bmatrix} -3 \\ -1 \end{bmatrix}\right]$ and $T_1 \circ T_2\begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

Example 12

3 $T_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 4x+y \\ 3x-5y \end{bmatrix}$ and $T_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x-3y \\ 3x-4y \end{bmatrix}$.

a Find $T_2 \circ T_1$

b Find $T_1 \circ T_2$

c Comment on your answers.

4 $T_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x-2y \\ 3y-4x \end{bmatrix}$ and $T_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -x-y \\ 2x+4y \end{bmatrix}$. Find $T_2 \circ T_1$ and $T_1 \circ T_2$.

5 $T_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2y-4x \\ -x-3y \end{bmatrix}$ and $T_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x+4y \\ 4x+y \end{bmatrix}$. Find $T_2 \circ T_1$ and $T_1 \circ T_2$.

6 $T_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x+6y \\ 10x+5y \end{bmatrix}$ and $T_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+3y \\ 5x+2y \end{bmatrix}$. Find $T_2 \circ T_1$ and $T_1 \circ T_2$.

Example 13

7 A dilation **D** has factors $\lambda_1 = 0.5$ and $\lambda_2 = 3$. **M** is a reflection in the line through the origin with inclination 15°.

Use matrices to find $\mathbf{M} \circ \mathbf{D}$ and $\mathbf{D} \circ \mathbf{M}$ and comment on your answers.

8 A rotation **R** around the origin has inclination $\frac{2\pi}{3}$ and **D** is a dilation with x-factor 0.3 and y-factor 1.5.

Use matrices to find $\mathbf{R} \circ \mathbf{D}$ and $\mathbf{D} \circ \mathbf{R}$ and comment on your answers.

9 **M** is a reflection in the line $y = -\frac{1}{3}x$ and **R** is a rotation around the origin through the angle 60°.

Use matrices to find $\mathbf{M} \circ \mathbf{R}$ and $\mathbf{R} \circ \mathbf{M}$ and comment on your answers.

10 $\mathbf{M}_1$ is a reflection in the line $y = 2x$ and $\mathbf{M}_2$ is a reflection in a line through the origin with inclination −30°.

Use matrices to find $\mathbf{M}_1 \circ \mathbf{M}_2$ and $\mathbf{M}_2 \circ \mathbf{M}_1$ and comment on your answers.

ISBN 9780170412995

10.07 Geometric results of transformations

Under the transformation $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x+y \\ 2x+3y \end{bmatrix}$,

$O(0, 0), A(0, 1), B(1, 1), C(1, 0) \to O'(0, 0), A'(1, 3), B'(4, 5), C'(3, 2)$, as below.

The area of $OA'B'C'$ is given by:

$OEB'G - EB'A'D - ODA' - GB'C'F - OFC'$

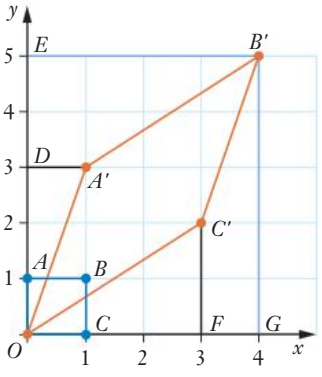
$= 4 \times 5 - \frac{1}{2} \times 2(4 + 1) - \frac{1}{2} \times 3 \times 1 - \frac{1}{2} \times 1(5 + 2) - \frac{1}{2} \times 3 \times 2$

$= 20 - 5 - 1.5 - 3.5 - 3$

$= 7$

Notice that $|\mathbf{T}| = 7$.

The area of $O'A'B'C'$ is $|\mathbf{T}|$ times the area of $OABC$.



Determinants and areas

The area of a 2D shape changes by the factor $\pm|\mathbf{T}|$ under a transformation $\mathbf{T}$.

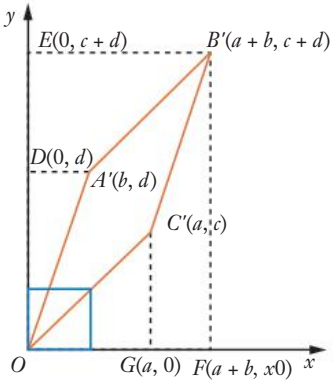
Proof

Any area will change by the same factor as the unit square.

Under the transformation $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$,

$O(0, 0), A(0, 1), B(1, 1), C(1, 0) \to O'(0, 0), A'(b, d), B'(a + b, c + d), C'(a, c)$,

as shown on the right.



Area of $OEB'G = (a + b)(c + d) = ac + ad + bc + bd$

Area of $EB'A'D = \frac{1}{2}(c + d - d)(b + a + b) = bc + \frac{1}{2}ac$

Area of $ODA' = \frac{1}{2}bd$

Area of $GB'C'F = \frac{1}{2}(a + b - a)(c + c + d) = bc + \frac{1}{2}bd$

Area of $OFC' = \frac{1}{2}ac$

Area of $OA'B'C' = ac + ad + bc + bd - bc - \frac{1}{2}ac - \frac{1}{2}bd - bc - \frac{1}{2}bd - \frac{1}{2}ac$

$= ad - bc$

$= |\mathbf{T}|$ QED

Since areas are always positive, the absolute value of $|\mathbf{T}|$ is used.

ISBN 9780170412995

EXAMPLE 14

The triangle $A(2, 1)$, $B(4, 5)$, $C(7, 1)$ is transformed by $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 2x - y \\ x + 3y \end{bmatrix}$ to $A'B'C'$.
What is the area of $A'B'C'$?

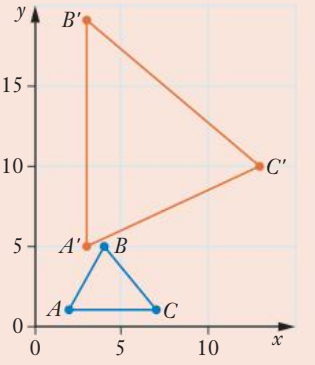
Solution

Use matrices to find $A'B'C'$.

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 4 & 7 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 13 \\ 5 & 19 & 10 \end{bmatrix},$$

so $A'B'C'$ is $A'(3, 5)$, $B'(3, 19)$, $C'(13, 10)$.

Sketch ABC and $A'B'C'$.



Write the height and base of ABC.

$h_{ABC} = 5 - 1 = 4$

$b_{ABC} = 7 - 2 = 5$

Find the area of ABC.

$\text{Area}_{ABC} = \frac{1}{2} \times 5 \times 4 = 10$ square units

Find the determinant.

Determinant of transformation $= ad - bc$

$= 2 \times 3 - 1 \times (-1)$

$= 7$

Use the determinant area property to find the area.

$\text{Area}_{A'B'C'} = 7 \times 10$

$= 70$ square units

Write the answer.

The area of $A'B'C'$ is 70 square units.

ISBN 9780170412995

What is the result of combining different kinds of transformations?

EXAMPLE 15

A rotation of $-\frac{\pi}{3}$ around the origin is followed by a reflection in the line $y = 3x$.

a Find the matrix of the composition of transformations.

b Find the image of the triangle $P(-2, -1)$, $Q(2, 3)$, $R(4, 1)$.

c Sketch the triangle and its image on the same axes.

d Describe the combined transformation.

Solution

a Find the parameters of the reflection.

$$\frac{1-m^2}{1+m^2} = \frac{1-9}{1+9} = -0.8$$
$$\frac{2m}{1+m^2} = \frac{2\times 3}{1+9} = 0.6$$

Multiply the matrices in the correct order.

$$\begin{bmatrix} -0.8 & 0.6 \\ 0.6 & -(-0.8) \end{bmatrix} \begin{bmatrix} \cos\left(-\frac{\pi}{3}\right) & -\sin\left(-\frac{\pi}{3}\right) \\ \sin\left(-\frac{\pi}{3}\right) & \cos\left(-\frac{\pi}{3}\right) \end{bmatrix}$$
$$= \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{-3\sqrt{3}-4}{10} & \frac{3-4\sqrt{3}}{10} \\ \frac{3-4\sqrt{3}}{10} & \frac{3\sqrt{3}+4}{10} \end{bmatrix}$$

b Apply to triangle.

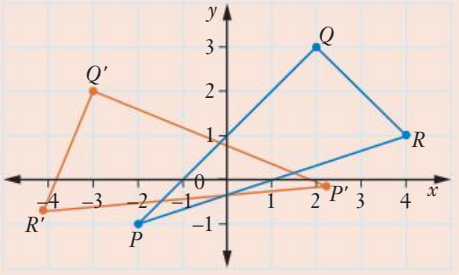
$$\begin{bmatrix} \frac{-3\sqrt{3}-4}{10} & \frac{3-4\sqrt{3}}{10} \\ \frac{3-4\sqrt{3}}{10} & \frac{3\sqrt{3}+4}{10} \end{bmatrix} \begin{bmatrix} -2 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1+2\sqrt{3}}{2} & \frac{1-18\sqrt{3}}{10} & \frac{-13-16\sqrt{3}}{10} \\ \frac{-2+\sqrt{3}}{2} & \frac{18+\sqrt{3}}{10} & \frac{16-13\sqrt{3}}{10} \end{bmatrix}$$

ISBN 9780170412995

Write the answer.

The image is $P'\left(\frac{1+2\sqrt{3}}{2}, \frac{-2+\sqrt{3}}{2} \approx (2.2, -0.1)\right)$, $Q'\left(\frac{1-18\sqrt{3}}{10}, \frac{18+\sqrt{3}}{10}\right)$, $R'\left(\frac{-13-16\sqrt{3}}{10}, \frac{16-13\sqrt{3}}{10}\right)$.

c Sketch PQR and $P'Q'R'$.



d Describe the result.

The combined transformation appears to be a reflection across a line through the origin with an inclination of about 102°.

Exercise 10.07 Geometric results of transformations

Example 14

1 The triangle $E(3, -2)$, $F(-1, -2)$, $G(1, 2)$ is transformed by $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+y \\ x-2y \end{bmatrix}$ to $E'F'G'$. What is the area of $E'F'G'$?

2 The parallelogram $A(-3, -2)$, $B(0, 2)$, $C(3, 2)$, $D(0, -2)$ is transformed by $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} y-2x \\ 3x-y \end{bmatrix}$ to $A'B'C'D'$. What is the area of $A'B'C'D'$?

3 The kite $A(-4, 5)$, $B(2, 8)$, $C(4, 1)$, $D(-1, 2)$ is transformed by $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 2y-x \\ 3x+2y \end{bmatrix}$ to $A'B'C'D'$. What is the area of $A'B'C'D'$?

4 The trapezium $M(2, 0)$, $G(0, -4)$, $T(2, -5)$, $D(5, 1)$ is transformed by a dilation with $\lambda_1 = 3$ and $\lambda_2 = 4$ to $M'G'T'D'$. What is the area of $M'G'T'D'$?

Example 15

5 Reflections in the lines $y = 2x$ and $y = -2x$ are performed in that order.

a Find the matrix of the composition of transformations.

b Find the image of the kite $A(3, 1)$, $B(-1, 3)$, $C(-3, -1)$, $D(2, -6)$.

c Sketch the kite and its image on the same axes.

d Describe the combined transformation.

ISBN 9780170412995

6 Dilations with x and y factors of 2 and 0.5, 1.5 and 0.3 respectively are performed in that order.

- **a** Find the matrix of the composition of transformations.
- **b** Find the image of the parallelogram $C(-1, 1)$, $D(-4, 5)$, $E(-5, -1)$, $F(-2, -5)$.
- **c** Sketch the parallelogram and its image on the same axes.
- **d** Describe the combined transformation.

7 Rotations around the origin through angles of $\frac{7\pi}{6}$ and $-\frac{3\pi}{4}$ are performed in that order.

- **a** Find the matrix on the composition of transformations.
- **b** Find the image of the rectangle $F(0, 2)$, $G(3, 1)$, $L(1, -5)$, $Q(-2, -4)$.
- **c** Sketch the rectangle and its image on the same axes.
- **d** Describe the combined transformation.

Problem solving

8 Show that the area of a shape remains unchanged under a rotation.

9 Show that the area of a shape remains unchanged under a reflection.

10 Show that the area of a shape under a dilation is the product of the original area and the dilation factors in the horizontal and vertical directions.

ISBN 9780170412995

10. CHAPTER SUMMARY

Transformations

Translations

- A **translation** moves every point in the plane the same distance in the same direction
- A translation (a, b) is given by $\mathbf{T}(x, y) \rightarrow (x', y')$, where $x' = x + a, y' = y + b$
- A translation can be treated as a row or column vector
- When a transformation is applied, the original point or shape is called the **object** and the result is called the **image**
- Translations are not linear transformations

Rotation around the origin

- Rotation, $\mathbf{R}_\theta$, through the angle θ around the origin is given by

$$\mathbf{R}_\theta : \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}$$

- The rotation is given by the matrix product $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- A rotation around the origin has the format $\mathbf{R} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2 + b^2 = 1$. The angle of rotation is given by $\sin^{-1}(b)$.
- Rotation around the origin is a linear transformation

Dilations

- Dilation, $\mathbf{D}$, by factors λ_1 and λ_2 in the x and y directions respectively is given by

$$\mathbf{D}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_1 x \\ \lambda_2 y \end{bmatrix}$$ for centre of dilation at the origin.

- The matrix for dilation with centre at the origin is $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
- Rotation around the origin is a linear transformation

Reflections

- Reflection, $\mathbf{M}$, in a line with inclination θ through the origin is given by

$$\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x\cos(2\theta) + y\sin(2\theta) \\ x\sin(2\theta) - y\cos(2\theta) \end{bmatrix}$$

- The matrix for the reflection is $\mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

ISBN 9780170412995

- Reflection in a line of slope m through the origin is given by

$$\mathbf{M}:\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x\left(\frac{1-m^2}{1+m^2}\right)+y\left(\frac{2m}{1+m^2}\right) \\ x\left(\frac{2m}{1+m^2}\right)-y\left(\frac{1-m^2}{1+m^2}\right) \end{bmatrix}$$

- The matrix for the reflection is $\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & -\frac{1-m^2}{1+m^2} \end{bmatrix}$
- A reflection in lines passing through the origin has the format $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. The angle of inclination of the line or reflection is given by $\frac{1}{2}\sin^{-1}(b)$.
- Reflection in a line through the origin is a linear transformation.

Linear transformations

- Any linear transformation is of the form $\mathbf{T}:\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$, where $a, b, c, d \in \boldsymbol{R}$
- Every linear transformation has a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Inverse transformations

- The identity transformation is $\mathbf{I}:\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix}$
- A transformation $\mathbf{T}:\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$, has an inverse if and only if the determinant of the transformation, $|\mathbf{T}| = ad - bc \neq 0$
- A transformation with no inverse is called **singular** and one with an inverse is called **non-singular**
- The **inverse** of a non-singular transformation $\mathbf{T}:\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ is given by $\mathbf{T}^{-1}:\begin{bmatrix} x \\ y \end{bmatrix} \to \frac{1}{ad-bc}\begin{bmatrix} dx-by \\ -cx+ay \end{bmatrix}$

Composition of transformations

- The **composition** of two transformations is the result of doing them one after the other
- The composition of $\mathbf{T}_1$ and $\mathbf{T}_2$ is shown as $\mathbf{T}_2 \circ \mathbf{T}_1$. $\mathbf{T}_1$ is done first, followed by $\mathbf{T}_2$

Determinants and areas

- The area of a 2D shape changes by the factor $\pm|\mathbf{T}|$ under a transformation $\mathbf{T}$

10. CHAPTER REVIEW

Transformations

Example 1

1 Write down the vector for a movement 3 units right and 2 units down and use it to translate $(-2, 3)$.

Example 2

2 Show the kite $A(-3, -1)$, $B(0, 3)$, $C(5, 3)$, $D(4, -5)$ and its translation under $\mathbf{T}(x, y) \rightarrow (x - 2, y + 4)$ on the same axes.

Example 3

3 **a** Write the rotation around the origin through the angle $-30°$.

b Find the image of the figure $P(3, 3)$, $Q(4, 5)$, $R(6, 2)$, $S(-1, 1)$ under the rotation.

c Show the figure and its image on the same axes.

Example 4

4 **a** Write the matrix for the rotation $\frac{2\pi}{5}$ around the origin correct to 4 decimal places.

b Find the image of the hexagon $A(-1, 1)$, $B(0, 2)$, $C(1, 2)$, $D(2, 1)$, $E(1, -2)$, $F(0, -2)$ under the rotation, correct to two decimal places.

Example 5

5 **a** Find the transformation for a dilation with vertical factor 3 and horizontal factor 2.5.

b Find the image of the rectangle $K(-3, -2)$, $L(1, 2)$, $M(2, 1)$, $N(-2, -3)$ under the transformation.

c What is the shape of the image?

Example 6

6 **a** Find the matrix for a dilation with factors $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$.

b Find the image of the triangle $P(-6, -8)$, $E(0, 4)$, $T(8, 0)$ under the dilation.

c Show the triangle and its image on the same set of axes.

Example 7

7 Find the transformation for a reflection in a line through the origin with inclination $\frac{3\pi}{8}$.

Example 7

8 Find the transformation for a reflection in the line $y = -7x$.

Example 7

9 **a** Find the matrix for a reflection in the line $y = 0.5x$.

b Find the image of the trapezium $A(-2, 1)$, $B(2, 5)$, $C(6, 5)$, $D(-1, -2)$ after a reflection in the line.

c Sketch the trapezium and its image on the same axes.

Example 8

10 **a** Find the matrix for reflection in the line through the origin with inclination $315°$.

b Find the image of the parallelogram $P(-4, 0)$, $K(2, 4)$, $C(3, 3)$, $M(-3, -1)$ after a reflection in the line.

Example 9

11 Use matrices to show that a reflection in the line $y = 4x$ is its own inverse.

ISBN 9780170412995

12 State whether each transformation is singular or non-singular and find any inverses.

Example 10

a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x-3y \\ 3x-5y \end{bmatrix}$ **b** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 6x+2y \\ 9x+4y \end{bmatrix}$ **c** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 6x-9y \\ 4x-6y \end{bmatrix}$

13 $\mathbf{G}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} y-3x \\ 2x-3y \end{bmatrix}$ and $\mathbf{H}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+3y \\ x-2y \end{bmatrix}$.

Example 11

a Find $\mathbf{G}\left[\mathbf{H}\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right]$. **b** Find $\mathbf{G} \circ \mathbf{H}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

14 $\mathbf{P}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+0.5y \\ x-0.5y \end{bmatrix}$ and $\mathbf{Q}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x-y \\ 2x-4y \end{bmatrix}$. Find $\mathbf{P} \circ \mathbf{Q}$.

Example 12

15 A rotation **R** around the origin has inclination $-\frac{\pi}{3}$ and a dilation **D** has factors $\lambda_1 = 2.5$ and $\lambda_2 = 0.8$. Use matrices to find $\mathbf{R} \circ \mathbf{D}$.

Example 13

16 The rectangle $U(1, 2)$, $V(5, 5)$, $W(11, -3)$, $X(7, -6)$ is transformed by $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 4x+3y \\ x+2y \end{bmatrix}$. Find the area of the image.

Example 14

17 Reflections in the lines through the origin with inclinations 45° and 120° are performed in that order.

Example 15

a Find the matrix of the composition of transformations.

b Find the image of the square $A(1, 2)$, $B(4, 4)$, $C(6, 1)$, $D(3, -1)$.

c Sketch the square and its image on the same axes.

d Describe the combined transformation.

Problem solving

18 Show that translation is not a linear transformation.

19 Show that if rotations through angles θ and φ are performed in succession, the order makes no difference to the composition.

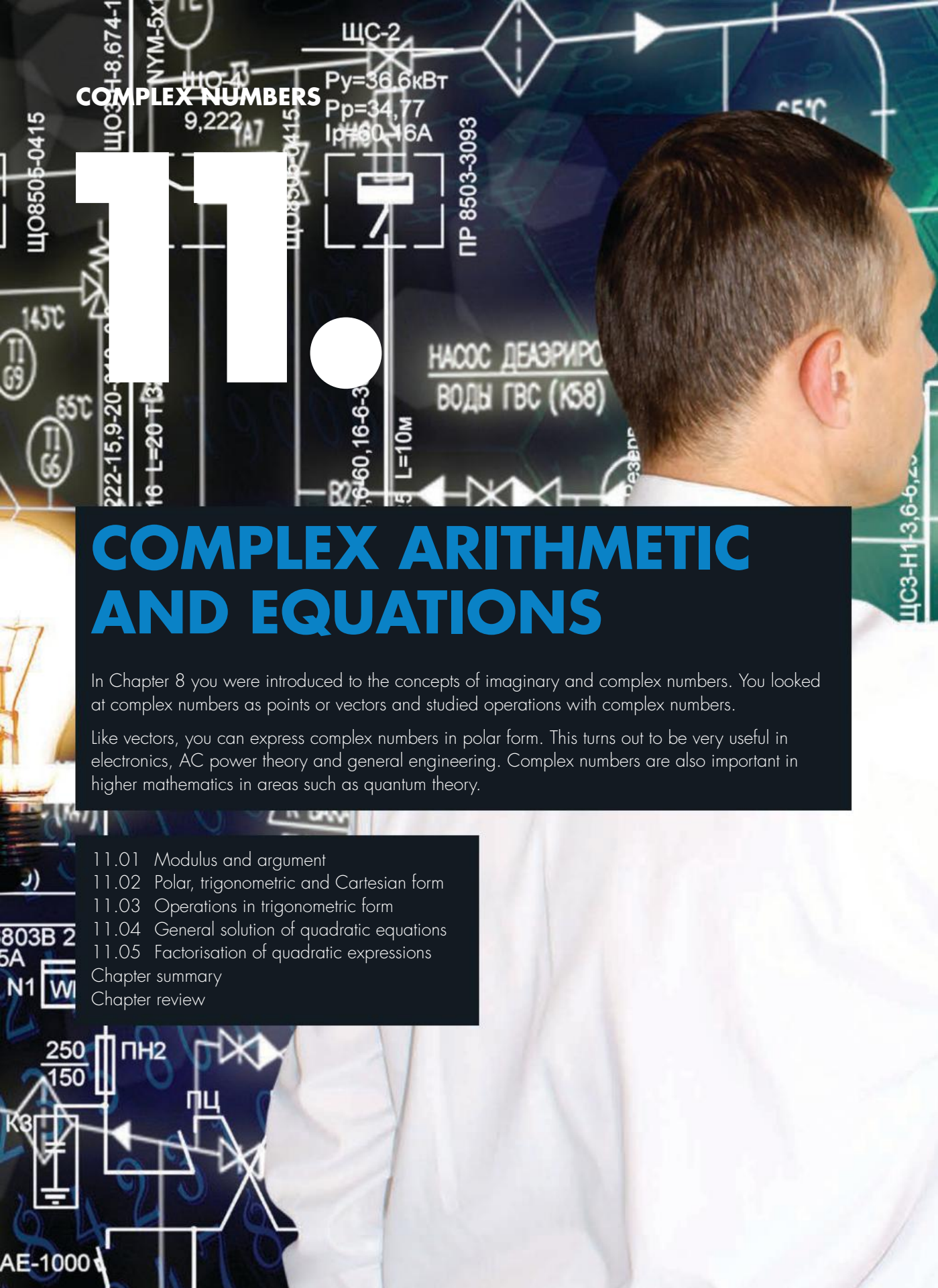
20 Show that a dilation with factors $\lambda_1 = 1.5$ and $\lambda_1 = 0.75$ is a linear transformation.

21 Show that composition of reflections are not commutative.

22 Prove that the inverse of a dilation with horizontal and vertical factors a and b is the dilation with horizontal factor $\frac{1}{a}$ and vertical factor $\frac{1}{b}$.

Practice quiz

ISBN 9780170412995



COMPLEX NUMBERS

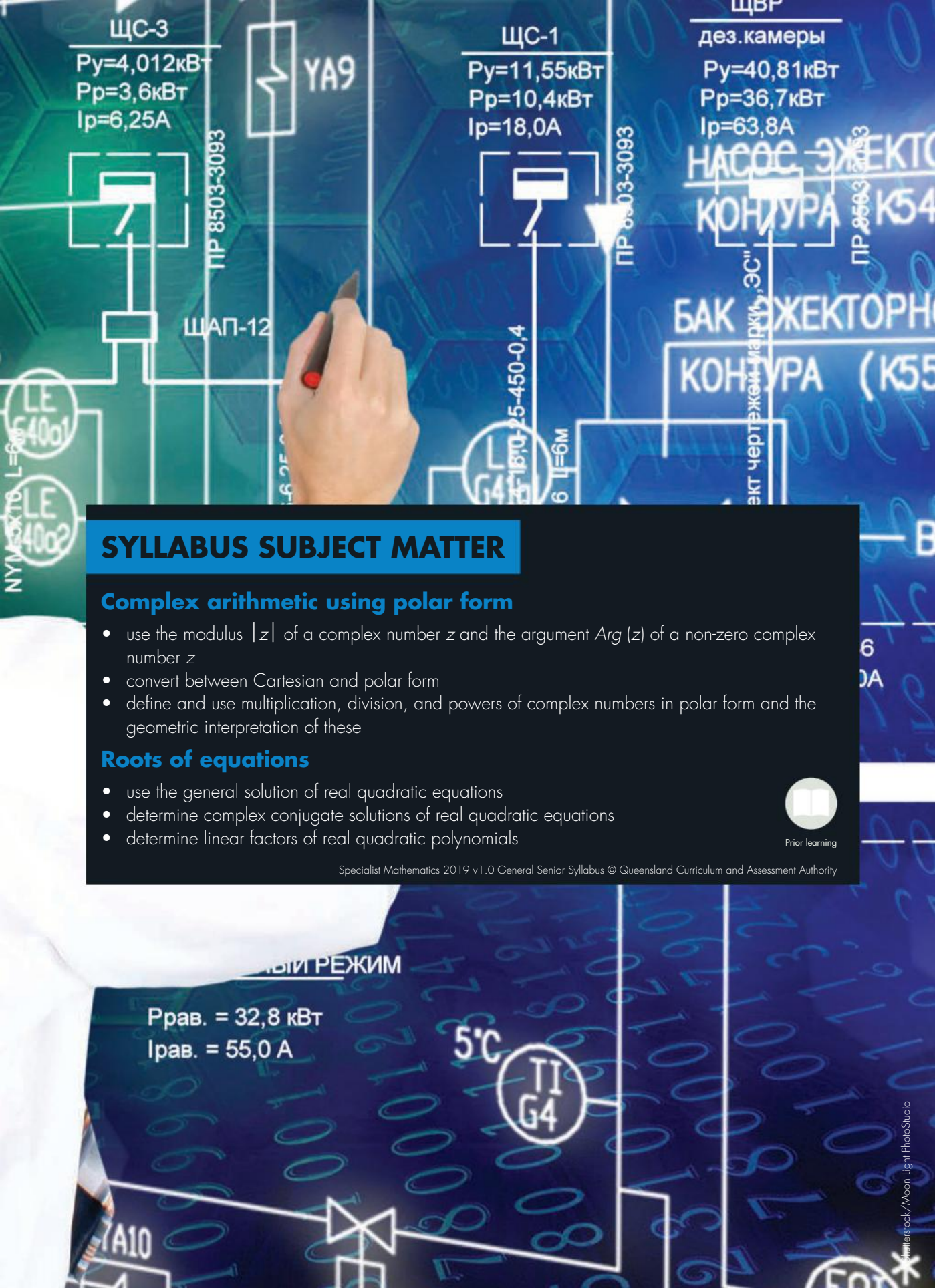
11.

COMPLEX ARITHMETIC AND EQUATIONS

In Chapter 8 you were introduced to the concepts of imaginary and complex numbers. You looked at complex numbers as points or vectors and studied operations with complex numbers.

Like vectors, you can express complex numbers in polar form. This turns out to be very useful in electronics, AC power theory and general engineering. Complex numbers are also important in higher mathematics in areas such as quantum theory.

11.01 Modulus and argument
11.02 Polar, trigonometric and Cartesian form
11.03 Operations in trigonometric form
11.04 General solution of quadratic equations
11.05 Factorisation of quadratic expressions
Chapter summary
Chapter review



SYLLABUS SUBJECT MATTER

Complex arithmetic using polar form

- use the modulus $|z|$ of a complex number z and the argument $Arg\,(z)$ of a non-zero complex number z
- convert between Cartesian and polar form
- define and use multiplication, division, and powers of complex numbers in polar form and the geometric interpretation of these

Roots of equations

- use the general solution of real quadratic equations
- determine complex conjugate solutions of real quadratic equations
- determine linear factors of real quadratic polynomials

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

Shutterstock/Moon Light PhotoStudio

TERMINOLOGY

Argand diagram
argument
Cartesian form
cis (θ)
complex conjugate
discriminant
factorise
inverse
modulus
polar form
principal argument
quadratic equation
quadratic formula
root
trigonometric form

Modulus and argument

11.01 Modulus and argument

You learned in Chapter 8 that the **modulus** and **argument** are the size and direction of complex numbers considered as vectors in the **Argand plane**.

Modulus and argument

- The **modulus** (size) of the complex number $z = a + bi$ is given by $\sqrt{a^2 + b^2}$
- The modulus of z is written as $|z|$ or mod (z)
- The **argument** (direction) of the complex number $z = a + bi$ is given by $\tan(\theta) = \frac{b}{a}$
- The argument of z is written as arg (z)

EXAMPLE 1

Find the modulus and argument of:

a $4 - 6i$ **b** $-5 + 5\sqrt{3}i$

Solution

a Write the formula and calculate the modulus.

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \\ &= \sqrt{4^2 + (-6)^2} \\ &= \sqrt{52} \\ &= 2\sqrt{13} \end{aligned}$$

Write the formula and calculate $\tan(\theta)$.

$$\begin{aligned} \tan(\theta) &= \frac{b}{a} \\ &= \frac{-6}{4} \\ &= -\frac{3}{2} \end{aligned}$$

ISBN 9780170412995

Write the reference angle. $\theta = \tan^{-1}\left(\frac{3}{2}\right)$

Find the angle (in the 4th quadrant). $\arg(z) = 2\pi - \tan^{-1}\left(\frac{3}{2}\right)$ $\left[\text{or} -\tan^{-1}\left(\frac{3}{2}\right)\right]$

b Write the formula and calculate the modulus.

$$|z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{(-5)^2 + (5\sqrt{3})^2}$$
$$= \sqrt{100}$$
$$= 10$$

Write the formula and calculate $\tan(\theta)$.

$$\tan(\theta) = \frac{b}{a}$$
$$= \frac{5\sqrt{3}}{-5}$$
$$= -\sqrt{3}$$

Write the reference angle. $\theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

Find the angle (in the 2nd quadrant).

$$\arg(z) = \pi - \frac{\pi}{3}$$
$$= \frac{2\pi}{3}$$

If you are not asked to find an approximate answer, give the exact answer. Unless otherwise stated, you should give answers in radians. You may find it easier to give approximate answers in degrees.

There are many angles that have the same tangent, but you usually give only one answer for the argument.

Principal argument

- The principal argument of a complex number is in the domain $-\pi < \arg(z) \leq \pi$
- Unless otherwise asked, always give the principal argument

EXAMPLE 2

What are the modulus and principal argument of $-8 - 5i$, correct to 2 decimal places?

Solution

Find the modulus and round your answer.

$$|z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{(-8)^2 + (-5)^2}$$
$$= \sqrt{89}$$
$$\approx 9.43$$

Find $\tan(\theta)$.

$$\tan(\theta) = \frac{-5}{-8}$$
$$= 0.625$$

Find the reference angle.

$$\theta = 0.5585\ldots \text{ (or } 32.005\ldots°)$$

Find the principal argument (3rd quadrant) and round your answer.

$$\arg(z) = -\pi + \tan^{-1}(0.625)$$
$$\approx -2.58 \text{ (or } -147.99°)$$

TI 84 Plus CE

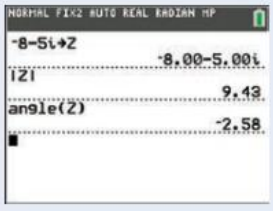
TI-Nspire CX Chapter 11

Press mode and use the cursor controls to set the angle measure and accuracy.

Enter $-8 - 5i$ and store in, say, z. Remember to use (-) for negative 8. Use 2nd . for i.

Press math and use the cursor controls to choose CMPLX and use 5: abs(for the modulus.

Use 4: angle(in CMPLX for the argument.



Casio fx-CG20AU

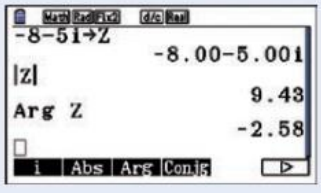
Use the Run–Matrix menu.

Press SHIFT MENU (SET UP) and use the cursor controls to set the angle measure and accuracy.

Enter $-8 - 5i$ and store in, say, z. Remember to use (-) for negative 8. Use SHIFT 0 for i.

Press OPTN F3 (COMPLEX) and F2 (Abs) for the modulus. Press F↔D to get the approximation.

Use Arg in COMPLEX for the argument.



ISBN 9780170412995

Exercise 11.01 Modulus and argument

Use your graphics calculator for the problems your teacher chooses.

Examples 1,2

1 Find the modulus of each complex number.

a $3 + 4i$ **b** $5 - 12i$ **c** $-5 + 5i$ **d** $-8 - 15i$

e $2 - 6i$ **f** $-5 - 10i$ **g** $7 + 7\sqrt{3}i$ **h** $-6 + 8i$

2 Find the principal argument of each complex number.

a $2 + 2i$ **b** $-1 - \sqrt{3}i$ **c** $\sqrt{3} + i$ **d** $6 - 6i$

e $4\sqrt{3} - 4i$ **f** $-2i$ **g** $4 + 5i$ **h** $-7 - 3i$

3 Find the modulus and argument in degrees of each complex number, correct to 1 decimal place.

a $6 + i$ **b** $-4 + 2i$ **c** $3 - i$ **d** $-9 - 6i$

e $-5 + 8i$ **f** $1 + 6i$ **g** $-3 - 9i$ **h** $2 - 3i$

4 Find the modulus and argument in radians of each complex number, correct to 2 decimal places.

a $-3 - 7i$ **b** $4 + 9i$ **c** $3 - 6i$ **d** $-2 + 8i$

e $10 + 7i$ **f** $-2 - 4i$ **g** $9 - 12i$ **h** $6i - 5$

Problem solving

5 Prove that $|z|\ |w| = |zw|$.

6 Prove that $\frac{|z|}{|w|} = \left|\frac{z}{w}\right|$.

7 Prove that $|z| = |\overline{z}|$.

8 Prove that $|z|^2 = |z^2|$.

9 Prove that $|z^2| = z\overline{z}$.

10 Prove that z is purely imaginary if and only if $\arg(z) = \pm\frac{\pi}{2}$.

ISBN 9780170412995

Complex number conversions

11.02 Polar, trigonometric and Cartesian form

The complex number $z = 6 - 6i$ is written in **Cartesian form** and its Cartesian coordinates on the Argand plane are $(6, -6)$.

$|z| = 6\sqrt{2}$ and $\arg(z) = -\frac{\pi}{4}$ (or $-45°$).

The polar coordinates of the point are $\left(6\sqrt{2}, -\frac{\pi}{4}\right)$, just like other vectors.

$6\sqrt{2}\cos\left(-\frac{\pi}{4}\right) = 6\sqrt{2} \times \frac{\sqrt{2}}{2} = 6$ and $6\sqrt{2}\sin\left(-\frac{\pi}{4}\right) = 6\sqrt{2} \times \left(-\frac{\sqrt{2}}{2}\right) = -6$.

This means that you could write $z = 6\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] = 6 - 6i$.

Polar, Cartesian and trigonometric forms

- The Cartesian form of a complex number is $z = x + iy$ for $x, y \in \boldsymbol{R}$
- The **polar form** is $z = (r, \theta)$, where $r = |z| = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \frac{y}{x}$
- The **trigonometric form** is $z = r[\cos(\theta) + i\sin(\theta)] = r\,\text{cis}(\theta)$
- $\text{cis}(\theta)$ is the abbreviation for $\cos(\theta) + i\sin(\theta)$

It doesn't matter whether you use $z = x + iy$ or $z = a + bi$.

You need to be able to change between different forms of complex numbers.

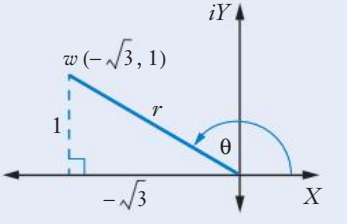
EXAMPLE 3

Write in polar and trigonometric forms.

a $w = -\sqrt{3} + i$ **b** $z = -1 - 2i$

Solution

a Draw a diagram.



ISBN 9780170412995

Find r.

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-\sqrt{3})^2 + 1^2}$$
$$= 2$$

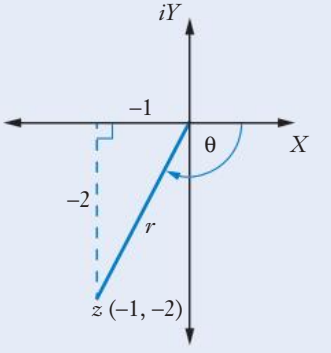
Find θ.

$$\tan(\theta) = \frac{y}{x}$$
$$= \frac{1}{-\sqrt{3}}$$
$$= -\frac{1}{\sqrt{3}}$$
$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Write the answers.

$$z = \left(2, \frac{5\pi}{6}\right) = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

b Draw a diagram. Show θ in the principal domain, $-\pi < \theta \leq \pi$.



Find r.

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-1)^2 + (-2)^2}$$
$$= \sqrt{5}$$

Find θ.

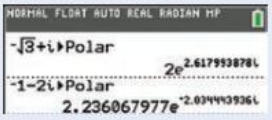
$$\tan(\theta) = \frac{y}{x}$$
$$= \frac{-2}{-1} = 2$$
$$\theta = -\pi + \tan^{-1}(2)$$

Write the answers.

$$z = (\sqrt{5}, \tan^{-1}(2))$$
$$= \sqrt{5}[\cos(\tan^{-1}(2)] + i\sin[\tan^{-1}(2)]$$

TI 84 Plus CE

Enter $-\sqrt{3}+i$ and press [math] and choose 7: Polar from the CMPLX menu, then press [enter].



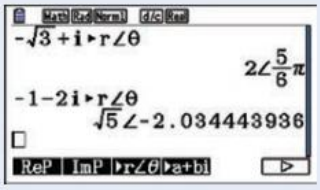
Repeat for $-1 - 2i$.

The TI-84 expresses cis (θ) as $e^{\theta i}$.

Casio fx-CG20AU

Use the Run–Matrix menu.

Enter $-\sqrt{3} + i$ and press [OPTN] and choose ▶r∠θ from the COMPLEX ([F3]) menu then press [EXE].



Repeat for $-1 - 2i$.

Don't use approximations unless they are requested.

EXAMPLE 4

Write $2\left[\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right]$ in Cartesian form, correct to 3 decimal places.

Solution

Change ratios from third to first quadrant.

$$2\left[\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right]$$

$$= 2\left[\cos\left(-\pi+\frac{\pi}{3}\right)+i\sin\left(-\pi+\frac{\pi}{3}\right)\right]$$

$$= 2\left[-\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)\right]$$

Substitute values and simplify.

$$= 2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$

$$= -1 - \sqrt{3}i$$

Express correct to 3 decimal places.

$$\approx -1 - 1.732i$$

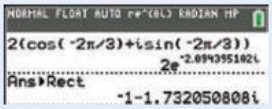
ISBN 9780170412995

TI 84 Plus CE

Type the trigonometric form and press enter.

Then press math and choose 6: ▶Rect from the CMPLX menu and press enter.

You can do 6: ▶Rect after the trigonometric form if you want.



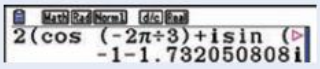
TI-Nspire CX Chapter 11

Casio fx-CG20AU

Use the Run–Matrix menu.

Type the trigonometric form and press EXE.

Press F↔D to get a decimal answer.



Exercise 11.02 Polar, trigonometric and Cartesian form

Use your graphics calculator for the problems your teacher chooses.

1 Write each complex number in polar and trigonometric forms. Example 3

a $z = 1 + 2i$ **b** $z = \sqrt{3} - i$ **c** $z = -3 - 5i$

d $z = -1 + \sqrt{3}i$ **e** $z = 3 + 4i$ **f** $z = -7 + 2i$

2 Write each complex number in trigonometric form.

a $z = 1 + i$ **b** $z = \sqrt{2} - \sqrt{2}i$ **c** $z = 3i$ **d** $z = -3\sqrt{2} + 3\sqrt{2}i$

e $z = 1 + \sqrt{3}i$ **f** $z = -6 + 6\sqrt{3}i$ **g** $z = 2 - i$ **h** $z = -2 + 3i$

i $z = 1 - 2i$ **j** $z = -1 - 3i$ **k** $z = -2 - i$ **l** $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

3 Write each complex number in trigonometric form, correct to 2 decimal places.

a $-4 + 7i$ **b** $7 - 8i$ **c** $2 + 5i$ **d** $-8 - 11i$

e $-3 + 12i$ **f** $16 + 11i$ **g** $-9 - 10i$ **h** $10 - 4i$

4 Write each complex number in Cartesian form. Example 4

a $5\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ **b** $8\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

c $2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$ **d** $6\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$

e $7\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$ **f** $11\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$

g $20\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$ **h** $15\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$

ISBN 9780170412995

5 Write each complex number in Cartesian form, correct to 2 decimal places.

a $6[\cos(1.2) + i\sin(1.2)]$ **b** $8[\cos(-2.9) + i\sin(-2.9)]$

c $5[\cos(142°) + i\sin(142°)]$ **d** $12[\cos(2) + i\sin(2)]$

e $17[\cos(-37°) + i\sin(-37°)]$ **f** $4[\cos(-110°) + i\sin(-110°)]$

g $2[\cos(2.3) + i\sin(2.3)]$ **h** $11[\cos(-1.85) + i\sin(-1.85)]$

i $9[\cos(24.3°) + i\sin(24.3°)]$ **j** $6[\cos(-3) + i\sin(-3)]$

6 Express each complex number in Cartesian and trigonometric forms.

a $\frac{1}{1-i}$ **b** $\frac{1}{2-i} \times \frac{1}{1+i}$ **c** $\frac{1}{3-i} \times \frac{1}{3+i}$ **d** $(3 + 2i)(1 - 3i)$

e $-2(1 - i)$ **f** $\frac{-3}{-\sqrt{3}+i}$ **g** $(1 + i)^3$ **h** $(1 + i)^2 + 1$

Polar complex number operations

11.03 Operations in trigonometric form

It is easy to add and subtract complex numbers in Cartesian form. It is more difficult to multiply and divide them in Cartesian form, so we use trigonometric or polar form.

Multiplication of complex numbers in trigonometric form

- You can multiply complex numbers by multiplying their moduli and adding their arguments
- For $z_1 = r_1 \text{ cis}(\theta_1)$ and $z_2 = r_2 \text{ cis}(\theta_2)$, $z_1 \times z_2 = r_1 r_2 \text{ cis}(\theta_1 + \theta_2)$
- Powers of $z = r \text{ cis}(\theta)$ are given by $z^n = r^n \text{ cis}(n\theta)$ for $n \in \boldsymbol{Z}^+$

You can prove the formula for multiplication using trigonometric identities (Section 9.08).

$$\begin{aligned} z_1 \times z_2 &= r_1[\cos(\theta_1) + i\sin(\theta_1)] \times r_2[\cos(\theta_2) + i\sin(\theta_2)] \\ &= r_1 r_2[\cos(\theta_1) + i\sin(\theta_1)][\cos(\theta_2) + i\sin(\theta_2)] \\ &= r_1 r_2[\cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + i^2\sin(\theta_1)\sin(\theta_2)] \\ &= r_1 r_2\{[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)] + i[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)]\} \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{QED} \end{aligned}$$

The rule for powers is obtained by repeated multiplication.

EXAMPLE 5

Find each product in trigonometric and Cartesian forms, correct to 2 decimal places where appropriate.

a $2 \text{ cis}\left(\frac{\pi}{3}\right) \times 4 \text{ cis}\left(\frac{5\pi}{6}\right)$ **b** $(1 + i)(\sqrt{3} - i)$ **c** $[3 \text{ cis}(2)]^4$

ISBN 9780170412995

Solution

a Use the rule for products.

$$2\operatorname{cis}\left(\frac{\pi}{3}\right)\times 4\operatorname{cis}\left(\frac{5\pi}{6}\right)=2\times 4\operatorname{cis}\left(\frac{\pi}{3}+\frac{5\pi}{6}\right)$$

Simplify.

$$=8\operatorname{cis}\left(\frac{7\pi}{6}\right)$$

Change to principal argument.

$$=8\operatorname{cis}\left(-\frac{5\pi}{6}\right)$$

Expand.

$$=8\cos\left(-\frac{5\pi}{6}\right)+8i\sin\left(-\frac{5\pi}{6}\right)$$

Substitute values.

$$=8\times\left(-\frac{\sqrt{3}}{2}\right)+8\times\left(-\frac{1}{2}\right)i$$

$$=-4\sqrt{3}-4i$$

b Change to trigonometric form.

$$(1+i)(\sqrt{3}-i)=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\times 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$$

Use the rule for products.

$$=2\sqrt{2}\operatorname{cis}\left[\frac{\pi}{4}+\left(-\frac{\pi}{6}\right)\right]$$

Simplify.

$$=2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)$$

Expand.

$$=2\sqrt{2}\cos\left(\frac{\pi}{12}\right)+2\sqrt{2}i\sin\left(\frac{\pi}{12}\right)$$

Substitute values.

$$=2.828\ldots\times 0.965\ldots+2.828\ldots\times 0.258\ldots\,i$$

Round as required.

$$\approx 2.73+0.67i$$

c Use the rule for powers.

$$[3\operatorname{cis}(2)]^4=3^4\operatorname{cis}(4\times 2)$$

Simplify.

$$=81\operatorname{cis}(8)$$

Change to principal argument.

$$=81\operatorname{cis}(8-2\pi)$$

Expand.

$$=81\cos(8-2\pi)+81i\sin(8-2\pi)$$

Substitute values.

$$=81\times(-0.1455\ldots)+81i\times 0.9893\ldots$$

Round as required.

$$=-11.79+80.14i$$

ISBN 9780170412995

You can divide complex numbers in trigonometric form in a similar way to multiplication.

Division of complex numbers in trigonometric form

- You can divide complex numbers by dividing their moduli and subtracting their arguments
- For $z_1 = r_1 \text{ cis } (\theta_1)$ and $z_2 = r_2 \text{ cis } (\theta_2)$, $\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2)$

You can prove the rule for division in a similar way to that for multiplication, but you have to multiply by the conjugate of the divisor.

$$\frac{z_1}{z_2} = \frac{r_1[\cos(\theta_1)+i\sin(\theta_1)]}{r_2[\cos(\theta_2)+i\sin(\theta_2)]}$$

$$= \frac{r_1}{r_2} \times \frac{[\cos(\theta_1)+i\sin(\theta_1)][\cos(\theta_2)-i\sin(\theta_2)]}{[\cos(\theta_2)+i\sin(\theta_2)][\cos(\theta_2)-i\sin(\theta_2)]}$$

$$= \frac{r_1}{r_2} \times \frac{\cos(\theta_1)\cos(\theta_2)-i\cos(\theta_1)\sin(\theta_2)+i\sin(\theta_1)\cos(\theta_2)-i^2\sin(\theta_1)\sin(\theta_2)}{\cos^2(\theta_2)-i^2\sin^2(\theta_2)}$$

$$= \frac{r_1}{r_2} \times \frac{[\cos(\theta_1)\cos(\theta_2)+\sin(\theta_1)\sin(\theta_2)]+i[\sin(\theta_1)\cos(\theta_2)-i\cos(\theta_1)\sin(\theta_2)]}{\cos^2(\theta_2)+\sin^2(\theta_2)}$$

$$= \frac{r_1}{r_2} \times \frac{\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)}{1}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)] \qquad \text{QED}$$

EXAMPLE 6

Find each quotient in trigonometric form.

a $\dfrac{8\text{cis}\left(\frac{3\pi}{4}\right)}{2\text{cis}\left(\frac{\pi}{3}\right)}$ **b** $\dfrac{4\text{cis}\left(\frac{\pi}{12}\right)}{2\text{cis}\left(-\frac{5\pi}{12}\right)}$ **c** $[4 \text{ cis } (30°)]^{-1}$ **d** $\left[3\text{cis}\left(\frac{\pi}{6}\right)\right]^{-2}$

ISBN 9780170412995

Solution

a Use the rule for division.

$$\frac{8\text{cis}\left(\frac{3\pi}{4}\right)}{2\text{cis}\left(\frac{\pi}{3}\right)} = \frac{8}{2}\text{cis}\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)$$

Simplify.

$$= 4\text{ cis}\left(\frac{5\pi}{12}\right)$$

b Use the rule for division.

$$\frac{4\text{cis}\left(\frac{\pi}{12}\right)}{2\text{cis}\left(-\frac{5\pi}{12}\right)} = \frac{4}{2}\text{cis}\left[\frac{\pi}{12} - \left(-\frac{5\pi}{12}\right)\right]$$

Simplify.

$$= 2\text{ cis}\left(\frac{\pi}{2}\right)$$

c Write as a division.

$$[4\text{ cis }(30°)]^{-1} = \frac{1}{4\text{cis}(30°)}$$

Write 1 in trigonometric form.

$$= \frac{1\text{cis}(0°)}{4\text{cis}(30°)}$$

Use the rule for division.

$$= \frac{1}{4}\text{cis}(0° - 30°)$$

Simplify.

$$= \frac{1}{4}\text{cis}(-30°)$$

d Write as a division.

$$\left[3\text{cis}\left(\frac{\pi}{6}\right)\right]^{-2} = \frac{1}{\left[3\text{cis}\left(\frac{\pi}{6}\right)\right]^{2}}$$

Use the rule for powers and write 1 in trigonometric form.

$$= \frac{1\text{cis}(0)}{9\text{cis}\left(\frac{\pi}{3}\right)}$$

Use the rule for division.

$$= \frac{1}{9}\text{cis}\left(-\frac{\pi}{3}\right)$$

You should not really do the algebra shown in parts **c** and **d** above because negative powers of complex numbers have not been defined. However, it does give meaning in the same way that the same algebra does to real numbers.

ISBN 9780170412995

Negative powers of complex numbers

- The multiplicative inverse of the complex number $z = r\,\text{cis}\,(\theta)$ is given by

  $z^{-1} = r^{-1}\,\text{cis}(-\theta) = \frac{1}{r}\,\text{cis}\,(-\theta)$

- A negative power of the complex number $z = r\,\text{cis}\,(\theta)$ is given by

  $z^{-n} = r^{-n}\,\text{cis}\,(-n\theta)$

INVESTIGATION

GEOMETRIC INTERPRETATION OF MULTIPLICATION AND DIVISION

1 Sketch the vector for $z = 3\,\text{cis}\left(\frac{\pi}{6}\right)$ on an Argand diagram.

2 Work out the product of z and $w = 2\,\text{cis}\left(\frac{\pi}{4}\right)$, zw.

3 Sketch the vector of zw on the same Argand diagram.

4 What is the geometric effect of multiplying by w?

5 Sketch the vector for $z = 6\,\text{cis}\left(\frac{2\pi}{3}\right)$ on an Argand diagram.

6 Work out the quotient of z and $w = 2\,\text{cis}\left(\frac{\pi}{6}\right)$, $\frac{z}{w}$.

7 Sketch the vector of $\frac{z}{w}$ on the same Argand diagram.

8 What is the geometric effect of dividing by w?

Geometric interpretation of multiplication and division of complex numbers

- Multiplication by $r\,\text{cis}\,(\theta)$ rotates the vector for a complex number around the origin through the angle θ and dilates it by the factor r in both directions
- Division by $r\,\text{cis}\,(\theta)$ rotates the vector for a complex number around the origin through the angle $-\theta$ and dilates it by the factor $\frac{1}{r}$ in both directions

ISBN 9780170412995

EXAMPLE 7

$z = 6\,\text{cis}\left(\frac{2\pi}{3}\right)$, $w = 3\,\text{cis}\left(\frac{\pi}{6}\right)$ and $u = 2\,\text{cis}\left(\frac{\pi}{4}\right)$. Without calculation:

a show z and zw on the same Argand plane

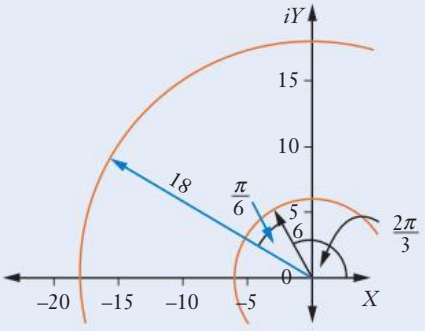
b show z and $\frac{z}{u}$ on the same Argand plane.

Solution

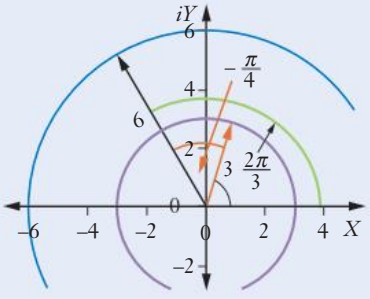
a Show z with length 6 and angle $\frac{2\pi}{3}$.

To show zw, rotate z through $\frac{\pi}{6}$ and triple its length to 18.

Using circles to show lengths of vectors makes it easy to see.



b To show $\frac{z}{u}$, rotate z through $-\frac{\pi}{4}$ and reduce its length by a factor of 2 to 3.



Exercise 11.03 Operations in trigonometric form

1 Calculate each product in trigonometric form.

Example 5

a $\left[3\text{cis}\left(\frac{\pi}{3}\right)\right]\left[2\text{cis}\left(\frac{\pi}{2}\right)\right]$

b $\left[2\text{cis}\left(\frac{\pi}{6}\right)\right]\left[\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right]$

c $\left[2\text{cis}\left(-\frac{\pi}{3}\right)\right]\left[2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right]$

d $\left[5\text{cis}\left(\frac{\pi}{6}\right)\right]\left[3\text{cis}\left(\frac{\pi}{3}\right)\right]$

e $\left[4\text{cis}\left(\frac{\pi}{6}\right)\right]\left[2\text{cis}\left(-\frac{2\pi}{3}\right)\right]$

f $\left[\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right]\left[2\text{cis}\left(-\frac{\pi}{3}\right)\right]$

g $\left[4\text{cis}\left(\frac{5\pi}{6}\right)\right]\left[2\text{cis}\left(\frac{2\pi}{3}\right)\right]$

h $\left[2\text{cis}\left(-\frac{\pi}{6}\right)\right]\left[\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\right]$

ISBN 9780170412995

2 Calculate each product in Cartesian form.

a $\left[2\text{cis}\left(\frac{\pi}{6}\right)\right]\left[3\text{cis}\left(\frac{2\pi}{3}\right)\right]$ **b** $\left[4\text{cis}\left(\frac{\pi}{4}\right)\right]\left[\sqrt{2}\,\text{cis}\left(-\frac{\pi}{2}\right)\right]$

c $\left[5\text{cis}\left(\frac{\pi}{2}\right)\right]\left[2\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right]$ **d** $\left[2\text{cis}\left(\frac{5\pi}{6}\right)\right]\left[2\text{cis}\left(\frac{2\pi}{3}\right)\right]$

e $\left[2\text{cis}\left(-\frac{\pi}{4}\right)\right]\left[2\text{cis}\left(\frac{\pi}{3}\right)\right]$ **f** $\left[\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right]\left[2\text{cis}\left(\frac{\pi}{3}\right)\right]$

g $\left[3\text{cis}\left(\frac{\pi}{6}\right)\right]\left[7\text{cis}\left(-\frac{\pi}{2}\right)\right]$ **h** $\left[2\text{cis}\left(-\frac{5\pi}{12}\right)\right]\left[2\text{cis}\left(\frac{2\pi}{3}\right)\right]$

3 Find each product by first converting them to trigonometric form.

a $(1+i)\left(\sqrt{3}+i\right)$ **b** $\left(1-\sqrt{3}\,i\right)\left(\frac{1}{2}+\frac{1}{2}i\right)$

c $\left(2+2\sqrt{3}\,i\right)(3-3i)$ **d** $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(-4-4i)$

e $(5-5i)\left(2\sqrt{3}-2i\right)$ **f** $\left(\frac{4}{5}-\frac{4\sqrt{3}}{5}i\right)(-1-i)$

g $\left(-1-\sqrt{3}i\right)(2-2i)$ **h** $(1+i)\left(2+2\sqrt{3}\,i\right)\left(\sqrt{3}+i\right)$

4 Find each expression in both trigonometric and Cartesian forms.

a $[2\text{ cis }(30°)]^5$ **b** $\left[3\text{cis}\left(\frac{\pi}{6}\right)\right]^2$ **c** $\left[2\text{cis}\left(\frac{5\pi}{6}\right)\right]^4$ **d** $[4\text{ cis }(-60°)]^4$

e $\left[5\text{cis}\left(-\frac{2\pi}{3}\right)\right]^3$ **f** $[3\text{ cis }(20°)]^3$ **g** $[\text{cis }(135°)]^3$ **h** $\left[\sqrt{2}\,\text{cis}\left(\frac{\pi}{16}\right)\right]^4$

Example 6

5 Find each quotient in trigonometric form.

a $\dfrac{3\text{cis}\left(\frac{\pi}{4}\right)}{2\text{cis}\left(\frac{\pi}{3}\right)}$ **b** $\dfrac{4\text{cis}\left(-\frac{\pi}{6}\right)}{3\text{cis}\left(\frac{4\pi}{9}\right)}$ **c** $\dfrac{5\text{cis}\left(-\frac{3\pi}{4}\right)}{2\text{cis}\left(\frac{5\pi}{6}\right)}$

d $\dfrac{1+i}{\sqrt{3}+i}$ **e** $\dfrac{2\sqrt{3}+2i}{-3-3i}$ **f** $\dfrac{\sqrt{2}-\sqrt{2}\,i}{2+2\sqrt{3}\,i}$

ISBN 9780170412995

6 Find each quotient in Cartesian form.

a $\dfrac{2\text{cis}\left(\frac{\pi}{6}\right)}{3\text{cis}\left(\frac{\pi}{3}\right)}$ **b** $\dfrac{4\text{cis}\left(-\frac{2\pi}{3}\right)}{2\text{cis}\left(\frac{5\pi}{6}\right)}$ **c** $\dfrac{4\text{cis}\left(-\frac{\pi}{4}\right)}{3\text{cis}\left(\frac{7\pi}{9}\right)}$

d $\dfrac{2+2i}{3\sqrt{3}-3i}$ **e** $\dfrac{5\sqrt{2}+5\sqrt{2}i}{-6+6i}$ **f** $\dfrac{-2-2\sqrt{3}\,i}{3\sqrt{3}-4i}$

7 Find the multiplicative inverse of each complex number in both trigonometric and Cartesian forms.

a $3\text{cis}\left(\frac{\pi}{6}\right)$ **b** $2\text{cis}\left(\frac{2\pi}{3}\right)$ **c** $2\text{cis}\left(-\frac{\pi}{3}\right)$ **d** $4\text{cis}\left(-\frac{5\pi}{6}\right)$

e $2+2i$ **f** $1-\sqrt{3}\,i$ **g** $-3\sqrt{3}+3i$ **h** $-2-2\sqrt{3}\,i$

8 Find each expression in both trigonometric and Cartesian form.

a $[4\text{ cis }(75°)]^{-2}$ **b** $\left[2\text{cis}\left(\frac{\pi}{9}\right)\right]^{-3}$ **c** $\left[5\text{cis}\left(\frac{\pi}{6}\right)\right]^{-2}$ **d** $[2\text{ cis }(-15°)]^{-4}$

e $[5\text{ cis }(60°)]^{-3}$ **f** $\left[\text{cis}\left(\frac{\pi}{12}\right)\right]^{-4}$ **g** $\left[3\text{cis}\left(\frac{5\pi}{16}\right)\right]^{-4}$ **h** $[3\text{ cis }(120°)]^{-2}$

9 $z = 8\text{ cis}\left(\frac{\pi}{3}\right)$, $w = 2\text{ cis}\left(\frac{3\pi}{4}\right)$ and $u = 4\text{ cis}\left(\frac{\pi}{2}\right)$. Without calculation, show on the same Argand plane:

Example 7

a z and zw **b** z and $\frac{z}{u}$

10 $z = 10\text{ cis }(100°)$, $w = 1.5\text{ cis }(40°)$ and $u = 2\text{ cis }(60°)$. Without calculation, show on the same Argand plane:

a z and zw **b** z and $\frac{z}{u}$

11 $z = 4\text{ cis}\left(-\frac{3\pi}{10}\right)$, $w = 2\text{ cis}\left(\frac{\pi}{5}\right)$ and $u = \text{cis}\left(\frac{3\pi}{5}\right)$. Without calculation, show on the same Argand plane:

a z and zw **b** z and $\frac{z}{u}$

Problem solving

12 Use $\text{cis }(75°) = \text{cis }(45° + 30°)$ to find exact expressions for sin (75°) and cos (75°).

13 Use $\text{cis}\left(\frac{\pi}{12}\right) = \text{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ to find exact expressions for $\sin\left(\frac{\pi}{12}\right)$ and $\cos\left(\frac{\pi}{12}\right)$.

ISBN 9780170412995

14 The alternating current in an electric inductor is given by $I = \frac{V}{Z}$ amperes, where V is voltage and $Z = R + X_L i$ is impedance. Find the current if $V = 10[\cos(20°) + i \sin(20°)]$, $R = 5$ and $X_L = 3$.

15 The current I in an electrical circuit with voltage V, resistance R, capacitive reactance X_c and inductive reactance X_L is given by $\frac{V}{R + (X_L - X_c)i}$.

Find I if $V = 10[\cos(25°) + i \sin(25°)]$, $R = 5$, $X_L = 3$ and $X_c = 7$. Give your answer in Cartesian form with real and imaginary parts correct to 2 decimal places.

11.04 General solution of quadratic equations

iStock/Leonid Andronov

You have already used the **quadratic formula** to solve equations.

The quadratic formula

The solutions (or roots) of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \boldsymbol{R}$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Look for factorisation first, but if you cannot spot one, use the quadratic formula. In some cases, the answers will be complex numbers.

ISBN 9780170412995

EXAMPLE 8

Solve each equation.

a $3v^2 + 13v = 10$ **b** $x^2 + 7 = 4x$

Solution

a Write the equation in the standard form and state the values of a, b and c.

$3v^2 + 13v - 10 = 0$

$a = 3, b = 13, c = -10$

Write the quadratic formula and substitute values.

$$v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-13 \pm \sqrt{13^2 - 4 \times 3 \times (-10)}}{2 \times 3}$$

$$= \frac{-13 \pm \sqrt{289}}{6}$$

$$= \frac{-13 \pm 17}{6}$$

The rational answers mean you could solve this equation by factorisation.

$$= -5 \text{ or } \frac{2}{3}$$

Write the answers.

$$v = -5 \text{ or } v = \frac{2}{3}$$

b Write the equation in the standard form and state the values of a, b and c.

