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VICmaths

VCE UNITS 1 + 2

Dirk Strasser

general mathematics 11

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general mathematics 11

Nelson

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Contents

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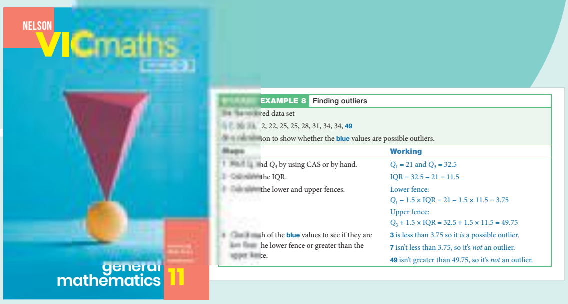
To the student

Nelson VICmaths is your best friend when it comes to studying General Mathematics in Year 11. It has been written to help you maximise your learning and success this year. Every explanation, every exam hack and every worked example has been written with the exams in mind.

The 3 steps to mastering each topic

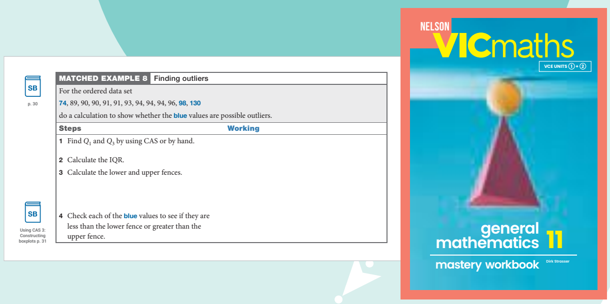
STEP 1

Study each Worked example



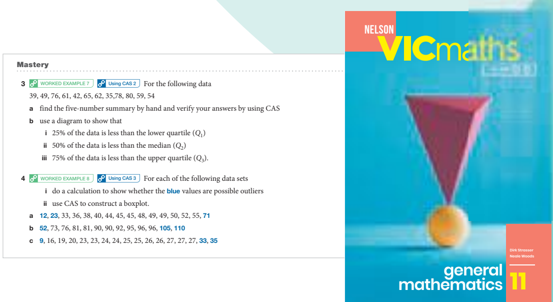
STEP 2

Complete the Matched example in the Mastery Workbook



STEP 3

Do the Mastery questions in the exercise that are linked to the Worked example



To the teacher

Now there's a better way to VCE maths mastery.

Nelson VICmaths 11–12 is a new VCE mathematics series that is backed by research into the science of learning. The design and structure of the series has been informed by teacher advice and evidence-based pedagogy, with the focus on preparing VCE students for their exams and maximising their learning achievement.

- Using **backwards learning design**, this series has been built by analysing past VCE exam questions and ensuring that all theory and examples are precisely mapped to the VCE Study Design.
- To reduce the **cognitive load** for learners, explanations are clear and concise, using the technique of **chunking** text with accompanying diagrams and infographics.
- The student book and workbook combination has been designed for **mastery** of the learning content.
- The exercise structure of **Recap**, **Mastery** and **Exam practice** leads students from procedural fluency to **higher-order thinking** using the learning technique of **interleaving**.
- **Exam practice** includes exam-style questions and graded past VCE exam questions with success percentages based on **VCAA performance data**.
- The cumulative structure of Exercise **Recaps** and chapter-based **Cumulative examinations** is built on the learning and memory techniques of **spacing** and **retrieval**.

About the authors

Dirk Strasser is an experienced teacher, a former Head of Mathematics and a lead author and senior publisher of mathematics series for over 30 years. He has published and co-written eight best-selling mathematics series for Heinemann and Pearson, and won several Australian Educational Publishing Awards. He is the Manager of Secondary Mathematics at Nelson Cengage.

Contributing authors

Neale Woods coordinated the *Using CAS* content. He is a retired mathematics teacher with over 45 years of teaching experience. Neale is a CAS specialist who has presented at numerous conferences over the years and continues to conduct technology workshops for teachers and students.

Darren Smyth (maffsguru.com) created the VCE question analysis videos.

Study Design grid

Area of study		*Nelson VICmaths General Mathematics 11* chapter
UNIT 1		
1 Data analysis, probability and statistics		
Investigating and comparing data distributions	1	Investigating and comparing data distributions
2 Algebra, number and structure		
Arithmetic and geometric sequences, first-order linear recurrence relations and financial mathematics	2	Arithmetic sequences and financial recurrence relations
	3	Geometric sequences and financial mathematics
3 Functions, relations and graphs		
Linear functions, graphs, equations and models	4	Linear functions, graphs, equations and models
4 Discrete mathematics		
Matrices	5	Matrices
UNIT 2		
1 Data analysis, probability and statistics		
Investigating relationships between two numerical variables	6	Relationships between numerical variables
2 Discrete mathematics		
Graphs and networks	7	Graphs and networks
3 Functions, relations and graphs		
Variation	8	Variation
4 Space and measurement		
Space, measurement and applications of trigonometry	9	Measurement, scale and similarity
	10	Applications of trigonometry

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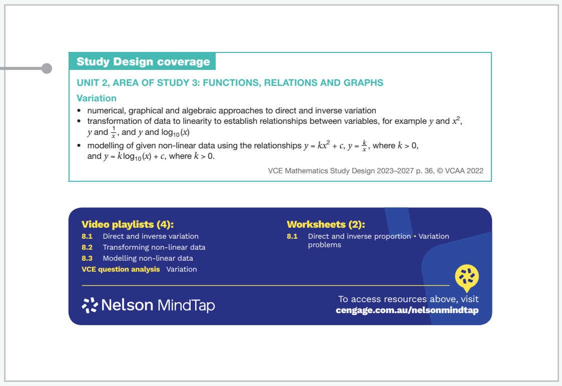
About this book

In each chapter

Study Design coverage and extracts are shown at the start of the chapter, along with a listing of **Nelson MindTap chapter resources**.



Study Design coverage

UNIT 2, AREA OF STUDY 3: FUNCTIONS, RELATIONS AND GRAPHS

Variation

- numerical, graphical and algebraic approaches to direct and inverse variation
- transformation of data to linearity to establish relationships between variables, for example y and x^2, y and $\frac{1}{x}$, and y and $\log_{10}(x)$
- modelling of given non-linear data using the relationships $y = kx^2 + c$, $y = \frac{k}{x}$, where $k > 0$, and $y = k\log_{10}(x) + c$, where $k > 0$.

VCE Mathematics Study Design 2023–2027 p. 36, © VCAA 2022

Video playlists (4):
8.1 Direct and inverse variation
8.2 Transforming non-linear data
8.3 Modelling non-linear data
VCE question analysis Variation

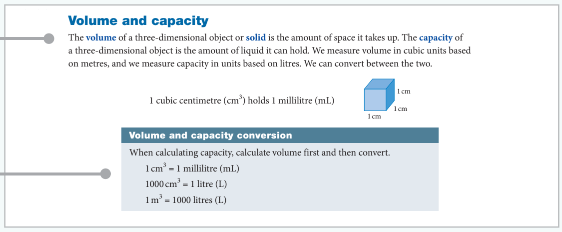
Worksheets (2):
8.1 Direct and inverse proportion • Variation problems

Nelson MindTap

To access resources above, visit cengage.com.au/nelsonmindtap

Important words and phrases are printed in blue and listed in the **Glossary and index** at the back of the book.

Important facts and formulas are highlighted in a shaded box.



Volume and capacity

The volume of a three-dimensional object or solid is the amount of space it takes up. The capacity of a three-dimensional object is the amount of liquid it can hold. We measure volume in cubic units based on metres, and we measure capacity in units based on litres. We can convert between the two.

1 cubic centimetre (cm³) holds 1 millilitre (mL)

Volume and capacity conversion

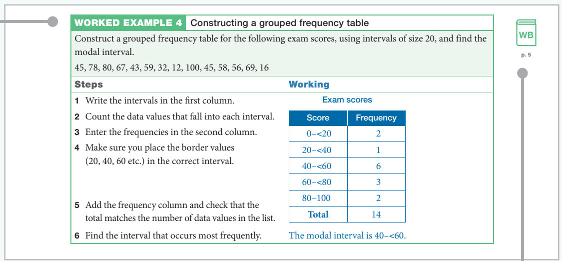
When calculating capacity, calculate volume first and then convert.

$1\text{ cm}^3 = 1$ millilitre (mL)

$1000\text{ cm}^3 = 1$ litre (L)

$1\text{ m}^3 = 1000$ litres (L)

Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.



WORKED EXAMPLE 4 Constructing a grouped frequency table

Construct a grouped frequency table for the following exam scores, using intervals of size 20, and find the modal interval.

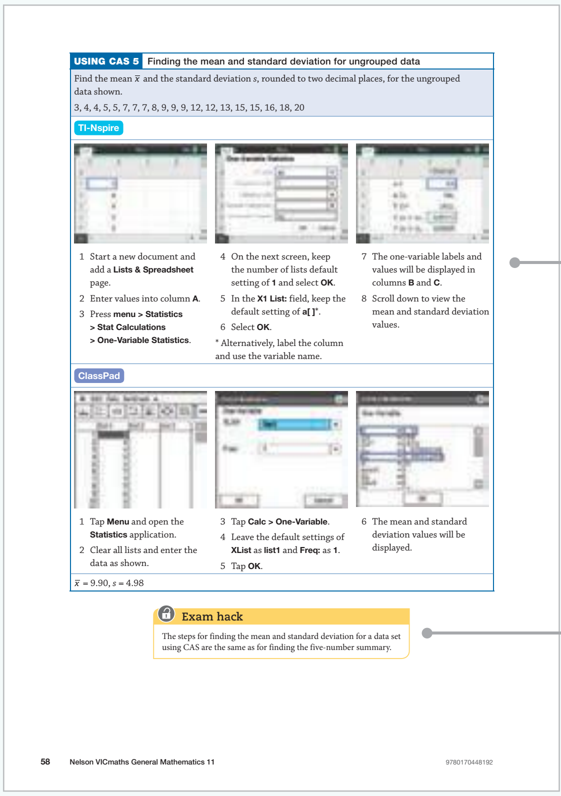
45, 78, 80, 67, 43, 59, 32, 12, 100, 45, 58, 56, 69, 16

Steps	Working
1 Write the intervals in the first column. 2 Count the data values that fall into each interval. 3 Enter the frequencies in the second column. 4 Make sure you place the border values (20, 40, 60 etc.) in the correct interval. 5 Add the frequency column and check that the total matches the number of data values in the list.	Exam scores
6 Find the interval that occurs most frequently.	The modal interval is 40–<60.

Score	Frequency
0–<20	2
20–<40	1
40–<60	6
60–<80	3
80–100	2
Total	14

WB p. 5

Links to scaffolded Matched examples in the Mastery Workbook (WB).



USING CAS 5 Finding the mean and standard deviation for ungrouped data

Find the mean $\bar{x}$ and the standard deviation s, rounded to two decimal places, for the ungrouped data shown.

3, 4, 4, 5, 5, 7, 7, 7, 8, 9, 9, 9, 12, 12, 13, 15, 15, 16, 18, 20

TI-Nspire

1 Start a new document and add a **Lists & Spreadsheet** page.
2 Enter values into column **A**.
3 Press **menu > Statistics > Stat Calculations > One-Variable Statistics**.
4 On the next screen, keep the number of lists default setting of **1** and select **OK**.
5 In the **X1 List:** field, keep the default setting of **a[ ]***.
6 Select **OK**.
7 The one-variable labels and values will be displayed in columns **B** and **C**.
8 Scroll down to view the mean and standard deviation values.

* Alternatively, label the column and use the variable name.

ClassPad

1 Tap **Menu** and open the **Statistics** application.
2 Clear all lists and enter the data as shown.
3 Tap **Calc > One-Variable**.
4 Leave the default settings of **XList** as **list1** and **Freq:** as **1**.
5 Tap **OK**.
6 The mean and standard deviation values will be displayed.

$\bar{x} = 9.90$, $s = 4.98$

Exam hack

The steps for finding the mean and standard deviation for a data set using CAS are the same as for finding the five-number summary.

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Using CAS provides clear instructions for TI-Nspire and Casio ClassPad calculators.

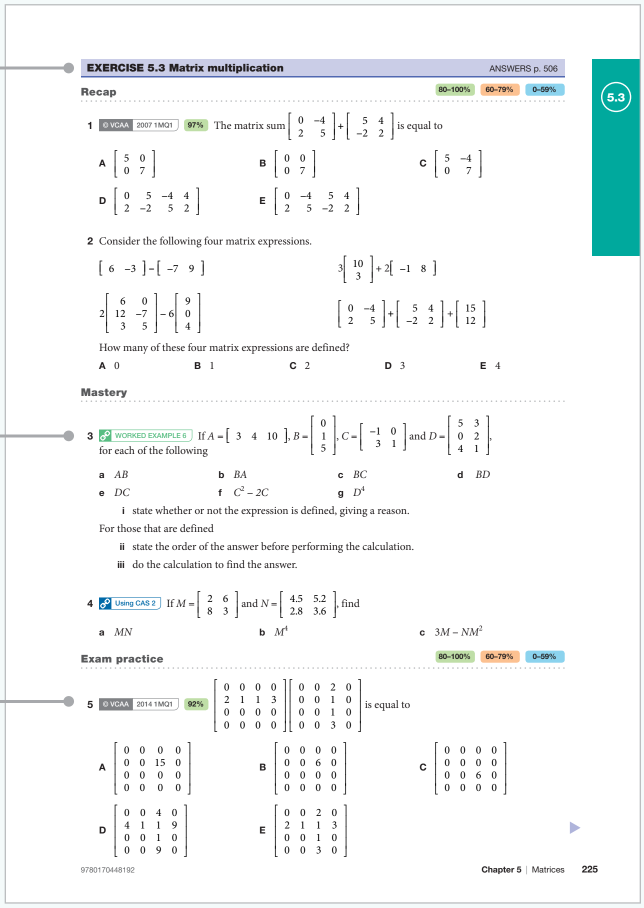
Exam hacks highlight valuable exam hints and common student errors.

Graded exercises, including **Recap**, **Mastery** and **Exam practice** questions, are linked to worked examples and Using CAS, and include past VCE exam and exam-style questions.

Recap questions revise skills from the previous exercise, **Mastery** questions provide skill practice, while **Exam practice** applies learned skills to VCE exam and exam-style problems.

© VCAA 2014 1MQ1 92%

Past VCE exam questions are clearly tagged and graded by colour-coded success percentages based on VCAA data, presented in the order: Exam 1, Exam 2.



EXERCISE 5.3 Matrix multiplication ANSWERS p. 506

Recap 80–100% 60–79% 0–59%

1 © VCAA 2007 1MQ1 97% The matrix sum $\begin{bmatrix} 0 & -4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ -2 & 2 \end{bmatrix}$ is equal to

A $\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 5 & -4 \\ 0 & 7 \end{bmatrix}$

D $\begin{bmatrix} 0 & 5 & -4 & 4 \\ 2 & -2 & 5 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & -4 & 5 & 4 \\ 2 & 5 & -2 & 2 \end{bmatrix}$

2 Consider the following four matrix expressions.

$\begin{bmatrix} 6 & -3 \end{bmatrix} - \begin{bmatrix} -7 & 9 \end{bmatrix}$ $3\begin{bmatrix} 10 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 & 8 \end{bmatrix}$

$2\begin{bmatrix} 6 & 0 \\ 12 & -7 \\ 3 & 5 \end{bmatrix} - 6\begin{bmatrix} 9 \\ 0 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 0 & -4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 15 \\ 12 \end{bmatrix}$

How many of these four matrix expressions are defined?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

Mastery

3 WORKED EXAMPLE 6 If $A = \begin{bmatrix} 3 & 4 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 3 \\ 0 & 2 \\ 4 & 1 \end{bmatrix}$, for each of the following

a AB **b** BA **c** BC **d** BD

e DC **f** $C^2 - 2C$ **g** D^4

i state whether or not the expression is defined, giving a reason.

For those that are defined

ii state the order of the answer before performing the calculation.

iii do the calculation to find the answer.

4 Using CAS 2 If $M = \begin{bmatrix} 2 & 6 \\ 8 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} 4.5 & 5.2 \\ 2.8 & 3.6 \end{bmatrix}$, find

a MN **b** M^4 **c** $3M - NM^2$

Exam practice 80–100% 60–79% 0–59%

5 © VCAA 2014 1MQ1 92% $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ is equal to

A $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 4 & 0 \\ 4 & 1 & 1 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

9780170448192 Chapter 5 | Matrices 225

KEY

2021	2021 exam year
<year>N	Northern hemisphere exam
<year>S	VCAA sample exam
1	Exam 1
2	Exam 2
C	Core
M	Matrices
N	Networks and decision mathematics
GM	Geometry and measurement
GR	Graphs and relations
GT	Geometry and trigonometry
BRM	Business-related mathematics
NP	Number patterns

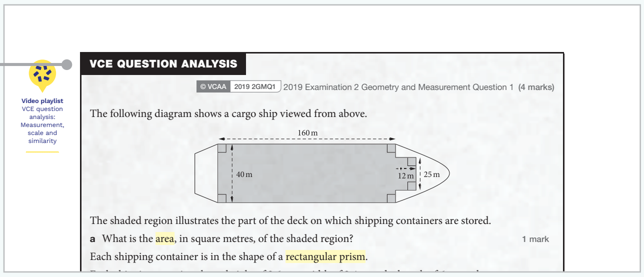
For multiple-choice questions and other 1-mark questions, the success percentage is the percentage of students who answered correctly. For questions worth more than 1 mark, the success percentage is the mean percentage result scored by students.

Success percentage range	Example
80–100% (straightforward)	83%
60–79% (standard)	62%
0–59% (complex)	45%

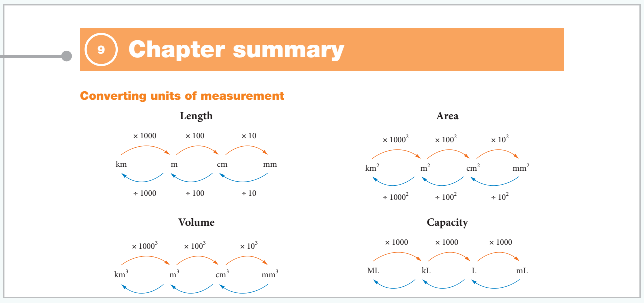
Note: Questions from Northern hemisphere and VCAA sample exams do not have success percentages.

At the end of each chapter

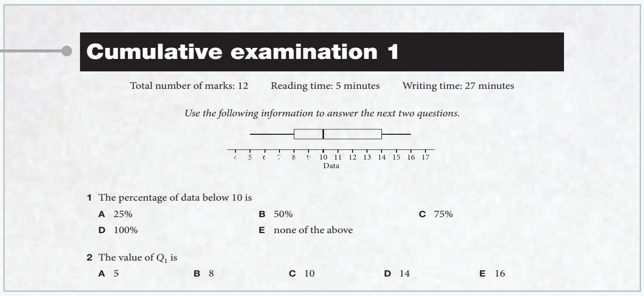
VCE question analysis leads students through a past VCE exam question that exemplifies the chapter, discussing how to approach the question, providing advice on interpreting the question, common student errors and statistics on student performance.



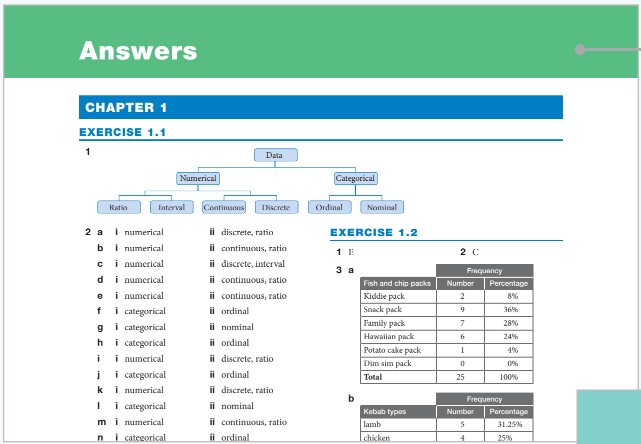
Chapter summary for easy reference.



Cumulative examinations 1 and **2** are mini-exams based on the format of the VCE examinations 1 and 2, with around 50% of questions focusing on the chapter in which they appear.



At the end of the book



Answers (with worked solutions provided on Nelson MindTap for teachers to allocate to students).

A combined **Glossary and index**.

Nelson MindTap

Nelson MindTap is an online learning space that provides students with tailored learning experiences. Access tools and content that make learning simpler yet smarter to help you achieve VCE maths mastery.

Nelson MindTap includes an eText with integrated interactives and online assessment.

Margin links in the student book signpost multimedia student resources found on MindTap.

Video playlists

Worksheets

Skillsheets

Nelson MindTap for students:

- **Watch** video tutorials featuring expert teacher advice to unpack new concepts and develop your understanding.
- **Revise** using quizzes, worksheets and skillsheets to practise your skills and build your confidence.
- **Navigate** your own path, accessing the content, analytics and support as you need it.

Nelson MindTap for teachers*:

- Tailor content to different learning needs – assign directly to the student, or the whole class.
- Monitor progress using assessment tools like Gradebook and Reports.
- Integrate content and assessment directly within your school's LMS for ease of access.
- Access topic tests, teaching plans and worked solutions to each exercise set.

*Complimentary access to these resources is only available to teachers who use this book as part of a class set, book hire or booklist. Contact your Cengage Education Consultant for information about access and conditions.

Nelson VICmaths 11–12 series

Companion resources

Mastery Workbook

WB

- Step-by-step scaffolding to guide students towards mastery of the course content.
- Write-in support that encourages students to show working.
- Matched examples for *every* worked example in the student book.
- Full integration between student book and workbook.
- Answers (worked solutions for teachers on Nelson MindTap)

Exam*plus*

- Create and simulate exam-like conditions in minutes.
- Save time with an extensive bank of filterable and difficulty-graded questions for Year 12.
- Over 1000 past VCAA and unseen exam-style questions and solutions all in the one place.
- Extensively researched and user-tested.

TI-Nspire CAS introduction

The latest TI-Nspire model is TI-Nspire CX II CAS. When purchasing a new handheld, you also gain access to the student software. If you purchase a used handheld, then you can pay an additional fee for the student software. Alternatively, you can connect your handheld to your computer using the TI-Nspire™ CX II Connect web-based app, which enables you to perform a variety of functions such as screen captures, file transfers and operating system updates.

Note that TI-Nspire non-CAS technology is also available. It is **vital** that you use the CAS technology.

TI-Nspire CX II CAS

Student book instructions

The instructions in this student book use words instead of symbols. Most keys on the keypad are clearly labelled with a word or an abbreviation. Four words that are used to represent less obvious keys are:

Word	Key
home	
catalog	
template	
Scratchpad	

Several keys have a second function highlighted in blue above the key. For example, press **ctrl + x²** to access the square root function $\sqrt{\ }$.

Applications

The applications available are outlined below.

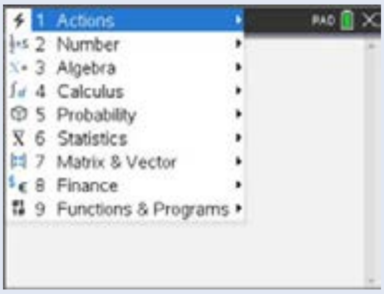
Press **home** to view the home page. The **Scratchpad** options on the left are available for quick calculations and graphing. The **Document** options on the right are used for navigation. The seven icons on the bottom are the main applications.

When you select **Documents > New** from the home page, a list of the seven applications plus three additional menu options will be displayed. From any application, press **ctrl + I** (for insert) to display this list and add a new page to the document.

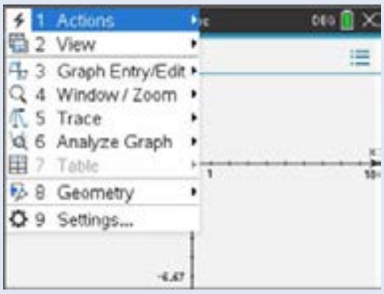
Menus

The instructions in this student book primarily use the **Calculator, Graphs, Lists & Spreadsheet** and **Data & Statistics** applications (see **Hints** on page xvii). The following figures show the initial menu options for these four applications. These menu options link to submenu options, which are not shown below. On the handheld and software, the applications are referred to as **Documents**. In the student book instructions, the applications are referred to as **pages** of the document (e.g. Add a **Graphs** page).

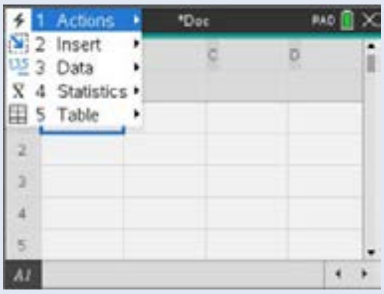
Calculator



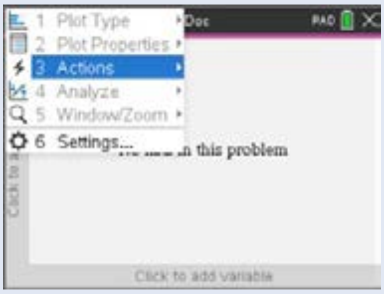
Graphs



Lists & Spreadsheet



Data & Statistics



If your document contains more than one page, move among them by clicking on the numbered tabs at the top of the screen. Alternatively, press **ctrl + left arrow** or **ctrl + right arrow**. To view all the pages of a document, press **ctrl + up arrow**.

All menu and submenu options include numbers. The student book instructions do not include numbers. For example, the instruction in a **Calculator** page for clearing all calculations is 'press **menu > Actions > Clear History**'. The shortcut is 'press **menu > 1 > 5**'. For efficiency, you are encouraged to learn the sequence of numbers for frequently used commands.

Document Settings

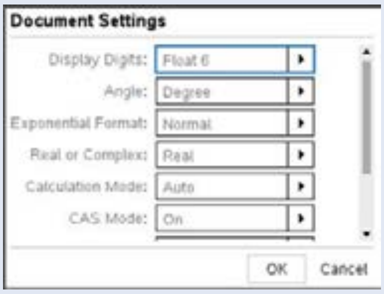
The document settings can be accessed in the following ways:

1 From the **home** page, press **Settings > Document Settings**.

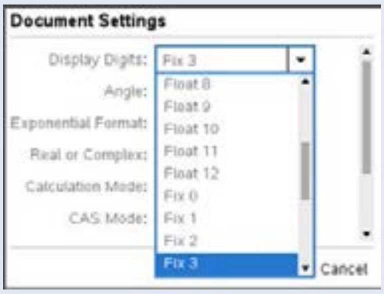
2 From a document page, press the **doc** key or click on **Doc** at the top of the page, then select **Settings & Status > Document Settings**.

3 Click on the **battery icon** in the top right-hand corner of the page, then select **Document Settings**.

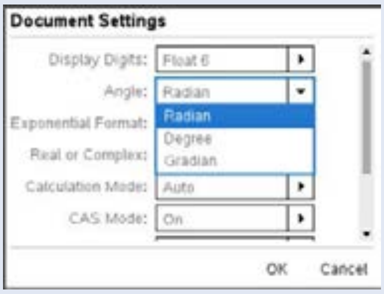
The document settings shown below are the primary ones you will be using.



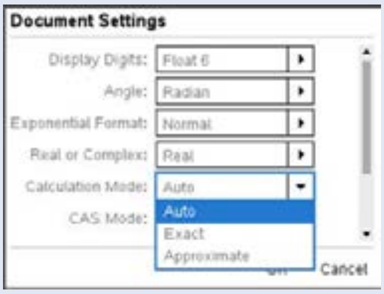
The screen above shows the default document settings. The **Display Digits** field is set to **Float 6**, which means up to six significant figures will be displayed.



Click in any field to display the options. The screen above shows the **Display Digits** options. Scroll down to select a specific number of decimal places.



There are three **Angle** options shown above. Select either **Radian** or **Degree**. These two angle settings can be toggled at any time by clicking on **DEG** or **RAD** in the top right-hand corner of the screen.

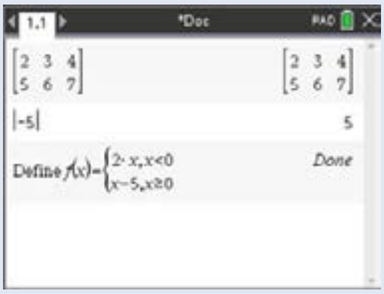


There are three **Calculation Mode** options shown above. It is recommended that you keep the default setting of **Auto**. If you require an approximate or decimal answer, press **ctrl + enter** or include a decimal point in your calculation.

Templates

The **template** key is located to the right of the **9** key.

Press **template** to view the template options. Most templates will be directly inserted but for some, you will be prompted with a dialogue box. For example, after selecting the **3×3** matrix template, you will be prompted for the number of rows and columns of the matrix.

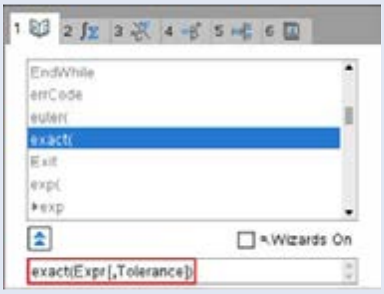


The screen above shows three examples using templates: a **2×3** matrix, an **absolute value** and a **piecewise function**. Many of the templates can be accessed using keys on the keyboard, for example, fraction, square root etc.

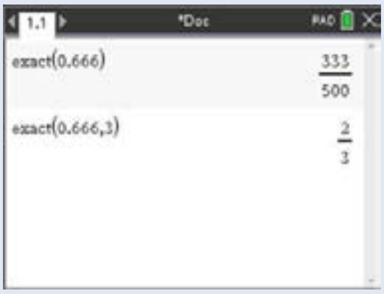
Catalog

The **catalog** key is located to the right of the **template** key. In addition to using the menus and submenus, you can access all the commands using the catalog. The advantage of using the catalog is that it shows the parameters required for each command at the bottom of the screen. Optional parameters appear in square brackets.

For example:



Press **catalog** and ensure tab **1** is selected. Press **E** to jump to the commands starting with E. Scroll down to **exact(**. The parameters for the **exact** command appear at the bottom of the screen (see **red** rectangle). **Expr** is a required parameter. The square brackets around **Tolerance** means it is an optional parameter.

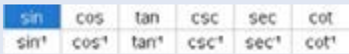


When you use the **exact** command to convert 0.666 to a fraction, the answer is $\frac{333}{500}$.

If you include the optional tolerance of 3 then 0.666 converts to $\frac{2}{3}$.

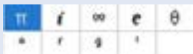
Symbols

All symbols are listed at the end of the **catalog**. To access the symbol palette, press **ctrl** + **catalog**. Frequently used symbols are available in mini-palettes by pressing the keys shown below.



Press **trig** to access the trigonometry functions.

Press **ctrl + =** to access the inequalities and the **constraint** symbol.

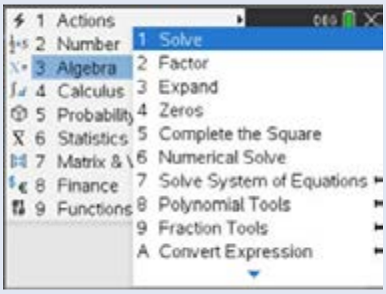


Press **π** to access commonly used symbols.

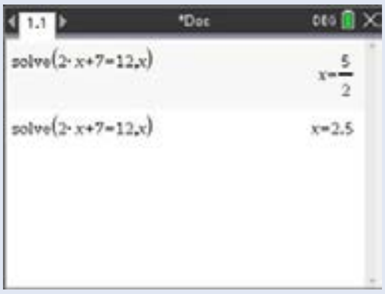
Hints

The instructions below provide a few hints to assist with using the four main applications.

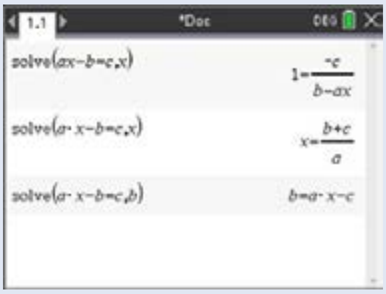
Calculator



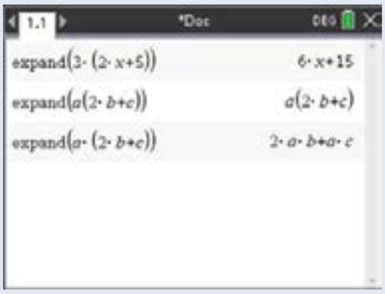
Press **menu > Algebra** to access the algebra functions. Select **Solve**. Other useful menu options include **Factor** and **Expand**.



Enter the equation followed by **,x**. Press **enter** for the exact answer. Press **ctrl + enter** for the decimal answer.

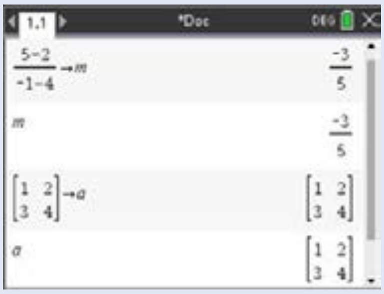


When multiplying variables, you *must* include a multiplication sign. The first answer above is incorrect as there is no multiplication sign between the **a** and **x**. CAS assumes this is the variable **ax**, not **a × x**. The second answer is correct. The third answer shows a **solve** example using the variable **b** instead of **x**.



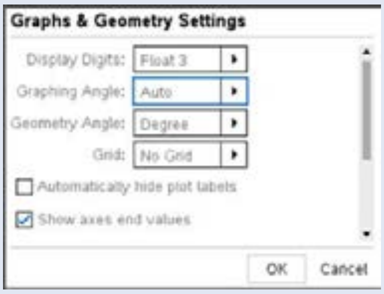
When you enter a left or right bracket, both brackets appear. When you enter a number in front of a bracket, CAS automatically inserts a multiplication sign as in the first example above. However, if you enter a variable in front of a bracket, you *must* include the multiplication sign or you will get an error, or an incorrect answer. The second example above is incorrect, whereas the third example is correct.

Press **ctrl + =** to access the mini-palette for the inequality symbols. This mini-palette also has the **constraint** symbol (|), which can be used for substitution and to restrict domains.

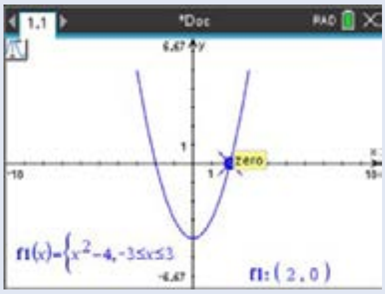


Press **ctrl + var** for the **store** symbol, which can be used to store values and matrices. Note that TI-Nspire converts all letters to lower case.

Graphs & Geometry

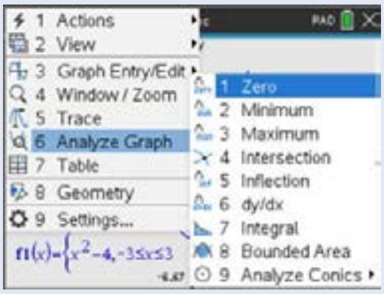


Press **menu > Settings** to view and/or change the settings for the **Graphs** and **Geometry** applications. The default setting for **Display Digits** is **Float 3** so change this if you need greater accuracy. The **Graphing Angle** is set to **Auto** but can be changed to **Degree** or **Radian**.

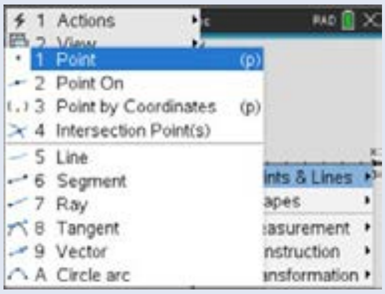


The **constraint** command can also be used in a **Graphs** page to specify a domain.

Press **menu > Trace > Graph Trace** to identify points on a graph.

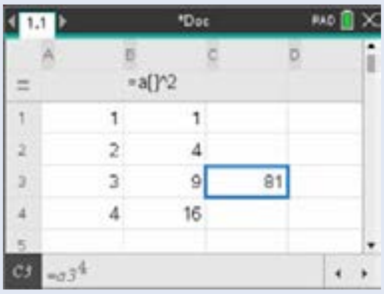


A second option for identifying key points on a graph is to press **menu > Analyze Graph**. These options prompt you for a **lower bound** and **upper bound** to locate the point.

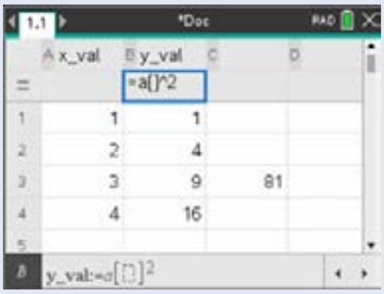


A third option is to press **menu > Geometry > Points & Lines**. Locate points of intersection or place points on the graphs and move them along the graph. Click on a coordinate to manually change a value.

Lists & Spreadsheet

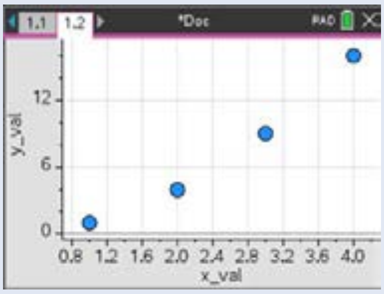


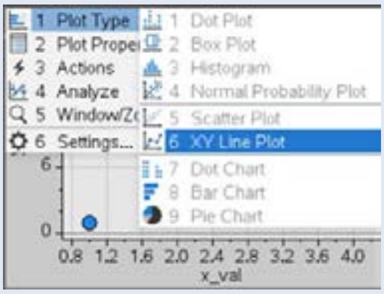
The columns in the **Lists & Spreadsheet** application can be used for lists. Above, the values in the list in column **B** are the squares of the values of the list in column **A**. It can also be used as a spreadsheet. The value in cell **C3** is the value in cell **A3** raised to the power of 4.



Calculations using the lists can be completed using the column headings **A**, **B**, **C** etc. Lists need to be labelled to be used in other applications. The list **A** above has been labelled **x_val** and list **B** labelled **y_val**. List names cannot have spaces so spaces are replaced by asterisks. Alternatively, press **ctrl > space bar** to insert the **underscore** character instead of a space.

Data & Statistics





The **Data & Statistics** application is reliant on lists generated in other applications. It is designed for ungrouped data. The plot above displays the **x_val** and **y_val** lists from the **Lists & Spreadsheet** application.

Press **menu > Plot Type** to view the various graphing options. Options 1 to 4 are for univariate data (one list). Options 5 and 6 are for bivariate data (two lists). Options 7 to 9 are for categorical data (one list).

Operating systems

Ensure the latest operating system is installed on your handheld and software.

Installing the latest operating system is relatively straightforward. Using the USB cable provided, connect the handheld to a computer with the student or computer link software installed. Select **Help > Check for OS Updates**. If you see a message that a new OS is available, follow the links to install it. Alternatively, go to the TI website at https://education.ti.com/ to download the latest operating system. Select **Tools > Install OS** then select the downloaded file.

To determine the version of your operating system, press **home > Settings > Status**. At the time of publication, the operating system for the CX II is version **5.4.0.259**.

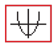
Casio ClassPad introduction

The latest model of the Casio ClassPad is the fx-CP400. The connectivity software Screen Receiver, Share Assistant and Program Link Software can be downloaded for free. The ClassPad Manager software emulator is a separate program available at an additional cost.

Casio ClassPad

Student book instructions

The instructions in this student book use words instead of symbols. ClassPad tools are located at the top of the screen. These tools vary with each application. Initially, these instructions will show a tool enclosed in a red rectangle with the corresponding word highlighted in red. Examples are shown.

Word	Tool
Graph	
View Window	
Table	
Table Input	

Applications

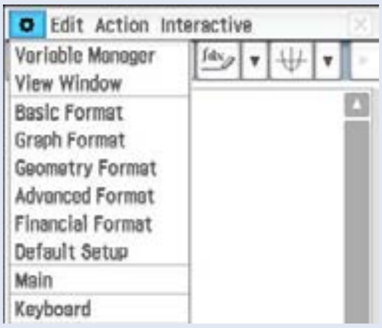
The applications available are outlined below.

Tap **Menu** to view the applications.

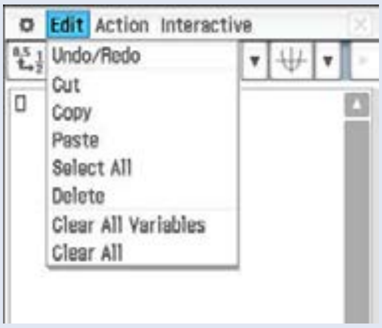
Slide the scroll bar at the bottom of the screen to access the full list.

Menus

The instructions in this student book will primarily use the Main, Statistics, Spreadsheet, Graph&Table, Sequence and Financial applications (see **Applications** on page xxiii). All these applications have the ⚙ (systems) and **Edit** menus available at the top left of the screen.



Tap ⚙ to view the system menu. The menu options allow you to manage variables and format applications.



Tap **Edit** to view the edit menu. The menu options allow you to cut, copy, paste and delete screen content and clear variables. The **Edit** menu varies with each application.

Document Settings

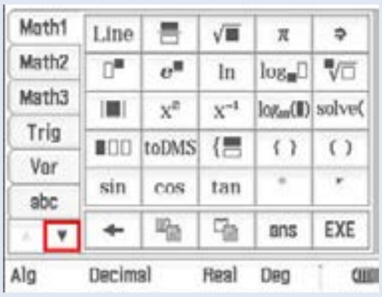
Tap ⚙ > **Basic Format**. The screen above shows the **default** document settings. The **Number Format** field is set to **Normal 1**.

Tap the **Number Format** field to display the options. The screen above shows the **Display Digits** options. Scroll down to fix a specific number of decimal places.

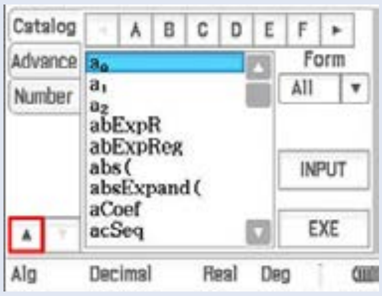
The settings **Standard/Decimal**, **Real/Cplx** and **Rad/Deg/Gra** can be toggled at the bottom of the screen. The recommended settings are **Decimal**, **Real** and **Deg**.

Keyboard

There are nine soft keyboards available.



Press **Keyboard** to view the soft keyboards.
The **Math1** soft keyboard is shown above.
Tap the left tabs to access the other keyboards.
Press the **down arrow** to view the second screen.



All functions can be accessed from **Catalog**.
Tap on the letters at the top to jump through the list. Press the **up arrow** to return to the first screen.

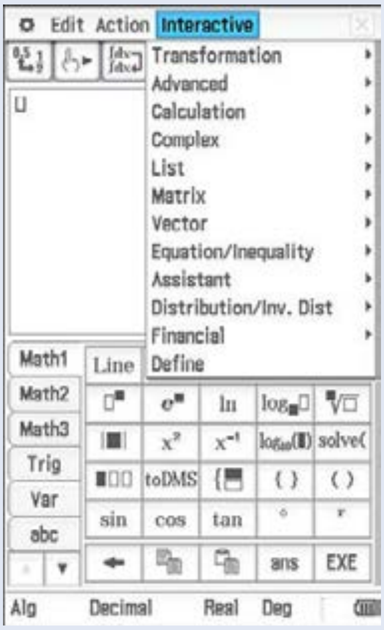
From the first screen, tap **Var** to access the variables. Use variables, not letters, in your algebraic calculations.

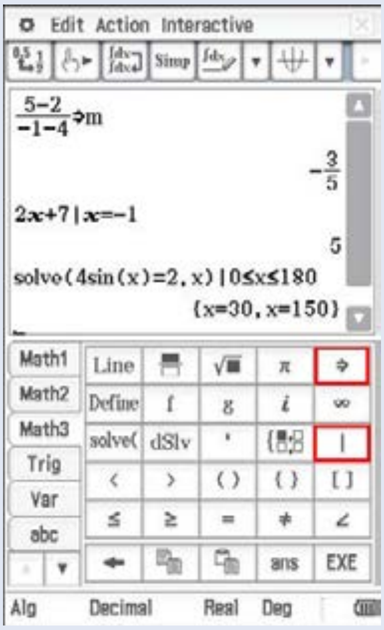
Tap **abc** to access letters and symbols. Use letters to name functions, matrices etc. Tap the tabs at the top of the screen to access the range of symbols. Press **back** to return to the main screen.

Applications

The instructions below provide a few hints to assist with using the main applications.

Main





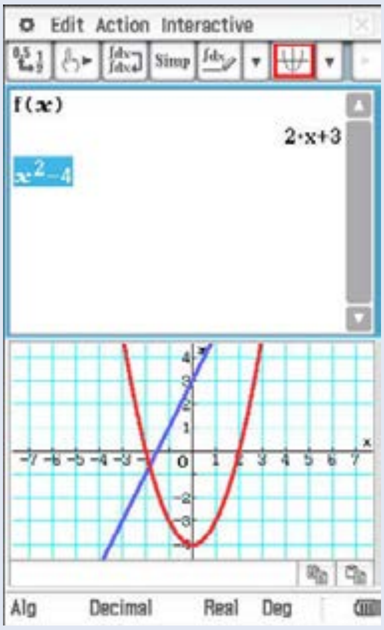
In **Main**, there are two menus available to enter functions: **Active** and **Interactive**. With the **Interactive** menu, enter the expression first then highlight and operate from there. With few exceptions, the instructions in this student book are written using the **Interactive** menu.

The **store** (→) arrow is available from the **Math** and **Trig** soft keyboards. **Store** can be used to store values and matrices. Use the **abc** soft keyboard to label them.

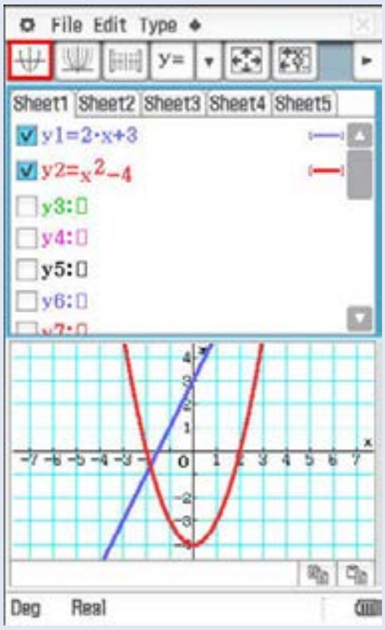
The **constraint** symbol **|** is available in the **Math3** soft keyboard. Use **constraint** for substitution and to restrict domains.

Graphing

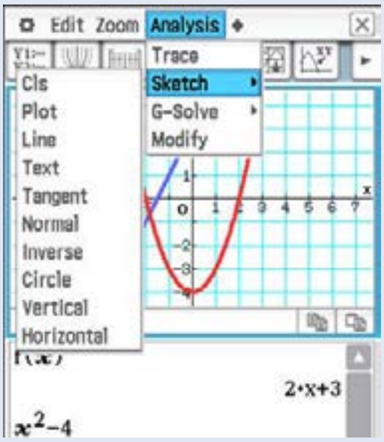
There are two main options for graphing functions and relations.



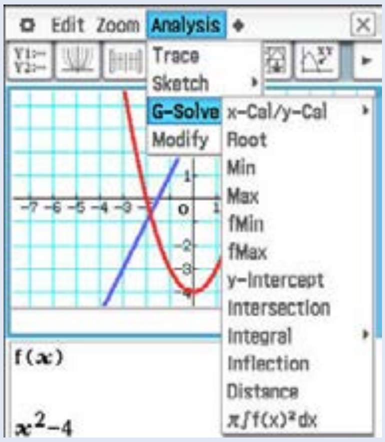
In **Main**, tap **Graph** to open the graph window. Enter and highlight the function or expression and drag it into the **Graph** window.



Tap **Menu > Graph&Table**. Enter the function then tap **Graph**. The instructions in this student book use the **Main** option because the **y=** is not required, but ensure you are familiar with both graphing methods.



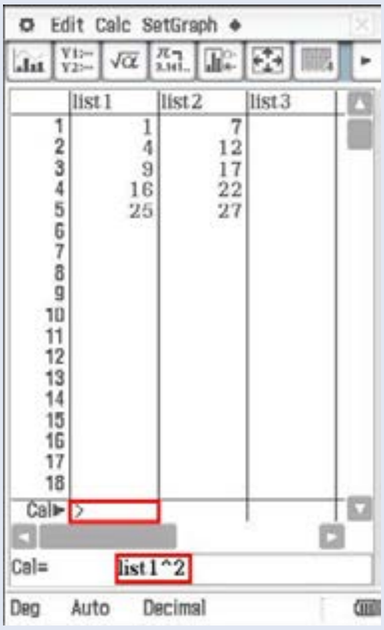
There are several menu options available to analyse graphs. Tap **Analysis > Trace** and press the **arrow** keys to move along the graph. Tap **Analysis > Sketch** for the options shown above.



Tap **Analysis > G-Solve** to identify key features of a graph. For example, select **Root** to find the *x*-intercepts. Select **Intersection** to locate points of intersection of two graphs.

9780170448192

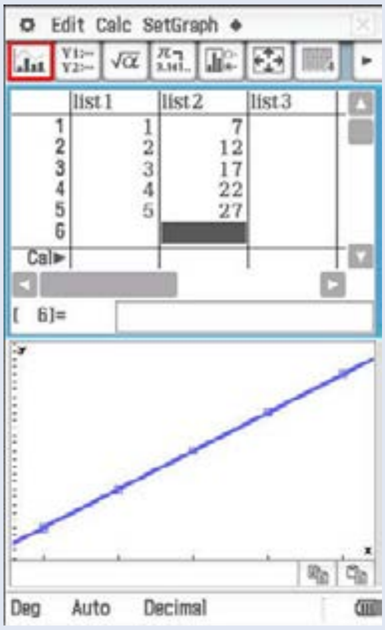
Statistics



Tap **Menu > Statistics** to open the Statistics application. The default settings show **list1**, **list2**, **list3** etc. If required, tap on the list name to enter a new heading. If you need to perform a calculation, tap in the **Cal** row at the bottom of the list to enter the formula.

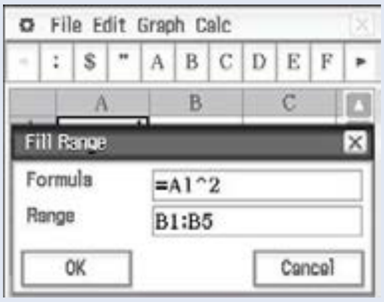
Tap **Calc** to view the Calculation menu options. Tap **One-Variable** for the dialogue box used to calculate statistical analysis of a list. Tap **Regression** to access the Regression submenu options shown above.

Tap **SetGraph** to set up the statistics graphs. Tap **Type:** to view the graphing options for statistical graphs. When finished, tap **Set**.

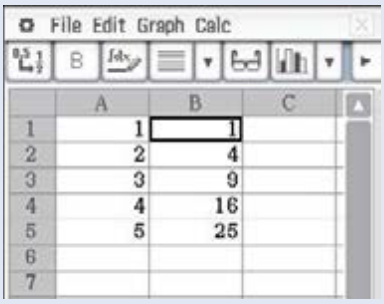


Tap **Graph** to display the statistical graph. The graph above displays a scatterplot of the **list1** and **list2** values with the linear regression line joining the points.

Spreadsheet



Tap **Menu > Spreadsheet** to open the Spreadsheet application. Tap **Edit > Fill > Fill Range**. In the dialogue box, enter the formula and range.



In the screen above, the values 1 to 5 were entered into column **A**. The formula and range entered squared these values and placed them in column **B**.

Operating systems

Ensure you have the latest operating system installed on your handheld.

Tap **Menu** then tap **Settings** (located in the bottom left corner of the screen). Select **Version**. At the time of publication, the latest version is **02.01.7001**.

To download the latest operating system, go to the Australian Shriro website at http://www.casio.edu.shriro.com.au/classpad.php. Using the USB cable provided, connect the handheld to a computer. Start the installation program and follow the prompts. Some of these prompts will be on the computer and others will be on the handheld.

CHAPTER

1 INVESTIGATING AND COMPARING DATA DISTRIBUTIONS

STUDY DESIGN COVERAGE

Study Design coverage

UNIT 1, AREA OF STUDY 1: DATA ANALYSIS, PROBABILITY AND STATISTICS

Investigating and comparing data distributions

- types of data, including categorical (nominal or ordinal) or numerical (discrete or continuous, interval, ratio)
- display and description of categorical data distributions of one or more groups using frequency tables and bar charts, and the mode and its interpretation
- display and description of numerical data distributions using histograms, stem plots and dot plots and choosing between plots according to context and purpose
- summarising numerical data distributions, including use of and calculation of the sample summary statistics, median, range, and interquartile range (IQR) or mean and standard deviation
- the five-number summary and the boxplot as its graphical representation and display, including the use of the lower fence (Q_1 − 1.5 × IQR) and upper fence (Q_3 + 1.5 × IQR) to identify possible outliers
- consideration of a range of distributions (symmetrical, asymmetrical), their summary statistics and the percentage of data lying within several standard deviations of the mean
- use of back-to-back stem plots or parallel boxplots, as appropriate, to compare the distributions of a single numerical variable across two or more groups in terms of centre (median) and spread (IQR and range), and the interpretation of any differences observed in the context of the data.

Video playlists (8):

1.1 Introduction to data distributions
1.2 Tables and charts
1.3 Histograms
1.4 Boxplots
1.5 Dot plots and stem plots
1.6 Back-to-back stem plots and parallel boxplots
1.7 The mean and standard deviation
VCE question analysis Investigating and comparing data distributions

Skillsheets (1):

1.1 Statistical measures

Worksheets (27):

1.1 Statistical data match-up • Mean, median, mode and range • Mode, median and mean • Measures of central tendency
1.2 Frequency distribution tables • Frequency tables
1.3 Histograms • Shapes of distributions
1.4 Five-number summaries 1 • Five-number summaries 2 • Boxplots • Boxplots 1 • Boxplots 2 • Interquartile range
1.5 Stem-and-leaf plots
1.6 Box-and-whisker plots
1.7 Comparing group measures • Comparing city temperatures • Comparing word lengths • Comparing sports scores • Investigating young drivers • Standard deviation • Statistical calculations • Statistics review • Calculating and interpreting • Data and statistics crossword • Statistics crossword

Puzzles (1):

1.4 Statistical measures puzzle

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Introduction to data distributions

Worksheet
Statistical data match-up

1.1 Introduction to data distributions

What is data?

We live in a world where we can access **data** to make informed decisions. How much of climate change is caused by human activity? Let's look at the data. What illnesses does vaping cause? Let's look at the data. How can we reduce crime? Let's look at the data.

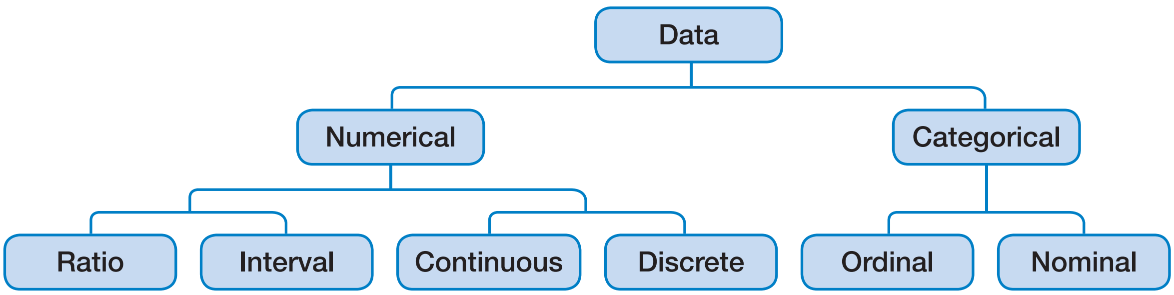
Data is information collected through observation that can then be used to make decisions. Here is some data based on the responses to survey questions by a group of ten students:

Survey question	Data
What is the colour of your best friend's eyes?	brown, brown, brown, blue, brown, green, blue, blue, hazel, brown
How many pets do you have at home?	0, 1, 1, 5, 2, 0, 3, 1, 3, 2
How would you rate the current most popular song on Spotify from 1 to 4, where 1 = great, 2 = okay, 3 = awful and 4 = haven't heard it?	2, 4, 2, 2, 1, 3, 1, 2, 4, 1
What is the length of your handspan?	8.0 cm, 7.3 cm, 6.8 cm, 9.2 cm, 8.0 cm, 8.9 cm, 9.2 cm, 7.1 cm, 9.3 cm, 6.5 cm
What is your house number?	23, 41, 6, 118, 51, 33, 2, 19, 12, 4
What was the coldest temperature (°C) you remember experiencing?	−1°C, 3°C, 5°C, 0°C, −8°C, 12°C, −4°C, 2°C, 7°C, −3°C

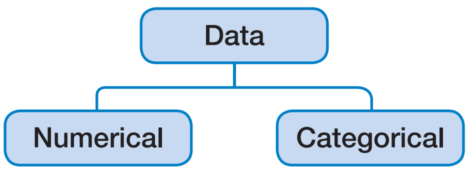
The information you are looking for such as *colour of eyes*, *number of pets* and *temperature* are called **variables**.

Types of data

We will be looking at eight data types. We need to know the data type before we can decide on the right way to work with data.



There are two main types of data:



Numerical data involves numbers that can be measured or counted. One way to test this is to ask, 'Does it makes sense to add the numbers together?' For example:

- *number of pets* in your home
  We are counting the number of pets, and adding the numbers makes sense: 2 pets + 4 pets = 6 pets
- *length* of your handspan
  We are measuring length, and adding the numbers makes sense: 8.1 cm + 7.3 cm = 15.4 cm

Categorical data involves either numbers where adding makes no sense or categories that don't involve any numbers. For example:

- *colour* of your best friend's eyes

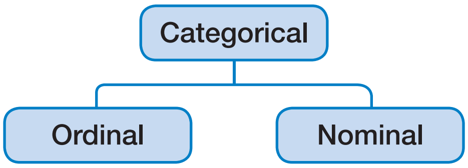
  The categories brown, blue etc. are not numbers.
- the number of your house

  Adding these numbers makes no sense:
  house number 23 + house number 41 ≠ house number 64

> **Exam hack**
>
> Don't fall into the trap of thinking that because numbers are involved, the data must be numerical!

Types of categorical data

There are two types of categorical data:



Ordinal data is categorical data that has a natural order. For example:

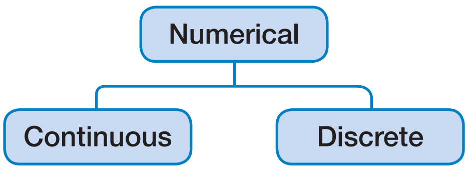
- *rating* of the current most popular song on Spotify from 1 to 4, where 1 = great, 2 = okay, 3 = awful and 4 = haven't heard it
- the *number* of your house

Nominal data is categorical data with *no* natural order. For example:

- *colour* of your best friend's eyes

Types of numerical data

Numerical data can be divided into these two types of data:



Continuous data is numerical data that can be measured to ever-increasing levels of accuracy. For example:

- *length* of your handspan

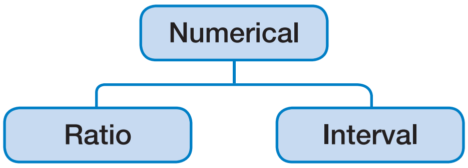
  This can be measured to ever-increasing levels of accuracy (8 cm, 7 cm, 7 cm … or 8.1 cm, 7.3 cm, 6.8 cm … or 8.14 cm, 7.31 cm, 6.79 cm … etc.).

Discrete data is numerical data that can only take specific values and *can't* be measured to ever-increasing levels of accuracy. For example:

- *number of pets* in your home

  This can only take whole number values 0, 1, 2, 3 … It is impossible, for example, to have 1.5 or 2.8 of a pet.

Numerical data can *also* be divided into these two types of data:



Ratio data is numerical data that has a fixed beginning. For example:

- *handspan* measurements start at 0; a negative handspan is impossible.
- *number of pets* in your home start at 0; a negative number of pets is impossible.