$x^2 - 4x + 7 = 0$

$a = 1, b = -4, c = 7$

Write the quadratic formula and substitute values.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 7}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{-12}}{2}$$

Simplify the surd if you can. Use $\sqrt{-1} = i$.

$$= \frac{4 \pm \sqrt{4 \times 3 \times (-1)}}{2}$$

$$= \frac{4 - 2\sqrt{3}i}{2} \text{ or } \frac{4 + 2\sqrt{3}i}{2}$$

Cancel any common factor.

$x = 2 - \sqrt{3}i$ or $x = 2 + \sqrt{3}i$

ISBN 9780170412995

The nature of the roots of a quadratic equation depend on the **discriminant**, Δ, the number under the square root sign in the quadratic formula.

The discriminant

- The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in Z$ is given by $\Delta = b^2 - 4ac$
- For $\Delta < 0$, the roots are complex numbers
- For $\Delta = 0$, the roots equal and rational
- For $\Delta > 0$, the roots are different real numbers
  - If Δ is a perfect square, the roots are rational
  - If Δ is a not perfect square, the roots are irrational

EXAMPLE 9

Find the nature of the roots of each equation.

a $2m^2 + 5m - 8 = 0$ **b** $3x^2 + 2x + 5 = 0$

c $2g^2 + 3g - 2 = 0$ **d** $9p^2 - 24p + 16 = 0$

Solution

a Find the discriminant.

$\Delta = b^2 - 4ac$

$= 5^2 - 4 \times 2 \times (-8)$

$= 89$

State the nature of the roots.

$\Delta > 0$ and not a perfect square so the roots are different and irrational.

b Find the discriminant.

$\Delta = b^2 - 4ac$

$= 3^2 - 4 \times 2 \times 5$

$= -31$

State the nature of the roots.

$\Delta < 0$, so the roots are complex.

ISBN 9780170412995

c Find the discriminant.

$\Delta = b^2 - 4ac$

$= 3^2 - 4 \times 2 \times (-2)$

$= 25$

State the nature of the roots.

$\Delta > 0$ and a perfect square, so the roots are different and rational.

d Find the discriminant.

$\Delta = b^2 - 4ac$

$= (-24)^2 - 4 \times 9 \times 16$

$= 0$

State the nature of the roots.

$\Delta = 0$, so the roots are equal and rational.

INVESTIGATION

SUM AND PRODUCT OF ROOTS

Consider the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \boldsymbol{R}$ with roots α and β.

- Simplify the expression $(x - \alpha)(x - \beta)$.
- Express the equation with an x^2 coefficient of 1.
- What can you say about $(x - \alpha)(x - \beta)$ and the expression on the LHS of the equation?
- Use the above to find expressions for $\alpha + \beta$ and $\alpha\beta$.
- Write a quadratic equation with roots α and β.

Sum and product of roots of a quadratic equation

- The roots α and β of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \boldsymbol{Z}$, have the properties that

  $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

- The quadratic equation $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ has roots α and β

For a quadratic equation with real coefficients, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ must be real.

When the roots are complex, the only way this can be true is if they are conjugates.

Therefore, complex roots of real quadratic equations must be **complex conjugates**. That is, they are of the form $a + bi$ and $a - bi$.

ISBN 9780170412995

EXAMPLE 10

Find a quadratic equation whose roots are triple the roots of $5x^2 + x - 2 = 0$.

Solution

Find the sum and product of the roots.

Sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{1}{5}$

Product roots, $\alpha\beta = \frac{c}{a} = -\frac{2}{5}$

Choose names for the new equation.

Let the roots of the new equation be θ and ϕ.

Find the sum and the new roots.

$\theta + \phi = 3\alpha + 3\beta$

$= 3(\alpha + \beta)$

$= -\frac{3}{5}$

Find the product of the new roots.

$\theta\phi = 3\alpha \times 3\beta$

$= 9\alpha\beta$

$= -\frac{18}{5}$

Write the new equation.

New equation is $x^2 - (\theta + \phi)x + \theta\phi = 0$

Substitute values and simplify.

$x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{18}{5}\right) = 0$

$5x^2 + 3x - 18 = 0$

Write the answer.

The required equation is $5x^2 + 3x - 18 = 0$.

Exercise 11.04 General solution of quadratic equations

Example 8

1 Solve each equation.

a $2x^2 - 7x + 9 = 0$ **b** $2x^2 - x - 6 = 0$ **c** $2x^2 + 4x + 1 = 0$ **d** $x^2 + 6x + 9 = 0$

e $2x^2 - 3x - 4 = 0$ **f** $x^2 - 3x + 9 = 0$ **g** $2x^2 - 7x - 1 = 0$ **h** $3x^2 - 4x + 4 = 0$

Example 9

2 Find the nature of the roots of each equation.

a $2x^2 - 4x - 5 = 0$ **b** $2x^2 - 7x + 8 = 0$ **c** $3x^2 + 5x - 2 = 0$

d $2x^2 - 3x - 4 = 0$ **e** $3x^2 - 4x - 7 = 0$ **f** $3x^2 - 4x + 11 = 0$

g $25x^2 - 90x + 81 = 0$ **h** $2x^2 - 6x + 13 = 0$

Example 10

3 Find a quadratic equation whose roots are double the roots of $x^2 - x - 5 = 0$.

4 Find a quadratic equation whose roots are five times the roots of $3x^2 + 8x + 2 = 0$.

ISBN 9780170412995

Problem solving

5 Find a quadratic equation whose roots are the squares of the roots of $4x^2 - 3x - 5 = 0$.

6 Find a quadratic equation whose roots are the reciprocals of the roots of $x^2 + x - 8 = 0$.

11.05 Factorisation of quadratic expressions

The expression $3v^2 + 13v - 10$ has the discriminant $\Delta = 289$.

289 is a perfect square.

This means that the roots of the equation $3v^2 + 13v - 10 = 0$ are rational.

In fact, they are -5 or $\frac{2}{3}$, so the equation is $(v + 5)\left(v - \frac{2}{3}\right) = 0$.

Multiplying by the LCD of 5 and $\frac{2}{3}$ gives $(v + 5)(3v - 2) = 0$.

Thus $3v^2 + 13v - 10 = (v + 5)(3v - 2)$.

The same logic applies to any quadratic expression with integer coefficients whose discriminant is a perfect square.

For $\Delta > 0$ but not a perfect square, the expression can only be factorised using surds. The roots must be conjugate surds to give integer coefficients.

For $\Delta < 0$, it must have conjugate complex roots.

Factors of quadratic expressions

The factors of quadratic expressions with integer coefficients are:

- linear factors for $\Delta = 0$ or $\Delta > 0$ and a perfect square
- factors with conjugate surds for $\Delta > 0$ but not a perfect square
- factors of the form $a(x - z)(x - \bar{z})$ for $\Delta < 0$

If you cannot find the factors, use the quadratic formula to find the roots. Then you will be able to write the factorisation.

ISBN 9780170412995

EXAMPLE 11

Factorise each expression.

a $3t^2 + 14t + 15$ **b** $40e^2 + 14e - 33$ **c** $x^2 - 4x - 23 = 0$

Solution

a Write with possible linear factors.

$3t^2 + 14t + 15 = (3t + ??)(t + ??)$

Try possible combinations.

Try $15 = 3 \times 5$ or $15 = 1 \times 15$

$15 = 3 \times 5$ works!

Write the factorisation.

$3t^2 + 14t + 15 = (3t + 5)(t + 3)$

b This has too many possibilities for guess and check, so find the zeros.

Use your calculator for the arithmetic!

$$e = \frac{-14 \pm \sqrt{14^2 - 4 \times 40 \times (-33)}}{2 \times 40}$$

$$= \frac{-14 \pm \sqrt{5476}}{80}$$

$$= \frac{-14 \pm 74}{80} = -\frac{11}{10} \text{ or } \frac{3}{4}$$

Take out the coefficient of e^2 to insert the roots and do the factorisation.

$$40e^2 + 14e - 33 = 40\left[e - \left(-\frac{11}{10}\right)\right]\left(e - \frac{3}{4}\right)$$

Use $40 = 10 \times 4$.

$$= (10e + 11)(4e - 3)$$

c This won't factorise, so find the zeros.

$$x = \frac{-(-4) \pm \sqrt{4^2 - 4 \times 1 \times (-23)}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{96}}{2}$$

$$= \frac{4 \pm 4\sqrt{6}}{2}$$

$$= 2 - 2\sqrt{6} \text{ or } 2 + 2\sqrt{6}$$

Complete the factorisation.

$x^2 - 4x - 23$

$= [x - (2 - 2\sqrt{6})][x - (2 + 2\sqrt{6})]$

$= (x - 2 + 2\sqrt{6})(x - 2 - 2\sqrt{6})$

In most cases, you will not be able to find complex factors by guess and check.

ISBN 9780170412995

EXAMPLE 12

Factorise $m^2 - 4m + 13$.

Solution

Check the discriminant. $\Delta = (-4)^2 - 52 = -36$

Zeros are complex so use the formula. $m = \dfrac{-(-4) \pm \sqrt{-36}}{2 \times 1}$

$= \dfrac{4 \pm 6i}{2}$

$= 2 - 3i$ or $2 + 3i$

Write the factorisation. $m^2 - 4m + 13 = (m - 2 + 3i)(m - 2 - 3i)$

Exercise 11.05 Factorisation of quadratic expressions

1 Factorise each expression. Example 11

a $20q^2 + 9q + 1$ **b** $12n^2 + 11n - 15$ **c** $3n^2 - 20n + 25$ **d** $28g^2 - 19g - 20$

e $5h^2 + 24h - 36$ **f** $8n^2 - 17n + 9$ **g** $60m^2 + m - 33$ **h** $50t^2 - 25t - 42$

2 Factorise each expression.

a $u^2 - 6u + 7$ **b** $z^2 + 2z - 9$ **c** $x^2 - 4x - 1$ **d** $a^2 + 10a + 22$

e $a^2 + 16a + 57$ **f** $g^2 - 4g - 2$ **g** $u^2 - 24u + 139$ **h** $m^2 - 20m + 90$

3 Factorise each expression. Example 12

a $v^2 - 2v + 2$ **b** $m^2 + 4m + 5$ **c** $n^2 + 6n + 13$ **d** $w^2 + 2w + 37$

e $n^2 + 6n + 45$ **f** $h^2 + 12h + 85$ **g** $r^2 + 16r + 73$ **h** $e^2 + 10e + 61$

4 Factorise each expression.

a $k^2 - 8k + 52$ **b** $10r^2 + 23r + 12$ **c** $q^2 - 4q + 29$ **d** $c^2 + 16c + 89$

e $70u^2 - 32u - 24$ **f** $g^2 - 22g + 111$ **g** $21t^2 - 4t - 32$ **h** $y^2 - 14y + 58$

5 Factorise each expression.

a $21h^2 + 50h + 24$ **b** $y^2 - 8y + 20$ **c** $q^2 - 16q + 80$ **d** $18e^2 + 15e - 7$

e $a^2 + 2a + 26$ **f** $e^2 - 12e + 28$ **g** $u^2 + 12u + 37$ **h** $r^2 + 20r + 104$

ISBN 9780170412995

11. CHAPTER SUMMARY

Complex arithmetic and equations

Modulus and argument

- The **modulus** (size) of the complex number $z = a + bi$ is given by $\sqrt{a^2 + b^2}$
- The modulus of z is written as $|z|$ or mod (z)
- The **argument** (direction) of the complex number $z = a + bi$ is given by $\tan(\theta) = \frac{b}{a}$
- The argument of z is written as arg (z)
- The principal argument of a complex number is in the domain $-\pi < \arg(z) \leq \pi$
- Unless otherwise asked, always give the principal argument

Polar, Cartesian and trigonometric forms

- The Cartesian form of a complex number is $z = x + iy$ for $x, y \in \boldsymbol{R}$
- The **polar form** is $z = (r, \theta)$, where $r = |z| = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \frac{y}{x}$
- The **trigonometric form** is $z = r[\cos(\theta) + i \sin(\theta)] = r \operatorname{cis}(\theta)$
- $\operatorname{cis}(\theta)$ is the abbreviation for $\cos(\theta) + i \sin(\theta)$

Operations on complex numbers in trigonometric form

- You can **multiply** complex numbers by multiplying their moduli and adding their arguments
- For $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$, $z_1 \times z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$
- **Powers** of $z = r \operatorname{cis}(\theta)$ are given by $z^n = r^n \operatorname{cis}(n\theta)$ for $n \in \boldsymbol{Z}^+$
- You can **divide** complex numbers by dividing their moduli and subtracting their arguments
- The **multiplicative inverse** of the complex number $z = r \operatorname{cis}(\theta)$ is given by $z^{-1} = r^{-1} \operatorname{cis}(-\theta) = \frac{1}{r} \operatorname{cis}(-\theta)$
- A **negative power** of the complex number $z = r \operatorname{cis}(\theta)$ is given by $z^{-n} = r^{-n} \operatorname{cis}(-n\theta)$

Geometric interpretation of multiplication and division of complex numbers

- Multiplication by $r \operatorname{cis}(\theta)$ rotates the vector for a complex number around the origin through the angle θ and dilates it by the factor r in both directions
- Division by $r \operatorname{cis}(\theta)$ rotates the vector for a complex number around the origin through the angle $-\theta$ and dilates it by the factor $\frac{1}{r}$ in both directions

ISBN 9780170412995

Quadratic equations

- The solutions of the **quadratic equation** $ax^2 + bx + c = 0$, where $a, b, c \in \boldsymbol{R}$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \boldsymbol{Z}$ is given by $\Delta = b^2 - 4ac$
- For $\Delta < 0$, the roots are complex numbers
- For $\Delta = 0$, the roots are equal and rational
- For $\Delta > 0$, the roots are different real numbers
  - If Δ is a perfect square, the roots are rational
  - If Δ is a not perfect square, the roots are irrational

Sum and product of roots of a quadratic equation

- The roots α and β of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \boldsymbol{Z}$ have the properties that

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

- The quadratic equation $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ has roots α and β
- Complex roots of real quadratic equations are **complex conjugates** of the form $a + bi$ and $a - bi$.

Factors of quadratic expressions

- The factors of quadratic expressions with integer coefficients are:
  - linear factors for $\Delta = 0$ or $\Delta > 0$ and a perfect square
  - factors with conjugate surds for $\Delta > 0$ but not a perfect square
  - factors of the form $a(x - z)(x - \bar{z})$ for $\Delta < 0$

ISBN 9780170412995

11. CHAPTER REVIEW

Complex arithmetic and equations

Examples 1,2

1 Find the modulus and argument of each complex number.

a $-5\sqrt{3} + 5i$ **b** $9 - 4i$ **c** $3 + 2i$ (correct to 3 decimal places)

Example 3

2 Write each complex number in polar and trigonometric forms.

a $-3\sqrt{2} - 3\sqrt{2}i$ **b** $-4 + 7i$

Example 4

3 Write $4\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$ in Cartesian form.

Example 4

4 Write $7[\cos(-125°) + i\sin(-125°)]$ in Cartesian form, correct to 2 decimal places.

Example 5

5 Find each product in trigonometric and Cartesian form, correct to 2 decimal places where appropriate.

a $4\text{ cis}\left(\frac{\pi}{12}\right) \times 5\text{ cis}\left(\frac{\pi}{6}\right)$ **b** $3\text{ cis}(-28°) \times 7\text{ cis}(-94°)$ **c** $[3\text{ cis}(75°)]^4$

Example 6

6 Find each quotient in trigonometric form and Cartesian form.

a $\frac{12\text{cis}(-125°)}{3\text{cis}(85°)}$ **b** $\left[4\text{cis}\left(-\frac{\pi}{6}\right)\right]^{-3}$

Example 7

7 $z = 12\text{ cis}(-70°)$, $w = 2\text{ cis}(55°)$ and $u = 4\text{ cis}(80°)$. Without calculation, show on the same Argand plane:

a z and zw **b** z and $\frac{z}{u}$

Example 8

8 Solve each equation.

a $3h^2 - 17h - 28$ **b** $a^2 - 16a + 54$ **c** $x^2 - 12x + 85$

Example 9

9 Find the nature of the roots of each equation.

a $3x^2 - 5x + 8$ **b** $4x^2 + x - 6$ **c** $2x^2 + 5x - 7$

ISBN 9780170412995

10 Find a quadratic equation whose roots are $\frac{1}{3}$ of those of $4x^2 + 3x - 8 = 0$.

Example 10

11 Factorise each expression.

Examples 11,12

a $h^2 - 10h + 29$ **b** $4r^2 + 13r - 12$ **c** $k^2 - 12k + 31$

Problem solving

12 Prove that $\frac{1}{|z|} = \left|\frac{1}{z}\right|$.

13 Prove that $z \in \boldsymbol{R}$ if and only if $\arg(z) = 0$ or π.

14 Use $\text{cis}\left(\frac{11\pi}{12}\right) = \text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$ to find exact expressions for $\sin\left(\frac{11\pi}{12}\right)$ and $\cos\left(\frac{11\pi}{12}\right)$.

15 Find a quadratic equation whose roots are the cubes of the roots of $4x^2 - 9x + 4 = 0$.

Practice quiz

12.

TRIGONOMETRIC IDENTITIES AND APPLICATIONS

In this chapter, you will extend the work you have already completed with trigonometric identities. You will also look at a special type of motion called periodic motion. Periodic motion is seen when motion repeats itself in regular intervals of time. Examples of periodic motion include the heights of waves over a 24-hour period, the motion of a car tyre and the movement of a weight on a spring or pendulum swinging backward and forward.

12.01 Products of sine and cosine (product to sums)
12.02 Sums of sine and cosine (sums to products)
12.03 $a \cos(x) + b \sin(x)$
12.04 Graphs of $a \sin(x) + b \cos(x)$
12.05 Equations of the form $a \sin(x) + b \cos(x) = c$
12.06 Multiple angle trigonometric identities
12.07 Periodic motion
12.08 Modelling with trigonometric functions
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Trigonometric identities

- prove and apply the identities for products of sines and cosines expressed as sums and differences
- convert sums $a\cos(x) + b\sin(x)$ to $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$ and apply these to sketch graphs, solve equations of the form $a\cos(x) + b\sin(x) = c$ and solve problems
- prove and apply multi-angle trigonometric identities up to $\sin(4x)$ and $\cos(4x)$

Applications of trigonometric functions to model periodic phenomena

- model periodic motion using sine and cosine functions and understand the relevance of the period and amplitude of these functions in the model

Prior learning

Specialist Mathematics 2019 v1.0 General Senior Syllabus © Queensland Curriculum and Assessment Authority

iStock/courtneyk

TERMINOLOGY

amplitude
period
quadruple angle identities
triple angle identities
equilibrium value
phase shift
simple harmonic motion (SHM)
periodic motion
product to sum identities
sum to product identities

12.01 Products of sine and cosine

You have already learned a number of identities useful for expanding or simplifying trigonometric expressions. There are other situations where additional identities are useful. Sometimes you may need to express the product of cosine and sine as a sum. The angle sum and difference identities you established in chapter 9 can be used to do this.

Write the angle sum and difference identities for sine and add them:

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\sin(x+y) + \sin(x-y) = 2\sin(x)\cos(y)$$

Reverse the result and divide both sides by 2:

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

This is one of the **product to sum identities** because it transforms a product of two trigonometric terms to a sum of trigonometric terms. The other product to sum identities can be obtained in a similar way and are listed below.

Product to sum identities

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$
$$\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$
$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$
$$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

ISBN 9780170412995

EXAMPLE 1

Express $\sin(7A)\cos(2A)$ as a sum or difference.

Solution

This is a sin × cos product. Select the appropriate identity.

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

Replace x with $7A$ and y with $2A$.

$$\sin(7A)\cos(2A) = \frac{1}{2}[\sin(7A+2A) + \sin(7A-2A)]$$

Simplify.

$$= \frac{1}{2}[\sin(9A) + \sin(5A)]$$

EXAMPLE 2

Use a product to sum formula to evaluate $\cos\left(\frac{11\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$.

Solution

This is a cos × cos product. Select the appropriate identity.

$$\cos(x)\cos(y)$$
$$= \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

Replace x with $\frac{11\pi}{12}$ and y with $\frac{\pi}{12}$.

$$\cos\left(\frac{11\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$$
$$= \frac{1}{2}\left[\cos\left(\frac{11\pi}{12}+\frac{\pi}{12}\right) + \cos\left(\frac{11\pi}{12}-\frac{\pi}{12}\right)\right]$$

Simplify.

$$= \frac{1}{2}\left[\cos(\pi) + \cos\left(\frac{5\pi}{6}\right)\right]$$

Use exact values.

$$= \frac{1}{2}\left[-1 + \left(-\frac{\sqrt{3}}{2}\right)\right]$$

Evaluate.

$$= -\left(\frac{\sqrt{3}+2}{4}\right)$$

ISBN 9780170412995

EXAMPLE 3

Given that $\cos(70°) \approx 0.342$, use a product to sum identity to evaluate $\sin(50°)\sin(20°)$.

Solution

This is a sin × sin product. Select the appropriate identity.	$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
Replace x with 50° and y with 20°.	$\sin(50°)\sin(20°)$ $= \frac{1}{2}[\cos(50° - 20°) - \cos(50° + 20°)]$
Simplify.	$= \frac{1}{2}[\cos(30°) - \cos(70°)]$
Substitute for known values.	$\approx \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 0.342\right)$
Evaluate.	$= 0.26201\ldots$
Round off and state the result.	$\sin(50°)\sin(20°) \approx 0.262$

Exercise 12.01 Products of sine and cosine

Example 1

1 Write each expression as a sum or difference.

a $\sin(4x)\cos(x)$ **b** $\sin(5x)\cos(3x)$ **c** $\cos(4x)\cos(x)$

d $2\cos(x)\cos(3x)$ **e** $2\cos(6x)\sin(5x)$ **f** $2\cos(3x)\sin(x)$

g $-5\sin(6x)\sin(3x)$ **h** $5\sin(3x)\sin(4x)$ **i** $3\sin(5x)\cos(2x)$

Example 2

2 Use a product to sum identity to find the exact value of each product.

a $\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{12}\right)$ **b** $\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right)$

c $\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{3}\right)$ **d** $\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$

e $\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right)$ **f** $\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$

Example 3

3 Given that $\sin(40°) \approx 0.643$ and $\cos(40°) \approx 0.766$, use a product to sum identity to evaluate:

a $\cos(50°)\cos(10°)$ **b** $\sin(50°)\sin(10°)$

c $\sin(50°)\cos(10°)$ **d** $\cos(50°)\sin(10°)$

4 Write each expression as a sum or difference.

a $2\cos(3x)\sin(x)$ **b** $\cos(a + 30°)\cos(15°)$

ISBN 9780170412995

c $-4 \sin(5x) \sin(2x)$

d $2 \sin\left[\frac{1}{2}(p+q)\right] \cos\left[\frac{1}{2}(p-q)\right]$

e $3 \cos(2a-b) \sin(a+2b)$

f $\sin(m-n) \sin(m+n)$

g $2 \sin(m+n) \cos(m-n)$

h $\cos(4c) \cos(4c)$

5 Find the exact value of the product $\sin(105°) \sin(15°)$ using a product to sum identity.

6 Find the exact value of the product $\cos\left(\frac{11\pi}{12}\right) \sin\left(\frac{5\pi}{12}\right)$ using a product to sum identity.

Problem solving

7 Use the identities for $\cos(x \pm y)$ to show that $\cos(x) \cos(y) = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

8 Show that $\sin(9x) \cos(5x) - \sin(7x) \cos(3x) = \sin(2x) \cos(12x)$.

9 Show that $\sin(11a) \cos(6a) - \sin(7a) \cos(2a) = \sin(4a) \cos(13a)$.

10 Show that $\sin(11x) \cos(7x) - \sin(8x) \cos(4x) = \sin(3x) \cos(15x)$.

11 Show that $4 \sin(3x) \sin(5x) \sin(8x) = \sin(6x) + \sin(10x) - \sin(16x)$.

12 Show that $\sin(x) + \sin(y) = 2 \sin\frac{1}{2}(x+y) \cos\frac{1}{2}(x-y)$.

12.02 Sums of sine and cosine

It is possible to transform the product to sum identities into equivalent ones called the **sum to product identities**. The sum to product identities are used to express sums and differences involving sines and cosines as products involving sines and cosines.

The following illustrates how this transformation is achieved.

Write the $\sin(x) \cos(y)$ product to sum identity in terms of A and B:

$$\sin(A) \cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \qquad [1]$$

Let $A = \frac{x+y}{2}$ and $B = \frac{x-y}{2}$

$$\sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) = \frac{1}{2}\left[\sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right) + \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right)\right]$$

$$= \frac{1}{2}\left[\sin\left(\frac{2x}{2}\right) + \sin\left(\frac{2y}{2}\right)\right]$$

$$2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) = \sin(x) + \sin(y)$$

$$\sin(x) + \sin(y) = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

The other 3 identities can be obtained by following similar procedures.

Sum to product identities

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

EXAMPLE 4

Write the expression $\sin(6a) - \sin(2a)$ as a product.

Solution

This is a 'sin – sin' expression. Select the appropriate identity.

$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Substitute.

$$\sin(6a) - \sin(2a) = 2\cos\left(\frac{6a+2a}{2}\right)\sin\left(\frac{6a-2a}{2}\right)$$

Evaluate and state the result.

$$= 2\cos(4a)\sin(2a)$$

EXAMPLE 5

Use a sum to product identity to find the exact value of $\cos(30°) - \cos(150°)$.

Solution

Select the appropriate identity.

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Substitute.

$$\cos(30°) - \cos(150°)$$

$$= -2\sin\left(\frac{30° + 150°}{2}\right)\sin\left(\frac{30° - 150°}{2}\right)$$

Simplify.

$$= -2\sin(90°)\sin(-60°)$$

Evaluate.

$$= -2 \times 1 \times \left(-\frac{\sqrt{3}}{2}\right)$$

Evaluate and state the result.

$$\cos(30°) - \cos(150°) = \sqrt{3}$$

ISBN 9780170412995

The sum to product identities can be used to prove other trigonometric identities.

EXAMPLE 6

Prove $\dfrac{\cos(3x)-\cos(5x)}{\sin(3x)+\sin(5x)} = \tan(x)$.

Solution

Write the LHS.

$$\text{LHS} = \frac{\cos(3x)-\cos(5x)}{\sin(3x)+\sin(5x)}$$

Apply appropriate sum to product identities.

$$= \frac{-2\sin\left(\dfrac{3x+5x}{2}\right)\sin\left(\dfrac{3x-5x}{2}\right)}{2\sin\left(\dfrac{3x+5x}{2}\right)\cos\left(\dfrac{3x-5x}{2}\right)}$$

Simplify bracketed terms.

$$= \frac{-\sin(4x)\sin(-x)}{\sin(4x)\cos(-x)}$$

Cancel and simplify using $\cos(-x) = \cos(x)$.

$$= \frac{-\sin(-x)}{\cos(x)}$$

Simplify.

$$= \frac{\sin(x)}{\cos(x)}$$

Use the definition of $\tan(x)$.

$$= \tan(x)$$

Compare with the RHS.

$$= \text{RHS} \quad \text{QED}$$

Exercise 12.02 Sums of sine and cosine

Example 4

1 Write each expression as a product.

a $\cos(2a) - \cos(8a)$ **b** $3[\cos(15m) - \cos(5m)]$ **c** $-5[\cos(3b) - \cos(9b)]$

d $\sin(7x) + \sin(3x)$ **e** $4[\sin(14q) - \sin(2q)]$ **f** $\cos(4x) - \cos(6x)$

2 Write each expression as a product.

a $\sin(3x) + \sin(7x)$ **b** $\sin(8x) - \sin(2x)$ **c** $\cos(5x) + \cos(9x)$

d $\cos(11x) - \cos(7x)$ **e** $\sin(4x) + \sin(9x)$ **f** $\sin(2x) - \sin(5x)$

g $\cos(x) + \cos(5x)$ **h** $\cos(3x) - \cos(11x)$ **i** $\sin(x) + \sin(2x)$

3 Use a sum to product identity to simplify each expression.

a $-4[\cos(94°) - \cos(118°)]$ **b** $2[\cos(102°) - \cos(246°)]$

c $\sin(53°) + \sin(17°)$ **d** $\sin(104°) - \sin(20°)$

e $\cos(70°) - \cos(140°)$ **f** $\sin(-100°) + \sin(-50°)$

ISBN 9780170412995

Example 5

4 Use a sum to product identity to find the exact value of:

a $\cos(195°) + \cos(105°)$ **b** $\cos(15°) - \cos(75°)$ **c** $\sin(105°) - \sin(15°)$

Example 6

5 Show that $\dfrac{\sin(4x)+\sin(2x)}{\sin(2x)} = \dfrac{\sin(3x)}{\sin(x)}$.

Problem solving

6 Prove each identity using the relevant product to sum identity.

a $\sin(x) - \sin(y) = 2\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

b $\cos(x) + \cos(y) = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

c $\cos(x) - \cos(y) = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

7 Show that $\dfrac{\sin(x)+\sin(y)}{\sin(x)-\sin(y)} = \tan\left(\dfrac{x+y}{2}\right)\cot\left(\dfrac{x-y}{2}\right)$.

8 Show that $\dfrac{\sin(6x)+\sin(4x)}{\sin(6x)-\sin(4x)} = \tan(5x)\cot(x)$.

9 Show that $\dfrac{\cos(2x)-\cos(4x)}{\sin(2x)+\sin(4x)} = \tan(x)$.

10 Show that $\cos(x) - \cos(3x) = 4\sin^2(x)\cos(x)$.

12.03 $a\sin(x) + b\cos(x)$

The function $y = a\sin(x) + b\cos(x)$ is the sum of a sine and cosine term. It can be expressed in the form $y = r\cos(x \pm \alpha)$ or $y = r\sin(x \pm \alpha)$ using the angle sum identity.

Consider the expression $r\cos(x - \alpha)$. You know that $\cos(x - \alpha)$ can be expanded using an angle sum identity.

$$r\cos(x - \alpha) = r[\sin(x)\sin(\alpha) + \cos(x)\cos(\alpha)]$$
$$= r\sin(x)\sin(\alpha) + r\cos(x)\cos(\alpha)$$

You can rearrange this expression as follows:

$r\cos(x - \alpha) = [r\sin(\alpha)]\sin(x) + [r\cos(\alpha)]\cos(x)$

Now compare:

$a\sin(x) + b\cos(x)$ with $[r\sin(\alpha)]\sin(x) + [r\cos(\alpha)]\cos(x)$

ISBN 9780170412995

You can see that:

$a = r \sin(\alpha)$ [1]

$b = r \cos(\alpha)$ [2]

Square [1] and [2] and add:

$$a^2 + b^2 = r^2 \sin^2(\alpha) + r^2 \cos^2(\alpha)$$
$$= r^2 [\sin^2(\alpha) + \cos^2(\alpha)]$$

Use the Pythagorean identity:

$a^2 + b^2 = r^2 \times 1$ $[\sin^2(\alpha) + \cos^2(\alpha) = 1]$

$a^2 + b^2 = r^2$

So $r = \sqrt{a^2 + b^2}$

Also $\dfrac{r\sin(\alpha)}{r\cos(\alpha)} = \dfrac{a}{b}$

$\tan(\alpha) = \dfrac{a}{b}$

So $\alpha = \tan^{-1}\left(\dfrac{a}{b}\right)$

The **cosine form** of the sum of a sine and cosine term is stated below.

Cosine form of *a* sin (*x*) + *b* cos (*x*)

$a \sin(x) + b \cos(x) = r \cos(x - \alpha)$

$a \sin(x) - b \cos(x) = -r \cos(x + \alpha)$

where $r = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\left(\dfrac{a}{b}\right)$.

The definition above is a very useful tool for reducing the sum of two trigonometric functions to one trigonometric function. This makes it much easier to perform some operations.

EXAMPLE 7

Express $3 \sin(x) + 4 \cos(x)$ in the form $r \cos(x - \alpha)$.

Solution

Write the identity for $r \cos(x - \alpha)$. $r \cos(x - \alpha) = a \sin(x) + b \cos(x)$

Compare with the given expression. $3 \sin(x) + 4 \cos(x) = a \sin(x) + b \cos(x)$

Identify the variables.	$a = 3$ and $b = 4$
Calculate r.	$r = \sqrt{3^2 + 4^2}$
Evaluate.	$= \sqrt{25}$
	$= 5$
Calculate α.	$\alpha = \tan^{-1}\left(\frac{a}{b}\right)$
	$= \tan^{-1}\left(\frac{3}{4}\right)$
Evaluate.	$= 0.6435\ldots$
Round off and state the result	$3 \sin(x) + 4 \cos(x) \approx 5 \cos(x - 0.644)$

Using a similar approach, you can establish the formula for the **sine form** of the sum of a sine and cosine term.

Consider the expression $\sin(x + \alpha)$.

$\sin(x + \alpha) = \sin(x) \cos(\alpha) + \cos(x) \sin(\alpha)$

Write the expansion for $r \sin(x + \alpha)$

$r \sin(x + \alpha) = r \sin(x) \cos(\alpha) + r \cos(x) \sin(\alpha)$

$= [r \cos(\alpha)] \sin(x) + [r \sin(\alpha)] \cos(x)$

Now compare:

$a \sin(x) + b \cos(x)$ with $[r \cos(\alpha)] \sin(x) + [r \sin(\alpha)] \cos(x)$

You can see that:

$a = r \cos(\alpha)$ [1]

$b = r \sin(\alpha)$ [2]

Square [1] and [2] and add:

$a^2 + b^2 = r^2 \cos^2(\alpha) + r^2 \sin^2(\alpha)$

$= r^2 [\cos^2(\alpha) + \sin^2(\alpha)]$

Use the Pythagorean identity:

$a^2 + b^2 = r^2 \times 1$ $[\sin^2(\alpha) + \cos^2(\alpha) = 1]$

$a^2 + b^2 = r^2$

So $r = \sqrt{a^2 + b^2}$

ISBN 9780170412995

Also $\dfrac{r\sin(\alpha)}{r\cos(\alpha)} = \dfrac{b}{a}$

$\tan(\alpha) = \dfrac{b}{a}$

So $\alpha = \tan^{-1}\left(\dfrac{b}{a}\right)$

The sine form of the sum of a sine and cosine term is $a\sin(x) + b\cos(x) = r\sin(x + \alpha)$.

Sine form of $a\sin(x) + b\cos(x)$

$$a\sin(x) + b\cos(x) = r\sin(x + \alpha)$$

$$a\sin(x) - b\cos(x) = r\sin(x - \alpha)$$

where $r = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\left(\dfrac{b}{a}\right)$.

Note that for the cosine form, $\alpha = \tan^{-1}\left(\dfrac{a}{b}\right)$, but for the sine form $\alpha = \tan^{-1}\left(\dfrac{b}{a}\right)$.

EXAMPLE 8

Express $\sqrt{3}\sin(x) + \cos(x)$ in the form $r\sin(x + \alpha)$ using radians.

Solution

Write the relevant formula.	$a\sin(x) + b\cos(x) = r\sin(x + \alpha)$
Compare with the given expression.	$\sqrt{3}\sin(x) + \cos(x) = a\sin(x) + b\cos(x)$
Identify the variables.	$a = \sqrt{3}$ and $b = 1$
Calculate $\tan(\alpha)$.	$\tan(\alpha) = \dfrac{b}{a} = \dfrac{1}{\sqrt{3}}$
Calculate α.	$\alpha = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$
Calculate r.	$r = \sqrt{\sqrt{3}^2 + 1^2}$
Evaluate.	$= \sqrt{4} = 2$
State the result.	$\sqrt{3}\sin(x) + \cos(x) = 2\sin\left(x + \dfrac{\pi}{6}\right)$

EXAMPLE 9

Express $2\cos(x) - 5\sin(x)$ in the form $r\cos(x - \alpha)$, where $0 < \alpha < 2\pi$.

Solution

Write the identity.

$$a\sin(x) + b\cos(x) = r\cos(x - \alpha)$$

Rearrange the expression.

$$2\cos(x) - 5\sin(x) = r\cos(x - \alpha)$$

$$-5\sin(x) + 2\cos(x) = r\cos(x - \alpha)$$

Identify a and b.

$$a = -5 \text{ and } b = 2$$

Calculate r.

$$r = \sqrt{(-5)^2 + 2^2}$$

Evaluate.

$$= \sqrt{29}$$

Calculate α.

$$\alpha = \tan^{-1}\left(\frac{a}{b}\right) = \tan^{-1}\left(-\frac{5}{2}\right)$$

Evaluate.

$$\approx 1.951 \text{ or } 5.093$$

Decide which quadrant α is in.

$$r\cos(x - \alpha) = [r\sin(\alpha)]\sin(x) + [r\cos(\alpha)]\cos(x)$$

$$= a\sin(x) + b\cos(x)$$

So

$$\sin(\alpha) = \frac{a}{r} = -\frac{5}{\sqrt{29}} < 0$$

$$\cos(\alpha) = \frac{b}{r} = \frac{2}{\sqrt{29}} > 0$$

$$\tan(\alpha) = \frac{a}{b} = -\frac{5}{2} < 0$$

α is in the 4th quadrant.

State the value of α.

$$\alpha \approx 5.093$$

State the result.

$$2\cos(x) - 5\sin(x) \approx \sqrt{29}\cos(x - 5.093)$$

Exercise 12.03 *a* sin (*x*) + *b* cos (*x*)

Work in radians unless otherwise directed.

Example 1

1 Write each expression in the form $r\cos(x \pm \alpha)$. Where necessary, give answers correct to 3 decimal places.

a $12\sin(x) + 5\cos(x)$ **b** $7\sin(x) + 12\cos(x)$ **c** $5\sin(x) + 6\cos(x)$

d $-\sin(x) + 3\cos(x)$ **e** $-3\sin(x) - 2\cos(x)$ **f** $-\sin(x) + 4\cos(x)$

g $12\sin(x) - 5\cos(x)$ **h** $3\sin(x) - \cos(x)$ **i** $2.3\sin(x) - 1.4\cos(x)$

ISBN 9780170412995

2 Write each expression in the form $r\cos(x \pm \alpha)$, correct to 3 decimal places where necessary.

a $3\cos(x) + 4\sin(x)$ **b** $2\sin(x) + \cos(x)$ **c** $\cos(x) + \sqrt{3}\sin(x)$

d $5\cos(x) - 12\sin(x)$ **e** $8\cos(x) - 15\sin(x)$ **f** $3\cos(x) + \sqrt{3}\sin(x)$

3 Write each expression in the form $r\sin(x \pm \alpha)$, correct to 3 decimal places where necessary.

Example 8

a $3\sin(x) + 5\cos(x)$ **b** $12\sin(x) + 5\cos(x)$ **c** $7\sin(x) - 6\cos(x)$

d $4\sin(x) + 3\cos(x)$ **e** $4\sin(x) - \cos(x)$ **f** $\sin(x) - 3\cos(x)$

4 Write each expression in the form $r\sin(x \pm \alpha)$, correct to 3 decimal places where necessary.

a $4\sin(x) + 2\cos(x)$ **b** $\sin(x) - 7\cos(x)$ **c** $2\sin(x) - 3\cos(x)$

d $\sin(x) + 0.5\cos(x)$ **e** $3\sin(x) - 2\cos(x)$ **f** $5\sin(x) - \cos(x)$

5 Express:

Example 9

a $-\sin(x) + \cos(x)$ in the form $r\cos(x - \alpha)$, where $0 < \alpha < 2\pi$

b $\sin(x) + \cos(x)$ in the form $r\cos(x - \alpha)$ for $-\pi \leq \alpha \leq \pi$

c $\sin(x) + \cos(x)$ in the form $r\sin(x + \alpha)$, where $0 < \alpha < \frac{\pi}{2}$

d $-\sin(x) + \cos(x)$ in the form $r\cos(x + \alpha)$ for $-\pi \leq \alpha \leq \pi$

e $-2\sin(x) + \sqrt{3}\cos(x)$ in the form $r\sin(x - \alpha)$, where $0 < \alpha < 2\pi$

f $\sin(x) - \cos(x)$ in the form $r\cos(x - \alpha)$ for $-\pi \leq \alpha \leq \pi$

g $4\sin(x) - 3\cos(x)$ in the form $r\sin(x - \alpha)$, where 0 and $0 < \alpha < \frac{\pi}{2}$

h $-[\sin(x) + \cos(x)]$ in the form $r\cos(x + \alpha)$ for $-\pi \leq \alpha \leq \pi$

6 Write the expression $4\cos(x) + 4\sin(x)$ in the form:

a $r\sin(x + \alpha)$ **b** $r\cos(x - \alpha)$

7 Write the expression $3\cos(x) - 5\sin(x)$ in the form:

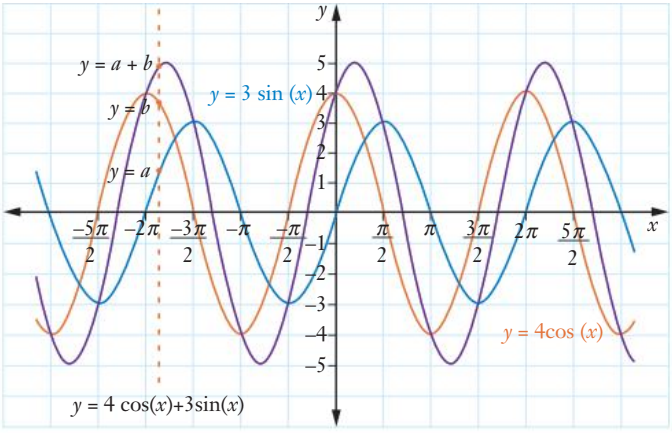
a $r\sin(x + \alpha)$ **b** $r\cos(x + \alpha)$

12.04 Graphs of $a\sin(x) + b\cos(x)$

The graph of $y = a\sin(x) + b\cos(x)$ could be constructed by adding the individual graphs of $y = a\sin(x)$ and $y = b\cos(x)$.

The diagram next page shows the graph of $y = 3\sin(x)$ in blue, the graph of $y = 4\cos(x)$ in orange and the graph of $y = 3\sin(x) + 4\cos(x)$ in purple.

ISBN 9780170412995



As shown in the diagram, each point on the graph of $y = 3 \sin(x) + 4 \cos(x)$ can be found by adding the corresponding values for the graphs of $y = 3 \sin(x)$ and $y = 4 \cos(x)$.

In practice, this would be a tedious procedure so your knowledge of the sum of a sine and cosine term can be used to simplify the process.

EXAMPLE 10

Use the function $y = r \cos(x - \alpha)$ to draw the graph of $y = 3 \sin(x) + 4 \cos(x)$ for $0 \leq x \leq 2\pi$, showing all key features.

Solution

Rewrite the RHS of the equation using a formula for the sum of a sine and cosine term.	$3 \sin(x) + 4 \cos(x) = r \cos(x - \alpha)$ where $\alpha = \tan^{-1}\left(\frac{a}{b}\right)$ and $r = \sqrt{a^2 + b^2}$
Calculate r and α.	$r = \sqrt{3^2 + 4^2} = 5$ $\alpha = \tan^{-1}\left(\frac{3}{4}\right) \approx 0.644$
Write the equation.	$y = 5 \cos(x - 0.644)$
Compare the graph of $y = 5 \cos(x - 0.644)$ with the graph of $y = \cos(x)$.	The graph of $y = 5 \cos(x - 0.644)$ is the same shape as the graph of $y = \cos(x)$, except that there has been a phase shift of 0.644 to the right and a vertical stretch with a scale factor of 5.

ISBN 9780170412995

Identify the key features of the graph.

The amplitude is increased by a factor of 5.

Amplitude $= 5 \times 1 = 5$

The period is unchanged. Period $= 2\pi$

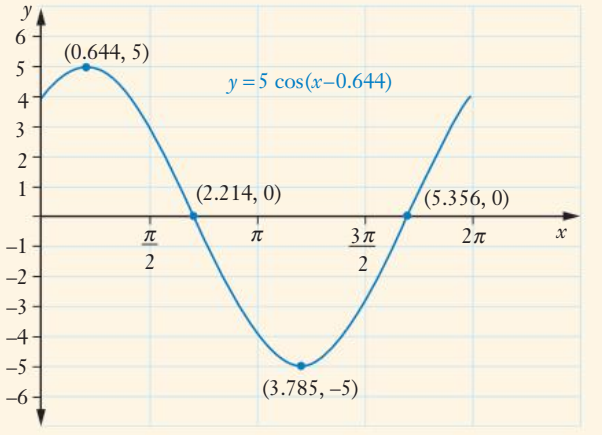
Maximum at $x \approx 0 + 0.644 = 0.644$

Minimum at $x \approx \pi + 0.644 \approx 3.785$

Zeros at $x \approx \frac{\pi}{2} + 0.644 \approx 2.214$

$$x \approx \frac{3\pi}{2} + 0.644 \approx 5.356$$

Use this information to draw the graph.



The same process can be used for graphs of functions of the form $y = r \sin(x \pm \alpha)$.

EXAMPLE 11

Draw the graph of $y = 3 \sin(x) - 3 \cos(x)$ for $0 \leq x \leq 2\pi$ using the function $y = r \sin(x - \alpha)$.

Solution

Rewrite the RHS of the equation using a formula for the sum of a sine and cosine term.

$3 \sin(x) - 3 \cos(x) = r \sin(x - \alpha)$

where $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

and $r = \sqrt{a^2 + b^2}$

Calculate r and α.

$r = \sqrt{3^2 + 3^2} = 3\sqrt{2} \approx 4.24$

$\alpha = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

Write the equation.

$y = 3\sqrt{2} \sin\left(x - \frac{\pi}{4}\right)$

ISBN 9780170412995

Compare the graph of $y = 3\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$ with the graph of $y = \sin(x)$.

The graph of $y = 3\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$ is the same shape as the graph of $y = \sin(x)$, except that there has been a phase shift of $\frac{\pi}{4}$ to the right and a vertical stretch with a scale factor of $3\sqrt{2}$.

Identify the key features of the graph.

The amplitude is increased by a factor of $3\sqrt{2}$.

Amplitude $= 3\sqrt{2} \times 1 = 3\sqrt{2}$

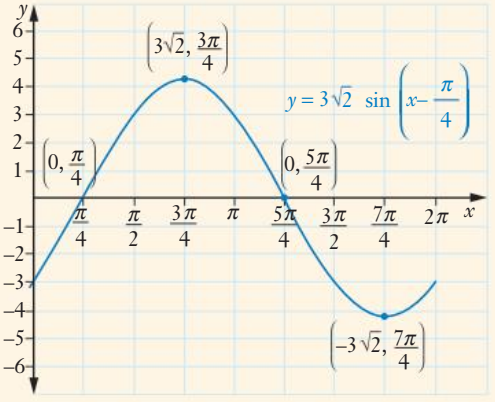
The period is unchanged. Period $= 2\pi$

Maximum at $x \approx \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

Minimum at $x \approx \frac{3\pi}{2} + \frac{\pi}{4} = \frac{7\pi}{4}$

Zeros at $x = 0 + \frac{\pi}{4} = \frac{\pi}{4}$ and $x = \pi + \frac{\pi}{2} = \frac{5\pi}{4}$.

Use this information to draw the graph.



Exercise 12.04 Graphs of *a* sin (*x*) + *b* cos (*x*)

Example 10

1 Use the function $y = r\cos(x \pm \alpha)$ to draw the graphs of each function for $0 \le x \le 2\pi$, showing all key features.

a $y = 4\sin(x) + 4\cos(x)$ **b** $y = 3\cos(x) - 5\sin(x)$ **c** $y = 5\sin(x) + 12\cos(x)$

Example 11

2 Use the function $y = r\sin(x \pm \alpha)$ to draw the graphs of each function for $0 \le x \le 2\pi$, showing all key features.

a $y = 3\sin(x) - 2\cos(x)$ **b** $y = 8\sin(x) + 6\cos(x)$ **c** $y = \sin(x) - 2\cos(x)$

ISBN 9780170412995

3 **a** Express $y = \sqrt{3}\sin(x) - \cos(x)$ in the form $y = r\sin(x \pm \alpha)$ and hence draw the graph of $y = \sqrt{3}\sin(x) - \cos(x)$ for $0 \leq x \leq 2\pi$.

b Express $y = \sqrt{3}\sin(x) - \cos(x)$ in the form $y = r\cos(x \pm \alpha)$ and hence draw the graph of $y = \sqrt{3}\sin(x) - \cos(x)$.

c Comment on the results of **a** and **b**.

4 Draw the graphs of each of the following for $0 \leq x \leq 2\pi$, showing all key features.

a $y = 2\sin(x) - \cos(x)$ **b** $y = 5\cos(x) + 2\sin(x)$

c $y = -3\sin(x) + \cos(x)$ **d** $y = -\sin(x) - 4\cos(x)$

Problem solving

5 For each function, calculate the maximum value and the smallest positive angle at which the maximum value occurs. State your answer correct to 3 decimal places.

a $f(x) = 4 + 4\cos(x) + 3\sin(x)$ **b** $f(x) = 5 - 3\cos(x) + 4\sin(x)$

c $f(x) = 2 - 5\cos(x) - 12\sin(x)$ **d** $f(x) = 7 + \cos(x) - \sin(x)$

6 Given the function $y = 3\sin(x) + 4\cos(x)$ for $0° \leq x \leq 360°$, calculate the value of x where the function has:

a a minimum **b** a maximum

12.05 Equations of the form $a\sin(x) + b\cos(x) = c$

The formulae for the sum of a sine and cosine term can also be used to solve trigonometric equations.

EXAMPLE 12

Solve the equation $\sin(x) + \sqrt{2}\cos(x) = 0$ for $-\pi \leq x \leq \pi$.

Solution

Express the LHS using the sum of a sine and cosine term formula.	$\sin(x) + \sqrt{2}\cos(x) = r\cos(x - \alpha)$
Identify the variables.	$a = 1$ and $b = \sqrt{2}$
Calculate r.	$r = \sqrt{a^2 + b^2} = \sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$
Calculate α.	$\alpha = \tan^{-1}\left(\frac{a}{b}\right) = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 0.6154\ldots$

Check for other solutions.	$\cos(\alpha) > 0$ $\sin(\alpha) > 0$ $\tan(\alpha) > 0$ So α must be in the 1st quadrant. $\alpha = 0.6154\ldots$
Rewrite the equation in terms of α.	$\sqrt{3}\cos(x - 0.6154\ldots) = 0$
Simplify.	$\cos(x - 0.6154\ldots) = 0$
Use $\cos^{-1}$ to solve.	$x - 0.6154\ldots = \cos^{-1}(0)$
Find all solutions for $-\pi \leq x \leq \pi$.	$x - 0.6154\ldots = -\frac{\pi}{2}, \frac{\pi}{2}$
Solve.	$x = -\frac{\pi}{2} + 0.6154\ldots$ or $\frac{\pi}{2} + 0.6154$
Evaluate.	$x = -0.9553\ldots$ or $2.1862\ldots$
Round off and state the result.	$x \approx -0.955$ or 2.186

EXAMPLE 13

Solve the equation $\sin(x) - \sqrt{3}\cos(x) = 2$ for $0 \leq x \leq 2\pi$.

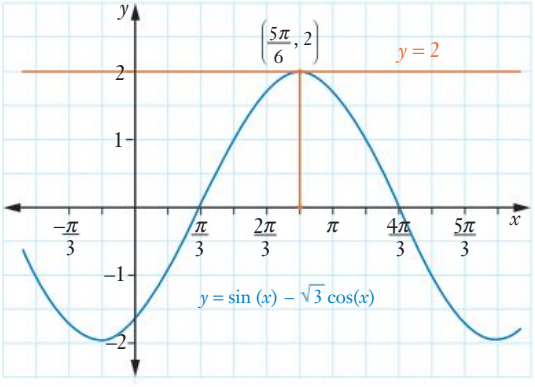
Solution

Express the LHS using the sum of a sine and cosine term formula.	$\sin(x) - \sqrt{3}\cos(x) = r\sin(x - \alpha)$
Identify the variables.	$a = 1$ and $b = \sqrt{3}$
Calculate r.	$r = \sqrt{a^2 + b^2} = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$
Rewrite the LHS in terms of α.	$\sin(x) - \sqrt{3}\cos(x) = 2\sin(x - \alpha)$
Calculate α.	$\alpha = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$
Write the equation $\sin(x) - \sqrt{3}\cos(x) = 2$ in terms of α.	$r\sin(x - \alpha) = 2$

ISBN 9780170412995

Substitute for known values.	$2\sin\left(x-\frac{\pi}{3}\right)=2$
Rearrange.	$\sin\left(x-\frac{\pi}{3}\right)=1$
Use $\sin^{-1}$ to solve.	$x-\frac{\pi}{3}=\sin^{-1}(1)$
Find all solutions for $0\le x\le 2\pi$.	$x-\frac{\pi}{3}=\frac{\pi}{2}$
Calculate the interval in which x lies.	If x lies in the interval $0<x<2\pi$, then $x-\frac{\pi}{3}$ lies in the interval $0-\frac{\pi}{3}<x-\frac{\pi}{3}<2\pi-\frac{\pi}{3}$
Solve.	$x=\frac{\pi}{2}+\frac{\pi}{3}$
Evaluate.	$x=\frac{5\pi}{6}$

You could have used a graphical approach to solve the equation shown in the previous example. This involves plotting the graph of $\sin(x)-\sqrt{3}\cos(x)$ and the graph $y=2$ on the same set of axes as shown in the following diagram and then finding the point of intersection of the graphs.



Exercise 12.05 Equations of the form *a* cos (*x*) + *b* sin (*x*) = *c*

1 a Express $\sqrt{3}\sin(x)+\cos(x)$ in the form $r\sin(x+\alpha)$, where $r>0$ and $0<\alpha<\frac{\pi}{2}$.

b Solve the equation $\sqrt{3}\sin(x)+\cos(x)=1$ for x in the interval $0\le x\le 2\pi$, giving your answers in terms of π.

ISBN 9780170412995

2 **a** Express $3\cos(x) + \sin(x)$ in the form $r\cos(x - \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Solve the equation $3\cos(x) + \sin(x) = 2$ for x in the interval $0 \leq x \leq 2\pi$, giving your answers correct to 3 decimal places.

Example 12

3 Solve each equation for $0 \leq x \leq 2\pi$, correct to 3 decimal places where necessary.

a $3\sin(x) + 4\cos(x) = 0$ **b** $5\sin(x) - 12\cos(x) = 0$

c $\sqrt{2}\sin(x) + \cos(x) = 0$ **d** $-3\sin(x) + 3\cos(x) = 0$

Example 13

4 Solve each equation, correct to 3 decimal places where necessary.

a $12\cos(x) + 5\sin(x) = 10$ for $0 \leq x \leq 2\pi$ **b** $\sin(x) - 2\cos(x) = 1$ for $0 \leq x \leq 2\pi$

c $2\sqrt{3}\cos(x) - 6\sin(x) = 5$ for $-\pi \leq x \leq \pi$ **d** $9\sin(x) + 4\cos(x) = 8$ for $-\pi \leq x \leq \pi$

5 Use a graphical approach to find approximate solutions for each equation.

a $2\sin(x) + 2\cos(x) = 1$ for $0 \leq x \leq 2\pi$ **b** $4\cos(x) - \sin(x) = 2$ for $0 \leq x \leq 2\pi$

c $6\sin(x) + 8\cos(x) = 5$ for $-\pi \leq x \leq \pi$ **d** $3\cos(x) + \sin(x) = -1$ for $-\pi \leq x \leq \pi$

Problem solving

6 Find the general solution of $\sin(\theta) + \sqrt{3}\cos(\theta) = 0$.

7 Find the general solution of $3\sin(\theta) - 4\cos(\theta) = 3$.

12.06 Multiple angle trigonometric identities

You were introduced to the double and half angle trigonometric identities in chapter 9.

Double angle identities

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

Half angle identities

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\tan(x) = \frac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}$$

It is possible to find identities for other multiples of angles. For example, the formula for $\sin(3x)$ can be developed by using both the double angle formula and the sum formula.

ISBN 9780170412995

$\sin(3x) = \sin(2x + x)$

$= \sin(2x)\cos(x) + \cos(2x)\sin(x)$ (Angle sum formula)

$= [2\sin(x)\cos(x)]\cos(x) + [\cos^2(x) - \sin^2(x)]\sin(x)$ (Double angle formulas)

$= 2\sin(x)\cos^2(x) + \cos^2(x)\sin(x) - \sin^3(x)$

$= 2\sin(x)[1 - \sin^2(x)] + [1 - \sin^2(x)]\sin(x) - \sin^3(x)$ ($\sin^2(x) + \cos^2(x) = 1$)

$= 2\sin(x) - 2\sin^3(x) + \sin(x) - \sin^3(x) - \sin^3(x)$

$= 3\sin(x) - 4\sin^3(x)$

You can use a similar approach to develop identities for $\cos(3x)$ and $\tan(3x)$.

Triple angle identities

$$\sin(3x) = 3\sin(x) - 4\sin^3(x)$$

$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

$$\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$$

EXAMPLE 14

Use a multiple angle formula to find the exact value for $\sin(135°)$.

Solution

Write 135° as a multiple of another angle. $\sin(135°) = \sin(3 \times 45°)$

Use a triple angle identity. $\sin(3 \times 45°) = 3\sin(45°) - 4\sin^3(45°)$

Use exact values. $= 3 \times \frac{\sqrt{2}}{2} - 4 \times \left(\frac{\sqrt{2}}{2}\right)^3$

Evaluate. $= \frac{3\sqrt{2}}{2} - \frac{4 \times 2\sqrt{2}}{8}$

Simplify. $= \frac{3\sqrt{2} - 2\sqrt{2}}{2}$

Evaluate. $= \frac{\sqrt{2}}{2}$

The quadruple angle trigonometric identities can be developed in a similar way to the triple angle identities. Consider $\cos(4x)$.

ISBN 9780170412995

$$\cos(4x) = \cos(2x + 2x)$$
$$= \cos(2x)\cos(2x) - \sin(2x)\sin(2x)$$
$$= [\cos^2(x) - \sin^2(x)][\cos^2(x) - \sin^2(x)] - [2\sin(x)\cos(x)][2\sin(x)\cos(x)]$$
$$= \cos^4(x) - 2\cos^2(x)\sin^2(x) + \sin^4(x) - 4\cos^2(x)\sin^2(x)$$
$$= \cos^4(x) - 2\cos^2(x)[1 - \cos^2(x)] + [1 - \cos^2(x)]^2 - 4\cos^2(x)[1 - \cos^2(x)]$$
$$= 8\cos^4(x) - 8\cos^2(x) + 1$$

A similar approach can be used to develop identities for $\sin(4x)$ and $\tan(4x)$.

Quadruple angle identities

$$\sin(4x) = 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)$$
$$\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$$
$$\tan(4x) = \frac{4\tan(x) - 4\tan^3(x)}{1 - 6\tan^2(x) + \tan^4(x)}$$

Multiple angle identities can also be used to help solve trigonometric equations.

EXAMPLE 15

Solve $2\sin(3x)\sin(x) = 1$ for $0 \leq x \leq 2\pi$.

Solution

Write the equation.	$2\sin(3x)\sin(x) = 1$
Use the definition for $\sin(3x)$.	$2[3\sin(x) - 4\sin^3(x)]\sin(x) = 1$
Expand.	$6\sin^2(x) - 8\sin^4(x) = 1$
Rearrange.	$8\sin^4(x) - 6\sin^2(x) + 1 = 0$
Let $u = \sin^2(x)$.	$8u^2 - 6u + 1 = 0$
Factorise.	$(4u - 1)(2u - 1) = 0$
Solve for u.	$4u - 1 = 0$ or $2u - 1 = 0$ $u = \frac{1}{4}$ or $\frac{1}{2}$
Replace u with $\sin^2(x)$.	$\sin^2(x) = \frac{1}{4}$ or $\sin^2(x) = \frac{1}{2}$

ISBN 9780170412995

Simplify.	$\sin(x) = \pm\frac{1}{\sqrt{2}}$ or $\sin(x) = \pm\frac{1}{2}$
Find all solutions for $0 \le x \le 2\pi$.	$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ or $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
State the result.	$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$

Exercise 12.06 Multiple angle trigonometric identities

1 Verify the identity $\sin(3x) = 3\sin(x) - 4\sin^3(x)$ for:

a $x = 30°$ **b** $x = \frac{\pi}{4}$ **c** $x = 60°$

2 Verify the identity $\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$ for:

a $x = 45°$ **b** $x = \frac{4\pi}{3}$ **c** $x = -\frac{\pi}{6}$

3 Verify the identity $\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$ for:

a $x = 0.5$ **b** $x = \frac{11\pi}{6}$ **c** $x = -1.24$

4 Use a multiple angle formula to find the exact value for: Example 14

a $\sin(120°)$ **b** $\cos(135°)$ **c** $\cos(240°)$

5 If $\sin(x) = \frac{3}{5}$ and x is an acute angle, find the exact values of:

a $\cos(2x)$ **b** $\cos(4x)$ **c** $\sin(2x)$ **d** $\sin(4x)$

6 Solve $\sin(2x) - \sin(x) = 0$ for $0 \le x \le 2\pi$. Example 15

Problem solving

7 Prove the identity $\cos(3x) = 4\cos^3(x) - 3\cos(x)$.

8 Prove the identity $\sin(4x) = 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)$.

9 Prove the identity $\tan(4x) = \frac{4\tan(x) - 4\tan^3(x)}{1 - 6\tan^2(x) + \tan^4(x)}$.

10 Solve $\sin(3x) = 0$ for $0 \le x \le 2\pi$.

11 Solve $\sin(4x) = 0$ for $0 \le x \le 2\pi$.

12 Solve $\cos(4x) - 3\cos(2x) = 4$ for $0 \le x \le 2\pi$.

13 Solve $\sin(2x) + 4\sin(x) + 2\cos(x) = -4$ for $0 \le x \le 2\pi$.

14 Solve $\cos(x) + \cos(3x) = 0$ for $0 \le x \le 2\pi$.

ISBN 9780170412995

12.07 Periodic motion

Consider a Ferris wheel that completes one rotation every 30 seconds.

This is an example of **periodic motion**.

Periodic motion is any motion that repeats itself in equal intervals of time. Other examples include the Earth rotating on its axis every 24 hours, the Earth revolving around the Sun every year, a vibrating spring and a swinging pendulum.

The **period** of the motion is the time taken for the motion to complete one cycle before repeating itself.

iStock/Vitally

A special type of periodic motion in which an object oscillates forwards and backwards is called **simple harmonic motion** (SHM). It should not be surprising that you can define simple harmonic motion using trigonometric functions.

Simple harmonic motion

The position of a particle undergoing simple harmonic motion is:

$$x = a\cos(nt + \varepsilon),$$

where x is the displacement of the particle from its equilibrium value at time, t.

The amplitude, $A = a$

The period, $T = \dfrac{2\pi}{n}$

The phase shift $= \varepsilon$

The position of x could also be expressed as a sine function: $x = a\sin(nt + \varepsilon)$.

EXAMPLE 16

The displacement, x, of a particle in metres over t seconds is given by $x = 4\cos(3t)$.

a Calculate the amplitude and period.

b Draw a graph of the motion of the particle.

c Describe the motion of the particle.

ISBN 9780170412995

Solution

a Write the equation of motion.

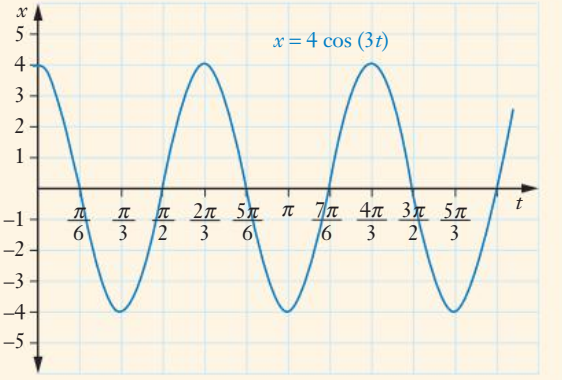
$x = 4 \cos (3t)$

Compare the equation with $x = a \cos (nt + \varepsilon)$.

Amplitude = 4

Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$

b The graph of $x = 4 \cos (3t)$ has the same shape as the graph of $x = \cos (t)$, except that it has been dilated vertically by a factor of 4 and compressed horizontally by a factor of $\frac{1}{3}$.



c Use the graph to describe the motion.

The initial position of the particle ($x = 0$) is 4 m above the equilibrium value (the origin). After $\frac{\pi}{6}$ s, the particle passes through the origin and moves away from it. At $\frac{\pi}{3}$ s, the particle is 4 m below the origin. The particle returns to the origin at $\frac{\pi}{2}$ s and continues until it is 4 m above the origin at $\frac{2\pi}{3}$ s. This motion is then repeated.

The graph in Example 16 shows that

- the maximum displacement of the particle is 4 m (amplitude)
- the cycle of the motion repeats every $\frac{2\pi}{3}$ s (period)
- the particle changes its direction of motion when it reaches its maximum displacement, so it comes to rest at these points.

ISBN 9780170412995

EXAMPLE 17

An object moves in a straight line such that its displacement, x, is given by

$$x = 4\sqrt{3}\sin\left(2t - \frac{\pi}{4}\right)$$

where x is measured in centimetres and t is in seconds.

a Draw a graph of the motion of the object.

b When does the object first come to rest?

c When is the first time the object is 4 cm from the origin?

d When is the next time the object is 4 cm from the origin?

e When is the first time the object returns to the origin after it reaches its maximum displacement?

Solution

a Write the equation of motion.

$x = 4\sqrt{3}\sin\left(2t - \frac{\pi}{4}\right)$

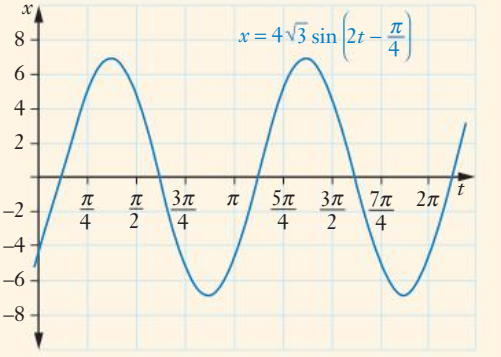
Compare the equation with $x = a\cos(nt + \varepsilon)$.

Amplitude $= 4\sqrt{3}$

Period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$

Phase shift $= -\frac{\pi}{4}$

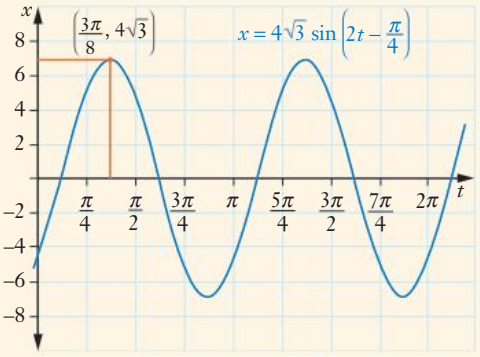
The graph of $x = 4\sqrt{3}\sin\left(2t - \frac{\pi}{4}\right)$ has the same shape as the graph of $x = \sin(t)$, except that it has been dilated vertically by a factor of $4\sqrt{3}$ and compressed horizontally by a factor of $\frac{1}{2}$ and it has been translated horizontally $\frac{\pi}{4}$ to the right.



ISBN 9780170412995

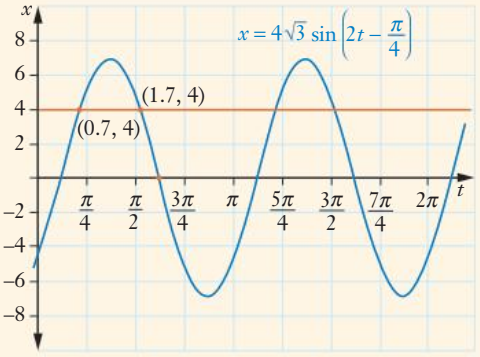
b The object is at rest when it first changes the direction of motion.

Read the value from the graph.



The object comes to rest at $\frac{3\pi}{8}$ s.

c Draw the function $x = 4$ and find the first point of intersection.



The object is first 4 cm from the origin after about 0.7 s.

d Read the value from the graph.

The object is next 4 cm from the origin after about 1.7 s.

e Read the value from the graph.

The object returns to the origin after $\frac{5\pi}{8}$ s.

Exercise 12.07 Periodic motion

1 In each of the following equations of the motion of a particle, x is measured in metres and t is measured in seconds. Calculate the maximum displacement of the particle in each case.

a $x = 3\cos(2t)$ **b** $x = 5\cos\left(3t + \frac{\pi}{3}\right)$ **c** $x = 2.5\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

d $x = 2\sqrt{3}\cos\left(3.4t + \frac{\pi}{2}\right)$ **e** $x = 3\sin\left(\frac{4}{3}t - \frac{3\pi}{8}\right)$ **f** $x = 1.25\cos(3.6t + 2.1)$

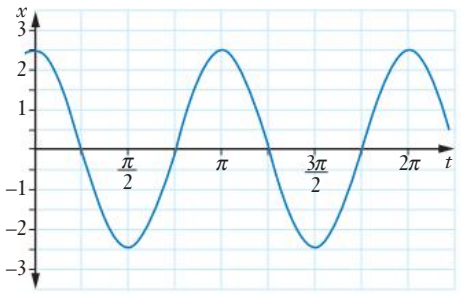
2 For each equation in question **1**, calculate the period of the motion.

3 For each equation in question **1**, calculate the first time ($t > 0$) that the particle is at the origin.

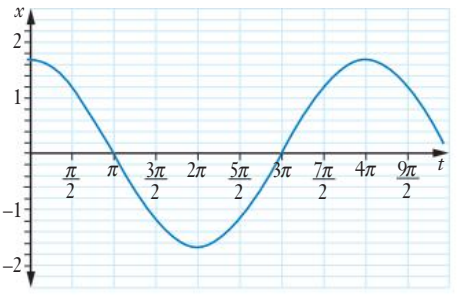
ISBN 9780170412995

4 Calculate the amplitude and period for each graph of the displacement (x) of a particle undergoing simple harmonic motion.

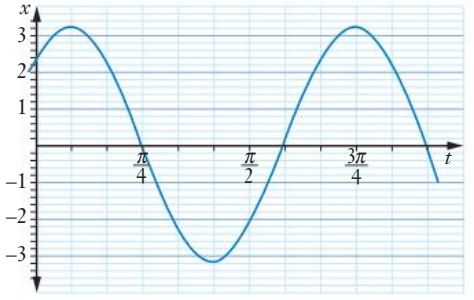
a



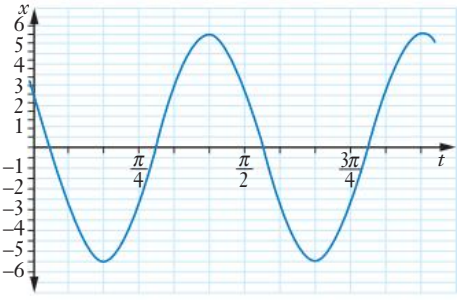
b



c



d



5 For each graph in question **4**, use an equation to find the first time ($t > 0$) that the particle is at the origin.

6 For each graph in question **4**, use an equation to find the first time ($t > 0$) that the particle is at the maximum displacement.

Example 16

7 The displacement, x, of a particle in metres over t seconds is given by $x = 6\cos(4t)$.

a Calculate the amplitude and period of motion.

b Draw a graph of the motion of the particle.

c Describe the motion of the particle.

8 The displacement, x, of a particle in centimetres over t seconds is given by $x = 5\cos(2.5t)$.

a Draw a graph of the motion of the particle.

b Describe the motion of the particle.

Example 17

9 An object moves in a straight line such that its displacement, x, is given by $x = 3.5\sin\left(2t - \frac{\pi}{3}\right)$, where x is measured in centimetres and t is in seconds.

a Draw a graph of the motion of the object.

b When does the object first come to rest?

c When is the first time the object is 2 cm from the origin?

d When is the second time the object is at rest?

e When is the first time the object returns to the origin after it reaches its maximum displacement?

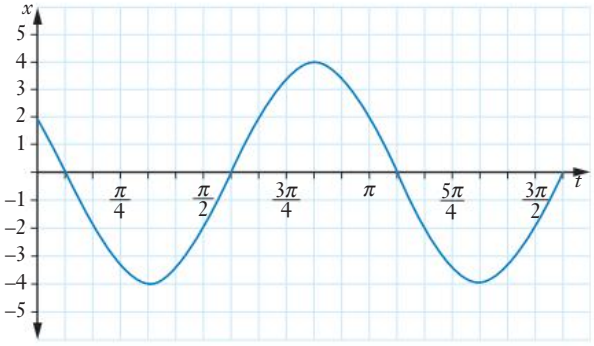
ISBN 9780170412995

10 An object moves in a straight line such that its displacement, x, is given by $x = 7\cos\left(1.3t + \frac{2\pi}{5}\right)$, where x is measured in metres and t is in seconds.

- **a** Draw a graph of the motion of the object.
- **b** When does the object first come to rest?
- **c** After coming to rest for the first time, when is the object 4 m from the origin?
- **d** When is the next time the object is 4 m from the origin?
- **e** When is the first time the object returns to the origin after it reaches its maximum displacement?
- **f** When is the second time the object first reaches its maximum displacement from the origin?

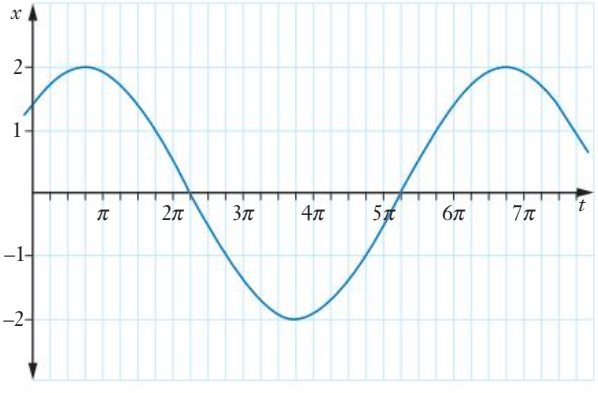
Problem solving

11 The graph below shows the displacement of an object undergoing simple harmonic motion, where x is measured in metres and t in seconds.



Determine a possible equation to describe the displacement of the object.

12 A particle is attached to the end of a spring. The particle oscillates forwards and backwards in a straight line on a frictionless surface. The displacement of the particle is represented by the graph below, where x is measured in centimetres and t is measured in seconds.



Determine a possible equation to describe the displacement of the particle.

12.08 Modelling with trigonometric functions

Many real world situations can be explained using functions that are periodic. In particular, situations involving quantities that oscillate can often be modelled by a sine or cosine function. Once a suitable model is created, you can use the equation and function values to interpret the situation.

EXAMPLE 18

The mean monthly maximum temperatures for a city over 12 months are listed below.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp (°C)	28	27	25.4	22	18.5	16	15	16	18	21.6	24	26

a Draw a graph of the mean temperatures by hand or by using a computer or calculator.

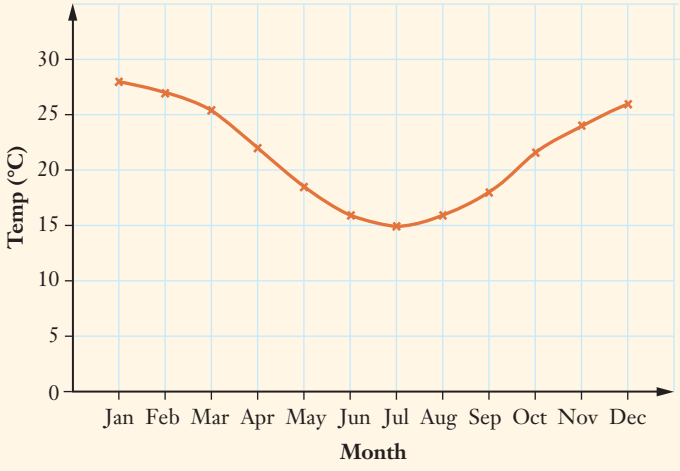
b Is the data periodic?

c Calculate the period and amplitude.

d What type of trigonometric function does the graph resemble?

Solution

a



b Compare the data with other periodic functions.

The graph shows a function that is approximately periodic. This is to be expected as the temperature oscillates between high and low values over the year.

ISBN 9780170412995

c State the period.

Period = 1 year = 12 months

Calculate the amplitude.

$$\text{Amplitude} = \frac{28 - 18.5}{2}$$
$$= 4.75\ ^\circ\text{C}$$

d Compare the graph with the graphs of $y = \sin(x)$ and $y = \cos(x)$.

The graph looks similar to the graph of a cosine function, but it could also be a sine function with a phase shift.

INVESTIGATION

MUSICAL BEATS

To tune a guitar, the frets are used to play a note on one string that should be the same as the base note of the next string. When the strings are in tune, the notes will be the same. When the strings are quite out of tune, they sound quite different. When they are close, but not quite the same, the sound will seem to beat (fade and strengthen rhythmically). This can be shown quite easily on a graphics calculator.

1 Consider a note that should have a frequency of 440 Hz. Store this in the memory in C. Store 442 in the memory in D.

2 Graph the functions $f_1(x) = \sin(2\pi Cx)$, $f_2(x) = \sin(2\pi Dx)$, and $f_3(x) = f_1(x) + f_2(x)$.

3 Turn $f_1(x)$ and $f_2(x)$ off.

4 Set the window settings to: $0 \leq x \leq 5$ and $-3 \leq y \leq 3$ and examine the graph.

5 Change D to 441 and graph the function again.

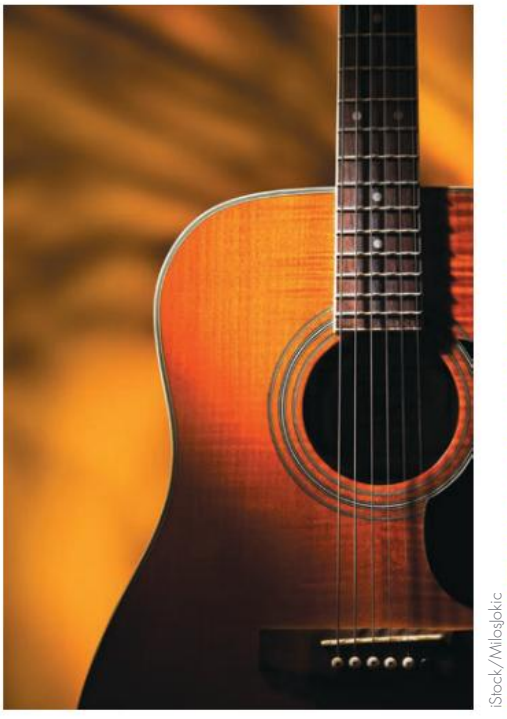
6 Change D to 440.2, change the window settings for x to $0 \leq x \leq 20$ and graph the function again.

7 Find what note corresponds to 440 Hz.

8 Try different values of C and D.

9 What do you find?

10 Write a short account of your results.



iStock/Milosjokic

ISBN 9780170412995

EXAMPLE 19

The depth of water in a harbour at a costal town changes each hour (midnight = 0).

Time (h)	0	1	2	3	4	5	6	7	8	9	10	11	12
Depth (m)	5.6	6.3	8.6	11.6	14.5	16.7	17.6	16.7	14.5	11.6	8.6	6.3	5.6

a Draw a graph of this data.

b Determine an equation that models the depth of the water over time.

c Use the equation to determine the depth of water at 8:30 a.m. and check this value using the graph.

d When is the water first at a depth of 8 m?

Solution

a Calculate the amplitude.

$$\text{Amplitude} = \frac{17.6 - 5.6}{2} = 6$$

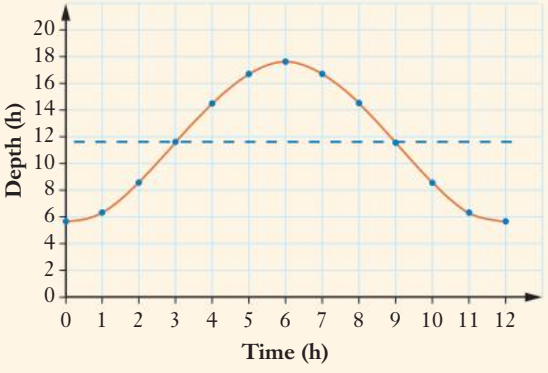
Calculate the period.

$$\text{Period} = 12 \text{ hours}$$

The equilibrium value is halfway between the high and low values.

$$\text{Equilibrium value} = \frac{17.6 + 5.6}{2} = 11.6$$

Draw a graph of the data showing the equilibrium value.



b The graph could be either a sine or cosine function. Choose the sine function.

$$y = a \sin(bt + c) + d \text{ or}$$
$$y = a \sin\left[b\left(t + \frac{c}{b}\right)\right] + d$$

Calculate a.

$$\text{Amplitude} = a = 6$$

Calculate b.

$$\text{Period} = \frac{2\pi}{b} = 12 \text{ so } b = \frac{\pi}{6}$$

ISBN 9780170412995

	Calculate c.	The graph of $y = \sin(x)$ has been translated 3 h to the right. Phase shift $= 3 = \frac{-c}{b}$ $c = -3b = -3 \times \frac{\pi}{6} = -\frac{\pi}{2}$
	Calculate d.	The graph of $y = \sin(x)$ has been translated vertically upwards 11.6 m. $d = 11.6$
	Write the equation.	$y = 6 \sin\left[\frac{\pi}{6}(t-3)\right] + 11.6$
c	Let $t = 8.5$.	$y = 6 \sin\left[\frac{\pi}{6}(8.5-3)\right] + 11.6$
	Evaluate.	$= 13.1529\ldots$
	State the result.	The depth of the water is about 13.2 m. This figure agrees with the value shown on the graph.
d	Let $y = 8$.	$8 = 6 \sin\left[\frac{\pi}{6}(t-3)\right] + 11.6$
	Evaluate.	$6 \sin\left[\frac{\pi}{6}(t-3)\right] = -3.6$ $\sin\left[\frac{\pi}{6}(t-3)\right] = -0.6$
	Let $x = \frac{\pi}{6}(t-3)$.	$\sin^{-1}(x) = -0.6$
	Evaluate.	$x = -0.6435\ldots$
	Let $x = \frac{\pi}{6}(t-3)$.	$\frac{\pi}{6}(t-3) = -0.6435\ldots$
	Solve.	$t = 1.771\ldots$
	So the depth of water is 8 m at 1.771… hours after midnight. Convert this to a time.	The water is at a depth of 8 m at about 1:46 a.m.

EXAMPLE 20

The hours of daylight in a city on the east coast of Australia oscillate from a high of 16.2 hours in January to a low of 8.2 hours in July.

a Determine an equation for the hours of daylight.

b When should you plant a garden if you want to do it during the month where there are 13 hours of daylight in a day?

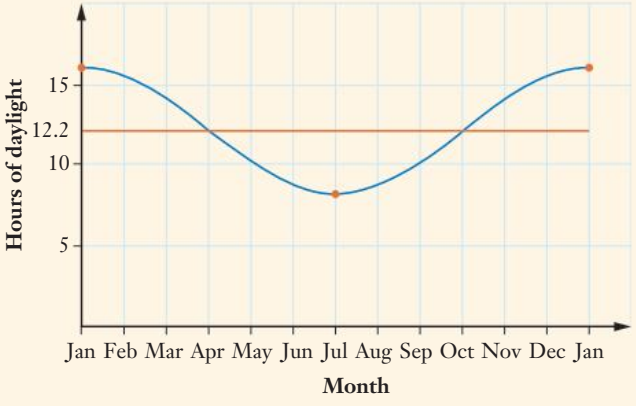
Solution

a The hours of daylight oscillate with a period of 12 months. With a high of 16.2 in January and a low of 8.2 in July.

The equilibrium value is halfway between the high and low values.

$$\text{Equilibrium value} = \frac{16.2+8.2}{2} = 12.2$$

Sketch a diagram to show how the hours of daylight oscillate.



Compare the graph to the graphs of periodic functions.

The graph could be modelled with the cosine function $y = a\cos(bt + c) + d$, where y is the hours of daylight.

Calculate the amplitude.

$$a = \frac{16.2-8.2}{2} = 4$$

Calculate b.

$$T = 12 = \frac{2\pi}{b}$$

$$b = \frac{\pi}{6}$$

ISBN 9780170412995

	There is no phase shift.	$c = 0$
	The graph has been translated 12.2 units vertically upwards.	$d = 12.2$
	Write the equation.	$y = 4\cos\left(\frac{\pi t}{6}\right) + 12.2$
b	Let y =13.	$13 = 4\cos\left(\frac{\pi t}{6}\right) + 12.2$
	Rearrange and simplify.	$\cos\left(\frac{\pi t}{6}\right) = \frac{0.8}{4} = 0.2$
	Use $\cos^{-1}$.	$\frac{\pi t}{6} = \cos^{-1}(0.2)$
	Evaluate.	$\frac{\pi t}{6} = 1.3694\ldots$
		$t = 2.6154\ldots$
	Interpret the result.	There will be 13 hours of daylight about 2.6 months after January, i.e., about halfway through March.

Exercise 12.08 Modelling with trigonometric functions

Example 18

1 An object on the end of a spring moves backwards and forwards in a straight line. The distance of the object from a fixed point at various times is shown below.

Time (s)	0	1	2	3	4	5	6	7	8	9	10	11	12
Distance (cm)	8.5	10	11.1	11.5	11.1	10	8.5	7	5.9	5.5	5.9	7	8.5

a Draw a graph of this data by hand or by using a computer or calculator.

b Is the data periodic?

c Calculate the period and amplitude.

d Calculate the equilibrium value.

e What type of trigonometric function does the graph resemble?

2 The height of a cork floating on the surface of a lake is given in the table below.

Time (s)	0	2	4	6	8	10	12	14	16	18	20
Height (cm)	10.1	6.7	4.0	3.0	4.1	6.9	10.3	13.0	14.0	12.9	10.1

a Draw a graph of this data.

b Is the data periodic?

c Calculate the period and amplitude.

d Calculate the equilibrium value.

e What type of trigonometric function does the graph resemble?

Example 19

3 The depth of water in a harbour at t hours after midnight is shown below.

Time (h)	0	1	2	3	4	5	6	7	8	9	10	11	12
Depth (m)	10.0	14.0	16.9	18.0	16.9	14.0	10.0	6.0	3.1	2.0	3.1	6.0	10.0

a Draw a graph of this data.

b Determine an equation that models the depth of water in the harbour.

c Use the equation to determine the depth of water at 5:30 a.m. and check this value using the graph.

d When is the water first at a depth of 7 m?

e When is the water first at a depth of 8.4 m?

Example 20

4 A wheel has a radius of 4 m and rotates once every 24 s. The bottom of the wheel is on a bridge 2 m above the ground. A piece of chewing gum is firmly stuck on the outside edge of the wheel, level with the centre axle of the wheel.

a Determine an equation that describes the position of the gum above the ground.

b In the first rotation of the wheel, when will the gum be 3 m above the ground?

Problem solving

5 Determine a possible equation for the periodic function represented in question **1**.

6 Determine a possible equation for the periodic function represented in question **2**.

ISBN 9780170412995

7 The number of daylight hours for the months that have elapsed since January 1 ($t = 0$) in a particular town is given in the following table.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Daylight (h)	16.9	15.8	13.7	11.1	8.8	7.3	7.1	8.2	10.3	12.9	15.2	16.7	16.9

a Draw a graph of this data.

b Determine an equation that models the hours of daylight in the town.

c Use the equation to determine the number of hours of daylight in the middle of September.

d When is the first time that there are 10 hours of daylight in a day?

8 In Kolkata, India, the highest mean monthly temperature is 29.45°C in June and the lowest is 18.3°C in December. Find a model for the temperature throughout the year and graph it.

9 In Upernivik, Greenland, the average temperature varies between –23°C in January and 5°C in July. Find and graph a model for the average temperature.

10 The time between successive high tides at a pier is 12 h 20 min. The average depth of water is 4 m, but at low tide it is 3.1 m. Write an equation to model this relationship and use it to find the lengths of time that two boats who require draughts (minimum water depths) of 3.5 m and 4.5 m respectively can use the pier.

11 An oscilloscope tracing signals from a microphone shows a sinusoidal curve. The amplitude relates to the volume of the sound while the frequency gives the pitch.

A violin string vibrates at 200 Hz (1 Hz is 1 vibration per second). The oscilloscope trace amplitude is 12 V.

a Find the period of the sound wave.

b Write an equation for the sound wave.

12 A spring that has a natural length of 20 cm, is connected to a vertical surface. An object is fixed to the end of the spring and rests on a frictionless horizontal surface. The object is pulled back 8 cm past the natural length and released. The object oscillates 3 times per second. Find an equation for x, the distance from the object to the wall.

ISBN 9780170412995

12. CHAPTER SUMMARY

Trigonometric identities and applications

- **Product to sum identities**

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

- **Sum to product identities**

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

- **Cosine form of $a\sin(x) + b\cos(x)$**

$a\sin(x) + b\cos(x) = r\cos(x-\alpha)$, or

$a\sin(x) - b\cos(x) = -r\cos(x+\alpha)$, where $r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\left(\frac{a}{b}\right)$.

- **Sine form of $a\sin(x) + b\cos(x)$**

$a\sin(x) + b\cos(x) = r\sin(x+\alpha)$ or

$a\sin(x) - b\cos(x) = r\sin(x-\alpha)$, where $r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

- **Triple angle identities**

$$\sin(3x) = 3\sin(x) - 4\sin^3(x)$$

$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

$$\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$$

ISBN 9780170412995

- **Quadruple angle identities**

  $\sin(4x) = 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)$

  $\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$

  $\tan(4x) = \dfrac{4\tan(x) - 4\tan^3(x)}{1 - 6\tan^2(x) + \tan^4(x)}$

- **Periodic motion** is any motion that repeats itself in equal intervals of time. The **period** of the motion is defined as the time for the motion to repeat itself.
- **Simple harmonic motion** (SHM) is motion in which an object oscillates forwards and backwards in a straight line.
- The position of a particle undergoing simple harmonic motion is $x = a\cos(nt + \varepsilon)$, where x is the displacement of the particle from its equilibrium value at time, t, amplitude = a, period = $\dfrac{2\pi}{n}$ and phase shift = ε. The position of x could also be expressed in the form $x = a\sin(nt + \varepsilon)$.

ISBN 9780170412995

12. CHAPTER REVIEW

Trigonometric identities and applications

Example 1
1 Write each expression as a sum or difference.

a $2\cos(4x)\sin(7x)$ **b** $\sin(8x)\sin(18x)$

Example 2
2 Use a product to sum formula to evaluate $\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{12}\right)$.

Example 3
3 Given that $\cos(100°) \approx -0.174$, use a product to sum identity to evaluate $\sin(80°)\sin(20°)$.

Example 4
4 Express each difference as a product.

a $\sin(5x) - \sin(8x)$ **b** $\cos(10x) - \cos(2x)$

Example 5
5 Use a sum to product identity to find the exact value of $\sin(75°) - \sin(15°)$.

Example 6
6 Prove that $\cot^2(2\theta) = \frac{1+\cos(4\theta)}{1-\cos(4\theta)}$.

Example 7
7 Express $4\cos(x) + 3\sin(x)$ in the form $r\cos(x-\alpha)$ using radians.

Example 8
8 Express $\sin(x) + \cos(x)$ in the form $r\sin(x+\alpha)$ using radians.

Example 9
9 Write each expression in the form $r\cos(x-\alpha)$, where $-\pi < \alpha < \pi$.

a $6\cos(x) + 5\sin(x)$ **b** $-2\cos(x) - 3\sin(x)$

Example 10
10 Use the function $y = r\cos(x-\alpha)$ to draw the graph of $y = -5\cos(x) + 12\sin(x)$ for $0 \le x \le 2\pi$, showing all key features.

Example 11
11 Use the function $y = r\sin(x-\alpha)$ to draw the graph of $y = 3\sin(x) - \cos(x)$ for $0 \le x \le 2\pi$, showing all key features.

Example 12
12 Solve the equation $2\cos(x) + \sin(x) = 0$ for $0 \le x \le 2\pi$.

Example 13
13 Solve the equation $4\cos(x) - \sin(x) = 2$ for $0 \le x \le 2\pi$.

Example 14
14 If $\sin(12°) \approx 0.208$ and $\cos(12°) \approx 0.978$, use a multiple angle formula to find an approximate value for $\sin(48°)$.

Example 15
15 Solve $\cos(2x) + \cos(4x) = 0$ for $0 \le x \le 2\pi$.

Example 16
16 The displacement, x, of a particle in metres over t seconds is given by $x = 5\cos(2t)$.

a Calculate the amplitude and period.

b Draw a graph of the motion of the particle.

c Describe the motion of the particle.

ISBN 9780170412995

17 An object moves in a straight line such that its displacement, x, is given by $x = 4\cos\left(1.33t - \frac{2\pi}{5}\right)$, where x is measured in metres and t is in seconds. Example 15

a Draw a graph of the motion of the object.

b When does the object first come to rest?

c What is the maximum displacement of the object?

d When is the first time the object is 3 m from the origin?

e When is the next time the object is 3 cm from the origin?

f When is the first time the object returns to the origin after it reaches its maximum displacement?

18 The mean monthly maximum temperatures for a city over 12 months are listed below. Example 18

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp (°C)	26.5	27.5	26.5	23.8	20.0	16.3	13.5	12.5	13.5	16.3	20.0	23.8

a Draw a graph of the mean temperatures by hand or by using a computer or calculator.

b Is the data periodic?

c Calculate the period and amplitude.

d What type of trigonometric function does the graph resemble?

19 The depth of water at the entrance to a harbour each hour is shown below, where midnight = 0. Example 19

Time (h)	0	1	2	3	4	5	6	7	8	9	10	11
Depth (m)	5.3	7.5	10.5	13.5	15.7	16.5	15.7	13.5	10.5	7.5	5.3	4.5

a Draw a graph of this data.

b Determine an equation that models the depth of water over time.

c Use the equation to determine the depth of water at 9:30 a.m. and check this value using the graph.

d When is the water first at a depth of 12 m?

20 A clock stands on a shelf. The centre of the clock, where the hands are attached, is 15 cm above the shelf, and the minute hand is 10 cm long. The distance of the tip of the hand from the shelf can be modelled by a periodic function. Write an equation for the model in terms of the time t, where t is the number of minutes past midnight. Hence find the number of times during the day when the tip is 10 cm above the shelf. Example 20

Problem solving

21 Prove that $\sin(3x) - \cos(3x) = [\cos(x) + \sin(x)][4\cos(x)\sin(x) - 1]$.

22 Show that $\sin(9x) + \sin(5x) - \sin(3x) + \sin(x) = 4\cos(2x)\sin(3x)\cos(4x)$.

23 Prove the identity $\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$.

Practice quiz

ISBN 9780170412995

Practice examination 4

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (2 marks)

a What is the vector for a translation 2 units up and 3 units to the left?

b Find the image of $P(4, 2)$ under this translation.

Question 2 (2 marks)

a What is the transformation for a rotation by 45° around the origin?

b Find the image of $Q(2, 6)$ under this rotation.

Question 3 (2 marks)

a What is the matrix for a dilation by factors of 2 horizontally and 1.5 vertically?

b Find the image of the triangle $A(-1, 6)$, $B(3, -2)$, $C(0, -4)$ under this dilation.

Question 4 (2 marks)

What is the matrix for reflection across the line $y = -3x$?

ISBN 9780170412995

Question 5 (2 marks)

Find the inverse of the transformation $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 4x+y \\ 3x+2y \end{bmatrix}$.

Question 6 (2 marks)

Find the modulus and argument of $3 - 4i$.

Question 7 (2 marks)

Write $6\sqrt{3}\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]$ in Cartesian form.

Question 8 (2 marks)

Evaluate each expression.

a $3\text{ cis}\left(\frac{\pi}{4}\right)\times 4\text{ cis}\left(-\frac{\pi}{3}\right)$ **b** $\frac{10\text{cis}(150°)}{2\text{cis}(60°)}$

Question 9 (1 mark)

Find the nature of the roots of the equation $2m^2 + 4m + 5 = 0$.

Question 10 (2 marks)

Express each expression as a sum or difference.

a $4\cos(5x)\sin(6x)$ **b** $\sin(8x)\sin(18x)$

Question 11 (1 mark)

Use a product to sum formula to find the exact value for $\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$.

Question 12 (2 marks)

Express each difference as a product.

a $\cos(11x) - \cos(7x)$ **b** $\sin(2x) - \sin(5x)$

Question 13 (1 mark)

Use a sum to product identity to find the exact value of $\cos(135°) - \cos(45°)$.

ISBN 9780170412995

Question 14 (1 mark)

Express $\sin(x) - \cos(x)$ in the form $r\cos(x - \alpha)$ using radians.

Question 15 (2 marks)

Express each of the following in the form $r\sin(x \pm \alpha)$ using radians.

a $3\sin(x) - 5\cos(x)$ **b** $4\sin(x) + 2\cos(x)$

Question 16 (2 marks)

Solve the equation $\sin(x) + \sqrt{2}\cos(x) = 1$ for $-\pi \leq x \leq \pi$.

Question 17 (2 marks)

Use the function $y = r\sin(x + \alpha)$ to draw the graph of $y = 4\cos(x) - 3\sin(x)$ for $0 \leq x \leq 2\pi$, showing all key features.

Question 18 (3 marks)

The trapezium $A(3, 5)$, $B(5, -1)$, $C(-1, -3)$, $D(-2, 0)$ is transformed by $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 3x - y \\ 2y - x \end{bmatrix}$ to $A'B'C'D'$. Find the area of $A'B'C'D'$.

Question 19 (3 marks)

Use $\text{cis}(105°) = \text{cis}(135° - 30°)$ to find exact expressions for $\sin(105°)$ and $\cos(105°)$.

Question 20 (4 marks)

The displacement of an object attached to the end of a spring is shown in the table below, where x is the displacement in centimetres and t is the time in seconds.

t	0	2	4	6	8	10	12	14
x	6.5	10.0	13.5	15.0	13.5	10.0	6.5	5.0

t	16	18	20	22	24	26	28	30
x	6.5	10.0	13.5	15.0	13.5	10.0	6.5	5.0

a Draw a graph of this data.

b Determine an equation that models the displacement of the object over time.

c Use the equation to determine the displacement of the object after 15 s.

d When is the object first at a displacement of 14 cm?

ISBN 9780170412995

Question 21 (3 marks)

Use composition of rotations around the origin through angles θ and ϕ to show that $\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$ and $\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$.

Question 22 (3 marks)

Find a quadratic equation whose roots are half the squares of the roots of $3x^2 + 3x - 4 = 0$.

Question 23 (4 marks)

A large Ferris wheel has a diameter of 135 m and it completes one rotation every 30 minutes. Passengers board the Ferris wheel from a platform at the bottom of the wheel, 5 m above the ground.

a Express a passenger's height as a periodic function of time.

b How high will the passengers be after 10 seconds?

c When will passengers be at a height of 100 m in the first rotation of the wheel?

END OF EXAMINATION

ISBN 9780170412995

ANSWERS

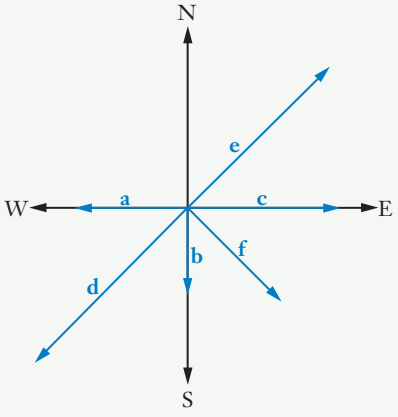
Chapter 1

Exercise 1.01

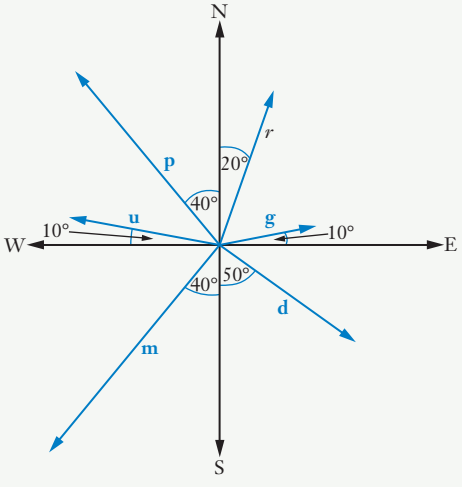
1 **b**, **d**, **f** and **g**

2 **a** 18 km east **b** 24 km north
c 16 km south-west **d** 28 km N 30° W
e 20 km N 75° E **f** 14 km S 25° E

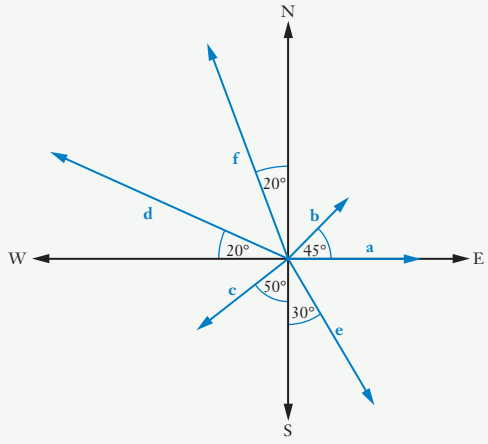
3



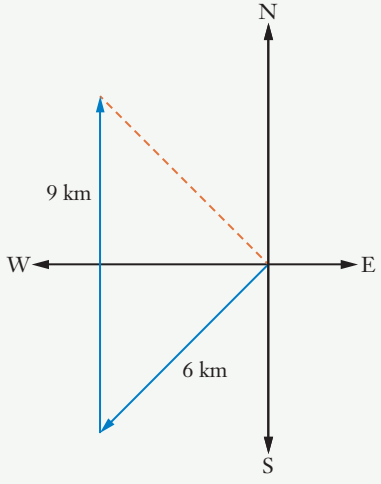
4



5



6 **a** **b** 318°T



7 25

8 **a** 033°T **b** About 13.2 km.

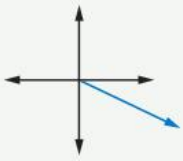
Exercise 1.02

1 **a** (5, 0) **b** (2, 4) **c** (–4, –5) **d** (0, –3)
e (–5, 4) **f** (3, –5) **g** (6, –2)

ISBN 9780170412995

2 Vectors should be in the same direction, but the scale could be different.

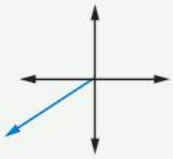
a



b

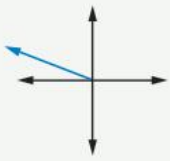
c

d

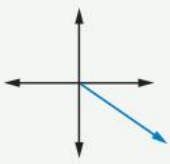


e

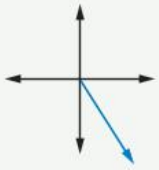
f



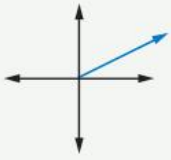
g



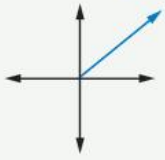
h



i



j



3 **a** 4.47 **b** 3.16 **c** 8.06 **d** 6.08
e 8.54 **f** 5.39 **g** 7.21 **h** 12.37

4 **a** 63.4° **b** 108.4° **c** 299.7° **d** 9.5°
e 159.4° **f** 338.2° **g** 236.3° **h** 104.0°

5 **a** (5, 306.9°) **b** (7.07, 135°)
c (12.21, 235.0°) **d** (12.65, 71.6°)
e (18.87, 122.0°) **f** (17, 61.9°)
g (13, 22.6°) **h** (15, 233.1°)
i (13.89, 239.7°) **j** (11.31, 315°)

6 **a** (7.07, 45°) **b** (7.21, 56.3°)
c (10.30, 60.9°) **d** (8.54, 159.4°)
e (12.04, 228.4°) **f** (12.21, 125.0°)

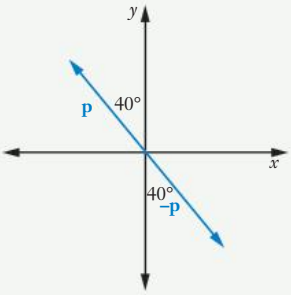
7 **a** (4.33, 2.5) **b** (5, –8.66)
c (0, 24) **d** (–11.31, 11.31)
e (–14, –24.25) **f** (0, –70)
g (35, 0) **h** (–22, 0)
i (–3.76, –1.37) **j** (4, –6.93)

8 **a** (3, 5), (5.83, 59.0°) **b** (10, 1), (10.05, 5.7°)
c (8, 1), (8.06, 7.1°) **d** (6, –6), (8.49, 225°)
e (–9, 9), (12.73, 135°) **f** (9, 11), (14.21, 50.7°)
g (–6, –5), (7.81, 219.8°)

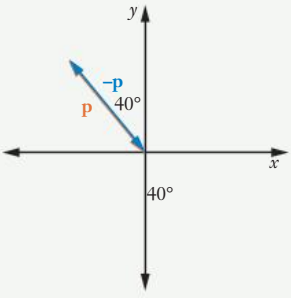
9 $a = 7, b = 4$

10 **a** **–p** has a magnitude of 6 and a direction of S 40° E.

b



c

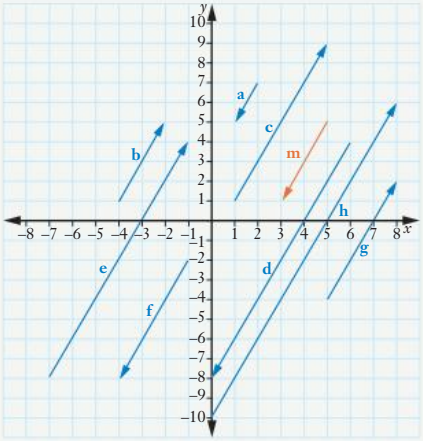


d The displacement vector is zero.

Exercise 1.03

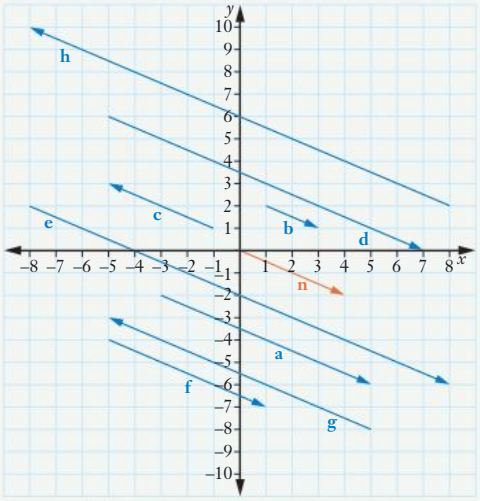
1 $\mathbf{b} = \mathbf{a}$, $\mathbf{c} = -\mathbf{a}$, $\mathbf{d} = 2\mathbf{a}$, $\mathbf{e} = -2\mathbf{a}$, $\mathbf{f} = -3\mathbf{a}$, $\mathbf{g} = \frac{1}{2}\mathbf{a}$, $\mathbf{h} = 4\mathbf{a}$

2 The vectors have the same length and direction as shown here, but their positions may vary.



ISBN 9780170412995

3 The vectors have the same length and direction as shown here but their positions may vary.



4 **a** (24, 6) **b** (−8, −2) **c** (4, 1)
d (12, 3) **e** (−20, −5) **f** (32, 8)
g (−2, −0.5) **h** (6, 1.5)

5 **a** $\begin{bmatrix}-24\\60\end{bmatrix}$ **b** $\begin{bmatrix}36\\-90\end{bmatrix}$ **c** $\begin{bmatrix}-4\\10\end{bmatrix}$
d $\begin{bmatrix}30\\-75\end{bmatrix}$ **e** $\begin{bmatrix}8\\-20\end{bmatrix}$ **f** $\begin{bmatrix}-18\\45\end{bmatrix}$
g $\begin{bmatrix}2\\-5\end{bmatrix}$ **h** $\begin{bmatrix}-6\\15\end{bmatrix}$

6 **a** (10, 58°) **b** (5, 238°) **c** (15, 58°)
d (2.5, 58°) **e** (7.5, 238°) **f** (12.5, 58°)
g (1, 238°) **h** (30, 238°)

7 **a** (−24, 16) **b** (−8, 20) **c** (3, 1)
d (−10, 25) **e** (9, −6) **f** (27, 9)
g (3, −7.5) **h** (1.5, −1)

8 **a** (18, 36°) **b** (9, 38°) **c** (8, 124°)
d (2, 216°) **e** (10, 304°) **f** (9, 36°)
g (3, 38°) **h** (20, 304°)

Exercise 1.04

1 **a** 13 right **b** 5 left **c** 5 right
d 0 **e** 17 left

2 **a** 40 N down **b** 51 N up **c** 72 N down
d 26 N up **e** 38 N left

3 **a** (3, 0°) **b** (2, 270°)
c (2, 180°) **d** (12, 90°)
e (5.4, 111.8°) **f** (8.6, 54.5°)
g (9.9, 225°) **h** (8.6, 144.5°)

4 **a** (23.3, 31.0°) **b** (62.5, 10.9°)
c (70.2, 341.5°) **d** (116.2, 152.1°)
e (38.0, 302.3°) **f** (68.4, 137.6°)

5 (14.3, 41.2°)

6 (38.7, 104.6°)

7 **a** (7.21, 146.1°) **b** (4.50, 12.6°)
c (10.93, 9.3°) **d** (10.82, 276.3°)
e (16.93, 77.7°)

8 **a** (14.79, 109.4°) **b** (21.62, 259.1°)
c (7.24, 217.6°) **d** (4.35, 221.4°)
e (5.71, 355.6°)

9 $\mathbf{a} + \mathbf{b}$

10 $\mathbf{p} + \mathbf{q}$ represents the diagonal AC.

11 About 9.6 km at 089.8°.

12 **a** $\mathbf{p} + \mathbf{r}$ **b** $\mathbf{q} + \mathbf{r}$ **c** $\mathbf{q} + \mathbf{r}$ **d** $-\mathbf{r} + \mathbf{q}$
e $-\mathbf{p} + (-\mathbf{q})$

Exercise 1.05

1 **a** (2, 0°) **b** (6, 0°)
c (1, 90°) **d** (8, 270°)
e (5, 180°) **f** (7, 270°)

2 **a** (8.6, 54.5°) **b** (5, 36.9°)
c (6.4, 128.7°) **d** (5.8, 149.0°)
e (6.4, 51.3°) **f** (8.1, 240.3°)

3 **a** (7.3, 344.1°) **b** (8.2, 256.0°)
c (7.3, 254.1°) **d** (7.1, 171.9°)
e (10.8, 33.7°) **f** (7.3, 164.1°)

4 **a** (9.32, 324.9°) **b** (173.28, 271.7°)
c (6, 310°)

5 **a** (15.77, 232.5°) **b** (16.42, 295.9°)
c (15.57, 92.9°) **d** (12.17, 74.7°)
e (12.17, 254.7°)

6 **a** $-\mathbf{c} - \mathbf{a}$ **b** $-\mathbf{b} - \mathbf{a}$ **c** $-\mathbf{b} - \mathbf{c}$
d $-\mathbf{c} + \mathbf{b}$ **e** $-\mathbf{b} + \mathbf{a}$

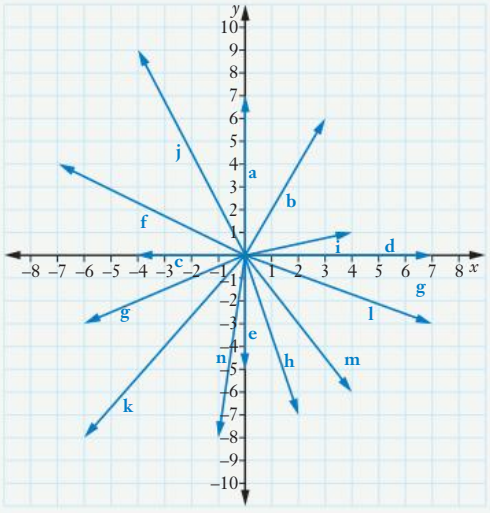
7 **a** $\mathbf{AB} = \mathbf{b} - \mathbf{a}$

b $\mathbf{MN} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{AB}$

ISBN 9780170412995

Exercise 1.06

1



2 7 units

3 **a** $\left(\frac{3}{5}, \frac{4}{5}\right)$ **b** $\left(-\frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17}\right)$

c $\left(\frac{5\sqrt{26}}{26}, -\frac{\sqrt{26}}{26}\right)$ **d** $\left(-\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$

e $\left(-\frac{5\sqrt{34}}{34}, \frac{3\sqrt{34}}{34}\right)$ **f** $\left(-\frac{\sqrt{10}}{10}, -\frac{3\sqrt{10}}{10}\right)$

g $\left(\frac{-\sqrt{17}}{17}, \frac{4\sqrt{17}}{17}\right)$ **h** $\left(-\frac{5}{13}, \frac{12}{13}\right)$

4 **a** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **b** $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **c** (0, 1)

d (–1, 0) **e** (0, –1) **f** (1, 0)

g (0.669, 0.743) **h** (–0.914, 0.407)

i (0.423, –0.906) **j** (0.848, –0.530)

k (0.616, –0.788) **l** (–0.799, –0.602)

5 **a** 5**i** – 3**j** **b** –6**i** + 4**j** **c** –4**i** – 7**j**

d 6**i** + 5**j** **e** –3.2**i** – 9.4**j** **f** **i** – 4**j**

g –2**i** – 8**j** **h** –3**i** + 5**j** **i** 7.59**i** + 3.68**j**

j 0.07**i** + 0.19**j**

6 **a** 1.93**i** – 6.73**j** **b** –5.16**i** + 7.37**j**

c 9.53**i** – 5.5**j** **d** –8**i** – 13.86**j**

e –2.33**i** – 4.20**j** **f** 0.70**i** + 7.97**j**

g –2.78**i** + 1.12**j** **h** 0.52**i** – 9.99**j**

i –3.88**i** + 14.49**j** **j** –9.70**i** – 17.49**j**

7 **a** 6**i** – 4**j** **b** –6**i** + 4**j**

8 $\left(\frac{5}{13}, \frac{-12}{13}\right)$

9 (–9, 12)

Exercise 1.07

1 **a** = (5, 3), **b** = (5, 7), **c** = (8, –5), **d** = (–5, 2), **e** = (–3, –8), **f** = (–5, 5) and **g** = (–5, –4).

a (10, 10) **b** (5, –13) **c** (0, 9)

d (0, 8) **e** (–10, 1) **f** (13, 2)

g (–10, –2)

2 **a** (–2, 13) **b** (–20, 1) **c** (6.1, 3.7)

d (13.16, –10.19) **e** $\left(-1\frac{1}{4}, \frac{5}{6}\right)$

f $\left(-5\frac{1}{10}, 2\frac{7}{12}\right)$

3 **a** $\begin{bmatrix} 6 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$ **c** $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

d $\begin{bmatrix} 11.3 \\ 3.1 \end{bmatrix}$ **e** $\begin{bmatrix} 2.38 \\ 2.18 \end{bmatrix}$ **f** $\begin{bmatrix} -\frac{5}{6} \\ -6\frac{3}{4} \end{bmatrix}$

4 **a** 6**i** + **j** **b** 13**i** + 5**j** **c** –2.2**i** – 15**j**

d –2.11**i** – 2.92**j** **e** 8.23**i** – 19.31**j**

f $-2\frac{2}{9}\mathbf{i} + -3\frac{7}{12}\mathbf{j}$

5 **a** (–4, 5) **b** (0, 10) **c** (5, 20)

d (12, –2) **e** (–9, 14) **f** (–30, 24)

g (5, –1) **h** (0, 0) **i** (–34, 69)

j (–4, –32)

6 **a** (4, –5) **b** (5, 13) **c** (9, 7)

d (–9, 16) **e** (0, 1) **f** (5, –10)

g (5, 3) **h** (–8, 14) **i** (–14, 5)

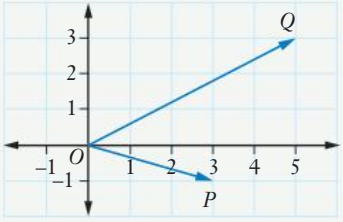
j (–12, 7)

7 **a** (14.33, 41.1°) **b** (38.65, 104.6°)

c (9.32, 324.9°) **d** (173.28, 271.7°)

e (6, 130°)

8 **a**



b **PQ** = 2**i** + 4**j** **c** **OM** = 4**i** + **j**

ISBN 9780170412995

9 $\mathbf{p} + \mathbf{q} = (x_1 + x_2, y_1 + y_2)$,

$k(\mathbf{p} + \mathbf{q}) = k(x_1 + x_2, y_1 + y_2)$

$= (kx_1 + kx_2, ky_1 + ky_2)$

$k\mathbf{p} + k\mathbf{q} = k(x_1, y_1) + k(x_2, y_2)$

$= (kx_1, ky_1) + (kx_2, ky_2)$

$= (kx_1 + kx2, ky_1 + ky_2)$

Exercise 1.08

1 **a** $5\mathbf{i} - 3\mathbf{j}$ **b** $-6\mathbf{i} + 4\mathbf{j}$ **c** $-4\mathbf{i} - 7\mathbf{j}$

d $6\mathbf{i} + 5\mathbf{j}$ **e** $-3.2\mathbf{i} - 9.4\mathbf{j}$ **f** $2\sqrt{3}\mathbf{i} + \sqrt{5}\mathbf{j}$

g $\frac{2}{3}\mathbf{i} + \left(-\frac{4}{9}\right)\mathbf{j}$ **h** $-5\sqrt{7}\mathbf{i} + 6\sqrt{2}\mathbf{j}$

2 **a** $\mathbf{i} - 4\mathbf{j}$ **b** $-2\mathbf{i} - 8\mathbf{j}$ **c** $-3\mathbf{i} + 5\mathbf{j}$

d $7.59\mathbf{i} + 3.68\mathbf{j}$ **e** $0.07\mathbf{i} + 0.19\mathbf{j}$

f $-3\sqrt{5}\mathbf{i} + 4\sqrt{7}\mathbf{j}$

g $\frac{6}{11}\mathbf{i} + \left(-\frac{5}{4}\right)\mathbf{j}$ **h** $-9\sqrt{11}\,\mathbf{i} + \left(-7\sqrt{15}\right)\mathbf{j}$

3 **a** $1.93\mathbf{i} - 6.73\mathbf{j}$ **b** $-5.16\mathbf{i} + 7.37\mathbf{j}$

c $9.53\mathbf{i} - 5.5\mathbf{j}$ **d** $-8\mathbf{i} - 13.86\mathbf{j}$

e $-2.33\mathbf{i} - 4.20\mathbf{j}$ **f** $7.82\mathbf{i} + 10.38\mathbf{j}$

g $7.75\mathbf{i} - 3.95\mathbf{j}$ **h** $2.92\mathbf{i} - 10.19\mathbf{j}$

4 **a** (46.1, 36.4) **b** (6.2, 57.8)

c (34.8, –33.3)

5 **a** (15.48, 217.1°) **b** (13.87, 214.2°)

c (20.01, 239.6)

6 **a** (63.91, 39.3°) **b** (54.60, 112.1°)

c (55.56, 65.8°) **d** (88.23, 247.8°)

e (73.1, 180.5°) **f** (49.71, 194.7°)

7 **a** (92.95, 39.3°) **b** (33.00, 232.9°)

c (110.53, 338.2°)

8 2000 N parallel to the banks.

9 47.95 N in the direction S 12° W.

10 N 82.8° E

Exercise 1.09

1 About 12 340 N at 11.6° to the passenger's side forward direction.

2 About 26.9 m/s at 042.0°.

3 About 7.8 knots at 120.6°.

4 About 1008 N at 13.4° towards the 500 N side of the 400 N force.

5 About 977 N each.

6 About 181.8 knots at 066.5°.

7 About 10.7 m/s at 082.5°.

8 $v\sqrt{2[1-\cos(\theta)]}$

9 About 9.3 knots at 216.1°.

10 About 124.9° and 10.1 knots.

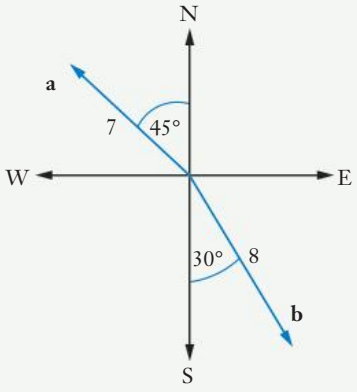
11 About 315.3° and 140.11 knots.

12 About 12:31:39 p.m.

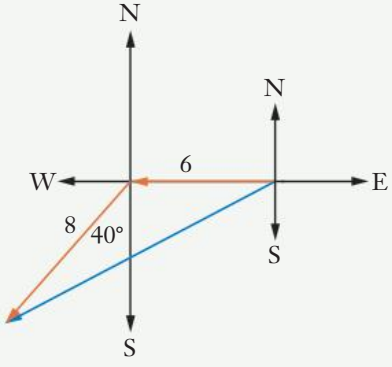
13 About 9.8 knots.

Chapter review

1



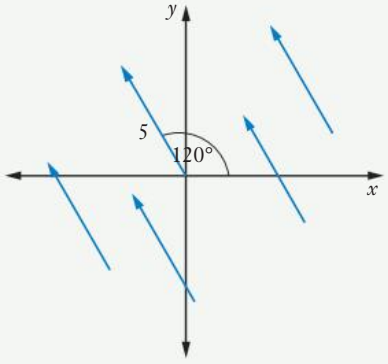
2 **a**



b 208.8°

3 **a** (5, 120°)

b



ISBN 9780170412995

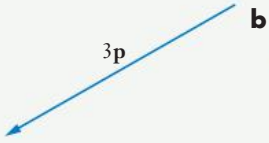
4 **a** $(-2, 6)$ **b** $(8, 5)$ **c** $(-4, 4)$
d $(-2, 4)$ **e** $(7, -12)$

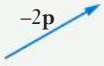
5 **a** 26 **b** 5 **c** 17
d 10.296 **e** 15.621

6 **a** $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ **b** $(-4\sqrt{3}, 4)$ **c** $(2, -2\sqrt{3})$
d $\left(\frac{7\sqrt{2}}{2}, \frac{-7\sqrt{2}}{2}\right)$ **e** $(-5.785, -6.894)$

7 **a** $(9.434, 58°)$ **b** $(5, 126.9°)$
c $(11.662, 239°)$ **d** $(9.487, 341.6°)$
e $(4, 180°)$

8 **a** $(5.831, 59°)$ **b** $(6.325, 251.6°)$
c $(4.123, 284°)$ **d** $(5, 90°)$
e $(8.246, 104°)$

9 **a**


b


c

10 **a** $\begin{bmatrix} 20 \\ -16 \end{bmatrix}$ **b** $\begin{bmatrix} -15 \\ 12 \end{bmatrix}$ **c** $(40, 353°)$
d $(48, 173°)$

11 $(7.81, 309.8°)$

12 18.3 N at 78.8° to the positive direction of the x-axis.

13 **a** $(10.39, 121.1°)$ **b** $(10.92, 176.1°)$
c $(4.99, 158.8°)$ **d** $(9.85, 55.9°)$
e $(7.13, 275.8°)$

14 $(9.22, 167.5°)$

15 **a** $(9.43, 208°)$ **b** $(16.86, 52.1°)$

16 **a** $(0.6, 0.8)$ **b** $(-0.55, 0.83)$

17 **a** $\mathbf{i} - 3\mathbf{j}$ **b** $4\mathbf{i} + 5\mathbf{j}$ **c** $-2\mathbf{i} + 5\mathbf{j}$
d $4\mathbf{i} - 3\mathbf{j}$ **e** $-2\mathbf{i} - 5\mathbf{j}$ **f** $3.36\mathbf{i} + 4.97\mathbf{j}$
g $-7.52\mathbf{i} - 2.74\mathbf{j}$ **h** $\frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$
i $-7.88\mathbf{i} + 6.16\mathbf{j}$ **j** $\frac{-5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}$

18 **a** $(3, 3)$ **b** $\begin{bmatrix} 17 \\ -10 \end{bmatrix}$ **c** $6\mathbf{i} + 6\mathbf{j}$

19 **a** $(2, 9)$ **b** $(-5, -8)$ **c** $(-8, -1)$
d $(21, 34)$ **e** $(0, 6)$

20 **a** $\begin{bmatrix} -6 \\ 4 \end{bmatrix}$ **b** $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$ **c** $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$
d $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ **e** $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$

21 **a** $-4\mathbf{i} + 3\mathbf{j}$ **b** $\frac{1}{8}\mathbf{i} - \frac{5}{7}\mathbf{j}$ **c** $3\sqrt{5}\mathbf{i} + \sqrt{6}\mathbf{j}$
d $13.4\mathbf{i} + 10.5\mathbf{j}$ **e** $-3.6\mathbf{i} - 10.4\mathbf{j}$

22 $(3.65, 36.88)$, $(37.1, 84.4°)$

23 $(67.2, 311.5°)$

24 86.4 N at 43.2° to the horizontal.

25 0.4013 m/s at 151.2° to the original direction.

26 12.37 kn at 239°.

27 1082 m, 303.7°.

28 39.0° and 134.9 kn.