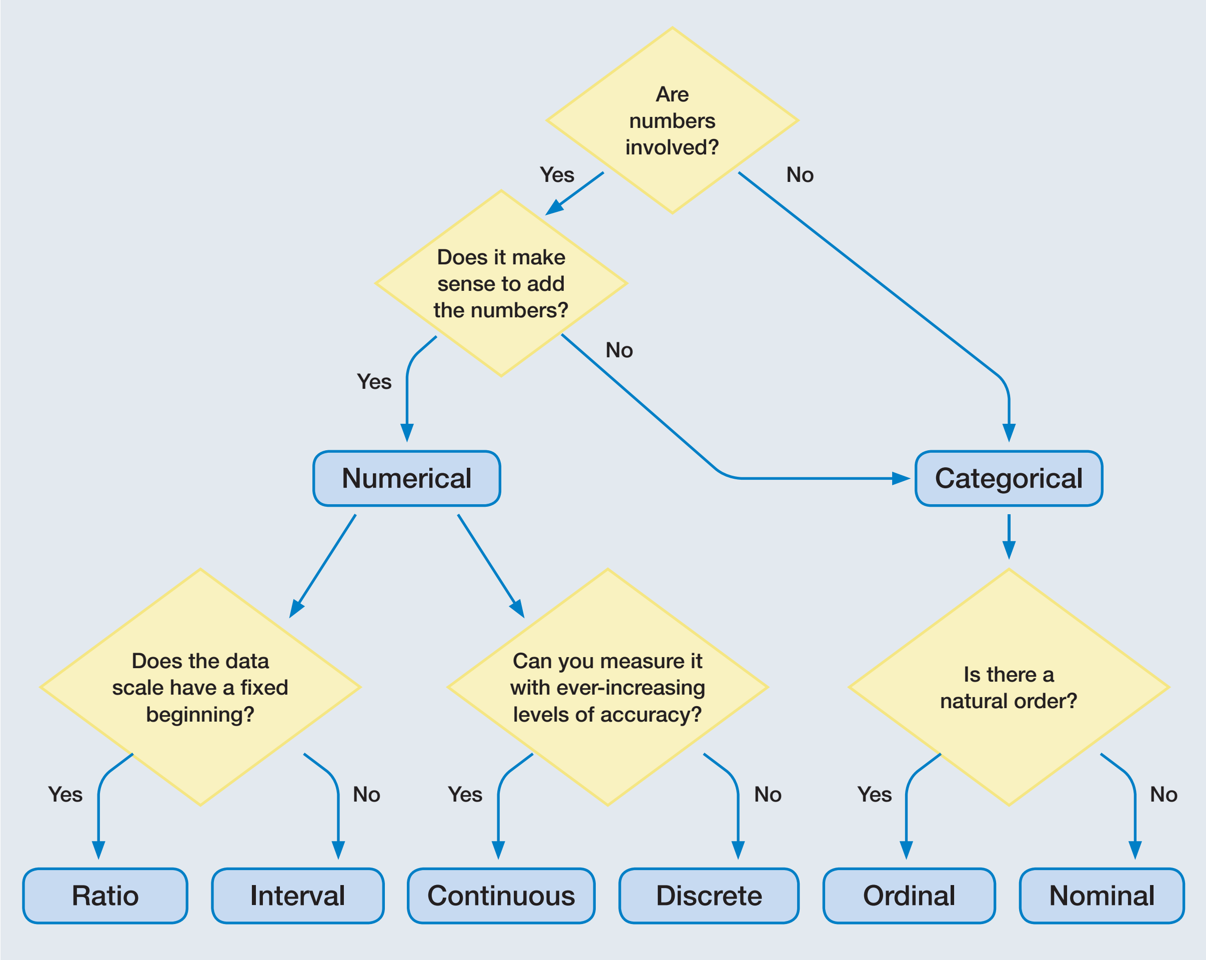
Interval data is numerical data that has **no** fixed beginning. For example:

- *temperature* (°C) doesn't begin at 0°C as we can have temperatures of −5°C, −70°C etc.
- *calendar years* don't have a beginning as we can have 45 BC, 2000 BC etc.

Exam hack

Ratio and interval data can be continuous or discrete.

Classifying data



WB p. 1

WORKED EXAMPLE 1 Deciding on the type of data

State whether the following data is

i categorical or numerical

ii nominal, ordinal, discrete, continuous, interval or ratio.

Steps		Working
a Height		
i	**1** Are numbers involved?	yes
	2 Does it make sense to add the numbers?	Yes, so it is numerical.
ii	**1** Can you measure it with increasing levels of accuracy?	Yes, so it is continuous.
	2 Does the data scale have a fixed beginning?	Yes, so it is ratio.
b Hair colour		
i	Are numbers involved?	No, so it is categorical.
ii	Is there a natural order?	No, so it is nominal.
c Finishing position in a 100-metre race		
i	**1** Are numbers involved?	yes
	2 Does it make sense to add the numbers?	No, so it is categorical.
ii	Is there a natural order?	Yes, so it is ordinal.
d Number of friends on a social media platform		
i	**1** Are numbers involved?	yes
	2 Does it make sense to add the numbers?	Yes, so it is numerical.
ii	**1** Can you measure it with increasing levels of accuracy?	No, so it is discrete.
	2 Does the data scale have a fixed beginning?	Yes, so it is ratio.
e Numbers worn by football players		
i	**1** Are numbers involved?	yes
	2 Does it make sense to add the numbers?	No, so it is categorical.
ii	Is there a natural order?	No, so it is nominal.
f The annual profit or loss of a business to the nearest cent		
i	**1** Are numbers involved?	yes
	2 Does it make sense to add the numbers?	Yes, so it is numerical.
ii	**1** Can you measure it with increasing levels of accuracy?	No, so it is discrete.
	2 Does the data scale have a fixed beginning?	No, so it is interval.

Measures of centre and spread

Two key features to look for when analysing data are centre and spread. The **centre of a distribution** is a single value that best describes the distribution. We will first look at two ways of measuring the centre of a distribution:

- The **mode** is the most frequently occurring data value and is often called the **modal category** for categorical data.
  - There can be more than one mode.
  - Data with two modes is called **bi-modal**.
  - If *every* data value appears exactly once, there is no mode.

1.1

Skillsheet Statistical measures

Worksheets Mean, median, mode and range

Mode, median and mean

Measures of central tendency

- The **median** is the middle value. If there are two middle values, add them and divide the sum by 2.

Odd number of ordered data values

Median is the 6th data value

3, 6, 7, 7, 7, 8, 13, 19, 20, 20, 22

5 data values ↑ 5 data values

Median

Even number of data values

Median is between the 6th and 7th data value

3, 6, 7, 7, 7, 8, 13, 17, 19, 20, 22, 26

6 data values ↑ 6 data values

$\text{Median} = \frac{8 + 13}{2} = 10.5$

When analysing data, we also look for the **spread of a distribution**. One measure of the spread of a distribution is the **range**:

range = largest value – smallest value

The more spread out the data is, the larger the range.

Exam hack

Not every measure of centre and spread can be used with every data type.

Centre, spread and data types

Categorical: Nominal data	Categorical: Ordinal data	Numerical data
Measures of centre		
mode	mode median	mode median
Measures of spread		
–	range	range

WB p. 3

WORKED EXAMPLE 2 Finding the median, mode and range

For each of the following, find the

i median **ii** mode **iii** range.

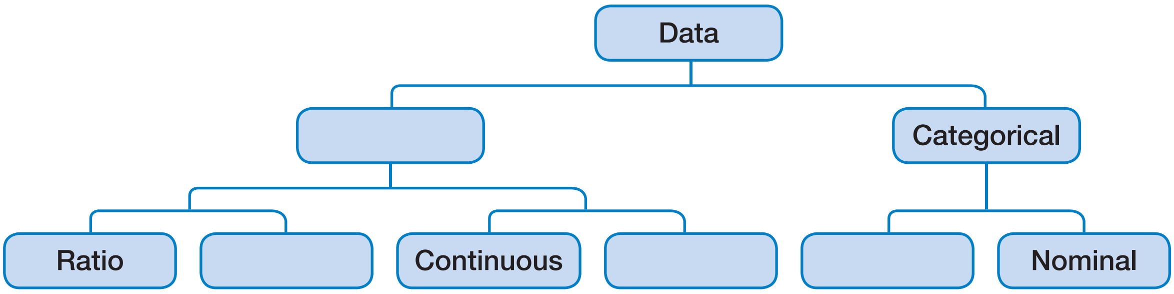
Steps	Working
a Number of pets ten students have at home: 0, 1, 1, 5, 2, 0, 3, 1, 3, 2	
i **1** Order the values from smallest to largest.	0, 0, 1, 1, 1, 2, 2, 3, 3, 5
2 Find the middle value (or two middle values). If there are two middle values, then add them and divide by 2.	middle values are 1 and 2 $\text{median} = \frac{1 + 2}{2} = 1.5$ pets
ii Find the most commonly occurring value.	mode = 1 pet
iii range = largest value – smallest value	range = 5 – 0 = 5 pets
b Marks in a General Mathematics test for nine students: 15, 19, 17, 6, 12, 18, 18, 17, 15	
i **1** Order the values from smallest to largest.	6, 12, 15, 15, 17, 17, 18, 18, 19
2 Find the middle value (or two middle values). If there are two middle values, then add them and divide by 2.	The median is the 5th value in the ordered list. median = 17 marks
ii Find the most commonly occurring value.	mode = 15, 17 and 18 marks
iii range = largest value – smallest value	range = 19 – 6 = 13 marks

EXERCISE 1.1 Introduction to data distributions

ANSWERS p. 490

Mastery

1 Copy and complete the following diagram:



2 WORKED EXAMPLE 1 State whether the following data is

i categorical or numerical

ii nominal, ordinal, discrete, continuous, interval or ratio.

a Number of books in a home

b Distance to the nearest train station from your home

c Years when a solar eclipse occurred

d Amount of time spent sleeping last night

e Speed of an aeroplane just after take-off

f Sizes of pizzas from a takeaway (regular, large, family)

g Numbers on the Matildas' soccer uniforms

h Opinion of chocolate on a scale of 1 to 5, where 1 is hate and 5 is love

i Salary in dollars

j Salary classified as high, medium or low

k Number of people watching the Australian Tennis Open

l Titles of surfing magazines

m Foot length of swimmers in a squad

n Classification ratings for movies (G, PG, M, MA, R)

o Maximum temperature (Celsius) for a series of days

p State of birth of each person born in Australia

3 WORKED EXAMPLE 2 For each of the following, find the

i median **ii** mode **iii** range.

a Number of screens (phones, tablets and television) twelve students have in their home:

5, 8, 3, 10, 4, 5, 7, 16, 9, 12, 4, 6

b Marks in a General Mathematics test for nine students:

12, 9, 14, 13, 20, 15, 18, 18, 15

c Number of minutes it takes for eight students to get to school on a particular day:

7, 3, 20, 9, 5, 3, 10, 12

d Thumb length of ten students in centimetres:

6, 6, 5, 5, 6, 7, 6, 7, 5, 5

4 A survey was completed to collect the heights of students within a group of Year 11 students. The type of data collected is best described as

A nominal categorical data.
B ordinal numerical data.
C ordinal categorical data.
D discrete numerical data.
E continuous numerical data.

5 The heights of players in a basketball team are measured. Which one of the following is correct?

A The data is both continuous and interval.
B The data is both discrete and interval.
C The data is both discrete and ratio.
D The data is both continuous and ratio.
E The data is both interval and ratio.

6 The marks for a spelling test of ten words were recorded as follows:

8, 6, 8, 4, 5, 6, 8, 5, 7, 4, 7, 8, 6, 8, 9

The median and mode respectively are

A 6 and 8 **B** 7 and 6.6 **C** 7 and 7 **D** 7 and 8 **E** 8 and 7

7 A survey was conducted about the colour of family cars, with the following results:

grey, grey, white, grey, white, red, white, blue, red, white, red, silver

What is the median of this data?

A white **B** grey **C** red
D blue **E** there is no median

Use the following information to answer the next two questions.

The percentage investment returns of seven superannuation funds for the year 2002 are

–4.6%, –4.7%, 2.9%, 0.3%, –5.5%, –4.4%, –1.1%

8 © VCAA 2003 1CQ1 83% The median investment return is

A –4.7% **B** –4.6% **C** –4.5% **D** –4.4% **E** 0.3%

9 © VCAA 2003 1CQ2 73% The range of investment returns is

A 2.6% **B** 3.5% **C** 4.0% **D** 5.5% **E** 8.4%

10 © VCAA 2002 1CQ1 79% Researchers conducted a survey of 403 school leavers who had recently entered the workforce. The aim was to determine whether the type of work they undertook was gender related. Work type was classified as 'trade', 'clerical', 'manual' or 'professional'. In this survey, the variables **work type** (trade, clerical, manual or professional) and **gender** (male or female) are

A both categorical variables.
B both numerical variables.
C categorical and numerical variables respectively.
D numerical and categorical variables respectively.
E neither categorical nor numerical variables.

11 © VCAA 2020 1CQ7 69% Data relating to the following five variables was collected from insects that were caught overnight in a trap:

- *colour*
- *name of species*
- *number of wings*
- *body length* (in millimetres)
- *body weight* (in milligrams)

The number of these variables that are discrete numerical variables is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

12 © VCAA 2003 1CQ7 62% The level of water usage of 250 houses was rated in a survey as low, medium or high, and the size of the houses as small, standard or large.

The variables **level of water usage** and **size of house**, as recorded in this survey, are

A both numerical variables.
B both categorical variables.
C neither numerical nor categorical variables.
D numerical and categorical variables respectively.
E categorical and numerical variables respectively.

13 (10 marks) The maximum daily temperatures (°C to one decimal place) in Melbourne during the first week of the 2020 Australian Open are shown in the table.

Day	Date	Max temperature (°C)
Monday	20/01/2020	21.8
Tuesday	21/01/2020	23.5
Wednesday	22/01/2020	31.7
Thursday	23/01/2020	22.6
Friday	24/01/2020	24.1
Saturday	25/01/2020	27.4
Sunday	26/01/2020	23.6

a Find, rounded to one decimal place, the

i median **ii** mode **iii** range. 3 marks

b Round the original data to the nearest whole degree. Use these values to find, rounded to one decimal place, the

i median **ii** mode **iii** range. 3 marks

c Explain why your answers for the median in parts **a** and **b** are different. Which of the two answers do you think is more accurate and why? 2 marks

d Which of your answers for the mode in parts **a** and **b** give more helpful information? Why would finding the mode of data rounded to a large number of decimal places usually not give helpful information? 2 marks

Video playlist Tables and charts

Worksheets Frequency distribution tables

Frequency tables

1.2 Tables and charts

When we're dealing with a large number of data values to see patterns or draw conclusions, we need to organise and present the data in a manageable form. When choosing a display for the data, we have to decide which one best shows what we want to communicate.

Frequency tables

A **frequency table** can be used to display both categorical and numerical data. It involves counting the number of times each data value occurs and the frequencies are often expressed as **percentages** of the total number of data values.

WB p. 4

WORKED EXAMPLE 3 Constructing a frequency table

The following is raw data collected by Patty's Hamburger Restaurant of the types of hamburgers home delivered to 20 people one evening, where A = Aussie burger, B = bacon burger, C = cheese burger, H = Hawaiian burger and V = vegie burger.

V, C, B, B, B, A, B, A, V, H, H, A, B, H, A, A, B, B, H, A

Set up a frequency table that includes both the number and percentage of each type of hamburger ordered.

Steps | **Working**

1 Set up a table with three columns and list the categories in the first column. Count the number in each category and record the frequency.

Check that the total frequency equals the total number of data values given in the question.

Burger type	Frequency	Percentage
Aussie burger	6	
bacon burger	7	
cheese burger	1	
Hawaiian burger	4	
vegie burger	2	
Total	20	

2 Calculate the percentage for each category using

$$\text{percentage} = \frac{\text{frequency}}{\text{total}} \times 100\%$$

Check that the total percentage is equal to 100% (or 99.9% or 100.1% if the percentages have been rounded).

Burger type	Frequency	Percentage
Aussie burger	6	30%
bacon burger	7	35%
cheese burger	1	5%
Hawaiian burger	4	20%
vegie burger	2	10%
Total	20	100%

Frequency tables

Frequency tables

- display both categorical and numerical data
- can include frequencies or percentages
- $\text{percentage} = \frac{\text{frequency}}{\text{total}} \times 100\%$

Grouped frequency tables

For continuous data, or discrete data with a large number of values, it's not practical to list each individual value in a frequency table. For example, if we have whole number data values ranging between 0 and 99, we would need to include 100 rows in a frequency table. If our data included decimals, we would need even more rows.

To reduce the number of rows, we use intervals that describe a range of data values such as 20–<30, meaning any number from 20 to 30 but not including 30. A **grouped frequency table** is a frequency table that uses intervals.

The following grouped frequency table shows the times (in minutes) taken by the 122 participants in a fun run.

Time (min)	Number
0–<30 (This means less than 30 minutes)	12
30–<60 (This means 30 minutes or more but less than 60 minutes)	52
60–<90 (This means 60 minutes or more but less than 90 minutes)	42
90–<120	16
Total	122

The **modal interval** is the interval that occurs most frequently.

Grouped frequency tables

- When numerical data is continuous, or discrete with a large number of values, numbers need to be grouped into intervals to form a grouped frequency table.
- The intervals must all be the same size.
- There should be a maximum of 15 intervals.

WORKED EXAMPLE 4 Constructing a grouped frequency table

WB p. 5

Construct a grouped frequency table for the following exam scores, using intervals of size 20, and find the modal interval.

45, 78, 80, 67, 43, 59, 32, 12, 100, 45, 58, 56, 69, 16

Steps

1 Write the intervals in the first column.
2 Count the data values that fall into each interval.
3 Enter the frequencies in the second column.
4 Make sure you place the border values (20, 40, 60 etc.) in the correct interval.
5 Add the frequency column and check that the total matches the number of data values in the list.
6 Find the interval that occurs most frequently.

Working

Exam scores

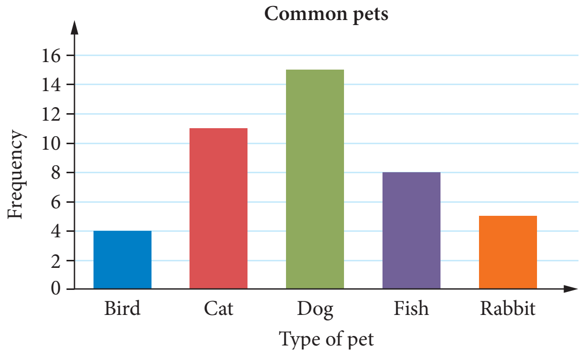
Score	Frequency
0–<20	2
20–<40	1
40–<60	6
60–<80	3
80–100	2
Total	14

The modal interval is 40–<60.

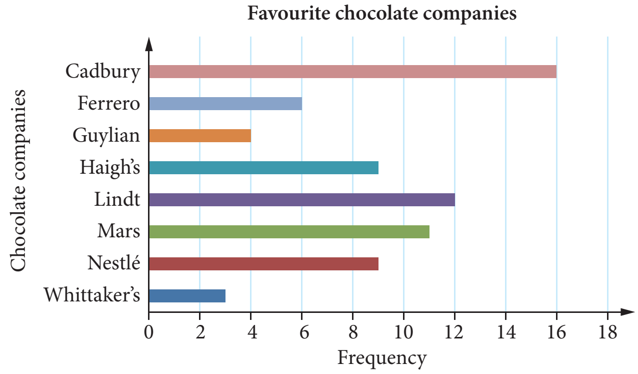
Bar charts

Bar charts help us to see patterns when dealing with categorical data. Categories can be represented on the horizontal or vertical axis, with their corresponding frequency on the other axis.

Bar chart with vertical bars



Bar chart with horizontal bars



Bar charts

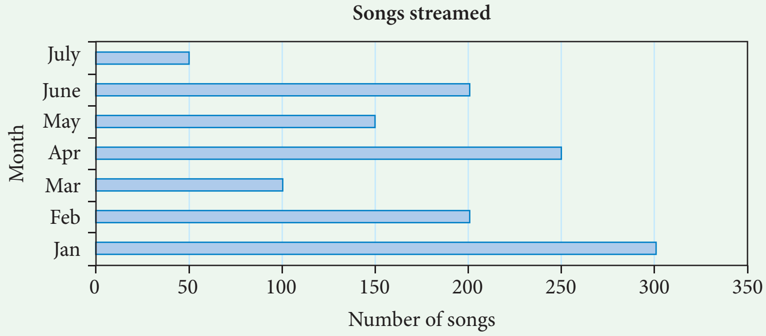
Bar charts

- display categorical data
- can have vertical or horizontal bars.

WB p. 6

WORKED EXAMPLE 5 Reading a bar chart

This bar chart shows the number of songs streamed over seven months by a student.



Use the graph to answer the following questions.

a In which month did the student stream the most songs?

b How many songs were streamed over the seven-month period?

c What percentage of songs did the student stream in March?

Steps	Working
a Find the longest bar.	The student streamed the most songs in January.
b Add the frequencies for each month.	$300 + 200 + 100 + 250 + 150 + 200 + 50 = 1250$ The student streamed 1250 songs over the seven-month period.
c Use percentage $= \frac{\text{frequency}}{\text{total}} \times 100\%$	$\frac{100}{1250} \times 100\% = 8\%$, therefore 8% of the songs were streamed in March.

9780170448192

EXERCISE 1.2 Tables and charts

ANSWERS p. 490

Recap

1 What is the correct data classification for a person's favourite ice-cream flavour?

A discrete **B** numerical **C** ordinal **D** continuous **E** nominal

2 The marks of 15 students for a maths test marked out of ten were recorded as follows:

9, 7, 7, 7, 5, 10, 6, 8, 3, 10, 9, 6, 8, 10, 4

The median and mode respectively are

A median = 7, mode = 7

B median = 10, mode = 7 and 10

C median = 7, mode = 7 and 10

D 7 and 8

E median = 7, mode = 10

Mastery

3 a WORKED EXAMPLE 3 The following is raw data collected by the Off the Hook Fish and Chippery of the type of packs that 25 people bought one day, where K = Kiddie pack, S = Snack pack, F = Family pack, H = Hawaiian pack, P = Potato cake pack and D = Dim sim pack.

S, K, S, S, H, K, F, S, F, S, H, H, P, F, H, S, F, H, S, S, S, F, F, H, F

Set up a frequency table that includes both the number and percentage of each type of hamburger ordered.

b The following is raw data collected by Kebabs R US of the types of kebabs that 16 people bought one day, where L = lamb, C = chicken, M = mix, F = falafel and V = vegan.

C, L, L, C, V, M, C, L, F, M, M, F, V, C, L, L

Set up a frequency table that includes both the number and percentage of each type of kebab ordered.

> **Exam hack**
>
> If you are not asked to round an answer, don't round it.

4 a WORKED EXAMPLE 4 Construct a grouped frequency table for the following test scores, using intervals of size 10, and find the modal interval.

28, 35, 41, 15, 8, 49, 30, 22, 39, 50, 47, 25, 38, 44, 12, 19

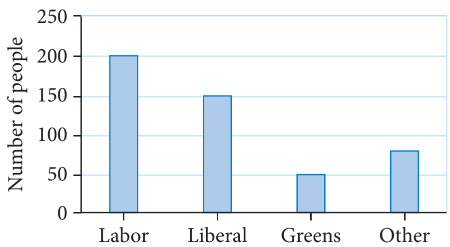
b Construct a grouped frequency table for the following test scores, using intervals of size 5, and find the modal interval.

22, 15, 4, 25, 10, 4, 19, 0, 22, 17, 13, 0, 2, 12, 14

c Construct a grouped frequency table for the following test scores, using intervals of size 10, and find the modal interval.

44, 32, 60, 37, 15, 58, 41, 31, 15, 9, 36, 0, 25, 30

5 WORKED EXAMPLE 5 This bar chart shows the results of a poll before an election.



a How many people favour the Greens?

b Which is the most popular party in the poll?

c How many more people favour the Labor Party than the Liberal Party?

d If there are 80 people who are in the 'Other' category, how many people were polled altogether?

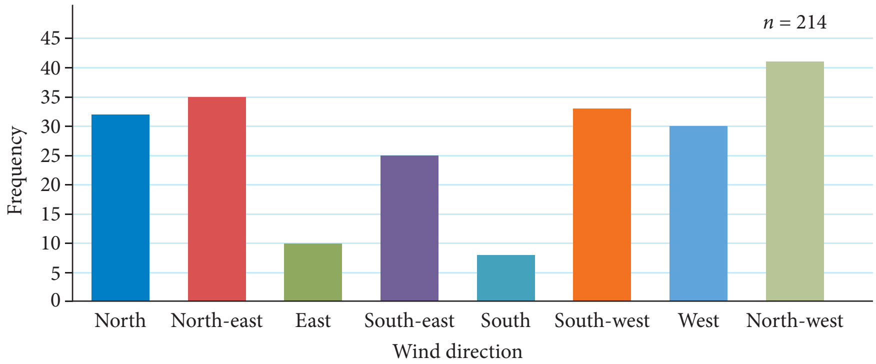
e What percentage of those polled favoured the Liberal Party?

Exam practice

80–100% 60–79% 0–59%

Use the following information to answer the next two questions.

The following bar chart shows the distribution of wind directions recorded at a weather station at 9.00 am on each of 214 days in 2011.



6 © VCAA 2012 1CQ1 99% According to the bar chart, the most frequently observed wind direction was

A south-east **B** south **C** south-west **D** west **E** north-west

7 © VCAA 2012 1CQ2 68% According to the bar chart, the percentage of the 214 days on which the wind direction was observed to be east or south-east is closest to

A 10% **B** 16% **C** 25% **D** 33% **E** 35%

Use the following information to answer the next two questions.

The heights of 82 mothers and their eldest daughters are classified as 'short', 'medium' or 'tall'. The results are displayed in the frequency table below.

		Mother		
		Short	Medium	Tall
Eldest daughter	Short	16	10	3
	Medium	8	14	11
	Tall	5	7	8

8 © VCAA 2013 1CQ3 79% The number of mothers whose height is classified as 'medium' is

A 7 **B** 10 **C** 14 **D** 31 **E** 33

9 © VCAA 2013 1CQ4 45% Of the mothers whose height is classified as 'tall', the percentage who have eldest daughters whose height is classified as 'short' is closest to

A 3% **B** 4% **C** 14% **D** 17% **E** 27%

Exam hack

Questions that include the words 'of the' always reduce the total you are looking at.

9780170448192

10 The missing information in the frequency table is

A a frequency of 3 and the interval 10–<20

B a frequency of 3 and the interval 15–<20

C a frequency of 5 and the interval 15–<20

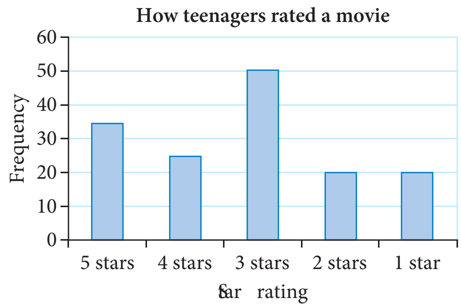
D a frequency of 10 and the interval 15–<20

E a frequency of 47 and the interval 15–<20.

Score	Frequency
0–<5	5
5–<10	
10–<15	15
	17
20–<25	10
Total	50

Use the following information to answer the next three questions.

A number of teenagers were surveyed on how they rated a particular movie. The results are shown in the following bar chart.



11 The most common star rating for the movie is

A 1 star **B** 2 stars **C** 3 stars **D** 4 stars **E** 5 stars

12 The number of teenagers who rated the movie is

A 75 **B** 100 **C** 150 **D** 175 **E** 200

13 The percentage of teenagers who rated the movie as 2 stars is closest to

A 9% **B** 13% **C** 25% **D** 33% **E** 50%

14 © VCAA 2008 2CQ1 95% (2 marks) In a small survey, twenty-five Year 8 girls were asked what they did (walked, sat, stood, ran) for most of the time during a typical school lunch time.

Their responses are recorded below.

sat	stood	sat	ran	sat
walked	walked	sat	walked	ran
sat	walked	walked	walked	ran
walked	ran	walked	ran	walked
ran	sat	ran	ran	walked

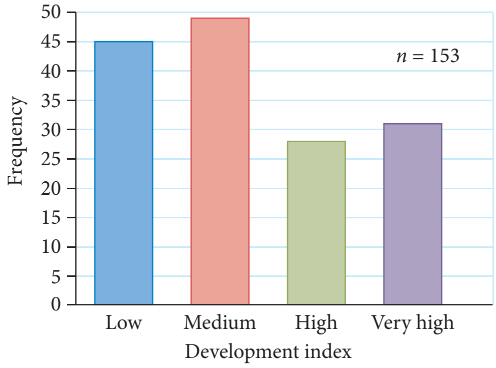
Use the data to

a copy and complete the frequency table 1 mark

Activity	Frequency
walked	
sat or stood	
ran	
Total	25

b determine the percentage of Year 8 girls who ran for most of the time during a typical school lunch time. 1 mark

15 © VCAA 2013 2CQ1 85% (2 marks) A development index is used as a measure of the standard of living in a country. The bar chart displays the development index for 153 countries in four categories: low, medium, high and very high.



a How many of these countries have a very high development index? 1 mark

b What percentage of the 153 countries has either a low or medium development index? Write your answer, correct to the nearest percentage. 1 mark

Video playlist Histograms

Worksheet Histograms

1.3 Histograms

Histograms and grouped frequency tables

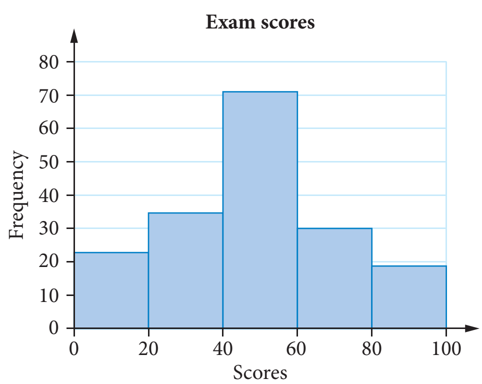
A **histogram** is a graphical way of displaying data from a grouped frequency table. It is effective when dealing with large amounts of data.

Grouped frequency table

Exam scores

Score	Frequency
0–<20	23
20–<40	35
40–<60	71
60–<80	30
80–<100	19
Total	178

Histogram



Following the pattern of the grouped frequency table, someone with a score of exactly 20 is recorded in the second interval, someone with a score of exactly 40 is recorded in the third interval, and so on.

Exam hack

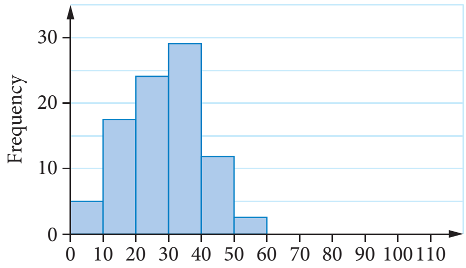
Although at first glance a histogram looks similar to a bar chart, there are a number of differences:

- Histograms display numerical data, whereas bar charts are best used to display categorical data.
- Histograms do not have spaces between the columns.
- Histograms are *always* vertical.

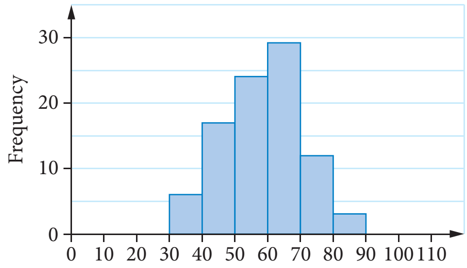
Centre and spread of histograms

Histograms make it easier to compare the centres or spreads of two distributions. We refer to the modal interval, or most frequently occurring interval, rather than the mode when looking at centres of histograms.

Here are two distributions with the same spread but different centres:

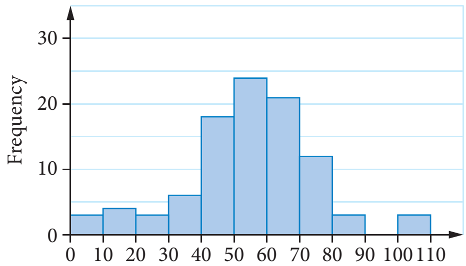


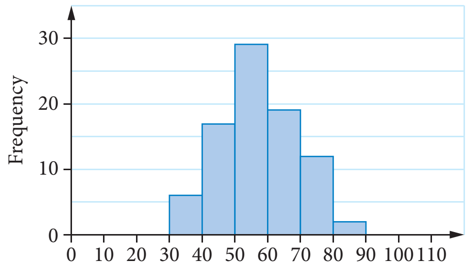
The modal interval is 30–<40.



The modal interval is 60–<70.

Here are two distributions with approximately the same centres but different spreads:





The modal interval for both of these is 50–<60.

Shapes of histograms

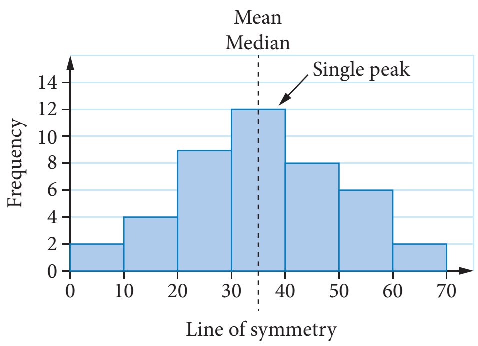
Worksheet Shapes of distributions

A histogram can also be used to describe the **shape of a distribution**. The shape of a distribution can be

- approximately symmetric
- **asymmetric**
  - **positively skewed**
  - **negatively skewed**.

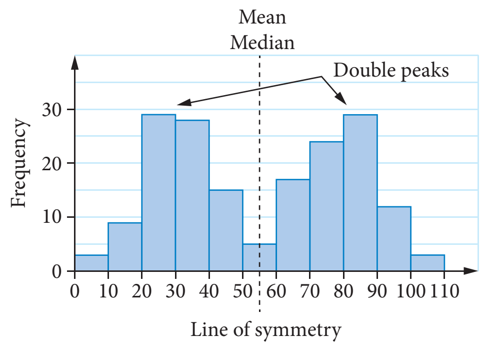
A histogram shows an approximately **symmetric distribution** when the left side closely mirrors the right side. For approximately symmetric distributions, the median is always near the line of symmetry but the modal interval can be somewhere else.

Single-peaked
Approximately symmetric distribution



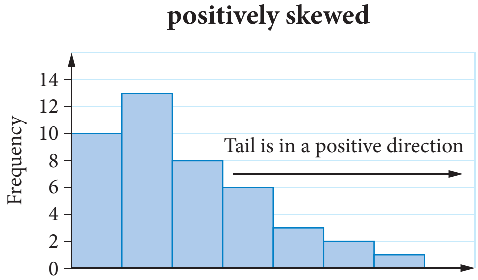
The modal interval is where a single peak occurs. Here it is 30–<40.

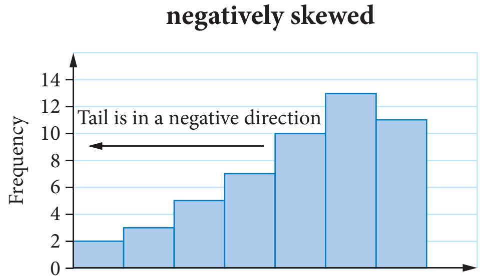
Double-peaked
Approximately symmetric distribution



When a histogram has two *equal* peaks, it is a bi-modal distribution and has two modal intervals. Here they are 20–<30 and 80–<90.

The following two histograms show asymmetric or **skewed distributions**.



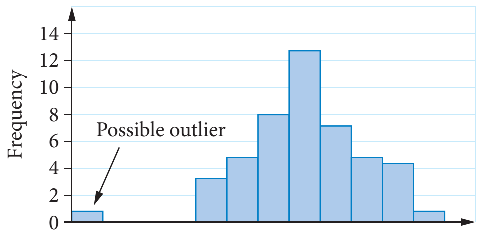


Exam hack

To help you remember which skew is positive and which is negative, identify where the 'tail' of the histogram is. Think of the positive and negative directions of a number line. If the tail is in the positive direction, the distribution is positively skewed. If the tail is in the negative direction, the distribution is negatively skewed.

Outliers

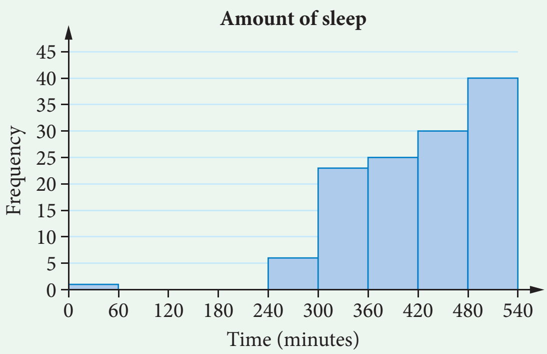
We can also comment on a distribution by referring to its **outliers**. An outlier is an extreme high or low value in the data. Outliers can indicate an error made in dealing with the data and can sometimes contaminate calculations and conclusions drawn from data sets. However, they can also occur without an error being involved. Histograms often make it easier to identify possible outliers.



WB
p. 7

WORKED EXAMPLE 6 Reading and describing histograms

The following table shows the amount of sleep (in minutes) a group of 125 students had the night before an exam.



Steps	Working
a How many intervals are there?	
Count the number of intervals.	There are nine intervals.
b How many students slept for more than 7 hours?	
1 Convert the time to hours.	7 hours = 420 minutes
2 Read from the histogram.	30 + 40 = 70
	70 students slept for more than 7 hours.
c What percentage of students slept for less than 5 hours? Give your answer rounded to one decimal place.	
1 Convert the time to hours.	5 hours = 300 minutes
2 Read from the histogram.	1 + 6 = 7
	Seven students slept for less than 5 hours.
3 Convert to a percentage using $\text{percentage} = \frac{\text{frequency}}{\text{total}} \times 100\%$, rounding to one decimal place.	$\text{percentage} = \frac{7}{125} \times 100\%$ $= 0.056 \times 100\%$ $= 5.6\%$
d Is the histogram approximately symmetric, positively skewed or negatively skewed? Does it have any possible outliers?	
The histogram has a negative tail.	The histogram is negatively skewed with a possible outlier.
e What is the modal interval?	
Which interval has the highest frequency?	The modal interval is 480–<540 minutes.
f Use the histogram to copy and complete the following grouped frequency table.	

Amount of sleep (min)	Frequency
0–<60	
60–<120	
120–<180	
Total	

Amount of sleep (min)	Frequency
0–<60	1
60–<120	0
120–<180	0
180–<240	0
240–<300	6
300–<360	23
360–<420	25
420–<480	30
480–<540	40
Total	125

Exam hack

A ruler often helps when you are reading values from histograms and other statistical charts.

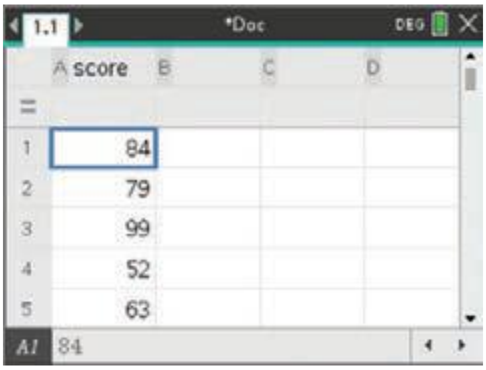
USING CAS 1 Constructing histograms for ungrouped data

Scores on an end-of-year mathematics examination, out of 100, achieved by 37 Year 11 students from a particular school were recorded as follows:

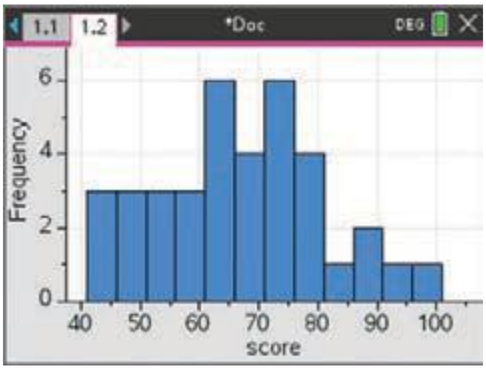
84, 79, 99, 52, 63, 70, 65, 78, 47, 72, 73, 60, 52, 76, 77, 65, 71, 61, 53,
62, 41, 88, 57, 71, 43, 89, 74, 50, 49, 61, 72, 70, 68, 95, 58, 67, 43

Use intervals of 40–<50, 50–<60 and so on to construct a histogram for this data.

TI-Nspire



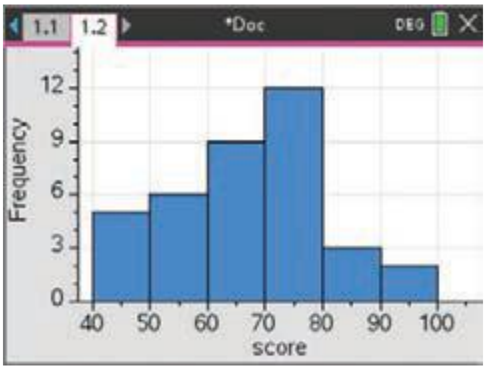
1. Start a new document and add a **Lists & Spreadsheet** page.
2. Label column **A** as **score**.
3. Enter the ungrouped scores as shown above.



4. Insert a **Data & Statistics** page.
5. For the horizontal axis, select score.
6. Press **menu > Plot Type > Histogram**.
7. A histogram of the data will be displayed.

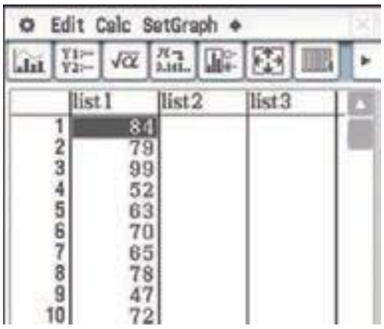
8. Press **menu > Plot Properties > Histogram Properties > Bin Settings > Equal Bin Width.**
9. Enter the following settings:

   Width: **10**
   Alignment: **30**
10. Select **OK**.



11. A histogram of the grouped data in intervals of 10 will be displayed.
12. Adjust the **Window/Zoom** settings or grab the vertical axis and drag it down to view the full histogram.

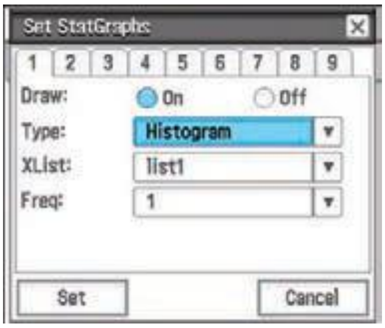
ClassPad



1 Tap **Menu** and open the **Statistics** application.

2 Clear all lists.

3 Enter the ungrouped scores into **list1** as shown above.

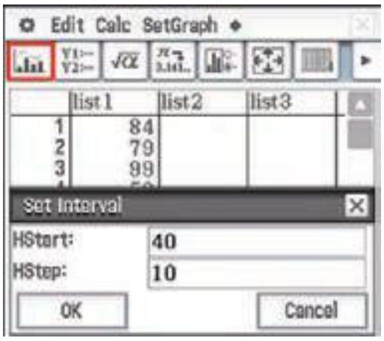


4 Tap **SetGraph > Setting**.

5 In the dialogue box, select the following:

Type: **Histogram**
XList: **list1**
Freq: **1**

6 Tap **Set**.

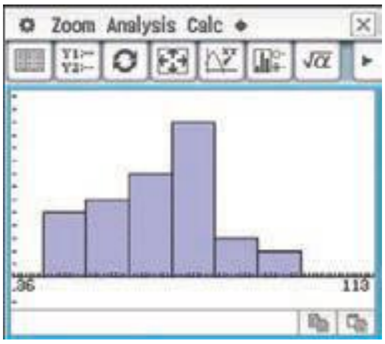


7 Tap **Graph**.

8 In the dialogue box, set the following:

HStart: **40**
HStep: **10**

9 Tap **OK**.



10 A histogram of the data will appear in the lower window.

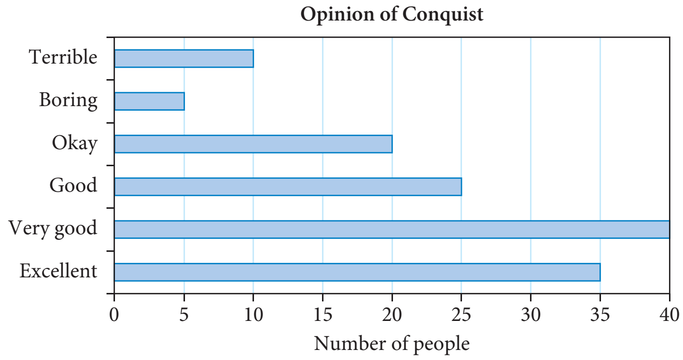
EXERCISE 1.3 Histograms

ANSWERS p. 491

Recap

Use the following information to answer the next two questions.

The bar chart shows the results of a survey of opinions on a new online game called *Conquist*.



1 The type of data collected is best described as

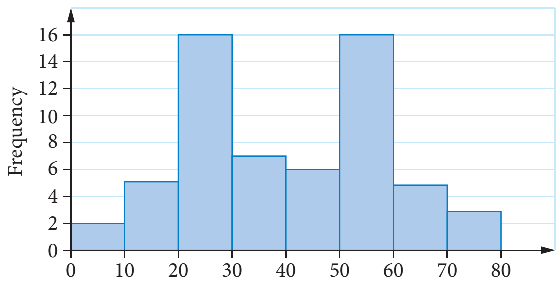
A ordinal numerical.
B nominal categorical.
C discrete numerical.
D ordinal categorical.
E continuous numerical.

2 The percentage of people that thought it was either very good or excellent, rounded to one decimal place, is

A 13.5%
B 25.9%
C 29.6%
D 55.6%
E 75.0%

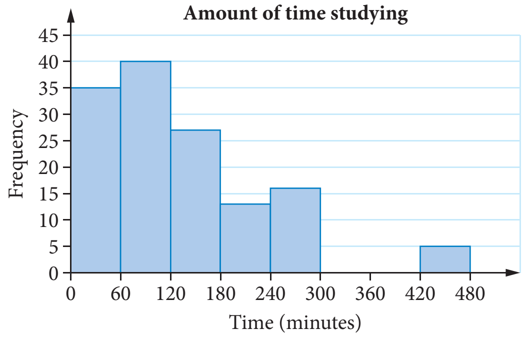
Mastery

3 For the histogram shown, state whether each of the statements are true or false.



a The histogram is approximately symmetric.
b The histogram has a possible outlier.
c The histogram is double-peaked.
d The histogram is negatively skewed.
e The modal interval is 70–<80.
f The centre is around 40.
g The histogram is bi-modal.

4 WORKED EXAMPLE 6 The following table shows the amount of time (in minutes) a group of students spent studying the night before an exam.



a How many intervals are there?

b How many students studied for less than 2 hours?

c What percentage of students studied for more than 4 hours? Give your answer rounded to one decimal place.

d Is the histogram approximately symmetric, positively skewed or a negatively skewed? Does it have any possible outliers?

e What is the modal interval?

f Use the histogram to copy and complete the following grouped frequency table.

Amount of time studying (min)	Frequency
0–<60	
60–<120	
120–<180	
Total	

5 **a** Draw a histogram by hand from the grouped frequency table of test results shown.

b Use the histogram from part **a** to determine whether it is approximately symmetric, positively skewed or negatively skewed.

Marks	Frequency
0–<10	0
10–<20	5
20–<30	16
30–<40	25
40–<50	14
50–<60	6
Total	66

6 Using CAS 1 Scores on an end-of-year science examination, out of 100, achieved by 25 Year 11 students from a particular school were recorded as follows:

52, 45, 40, 67, 60, 73, 88, 61, 64, 73, 71, 41, 57, 48, 76, 97, 79, 59, 83, 73, 66, 71, 76, 32, 67

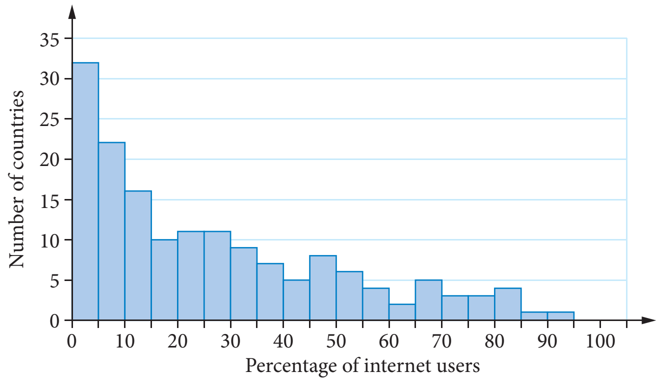
Use intervals of 30–<40, 40–<50 and so on to construct a histogram for this data.

Exam practice

80–100% 60–79% 0–59%

Use the following information to answer the next two questions.

The histogram below displays the distribution of the percentage of internet users in 160 countries in 2007.



7 © VCAA 2011 1CQ1 MODIFIED **87%** The shape of the histogram is best described as

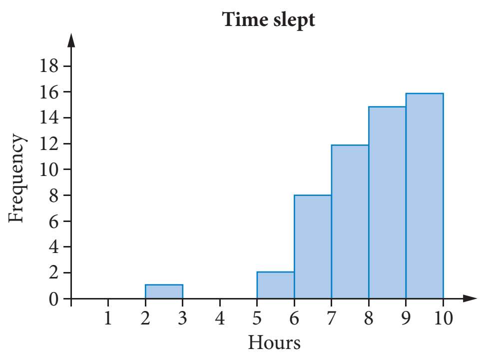
A approximately symmetric. **B** double-peaked. **C** positively skewed.

D negatively skewed. **E** bi-modal.

8 © VCAA 2011 1CQ2 **86%** The number of countries in which less than 10% of people are internet users is closest to

A 10 **B** 16 **C** 22 **D** 32 **E** 54

Use the following information to answer the next two questions.



9 What is the most appropriate description for the histogram showing how long students slept the previous night?

A approximately symmetric with a possible outlier

B symmetric with no possible outlier

C positively skewed with a possible outlier

D negatively skewed with a possible outlier

E double-peaked with a possible outlier

10 What is the modal interval?

A students who had 9 hours sleep

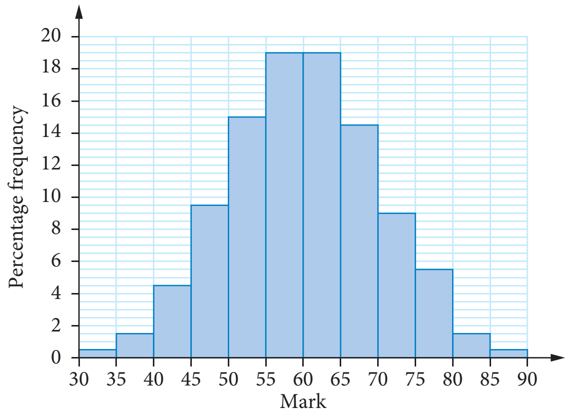
B students who had 10 hours sleep

C students who had more than 9 but less than 10 hours sleep

D students who had more than 2 but less than 3 hours sleep

E students who had 16 hours sleep

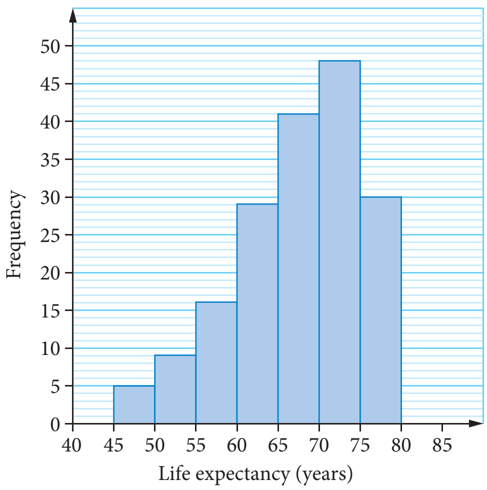
11 © VCAA 2002 1CQ5 MODIFIED 1526 students sat for an examination. The histogram shows the distribution of marks.



The percentage of students with marks 80 or higher is closest to

A 1% **B** 2% **C** 3% **D** 4% **E** 5%

12 © VCAA 2015 2CQ1bc (2 marks) The histogram shows the distribution of life expectancy of people for 183 countries.



a 77% In how many of these countries is life expectancy less than 55 years? 1 mark

b 68% In what percentage of these 183 countries is life expectancy between 75 and 80 years? Write your answer correct to one decimal place. 1 mark

13 (6 marks) The age, in years, of employees at Burger Heaven were recorded in the following grouped frequency table.

Age (years)	Frequency
15–<20	16
20–<25	20
25–<30	4
30–<35	5
35–<40	1
40–<45	2
45–<50	2

a How many people work at Burger Heaven? 1 mark

b Construct a histogram to display this data by hand. 2 marks

c How many employees are aged 40 or over? 1 mark

d What percentage of employees are aged 40 or over? 1 mark

e Describe what the histogram shows about Burger Heaven's employment practices. 1 mark

Video playlist
Boxplots

Worksheets
Five-number summaries 1
Five-number summaries 2
Boxplots
Boxplots 1
Boxplots 2

1.4 Boxplots

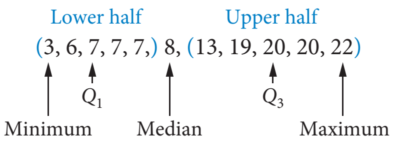
The five-number summary

The median is the value that divides an ordered data set in half. The **quartiles** are three values that divide an ordered data set in quarters.

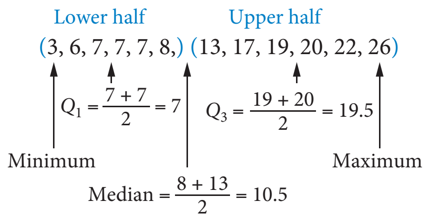
- Q_1 or the **lower quartile** is the median of the lower half of the data.
- Q_2 is the **median** of the whole data set.
- Q_3 or the **upper quartile** is the median of the upper half of the data.

The **five-number summary** provides a good overview of a distribution. It is created using the maximum and minimum data values and the three quartiles.

Example with an odd number of data values:



Example with an even number of data values:

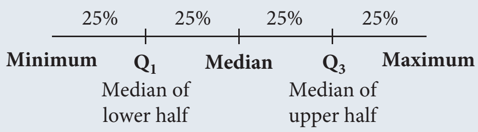


The five-number summary

The five-number summary divides the data into segments of 25%:

- 25% of the data is less than Q_1.
- 50% of the data is less than the median (Q_2).
- 75% of the data is less than Q_3.

These percentages are not exact for an odd-numbered data set because the median is one of the data values and we can't split a data value in half.



WB p. 9

WORKED EXAMPLE 7 Finding the five-number summary by hand

For the following data

23, 25, 35, 35, 22, 49, 7, 26, 24, 31, 22, 30

a find the five-number summary by hand

b use a diagram to show that

i 25% of the data is less than the lower quartile (Q_1)

ii 50% of the data is less than the median (Q_2)

iii 75% of the data is less than the upper quartile (Q_3).

Steps	Working
a **1** Order the data from smallest to largest.	7, 22, 22, 23, 24, 25, 26, 30, 31, 35, 35, 49
2 Find the minimum and maximum value.	min = 7, max = 49
3 Find the median. There is an even number of data values, so find the average of the two middle points.	The median is between the 6th and 7th data values in the ordered list. $Q_2 = \text{median} = \frac{25 + 26}{2} = 25.5$

4 Find Q_1, the median of the lower half of the data.

The lower half of the data is 7, 22, 22, 23, 24, 25.

$$Q_1 = \text{lower quartile} = \frac{22 + 23}{2} = 22.5$$

5 Find Q_3, the median of the upper half of the data.

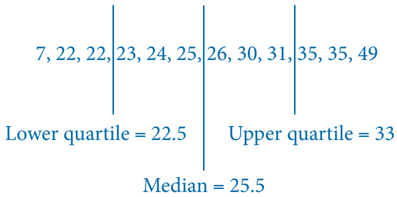
The upper half of the data is 26, 30, 31, 35, 35, 49

$$Q_3 = \text{upper quartile} = \frac{31 + 35}{2} = 33$$

6 List the five-number summary.

min = 7, Q_1 = 22.5, median = 25.5, Q_3 = 33, max = 49

b 1 Draw a diagram showing the three quartiles.



2 Use the diagram to calculate the percentage of data less than each quartile.

i 3 out of a total of 12 data values are less than the lower quartile. $\frac{3}{12} = 25\%$

ii 6 out of a total of 12 data values are less than the median. $\frac{6}{12} = 50\%$

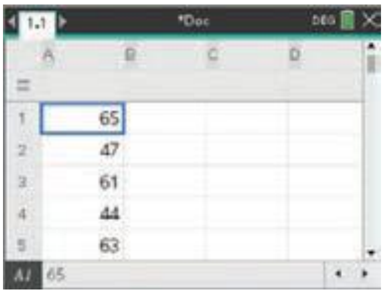
iii 9 out of a total of 12 data values are less than the upper quartile. $\frac{9}{12} = 75\%$

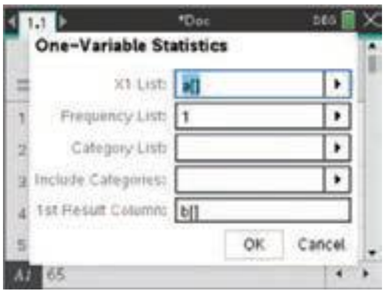
USING CAS 2 Finding the five-number summary

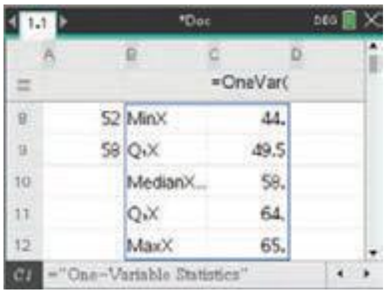
Use CAS to calculate the five-number summary for the following data:

65, 47, 61, 44, 63, 56, 65, 52, 58

TI-Nspire







1 Start a new document and add a **Lists & Spreadsheet** page.

2 Enter the data into column **A** as shown.

3 Press **menu > Statistics > Stat Calculations > One-Variable Statistics**.

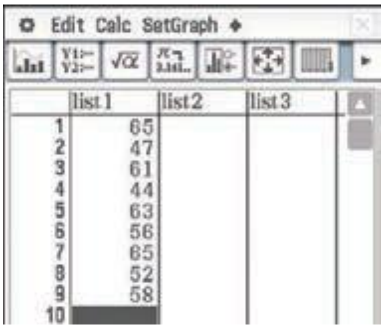
4 On the next screen, keep the number of lists default setting of **1** and select **OK**.

5 Leave the **X1 List** default setting of **a[ ]** then select **OK**.

6 The labels will appear in column **B** and the corresponding one-variable statistics will appear in column **C**.

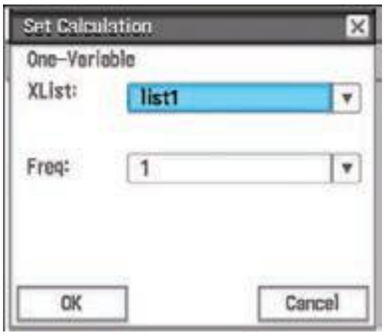
7 Scroll down to view the five-number summary values.

ClassPad



1 Tap **Menu** and open the **Statistics** application.

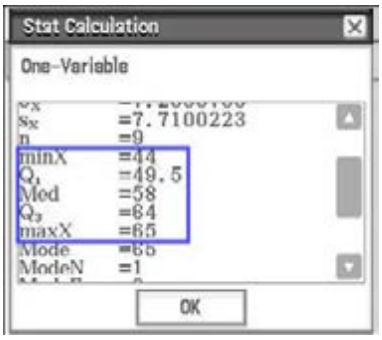
2 Clear all lists and enter the data as shown.



3 Tap **Calc > One-Variable**.

4 Leave the **XList** default setting of **list1**.

5 Tap **OK**.



6 The one-variable statistics will be displayed.

7 Scroll down to view the five-number summary values.

Exam hack

If a data set is small and already ordered, it can sometimes be quicker to find the five-number summary by hand than by using CAS.

Worksheet Interquartile range

Puzzle Statistical measures puzzle

IQR, outliers and fences

The **interquartile range (IQR)** is the measure of the spread of the middle 50% of the data values.

$\text{IQR} = Q_3 - Q_1$

The IQR is usually a better measure of spread than the range because, by looking at only the middle 50% of data, we avoid taking outliers into account. The IQR is also used in a calculation to identify possible outliers, which allows us to do more than simply say something 'looks like an outlier'.

Interquartile range and fences

$\text{IQR} = Q_3 - Q_1$

A data value is a possible outlier if it is

less than the **lower fence**: $Q_1 - 1.5 \times \text{IQR}$

or

greater than the **upper fence**: $Q_3 + 1.5 \times \text{IQR}$

WB p. 10

WORKED EXAMPLE 8 Finding outliers

For the ordered data set

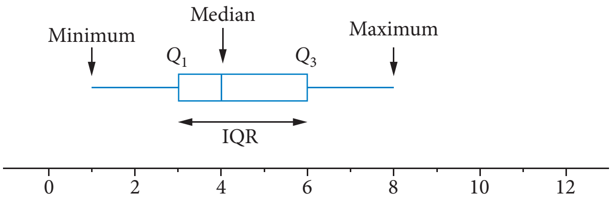
3, **7**, 20, 22, 22, 22, 25, 25, 28, 31, 34, 34, **49**

do a calculation to show whether the **blue** values are possible outliers.

Steps	Working
1 Find Q_1 and Q_3 by using CAS or by hand.	$Q_1 = 21$ and $Q_3 = 32.5$
2 Calculate the IQR.	$\text{IQR} = 32.5 - 21 = 11.5$
3 Calculate the lower and upper fences.	Lower fence: $Q_1 - 1.5 \times \text{IQR} = 21 - 1.5 \times 11.5 = 3.75$ Upper fence: $Q_3 + 1.5 \times \text{IQR} = 32.5 + 1.5 \times 11.5 = 49.75$
4 Check each of the **blue** values to see if they are less than the lower fence or greater than the upper fence.	**3** is less than 3.75 so it *is* a possible outlier. **7** isn't less than 3.75, so it's *not* an outlier. **49** isn't greater than 49.75, so it's *not* an outlier.

Boxplots

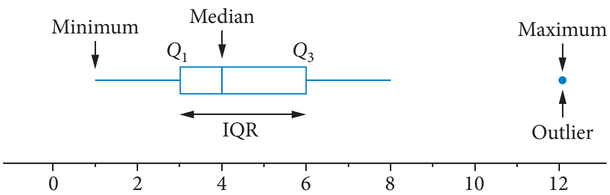
Boxplots, also known as **box-and-whisker plots**, display numerical data based on the five-number summary, IQR and outliers. If there are no outliers, the **whiskers** show the minimum and maximum values.



Exam hack

The lengths of the boxes and whiskers depend on the spread of the data values in each quartile.

Outliers are shown as dots. If there are outliers, the lowest outlier is the minimum value and the highest outlier is the maximum value.

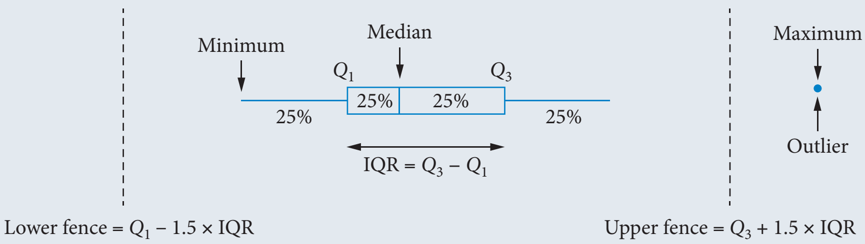


Exam hack

You need to include outliers when finding the minimum and maximum values.

Boxplots

Boxplots provide the following information:



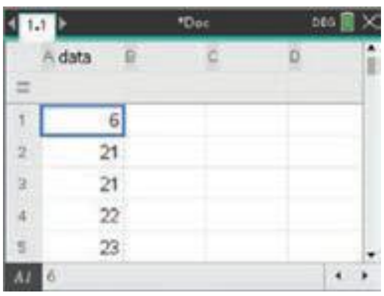
Boxplots can also be displayed vertically.

USING CAS 3 Constructing boxplots

Construct a boxplot for the data set

6, 21, 21, 22, 23, 24, 25, 29, 30, 34, 34, 48

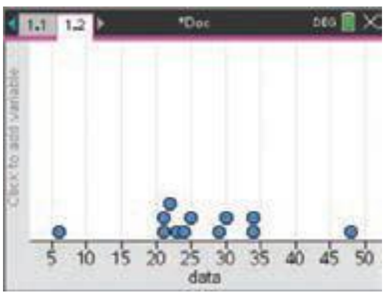
TI-Nspire



1 Start a new document and add a **Lists & Spreadsheet** page.

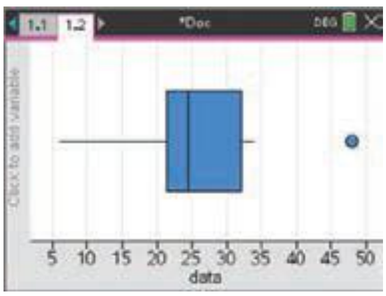
2 Label column **A** with the heading data.

3 Enter the data as shown above.



4 Insert a **Data & Statistics** page.

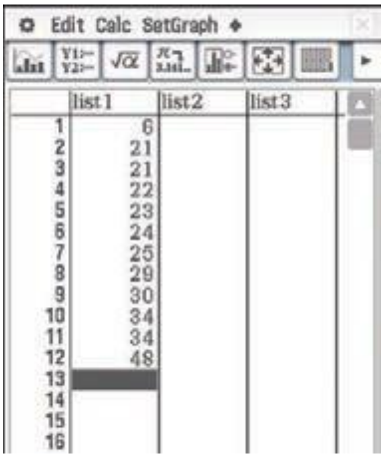
5 For the horizontal axis, select **data**. The data will be displayed as a dot plot.

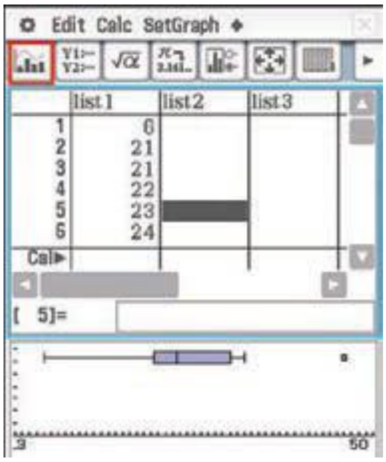


6 Press **menu > Plot Type > Box Plot**. The data will be displayed as a boxplot.

7 Move the cursor over the boxplot to display the key values.

ClassPad





1 Tap **Menu** and open the **Statistics** application.

2 Clear all lists.

3 Enter the data into **list1** as shown above.

4 Tap **SetGraph > Setting**.

5 In the dialogue box, change the **Type**: field to **MedBox**.

6 Tap the box to select **Show Outliers**.

7 Tap **Set**.

8 Tap on the **Graph** tool to display the data as a boxplot in the lower window.

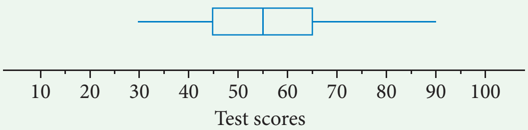
9 With the graph window highlighted, tap **Analysis > Trace** to display the key values.

WB

p. 11

WORKED EXAMPLE 9 Reading boxplots

The boxplot shows the distribution of 48 student test scores marked out of 100.



Find the

a five-number summary

b percentage of students who scored more than 65

c percentage of students who scored less than 55

d percentage of students who scored between 30 and 65

e number of students who scored less than 45

f scores at the lower end that would be considered outliers

g scores at the upper end that would be considered outliers.

Steps	Working
a Read directly from the boxplot.	min = 30, Q_1 = 45, median = 55, Q_3 = 65, max = 90
Use the fact that quartiles divide data into four equal groups, so 25% of the data is in each group. 	**b** Q_3 = 65, so 25% of students scored more than 65. **c** median = 55, so 50% of students scored less than 55. **d** Q_3 = 65, so 75% of students scored between 30 and 65.

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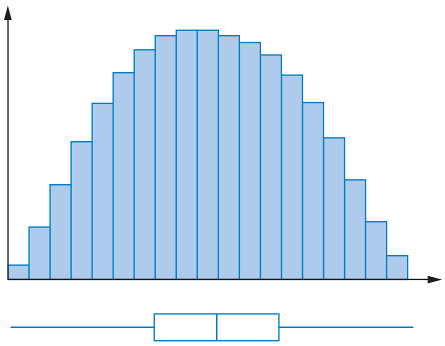
e	Find the percentage first and then multiply by the total number.	$Q_1 = 45$, so 25% of students scored less than 45. Total number of students = 48 Number of students who scored less than 45 = 48 × 25% = 12
f	Use the IQR to calculate the lower fence. lower fence = $Q_1 - 1.5 \times \text{IQR}$	$\text{IQR} = Q_3 - Q_1 = 65 - 45 = 20$ lower fence = 45 − 1.5 × 20 = 45 − 30 = 15 Scores less than 15 would be considered outliers.
g	Use the IQR to calculate the upper fence. upper fence = $Q_3 + 1.5 \times \text{IQR}$	upper fence = 65 + 1.5 × 20 = 65 + 30 = 95 Scores greater than 95 would be considered outliers.

Comparing boxplots and histograms

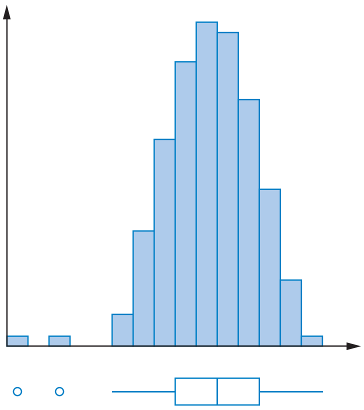
If we know what the histogram of a distribution looks like, we can often have some idea of what the boxplot will look like.

Approximately symmetric distributions



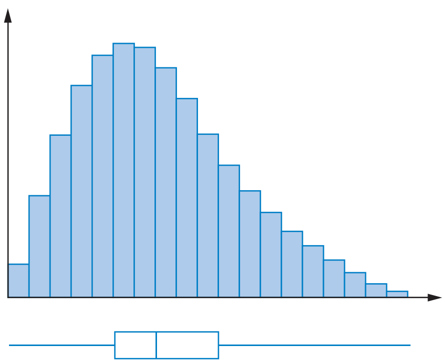
The median is approximately in the middle of the box and whiskers are about the same length.

Distributions with outliers



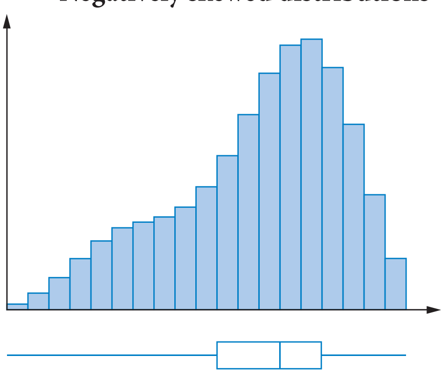
The boxplot matches the histogram with the outliers shown by dots.

Positively skewed distributions



The box and whisker in the positive direction are longer than the box and whisker in the negative direction.

Negatively skewed distributions



The box and whisker in the negative direction are longer than the box and whisker in the positive direction.

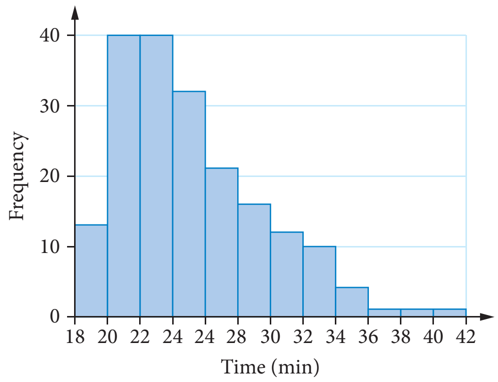
EXERCISE 1.4 Boxplots

ANSWERS p. 491

Recap

Use the following information to answer the next two questions.

The times of runners in a long-distance race were recorded in minutes in the histogram shown.



1 Which of the following best describes the histogram?

A negatively skewed

B positively skewed with three outliers

C approximately symmetric

D bi-modal

E single-peaked

2 Which of the following is **not** true?

A The two modal intervals are 20–<22 and 22–<24.

B The histogram is double-peaked.

C There are no runners with times under 18 minutes.

D The number of runners with times greater than 30 minutes is over 20.

E Under 100 runners were in the race.

Mastery

3 WORKED EXAMPLE 7 Using CAS 2 For the following data

39, 49, 76, 61, 42, 65, 62, 35,78, 80, 59, 54

a find the five-number summary by hand and verify your answers by using CAS

b use a diagram to show that

i 25% of the data is less than the lower quartile (Q_1)

ii 50% of the data is less than the median (Q_2)

iii 75% of the data is less than the upper quartile (Q_3).

4 WORKED EXAMPLE 8 Using CAS 3 For each of the following data sets

i do a calculation to show whether the **blue** values are possible outliers

ii use CAS to construct a boxplot.

a **12**, **23**, 33, 36, 38, 40, 44, 45, 45, 48, 49, 49, 50, 52, 55, **71**

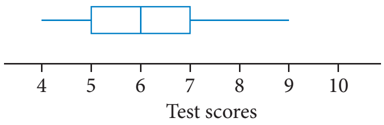
b **52**, 73, 76, 81, 81, 90, 90, 92, 95, 96, 96, **105**, **110**

c **9**, 16, 19, 20, 23, 23, 24, 24, 25, 25, 26, 26, 27, 27, 27, **33**, **35**

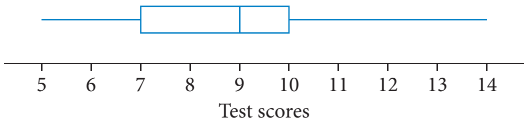
5 WORKED EXAMPLE 9 For each of the boxplots below showing the distribution of 60 student test scores marked out of 15, find the following:

- **i** five-number summary
- **ii** percentage of students who scored more than 7
- **iii** percentage of students who scored less than 10
- **iv** percentage of students who scored between 7 and 9
- **v** number of students who scored less than 5
- **vi** scores at the lower end that would be considered outliers
- **vii** scores at the upper end that would be considered outliers.

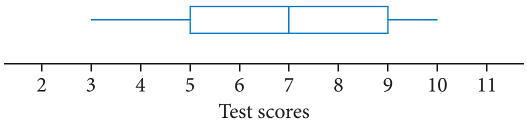
a



b



c



6 For each of the following boxplots, state whether the distribution is approximately symmetric, positively skewed or negatively skewed, and whether it has outliers, giving a reason for your answer.

a 

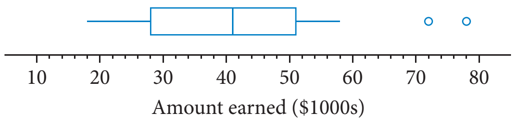
b 

c 

Exam practice

80–100% 60–79% 0–59%

7 The boxplot shows the annual amounts earned by workers in a fast-food franchise.

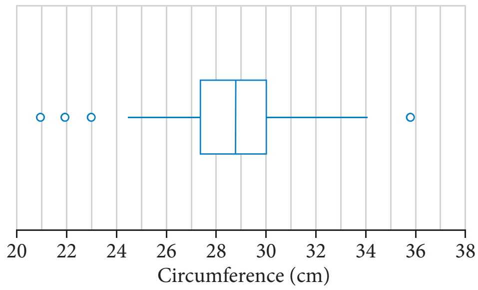


What percentage of workers earn less than $28 000 per year?

A 0% **B** 20% **C** 25% **D** 50% **E** 75%

Use the following information to answer the next two questions.

The boxplot shows the distribution of the forearm *circumference*, in centimetres, of 252 people.



8 © VCAA 2017 1CQ1 93% The percentage of these 252 people with a forearm *circumference* of less than 30 cm is closest to

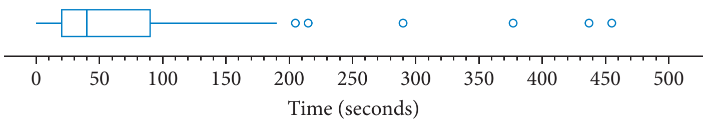
A 15% **B** 25% **C** 50% **D** 75% **E** 100%

9 © VCAA 2017 1CQ2 58% The five-number summary for the forearm *circumference* of these 252 people is closest to

A 21, 27.4, 28.7, 30, 34
B 21, 27.4, 28.7, 30, 35.9
C 24.5, 27.4, 28.7, 30, 34
D 24.5, 27.4, 28.7, 30, 35.9
E 24.5, 27.4, 28.7, 30, 36

Use the following information to answer the next three questions.

The boxplot shows the distribution of the time, in seconds, that 79 customers spent moving along a particular aisle in a large supermarket.



Data source: www.stars.ac.uk

10 © VCAA 2008 1CQ2 83% The shape of the distribution is best described as

A symmetric.
B negatively skewed.
C negatively skewed with outliers.
D positively skewed.
E positively skewed with outliers.

11 © VCAA 2008 1CQ1 82% The longest time, in seconds, spent moving along this aisle is closest to

A 40 **B** 60 **C** 190 **D** 450 **E** 500

12 © VCAA 2008 1CQ3 42% The number of customers who spent more than 90 seconds moving along this aisle is closest to

A 7 **B** 20 **C** 26 **D** 75 **E** 79

Exam hack

With boxplot questions, always check whether the value you are being asked about is one of the quartiles.

13 © VCAA 2002 1CQ7 76% The following data was recorded from measurements made on 12 men.

For these men, the median age (M) and the interquartile range (IQR), in years, are respectively

A $M = 37$ and IQR = 10

B $M = 37.4$ and IQR = 7.2

C $M = 37.4$ and IQR = 6.9

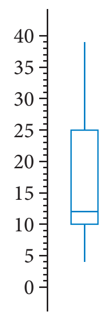
D $M = 38$ and IQR = 10

E $M = 38$ and IQR = 25

Age (years)	Mass (kg)	Waist (cm)
26	84	84
29	72	74
32	67	89
32	59	75
34	97	106
37	112	114
39	67	80
40	91	101
41	98	101
43	89	94
45	117	126
51	62	82

1.4

14 (5 marks) The boxplot shows the number of cigarettes smoked per day by a sample of 120 smokers who were trying to quit.



a What is the median number of cigarettes smoked per day? 1 mark

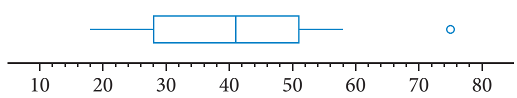
b What is the interquartile range? 1 mark

c Is the distribution approximately symmetric, positively skewed or negatively skewed? 1 mark

d What percentage of people smoked less than 12 cigarettes per day? 1 mark

e Estimate how many people smoked more than 10 cigarettes per day. 1 mark

15 (5 marks) For the following boxplot



a find Q_3 1 mark

b calculate the IQR 1 mark

c state the value of the outlier shown 1 mark

d explain why the outlier is incorrect. 2 marks

Video playlist
Dot plots and stem plots

1.5 Dot plots and stem plots

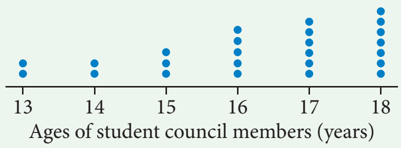
Dot plots

Dot plots can be used for both categorical and numerical data. They are useful as long as there are not too many data values involved and the values are not too spread out.

WB p. 12

WORKED EXAMPLE 10 Using dot plots

The dot plot shows the ages of student council members at a school.

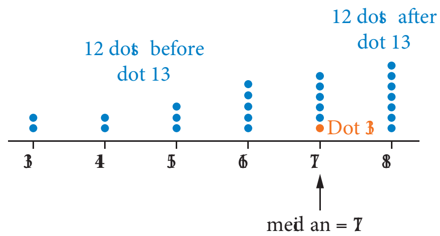
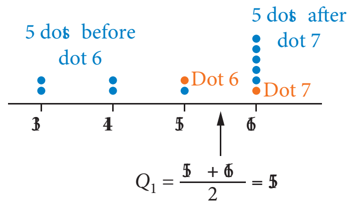


a Find the

i mode **ii** range **iii** median

iv lower quartile (Q_1) **v** upper quartile (Q_3) **vi** interquartile range (IQR).

b What could best describe the shape of the distribution: approximately symmetric, positively skewed or negatively skewed?

Steps	Working
a **i** Find the most common value.	mode = 18 years
ii range = largest value − smallest value	range = 18 − 13 = 5 years
iii **1** Count the number of dots n, note whether it's odd or even, and find the position of the median.	$n = 25$; odd $\frac{n+1}{2} = \frac{25+1}{2} = \frac{26}{2} = 13$ The median is the 13th ordered data value.
2 If n is odd, find data value of the middle dot. If n is even, find the average of the data values for the two middle dots. Count each column of dots from the bottom up to reach the median.	
iv **1** To find the lower quartile Q_1, find the median of the lower half. Count the number of dots n, note whether it's odd or even, and find the position of Q_1.	median = 17 years $n = 12$; even $\frac{n+1}{2} = \frac{12+1}{2} = \frac{13}{2} = 6.5$ Q_1 is between the 6th and 7th ordered data values in the lower half.
2 If n is odd, find data value of the middle dot. If n is even, find the average of the data values for the two middle dots. Count each column of dots from the bottom up to reach the lower quartile.	 $Q_1 = 15.5$ years

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v **1** To find the upper quartile Q_3, find the median of the upper half.

Count the number of dots n, note whether it's odd or even, and find the position of Q_3.

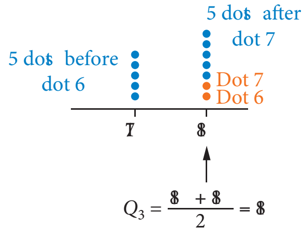
$n = 12$; even

$\frac{n+1}{2} = \frac{12+1}{2} = \frac{13}{2} = 6.5$

Q_3 is between the 6th and 7th ordered data values in the upper half.

2 If n is odd, find data value of the middle dot.

If n is even, find the average of the data values for the two middle dots. Count each column of dots from the bottom up to reach the upper quartile.

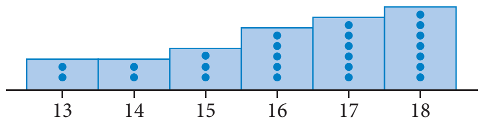


vi Use Q_3 and Q_1 to calculate the interquartile range.

$Q_3 = 18$ years

$IQR = Q_3 - Q_1 = 18 - 15.5 = 2.5$ years

b Picture the dot plot as a histogram.



The distribution is negatively skewed.

Stem plots

Stem plots, also known as **stem-and-leaf plots**, are an alternative to histograms where actual data values appear. The data values are ordered from smallest to largest, where the stem is made up of the leading digits and the leaf is the last digit. Stem plots are best used up to a maximum of 50 data values. When there are a small number of stems, we split the stem to see the distribution more clearly.

Worksheet Stem-and-leaf plots

The following stem plots show the ordered data values:

80, 81, 84, 85, 89, 89, 91, 92, 92, 96, 105, 107, 108, 109, 109, 112, 114, 118

Stem plot

Stem	Leaf	
8	0 1 4 5 9 9	
9	1 2 2 6	
10	5 7 8 9 9	← Data values from 100 to 109
11	2 4 8	

Key: 9 | 1 means 91

Split stem plot

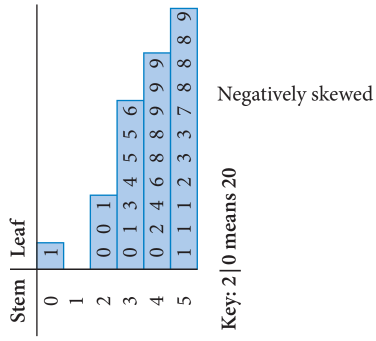
Stem	Leaf	
8	0 1 4	
8	5 9 9	
9	1 2 2	
9	6	
10		← Data values from 100 to 104
10	5 7 8 9 9	← Data values from 105 to 109
11	2 4	
11	8	

Key: 9 | 1 means 91

To decide whether a stem plot is approximately symmetric, positively skewed or negatively skewed, rotate it so that the stem forms the horizontal axis, and picture it as a histogram. For example, we can see after rotating the following stem plot that the distribution is negatively skewed.

Stem	Leaf
0	1
1	
2	0 0 1
3	0 1 3 4 5 5 6
4	0 2 4 6 8 8 9 9 9
5	1 1 1 2 3 3 7 8 8 8 9

Key: 2|0 means 20



Stem plots

Stem plots

- can be used with both continuous and discrete data
- are best used with up to a maximum of 50 data values
- always require a key.

WB p. 14

WORKED EXAMPLE 11 Using stem plots

The stem plot shows the test scores out of 70 for a class of 37 students.

a Find the

i mode **ii** range

iii median **iv** lower quartile (Q_1)

v upper quartile (Q_3) **vi** interquartile range (IQR).

b Is there a possible outlier? Justify your answer.

Stem	Leaf
0	5
1	
2	5 6 6 7 9 9
3	2 3 3 4 6 6 6 8 9
4	0 1 2 3 3 4 5 5 6 8
5	1 1 6 6 6 7 8
6	0 5 7 9

Key: 2|5 means 25

Steps	Working
a **i** Find the most common value.	Test scores of 36 and 56 each appear three times, so this data set is bi-modal. Modes are scores of 36 and 56.
ii range = largest value – smallest value	range = 69 – 5 = 64
iii **1** Count the number of data values n, note whether it's odd or even, and find the position of the median.	$n = 37$; odd $\frac{n+1}{2} = \frac{37+1}{2} = \frac{38}{2} = 19$ The median is the 19th ordered data value.
2 If n is odd, find the middle data value. If n is even, find the average of the two middle data values.	(see stem plot below) median = 42

Stem	Leaf	
0	5	18 values before the median
1		
2	5 6 6 7 9 9	
3	2 3 3 4 6 6 6 8 9	
4	0 1 **2** 3 3 4 5 5 6 8	median = 42
5	1 1 6 6 6 7 8	
6	0 5 7 9	18 values after the median

median = 42

iv **1** To find the lower quartile Q_1, find the median of the lower half.

Count the number of data values n, note whether it's odd or even, and find the position of Q_1.

$n = 18$; even

$$\frac{n+1}{2} = \frac{18+1}{2} = \frac{19}{2} = 9.5$$

Q_1 is between the 9th and 10th ordered data values in the lower half.

2 If n is odd, find the middle data value.

If n is even, find the average of the two middle data values.

Stem	Leaf
0	5
1	
2	5 6 6 7 9 9
3	2 3 3 4 6 6 6 8 9
4	0 1

8 values before h e 9h value

$Q_1 = \frac{3 + 3}{2} = 3$

8 values after h e 10h value

$Q_1 = 33$

v **1** To find the upper quartile Q_3, find the median of the upper half.

Count the number of data values n, note whether it's odd or even, and find the position of Q_3.

$n = 18$; even

$$\frac{n+1}{2} = \frac{18+1}{2} = \frac{19}{2} = 9.5$$

Q_3 is between the 9th and 10th ordered data values in the upper half.

2 If n is odd, find the middle data value.

If n is even, find the average of the two middle data values.

Stem	Leaf
4	3 3 4 5 5 6 8
5	1 1 6 6 6 7 8
6	0 5 7 9

8 values before h e 9h value

$Q_3 = \frac{5 + 6}{2} = 5$

8 values after h e 10h value

$Q_3 = 53.5$

vi Use Q_3 and Q_1 to calculate the interquartile range.

$IQR = Q_3 - Q_1 = 53.5 - 33 = 20.5$

b Check any value that appears to be an outlier against the upper or lower fence.

The 5 score may be an outlier. Check using the lower fence.

$Q_1 - 1.5 \times IQR = 33 - 1.5 \times 20.5 = 2.25$

5 isn't less than 2.25, so it's not an outlier.

Exam hack

Always look to see if the total number of data values is given in the question, so you don't waste time counting them.

Displays and data types

Categorical: Nominal data	Categorical: Ordinal data	Numerical data
Displays		
dot plot bar chart	dot plot bar chart	dot plot histogram boxplot stem plot

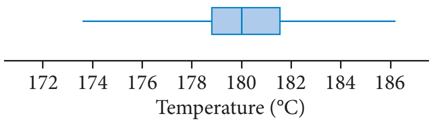
EXERCISE 1.5 Dot plots and stem plots

ANSWERS p. 492

Recap

Use the following information to answer the next two questions.

To test the temperature control on an oven, the control is set to 180°C and the oven is heated for 15 minutes. The temperature of the oven is then measured. Three hundred ovens were tested in this way. Their temperatures were recorded and are displayed using a boxplot.



1 © VCAA 2010 1CQ2 **70%** The interquartile range for temperature is closest to

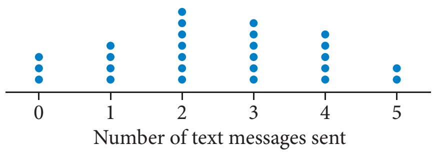
A 1.3°C **B** 1.5°C **C** 2.0°C **D** 2.7°C **E** 4.0°C

2 The range is closest to

A 2.7°C **B** 11.5°C **C** 12.7°C **D** 14.0°C **E** 18.0°C

Mastery

3 WORKED EXAMPLE 10 The dot plot shows the number of text messages sent by a group of students during a one-hour period.



a Find the

i mode **ii** range **iii** median

iv lower quartile (Q_1) **v** upper quartile (Q_2) **vi** interquartile range (IQR).

b What could best describe the shape of the distribution: approximately symmetric, positively skewed, or negatively skewed?

4 WORKED EXAMPLE 11 The stem plot shows the ages of 33 customers in a store recorded one day.

Stem	Leaf
2	3 5 7 7 7 7 8 9 9
3	0 1 1 1 1 1 3 4 4 5 6 7 8 8
4	0 2 3 4 4 8
5	7 9
6	3
7	7

Key: 2 | 3 means 23

a Find the

i mode **ii** range **iii** median

iv lower quartile (Q_1) **v** upper quartile (Q_2) **vi** interquartile range (IQR).

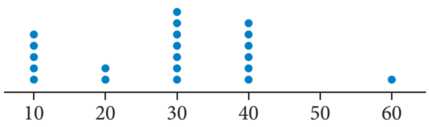
b Is there a possible outlier? Justify your answer.

5 For each of the following

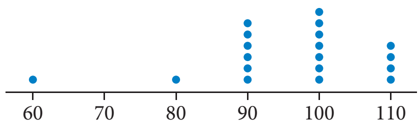
i find the IQR

ii do a calculation to show whether 60 is an outlier.

a



b



c

Stem	Leaf
1	2 5 6 7
2	6 6 7 7 8 9 9
3	0 3 5 7 8 8
4	3
5	
6	0

Key: 1|2 means 12

d

Stem	Leaf
6	0
7	9
8	3 3 4 5 5 5 7 8
9	0 4 4 4 5 6 7 7 7
	8 9

Key: 7|9 means 79

6 This stem plot represents the number of goals scored per match by the Sapphires in a netball season. The leaves in this stem plot are not in the right order.

Stem	Leaf
3	7 5 8 4
4	9 0 4 1
5	3 6 3 7 4 3 1
6	2 8 2 0
7	9 3 5

Key: 6|8 means 68

a Rewrite the stem plot so that the leaves are in the right order.

b How many matches were played in the season?

c What was the Sapphires' highest score for a match?

d In what percentage of matches did the Sapphires score below 55 goals? Round your answer to the nearest percentage.

e What could best describe the shape of the distribution: approximately symmetric, positively skewed or negatively skewed?

Exam practice

80–100% 60–79% 0–59%

Use the following information to answer the next two questions.

The stem plot shows the percentage of homes connected to broadband internet for 24 countries in 2007.

Stem	Leaf
1	
1	6 7
2	0 1 1 3 4 4
2	5 7 8 9
3	0 0 1 1 1 2 2 3
3	5 7 8 8
4	

Key: 1|6 means 16%

7 © VCAA 2013 1CQ1 93% (2 marks) The number of these countries with more than 22% of homes connected to broadband internet in 2007 is

A 4 **B** 5 **C** 19 **D** 20 **E** 22

8 © VCAA 2013 1CQ2 86% Which one of the following statements relating to the data in the stem plot is **not** true?

A The minimum is 16%.

B The median is 30%.

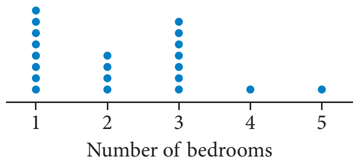
C The first quartile is 23.5%.

D The third quartile is 32%.

E The maximum is 38%.

Use the following information to answer the next two questions.

The dot plot shows the distribution of the number of bedrooms in each of 21 apartments advertised for sale in a new high-rise apartment block.



9 © VCAA 2007 1CQ1 **91%** The mode of this distribution is

A 1 **B** 2 **C** 3 **D** 7 **E** 8

10 © VCAA 2007 1CQ2 **82%** The median of this distribution is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

Use the following information to answer next two questions.

The marks obtained by students who sat for a test are displayed as a stem plot, as shown.

Stem	Leaf
0	0
1	
2	0 1 2 5 6
3	0 1 1 1 3 5 5 7 8 9 9
4	1 2 3 4 4 6 7 7
5	0

Key: 2 | 1 means 21

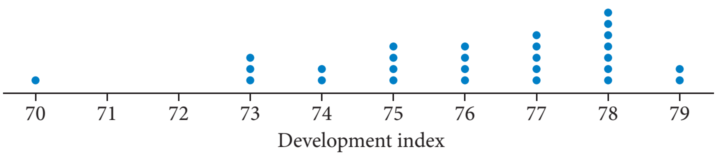
11 © VCAA 2004 1CQ1 **94%** The number of students who sat the test is

A 25 **B** 26 **C** 27 **D** 32 **E** 50

12 © VCAA 2004 1CQ2 **80%** The interquartile range of these test marks is closest to

A 9 **B** 13 **C** 30 **D** 36 **E** 41

13 © VCAA 2013 2CQ2 (3 marks) The development index for each country is a whole number between 0 and 100. The dot plot displays the values of the development index for each of the 28 countries that has a high development index.



a **70%** Using the information in the dot plot, determine the mode and the range. 1 mark

b **70%** Write down an appropriate calculation and use it to explain why the country with a development index of 70 is an outlier for this group of countries. 2 marks

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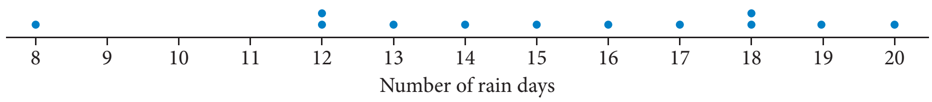
14 © VCAA 2009 2CQ1 (3 marks) Table 1 shows the number of rainy days recorded in a high rainfall area for each month during 2008.

Table 1

Month	Number of rainy days
January	12
February	8
March	12
April	14
May	18
June	18
July	20
August	19
September	17
October	16
November	15
December	13

The dot plot below displays the distribution of the number of rainy days for the 12 months of 2008.



a Copy the dot plot and **circle** the dot that represents the number of rainy days in April 2008. 1 mark

b For the year 2008, determine

i the median number of rainy days per month 1 mark

ii the percentage of months that have more than 10 rainy days. Write your answer correct to the nearest per cent. 1 mark

15 © VCAA 2011 2CQ1 (5 marks) The stem plot in Figure 1 shows the distribution of the average age, in years, at which women first marry in 17 countries.

Figure 1
Average age, in years, of women at first marriage

Stem	Leaf
24	
25	0
26	6
27	1 1 3 4 7
28	2 2 2 3 3 6
29	1 1
30	1 4
31	

Key: 27 | 3 means 27.3 years

a For these countries, determine

i the lowest average age of women at first marriage 1 mark

ii the median average age of women at first marriage. 1 mark

The stem plot in Figure 2 shows the distribution of the average age, in years, at which men first marry in 17 countries.

Figure 2
Average age, in years, of men at first marriage

Stem	Leaf
25	
26	0
27	
28	9
29	0 9 9
30	0 0 3 5 6 7 9
31	0 0 2
32	5 9
33	

Key: 32 | 5 means 32.5 years

b For these countries, determine the interquartile range (IQR) for the average age of men at first marriage. 1 mark

c If the data values displayed in Figure 2 were used to construct a boxplot with outliers, then the country for which the average age of men at first marriage is 26.0 years would be shown as an outlier. Explain why this is so. Show an appropriate calculation to support your explanation. 2 marks

1.6 Back-to-back stem plots and parallel boxplots

Video playlist
Back-to-back stem plots and parallel boxplots

Worksheet
Box-and-whisker plots

Back-to-back stem plots

Sometimes we want to compare the distribution of numerical data for two groups. For example, we may want to compare the test scores for two classes or the heights of players in two football teams. One way of displaying and comparing two sets of data is to construct **back-to-back stem plots**.

A back-to-back stem plot has two sets of leaves, one on the left of the stem and one on the right. This allows us to display all the data values for the two groups being compared, as in the example below.

Test results for two classes

This represents the data value 52%

11A	Stem	11B
5 4 2	5	
7 5 5 4 1	6	0 2 3 5
9 6 5 5 4 0 0	7	4 5 6 7 7 8 9 9
7 6 5 5 4 2 0	8	0 2 5 6 6
1 0	9	2 4 5 6 7 7 9

4 | 7 = 74% 7 | 4 = 74%

Largest value in th s leaf

Smallest value in this leaf

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WORKED EXAMPLE 12 Working with back-to-back stem plots

Two speed cameras on different roads recorded the following car speeds (in km/h).

Camera 1: 136, 140, 123, 135, 112, 120, 116, 131, 127, 125, 130, 116, 131, 120, 130, 117, 130, 134, 123, 148

Camera 2: 67, 72, 73, 78, 90, 84, 63, 69, 71, 89, 102, 86, 69, 71, 93, 83, 80, 65, 73, 69

a Display the data with a back-to-back stem plot.

b Comment on the shape of the data for Camera 2.

c Calculate the median, range and IQR for the car speeds recorded by each of the two speed cameras.

d If the speed limit on the first road is 100 km/h and on the second is 80 km/h, which road has the greater speeding problem? Provide statistical evidence for your answer.

Steps | **Working**

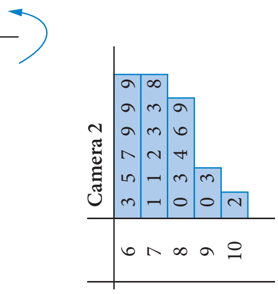
a Order the data and display it using a back-to-back stem plot.

Camera 1		Camera 2
	6	3 5 7 9 9 9
	7	1 1 2 3 3 8
	8	0 3 4 6 9
	9	0 3
	10	2
7 6 6 2	11	
7 5 3 3 0 0	12	
6 5 4 1 1 0 0 0	13	
8 0	14	

2 | 11 = 112 km/h   10 | 2 = 102 km/h

b To see the shape of the data in the right leaf, rotate the page 90° anticlockwise so that the stem forms the horizontal axis and picture it as a histogram.

	Camera 2
6	3 5 7 9 9 9
7	1 1 2 3 3 8
8	0 3 4 6 9
9	0 3
10	2
11	
12	
13	
14	



The data for Camera 2 is positively skewed.

c Calculate the median, range and IQR for each set of data in the back-to-back stem plot.

Camera 1: median = 128.5 km/h, range = 36 km/h, IQR = 12.5 km/h

Camera 2: median = 73 km/h, range = 39 km/h, IQR = 16 km/h

Compare the medians in relation to the speed limits to see if there are any noticeable differences.

The first road has the greater speeding problem. The median speed for the first road is 128.5 km/h, which is 28.5 km/h *above* the 100 km/h speed limit. The median speed for the second road is 73 km/h, which is 7 km/h *under* the 80 km/h speed limit.

1.6

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WB

p. 17

WORKED EXAMPLE 13 Interpreting back-to-back stem plots

Arthur and Stella deliver flyers. The number of flyers delivered per hour over 12 hours is shown below.

Arthur: 20, 38, 23, 31, 14, 38, 28, 30, 37, 30, 24, 37

Stella: 35, 27, 25, 31, 27, 30, 35, 31, 24, 31, 23, 26

a Display the data using a back-to-back stem plot with split stems.

b Comment on the shape of Stella's data.

c Calculate the median, range and IQR for the number of flyers delivered by each person.

d Who would you say is the better delivery person? Justify your answer by quoting appropriate data statistics.

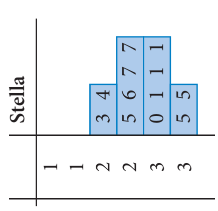
Steps	Working
a Order the data and display it with a back-to-back stem plot, splitting the stems. List leaves in the range 0–4 in the first half of the split stem, and leaves in the range 5–9 in the second half.	(see stem plot below)

Arthur		Stella
4	1	
	1	
3 0	2	3 4
8 4	2	5 6 7 7
1 0 0	3	0 1 1 1
8 8 7 7	3	5 5

3 | 2 = 23 flyers 2 | 3 = 23 flyers

b To see the shape of the data in the right leaf, rotate the page 90° anticlockwise so that the stem forms the horizontal axis and picture it as a histogram.

	Stella
1	
1	
2	3 4
2	5 6 7 7
3	0 1 1 1
3	5 5



Stella's data is symmetric.

c Calculate the medians, ranges and IQRs from the back-to-back stem plot.

Arthur: median = 30, range = 24, IQR = 14.5

Stella: median = 28.5, range = 12, IQR = 5.5

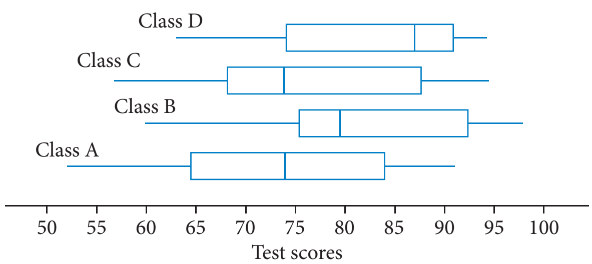
d Use the results to decide who is the better delivery person.

Although the medians (30 and 28.5) are similar, Arthur's range (24) and IQR (14.5) are considerably higher than Stella's range (12) and IQR (5.5). This means Stella's deliveries have less variability and are more consistent than Arthur's, which indicates that Stella is the better delivery person.

Parallel boxplots

Parallel boxplots are a good choice if we want to compare the distribution of numerical data for two or more groups, particularly when the data set is large. It is also easier to compare medians and quartiles from parallel boxplots than from back-to-back stem plots.

Here's an example based on the test results of four classes. It's relatively easy to find which class has the highest median, lowest Q_1 or highest maximum value etc.



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USING CAS 4 Constructing parallel boxplots

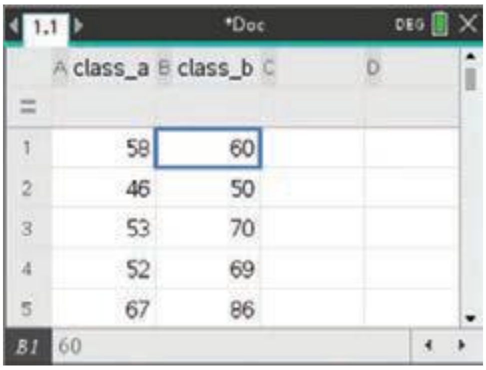
The test results of two Year 11 classes in General Mathematics are shown below.

Class A: 58, 46, 53, 52, 67, 36, 61, 49, 47, 59, 66, 53, 94, 69, 46, 44, 57

Class B: 60, 50, 70, 69, 86, 43, 60, 60, 44, 56, 49, 50, 56, 56, 42, 65, 47, 67, 25, 46

Construct parallel boxplots for the data.

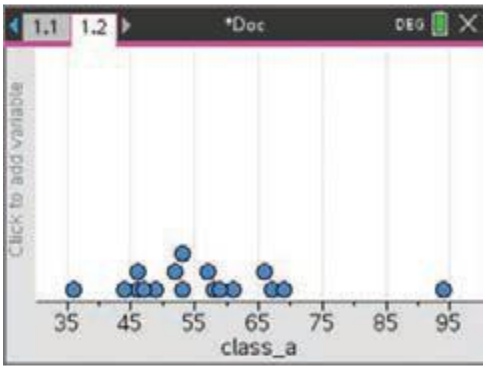
TI-Nspire



1 Start a new document and add a **Lists & Spreadsheet** page.

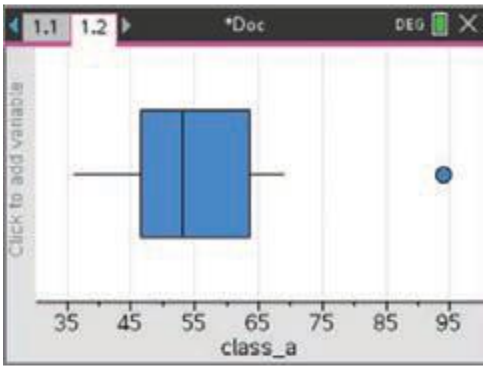
2 Label column **A** as **class_a** and column **B** as **class_b**.

3 Enter the data as shown above.

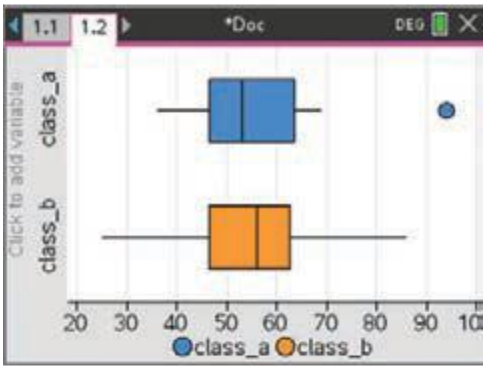


4 Insert a **Data & Statistics** page.

5 For the horizontal axis, select **class_a**. A dot plot of the data will be displayed.



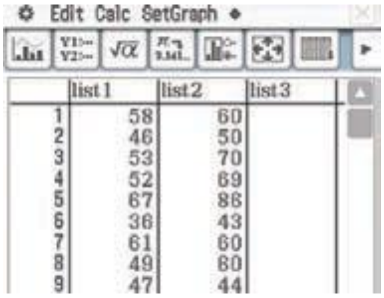
6 Press **menu > Plot Type > Box Plot**. A boxplot of the data will be displayed.

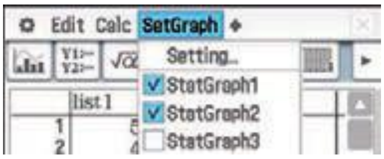


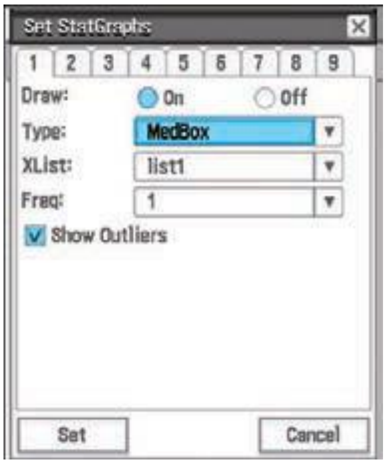
7 Press **menu > Plot Properties > Add X Variable**.

8 Select **class_b**. Parallel boxplots of the data will be displayed.

ClassPad







1 Tap **Menu** and open the **Statistics** application.

2 Clear all lists.

3 Enter the data into **list1** and **list2**.

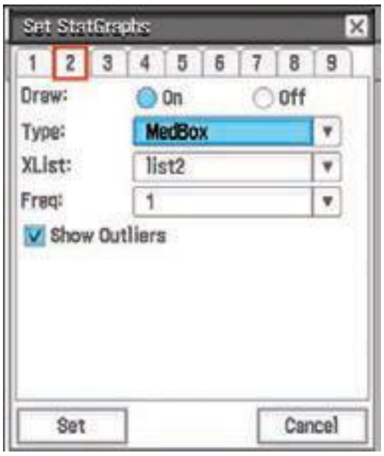
4 Tap **SetGraph** to ensure both **StatGraph1** and **StatGraph2** are selected.

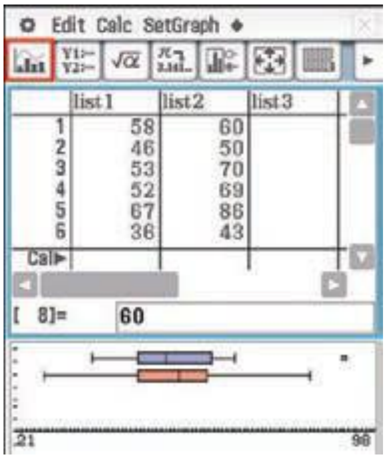
5 Tap **SetGraph > Setting**.

6 In the dialogue box, change the **Type:** field to **MedBox**.

7 Leave **XList:** as **list1**.

8 Tap the box to select **Show Outliers**.





9 Tap the 2 tab at the top of the window.

10 Ensure **Draw:** is set to **On**.

11 Change the **Type:** field to **MedBox**.

12 Change **XList:** to **list2**.

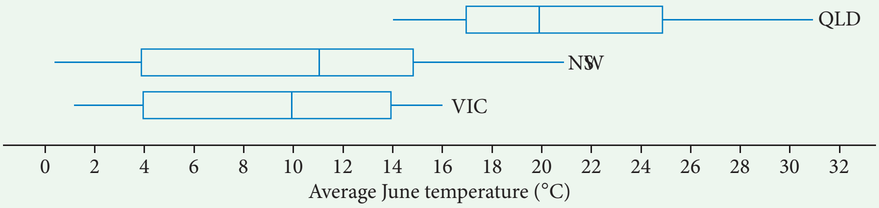
13 Tap the box to select **Show Outliers**.

14 Tap **Set**.

15 Tap the **Graph** tool. The parallel boxplots will be displayed in the lower window.

WORKED EXAMPLE 14 **Working with parallel boxplots**

The parallel boxplots below represent the average temperature (°C) in June for Victoria, New South Wales and Queensland over a number of years.



a Which state has the highest median average June temperature?

b Which state has the largest range of average June temperatures?

c Which state's data is best described as positively skewed?

d Which state had the lowest average June temperature?

e Which of the three states has noticeably higher average June temperatures than the other two? Refer to medians as evidence in your answer.

Steps	Working
a Look for the state whose median line is furthest along the scale.	Queensland
b Look for the longest boxplot, including whiskers.	New South Wales
c Look for the boxplot with its median to the left of the box and the right whisker being longer than the left one.	Queensland
d Look for the lowest left endpoint.	New South Wales
e Compare the medians shown on the states' boxplots.	Queensland has noticeably higher average June temperatures than Victoria and New South Wales. Queensland's median (20°C) is much higher than Victoria's (10°C) and NSW's (11°C).

WB p. 18

Which display do we use?

Often there is more than one suitable display. Here are some guidelines to help us decide which statistical display to use.

Display	Type of data	Guidelines
Bar chart	Categorical data	Categories can be represented on the horizontal or vertical axis.
Histogram	Numerical data	Best if data has been grouped into between 5 and 15 intervals. Can be used for a large number of data values.
Dot plot	Numerical data	Best used with a maximum of 50 data values and when the data values are not too spread out.
Boxplot	Numerical data	Best to read the five-number summary easily.
Stem plot	Numerical data	Best used with a maximum of 50 data values and to see all the actual data values.
Back-to-back stem plots	Categorical and numerical data	Used to compare the distribution of numerical data for two groups, where the two data sets are small.
Parallel boxplots	Categorical and numerical data	Used to compare the distribution of numerical data for two or more groups, where the data sets may be large and we want to compare the five-number summaries.

Recap

Use the following information to answer the next three questions.

The stem plot shows the times (in seconds) of skiers who finished a slalom ski race.

Stem	Leaf
9	1 5 7 9
10	2 4 5 6 6 8
11	0 2 2 3 4 4 5
12	1 2 3 3 3 7 9
13	2 3 4 5 7 7
14	3 6 9
15	0 1 2

Key: 14|3 means 143 seconds

1 The number of skiers who finished the race is

A 7 **B** 15 **C** 23 **D** 36 **E** 152

2 If skiers with times under 110 seconds were of Olympic standard, the percentage of skiers of this standard is closest to

A 12% **B** 22% **C** 25% **D** 27% **E** 28%

Mastery

3 WORKED EXAMPLE 12 Two speed cameras on different roads recorded the following car speeds (in km/h).

Camera 1: 75, 83, 75, 84, 91, 82, 64, 69, 73, 89, 105, 88, 76, 72, 95, 84, 82, 68, 74, 68

Camera 2: 92, 100, 96, 119, 109, 109, 116, 84, 109, 100, 110, 97, 110, 114, 115, 118, 96, 111, 103, 88

a Display the data with a back-to-back stem plot.

b Comment on the shape of the data for Camera 2.

c Calculate the median, range and IQR for the car speeds recorded by each of the two speed cameras.

d If the speed limit on the first road is 80 km/h and on the second is 100 km/h, which road has the greater speeding problem? Provide statistical evidence for your answer.

4 WORKED EXAMPLE 13 Irina and Steven deliver pamphlets. The number of pamphlets delivered per hour over 12 hours is shown below.

Irina: 32, 29, 32, 37, 33, 36, 29, 27, 24, 33, 22, 28

Steven: 25, 33, 16, 38, 30, 32, 22, 38, 39, 32, 26, 39

a Display the data with a back-to-back stem plot with split stems.

b Comment on the shape of Steven's data.

c Calculate the median, range and IQR for the number of pamphlets delivered by each person.

d Who would you say is the better delivery person? Justify your answer by quoting appropriate data statistics.

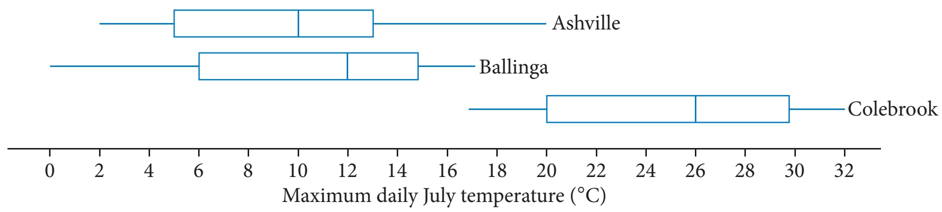
5 Using CAS 4 Two speed cameras on different roads recorded the following car speeds (in km/h).

Camera 1: 83, 80, 69, 94, 92, 98, 63, 95, 69, 91, 90, 83, 65, 98, 69, 91, 93, 69, 89, 96, 132, 83

Camera 2: 91, 120, 116, 98, 55, 116, 96, 106, 118, 112, 98, 112, 100, 60, 120, 116, 95, 125, 90, 90, 94, 123

Construct parallel boxplots for the data.

6 WORKED EXAMPLE 14 The parallel boxplots below represent the maximum daily temperatures (°C) recorded in July for three towns, Ashville, Ballinga and Colebrook, over a number of years.



a Which town has the highest median average July temperature?

b Which town has the largest range of average July temperatures?

c Which town's data is best described as negatively skewed?

d Which town had the lowest average July temperature?

e Which of the three towns has noticeably higher average July temperatures than the other two? Refer to medians as evidence in your answer.

Exam practice

80–100% 60–79% 0–59%

Use the following information to answer the next three questions.

The back-to-back stem plot shows the female and male smoking rates, expressed as a percentage, in 18 countries.

Female		Male
9 9 9 7 7 6 5	1	7 9
8 6 5 5 5 5 5 3 2 1 0	2	2 4 4 4 5 6 7 7 7
	3	0 0 1 1 6 9
	4	7

Key: 2 | 7 means 27%

7 © VCAA 2009 1CQ1 94% For these 18 countries, the lowest female smoking rate is

A 5% **B** 7% **C** 9% **D** 15% **E** 19%

8 © VCAA 2009 1CQ2 85% For these 18 countries, the interquartile range (IQR) of the female smoking rates is

A 4 **B** 6 **C** 19 **D** 22 **E** 23

9 © VCAA 2009 1CQ3 85% For these 18 countries, the smoking rates for females are generally

A lower and less variable than the smoking rates for males.

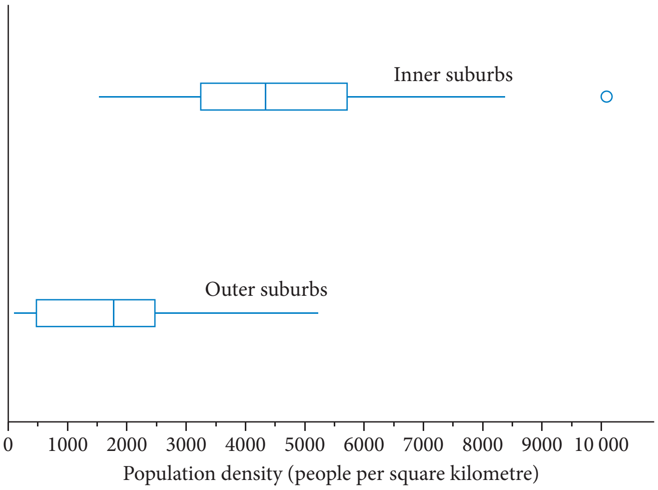
B lower and more variable than the smoking rates for males.

C higher and less variable than the smoking rates for males.

D higher and more variable than the smoking rates for males.

E about the same as the smoking rates for males.

10 © VCAA 2014 1CQ7 77% The parallel boxplots summarise the distribution of population density, in people per square kilometre, for 27 inner suburbs and 23 outer suburbs of a large city.



Which one of the following is **not** true?

A More than 50% of the outer suburbs have population densities below 2000 people per square kilometre.

B More than 75% of the inner suburbs have population densities below 6000 people per square kilometre.

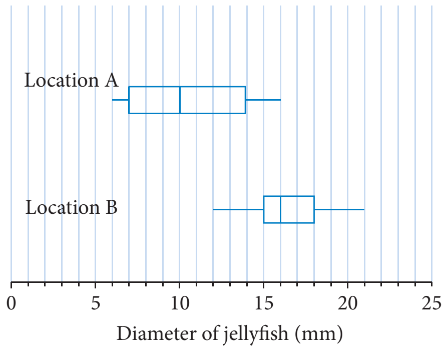
C Population densities are more variable in the outer suburbs than in the inner suburbs.

D The median population density of the inner suburbs is approximately 4400 people per square kilometre.

E Population densities are, on average, higher in the inner suburbs than in the outer suburbs.

Use the following information to answer the next two questions.

Samples of jellyfish were selected from two different locations, A and B. The diameter (in mm) of each jellyfish was recorded and the resulting data is summarised in the boxplots shown.



11 © VCAA 2007 1CQ6 75% From the boxplots, it can be concluded that the diameters of the jellyfish taken from location A are generally

A similar to the diameters of the jellyfish taken from location B.

B less than the diameters of the jellyfish taken from location B and less variable.

C less than the diameters of the jellyfish taken from location B and more variable.

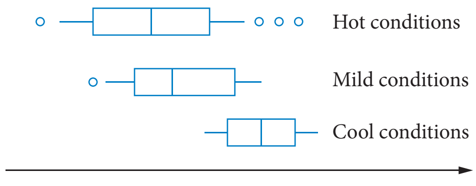
D greater than the diameters of the jellyfish taken from location B and less variable.

E greater than the diameters of the jellyfish taken from location B and more variable.

12 © VCAA 2007 1CQ5 66% The percentage of jellyfish taken from location A with a diameter greater than 14 mm is closest to

A 2% **B** 5% **C** 25% **D** 50% **E** 75%

13 © VCAA 2005 1CQ7 62% As part of an experiment, three samples of pine trees were planted. Each sample contained 50 trees. One sample was grown under hot conditions, one sample was grown under mild conditions and one sample was grown under cool conditions. The parallel boxplots show the rate of growth (in centimetres per year) of these three samples.



From the parallel boxplots it can be concluded that, as conditions change from hot to mild to cool, the rate of growth for these trees

A decreases on average and becomes less variable.

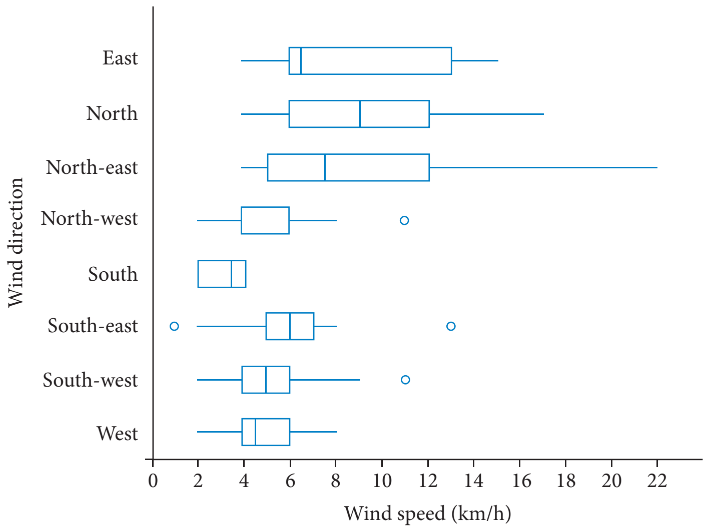
B decreases on average and becomes more variable.

C does not change on average but becomes more variable.

D increases on average and becomes less variable.

E increases on average and becomes more variable.

14 © VCAA 2012 2CQ3 MODIFIED (4 marks) A weather station records the wind speed and the wind direction each day at 9.00 am. The wind speed is recorded, correct to the nearest whole number. The parallel boxplots have been constructed from data that was collected on the 214 days from June to December in 2011.



a Copy and complete the following statements.

The wind direction with the lowest recorded wind speed was ________.

The wind direction with the largest range of recorded wind speeds was ________. 1 mark

b Which wind direction had the greatest variability? 1 mark

c Which wind directions had 50% of their speeds between 4 and 6 km/h? 1 mark

d Which wind directions had outliers of 11 km/h? 1 mark

Video playlist The mean and standard deviation

Worksheets Comparing group measures

Comparing city temperatures

Comparing word lengths

Comparing sports scores

Investigating young drivers

1.7 The mean and standard deviation

The mean

The **mean** is another measure of the centre of a distribution. It is often referred to as the 'average'. The symbol for the mean of a set of data is $\bar{x}$ (called 'x bar'). There are shortcuts for calculating the mean depending on how the data is displayed.