Chapter 2

Exercise 2.01

1 Let $x > y$
$\Rightarrow x - y$ is positive (meaning of $>$)
$\Rightarrow x - y + c - c$ is positive (adding 0)
$\Rightarrow (x - c) - (y - c)$ is positive (rearrangement)
$\Rightarrow x - c > y - c$ (meaning of $>$) QED

2 Let $x > y$ and $y > z$
$\Rightarrow x - y$ and $y - z$ are positive (meaning of $>$)
$\Rightarrow (x - y) + (y - z)$ is positive (adding positives)
$\Rightarrow x - z > 0$ (rearrangement using $y - y = 0$)
$\Rightarrow x > z$ (meaning of $>$) QED

3 Let $x > y$ and $c > 0$
$\Rightarrow x - y$ and c are both positive (meaning of $>$)
$\Rightarrow c(x - y)$ is positive (product of positives)
$\Rightarrow cx - cy$ is positive (rearrangement)
$\Rightarrow cx > cy$ (meaning of $>$) QED

4 Let $x > y$ and $c > 0$
$\Rightarrow x - y$ and c are both positive (meaning of $>$)
$\Rightarrow \frac{x - y}{c}$ is positive (division of positives)
$\Rightarrow \frac{x}{c} - \frac{y}{c}$ is positive (rearrangement)
$\Rightarrow \frac{x}{c} > \frac{y}{c}$ (meaning of $>$) QED

5 **a** Assumption is not reversible
b False premise

ISBN 9780170412995

c False premise: Not all Native Americans eat a lot of rice.

d False premise; David may be truthful.

6 a Let $x + c > y + c$

$\Rightarrow (x + c) - (y + c)$ is positive (meaning of >)

$\Rightarrow x - y$ is positive (rearrangement, using $c - c = 0$)

$\Rightarrow x > y$ (meaning of >) QED

b It is true both ways: $x > y \Leftrightarrow x + c > y + c$

7 Let $x > y > 1$

$\Rightarrow x - y, y - 1$ are positive

$\Rightarrow x(x - y), y(x - y), y(y - 1)$ are positive (product of positives)

$\Rightarrow x^2 - xy, xy - y^2, y^2 - y$ are positive (rearrangements)

$\Rightarrow x^2 - xy + xy - y^2$ is positive (adding positives)

$\Rightarrow x^2 - y^2$ is positive

$\Rightarrow x^2 > y^2, y^2 > y$ (meaning of >, second from 2 steps above)

$\Rightarrow x^2 > y^2 > y > 1$ (combining, also with assumption)

$\Rightarrow x^2 > y^2 > 1$ (using transitivity) QED

8 The 4th step is division by $x - 1 = 0$, which is undefined.

Exercise 2.02

1 $(-3)^2 > 1^2$ but $-3 < 1$

2 $(-4)^2 = 16$, but $-4 \neq 4$

3 For $n = 10$, $n^3 - 6n^2 + 11n - 6 = 1000 - 600 + 110 - 6 = 504 \neq 0$

4 For $n = 17$, $n^2 + n + 17 = 17 \times 19$, which is not prime.

5 For $n = 23$, $3n^2 + 3n + 23 = 23 \times 73$, which is not prime.

6 a For $x = 1$ and $y = -1$, $x > y$ but $\frac{1}{x} > \frac{1}{y}$.

b It is true for $x, y > 0$.

7 a For $x = 3$ and $y = 0$, $3^2 + (0)^2 = (3 + 0)^2$

b It is true for $x, y > 0$

8 a For $x = 3$ and $y = 0$, $3^2 - (0)^2 = (3 - 0)^2$

b It is true for $x, y > 0$

9 a For $n = 2$, $m = 3$ and $p = 0$, $np = mp$, but $n \neq m$.

b It is true for $p \neq 0$

10 a For $n = -0.5$, $\frac{1}{n} < \frac{1}{n+1}$

b It is true for $n < -1$ or $n > 0$.

Exercise 2.03

1 Dangerous animals are sharks.

2 x, y integers $\Rightarrow x + y$ is an integer.

3 $x + y = 180° \Rightarrow x, y$ are adjacent angles on a straight line.

4 People sometimes eat crocodiles. Both the statement and the converse are true.

5 The diagonals of a quadrilateral cross at 90° $\Rightarrow$ it is a kite. The statement is true, but the converse is not.

6 For integers, $x + y$ is odd $\Rightarrow$ one is even and one is odd. Both the statement and its converse are true.

7 $x = 0$ or $y = 0 \Rightarrow xy = 0$. Both the statement and its converse are true.

8 x is an integer $\Rightarrow 2x + 1$ is an odd integer. Both the statement and its converse are true.

9 $2x$ is an even integer $\Rightarrow x$ is an integer. Both the statement and its converse are true.

10 a Yes **b** Yes

11 a Yes **b** Yes

12 a Yes **b** No

Exercise 2.04

1 a Equivalent **b** Not equivalent

c Equivalent **d** Equivalent

e Equivalent

2 No, because 1 is neither prime nor composite.

3 a x is not an integer

b At least one of the angles in a quadrilateral is more than 180°.

c At least one number that can be shown on a number line is neither an integer nor a fraction.

d At least one number that can be shown on a number line is neither a rational nor irrational number.

e There is at least one value of $y \geq 0$ such that $(x - y)^2 > x^2 + y^2$

4 a 0.625 **b** $0.\overline{714285}$ **c** $0.\overline{461538}$

d 0.425 **e** $0.9\overline{4}$

5 a $\frac{9}{40}$ **b** $\frac{419}{990}$ **c** $\frac{514}{999}$

d $\frac{281}{500}$ **e** $\frac{707}{1100}$

6 Yes, because any terminating decimal can be written as a recurring decimal by putting 0s on the end.

ISBN 9780170412995

7 No, because there are numbers that are neither fractions nor integers (irrational numbers).

8 It is neither the converse nor inverse because there are (very rare) hermaphrodites.

9 No, because there are fractions, etc. If it was just integers, then it would be.

10 No, the equivalent would be 'If you get a poor result in Maths, you didn't work hard'.

11 They are equivalent because all integers are either odd or even.

Exercise 2.05

1 n is not even

$\Rightarrow n = 2m + 1$ for some $m \in \mathbf{Z}$

$\Rightarrow n^2 = (2m + 1)^2$

$= 4m^2 + 4m + 1$

$= 2(2m^2 + 2m) + 1$

$\Rightarrow n^2$ is not even

Thus n^2 is even $\Rightarrow n$ is even QED

2 Suppose the sides AB and DC of a quadrilateral $ABCD$ are not parallel.

$\Rightarrow$ Cointerior angles $\angle BAD$ and $\angle ADC$ are not supplementary.

$\Rightarrow \angle BAD + \angle ADC \neq 180°$

$\Rightarrow \angle BAD \neq 90°$ or $\angle ADC \neq 90°$

$\Rightarrow ABCD$ is not a rectangle

Since opposite sides not parallel $\Rightarrow ABCD$ is not a rectangle

$ABCD$ is a rectangle $\Rightarrow$ opposite sides parallel QED

3 Suppose n is not odd

$\Rightarrow n = 2m$ for some $m \in \mathbf{Z}$

$\Rightarrow 3n + 1 = 3 \times 2m + 1$

$= 2(3m) + 1$, which is not even

Since n is not odd $\Rightarrow 3n + 1$ is not even

$3n + 1$ is even $\Rightarrow n$ is odd

4 Suppose n is not odd

$\Rightarrow n = 2p$ for some $p \in \mathbf{Z}$

$\Rightarrow nm = 2pm$, which is not odd

Similarly, m not odd $\Rightarrow nm$ is not odd

Since n or m even $\Rightarrow nm$ is not odd

nm odd $\Rightarrow n, m$ are both odd

5 Suppose both n, m are not even

$\Rightarrow n = 2p + 1, m = 2q + 1$ for some $p, q \in \mathbf{Z}$

$\Rightarrow nm = (2p + 1)(2q + 1)$

$= 4pq + 2p + 2q + 1$

$= 2(2pq + p + q) + 1$, which is not even

Since both n, m not even $\Rightarrow nm$ is not even

nm even $\Rightarrow$ at least one of n, m is even

6 Suppose n is even and m is odd

$\Rightarrow n = 2p, m = 2q + 1$ for some $p, q \in \mathbf{Z}$

$\Rightarrow n + m = 2p + 2q + 1$

$= 2(p + q) + 1$, which is not even

Similarly, n is odd and m is even $\Rightarrow n + m$ is not even.

Since n, m not both even or both odd $\Rightarrow nm$ is not even.

$n + m$ even $\Rightarrow n, m$ are both even or both odd.

7 $n \leq 10, m \leq 10$

$\Rightarrow nm \leq 10m$ and $10m \leq 100$ (multiplying by m, $10 > 0$ resp.)

$\Rightarrow nm \leq 100$

Since $n \leq 10, m \leq 10 \Rightarrow nm \leq 100$

$nm > 100 \Rightarrow n > 10$ or $m > 10$

8 Suppose n is divisible by a

$\Rightarrow n = ab$ for some $b \in \mathbf{Z}$

$\Rightarrow nm = abm$

$= a(bm)$, so nm is divisible by a

Since n divisible by $a \Rightarrow nm$ divisible by a

nm not divisible by $a \Rightarrow n$ not divisible by a

9 Suppose x, y are rational

$\Rightarrow x = \frac{a}{b}, y = \frac{c}{d}$ for some $a, b, c, d \in \mathbf{Z}$

$\Rightarrow xy = \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

But $ac, bd \in \mathbf{Z}$, so xy is rational

Since x, y rational $\Rightarrow xy$ is rational

xy irrational $\Rightarrow$ at least one of x, y irrational

10 Suppose $\sqrt{x}$ is rational

$\Rightarrow \sqrt{x} = \frac{a}{b}$ for some $a, b \in \mathbf{Z}$

$\Rightarrow x = \frac{a^2}{b^2}$

But $a^2, b^2 \in \mathbf{Z}$, so x is rational.

Since $\sqrt{x}$ rational $\Rightarrow x$ rational.

x irrational $\Rightarrow \sqrt{x}$ irrational

ISBN 9780170412995

Exercise 2.06

1 Suppose there is a greatest even number, say n.

$\Rightarrow n + 2$ is also even.

But $n + 2 > n$

Contradiction

$\Rightarrow$ There is no greatest even number

2 Suppose there is a greatest odd number, say n.

$\Rightarrow n + 2$ is also odd

But $n + 2 > n$

Contradiction

$\Rightarrow$ There is no greatest odd number.

3 Suppose there are a finite number of primes, say $p_1, p_2, p_3, \ldots, p_n$

Write $m = p_1 p_2 p_3 \ldots p_n + 1$

m divided by any of $p_1\ p_1, p_2, p_3, \ldots, p_n$ has remainder 1.

Thus m is prime.

But m is larger than any of $p_1, p_2, p_3, \ldots, p_n$ so it is another prime number.

Contradiction

Hence, there are an infinite number of prime numbers.

4 Suppose $\sqrt{2}$ is rational

$\Rightarrow \sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbf{Z}$

If $\frac{a}{b}$ is not in the lowest form with $a, b > 0$, cancel until it is.

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 2b^2$

$\Rightarrow a^2$ is divisible by 2

$\Rightarrow a$ is divisible by 2

$\Rightarrow a = 2c$ for some $c \in \mathbf{Z}$

$\Rightarrow (2c)^2 = 2b^2$

$\Rightarrow 2b^2 = 4c^2$

$\Rightarrow b^2 = 2c^2$

$\Rightarrow b^2$ is divisible by 2

$\Rightarrow b$ is divisible by 2

$\Rightarrow \frac{a}{b}$ is not in the lowest form (a and b both divide by 2)

This contradicts the assumption that $\frac{a}{b} = \sqrt{2}$ is rational.

Thus $\sqrt{2}$ is irrational.

5 Suppose $\sqrt{5}$ is rational.

$\Rightarrow \sqrt{5} = \frac{a}{b}$ for some $a, b \in \mathbf{Z}$

If $\frac{a}{b}$ is not in the lowest form with $a, b > 0$, cancel until it is.

$\Rightarrow 5 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 5b^2$

$\Rightarrow a^2$ is divisible by 5

$\Rightarrow a$ is divisible by 5

$\Rightarrow a = 5c$ for some $c \in \mathbf{Z}$

$\Rightarrow (5c)^2 = 5b^2$

$\Rightarrow 5b^2 = 25c^2$

$\Rightarrow b^2 = 5c^2$

$\Rightarrow b^2$ is divisible by 5

$\Rightarrow b$ is divisible by 5

$\Rightarrow \frac{a}{b}$ is not in the lowest form (a and b both divide by 5)

This contradicts the assumption that $\frac{a}{b} = \sqrt{5}$ is rational.

Thus $\sqrt{5}$ is irrational.

6 Suppose there are 2 obtuse angles $\angle A$ and $\angle B$ in a triangle $\triangle ABC$.

$\Rightarrow \angle A > 90°, \angle B > 90°$

$\Rightarrow \angle A + \angle B + \angle C > 180°$

But $\angle A + \angle B + \angle C = 180°$ (angle sum of a triangle)

Contradiction

Thus, there is at most one obtuse angle in a triangle.

7 Suppose $a, b \in \mathbf{Z}$ and $a^2 - 4b = 2$

$\Rightarrow a^2 = 2 + 4b = 2(1 + 2b)$, so a is even.

$\Rightarrow a = 2c$ for some $c \in \mathbf{Z}$

$\Rightarrow (2c)^2 - 4b = 2$

$\Rightarrow 4c^2 - 4b = 2$

$\Rightarrow c^2 - b = \frac{1}{2}$

But $c, b \in \mathbf{Z}$ so $c^2 - b \in \mathbf{Z}$

Contradiction

Thus if $a, b \in \mathbf{Z}$, then $a^2 - 4b \neq 2$.

8 Suppose a, b are both odd.

$\Rightarrow a = 2p + 1, b = 2q + 1$ for some $p, q \in \mathbf{Z}$

$\Rightarrow c^2 = (2p + 1)^2 + (2q + 1)^2$

$= 4p^2 + 4p + 1 + 4q^2 + 4q + 1$

ISBN 9780170412995

$= 4(p^2 + p + q^2 + q) + 2$

$\Rightarrow c^2$ is even $\Rightarrow c$ is even, so $c = 2r$ for some $r \in \mathbf{Z}$.

$\Rightarrow c^2 = (2r)^2 = 4r^2$

$\Rightarrow 4r^2 = 4(p^2 + p + q^2 + q) + 2$

$\Rightarrow 2r^2 = 2(p^2 + p + q^2 + q) + 1$

$\Rightarrow 2r^2 - 2(p^2 + p + q^2 + q) = 1$

But $2r^2 - 2(p^2 + p + q^2 + q)$ is even.

Contradiction

$\Rightarrow a$ or b is even.

9 Suppose $-x$ is rational

$\Rightarrow -x = \frac{a}{b}$ for some $a, b \in \mathbf{Z}$

Now $-(-x) = -\frac{a}{b}$ and $a \in \mathbf{Z} \Rightarrow -a \in \mathbf{Z}$

$\Rightarrow x = -\frac{a}{b}$ for $-a, b \in \mathbf{Z}$

$\Rightarrow x$ is rational

Contradiction

$\Rightarrow -x$ is irrational

10 For x rational and y irrational, assume $x + y$ is rational.

$\Rightarrow x = \frac{a}{b}, x + y = \frac{c}{d}$ for some $a, b, c, d \in \mathbf{Z}$

$\Rightarrow \frac{a}{b} + y = \frac{c}{d}$

$\Rightarrow y = \frac{c}{d} - \frac{a}{b} = \frac{cb - ad}{db}$

But $cb - ad, db \in \mathbf{Z}$

$\Rightarrow y$ is rational

Contradiction

So $x + y$ is irrational.

11 For x rational and y irrational, assume that xy is rational.

$\Rightarrow x = \frac{a}{b}, xy = \frac{c}{d}$ for some $a, b, c, d \in \mathbf{Z}$

$\Rightarrow \frac{a}{b} y = \frac{c}{d}$

$\Rightarrow y = \frac{bc}{ad}$

But $bc, ad \in \mathbf{Z}$

$\Rightarrow y$ is rational.

Contradiction

So xy is irrational.

12 For x rational and y irrational, assume that $\frac{x}{y}$ is rational.

$\Rightarrow x = \frac{a}{b}, \frac{x}{y} = \frac{c}{d}$ for some $a, b, c, d \in \mathbf{Z}$

$\Rightarrow x = \frac{c}{d} y$

$\Rightarrow y = x\frac{d}{c} = \frac{ad}{bc}$

But $ad, bc \in \mathbf{Z}$

$\Rightarrow y$ is rational.

Contradiction

So $\frac{x}{y}$ is irrational.

13 In the parallelogram $ABCD$ assume that $AB \neq DC$.

Draw a diagram including the diagonal AC.

Now $\angle BCA = \angle DAC$ and $\angle BAC = \angle DCA$ (Alternate)

Also $AC = CA$ (same)

$\Rightarrow \triangle ABC \equiv \triangle ADC$ (ASA)

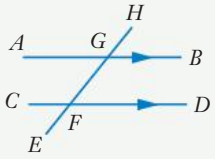
$\Rightarrow AB = DC$

Contradiction

$\Rightarrow$ Opposite sides of a parallelogram are equal.

Exercise 2.07

1 Draw parallel lines AB and CD with a transversal $EFGH$ intersecting the lines at F and G.



RTP: $\angle AGH = \angle CFH$

$\angle AGH + \angle AGF =$ 180° (straight line)

$\angle CFH + \angle AGF = 180°$ (co-interior)

$\Rightarrow (\angle AGH + \angle AGF) - (\angle CFH + \angle AGF) = 0°$

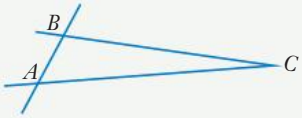
$\Rightarrow \angle AGH - \angle CFH = 0°$

$\Rightarrow \angle AGH = \angle CFH$ QED

2 Suppose two lines are not parallel.

$\Rightarrow$ The lines meet at, say, C.

Draw a transversal through the lines, intersecting at A and B.



RTP $\angle ABC + \angle BAC \neq 180°$

$\angle ABC + \angle BAC + \angle BCA = 180°$ (sum of angles in $\triangle ABC$)

$\Rightarrow \angle ABC + \angle BAC < 180°$

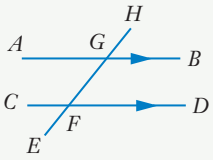
$\Rightarrow$ The co-interior angles do not sum to 180°.

ISBN 9780170412995

⇒ The contrapositive is true.

⇒ If co-interior angles sum to 180°, then the lines are parallel. QED

3 Draw parallel lines AB and CD with a transversal $EFGH$ intersecting the lines at F and G.



RTP: $\angle BGE = \angle CFH$

$\angle BGE + \angle AGF = 180°$ (straight line)

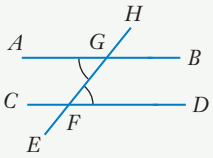
$\angle CFH + \angle AGF = 180°$ (co-interior)

$\Rightarrow (\angle BGE + \angle AGF) - (\angle CFH + \angle AGF) = 0°$

$\Rightarrow \angle BGE - \angle CFH = 0°$

$\Rightarrow \angle BGE = \angle CFH$ QED

4 Draw two lines with equal alternate angles on a transversal.



RTP: $AB \,||\, CD$

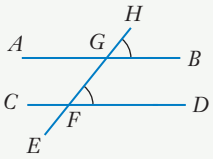
$\angle AGE + \angle EGB = 180°$ (straight line)

$\angle AGE = \angle GFD$ (given)

$\Rightarrow \angle GFD + \angle EGB = 180°$

$\Rightarrow AB \,||\, CD$ (Converse of co-interior angles) QED

5 Draw two lines with equal corresponding angles on a transversal.



RTP: $AB \,||\, CD$

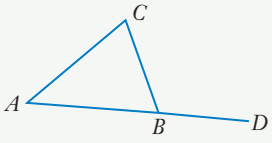
$\angle BGH + \angle BGE = 180°$ (straight line)

$\angle BGH = \angle GFD$ (given)

$\Rightarrow \angle GFD + \angle BGE = 180°$

$\Rightarrow AB \,||\, CD$ (Converse of co-interior angles) QED

6 Draw a triangle ABC and extend AB to D to make the external angle CBD.



RTP: $\angle CBD = \angle CBA + \angle ACB$

$\angle CBD + \angle CBA = 180°$ (straight line)

$\angle CAB + \angle ACB + \angle CBA = 180°$ (sum of angles in $\triangle ABC$)

$\Rightarrow (\angle CBD + \angle CBA) - (\angle CAB + \angle ACB + \angle CBA) = 0°$

$\Rightarrow \angle CBD - \angle CAB - \angle ACB = 0°$

$\Rightarrow \angle CBD = \angle CAB + \angle ACB$ QED

7 **a** $\angle IGL = 111°$, corresponding

$\Rightarrow \angle SGL = 69°$, supplementary

$\Rightarrow h = 42°$, angle sum of isosceles triangle

b $\angle GHT = 117°$, co-interior and $\angle RGH = 104°$, supplementary

$\Rightarrow i = 56°$, angle sum of quadrilateral $RGHT$

c $\angle INT = 73°$, angle sum of $\triangle ITN$

$\Rightarrow \angle NAC = 73°$, corresponding

$\Rightarrow j = 43°$, exterior angle of $\triangle CAE$

8 **a** $\angle IFE = 91°$, supplementary and $\angle HEF = 65°$, supplementary

$\Rightarrow g = 123°$, angle sum of quadrilateral

b $\angle CGA = 86°$, corresponding

$\Rightarrow h = 53°$, angle sum of $\triangle CAG$

c $\angle IST = 77°$, alternate

$\Rightarrow \angle FTS = 75°$, angle sum of $\triangle FST$

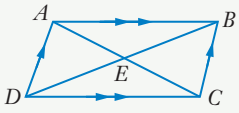
$\Rightarrow f = 75°$, corresponding

d $\angle TAE = 60°$, angle sum equilateral of triangle

$\Rightarrow i = 60°$, alternate

9 **Proposition**

Assume that $ABCD$ is a parallelogram.



Draw the diagonals to intersect at E.

RTP: $AE = EC$ and $BE = ED$

$\angle BAC = \angle ACD$ (alternate)

$\angle ABD = \angle BDC$ (alternate)

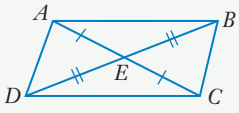
$AB = DC$ (opposite sides of parallelogram)

$\Rightarrow \triangle BEA \equiv \triangle CED$ (ASA)

$\Rightarrow AE = EC$ and $BE = ED$

Converse

Draw a quadrilateral $ABCD$ with diagonals that bisect each other at E.



$AE = EC$ (given)

$\angle AEB = \angle CED$ (vertically opposite)

$BE = ED$ (given)

$\Rightarrow \triangle BEA \equiv \triangle CED$ (SAS)

$\Rightarrow \angle BAE = \angle ECD$

$\Rightarrow AB$ is parallel to CD (converse of alternate)

Simlilarly, AD is parallel to BC.

$\Rightarrow ABCD$ is a parallelogram

Both the propositon and converse are proven. QED

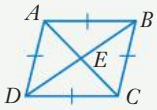
10 **Proposition**

Assume that $ABCD$ is a rhombus.

Draw the diagonals to intersect at E.

ISBN 9780170412995

RTP $DE = EB$, $AE = EC$, $\angle BEA = 90°$



$AD = DC$ (rhombus)

$AB = BC$ (rhombus)

$AC = AC$ (same)

$\Rightarrow \triangle ADC \equiv \triangle ABC$ (SSS)

$\Rightarrow \angle DAE = \angle BAE$

$AD = AB$ (rhombus)

$\angle ADB = \angle ABD$ ($\triangle DAB$ is isosceles)

$\Rightarrow \triangle DAE \equiv \triangle BAE$ (ASA)

$\Rightarrow DE = EB$, $\angle DEA = \angle BEA$

Also $\angle DEA + \angle BEA = 180°$ (straight line)

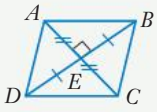
$\Rightarrow \angle BEA = 90°$

Similarly, $AE = EC$ and $\angle DEC = 90°$.

So the proposition is proven.

Converse

Assume that $ABCD$ is a quadriateral with diagonals that bisect each other at E at 90°.



$AE = EC$ (given)

$\angle BEA = \angle BEC$ (given)

$BE = BE$ (same)

$\Rightarrow \triangle AEB \equiv \triangle CEB$ (SAS)

$\Rightarrow AB = BC$

Similarly, $BC = CD$ and $CD = DA$

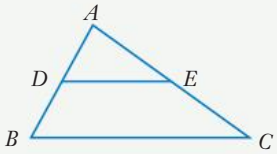
$\Rightarrow$ All sides are equal

$\Rightarrow ABCD$ in a rhombus

The proposition and converse are proven. QED

11 Draw a line segment DE between the midpoints D and E of the sides AB and AC of $\triangle ABC$.

RTP: $DE = \frac{1}{2}BC$ and $DE \parallel BC$



$AD = \frac{1}{2}AB$ (given)

$\angle BAC = \angle BAC$ (same)

$AE = \frac{1}{2}AC$ (given)

$\Rightarrow \triangle DAE \vert\vert\vert \triangle BAC$ (SAS)

$\Rightarrow \angle ADE = \angle ABC$

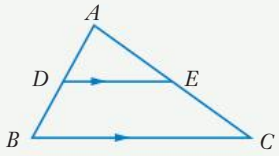
$DE = \frac{1}{2}BC$ (scale $\frac{1}{2}$)

$\Rightarrow DE \parallel BC$ (converse of corresponding)

Since $AE = \frac{1}{2}AC$, the scale factor is $\frac{1}{2}$.

$\Rightarrow DE = \frac{1}{2}BC$ QED

12 If a line segment between points on the sides of a triangle is parallel to and half the length of the third side, then they are the midpoints



Draw a line segment between the points D and E of the sides A and B of $\triangle ABC$ so that it is parallel to and half the length of the third side BC.

RTP $AD = DE$, $AE = EC$

$\angle ADE = \angle ABC$ (corresponding)

$\angle AED = \angle ACB$ (corresponding)

$\Rightarrow \triangle DAE \vert\vert\vert \triangle BAC$ (AA)

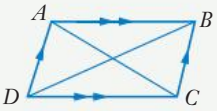
Since $DE = \frac{1}{2}BC$, the scale factor is $\frac{1}{2}$.

$\Rightarrow AD = \frac{1}{2}AB$ and $AE = \frac{1}{2}AC$

$\Rightarrow AD = DE$, $AE = EC$ QED

13 **Proposition**

Suppose that $ABCD$ is a parallelogram.



Draw the diagonals.

RTP: $\angle ABC = \angle CDA$ and $\angle BCD = \angle DAB$.

$\angle ADB = \angle CBD$ (alternate)

$DB = DB$ (same)

$\angle ABD = \angle CDB$ (alternate)

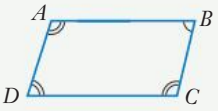
$\Rightarrow \triangle ADB \vert\vert\vert \triangle CBD$ (ASA)

$\Rightarrow \angle BCD = \angle DAB$

Similarly, $\angle ABC = \angle CDA$.

Converse

Suppose that $ABCD$ is a quadrilateral with opposite angles equal.



RTP $AB \parallel DC$ and $AD \parallel BC$

$\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360°$ (angle sum of quadrilateral)

$\angle ABC = \angle CDA$ (given)

$\angle BCD = \angle DAB$ (given)

$\Rightarrow \angle ABC + \angle BCD + \angle ABC + \angle BCD = 360°$

$\Rightarrow 2(\angle ABC + \angle BCD) = 360°$

$\Rightarrow \angle ABC + \angle BCD = 180°$

$\Rightarrow AB \parallel DC$

Similarly, $AD \parallel BC$

$\Rightarrow ABCD$ is a parallelogram.

Both the proposition and the converse are proven. QED

ISBN 9780170412995

Chapter review

1 $w > x, y > z$

$\Rightarrow w - x, y - z$ are positive

$\Rightarrow y(w - x), x(y - z)$ are positive

$\Rightarrow yw - xy, xy - xz$ are positive

$\Rightarrow yw - xy + xy - xz$ is positive

$\Rightarrow yw - xz$ is positive

$\Rightarrow yw > xz$

2 **a** Boys like to wrestle is a generalisation that is not true in every case. False premise.

b Reversing a non-reversible statement.

3 **a** $x = 2, y = -3$: $x > y$ but $x^2 < y^2$

b $x = 5$: LHS $= 5^2 + 24 = 49$, RHS $= 10 \times 5 = 50$, but $49 < 50$.

4 **a** All carnivores are lions

b xy is an integer $\Rightarrow x, y$ integers

5 x, y are perfect squares $\Rightarrow xy$ is a perfect square.

The statement is false, but the converse is true.

6 **a** Equivalent **b** Equivalent

c Not equivalent

7 **a** There is at least one rational number that cannot be written as a recurring decimal.

b There is at least one Australian who does not live in the southern hemisphere.

c An acute angled triangle has at least one angle that is a right or obtuse angle.

8 **a** $0.41\dot{6}, 0.36$ **b** $\frac{149}{200}, \frac{124}{330}$

9 Suppose x is rational.

$\Rightarrow x = \frac{a}{b}$ for $a, b \in \mathbf{Z}$

$\Rightarrow x^2 = \frac{a^2}{b^2}$

Now $a^2, b^2 \in \mathbf{Z}$

$\Rightarrow x^2$ is rational

Using the contrapositive, x^2 irrational $\Rightarrow x$ irrational.

10 Suppose a quadrilateral is a rectangle.

$\Rightarrow$ all the angles are 90°

$\Rightarrow$ none of the angles are acute

Using the contrapositive, an acute angle in a quadrilateral $\Rightarrow$ it is not a rectangle.

11 Assume there is a least integer n.

$\Rightarrow n - 1$ is an integer

But $n - 1 < n$

Contradiction, so the assumption is false.

$\Rightarrow$ There is no least integer.

12 Suppose $\sqrt{7}$ is rational.

$\Rightarrow \sqrt{7} = \frac{a}{b}$ for some $a, b \in \mathbf{Z}$

If $\frac{a}{b}$ is not in the lowest form with $a, b > 0$, cancel until it is.

$\Rightarrow 7 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 7b^2$

$\Rightarrow a^2$ is divisible by 7

$\Rightarrow a$ is divisible by 7

$\Rightarrow a = 7c$ for some $c \in \mathbf{Z}$

$\Rightarrow (7c)^2 = 7b^2$

$\Rightarrow 7b^2 = 49c^2$

$\Rightarrow b^2 = 7c^2$

$\Rightarrow b^2$ is divisible by 7

$\Rightarrow b$ is divisible by 7

$\Rightarrow \frac{a}{b}$ is not in the lowest form (a and b both divide by 7)

This contradicts the assumption that $\frac{a}{b} = \sqrt{7}$ is rational.

Thus $\sqrt{7}$ is irrational.

13 Suppose there are two right angles in $\triangle ABC$, at, say, A and B.

$\Rightarrow \angle ABC + \angle BAC = 90° + 90° = 180°$

$\Rightarrow BC \,||\, AC$

$\Rightarrow BC, AC$ do not meet

Contradiction

Assumption is false.

$\Rightarrow$ There is at most one right angle in a triangle. QED

14 Draw a trapezium $ABCD$ with $AB \,||\, DC$.

$\Rightarrow \angle CDA + \angle DAB = 180°$

$\angle ABD + \angle BCD = 180°$ (cointerior)

$\Rightarrow \angle CDA + \angle DAB + \angle ABD + \angle BCD = 180° + 180° = 360°$ QED

15 **a** $\angle BUT = 42°$, supplementary

$\Rightarrow g = 120°$, angle sum of quadrilateral

ISBN 9780170412995

b $\angle BON = 50°$, isosceles triangle

$\Rightarrow \angle YOF = 50°$, vertically opposite and

$\angle OFE = 81°$, supplementary

$\Rightarrow i = 111°$, angle sum of quadrilateral

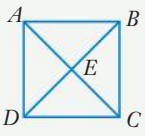
c $\angle OTH = 63°$, angles at a point

$\Rightarrow j = 58°$, angle sum of triangle

d $\angle NOT = 69°$, opposite angles of parallelogram

$\Rightarrow h = 111°$, supplementary

16 **Proposition** Draw a square $ABCD$ with diagonals that intersect at E.



$\triangle ABC$ is isosceles right-angled.

$\Rightarrow \angle CAB = \angle BCA = 45°$

Similarly $\angle DBA = 45°$, $\angle BDC = 45°$, $\angle ACD = 45°$

$\Rightarrow \angle EAB = \angle EDC$ (both 45°)

$\angle EBA = \angle ECD$ (both 45°)

$AB = DC$

$\Rightarrow \triangle ABE = \triangle CDE$ (AAS)

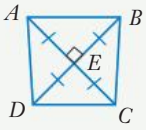
$\Rightarrow AE = EC = BE = ED$

Also $\angle AEB = 90°$ (sum of angles in $\triangle ABE$)

$\Rightarrow$ The diagonals are equal and bisect each other at 90°.

Converse

Draw a quadrilateral $ABCD$ with equal diagonals that bisect each other at 90° at E.



$\triangle ABE$ is isosceles right-angled.

$\Rightarrow \angle EAB = \angle EBA = 45°$

Similarly $\angle EBC = \angle ECB = 45°$

$\Rightarrow \angle ABC = \angle EBA + \angle EBC = 45° + 45° = 90°$

Similarly, $\angle BCD = \angle CDA = \angle DAB = 90°$

$AE = EB = EC$ and $\angle AEB = \angle BEC = 90°$

$\Rightarrow AB = BC$

Similarly, $BC = CD = DA$

$\Rightarrow ABCD$ is a square.

The proposition and converse have been proven. QED

17 For any $x, y \in \boldsymbol{R}$, $(x - y)^2 \geq 0$

$\Rightarrow x^2 - 2xy + y^2 \geq 0$

$\Rightarrow x^2 + y^2 \geq 2xy$ QED

18 **a** For $n = 2$ and $m = -3$,

$\frac{1}{n} = 0.5$, $\frac{1}{n+m} = -1$, so $\frac{1}{n} < \frac{1}{n+m}$ is not true.

b It is true for $n > 1$ and $-n < m < 0$

19 None of these.

There is no conditional so they are not converses.

The negation of $x > 0$ is $x \leq 0$, not $x < 0$.

They do not mean the same thing so ar not equivalents.

20 Suppose n is not divisible by 2 or 3.

$\Rightarrow n = 6m + 1$ or $6m + 5$ for some $m \in \boldsymbol{Z}$

$\Rightarrow n^2 = (6m + 1)^2$ or $(6m + 5)^2$

$= 6(m^2 + 2m) + 1$ or $6(m^2 + 10m + 4) + 1$

$\Rightarrow n^2$ is not divisible by 6

Using contrapositive,

n^2 divisible by 6 $\Rightarrow n$ is divisible by both 2 and 3. QED

21 Suppose x, y are both rational

$\Rightarrow x = \frac{a}{b}, y = \frac{c}{d}$, for $a, b, c, d \in \boldsymbol{Z}$

$\frac{x}{y} = \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

Now $ad, bc \in \boldsymbol{Z}$

$\Rightarrow \frac{x}{y}$ is rational

Using contrapositive,

$\frac{x}{y}$ not rational $\Rightarrow$ at least one of x, y is not rational. QED

22 For x rational and y irrational, assume $x - y$ is rational.

$\Rightarrow x = \frac{a}{b}, x - y = \frac{c}{d}$ for some $a, b, c, d \in \boldsymbol{Z}$

$\Rightarrow \frac{a}{b} - y = \frac{c}{d}$

$\Rightarrow y = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$

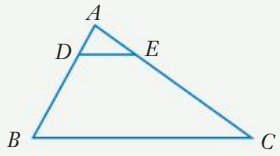
But $ad - bc, bd \in \boldsymbol{Z}$

$\Rightarrow y$ is rational

Contradiction

So $x - y$ is irrational.

23 Draw a line segment DE between the points D and E that divides the sides AB and AC of $\triangle ABC$ in the ratio 1 : 4.



RTP: $DE = \frac{1}{5}BC$ and $DE \parallel BC$

$\frac{1}{5}$ of the distance from A to B and C.

ISBN 9780170412995

$AD = \frac{1}{5} AB$ (given)

$\angle BAC = \angle BAC$ (same)

$AE = \frac{1}{5} AC$ (given)

$\Rightarrow \triangle DAE \,|||\, \triangle BAC$ (SAS)

$\Rightarrow \angle ADE = \angle ABC$

$\Rightarrow DE \,||\, BC$ (converse of corresponding)

Since $AE = \frac{1}{5} AC$, the scale factor is $\frac{1}{5}$.

$\Rightarrow DE = \frac{1}{5} BC$ QED

Chapter 3

Exercise 3.01

1 **a** A1, A2, B1, B2, C1, C2 **b** 6
2 512 **3** 8 ways
4 45 697 600 **5** 24
6 168 **7** 12
8 128 **9** 128
10 624 **11** 40 320
12 336
13 **a** 360 **b** 1296
14 2600
15 **a** 2652 **b** 156
16 506
17 **a** 17 576 000
b No, more than 18 million would be needed.
18 392 **19** 4 digits
20 990 000

Exercise 3.02

1 15
2 14
3 10
4 **a** 120 **b** 48
5 45 656
6 **a** 1680 **b** 4096 **c** 720 **d** 720
e 120 **f** 1560 **g** 18 000
7 **a** 40 320 **b** 1152 **c** 2880
d 10 080 **e** 30 240
8 **a** 18 **b** 6 **c** 4
d 10 **e** 8 **f** 10
9 **a** 5 **b** 6 **c** 12
10 **a** 60 **b** 720 **c** 30 240
11 64 **12** 72 **13** 36
14 **a** 25 **b** 20

Exercise 3.03

1 **a** 86 **b** 84
2 26 **3** 40 **4** 40
5 10 **6** 80 **7** 19
8 16
9 **a** 7 **b** 11 **c** 5
10 26
11 **a** 2 **b** 7 **c** 1 **d** 9
12 **a** 14 **b** 49 **c** 59 **d** 47
13 **a** 31 **b** 4 **c** 59 **d** 36
14 55
15 **a** 2 **b** 12 **c** 43 **d** 1

Exercise 3.04

1 **a** $4 \times 3 \times 2 \times 1$
b $6 \times 5 \times 4 \times 3 \times 2 \times 1$
c $5 \times 4 \times 3 \times 2 \times 1$
d $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
e $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
2 **a** 6 **b** 24 **c** 720
d 40 320 **e** 3 628 800
3 **a** 6 **b** 120 **c** 720
d 24 **e** 40 320
4 **a** 120 **b** 5040 **c** 360
d 40 320 **e** 24
5 24
6 5040
7 **a** 6720 **b** 15 120 **c** 720
d 7 **e** 120
8 **a** $n(n-1)(n-2)(n-3)$ **b** $(n+4)(n+3)$
c $\frac{1}{n(n-1)(n-2)(n-3)}$
d $(n-2)(n-3)(n-4)(n-5)$
e $\frac{1}{(n+3)(n+2)(n+1)n(n-1)(n-2)}$
9 **a** 6 **b** 60 **c** 42
d 362 880 **e** 1680 **f** 120
g 110 **h** 30 240 **i** 592 620
j 240 240

ISBN 9780170412995

10 **a** $n = 10$ **b** $n = 3$ **c** $n = 15$
d $n = 29$ **e** $n = 5$ **f** $n = 10$

11 **a** 60 ways **b** 125 ways

12 5040 **13** 60

14 5040 **15** 2184

16 720

17 3600 (Consider if Riley is in bed 1, bed 7 or beds 2, 3, 4, 5, 6 as 3 separate situations.)

Exercise 3.05

1 360 **2** 4 989 600

3 35 **4** 151 200

5 27 720 **6** 1 081 080

7 362 880 **8** 120

9 **a** 12 **b** 12 **c** 8

10 **a** 39 916 800 **b** 86 400

11 **a** 7 257 600 **b** 2 177 280

12 **a** 5040 **b** 1440 **c** 3600

13 **a** 24 **b** 696

14 192 **15** 1440

16 2880 **17** 30 240

18 **a** 120 **b** 150

Exercise 3.06

1 **a** $\frac{1}{54145} \approx 0.000\,018$ **b** $\frac{33}{16660} \approx 0.001\,98$

2 $\frac{1}{336} \approx 0.3\%$

3 $\frac{13}{14}$

4 **a** 64 **b** 15 **c** $\frac{15}{64}$

5 **a** $\frac{5}{16}$ **b** $\frac{7}{64}$ **c** $\frac{7}{64}$

d $\frac{3}{8}$ **e** $\frac{184756}{1048576} \approx 0.176$

6 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$

7 **a** $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** $\frac{2}{3}$ **d** $\frac{1}{6}$

8 **a** $\frac{1}{7}$ **b** $\frac{6}{7}$ **c** $\frac{1}{7}$

9 $\frac{1}{5}$ **10** $\frac{1}{28}$

11 **a** 5040 **b** **i** $\frac{2}{7}$ **ii** $\frac{5}{7}$ **iii** $\frac{1}{42}$

12 **a** $\frac{1}{5}$ **b** $\frac{1}{5}$ **c** $\frac{1}{10}$

13 **a** $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** $\frac{1}{5}$ **d** $\frac{1}{15}$

14 $\frac{1}{462}$

15 $\frac{1}{126}$

16 **a** $\frac{3}{20}$ **b** $\frac{1047}{8000}$ **c** $\frac{51}{400}$

17 **a** $\frac{2}{9}$ **b** $\frac{5}{9}$

Chapter review

1 **a** 9 **b** 20 **a** 84 **b** 504

2 42

3 **a** 5040 **b** 30 240 **c** 3000
d 12 000 **e** 12 000 **f** 30 120

4 **a** 14 **b** 86 **c** 67

5 **a** 26 **b** 4

6 **a** 5040 **b** 720

7 **a** 210 **b** 132 **c** 116 280

8 **a** 1320 **b** $\frac{1}{(n-1)(n-2)}$ **c** $n = 9$

9 **a** $\frac{2}{5}$ **b** $\frac{3}{5}$ **c** $\frac{1}{5}$

10 336

11 6720

12 15 120

13 30

14 720

15 **a** 182 in order, otherwise 91 **b** $\frac{3}{91}$
c $\frac{71}{91}$

16 **a** 840 **b** **i** $\frac{1}{7}$ **ii** $\frac{1}{21}$ **iii** $\frac{2}{21}$ **iv** $\frac{1}{7}$

17 $\frac{1}{720}$

18 $\frac{1}{15}$

19 \$725 760

20 120 with the engine at the front, but only 24 with the little red caboose last as well.

ISBN 9780170412995

21 $\frac{30!}{5! \times 200} \approx 10^{28}$

22 $\frac{105}{512}$

23 $\frac{1}{1296}$

Practice examination 1

1 **a** (–4.33, –2.5) **b** (8.06, 119.7°)

2 4.36

3 **a**

b

4 **a** (40, –40) **b** (–13, 10)

5 **a** (29.45, 85.7°) **b** (37.4, 62.7°)

6 **AB** = (–5, 8)

7 Horizontal 141.0 N, vertical –51.3 N

8 For $x = 3$, $x^2 + 8 = 17$, $6x = 18$.
But $17 < 18$, so $x^2 + 8 > 6x$ is not true.

9 Even numbers are divisible by 4.

10 $\exists\, a \in Z$ such that $b - 2a = 0 \Rightarrow b \notin Z$

11 A quadrilateral that has a pair of non-supplementary co-interior angles is not a parallelogram.

12 Let $a = 2n + 1$, $b = 2m + 1$ for $n, m \in Z$ (1 mark)
$\Rightarrow a + b = 2n + 1 + 2m + 1$
$= 2n + 2m + 2$ (1 mark)
$= 2(n + m + 1)$ which is even (1 mark)
Thus a, b odd $\Rightarrow a + b$ is even. QED

13 25

14 **a** 1680 **b** 6720 **c** 5400
d 840 **e** 4200

15 **a** 24 **b** 256

16 1 048 576

17 About 8.3 kn at 071.9°.

18 225° and 95 kn

19 Suppose x, y not irrational
$\Rightarrow x = \frac{a}{b}\ y = \frac{c}{d}$ for some $a, b, c, d \in Z$ (1 mark)
$\Rightarrow x - y = \frac{a}{b} - \frac{c}{d}$
$= \frac{ad - bc}{bd}$ (1 mark)
But $ad - bc, bd \in Z$
$\Rightarrow x - y$ is rational (1 mark)
Using the contrapositive,
$x - y$ not rational $\Rightarrow$ at least one of x, y is irrational. QED

20 0.3

21 About 743 N

22 Suppose $\sqrt{6}$ is rational
$\Rightarrow \sqrt{6} = \frac{a}{b}$ for some $a, b \in Z$
If $\frac{a}{b}$ is not in the lowest form with $a, b > 0$, cancel until it is.
$\Rightarrow 6 = \frac{a^2}{b^2}$
$\Rightarrow a^2 = 6b^2$
$\Rightarrow a^2$ is divisible by 6
$\Rightarrow a$ is divisible by 6
$\Rightarrow a = 6c$ for some $c \in Z$
$\Rightarrow (6c)^2 = 6b^2$
$\Rightarrow 6b^2 = 36c^2$
$\Rightarrow b^2 = 6c^2$
$\Rightarrow b^2$ is divisible by 6
$\Rightarrow b$ is divisible by 6
$\Rightarrow \frac{a}{b}$ is not in the lowest form (a and b both divide by 6)
This contradicts the assumption that $\frac{a}{b} = \sqrt{6}$ is rational.
Thus $\sqrt{6}$ is irrational.

23 $\frac{2}{7}$

24 12

Chapter 4

Exercise 4.01

1 **a** **OC** **b** **AE** **c** **MR**

2 Complete the parallelogram of forces to show that the resultant of $\mathbf{p} + \mathbf{q}$ is the same as the resultant of $\mathbf{q} + \mathbf{p}$. Head-to-tail addition could also be used to demonstrate the same thing.

3 **a** $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{c}$ **c** $-(\mathbf{a} + \mathbf{b})$
d $-\mathbf{a} - \mathbf{b} + \mathbf{c}$ **e** $\mathbf{c} - \mathbf{a}$ **f** $-\mathbf{c} + \mathbf{a} + \mathbf{b}$

4 **a** $2\mathbf{a}$ **b** $3\mathbf{a}$ **c** $2\mathbf{a} + \mathbf{b}$
d $-(\mathbf{b} + 3\mathbf{a})$

ISBN 9780170412995

5 **a** $-\mathbf{r}-\mathbf{t}$ **b** $-(\mathbf{t}+\mathbf{u})$ **c** $\mathbf{s}-\mathbf{r}$
d $\mathbf{r}+\mathbf{t}-\mathbf{s}$ **e** $-\mathbf{r}-\mathbf{t}-\mathbf{u}+\mathbf{s}$

6 $(\mathbf{r}+\mathbf{t})+\mathbf{u}=\mathbf{r}+(\mathbf{t}+\mathbf{u})=\mathbf{AD}$

7 **a** $\mathbf{a}$ **b** $2\mathbf{a}$ **c** $\mathbf{c}$
d $\mathbf{c}+2\mathbf{a}$ **e** $\mathbf{c}+2\mathbf{a}$ **f** $\frac{1}{2}\mathbf{c}$
g $\frac{1}{2}\mathbf{c}, \frac{1}{2}\mathbf{a}, \frac{1}{2}\mathbf{c}+1\frac{1}{2}\mathbf{a}, \mathbf{c}+\mathbf{a}, \frac{1}{2}\mathbf{c}+\mathbf{a}$

8 **a** $\mathbf{AB}=\frac{1}{2}\mathbf{PR}$ **b** $\mathbf{CD}=\frac{1}{2}\mathbf{PR}$
c $\mathbf{AB}=\mathbf{CD}$
d $\mathbf{AD}=\frac{1}{2}\mathbf{QS}$ and $\mathbf{BC}=\frac{1}{2}\mathbf{QS}$, so $\mathbf{AD}=\mathbf{BC}$
e *ABCD* is a parallelogram because opposite sides are parallel.

9 **a** $\mathbf{b}-\mathbf{a}$ **b** $\mathbf{a}-\mathbf{b}$ **c** $\frac{1}{2}(\mathbf{a}+\mathbf{b})$
d $\frac{1}{2}(\mathbf{b}-\mathbf{a})$ **e** $\frac{1}{2}(\mathbf{a}-\mathbf{b})$

10 **a** $\mathbf{p}+\mathbf{q}$ **b** $-\mathbf{q}-\mathbf{p}+\mathbf{r}$
c $\mathbf{q}+\frac{1}{2}(-\mathbf{q}-\mathbf{p}+\mathbf{r})=\frac{1}{2}\mathbf{q}-\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{r}$
d $\mathbf{p}+\mathbf{q}+\frac{1}{2}(-\mathbf{q}-\mathbf{p}+\mathbf{r})=\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{q}+\frac{1}{2}\mathbf{r}$

11 $\frac{3\mathbf{b}+\mathbf{a}}{4}$

12 $\frac{3\mathbf{p}+2\mathbf{q}}{5}$

13 **a** $\mathbf{XY}=-\mathbf{a}+\mathbf{b}$ **b** $\mathbf{XZ}=-2\mathbf{a}+2\mathbf{b}$
c $\mathbf{XC}=-\mathbf{a}+2\mathbf{b}$ **d** $\mathbf{ZC}=\mathbf{a}$
e The result from part **d** shows $BX \parallel ZC$. Show that $\mathbf{BZ}=\mathbf{XC}$, i.e. $BZ \parallel ZC$.
So *XBZC* is a parallelogram.

14 **a** $\mathbf{OZ}=\mathbf{OB}+\mathbf{BZ}=\frac{2}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$
b $\mathbf{OY}=\mathbf{OB}+\mathbf{BY}=\frac{4}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}$
c $\mathbf{OY}=2\mathbf{OZ}$, so *Z* is the midpoint of *OY*.

15 $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{PR}+\mathbf{c}=\mathbf{PS}$ and $\mathbf{a}+(\mathbf{b}+\mathbf{c})=\mathbf{a}+\mathbf{QS}=\mathbf{PS}$ so $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$.

16 **a** Begin with $\mathbf{QR}=\mathbf{QO}+\mathbf{OR}$.
b Begin with $\mathbf{XY}=\mathbf{XO}+\mathbf{OY}$.

17 $\mathbf{OR}=\frac{5}{3}\mathbf{a}$

Exercise 4.02

1 **a** 35.46 **b** 21.21 **c** 0 **d** 180
e 12.12 **f** −57.16 **g** −70.15 **h** 48

2 **a** 44.50 **b** 46.36 **c** 57
d 15 **e** 27.58 **f** −112.76

3 **a** −14 **b** 0 **c** −96 **d** 70
e 39 **f** 90 **g** −3 **h** −4

4 **a** −30 **b** −85 **c** 73
d −69 **e** −68 **f** 74

5 **a** 167.9° **b** 93.2° **c** 117.4°
d 81.9° **e** 45° **f** 30°

6 **a** 38.1° **b** 52.4° **c** 29.2°
d 80.4° **e** 99.1° **f** 84.6°

7–10 Demonstration by substitution.

11 **a** 8 **b** 11 **c** 2 **d** −8

12–14 Proofs

15 51.9°

Exercise 4.03

1 In each case, the scalar product = 0.

2 In each case, the scalar product = 0.

3 **a** $|\mathbf{s}||\mathbf{t}|=\mathbf{s}\bullet\mathbf{t}$ **b** $|\mathbf{g}||\mathbf{h}|=-\mathbf{g}\bullet\mathbf{h}$
c $|\mathbf{u}||\mathbf{v}|=\mathbf{u}\bullet\mathbf{v}$
d $|\mathbf{a}||\mathbf{b}|=-\mathbf{a}\bullet\mathbf{b}$ **e** $|\mathbf{c}||\mathbf{d}|=-\mathbf{c}\bullet\mathbf{d}$
f $|\mathbf{p}||\mathbf{q}|=\mathbf{p}\bullet\mathbf{q}$
g $|\mathbf{r}||\mathbf{s}|=\mathbf{r}\bullet\mathbf{s}$ **h** $|\mathbf{w}||\mathbf{v}|=-\mathbf{w}\bullet\mathbf{v}$

4 **a** $|\mathbf{m}||\mathbf{n}|=-\mathbf{m}\bullet\mathbf{n}$ **b** $|\mathbf{p}||\mathbf{q}|=\mathbf{p}\bullet\mathbf{q}$
c $|\mathbf{a}||\mathbf{b}|=-\mathbf{a}\bullet\mathbf{b}$
d $|\mathbf{m}||\mathbf{n}|=\mathbf{m}\bullet\mathbf{n}$ **e** $|\mathbf{c}||\mathbf{d}|=\mathbf{c}\bullet\mathbf{d}$
f $|\mathbf{v}||\mathbf{w}|=-\mathbf{v}\bullet\mathbf{w}$

5 **a** $\mathbf{a} \parallel \mathbf{b}$ (opposite directions) **b** Neither
c Perpendicular
d $\mathbf{p} \parallel \mathbf{q}$ (opposite directions)
e Neither **f** $\mathbf{v} \parallel \mathbf{w}$

6 $a=-\frac{2}{3}$

7 $\mathbf{b}=(6, 15)$

8 $|\mathbf{MN}||\mathbf{NP}|=\mathbf{MN}\bullet\mathbf{NP}$

9 $\mathbf{AC}\bullet\mathbf{BD}=0$, so $\mathbf{AC}\perp\mathbf{BD}$

10 Calculate $|\mathbf{AC}|^2$ and $|\mathbf{BD}|^2$. Examine the condition for $|\mathbf{AC}|^2=|\mathbf{BD}|^2$.
$|\mathbf{AC}|^2=|\mathbf{BD}|^2$ if and only if $4(\mathbf{AB}\bullet\mathbf{BC})=0$ so $\mathbf{AB}\perp\mathbf{BC}$, i.e. *ABCD* is a rectangle.

Exercise 4.04

1 16.18 **2** 12.61
3 −2.21 **4** 29.67
5 0 (perpendicular) **6** 19 (parallel)

ISBN 9780170412995

7 **a** 9.71 **b** –0.12 **c** 4.21
d –3.13 **e** 3.02 **f** 2.79
8 (10.47, 36.53) **9** (14.00, 5.38)
10 (7.47, 7.82) **11** 0.91
12 –7.71 **13** 8.69
14 About 1097 N parallel and 8933 N perpendicular to the slope.
15 About 1.25 N parallel and 5.87 N perpendicular, so the friction is about 0.55 N.

Exercise 4.05

1 **a** 84.8 N **b** 18.7 N **c** 247.6 N
2 **a** 46.1 N down the incline, 63.4 perpendicular to the incline.
b 34.4 N down the incline, 112.5 perpendicular to the incline.
c 91.8 N down the incline, 227.2 perpendicular to the incline.
3 **a** 47.1 N in the direction of A
b 48.0 N in the direction of B
c 11.4 N in the direction of B
4 **a** 180 J **b** 192.4 J **c** 1291.5 J
5 **a** 171.2 J **b** 120.9 J **c** 597.1 J
d 676.5 J
6 **a** 32 J **b** 14 J **c** 13 J **d** 163 J
7 **a** 120 J **b** 88 J **c** 59.5 J
8 **a** 147 J **b** 987.84 J **c** 1687.2 J
9 **a** 82.2 J **b** 236.2 J **c** 231.2 J
10 **a** 169.9 J **b** 178.2 J **c** 1369.5 J
11 5.621 ms^{-2}
12 282.8 N
13 **a** 605.7 J **b** 151.4 N **c** 79.2 N
14 $\cos(A - B) = (\cos(A), \sin(A)) \bullet (\cos(B), \sin(B))$
$= \cos(A)\cos(B) + \sin(A)\sin(B)$
15 **a** 76.53 N at 13° to the 50 N force
b **i** 488.9 J **ii** 231.8 J **iii** 257.1 J
c 488.9 = 231.8 + 257.1, but because of the angles, the contributions of the 30 N and 50 N forces are almost the same.

Chapter review

1 **a** $-\mathbf{a}$ **b** $-\mathbf{a} + \mathbf{c}$ **c** $\mathbf{a} + \mathbf{c}$
d $\frac{1}{2}(\mathbf{a} + \mathbf{c})$ **e** $\frac{1}{2}(-\mathbf{a} + \mathbf{c})$ **f** $\frac{1}{2}(-\mathbf{a} + \mathbf{c})$
2 **a** $\mathbf{QR} = \mathbf{b} - \mathbf{a}$ **b** $\mathbf{QM} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$
c $\mathbf{PM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$
3 $\mathbf{KC} = \mathbf{m} - \frac{1}{2}\mathbf{n}$
4 $\mathbf{MN} = \frac{1}{2}\mathbf{b} = \mathbf{XY}$
5 **a** 6.71 **b** –49.56 **c** –44
6 **a** 81.9° **b** 132.9° **c** 176.3° **d** 69.0°
7 Proof.
8 **a** 8 **b** 11 **c** 2 **d** –8
9 $\mathbf{m} \bullet \mathbf{n} = \mathbf{0}$
10 $\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = -1$
11 **a** Parallel **b** Neither
c Neither **d** Perpendicular
12 16.38
13 17.10
14 –1.79
15 Parallel 63.39 N, perpendicular –135.95 N.
16 9189.5 N
17 **a** 1500 J **b** 188.16 J
18 602.9 J
19 $\mathbf{OB} = \mathbf{a} + \mathbf{c}$ and $\mathbf{CA} = \mathbf{a} - \mathbf{c}$. Find $|\mathbf{OB}|^2 + |\mathbf{CA}|^2$ in terms of $\mathbf{a}$ and $\mathbf{c}$. Then compare with $(\mathbf{a} + \mathbf{c}) \bullet (\mathbf{a} + \mathbf{c}) + (\mathbf{a} - \mathbf{c}) \bullet (\mathbf{a} - \mathbf{c})$.
20 Find an expression for $\mathbf{AC}$ and $\mathbf{BC}$ and then find $\mathbf{AC} \bullet \mathbf{BC}$.
21 47.5°, 71.6° and 60.9°
22 Demonstration.