The mean

Mean for ungrouped data:

$$\bar{x} = \frac{\text{sum of all values}}{\text{number of values}} = \frac{\Sigma x}{n}$$

where Σ means 'sum of'

Mean for data in a grouped frequency table:

$$\bar{x} = \frac{\text{sum of (each value} \times \text{its corresponding frequency)}}{\text{sum of frequencies}} = \frac{\Sigma xf}{\Sigma f}$$

Comparing the mean and median

We often need to choose between the mean and the median for the best measure of the centre.

- For symmetric distributions, the mean = the median.
- For distributions that are approximately symmetric, the mean and the median will be very close in value.
- The mean is greater than the median for **positively skewed distributions**.
- The mean is less than the median for **negatively skewed distributions**.
- Outliers usually don't affect the median, but they can often significantly affect the mean.

Mean vs Median

Choosing between the mean and the median as the measure of the centre of a distribution:

Shape of distribution	Choose
Approximately symmetric distributions with no outliers	mean or median
Approximately symmetric distributions with outliers	median
Skewed distributions	median

WB
p. 19

WORKED EXAMPLE 15 Calculating the mean

The scores for the players in two nine-hole golf tournaments were recorded. For each one

i find how many players were in each tournament

ii calculate the mean score, correct to one decimal place.

a Tournament 1 scores: 36, 44, 35, 47, 42, 37, 43, 39, 40, 38

b Tournament 2 scores:

Score (x)	Frequency
37	2
38	4
39	7
40	4
41	1

Steps		Working
a	**i** Count the number of data values.	There are 10 players in tournament 1.
	ii Use the formula $\bar{x} = \frac{\Sigma x}{n}$, giving your answer correct to one decimal place.	$\bar{x} = \frac{36 + 44 + 35 + 47 + 42 + 37 + 43 + 39 + 40 + 38}{10}$ $= \frac{401}{10} = 40.1$ The mean score for tournament 1 is 40.1.
b	**i** Find the sum of the frequencies.	$\Sigma f = 2 + 4 + 7 + 4 + 1 = 18$ There were 18 players in tournament 2.
	ii 1 Add an extra column to the table and an extra row for totals. Fill in the $x \times f$ column and totals.	see table below
	2 Use the formula $\bar{x} = \frac{\Sigma xf}{\Sigma f}$, giving your answer correct to one decimal place.	$\bar{x} = \frac{74 + 152 + 273 + 160 + 41}{2 + 4 + 7 + 4 + 1} = \frac{700}{18} = 38.9$ The mean score for tournament 2 is 38.9.

Score (x)	Frequency (f)	$x \times f$
37	2	74
38	4	152
39	7	273
40	4	160
41	1	41
Total	**18**	**700**

The standard deviation

The **standard deviation**, like the range and the interquartile range, is a measure of the spread of the data. The interquartile range measures the spread of data around the median, whereas the standard deviation measures the spread of data around the mean. The symbol for the standard deviation for a set of data is s.

Standard deviation

The formula for the standard deviation is

$$s = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}}$$

although calculations are done using CAS.

Standard deviation verses IQR

Choosing between the standard deviation and IQR as the measure of the spread of a distribution:

Shape of distribution	Choose
Approximately symmetric distributions with no outliers	standard deviation or IQR
Approximately symmetric distributions with outliers	IQR
Skewed distributions	IQR

1.7

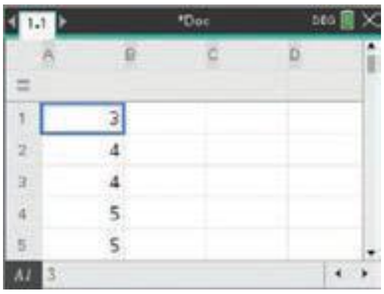
Worksheets
Standard deviation
Statistical calculations
Statistics review
Calculating and interpreting summary statistics
Data and statistics crossword
Statistics crossword

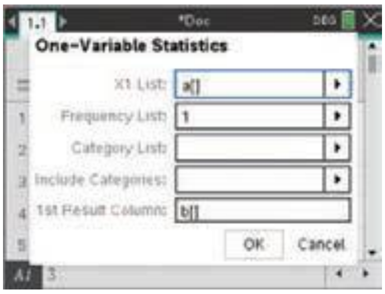
USING CAS 5 Finding the mean and standard deviation for ungrouped data

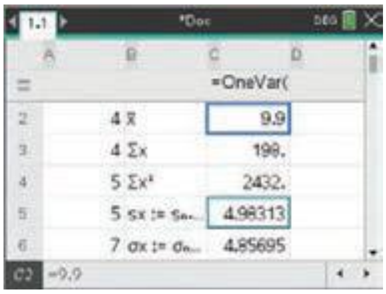
Find the mean $\bar{x}$ and the standard deviation s, rounded to two decimal places, for the ungrouped data shown.

3, 4, 4, 5, 5, 7, 7, 7, 8, 9, 9, 9, 12, 12, 13, 15, 15, 16, 18, 20

TI-Nspire







1 Start a new document and add a **Lists & Spreadsheet** page.

2 Enter values into column **A**.

3 Press **menu > Statistics > Stat Calculations > One-Variable Statistics**.

4 On the next screen, keep the number of lists default setting of **1** and select **OK**.

5 In the **X1 List:** field, keep the default setting of **a[ ]***.

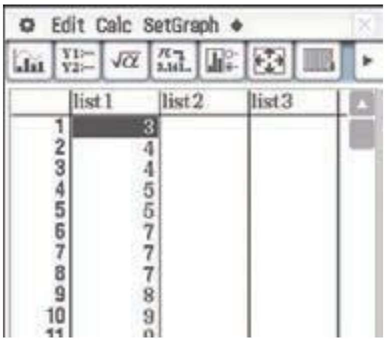
6 Select **OK**.

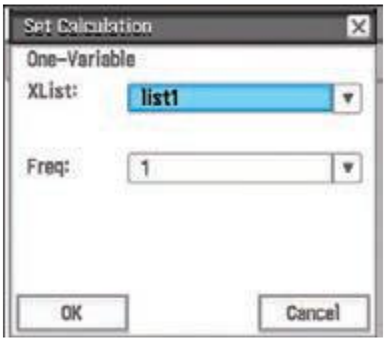
* Alternatively, label the column and use the variable name.

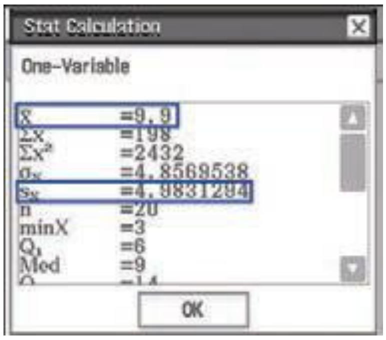
7 The one-variable labels and values will be displayed in columns **B** and **C**.

8 Scroll down to view the mean and standard deviation values.

ClassPad







1 Tap **Menu** and open the **Statistics** application.

2 Clear all lists and enter the data as shown.

3 Tap **Calc > One-Variable**.

4 Leave the default settings of **XList** as **list1** and **Freq:** as **1**.

5 Tap **OK**.

6 The mean and standard deviation values will be displayed.

$\bar{x}$ = 9.90, s = 4.98

Exam hack

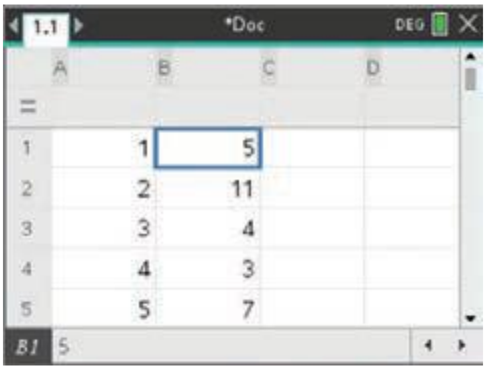
The steps for finding the mean and standard deviation for a data set using CAS are the same as for finding the five-number summary.

USING CAS 6 Finding the mean and standard deviation for grouped data

Find the mean $\bar{x}$ and the standard deviation s, rounded to two decimal places, for the grouped data shown.

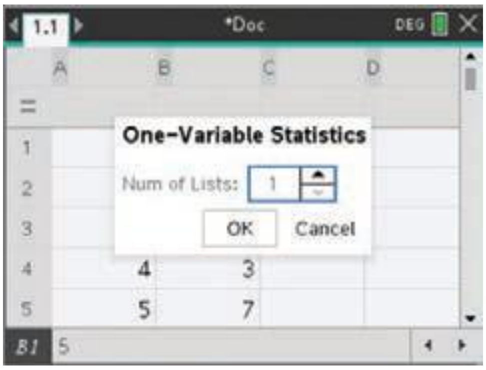
Score	Frequency
1	5
2	11
3	4
4	3
5	7
6	3

TI-Nspire



1 Start a new document and add a **Lists & Spreadsheet** page.

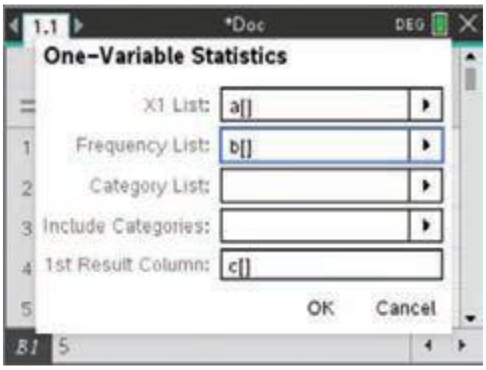
2 Enter the scores into column **A** and the frequencies into column **B**, as shown.



3 Press **menu > Statistics > Stat Calcuations > One–Variable Statistics**.

4 In the dialogue box, keep the default setting of the **Num of Lists:** as **1**.

5 Select **OK**.

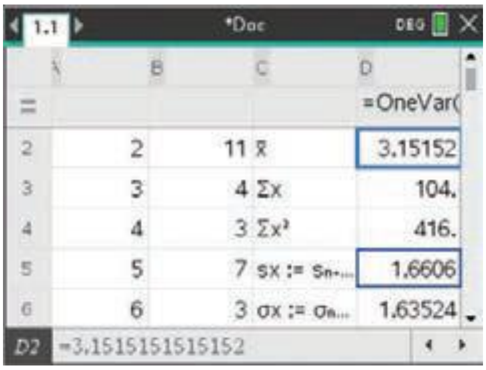


6 In the dialogue box, enter the following:

X1 List: a[ ]

Frequency List: b[ ].

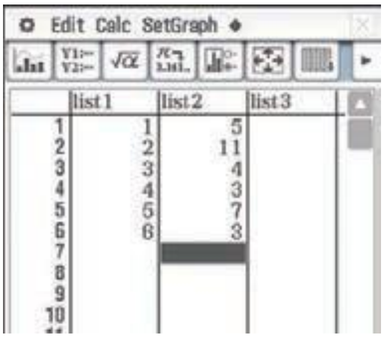
7 Select **OK**.

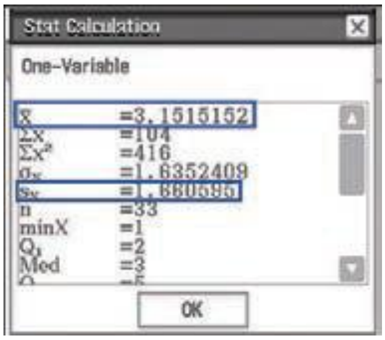


8 The labels and values will be displayed in columns **C** and **D**.

9 Scroll down to view the mean ($\bar{x}$) value of 3.15 and standard deviation (**sx**) value of 1.66.

ClassPad





1 Tap **Menu** and open the **Statistics** application.

2 Clear all lists.

3 Enter the scores into **list1** and the frequencies into **list2** as shown above.

4 Tap **Calc > One-Variable**.

5 In the dialogue box, select the following:

XList: **list1**

Freq: **list2**

6 Tap **OK**.

7 The mean ($\bar{x}$) value of 3.15 and standard deviation (**s_x**) value of 1.66 will be displayed.

WB

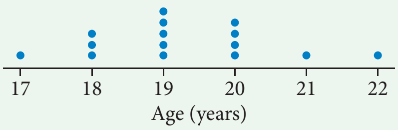
p. 20

WORKED EXAMPLE 16 Working with the mean and standard deviation from a display

For each of the following displays

i find the mean and standard deviation, correct to two decimal places

ii find the number of data values that are within one standard deviation from the mean.

Steps	**Working**
a 	
i Use CAS by entering the data values from the graph and selecting the mean and standard deviation. Round to two decimal places.	17, 18, 18, 18, 19, 19, 19, 19, 19, 20, 20, 20, 20, 21, 22 $\bar{x}$ = 19.27, s = 1.28
ii **1** Find $\bar{x} - s$ and $\bar{x} + s$.	$\bar{x} - s$ = 19.27 − 1.28 = 17.99 $\bar{x} + s$ = 19.27 + 1.28 = 20.55
2 Count the number of data values between $\bar{x} - s$ and $\bar{x} + s$.	The values between 17.99 and 20.55 are: 18, 18, 18, 19, 19, 19, 19, 19, 20, 20, 20, 20 There are 12 data values that are one standard deviation from the mean.

b

Stem	Leaf
1	5 6 7 8 9
2	0 1 7
3	2 4 6 6
4	4 8

Key: 1 | 5 means 15 years

i Use CAS by entering the data values from the graph and selecting the mean and standard deviation. Round to two decimal places.

15, 16, 17, 18, 19, 20, 21, 27, 32, 34, 36, 36, 44, 48

$\bar{x} = 27.36$, $s = 10.95$

ii **1** Find $\bar{x} - s$ and $\bar{x} + s$.

$\bar{x} - s = 27.36 - 10.95 = 16.41$

$\bar{x} + s = 27.36 + 10.95 = 38.31$

2 Count the number of data values between $\bar{x} - s$ and $\bar{x} + s$.

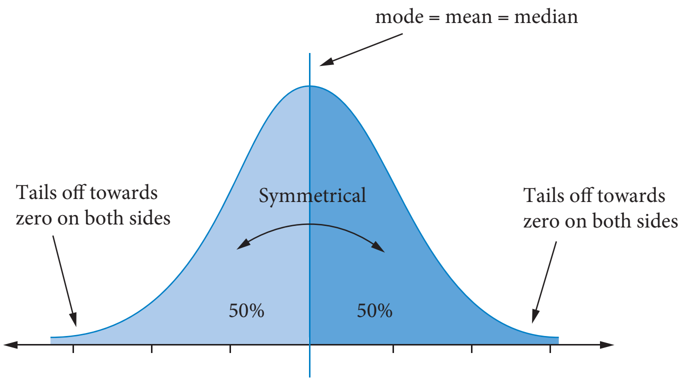
The values between 16.41 and 38.31 are:

17, 18, 19, 20, 21, 27, 32, 34, 36, 36

There are 10 data values that are one standard deviation from the mean.

Standard deviations from the mean

A lot of data in real life has what is known as a **normal distribution**, which has the following bell shape:



For normal distributions, we know that the following mean-standard deviation scale gives the percentages of data lying within 1, 2 or 3 standard deviations from the mean:

0.15%	−3s	2.35%	−2s	13.5%	−1s	34%	$\bar{x}$	34%	+1s	13.5%	+2s	2.35%	+3s	0.15%

If $\bar{x} = 10$ and $s = 2$, then this becomes

0.15%	−3s	2.35%	−2s	13.5%	−1s	34%	$\bar{x}$	34%	+1s	13.5%	+2s	2.35%	+3s	0.15%
	4		6		8		10		12		14		16	

So we know, for example, that

- 34% of the data lies between 10 and 12
- 13.5% of the data lies between 12 and 14
- 2.35% of the data lies between 4 and 6
- 0.15% of the data is less than 4

WB

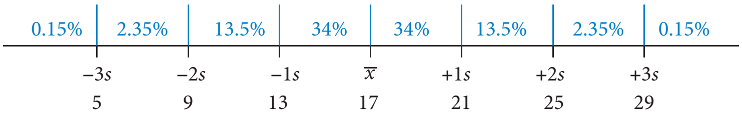
p. 21

WORKED EXAMPLE 17 Working with the mean-standard deviation scale

A study found that the number of orange hundreds and thousands on a freckle chocolate button has a normal distribution with a mean of 17 and a standard deviation of 4. Find the percentage of freckles that have

a between 9 and 13 **b** more than 29 **c** between 17 and 25 **d** less than 13

orange hundreds and thousands.

Steps	Working
a 1 Write a mean-standard deviation scale that includes the mean and standard deviation values given.	
2 Read from the mean-standard deviation scale.	13.5% of freckles have between 9 and 13 orange hundreds and thousands.
b Read from the mean-standard deviation scale.	0.15% of freckles have more than 29 orange hundreds and thousands.
c Add the required percentages from the mean-standard deviation scale.	Add the percentages between 17 and 25: 34% + 13.5% = 47.5% 47.5% of freckles have between 17 and 25 orange hundreds and thousands.
d Add the required percentages from the mean-standard deviation scale.	Add the percentages less than 13: 0.15% + 2.35% + 13.5% = 16% 16% of freckles have less than 13 orange hundreds and thousands.

Video playlist VCE question analysis: Investigating and comparing data distributions

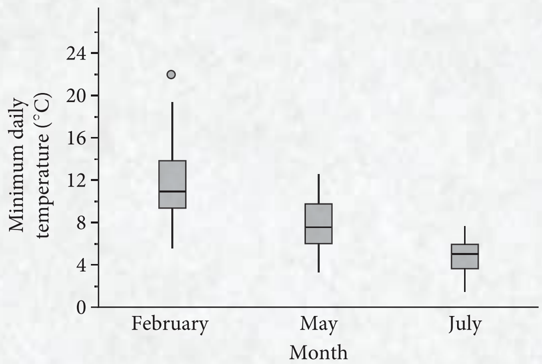
VCE QUESTION ANALYSIS

© VCAA 2019 2CQ3 MODIFIED 2019 Examination 2 Core Question 3 (6 marks)

The five-number summary for the distribution of *minimum daily temperature* for the months of February, May and July in 2017 is shown in the table. The associated boxplots are shown below the table.

Five-number summary for *minimum daily temperature*

Month	Minimum	Q_1	Median	Q_3	Maximum
February	5.9	9.5	10.9	13.9	22.2
May	3.3	6.0	7.5	9.8	12.7
July	1.6	3.7	5.0	5.9	7.7



Data: Australian Government, Bureau of Meteorology, <www.bom.gov.au/>

a Copy and complete the following sentence.

The middle 50% of the July minimum daily temperatures are between ______°C and ______°C. 1 mark

b Describe the shape of the distribution of the minimum daily temperatures (including outliers) for February. 1 mark

c What is the range of the May minimum daily temperatures? 1 mark

d Is there a noticeable difference in minimum daily temperatures between the three months? Refer to medians as evidence in your answer. 1 mark

Exam hack

Don't give more information than asked for. If the extra information is wrong, you will lose marks.

e Determine the value of the upper fence for the February boxplot and hence show that the outlier is correct. 2 marks

Reading the question

- The data is presented in two ways: as five-number summaries and as boxplots.
- Make sure you are clear on what the 'middle 50%' is referring to.
- Note you are asked to 'refer' to something in part **d** and to 'show' something in part **e**.

Thinking about the question

- Decide whether it's best to use the five-number summaries or the boxplots for each question part.
- A 'show' answer must have whatever needs to be shown at the very end.
- Think about why part **e** is worth two marks.

Worked solution (✓ = 1 mark)

a The middle 50% lies between Q_1 and Q_3.

From the five-number summary table: for July $Q_1 = 3.7$ and $Q_3 = 5.9$. This means:

The middle 50% of the July minimum daily temperatures are between **3.7°C** and **5.9°C.** ✓

b The box and whisker in the positive direction are longer than the box and whisker in the negative direction, so the shape is: **positively skewed with an outlier** ✓

c range = largest value − smallest value = 12.7 − 3.3 = **9.4°C** ✓

d Yes, there is a noticeable difference. The February median (10.9°C) is noticeably higher than the May median (7.5°C), which is noticeably higher than the July median (5.0°C). ✓

e upper fence = Q_3 + 1.5 × IQR = 13.9 + 1.5 × 6.6 = **20.5.** ✓ The outlier is 22.2, the maximum given in the five-number summary table. **This outlier is correct because 22.2 > 20.5.** ✓

Student performance

a Students needed to use the exact values given in the five-number summary table rather than relying on estimating the values from the boxplot.

b The question specifically said to refer to any outliers, so this needed to be done to get the mark.

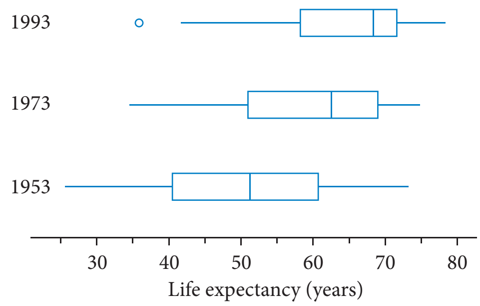
c Students again needed to use the exact values given in the five-number summary table rather than relying on estimating the values from the boxplot.

d Questions of this type require reasoning rather than comments about which season each month falls in. Students should not use the word 'average' in their answer. They should refer to all three medians, not just two. Referring to the difference in maximums and minimums doesn't answer the question.

e The first mark was for the correct calculation of the upper fence and the second for a statement, with specific values, that the outlier was greater than the upper fence. Don't round if not asked to: 21 is incorrect.

Recap

1 © VCAA 2015 2CQ2 MODIFIED The parallel boxplots shown compare the distribution of life expectancy for 183 countries for the years 1953, 1973 and 1993.



The shape of the distribution of life expectancy for 1973 can best be described as

A positively skewed with no outliers.

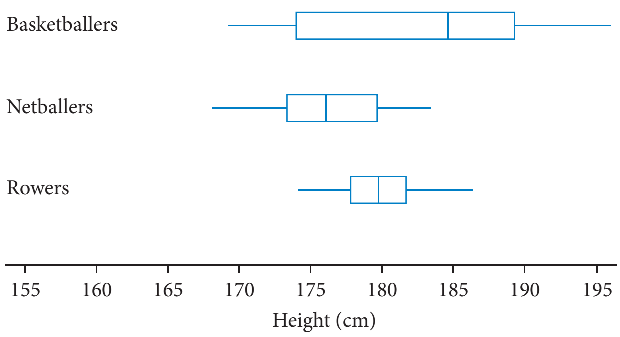
B negatively skewed with no outliers.

C approximately symmetric.

D positively skewed with outliers.

E negatively skewed with outliers.

2 © VCAA 2018N 1CQ6 The parallel boxplots display the distribution of height for three groups of athletes: rowers, netballers and basketballers.



Which one of the following statements is **not** true?

A The shortest athlete is a netballer.

B The rowers have the least variable height.

C More than 25% of the netballers are shorter than all rowers.

D The basketballers are the tallest athletes in terms of median height.

E More than 50% of the basketballers are taller than any of the rowers or netballers.

Mastery

3 WORKED EXAMPLE 15 Two groups of students were surveyed about the number of movies they had streamed in the last month. For each group

i find how many students were surveyed

ii calculate the mean number of movies streamed, correct to one decimal place.

a Group 1 movies streamed: 4, 7, 2, 0, 8, 3, 6, 2, 1, 0, 4, 5, 3

b Group 2 movies streamed:

Movies (x)	Frequency
0	6
1	7
2	8
3	10
4	9
5	5
6	5

4 Using CAS 5 Find the mean $\bar{x}$ and the standard deviation s, rounded to two decimal places, for the ungrouped data shown.

32, 43, 35, 45, 31, 43, 34, 34, 35, 47, 37, 39, 42, 38, 36

5 Using CAS 6 Find the mean $\bar{x}$ and the standard deviation s, rounded to two decimal places, for the grouped data shown.

Score	Frequency
5	3
6	5
7	1
8	0
9	5
10	2

6 Find the mean, median, mode, range, interquartile range and standard deviation of the following data. Where necessary, round answers to one decimal place.

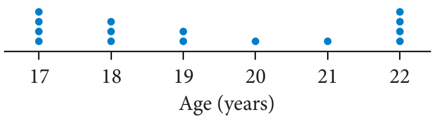
23, 28, 29, 25, 26, 25, 29, 28, 22, 24, 21, 31, 32, 24, 27, 24, 26

7 WORKED EXAMPLE 16 Using CAS 5 For each of the following displays

i find the mean and standard deviation, correct to two decimal places

ii find the number of data values that are within one standard deviation from the mean.

a



b

Stem	Leaf
1	2 4 6 9
2	1 1 4 6 7
3	0 3 5 8 9 9
4	3

Key: 1 | 2 means 12 years

8 WORKED EXAMPLE 17 A study found that the number of cheese puffs in a packet has a normal distribution with a mean of 55 and a standard deviation of 3. Find the percentage of packets that have

a between 55 and 58

b less than 46

c between 55 and 64

d more than 55 of cheese puffs.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2012 1CQ3 88% The total mass of nine oranges is 1.53 kg. Using this information, the mean mass of an orange would be calculated to be closest to

A 115 g **B** 138 g **C** 153 g **D** 162 g **E** 170 g

10 © VCAA 2008 1CQ6 71% The pulse rates of a large group of 18-year-old students are approximately normally distributed with a mean of 75 beats/minute and a standard deviation of 11 beats/minute. The percentage of 18-year-old students with pulse rates less than 75 beats/minute is closest to

A 32% **B** 50% **C** 68% **D** 84% **E** 97.5%

Use the following information to answer the next two questions.

The number of DVD players in each of 20 households is recorded in the frequency table.

Number of DVD players	Frequency
0	6
1	9
2	3
3	1
4	0
5	1
Total	**20**

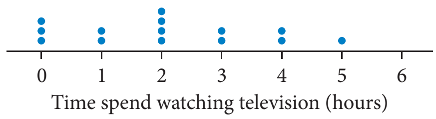
11 © VCAA 2004 1CQ4 62% For this sample of households, the percentage of households with **at least** one DVD player is

A 30% **B** 45% **C** 50% **D** 70% **E** 90%

12 © VCAA 2004 1CQ5 54% For this sample of households, the mean number of DVD players in these 20 households is

A 0.75 **B** 1.00 **C** 1.15 **D** 1.64 **E** 2.00

13 © VCAA 2008 1CQ5 60% A sample of 14 people were asked to indicate the time (in hours) they had spent watching television on the previous night. The results are displayed in the dot plot.



Correct to one decimal place, the mean and standard deviation of these times are respectively

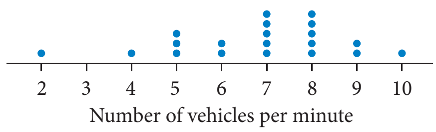
A $\bar{x} = 2.0$ $s = 1.5$ **B** $\bar{x} = 2.1$ $s = 1.5$ **C** $\bar{x} = 2.1$ $s = 1.6$

D $\bar{x} = 2.6$ $s = 1.2$ **E** $\bar{x} = 2.6$ $s = 1.3$

14 © VCAA 2004 1CQ3 58% The distribution of the weights of eggs produced by a chicken farm is approximately normal with a mean of 85 g and a standard deviation of 5 g. Eggs weighing 95 g or more are classified as Extra Large. The percentage of eggs that would be classified as Extra Large is closest to

A 0.15% **B** 0.35% **C** 2.5% **D** 5% **E** 16%

15 The dot plot shows the number of vehicles driving past Westvale High School every minute for a 20-minute period.



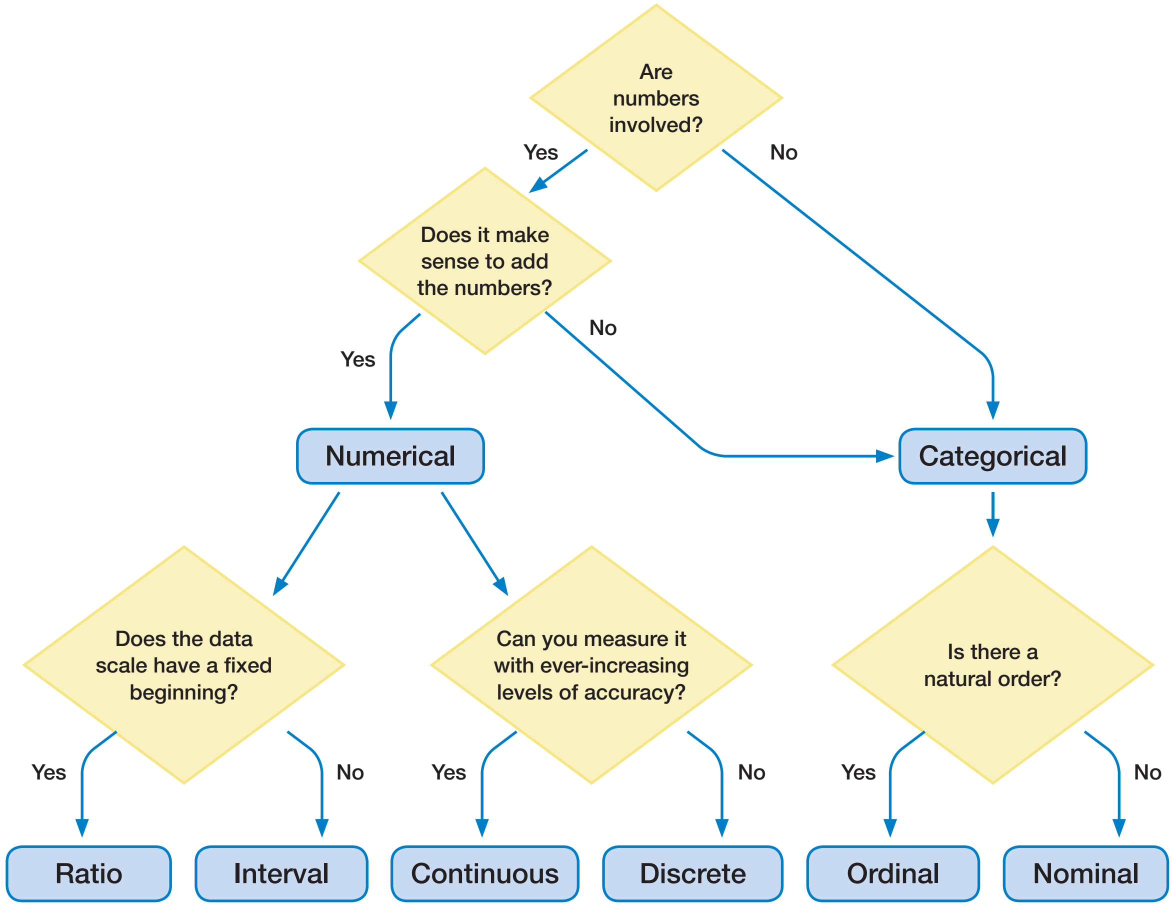
a Find the mean.

b Calculate the standard deviation, correct to two decimal places.

c How many data values were within one standard deviation from the mean?

1 Chapter summary

Types of data

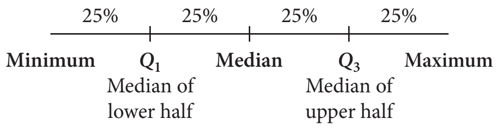


Describing data

Measure of centre

Measure of centre	Use for	Description
mode	numerical categorical	• most frequently occurring data value • is called the **modal category** for categorical data • data with two modes is called **bi-modal** • if every data value appears exactly once, there's no mode
median	numerical ordinal	• the middle value • if there are two middle values, add them and divide by 2.
mean	numerical	• the average of all the data values • $\bar{x} = \dfrac{\text{sum of all values}}{\text{number of values}}$ for a list of data values • $\bar{x} = \dfrac{\text{sum of (each value} \times \text{its corresponding frequency)}}{\text{sum of frequencies}}$ for data in a frequency table

The five-number summary



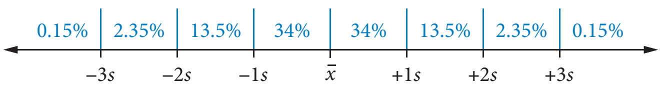
Measures of spread

Measures of spread	Use for	Description
Range	numerical	• measures the spread of the entire data set • range = largest value − smallest value
Interquartile range	numerical	• measures the spread of the middle 50% of the data values • IQR = $Q_3 - Q_1$
Standard deviation	numerical	• measures the spread around the mean • $s = \sqrt{\dfrac{\Sigma(x - \bar{x})^2}{n - 1}}$ (use CAS)

Outliers

- An **outlier** is an extreme high or low data value.
- A data value is a possible outlier if it is less than the **lower fence** $Q_1 - 1.5 \times \text{IQR}$ or greater than the **upper fence** $Q_3 + 1.5 \times \text{IQR}$

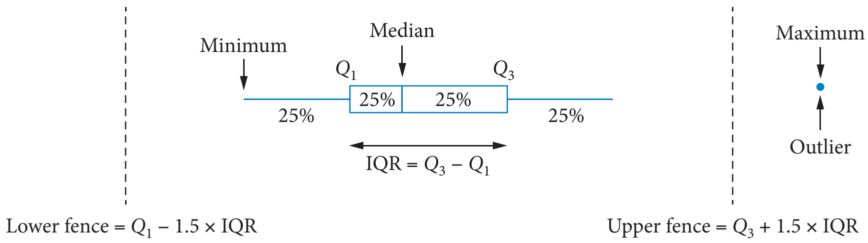
Standard deviations from the mean



Centre, spread, display and data type summary

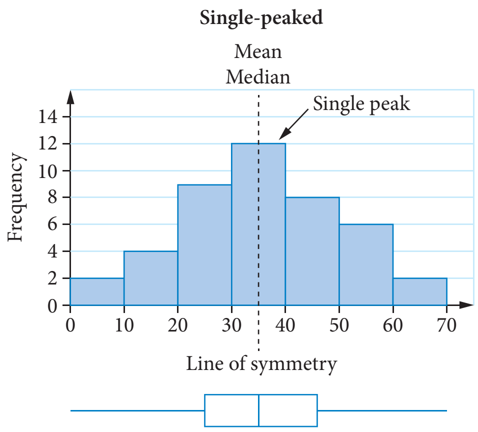
Categorical: Nominal data	Categorical: Ordinal data	Numerical data
Measures of centre		
mode	mode median	mode median mean
Measures of spread		
–	range IQR	range IQR standard deviation
Displays		
dot plot bar chart	dot plot bar chart	dot plot histogram boxplot stem plot

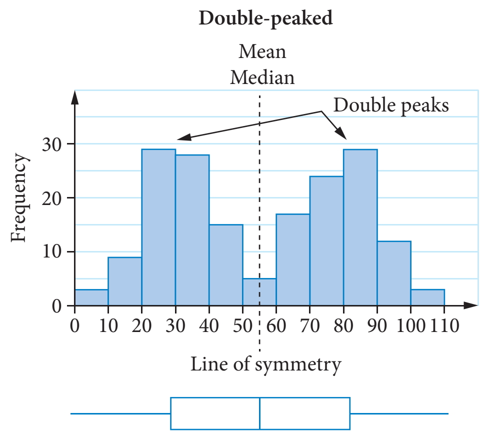
Data displays

Display	Description
Frequency table	• involves counting the number of times each data value occurs • frequencies often shown as percentages • percentage = $\frac{\text{frequency}}{\text{total}} \times 100\%$
Bar chart	• bars can be horizontal or vertical
Grouped frequency table	• involves numerical data that has been grouped into regular intervals • makes it easier to deal with large amounts of data
Histogram	• graphical display of data from a grouped frequency table • best if data is grouped into 5 to 15 intervals
Dot plot	• used for both categorical and numerical data • should not involve too many data values or a large data spread
Stem plot	• involves actual data values and requires a key • best used with a maximum of 50 data values and to see all the data values • back-to-back stem plots allow us to compare the distribution of numerical data for two groups
Boxplot	• best if we want to read the five-number summary easily • helps to find IQR and outliers • parallel boxplots allow us to compare the distribution of numerical data for several groups 

Describing distributions

Symmetric distributions



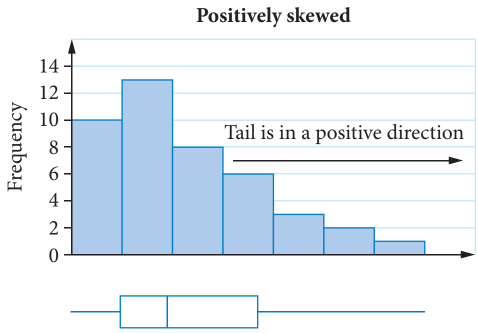


Modal interval: 30–<40

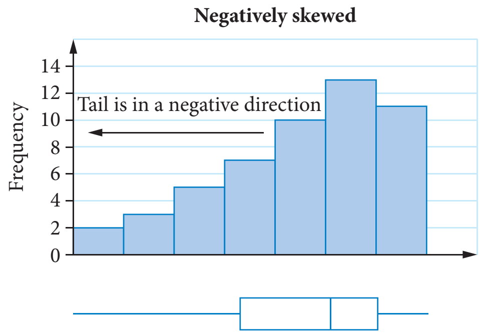
Bi-modal: 20–<30 and 80–<90

The mean and median are both close to the line of symmetry.

Skewed distributions

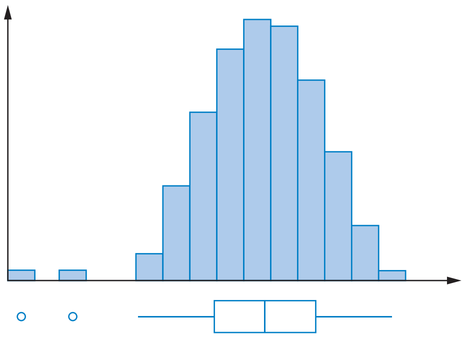


The mean is greater than the median.



The mean is less than the median.

Distributions with outliers



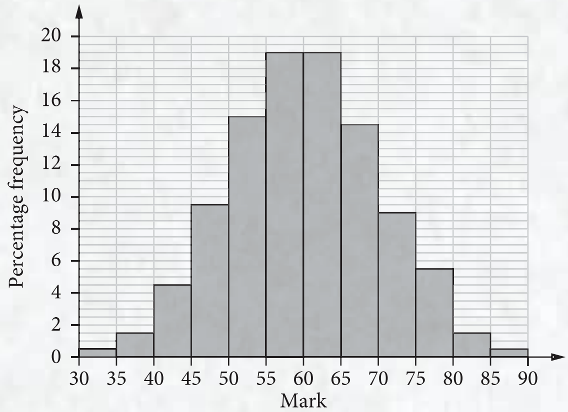
Mean vs median / Standard deviation vs IQR

Shape of distribution	Choose	
Approximately symmetric distributions with no outliers	mean or median	standard deviation or IQR
Approximately symmetric distributions with outliers	median	IQR
Skewed distributions	median	IQR

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 23 minutes

1 © VCAA 2002 1CQ5 MODIFIED 1526 students sat for an examination. The histogram shows the distribution of marks.



The centre of this distribution is closest to

A 50 **B** 55 **C** 60 **D** 65 **E** 70

2 © VCAA 2015 1CQ2 For an ordered set of data containing an odd number of values, the middle value is always

A the mean.

B the median.

C the mode.

D the mean and the median.

E the mean, the median and the mode.

Use the following information to answer the next three questions.

The back-to-back stem plot shows the distribution of maximum temperatures (in °Celsius) of two towns, Beachside and Flattown, over 21 days in January.

Beachside		Flattown
9 8 7 5	1	8 9
4 3 2 2 1 1 0 0	2	
9 9 8 7 6 5	2	8 9
3 2	3	3 3 4
8	3	5 5 6 7 7 7 8 8
	4	0 0 1 2
	4	5 6

3 © VCAA 2006 1CQ1 The variables temperature (°Celsius) and town (Beachside or Flattown) are

A both categorical variables.

B both numerical variables.

C categorical and numerical variables respectively.

D numerical and categorical variables respectively.

E neither categorical nor numerical variables.

4 © VCAA 2006 1CQ2 For **Beachside**, the range of maximum temperatures is

A 3°C **B** 23°C **C** 32°C **D** 33°C **E** 38°C

5 © VCAA 2006 1CQ3 The distribution of maximum temperatures for **Flattown** is best described as

A negatively skewed.
B positively skewed.
C positively skewed with outliers.
D approximately symmetric.
E approximately symmetric with outliers.

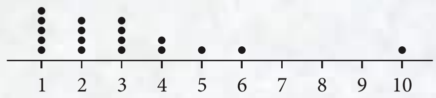
6 Which of these displays could be described as positively skewed with a possible outlier?

A

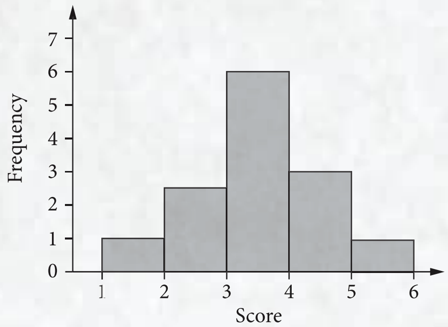
Stem	Leaf
7	1 5 6 8 9 9
8	0 0 2 3 4 6 8 9
9	0 2 5 7 9
10	2 3 5 6 7 7 8

Key: 7 | 9 means 79

B



C

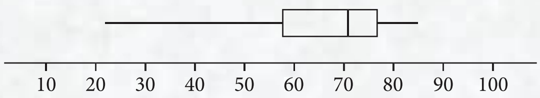


D

Stem	Leaf
12	3
13	
14	
15	4 7 9 9 9
16	0 2 3 7 8 8
17	0 0 2 4 4 5 5 7 7 7 9
18	3 9
19	0 1 2

Key: 12 | 3 means 123

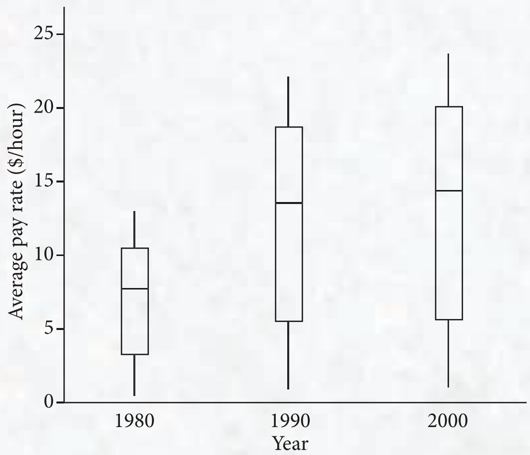
E



7 © VCAA 2005 1CQ6 The distribution of fuel consumption of a particular model of car is approximately normal with a mean of 8.8 km per litre and a standard deviation of 2.2 km per litre. The percentage of this model of car that has a fuel consumption less than 6.6 km per litre is closest to

A 2.5% **B** 5% **C** 16% **D** 32% **E** 68%

8 © VCAA 2011 1CQ5 The boxplots display the distribution of average pay rates, in dollars per hour, earned by workers in 35 countries for the years 1980, 1990 and 2000. Based on the information contained in the boxplots, which one of the following statements is **not** true?



A In 1980, over 50% of the countries had an average pay rate less than $8.00 per hour.
B In 1990, over 75% of the countries had an average pay rate greater than $5.00 per hour.
C In 1990, the average pay rate in the top 50% of the countries was higher than the average pay rate for any of the countries in 1980.
D In 1990, over 50% of the countries had an average pay rate less than the median average pay rate in 2000.
E In 2000, over 75% of the countries had an average pay rate greater than the median average pay rate in 1980.

Use the following information to answer the next two questions.

Joel is training for a triathlon. He swam the following times, in minutes, in his last ten races.

28, 34, 22, 24, 25, 24, 26, 26, 24, 27

9 Joel's mean swim time is

A 24 min **B** 25 min **C** 25.5 min **D** 26 min **E** 27 min

10 If the outlier score of 34 was removed from the set of data, which of the following would happen?

A The mean would stay the same.

B The mean would have 34 added to it.

C The mean would decrease.

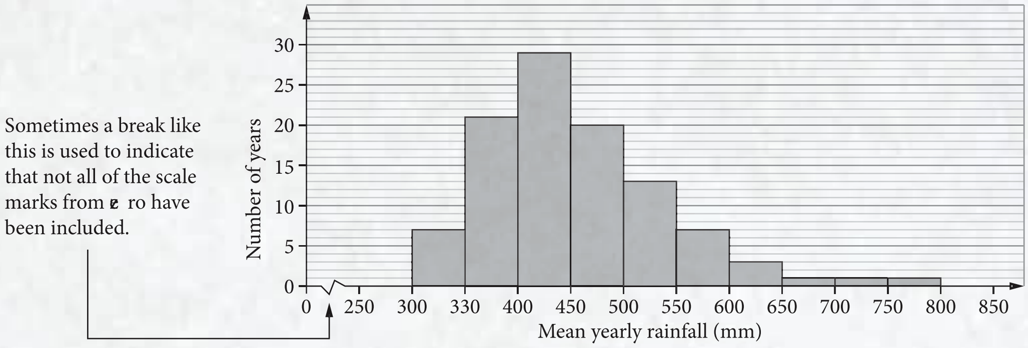
D The mean would increase.

E None of the above.

Cumulative examination 2

Total number of marks: 15 Reading time: 4 minutes Writing time: 23 minutes

1 © VCAA 2007 2CQ1 (3 marks) The histogram shows the distribution of mean yearly rainfall (in mm) for Australia over 103 years.



Data source: ABS 2007

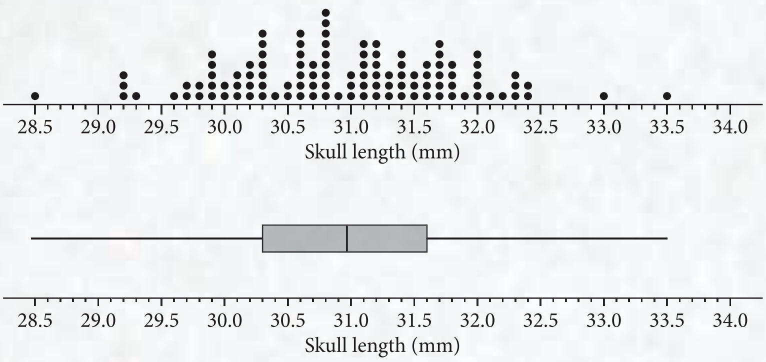
a Describe the shape of the histogram. 1 mark

b Use the histogram to determine

i the number of years in which the mean yearly rainfall was 500 mm or more 1 mark

ii the percentage of years in which the mean yearly rainfall was between 500 mm and 600 mm. Write your answer correct to one decimal place. 1 mark

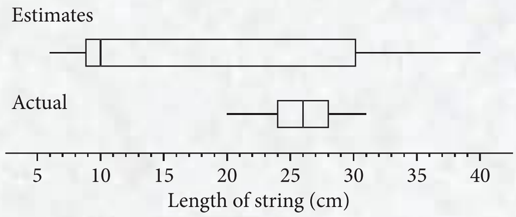
2 © VCAA 2018N 2CQ1 (3 marks) The dot plot and boxplot display the distribution of *skull length*, in millimetres, for a sample of the same species of bird.



a Write down the modal skull length. 1 mark

b Use information from the plots above to show why the bird with a skull length of 33.5 mm is **not** plotted as an outlier in the boxplot. 2 marks

3 (5 marks) Edgar has a theory that everyone always underestimates the length of a piece of string. He asked his friends to estimate the lengths of several pieces of string and then measured their actual lengths. He then showed the results as the following parallel boxplots.



Find the following for both the estimated lengths and the actual lengths.

a median — 1 mark

b range — 1 mark

c IQR — 1 mark

d Describe the shape of each data set. — 1 mark

e Do Edgar's results support his theory? Justify your answer by referring to medians. — 1 mark

4 (4 marks) The stem plot shows the number of customers in a store each day over a two-week period.

Stem	Leaf
2	3 7
3	1 1 4 6 7
4	0 3 5 8 9
5	2 6

Key: 1 | 2 means 12 years

a Use the data to find

i the mean and standard deviation, correct to two decimal places — 1 mark

ii the number of data values that are one standard deviation from the mean. — 1 mark

b Assuming the data has a normal distribution, what percentage of data values would lie one standard deviation from the mean? — 1 mark

c What is the actual percentage of data values that lie within one standard deviation from the mean? — 1 mark

CHAPTER

2 ARITHMETIC SEQUENCES AND FINANCIAL RECURRENCE RELATIONS

Study Design coverage

UNIT 1, AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

Arithmetic and geometric sequences, first-order linear recurrence relations and financial mathematics

- the concept of an arithmetic sequence as a function with the set of non-negative integers as its domain
- tabular and graphical display of sequences, investigation of their behaviour (increasing, decreasing, constant, oscillating, limiting values)
- use of a first-order linear recurrence relation of the form $u_0 = a$, $u_{n+1} = u_n + d$ where a and d are constants, to generate the values of an arithmetic sequence
- use of a first-order linear recurrence relation of the form $u_0 = a$, $u_{n+1} = u_n + d$ where a and d are constants, to model and analyse practical situations involving discrete linear growth or decay such as a simple interest loan or investment, the depreciating value of an asset using the unit cost or flat-rate method
- generation of the explicit rule, u_n, of an arithmetic sequence, its use and evaluation, including various practical and financial contexts.

Video playlists (5):

2.1 Sequences
2.2 Arithmetic recurrence relations
2.3 Simple interest recurrence relations
2.4 Depreciation recurrence relations
VCE question analysis Arithmetic sequences and financial recurrence relations

Worksheets (3):

2.1 Arithmetic sequences • Arithmetic progressions
2.2 First-order difference equations

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Sequences

2.1 Sequences

Sequences, values and rules

A **sequence** is a list of numbers called **values**. Sequences can be randomly generated and have no pattern; however, we will only be looking at sequences generated by a rule.

An example of a rule-based sequence is 5, 8, 11, 14 … where the three dots indicate the sequence continues. We can see that the rule involves adding 3 to generate each new value.

To analyse more complex sequences, we need to label the values in a sequence. Label the starting value u_0, the next one u_1, the one after that u_2 and so on. So, the sequence above becomes

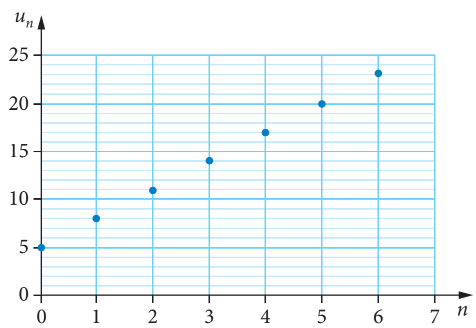
$u_0 = 5, u_1 = 8, u_2 = 11, u_3 = 14 \ldots$

We can continue the sequence, recording it in table form and graphing it where we identify the values using $n = 0, 1, 2, 3 \ldots$

Table

n	0	1	2	3	4	5	6	…
u_n	5	8	11	14	17	20	23	…

Graph



Exam hack

Graphs of sequences are always a series of dots. Don't join them.

Labelling the values allows us to write the rule in a more efficient way, rather than saying 'add 3 to generate each new value'.

The rule for this sequence is $u_n = 5 + 3n$, where $n = 0, 1, 2, 3 \ldots$ giving

$u_0 = 5 + 3 \times 0 = 5$

$u_1 = 5 + 3 \times 1 = 8$

$u_2 = 5 + 3 \times 2 = 11$

$u_3 = 5 + 3 \times 3 = 14$

and so on.

Exam hack

We can use letters other than u for values in a sequence.

WB
p. 22

WORKED EXAMPLE 1 Finding sequences from rules

For each of the following sequence rules

- **i** find the first five values
- **ii** describe in words how each new value is generated.

a $u_n = 6 + 10n$, where $n = 0, 1, 2 \ldots$

b $u_n = 5 - 2n$, where $n = 0, 1, 2 \ldots$

c $u_n = 2^n$, where $n = 0, 1, 2 \ldots$

d $T_n = (-10)^n$, where $n = 0, 1, 2 \ldots$

Steps		Working
a	**i** Substitute $n = 0, 1, 2, 3, 4$ into the rule.	$u_0 = 6 + 10 \times 0 = 6$ $u_1 = 6 + 10 \times 1 = 16$ $u_2 = 6 + 10 \times 2 = 26$ $u_3 = 6 + 10 \times 3 = 36$ $u_4 = 6 + 10 \times 4 = 46$
	ii Find the pattern from one value to the next.	Add 10 to generate each new value.
b	**i** Substitute $n = 0, 1, 2, 3, 4$ into the rule.	$u_0 = 5 - 2 \times 0 = 5$ $u_1 = 5 - 2 \times 1 = 3$ $u_2 = 5 - 2 \times 2 = 1$ $u_3 = 5 - 2 \times 3 = -1$ $u_4 = 5 - 2 \times 4 = -3$
	ii Find the pattern from one value to the next.	Subtract 2 to generate each new value.
c	**i** Substitute $n = 0, 1, 2, 3, 4$ into the rule.	$u_0 = 2^0 = 1$ $u_1 = 2^1 = 2$ $u_2 = 2^2 = 4$ $u_3 = 2^3 = 8$ $u_4 = 2^4 = 16$
	ii Find the pattern from one value to the next.	Multiply by 2 to generate each new value.
d	**i** Substitute $n = 0, 1, 2, 3, 4$ into the rule.	$T_0 = (-10)^0 = 1$ $T_1 = (-10)^1 = -10$ $T_2 = (-10)^2 = 100$ $T_3 = (-10)^3 = -1000$ $T_4 = (-10)^4 = 10\,000$
	ii Find the pattern from one value to the next.	Multiply by −10 to generate each new value.

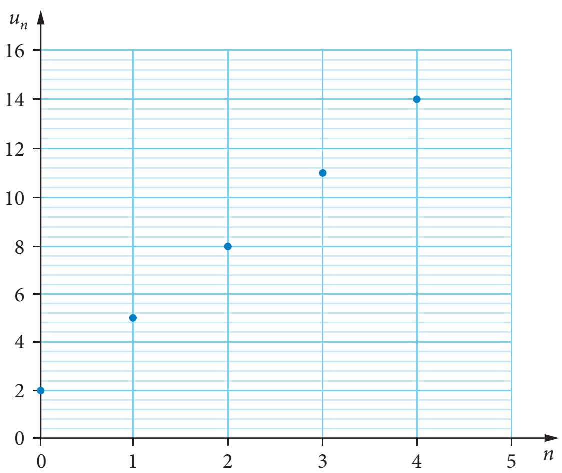
Types of sequences

There are five types of sequences, as seen in the following examples.

Increasing sequence

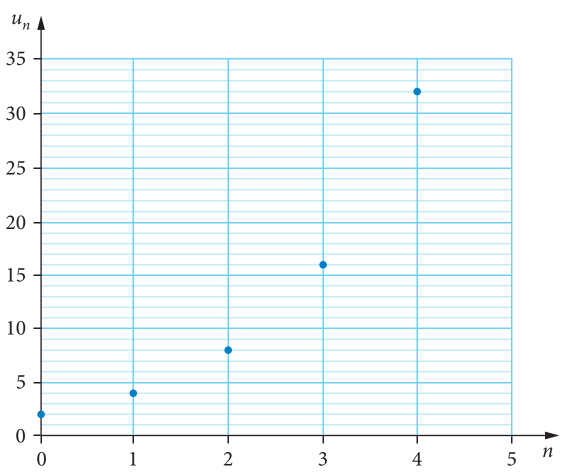
Add 3 to generate each new value.

n	0	1	2	3	4	…
u_n	2	5	8	11	14	…



Multiply by 2 to generate each new value.

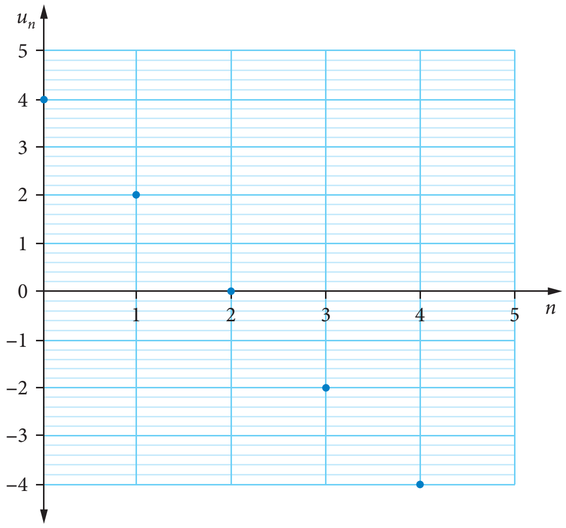
n	0	1	2	3	4	…
u_n	2	4	8	16	32	…



Decreasing sequence

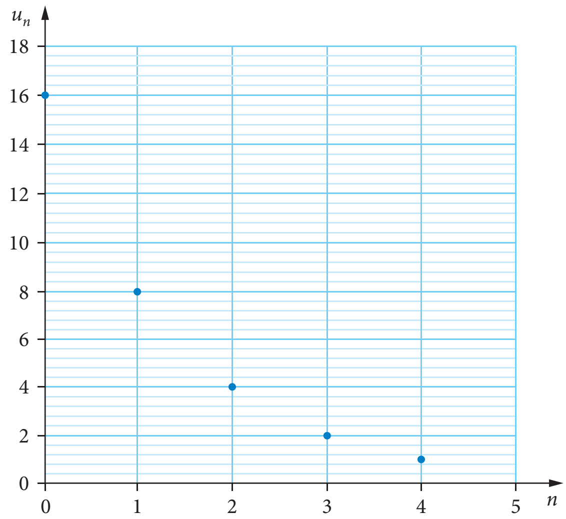
Subtract 2 to generate each new value.

n	0	1	2	3	4	…
u_n	4	2	0	−2	−4	…



Divide by 2 to generate each new value.

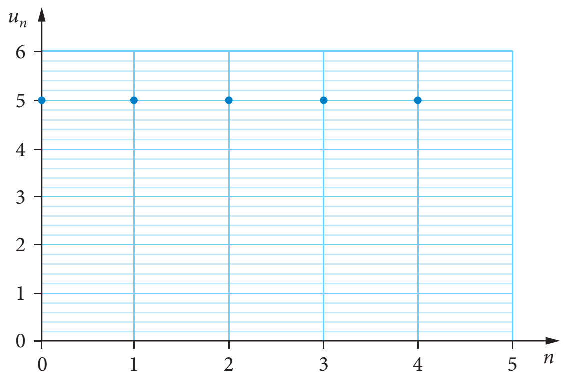
n	0	1	2	3	4	…
u_n	16	8	4	2	1	…



Constant sequence

Multiply by 2 and subtract 5 to generate each new value.

n	0	1	2	3	4	…
u_n	5	5	5	5	5	…

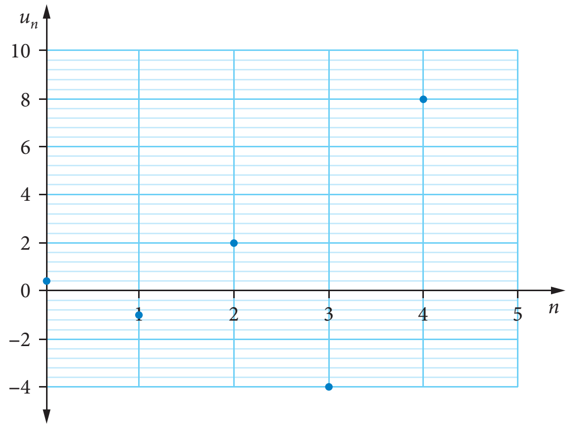


The values of this sequence are always the same.

Oscillating sequence

Multiply by −2 to generate each new value.

n	0	1	2	3	4	…
u_n	0.5	−1	2	−4	8	…



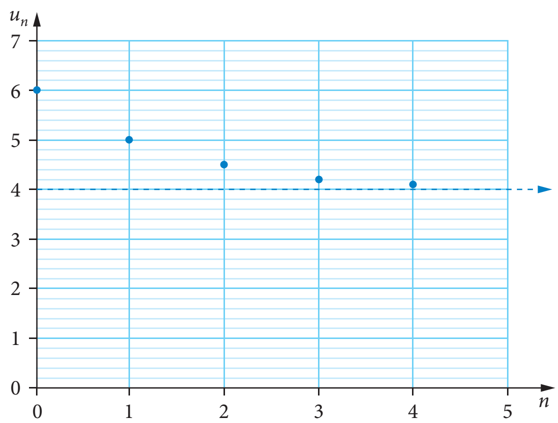
The values of this sequence switch between positive and negative.

Limiting value sequence

n	0	1	2	3	4	…
u_n	6	5	4.5	4.25	4.125	…

Divide by 2 and add 2 to generate each new value.

This sequence has a limiting value of 4. The values tend towards 4 but never reach it.



9780170448192

WORKED EXAMPLE 2 Identifying sequence types

Identify whether each of the following are increasing, decreasing, constant, oscillating or limiting value sequences, giving more than one option where relevant.

a

n	0	1	2	3	4	…
u_n	9	5	1	−3	−7	…

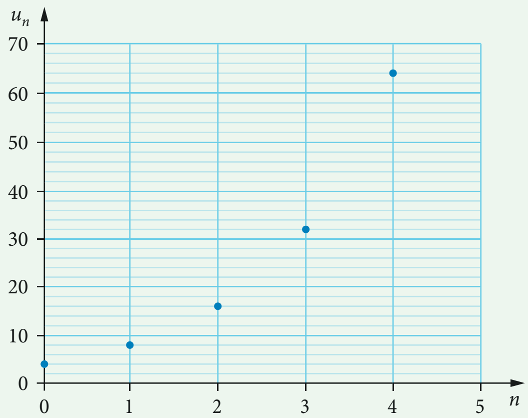
b

n	0	1	2	3	4	…
u_n	−2	2	−2	2	−2	…

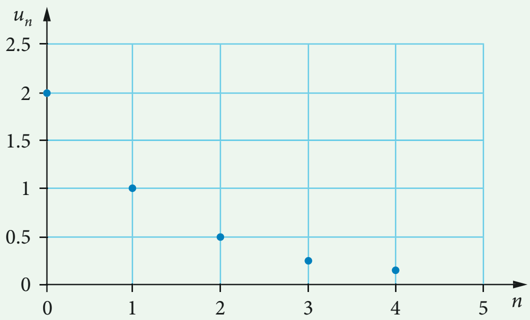
c

n	0	1	2	3	4	…
u_n	−8	−8	−8	−8	−8	…

d



e



Steps	Working
Are the values always increasing or decreasing?	**a** decreasing
Do the values stay the same?	**b** oscillating
Do the values alternate between a positive number and a negative number?	**c** constant
	d increasing
Do the values tend towards a particular value without ever reaching it?	**e** decreasing, limiting value

WB p. 24

Arithmetic sequences

A sequence where we add or subtract a fixed amount to generate each new value is called an **arithmetic sequence**. The points in a graph of an arithmetic sequence are always in a straight line. We will be looking at arithmetic sequences in this chapter.

Worksheets
Arithmetic sequences
Arithmetic progressions

WORKED EXAMPLE 3 Graphing arithmetic sequences

For each of the following sequences

a 3, 9, 15, 21 …

b 20, 17, 14, 11 …

- **i** explain why it's an arithmetic sequence
- **ii** find u_2 and u_5
- **iii** make a table showing all the values of u_n up to $n = 6$
- **iv** sketch a graph of the table of values in part **iii**
- **v** find the slope of the straight line created by joining the points and comment on the result.

WB p. 25

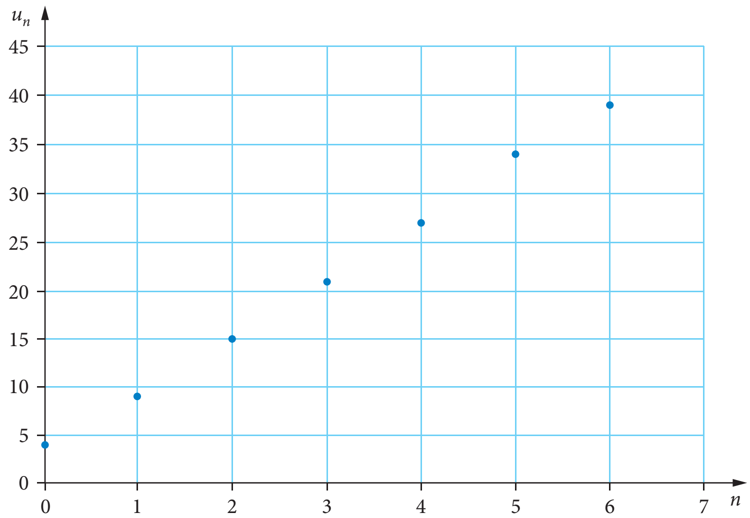
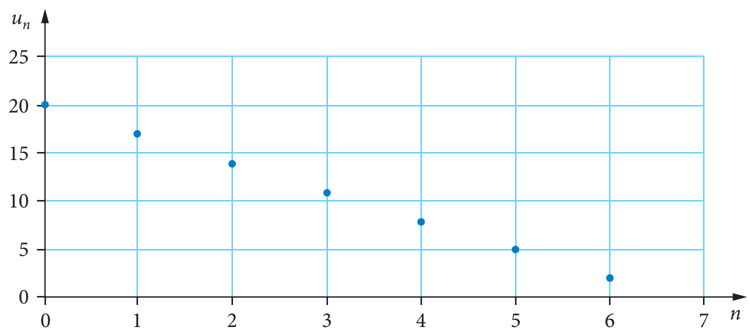
Steps		Working
a	**i** Is a fixed amount being added or subtracted each time?	6 is being added to generate each new value, so this is an arithmetic sequence.
	ii Extend the sequence by continuing the rule if necessary.	3, 9, 15, 21, 27, 33 … $u_2 = 15$, $u_5 = 33$
	iii List n in the first row of the table and u_n in the second. Find u_n for $n = 0$ to 6.	see table below
	iv Sketch the table of values with n on the horizontal axis and u_n on the vertical axis.	
	v Use any two points to calculate the slope and compare it to the sequence. Often the first two points are the easiest to use.	(0, 3) and (1, 9) $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{9-3}{1-0} = \frac{6}{1} = 6$ The slope is the amount being added to generate each new value.
b	**i** Is a fixed amount being added or subtracted each time?	3 is being subtracted to generate each new value, so this is an arithmetic sequence.
	ii Extend the sequence by continuing the rule if necessary.	20, 17, 14, 11, 8, 5 … $u_2 = 14$, $u_5 = 5$
	iii List n in the first row of the table and u_n in the second. Find u_n for $n = 0$ to 6.	see table below
	iv Sketch the table of values with n on the horizontal axis and u_n on the vertical axis.	
	v Use any two points to calculate the slope and compare it to the sequence. Often the first two points are the easiest to use.	(0, 20) and (1, 17) $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{17-20}{1-0} = \frac{-3}{1} = -3$ Here we are adding −3. The slope is the amount being added to generate each new value.

Table for a iii:

n	0	1	2	3	4	5	6
u_n	3	9	15	21	27	33	39

Table for b iii:

n	0	1	2	3	4	5	6
u_n	20	17	14	11	8	5	2

9780170448192

EXERCISE 2.1 Sequences

ANSWERS p. 494

2.1

Mastery

1 WORKED EXAMPLE 1 For each of the following sequence rules

i find the first five values

ii describe in words how each new value is generated.

a $u_n = 4 + 5n$, where $n = 0, 1, 2 \ldots$

b $u_n = 1.5 + 0.5n$, where $n = 0, 1, 2 \ldots$

c $u_n = 2 - n$, where $n = 0, 1, 2 \ldots$

d $V_n = 3^n$, where $n = 0, 1, 2 \ldots$

e $T_n = 20 - 5n$, where $n = 0, 1, 2 \ldots$

f $u_n = (-3)^n$, where $n = 0, 1, 2 \ldots$

2 WORKED EXAMPLE 2 Identify whether each of the following are increasing, decreasing, constant, oscillating or limiting value sequences, giving more than one option where relevant.

a

n	0	1	2	3	4	…
u_n	6	6	6	6	6	…

b

n	0	1	2	3	4	…
u_n	12	22	32	42	52	…

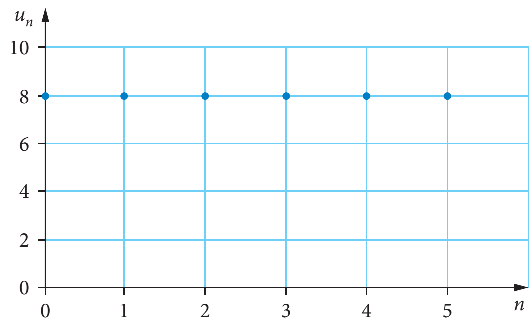
c

n	0	1	2	3	4	…
u_n	11	5	−1	−7	−13	…

d

n	0	1	2	3	4	…
u_n	−1	3	−9	27	−81	…

e



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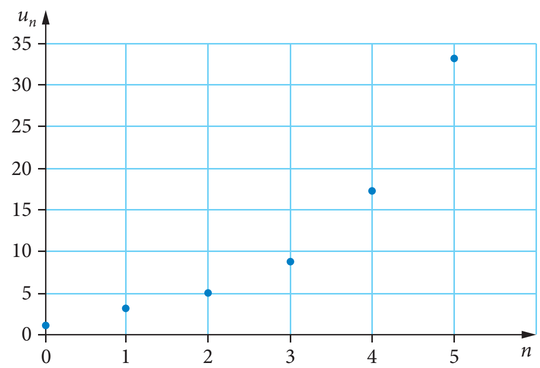

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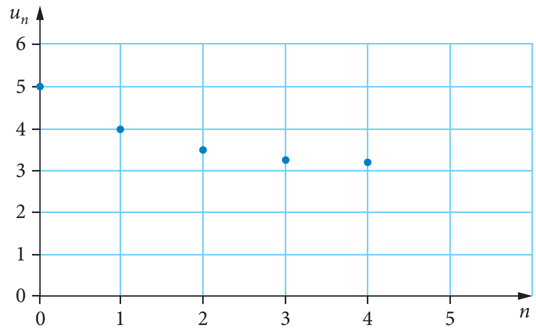

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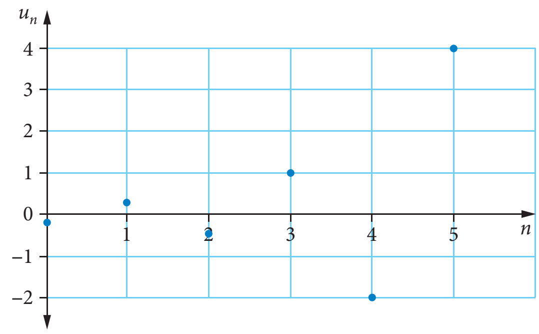


3 WORKED EXAMPLE 3 For each of the following sequences

a 105, 95, 85, 75 …

b 1, 7, 13, 19 …

i explain why it's an arithmetic sequence

ii find u_2 and u_5

iii make a table showing all the values of u_n up to $n = 6$

iv sketch a graph of the table of values in part **iii**

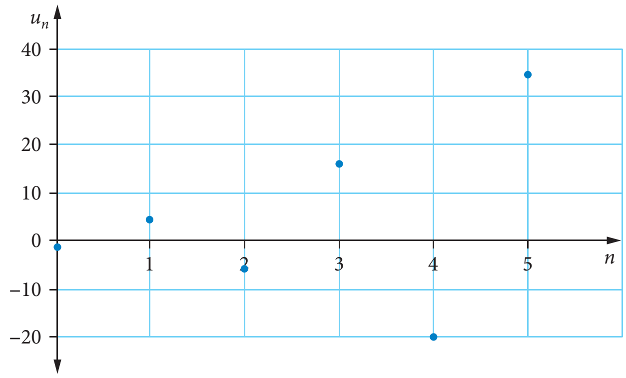
v find the slope of the straight line created by joining the points and comment on the result.

4 The 7th value of the arithmetic sequence 0, −2, −4, −6 … is

A −12 **B** −10 **C** −6 **D** −5 **E** 0

5 Which of these types of sequences could be represented by this graph?

A increasing **B** decreasing **C** constant **D** oscillating **E** limiting value

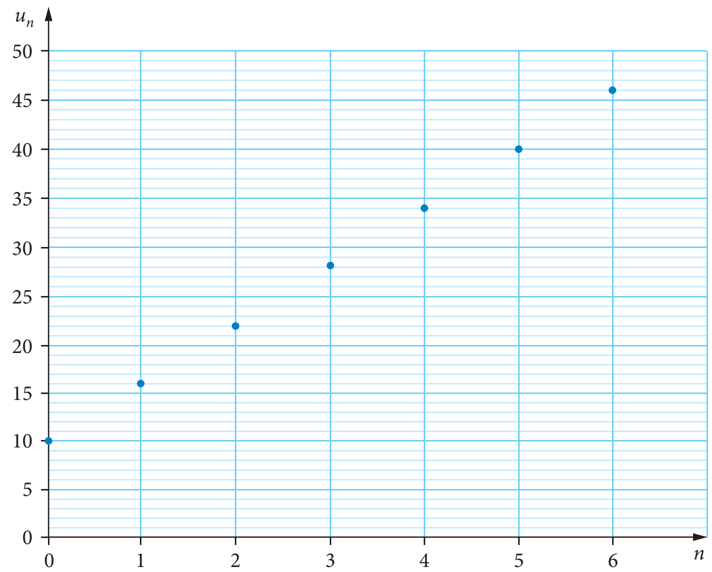


6 The first three values in the sequence with the rule $u_n = 4 - 5n$, where $n = 0, 1, 2 \ldots$ are

A 4, 9, 14 **B** 4, −1, −6 **C** −1, −6, −11 **D** 4, −9, 14 **E** 4, −5, −9

Use the following information to answer the next three questions.

The graph of an arithmetic sequence with values u_n, where $n = 0, 1, 2 \ldots$ is shown.



7 What is the value u_2?

A 2 **B** 16 **C** 20 **D** 21 **E** 22

8 What amount is being added to generate each new value?

A 3 **B** 6 **C** 8 **D** 10 **E** 16

9 If a straight line is drawn through the points, what would be the slope of the line?

A 6 **B** 8 **C** 10 **D** 16 **E** 46

10 Which of the following rules generates the sequence 1, −5, 25, −125 … ?

A $u_n = 1 + 5n$, where $n = 0, 1, 2 \ldots$

B $u_n = 1 - 5n$, where $n = 0, 1, 2 \ldots$

C $u_n = -1 + 5n$, where $n = 0, 1, 2 \ldots$

D $u_n = 5^n$, where $n = 0, 1, 2 \ldots$

E $u_n = (-5)^n$, where $n = 0, 1, 2 \ldots$

11 (6 marks) For the following sequence

n	0	1	2	3	4	5	6	7	…
u_n	10	6	4	3	2.5	2.25	2.125	2.0625	…

a explain why it is not an arithmetic sequence — 1 mark

b find u_8 given that each new value is generated by dividing the previous value by 2 and adding 1 — 1 mark

c sketch the graph of the table of values given — 2 marks

d state the value this sequence is tending towards — 1 mark

e give the name of this type of sequence. — 1 mark

Exam hack

If you are not told to round an answer, give the unrounded answer.

2.2 Arithmetic recurrence relations

Video playlist Arithmetic recurrence relations

Worksheet First-order difference equations

Recurrence relations

The sequences we will be looking at can be written as a **recurrence relation**. A recurrence relation tells us how a particular value in a sequence can be found from the previous value in the same sequence. It consists of two equations:

1 An equation telling us the first value of the series, a.

2 An equation telling us how to find a value if we know the previous value.

Calculations like this where we continually use the previous value to generate the next value are called **recursive computation** (or **recursion**).

To find the recurrence relation for an arithmetic sequence, we need to know the **common difference**, d, between the values, which is the fixed amount that is being added to generate each new value.

For example, for the arithmetic sequence 3, 8, 13, 18 …

$a = 3$ and $d = 5$

This sequence has the recurrence relation:

$u_0 = 3, \quad u_{n+1} = u_n + 5$

This recurrence relation is saying:

1 start with 3

2 add 5 to each value to generate the next value

3 keep going.

It generates the sequence in this way:

$u_0 = 3$

$u_1 = u_0 + 5 = 3 + 5 = 8$

$u_2 = u_1 + 5 = 8 + 5 = 13$

$u_3 = u_2 + 5 = 13 + 5 = 18$

and so on.

The arithmetic sequence recurrence relation

The arithmetic sequence $u_0, u_1, u_2, u_3 \ldots$ has the recurrence relation

$u_0 = a, \quad u_{n+1} = u_n + d$

where

u_{n+1} is the value after u_n

a is the first value

d is the common difference between values.

WB
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WORKED EXAMPLE 4 Finding arithmetic sequences from recurrence relations

For the recurrence relation

$u_0 = 5, \quad u_{n+1} = u_n - 2$

find

a a and d

b the first four values of the arithmetic sequence generated by the recurrence relation, showing all the calculation steps.

Steps	Working
a Use $u_0 = a$, $u_n + 1 = u_n + d$ where a is the first value d is the common difference between values.	$a = 5$ $d = -2$
b **1** The first part of the recurrence relation gives us u_0.	$u_0 = 5$
2 The second part of the recurrence relation tells us what to add or subtract to generate the next value.	$u_1 = u_0 - 2 = 5 - 2 = 3$
3 Repeat for the next two values.	$u_2 = u_1 - 2 = 3 - 2 = 1$ $u_3 = u_2 - 2 = 1 - 2 = -1$
4 Write down the first four values of the sequence.	The first four values of the sequence are 5, 3, 1, −1.

USING CAS 1 Generating sequences through recursive computation

Find the first six values of the sequences defined by each of the following recurrence relations using repeated steps and state whether it is an arithmetic sequence or not.

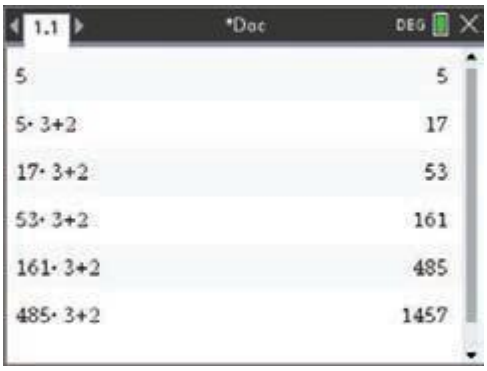
a $u_0 = 12, \quad u_{n+1} = u_n - 4$ **b** $u_0 = 5, \quad u_{n+1} = 3u_n + 2$

TI-Nspire

a

1 Start a new document and add a **Calculator** page.
2 Enter **12** then press **enter**.
3 Enter **–4** then press **enter**.
4 Continue to press **enter** until the first six values are displayed.

b

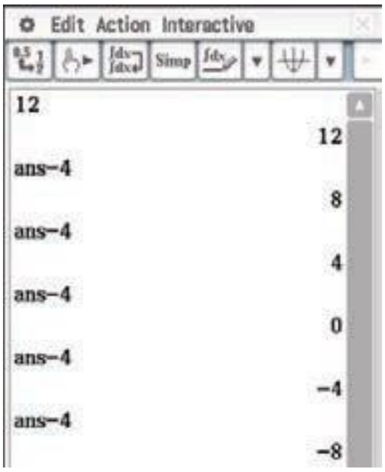


1 Press **menu** > **Actions** > **Clear History** to remove the previous calculations.
2 Enter **5** then press **enter**.
3 Enter ×**3** + **2** then press **enter**.
4 Continue to press **enter** until the first six values are displayed.

a This is an arithmetic sequence. **b** This is not an arithmetic sequence.

ClassPad

a



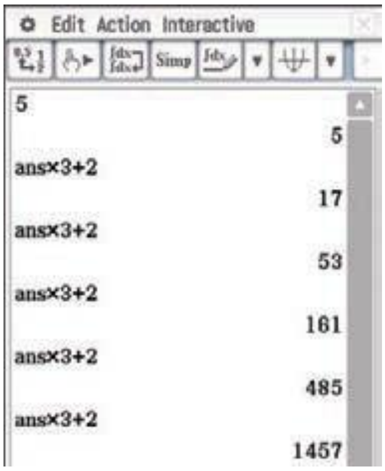
1 Tap **Main** and clear all entries.

2 Enter **12** then press **EXE**.

3 Enter **–4** then press **EXE**.

4 Continue to press **EXE** until the first six values are displayed.

b



1 Clear all entries.

2 Enter **5** then press **EXE**.

3 Enter ×**3** + **2** then press **EXE**.

4 Continue to press **EXE** until the first six values are displayed.

a This is an arithmetic sequence.

b This is not an arithmetic sequence.

The *n*th value arithmetic sequence rule

An ***n*th value rule** is a rule to find any value of a sequence without listing all the values up to the one we want.

The arithmetic sequence *n*th value rule

The arithmetic sequence u_0, u_1, u_2, u_3 … has the *n*th value rule

$u_n = a + nd$

where

u_n is the ***n*th value**

a is the first value

d is the common difference between values.

Exam hack

Don't confuse a rule with a recurrence relation. Rules have $u_n =$, whereas recurrence relations have $u_{n+1} =$.

WB

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WORKED EXAMPLE 5 Finding the *n*th value of an arithmetic sequence

For each of the following sequences

i explain how we know that it is an arithmetic sequence

ii find the *n*th value rule

iii use the rule to find u_{20} and u_{100}.

Steps	Working
a 2, 7, 12, 17 …	
i Are we adding or subtracting a fixed amount to generate each new value?	We are adding 5 to generate each new value, so this is an arithmetic sequence.
ii 1 Find the first value a and the common difference d.	$a = 2, d = 5$
2 Substitute the values for a and d into the nth value rule for arithmetic sequences $u_n = a + nd$ and simplify.	$u_n = a + nd$ $u_n = 2 + n \times 5 = 2 + 5n$ The rule is $u_n = 2 + 5n$.
iii Substitute the values for n into the nth value rule of the sequence, $u_n = a + nd$.	$u_n = 2 + 5n$ $u_{20} = 2 + 5 \times 20 = 2 + 100 = 102$ $u_{100} = 2 + 5 \times 100 = 2 + 500 = 502$
b $u_0 = 120, \quad u_{n+1} = u_n - 6$	
i Are we adding or subtracting a fixed amount to generate each new value?	We are subtracting 6 to generate each new value, so this is an arithmetic sequence.
ii 1 Find the first value a and the common difference d.	$a = 120, d = -6$
2 Substitute the values for a and d into the nth value rule for arithmetic sequences $u_n = a + nd$ and simplify.	$u_n = a + nd$ $u_n = 120 + n \times -6 = 120 - 6n$ The rule is $u_n = 120 - 6n$.
iii Substitute the values for n into the nth value rule of the sequence, $u_n = a + nd$.	$u_n = 120 - 6n$ $u_{20} = 120 - 6 \times 20 = 120 - 120 = 0$ $u_{100} = 120 - 6 \times 100 = 120 - 600 = -480$

Exam hack

Always check if the question is asking you to use a recurrence relation or a rule.

EXERCISE 2.2 Arithmetic recurrence relations

ANSWERS p. 495

Recap

1 The value u_6 in the sequence −3, 1, 5, 9 … is

A 7 **B** 12 **C** 13 **D** 17 **E** 21

2 How many of the following show arithmetic sequences?

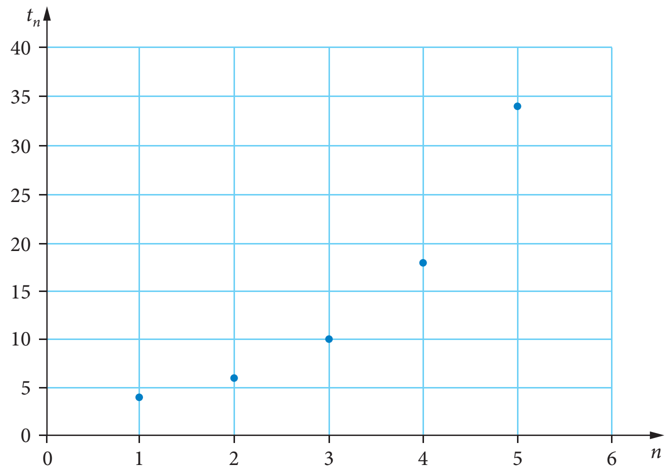
i 100, 95, 90 …

ii 2, 4, 8, 16 …

iii

n	1	2	3	4	5
u_n	1	7	13	19	25

iv



A 0 **B** 1 **C** 2 **D** 3 **E** 4

Mastery

3 WORKED EXAMPLE 4 For each of the following recurrence relations, find

i a and d

ii the first four values of the arithmetic sequence generated by the recurrence relation, showing all the calculation steps.

a $u_0 = 8, \quad u_{n+1} = u_n + 2$

b $u_0 = 1, \quad u_{n+1} = u_n - 3$

c $u_0 = 20, \quad u_{n+1} = u_n + 5$

d $u_0 = -3, \quad u_{n+1} = u_n - 4$

4 Using CAS 1 Find the first six values of the sequences defined by each of the following recurrence relations using repeated steps and state whether it is an arithmetic sequence or not.

a $u_0 = 15, \quad u_{n+1} = u_n - 2$

b $u_0 = 2, \quad u_{n+1} = 4u_n$

c $u_0 = -8, \quad u_{n+1} = u_n + 7$

d $u_0 = 40, \quad u_{n+1} = 0.5u_n + 2$

5 WORKED EXAMPLE 5 For each of the following sequences

i explain how we know that it is an arithmetic sequence

ii find the nth value rule

iii use the rule to find u_{10} and u_{50}.

a 5, 7, 9, 11 …

b 100, 93, 86, 79 …

c $u_0 = 20, \quad u_{n+1} = u_n - 4$

d $u_0 = 45, \quad u_{n+1} = u_n + 10$

6 For each of these arithmetic sequences, find the first value a and the common difference d.

a 11, 19, 27, 35, 43 …

b −10, −4, 2, 8, 14 …

c 31, 29, 27, 25, 23 …

d −9, −13, −17 …

7 Generate the first five values of the arithmetic sequence if

a $a = 1, d = 7$

b $a = 10, d = -6$

c $a = 9, d = 3$

d $a = -2, d = -4$

8 For each of these arithmetic sequences, state the value of u_{10} and u_{20}.

a 50, 42, 36, 28 …

b −16, −25, −34, −43 …

c 4, 15, 26, 37 …

d −20, −13, −6, 1, 8 …

9 Find a recurrence relation that generates each of the following arithmetic sequences.

a −4, −1, 2, 5 …

b 30, 21, 12, 3 …

c 1, 3.5, 6, 8.5 …

d 3.8, 2.7, 1.6, 0.5 …

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2020 1CQ21 85% The following recurrence relation can generate a sequence of numbers.

$T_0 = 10, \quad T_{n+1} = T_n + 3$

The number 13 appears in this sequence as

A T_1 **B** T_2 **C** T_3 **D** T_{10} **E** T_{13}

11 © VCAA 2006 1NPQ5 MODIFIED A recurrence relation is defined by

$u_0 = -1, \quad u_{n+1} = u_n + 5$

The sequence u_0, u_1, u_2 … is

A 5, 4, 3 …

B 4, 9, 14 …

C −1, −6, −11 …

D −1, 4, 9 …

E −1, 6, 11 …

12 The common difference, d, of the arithmetic sequence 10, 6, 2, −2 … is

A −4 **B** −2 **C** 4 **D** 6 **E** 10

13 For the arithmetic sequence described by the recurrence relation

$u_0 = 3, \quad u_{n+1} = u_n + 12$

A $d = 12, u_2 = 18$

B $d = 12, u_2 = 27$

C $d = 3, u_2 = 12$

D $d = 3, u_2 = 36$

E $d = 12, u_2 = 15$

14 Which of the following recurrence relations generates an arithmetic sequence?

A $V_0 = 1, \quad V_{n+1} = 3V_n - 2$

B $V_0 = 1, \quad V_{n+1} = 5V_n$

C $V_0 = 1, \quad V_{n+1} = 6V_n - 4$

D $V_0 = 1, \quad V_{n+1} = V_n + 3$

E $V_0 = 1, \quad V_{n+1} = 7V_n$

15 A sequence is defined by

$u_0 = 8, \quad u_{n+1} = u_n + 5$

The rule for the nth value is

A $u_n = 5n + 3$

B $u_n = 5n - 1$

C $u_n = 8n + 5$

D $u_n = n + 5$

E $u_n = 5n + 8$

16 (6 marks) A sequence is defined by the recurrence relation

$u_0 = 100, \quad u_{n+1} = u_n - 40$

a State the type of sequence defined by this recurrence relation, giving a reason for your answer. 1 mark

b State the common difference for this sequence. 1 mark

c Make a table of values for this sequence showing the first four values of the sequence. 1 mark

d Find a rule for the nth value of the sequence in simplified form. 1 mark

e Use the nth value rule to show a calculation for the 48th value of the sequence after the first value. 2 marks

2.3 Simple interest recurrence relations

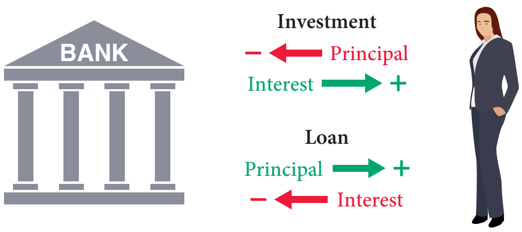
Recurrence relations can be used to model financial application such as investments, loans and depreciation.

Simple interest investments and loans

Interest is the fee for using someone else's money. It applies to both investments and loans.

- For investments, the bank uses our money and pays us the interest.
- For loans, we use the bank's money and we pay the bank the interest.

The amount that is invested or borrowed is called the **principal**.



Money coming *to the person* is positive.

Money going *away from the person* is negative.

Video playlist Simple interest recurrence relations

Simple interest is a fixed amount of interest that is paid at regular time periods. When these time periods are years, we use the term **per annum** (p.a.), which means per year.

We can use a recurrence relation based on the arithmetic sequence recurrence relation to show how simple interest works for investments and loans. The starting value in the recurrence relation is the principal. The value of the investment or loan at any time is called the **balance**.

Loans involve extra payments to the bank in addition to the interest payments. We will focus on investments in this chapter.

Simple interest recurrence relations

The recurrence relation for the value u_n of a simple interest **investment** after n years is

$$u_0 = \text{principal}, \quad u_{n+1} = u_n + d$$

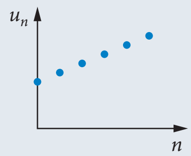
where

$d = \frac{r}{100} \times u_0$ is the fixed amount of interest paid each year

r = the percentage interest rate per year

n = the number of years.

The graph of u_n would look like this.



Total amount of interest after n years = $u_n - u_0$.

To find r use:

$$r = \frac{d}{u_0} \times 100\%$$

WB
p. 29

WORKED EXAMPLE 6 Using simple interest recurrence relations

Suri invests $4000 in an account earning 5% per annum simple interest.

Steps	Working

a What is the fixed amount of interest paid for each year?

Use $d = \frac{r}{100} \times u_0$ to find the fixed amount of interest each year. Round the answer if required.

$r = 5,\ u_0 = 4000$

$$d = \frac{r}{100} \times u_0$$
$$= \frac{5}{100} \times 4000$$
$$= 200$$

The fixed amount of interest paid for each year is $200.

b Copy and complete the following table to find

i Suri's bank account balance after four years

ii the first year when her balance is greater than $4500

iii the total amount of interest earned after five years.

n	Account balance after n years ($)
0	4000
1	4000 + =
2	+ =
3	+ =
4	+ =
5	+ =

Complete the table by using CAS recursive computation to find the bank account balance after five years.

n	Account balance after n years ($)
0	4000
1	4000 + 200 = 4200
2	4200 + 200 = 4400
3	4400 + 200 = 4600
4	4600 + 200 = 4800
5	4800 + 200 = 5000

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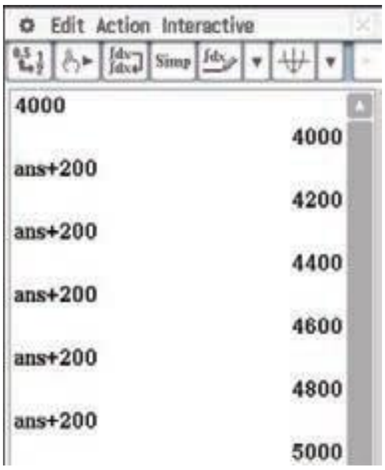
ClassPad



i Read the answer from the table.

Suri's bank account balance after four years is $4800.

ii Read the answer from the table.

Suri's balance is first greater than $4500 after three years.

iii Total amount of interest earned after n years $= u_n - u_0$.

Total amount of interest earned after five years is

$u_5 - u_0 = 5000 - 4000$

$= \$1000$

c Write a recurrence relation, u_n, for the account balance for n years.

Identify the starting value. Each value is calculated by adding d to the previous value.

Let u_n = the account balance after n years.

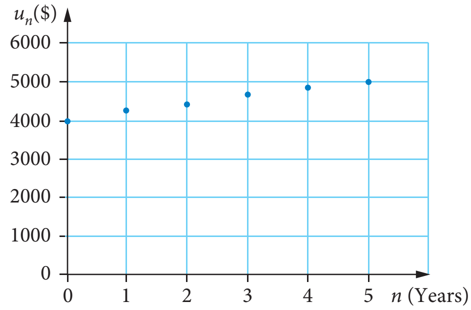
The recurrence relation is

$u_0 = 4000, \quad u_{n+1} = u_n + 200$

d Sketch the graph of the recurrence relation up to $n = 5$ and describe the line made by joining the points.

The horizontal axis is n (Years) and the vertical axis is u_n (\$).

Plot the values from the table.



The points form an increasing straight line.

WORKED EXAMPLE 7 Using recursive computation for simple interest

Find following balances using CAS recursive computation.

a Elton invests \$3500 in a bank account earning 3% per annum simple interest.

b Edwina takes out a simple interest loan of \$3500 from a bank and makes monthly payments of \$105. What is the balance after five months?

Steps	**Working**

a 1 Use $d = \frac{r}{100} \times u_0$ to find the fixed amount of interest for each year. Add d for an investment and subtract d for a loan.

$r = 3, u_0 = 3500$

$d = \frac{r}{100} \times u_0$

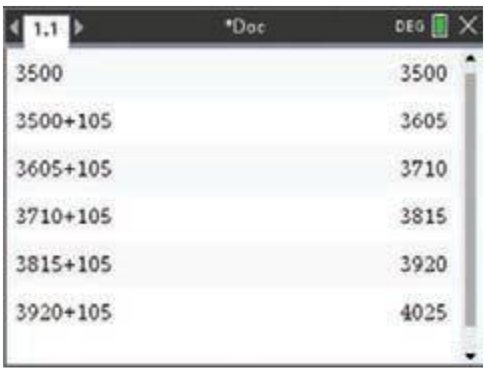
$= \frac{3}{100} \times 3500$

$= \$105$

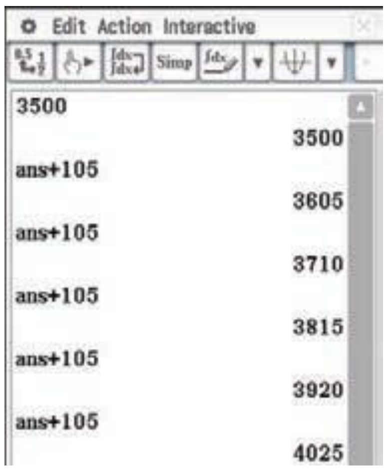
This is an investment, so we *add* \$105.

2 Use CAS recursive computation to find the balance after five years.

TI-Nspire



ClassPad



3 Write the answer.

The balance after five years is \$4025.

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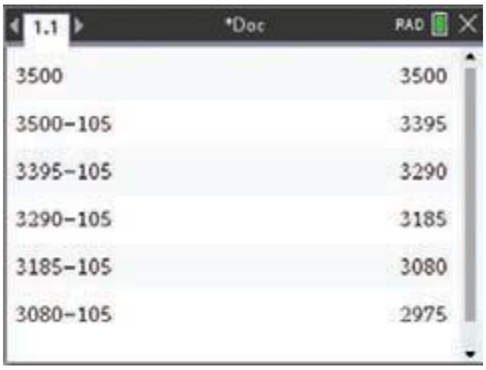
b **1** Add for an investment and subtract for a loan. This is a loan, so we *subtract* \$105.

2 Use CAS recursive computation to find the balance after five years.

TI-Nspire



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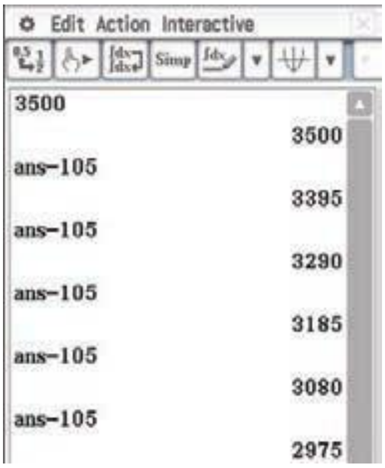

3 Write the answer. The balance after five months is \$2975.

Simple interest general rule

Recurrence relations can help us to understand how financial situations work but the following general rules can be used to solve problems more quickly.

Simple interest general rule

The general rule for the value u_n after n years of a simple interest investment is

$$u_n = u_0 + nd$$

where

u_0 = principal

$d = \frac{r}{100} \times u_0$ is the fixed amount of interest paid each year

r = the percentage interest rate per year

n = the number of years.

When solving for n, always round *up*, never down, to the nearest whole number.

WORKED EXAMPLE 8	Using the simple interest general rule
Elouise invests \$7000 in an account earning 5% p.a. simple interest.	
Steps	**Working**
a Find the fixed amount of interest paid each year.	
Find the value of d, using $d = \frac{r}{100} \times u_0$, rounding the answer if required.	$r = 5, u_0 = 7000$ $d = \frac{r}{100} \times u_0$ $= \frac{5}{100} \times 7000$ $= 350$ The fixed amount of interest paid each year is \$350.
b Write a rule that will calculate the balance of the account after n years.	
Decide if this is an investment or a loan	This is an investment so use $u_n = u_0 + nd$: $d = 350, u_0 = 7000$ $u_n = u_0 + nd = 7000 + 350n$
c Use the rule to find the balance of the account after seven years.	
Substitute the value of n into the rule.	$n = 7$ Substituting into $u_n = 7000 + 350n$: $u_7 = 7000 + 350 \times 7$ $= 9450$ The balance of Elouise's account after seven years is \$9450.

2.3

WB

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Exam hack

Remember, the general rule is *not* a recurrence relation. It *doesn't* link one value to the next. If a question asks you to answer using a recurrence relation, you can't answer it with a rule.

EXERCISE 2.3 Simple interest recurrence relations

ANSWERS p. 496

Recap

1 A recurrence relation is defined by

$u_0 = -10, \quad u_{n+1} = u_n + 6$

The sequence $u_0, u_1, u_2 \ldots$ it generates is

A 10, 16, 22 … **B** −10, −4, 2 … **C** 6, −16, −26 …

D −10, −16, −22 … **E** 10, 4, −2 …

2 Which of the following recurrence relations **doesn't** generate an arithmetic sequence?

A $u_0 = 4, \quad u_{n+1} = u_n - 2$ **B** $u_0 = 1, \quad u_{n+1} = 5u_n$ **C** $u_0 = -1, \quad u_{n+1} = u_n - 4$

D $u_0 = 8, \quad u_{n+1} = u_n + 3$ **E** $u_0 = 3, \quad u_{n+1} = u_n + 3$

Mastery

3 WORKED EXAMPLE 6 Lexi invests $3000 in an account earning 4% p.a. simple interest.

a What is the fixed amount of interest paid for each year?

b Copy and complete the following table to find

i Lexi's bank account balance after four years

ii the first year in which the balance is greater than $3250

iii the total amount of interest earned after four years.

n	Account balance after n years ($)
0	3000
1	3000 + =
2	+ =
3	+ =
4	+ =

c Write a recurrence relation, u_n, for the account balance after n years.

d Sketch the graph of the recurrence relation up to $n = 5$ and describe the pattern made by joining the points.

4 WORKED EXAMPLE 7 Find the balance after four years for each of the following using CAS recursive computation.

a Bella invests $4800 in a bank account earning 2% per annum simple interest.

b Barry takes out a simple interest loan of $4800 from a bank and makes monthly payments of $96.

5 WORKED EXAMPLE 8 Hugh invests $11 000 in an account earning 3% p.a. simple interest.

a Find the fixed amount of interest paid each year.

b Write a rule that will calculate the balance of the account after n years.

c Use the rule to find the balance of the account after eight years.

6 If u_n is the simple interest account balance after n years, for each of the following

i write the recurrence relation for the account balance

ii write the rule for the account balance

iii find the balance after nine years using the rule.

a $9000 invested at 12% p.a.

b $6500 invested at 4% p.a.

c $8400 invested at 3% p.a.

d $7000 invested at 11% p.a.

Exam practice

80–100% 60–79% 0–59%

7 $48 000 is invested at a simple interest rate of 4% per annum. Which of the following recurrence relations can be used to model the balance of the investment after n years?

A $u_0 = 48\,000, \quad u_{n+1} = u_n + 192$

B $u_0 = 48\,000, \quad u_{n+1} = u_n - 1920$

C $u_0 = 48\,000, \quad u_{n+1} = u_n + 1920$

D $u_0 = 48\,000, \quad u_{n+1} = u_n + 19\,200$

E $u_0 = 48\,000, \quad u_{n+1} = u_n - 192$

8 Which of the following rules could be used to find the balance after n years of a $14 000 investment at a simple interest rate of 2% p.a.?

A $u_0 = 14\,000, \quad u_{n+1} = u_n + 280$

B $u_n = 14\,000 - 280 \times n$

C $u_n = 13\,920 + 280 \times n$

D $u_n = 14\,280 + 280 \times n$

E $u_n = 14\,000 + 280 \times n$

9 © VCAA 2006 1BRMQ1 91% $4000 is invested at a simple interest rate of 5% per annum. The amount of interest earned in the first year is

A $20 **B** $200 **C** $220 **D** $420 **E** $2000

10 © VCAA 2012 1BRMQ2 90% $3000 is invested at a simple interest rate of 6.5% per annum. The total interest earned in three years is

A $195.00 **B** $580.50 **C** $585.00 **D** $3585.00 **E** $3623.85

11 © VCAA 2004 1BRMQ1 90% Sarah invests $37 000 at a simple interest rate of 4% per annum. The total amount of interest earned in two years is

A $1480 **B** $2960 **C** $5920 **D** $38 480 **E** $39 960

12 (4 marks) Stephen has an investment of $10 000 at a simple interest rate of 4% p.a.

a Write a recurrence relation for the balance of this investment after n years. 1 mark

b Describe what the shape of the graph of the balance will be? 1 mark

c Find the rule for the value of the investment after n years. 1 mark

d What is the balance of the investment after four years? 1 mark

13 (5 marks) Ashwini borrows $5000 from an unauthorised lender and is unable to pay it back. After missing her first monthly payment, she is charged another $3000 for every month after the first missed payment. She continues to miss her payments.

a Write the values of the sequence for the money she owes after each of the first three months. 1 mark

b What is the name of this type of sequence. 1 mark

c If A_n is the amount of money she owes after n months, write a recurrence relation that generates the sequence. 1 mark

d Write the rule for A_n which calculates the amount of money she owes after n months. 1 mark

e How much does Ashwini owe after five months if she continues to be unable to pay the money back? 1 mark

2.4 Depreciation recurrence relations

Video playlist Depreciation recurrence relations

Depreciation

Businesses need to purchase items to help them function, such as machines, equipment and computers. These items are called **assets**. Most assets used by businesses decrease in value over time. We use the term **depreciation** to describe this decrease in value. Depreciation occurs due to age, amount of use or lack of demand. The estimate of the value of an asset at any point in time is called the **future value**.

We will be looking at two ways of depreciating assets in this chapter:

1 Flat rate depreciation where the value decreases with age

2 Unit cost depreciation where the value decreases with use

Flat rate depreciation recurrence relations

Flat rate depreciation calculates the future value of an asset by reducing the value every year by a fixed amount. The amount is usually given as a fixed percentage of the purchase price. Flat rate depreciation is the same as a simple interest loan where we subtract a fixed amount each time period.

Flat rate depreciation recurrence relation

The recurrence relation for the value of an asset u_n after n years using flat rate depreciation is

u_0 = initial value of the asset, $\quad u_{n+1} = u_n - d$

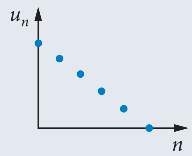
where

$d = \frac{r}{100} \times u_0$ is the fixed amount of depreciation each year

r = the percentage depreciation rate per year

n = the number of years.

The graph will look like this:



Total amount of depreciation after n years = $u_n - u_0$.

To find r use:

$$r = \frac{d}{u_0} \times 100\%$$

WB

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WORKED EXAMPLE 9 Using flat rate depreciation recurrence relations

A small industrial truck is purchased by a business for $10 500. Its value depreciates at a flat rate of 20% each year.

Steps | **Working**

a What is the fixed amount of depreciation each year?

Use $d = \frac{r}{100} \times u_0$ to find the fixed amount of depreciation each year.

$r = 20,\ u_0 = 10\,500$

$$d = \frac{r}{100} \times u_0$$
$$= \frac{20}{100} \times 10\,500$$
$$= \$2100$$

b Copy and complete the following table to find

i the value of the industrial truck after two years

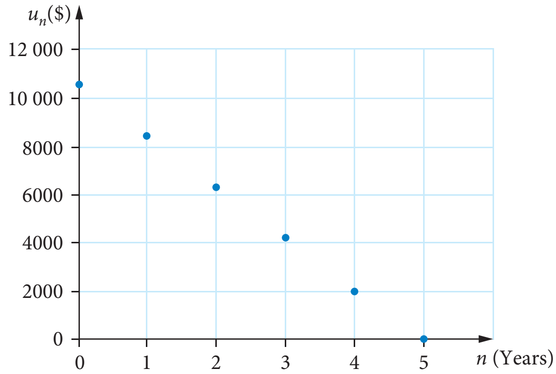
ii when the value of the industrial truck first falls below $4500

iii when the industrial truck depreciates to zero.

n	Value after n years ($)
0	10 500
1	10 500 – =
2	– =
3	– =
4	– =
5	– =

1 Complete the table by using CAS recursive computation to find the value of the asset after five years.

n	Value after n years ($)
0	10 500
1	10 500 – 2100 = 8400
2	8400 – 2100 = 6300
3	6300 – 2100 = 4200
4	4200 – 2100 = 2100
5	2100 – 2100 = 0

2 **i** Read the answer from the table.	The value of the industrial truck after two years is \$6300.
ii Read the answer from the table.	The value of the industrial truck first falls below \$4500 after three years.
iii Read the answer from the table.	The industrial truck depreciates to zero after five years.
c Write a recurrence relation for the value of the industrial truck.	
Identify the initial value of the asset, u_0. Each value is calculated by subtracting d from the previous value.	Let u_n = value of the industrial truck after n years. The recurrence relation is $u_0 = 10\,500, \quad u_{n+1} = u_n - 2100$
d Sketch the graph of the recurrence relation up to $n = 5$.	
The horizontal axis is n (Years) and the vertical axis is u_n (\$). Plot the values from the table.	

WORKED EXAMPLE 10 Finding the rate for flat rate depreciation

Green Acres Gardening Service owns a mulcher that is depreciated in value using flat rate depreciation. The value of the mulcher, in dollars, after n years, u_n, can be modelled by the recurrence relation

$$u_0 = 120\,000, \qquad u_{n+1} = u_n - 18\,000$$

a By what amount, in dollars, does the value of the mulcher decrease each year?

b Showing recursive calculations, determine the value of the mulcher, in dollars, after two years.

c What annual flat rate percentage of depreciation is used by Green Acres Gardening Service?

Steps	**Working**
a Use the recurrence relation $u_{n+1} = u_n - d$ to identify d.	$d = 18\,000$ The mulcher decreases by \$18 000 each year.
b Show the stepped calculations for u_1 and u_2 from the recurrence relation.	$u_1 = 120\,000 - 18\,000 = 102\,000$ $u_2 = 102\,000 - 18\,000 = 84\,000$
c **1** Identify what we know and what we need to find.	$d = 18\,000,\ u_0 = 120\,000,\ r = ?$
2 Substitute the values into $r = \dfrac{d}{u_0} \times 100\%$ and evaluate.	$r = \dfrac{18\,000}{120\,000} \times 100\%$ $= 15$
3 Write the answer.	The annual flat rate of depreciation is 15%.

2.4

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Flat rate depreciation general rule

The flat rate depreciation general rule is the same as the simple interest loan rule, where we subtract a fixed amount each year.

Flat rate depreciation general rule

The general rule for the value u_n of an asset after n years using flat rate depreciation is

$$u_n = u_0 - nd$$

where

u_0 = initial value of the asset

$d = \frac{r}{100} \times u_0$ is the fixed amount of depreciation each year

r = the percentage depreciation rate per year

n = the number of years.

When solving for n, always round *up*, never down, to the nearest whole number.

The last year of depreciation often involves a partial amount.

WB

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WORKED EXAMPLE 11 Using the flat rate depreciation general rule

A company buys a building for $2 000 000 and depreciates it at a flat rate of 3% per year.

Steps	Working
a Find the fixed amount of depreciation each year.	
Use $d = \frac{r}{100} \times u_0$ to find the fixed amount of depreciation each year.	$r = 3,\ u_0 = 2\,000\,000$ $d = \frac{r}{100} \times u_0$ $= \frac{3}{100} \times 2\,000\,000$ $= 60\,000$ The fixed amount of depreciation paid each year is \$60 000.
b Write a rule that will calculate the value of the building after n years.	
Substitute the values of d and u_n into the flat rate depreciation general rule: $u_n = u_0 - nd$.	$u_0 = 2\,000\,000,\ d = 60\,000$ $u_n = u_0 - nd = 2\,000\,000 - n \times 60\,000$ The rule is $u_n = 2\,000\,000 - 60\,000n$
c Use the rule to find the value of the building after nine years.	
Substitute the value of n into the rule.	$n = 9$ Substituting into $u_n = 2\,000\,000 - 60\,000n$ $u_9 = 2\,000\,000 - 60\,000 \times 9$ $= 1\,460\,000$ The value of the building after nine years is \$1 460 000.

d Use the rule to find how many years it would take for the building to depreciate to zero.	
1 Substitute the known values into $u_n = u_0 - nd$. Let $u_n = 0$.	$u_0 = 2\,000\,000$, $d = 60\,000$, $u_n = 0$, $n = ?$ From part **b** the rule is: $u_n = 2\,000\,000 - 60\,000n$ $0 = 2\,000\,000 - 60\,000n$
2 Solve for n, using CAS if necessary.	$60\,000n = 2\,000\,000$ $n = \frac{2\,000\,000}{60\,000} = 33.333\ldots$
3 Write the answer. When solving for n always round *up* to the nearest year.	It would take 34 years for the building to depreciate to zero. (At 33 years it hasn't quite depreciated to zero yet.)

2.4

Unit cost depreciation recurrence relations

Unit cost depreciation occurs when an asset is depreciated according to the amount of use it has had, not according to its age. When using the unit cost method of depreciation, the amount of depreciation is determined by applying a rate per unit of use. For example, cars are often depreciated by the number of kilometres travelled, rather than by how old they are.

Unit cost depreciation recurrence relation

The recurrence relation for the value of an asset u_n after n uses using unit cost depreciation is

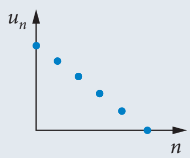
u_0 = initial value of the asset, $u_{n+1} = u_n - d$

where

d = cost per unit of use

n = the number of units of use.

The graph would look like this:



Exam hack

The key difference between unit cost and flat rate depreciation is:

- for unit cost depreciation, n is the number of *units of use*
- for flat rate depreciation, n is the number of years.

WORKED EXAMPLE 12	Using unit cost depreciation recurrence relations
A collector buys a limited edition original vinyl record of the Beatles' Hey Jude for \$28 000, which depreciates by \$4000 every time it's played.	
Steps	**Working**
a Explain why this is unit cost depreciation and not flat rate depreciation.	
1 Refer to 'use' in the answer.	The amount of depreciation is determined by applying a rate per unit of use: \$4000 every time the record is played.

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b Copy and complete the following table to find

i the value of the record after three plays

ii how many plays it will take for the value of the record to first fall below $10 000.

n	Value after n uses ($)
0	28 000
1	28 000 – =
2	– =
3	– =
4	– =
5	– =

1 Complete the table by using CAS recursive computation to find the value after five plays.

n	Value after n uses ($)
0	28 000
1	28 000 – 4000 = 24 000
2	24 000 – 4000 = 20 000
3	20 000 – 4000 = 16 000
4	16 000 – 4000 = 12 000
5	12 000 – 4000 = 8000

2 **i** Read the answer from the table.

The value of the record after three plays is $16 000.

ii Read the answer from the table.

The value of the record first falls below $10 000 after five plays.

c Write a recurrence relation for the value of the record.

Identify the initial value of the asset and the cost per unit of use.

Let u_n = value of the record after n plays.

cost per play = $4000

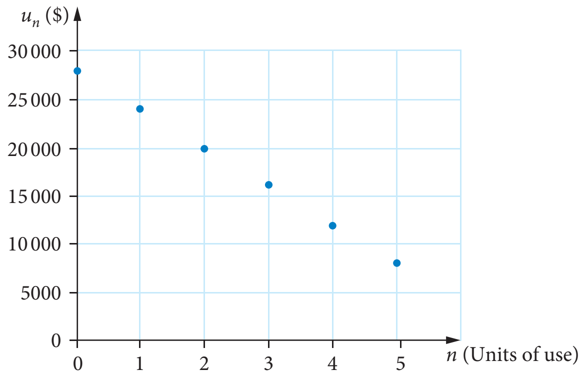
The recurrence relation is

$u_0 = 28\,000, \quad u_{n+1} = u_n - 4000$

d Sketch the graph of the recurrence relation up to $n = 5$.

The horizontal axis is n (Units of use) and the vertical axis is u_n ($).

Plot the values from the table.



Unit cost depreciation general rule

Unit cost depreciation general rule

The general rule for the value of an asset u_n using unit cost depreciation is

$$u_n = u_0 - nd$$

where

u_0 = initial value of the asset

d = cost per unit of use

n = the number of units of use.

When solving for n, always round *up*, never down, to the nearest whole number.

WORKED EXAMPLE 13 Using the unit cost depreciation general rule

A delivery van was purchased for $70 000. The van's value depreciates at a rate of 62 cents per kilometre.

Steps	Working
a Write a rule that will calculate the value of the van after n kilometres of travel.	
Substitute the values of d (in dollars) and u_0 into the unit cost depreciation general rule $u_n = u_0 - nd$.	$d = 0.62$, $u_0 = 70\,000$ $u_n = 70\,000 - 0.62n$
b Use the rule to find the value of the van after it has travelled a total distance of 30 000 kilometres.	
Substitute the value of n into the rule.	$n = 30\,000$ Substituting into $u_n = 70\,000 - 0.62n$ $u_{30\,000} = 70\,000 - 0.62 \times 30\,000$ $= 51\,400$ The value of the van after it has travelled a total distance of 30 000 kilometres is $51 400.

2.4

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VCE QUESTION ANALYSIS

© VCAA 2020 2CQ7 2020 Examination 2 Core Question 7 (4 marks)

Samuel owns a printing machine. The printing machine is depreciated in value by Samuel using flat rate depreciation. The value of the machine, in dollars, after n years, u_n, can be modelled by the recurrence relation

$$u_0 = 120\,000, \quad u_{n+1} = u_n - 15\,000$$

a By what amount, in dollars, does the value of the machine decrease each year? 1 mark

b Showing recursive calculations, determine the value of the machine, in dollars, after two years. 1 mark

c What annual flat rate percentage of depreciation is used by Samuel? 1 mark

d The value of the machine, in dollars, after n years, u_n, could also be determined using a rule of the form $u_n = a + nd$. Write down this rule for u_n. 1 mark

Video playlist VCE question analysis: Arithmetic sequences and financial recurrence relations

Reading the question

- Note when a recurrence relation is mentioned and when a rule is mentioned.
- Part **a** is asking for a dollar amount, whereas part **c** is asking for a percentage.

Thinking about the question

- Be aware that the word 'show' in a question means you need to include calculations.
- Are you clear on what 'recursive' means?

Worked solution (✓ = 1 mark)

a Use the recurrence relation $u_{n+1} = u_n - d$ to identify d.

$d = 15\,000$, so the printing machine decreases by **$15 000** ✓ each year.

b $u_1 = 120\,000 - 15\,000$ $\quad$ $u_2 = 105\,000 - 15\,000$

$= 105\,000$ $\quad$ $= 90\,000$ ✓

c Use $r = \dfrac{d}{u_0} \times 100\%$

Substituting $d = 15\,000$, $u_0 = 120\,000$ gives

$r = \dfrac{15\,000}{120\,000} \times 100\% = \mathbf{12.5\%}$ ✓

d Use the rule $u_n = u_0 - nd$

$u_n = 120\,000 - n \times 15\,000$

The rule is $\boldsymbol{u_n = 120\,000 - 15\,000n}$. ✓

Student performance 80–100% 60–79% 0–59%

a 94% This question was answered correctly by almost all students.

b 72% Students needed to show each detailed step of the two separate calculations. The final answer on its own, or a statement using the rule rather than the recurrence relation, wouldn't get a mark. Some students copied numbers incorrectly and left out zeros.

c 75%

d 44% Some students were unsure of the distinction between a rule and a recurrence relation.

EXERCISE 2.4 Depreciation recurrence relations

ANSWERS p. 497

Recap

1 © VCAA 2009 1BRMQ1 91% An amount of \$800 is invested for two years at a simple interest rate of 4% per annum. The total amount of interest earned after two years is

A \$32 **B** \$64 **C** \$160 **D** \$320 **E** \$640

2 A loan of \$31 000 is taken out at a simple interest rate of 5% per annum. Which of the following recurrence relations models the balance of the loan after n years?

A $u_0 = 31\,000,\quad u_{n+1} = u_n + 155$

B $u_0 = 31\,000,\quad u_{n+1} = u_n - 1550$

C $u_0 = 31\,000,\quad u_{n+1} = u_n + 1550$

D $u_0 = 31\,000,\quad u_{n+1} = u_n + 15\,500$

E $u_0 = 31\,000,\quad u_{n+1} = u_n - 155$

Mastery

3 WORKED EXAMPLE 9 A carpet shampooer is purchased by a cleaning business for \$8500. Its value depreciates at a flat rate of 25% each year.

a What is the fixed amount of depreciation each year?

b Copy and complete the following table to find

i the value of the carpet shampooer after three years

ii when the value of the carpet shampooer first falls below \$5000

ii when the carpet shampooer depreciates to zero.

c Write a recurrence relation for the value of the carpet shampooer.

d Sketch the graph of the recurrence relation up to $n = 4$.

n	Value after n years (\$)
0	8500
1	8500 – =
2	– =
3	– =
4	– =

4 WORKED EXAMPLE 10 High Tops Tree Lopping Service owns a crane which is depreciated in value using flat rate depreciation. The value of the crane, in dollars, after n years, u_n, can be modelled by the recurrence relation

$$u_0 = 100\,000,\quad u_{n+1} = u_n - 22\,000$$

a By what amount, in dollars, does the value of the crane decrease each year?

b Showing recursive calculations, determine the value of the crane, in dollars, after two years.

c What annual flat rate percentage of depreciation is used by High Tops Tree Lopping Service?

2.4

5 WORKED EXAMPLE 11 For each of the following

i find the fixed amount of depreciation each year

ii write a rule that will calculate the value of the asset after n years

iii use the rule to find the value of the asset after six years

iv use the rule to find how many years it would take for the asset to depreciate to zero.

a A company buys a commercial building for $1 800 000 and depreciates it at a flat rate of 4% per year.

b A business buys a car for $50 000 and depreciates it at a flat rate of 15% per year.

c A commercial dishwasher is purchased for $22 500 and is depreciated at a flat rate of 9% per year.

6 WORKED EXAMPLE 12 A collector buys a limited edition original vinyl record of Ed Sheeran's One Life for $14 000, which depreciates by $750 every time it is played.

a Explain why this is unit cost depreciation and not flat rate depreciation.

b Copy and complete the following table to find

i the value of the record after four plays

ii how many plays it will take for the value of the record to first fall below $12 000.

c Write a recurrence relation for the value of the record.

d Sketch the graph of the recurrence relation up to $n = 6$.

n	Value after n uses ($)
0	14 000
1	14 000 – =
2	– =
3	– =
4	– =
5	– =

7 WORKED EXAMPLE 13 A company car was purchased for $60 000. The car's value depreciates at a rate of 54 cents per kilometre.

a Write a rule that will calculate the value of the car after n kilometres of travel.

b Use the rule to find the value of the car after it has travelled a total distance of 40 000 kilometres.

8 For each of the following, write a recurrence relation for unit cost depreciation which includes the definition of u_n.

a A concert violin purchased for $8540 depreciates at a rate of $6 per concert.

b A truck purchased for $170 000 depreciates at a rate of $2 per kilometre.

c A van purchased for $90 000 depreciates at a rate of 84 cents per kilometre.

d A photocopier purchased for $18 000 depreciates at a rate of 1 cent per copy.

9 For each of the following

i state whether the depreciation is flat rate or unit cost

ii write a recurrence relation for u_n, stating what n represents

ii find the rule for u_n.

a A piece of jewellery is purchased for $10 000. It depreciates by $1500 every time it is worn.

b A piece of jewellery is purchased for $10 000. It depreciates by 5% of its initial value each year.