Chapter 5

Exercise 5.01

1 **a** $x = 164°$ (angle at centre)
b $x = 224°$ (angle at centre)
c $x = 74°$ (angle sum in triangle)
2 **a** $\theta = 39°$ (angle at centre)
b $\phi = 72\frac{1}{2}°$ (angle at centre)
c $\alpha = 133°$ (angle at centre)
3 **a** $p = 114°$ (angle at centre)
$q = 57°$ (angle at centre)
b $m = 154°$ (angle at centre)
$n = 103°$ (angle at centre)

ISBN 9780170412995

c $y = 50°$ (co-interior)
$z = 7°$ (sum of angles in $\triangle BPX$)

4 a $x = 68°$ (straight line)
b $k = 128°$ (angle at centre)
c $a = 124°$ (straight line)

5–9 Proofs

Exercise 5.02

1 a $x = 110°$ (angles at circumference)
b $\theta = 38°$ (angles at circumference), $\phi = 83°$ (angles at circumference)
c $w = 49°$ (angles at circumference), $y = 19°$ (angles at circumference)

2 a $e = 64°$ (angles at circumference), $f = 67°$ (angles at circumference), $g = 49°$ (angle sum of $\triangle XYV$)
b $\alpha = 46°$ (angles at circumference), $\beta = 53°$ (angles at circumference)
c $p = 53°$ (angles at circumference), $q = 34°$ (angles at circumference), $r = 53°$ (angles at circumference)

3 a $\theta = 67°$ (straight line) **b** $\phi = 52°$ (angle sum of $\triangle LMP$)
c $\theta = 67°$ (angles at circumference)

4, 5 Proofs **6** $ST = 7.2$ cm
7 $BC = 6.12$ m **8** Proof

Exercise 5.03

1 a $p = 90°$ (semicircle angle), $q = 14°$ (angle sum of $\triangle DEF$)
b $\theta = 90°$ (semicircle angle), $\phi = 58°$ (angle sum of $\triangle KMN$)
c $x = 90°$ (semicircle angle), $y = 45°$ (angle sum of isosceles $\triangle RST$)

2 a $x = 25$ cm **b** $y = 15$ mm
c $z = 9\sqrt{2}$ m

3 a $\alpha = 28°$ (complementary angles)
b $\beta = 49°$ (angle sum of $\triangle ABC$)
c $\gamma = 30°$ (vertically opposite)

4 a $a = 35°$ (complementary with $\angle YWV$)
b $x = 27°$ (isosceles $\triangle OBC$)
c $\theta = 44°$ (angle sum of $\triangle PQT$)

5–8 Proofs

Exercise 5.04

1 a $g = 119°$ (opposite angles in cyclic quad)
b $\beta = 123°$ (interior opposite of cyclic quad)
c $y = 78°$ (exterior angle of cyclic quad), $x = 97°$ (opposite angles in cyclic quad)

2 a $m = 107°$ (opposite angles in cyclic quad), $n = 73°$ (co-interior)
b $w = 132°$ (exterior angle of cyclic quad)
c $z = 135°$ (opposite angles in cyclic quad)

3 a $\alpha = 117°$ (opposite angles in cyclic quad)
b $\beta = 87°$ (opposite angles in cyclic quad)
c $\gamma = 13°$ (by subtraction)

4–9 Proofs

Exercise 5.05

1 a $\theta = 37°$ (equal arcs)
b $a = 50°$ (vertically opposite), $b = 8$ cm (equal angles)
c $x = 84°$ (equal arcs), $y = 42°$ (angle at centre)

2 a $a = 4$ cm (intersecting chords)
b $b = 26\frac{2}{3}$ m (intersecting chords)
c $x = 2$ m (intersecting chords)

3 a $x = 48°$ (angle sum of $\triangle AON$), $y = 69°$ (angle sum of isosceles $\triangle POH$)
b $a = 12$ cm (Pythagoras in $\triangle TOM$), $b = 4\sqrt{13}$ cm (Pythagoras in $\triangle MET$)
c $c = 90°$ (chord bisector), $a = 70°$ (angle sum of $\triangle VOW$), $e = 55°$ (angle sum of isosceles $\triangle VOS$)

4 a $a = 41°$ (straight line)
b $b = 2\sqrt{333}$ (Pythagoras twice)
c $c = \sqrt{69}$ (Pythagoras)

5–11 Proofs

Exercise 5.06

1 a $m = 90°$ (tangent and radius), $n = 45°$ (angle sum of isosceles $\triangle EOF$), $x = 10\sqrt{2}$ cm (Pythagoras)
b $x = 90°$ (tangent and radius), $y = 36°$ (angle sum of $\triangle ABO$), $z = 3\sqrt{41}$ cm (Pythagoras)
c $x = 90°$ (tangent and radius), $y = 68°$ (angle sum of $\triangle PQO$), $z = 20°$ (angle sum of $\triangle RQO$)

2 a $p = 18$ cm (tangents from a common point), $q = 90°$ (tangent and radius), $r = 102°$ (angle sum of $SMOG$)

ISBN 9780170412995

b $x = 67°$ (angle sum of $\triangle POD$), $y = 28$ m (tangents from a common point), $z = 4\sqrt{58}$ (Pythagoras)

3 a $y = 90°$ (tangent and radius), $x = 31$ cm (tangents from common points)

b $x = 28°$ (angle sum of $\triangle YOU$), $y = 14°$ (angle at centre)

c $x = 104°$ (angle at centre and angle sum of $\triangle NOI$)

4–5 Proofs

Exercise 5.07

1 a $a = 85°$ (alternate segment)

b $b = 56°$ (alternate segment), $c = 112°$ (angle at centre), $d = 34°$ (angle sum of isosceles $\triangle QOT$)

c $v = 62°$ (alternate segment), $u = 72°$ (angle sum of isosceles $\triangle BCD$), $w = 52°$ (angle sum of $\triangle ACE$)

2 a $x = 4(\sqrt{29} - 2)$ cm (tangent and secant)

b $x = 1.17$ m (tangent and secant)

c $x = 11.22$ mm (tangent and secant)

d $x = 20\frac{6}{7}$ cm (tangent and secant)

3 $AC = 42$ cm (tangents from common point + tangent and secant)

4–5 Proofs

Exercise 5.08

1 a $x = 105°$ (alternate segment), $v = 150°$ (angles at a point), $y = 35°$ (alternate segment), $z = 70°$ (angle at centre)

b $x = 12$ cm (tangent and secant), $y = 20$ cm (tangent and secant)

2 a $\angle QAR = 50°$ (cyclic quadrilateral), $\angle EAC = 50°$ (vertically opposite), $x = 130°$ (cyclic quadrilateral)

b $z = 70°$ (interior opposite in cyclic quadrilateral), $y = 70°$ (interior opposite in cyclic quadrilateral), $\angle DME = 100°$ (opposite in cyclic quadrilateral), $v = 100°$ (interior opposite in cyclic quadrilateral)

3 a $m = 141°$ (angle sum of $\triangle RWD$ and opposite angles in cyclic quad $SCRD$), $n = 90°$ (opposite angles in cyclic quad $SCRD$)

b $\theta = 20°$ (angle sum of $\triangle OAP$)

4 a $x = \sqrt{230}$ cm (tangent and secant), $y = 8\frac{2}{5}$ cm (intersecting chords)

b $x = 15$ cm (Pythagoras twice and tangents from a common point twice)

5 a $x = 100°$ (angle sum of $LOEU$), $y = 47°$ (angle sum of $\triangle COL$)

b $x = 40°$ (angle sum of $\triangle TAN$)

6 a $a = 60°$ (opposite angles in cyclic quad), $b = 100°$ (opposite angles in cyclic quad)

b $x = 28°$ (angle at centre)

c $x = 20°$ (angles at the circumference)

7 a $\alpha = 54°$ (angle sum of $\triangle MLK$)

b $x = 16°$ (exterior angle of cyclic quad)

c $x = 1\frac{2}{3}$ (intersecting chords)

8 a Proof **b** $QP = 25$ cm

9 a $x = 12°$ (angle sum of $\triangle LNP$ and angles at the circumference)

b $x = 19°$ (angle sum of isosceles $\triangle HOL$, angle at centre and alternate segment)

c $x = 14°$ (angle sums of $\triangle SEM$, $\triangle MIL$ and vertically opposite)

10 Proof

Exercise 5.09

Proofs

Exercise 5.10

Proofs

Chapter review

1 a $a = 128°$, angle at the centre

b $b = 58°$, angle at the centre

c $c = 160°$, opposite angles of parallelogram, exterior angle of triangle, straight line

2 a $\beta = 59°$, angles at the circumference

b $x = 29°$, straight line, angles at the circumference

c $z = 46°$, vertically opposite, angle sum of $\triangle MEP$, angles at the circumference

3 a $a = 90°$, semicircle angle

b $\phi = 11°$, semicircle angle, angle sum of $\triangle CUB$

c $\theta = 41°$, semicircle angle, angle sum of $\triangle ABD$, angles at the circumference

4 a $a = 120°$, opposite angles in cyclic quadrilateral

b $b = 110°$, exterior angle of cyclic quadrilateral

ISBN 9780170412995

c $c = 112°$, angle sum of $\triangle CAN$, vertically opposite, opposite angles in cyclic quadrilateral

5 a $p = 61°$, angles at the centre subtended by equal chords, angle sum of $\triangle BLO$

b $HM = 12$ cm (Pythagoras), $q = 10\frac{2}{3}$ (intersecting chords)

c $\angle CEO = 90°$ (chord bisector), $\angle ECO = 32°$ (angle sum of $\triangle ECO$), $r = 32°$ (symmetry)

6 a $\angle CBO = 90°$ (tangent and radius), $x = 43°$ (angle sum of $\triangle BCO$)

b $d = 14$ cm, common tangents twice

c $k = 21$ cm, common tangents, tangent and radius, Pythagoras

7 a $\theta = 40°$, alternate segment, angle sum of $\triangle SUT$

b $x = 4\sqrt{14}$ m, tangent and secant

8 $FB = 6$ cm, symmetry

$AB = 18$ cm, Pythagoras

$UB = 2\sqrt{7}$ cm, Pythagoras

Distance between centres $= 8 + 2\sqrt{7}$ cm

$BT = 2\sqrt{29}$, Pythagoras

$x = 2\sqrt{29} - 6$ cm

9 a $11x + 25° = 2(7x + 5°)$, angle at centre

$x = 5°$

b $x + 5° + x + 25° = 180°$, opposite angles of cyclic quadrilateral are supplementary

$x = 70°$

$y = 20°$, angle sum of $\triangle CED$

10–12 Proofs

Chapter 6

Exercise 6.01

1 a 1 **b** 120 **c** 360
d 10 **e** 1512

2 a 24 **b** 120 **c** 120
d 1 **e** 5040 **f** 360
g 210 **h** 151 200 **i** 20
j 39 916 800 **k** 3024 **l** 132
m 6 **n** 704 880 **o** 360 360

3 72 different pizzas

4 480

5 17 576 000 possible combinations. With 2 letters and 4 numbers, only 6 760 000 possibilities exist – and would have become depleted over time.

6 3 628 800 **7** 120 **8** 720
9 10 080 **10** 720 **11** 2880
12 a 1000 **b** 8000 **c** 27 000
13 205 920

Exercise 6.02

1 a 15 **b** 56 **c** 210
d 495 **e** 1 **f** 25
g 1287 **h** 1287 **i** 84
j 66

2 a 4 **b** 2 **c** 10
d 6 **e** 12 **f** 16

3 2 598 960 **4** 4845 **5** 10

6 a 45 **b** $\frac{n(n-1)}{2}$

7 10

8 252 **9** 792 **10** 3 838 380

Exercise 6.03

1 a 715 **b** 286 **c** 78

2 3744

3 a 126 **b** 45 **c** 120

4 153 **5** 126 **6** 2100

7 a 48 **b** 210

8 60 **9** 246

10 5005 **11** 560 **12** 1.58%

13 a 20 **b** 21

14 1260

Exercise 6.04

1 54 000 **2** 5.26% **3** 4.83%

4 a 20% **b** 20% **c** 20%

5 14.99% **6** 0.144%

7 a 2.08% **b** 5.56% **c** 18.75%
d 36.67% **e** 36.81%

8 a 6 **b** 15 **c** 63

9 a 5 diagonals, 5 intersections

b 9 diagonals, 13 intersections (count on diagram)

c 14 diagonals, 35 intersections

d 20 diagonals, 49 intersections (count on diagram)

10 a 70 **b** 315 **c** 1386 **d** 25 740

ISBN 9780170412995

Exercise 6.05

1 1, 7, 21, 35, 35, 21, 7, 1; ${}^7C_3 = 35$, ${}^7C_5 = 21$

2 1, 8, 28, 56, 70, 56, 28, 8, 1; ${}^8C_2 = 28$, ${}^8C_6 = 28$

3 120

4 **a** **i** $8 + 12x + 6x^2 + x^3$

ii $1 - 25x + 250x^2 - 1250x^3 + 3125x^4 - 3125x^5$

iii $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$

iv $1 + \frac{12}{a} + \frac{54}{a^2} + \frac{108}{a^3} + \frac{81}{a^4}$

b $1 - 4x + 7x^2 + \ldots$

5 $1 + 6x + 12x^2 + 8x^3$

6 $1 - 12x + 54x^2 - 108x^3 + 81x^4$

7 129 024

8 -8

Exercise 6.06

1 13 people in 12 months – pigeon hole proof

2 27 words with 26 possible first letters – pigeon hole proof

3 22 possible birthdays from 7 possible days – pigeon hole proof

4 25 grades from three options – pigeon hole proof

5 3 different possible remainders from four numbers – pigeon hole proof

6 9

7 **a** 3 **b** 5

8 7

9 The two numbers have to average more than 20, therefore at least one must be > 20.

10 $\frac{100}{101} < 1$. With an average distance of < 1 cm, at least two numbers must be less than 1 cm apart.

11 A 3×3 square contains 9 unit squares. The maximum distance in a unit square (across diagonals) is $\sqrt{2}$ units by Pythagoras' theorem. Therefore 10 points within a total of 9 squares guarantees at least 2 points in the same unit square, and within $\sqrt{2}$ units of each other.

12 The average value of a group of integers will lie between the lowest and highest value, except where all the integers are equal, in which case the average is equal to the value of each of the integers.

13 12 places equates to 6 pairs of opposite places. 7 girls into 6 places guarantees at least one pair of opposites – pigeon hole proof.

Chapter review

1 **a** 24 **b** 840 **c** 3360

2 220

3 **a** 120 **b** 20 **c** 30 **d** 840 **e** 15 120

4 **a** 495 **b** 560 **c** 53 130

5 **a** 6 **b** 9

6 4845

7 **a** 44 **b** 208

8 1 764 000

9 **a** 23.81% **b** 7.14% **c** 1.19%

10 **a** 1, 8, 28, 56, 70, 56, 28, 8, 1 **b** 70

11 560 **12** $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$

13 15 **14** 19

15 **a** 66 **b** 924 **c** 3172

16 10

17 **a** 56 **b** 105 **c** 1680

18 10

19 **a** 495 **b** 220 **c** 66

20 5% **21** 1680

22 **a** 4 **b** 8 **c** 16 **d** 32 **e** 2^n

23 28

Practice examination 2

1 **a** 13.89 **b** 22.64 **c** 11

2 **a** 63.43° **b** 93.2°

3 **a**, **b** = 0

4 6.957

5 **a** $a = 152°$, angle at the centre

b $b = 38°$, angles at the circumference

c $c = 90°$, semicircle angle

6 **a** $\alpha = 69°$, opposite angles in cyclic quadrilateral

b $\beta = 82°$, external angle of cyclic quadrilateral

c $\gamma = 31°$, alternate segment

7 **a** $\alpha = 58°$, angle at the centre subtended by equal chords

b $x = 7\frac{1}{2}$ m

c $\phi = 65°$, corresponding angles in congruent triangles

8 **a** $t = 22°$, tangent and radius and angle sum of $\triangle ONI$

ISBN 9780170412995

b $x = 54°$, tangent and radius and angle sum of $\triangle SEA$

$y = 16$ cm, tangents from a common point

9 $x = 18$ mm

10 **a** 30 **b** 84

11 **a** 210 **b** 120

12 20

13 $\mathbf{c} = \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$

14 21 m

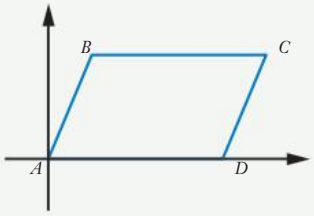
15 $x = 45°$

16 14.29%

17 22

18 Proof

19 Construct the parallelogram $ABCD$. Place the origin at A with the x-axis aligned with AB.



RTP: $\angle BAD = \angle BCD$

$\mathbf{c} = \mathbf{b} + \mathbf{d}$

$$\cos(\angle BAD) = \frac{\mathbf{AD} \bullet \mathbf{AB}}{|\mathbf{AD}||\mathbf{AB}|} = \frac{\mathbf{d} \bullet \mathbf{b}}{|\mathbf{d}||\mathbf{b}|}$$

$$\cos(\angle BCD) = \frac{\mathbf{CB} \bullet \mathbf{CD}}{|\mathbf{CB}||\mathbf{CD}|} = \frac{-\mathbf{d} \bullet (-\mathbf{b})}{|\mathbf{d}||\mathbf{b}|} = \frac{\mathbf{d} \bullet \mathbf{b}}{|\mathbf{d}||\mathbf{b}|}$$

$\Rightarrow \cos(\angle BAD) = \cos(\angle BCD)$

$\Rightarrow \angle BAD = \angle BCD$ QED

20 394

Chapter 7

Exercise 7.01

1 **a**, **c**, **e** and **f** represent matrices.

2 **a** 4 **b** 9 **c** 5 **d** 0

e 2 **f** There is no a_{43}. **g** 3×4

3 **a** row of order 5 **b** 3×4 rectangular

c column of order 4 **d** square of order 3

e 3×4 zero **f** zero row of order 6

4 **a** $\begin{bmatrix} 5 & 6 \\ 8 & -5 \end{bmatrix}$ **b** $\begin{bmatrix} -3 & 10 \\ 17 & -13 \end{bmatrix}$ **c** $\begin{bmatrix} 8 & -1 \\ 6 & -2 \end{bmatrix}$

d $\begin{bmatrix} -1 & -3 \\ 5 & -9 \end{bmatrix}$ **e** $\begin{bmatrix} -3 & -6 \\ -9 & 3 \end{bmatrix}$

f $\begin{bmatrix} -8 & 4 \\ 9 & -8 \end{bmatrix}$ **g** $\begin{bmatrix} -2 & 13 \\ 12 & -4 \end{bmatrix}$

h $\begin{bmatrix} -9 & -2 \\ -1 & -7 \end{bmatrix}$

5 **a** $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & 7 \\ 9 & 2 & 3 \\ -4 & 2 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & 8 & 9 \\ 0 & 5 & 10 \\ 5 & 6 & 1 \\ 7 & 5 & -2 \end{bmatrix}$

c $\begin{bmatrix} 0 & 1 & 5 \\ 2 & -1 & 5 \\ 4 & 8 & 6 \\ 3 & -3 & -2 \end{bmatrix}$ **d** $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & 7 \\ 9 & 2 & 3 \\ -4 & 2 & -4 \end{bmatrix}$

e $\begin{bmatrix} 1 & 4 & 3 \\ -2 & 3 & -1 \\ -4 & -2 & -4 \\ 4 & 3 & 2 \end{bmatrix}$ **f** $\begin{bmatrix} -3 & 2 & 7 \\ 0 & -1 & -2 \\ -5 & 6 & 3 \\ 7 & -5 & 2 \end{bmatrix}$

g $\begin{bmatrix} 3 & -2 & -7 \\ 0 & 1 & 2 \\ 5 & -6 & -3 \\ -7 & 5 & -2 \end{bmatrix}$ **h** $\begin{bmatrix} -2 & -9 & -11 \\ 2 & -5 & -3 \\ 4 & -4 & 2 \\ -11 & -3 & -2 \end{bmatrix}$

6 **a** $\mathbf{M} = \begin{bmatrix} 3 & 2 & 4 & 1 & 5 & 0 & 3 & 6 & 1 \\ 1 & 8 & 1 & 7 & 1 & 8 & 0 & 5 & 8 \\ 4 & 0 & 5 & 6 & 4 & 5 & 9 & 1 & 7 \\ 7 & 9 & 2 & 0 & 1 & 1 & 10 & 9 & 1 \end{bmatrix}$

b 0 **c** 1 **d** 0 **e** 4×9

7 **a** Because there is no entry in one cell.

b Place a very high cost in that square to indicate secondary sourcing – say 1000.

c $\mathbf{T} = \begin{bmatrix} 4.20 & 4.30 & 1000 & 4.40 \\ 9.10 & 8.80 & 8.70 & 9.00 \\ 6.20 & 6.40 & 6.50 & 6.10 \\ 3.80 & 3.90 & 3.60 & 3.70 \end{bmatrix}$

d 9.00

8 **a** 0.7324 USD

b $\mathbf{E} = \begin{bmatrix} 1 & 0.7324 & 0.5874 & 86.24 \\ 1.3654 & 1 & 0.8020 & 117.75 \\ 1.7024 & 1.2469 & 1 & 146.82 \\ 0.011\,59 & 0.008\,493 & 0.006\,811 & 1 \end{bmatrix}$

c 0.8020 **d** 1.2469 **e** $e_{ij} = \frac{1}{e_{ji}}$

9 **a** $\mathbf{B} = \begin{bmatrix} 4 & 9 & 20 & 40 \\ 1 & 2 & 4 & 9 \\ 0.2 & 0.3 & 0.5 & 1 \end{bmatrix}$

ISBN 9780170412995

b 4 **c** 0.3

d It is the amount of epoxy needed to make a shower base.

10 a $\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 6 & 4 & 0 \end{bmatrix}$

b Mosquito larvae do not eat the other animals.

c Water beetles do not eat each other.

11 $\mathbf{Q} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

12 $\mathbf{P} = [3 \ \ 2 \ \ 1 \ \ 0 \ \ -1 \ \ -2]$

13 a $x = -5$ **b** $x = -1$ **c** $x = 2$

14 $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix}$

15 $\mathbf{M} = \begin{bmatrix} -8 & 5 \\ -8 & -5 \end{bmatrix}$

16 a $\begin{bmatrix} 300 & 400 & 200 & 150 \\ 400 & 220 & 300 & 200 \\ 250 & 150 & 400 & 150 \end{bmatrix}$

b $\begin{bmatrix} 30 & 70 & 20 & 20 \\ 20 & 30 & 40 & 30 \\ 40 & 10 & 50 & 10 \end{bmatrix}$

c $\begin{bmatrix} 50 & 100 & 10 & 15 \\ 15 & 50 & 30 & 10 \\ 30 & 25 & 20 & 40 \end{bmatrix}$

d $\mathbf{A} - \mathbf{B} + \mathbf{C} = \begin{bmatrix} 320 & 430 & 190 & 145 \\ 395 & 240 & 290 & 180 \\ 240 & 165 & 370 & 180 \end{bmatrix}$

17 a $\mathbf{F} + \mathbf{M} + \mathbf{L}$ **b** $\begin{bmatrix} 10 & 1 & 4 \\ 6 & 0 & 9 \\ 4 & 3 & 8 \\ 8 & 2 & 5 \\ 4 & 2 & 9 \\ 9 & 0 & 6 \end{bmatrix}$ **c** 1st team

Exercise 7.02

1 a $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ **b** $\begin{bmatrix} -7 & 4 \end{bmatrix}$

c $\begin{bmatrix} 6 & -2 & 4 \\ -9 & 3 & -6 \\ 12 & -4 & 8 \end{bmatrix}$ **d** $\begin{bmatrix} -2 & 1 \\ 6 & 3 \end{bmatrix}$

e $\begin{bmatrix} -17 & 9 \\ 8 & -16 \end{bmatrix}$ **f** $\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$

g $\begin{bmatrix} 6 & -15 \\ -2 & -14 \end{bmatrix}$ **h** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i $\begin{bmatrix} -12 & 6 \\ -8 & 4 \end{bmatrix}$ **j** $\begin{bmatrix} -1 & 5 \\ -4 & 7\frac{1}{2} \end{bmatrix}$

k $\begin{bmatrix} 0 & 3 \\ -3\frac{2}{3} & -2 \end{bmatrix}$ **l** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2 a $\begin{bmatrix} 4 & 8 & -11 \end{bmatrix}$ **b** $\begin{bmatrix} -7 \\ -2 \\ -10 \end{bmatrix}$

c $\begin{bmatrix} -4 & 14 & -4 \\ 2 & 2 & 8 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 13 \\ -2 & 14 \end{bmatrix}$

e $\begin{bmatrix} 18 & 10 & 4 \\ -17 & 0 & -11 \\ 6 & 9 & -3 \end{bmatrix}$ **f** $\begin{bmatrix} 16 & 22 \\ 10 & -4 \\ 2 & -2 \end{bmatrix}$

g $\begin{bmatrix} 12 & -4 & 12 & 12 \\ 17 & 8 & 7 & 11 \\ 3 & 6 & 3 & 27 \end{bmatrix}$ **h** $\begin{bmatrix} 13 & 0 \\ 13 & -2 \\ -19 & 22 \\ -9 & 15 \end{bmatrix}$

3 $\begin{bmatrix} 33 & 21 \\ 35 & 40 \end{bmatrix}$ and $\begin{bmatrix} 33 & 21 \\ 35 & 40 \end{bmatrix}$, the same answer in both orders.

4 $\begin{bmatrix} -11 & 7 & -11 \\ 7 & 13 & 7 \\ -11 & 7 & -11 \end{bmatrix}$ and $\begin{bmatrix} -11 & 7 & -11 \\ 7 & 13 & 7 \\ -11 & 7 & -11 \end{bmatrix}$, the same answer both ways.

5 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The product is a zero matrix. This doesn't happen for non-zero numbers.

6 They are not conformable. The number of columns in the first is not the same as the number of rows in the second. In the reverse order, they are conformable so can be multiplied.

ISBN 9780170412995

7 **a** $\begin{bmatrix} 6 & 0 \\ -3 & -6 \end{bmatrix}$ **b** $\begin{bmatrix} 6 & 0 \\ -3 & -6 \end{bmatrix}$

c $\begin{bmatrix} 15 & 30 \\ 45 & -15 \end{bmatrix}$ **d** $\begin{bmatrix} -24 & 16 \\ 32 & -40 \end{bmatrix}$

e $\begin{bmatrix} -3 & 2 \\ 4 & -5 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

g $\begin{bmatrix} 1.2 & 2.4 \\ 3.6 & -1.2 \end{bmatrix}$ **h** $\begin{bmatrix} 7 & -6 \\ -14 & -7 \end{bmatrix}$

i $\begin{bmatrix} 5 & 18 \\ 29 & -5 \end{bmatrix}$ **j** $\begin{bmatrix} 2 & -10 \\ 3 & -6 \end{bmatrix}$

k $\begin{bmatrix} 25 & 14 \\ 14 & -9 \end{bmatrix}$ **l** $\begin{bmatrix} 2 & 20 \\ 30 & -34 \end{bmatrix}$

8 **a** $\begin{bmatrix} 15 & 0 & -5 & 10 \\ 25 & -20 & 15 & 5 \\ -5 & 10 & 0 & 20 \end{bmatrix}$

b $\begin{bmatrix} -18 & 15 & -12 & 9 \\ 3 & -24 & 27 & -6 \\ -12 & -21 & 6 & -15 \end{bmatrix}$

c $\begin{bmatrix} 4 & 16 & 12 & -20 \\ -12 & 8 & -4 & -24 \\ 0 & 20 & -8 & 12 \end{bmatrix}$

d $\begin{bmatrix} -15 & 0 & 5 & -10 \\ -25 & 20 & -15 & -5 \\ 5 & -10 & 0 & -20 \end{bmatrix}$

e $\begin{bmatrix} 3 & -2\frac{1}{2} & 2 & -1\frac{1}{2} \\ -\frac{1}{2} & 4 & -4\frac{1}{2} & 1 \\ 2 & 3\frac{1}{2} & -1 & 2\frac{1}{2} \end{bmatrix}$

f $\begin{bmatrix} 0.1 & 0.4 & 0.3 & -0.5 \\ -0.3 & 0.2 & -0.1 & -0.6 \\ 0 & 0.5 & -0.2 & 0.3 \end{bmatrix}$

g $\begin{bmatrix} 2 & -1\frac{2}{3} & 1\frac{1}{3} & -1 \\ -\frac{1}{3} & 2\frac{2}{3} & -3 & \frac{2}{3} \\ 1\frac{1}{3} & 2\frac{1}{3} & -\frac{2}{3} & 1\frac{2}{3} \end{bmatrix}$

h $\begin{bmatrix} -6 & 15 & -16 & 17 \\ 23 & -40 & 39 & -2 \\ -16 & -13 & 6 & 1 \end{bmatrix}$

i $\begin{bmatrix} 9 & -20 & 21 & -22 \\ -29 & 52 & -51 & 3 \\ 21 & 18 & -8 & 0 \end{bmatrix}$

j $\begin{bmatrix} 35 & -29 & 15 & -6 \\ 8 & 30 & -38 & 18 \\ 18 & 34 & -8 & 30 \end{bmatrix}$

k $\begin{bmatrix} -43 & 38 & -15 & 2 \\ -15 & -32 & 43 & -27 \\ -21 & -38 & 8 & -36 \end{bmatrix}$

l $\begin{bmatrix} -3 & -39 & -1 & 11 \\ -10 & 32 & -36 & 37 \\ 17 & -19 & 6 & -23 \end{bmatrix}$

9 Any 2×2 matrices of the form $\begin{bmatrix} a & b \\ ac & bc \end{bmatrix}$ and $\begin{bmatrix} b & -bc \\ -a & ac \end{bmatrix}$.

10 **a** $\begin{bmatrix} 12 & 10 & 8 \\ 15 & 8 & 4 \\ 10 & 10 & 10 \\ 12 & 12 & 4 \\ 16 & 6 & 12 \end{bmatrix}$ **b** $\begin{bmatrix} 117.80 \\ 155 \\ 190 \end{bmatrix}$

c $\begin{bmatrix} 4483.6 \\ 3767 \\ 4628 \\ 4033.6 \\ 5094.8 \end{bmatrix}$ B
L
Ch
C
GC

11 **a** $\begin{bmatrix} 7160 \\ 9640 \\ 7140 \\ 7200 \\ 11\,580 \\ 9180 \end{bmatrix}$ **b** $\begin{bmatrix} 2506 \\ 3374 \\ 2499 \\ 2520 \\ 4053 \\ 3213 \end{bmatrix}$ **c** $\begin{bmatrix} 9666 \\ 13\,014 \\ 9639 \\ 9720 \\ 15\,633 \\ 12\,393 \end{bmatrix}$

d $\begin{bmatrix} 2 & 1 & 0 & 3 & 2 & 1 \\ 4 & 2 & 2 & 4 & 1 & 2 \\ 3 & 1 & 5 & 3 & 3 & 2 \end{bmatrix}$

e $\begin{bmatrix} 77\,900 \\ 120\,940 \\ 141\,520 \end{bmatrix}$

12 **a** $\mathrm{P} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

ISBN 9780170412995

b $\mathbf{FP} = \begin{bmatrix} 12 \\ 9 \\ 4 \\ 7 \\ 3 \\ 9 \end{bmatrix}$, $\mathbf{MP} = \begin{bmatrix} 10 \\ 3 \\ 5 \\ 9 \\ 7 \\ 9 \end{bmatrix}$, $\mathbf{LP} = \begin{bmatrix} 9 \\ 6 \\ 6 \\ 10 \\ 4 \\ 9 \end{bmatrix}$

13 a $\begin{bmatrix} 4.08 & 8.33 & 14.28 \\ 7.14 & 13.43 & 20.06 \\ 6.46 & 14.28 & 18.70 \\ 10.03 & 20.06 & 24.48 \end{bmatrix}$

b $\begin{bmatrix} 4.50 & 9.15 & 15.70 \\ 7.85 & 14.75 & 22.05 \\ 7.10 & 15.70 & 20.55 \\ 11.05 & 22.05 & 26.95 \end{bmatrix}$

Customer prices are rounded to the nearest 5 cents.

Exercise 7.03

1 a $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$

b $\begin{bmatrix} 1 & 3 & -4 & 0 & 6 \\ 2 & 4 & -3 & 1 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} -3 & 4 \end{bmatrix}$

d $\begin{bmatrix} 2 & 1 & -3 \\ -5 & 0 & 2 \\ 4 & 0 & 4 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & -2 \\ 6 & -3 \\ 0 & 7 \end{bmatrix}$

f $\begin{bmatrix} 3 & -3 & -6 \\ 0 & 4 & 6 \\ -2 & 1 & 6 \\ 1 & 0 & 2 \end{bmatrix}$

g $\begin{bmatrix} 2 & -30 & 5 & 20 \\ 5 & 15 & 2 & 0 \\ 7 & 2 & -11 & 0 \\ 0 & -6 & 6 & 5 \end{bmatrix}$ **h** $\begin{bmatrix} 4 & 4 & 2 & 6 \\ -4 & -3 & 0 & 0 \\ 6 & 0 & 7 & 5 \\ -5 & 7 & 8 & 6 \end{bmatrix}$

2 a 8, non-singular **b** 5, non-singular
c 18, non-singular **d** 0, singular
e −3, non-singular **f** 0, singular
g −20, non-singular **h** −7, non-singular

3 a $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

c $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ **d** $-\frac{1}{34}\begin{bmatrix} 5 & -2 \\ -7 & -4 \end{bmatrix}$

e $\begin{bmatrix} -4 & 0 \\ -2 & -8 \end{bmatrix}$ **f** $\begin{bmatrix} -1\frac{1}{2} & \frac{1}{2} \\ -2 & \frac{1}{2} \end{bmatrix}$

g singular **h** $\begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$

4 a 6, non-singular **b** 0, singular
c −12, non-singular **d** −9, non-singular
e 27, non-singular **f** 0, singular
g 11, non-singular **h** 4, non-singular

5 a $\begin{bmatrix} 3 & 5 & -4 \\ -3 & -6 & 5 \\ -1 & -1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 9 & -6 & -1 \\ -6 & 4 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} \frac{8}{15} & -\frac{4}{15} & -\frac{1}{5} \\ -\frac{3}{5} & -\frac{1}{5} & \frac{3}{5} \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 0 & 2 & -1 \\ -5 & 8 & 11 & -6 \\ 3 & -5 & -7 & 4 \\ 1 & -2 & -6 & 4 \end{bmatrix}$

e $\begin{bmatrix} 1 & 0 & 14 & -8 \\ 0 & -0.5 & 0 & -0.5 \\ -1 & -1.5 & -11.5 & 6 \\ 0 & -2 & 4 & -3 \end{bmatrix}$

f $\begin{bmatrix} 0 & -0.5 & 0 & 0 & 0.5 \\ 0.1 & 0.6 & 1.3 & -0.3 & 0.4 \\ -0.1 & -0.1 & -0.8 & 0.3 & -0.4 \\ 0.3 & 2.3 & 2.4 & -0.9 & 0.2 \\ 0 & -2 & -2.5 & 1 & -0.5 \end{bmatrix}$

g $\begin{bmatrix} 3\frac{4}{9} & \frac{8}{9} & \frac{1}{3} \\ 2\frac{7}{9} & \frac{5}{9} & \frac{1}{3} \\ -1\frac{2}{3} & -\frac{1}{3} & 0 \end{bmatrix}$

h $\begin{bmatrix} -6\frac{1}{2} & -11\frac{1}{2} & -8 & 1 \\ -2\frac{2}{3} & -4\frac{2}{3} & -3\frac{1}{3} & \frac{1}{3} \\ -3 & -6 & -5 & 0 \\ 5 & 8 & 6 & 0 \end{bmatrix}$

6, 7 Proofs

8 $\mathbf{A}^2 = \mathbf{I}$, so $\mathbf{A}$ is its own inverse. There are many others.

ISBN 9780170412995

Exercise 7.04

1 **a** $\begin{bmatrix} -15 \\ 17 \end{bmatrix}$ **b** $\begin{bmatrix} -11 \\ 10\frac{1}{2} \end{bmatrix}$

c No solution **d** $[11 \quad -1]$

e $\begin{bmatrix} -8 \\ -3 \end{bmatrix}$

2 **a** $\begin{bmatrix} -31 \\ -12 \\ 21 \end{bmatrix}$ **b** $\begin{bmatrix} -41\frac{1}{2} \\ 56\frac{1}{2} \\ 67 \end{bmatrix}$ **c** $\begin{bmatrix} 8\frac{1}{3} \\ -3\frac{2}{3} \\ 14\frac{1}{3} \end{bmatrix}$

d $\begin{bmatrix} -860 \\ 538 \\ -279 \\ 111 \end{bmatrix}$ **e** $\begin{bmatrix} 8\frac{6}{7} \\ 62\frac{3}{7} \\ -37\frac{2}{7} \\ -47\frac{4}{7} \end{bmatrix}$

3 $\begin{bmatrix} 19 & 37 \\ -13 & -26 \end{bmatrix}$ **4** $\begin{bmatrix} 0.5 & -1.5 \\ 0 & -1 \end{bmatrix}$

5 $\begin{bmatrix} -16 \\ -12 \end{bmatrix}$

6 **a** $\begin{bmatrix} \frac{3}{4} \\ -2\frac{1}{4} \end{bmatrix}$ **b** $\begin{bmatrix} 0.8 \\ 0.3 \end{bmatrix}$

Chapter review

1 **a** 3×5

b $m_{23} = -3, m_{34} = 7, m_{42}$ does not exist

2 **a** $\mathbf{P} + \mathbf{R} = \begin{bmatrix} 2 & 0 & 1 \\ 7 & 5 & -4 \end{bmatrix}$

b $\mathbf{Q} - \mathbf{R} = \begin{bmatrix} -3 & 0 & 2 \\ 2 & 4 & 2 \end{bmatrix}$

3 **a** $\begin{bmatrix} 12 & 11 \\ 5 & 33 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & -5 & -1 \\ -20 & 3 & 3 \end{bmatrix}$

c not conformable, no product

4 **a** $\begin{bmatrix} -7 & -8 & 2 \\ -51 & 20 & -16 \\ 45 & -16 & 13 \end{bmatrix}$ **b** $\begin{bmatrix} 34 & 3 & -25 \\ 18 & -4 & 5 \\ -16 & 3 & -4 \end{bmatrix}$

5 **a** $\begin{bmatrix} 16 & 8 \\ -12 & 20 \\ 0 & -8 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & -1 \\ -4 & 1 \\ 2 & -4 \end{bmatrix}$

c $\begin{bmatrix} -17 & -1 \\ 18 & -13 \\ -6 & 16 \end{bmatrix}$

6 $\begin{bmatrix} 3 & 4 & -4 \\ -2 & 1 & 5 \end{bmatrix}$

7 13

8 **a** $\begin{bmatrix} 7 & -5 \\ 3 & -2 \end{bmatrix}$

b Singular, no inverse **c** $\begin{bmatrix} 2 & 1 \\ 0.6 & 0.4 \end{bmatrix}$

9 **a** Singular, no inverse

b $\begin{bmatrix} -\frac{14}{9} & -\frac{13}{9} & -\frac{8}{3} & 0 & \frac{2}{3} \\ -\frac{239}{270} & -\frac{83}{135} & -\frac{119}{90} & -\frac{1}{15} & \frac{7}{18} \\ -\frac{143}{135} & -\frac{127}{135} & -\frac{68}{45} & \frac{1}{15} & \frac{4}{9} \\ \frac{217}{135} & \frac{188}{135} & \frac{112}{45} & \frac{1}{15} & -\frac{5}{9} \\ -\frac{226}{135} & -\frac{179}{135} & -\frac{121}{45} & \frac{2}{15} & \frac{5}{9} \end{bmatrix}$

10 **a** $\begin{bmatrix} 6 & 2 \\ 12 & 4 \end{bmatrix}$ is singular, no solution

b $\mathbf{X} = \begin{bmatrix} -16 \\ 44 \end{bmatrix}$ **c** $\mathbf{X} = [-2.5 \quad 0.5]$

11 $\mathbf{X} = \begin{bmatrix} 33 \\ -21.2 \\ -23.8 \\ -6.2 \end{bmatrix}$ **12** **a** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 3 & 0 & 5 & 1 & 3 & 0 \end{bmatrix}$

b [25 000 70 000 4000 500 100 6]

c [75 000 210 000 12 000 1500 300 18]

d [4618 21500 730 106 18 0] and [13 854 64 500 2190 318 54 0]

13 **a** They do not truck parcels within a city.

b $\mathbf{H} = \begin{bmatrix} 0 & 4 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 & 4 \\ 4 & 4 & 0 & 4 & 4 \\ 4 & 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 4 & 0 \end{bmatrix}$

ISBN 9780170412995

$$\mathbf{T} = \begin{bmatrix} 0 & 14.80 & 13.80 & 10.40 & 9.00 \\ 14.80 & 0 & 11.20 & 20.40 & 19.40 \\ 13.80 & 11.20 & 0 & 20.20 & 18.40 \\ 10.40 & 20.40 & 20.20 & 0 & 15.00 \\ 9.00 & 19.40 & 18.40 & 15 & 0 \end{bmatrix}$$

c $1.5\mathbf{C} + \mathbf{H}$ **d** $3\mathbf{C} + 2\mathbf{H}$

14 a [12.5 13.2 15 16.4 17.5 18]

b $\begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 5 \\ 2 & 3 & 0 \\ 3 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

c George's: $147.40, Frank's: $201.30, Maria's: $168.40

15 $\begin{bmatrix} 8 \\ 12 \end{bmatrix}$

Chapter 8

Exercise 8.01

1 a $\pm 2i$ **b** $\pm 3i$ **c** $\pm 2\sqrt{3}i$
d $\pm\sqrt{7}i$ **e** $\pm 2\sqrt{2}i$ **f** $\pm\frac{1}{3}i$
g $\pm\frac{1}{\sqrt{3}}i = \pm\frac{\sqrt{3}}{3}i$ **h** $\pm\frac{3\sqrt{2}}{2}i$

2 a $3 + 4i$ or $3 - 4i$ **b** $-2 + \sqrt{3}i$ or $-2 - \sqrt{3}i$
c $\frac{2+2\sqrt{5}i}{3}$ or $\frac{2-2\sqrt{5}i}{3}$
d $\frac{-5+\sqrt{3}i}{2}$ or $\frac{-5-\sqrt{3}i}{2}$
e $\frac{1+3\sqrt{3}i}{2}$ or $\frac{1-3\sqrt{3}i}{2}$
f $\frac{3+\sqrt{7}i}{2}$ or $\frac{3-\sqrt{7}i}{2}$
g $\frac{-1+\sqrt{26}i}{3}$ or $\frac{-1-\sqrt{26}i}{3}$
h $\frac{5+\sqrt{103}i}{8}$ or $\frac{5-\sqrt{103}i}{8}$

3 a $\text{Re}(z) = 3, \text{Im}(z) = -2$
b $\text{Re}(z) = 7, \text{Im}(z) = 1$
c $\text{Re}(z) = -5, \text{Im}(z) = 6$
d $\text{Re}(z) = -4, \text{Im}(z) = -7$
e $\text{Re}(z) = \sqrt{5}, \text{Im}(z) = -\pi$
f $\text{Re}(z) = -2\sqrt{6}, \text{Im}(z) = 3$
g $\text{Re}(z) = 4\pi, \text{Im}(z) = \sqrt{5}$
h $\text{Re}(z) = -\pi, \text{Im}(z) = 2$

4 a $9i$ **b** $7i$ **c** -4 **d** $-8\sqrt{3}$
e $\frac{\sqrt{3}}{5}$ **f** $-10\sqrt{3}$ **g** $\frac{2(a^2-3b)}{5b^2}$
h $\sqrt{2} - \sqrt{6}c$

5 a $5i$ **b** $\sqrt{5}i$ **c** $2\sqrt{2}i$ **d** $2\sqrt{10}i$
e $\frac{4}{5}i$ **f** $\frac{1}{7}\sqrt{11}i$ **g** $\frac{3}{\sqrt{19}}i$
h $\frac{3\sqrt{3}}{2\sqrt{2}}i$

6 a i **b** -1 **c** 1 **d** i
e -1 **f** $-i$ **g** -1 **h** i

7 a -20 **b** $-6\sqrt{2}$ **c** $-15\sqrt{15}$ **d** -16
e $-12i$ **f** $-18\sqrt{2}i$ **g** $-10\sqrt{3}i$ **h** 36

8 a -4 **b** 81 **c** $-10i$ **d** $-3i$
e $6(1 + i)$ **f** -3 **g** $-24i$ **h** $-5i$

9 a Real, different, irrational
b Complex, different
c Complex, different
d Real, different, rational

10 a $m < 1\frac{1}{24}$ **b** $m = 1\frac{1}{24}$ **c** $m > 1\frac{1}{24}$

11 a $p < -3\frac{5}{9}$ or $p > 0$ **b** $p = -3\frac{5}{9}$ or $p = 0$
c $-3\frac{5}{9} < p < 0$

12 $3x^2 + 10x + 32 = 0$

Exercise 8.02

1 a $7 + 11i$ **b** $15 - 8i$ **c** $5 + i$ **d** 13
e $1 + i$ **f** $12 + 4i$ **g** $-3 + 4i$ **h** $-5 + 9i$
i $18 + 9i$ **j** $-10 + 3i$

2 a $9 + 3i$ **b** $-6 + 12i$ **c** $11 + 27i$
d $-1 - 33i$ **e** $33 - 3i$ **f** $8 + 54i$

3 a $x = 3, y = 2$ **b** $x = \frac{7}{3}, y = -2$
c $x = 2, y = \frac{5}{9}$ **d** $x = 5, y = -6$
e $x = 6, y = 4$ **f** $x = -1, y = -2$
g $x = -2, y = -11$ **h** $x = 2, y = \frac{1}{2}$

4 a $x = 2, y = -3$ **b** $x = 5\frac{1}{2}, y = -\frac{4}{7}$
c $x = 0, y = -2\frac{1}{2}$ **d** $x = -\frac{5}{3}, y = 16\frac{1}{3}$

ISBN 9780170412995

5 **a** None **b** $3i, \sqrt{-7}$ **c** $-7, 0$ **d** $-\sqrt{5}, \pi$
e $-7, -\sqrt{5}, 0, \pi, \frac{5}{8}$ **f** All of them
g $-7, 0, \frac{5}{8}$

6 **a** $-5 + 2i$ **b** $2 + 6i$ **c** $1 - 6i$ **d** $2 + 3i$
e $3 - 3i$ **f** $4 - 8i$ **g** $4 + 4i$ **h** $10 + 3i$

Exercise 8.03

1 **a** $-13 + 13i$ **b** $14 + 13i$ **c** $23 - i$
d $5 + 10i$ **e** $-7 - 17i$ **f** $-5 + 10i$
g $17 + 19i$ **h** $19 - 9i$
i $-17i$ **j** $9 + 20i$

2 **a** 0 **b** $11 - 10i$
c $11 + 3i$ **d** $3 - 11i$
e $-11 - 27i$ **f** $3\sqrt{2} + 4 + 2(\sqrt{6} - \sqrt{3})i$
g $5\sqrt{3} + 2 + (2\sqrt{5} - \sqrt{15})i$
h $4\sqrt{2} + \sqrt{6} + (2 - 4\sqrt{3})i$
i $-1 - 17\sqrt{3}i$ **j** $26 + 13i$

3 **a** $-9 - 46i$ **b** $476 - 480i$ **c** 16
d $-119 + 120i$ **e** $278 - 29i$ **f** -64
g $-122 - 597i$ **h** $-177 - 44i$ **i** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

4 **a** 13 **b** 26 **c** $77 + 36i$
d $-5 - 12i$ **e** 90 **f** $18 - 26i$
g $46 + 9i$ **h** $-10 - 5i$ **i** $142 + 65i$

5 **a** $47 - i$ **b** $-16 + 30i$ **c** $-524 + 7i$
d $282 - 6i$ **e** $45 - 28i$ **f** $-143 - 24i$

6 **a** $a = -5, b = 21$ **b** $a = 3, b = -2$
c $a = -0.04, b = -0.72$ **d** $a = \frac{1}{2}, b = \frac{3}{2}$
e $a = 2\sqrt{2}, b = \sqrt{2}$ or $a = -2\sqrt{2}, b = -\sqrt{2}$
f $a = 1, b = \sqrt{3}$ or $a = -1, b = -\sqrt{3}$
g $a = \frac{3\sqrt{2}}{2}, b = \frac{\sqrt{2}}{2}$ or $a = -\frac{3\sqrt{2}}{2}, b = -\frac{\sqrt{2}}{2}$
h $a = 2, b = -1$ **i** $a = 0, b = 1$
j $a = \frac{11}{17}, b = -\frac{7}{17}$

7, 8 Proofs

Exercise 8.04

1 **a** $3 - 4i$ **b** $5 + 2i$ **c** 6 **d** $-8i$
e $\frac{1}{2} - \frac{3}{4}i$ **f** $1 + i$ **g** $7i$
h $-3 + 8i$ **i** $-2 - 3i$ **j** $2 - \frac{8}{3}i$
k $-2i$ **l** $5 + i$

2 **a** $\frac{3}{13} - \frac{2}{13}i$ **b** $\frac{2}{13} + \frac{3}{13}i$
c $\frac{3}{25} + \frac{4}{25}i$ **d** $\frac{4}{25} - \frac{3}{25}i$
e $-\frac{7}{34} + \frac{11}{34}i$ **f** $-\frac{3}{5} + \frac{4}{5}i$
g $\frac{5}{13} - \frac{1}{13}i$ **h** $\frac{1}{2}$
i $\frac{1}{26} - \frac{3}{52}i$ **j** $\frac{9}{5} + \frac{7}{5}i$
k $-\frac{1}{85} + \frac{18}{85}i$ **l** $\frac{14}{25} + \frac{73}{25}i$

3 **a** $(a + bi)(a - bi)$ **b** $(x + 4yi)(x - 4yi)$
c $(3m + 5ni)(3m - 5ni)$ **d** $(x + \sqrt{3}i)(x - \sqrt{3}i)$
e $(\sqrt{2}x + i)(\sqrt{2}x - i)$
f $(\sqrt{3}p + \sqrt{7}qi)(\sqrt{3}p - \sqrt{7}qi)$
g $(2\sqrt{2}x + 3\sqrt{3}yi)(2\sqrt{2}x - 3\sqrt{3}yi)$
h $3(2g + 5hi)(2g - 5hi)$

4 **a** $-8 - 27i$ **b** $-5 - 12i$ **c** $-415 - 234i$
d $-48 - 162i$ **e** $-32 - 126i$ **f** $-18i$

5 **a** $\frac{1}{2}$ **b** $\frac{10}{17}$ **c** $\frac{1}{26}$

6 **a** $\frac{41}{65} - \frac{23}{65}i$ **b** $-\frac{5}{13}$ **c** $\frac{8}{5} - 2i$
d $\frac{254}{145} + \frac{258}{145}i$

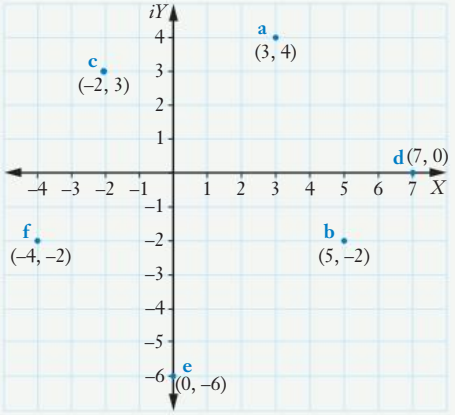
7 **a** $\frac{12}{5} - \frac{4}{5}i$ **b** $\frac{3}{2} - \frac{1}{2}i$ **c** $-\frac{1}{5} - \frac{3}{5}i$

8 2

9–14 Proofs

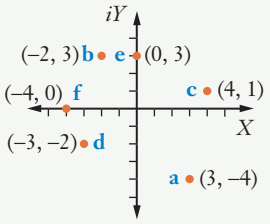
Exercise 8.05

1



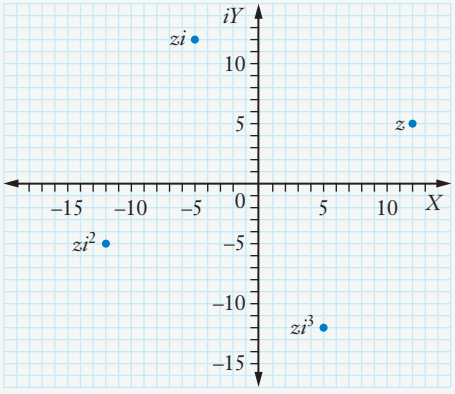
ISBN 9780170412995

2



3 **a** $zi = -5 + 12i, zi^2 = -12 - 5i, zi^3 = 5 - 12i$

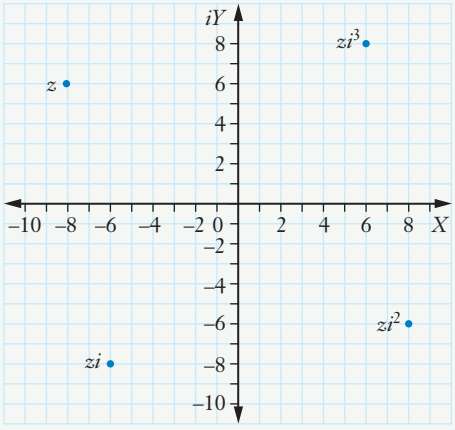
b



c z, zi, zi^2 and zi^3 are each rotated 90° from the previous

4 **a** $zi = -6 - 8i, zi^2 = 8 - 6i, zi^3 = 6 + 8i$

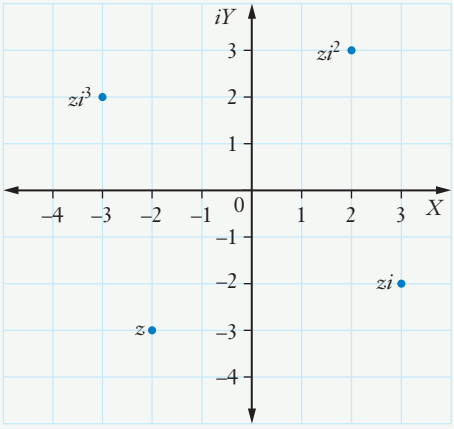
b



c z, zi, zi^2 and zi^3 are each rotated 90° from the previous

5 **a** $zi = 3 - 2i, zi^2 = 2 + 3i, zi^3 = -3 + 2i$

b



c z, zi, zi^2 and zi^3 are each rotated +90° from the previous

6 **a** z, zi, zi^2 and zi^3 are each rotated +90° from the previous

b Put $z = a + bi$, find zi, zi^2 and zi^3. Then find the angles at the origin as per Example 12.

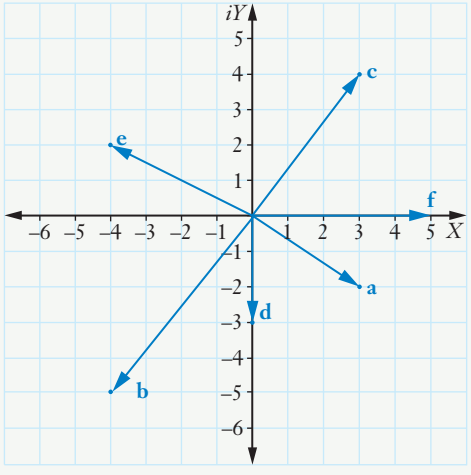
z, zi, zi^2 and zi^3 are each rotated +90° from the previous

7 They are reflections of each other in the X (real) axis.

8 They are rotations of each other by 180° around the origin.

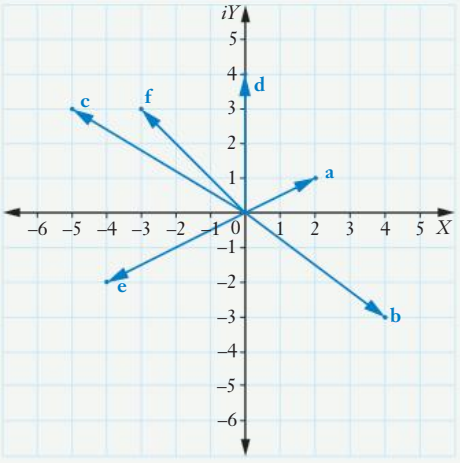
Exercise 8.06

1



ISBN 9780170412995

2



3 **a** $10, \frac{\pi}{3} = 60°$

b $\sqrt{113} \approx 10.6, -0.8 \approx -48.8°$

c $6\sqrt{2} \approx 8.5, -\frac{3\pi}{4} = -135°$

d $8, \frac{3\pi}{4} = 135°$

e $14, -\frac{\pi}{6} = -30°$

4 **a** $8\sqrt{2} \approx 11.3, -\frac{\pi}{4} = -45°$ **b** $5, 0.9 \approx 53.1°$

c $13, -2.0 \approx -112.6°$ **d** $4, -\frac{5\pi}{6} = -150°$

e $4\sqrt{2} \approx 5.7, \frac{3\pi}{4} = 135°$

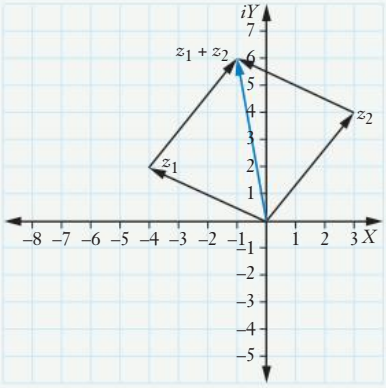
5 30°

6 Arguments are 150° and 105° ⇒ angle = 45°

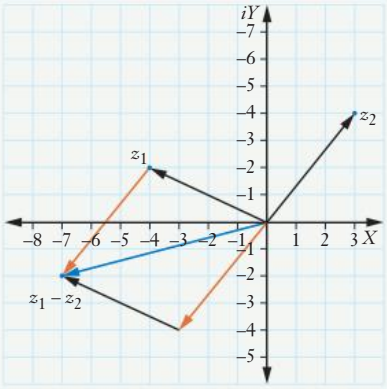
7, 8 Proofs

Exercise 8.07

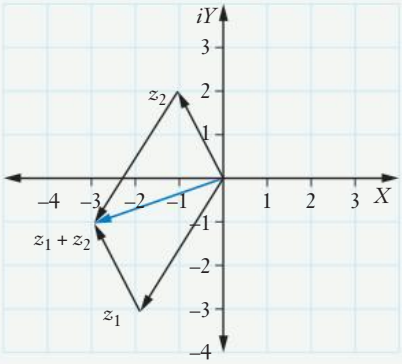
1 **a**



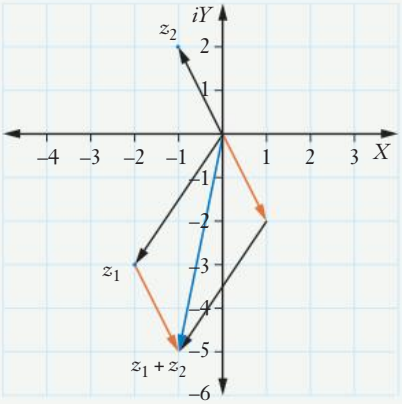
b



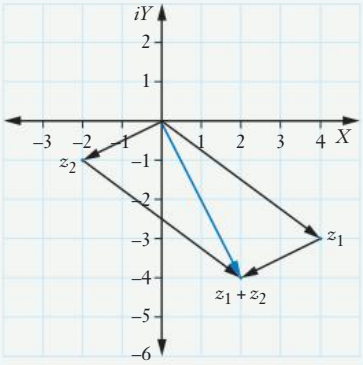
2 **a**



b

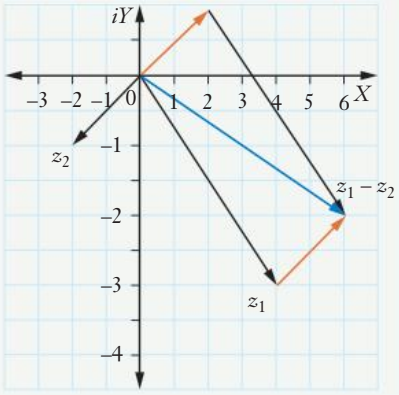


3 **a**

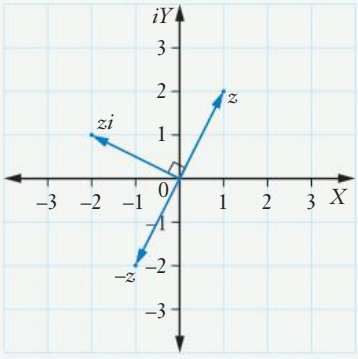


ISBN 9780170412995

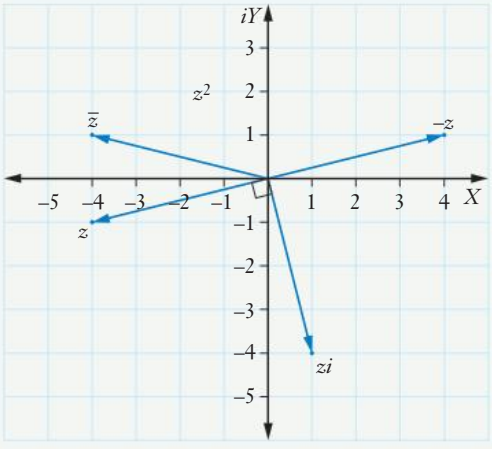
b



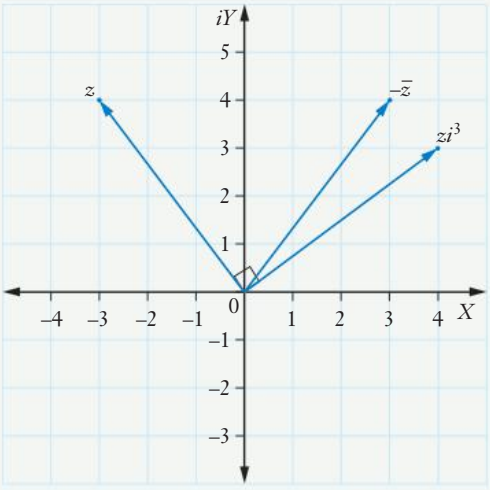
4



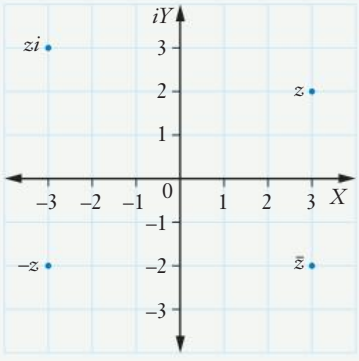
5



6



7



8–12 Proofs

Chapter review

1 **a** $x = \pm 7i$ **b** $m = \frac{3+\sqrt{15}i}{2}, \frac{3-\sqrt{15}i}{2}$

2 **a** $\text{Re}(z) = -3, \text{Im}(z) = 5$

b $\text{Re}(z) = \sqrt{5}, \text{Im}(z) = 2$

3 **a** $20i$ **b** $7i$ **c** -15

d $-i$ **e** $-\sqrt{2}$

4 **a** $-2 + 5i$ **b** $6 - 8i$ **c** 4 **d** $-3 + 7i$

5 $x = -1, y = 1$

6 **a** $23 + 11i$ **b** 13

7 $28 + 96i$

8 $x = -34, y = -6$

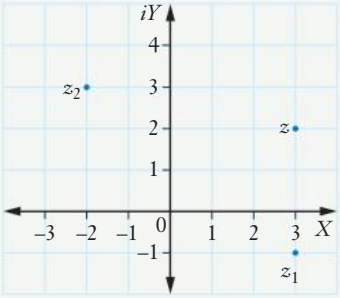
9 **a** $8 + 6i$ **b** $-2 - 3i$

10 **a** $5 - 5i$ **b** $-i$

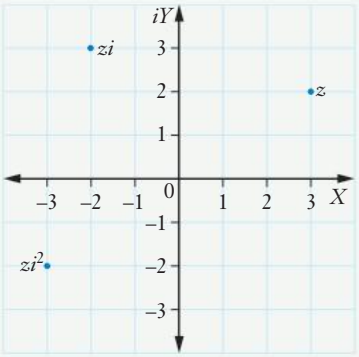
11 $(x + 4yi)(x - 4yi)$

ISBN 9780170412995

12

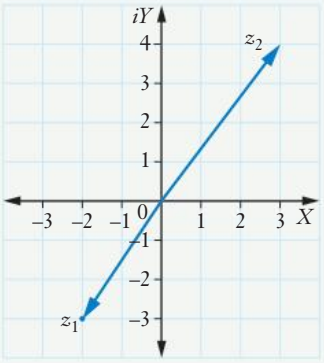


13



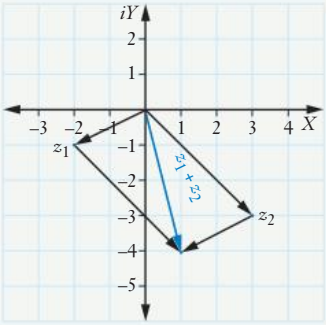
zi is rotated 90° around the origin from z, and zi^2 is rotated 180° from z

14

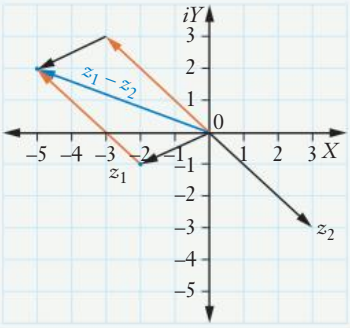


15 $|5+4i| = \sqrt{41} \approx 6.4$, $\arg(5+4i) = \tan^{-1}(0.8) \approx 38.7°$

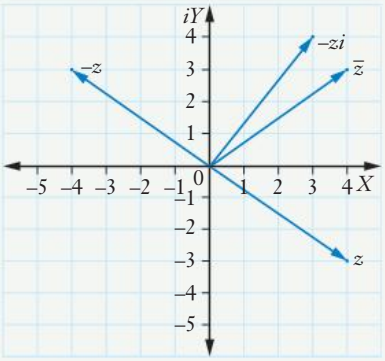
16 **a** $z_1 + z_2 = 1 - 4i$



b $z_1 - z_2 = -5 + 2i$



17



18 Proof

19 **a** $p < -\frac{4}{9}$ or $p > 0$ **b** $p = -\frac{4}{9}$ or $p = 0$

c $-\frac{4}{9} < p < 0$

20 $4x^2 - 21x + 1 = 0$

21–23 Proofs

Chapter 9

Exercise 9.01

1 **a** $x = \dfrac{\left[3n + (-1)^n\right]\pi}{3}$ **b** $x = \dfrac{(8n \pm 1)\pi}{4}$

c $x = \dfrac{(6n+1)\pi}{6}$ **d** $x = \dfrac{(12n \pm 5)\pi}{6}$

e $x = n\pi - 0.927$ **f** $x = n\pi + (-1)^n \times 0.253$

g $x = n\pi + 0.620$ **h** $x = 2n\pi \pm 0.952$

2 **a** $x = (2n \pm 1)\pi$ **b** $x = \dfrac{\left[2n + (-1)^n\right]\pi}{2}$

c $x = \dfrac{(4n+1)\pi}{4}$ **d** $x = 2n\pi$

e $x = (2n \pm 1)\pi$ **f** $x = \dfrac{(4n \pm 1)\pi}{2}$

g $x = \dfrac{(4n+3)\pi}{4}$ **h** $x = \dfrac{\left[6n + (-1)^n\right]\pi}{6}$

ISBN 9780170412995

i $x = \dfrac{(6n \pm 1)\pi}{3}$ **j** $x = \dfrac{(6n \pm 1)\pi}{3}$

k $x = \dfrac{\left[6n - (-1)^n\right]\pi}{6}$

3 **a** $x = n\pi + 0.3398(-1)^n$ **b** $x = 2n\pi \pm 1.8235$

c $x = n\pi + 0.3805$ **d** $x = 2n\pi \pm 0.9818$

e $x = n\pi + 0.4429(-1)^n$ **f** $x = n\pi - 1.4219$

4 **a** $x = \dfrac{\left[6n + (-1)^n\right]\pi}{12}$ **b** $x = \dfrac{(2n \pm 1)\pi}{2}$

c $x = \dfrac{\left[2n + (-1)^n\right]\pi}{4}$ **d** $x = \dfrac{\left[6n - (-1)^n\right]\pi}{12}$

e $x = \dfrac{(3n + 1)\pi}{6}$ **f** $x = \dfrac{(4n + 1)\pi}{8}$

5 **a** $x = \dfrac{\left[2n + (-1)^n\right]\pi}{4}$ **b** $x = \dfrac{(4n + 1)\pi}{12}$

c $x = \dfrac{\left[n\pi - 0.2527(-1)^n\right]}{2}$ **d** $x = n\pi \pm 0.5520$

e $x = \dfrac{n\pi + 1.2358}{4}$

6 **a** $m = \dfrac{n\pi}{3}$ or $m = \dfrac{\left[6n - (-1)^n\right]\pi}{18}$

b $q = \dfrac{(2n \pm 1)\pi}{5}$ or $q = \dfrac{(4n \pm 1)\pi}{10}$

c $t = \dfrac{n\pi}{4}$ or $t = \dfrac{(4n + 1)\pi}{16}$

d $t = \dfrac{\left[6n - (-1)^n\right]\pi}{36}$

e $t = \dfrac{n\pi}{2}$ or $t = \dfrac{n\pi + 0.4636}{2}$

f $t = n\pi$ or $t = (2n \pm 1)\dfrac{\pi}{2}$

7 **a** $\alpha = \dfrac{\left[2n - (-1)^n\right]\pi}{6}$ **b** $\alpha = \dfrac{(6n \pm 1)\pi}{12}$

c $\alpha = \dfrac{(4n - 1)\pi}{20}$ or $\alpha = \dfrac{n\pi + 0.9828}{5}$

d $\alpha = \dfrac{\left[2n + (-1)^n\right]\pi}{4}$ or $\alpha = \dfrac{n\pi - 0.8481(-1)^n}{2}$

e $\alpha = \dfrac{\left[6n - (-1)^n\right]\pi}{18}$ or $\alpha = \dfrac{n\pi + (-1)^n(0.3398)}{3}$

8 2 h, 10 h, 14 h and 22 h after high tide.