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2019 1CQ19 76% Geoff purchased a computer for $4500. He will depreciate the value of his computer by a flat rate of 10% of the purchase price per annum. A recurrence relation that Geoff can use to determine the value of the computer after n years, V_n, is

A $V_0 = 4500, \quad V_{n+1} = V_n - 450$

B $V_0 = 4500, \quad V_{n+1} = V_n + 450$

C $V_0 = 4500, \quad V_{n+1} = 0.9V_n$

D $V_0 = 4500, \quad V_{n+1} = 1.1V_n$

E $V_0 = 4500, \quad V_{n+1} = 0.1(V_n - 450)$

11 A photocopier bought for \$20 000 is depreciated at a rate of \$3000 every year. Which of the following is *not* true?

A The depreciation method used is flat rate depreciation.

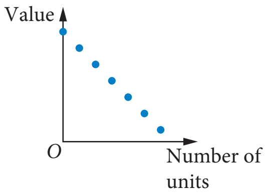
B The recurrence relation that can be used to model this is $u_0 = 20\,000$, $u_{n+1} = u_n - 3000$, where u_n is the future value of the photocopier after n years.

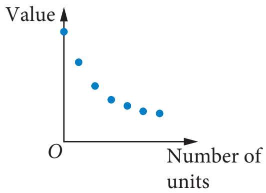
C The rule for finding the photocopier's value after n years is $u_n = u_0 - 3000n$, where u_n is the value of the photocopier after n years.

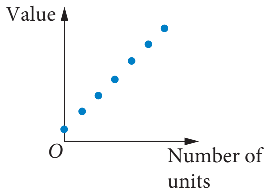
D The value of the photocopier after two years is \$17 000.

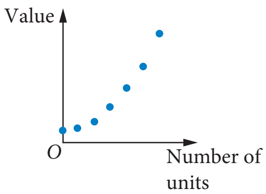
E The graph of the recurrence relation follows a straight line.

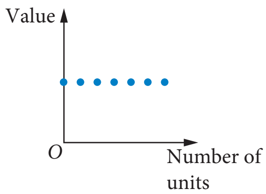
12 © VCAA 2018N 1CQ20 The value of a photocopier is depreciated using a unit cost method. Which one of the following graphs could show the value of the photocopier as it depreciates?

A


B


C


D


E


13 © VCAA 2017 2CQ5ab (4 marks) Alex is a mobile mechanic. He uses a van to travel to his customers to repair their cars. The value of Alex's van is depreciated using the flat rate method of depreciation. The value of the van, in dollars, after n years, u_n, can be modelled by the recurrence relation

$$u_0 = 75\,000, \qquad u_{n+1} = u_n - 3375$$

a 97% Recursion can be used to calculate the value of the van after two years. Copy and complete the calculations shown. 2 marks

$u_0 = 75\,000$

$u_1 = 75\,000 -$ ________ $= 71\,625$

$u_2 =$ ________ $-$ ________ $=$ ________

b **i** 96% By how many dollars is the value of the van depreciated each year? 1 mark

ii 69% Calculate the annual flat rate of depreciation in the value of the van. Write your answer as a percentage. 1 mark

14 © VCAA 2019N 2CQ6 MODIFIED (3 marks) Marlon plays guitar in a band. He paid \$3264 for a new guitar. The value of Marlon's guitar will be depreciated by a fixed amount for each concert that he plays. After 25 concerts, the value of the guitar will have decreased by \$200.

a Write a calculation that shows that the value of Marlon's guitar will depreciate by \$8 per concert. 1 mark

b The value of Marlon's guitar after n concerts, G_n, can be determined using a rule. Copy and complete the rule below by writing the appropriate numbers in the boxes.

$G_n = \square - \square \times n$ 1 mark

c What will be the value of Marlon's guitar after he has played in 30 concerts? 1 mark

 9780170448192

2 Chapter summary

Sequences

- A **sequence** is a list of numbers called **values** (or terms).
- A sequence u_n can be written in table form and graphed, where the x-axis is $n = 0, 1, 2, 3 \ldots$ and the y-axis is $u_n = u_0, u_1, u_2, u_3 \ldots$, the values of the sequence.
- The sequences looked at in this chapter can be written as a **recurrence relation** where each new value is generated from the previous value.
- Calculations that continually use the previous answer to generate the next answer are called **recursive computation**.

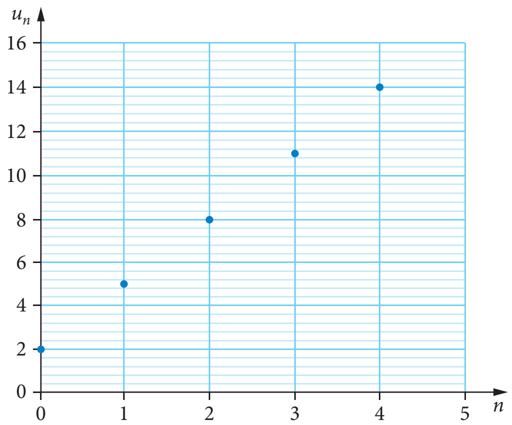
Sequence types

There are five types of sequences, as seen in the examples below.

Increasing sequence

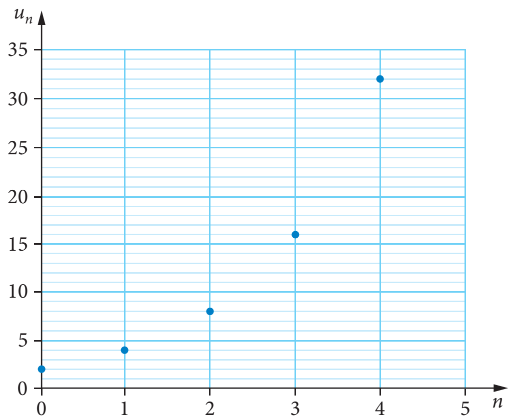
Add 3 to generate each new value.

n	0	1	2	3	4	…
u_n	2	5	8	11	14	…



Multiply by 2 to generate each new value.

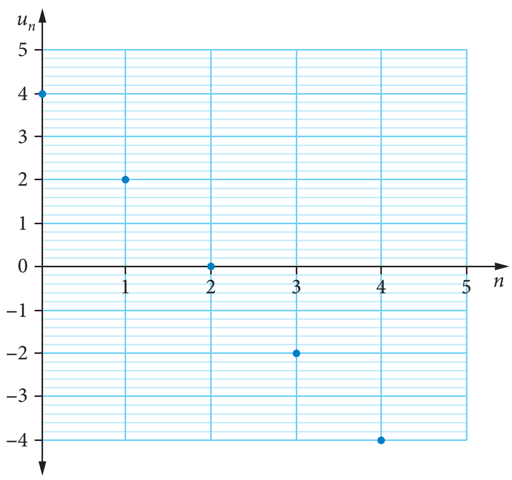
n	0	1	2	3	4	…
u_n	2	4	8	16	32	…



Decreasing sequence

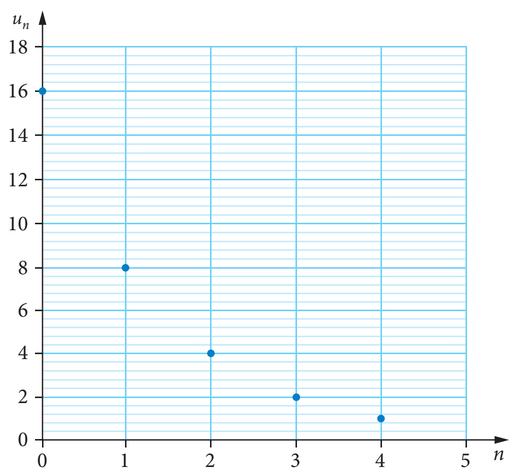
Subtract 2 to generate each new value.

n	0	1	2	3	4	…
u_n	4	2	0	–2	–4	…



Divide by 2 to generate each new value.

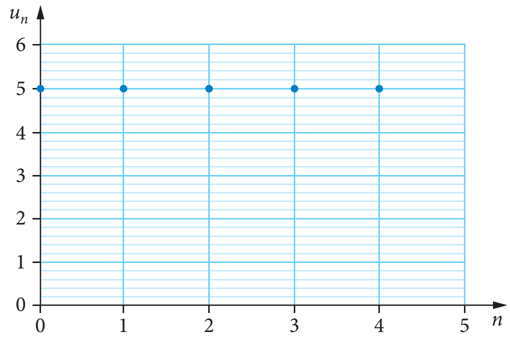
n	0	1	2	3	4	…
u_n	16	8	4	2	1	…



Constant sequence

Multiply by 2 and subtract 5 to generate each new value.

n	0	1	2	3	4	…
u_n	5	5	5	5	5	…

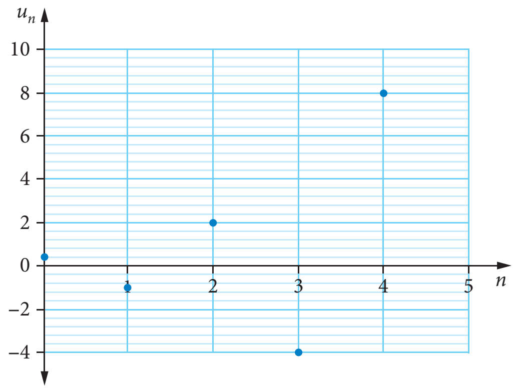


The values of this sequence are always the same.

Oscillating sequence

Multiply by −2 to generate each new value.

n	0	1	2	3	4	…
u_n	0.5	−1	2	−4	8	…



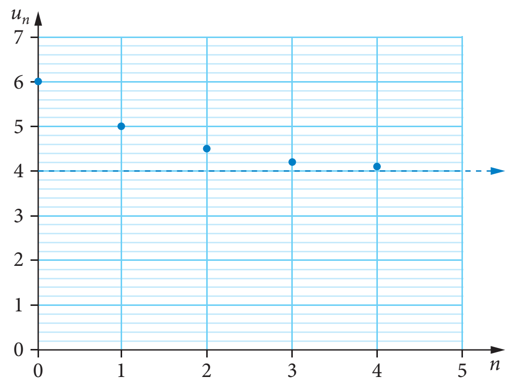
The values of this sequence switch between positive and negative.

Limiting value sequence

n	0	1	2	3	4	…
u_n	6	5	4.5	4.25	4.125	…

Divide by 2 and add 2 to generate each new value.

This sequence has a limiting value of 4. The values tend towards 4 but never reach it.



Arithmetic sequences

- An **arithmetic sequence** is a sequence where a fixed amount is added or subtracted to generate each new value.
- For an arithmetic sequence $u_0, u_1, u_2, u_3 \ldots$

  recurrence relation — $u_0 = a, \quad u_{n+1} = u_n + d$

  ***n*th value rule** — $u_n = a + nd$

  where

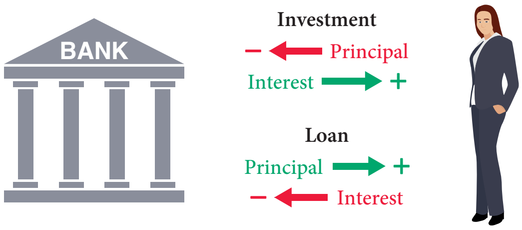
  u_{n+1} is the value after u_n

  a is the first value

  d is the common difference between values.

Simple interest investments and loans

- **Interest** is the fee for using someone else's money.
- **Simple interest** is a fixed amount of interest that is paid at regular time periods.
- When these time periods are years, we use the term **per annum** (or p.a.) which means per year.
- The amount that is borrowed or invested is called the **principal.**
- The value of the investment or loan at any time is called the **balance.**

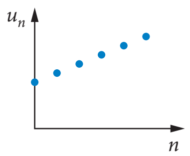
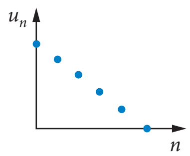
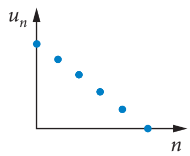
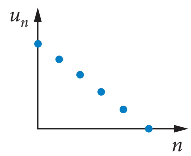


Money coming *to the person* is positive.
Money going *away from the person* is negative.

Depreciation

- **Depreciation** is the decrease in value of items or **assets** used by businesses over time.
- The estimate of the value of an asset at any point in time is called the **future value**.
- **Flat rate depreciation** calculates the future value of an asset by reducing the value every year by a fixed amount.
- **Unit cost depreciation** occurs when an asset is depreciated according to the amount of use it has had, not according to its age.

Simple interest and depreciation summary

	Simple interest investment	Simple interest loan	Flat rate depreciation	Unit cost depreciation
Recurrence relation for balance	u_0 = principal, $u_{n+1} = u_n + d$	u_0 = principal, $u_{n+1} = u_n - d$	u_0 = initial asset value, $u_{n+1} = u_n - d$	u_0 = initial asset value, $u_{n+1} = u_n - d$
Type	increasing arithmetic	decreasing arithmetic	decreasing arithmetic	decreasing arithmetic
Graph				
Rule	$u_n = u_0 + nd$	$u_n = u_0 - nd$	$u_n = u_0 - nd$	$u_n = u_0 - nd$
d	$d = \frac{r}{100} \times u_0$ fixed amount each year	$d = \frac{r}{100} \times u_0$ fixed amount each year	$d = \frac{r}{100} \times u_0$ fixed amount each year	d = cost per unit of use
***r*%**	interest rate per year $r = \frac{d}{u_0} \times 100\%$	depreciation rate per year $r = \frac{d}{u_0} \times 100\%$	depreciation rate per year $r = \frac{d}{u_0} \times 100\%$	–
n	number of years	number of years	number of years	number of units of use

- Total amount of interest/depreciation after n years = $u_n - u_0$.
- The last year of a depreciation/loan often involves a partial amount.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 23 minutes

1 Data relating to the following variables was collected from birds from a particular region:

- *beak length* (in millimetres)
- *name* of species
- *location found*
- *wingspan* (in millimetres)
- *body weight* (in grams)

The number of these variables that are continuous numerical variables is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

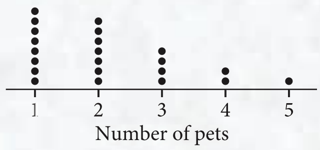
2 According to the frequency table shown, the percentage of people aged in their thirties is

A 0.4% **B** 4% **C** 16%

D 30% **E** 40%

Age	Frequency
0–<10	3
10–<20	11
20–<30	4
30–<40	
40–<50	6
Total	40

3 The dot plot shows the distribution of the number of pets in the 22 homes in a street.



The five-number summary of the number of pets is

A min = 1, Q_1 = 1, median = 2, Q_3 = 3, max = 5

B min = 1, Q_1 = 1, median = 3, Q_3 = 4, max = 8

C min = 1, Q_1 = 2, median = 3, Q_3 = 4, max = 5

D min = 1, Q_1 = 2, median = 2, Q_3 = 3, max = 5

E min = 1, Q_1 = 5, median = 3, Q_3 = 4, max = 5

4 The following back-to-back stem plot shows the female and male participation in sport in 19 country towns.

Female		Male
9 9 8 8 7 6 5 4	1	1 8 8
8 7 5 5 5 4 4 1 0 0 0	2	0 3 3 4 5 6 6 7 8
	3	0 1 1 4 6 9
	4	2

Key: 5 | 2 means 25% **Key: 3 | 4 means 34%**

What is the lowest female participation rate?

A 0% **B** 11% **C** 14% **D** 20% **E** 41%

5 A study found that in a packet of jelly beans the number of red jelly beans has a normal distribution with a mean of 14 and a standard deviation of 3. What is the percentage of packets that have between 8 and 17 red jelly beans?

A 2.5% **B** 47.5% **C** 50% **D** 68% **E** 81.5%

6 How many of the following descriptions apply to $u_0 = 5, u_{n+1} = u_n + 4$?

- recurrence relation
- arithmetic
- increasing
- decreasing

A 0 **B** 1 **C** 2 **D** 3 **E** 4

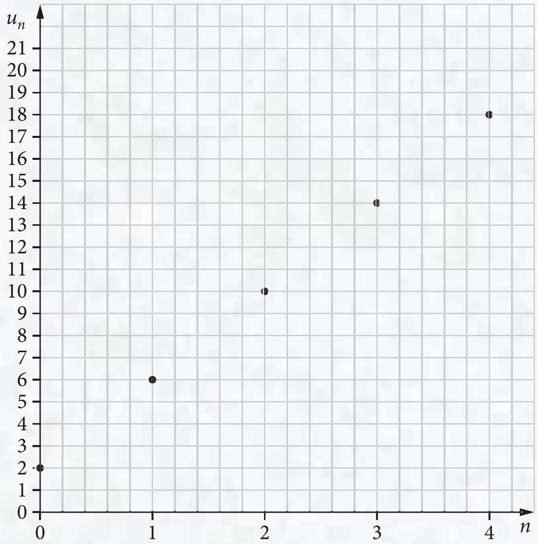
7 Amanda earned $20 000 in one year. At the beginning of the next year, she received a salary increase of $450. She now receives the same increase at the beginning of each year. What will her salary be at the beginning of the 10th year?

A $450 **B** $20 450 **C** $23 600 **D** $24 050 **E** $24 500

8 © VCAA 2009 1BRMQ4 A delivery truck when new was valued at $65 000. The truck's value depreciates at a rate of 22 cents per kilometre travelled. After it has travelled a total distance of 132 600 km, the value of the truck will be

A $14 300 **B** $22 100 **C** $22 516 **D** $29 172 **E** $35 828

9 The graph shows the first five values of a sequence.



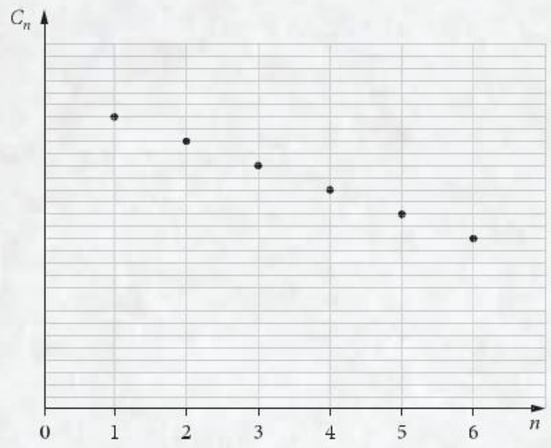
A recurrence relation that generates the values of this sequence is

A $u_0 = 2, \quad u_{n+1} = u_n + 4$ **B** $u_0 = 2, \quad u_{n+1} = 3u_n$ **C** $u_0 = 4, \quad u_{n+1} = u_n + 2$

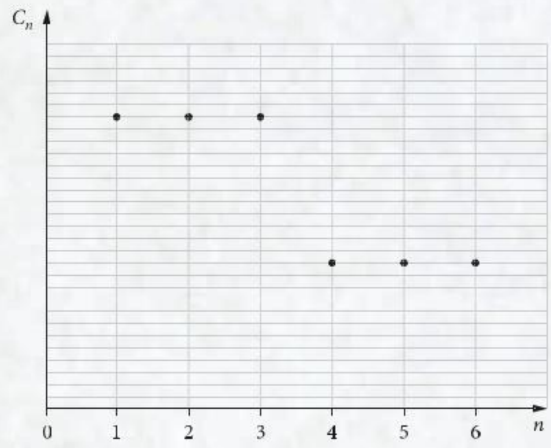
D $u_0 = 4, \quad u_{n+1} = 6 - u_n$ **E** $u_0 = 2, \quad u_{n+1} = 4u_n$

10 © VCAA 2019N 1CQ20 Marty has been depreciating the value of his car each year using flat rate depreciation. After three years of ownership, the value of the car was halved due to an accident. Marty continued to depreciate the value of his car by the same amount each year after the accident. Which one of the following graphs could show the value of Marty's car after n years, C_n?

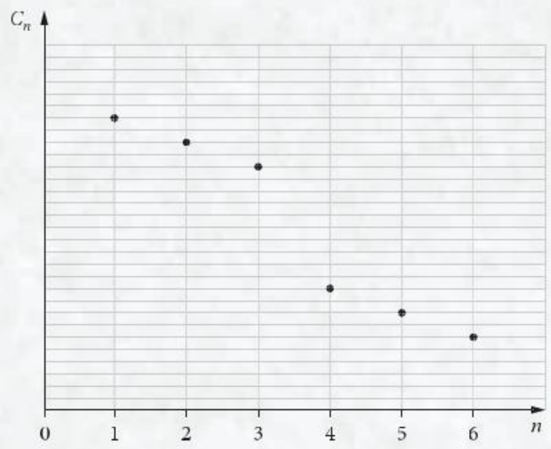
A



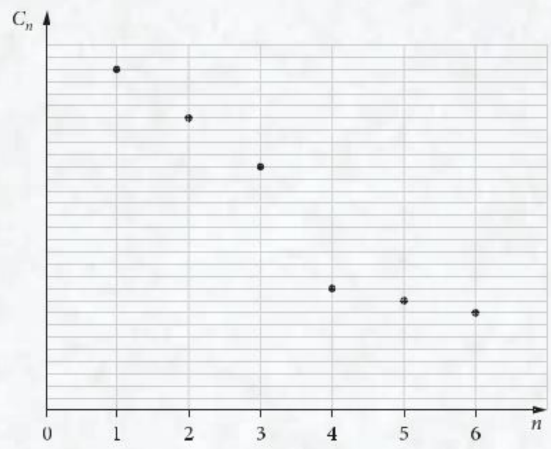
B



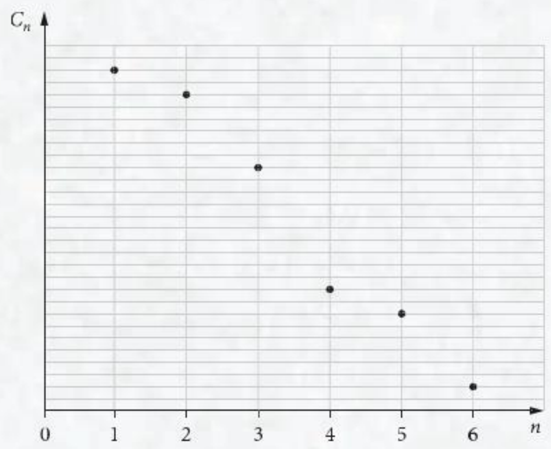
C



D



E



Cumulative examination 2

Total number of marks: 15 Reading time: 4 minutes Writing time: 23 minutes

1 © VCAA 2010 2CQ1 (5 marks) This table shows the percentage of women ministers in the parliaments of 22 countries in 2008.

Country	Percentage of women ministers	Country	Percentage of women ministers
Norway	56	Australia	24
Sweden	48	Italy	24
France	47	United States	24
Spain	44	Belgium	23
Switzerland	43	United Kingdom	23
Austria	38	Ireland	21
Denmark	37	Liechtenstein	20
Iceland	36	Canada	16
Germany	33	Luxembourg	14
Netherlands	33	Japan	12
New Zealand	32	Singapore	0

a What proportion of these 22 countries have a higher percentage of women ministers in their parliament than Australia? 1 mark

b Determine the median, range and interquartile range of this data. 2 marks

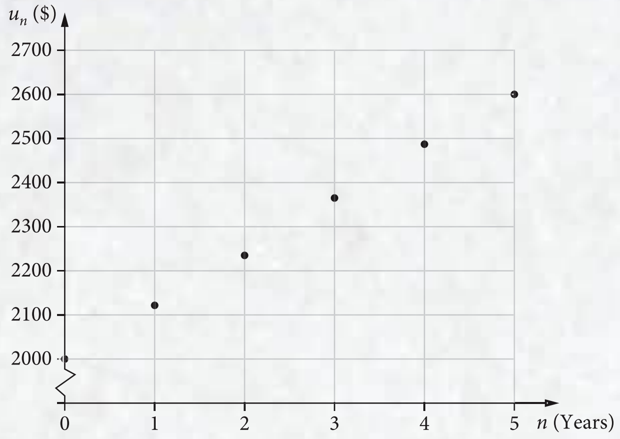
The stem plot displays the distribution of the percentage of women ministers in parliament for 21 of these countries. The value for **Canada** is missing.

Stem (10s)	Leaf (units)
0	0
1	2 4
2	0 1 3 3 4 4 4
3	2 3 3 6 7 8
4	3 4 7 8
5	6

c Copy and complete the stem plot by adding the value for Canada. 1 mark

d Both the median and the mean are appropriate measures of centre for this distribution. Explain why. 1 mark

2 (7 marks) The graph shows the value of an investment at 6% interest p.a.



a Explain what the crooked line under 2000 indicates. 1 mark

b How do we know from the graph that this is a simple interest investment? 2 marks

Use the graph to find

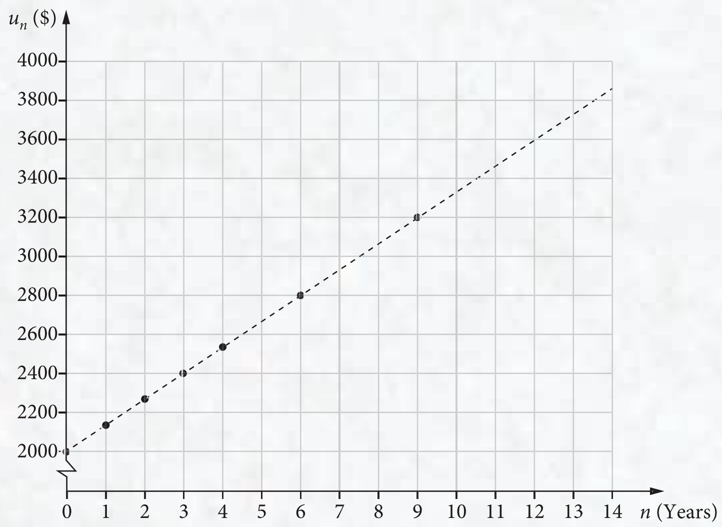
c the principal 1 mark

d the value of the investment after five years. 1 mark

Use the following extended version of the graph to find

e u_{10} 1 mark

f after how many years the value of the investment was first greater than $3000. 1 mark



3 © VCAA 2018N 2CQ8ai–ii MODIFIED (3 marks) Richard is selling his stereo system to help pay for a holiday. The stereo system was originally purchased for $8500. He will sell the stereo system at a depreciated value of $867 per year using a flat rate depreciation method. Let S_n be the value, in dollars, of Richard's stereo system n years after it was purchased.

a What is the recurrence relation that models this depreciation? 1 mark

b Using this depreciation method, what is the value of the stereo system four years after it was purchased? 1 mark

c Calculate the annual percentage flat rate of depreciation for this depreciation method. 1 mark

CHAPTER

3

GEOMETRIC SEQUENCES AND FINANCIAL MATHEMATICS

Study Design coverage

UNIT 1, AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

Arithmetic and geometric sequences, first-order linear recurrence relations and financial mathematics

- the concept of a geometric sequence as a function with the set of non-negative integers as its domain
- use of a first-order linear recurrence relation of the form $u_0 = a$, $u_{n+1} = Ru_n$ where a and R are constants, to generate the values of a geometric sequence
- use of a first-order linear recurrence relation of the form $u_0 = a$, $u_{n+1} = Ru_n$ where a and R are constants, to model growth and decay and analyse practical situations involving geometric sequences such as the reducing height of a bouncing ball, reducing balance depreciation, compound interest loans or investments
- generation of the explicit rule, u_n, of a geometric sequence, its use and evaluation, including various practical and financial contexts
- percentage increase and decrease, mark-ups and discounts, and calculating GST in various financial contexts
- determining the impact of inflation on costs and the spending power of money over time
- the unitary method and its use in making comparisons and solving practical problems involving percentages and finance
- comparison of purchase options including cash, credit and debit cards, personal loans, buy now and pay later schemes.

3.1 Geometric sequences and recurrence relations

Geometric sequences

In Chapter 2 we looked at arithmetic sequences where we add or subtract a fixed amount to generate each new value. In this chapter, we will look at **geometric sequences** where we *multiply* by a fixed amount, called the **common ratio** R, to generate each new value. For example,

the geometric sequence 1, 2, 4, 8, 16 … has a common ratio $R = 2$

the geometric sequence 60, 30, 15, 7.5, 3.75 … has a common ratio $R = 0.5$.

We will only be looking at positive ratios in this chapter.

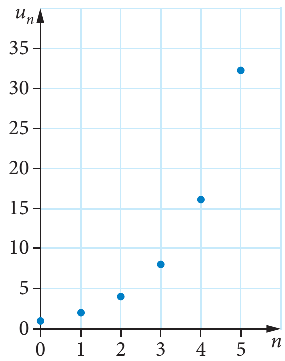
As with arithmetic sequences, we can continue the sequence, recording it in table form and graphing it where we identify the values using $n = 0, 1, 2, 3$ …

Table

n	0	1	2	3	4	5	…
u_n	1	2	4	8	16	32	…

$R = 2$

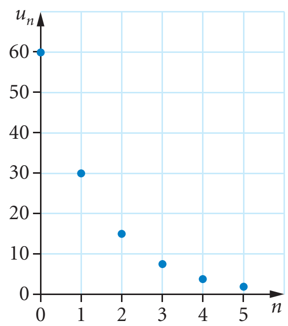
Graph



When $R > 1$, the graph increases by increasing amounts.

n	0	1	2	3	4	5	…
u_n	60	30	15	7.5	3.75	1.875	…

$R = \frac{1}{2}$



When R is between 0 and 1, the graph decreases by decreasing amounts and u_{n+1} tends towards zero.

Geometric sequences and the common ratio *R*

For a geometric sequence where we multiply by a fixed amount, R, to generate each new value

$$R = \frac{\text{any value}}{\text{previous value}}$$

Video playlist
Geometric sequences and recurrence relations

Worksheets
Geometric sequences

Geometric progressions 1

WB

p. 38

WORKED EXAMPLE 1 Graphing geometric sequences

For each of the following sequences

a 2, 6, 18, 54 …

b 64, 16, 4, 1 …

i explain why it's a geometric sequence

ii show three calculations for finding the common ratio, R.

iii find u_2 and u_4

iv make a table of values showing all the values of u_n for $n = 0$ to 5

v sketch a graph of the table of values in part **iv**.

Steps | **Working**

a i Is a fixed value being multiplied each time?

We are multiplying by 3 to generate each new value, so this is a geometric sequence.

ii Use $R = \dfrac{\text{any value}}{\text{previous value}}$

$R = \dfrac{6}{2} = 3, R = \dfrac{18}{6} = 3, R = \dfrac{54}{18} = 3$

iii Extend the sequence by continuing the rule if necessary.

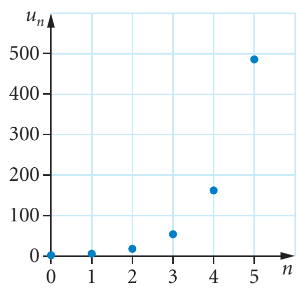
2, 6, 18, 54, 162 …

$u_2 = 18, u_4 = 162$

iv List n in the first row of the table and u_n in the second. Find u_n for $n = 0$ to 5.

n	0	1	2	3	4	5
u_n	2	6	18	54	162	486

v Sketch the table of values with n on the horizontal axis and u_n on the vertical axis.



b i Is a fixed value being multiplied each time?

We are dividing by 4 to generate each new value (or multiplying by $\frac{1}{4}$ or 0.25), so this is a geometric sequence.

ii Use $R = \dfrac{\text{any value}}{\text{previous value}}$

$R = \dfrac{16}{24} = 0.25, R = \dfrac{4}{16} = 0.25, R = \dfrac{1}{4} = 0.25$

iii Extend the sequence by continuing the rule if necessary.

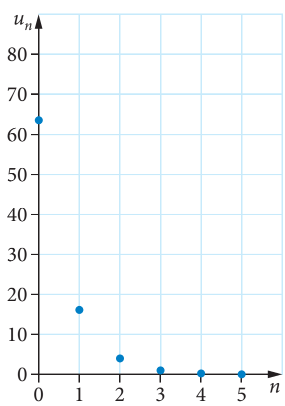
64, 16, 4, 1, 0.25 …

$u_2 = 4, u_4 = 0.25$

iv List n in the first row of the table and u_n in the second. Find u_n for $n = 0$ to 5.

n	0	1	2	3	4	5
u_n	64	16	4	1	0.25	0.0625

v Sketch the table of values with n on the horizontal axis and u_n on the vertical axis.



Geometric recurrence relations

To find the recurrence relation for a geometric sequence, we need to know the common ratio, R, and the first value, a.

For example, the geometric sequence 4, 8, 16, 32 …, where $a = 4$ and $R = 2$, has the recurrence relation

$$u_0 = 4, \quad u_{n+1} = 2u_n$$

The calculations required to generate the sequence for this recurrence relation are:

1 start with 4

2 multiply each value by 2 to generate the next value

3 keep going.

It generates the sequence in this way:

$u_0 = 4$

$u_1 = 2u_0 = 2 \times 4 = 8$

$u_2 = 2u_1 = 2 \times 8 = 16$

$u_3 = 2u_2 = 2 \times 16 = 32$

and so on.

3.1

The geometric sequence recurrence relation

The geometric sequence u_0, u_1, u_2, u_3 … has the recurrence relation

$$u_0 = a, \quad u_{n+1} = Ru_n$$

where

u_{n+1} is the value after u_n

a is the first value

$R = \dfrac{\text{any value}}{\text{previous value}}$ is the common ratio between values.

WORKED EXAMPLE 2 Finding geometric sequences from recurrence relations

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For the recurrence relation

$$u_0 = -5, \quad u_{n+1} = 3u_n,$$

find

a a and R

b the first four values of the geometric sequence generated by the recurrence relation, showing all the calculation steps.

Steps	Working
a Use $u_0 = a, \quad u_{n+1} = Ru_n$ where	
a is the first value	$a = -5$
R is the common ratio between values.	$R = 3$
b **1** The first part of the recurrence relation gives u_0.	$u_0 = -5$
2 The second part of the recurrence relation tells us what to multiply to generate the next value.	$u_1 = 3 \times u_0 = 3 \times -5 = -15$
3 Repeat for the next two values.	$u_2 = 3 \times u_1 = 3 \times -15 = -45$ $u_3 = 3 \times u_2 = 3 \times -45 = -135$
4 Write down the first four values of the sequence.	The first four values of the sequence are −5, −15, −45, −135.

USING CAS 1 Generating geometric sequences through recursive computation

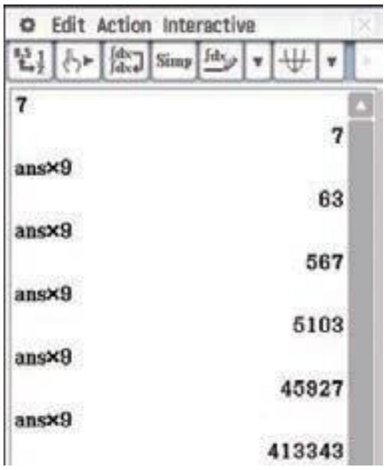
Find the first six values of the sequence defined by the following recurrence relation using repeated steps.

$u_0 = 7, \quad u_{n+1} = 9u_n$

TI-Nspire

1. Start a new document and add a **Calculator** page.
2. Enter **7** then press **enter**.
3. Enter **×9** then press **enter**.
4. Continue to press **enter** until the first six values are displayed.

ClassPad



1. Tap **Main** and clear all entries.
2. Enter **7** then press **EXE**.
3. Enter **×9** then press **EXE**.
4. Continue to press **EXE** until the first six values are displayed.

The *n*th value geometric sequence rule

We can find an ***n*th value rule** for geometric sequences, just as we did for arithmetic sequences.

The geometric sequence *n*th value rule

The geometric sequence $u_0, u_1, u_2, u_3 \ldots$ has the nth value rule

$$u_n = aR^n$$

where

u_n is the nth value

a is the first value

$R = \dfrac{\text{any value}}{\text{previous value}}$ is the common ratio between values.

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WORKED EXAMPLE 3 Finding the nth value of a geometric sequence

For each of the following sequences

i explain how we know that it is a geometric sequence

ii find the nth value rule

iii use the rule to find u_5 and u_8, rounding to two decimal places if necessary.

Steps	Working
a 2, 14, 98, 686 …	
i Are we multiplying the same amount to generate each new value?	$\frac{14}{2} = 7, \frac{98}{14} = 7, \frac{686}{98} = 7$ We are multiplying by 7 to generate each new value, so this is a geometric sequence.
ii 1 Find the first value a and the common ratio R.	$a = 2, R = 7$
2 Substitute the values for a and R into the nth value rule for geometric sequences $u_n = aR^n$.	$u_n = aR^n$ $u_n = 2 \times 7^n$
iii Substitute the values for n into the nth value rule of the sequence, rounding to two decimal places if necessary.	$u_5 = 2 \times 7^5 = 33\,614$ $u_8 = 2 \times 7^8 = 5\,764\,801$
b $u_0 = 1200, \quad u_{n+1} = 0.75u_n$	
i Are we multiplying the same amount to generate each new value?	We are multiplying by 0.75 to generate each new value, so this is a geometric sequence.
ii 1 Find the first value a and the common ratio R.	$a = 1200, R = 0.75$
2 Substitute the values for a and R into the nth value rule for geometric sequences $u_n = aR^n$.	$u_n = ar^n$ $u_n = 1200 \times (0.75)^n$
iii Substitute the values for n into the nth value rule of the sequence, rounding to two decimal places if necessary.	$u_5 = 1200 \times (0.75)^5$ $= 284.765\ldots$ ≈ 284.77 $u_8 = 1200 \times (0.75)^8$ $= 120.135\ldots$ ≈ 120.14

3.1

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EXERCISE 3.1 Geometric sequences and recurrence relations

ANSWERS p. 498

Mastery

1 WORKED EXAMPLE 1 For each of the following sequences

a 3, 12, 48, 192 …

b 162, 54, 18, 6 …

i explain why it's a geometric sequence

ii show three calculations for finding the common ratio R

iii find u_2 and u_4

iv make a table of values showing all the values of u_n for $n = 0$ to 5

v sketch a graph of the table of values in part **iv**.

2 Find the common ratio, R, for each of these geometric sequences.

a 11, 55, 275, 1375 …

b 240, 48, 9.6, 1.92 …

c 3.2, 5.12, 8.192, 13.1072 …

d 8, 7.2, 6.48, 5.832 …

3 WORKED EXAMPLE 2 For each of the following recurrence relations, find

i a and R

ii the first four values of the geometric sequence generated by the recurrence relation, showing all the calculation steps.

a $u_0 = 4, \quad u_{n+1} = 10u_n$

b $u_0 = -1, \quad u_{n+1} = 6u_n$

c $u_0 = 80, \quad u_{n+1} = 0.25u_n$

d $u_0 = 12, \quad u_{n+1} = 1.5u_n$

4 Using CAS 1 Find the first six values of the sequence defined by the following recurrence relation using repeated steps.

$u_0 = 8, \quad u_{n+1} = 6u_n$

5 WORKED EXAMPLE 3 For each of the following sequences

i explain how we know that it is a geometric sequence

ii find the nth value rule

iii use the rule to find u_6 and u_9, rounding to three decimal places if necessary.

a 8, 24, 72, 216 …

b 16 000 000, 2 000 000, 250 000, 31 250 …

c $u_0 = 2240, \quad u_{n+1} = 0.9u_n$

d $u_0 = 4, \quad u_{n+1} = 1.5u_n$

6 Find a recurrence relation that generates each of the following geometric sequences.

a 2, 4, 8 …

b 3, 15, 75 …

c 5, 30, 180 …

d 48, 24, 12 …

e 2, 5.6, 15.68 …

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2016S 1CQ17 MODIFIED The first three values of a sequence generated by the recurrence relation

$P_0 = 2000, \quad P_{n+1} = 1.5P$

are

A 2000, 3000, 4500 …

B 2000, 1000, 500 …

C 2000, 2500, 3000 …

D 2000, 3000, 4000 …

E 2000, 1500, 1000 …

8 © VCAA 2019N 1CQ17 MODIFIED A sequence of numbers is generated by the recurrence relation

$P_0 = 2, \quad P_{n+1} = 3P_n$

What is the value of P_3?

A 2

B 3

C 6

D 18

E 54

9 A recurrence relation is defined by

$u_0 = -1, \quad u_{n+1} = 5u_n$

The sequence u_0, u_1, u_2 … is

A −1, −5, −25 …

B −1, 5, −25 …

C −1, −6, −11 …

D −1, 4, 9 …

E −1, 6, 11 …

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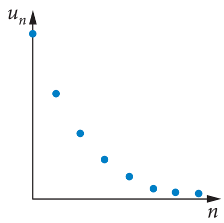
10 The following recurrence relation can generate a sequence of numbers:

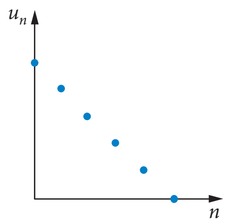
$u_0 = 10, \quad u_{n+1} = 2u_n$

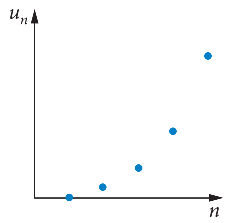
The number 80 appears in this sequence as

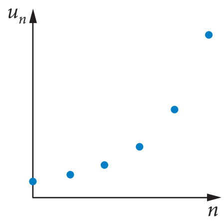
A u_1 **B** u_2 **C** u_3 **D** u_7 **E** u_{10}

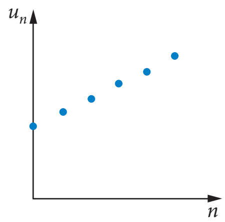
11 Which of the following graphs could best represent the recurrence relation $u_0 = 80$, $u_{n+1} = 2.5u_n$?

A


B


C


D


E


12 A sequence is defined by the recurrence relation

$u_0 = 20, \quad u_{n+1} = 0.2u_n$

The rule that gives u_n, the nth value of the sequence, is

A $u_{n+1} = 0.2(20)^n$ **B** $u_{n+1} = 20(0.2)^n$ **C** $u_n = 20(0.2)^{n-1}$

D $u_n = 0.2(-20)^{n-1}$ **E** $u_n = 20(0.2)^n$

13 The rule $u_n = 0.5(12)^n$ could be used to find the 100th value of which of the following sequences:

A 12, 144, 1728 … **B** 0.5, 6, 12 … **C** 12, 6, 3 …

D 0.5, 6, 72 … **E** 0.5, 12.5, 24.5 …

14 (6 marks) A sequence is defined by the recurrence relation $u_0 = 3$, $u_{n+1} = 2u_n$.

a State the type of sequence defined by this recurrence relation, giving a reason for your answer. 1 mark

b State the common ratio for this sequence. 1 mark

c Make a table of values for this sequence showing the first four values of the sequence. 1 mark

d Find a rule for the nth value of the sequence. 1 mark

e Show a calculation to find u_{10} using the nth value rule. 2 marks

15 (3 marks) A rubber ball is dropped from a height of 96 metres and each time it hits the ground, it bounces up to a quarter of the previous height.

a If h_n is the maximum height the ball reaches after each bounce, write a recurrence relation for the height of the ball after the nth bounce. 1 mark

b Show the calculations needed to find the height after the second bounce using the recurrence relation. 1 mark

c What height does the ball reach after the third bounce? 1 mark

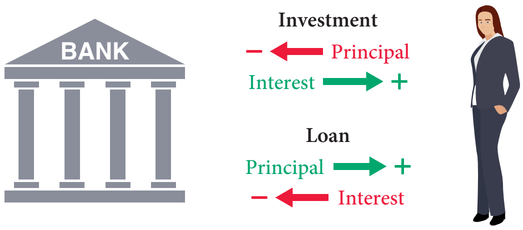
Video playlist
Compound interest recurrence relations

3.2 Compound interest recurrence relations

Compound interest vs simple interest

In Chapter 2 we looked at simple interest and how it is based on arithmetic sequences. In this chapter we will be looking at **compound interest** and how is it based on geometric sequences.

Again, keep in mind that **interest** is the fee for using someone else's money, and the amount that is invested or borrowed is called the **principal**.



Money coming *to the person* is positive.

Money going away *from the person* is negative.

Most investments and loans use compound interest where the interest is added to the principal, and the interest for the next time period is calculated using this new balance. The interest is regularly calculated at the end of a certain time period, which is called a **compounding period**.

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WORKED EXAMPLE 4 Comparing compound and simple interest

Najina is investing \$4000 for four years and wants to compare an investment at 10% p.a (per year) compounding yearly to 10% p.a. simple interest.

a Copy and complete the following table for $n = 3$ and $n = 4$.

	Compound		Simple	
n	Interest (\$)	Value of investment (\$)	Interest (\$)	Value of investment (\$)
0	–	4000	–	4000
1	$\frac{10}{100} \times 4000 = 400$	$4000 + 400 = 4400$	$\frac{10}{100} \times 4000 = 400$	$4000 + 400 = 4400$
2	$\frac{10}{100} \times 4400 = 440$	$4400 + 440 = 4840$	$\frac{10}{100} \times 4000 = 400$	$4400 + 400 = 4800$
3				
4				

b What is the value of the compound interest investment after four years?

c After four years, how much more is the value of the compound interest investment compared to the simple interest investment?

Steps	Working
a	

	Compound		Simple	
n	Interest ($)	Value of investment ($)	Interest ($)	Value of investment ($)
0	–	4000	–	4000
1	$\frac{10}{100} \times 4000 = 400$	$4000 + 400 = 4400$	$\frac{10}{100} \times 4000 = 400$	$4000 + 400 = 4400$
2	$\frac{10}{100} \times 4400 = 440$	$4400 + 440 = 4840$	$\frac{10}{100} \times 4000 = 400$	$4400 + 400 = 4800$
3	$\frac{10}{100} \times 4840 = 484$	$4840 + 484 = 5324$	$\frac{10}{100} \times 4000 = 400$	$4800 + 400 = 5200$
4	$\frac{10}{100} \times 5324 = 532.40$	$5324 + 532.40 = 5856.40$	$\frac{10}{100} \times 4000 = 400$	$5200 + 400 = 5600$

Steps	Working
b Read from the table.	The value of the compound interest investment after four years is $5856.40.
c Compare the last entries in the two interest ($) columns in the table.	5856.40 – 5600 = 256.40 The compound interest investment has $256.40 more than the simple interest investment.

3.2

Compounding periods

Compound interest is always given as a rate per year, but compounding periods can vary:

- Daily compounding means the interest is calculated every day and added to the account.
- Weekly compounding means the interest is calculated every week and added to the account.
- Further compounding periods are shown in the table.

Compounding period	Number of compounding periods per year
Daily	365
Weekly	52
Fortnightly	26
Monthly	12
Quarterly	4
Six-monthly	2
Yearly	1

Interest rates per compounding period

$$\text{percentage interest rate per compounding period} = \frac{\text{percentage interest rate per year}}{\text{number of compounding periods per year}}$$

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WORKED EXAMPLE 5 Working with compounding periods

For each of the following investments, find

- **i** the number of compounding periods per year
- **ii** the number of compounding periods over eight years
- **ii** the percentage interest rate per compounding period, written as a fraction
- **iv** the amount of interest earned in the first compounding period to the nearest cent.

a Sher-Li invests $50 000 at 4% compound interest per annum compounding daily.

b Callum invests $16 000 at 7% compound interest per annum compounding weekly.

Steps	Working
a **i** How many compounding periods are there per year?	There are 365 daily compounding periods per year.
ii Multiply the number of compounding periods per year by the number of years.	There are 365 × 8 = 2920 daily compounding periods over eight years.
iii Divide the percentage interest rate per year by the number of compounding periods per year.	The percentage interest rate per day = $\frac{4}{365}$%.
iv Convert the compounding period interest rate to a decimal and multiply by the principal. Round to the nearest cent.	The amount of interest earned on the first day is $\frac{4}{365} \times \frac{1}{100} \times 50\,000 = \$5.48$.
b **i** How many compounding periods are there per year?	There are 52 weekly compounding periods per year.
ii Multiply the number of compounding periods per year by the number of years.	There are 52 × 8 = 416 weekly compounding periods over eight years.
iii Divide the percentage interest rate per year by the number of compounding periods per year.	The percentage interest rate per week = $\frac{7}{52}$%.
iv Convert the compounding period interest rate to a decimal and multiply by the principal. Round to the nearest cent.	The amount of interest earned in the first week is $\frac{7}{52} \times \frac{1}{100} \times 16\,000 = \$21.54$.

Compound interest recurrence relations

Compound interest investments and loans can be modelled by recurrence relations based on geometric sequences.

Compound interest investment recurrence relation

The recurrence relation for the value u_n of a compound interest investment, after n compounding periods, is

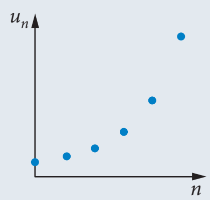
$$u_0 = \text{principal}, \quad u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$$

where

r = the percentage interest rate per compounding period

n = the number of compounding periods.

The graph of a compound interest investment recurrence relation will look like this:



Total amount of interest after n compounding periods = $u_n - u_0$.

WORKED EXAMPLE 6 Finding compound interest recurrence relations

Write a recurrence relation for the account balance, after n compounding periods, for each of the following in simplest form.

a Adamma deposited \$18 000 into a savings account earning compound interest at the rate of 5.6% per annum, compounding annually.

b Astrid deposited \$14 000 into a savings account earning compound interest at the rate of 4.8% per annum, compounding quarterly.

c Aroha deposited \$15 000 into a savings account earning compound interest at the rate of 3% per annum, compounding monthly.

Steps	Working
a **1** Find the number of compounding periods per year.	There is one compounding period per year.
2 Identify u_n, u_0 and r.	Let u_n = the account balance after n compounding periods. $u_0 = 18\,000,\ r = \frac{5.6}{1} = 5.6\%$
3 Substitute the values into u_0 = principal, $u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$ and simplify.	$u_0 = 18\,000,\quad u_{n+1} = \left(1 + \frac{5.6}{100}\right)u_n$ $u_0 = 18\,000,\quad u_{n+1} = (1 + 0.056)u_n$ $u_0 = 18\,000,\quad u_{n+1} = 1.056u_n$
b **1** Find the number of compounding periods per year.	There are four compounding periods per year.
2 Identify u_n, u_0 and r.	Let u_n = the account balance after n compounding periods. $u_0 = 14\,000,\ r = \frac{4.8}{4} = 1.2\%$
3 Substitute the values into u_0 = principal, $u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$ and simplify.	$u_0 = 14\,000,\quad u_{n+1} = \left(1 + \frac{1.2}{100}\right)u_n$ $u_0 = 14\,000,\quad u_{n+1} = (1 + 0.012)u_n$ $u_0 = 14\,000,\quad u_{n+1} = 1.012u_n$
c **1** Find the number of compounding periods per year.	There are 12 compounding periods per year.
2 Identify u_n, u_0 and r.	Let u_n = the account balance after n compounding periods. $u_0 = 15\,000,\ r = \frac{3}{12} = 0.25\%$
3 Substitute the values into u_0 = principal, $u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$ and simplify.	$u_0 = 15\,000,\quad u_{n+1} = \left(1 + \frac{0.25}{100}\right)u_n$ $u_0 = 15\,000,\quad u_{n+1} = (1 + 0.0025)u_n$ $u_0 = 15\,000,\quad u_{n+1} = 1.0025u_n$

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Compound interest general rule

As with simple interest, there is a compound interest general rule that can be used to solve problems more quickly.

Compound interest general rule

The general rule for the value u_n, after n compounding periods, of a compound interest investment is

$$u_n = \left(1 + \frac{r}{100}\right)^n \times u_0$$

where

u_0 = principal

r = the percentage interest rate per compounding period

n = the number of compounding periods.

Total amount of interest after n compounding periods = $u_n - u_0$

WB

p. 45

WORKED EXAMPLE 7 Using the compound interest rule

Ricky invests $38 000 in an account where he earns interest of 3% p.a. compounded monthly.

a Find r, the percentage interest rate per compounding period.

b Write a rule that will calculate the value of the investment after n compounding periods.

c Use the rule to find the value of the investment after 15 compounding periods to the nearest cent.

d Calculate the total amount of interest paid after 15 compounding periods to the nearest cent.

Steps	Working
a Divide the yearly interest rate by the number of compounding periods per year.	The percentage interest rate per month $= \frac{3}{12}\%$ $= 0.25\%$
b Substitute the values of u_0 and r into the compound interest general rule $u_n = \left(1 + \frac{r}{100}\right)^n \times u_0$ and simplify.	$u_n = \left(1 + \frac{0.25}{100}\right)^n \times 38\,000$ $u_n = 1.0025^n \times 38\,000$
c **1** Substitute the value of n, the number of compounding periods, into the rule.	$n = 15$ $u_{15} = 1.0025^{15} \times 38\,000$ $= 39\,450.209\ldots$
2 Write the answer, rounding to the nearest cent.	The value of the investment after 15 months is $39 450.21.
d Total amount of interest after n compounding periods = $u_n - u_0$. Write the answer, rounding to the nearest cent.	Find $u_{15} - u_0$. Total amount of interest after 15 months = $39 450.21 – $38 000.00 = $1450.21

EXERCISE 3.2 Compound interest recurrence relations

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Recap

1 The first three values of a sequence generated by the recurrence relation $u_0 = 8800$, $u_{n+1} = 0.5u_n$ are

A 8800, 1760, 352 …
B 8800, 17 600, 35 200 …
C 8800, 4400, 2200 …
D 8800, 6600, 4400 …
E 8800, 4400, 1100 …

2 The recurrence relation $u_0 = 2$, $u_{n+1} = 3u_n$ can generate a sequence of numbers.

The number 162 appears in this sequence as

A u_1 **B** u_2 **C** u_3 **D** u_4 **E** u_5

3.2

Mastery

3 WORKED EXAMPLE 4 Talia is investing \$3000 for four years and wants to compare an investment at 5% p.a. compounding yearly to 5% p.a. simple interest.

a Copy and complete the following table for $n = 3$ and $n = 4$.

	Compound		Simple	
n	Interest (\$)	Value of investment (\$)	Interest (\$)	Value of investment (\$)
0	–	3000	–	3000
1	$\frac{5}{100} \times 3000 = 150$	$3000 + 150 = 3150$	$\frac{5}{100} \times 3000 = 150$	$3000 + 150 = 3150$
2	$\frac{5}{100} \times 3150 = 157.50$	$3150 + 157.50 = 3307.50$	$\frac{5}{100} \times 3000 = 150$	$3150 + 150 = 3300$
3				
4				

b What is the value of the compound interest investment after four years?

c After four years, how much more is the value of the compound interest investment compared to the simple interest investment?

4 WORKED EXAMPLE 5 For each of the following investments, find

i the number of compounding periods per year

ii the number of compounding periods over nine years

iii the percentage interest rate per compounding period, written as a fraction

iv the amount of interest earned in the first compounding period to the nearest cent.

a Ross invests \$62 000 at 3% compound interest per annum compounding weekly.

b Rachel invests \$110 000 at 7% compound interest per annum compounding daily.

c Monica invests \$10 000 at 5% compound interest per annum compounding monthly

d Joey invests \$14 000 at 6% compound interest per annum compounding quarterly.

e Phoebe invests \$22 000 at 8% compound interest per annum compounding fortnightly.

5 WORKED EXAMPLE 6 Write a recurrence relation for the account balance, after n compounding periods, for each of the following in simplest form.

a Ari deposited \$20 000 into a savings account earning compound interest at the rate of 3.1% per annum, compounding annually.

b Almir deposited \$16 000 into a savings account earning compound interest at the rate of 2.4% per annum, compounding monthly.

c Aksel deposited \$21 000 into a savings account earning compound interest at the rate of 4.8% per annum, compounding quarterly.

6 WORKED EXAMPLE 7 For each of the following

i find r, the percentage interest rate per compounding period

ii write a rule that will calculate the value of the investment after n compounding periods

iii use the rule to find the value of the investment after 19 compounding periods to the nearest cent

iv calculate the total amount of interest paid after 19 compounding periods to the nearest cent.

a Linus invests \$75 000 in an account where he earns interest of 6% p.a. compounded monthly.

b Elke invests \$45 000 in an account where she earns interest of 4% p.a. compounded quarterly.

c Ingrid invests \$6000 in an account where she earns interest of 8.5% p.a. compounded yearly.

d Trevor invests \$34 000 in an account where he earns interest of 5% p.a. compounded six-monthly.

Exam practice

80–100% 60–79% 0–59%

7 \$6000 has been invested at 10% p.a. compound interest compounding yearly. What are the two values for $n = 2$ in the compound interest table below, where interest is paid after n years?

n	Compound interest (\$)	Value of investment (\$)
0	–	6000
1	600	6000 + 600 = 6600
2		

A 660 and 6600
B 600 and 7200
C 660 and 7200
D 600 and 6000
E 660 and 7260

8 The recurrence relation $u_0 = 15\,000$, $u_{n+1} = u_n \times 1.06$ can represent the value of an investment of

A \$6000 at 15% p.a. compounded yearly
B \$15 000 at 10.6% p.a. compounded yearly
C \$10 600 at 15% p.a. compounded yearly
D \$15 000 at 1.06% p.a. compounded yearly
E \$15 000 at 6% p.a. compounded yearly

9 © VCAA 2020 1CQ24 80% Manu invests \$3000 in an account that pays interest compounding monthly. The balance of his investment after n months, B_n, can be determined using the recurrence relation

$B_1 = 3000, \quad B_{n+1} = 1.0048 \times B_n$

The total interest earned by Manu's investment after the first five months is closest to

A \$57.60
B \$58.02
C \$72.00
D \$72.69
E \$87.44

10 © VCAA 2002 1BRMQ2 79% An investment of \$16 000 is made at 4% interest per annum, compounding yearly. The value of the investment after two years is

A \$17 280.71
B \$17 305.60
C \$17 325.71
D \$21 120.00
E \$21 897.10

11 © VCAA 2017N 1CQ18 Andre deposited \$20 000 into a savings account earning compound interest at the rate of 3.1% per annum, compounding annually. Which one of the following recurrence relations can be used to determine the amount in the savings account, S_n, after n years?

A $S_0 = 20\,000, \quad S_{n+1} = S_n + 620$
B $S_0 = 20\,000, \quad S_{n+1} = 1.031 \times S_n$
C $S_0 = 20\,000, \quad S_{n+1} = 620 \times S_n$
D $S_0 = 20\,000, \quad S_{n+1} = 3.1 \times S_n + 620$
E $S_0 = 20\,000, \quad S_{n+1} = S_n + 3.1 \times 620$

12 © VCAA 2019 1CQ18 66% The value of a compound interest investment, in dollars, after n years, V_n, can be modelled by the recurrence relation $V_0 = 100\,000$, $V_{n+1} = 1.01\ V_n$. The interest rate, per annum, for this investment is

A 0.01%
B 0.101%
C 1%
D 1.01%
E 101%

13 © VCAA 2003 1BRMQ3 **51%** Heather invests \$45 000 at 4% per annum for five years compounding annually. The total amount of interest earned after five years is

A \$1800 **B** \$2100 **C** \$9000 **D** \$9750 **E** \$54 750

14 © VCAA 2007 1BRMQ6 **42%** \$10 000 is invested at a rate of 10% per annum compounding half yearly. The value, in dollars, of this investment after five years, is given by

A $10\,000 \times 0.10 \times 5$ **B** $10\,000 \times 0.05 \times 10$ **C** $10\,000 \times 0.05^{10}$

D $10\,000 \times 1.05^{10}$ **E** $10\,000 \times 1.10^{5}$

15 © VCAA 2009 2BRMQ3 (5 marks) A golf club's social committee has \$3400 invested in an account that pays interest at the rate of 4.4% per annum compounding quarterly.

a Show that the interest rate per quarter is 1.1%. 2 marks

b Determine the value of the \$3400 investment after three years. Write your answer in dollars correct to the nearest cent. 1 mark

c Calculate the interest the \$3400 investment will earn over **six** years. Write your answer in dollars correct to the nearest cent. 2 marks

3.3 Reducing balance depreciation

Video playlist Reducing balance depreciation

Worksheets Modelling with sequences

Modelling arithmetic and geometric sequences

Reducing balance depreciation recurrence relations

In Chapter 2 we looked at two ways businesses deal with assets that reduce in value over time: flat rate depreciation and unit cost depreciation. These two ways involve simple interest calculations by reducing the value of the asset by a *fixed amount* each time.

We will now look at a third type of depreciation called **reducing balance depreciation**, where the value of the asset is reduced by a fixed percentage of its value in the preceding year. With flat rate depreciation, the asset depreciates by the *same* amount each year, whereas with reducing balance depreciation, the asset depreciates by smaller amounts each year. So reducing balance depreciating depreciation involves compound interest payments.