9 **a** 0.34 m **b** 0.37 m

c 0.38 m when $t = \dfrac{\pi}{8}$

10 $x = 0.468, -1.515$

11 **a** 5 m **b** 5 m

c 9 a.m. **d** 3 a.m and 3 p.m.

Exercise 9.02

1 **a** Amplitude = 4, period = π

b Amplitude = 2, period = $\dfrac{2\pi}{3}$

c Amplitude = 3, period = π

d Amplitude = 5, period = $\dfrac{2\pi}{3}$

2 **a** π **b** $\dfrac{\pi}{4}$

c $\dfrac{\pi}{3}$ **d** $\dfrac{\pi}{2}$

e $\dfrac{\pi}{2}$ **f** $\dfrac{\pi}{3}$

3 **a** Amplitude = 2, period = $\dfrac{2\pi}{3}$, phase shift = 0

b Amplitude = 3, period = 4π, phase shift = 0

c Amplitude = 4, period = 2π, phase shift = $\dfrac{\pi}{2}$

d Amplitude = 5, period = 4π, phase shift = $\dfrac{\pi}{3}$

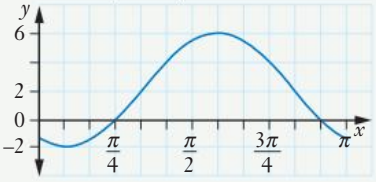
e Amplitude = 2.4, period = π, phase shift = $\dfrac{\pi}{4}$

f Amplitude = 4.7, period = $\dfrac{2\pi}{3}$, phase shift = $-\dfrac{\pi}{4}$

4 **a** $\dfrac{\pi}{2}$ **b** 4π **c** 3π

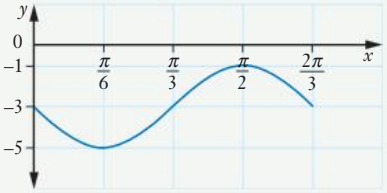
d $\dfrac{2\pi}{3}$ **e** 1 **f** 6π

5 **a** $y = 4 \sin 2\left(x - \dfrac{\pi}{3}\right) + 2$

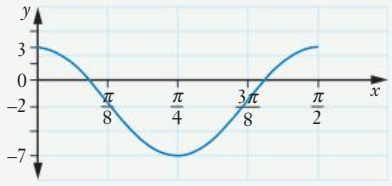


ISBN 9780170412995

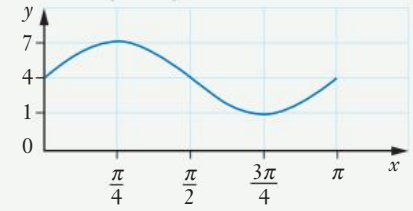
b $y = 2\cos 3\left(x + \frac{\pi}{6}\right) - 3$



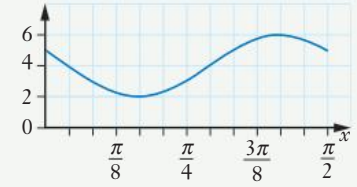
c $y = 5\sin 4\left(x + \frac{\pi}{8}\right) - 2$



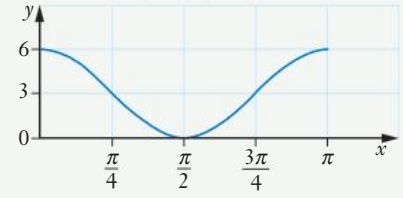
d $y = 3\cos 2\left(x - \frac{\pi}{4}\right) + 4$



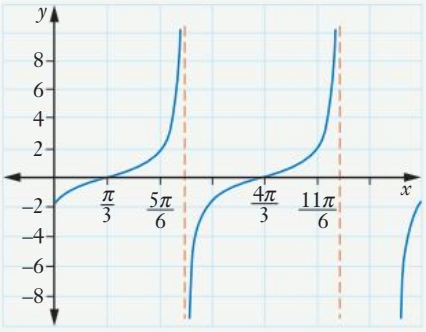
e $y = 4 - 2\cos 4\left(x + \frac{\pi}{3}\right)$



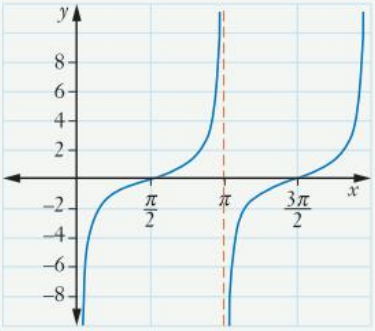
f $y = 3 - 3\sin 2\left(x - \frac{\pi}{4}\right)$



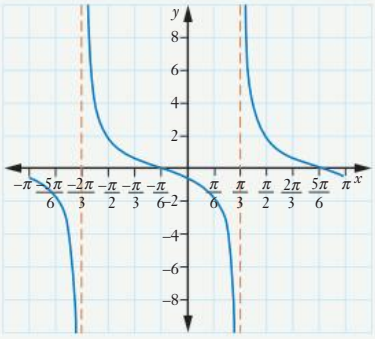
6 a $y = \tan\left(x - \frac{\pi}{3}\right)$



b $y = \tan\left(x + \frac{\pi}{2}\right)$

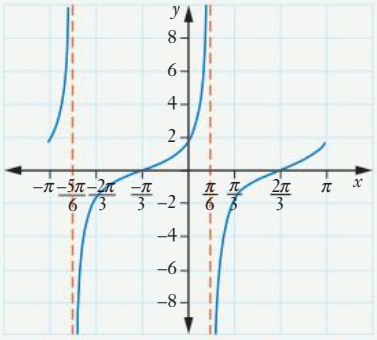


c $y = -\tan\left(x + \frac{\pi}{6}\right)$

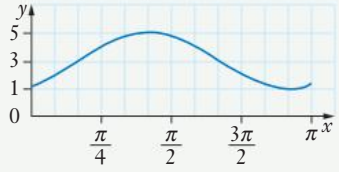


ISBN 9780170412995

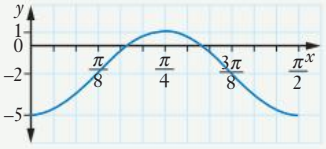
d $y = \tan\left(x - \frac{2\pi}{3}\right)$



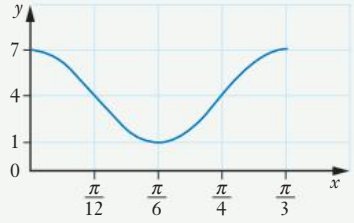
7 **a** $y = 2\sin\left(2x - \frac{\pi}{3}\right) + 3$



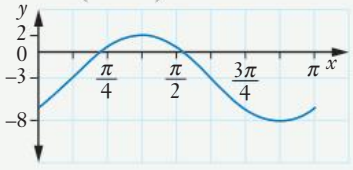
b $y = 3\cos(4x + \pi) - 2$



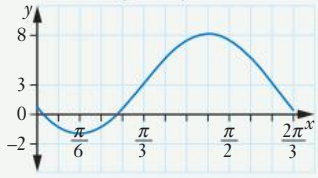
c $y = 4 - 3\sin\left(6x + \frac{3\pi}{2}\right)$



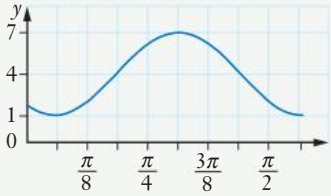
d $y = 5\cos\left(2x - \frac{3\pi}{4}\right) - 3$



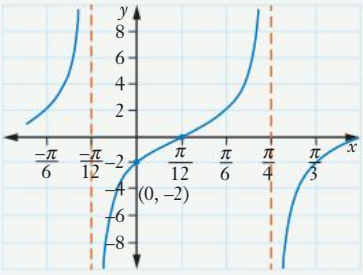
e $y = 3 - 5\cos\left(3x - \frac{\pi}{3}\right)$



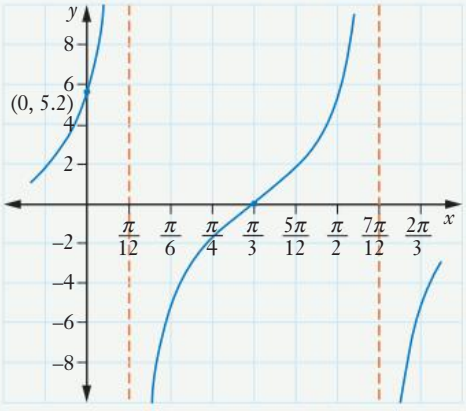
f $y = 4 - 3\sin\left(4x + \frac{\pi}{4}\right)$



8 **a** $y = 2\tan\left(3x - \frac{\pi}{4}\right)$

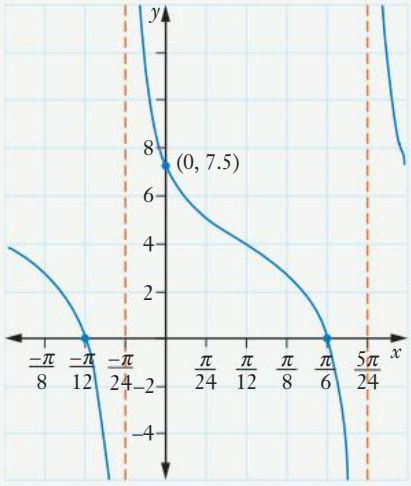


b $y = 3\tan\left(2x + \frac{\pi}{3}\right)$

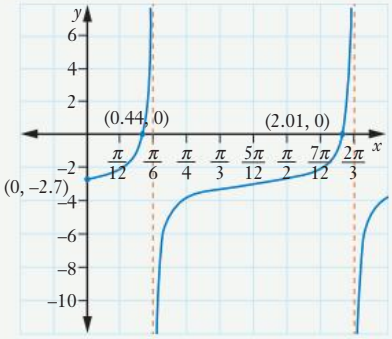


ISBN 9780170412995

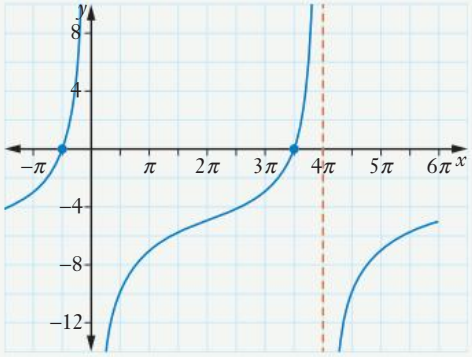
c $y = -2\tan\left(4x - \frac{\pi}{3}\right) + 4$



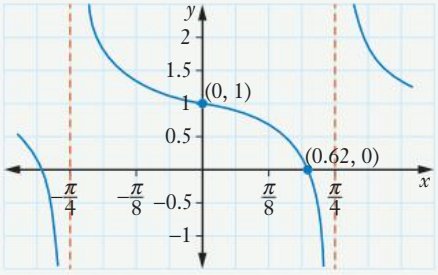
d $y = \frac{1}{2}\tan\left(2x + \frac{\pi}{6}\right) - 3$



e $y = 2\tan\left(\frac{1}{4}x - \frac{\pi}{2}\right) - 5$



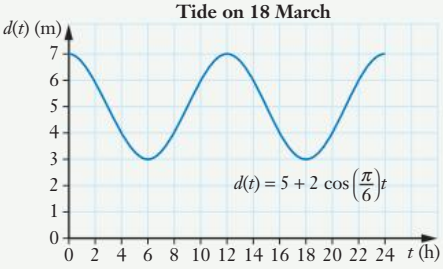
f $y = \frac{1}{3}\tan(-2x - \pi) + 1$



9 $y = 3\sin\left(x - \frac{5\pi}{6}\right), y = 3\cos\left(x + \frac{2\pi}{3}\right)$ (Other answers are possible.)

10 $y = 2\cos\left(\frac{2}{3}x\right), y = 2\sin\left(\frac{2}{3}\left[x + \frac{3\pi}{4}\right]\right)$ (Other answers are possible.)

11 a

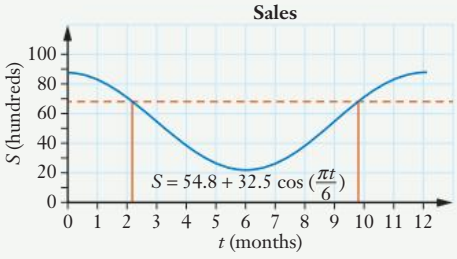


b 1:30 a.m.

c 5:30 a.m., 9:30 a.m., 5:30 p.m., 9:30 p.m.

d 6 hours

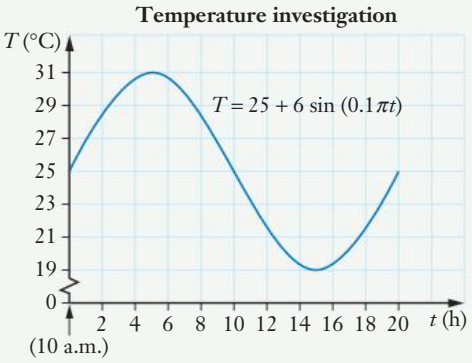
12 a



b Jan, Feb, Oct, Nov and Dec

ISBN 9780170412995

13 a



b 31°C, 19°C **c** 3 p.m., 1 a.m.

d i 11:05 a.m., 6:55 p.m., …

ii 11:08 p.m., 2:52 a.m., …

e Because the period was not 24 hours.

14 About 0.352 s, 0.648 s and 1.352 s.

Exercise 9.03

1 a 7 **b** 2 **c** –20

d 20 **e** –25 **f** 6

g 45 **h** –18 **i** 22

2 a $x = \pm 5$ **b** $x = -\frac{1}{3}, -3$ **c** $x = -3, -5$

d $x = \frac{5}{3}, 3$ **e** $x = -1, 2$ **f** $x = -2, \frac{22}{5}$

g $x = -4, \frac{8}{5}$ **h** $x = -\frac{82}{9}, 10$

i $x = -\frac{76}{7}, 10$

3 a $x = \frac{5}{3}, \frac{4}{3}$ **b** No solution **c** $x = -7, -5$

d $x = -8, -6$ **e** $x = -\frac{16}{3}, 6$ **f** $x = 0, \frac{2}{5}$

g $x = \frac{2}{3}$ **h** No solution **i** $x = \frac{1}{6}, \frac{5}{2}$

4 a $x = -\frac{1}{2}$ **b** $x = -\frac{2}{7}$ **c** $x = \frac{5}{3}$

d $x = \frac{2}{3}, 4$ **e** $x = -5$ **f** $x = -\frac{3}{4}, \frac{1}{2}$

5 a $x = -3$ **b** $x = \frac{1}{5}, 7$

c $x = -\frac{4}{3}, -\frac{2}{7}$ **d** $x = -6, \frac{2}{5}$

e $x = \frac{1}{5}, -9,$ **f** $x = -\frac{5}{2}, -\frac{1}{10}$

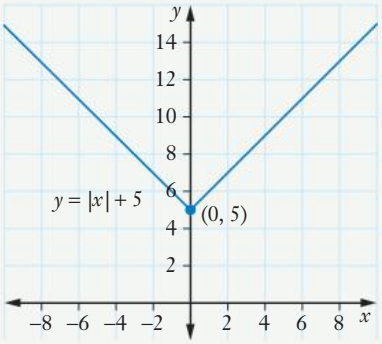
g $x = -5, \frac{1}{5}$

6 a $x = \frac{3}{8}, \frac{11}{4}$ **b** $x = 1, 5$ **c** $x = 1, 5$

d $x = 0, 1$ **e** $x = \frac{7}{5}, 5$ **f** $x = \frac{-9}{5}, \frac{13}{5}$

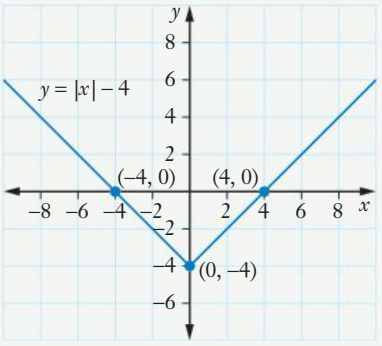
7 a $y = |x| + 5$

Domain: $x \in \boldsymbol{R}$, range: $y \geq 5$



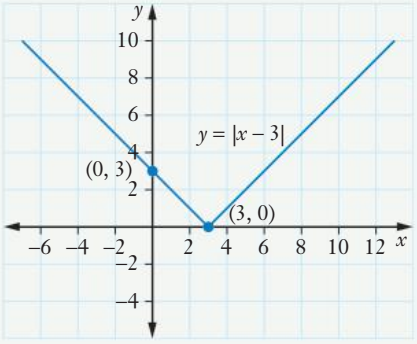
b $y = |x| - 4$

Domain: $x \in \boldsymbol{R}$, range: $y \geq -4$



c $y = |x - 3|$

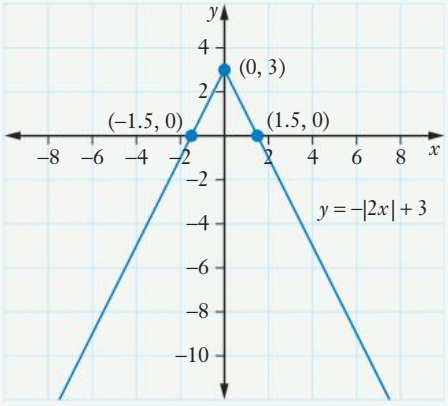
Domain: $x \in \boldsymbol{R}$, range: $y \geq 0$



ISBN 9780170412995

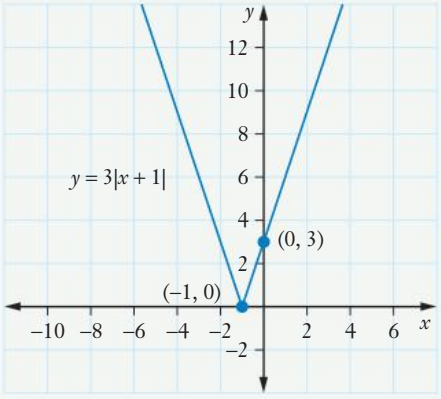
d $y = -2|x| + 3$

Domain: $x \in \boldsymbol{R}$, range: $y < 3$



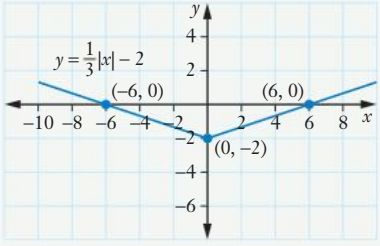
e $y = 3|x + 1|$

Domain: $x \in \boldsymbol{R}$, range: $y \geq 0$



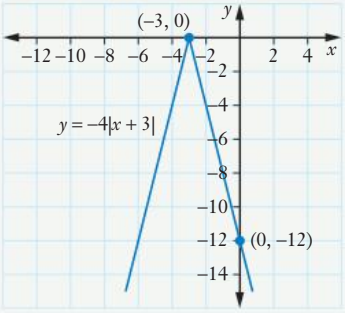
f $y = \frac{1}{3}|x| - 2$

Domain: $x \in \boldsymbol{R}$, range: $y \geq -2$



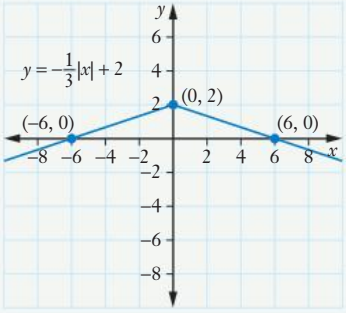
g $y = -4|x + 3|$

Domain: $x \in \boldsymbol{R}$, range: $y \leq 0$



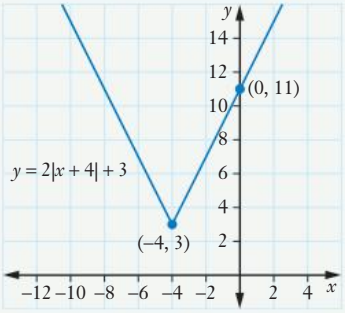
h $y = -\frac{1}{3}|x| + 2$

Domain: $x \in \boldsymbol{R}$, range: $y \leq 2$



8 a $y = 2|x + 4| + 3$

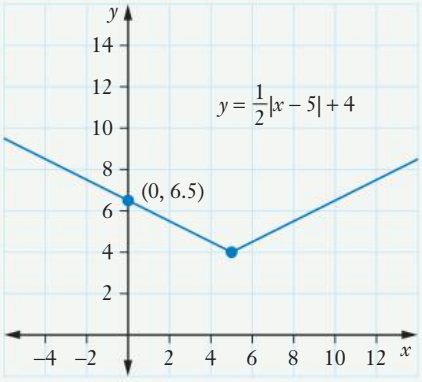
Domain: $x \in \boldsymbol{R}$, range: $y \geq 3$



ISBN 9780170412995

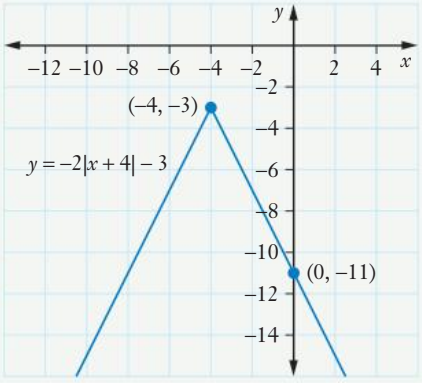
b $y = \frac{1}{2}|x - 5| + 4$

Domain: $x \in \boldsymbol{R}$, range: $y \geq 4$

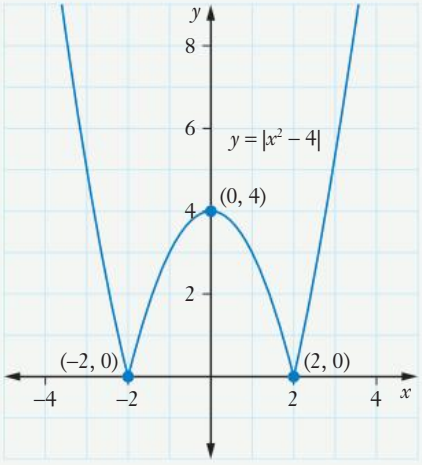


c $y = -2|x + 4| - 3$

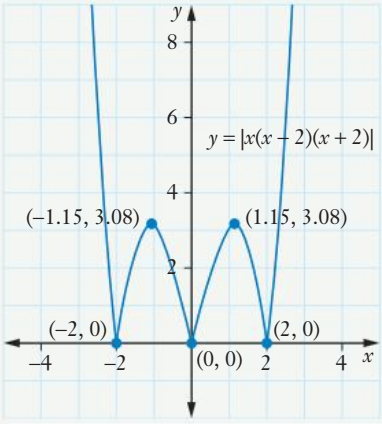
Domain: $x \in \boldsymbol{R}$, range: $y \leq -3$



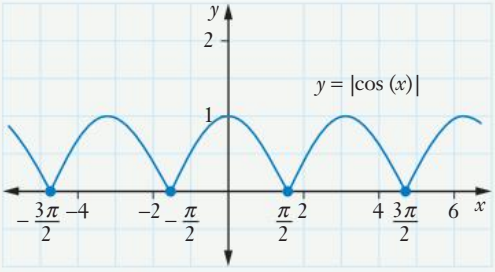
9 a $y = |x^2 - 4|$



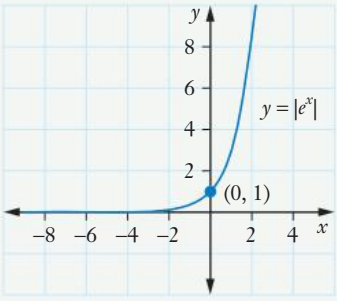
b $y = |x(x - 2)(x + 2)|$



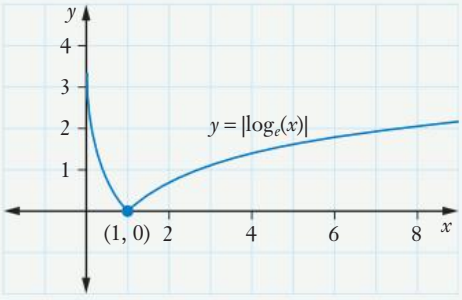
c $y = |\cos(x)|$



d $y = |e^x|$

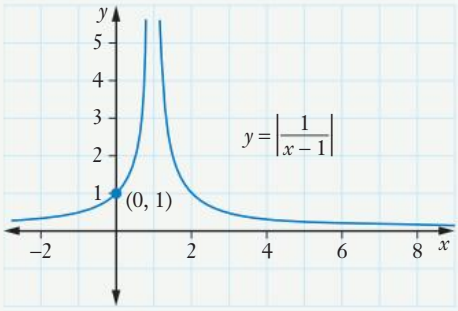


e $y = |\log_e(x)|$

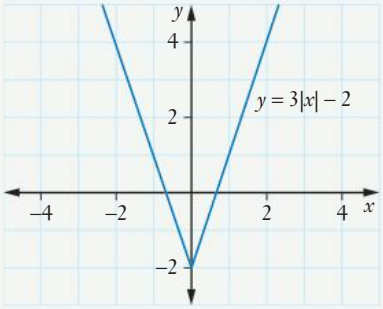


ISBN 9780170412995

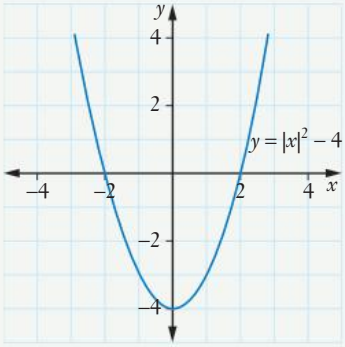
f $y = \left|\frac{1}{x-1}\right|$



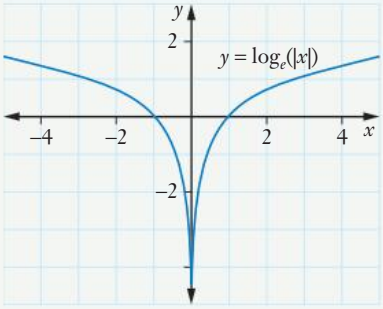
10 a $y = 3|x| - 2$



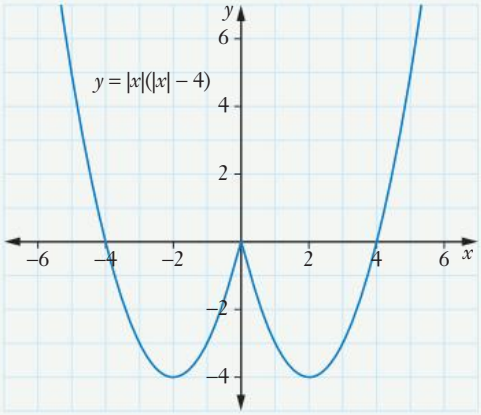
b $y = |x|^2 - 4$



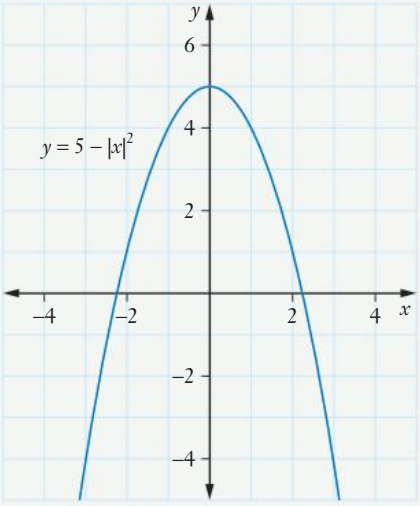
c $y = \log_e|x|$



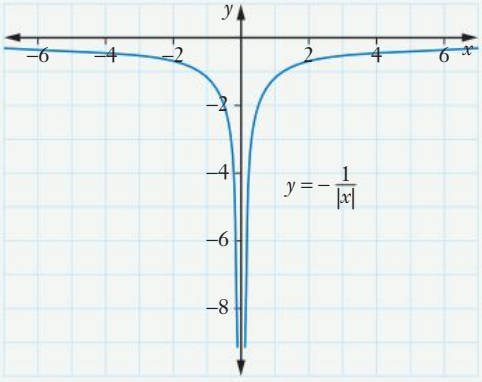
d $y = |x|(|x| - 4)$



e $y = 5 - |x|^2$



f $y = -\frac{1}{|x|}$



ISBN 9780170412995

11 **a** $|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$

b $|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b|$

12 There is no point on the roof that is 10 m high.

13 (–4, 0) and (10, 0)

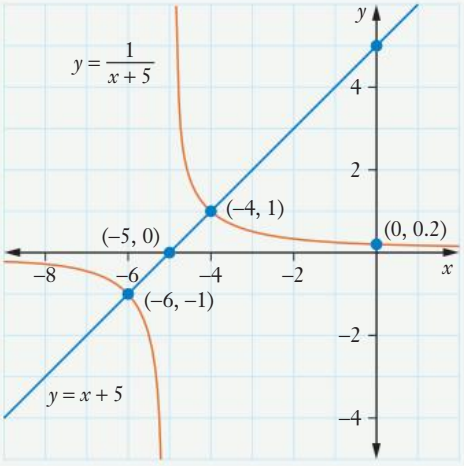
14 $x^2 - 7x \geq -12$

15 $x = -1, 11$

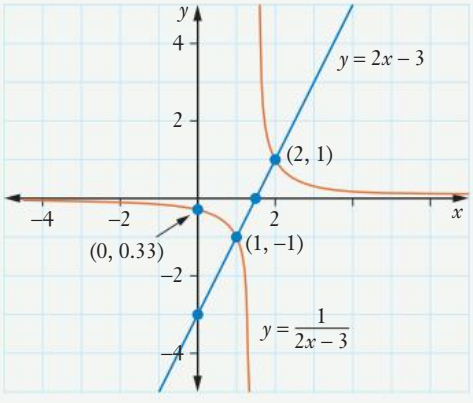
16 $x < -\sqrt{14}, -2 < x < 2, x > \sqrt{14}$

Exercise 9.04

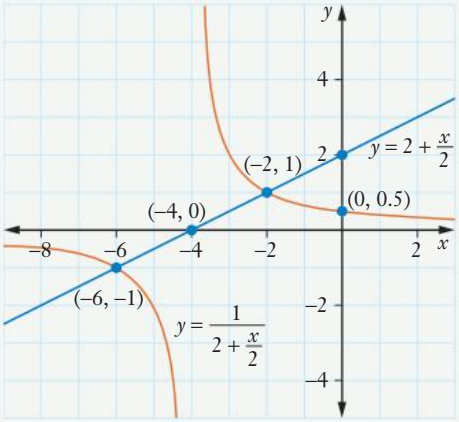
1 **a** $y = x + 5$



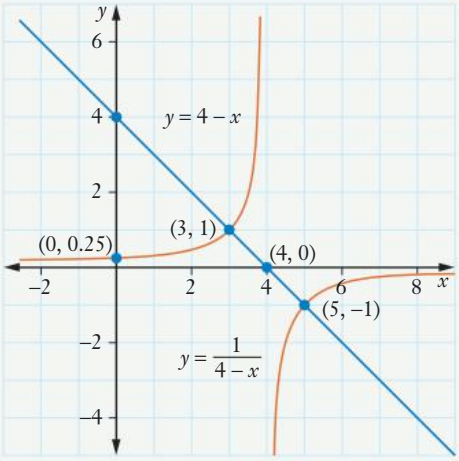
b $y = 2x - 3$



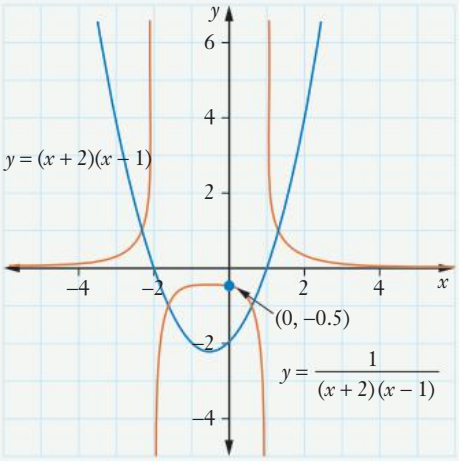
c $y = 2 + \frac{1}{2}x$



d $y = 4 - x$

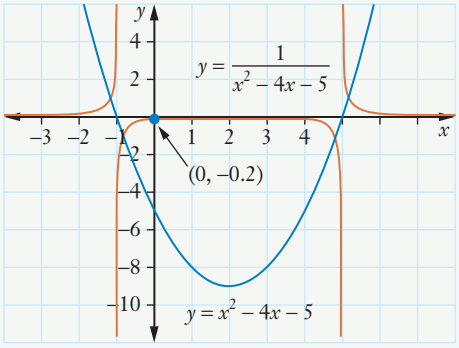


2 **a** $y = (x + 2)(x - 1)$

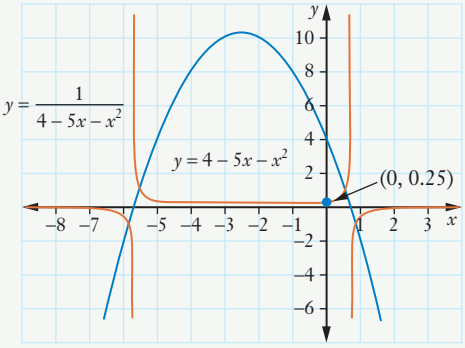


ISBN 9780170412995

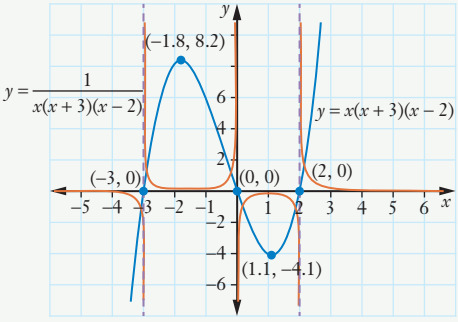
b $y = x^2 - 4x - 5$



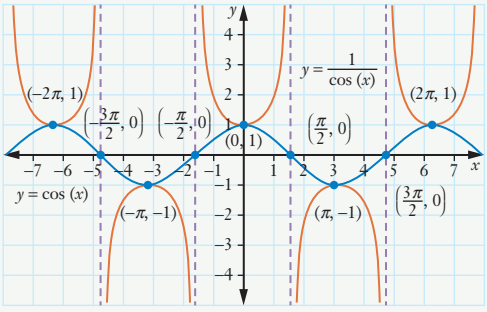
c $y = 4 - 5x - x^2$



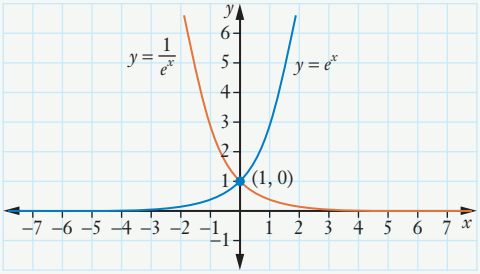
d $y = x(x + 3)(x - 2)$



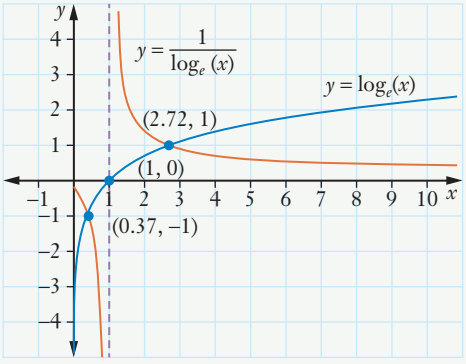
3 a $y = \cos(x)$



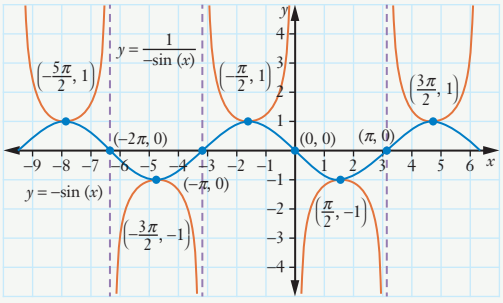
b $y = e^x$



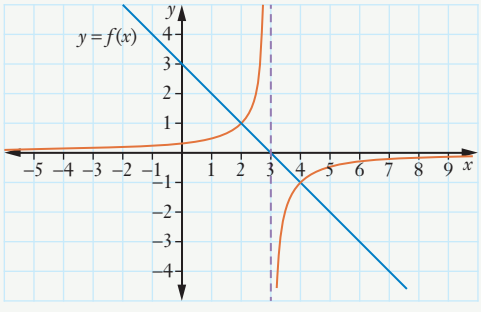
c $y = \log_e(x)$



d $y = -\sin(x)$

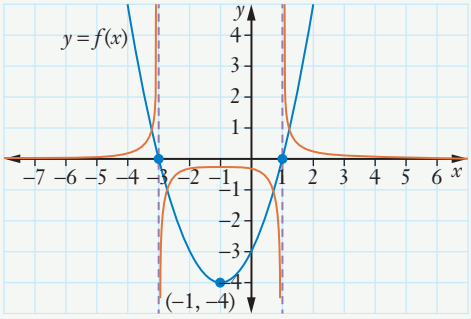


4



ISBN 9780170412995

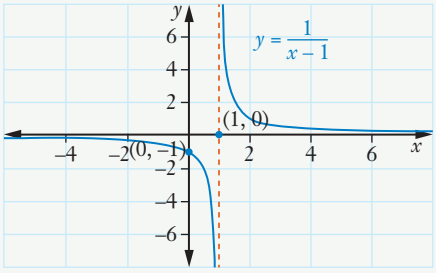
5



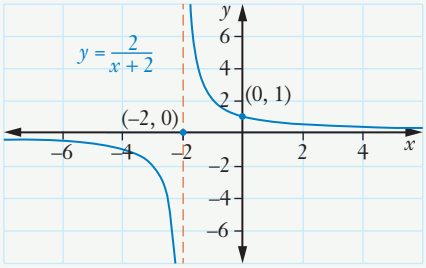
6 $x < -1, 0 < x < 3$

Exercise 9.05

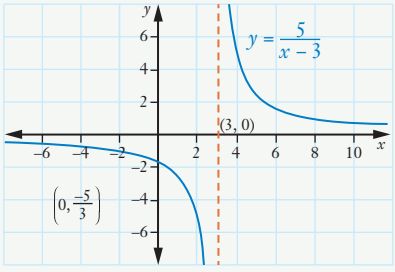
1 a



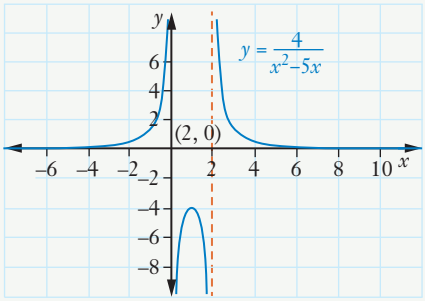
b



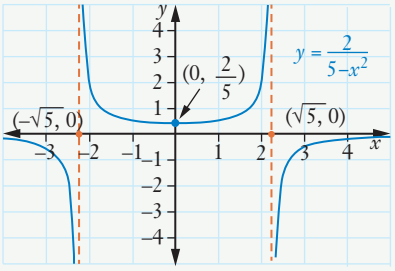
c



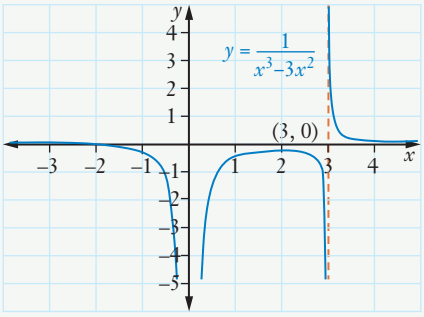
d



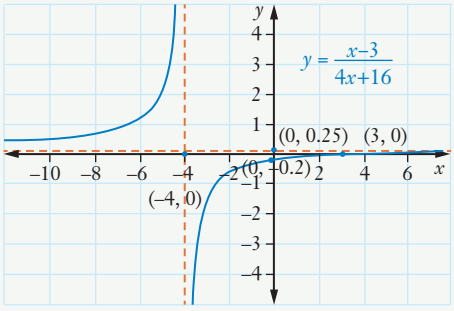
e



f

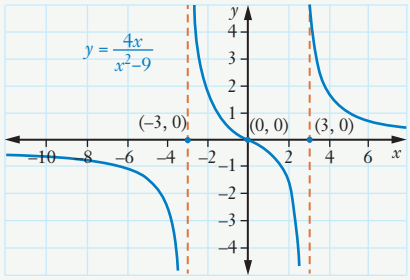


2 a

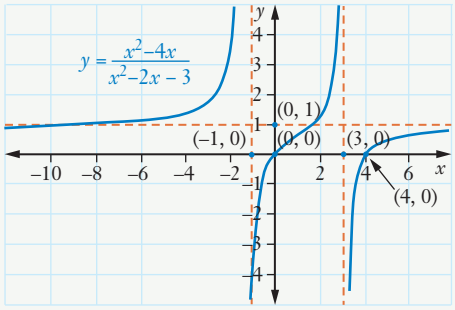


ISBN 9780170412995

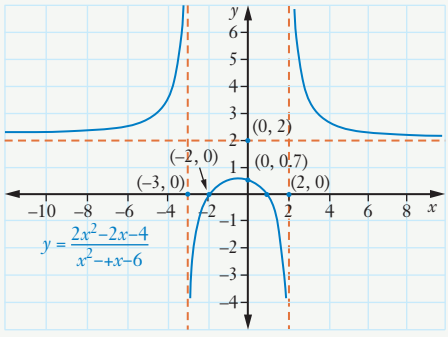
b



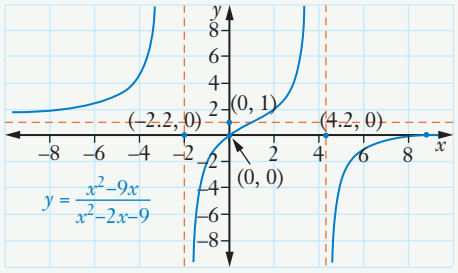
c



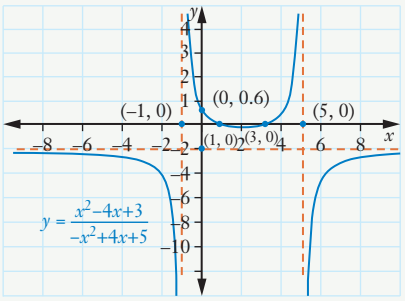
d



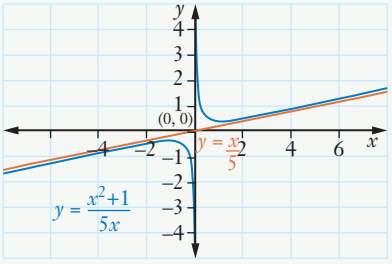
e



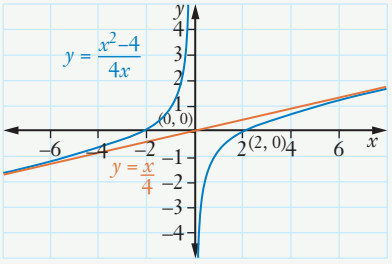
f



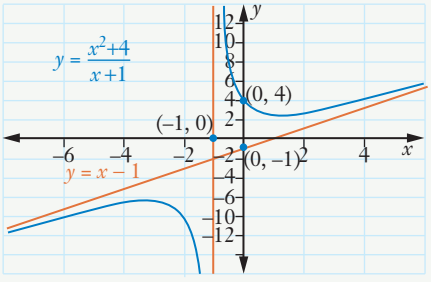
3 a



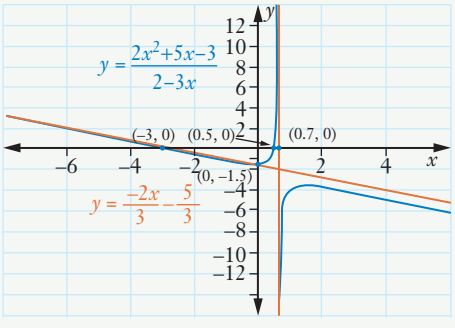
b



c

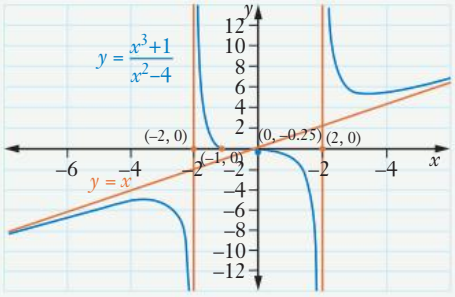


d

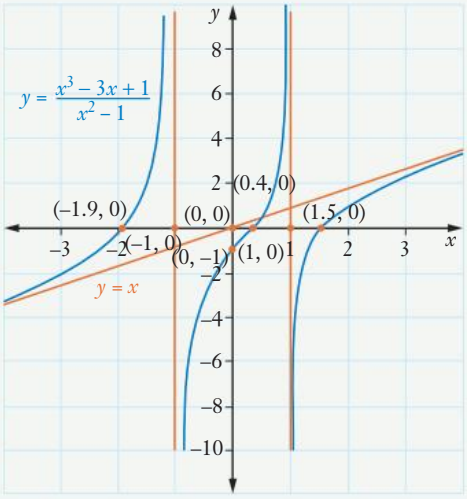


ISBN 9780170412995

e



f



4 $-4 < x < -2,\ 0 < x < 3,\ x > 4$

5 **a** $C(30) \approx \$801$ and $C(90) \approx \$16\,830$ are the costs to remove 30% and 90% of the carp respectively.

b The vertical asymptote at $n = 100$ means that as you try to remove 100% of the carp from the waterway, the cost increases without bound, so it is not possible to remove 100% of the carp.

c About 93%

6 **a** $A(n) = \dfrac{150n + 3000}{n}$

b $A(2) = \$1650$ and $A(90) \approx \$183$. The average cost of producing 2 units is much greater than the average cost when 90 units are produced.

c 40

d As $n \to \infty, A(n) \to \$150$. This means that as more and more components are produced, the cost per unit approaches \$150 and is always greater than \$150.

Exercise 9.06

		sin (x)	cos (x)	tan (x)	cosec (x)	sec (x)	cot (x)
1	**a**	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{4}{3}$
	b	$\frac{5}{13}$	$\frac{12}{13}$	$\frac{5}{12}$	$\frac{13}{5}$	$\frac{13}{12}$	$\frac{12}{5}$
	c	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

2 **a** 51.3° **b** 23.6° **c** 70.5°
d 36.9° **e** 64.6° **f** 82.9°
g 77.2° **h** 68.2° **i** 46.7°

3 **a** $\sqrt{2}$ **b** $\sqrt{3}$
c $\frac{2\sqrt{3}}{3}$ **d** $-\sqrt{3}$
e $-\frac{1}{2}$ **f** $\frac{2\sqrt{3}}{3}$
g -2 **h** $\frac{2\sqrt{3}}{3}$
i $-\sqrt{3}$ **j** -2
k 1 **l** 2

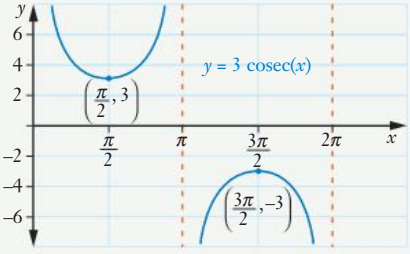
4 **a** $-\frac{\sqrt{3}}{2}$ **b** 2 **c** $-\frac{\sqrt{3}}{3}$

5 **a** $-\frac{\sqrt{7}}{4}$ **b** $-\frac{\sqrt{7}}{3}$ **c** $-\frac{4\sqrt{7}}{7}$

6 **a** $-\frac{2\sqrt{29}}{29}$ **b** $-\frac{\sqrt{29}}{5}$ **c** $-\frac{\sqrt{29}}{2}$

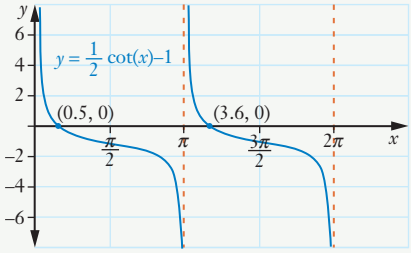
7 **a** $-\frac{4}{7}$ **b** $\frac{\sqrt{33}}{7}$ **c** $-\frac{4\sqrt{33}}{33}$

8 **a**

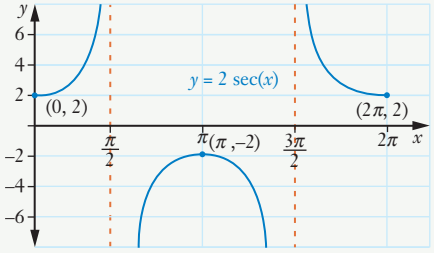


ISBN 9780170412995

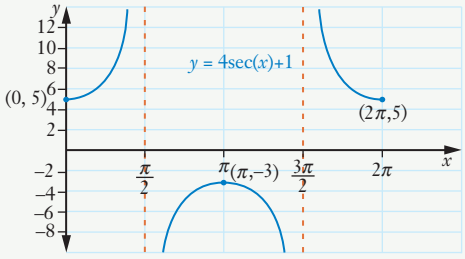
b



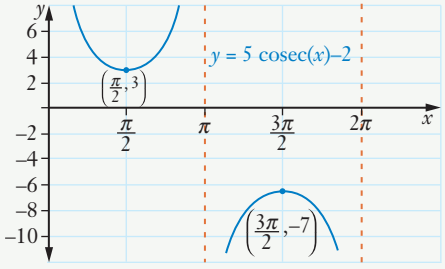
c



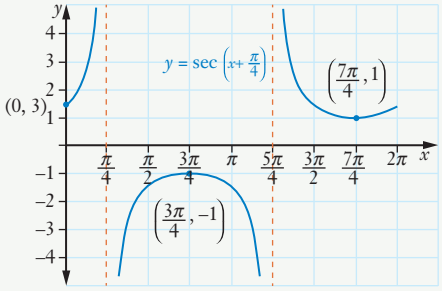
d



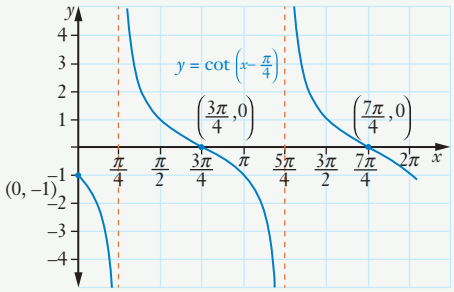
e



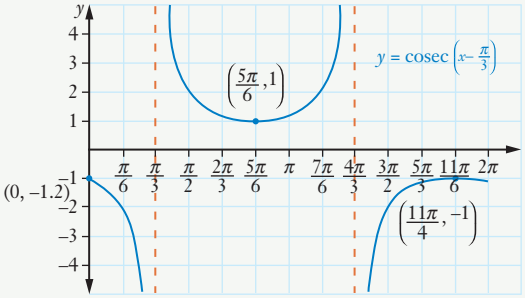
f



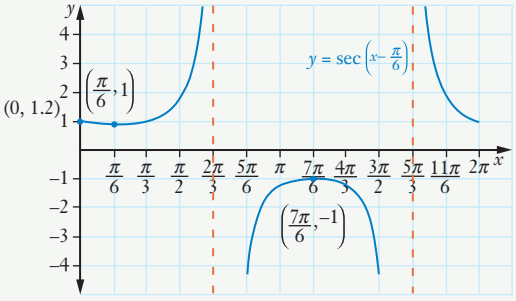
g



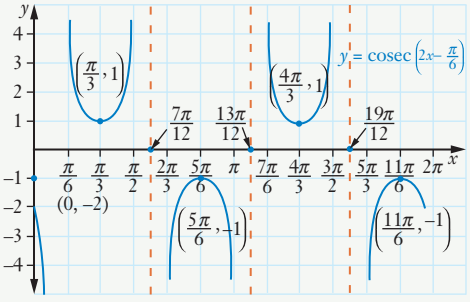
h



i

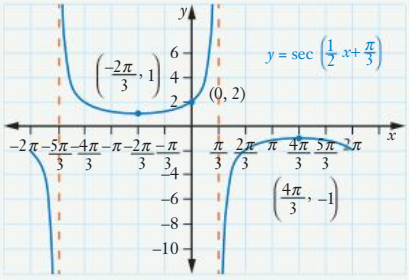


9 a

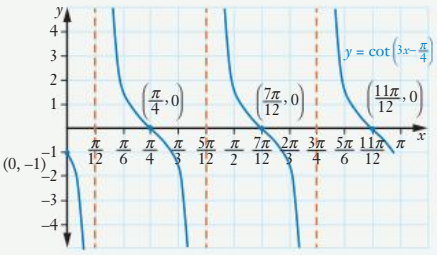


ISBN 9780170412995

b



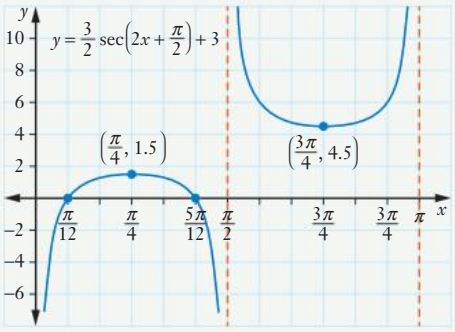
c



10 **a** $x = 0, x = \frac{\pi}{2}, x = \pi$ **b** $\left(\frac{\pi}{12}, 0\right), \left(\frac{5\pi}{12}, 0\right)$

c Min $= \left(\frac{\pi}{4}, 1.5\right)$, Max $= \left(\frac{3\pi}{4}, 4.5\right)$

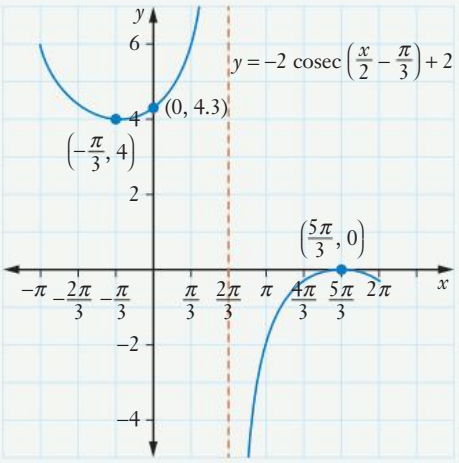
d



11 **a** $x = \frac{2\pi}{3}$ **b** $\left(\frac{5\pi}{3}, 0\right)$

c (0, 4.3) **d** Min $= \left(-\frac{\pi}{3}, 4\right)$, Max $= \left(\frac{5\pi}{3}, 0\right)$

e



12 $2\tan(x)$ **13** $-\sin^2(\theta)$

Exercise 9.07

1 **a** $\tan(x)$ **b** $\sin(x)$ **c** $\sin^2(x)$
d -1 **e** $\cos^2(x)$ **f** $\sin^2(x)$

2 **a** $\sin^2(A)$ **b** 1
c $\sin^2(\theta)$ **d** $\sin^2(A) - \cos^2(A)$
e $\sin^2(x)$ **f** $\tan^2(\theta)$

3 **a** 1 **b** $\cot(x)$ **c** $\text{cosec}(x)$
d $\sin^2(\theta)$ **e** 1 **f** $\sec(x)$

4 $\frac{4}{5}$

5 $\frac{25}{16}$

6 **a** $-\frac{1}{\sqrt{5}}$ **b** $-\frac{\sqrt{5}}{2}$ **c** $-\sqrt{5}$

7–15 Proofs

Exercise 9.08

1 **a** $\sin(2x)\cos(y) + \cos(2x)\sin(y)$
b $\cos(4x)\cos(3y) + \sin(4x)\sin(3y)$
c $\sin(3p)\cos(5q) - \cos(3p)\sin(5q)$
d $\frac{\tan(2e) - \tan(f)}{1 + \tan(2e)\tan(f)}$
e $\cos(3\alpha)\cos(2\beta) - \sin(3\alpha)\sin(2\beta)$
f $\sin(4m)\cos(n) + \cos(4m)\sin(n)$
g $\frac{\tan(w) + \tan(3x)}{1 - \tan(w)\tan(3x)}$
h $\sin(2D)\cos(5E) - \cos(2D)\sin(5E)$
i $\cos(p)\cos(4Q) + \sin(p)\sin(4Q)$

ISBN 9780170412995

2 **a** $\sin(x)$ **b** $-\cos(x)$ **c** $\cot(x)$

d $\sec(x)$ **e** $\tan(x)$ **f** $-\operatorname{cosec}(x)$

g $-\sin(x)$ **h** $-\sin(x)$ **i** $\tan(x)$

3 **a** $\sin(92°)$ **b** $\cos(47°)$ **c** $\tan(101°)$

d $\cos\left(\frac{11\pi}{6}\right)$ **e** $\sin\left(\frac{\pi}{12}\right)$ **f** $\tan\left(\frac{4\pi}{21}\right)$

4 **a** $2-\sqrt{3}$ **b** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **c** $-(2+\sqrt{3})$

d $\frac{\sqrt{6}+\sqrt{2}}{4}$ **e** $-(2+\sqrt{3})$ **f** $\frac{\sqrt{6}+\sqrt{2}}{4}$

g $\frac{-\sqrt{2}-\sqrt{6}}{4}$ **h** $\frac{\sqrt{6}+\sqrt{2}}{4}$ **i** $\frac{\sqrt{2}-\sqrt{6}}{4}$

5 Proof

6 $x=\frac{\pi}{3},\frac{5\pi}{3}$

7–13 Proofs

Exercise 9.09

1 Verification – teacher to check.

2 **a** $\sin(2x)=-\frac{24}{25},\ \cos(2x)=\frac{7}{25}$

b $\sin(2x)=\frac{120}{169},\ \cos(2x)=-\frac{119}{169}$

c $\sin(2x)=-\frac{120}{169},\ \cos(2x)=\frac{119}{169}$

d $\sin(2x)=\frac{\sqrt{3}}{2},\ \cos(2x)=\frac{1}{2}$

3 Verification – teacher to check.

4 **a** $\frac{\sqrt{26}}{26}$ **b** $\frac{\sqrt{18+12\sqrt{2}}}{6}$ **c** $\frac{1}{5}$

d -2 **e** -4 **f** $\frac{1}{3}$

5–9 Proofs

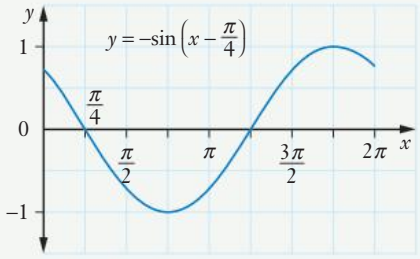
Chapter review

1 **a** $n\pi+\frac{\pi}{3}$ **b** $\frac{\pi(8n\pm3)}{4}$

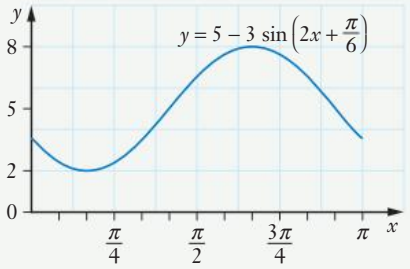
c $-\frac{23\pi}{24},-\frac{7\pi}{24},\frac{\pi}{24},\frac{17\pi}{24}$

2 **a** $x\approx2.68,5.82,0.46,3.61$ **b** $\frac{\pi}{2},\frac{3\pi}{2}$

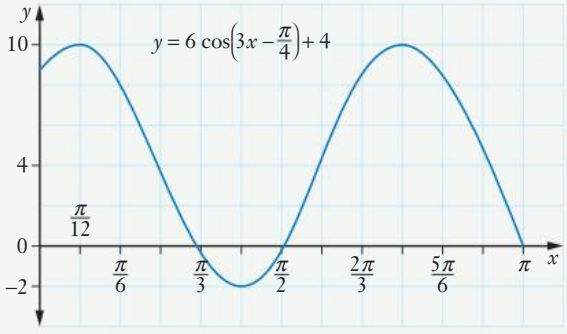
3 **a**



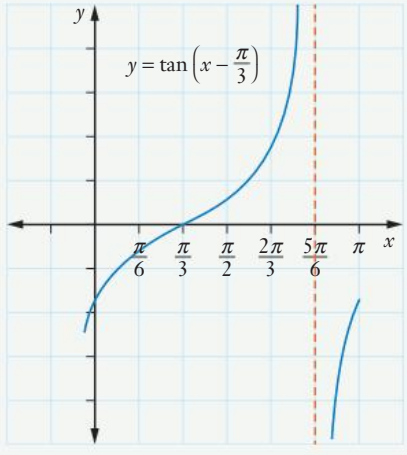
b



c

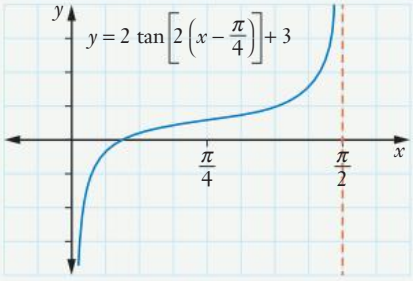


4 **a**

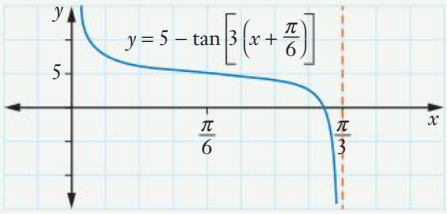


ISBN 9780170412995

b



c

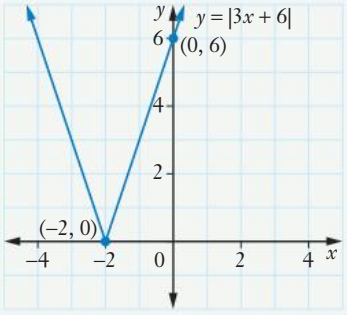


5 $y = 2\cos\left(\frac{x}{2}\right), y = 2\sin\left(\frac{x}{2} + \frac{\pi}{2}\right)$ (other equations are possible.)

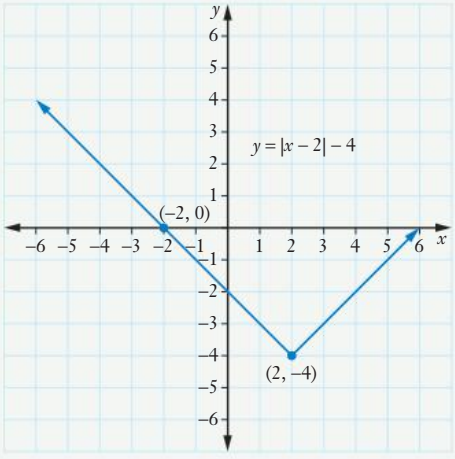
6 **a** $-4, 1$ **b** $10, 3$

c $-3, -\frac{1}{4}$ **d** $-11, \frac{1}{3}$

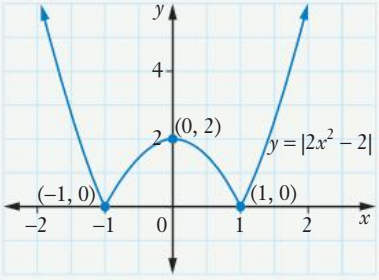
7 **a**



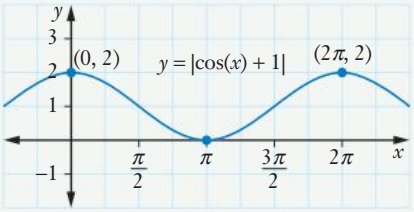
b



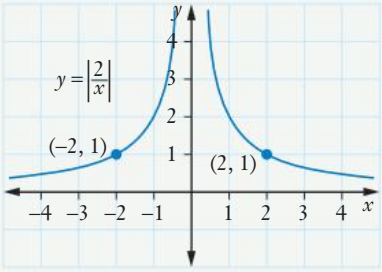
c



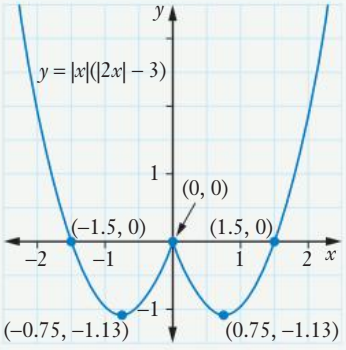
d



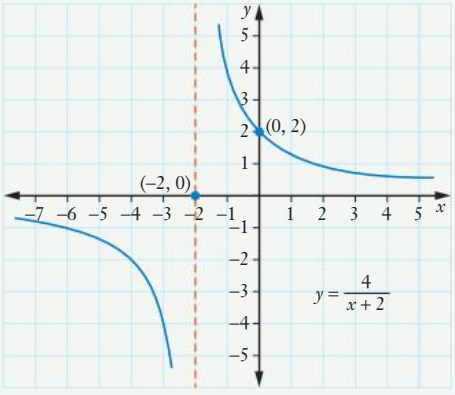
e



f

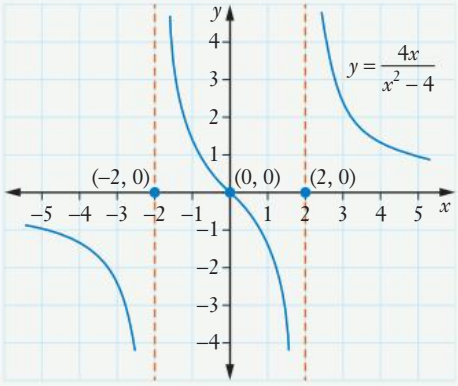


8 **a**

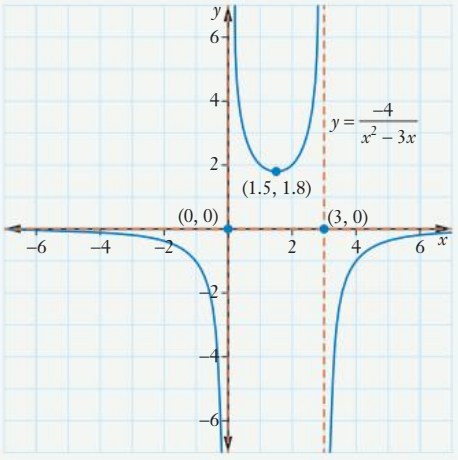


ISBN 9780170412995

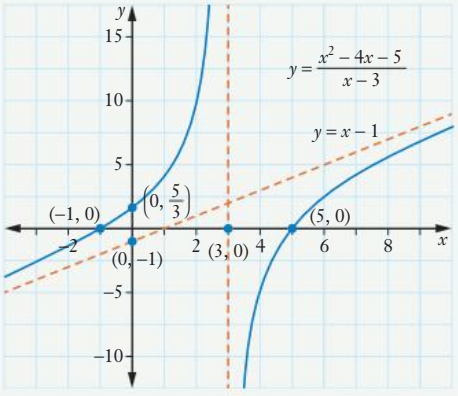
b



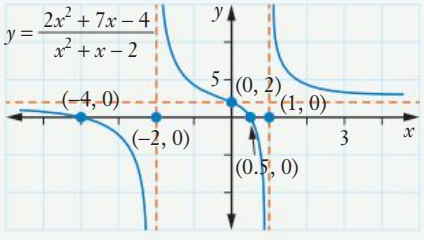
c



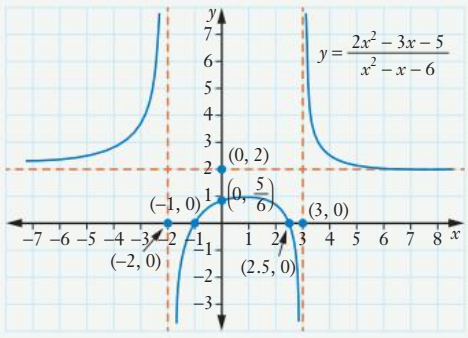
d



e



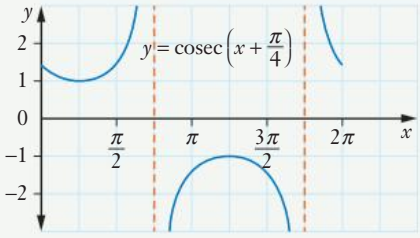
f



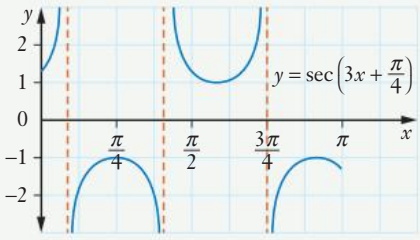
9 **a** $\frac{7\sqrt{74}}{74}$ **b** $\frac{\sqrt{74}}{5}$ **c** $\frac{7}{5}$

10 **a** 32.6° **b** $-\sqrt{2}$ **c** $\sqrt{5}$

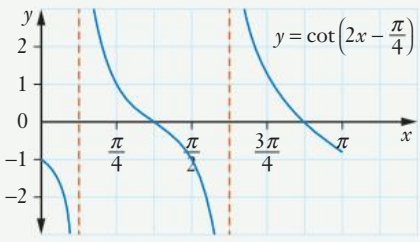
11 **a**



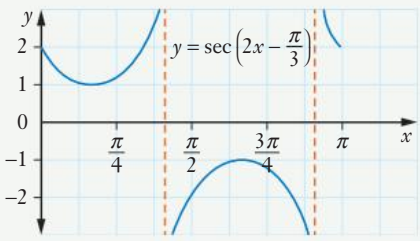
b



c



d



ISBN 9780170412995

12 **a** 1 **b** $-\frac{4}{5}$

13 Proof

14 **a** $\sin(5x)\cos(2y) - \cos(5x)\sin(2y)$

b $\frac{\tan(3x)+\tan(y)}{1-\tan(3x)\tan)(y)}$

c $-\cos(x)$ **d** $\tan(x)$

15 **a** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\frac{\sqrt{6}+\sqrt{2}}{4}$ **c** $\sqrt{3}-2$

16 Proof

17 **a** $-\frac{7}{25}$ **b** $-\frac{5\sqrt{26}}{26}$

18 Proof

19 **a** Low tide: 12:00 a.m., 12:24 p.m., high tide: 6:12 a.m., 6:36 p.m.

b 2:14 a.m., 10:12 a.m., 2:32 p.m., 10:36 p.m.

20–24 Proofs

Practice examination 3

1 **a** 3×4 **b** $a_{23} = 4$

2 **a** $\begin{bmatrix} -1 & 1 \\ 2 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} -21 & 14 \\ 18 & 5 \end{bmatrix}$ **c** $\begin{bmatrix} 6 & -6 \\ 3 & 7 \end{bmatrix}$

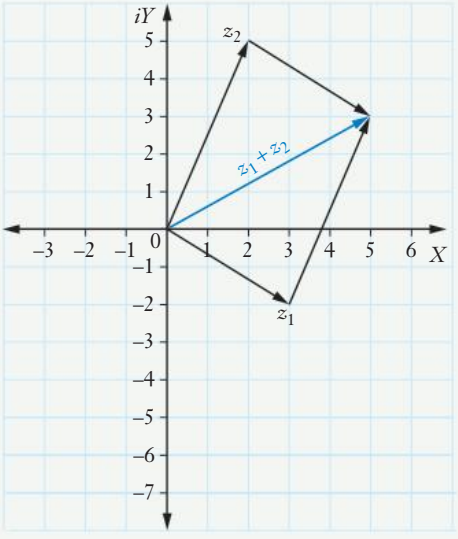
3 $\mathbf{X} = \begin{bmatrix} 10.5 \\ 11.25 \end{bmatrix}$

4 $x = \pm 3i$ **5** Re (z) = 7, Im (z) = –3

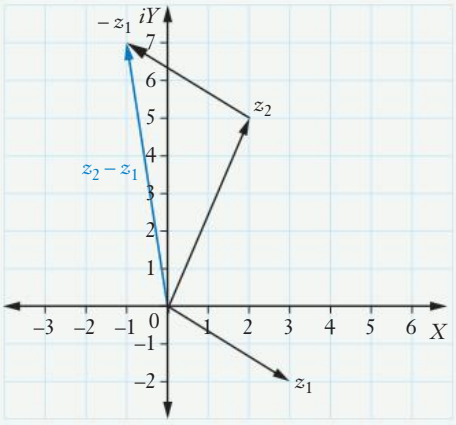
6 **a** $7 + 3i$ **b** $-9 + 13i$ **c** $\frac{7}{17} + \frac{11}{17}i$

7 $(3x + 5yi)(3x - 5yi)$

8 **a**



b

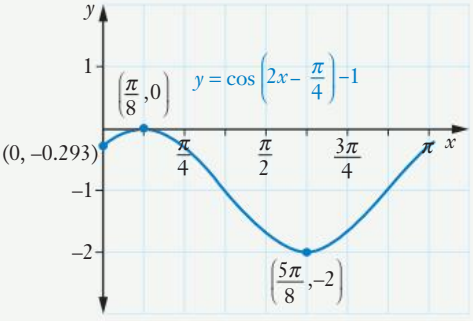


9 **a** $x = \frac{\left[6n - (-1)^n\right]\pi}{6}$

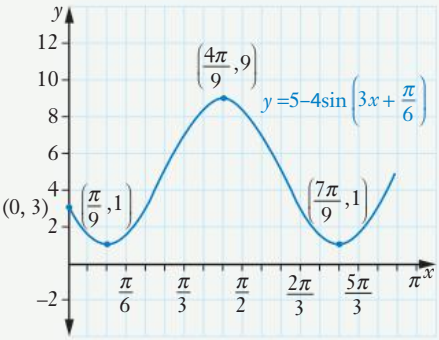
b $x = \frac{(6n \pm 1)\pi}{3}$ or $\frac{(3n \pm 1)2\pi}{3}$

c $x = \frac{(4n+1)\pi}{4}$

10 **a**

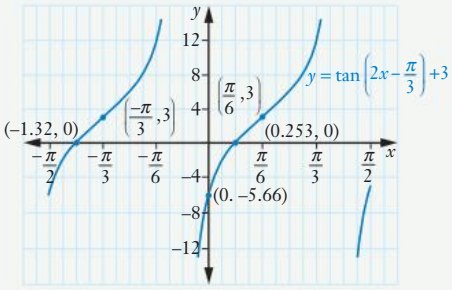


b



ISBN 9780170412995

c



11 **a** $-\frac{8}{3}$ or $\frac{2}{5}$ **b** -10 or $-\frac{2}{3}$

12 $-\frac{24}{25}$

13 $\frac{\sqrt{2-\sqrt{3}}}{2}$

14 cos (1)

15 $\begin{bmatrix} 0.25 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.15 & 0.1 & 0.12 \end{bmatrix}\begin{bmatrix} 16 \\ 18 \\ 42 \end{bmatrix} = \begin{bmatrix} 11.80 \\ 20.40 \\ 9.42 \end{bmatrix}$

16 Let $a, b, c, d, \alpha, \beta \in \boldsymbol{R}$ and write $z_1 = a + bi$ and $z_2 = c + di$.

Let $\mathbf{T}(z) = z(3 - 5i)$

Then $\mathbf{T}(\alpha z_1 + \beta z_2) = (\alpha z_1 + \beta z_2)(3 - 5i)$

$= 3\alpha z_1 + 3\beta z_2 - 5\alpha z_1 i - 5\beta z_2 i$

$= 3\alpha z_1 - 5\alpha z_1 i + 3\beta z_2 - 5\beta z_2 i$

$= \alpha z_1(3 - 5i) + \beta z_2(3 - 5i)$

$= \alpha\mathbf{T}(z_1) + \beta\mathbf{T}(z_2)$

Thus $\mathbf{T}$ is a linear transformation. QED

17 **a**, **b** Proofs

18 $\begin{bmatrix} 3\frac{1}{2} \\ 1 \\ -2 \end{bmatrix}$

19 Let the roots of $x^2 + 3x + 4 = 0$ be α and β.

Then $\alpha + \beta = -\frac{3}{1} = -3$ and $\alpha\beta = \frac{4}{1} = 4$

The roots of the new equation are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta} = -\frac{3}{4}$

$\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{4}$

The new equation is $x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha} \times \frac{1}{\beta} = 0$

i.e., $x^2 + \frac{3}{4}x + \frac{1}{4} = 0$

The equation $4x^2 + 3x + 1 = 0$ has roots that are the reciprocals of the roots of $x^2 + 3x + 4 = 0$.

20 **a**, **b** Proofs

Chapter 10

Exercise 10.01

1 **a** $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ **b** $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$ **d** $\begin{bmatrix} -6 \\ 0 \end{bmatrix}$

e $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ **f** $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ **g** $\begin{bmatrix} -2 \\ -6 \end{bmatrix}$ **h** $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

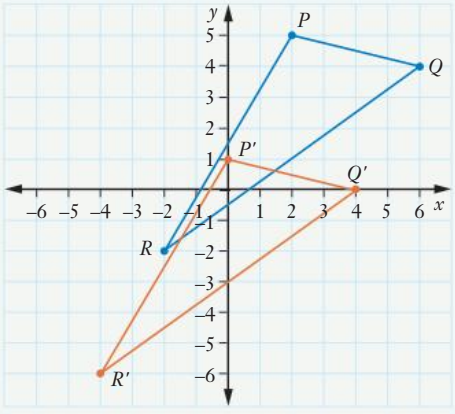
2 **a** (2, 3) **b** (0, 1)
c (4, −8) **d** (−5, −7)

3 **a** (1, 8) **b** (3, 7)
c (1, 0) **d** (10, 3)

4 **a** (−1, 13) **b** (−2, 1)
c (−7, −14) **d** (−10, 14)

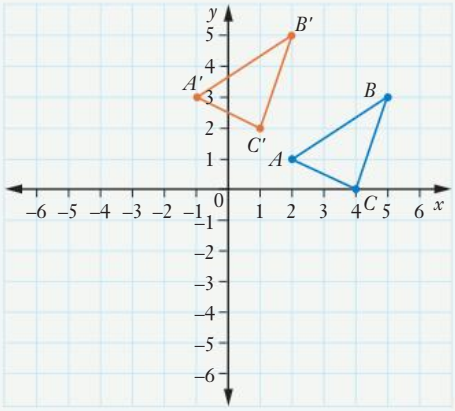
5 **a** (−9, −11) **b** (0, −2)
c (6, −13) **d** (−7, 3)

6

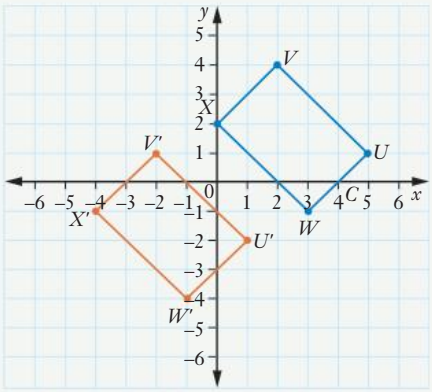


ISBN 9780170412995

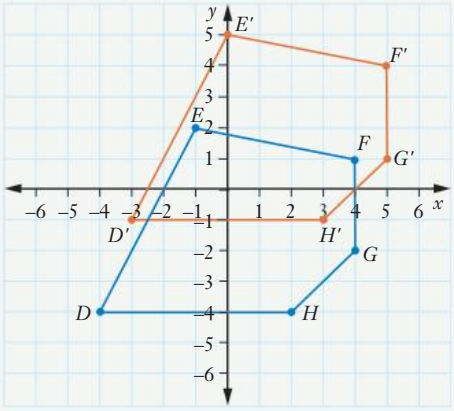
7



8



9



10 $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+4 \\ y+7 \end{bmatrix}$. Applying $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+a \\ y+b \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+c \\ y+d \end{bmatrix}$ successively gives the translation $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+a+c \\ y+b+d \end{bmatrix}$

11 Proof

Exercise 10.02

1 a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{\sqrt{2}x-\sqrt{2}y}{2} \\ \frac{\sqrt{2}x+\sqrt{2}y}{2} \end{bmatrix}$

b $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{x-\sqrt{3}y}{2} \\ \frac{\sqrt{3}x+y}{2} \end{bmatrix}$

c $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{\sqrt{3}x+y}{2} \\ \frac{-x+\sqrt{3}y}{2} \end{bmatrix}$

d $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{-\sqrt{3}x-y}{2} \\ \frac{x-\sqrt{3}y}{2} \end{bmatrix}$

e $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{x+\sqrt{3}y}{2} \\ \frac{-\sqrt{3}x+y}{2} \end{bmatrix}$

2 a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0.88x-0.47y \\ 0.47x+0.88y \end{bmatrix}$

b $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -0.99x-0.14y \\ 0.14x-0.99y \end{bmatrix}$

c $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -0.94x+0.34y \\ -0.34x-0.94y \end{bmatrix}$

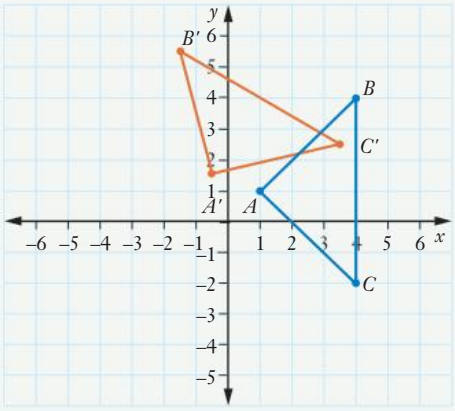
d $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -0.74x+0.68y \\ -0.68x-0.74y \end{bmatrix}$

e $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0.79x-0.62y \\ 0.62x+0.79y \end{bmatrix}$

3 a $A'\left(\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$, $B'(2-2\sqrt{3}, 2+2\sqrt{3})$, $C'(2+\sqrt{3}, 2\sqrt{3}-1)$

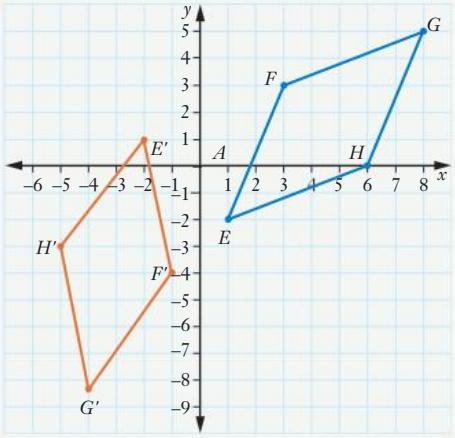
ISBN 9780170412995

b



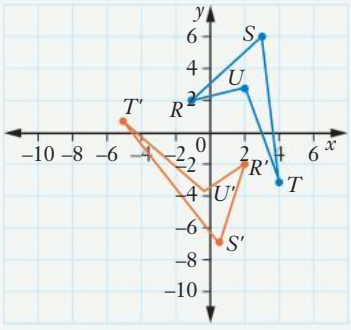
4 a $E'\left(-\frac{2+\sqrt{3}}{2}, \frac{-1+2\sqrt{3}}{2}\right)$, $F'\left(\frac{3-3\sqrt{3}}{2}, -\frac{3+3\sqrt{3}}{2}\right)$, $G'\left(\frac{5-8\sqrt{3}}{2}, -\frac{8+5\sqrt{3}}{2}\right)$, $H'(-3\sqrt{3}, -3)$

b



5 a $R'\left(\frac{2+\sqrt{3}}{2}, \frac{1-2\sqrt{3}}{2}\right)$, $S'\left(\frac{6-3\sqrt{3}}{2}, \frac{-3-6\sqrt{3}}{2}\right)$, $T'\left(\frac{-3-4\sqrt{3}}{2}, \frac{-4+3\sqrt{3}}{2}\right)$, $U'\left(\frac{3-2\sqrt{3}}{2}, \frac{-2-3\sqrt{3}}{2}\right)$

b



6 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

c $\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

e $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

7 a $\begin{bmatrix} -0.64 & 0.77 \\ -0.77 & -0.64 \end{bmatrix}$ **b** $\begin{bmatrix} 0.70 & -0.72 \\ 0.72 & 0.70 \end{bmatrix}$

c $\begin{bmatrix} -0.80 & -0.60 \\ 0.60 & -0.80 \end{bmatrix}$ **d** $\begin{bmatrix} -0.88 & 0.47 \\ -0.47 & -0.88 \end{bmatrix}$

e $\begin{bmatrix} 0.17 & 0.99 \\ -0.99 & 0.17 \end{bmatrix}$

8 a $P'\left(\frac{5\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $Q'(\sqrt{2}, -\sqrt{2})$, $R'\left(\frac{3\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$

b $P'\left(\frac{2-3\sqrt{3}}{2}, -\frac{3+2\sqrt{3}}{2}\right)$, $Q'(-\sqrt{3}, -1)$, $R'\left(-\frac{1+4\sqrt{3}}{2}, \frac{-4+\sqrt{3}}{2}\right)$

c $P'(1.22, -3.39)$, $Q'(-0.48, -1.94)$, $R'(-1.93, -3.65)$

9 a $A'\left(\frac{3+3\sqrt{3}}{2}, \frac{-3+3\sqrt{3}}{2}\right)$, $B'\left(\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$, $C'\left(-\frac{4+\sqrt{3}}{2}, \frac{1-4\sqrt{3}}{2}\right)$, $D'\left(-\frac{1+3\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right)$

ISBN 9780170412995

b $A'\left(\frac{3+3\sqrt{3}}{2}, \frac{-3+\sqrt{3}}{2}\right)$, $B'\left(\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$, $C'\left(-\frac{4+\sqrt{3}}{2}, \frac{1-4\sqrt{3}}{2}\right)$, $D'\left(-\frac{1+3\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right)$

c $A'(-2.43, -3.48)$, $B'(-4.92, -0.87)$, $C'(0.29, 4.11)$, $D'(2.78, 1.51)$

10 a $P'\left(\frac{3+2\sqrt{3}}{2}, \frac{-2+3\sqrt{3}}{2}\right)$, $R'\left(-\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$, $G'(-1-2\sqrt{3}, 2-\sqrt{3})$, $M'\left(\frac{3-\sqrt{3}}{2}, \frac{1+3\sqrt{3}}{2}\right)$

b $P'\left(\frac{3+2\sqrt{3}}{2}, \frac{-2+3\sqrt{3}}{2}\right)$, $R'\left(-\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$, $G'(-1-2\sqrt{3}, 2-\sqrt{3})$, $M'\left(\frac{3-\sqrt{3}}{2}, \frac{1+3\sqrt{3}}{2}\right)$

c $P'(-0.67, -3.54)$, $R'(0.53, 1.31)$, $G'(2.90, 3.40)$, $M'(2.09, -2.37)$

11 A rotation through an angle $-\theta$ around the origin.

12 No, you get the same result.

13–14 Proofs

Exercise 10.03

1 a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x \\ 2y \end{bmatrix}$ **b** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0.5x \\ 0.5y \end{bmatrix}$

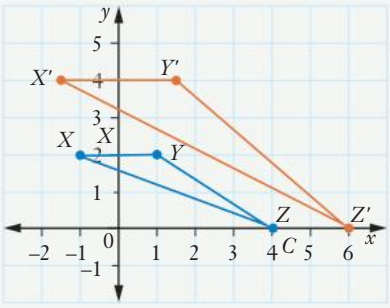
c $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x \\ 0.6y \end{bmatrix}$ **d** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 5x \\ 3y \end{bmatrix}$

e $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1.5x \\ 2.5y \end{bmatrix}$

2 a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1.5x \\ 2y \end{bmatrix}$

b $X'(-1.5, 4)$, $Y'(1.5, 4)$, $Z'(6, 0)$

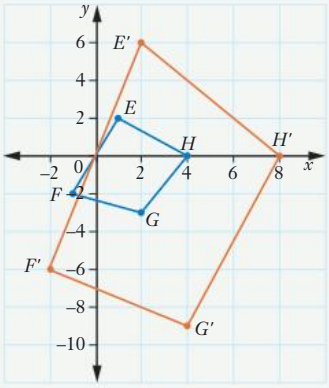
c



3 a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x \\ 3y \end{bmatrix}$

b $E'(2, 6)$, $F'(-2, -6)$, $G'(4, -9)$, $H'(8, 0)$

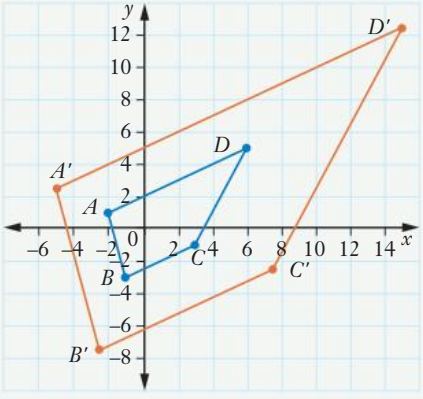
c



4 a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2.5x \\ 2.5y \end{bmatrix}$

b $A'(-5, 2.5)$, $B'(-2.5, -7.5)$, $C'(7.5, -2.5)$, $D'(15, 12.5)$

c



ISBN 9780170412995

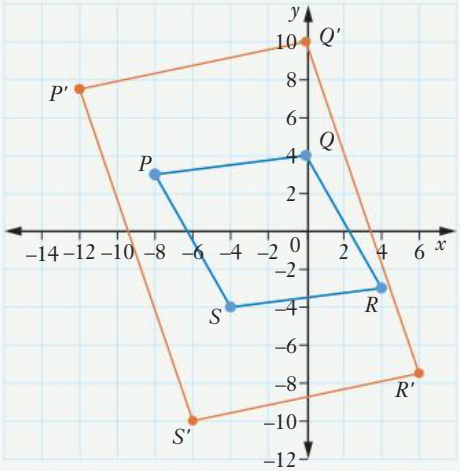
5 **a** $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

d $\begin{bmatrix} 3 & 0 \\ 0 & 0.6 \end{bmatrix}$ **e** $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}$

6 **a** $\begin{bmatrix} 1.5 & 0 \\ 0 & 2.5 \end{bmatrix}$

b $P'(-12, 7.5)$, $Q'(0, 10)$, $R'(6, -7.5)$, $S'(-6, -10)$

c

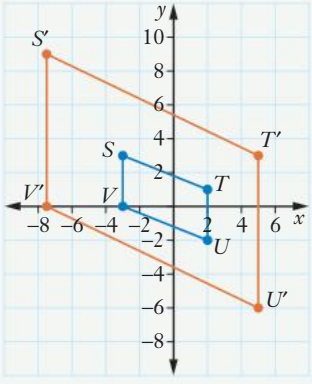


d The image is a parallelogram.

7 **a** $\begin{bmatrix} 2.5 & 0 \\ 0 & 3 \end{bmatrix}$

b $S'(-7.5, 9)$, $T'(5, 3)$, $U'(5, -6)$, $V'(-7.5, 0)$

c

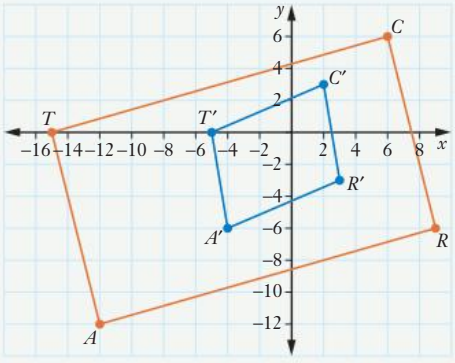


d The image is still a parallelogram.

8 **a** $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

b $R'(3, -3)$, $C'(2, 3)$, $T'(-5, 0)$, $A'(-4, -6)$

c



d The image is a parallelogram.

9 A dilation with factors $\lambda_1 = 2$ and $\lambda_1 = \frac{1}{3}$ with centre at the origin.

10 No

11–12 Proofs

Exercise 10.04

1 **a** $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{x+\sqrt{3}y}{2} \\ \frac{\sqrt{3}x-y}{2} \end{bmatrix}$

b $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -0.6x-0.8y \\ -0.8x+0.6y \end{bmatrix}$

c $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -y \\ -x \end{bmatrix}$

d $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0.6x+0.8y \\ 0.8x-0.6y \end{bmatrix}$

e $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{-x+\sqrt{3}y}{2} \\ \frac{\sqrt{3}x+y}{2} \end{bmatrix}$

f $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -y \\ -x \end{bmatrix}$

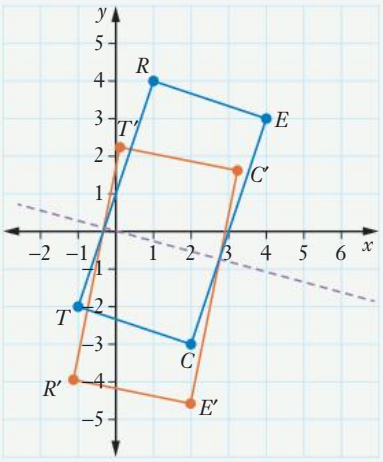
ISBN 9780170412995

2 a $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{\sqrt{3}x - y}{2} \\ \frac{-x - \sqrt{3}y}{2} \end{bmatrix}$

b $R'\left(\frac{\sqrt{3}-4}{2}, \frac{-1-4\sqrt{3}}{2}\right), E'\left(\frac{4\sqrt{3}-3}{2}, \frac{-4-3\sqrt{3}}{2}\right),$

$C'\left(\frac{2\sqrt{3}+3}{2}, \frac{3\sqrt{3}-2}{2}\right), D'\left(\frac{2-\sqrt{3}}{2}, \frac{1+2\sqrt{3}}{2}\right)$

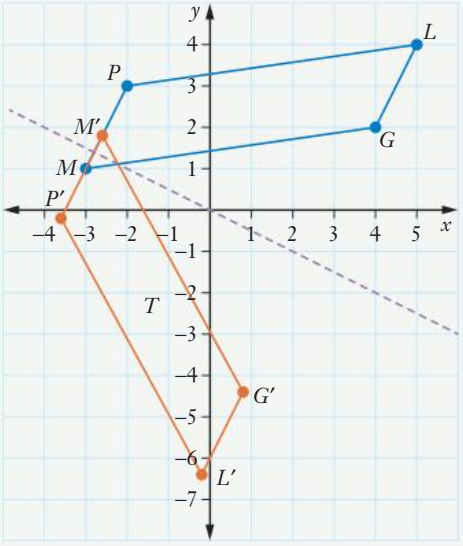
c



3 a $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0.6x - 0.8y \\ -0.8x - 0.6y \end{bmatrix}$

b $P'(-3.6, -0.2), L'(-0.2, -6.4), G'(0.8, -4.4),$ $M'(-2.6, 1.8)$

c

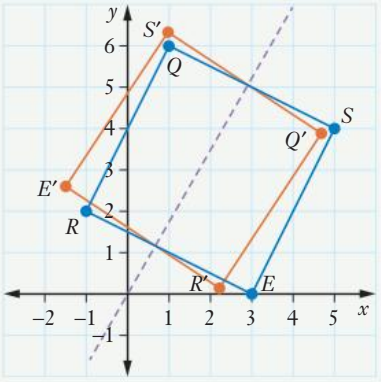


4 a $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{-x + \sqrt{3}y}{2} \\ \frac{\sqrt{3}x + y}{2} \end{bmatrix}$

b $S'\left(\frac{4\sqrt{3}-5}{2}, \frac{5\sqrt{3}+4}{2}\right), Q'\left(\frac{6\sqrt{3}-1}{2}, \frac{\sqrt{3}+6}{2}\right),$

$R'\left(\frac{2\sqrt{3}+1}{2}, \frac{2-\sqrt{3}}{2}\right), E'\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

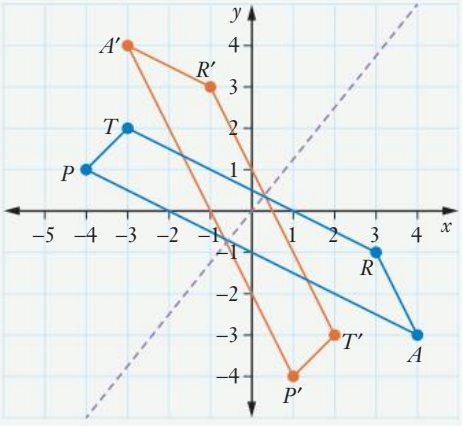
c



5 a $\mathbf{M}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} y \\ x \end{bmatrix}$

b $T'(2, -3), R'(-1, 3), A'(-3, 4), P'(1, -4)$

c



6 a $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **b** $\begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$

ISBN 9780170412995

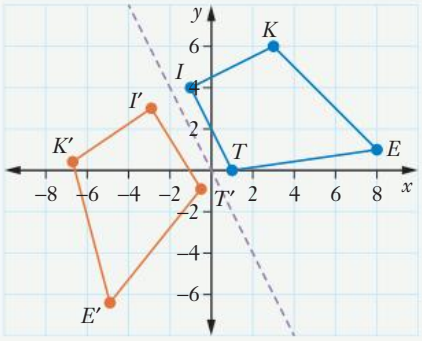
c $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} 0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix}$

e $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **f** $\begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$

7 a $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

b $K'(6.7, 0.4)$, $I'(-3.0, 2.9)$, $T'(-0.5, -0.9)$, $E'(-4.9, -6.4)$

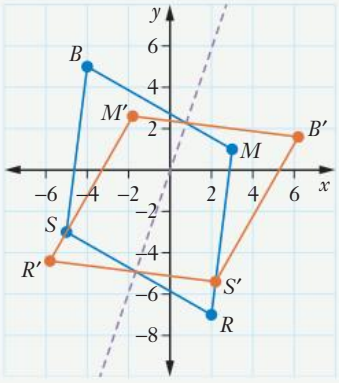
c



8 a $\begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$

b $R'(-5.8, -4.4)$, $M'(-1.8, 2.6)$, $B'(6.2, 1.6)$, $S'(2.2, -5.4)$

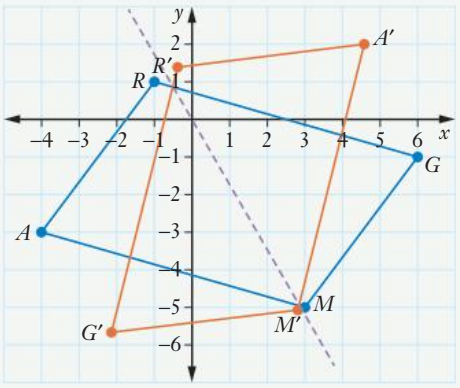
c



9 a $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

b $G'\left(\frac{\sqrt{3}-6}{2}, \frac{-1-6\sqrt{3}}{2}\right)$, $R'\left(\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$, $A'\left(\frac{3\sqrt{3}+4}{2}, \frac{4\sqrt{3}-3}{2}\right)$, $M'\left(\frac{5\sqrt{3}-3}{2}, -\frac{3\sqrt{3}+5}{2}\right)$

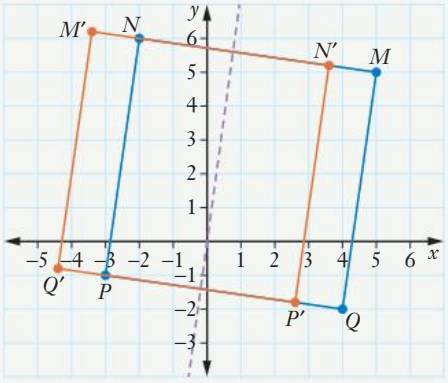
c



10 a $\begin{bmatrix} -0.96 & 0.28 \\ 0.28 & 0.96 \end{bmatrix}$

b $M'(-3.4, 6.2)$, $N'(3.6, 5.2)$, $P'(2.6, -1.8)$, $Q'(-4.4, -0.8)$

c



11 Reflection in the line $y = -3x$.

12 Yes

13, 14 Proofs

ISBN 9780170412995

Exercise 10.05

1 $\mathbf{R}_{120^\circ} \times \mathbf{R}_{-120^\circ} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2 $\mathbf{R}_{-60^\circ} \times \mathbf{R}_{60^\circ} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3 $\mathbf{D}_1 \times \mathbf{D}_2 = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4 $\mathbf{D}_1 \times \mathbf{D}_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3\frac{1}{3} & 0 \\ 0 & 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5 $\mathbf{M} \times \mathbf{M} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6 $\mathbf{M} \times \mathbf{M} = \begin{bmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7 $\begin{bmatrix} -5 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

8 $\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

9 Rotation around the origin through the angle –80°.

10 Dilation with x-factor $\frac{1}{3}$ and y-factor 4.

11 Reflection in the line $y = 0.6x$.

12 Translation with vector $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

13 **a** Singular

b Non-singular, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0.2x + 0.2y \\ 0.6x - 0.4y \end{bmatrix}$

c Non-singular, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -0.4x + 0.6y \\ 0.6x - 0.4y \end{bmatrix}$

d Non-singular, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$

e Singular

f Non-singular, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -5x + y \\ -6x + y \end{bmatrix}$

14–16 Proofs

Exercise 10.06

1 **a** $\begin{bmatrix} 39 \\ 47 \end{bmatrix}$ **b** $\begin{bmatrix} 39 \\ 47 \end{bmatrix}$ **c** they are the same

d $\begin{bmatrix} 20 \\ 45 \end{bmatrix}$ **e** $\begin{bmatrix} 20 \\ 45 \end{bmatrix}$

f They are the same, but different to the other order

2 **a** Both $\begin{bmatrix} -10 \\ -23 \end{bmatrix}$ **b** Both $\begin{bmatrix} 17 \\ -32 \end{bmatrix}$

3 **a** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -x + 17y \\ 23y \end{bmatrix}$

b $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 11x - 16y \\ -9x + 11y \end{bmatrix}$

c They are different.

4 $\mathbf{T}_2 \circ \mathbf{T}_1$: $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x - y \\ -14x + 8y \end{bmatrix}$ and

$\mathbf{T}_1 \circ \mathbf{T}_2$: $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -5x - 9y \\ 10x + 16y \end{bmatrix}$

5 $\mathbf{T}_2 \circ \mathbf{T}_1$: $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -16x - 6y \\ -17x + 5y \end{bmatrix}$ and

$\mathbf{T}_1 \circ \mathbf{T}_2$: $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -4x - 14y \\ -15x - 7y \end{bmatrix}$

ISBN 9780170412995

6 $\mathbf{T}_2 \circ \mathbf{T}_1: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 33x+21y \\ 35x+40y \end{bmatrix}$ and

$\mathbf{T}_1 \circ \mathbf{T}_2: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 33x+21y \\ 35x+40y \end{bmatrix}$; the same!

7 $\mathbf{D} \circ \mathbf{M} = \begin{bmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{3}{2} & \frac{-3\sqrt{3}}{2} \end{bmatrix}$ and

$\mathbf{M} \circ \mathbf{D} = \begin{bmatrix} \frac{\sqrt{3}}{4} & \frac{3}{2} \\ \frac{1}{4} & \frac{-3\sqrt{3}}{2} \end{bmatrix}$, which are different.

8 $\mathbf{R} \circ \mathbf{D} = \begin{bmatrix} -\frac{3}{20} & -\frac{3\sqrt{3}}{4} \\ \frac{3\sqrt{3}}{20} & -\frac{3}{4} \end{bmatrix}$ and

$\mathbf{D} \circ \mathbf{R} = \begin{bmatrix} -\frac{3}{20} & -\frac{3\sqrt{3}}{20} \\ \frac{3\sqrt{3}}{4} & -\frac{3}{4} \end{bmatrix}$, which are different.

9 $\mathbf{M} \circ \mathbf{R} = \begin{bmatrix} \frac{-3\sqrt{3}+4}{10} & \frac{-3-4\sqrt{3}}{10} \\ \frac{-4\sqrt{3}-3}{10} & \frac{3\sqrt{3}-4}{10} \end{bmatrix}$ and

$\mathbf{R} \circ \mathbf{M} = \begin{bmatrix} \frac{3\sqrt{3}+4}{10} & \frac{-3+4\sqrt{3}}{10} \\ \frac{4\sqrt{3}-3}{10} & \frac{-3\sqrt{3}-4}{10} \end{bmatrix}$,

which are different.

10 $\mathbf{M}_1 \circ \mathbf{M}_2 = \begin{bmatrix} \frac{-4\sqrt{3}-3}{10} & \frac{-3\sqrt{3}-4}{10} \\ \frac{4-3\sqrt{3}}{10} & \frac{-4\sqrt{3}-3}{10} \end{bmatrix}$ and

$\mathbf{M}_2 \circ \mathbf{M}_1 = \begin{bmatrix} \frac{-4\sqrt{3}-3}{10} & \frac{-3\sqrt{3}+4}{10} \\ \frac{3\sqrt{3}-4}{10} & \frac{-4\sqrt{3}-3}{10} \end{bmatrix}$,

which are different.

Exercise 10.07

1 $8 \times |-3| = 24$

2 $12 \times 1 = 12$

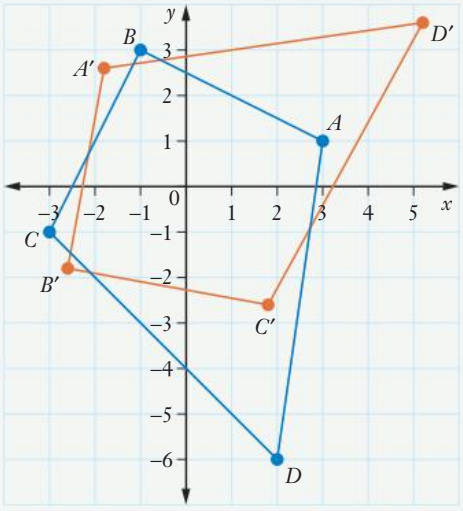
3 $60 \times 8 = 480$

4 $12.5 \times 12 = 150$

5 a $\begin{bmatrix} -0.28 & -0.96 \\ 0.96 & -0.28 \end{bmatrix}$

b $A'(-1.8, 2.6)$, $B'(-2.6, -1.8)$, $C'(1.8, -2.6)$, $D'(5.2, 3.6)$

c

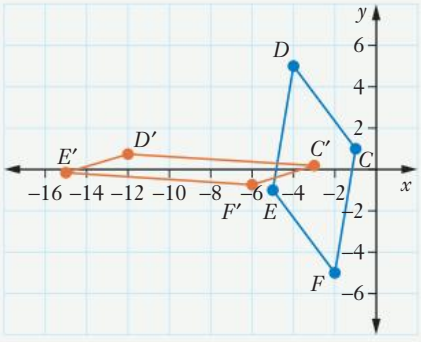


d A rotation of $\cos^{-1}(-0.28) \approx 106.3°$

6 a $\begin{bmatrix} 3 & 0 \\ 0 & 0.15 \end{bmatrix}$

b $C'(-3, 0.15)$, $D'(-12, 0.75)$, $E'(-15, -0.15)$, $F'(-6, -0.75)$

c



d A dilation with $\lambda_1 = 3$ and $\lambda_2 = 0.15$.

ISBN 9780170412995

7 a $\begin{bmatrix} \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{-\sqrt{6}-\sqrt{2}}{4} \\ \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4} \end{bmatrix}$

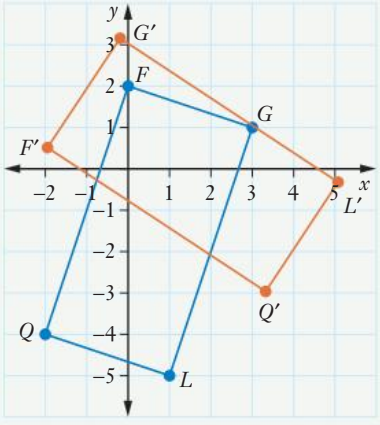
b $F'\left(\frac{-\sqrt{6}-\sqrt{2}}{2}, \frac{\sqrt{6}-\sqrt{2}}{2}\right)$,

$G'\left(\frac{\sqrt{6}-2\sqrt{2}}{2}, \frac{2\sqrt{6}+\sqrt{2}}{2}\right)$,

$L'\left(\frac{3\sqrt{6}+2\sqrt{2}}{2}, \frac{3\sqrt{2}-2\sqrt{6}}{2}\right)$,

$Q'\left(\frac{\sqrt{6}+3\sqrt{2}}{2}, \frac{\sqrt{2}-3\sqrt{6}}{2}\right)$

c



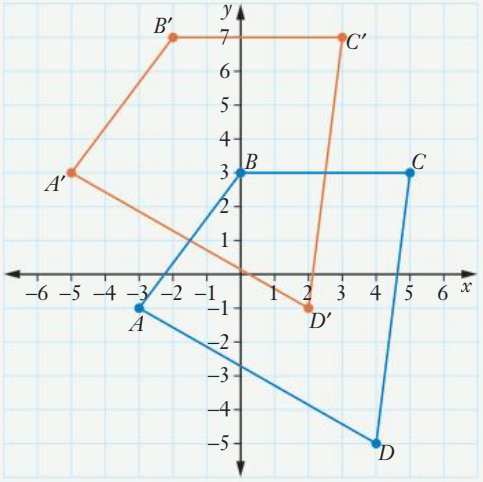
d Rotation through $\frac{5\pi}{12}$.

8–10 Proofs

Chapter review

1 $\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2

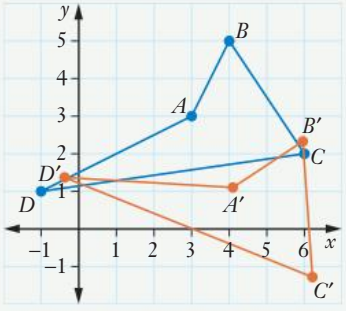


3 a $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

b $P'\left(\frac{3\sqrt{3}+3}{2}, \frac{3\sqrt{3}-3}{2}\right)$, $Q'\left(\frac{4\sqrt{3}+5}{2}, \frac{5\sqrt{3}-4}{2}\right)$,

$R'(3\sqrt{3}+1, \sqrt{3}-3)$, $S'\left(\frac{-\sqrt{3}+1}{2}, \frac{\sqrt{3}+1}{2}\right)$

c



4 a $\begin{bmatrix} 0.3090 & -0.9511 \\ 0.9511 & 0.3090 \end{bmatrix}$

b $A'(-1.26, -0.64)$, $B'(-1.9, 0.62)$, $C'(-1.59, 1.57)$, $D'(-0.33, 2.21)$, $E'(2.21, 0.33)$, $F'(1.9, -0.62)$

ISBN 9780170412995

5 **a** $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2.5x \\ 3y \end{bmatrix}$

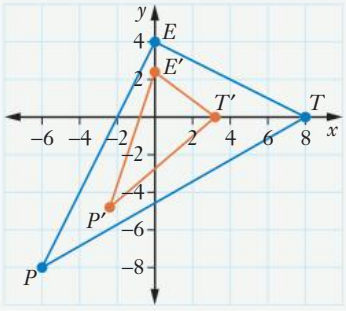
b $K'(-7.5, -6)$, $L'(2.5, 6)$, $M'(5, 3)$, $N'(-5, -9)$

c A parallelogram

6 **a** $\begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix}$

b $P'(-2.4, -4.8)$, $E'(0, 2.4)$, $T'(3.2, 0)$

c



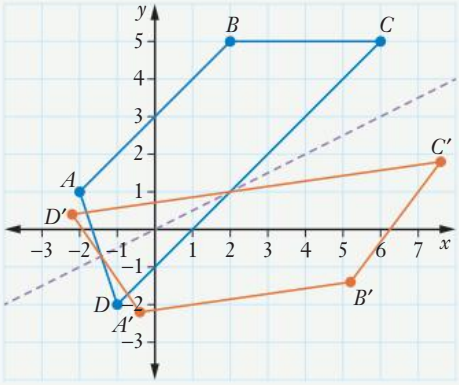
7 $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \dfrac{-\sqrt{2}x + \sqrt{2}y}{2} \\ \dfrac{\sqrt{2}x + \sqrt{2}y}{2} \end{bmatrix}$

8 $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -0.96x - 0.28y \\ -0.28x + 0.96y \end{bmatrix}$

9 **a** $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$

b $A'(-0.4, -2.2)$, $B'(5.2, -1.4)$, $C'(7.6, 1.8)$, $D'(-2.2, 0.4)$

c



10 **a** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

b $P'(0, 4)$, $K'(-4, -2)$, $C'(-3, -3)$, $M'(1, 3)$

11 $\begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix}\begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

12 **a** Non-singular, $\begin{bmatrix} 5 & -3 \\ 3 & -2 \end{bmatrix}$

b Non-singular, $\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -1\frac{1}{2} & 1 \end{bmatrix}$

c Singular

13 **a** $\begin{bmatrix} -16 \\ 27 \end{bmatrix}$ **b** $\begin{bmatrix} -16 \\ 27 \end{bmatrix}$

14 $\mathbf{P} \circ \mathbf{Q}$: $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 4x - 3y \\ 2x + y \end{bmatrix}$

15 $\mathbf{R} \circ \mathbf{D} = \begin{bmatrix} \frac{5}{4} & \frac{2\sqrt{3}}{5} \\ -\frac{5\sqrt{3}}{4} & \frac{2}{5} \end{bmatrix}$

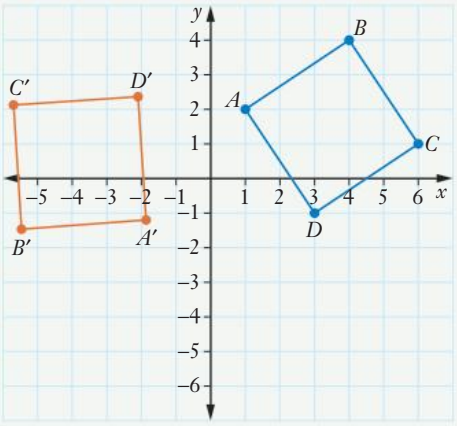
16 $50 \times 5 = 250$

17 **a** $\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

b $A'\left(\dfrac{-\sqrt{3}-2}{2}, \dfrac{1-2\sqrt{3}}{2}\right)$, $B'(-2\sqrt{3} - 2, 2 - 2\sqrt{3})$, $C'\left(\dfrac{-6\sqrt{3}-1}{2}, \dfrac{6-\sqrt{3}}{2}\right)$, $D'\left(\dfrac{-3\sqrt{3}+1}{2}, \dfrac{3+3\sqrt{3}}{2}\right)$

ISBN 9780170412995

c



d Rotation around the origin through the angle 150°.

18–22 Proofs

Chapter 11

Exercise 11.01

1 **a** 5 **b** 13 **c** $5\sqrt{2}$ **d** 17
e $2\sqrt{10}$ **f** $5\sqrt{5}$ **g** 14 **h** 10

2 **a** $\frac{\pi}{4}$ **b** $-\frac{2\pi}{3}$ **c** $\frac{\pi}{6}$ **d** $-\frac{\pi}{4}$
e $-\frac{\pi}{6}$ **f** $-\frac{\pi}{2}$ **g** $\tan^{-1}\left(\frac{5}{4}\right)$
h $-\pi+\tan^{-1}\left(\frac{3}{7}\right)$

3 **a** 6.1, 9.5° **b** 4.5, 153.4° **c** 3.2, –18.4°
d 10.8, –146.3° **e** 9.4, 122.0° **f** 6.1, 80.5°
g 9.5, –108.4° **h** 3.6, –56.3°

4 **a** 7.62, –1.98 **b** 9.85, 1.15 **c** 6.71, –1.11
d 8.25, 1.82 **e** 12.21, 0.61 **f** 4.47, –2.03
g 15, –0.93 **h** 7.81, 2.27

5–10 Proofs

Exercise 11.02

1 **a** $(\sqrt{5}, \tan^{-1}(2))$, $\sqrt{5}(\cos[\tan^{-1}(2)] + i\sin[\tan^{-1}(2)])$

b $\left(2, -\frac{\pi}{6}\right), 2\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$

c $\left(\sqrt{34}, \tan^{-1}\left(\frac{5}{3}\right)-\pi\right)$, $\sqrt{34}[\cos(\theta)+i\sin(\theta)]$, where $\theta=\tan^{-1}\left(\frac{5}{3}\right)-\pi$

d $\left(2, \frac{2\pi}{3}\right), 2\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

e $\left(5, \tan^{-1}\left(\frac{4}{3}\right)\right), 5\left(\cos\left[\tan^{-1}\left(\frac{4}{3}\right)\right]+i\sin\left[\tan^{-1}\left(\frac{4}{3}\right)\right]\right)$

f $\left(\sqrt{53}, \pi-\tan^{-1}\left(\frac{2}{7}\right)\right)$, $\sqrt{53}\,[\cos(\theta)+i\sin(\theta)]$, where $\theta=\pi-\tan^{-1}\left(\frac{2}{7}\right)$

2 **a** $\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$

b $2\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$

c $3\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$

d $6\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$

e $2\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$

f $12\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

g $\sqrt{5}\left(\cos\left[-\tan^{-1}\left(\frac{1}{2}\right)\right]+i\sin\left[-\tan^{-1}\left(\frac{1}{2}\right)\right]\right)$

h $\sqrt{13}\,[\cos(\theta)+i\sin(\theta)]$, where $\theta=\pi-\tan^{-1}\left(\frac{3}{2}\right)$

i $\sqrt{5}(\cos[-\tan^{-1}(2)] + i\sin[-\tan^{-1}(2)])$

j $\sqrt{10}[\cos(\theta)+i\sin(\theta)]$, where $\theta=\tan^{-1}(3)-\pi$

k $\sqrt{5}\,[\cos(\theta)+i\sin(\theta)]$, where $\theta=\tan^{-1}\left(\frac{1}{2}\right)-\pi$

l $\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]$

3 **a** 8.06[cos (2.09) + *i* sin(2.09)]
b 10.63[cos (–0.85) + *i* sin (–0.85)]
c 5.39[cos (1.19) + *i* sin (1.19)]
d 13.60[cos (–2.20) + *i* sin (–2.20)]
e 12.37[cos (1.82) + *i* sin (1.82)]
f 19.42[cos (0.60) + *i* sin (0.60)]
g 13.45[cos (–2.30) + *i* sin (–2.30)]
h 10.77[cos (–0.38) + *i* sin (–0.38)]

4 **a** $\frac{5}{2}+\frac{5\sqrt{3}}{2}i$ **b** $4\sqrt{3}-4i$
c $-\sqrt{2}+\sqrt{2}i$ **d** $-3\sqrt{3}-3i$

ISBN 9780170412995

e $-\frac{7}{2}+\frac{7\sqrt{3}}{2}i$ **f** $\frac{11\sqrt{3}}{2}+\frac{11\sqrt{3}}{2}i$

g $-10\sqrt{2}-10\sqrt{2}i$ **h** $\frac{15}{2}-\frac{15\sqrt{3}}{2}i$

5 **a** $2.17+5.59i$ **b** $-7.77-1.91i$

c $-3.94+3.08i$ **d** $-4.99+10.91i$

e $13.58-10.23i$ **f** $-1.37-3.76i$

g $-1.33+1.49i$ **h** $-3.03-10.57i$

i $8.20+3.70i$ **j** $-5.94-0.85i$

6 **a** $\frac{1}{2}+\frac{1}{2}i, \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{4}\right)$

b $0.3-0.1i, \frac{\sqrt{10}}{10}\text{cis}\left[\tan^{-1}\left(-\frac{1}{3}\right)\right]$

c $0.1, 0.1\ \text{cis}(0)$

d $9-7i, \sqrt{130}\,\text{cis}\left[-\tan^{-1}\left(\frac{7}{9}\right)\right]$

e $-2+2i, 2\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$

f $-\frac{3\sqrt{3}}{4}+\frac{3}{4}i, \frac{3}{2}\text{cis}\left(\frac{5\pi}{6}\right)$

g $-2+2i, 2\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$

h $1+2i, \sqrt{5}\,\text{cis}\,[\tan^{-1}(2)]$

Exercise 11.03

1 **a** $6\,\text{cis}\left(\frac{5\pi}{6}\right)$ **b** $2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$

c $4\sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$ **d** $15\,\text{cis}\left(\frac{\pi}{2}\right)$

e $8\,\text{cis}\left(-\frac{\pi}{2}\right)$ **f** $2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$

g $8\,\text{cis}\left(\frac{3\pi}{2}\right)$ **h** $2\sqrt{2}\,\text{cis}\left(-\frac{11\pi}{12}\right)$

2 **a** $-3\sqrt{3}+3i$ **b** $4-4i$

c $-10-10i$ **d** $0-4i$

e $3.86+1.04i$ **f** $5.46+1.46i$

g $\frac{21}{2}-\frac{21\sqrt{3}}{2}i$ **h** $2\sqrt{2}+2\sqrt{2}i$

3 **a** $2\sqrt{2}\ \text{cis}\left(\frac{5\pi}{12}\right)$ **b** $\sqrt{2}\ \text{cis}\left(-\frac{\pi}{12}\right)$

c $12\sqrt{2}\ \text{cis}\left(\frac{\pi}{12}\right)$ **d** $4\sqrt{2}\ \text{cis}\left(-\frac{\pi}{12}\right)$

e $20\sqrt{2}\ \text{cis}\left(-\frac{5\pi}{12}\right)$ **f** $\frac{8\sqrt{2}}{5}\text{cis}\left(\frac{11\pi}{12}\right)$

g $4\sqrt{2}\ \text{cis}\left(-\frac{11\pi}{12}\right)$ **h** $8\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$

4 **a** $32\ \text{cis}\ (150°) = -16\sqrt{3}+16i$

b $9\ \text{cis}\left(\frac{\pi}{3}\right) = 4.5+4.5\sqrt{3}i$

c $16\ \text{cis}\left(-\frac{2\pi}{3}\right) = -8-8\sqrt{3}i$

d $256\ \text{cis}\ (120°) = -128+128\sqrt{3}i$

e $125\ \text{cis}\ (0) = 125$

f $27\ \text{cis}\ (60°) = 13.5+13.5\sqrt{3}i$

g $\text{cis}\ (45°) = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$

h $4\ \text{cis}\left(\frac{\pi}{4}\right) = 2\sqrt{2}+2\sqrt{2}i$

5 **a** $\frac{3}{2}\ \text{cis}\left(-\frac{\pi}{12}\right)$ **b** $\frac{4}{3}\ \text{cis}\left(-\frac{11\pi}{18}\right)$

c $\frac{5}{2}\ \text{cis}\left(\frac{5\pi}{12}\right)$ **d** $\frac{\sqrt{2}}{2}\ \text{cis}\left(\frac{\pi}{12}\right)$

e $\frac{2\sqrt{2}}{3}\ \text{cis}\left(\frac{11\pi}{12}\right)$ **f** $\frac{1}{2}\ \text{cis}\left(-\frac{7\pi}{12}\right)$

6 **a** $\frac{\sqrt{3}}{3}-\frac{1}{3}i$ **b** $2i$

c $-1.3283+0.1162i$ **d** $0.1220+0.4553i$

e $-1.17i$ **f** $0.0806-0.6047i$

7 **a** $\frac{1}{3}\ \text{cis}\left(-\frac{\pi}{6}\right), \frac{\sqrt{3}}{6}-\frac{1}{6}i$

b $\frac{1}{2}\ \text{cis}\left(-\frac{2\pi}{3}\right), -\frac{1}{4}-\frac{\sqrt{3}}{4}i$

c $\frac{1}{2}\ \text{cis}\left(\frac{\pi}{3}\right), \frac{1}{4}+\frac{\sqrt{3}}{4}i$

d $\frac{1}{4}\ \text{cis}\left(\frac{5\pi}{6}\right), -\frac{\sqrt{3}}{8}+\frac{1}{8}i$

e $\frac{\sqrt{2}}{4}\ \text{cis}\left(-\frac{\pi}{4}\right), \frac{1}{4}-\frac{1}{4}i$

f $\frac{1}{2}\ \text{cis}\left(\frac{\pi}{3}\right), \frac{1}{4}+\frac{\sqrt{3}}{4}i$

g $\frac{1}{6}\ \text{cis}\left(-\frac{5\pi}{6}\right), -\frac{\sqrt{3}}{12}-\frac{1}{12}i$

ISBN 9780170412995

h $\frac{1}{4}\text{cis}\left(\frac{2\pi}{3}\right), -\frac{1}{8}+\frac{\sqrt{3}}{8}i$