Reducing balance depreciation recurrence relation

The recurrence relation for the value of an asset u_n, after n years, using reducing balance depreciation is

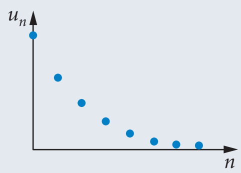
$$u_0 = \text{initial value of the asset, } u_{n+1} = \left(1 - \frac{r}{100}\right) \times u_n$$

where

r = the percentage depreciation rate per year

n = the number of years.

The graph of u_n would look like this:



Total amount of depreciation after n years = $u_n - u_0$.

WB

p. 46

WORKED EXAMPLE 8 Finding reducing balance depreciation recurrence relations

A factory owner purchased a machine for \$12 000. It is depreciated using reducing balance depreciation at a rate of 16% per annum. Give all answers to the nearest dollar.

a Copy and complete the table to find the value of the machine after five years.

n	Depreciation after n years (\$)	Value after n years (\$)
0	–	12 000
1	$\frac{16}{100} \times 12\,000 = 1920$	$12\,000 - 1920 = 10\,080$
2	$\frac{16}{100} \times 10\,080 = 1613$	$10\,080 - 1613 = 8467$
3		
4		
5		

b How much is the machine depreciated by after four years?

c Write down a recurrence relation that gives the value of the machine after n years.

Steps	**Working**
a **1** Calculate the percentage of successive values and subtract from the previous value. Use CAS recursive computation where possible. Give all values to the nearest dollar, but don't round until after all the calculations have been done. [Note: Answers can vary slightly depending on when values are rounded.]	(see table below)
2 Read the answer from the table.	The value of the machine after five years is \$5018.
b Read the answer from the table.	The machine is depreciated by \$1138 after four years.
c **1** Identify u_n, u_0 and r.	Let u_n = the value of the photocopier after n years. $u_0 = 12\,000$, $r = 16$
2 Substitute the values into u_0 = initial value of the asset, $u_{n+1} = \left(1 - \frac{r}{100}\right)u_n$ and simplify.	$u_0 = 12\,000, \quad u_{n+1} = \left(1 - \frac{16}{100}\right)u_n$ $u_0 = 12\,000, \quad u_{n+1} = (1 - 0.16)u_n$ $u_0 = 12\,000, \quad u_{n+1} = 0.84u_n$

n	Depreciation after n years (\$)	Value after n years (\$)
0	–	12 000
1	$\frac{16}{100} \times 12\,000 = 1920$	$12\,000 - 1920 = 10\,080$
2	$\frac{16}{100} \times 10\,080 = 1613$	$10\,080 - 1613 = 8467$
3	$\frac{16}{100} \times 8467 = 1355$	$8467 - 1355 = 7112$
4	$\frac{16}{100} \times 7112 = 1138$	$7112 - 1138 = 5974$
5	$\frac{16}{100} \times 5974 = 956$	$5974 - 956 = 5018$

WORKED EXAMPLE 9 Using reducing balance depreciation recurrence relations

A ute is depreciated using the reducing balance method of depreciation. The value of the ute u_n, in dollars, after n years, can be modelled by the recurrence relation

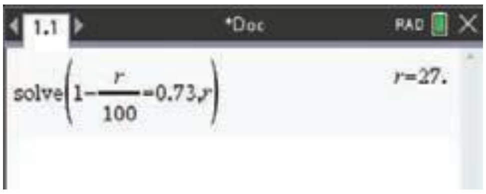
$$u_0 = 55\,000, \quad u_{n+1} = 0.73u_n$$

a At what annual percentage rate is the value of the ute depreciated each year?

b When does the ute first depreciate to under \$20 000?

Steps	Working
a 1 Use u_0 = initial value of the asset, $u_{n+1} = \left(1 - \frac{r}{100}\right) \times u_n$	Compare $u_{n+1} = \left(1 - \frac{r}{100}\right) \times u_n$ to $u_{n+1} = 0.73u_n$. $\left(1 - \frac{r}{100}\right) = 0.73$
2 Solve for r, using CAS if necessary.	$-\frac{r}{100} = -1 + 0.73$ $-\frac{r}{100} = -0.27$ $\frac{r}{100} = 0.27$ $r = 27$

TI-Nspire



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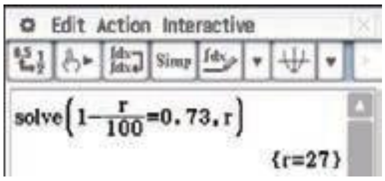
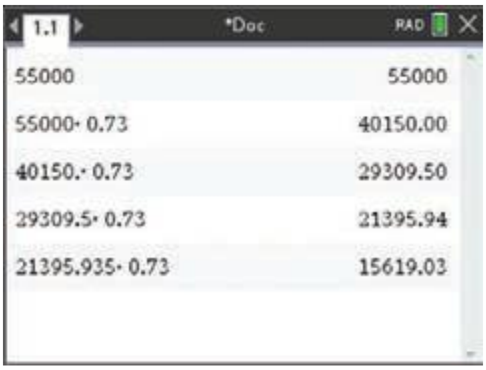


Write the answer. — The annual percentage rate is 27%.

b Use CAS recursive computation.

TI-Nspire



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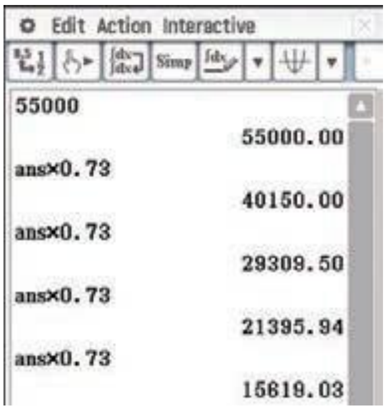


Find when the value is first below \$20 000. — The ute first depreciates to under \$20 000 after four years.

3.3

WB p. 47

Reducing balance depreciation general rule

The reducing balance depreciation general rule is similar to the compound interest general rule.

Reducing balance depreciation general rule

The general rule for the value u_n after n years of a reducing balance depreciation is

$$u_n = \left(1 - \frac{r}{100}\right)^n \times u_0$$

where

u_0 = initial value of the asset

r = the percentage interest rate per year

n = the number of years.

Total amount of interest after n years = $u_n - u_0$

WB

p. 48

WORKED EXAMPLE 10 Using the reducing balance depreciation rule

A commercial fishing company bought a boat for \$200 000 and is depreciating it by 25% of its value each year.

Steps	Working
a Explain why this involves reducing balance depreciation and not flat rate depreciation.	
Does the depreciation involve a changing amount or a fixed amount each year?	The boat is being depreciated by 25% of its *value* each year. If it was flat rate depreciation, it would be depreciated by 25% of its *initial value* each year.
b Write a rule that will calculate the value of the boat after n years.	
Substitute the values of u_0 and r into the reducing balance depreciation general rule $u_n = \left(1 - \frac{r}{100}\right)^n \times u_0$ and simplify.	$u_0 = 200\,000,\ r = 25$ $u_n = \left(1 - \frac{r}{100}\right)^n \times u_0$ $u_n = \left(1 - \frac{25}{100}\right)^n \times 200\,000$ $u_n = 0.75^n \times 200\,000$
c Use the rule to find the value of the boat after eight years to the nearest dollar.	
Substitute the value of n into the rule and solve.	$n = 8$ $u_8 = 0.75^8 \times 200\,000 = 20\,022.58\ldots$ The value of the boat after eight years is \$20 023.

EXERCISE 3.3 Reducing balance depreciation

ANSWERS p. 500

Recap

1 \$10 000 has been invested at 5% p.a. compound interest, compounding yearly. What are the two values for $n = 2$ in the compound interest table below where interest is paid after n years?

n	Compound interest (\$)	Value of investment (\$)
0	–	10 000
1	500	10 000 + 500 = 10 500
2		

A 500 and 11 000 **B** 525 and 11 025 **C** 500 and 11 025

D 525 and 11 000 **E** 500 and 10 500

2 The recurrence relation $u_0 = 27\,000$, $u_{n+1} = 1.05u_n$, represents the value of an investment of

A \$5000 at 27% p.a. compounded yearly
B \$27 000 at 10.5% p.a. compounded yearly
C \$10 500 at 27% p.a. compounded yearly
D \$27 000 at 5% p.a. compounded yearly
E \$27 000 at 1.05% p.a. compounded yearly

3.3

Mastery

3 WORKED EXAMPLE 8 A business purchased a workstation for \$10 000. It is depreciated using reducing balance depreciation at a rate of 18% per annum. Give all answers to the nearest dollar.

a Copy and complete the table to find the value of the workstation after five years.

n	Depreciation after n years (\$)	Value after n years (\$)
0	–	10 000
1	$\frac{18}{100} \times 10\,000 = 1800$	10 000 – 1800 = 8200
2	$\frac{18}{100} \times 8200 = 1476$	8200 – 1476 = 6724
3		
4		
5		

b How much is the workstation depreciated by after four years?

c Write down a recurrence relation that gives the value of the workstation after n years.

4 WORKED EXAMPLE 9 A car is depreciated using the reducing balance method of depreciation. The value of the car u_n, in dollars, after n years, can be modelled by the recurrence relation

$$u_0 = 58\,000, \quad u_{n+1} = 0.79u_n$$

a At what annual percentage rate is the value of the car depreciated each year?

b When does the car first depreciate to under \$20 000?

5 WORKED EXAMPLE 10 A removalist bought a truck for \$250 000 and is depreciating it by 20% of its value each year.

a Explain why this involves reducing balance depreciation and not flat rate depreciation.

b Write a rule that will calculate the value of the truck after n years.

c Use the rule to find the value of the boat after nine years to the nearest dollar.

6 For each of the following

- **i** state whether the depreciation is flat rate, unit cost or reducing balance
- **ii** write a recurrence relation for u_n, stating what n represents
- **iii** find the rule for u_n in simplest form.

a A colour laser printer is purchased for \$20 000. It is depreciated by 15% of its value each year.

b A colour laser printer is purchased for \$20 000. It is depreciated by 15% of its initial value each year.

c A colour laser printer is purchased for \$20 000. It is depreciated by 10 cents per page used.

80–100% 60–79% 0–59%

7 © VCAA 2016 1CQ19 74% The purchase price of a car was \$26 000. Using the reducing balance method, the value of the car is depreciated by 8% each year. A recurrence relation that can be used to determine the value of the car after n years, C_n, is

A $C_0 = 26\,000, \quad C_{n+1} = 0.92C_n$

B $C_0 = 26\,000, \quad C_{n+1} = 1.08C_n$

C $C_0 = 26\,000, \quad C_{n+1} = C_n + 8$

D $C_0 = 26\,000, \quad C_{n+1} = C_n - 8$

E $C_0 = 26\,000, \quad C_{n+1} = 0.92C_n - 8$

8 © VCAA 2019N 1CQ18 A truck was purchased for \$134 000. Using the reducing balance method, the value of the truck is depreciated by 8.5% each year. Which one of the following recurrence relations could be used to determine the value of the truck after n years, u_n?

A $u_0 = 134\,000, \quad u_{n+1} = 0.915 \times u_n$

B $u_0 = 134\,000, \quad u_{n+1} = 1.085 \times u_n$

C $u_0 = 134\,000, \quad u_{n+1} = u_n - 11\,390$

D $u_0 = 134\,000, \quad u_{n+1} = 0.915 \times u_n - 8576$

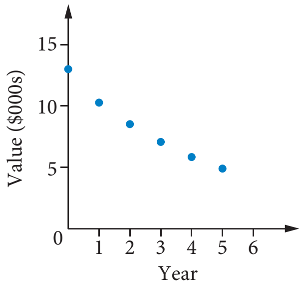
E $u_0 = 134\,000, \quad u_{n+1} = 1.085 \times u_n - 8576$

9 © VCAA 2010 1BRMQ5 57% A file server costs \$30 000. The file server depreciates by 20% of its value each year. After three years, its value is

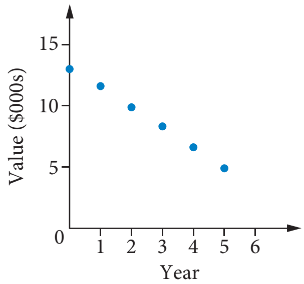
A \$6000 **B** \$12 000 **C** \$15 360 **D** \$19 200 **E** \$24 000

10 © VCAA 2004 1BRMQ8 modified 49% A machine is purchased for \$13 000. Using the reducing balance method of depreciation, its value after five years will be \$5000. The graph that best represents the value of the machine after each year over the five-year period is

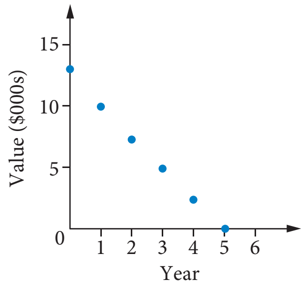
A



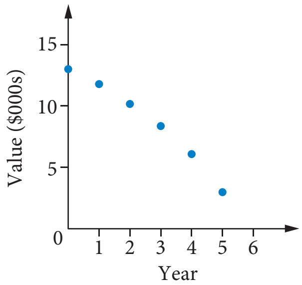
B



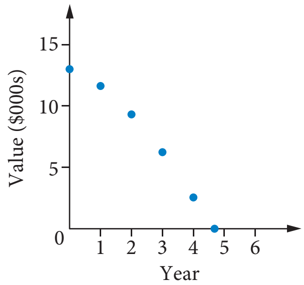
C



D



E



11 A business purchased a computer for \$4000. It is depreciated using reducing balance depreciation at a rate of 15% per annum. What are the two values for $n = 2$ in the depreciation table?

n	Depreciation after n years (\$)	Value after n years (\$)
0	–	4000
1		
2		

A 600 and 3400 **B** 510 and 2890 **C** 600 and 510

D 15 and 4000 **E** $\frac{15}{100}$ and 3400

12 © VCAA 2019 2CQ7abc (3 marks) Phil is a builder who has purchased a large set of tools. The value of Phil's tools is depreciated using the reducing balance method. The value of the tools, in dollars, after n years, u_n, can be modelled by the recurrence relation

$$u_0 = 60\,000, \quad u_{n+1} = 0.9u_n$$

a 74% Use recursion to show that the value of the tools after two years, u_2, is \$48 600. 1 mark

b 66% What is the annual percentage rate of depreciation used by Phil? 1 mark

c 79% Phil plans to replace these tools when their value first falls below \$20 000. After how many years will Phil replace these tools? 1 mark

13 © VCAA 2017 2CQ5c 58% (1 mark) Alex is a mobile mechanic. The value of his van is depreciated using the reducing balance method of depreciation. The value of the van, in dollars, after n years, R_n, can be modelled by the recurrence relation

$$R_0 = 75\,000, \quad R_{n+1} = 0.943R_n$$

At what annual percentage rate is the value of the van depreciated each year?

14 © VCAA 2009 2BRMQ4 (4 marks) A golf club management purchased new lawn mowers for \$22 000.

a Use the flat rate depreciation method with a depreciation rate of 12% per annum to find the depreciated value of the lawn mowers after four years. 2 marks

b Use the reducing balance depreciation method with a depreciation rate of 16% per annum to calculate the depreciated value of the lawn mowers after four years. Write your answer in dollars correct to the nearest cent. 1 mark

c After four years, which method, flat rate depreciation or reducing balance depreciation, will give the greater depreciation? Write down the greater depreciation amount in dollars correct to the nearest cent. 1 mark

3.4 Percentages and financial mathematics

Video playlist Percentages and financial mathematics

Financial percentage change

As we have seen, an understanding of percentages is crucial for financial mathematics. For example, sellers often change the price of an item for sale. When the price of an item is increased by a percentage, it's called a **mark-up**. When the price of an item is reduced by a percentage, it's called a **discount**. For a mark-up the change in price is positive, and for a discount the change in price is negative.

Mark-ups and discounts

For an item marked up by r%:

$$\text{change in price} = \text{original price} \times \frac{r}{100}$$

$$\text{new price} = \text{original price} + \left(\text{original price} \times \frac{r}{100}\right)$$

For an item discounted by r%:

$$\text{change in price} = -\left(\text{original price} \times \frac{r}{100}\right)$$

$$\text{new price} = \text{original price} - \left(\text{original price} \times \frac{r}{100}\right)$$

WB

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WORKED EXAMPLE 11 Calculating mark-ups and discounts

The original price of a pair of jeans is $80.

a What is the change in price if the pair of jeans is

i marked up by 20%

ii discounted by 12%.

b Find the new price if the pair of jeans is

i marked up by 8%

ii discounted by 15%.

Steps	Working
a **i** change in price = original price × $\frac{r}{100}$	change in price = $80 \times \frac{20}{100} = 16$ The change in price is $16.00.
ii change in price = $-\left(\text{original price} \times \frac{r}{100}\right)$	change in price = $-\left(80 \times \frac{12}{100}\right) = -9.6$ The change in price is −$9.60.
b **i** new price = original price + $\left(\text{original price} \times \frac{r}{100}\right)$	new price = $80 + \left(80 \times \frac{8}{100}\right)$ $= 80 + 6.4$ $= 86.4$ The new price is $86.40.
ii new price = original price − $\left(\text{original price} \times \frac{r}{100}\right)$	new price = $80 - \left(80 \times \frac{15}{100}\right)$ $= 80 - 12$ $= 68$ The new price is $68.

Exam hack

Don't round your answer unless you are asked to.

We can calculate the **percentage change**, r, for a discount and mark-up if we know the original price and new price. Percentage change is always a positive number.

Finding the percentage change, *r*

If the price of an item is marked up, then:

new price > original price

For an item marked up by r%:

$$r = \frac{\text{new price} - \text{original price}}{\text{original price}} \times 100\%$$

If the price of an item is discounted, then:

original price > new price

For an item discounted by r%:

$$r = \frac{\text{original price} - \text{new price}}{\text{original price}} \times 100\%$$

Exam hack

Make sure to divide by the original price, not the new price, when calculating percentage change.

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3.4

WORKED EXAMPLE 12 Finding the percentage change, *r*

The original price of a pair of jeans is \$80.

a Find the percentage mark-up if it has been increased to \$100.

b Find the percentage discount if it has been discounted to \$60.

Steps	Working
a $r = \dfrac{\text{new price} - \text{original price}}{\text{original price}} \times 100\%$	$r = \dfrac{100 - 80}{80} \times 100\%$ $= \dfrac{20}{80} \times 100\%$ $= 25\%$ The percentage mark-up is 25%.
b $r = \dfrac{\text{original price} - \text{new price}}{\text{original price}} \times 100\%$	$r = \dfrac{80 - 60}{80} \times 100\%$ $= \dfrac{20}{80} \times 100\%$ $= 25\%$ The percentage discount is 25%.

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We can also find the original price if we know the new price and *r*.

Finding the original price

For an item marked up by *r*%:

$$\text{original price} = \text{new price} \times \frac{100}{(100 + r)}$$

For an item discounted by *r*%:

$$\text{original price} = \text{new price} \times \frac{100}{(100 - r)}$$

WORKED EXAMPLE 13 Finding the original price

WB p. 51

Find the original price of a pair of jeans, to the nearest cent, if

a the new price is \$120 and the mark-up is 30%

b the new price is \$44 and the discount is 60%.

Steps	Working
a **1** original price = new price × $\dfrac{100}{(100 + r)}$	original price = $120 \times \dfrac{100}{(100 + 30)}$ $= 120 \times 0.769\ldots$ $= 92.307\ldots$
2 Give the answer to the nearest cent.	The original price was \$92.31.
b **1** original price = new price × $\dfrac{100}{(100 - r)}$	original price = $44 \times \dfrac{100}{(100 - 60)}$ $= 44 \times 2.5$ $= 110$
2 Give the answer to the nearest cent.	The original price was \$110.00.

GST

The **Goods and Services Tax** (**GST**) is currently a 10% tax on most sales and services in Australia. We can think of this as a mark-up where

price without GST = original price

price with GST = new price

GST amount = change in price

$r = 10\%$

Working with GST

$r = 10\%$

GST amount = price without GST × 0.1

$$\text{GST amount} = \frac{\text{price with GST}}{11}$$

price with GST = price without GST × 1.1

$$\text{price without GST} = \frac{\text{price with GST}}{1.1}$$

WB

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WORKED EXAMPLE 14 Working with GST

Round the following answers to the nearest dollar.

a A car is advertised at a price of $34 500, which includes GST.

i What is the price excluding GST?

ii How much of the advertised price is GST?

b A piece of software has a GST-free price of $190.

i What is the price with GST?

ii How much GST is payable?

Steps	Working
a **i** Use price without GST = $\frac{\text{price with GST}}{1.1}$, rounding to the nearest dollar.	price without GST = $\frac{34\,500}{1.1} = 31\,363.636\ldots$ The price without GST is $31 364.
ii Use GST amount = $\frac{\text{price with GST}}{11}$, rounding to the nearest dollar.	GST = $\frac{34\,500}{11} = 3136.363\ldots$ The GST amount is $3136.
b **i** Use price with GST = price without GST × 1.1, rounding to the nearest dollar.	Use price with GST = 190 × 1.1 = 209 The price with GST is $209.
ii Use GST amount = price without GST × 0.1, rounding to the nearest dollar.	GST = 190 × 0.1 = 19 The GST amount is $19.

The unitary method

The **unitary method** is a way of solving problems by calculating the value of one unit.

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WORKED EXAMPLE 15 Using the unitary method

Answer the following questions using the unitary method.

a If 12 doughnuts cost $26.40, find the cost of

i one doughnut **ii** 200 doughnuts.

b If 6 packets of crackers weigh 570 grams, find the weight of

i one packet **ii** 20 packets.

Steps				Working
a	i	1	Write the known equality.	12 doughnuts cost \$26.40.
		2	Identify the quantity being asked for.	cost
		3	Find the cost of one doughnut: the unit amount.	$\text{unit amount} = \frac{26.40}{12} = 2.2$
		4	Write the answer in words.	One doughnut costs \$2.20.
	ii	1	Multiply the unit amount by the required number.	$2.2 \times 200 = 440$
		2	Write the answer in words.	200 doughnuts cost \$440.
b	i	1	Write the known equality.	6 packets weigh 570 grams.
		2	Identify the quantity being asked for.	weight
		3	Find the mass of one packet: the unit amount.	$\text{unit amount} = \frac{570}{6} = 95$
		4	Write the answer in words.	One packet weighs 95 grams.
	ii	1	Multiply the unit amount by the required number.	$20 \times 95 = 1900$
		2	Write the answer in words.	20 packets of crackers weigh 1900 grams.

3.4

If a percentage of an amount is known, then the unitary method can be used by dividing to find 1% (i.e. one unit). The whole amount can then be found by multiplying by 100 to find 100%.

WORKED EXAMPLE 16 Using the unitary method for percentages

A town has 4745 adults and its population is 65% adults.

a What is the total population of the town?

b If 45% of the town's population are female, how many females are there?

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Steps			Working
a	1	Write the known equality.	65% of the population = 4745.
	2	Identify the quantity being asked for.	population
	3	Divide this amount by the number of units to find the unit amount (i.e. 1% of the population).	$\text{unit amount} = \frac{4745}{65} = 73$ 1% of the population is 73.
	4	Multiply the unit amount by 100 to find the whole amount (i.e. 100% of the population).	$100 \times 73 = 7300$
	5	Write the answer in words.	The total population of the town is 7300.
b	1	Multiply the unit amount by the required number.	45% of the population is $45 \times 73 = 3285$.
	2	Write the answer in words.	There are 3285 females in the town.

The unitary method is particularly useful when comparing the prices of the same product in different sizes to see which one is the best value.

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WORKED EXAMPLE 17 Using the unitary method to make comparisons

Paul is buying washing powder. His brand comes in three different sizes: 350 g for \$1.86, 800 g for \$3.28 and 2 kg for \$8.70.

a What is the unit cost of each of the sizes? Give your answer correct to four decimal places.

b Which size is the best value?

Steps	Working
a **1** Write the known equality for each of the sizes. If necessary, convert units of measurement so that all the units are the same.	350 g is \$1.86 800 g is \$3.28 2 kg is \$8.70
2 Identify the quantity being asked for.	cost
3 Divide this quantity to find the unit amounts for each of the sizes (i.e. the cost of one gram). Give your answer correct to four decimal places.	The cost of one gram in each case is: $\frac{\$1.86}{350} = \$0.0053$ $\frac{\$3.28}{800} = \$0.0041$ $\frac{\$8.70}{2000} = \$0.0044$
b Compare to find the cheapest and write the answer in words.	The 800 g for \$3.28 is the best value because it is the cheapest per gram.

Exam hack

When calculating the best value, always makes sure the units of measurement you are comparing are the same.

EXERCISE 3.4 Percentages and financial mathematics

ANSWERS p. 500

Recap

1 Which of the following recurrence relations best models a car purchased at \$32 000 being depreciated using reducing balance depreciation at a rate of 11% per annum?

A $u_n = 0.11^n \times 32\,000$

B $u_n = 0.89^n \times 32\,000$

C $u_0 = 32\,000, \quad u_{n+1} = 0.11u_n$

D $u_0 = 32\,000, \quad u_{n+1} = 0.89u_n$

E $u_0 = 32\,000, \quad u_n = 0.89u_{n+1}$

2 A car costs \$32 000. The car depreciates by 15% of its value each year. After five years, its value to the nearest cent is

A \$14 198.57 **B** \$16 704.20 **C** \$19 652.00 **D** \$24 000.00 **E** \$27 200.00

Mastery

3 WORKED EXAMPLE 11 The original price of a shirt is \$60.

a What is the change in price if the shirt is

i marked up by 20% **ii** discounted by 12%

iii discounted by 9% **iv** marked up by 35%.

b Find the new price if the shirt is

i marked up by 25% **ii** discounted by 15%

iii marked up by 6% **iv** discounted by 8%.

3.4

4 WORKED EXAMPLE 12 The original price of a pair of shorts is $50.

a Find the percentage mark-up if it has been increased to

i $70 **ii** $85 **iii** $60

b Find the percentage discount if it has been discounted to

i $40 **ii** $45 **iii** $25

5 WORKED EXAMPLE 13 Find the original price of a jacket to the nearest cent if

a the new price is $150 and the mark-up is 20%

b the new price is $45 and the discount is 60%

c the new price is $99 and the discount is 25%

d the new price is $200 and the mark-up is 35%.

6 WORKED EXAMPLE 14 Round the following answers to the nearest dollar.

a A television set is advertised at a price of $2850, which includes GST.

i What is the price excluding GST? **ii** How much of the advertised price is GST?

b A laptop has a GST-free price of $1795.

i What is the price with GST? **ii** How much GST is payable?

c A haircut has a GST-free price of $89.

i What is the price with GST? **ii** How much GST is payable?

d A boat is advertised at a price of $159 106, which includes GST.

i What is the price excluding GST? **ii** How much of the advertised price is GST?

7 WORKED EXAMPLE 15 Answer the following questions using the unitary method.

a If 12 cupcakes cost $14.64, find the cost of

i one cupcake **ii** 100 cupcakes.

b If 8 packets of nuts weighs 520 grams, find the weight of

i one packet **ii** 30 packets.

c If a factory can make 600 widgets in 5 days, find the number of widgets it can make in

i one day **ii** 30 days.

d If a theatre buys 120 seats for $10 440, find the cost of

i one seat **ii** 600 seats.

8 WORKED EXAMPLE 16

a A school has 242 Year 11 students and Year 11 students make up 22% of its student numbers.

i How many students are at the school?

ii If 35% take public transport to school, how many students take public transport to school?

b A computer file was being downloaded. The computer showed that it had downloaded 28% and this was 112 megabytes.

i How many megabytes is the total size of the file?

ii How many megabytes have been downloaded when the download is 72% complete?

9 WORKED EXAMPLE 17 For each pair of sizes

i what is the unit cost of each size? Give your answer correct to four decimal places.

ii which size is the better value?

a 150 mL for $2.36 or 1 L for $14.80

b 36 cans for $20 or 15 cans for $8

c $8.40 for 500 g or $20.99 for 1.2 kg

Exam practice

10 © VCAA 2013 1BRMQ1 92% A phone that normally retails for $200 is discounted to $170. The percentage discount is

A 10% **B** 15% **C** 20% **D** 25% **E** 30%

11 © VCAA 2014 1BRMQ1 85% This month, a business charges $1500 to install a water tank. Next month, the charge will increase by 3.5%. The charge next month will be

A $45.00 **B** $52.50 **C** $1545.00 **D** $1552.50 **E** $1950.00

12 © VCAA 2014 1BRMQ4 83% The cost of hiring a plasterer is $86.00 per hour plus GST of 10%. The cost of hiring a plasterer for four hours, including GST, is

A $120.40 **B** $309.60 **C** $344.00 **D** $352.60 **E** $378.40

13 © VCAA 2015 1BRMQ3 77% The closing price of a share on Wednesday was $160. The closing price of the same share on Thursday was 3% less than its closing price on Wednesday. The closing price of the same share on Friday was 4.5% more than its closing price on Thursday. The closing price of the share on Friday is closest to

A $157.38 **B** $161.98 **C** $162.18 **D** $162.40 **E** $172.22

14 © VCAA 2011 1BRMQ1 77% An electrician charges $68 per hour to complete a job. A Goods and Services Tax (GST) of 10% is added to the charge. Including GST, the cost of a job that takes three hours is

A $6.80 **B** $20.40 **C** $204.00 **D** $210.80 **E** $224.40

15 Which is the best buy for packets of a particular pasta?

A 2 kg for $17.80 **B** 250 g for $2.40 **C** 1.2 kg for $10.85
D 700 g for $6.30 **E** 400 g for $3.65

16 © VCAA 2014 2BRMQ1ab (2 marks) The adult membership fee for a cricket club is $150. Junior members are offered a discount of $30 off the adult membership fee.

a Write down the discount for junior members as a percentage of the adult membership fee. 1 mark

Adult members of the cricket club pay $15 per match in addition to the membership fee of $150.

b If an adult member played 12 matches, what is the total this member would pay to the cricket club? 1 mark

17 © VCAA 2015 2BRMQ1 63% (3 marks) Jane and Michael have started a business that provides music at parties. The business charges customers $88 per hour. The $88 per hour includes a 10% Goods and Services Tax (GST).

a Calculate the amount of GST included in the $88 hourly rate. 1 mark

Jane and Michael's first booking was a party where they provided music for four hours.

b Calculate the total amount they were paid for this booking. 1 mark

After six months of regular work, Jane and Michael decided to increase the hourly rate they charge by 12.5%.

c Calculate the new hourly rate (including GST). 1 mark

3.5 Inflation and purchasing options

Inflation

Inflation occurs when prices rise over time. We can think of single item inflation rate calculations as a mark-up calculation, where the original price and new price can be years apart.

Single item inflation rates

To calculate the single item inflation rate, r, over time, use

$$r = \frac{\text{new price} - \text{original price}}{\text{original price}} \times 100\%$$

WORKED EXAMPLE 18 Finding single item inflation rates

What is the single item inflation rate, r, that measures the change in price of the following items over the time periods given? Give your answer rounded to one decimal place.

a A loaf of white bread cost 12 cents in 1952 and \$2.80 in 2022.

b Melbourne's median house price was \$177 500 in 1997 and \$974 397 in 2021.

Steps	Working
a 1 Use $r = \frac{\text{new price} - \text{original price}}{\text{original price}} \times 100\%$	$r = \frac{2.8 - 0.12}{0.12} \times 100\% = 2233.333\ldots$
2 Give your answer rounded to one decimal place.	The single item inflation rate is 2233.3%.
b 1 Use $r = \frac{\text{new price} - \text{original price}}{\text{original price}} \times 100\%$	$r = \frac{974\,397 - 177\,500}{177\,500} \times 100\% = 448.956\ldots$
2 Give your answer rounded to one decimal place.	The single item inflation rate is 449.0%.

The official yearly inflation rate of a country is an average based on the individual inflation rates of many items. Each year's inflation compounds with the previous year's inflation, so it works the same way as a compound interest investment with the rate changing every year.

The effect of the yearly inflation rate on a single item

To find the effect of the yearly inflation rate on the price of a single item u_n, after n years, use the compound interest recurrence relation

$$u_0 = \text{original price}, \; u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$$

and change the value of r each year.

Video playlist Inflation and purchasing options

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WORKED EXAMPLE 19 Working with inflation rates

A country has an official inflation rate of 3.2% in 2023, 2.9% in 2024, and 2.1% in 2025.

a If the price of a loaf of white bread at the start of 2023 is $2.80, and its price rises match the official inflation rate exactly, show the calculation that will give the price of the loaf to the nearest cent

i at the end of 2023

ii at the end of 2024

iii at the end of 2025.

b What is the single item inflation rate of the loaf of bread from the start of 2023 to the end of 2025, rounded to one decimal place?

c Show that the single item inflation rate is not the same as adding together the inflation rates in each of the three years, and explain why this is so.

Steps	Working
a i 1 Use the recurrence relation u_0 = original price, $u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$.	$n = 1, r = 3.2, u_0 = 2.8$ $u_1 = \left(1 + \frac{r}{100}\right)u_0$ $= \left(1 + \frac{3.2}{100}\right) \times 2.8 = 2.8896$
2 Write the answer, rounding to the nearest cent.	The price of the loaf of bread at the end of 2023 is $2.89.
ii 1 Use the recurrence relation u_0 = original price, $u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$. Use the unrounded previous answer.	$n = 2, r = 2.9$ $u_2 = \left(1 + \frac{r}{100}\right)u_1$ $= \left(1 + \frac{2.9}{100}\right) \times 2.8896 = 2.9733\ldots$
2 Write the answer, rounding to the nearest cent.	The price of the loaf of bread at the end of 2024 is $2.97.
iii 1 Use the recurrence relation u_0 = original price, $u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$. Use the unrounded previous answer.	$n = 3, r = 2.1$ $u_3 = \left(1 + \frac{r}{100}\right)u_2$ $= \left(1 + \frac{2.1}{100}\right) \times 2.9733\ldots = 3.0358\ldots$
2 Write the answer, rounding to the nearest cent.	The price of the loaf of bread at the end of 2025 is $3.04.
b 1 Use $r = \frac{\text{new price} - \text{original price}}{\text{original price}} \times 100\%$	$r = \frac{3.0358\ldots - 2.8}{2.8} \times 100\% = 8.5714\ldots$
2 Give your answer rounded to one decimal place.	The single item inflation rate is 8.6%.
c 1 Add the inflation rates in each of the three years and compare this with the single item inflation rate.	3.2% + 2.9% + 2.1% = 8.2% The single item inflation rate is 8.6%.
2 Refer to the compounding nature of inflation.	Each year's inflation compounds with the previous year's inflation.

Exam hack

Remember to use the unrounded calculations at each step.

The purchasing power of money

The result of inflation is that the **purchasing power** of money reduces, so that the same amount of money one year is actually worth less in the following year because it can no longer buy the same amount of goods and services.

Purchasing power of money

To find the purchasing power of money, u_n, after n years, where the inflation rates are averaged over a number of years, use the reducing balance depreciation rule

$$u_n = \left(1 - \frac{r}{100}\right)^n \times u_0$$

where

u_0 = original amount of money

r = the average percentage inflation rate per year

n = the number of years.

WORKED EXAMPLE 20 Calculating the purchasing power of money

Quentin has kept a $100 note in his pocket for eight years. Over that time, inflation has averaged 2.7% each year. Find the purchasing power of the $100 note after eight years to the nearest cent?

Steps	Working
a **1** The inflation rate is given as an average over a number of years, so use the reducing balance depreciation rule $u_n = \left(1 - \frac{r}{100}\right)^n \times u_0$	$r = 2.7$, $u_0 = 100$, $n = 8$ $u_8 = \left(1 - \frac{2.7}{100}\right)^8 \times 100$ $= 80.3346$
2 Write you answer, rounding to the nearest cent.	The purchasing power of the $100 note is $80.33 after eight years.

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Comparing purchasing options

There are a number of options when it comes to making purchases:

Cash – A cash payment is made with actual notes and coins.

Debit cards – A debit card payment is made using money in a bank account.

Credit cards – A credit card payment is made with the bank's money. The payment will need to be repaid when the credit card payment is due (usually a month later). If the amount is not repaid in full, the bank will charge interest on the amount, which compounds daily.

Buy now pay later service (**BNPL**) – A BNPL platform payment is where only a percentage of the purchase price (e.g. 25%) is paid at the time of purchase. The rest of the purchase price is paid in regular instalments (e.g. 25% per fortnight). If payments are not made, a late fee is charged.

Personal loans – A personal loan payment involves a signed contract with a lender stating that the purchased item will be paid for in instalments, usually longer term than a BNPL platform, based on a quoted percentage rate of interest per annum. If a payment is missed, the amount of interest increases. Failure to make the payments can result in the lender taking back the purchased item.

Personal loans

To calculate a new balance on a personal loan, use

$$\text{new balance} = \text{old balance} + \left(\frac{r}{100} \times \text{old balance}\right) - \text{regular payment}$$

where

r is the percentage interest rate per compounding period.

total amount paid = number of payments × regular payment amount

total interest = total amount paid – loan amount

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WORKED EXAMPLE 21 Understanding personal loans

Ernestine has taken out a two-year \$8000 personal loan at 6% per annum, compounding monthly, with annual payments of \$354.56.

a Find the monthly interest rate.

Show the following calculations to the nearest cent.

b What is the balance of the loan

i after one month? **ii** after two months? **iii** after three months?

c Find the total amount of money Ernestine will have paid on this loan at the end of the two years.

d What is the total interest Ernestine will pay on this loan?

Steps	Working
a Divide the yearly interest rate by the number of compounding periods per year.	percentage interest rate per month $= \frac{6}{12}\%$ $= 0.5\%$
b new balance = old balance + $\left(\frac{r}{100} \times \text{old balance}\right)$ – regular payment, where r is the percentage interest rate per compounding period.	**i** \$8000 + (0.005 × \$8000) – \$354.56 = \$7685.44 **ii** \$7685.44 + (0.005 × \$7685.44) – \$354.56 = \$7369.31 **iii** \$7369.31 + (0.005 × \$7369.31) – \$354.56 = \$7051.60
c total amount paid = number of payments × regular payment amount	total amount paid = 24 × \$354.56 = \$8509.44
d total interest = total amount paid – loan amount	total interest = \$8509.44 – \$8000 = \$509.44

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WORKED EXAMPLE 22 Comparing purchasing options

Bruce wants to buy a bicycle from Cycles Unlimited. He doesn't have the \$1500, so he is looking at his options.

a Calculate the cost of each of the following over one year to the nearest dollar.

i Bruce decides to wait six months until he has enough cash to pay for the bicycle, and it ends up being on sale at a 12% discount.

ii Bruce gets a debit card with a \$5 monthly bank fee, and Cycles Unlimited has a 1% merchant surcharge.

iii Bruce uses a buy now pay later service (BNPL) of four fortnightly payments but ends up paying a \$68 late fee.

iv Bruce gets a credit card and ends up paying 22% per annum interest for 30 days, compounding daily, because he couldn't make the payment.

v Bruce takes out a personal loan at 20% per annum compound interest, compounding monthly, which involves making 12 monthly payments of \$138.95.

b Which option has the lowest cost?

Steps	Working
a **i** **1** For an item discounted by r%: new price = original price − $\left(\text{original price} \times \frac{r}{100}\right)$	new price = $1500 - \left(1500 \times \frac{12}{100}\right)$ = \$1320
2 State the answer to the nearest dollar.	cost of bicycle = \$1320
ii **1** For an item marked up by r%: new price = original price + $\left(\text{original price} \times \frac{r}{100}\right)$	new price = $1500 + \left(1500 \times \frac{1}{100}\right)$ = \$1515
2 Calculate the total bank fee for the year.	total bank fee for year = 12 × \$5 = \$60
3 State the answer to the nearest dollar.	cost of bicycle = \$1515 + \$60 = \$1575
iii Add the late fee to the price.	cost of bicycle = \$1500 + \$68 = \$1568
iv **1** Divide the yearly interest rate by the number of compounding periods per year.	percentage interest rate per day = $\frac{22}{365}$% = 0.06%
2 Use the interest general rule for compound interest $u_n = \left(1 + \frac{r}{100}\right)^n \times u_0$	$n = 30$, $r = 0.06\%$, $u_0 = 1500$; $u_{30} = \left(1 + \frac{0.06}{100}\right)^{30} \times 1500$ = 1527.236…
3 State the answer to the nearest dollar.	cost of bicycle = \$1527
v Multiply the payment by the number of months.	cost of bicycle = 12 × \$138.95 = \$1667.40
b Which cost is the lowest?	The lowest cost is to wait 6 months and pay cash for the item when on sale.

VCE QUESTION ANALYSIS

 2006 2BRMQ1ab MODIFIED 2006 Examination 2 Business-related mathematics Question 1ab modified (7 marks)

A company purchased a machine for \$60 000. For taxation purposes the machine is depreciated over time. Two methods of depreciation are considered.

a **Flat rate depreciation**

The machine is depreciated at a flat rate of 10% of the purchase price each year.

- **i** By how many dollars will the machine depreciate annually? 1 mark
- **ii** Calculate the value of the machine after three years. 1 mark
- **iii** When will the machine be \$30 000 in value? 1 mark

b **Reducing balance depreciation**

The value, u_n, of the machine after n years is given by the rule $u_n = 60\,000 \times (0.85)^n$.

- **i** By what percentage will the machine depreciate annually? 1 mark
- **ii** Calculate the value of the machine after three years. 1 mark
- **iii** When will the machine's value first fall below \$30 000? 1 mark

c Which of the two methods results in the machine depreciating faster? 1 mark

Video playlist VCE question analysis: Geometric sequences and financial mathematics

Reading the question

- The question is comparing two methods of depreciation.
- The reducing balance rule is given in part **b**.

Thinking about the question

- 'First fall below' indicates you should use a recurrence relation.
- What does 'depreciating faster' mean?

Worked solution (✓ = 1 mark)

a **i** $\frac{10}{100} \times \$60\,000 = \mathbf{\$6000}$ ✓

ii Use the flat rate depreciation rule $u_n = u_0 - nd$

$u_0 = 60\,000$, $d = 6000$, $n = 3$

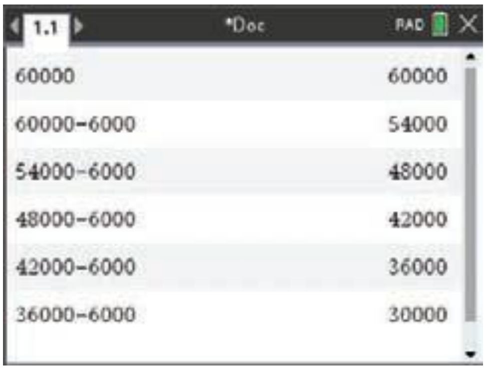
The value of the machine after three years is $u_3 = 60\,000 - 3 \times 6000 = \mathbf{\$42\,000}$ ✓

iii Use the flat rate depreciation recurrence relation u_0 = initial value of the asset, $u_{n+1} = u_n - d$

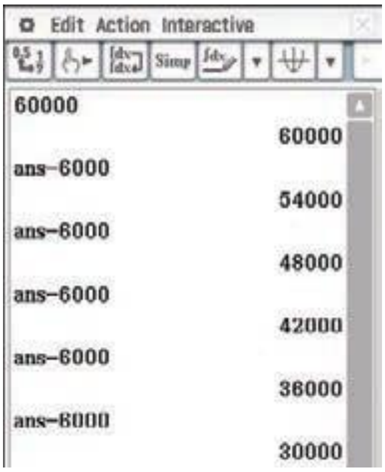
$u_0 = 60\,000$, $\quad u_{n+1} = u_n - 6000$

Use CAS recursive computation to find when the value is \$30 000.

TI-Nspire



ClassPad



The machine will be \$30 000 in value after **five years.** ✓

b **i** Use the reducing balance depreciation rule $u_n = \left(1 - \frac{r}{100}\right)^n \times u_0$.

Comparing $u_{n+1} = 60\,000 \times (0.85)^{n-1}$ with this, we can see $\left(1 - \frac{r}{100}\right) = 0.85$.

Solve for r, using CAS if necessary.

$\frac{r}{100} = 1 - 0.85 = 0.15$

$r = 15$

The machine will be depreciated annually by **15%**. ✓

ii Use the rule given: $u_n = 60\,000 \times (0.85)^n$

The value of the machine after three years is $u_3 = 60\,000 \times (0.85)^3 =$ **\$36 847.50** ✓

iii Use the reducing balance depreciation recurrence relation

u_0 = initial value of the asset, $u_{n+1} = \left(1 - \frac{r}{100}\right) \times u_n$

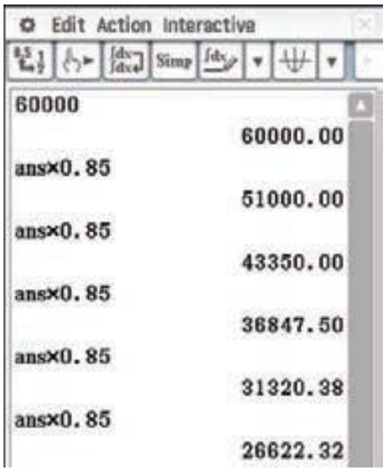
From part **b i**:

$\left(1 - \frac{r}{100}\right) = 0.85$, so $u_{n+1} = 0.85 \times u_n$

Use CAS recursive computation to find when the value is first below \$30 000.

TI-Nspire

ClassPad



The machine's value will first fall below \$30 000 after **five years.** ✓

c Compare the value of the machine for each of the two methods after each year.

n	Flat rate value	Reducing balance value	Difference
1	\$54 000	\$51 000.00	\$3000.00
2	\$48 000	\$43 350.00	\$4650.00
3	\$42 000	\$36 847.50	\$5152.50

The reducing balance value is always less than the flat rate value and the difference is increasing, so **reducing balance depreciation** ✓ results in the the machine depreciating faster.

Student performance

a **i** This was a straightforward use of $d = \frac{r}{100} \times u_0$

ii This was a straightforward use of the flat rate depreciation rule $u_n = u_0 - nd$

iii The fastest method for this is to use CAS recursive computation.

b **i** An incorrect answer of 85% was common.

ii Many students achieved this answer despite having part **i** incorrect.

iii The fastest method for this is to use CAS recursive computation.

c Students needed to compare the two sets of recursive computation values on CAS.

EXERCISE 3.5 Inflation and purchasing options

ANSWERS p. 500

Recap

80–100% 60–79% 0–59%

1 © VCAA 2015 1BRMQ1 83% Fong's gas bill is $368.40. If he pays this bill on time, it will be reduced by 5%. In this case, the bill would be reduced by

A $1.84 **B** $5.00 **C** $18.42 **D** $184.20 **E** $349.98

2 Sam has driven for 858 km. Calculate the total length of his journey if this distance represents 65% of his journey.

A 557.7 km **B** 1320 km **C** 1440 km **D** 1630 km **E** 55 770 km

Mastery

3 WORKED EXAMPLE 18 What is the single item inflation rate *r* that measures the change in price of the following items over the time periods given? Give your answer rounded to one decimal place.

a The average price of a new car in 1984 was $14 000 and $40 472 in 2021.

b A litre of milk cost 19 cents in 1970 and $1.29 in 2022.

c A kilo of sugar cost 42 cents in 1970 and $2.40 in 2022.

d The average cost of a television set in 1955 was $250 and $1277 in 2022.

4 WORKED EXAMPLE 19 A country has an official inflation rate of 2.7% in 2023, 3.5% in 2024, and 3.3% in 2025.

a If the price of a litre of milk at the start of 2023 is $1.29, and its price rises match the official inflation rate exactly, show the calculation that will give the price of the litre of milk to the nearest cent

i at the end of 2023

ii at the end of 2024

iii at the end of 2025.

b What is the single item inflation rate of a litre of milk from the start of 2023 to the end of 2025, rounded to one decimal place?

c Show that the single item inflation rate is not the same as adding together the inflation rates in each of the three years, and explain why this is so.

5 WORKED EXAMPLE 20 For each of the following, find the purchasing power of the amounts after the given number of years to the nearest cent.

a Gloria kept a $50 note in her pocket for nine years. Over that time inflation has averaged 2.1% each year.

b Mason hid $1000 in a shoe box for five years. Over that time inflation has averaged 3.6% each year.

c The Wallace family buried $26 000 for sixteen years. Over that time inflation has averaged 3.3% each year.

6 WORKED EXAMPLE 21 Max has taken out a two-year $10 000 personal loan at 9% per annum, compounding monthly, with annual payments of $456.85.

a Find the monthly interest rate.

Show the following calculations to the nearest cent.

b What is the balance of the loan

i after one month? **ii** after two months? **iii** after three months?

c Find the total amount of money Max will have paid on this loan at the end of the two years.

d What is the total interest Max will pay on this loan?

7 WORKED EXAMPLE 22 Zoltan wants to buy a gaming computer from TickTech. He doesn't have the $2500, so he is looking at his options.

a Calculate the cost of each of the following over one year to the nearest dollar.

i Zoltan decides to wait six months until he has enough cash to pay for the computer, and it ends up being on sale at a 15% discount.

ii Zoltan gets a debit card with a $6 monthly bank fee, and Cycles Unlimited has a 1.5% merchant surcharge.

iii Zoltan uses a buy now pay later service (BNPL) of four fortnightly payments but ends up paying a $68 late fee.

iv Zoltan gets a credit card and ends up paying 22% per annum interest for 30 days, compounding daily, because he couldn't make the payment.

v Zoltan takes out a personal loan at 24% per annum compound interest, compounding monthly, which involves making 12 monthly payments of $236.40.

b Which option has the lowest cost?

3.5

Exam practice

80–100% 60–79% 0–59%

8 The average price of 180 grams of tea cost 6 cents in 1901 and $3.50 in 2021. The single item inflation rate is closest to

A 3% **B** 98% **C** 99% **D** 5733% **E** 5734%

9 Which of the following will have a single item inflation rate closest to 150%?

A A product that was $60 ten years ago and is now $90.

B A product that was $60 ten years ago and is now $110.

C A product that was $60 ten years ago and is now $120.

D A product that was $60 ten years ago and is now $150.

E A product that was $60 ten years ago and is now $180.

Use the following information to answer the next two questions.

A country has an official inflation rate of 2.2% in 2022 and 3.8% in 2023. The price of a block of 100 grams of chocolate was $3.50 at the start of 2022 and its price rises match the inflation rate.

10 Which of the following calculations will give the price of the block of chocolate to the nearest cent at the end of 2023?

A $3.5 \times 2.2 + 3.5 \times 3.8$ **B** $3.5 \times 1.022 \times 1.038$ **C** $3.5 \times 0.078 \times 0.062$

D $3.5 \times 0.022 \times 0.038$ **E** $3.5 \times 2.2 \times 3.8$

11 By how much did the price of the block of chocolate rise from the start of 2022 to the end of 2023?

A $0.08 **B** $0.21 **C** $0.71 **D** $3.58 **E** $3.71

Use the following information to answer the next two questions.

Twenty \$100 notes were hidden in a mattress for ten years during which inflation has averaged 3.4% each year.

12 The calculation to find the purchasing power after ten years is

A $(0.966)^{10} \times 2000$ **B** $(1.034)^{10} \times 100$ **C** $(1.034)^{10} \times 2000$

D $(0.966)^{10} \times 100$ **E** $(1.034)^{10} \times 20$

13 What is the dollar amount of purchasing power that has been lost?

A \$29 **B** \$584 **C** \$585 **D** \$1415 **E** \$1416

14 (7 marks) Laura and Luisa are planning a holiday that costs \$6000. Laura will pay for the holiday using her credit card and expects to pay the 20% p.a interest, compounding daily, on the amount for 60 days.

a What is the daily interest rate, rounded to four decimal places? 1 mark

b What will Laura owe on her credit card after the 60 days, rounded to the nearest cent? 1 mark

c How much interest will Laura end up paying, rounded to the nearest cent? 1 mark

Luisa will take out a personal loan for the \$6000 and pay 8% p.a. interest, compounding monthly, for two years. She will pay monthly instalments of \$271.36.

d What is the monthly interest rate, rounded to three decimal places? 1 mark

e Show a calculation that gives the balance of Luisa's loan after one month. 1 mark

f How much interest will Luisa end up paying, rounded to the nearest cent? 1 mark

g Which of the two will end up paying less for their holiday? 1 mark

3 Chapter summary

Geometric sequences

- A **geometric sequence** is a sequence where we *multiply* by a fixed amount, called the **common ratio**, R, to generate each new value.
- For a geometric sequence $u_0, u_1, u_2, u_3 \ldots$

recurrence relation	***n*th value rule**
$u_0 = a, \quad u_{n+1} = Ru_n$	$u_n = aR^n$

where

u_{n+1} is the value after u_n

a is the first value

$R = \dfrac{\text{any value}}{\text{previous value}}$ is the common ratio between values.

Compound interest

- **Compound interest** is where the interest is added to the principal, and the interest for the next time period is calculated using this new balance.
- The time period when the interest is added is called a **compounding period**.
- Compound interest is always given as a rate per year, but compounding periods can vary.

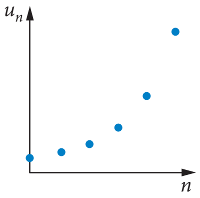
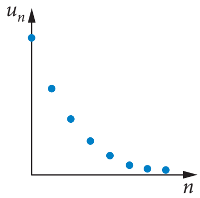
Compounding period	Number of compounding periods per year
Daily	365
Weekly	52
Fortnightly	26
Monthly	12
Quarterly	4
Six-monthly	2
Yearly	1

- percentage interest rate per compounding period = $\dfrac{\text{percentage interest rate per year}}{\text{number of compounding periods per year}}$

Depreciation

- **Depreciation** is the decrease in value of assets used by a business over time.
- **Reducing balance depreciation** involves compound interest calculations where the value of the asset is reduced by a fixed percentage of its value in the preceding year.

Compound interest and reducing balance depreciation summary

	Compound interest investment	Reducing balance depreciation
Recurrence relation for balance	u_0 = principal, $u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$	u_0 = initial asset value, $u_{n+1} = \left(1 - \frac{r}{100}\right) \times u_n$
Type	increasing geometric	decreasing geometric
Graph		
Rule	$u_n = \left(1 + \frac{r}{100}\right)^n \times u_0$	$u_n = \left(1 - \frac{r}{100}\right)^n \times u_0$
r%	interest rate per compounding period	depreciation rate per year
n	number of compounding periods	number of years

- Total amount of interest/depreciation after n years = $u_n - u_0$.

Mark-ups and discounts

- For an item **marked up** by r%:

$$\text{change in price} = \text{original price} \times \frac{r}{100}$$

$$\text{new price} = \text{original price} + \left(\text{original price} \times \frac{r}{100}\right)$$

$$r = \frac{\text{new price} - \text{original price}}{\text{original price}} \times 100\%$$

$$\text{original price} = \text{new price} \times \frac{100}{(100 + r)}$$

- For an item **discounted** by r%:

$$\text{change in price} = -\left(\text{original price} \times \frac{r}{100}\right)$$

$$\text{new price} = \text{original price} - \left(\text{original price} \times \frac{r}{100}\right)$$

$$r = \frac{\text{original price} - \text{new price}}{\text{original price}} \times 100\%$$

$$\text{original price} = \text{new price} \times \frac{100}{(100 - r)}$$

GST

The GST rate $r = 10\%$

GST amount = price without GST × 0.1

$$\text{GST amount} = \frac{\text{price with GST}}{11}$$

Price with GST = price without GST × 1.1

$$\text{Price without GST} = \frac{\text{price with GST}}{1.1}$$

The unitary method

- Calculate the value of one unit.
- For percentages, calculate the value of 1% and then multiply by 100 to find the whole amount.

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Inflation

- **Inflation** occurs when prices rise over time.
- To calculate the single item inflation rate r over time, use
- $r = \dfrac{\text{new price} - \text{original price}}{\text{original price}} \times 100\%$
- To find the effect of the yearly inflation rate on the price of a single item u_n, after n years, use the compound interest recurrence relation

  $$u_0 = \text{original price},\ u_{n+1} = \left(1 + \frac{r}{100}\right)u_n$$

  and change the value of r each year.
- To find the purchasing power of money u_{n+1} after n years, where the inflation rates are averaged over a number of years, use the reducing balance depreciation rule:

  $$u_n = \left(1 - \frac{r}{100}\right)^n \times u_0$$

  where

  u_0 = original amount of money

  r = the average percentage inflation rate per year.

  n = the number of years.

Comparing purchasing options

- **Cash** – A cash payment is made with actual notes and coins.
- **Debit cards** – A debit card payment is made using money in a bank account.
- **Credit cards** – A credit card payment is made with the bank's money. The payment will need to be repaid when the credit card payment is due (usually a month later). If the amount is not repaid in full, the bank will charge interest on the amount, which compounds daily.
- **Buy now pay later services (BNPL)** – A BNPL platform payment means only a percentage of the purchase price (e.g. 25%) is paid at the time of purchase. The rest of the purchase price is paid in regular instalments (e.g. 25% per fortnight). If payments are not made, a late fee is charged.
- **Personal loans** – A personal loan payment involves a signed contract with a lender stating that the purchased item will be paid for in instalments, usually longer term than a BNPL platform, based on a quoted percentage rate of interest per annum. If a payment is missed, the amount of interest increases. Failure to make the payments can result in the lender taking back the purchased item.

  To calculate a new balance on a personal loan, use

  $$\text{new balance} = \text{old balance} + \left(\frac{r}{100} \times \text{old balance}\right) - \text{regular payment}$$

  where

  r is the percentage interest rate per compounding period.

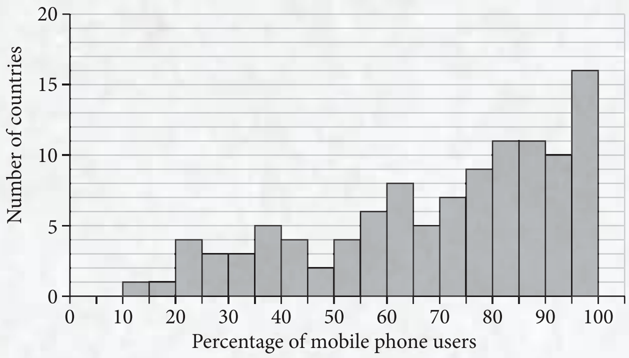
  Total amount paid = number of payments × regular payment amount

  Total interest = total amount paid – loan amount

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 23 minutes

1 The histogram displays the distribution of the percentage of mobile phone users in a number of countries.

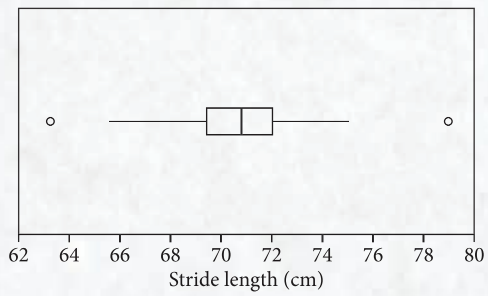


The shape of the histogram is best described as

A bi-modal. **B** positively skewed. **C** negatively skewed.

D double-peaked. **E** approximately symmetric.

2 The boxplot shows the distribution of the stride length of a sample of Year 11 students, in centimetres. The five-number summary for the *stride length* of these students is closest to



A 65.5, 69.4, 70.7, 72, 75 **B** 63.1, 69.4, 70.7, 72, 75 **C** 65.5, 69.4, 70.7, 72, 79

D 63.1, 69.4, 70.7, 72, 79 **E** 65.5, 69.4, 70.7, 72, 79

3 The sequence generated by the recurrence relation

$u_0 = 10, \quad u_{n+1} = u_n - 3$

is

A a decreasing geometric sequence. **B** an increasing geometric sequence.

C a decreasing arithmetic sequence. **D** an increasing arithmetic sequence.

E neither arithmetic nor geometric.

4 Which of the following is modelled by the recurrence relation $u_0 = 23\,000$, $u_{n+1} = u_n - 1200$?

A A car originally purchased for $23 000 is depreciated at a reducing balance rate of $1200 each year.

B A car originally purchased for $23 000 is depreciated at a flat rate of 12% each year.

C A car originally purchased for $23 000 is depreciated at a flat rate of $1200 each year.

D A car originally purchased for $23 000 has been depreciated to $1200.

E A car originally purchased for $21 800 is depreciated at a rate of $1200 each year.

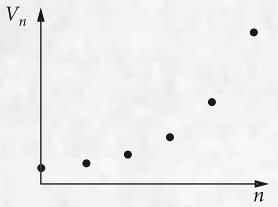
5 Consider the recurrence relation

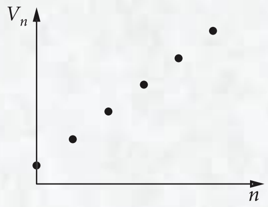
$$u_0 = 4,\ u_{n+1} = \frac{1}{2}u_n$$

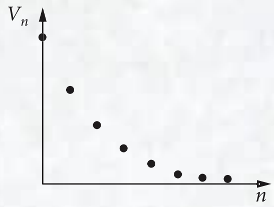
The first four values of this recurrence relation are

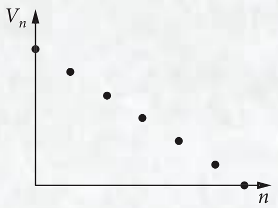
A $\frac{1}{2}$, 1, 2, 4 **B** 4, 8, 16, 32 **C** 4, 2, 1, 0.5 **D** $\frac{1}{2}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ **E** $\frac{1}{2}$, 2, 8, 16

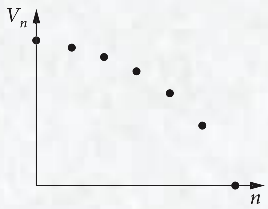
6 A machine is depreciated at a reducing balance rate of 12% per annum. What is the shape of the graph of V_n, the value of the machine after n years?