8 a $\frac{1}{16}\text{cis}\,(-150°) = -\frac{1}{32}\sqrt{3}+\frac{1}{32}i$

b $\frac{1}{8}\text{cis}\left(-\frac{\pi}{3}\right) = \frac{1}{16}-\frac{1}{16}\sqrt{3}i$

c $\frac{1}{25}\text{cis}\left(-\frac{\pi}{3}\right) = \frac{1}{50}-\frac{1}{50}\sqrt{3}i$

d $\frac{1}{16}\text{cis}\,(60°) = \frac{1}{32}+\frac{1}{32}\sqrt{3}i$

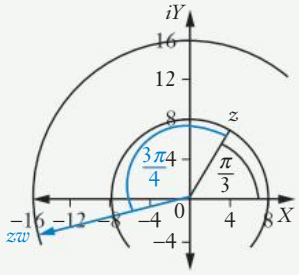
e $\frac{1}{125}\text{cis}\,(180°) = -\frac{1}{125}$

f $\text{cis}\left(-\frac{\pi}{3}\right) = \frac{1}{2}-\frac{1}{2}\sqrt{3}\,i$

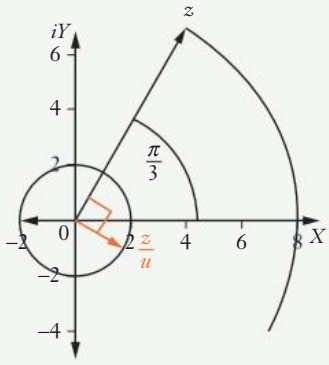
g $\frac{1}{81}\text{cis}\left(\frac{3\pi}{4}\right) = -\frac{1}{162}\sqrt{2}+\frac{1}{162}\sqrt{2}i$

h $\frac{1}{9}\text{cis}\,(120°) = -\frac{1}{18}+\frac{1}{18}\sqrt{3}i$

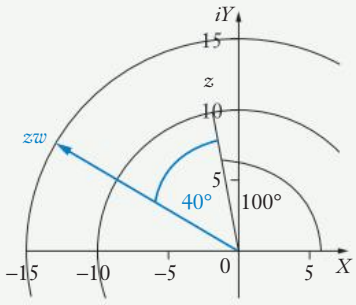
9 a



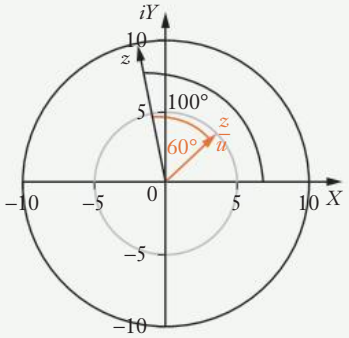
b



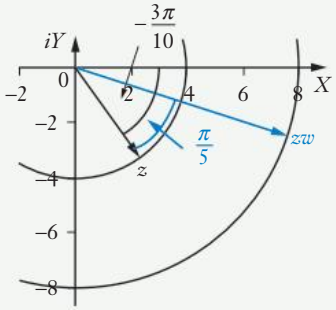
10 a



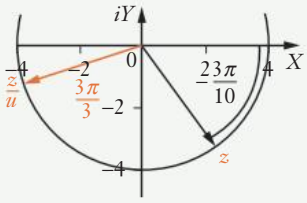
b



11 a



b



12 $\sin(75°) = \frac{\sqrt{6}+\sqrt{2}}{4}, \cos(75°) = \frac{\sqrt{6}-\sqrt{2}}{4}$

13 $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}, \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

14 $1.68 - 0.33i$

15 $0.69 + 1.40i$

Exercise 11.04

1 a $\frac{7}{4}+\frac{\sqrt{23}}{4}i, \frac{7}{4}-\frac{\sqrt{23}}{4}i$ **b** $-\frac{3}{2}, 2$

c $\frac{-2-\sqrt{2}}{2}, \frac{-2+\sqrt{2}}{2}$ **d** -3 only

e $\frac{3-\sqrt{41}}{4}, \frac{3+\sqrt{41}}{4}$ **f** $\frac{3}{2}-\frac{3\sqrt{3}}{2}i, \frac{3}{2}+\frac{3\sqrt{3}}{2}i$

g $\frac{7-\sqrt{57}}{4}, \frac{7+\sqrt{57}}{4}$ **h** $\frac{2}{3}-\frac{2\sqrt{2}}{3}i, \frac{2}{3}+\frac{2\sqrt{2}}{3}i$

ISBN 9780170412995

2 **a** $\Delta = 56$, so roots are irrational and different
b $\Delta = -15$, so roots are complex
c $\Delta = 49$, so roots are rational and different
d $\Delta = 41$, so roots are irrational and different
e $\Delta = 100$, so roots are rational and different
f $\Delta = -116$, so roots are complex
g $\Delta = 0$, so roots are the same and rational
h $\Delta = -68$, so roots are complex

3 $x^2 - 2x - 20 = 0$

4 $3x^2 + 40x + 50 = 0$

5 $16x^2 - 49x + 25 = 0$

6 $8x^2 - x - 1 = 0$

Exercise 11.05

1 **a** $(4q + 1)(5q + 1)$ **b** $(3n + 5)(4n - 3)$
c $(3n - 5)(n - 5)$ **d** $(4g - 5)(7g + 4)$
e $(5h - 6)(h + 6)$ **f** $(8n - 9)(n - 1)$
g $(15m - 11)(4m + 3)$ **h** $(5t - 6)(10t + 7)$

2 **a** $(u - 3 + \sqrt{2})(u - 3 - \sqrt{2})$
b $(z + 1 + \sqrt{10})(z + 1 - \sqrt{10})$
c $(x - 2 + \sqrt{5})(x - 2 - \sqrt{5})$
d $(a + 5 + \sqrt{3})(a + 5 - \sqrt{3})$
e $(a + 8 + \sqrt{7})(a + 8 - \sqrt{7})$
f $(g - 2 + \sqrt{6})(g - 2 - \sqrt{6})$
g $(u - 12 + \sqrt{5})(u - 12 - \sqrt{5})$
h $(m - 10 + \sqrt{10})(m - 10 - \sqrt{10})$

3 **a** $(v - 1 + i)(v - 1 - i)$
b $(m + 2 + i)(m + 2 - i)$
c $(n + 3 + 2i)(n + 3 - 2i)$
d $(w + 1 + 6i)(w + 1 - 6i)$
e $(n + 3 + 6i)(n + 3 - 6i)$
f $(h + 6 + 7i)(h + 6 - 7i)$
g $(r + 8 + 3i)(r + 8 - 3i)$
h $(e + 5 + 6i)(e + 5 - 6i)$

4 **a** $(k - 4 + 6i)(k - 4 - 6i)$ **b** $(2r + 3)(5r + 4)$
c $(q - 2 + 5i)(q - 2 - 5i)$ **d** $(c + 8 + 5i)(c + 8 - 5i)$
e $(10u - 4)(7u + 6)$
f $(g - 11 + \sqrt{10})(g - 11 - \sqrt{10})$
g $(3t - 4)(7t + 8)$
h $(y - 7 + 3i)(y - 7 - 3i)$

5 **a** $(3h + 2)(7h + 12)$ **b** $(y - 4 + 2i)(y - 4 - 2i)$
c $(q - 8 + 4i)(q - 8 - 4i)$ **d** $(3e - 1)(6e + 7)$
e $(a + 1 + 5i)(a + 1 - 5i)$
f $(e - 6 + 2\sqrt{2})(e - 6 - 2\sqrt{2})$
g $(u + 6 + i)(u + 6 - i)$
h $(r + 10 + 2i)(r + 10 - 2i)$

Chapter review

1 **a** $|z| = 10$, $\text{Arg}(z) = \dfrac{5\pi}{6}$

b $|z| = \sqrt{97}$, $\text{Arg}(z) = -\tan^{-1}\left(\frac{4}{9}\right)$

c $|z| \approx 3.606$, $\text{Arg}(z) \approx 0.588$

2 **a** $\left(6, -\dfrac{3\pi}{4}\right)$, $6\left[\cos\left(-\dfrac{3\pi}{4}\right) + i\sin\left(-\dfrac{3\pi}{4}\right)\right]$

b $\left(\sqrt{65}, \pi - \tan^{-1}\left(\frac{7}{4}\right)\right)$,
$\sqrt{65}\left(\cos\left[\pi - \tan^{-1}\left(\frac{7}{4}\right)\right] + i\sin\left[\pi - \tan^{-1}\left(\frac{7}{4}\right)\right]\right)$

3 $-2\sqrt{3} - 2$

4 $-4.02 - 5.73i$

5 **a** $20\ \text{cis}\left(\dfrac{\pi}{4}\right) = 10\sqrt{2} + 10\sqrt{2}i$

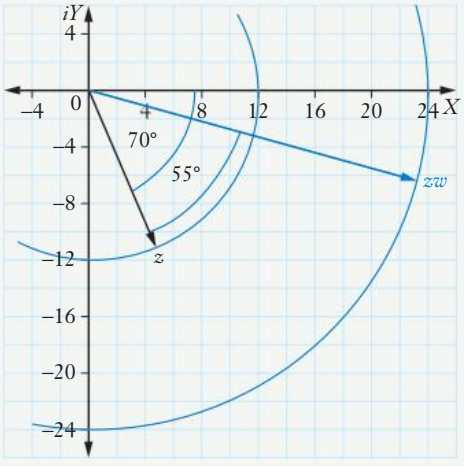
b $21\ \text{cis}(-122°) \approx -11.13 - 17.81i$

c $81\ \text{cis}(-60°) = 40.5 - 40.5\sqrt{3}i$

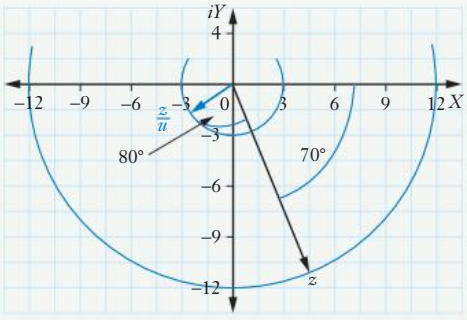
6 **a** $4\ \text{cis}(150°) = -2\sqrt{3} + 2i$

b $\frac{1}{64}\text{cis}\left(\dfrac{\pi}{2}\right) = \frac{1}{64}i$

7 **a**



b



8 **a** $7, -\frac{4}{3}$ **b** $8-\sqrt{10}, 8+\sqrt{10}$

c $6+7i, 6-7i$

9 **a** Complex **b** Different and irrational

c Different and rational

10 $36x^2+9x-8=0$

11 **a** $(h-5+2i)(h-5-2i)$ **b** $(r+4)(4r-3)$

c $(k-6+\sqrt{5})(k-6-\sqrt{5})$

12, 13 Proofs

14 $\sin\left(\frac{11\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}, \cos\left(\frac{11\pi}{12}\right)=-\frac{\sqrt{6}+\sqrt{2}}{4}$

15 $64x^2-297x+64=0$

Chapter 12

Exercise 12.01

1 **a** $\frac{1}{2}[\sin(5x)+\sin(4x)]$

b $\frac{1}{2}[\sin(8x)+\sin(2x)]$

c $\frac{1}{2}[\cos(3x)+\cos(5x)]$

d $\cos(4x)+\cos(2x)$

e $\sin(11x)-\sin(x)$

f $\sin(4x)-\sin(2x)$

g $-\frac{5}{2}[\cos(3x)-\cos(9x)]$

h $-\frac{5}{2}[\cos(7x)-\cos(x)]$

i $\frac{3}{2}[\sin(7x)+\sin(3x)]$

2 **a** $\frac{\sqrt{3}-1}{4}$ **b** $\frac{\sqrt{3}+1}{4}$ **c** $\frac{1}{2}$

d 0 **e** $\frac{\sqrt{3}}{4}$ **f** $\frac{\sqrt{3}}{4}$

3 **a** 0.633 **b** 0.133

c 0.755 **d** 0.112

4 **a** $\sin(4x)-\sin(2x)$

b $\frac{1}{2}[\cos(a+45°)+\cos(a+15°)]$

c $-2[\cos(3x)-\cos(7x)]$ **d** $\sin(p)+\sin(q)$

e $\frac{3}{2}[\sin(3a+b)-\sin(a-3b)]$

f $-\frac{1}{2}[\cos(2n)-\cos(2m)]$

g $\sin(2m)+\sin(2n)$ **h** $\frac{1}{2}[\cos(8c)+1]$

5 0.25

6 $\frac{-\left(\sqrt{3}+2\right)}{4}$

7–12 Demonstrations

Exercise 12.02

1 **a** $2\sin(5a)\sin(3a)$ **b** $-6\sin(10m)\sin(5m)$

c $-10\sin(6b)\sin(3b)$ **d** $2\sin(5x)\cos(2x)$

e $8\cos(8q)\sin(6q)$ **f** $2\sin(5x)\sin(x)$

2 **a** $2\sin(5x)\cos(2x)$ **b** $2\cos(5x)\sin(3x)$

c $2\cos(7x)\cos(2x)$ **d** $-2\sin(9x)\sin(2x)$

e $2\sin\left(6\frac{1}{2}x\right)\cos\left(2\frac{1}{2}x\right)$

f $-2\cos\left(3\frac{1}{2}x\right)\sin\left(1\frac{1}{2}x\right)$

g $2\cos(3x)\cos(2x)$

h $2\sin(7x)\sin(4x)$

i $2\sin\left(1\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)$

3 **a** $-8\sin(106°)\sin(12°)$

b $4\sin(174°)\sin(72°)$

c $2\sin(35°)\cos(18°)$

d $2\cos(62°)\sin(42°)$

e $2\sin(105°)\sin(35°)$

f $-2\sin(75°)\cos(25°)$

4 **a** $-\frac{\sqrt{6}}{2}$ **b** $\frac{\sqrt{2}}{2}$ **c** $\frac{\sqrt{2}}{2}$

5–10 Demonstrations

ISBN 9780170412995

Exercise 12.03

1 **a** $13\cos(x-1.176)$ **b** $\sqrt{193}\cos(x-0.528)$
c $\sqrt{61}\cos(x-0.695)$ **d** $\sqrt{10}\cos(x+0.322)$
e $\sqrt{13}\cos(x+2.159)$ **f** $\sqrt{17}\cos(x+0.245)$
g $13\cos(x-1.966)$ **h** $\sqrt{10}\cos(x+1.893)$
i $-2.69\cos(x+1.024)$

2 **a** $5\cos(x-0.927)$ **b** $\sqrt{5}\cos(x-1.107)$
c $2\cos\left(x-\frac{\pi}{3}\right)$ **d** $13\cos(x+1.176)$
e $17\cos(x+1.080)$ **f** $2\sqrt{3}\cos\left(x-\frac{\pi}{6}\right)$

3 **a** $\sqrt{34}\sin(x+1.030)$ **b** $13\sin(x+0.395)$
c $\sqrt{85}\sin(x-0.709)$ **d** $5\sin(x+0.644)$
e $\sqrt{17}\sin(x-0.245)$ **f** $\sqrt{10}\sin(x-1.249)$

4 **a** $2\sqrt{5}\sin(x+0.464)$ **b** $5\sqrt{2}\sin(x-1.429)$
c $\sqrt{13}\sin(x-0.983)$ **d** $1.118\sin(x+0.464)$
e $\sqrt{13}\sin(x-0.588)$ **f** $\sqrt{26}\sin(x-0.197)$

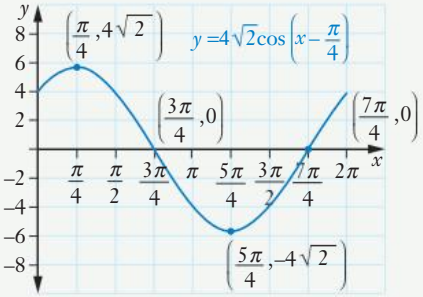
5 **a** $\sqrt{2}\cos\left(x-\frac{7\pi}{4}\right)$ **b** $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$
c $\sqrt{2}\sin(x+45°)$ **d** $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$
e $\sqrt{7}\sin(x+2.428)$ **f** $\sqrt{2}\cos\left(x-\frac{3\pi}{4}\right)$
g $5\sin(x-0.644)$ **h** $\sqrt{2}\cos\left(x+\frac{3\pi}{4}\right)$

6 **a** $4\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ **b** $4\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$

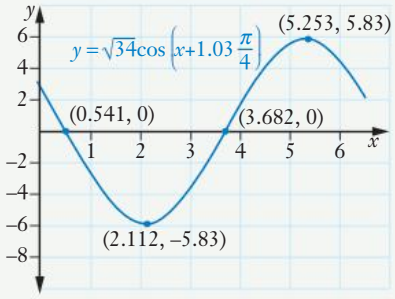
7 **a** $\sqrt{34}\sin(x+2.601)$ **b** $\sqrt{34}\cos(x+1.030)$

Exercise 12.04

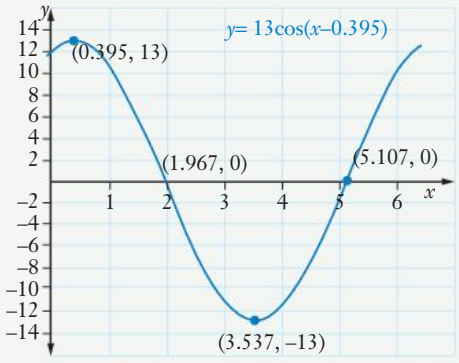
1 **a** $y=4\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$



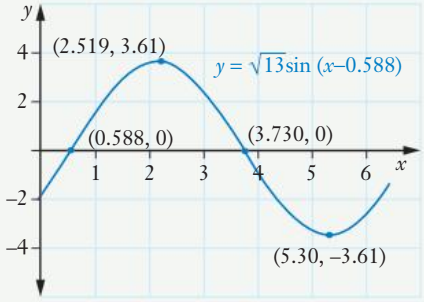
b $y=\sqrt{34}\cos(x+1.030)$



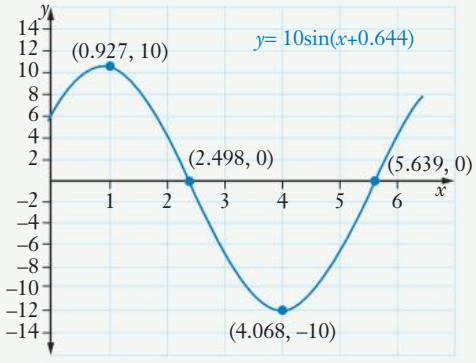
c $y=13\cos(x-0.395)$



2 **a** $y=\sqrt{13}\sin(x-0.588)$

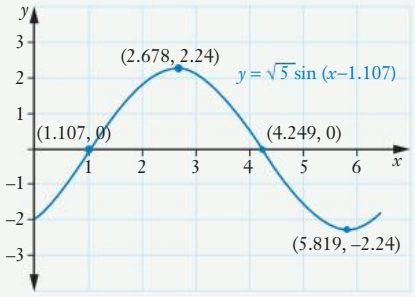


b $y=10\sin(x+0.644)$

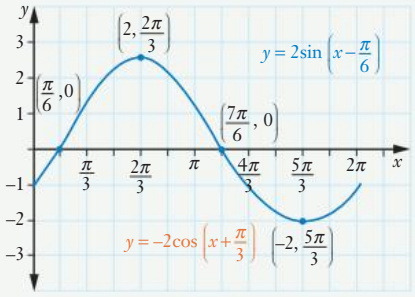


ISBN 9780170412995

c $y = \sqrt{5}\sin(x - 1.107)$



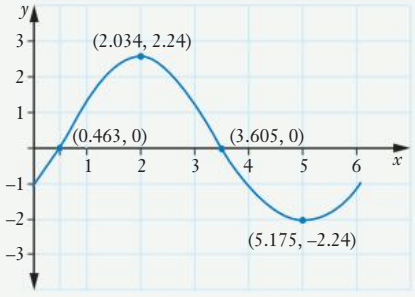
3 a $y = 2\sin\left(x - \frac{\pi}{6}\right)$



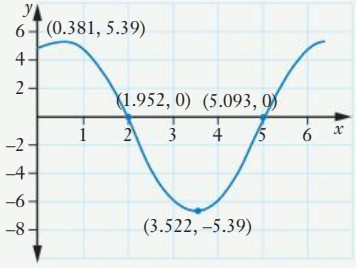
b $y = -2\cos\left(x + \frac{\pi}{3}\right)$

c The graphs of $y = 2\sin\left(x - \frac{\pi}{6}\right)$ and $y = -2\cos\left(x + \frac{\pi}{3}\right)$ are the same.

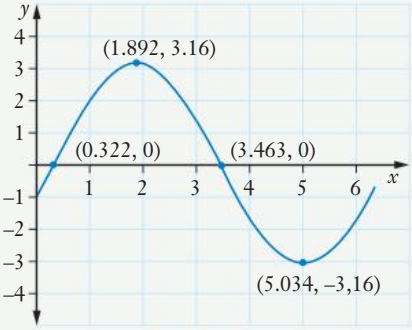
4 a $y = 2\sin(x) - \cos(x)$



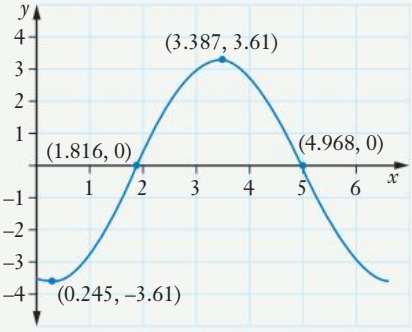
b $y = 5\cos(x) + 2\sin(x)$



c $y = -3\sin(x) + \cos(x)$



d $y = -\sin(x) - 4\cos(x)$



5 a 9 at 0.644 **b** 10 at 2.214
c 15 at 4.318 **d** $7 + \sqrt{2}$ at 5.498

6 a $x \approx 216.9°$ **b** $x \approx 36.9°$

Exercise 12.05

1 a $2\sin\left(x + \frac{\pi}{6}\right)$ **b** $x = 0, \frac{2\pi}{3}, 2\pi$

2 a $\sqrt{10}\cos(x - 0.322)$ **b** $x = 1.208$ or 5.719

3 a 2.214 or 5.356 **b** 1.176 or 4.318
c 2.526 or 5.668 **d** 0.785 or 3.927

4 a 1.088 or 5.985 **b** 1.571 or 3.785
c –1.812 or –0.283 **d** 0.530 or 1.775

5 a 1.995 or 5.959 **b** 0.819 or 4.974
c –0.404 or 1.691 **d** –1.571 or 2.214

6 $\frac{2\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in Z.$

7 $1.571 + 2n\pi, 3.425 + 2n\pi, n \in Z.$

Exercise 12.06

1–3 Demonstrations

4 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{2}}{2}$ **c** $-\frac{1}{2}$

ISBN 9780170412995

5 **a** $\frac{7}{25}$ **b** $-\frac{527}{625}$ **c** $\frac{24}{25}$ **d** $\frac{336}{625}$

6 $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

7–9 Proofs

10 $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

11 $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

12 $x = \frac{\pi}{2}, \frac{3\pi}{2}$

13 $x = \frac{3\pi}{2}$

14 $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

Exercise 12.07

1 **a** 3 m **b** 5 m **c** 2.5 m
d $2\sqrt{3}$ m **e** 3 m **f** 1.25 m

2 **a** π s **b** $\frac{2\pi}{3}$ s **c** 4π s
d 1.85 s **e** $\frac{3\pi}{2}$ s **f** 1.75 s

3 **a** $\frac{\pi}{4}$ s **b** 0.175 s **c** $\frac{\pi}{2}$ s
d 0.924 s **e** 0.884 s **f** 0.726 s

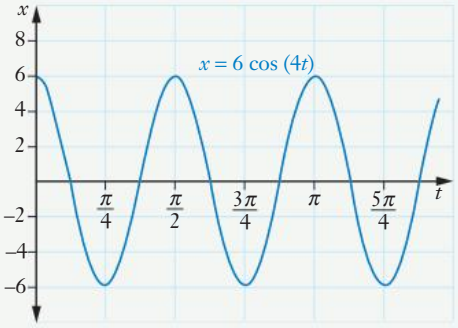
4 **a** $A = 2.5, T = \pi$ **b** $A = 1.7, T = 4\pi$
c $A = 3.2, T = \frac{2\pi}{3}$ **d** $A = 5.5, T = \frac{\pi}{2}$

5 **a** $\frac{\pi}{4}$ s **b** π s **c** $\frac{\pi}{4}$ s
d 0.131 s

6 **a** $\frac{\pi}{2}$ s **b** 2π s
c $\frac{\pi}{12}$ s **d** $\frac{\pi}{6}$ s

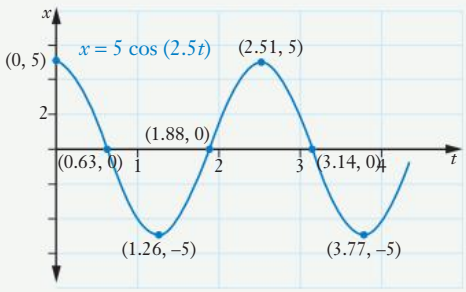
7 **a** $A = 6, T = \frac{\pi}{2}$

b



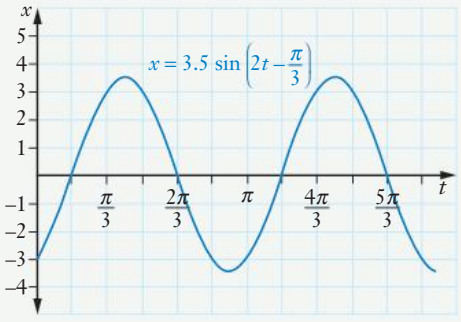
c The initial position of the particle ($t = 0$) is 6 m above the equilibrium point (the origin). After $\frac{\pi}{8}$ s, the particle passes through the origin and moves away from it. After $\frac{\pi}{4}$ s, the particle is 6 m below the origin. The particle returns to the origin after $\frac{3\pi}{8}$ s and continues until it is 6 m above the origin after $\frac{\pi}{2}$ s. This motion is then repeated.

8 **a**



b The initial position of the particle ($t = 0$) is 5 cm above the equilibrium point (the origin). After 0.63 s, the particle passes through the origin and moves away from it. After 1.26 s, the particle is 5 cm below the origin. The particle returns to the origin after 1.88 s and continues until it is 5 cm above the origin after 2.51 s. This motion is then repeated.

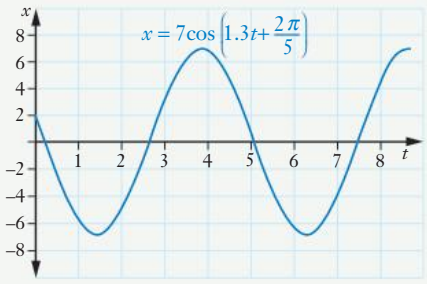
9 **a**



b $\frac{5\pi}{12}$ s **c** About 0.24 s
d $\frac{11\pi}{12}$ s **e** $\frac{2\pi}{3}$ s

ISBN 9780170412995

10 a



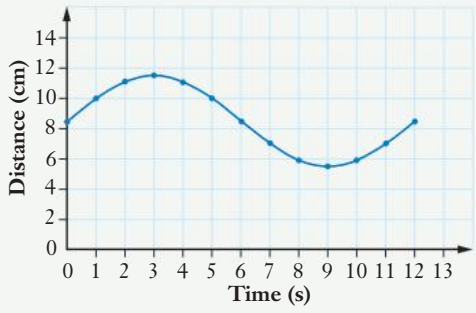
b About 1.45 s **c** About 2.19 s
d About 3.13 s **e** About 2.66 s
f About 3.87 s

11 $x = 4\cos\left(2t + \frac{\pi}{3}\right)$. Other answers are possible.

12 $x = 2\cos\left(\frac{t}{3} - \frac{\pi}{4}\right)$. Other answers are possible.

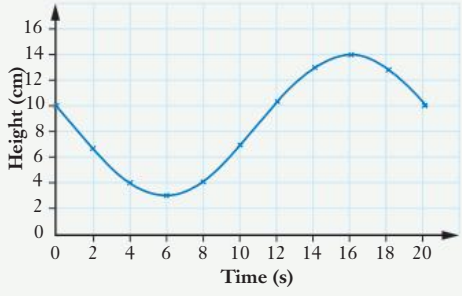
Exercise 12.08

1 a



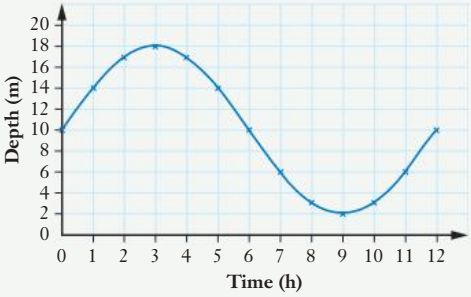
b The data is periodic.
c Period = 12 s, amplitude = 3
d Equilibrium value = 8.5
e The graph looks like a sine function. It could also be a cosine function with a phase shift.

2 a



b The data is periodic.
c Period = 20 s, amplitude = 5.5
d Equilibrium value = 8.5
e The graph looks like a cosine function with a phase shift of 4 units to the left. It could also be a sine function with a phase shift.

3 a



b $y = 8\sin\left(\frac{\pi t}{6}\right) + 10$ **c** 12 m
d 6:44 a.m. **e** 6:23 a.m.

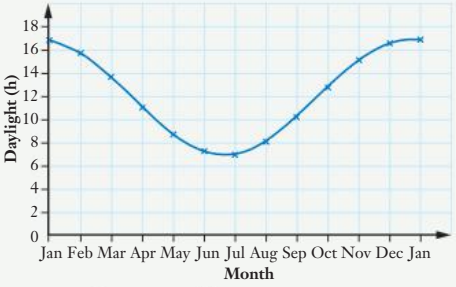
4 a $y = 4\sin\left(\frac{\pi t}{12}\right) + 6$

b About 15.2 s and 20.8 s.

5 $y = 3\sin\left(\frac{\pi t}{6}\right) + 8.5$

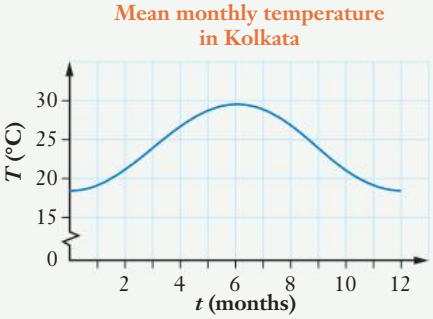
6 $y = 5.5\cos\left(\frac{\pi t}{10} - 4\right) + 8.5$

7 a



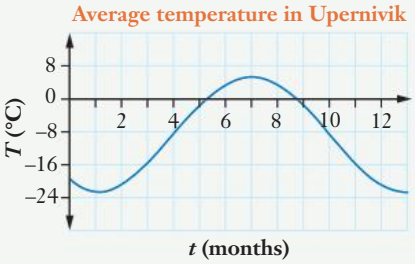
b $y = 5\cos\left[\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right] + 12$ **c** 11.6 h
d Just before the middle of April.

8 $T = -5.575\cos\left(\frac{\pi t}{6}\right) + 23.875$ (Other answers are possible.)



ISBN 9780170412995

9 $T = -14\cos\left[\frac{\pi}{6}(t-1)\right] - 9$ (Other answers are possible.)



10 $d = 0.9\sin\left(\frac{6\pi t}{37}\right) + 4$. The boat with a draught of 3.5 m can enter for 8.479 hours, then cannot for 3.854 hours. The boat with a draught of 4.5 m can enter for 3.854 hours, then cannot for 8.479 hours.

11 **a** $\frac{1}{200}$ s **b** $y = 12\sin(400\pi t)$

12 $x = 8\cos(6\pi t) + 20$

Chapter review

1 **a** $\sin(11x) + \sin(3x)$

b $\frac{1}{2}[\cos(10x) - \cos(26x)]$

2 $\frac{\sqrt{3}+1}{4}$

3 0.337

4 **a** $-2\cos\left(\frac{13x}{2}\right)\sin\left(\frac{3x}{2}\right)$ **b** $-2\sin(6x)\sin(4x)$

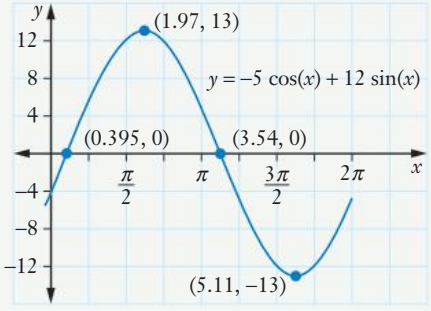
5 $\frac{\sqrt{2}}{2}$ **6** Proof

7 $5\cos(x - 0.644)$

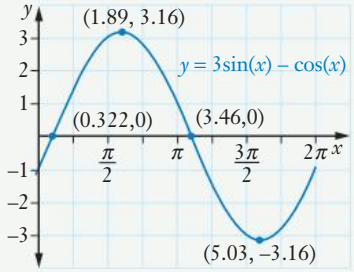
8 $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

9 **a** $\sqrt{61}\cos(x - 0.695)$ **b** $\sqrt{13}\cos(x + 2.159)$

10



11 $y = \sqrt{10}\sin(x - 0.322)$



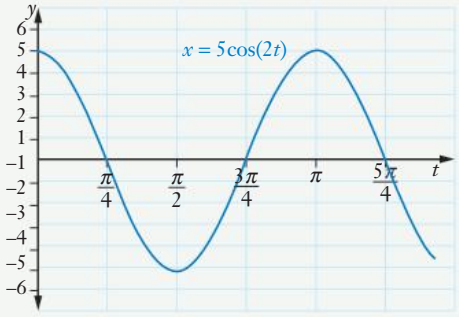
12 $x = 2.035$ or 5.176 **13** $x = 0.819$ or 4.974

14 0.743

15 $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

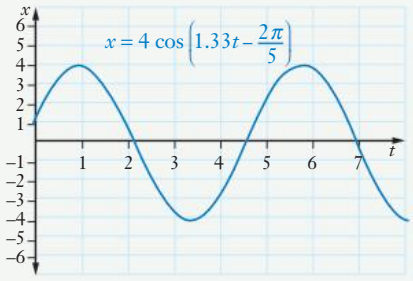
16 **a** Amplitude = 5, period = π

b



c The initial position of the particle ($t = 0$) is 5 m above the equilibrium value (the origin). After $\frac{\pi}{4}$ s, the particle passes through the origin and moves away from it. After $\frac{\pi}{2}$ s, the particle is 5 cm below the origin. The particle returns to the origin after $\frac{3\pi}{4}$ s and continues until it is 5 cm above the origin after π s. This motion is then repeated.

17 **a**



b About 0.945 s **c** 4 m

d About 0.401 s **e** About 1.49 s

f About 2.13 s

ISBN 9780170412995

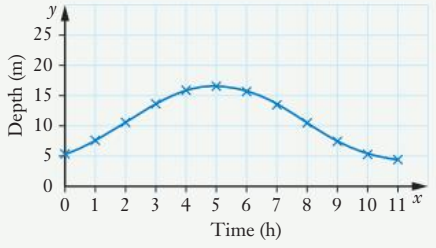
18 a

b The graph is periodic.

c Amplitude = 7.5, Period = 12 (months)

d The graph could be either a sine or cosine function depending on the phase shift.

19 a



b $y = 6\sin\left[\frac{\pi}{5}(t-2)\right] + 10.5$

c About 6.25 m **d** About 2:29 a.m.

20 $d = 15 + 10\cos\left(\frac{\pi t}{30}\right)$. The tip is 10 cm above the shelf 48 times a day (20 past and 20 to the hour).

21–23 Proofs

Practice examination 4

1 a $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ **b** $(1, 4)$

2 a $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{bmatrix}$

b $Q'(-2\sqrt{2}, 4\sqrt{2})$

3 a $\begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$

b $A'(-2, 9)$, $B'(6, -3)$, $C'(0, -6)$

4 $\begin{bmatrix} -0.8 & -0.6 \\ -0.6 & 0.8 \end{bmatrix}$

5 $\mathbf{T}^{-1}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0.4 & -0.2 \\ -0.6 & 0.8 \end{bmatrix}$

6 $|3 - 4i| = \sqrt{3^2 + (-4)^2} = 5$

$\arg(3 - 4i) = -\tan\left(\frac{4}{3}\right)$

7 $3\sqrt{3} - 9i$

8 a $12\,\text{cis}\left(-\frac{\pi}{12}\right)$ **b** $5i$

9 $\Delta = b^2 - 4ac = 16 - 40 = -24 > 0$, so the roots are different, irrational and complex.

10 a $2\sin(11x) + 2\sin(x)$

b $\frac{1}{2}\cos(10x) - \frac{1}{2}\cos(26x)$

11 $\frac{\sqrt{3}}{4}$

12 a $-2\sin(9x)\sin(2x)$

b $-2\cos\left(\frac{7x}{2}\right)\sin\left(\frac{3x}{2}\right)$

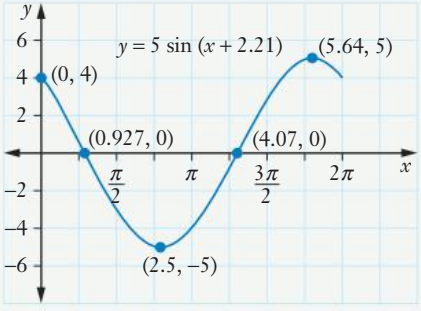
13 $-\sqrt{2}$

14 $\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right)$

15 a $\sqrt{34}\sin(x - 1.03)$ **b** $2\sqrt{5}\sin(x + 0.464)$

16 $x \approx -0.340, 1.571$

17

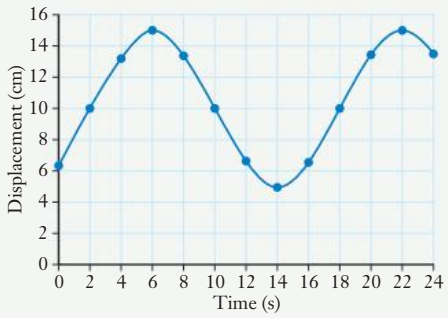


18 150

19 $\sin(105°) = \frac{\sqrt{2} + \sqrt{6}}{4}$, $\cos(105°) = \frac{\sqrt{2} - \sqrt{6}}{4}$

ISBN 9780170412995

20 **a**



b $x = 5\sin\left[\frac{\pi}{8}(t-2)\right] + 10$

c 5.38 cm **d** 4.36 s

21 $\mathbf{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ and

$\mathbf{R}_\phi = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$

Rotations of θ and ϕ applied one after the other give a rotation of $\theta + \phi$.

$\Rightarrow \mathbf{R}_{\theta+\phi} = \mathbf{R}_\theta \bullet \mathbf{R}_\phi$

Now $\mathbf{R}_{\theta+\phi} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$

$= \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

$= \begin{bmatrix} \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta) & -\cos(\phi)\sin(\theta) - \sin(\phi)\cos(\theta) \\ \sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta) & -\sin(\phi)\sin(\theta) + \cos(\phi)\cos(\theta) \end{bmatrix}$

$= \begin{bmatrix} \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) & -[\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)] \\ \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) & \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \end{bmatrix}$

$\Rightarrow \cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$ and

$\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$ QED

22 The roots α and β of $3x^2 + 3x - 4 = 0$ are given by $\alpha + \beta = -\frac{3}{3} = -1$ and $\alpha\beta = \frac{-4}{3} = -\frac{4}{3}$.

Suppose the roots of the new equation are $\theta = \frac{1}{2}\alpha^2$ and $\phi = \frac{1}{2}\beta^2$.

Then $\theta\phi = \frac{1}{2}\alpha^2 \times \frac{1}{2}\beta^2 = \frac{1}{4}(\alpha\beta)^2 = \frac{4}{9}$ and

$\theta + \phi = \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 = \frac{1}{2}(\alpha^2 + \beta^2) =$

$\frac{1}{2}[(\alpha+\beta)^2 - 2\alpha\beta] = \frac{1}{2}\left[1 - 2\times\left(-\frac{4}{3}\right)\right] = \frac{11}{6}$

The new equation if given by $x^2 - (\theta+\phi)x + \theta\phi = 0$

i.e., $x^2 - \frac{11}{6}x + \frac{4}{9} = 0$.

The equation $18x^2 - 33x + 8 = 0$ has roots that are half the squares of the roots of $3x^2 + 3x - 4 = 0$.

23 **a** $y = -67.5\cos\left(\frac{\pi t}{15}\right) + 72.5$

b 106.25 m **c** 9.5 s and 21.0 s

ISBN 9780170412995

GLOSSARY AND INDEX

absolute value function: a function that contains an absolute value. (p. 352)

absolute value: the value $|x|$ for $x \in \boldsymbol{R}$ given by $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$. (p. 348)

acute-angled triangle: a triangle with only acute angles. (p. 87)

addition principle: if two mutually exclusive operations can be performed in n and m ways, there are $n + m$ ways of performing one or the other. (p. 105)

air speed: speed of an aircraft as it would be measured in still air—the apparent speed of an aircraft relative to the air. (p. 53)

alternate angles: internal angles on opposite sides of a transversal crossing two lines; the rule that alternate angles on parallel lines are equal. (p. 87)

alternate segment: the segment on the opposite side of one of the angles between a tangent and a chord from the point of contact; the rule the angle is equal to the angle in the alternate segment. (p. 203)

altitude: a line passing through the vertex of a triangle perpendicular to the opposite side. (p. 215)

amplitude: distance from the average position to the extreme values of a periodic function. (p. 335, 498)

angle at the centre: the rule that the angle at the centre of a triangle is twice the angle at the circumference subtended by the same arc or chord. (p. 182)

angle sum and difference identities: trigonometric identities that are used to find the sine, cosine, or tangent of the sum or difference of two given angles. (p. 382)

angle-angle (AA): a test of similarity: two triangles are similar if two of their angles are equal. (p. 88)

angle-angle-side (AAS): a test of congruence: two triangles are congruent if two of their angles and a corresponding side are equal. (p. 88)

angles at a point: the rule that the angles at a point sum to 360°. (p. 87)

angles at the circumference: the rule that angles at the circumference subtended by the same arc or chord are equal. (p. 186)

Argand diagram, Argand plane: a rectangular coordinate system for complex numbers as points or vectors with the real axis horizontal and the complex axis vertical, *also* complex plane. (p. 313, 446)

argument: the direction (θ) of the polar, trigonometric or vector form of the complex number $z = a + bi$ given by $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$. (p. 316, 446)

binomial theorem: the expansion $(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-2}a^2b^{n-2} + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$, where ${}^nC_r$ is the rth entry in the nth row of Pascal's triangle. (p. 244)

Cartesian form: the rectangular form of a complex number $z = a + bi$, where $a, b \in \boldsymbol{R}$, *also* rectangular form. (p. 299, 450)

centroid: the intersection of the medians of a triangle. (p. 215)

chord: a line segment between two points on the circumference of a circle. (p. 195–7)

circular functions: see *trigonometric functions*. (p. 335–44)

circular permutation: the number of different ways n things can be arranged in a circle, given by $(n - 1)!$. (p. 122)

circumcentre: the intersection of the perpendicular bisectors of the sides of a triangle; the centre of the circumcircle. (p. 215)

circumcircle: the circle that passes through the vertices of a triangle. (p. 215)

cis (θ): abbreviation of $\cos(\theta) + i\sin(\theta)$. (p. 450)

ISBN 9780170412995

co-interior angles: internal angles on the same side of a transversal crossing two lines; the rule that co-interior angles on parallel lines are supplementary. (p. 85, 87)
column matrix: matrix with one column. (p. 261)
combination: selection of some or all of a set of objects where order doesn't matter. The number of selections of r objects from n objects is written as ${}^nC_r$. (p. 229)
combinatorics: systematic counting of a finite collection, particularly selections and orders. (p. 98)
commutative property: an operation is commutative if it gives the same answer with the order reversed; $a * b = b * a$. (p. 154, 272, 286)
complementary angles: angles that add up to 90°. (p. 370)
complex conjugate: the complex number $a - bi$ corresponding to the complex number $a + bi$. (p. 309)
complex number: a number that can be expressed as $z = a + bi$, where $i = \sqrt{-1}$ and $a, b \in \boldsymbol{R}$. (p. 298)
complex plane: A rectangular coordinate system for complex numbers with the real axis horizontal and the complex axis vertical, *also* Argand diagram. (p. 313)
component form: the components of a vector are its projections in the x- and y-directions. The component form may be shown by an ordered pair or by a linear combination of unit vectors. (p. 13)
composition: $\mathbf{T}_2$ o $\mathbf{T}_1$, the composition of transformations $\mathbf{T}_1$ and $\mathbf{T}_2$ is the result of doing $\mathbf{T}_1$ and then $\mathbf{T}_2$. It can be calculated using the product of the matrices. (p. 430)
concyclic: 4 or more points that lie on the circumference of a circle. (p. 192)
conformable: matrices that can be multiplied; the number of columns in the first is the same as the number of rows in the second. (p. 270)
congruent: exactly the same shape and size; capable of being superimposed exactly on one another. (p. 88)
conjugate: the conjugate of the complex number $z = a + bi$ is $\overline{z} = a - bi$. (p. 309)
contradiction: inconsistent; the method of proof where the negation of the desired result is shown to contradict itself or well-established knowledge. (p. 81)
contrapositive: the equivalent reverse negation $\overline{P} \Rightarrow \overline{Q}$ of a statement of the nature $P \Rightarrow Q$; the method of proof using their equivalence. (p. 78)
converse: the reverse $Q \Rightarrow P$ of a statement of the nature $P \Rightarrow Q$. (p. 70)
corollary: a result that follows easily from a theorem (major result). (p. 64)
corresponding angles: angles on the same sides of a transversal and the two crossing lines; the rule that corresponding angles on parallel lines are equal. (p. 87)
counterexample: an example that shows that a proposition is not true. (p. 68)
cyclic quadrilateral: a quadrilateral whose vertices are on the circumference of a circle. (p. 192)

determinant: a number determined by a matrix, for the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(\mathbf{A}) = ad - bc$, *see also* non-singular, singular. (p. 281, 426, 435)
dilation: a change of scale, usually shown by the factors λ_1 and λ_2 in the horizontal and vertical directions respectively. (p. 414)
dimension: the number of rows by the number of columns of a matrix, *also* size, order. (p. 261)
direct proof: proof of a result by logical steps from known or assumed facts. (p. 64)
directed line segment: a line segment in a specified direction. (p. 4)
direction: the direction of a vector measured counter-clockwise from the positive direction of the x-axis. (p. 4, 10)
discriminant: the value $\Delta = b^2 - 4ac$ for the quadratic equation $ax^2 + bx + c = 0$. (p. 299, 464)
displacement vector: the vector from one point to another. The change from position A to position B is written as **AB**. (p. 10, 12)
dot product: the value $\mathbf{p} \bullet \mathbf{q} = p_1q_1 + p_2q_2$, where $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j}$ and $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j}$, *also* scalar product. (p. 150, 211)
double angle identities: identities expressing trigonometric functions of $2x$ in terms of trigonometric functions of x. (p. 387)

element: one of the numbers in a matrix. (p. 260)
enumeration: process of listing all the elements of a collection. (p. 100)
equation: statement that the values of two mathematical expressions are equal, shown by =.
equilateral triangle: a triangle whose three sides are the same length. (p. 87)
equilibrium value: the average of the extreme values of a periodic function. (p. 335, 498)
equivalent: in logic, two statements that imply each other; $P \Leftrightarrow Q$, *also* if and only if, necessary and sufficient. (p. 72)

ISBN 9780170412995

Euclid's postulates: 5 assumptions used by Euclid of Alexandria to logically develop plane geometry. (p. 85)
exterior angle: an angle on the exterior of a polygon obtained by extending one side. (p. 192)

factorial notation: for an integer $n \geq 0$, factorial n (or n factorial), $n! = n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$ with $0! = 1$. (p. 116)
factorise: express as a product of factors, particularly linear factors.
force: a force is an interaction that, when unopposed, will change the motion of an object. A force can cause an object to change its velocity.

general solution: expression or set of expressions that represents all possible solutions to a trigonometric equation. (p. 332)
ground speed: the true speed of an aircraft as measured relative to the Earth. (p. 53)

half angle identities: identities that express trigonometric functions of angles x in terms of trigonometric functions of $\frac{x}{2}$. (p. 388)
head: final point of a vector. (p. 4)
heading: direction in which a boat or aircraft is steered. (p. 52)
head-to-tail: method of vector addition by placing the tail of the second vector at the head of the first. (p. 22)
horizontal asymptote: a horizontal line that the graph of the function $f(x)$ approaches as x tends to $+\infty$ or $-\infty$. (p. 364)

identity 1: a statement of equality that is true for all values of the variables. (p. 00)
identity 2: a matrix $\mathbf{I}$ with all entries zero except those in the top left to bottom right diagonal, which are all 1; a transformation that leaves all points unchanged, $\mathbf{I}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix}$; the matrix $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (p. 280, 482)
if and only if (iff): in logic, two statements that imply each other; $P \Leftrightarrow Q$, *also* equivalent, necessary and sufficient. (p. 66, 72)
image: the point or shape that a transformation produces. (p. 404)
imaginary number: a number of the form bi, where $i = \sqrt{-1}$ and $b \in \boldsymbol{R}$. (p. 298)
imaginary part: the imaginary part of the complex number $z = a + bi$ is $\text{Im}(z) = b$. (p. 299)
in-centre: the intersection of the angle bisectors of a triangle; the centre of the in-circle. (p. 215)
in-circle: the centre of the circle inside a triangle that is tangent to the 3 sides. (p. 215)
inclusion-exclusion principle: for two (finite) sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$; for 3 sets $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. (p. 110)
inner product: *See* 'scalar product'. (p. 150)
integer: a whole number, positive, zero or negative; see also natural number, rational, irrational.
inverse matrix : for a matrix $\mathbf{A}$, the matrix $\mathbf{A}^{-1}$, if it exists, such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, the identity. (p. 281, 458)
inverse transformation: for a transformation $\mathbf{T}$, the transformation $\mathbf{T}^{-1}$, if it exists, such that $\mathbf{T}^{-1}\left(\mathbf{T}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ for all points in the plane. (p. 426)
irrational number: a number on the number line that is not rational. (p. 75)
isosceles triangle: a triangle with two equal sides. (p. 87)

joule: one joule (J) is the amount of energy exerted when a force of one newton is applied over a displacement of one metre. (p. 169)

kite: a quadrilateral with two pairs of equal adjacent sides. (p. 87)
Kronecker delta function: the function $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ (p. 280)
lemma: a small result used to help prove a theorem (major result). (p. 64)
linear combination: a linear combination of quantities is a sum of the product of a real number with each quantity. (p. 35)
linear transformation: a linear transformation preserves linear combinations. (p. 322, 405)
linear transformation of vectors: a transformation of vectors that preserves addition and scalar multiplication; the transformation $\mathbf{T}$ is linear if and only if $\mathbf{T}(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha\mathbf{T}(\mathbf{v}) + \beta\mathbf{T}(\mathbf{w})$ for any vectors $\mathbf{v}$, $\mathbf{w}$ and any $\alpha, \beta \in \boldsymbol{R}$. (p. 425)

matrix equation: an equation containing matrices. (p. 286)
matrix, matrices (plural): a rectangle of numbers enclosed in brackets. (p. 260)
mean value: the equilibrium or middle value of the extreme values of a trigonometric function. (p. 335)

ISBN 9780170412995

median: a line joining the vertex of a triangle to the midpoint of the opposite side. (p. 215)
member of: being one of; in a set. (p. 79)
midpoint theorem: the segment joining two sides of a triangle at the midpoints of those sides is parallel to the third side and is half the length of the third side. (p. 145)
modulus 1: See *absolute value*. (p. 348)
modulus 2: the magnitude of a complex number $z = a + bi$ given by mod $(z) = |z| = \sqrt{a^2 + b^2}$. (p. 316)
modulus 3: the size of the vector that represents a complex number $z = a + bi$ in the plane, given by $|z| = \sqrt{a^2 + b^2}$. (p. 446)
multiplication principle: if one operation can be performed in n ways, and another in m ways, then the combined operation can be performed in $n \times m$ ways. (p. 101)
mutually exclusive: events or outcomes that have no common elements. (p. 105)

necessary and sufficient: in logic, two statements that imply each other; $P \Leftrightarrow Q$, *also* equivalent, if and only if. (p. 66, 72)
negation: the opposite of a statement. (p. 73)
newton: one newton (N) is equal to the force needed to accelerate a mass of one kilogram one metre per second per second. (p. 168)
non-singular: a matrix or transformation that has an inverse; the determinant is not zero. (p. 281, 426)
norm: the magnitude of a vector **v** is called its norm, represented as $v = |\mathbf{v}|$. (p. 4)

object: the point or shape to which a transformation is applied, *see also* image. (p. 404)
obtuse angled triangle: a triangle with an obtuse angle (an angle more than 90°). (p. 87)
order: the number of rows by the number of columns of a matrix, *also* dimension, size. (p. 261)
orthocentre: the intersection of the altitudes of a triangle. (p. 215)
orthogonal components: components that are perpendicular to each other. (p. 158)

parallel: lines in the same plane that never meet; lines in the same plane that are a constant distance apart; lines in the same direction in the same plane. (p. 79, 85)
parallelogram: a quadrilateral with opposite sides parallel. (p. 87)
parallelogram rule: method of vector addition where the vectors being added form the sides of a parallelogram and the resultant is the diagonal of the parallelogram. (p. 24)
Pascal's triangle: the triangular array of numbers such that the number in position r of row n is ${}^nC_r$, where r and n both start from zero. (p. 242)
period: the interval of the independent variable (often time) needed to complete one cycle of a periodic function. (p. 335, 498)
periodic motion: motion repeated in equal intervals of time. (p. 498)
permutation: an ordered arrangement of some or all of a set of objects. The number of permutations of r objects from n objects is written as ${}^nP_r$. (p. 115, 226)
phase shift: the value a periodic function is shifted along the x-axis from the corresponding standard function. (p. 335, 498)
pigeon-hole principle: if m objects are distributed among n boxes where $m > n$, then at least one box must contain more than one object. (p. 245)
polar form[1]: the form (r, θ) f a 2D vector **v** where r is the magnitude of **v** and θ is the direction of **v** measured anticlockwise from the x-axis. (p. 10, 450)
polar form[2]: the form $z = (r, \theta)$ of the complex number $x + iy$, where $r = |z| = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \frac{y}{x}$. (p. 9, 450)
position vector: the position vector of a point P is the displacement vector of P from the origin: $\mathbf{p} = \mathbf{OP}$. (p. 9)
principal argument: the argument of a complex number z in the domain $-\pi < \arg(z) \leq \pi$. (p. 447)
probability: the probability of an event is given by $P(\text{event}) = \frac{n(\text{event})}{n(\text{sample space})}$. (p. 126)
product to sum identities: identities for the conversion of products of trigonometric terms to sums of trigonometric terms. (p. 476)
projection of a on b: the vector is the orthogonal component of **a** in the direction of **b**, the scalar projection is its magnitude. (p. 163, 165)
proof: a logical sequence of steps that establishes the truth of a proposition from known or assumed facts. (p. 64)
Pythagorean identities: identities arising from $r^2 = x^2 + y^2$ in the defining circle of trigonometric ratios. (p. 379)

QED: Quod erat demonstrandum; written at the end of a proof to indicate it is finished. (p. 64)
quadratic equation: an equation of the form $ax^2 + bx + c = 0$, where $a, b, c \in \boldsymbol{R}$. (p. 462)

ISBN 9780170412995

quadratic formula: the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the roots of a quadratic equation. (p. 462)

quadruple angle identities: formulas used to convert a trigonometric term involving a quadruple angle (e.g. 4θ) to a trigonometric expression involving single angles (e.g. θ). (p. 496)

rational function: a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$. (p. 362)

rational number: a number that can be expressed in the form $\dfrac{a}{b}$, where a and b are integers. (p. 75)

real numbers: numbers on the number line; all the rational and irrational numbers. (p. 79)

real part: the real part of the complex number $z = a + bi$ is $\text{Re}(z) = a$. (p. 299)

reciprocal function: a function of the form $f(x) = \dfrac{1}{g(x)}$. (p. 358)

reciprocal trigonometric functions: reciprocal functions of trigonometric functions. (p. 368)

rectangular form: a complex number in the form $z = a + bi$, where $a, b \in \boldsymbol{R}$, *also* Cartesian form. (p. 299)

reference angle: the acute angle between the terminal side of an angle and the x-axis. (p. 369)

reflection: a transformation in which the image of each point is the same distance from a straight line through the origin as the original point, but on the opposite side of the line. (p. 420)

resolution: the process of changing a vector into component form. (p. 42)

resolve: express a vector in terms of orthogonal components. (p. 42)

rhombus: a quadrilateral whose sides are all equal. (p. 87)

right angle-hypotenuse-side (RHS): a test of congruence or similarity: two right-angled triangles are congruent if their hypotenuses and another side are equal; two right-angled triangles are similar if their hypotenuses and another side are in proportion. (p. 88)

right-angled triangle: a triangle with a right angle (an angle equal to 90°). (p. 87)

root: a value for the variable that makes an equation true. (p. 462)

rotation: a transformation in which each point in the plane is turned through the same angle around the origin, the centre of rotation. (p. 408)

row matrix: a matrix with only one row. (p. 261)

RTP: required to prove; a succinct statement of a desired result. (p. 64)

sample point: an element of the sample space. (p. 126)

sample space: the set of all possible outcomes of an experiment or trial. (p. 126)

scalar: a quantity having only magnitude, usually a real number. (p. 4)

scalar multiple[1]: a matrix that is a real number multiplied by another matrix. (p. 273)

scalar multiple[2]: the result of multiplying a vector by a real number (scalar). (p. 18)

scalar product: the sum of the products of corresponding elements. It is also called the dot product or inner product and is shown by a dot (•). (p. 150)

secant: a straight line that intersects a circle at two points. (p. 203)

segment: part of a circle on one side of a chord. (p. 203)

side-angle-side (SAS): a test of congruence or similarity: two triangles are congruent if two sides and the included angle are equal; two triangles are similar if two sides are in proportion and the included angles are equal. (p. 88)

side-side-side (SSS): a test of congruence or similarity: two triangles are congruent if all the sides are equal; two triangles are similar if all the sides are in proportion. (p. 88)

similar: having exactly the same shape, but perhaps different sizes. (p. 88)

simple harmonic motion (SHM): type of periodic motion in which an object oscillates between extreme positions in a straight line. (p. 498)

singular: a matrix or transformation with no inverse; the determinant is 0. (p. 281, 426)

size: the number of rows by the number of columns of a matrix, *also* dimension, order. (p. 261)

slant asymptote: a sloping asymptote that occurs in a rational function when the degree of the polynomial in the numerator is one higher than the degree of polynomial in the denominator. (p. 365)

speed and course made good: true speed and direction of a boat or aircraft, taking account of currents or wind. (p. 52)

speed through the water: speed of a boat as it would be measured in still water—the apparent speed of a boat relative to the water. (p. 52)

square matrix: a matrix with the same number of rows and columns. (p. 261)

subtend: an object that just touches the arms of an angle subtends the angle.

ISBN 9780170412995

sum to product identities: identities for the conversion of sums of trigonometric terms to products of trigonometric terms. (p. 480)
supplementary: angles that sum to 180° (two right angles). (p. 87)

tail: original point of a vector. (p. 4)
tangent: a straight line that intersects a circle at only one point. (p. 200)
theorem: a major proposition that is known to be true. (p. 64)
transformation: a change of points in the plane to new points; a rule such as rotation or translation that changes the vectors that it applies to into other vectors. (p. 321)
translation: a slide in the plane; every point is moved the same distance in the same direction. (p. 404)
trapezium: a quadrilateral with a pair of parallel sides. (p. 87)
tree diagram: a diagram that shows all the possible outcomes of an event as a branches. (p. 100)
triangle rule: adding two vectors by placing them head-to-tail to get the resultant from the first tail to the second head. (p. 22)
trigonometric form: the form
$z = r[(\cos(\theta) + i \sin(\theta)] = r \operatorname{cis}(\theta)$
of a complex number. (p. 450)
trigonometric function: Any of the functions sine, cosine and tangent of an angle (also called the circular functions). (p. 335)
triple angle identities: formulas used to convert a trigonometric expression involving a triple angle (e.g. 3θ) to a trigonometric expression involving single angles (e.g. θ). (p. 495)
true course: true direction of a boat or aircraft, taking account of currents or wind. (p. 52, 53)

unit vector: the vector in the direction of **p** with a magnitude 1, shown by $\hat{\mathbf{p}}$. (p. 34)

vector projection of (a on b): the orthogonal component of **a** in the direction of **b**. It is parallel to **b**. (p. 165)
vector: a quantity with both magnitude and direction. A geometric vector may be represented by a directed line segment in the direction of the vector with its length as the magnitude. (p. 4)
vertical asymptote: a vertical line corresponding to the zeroes of the denominator of a rational function. (p. 363)
vertically opposite: angles opposite each other on a pair of intersecting lines. (p. 87)

work: the scalar product of force and displacement given by $W = \mathbf{F} \cdot \mathbf{s}$ for a constant force of **F** on a point that moves a displacement of **s**. (p. 169)

zero matrix: a matrix in which every element is equal to 0. (p. 261)

ISBN 9780170412995

OVERVIEW

Nelson QMaths 11 & 12 has been rewritten and updated to cover the new Queensland senior courses: *Essential Mathematics, General Mathematics, Mathematical Methods* and *Specialist Mathematics*. ***Specialist Mathematics*** is an advanced calculus-based course taken in conjunction with Mathematical Methods.

nelsonnet

Register online at:
www.nelsonnet.com.au
to access digital resources to support improved learning & teaching*

9780170412650
Nelson QMaths 11 Essential Mathematics

9780170412711
Nelson QMaths 11 General Mathematics

9780170412858
Nelson QMaths 11 Mathematical Methods

9780170412995
Nelson QMaths 11 Specialist Mathematics

9780170412681
Nelson QMaths 12 Essential Mathematics

9780170412780
Nelson QMaths 12 General Mathematics

9780170412926
Nelson QMaths 12 Mathematical Methods

9780170413060
Nelson QMaths 12 Specialist Mathematics

FEATURES

- Written by a successful Queensland author team of experienced teachers, including the leading authors of **Nelson Senior Maths for the Australian Curriculum**
- Each chapter begins with a chapter table of contents, syllabus subject matter and a **Terminology** word list
- Graded exercises are linked to worked examples and feature **Problem Solving** sections
- **Investigations** explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks
- **Technology** sections promote ICT in the classroom, featuring Casio and Texas Instruments graphics calculator activities, spreadsheets and the Internet
- Each chapter ends with a **Chapter summary** and **Chapter review** exercise
- **Practice examinations** revise the skills and knowledge of previous chapters
- Syllabus grid, answers and glossary/index included

NELSON
A Cengage Company

*Please note that complimentary access to NelsonNet is only available to teachers who use the accompanying student textbook as a core educational resource in their classroom. Contact your Education Consultant for information about access codes and conditions.

ISBN 978-0170412995