A


B


C


D


E


7 A party caterer has a GST-free price of $5260. How much GST is payable to the nearest dollar?

A $478 **B** $526 **C** $4781 **D** $4782 **E** $5786

8 A certain brand of olive oil comes in different sizes. Which is the best value?

A 200 mL for $5.60 **B** 450 mL for $7.85 **C** 800 mL for $24

D 2.4 L for $49 **E** 5 L for $89

9 Milton deposited $35 000 into a savings account earning compound interest at a rate of 4.1% per annum, compounding annually. Which one of the following recurrence relations can be used to determine the balance of the savings account, B_n, up to the start of the nth year?

A $B_1 = 35\,000,\quad B_n = 1435 \times B_{n-1}$ **B** $B_1 = 35\,000,\quad B_n = B_{n-1} + 1435$

C $B_1 = 35\,000,\quad B_n = B_{n-1} + 4.1 \times 1435$ **D** $B_1 = 35\,000,\quad B_n = 1.041 \times B_{n-1}$

E $B_1 = 35\,000,\quad B_n = 4.1 \times B_{n-1} + 1435$

10 A country has an official inflation rate of 3.3% last year and 4.9% this year. The price of a packet of salt and vinegar chips was $2.50 at the start of last year and its price rises match the inflation rate. Which of the following calculations will give the price of the packet of chips, to the nearest cent, at the start of next year?

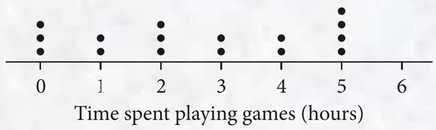
A 2.5 × 1.033 × 1.049 **B** 2.5 × 3.3 + 2.5 × 4.9 **C** 2.5 × 0.033 × 0.049

D 2.5 × 3.3 × 4.9 **E** 2.5 × 0.067 × 0.051

Cumulative examination 2

Total number of marks: 15 Reading time: 4 minutes Writing time: 23 minutes

1 (2 marks) A sample of 16 students were asked to indicate the time (in hours) they had spent playing computer games on the previous night. The results are displayed in the dot plot.



a What is the median time spent playing computer games? 1 mark

b What is the mean time spent playing computer games? 1 mark

2 (3 marks) Alicia invests \$16 000 at 7.5% p.a. simple interest.

a Write a recurrence relation for the balance of this loan after *n* years. 1 mark

b What will the investment be worth after three years? 1 mark

c After how many years will the balance reach \$22 000? 1 mark

3 © VCAA 2012 2BRMQ1abic (3 marks) A club purchased new equipment priced at \$8360. A 15% deposit was paid.

a Calculate the deposit. 1 mark

b Determine the amount of money that the club still owes on the equipment after the deposit is paid. 1 mark

c The price, \$8360, included 10% GST (Goods and Services Tax). Calculate the price of the equipment before the GST was added. 1 mark

4 © VCAA 2011 2BRMQ1 (3 marks) Tony plans to take his family on a holiday. The total cost of \$3630 includes a 10% Goods and Services Tax (GST).

a Determine the amount of GST that is included in the total cost. 1 mark

During the holiday, the family plans to visit some theme parks. The prices of family tickets for three popular theme parks are shown in the table.

	Wet World	Movie Journey	Outback Adventure
Family ticket	\$82	\$220	\$160

b What is the total cost for the family if it visits all three theme parks? 1 mark

If Tony purchases the **Movie Journey** family ticket online, the cost is discounted to \$202.40.

c Determine the percentage discount. 1 mark

5 © VCAA 2003 2BRMQ2ab MODIFIED (4 marks) Brad buys a coffee machine with an initial value of \$3100. He considers two methods of depreciating the value of the coffee machine.

a Suppose the value of the machine is depreciated using the reducing balance method at a rate of 15% per annum.

i What is the depreciated value of the machine after three years? Write your answer correct to the nearest dollar. 1 mark

ii After how many years will the value of the coffee machine first fall below \$1500? 1 mark

b Alternatively, suppose that the machine is depreciated using the unit cost depreciation method. Brad sells 15 000 cups of coffee per year and the unit cost per cup is 3.0 cents.

i Write the rule for the value, u_n, of the coffee machine after *n* uses using unit cost depreciation in simplest form. 1 mark

ii What is the value of the coffee machine after three years using unit cost depreciation, rounded to the nearest dollar? 1 mark

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CHAPTER

4

LINEAR FUNCTIONS, GRAPHS, EQUATIONS AND MODELS

Study Design coverage

UNIT 1, AREA OF STUDY 3: FUNCTIONS, RELATIONS AND GRAPHS

Linear functions, graphs, equations and models

- the linear function $y = a + bx$, its graph, and interpretation of the parameters, a and b in terms of initial value and constant rate of change respectively
- graphing linear relations $Ax + By = C$ and equivalent forms
- formulation and analysis of linear models from worded descriptions or relevant data (including simultaneous linear equations in two variables) and their application to solve practical problems including domain of interpretation
- piecewise linear (line segment, step) graphs and their application to modelling practical situations, including tax scales and charges and payment.

Video playlists (5):

4.1 Linear functions in the form $y = a + bx$

4.2 Interpreting linear functions in the form $y = a + bx$

4.3 Simultaneous linear equations

4.4 Piecewise linear graphs

VCE question analysis Linear functions, graphs, equations and models

Skillsheets (1):

4.1 Graphing linear equations

Worksheets (16):

4.1 Graphing linear functions 1 • Graphing linear functions 2 • Linear graphs • A page of number planes • Number plane grid paper • Drawing gradients • Finding the gradient between 2 points • Gradient and y-intercept • Finding the equation of a line

4.2 Practical applications

4.3 General equation of a straight line • Sketching simultaneous equations • Intersection of lines • Solving simultaneous equations • Simultaneous equations problems

4.4 Step graphs

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

4.1 Linear functions in the form $y = a + bx$

Graphing $y = a + bx$

A **linear function** (or **linear equation**) is an equation that can be written in the form $y = a + bx$, where x and y are variables and a and b are constants. For example:

Linear functions	Non-linear functions
$y = 11 - 7x$	$y = 11 - 7x^2$
$y = -5 + 2x$	$y = -5 + 2x + x^3$
$y = 6x + 3$ or $y = 3 + 6x$	$y = \frac{6}{x} + 3$ or or $y = 3 + \frac{6}{x}$

The graph of a linear function is a straight line. The **coordinates** of points on the line are found by substituting x values into the equation to calculate corresponding y values.

Video playlist Linear functions in the form $y = a + bx$

Linear graphs and tables of values

Functions can be drawn using a table of values.

WB p. 61

Skillsheet Graphing linear equations

Worksheets Graphing linear functions 1; Graphing linear functions 2; Linear graphs; A page of number planes; Number plane grid paper

WORKED EXAMPLE 1 Drawing linear graphs from tables of values

For the linear function $y = 5 - 3x$

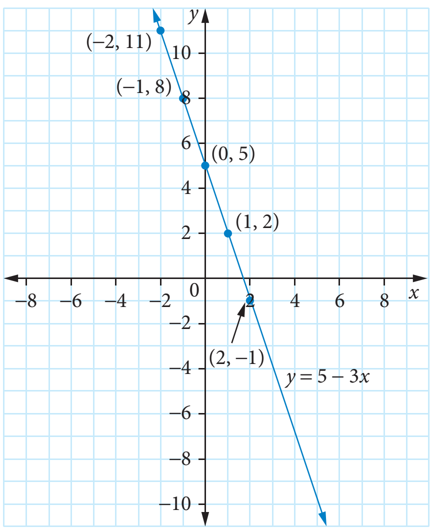
a find the y values for each of $x = -2, -1, 0, 1, 2$

b use the y values to construct a table of values for the linear function

c use the table of values to draw the linear function by hand.

Steps	Working
a Use the linear function to calculate the y value for each x value.	When $x = -2$, $y = -3 \times (-2) + 5 = 6 + 5 = 11$ When $x = -1$, $y = -3 \times (-1) + 5 = 3 + 5 = 8$ When $x = 0$, $y = -3 \times (0) + 5 = 0 + 5 = 5$ When $x = 1$, $y = -3 \times (1) + 5 = -3 + 5 = 2$ When $x = 2$, $y = -3 \times (2) + 5 = -6 + 5 = -1$
b Set up a table of values for the linear function to show the x values and their corresponding y values.	see table below
c Draw a Cartesian plane and plot the points from the table. Draw a line through the points, extending it and placing arrows on both ends of the line. Label the graph with its equation.	see graph below

x	−2	−1	0	1	2
y	11	8	5	2	−1



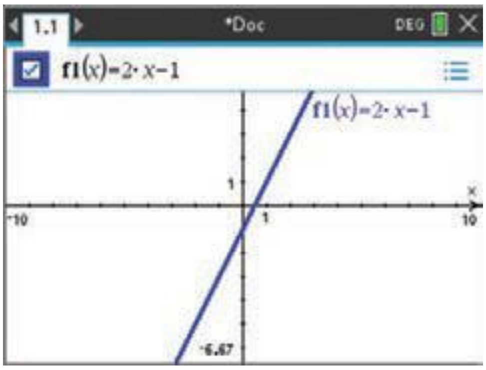
Exam hack

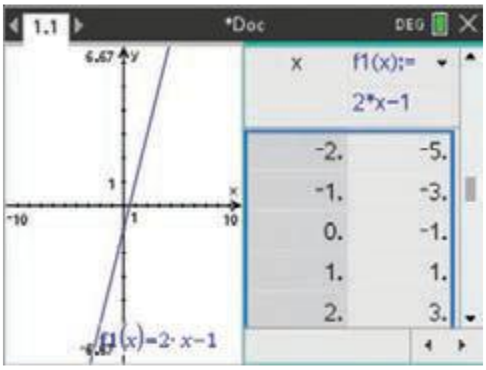
You will often see linear equations written in a different order. For example $y = -3x + 5$ rather than $y = 5 - 3x$.

USING CAS 1 Drawing linear graphs and generating tables of values

For the linear function $y = 2x - 1$, draw the graph and generate a table of values for $x = -2, -1, 0, 1, 2$.

TI-Nspire



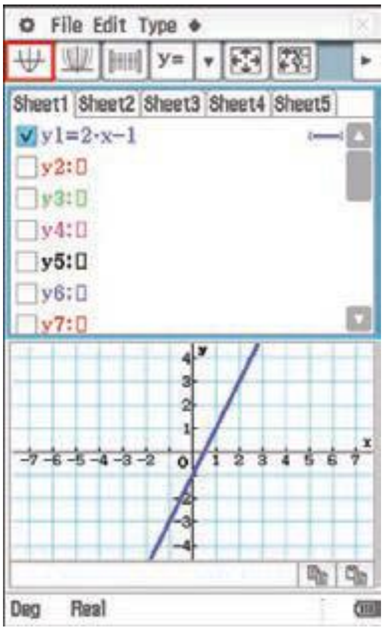


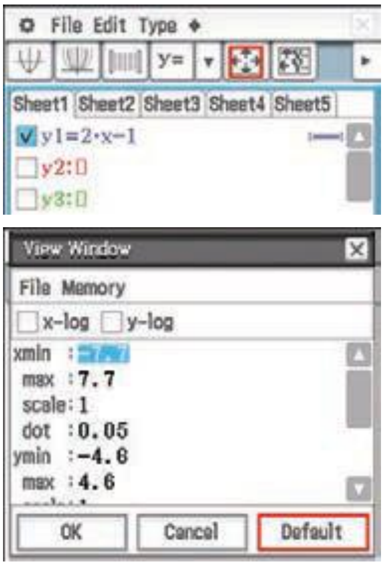
1. Start a new document and add a **Graphs** page.
2. In the **Graph Entry Line** at the top of the page, enter **2x–1**.
3. Press **enter**. The graph of the function will be displayed.

Note: after pressing **enter**, the **Graph Entry Line** will be hidden.

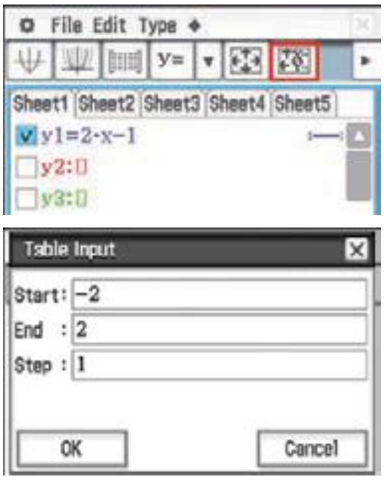
4. Press **menu** > **Table** > **Split-screen Table**. The table of values will be displayed in the right-hand window.
5. Scroll up to view the table for $x = -2, -1, 0, 1$ and 2.

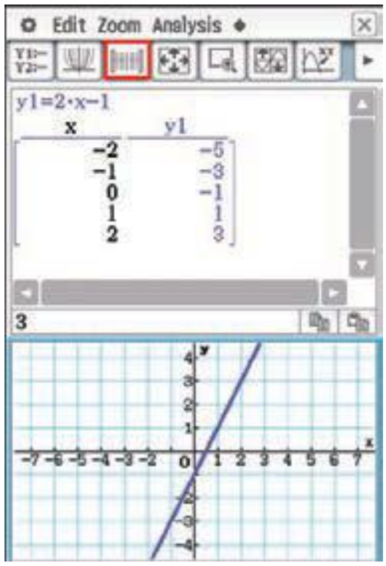
ClassPad





1. Tap **Menu** and then tap **Graph&Table**.
2. Clear all equations and enter **2x–1** into the line for **y1**.
3. Press **EXE** to select the equation.
4. Tap the **Graph** tool to graph it in the lower window.
5. If the graph does not display, tap the **View Window** tool.
6. In the **View Window** screen, tap **Default**.
7. Tap **OK**.





8 With the Equation window highlighted, tap the **Table Input** tool.

9 In the dialogue box, set **Start:** to **−2** and **End:** to **2**. Keep the Step: setting of 1.

10 Tap **OK**.

11 Tap the **Table** tool.

12 The table of values will be displayed when $x = -2, -1, 0, 1$ and 2.

Determining if a point lies on a line

To determine whether a point lies on a line, **substitute** the coordinates into the equation of the line and check if the result is true or false. If it's true, then the point lies on the line, and if it's false, then the point doesn't lie on the line.

WORKED EXAMPLE 2 Determining if a point lies on a line

WB p. 62

Determine whether the point (−2, 3) lies on each of the following lines, showing a calculation to justify your answer.

a $y = 7 + 2x$ **b** $y = 3x - 8$

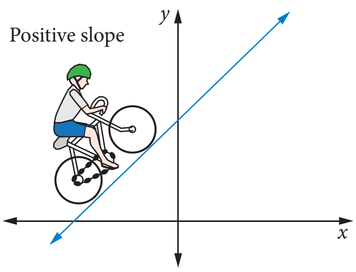
Steps	Working
a 1 Substitute the coordinates into the equation of the line, evaluate, and state whether the equation is true or false.	$x = -2, y = 3$ $y = 7 + 2x$ Does $3 = 7 + 2 \times (-2)$? $3 = 7 - 4$ true
2 Write the answer.	(−2, 3) lies on the line $y = 7 + 2x$ as $3 = 7 - 4$.
b 1 Substitute the coordinates into the equation of the line, evaluate, and state whether the equation is true or false.	$x = -2, y = 3$ $y = 3x - 8$ Does $3 = 3 \times (-2) - 8$? $3 \neq -6 - 8$ false
2 Write the answer.	(−2, 3) doesn't lie on the line $y = 3x - 8$ as $3 \neq -6 - 8$.

Worksheet
Drawing gradients

Identifying the slope

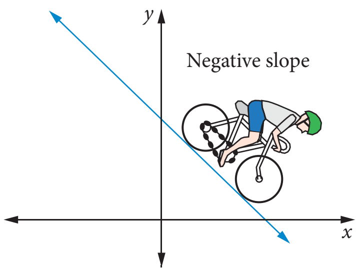
When a straight line is written in the form $y = a + bx$, a represents the **slope** of the line. Slopes can be positive, negative, zero or not defined.

Positive slope



Line sloping up from left to right

Negative slope



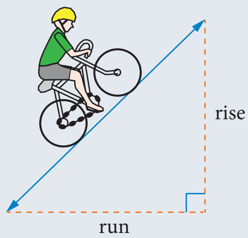
Line sloping down from left to right

The slope of a linear function

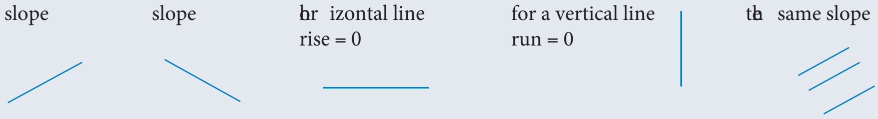
For a linear function $y = a + bx$

$$b = \text{slope} = \frac{\text{rise} \uparrow}{\text{run} \rightarrow}$$

$$= \frac{\text{vertical distance between two points}}{\text{horizontal distance between the same two points}}$$



Positive slope	Negative slope	Zero slope for a horizontal line rise = 0	Slope not defined for a vertical line run = 0	Parallel lines have the same slope



WB
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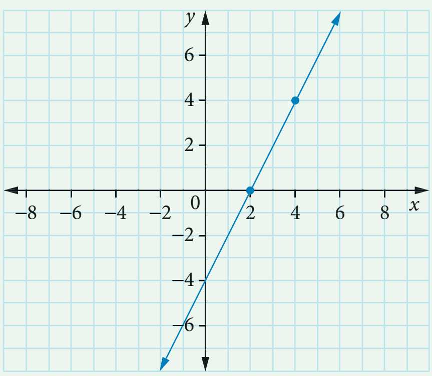
WORKED EXAMPLE 3 Finding the slope using $\frac{\text{rise}}{\text{run}}$

For each of the lines shown below

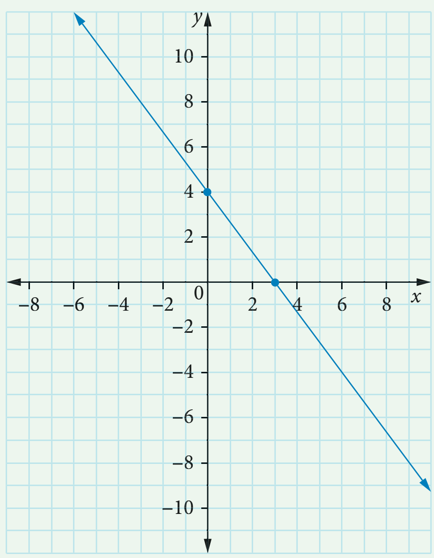
i state whether the slope is positive, negative, zero or not defined

ii if it is positive or negative, calculate the slope of the line using $\frac{\text{rise}}{\text{run}}$ for the two points shown.

a



b



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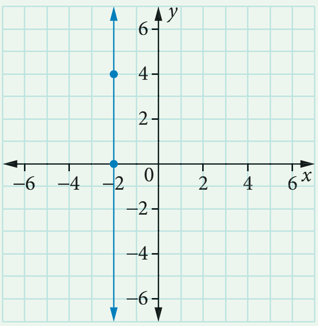
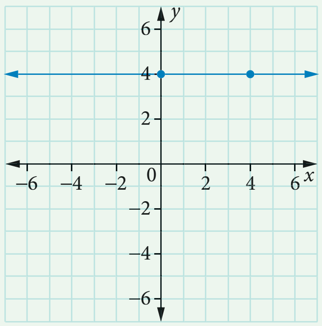
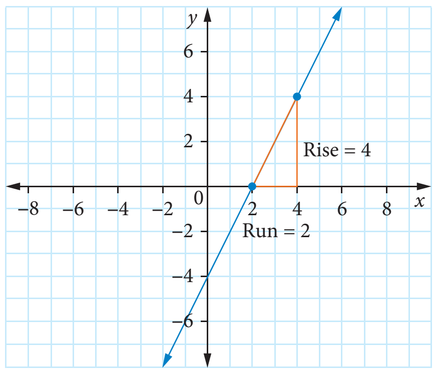
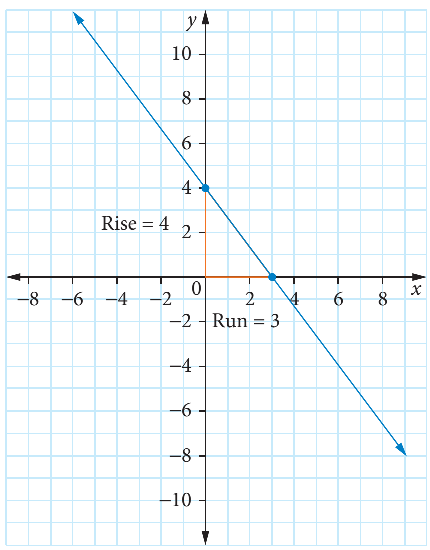
c



d



4.1

Steps	Working
a **i** Is the line sloping up or down from left to right?	The line is sloping up from left to right, so the slope is positive.
ii **1** Draw in a right-angled triangle using the two points and find the rise and run between them.	
2 Use slope $= \dfrac{\text{rise}}{\text{run}}$ and simplify.	slope $= \dfrac{4}{2} = 2$
b **i** Is the line sloping up or down from left to right?	The line is sloping down from left to right, so the slope is negative.
ii **1** Draw in a right-angled triangle using the two points and find the rise and run between them.	
2 Use slope $= \dfrac{\text{rise}}{\text{run}}$ and simplify.	slope $= -\dfrac{4}{3}$
c **i** The line is vertical.	The slope is not defined.
d **i** The line is horizontal.	The slope is 0.

Worksheets
Finding the gradient between 2 points

Gradient and y-intercept

The slope formula

If you know any two points on a straight line, the slope of a line can be found without drawing a graph.

The slope formula

If two points on a straight line are (x_1, y_1) and (x_2, y_2)

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Exam hack

If you have a choice of two points on a straight line to select from, choose the two that will make the calculation the easiest. This will often be the points that have a zero coordinate or lie on a grid line.

WB

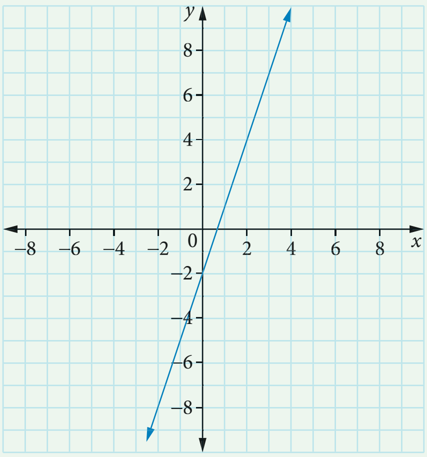
p. 65

WORKED EXAMPLE 4 Finding the slope from two points

Calculate the slope of the line for each of the following:

a a straight line through the points (2, 5) and (4, 13)

b



c a straight line drawn from the following table of values

x	−2	−1	0	1	2
y	16	11	6	1	−4

Steps	Working
a Use slope $= \frac{y_2 - y_1}{x_2 - x_1}$ for (x_1, y_1) and (x_2, y_2), and simplify.	$\text{slope} = \frac{13-5}{4-2} = \frac{8}{2} = 4$
b **1** Select two points on the line that can be clearly read from the graph.	(0, −2) and (2, 4)
2 Use slope $= \frac{y_2 - y_1}{x_2 - x_1}$ for (x_1, y_1) and (x_2, y_2), and simplify.	$\text{slope} = \frac{4-(-2)}{2-0} = \frac{6}{2} = 3$
c **1** Select two points from the table.	(0, 6) and (1, 1)
2 Use slope $= \frac{y_2 - y_1}{x_2 - x_1}$ for (x_1, y_1) and (x_2, y_2), and simplify.	$\text{slope} = \frac{1-6}{1-0} = \frac{-5}{1} = -5$

Identifying the y-intercept

When a straight line is written in the form $y = a + bx$, a is the ***y*-intercept**, which is the point where the graph crosses the y-axis.

Straight lines in the form $y = a + bx$

For a straight line in the form $y = a + bx$:

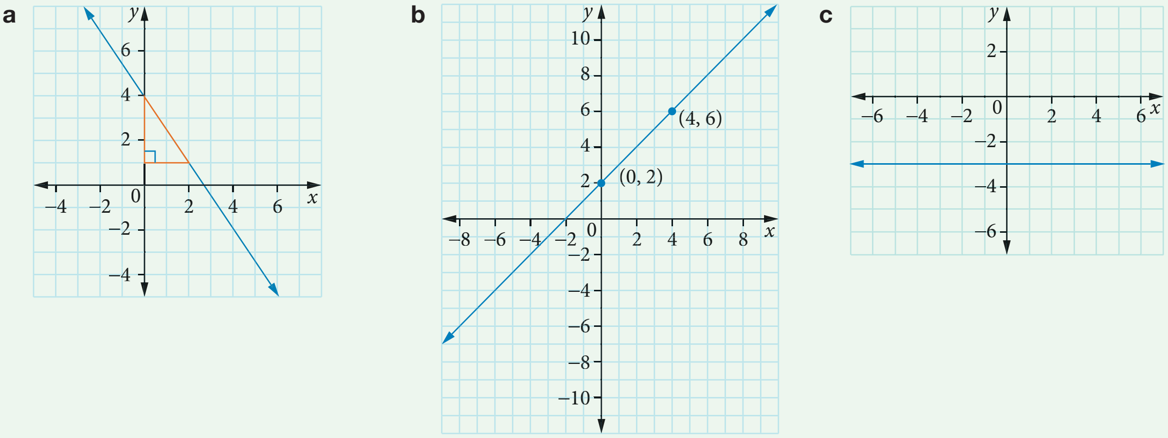
a = the y-intercept

b = the slope

4.1

WORKED EXAMPLE 5 Finding the straight line equation from the intercept and slope

Find the equation of each of the following straight lines by finding the y-intercept and slope.



Steps	Working
a 1 Where does the graph cross the y-axis?	The y-intercept = 4
2 Is the line sloping up or down from left to right?	The line is sloping down from left to right, so the slope is negative.
3 Use slope = $\frac{\text{rise}}{\text{run}}$ and simplify.	slope = $-\frac{3}{2}$
4 Identify the values for a and b in the equation $y = a + bx$.	$a = 4$ $b = -\frac{3}{2}$
5 Write the equation of the line.	The equation of the line is $y = 4 - \frac{3}{2}x$.
b 1 Where does the graph cross the y-axis?	y-intercept = 2
2 Use slope = $\frac{y_2 - y_1}{x_2 - x_1}$ for (x_1, y_1) and (x_2, y_2) and simplify.	slope = $\frac{6-2}{4-0} = \frac{4}{4} = 1$
3 Identify the values for a and b in the equation $y = a + bx$.	$a = 2$ $b = 1$
4 Write the equation of the line.	The equation of the line is $y = 2 + x$.
c 1 Where does the graph cross the y-axis?	y-intercept = −3
2 Is the line sloping up or down from left to right?	The line is neither sloping up or down. It's a horizontal line. slope = 0
3 Identify the values for a and b in the equation $y = a + bx$.	$a = -3$ and $b = 0$
4 Write the equation of the line.	The equation of the line is $y = -3$.

Worksheets Gradient and y-intercept

Finding the equation of a line

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EXERCISE 4.1 Linear functions in the form $y = a + bx$

ANSWERS p. 501

Mastery

1 WORKED EXAMPLE 1 For each of the following linear functions

i find the y values for each of $x = -2, -1, 0, 1, 2$

ii use the y values to construct a table of values for the linear function

iii use the table of values to draw the linear function by hand.

a $y = 1 + 3x$ **b** $y = 4 - 2x$ **c** $y = 3 - 4x$ **d** $y = -6 + 2x$

2 Using CAS 1 For the following linear functions, draw the graph and generate a table of values for $x = -2, -1, 0, 1, 2$.

a $y = -2x + 5$ **b** $y = x - 1$ **c** $y = \frac{1}{2}x + 2$ **d** $y = 3x - \frac{1}{2}$

3 WORKED EXAMPLE 2 Determine whether the point $(-1, 5)$ lies on each of the following lines, showing a calculation to justify your answer.

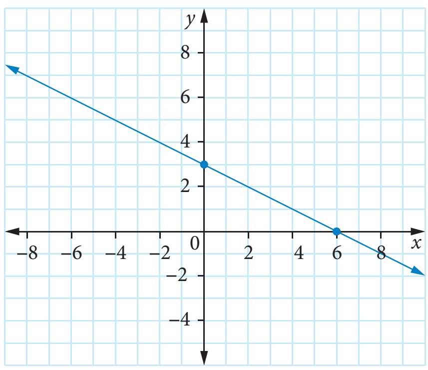
a $y = -5 + 15x$ **b** $y = -1 - 5x$ **c** $y = -10x - 5$

4 WORKED EXAMPLE 3 For each of the lines shown

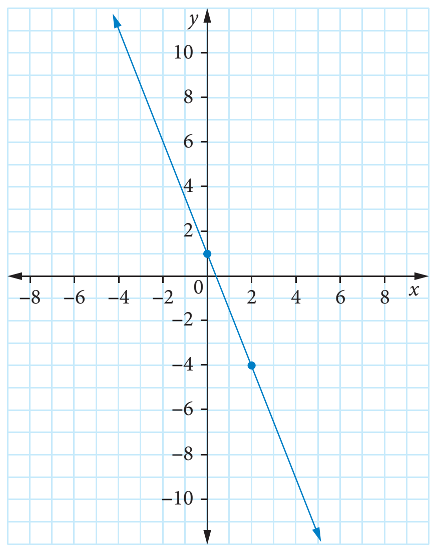
i state whether the slope is positive, negative, zero or not defined

ii if it is positive or negative, calculate the slope of the line using $\frac{\text{rise}}{\text{run}}$ for the two points shown.

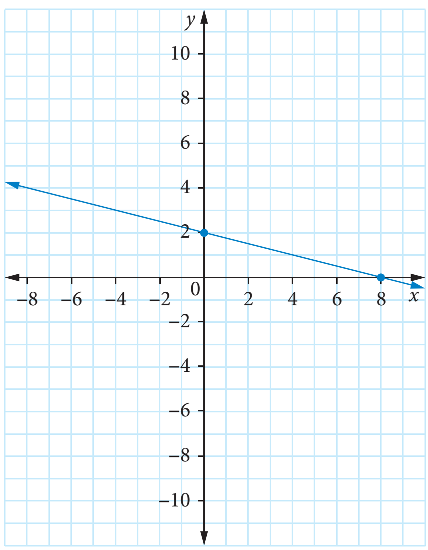
a



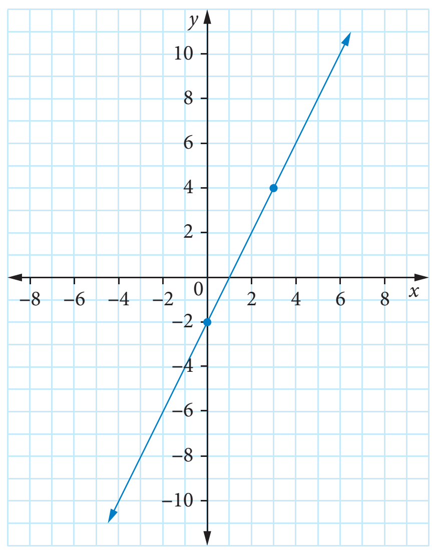
b



c



d



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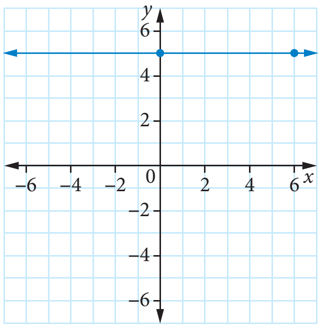
e



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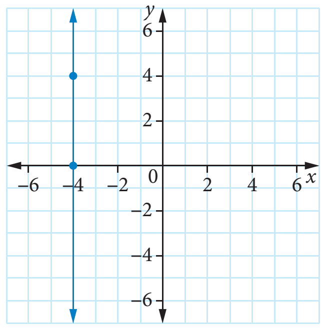
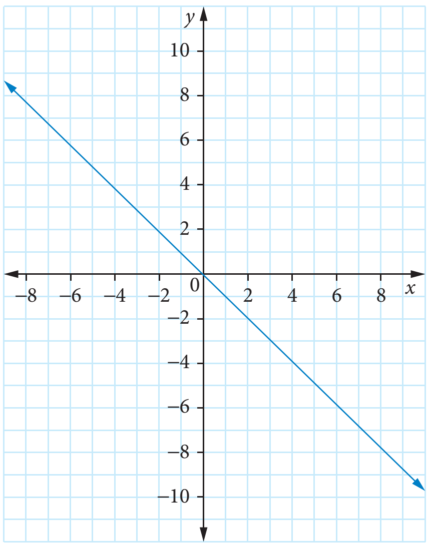


5 WORKED EXAMPLE 4 Calculate the slope of the line for each of the following:

a a straight line through the points (5, 3) and (2, 9)

b

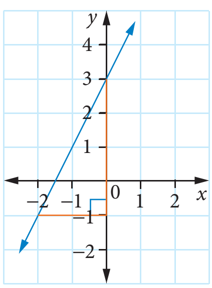


c a straight line drawn from the following table of values

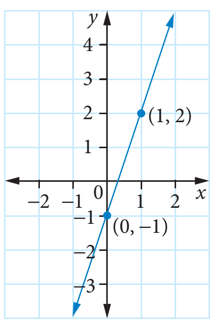
x	0	4	7	10
y	−1	11	20	29

6 WORKED EXAMPLE 5 Find the equation of each of the following straight lines by finding the slope and y-intercept.

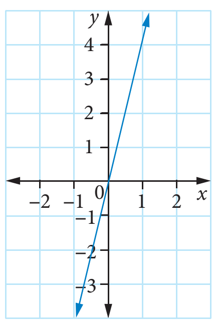
a



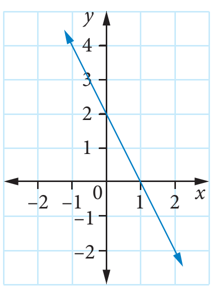
b



c



d



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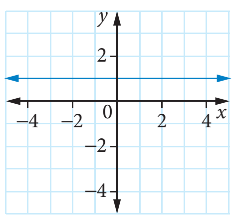
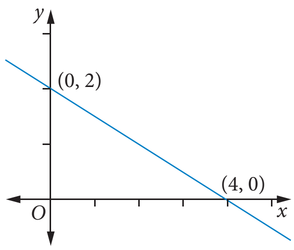


80–100% 60–79% 0–59%

7 Which one of the following equations is linear?

A $y = 2 + 4x^2$ **B** $y = \frac{5}{6}x - \frac{3}{8}$ **C** $y = 6 + 4x^3$ **D** $y = \sqrt{x} + 4$ **E** $y = 2 + \frac{4}{x}$

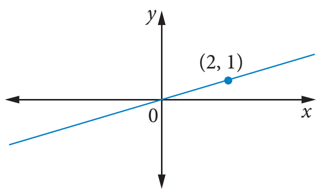
8 © VCAA 2018 1GRQ1 82% The graph shows a line intersecting the x-axis at $(4, 0)$ and the y-axis at $(0, 2)$.



The slope of this line is

A -4 **B** -2 **C** $-\frac{1}{2}$ **D** $\frac{1}{2}$ **E** 4

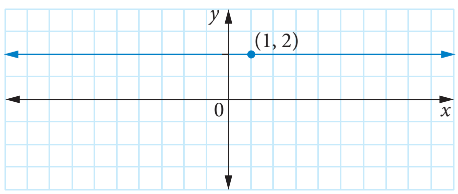
9 © VCAA 2007 1GRQ1 71%



The line passes through the origin $(0, 0)$ and the point $(2, 1)$. The slope of this line is

A -2 **B** -1 **C** $-\frac{1}{2}$ **D** $\frac{1}{2}$ **E** 2

10 © VCAA 2006 1GRQ1 70%



On the graph the equation of the line passing through the point $(1, 2)$ is

A $x = 1$ **B** $y = 1$ **C** $x = 2$ **D** $y = 2$ **E** $y = x + 1$

11 © VCAA 2017 1GRQ1 69% The equation of the line that passes through the points $(0, 4)$ and $(2, 4)$ is

A $x = 4$ **B** $y = 4$ **C** $y = 4x$ **D** $y = 4x + 2$ **E** $y = 2x + 4$

12 Which one of the following points does **not** lie on the straight line with equation $y = 1 - 3x$?

A $(0, 1)$ **B** $(2, -5)$ **C** $(-1, 4)$ **D** $(-4, -11)$ **E** $(-10, 31)$

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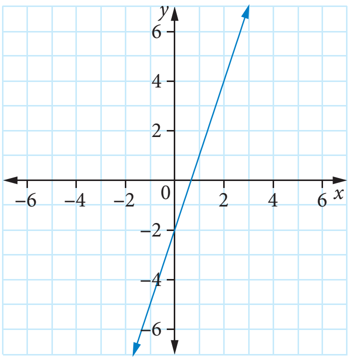
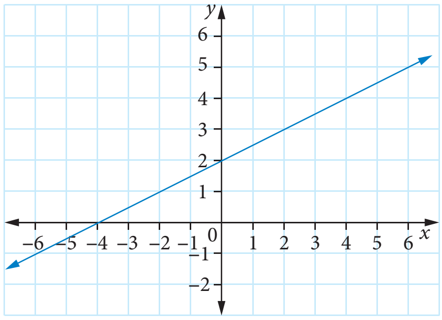
13 Which of the following is the graph for $y = 2 + \frac{1}{2}x$?

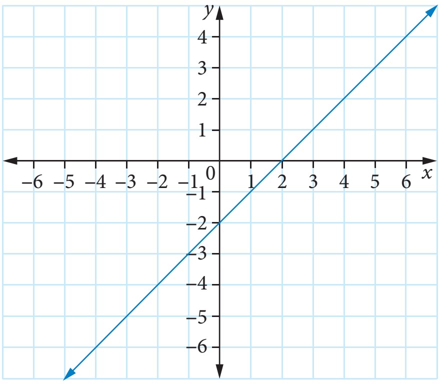
A



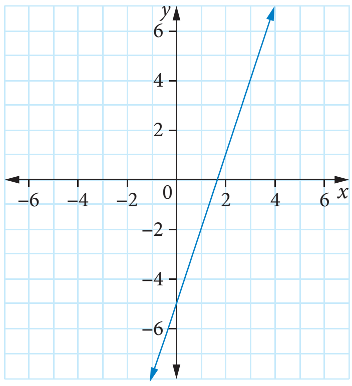
B



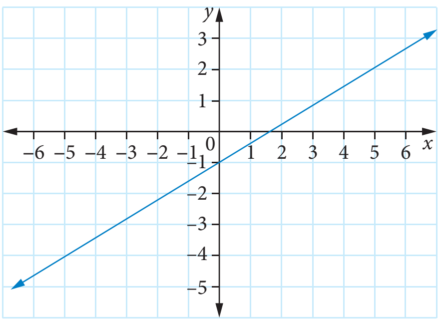
C



D



E



4.2 Interpreting linear functions in the form $y = a + bx$

Video playlist
Interpreting linear functions in the form $y = a + bx$

Worksheet
Practical applications

Initial value and constant rate of change

Linear functions in the form $y = a + bx$ can be used to model real-life problems where a is the **initial value** and b is the **constant rate of change** (or **rate**). The initial value is the value of y at the start. The constant rate of change measures the change in y as x changes. For example:

A computer support service company charges a $50 call-out fee plus $80 per hour.

This can be modelled by the linear equation

$C = 50 + 80n$

where

C is the total charge

n is the number of hours.

The initial value is \$50, the charge before any time has passed. The constant rate of change is \$80 per hour. For each hour n increases, the charge C increases by \$80.

Linear models using initial value and constant rate of change

For a linear function in the form $y = a + bx$:

a = the initial value

b = the constant rate of change

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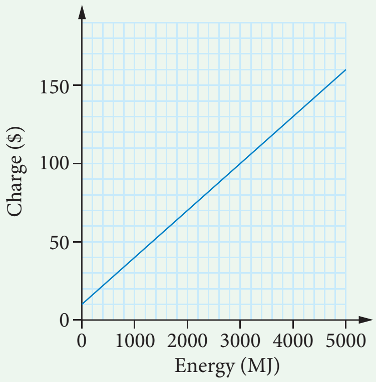
WORKED EXAMPLE 6 Working with the initial value and constant rate of change

The TruBlu Energy Company charges its customers a combination of an upfront cost (in dollars) and a rate per megajoule (MJ) of electricity used, according to the graph shown.



Exam hack

To identify the constant rate of change, look for the word 'per'.

Find the

a vertical intercept of the graph

b slope of the graph

c equation of the graph using C for the total charge and m as amount of electricity used in megajoules.

From the equation, find the

d upfront cost

e rate of change per megajoule

f total cost of using 2200 megajoules of electricity

g number of megajoules of electricity that have a total cost of \$90, rounding your answer to the nearest megajoule.

Steps	Working
a Read from the graph.	The vertical intercept is 10.
b **1** Select two points on the line that can be clearly read from the graph.	(0, 10) and (5000, 160)
2 Use slope = $\frac{y_2 - y_1}{x_2 - x_1}$ for (x_1, y_1) and (x_2, y_2) and simplify.	$\text{slope} = \frac{160 - 10}{5000 - 0} = \frac{150}{5000} = 0.03$
c **1** Identify a and b in the equation $y = a + bx$.	$a = 10$ $b = 0.03$
2 Write the equation $y = a + bx$ using the variables given.	$C = 10 + 0.03m$
d Find the initial value (vertical intercept).	The upfront cost is \$10.
e Find the constant rate of change (slope).	The rate of change is \$0.03 /MJ or 3 c per megajoule.

f Substitute the value into the equation and solve, using CAS if necessary.	$m = 2200$ $C = 10 + 0.03 \times 2200 = 76$ The total cost of using 2200 megajoules of electricity is \$76.
g **1** Substitute the value into the equation and solve, using CAS if necessary.	$C = 90$ $90 = 10 + 0.03m$ $0.03m = 80$ $m = \frac{80}{0.03} = 2666.666\ldots$
2 Write the answer rounding to the nearest megajoule.	2667 megajoules of electricity have a total cost of \$90.

4.2

Exam hack

When modelling real-life problems, make sure you always use the **pronumerals** given in the question, rather than x and y.

Domain of interpretation

We can use linear equations to make predictions about real-life situations. However, in real life not every value makes sense. We need to watch out for answers that are mathematically correct but impossible in real life. Linear models only apply to a specific range of values. These values are called the **domain of interpretation**.

WORKED EXAMPLE 7 Dealing with the domain of interpretation in real-life problems

WB p. 69

Sabine is organising the annual Year 11 dance. The total cost for the event will include \$500 venue hire, \$280 for the DJ and \$25 per head for food. The maximum capacity of the venue is 200.

a Find the linear equation in the form $C = a + bn$ for the total cost of the event, C, for n students.

b Use the equation to find the total cost of the dance if two students attend.

c Explain why your answer to part **b** makes no sense in real life.

d Use the equation to find the total cost of the dance if 1000 students attend.

e Explain why your answer to part **d** makes no sense in real life.

Steps	Working
a **1** Identify the initial value and constant rate of change.	The initial value = \$500 + \$280 = \$780 So $a = 780$. The constant rate of change is \$25. So $b = 25$.
2 Write the total cost equation.	$C = 780 + 25n$
b Substitute the value of n into the equation and evaluate. Write the answer in words.	$n = 2$ $C = 780 + 25 \times 2$ $= 830$ The total cost if 2 students attended is \$830.
c What would happen in real life?	The dance would be cancelled if so few students were coming.

d Substitute the value of n into the equation and evaluate. Write the answer in words.	$n = 1000$ $C = 780 + 25 \times 1000 = 25\,780$ The total cost if 1000 students attended is $25 780.
e What would happen in real life?	The venue has a maximum capacity of 200, so Sabine would have to find another venue and the cost equation would be different.

Modelling profit and loss

Linear functions can often be used to model business **cost** (money going out) and **revenue** (money coming in). We can combine these to form the linear function

profit = revenue − cost

If the profit is negative according to this equation, then the business will have made a loss.

WB
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WORKED EXAMPLE 8 Modelling profit and loss

Snow domes are sold for $6.50 each, and the cost C of making n snow domes is given by the equation

$C = 130 + 5n$

a Find the revenue equation in terms of n.

b Find the profit equation in terms of n.

c How much profit would be made if 100 snow domes were sold?

d How much profit would be made if 80 snow domes were sold?

e How many snow domes need to be sold to make at least $1000 profit? Explain why you need to round *up* for this calculation.

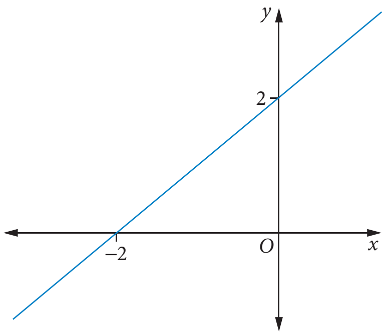
Steps	Working
a Use the price of one item to calculate the revenue from selling n items.	revenue = 6.50 × number of snow domes sold $= 6.5n$
b Use the profit equation and simplify.	profit = revenue − cost $= 6.5n - (130 + 5n)$ $= 1.5n - 130$
c Substitute $n = 100$ into the profit equation and write the answer.	profit $= 1.5 \times 100 - 130$ $= 20$ 100 snow domes would make a profit of $20.
d Substitute $n = 80$ into the profit equation and write the answer.	profit $= 1.5 \times 80 - 130$ $= -10$ 80 snow domes would make a loss of $10.
e **1** Let profit equal 1000 and solve for n, using CAS if necessary.	$1.5n - 130 = 1000$ $n \approx 753.33$
2 Round the answer according to the question. The answer needs to be a whole number as n represents the number of snow domes.	The profit needs to be *at least* $1000 and selling 753 snow domes won't quite make that profit. So, the number of snow domes that need to be sold is 754.

Exam hack

Always look at the context of the question when rounding. Sometimes the real-life context means that it makes sense to round up to the nearest whole number, even if the decimal indicates to round down.

Recap

1 © VCAA 2018N 1GRQ1 A straight line passes through the points $(-2, 0)$ and $(0, 2)$, as shown in the diagram.



The equation of this straight line is

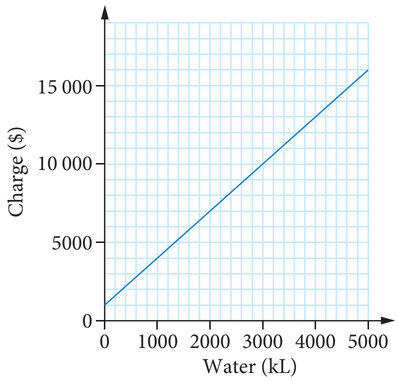
A $y = 2x$ **B** $y = x + 2$ **C** $y = 2x + 2$ **D** $y = x - 2$ **E** $y = 2x - 2$

2 Which one of the following points is **not** on the line with the equation $y = 5x - 2$?

A $(4, 18)$ **B** $(6, 28)$ **C** $(3, 13)$ **D** $(7, 23)$ **E** $(1, 3)$

Mastery

3 WORKED EXAMPLE 6 The United Water Company charges its commercial customers a combination of an upfront cost (in dollars) and a rate per kilolitre (kL) used, according to the graph shown.



Find the

a vertical intercept of the graph

b slope of the graph

c equation of the graph using C for the total charge and k as the amount of water used in kilolitre.

From the equation, find the

d upfront cost

e rate of charge per kilolitre

f total cost of using 4200 kilolitres of water

g number of kilolitres of water that have a total cost of $12 000, rounding your answer to the nearest kilolitre.

4 WORKED EXAMPLE 7 During Australian summers, crickets chirp faster at night if the temperature is higher. The linear equation for calculating the chirp rate, R (chirps/min), for a given temperature, t (°C), is $R = -24 + 8t$.

a Use the equation to find the chirp rate of crickets when the temperature is 26°C.

b Use the equation to find the chirp rate of crickets when the temperature is 4°C.

c Explain why your answer to part **b** makes no sense in real life.

5 WORKED EXAMPLE 8 Key rings are sold for $5.50 each, and the cost C of making n key rings is given by the equation

$$C = 110 + 4n$$

a Find the revenue equation in terms of n.

b Find the profit equation in terms of n.

c How much profit would be made if 200 key rings were sold?

d How much profit would be made if 50 key rings were sold?

e How many key rings need to be sold to make at least $600 profit? Explain why you need to round *up* for this calculation.

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2018 1GRQ2 96% Steven is a wedding photographer. He charges his clients a fixed fee of $500, plus $250 per hour of photography. The equation that represents the total amount, \$$C$, Steven charges, for t hours of photography is

A $C = 250t$ **B** $C = 500t$ **C** $C = 750t$

D $C = 500 + 250t$ **E** $C = 250 + 500t$

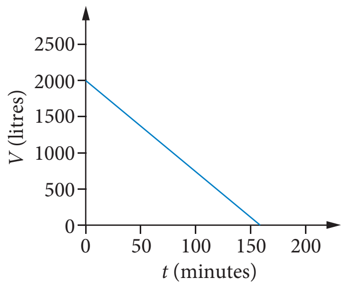
7 © VCAA 2008 1GRQ2 93% Initially there are 5000 litres of water in a tank. Water starts to flow out of the tank at the constant rate of 2 litres per minute until the tank is empty. After t minutes, the number of litres of water in the tank, V, will be

A $V = 5000 - 2t$ **B** $V = 2t - 5000$ **C** $V = 5000 + 2t$

D $V = 2 - 5000t$ **E** $V = 5000t - 2$

8 © VCAA 2013 1GRQ3 87% A full tank holds 2000 litres of water.

Water is pumped out of the tank at a constant rate. The graph shows how the volume of water in the tank, V, changes with time, t.



The constant rate, in litres per minute, at which the water is being pumped out of the tank is

A 0.8 **B** 2.0 **C** 12.5 **D** 80.0 **E** 160.0

9 © VCAA 2020 1GRQ6 74% Justin makes and sells electrical circuit boards. He has one fixed cost of $420 each week. Each circuit board costs $15 to make. The selling price of each circuit board is $27. The weekly profit if Justin makes and sells 200 circuit boards per week is

A $1980 **B** $2400 **C** $2820 **D** $4980 **E** $5400

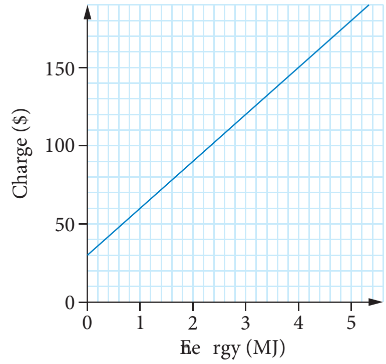
10 © VCAA 2015 1GRQ8 62% To raise funds, a club plans to sell lunches at a weekend market. The club will pay $190 to rent a stall. Each lunch will cost $12 to prepare and will be sold for $35. To make a profit of at least $1000, the minimum number of lunches that must be sold is

A 22 **B** 35 **C** 36 **D** 51 **E** 52

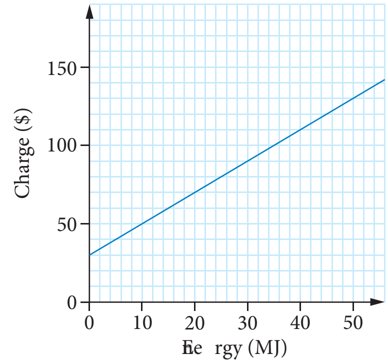
9780170448192

11 © VCAA 2013 1GRQ6 **52%** In one month, an energy company charges a $30 service fee, plus a supply charge of two cents per megajoule (MJ) of energy used. The graph that best models this situation is

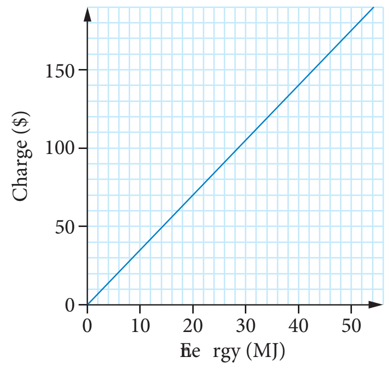
A



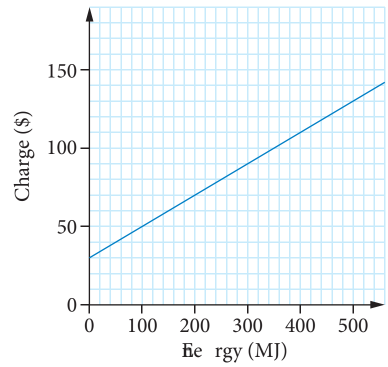
B



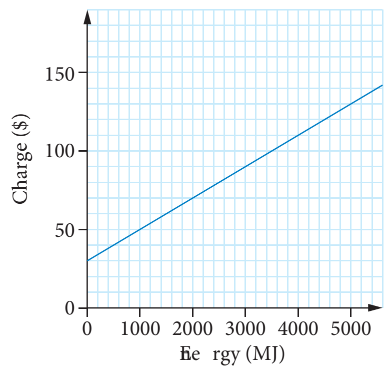
C



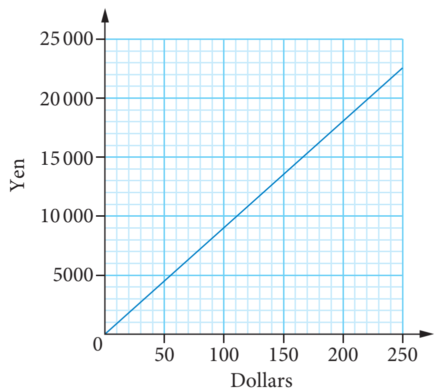
D



E



12 © VCAA 2015 2GRQ2 (2 marks) Ben will use a currency exchange agency to buy some Japanese yen (the Japanese currency unit). The graph shows the relationship between Japanese *yen* and Australian *dollars* on a particular day. This graph can be used to calculate a conversion between dollars and yen on that day.



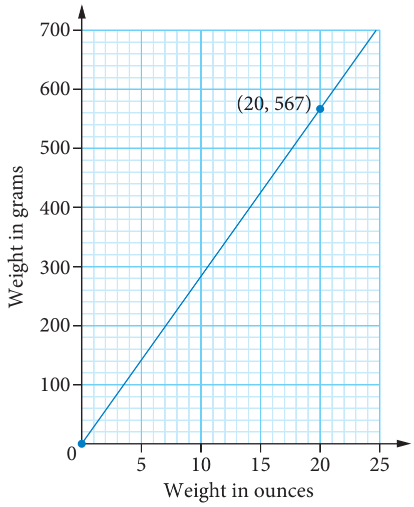
a Ben converts his dollars into yen using this graph. How many yen does he receive for $200? 1 mark

b The slope of this graph is the exchange rate for converting dollars into yen on that particular day. How many yen will Ben receive for each dollar? 1 mark

13 © VCAA 2018 2GRQ2 (3 marks) The weight of gold can be recorded in either grams or ounces. The following graph shows the relationship between *weight in grams* and *weight in ounces*.

The relationship between weight measured in grams and weight measured in ounces is shown in the equation

$$\textit{weight in grams} = M \times \textit{weight in ounces}$$



a 53% Show that $M = 28.35$ 1 mark

b 88% Robert found a gold nugget weighing 0.2 ounces. Using the equation above, calculate the weight, in grams, of this gold nugget. 1 mark

c 67% Last year Robert sold gold to a buyer at \$55 per gram. The buyer paid Robert a total of \$12 474. Using the equation above, calculate the weight, in ounces, of this gold. 1 mark

14 © VCAA 2013 2GRQ3 51% (2 marks) A rock-climbing activity will be offered to students at a camp on one afternoon. Each student who participates will pay \$24. The organisers have to pay the rock-climbing instructor \$260 for the afternoon. They also have to pay an insurance cost of \$6 per student. Let n be the total number of students who participate in rock climbing.

a Write an expression for the profit that the organisers will make in terms of n. 1 mark

b The organisers want to make a profit of at least \$500. Determine the minimum number of students who will need to participate in rock climbing. 1 mark

15 (6 marks) This table shows the cost, C cents, of mobile phone calls under the Oz-Zone Budget Plan, for calls of length t minutes.

Length of call, t (min)	1	2	5	10	15
Cost, C (cents)	102	182	422	822	1222

a Find the linear relationship in the form $C = a + bt$. 1 mark

b If this rule was graphed on a number plane, which variable would be shown on the vertical axis? 1 mark

c Use the relationship you found in part **a** to calculate the cost of an 18-minute call. 1 mark

d What is the vertical axis intercept of the graph and what does it represent? 1 mark

e If a phone call is extended by 3 minutes, by how much would its cost increase? 1 mark

f How long is a phone call under this plan if it costs \$5.82? 1 mark

4.3 Simultaneous linear equations

Graphing linear relations in the form $Ax + By = C$

Linear equations are often written in the form $Ax + By = C$ instead of $y = a + bx$. We can rewrite the equation to make y the **subject** and then graph it by finding the y-intercept and slope as we've done previously, but there is an easier way.

Graphing linear relations in the form $Ax + By = C$

To graph linear relations in the form $Ax + By = C$:

1. find the x-intercept by substituting $y = 0$ and solving for x
2. find the y-intercept by substituting $x = 0$ and solving for y
3. draw a straight line through the two intercepts.

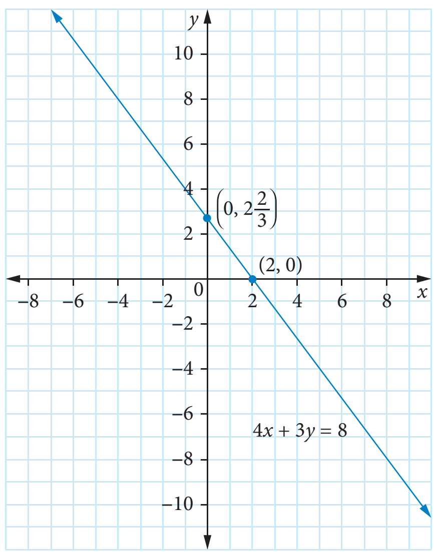
When A = 0, the graph is a horizontal line.

When $B = 0$, the graph is a vertical line.

WORKED EXAMPLE 9 Graphing linear relations in the form $Ax + By = C$ by hand

For the equation $4x + 3y = 8$

a find the x-intercept **b** find the y-intercept **c** and hence, sketch the graph by hand.

Steps	Working
a Substitute $y = 0$ and solve the equation. Write the coordinates of the x-intercept.	Substituting $y = 0$ into $4x + 3y = 8$ gives $4x + 0 = 8$ $4x = 8$ $x = 2$ The coordinates of the x-intercept are $(2, 0)$.
b Substitute $x = 0$ and solve the equation. Write the coordinates of the y-intercept.	Substituting $x = 0$ into $4x + 3y = 8$ gives $0 + 3y = 8$ $3y = 8$ $y = \frac{8}{3}$ $= 2\frac{2}{3}$ The coordinates of the y-intercept are $\left(0, 2\frac{2}{3}\right)$.
c Sketch the graph on a Cartesian plane by marking in the x- and y-intercepts and drawing a straight line through the two intercepts.	

4.3

Video playlist Simultaneous linear equations

Worksheets General equation of a straight line

Sketching simultaneous equations

Intersection of lines

Solving simultaneous equations

Simultaneous equations problems

WB

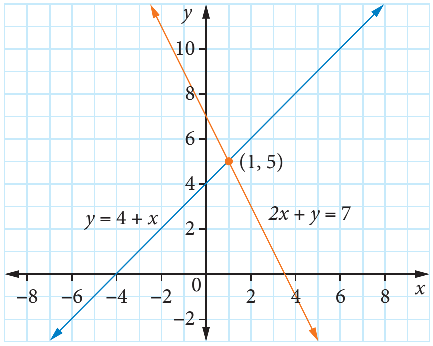
p. 71

Graphing and solving simultaneous linear equations

If two linear equations are graphed on the same axes, the **intersection** of the two lines is the point that lies on both lines. This point is the solution to the two **simultaneous equations**.

For example, the solution to the two simultaneous equations $y = 4 + x$ and $2x + y = 7$ is shown as the point $(1, 5)$ in the graph.

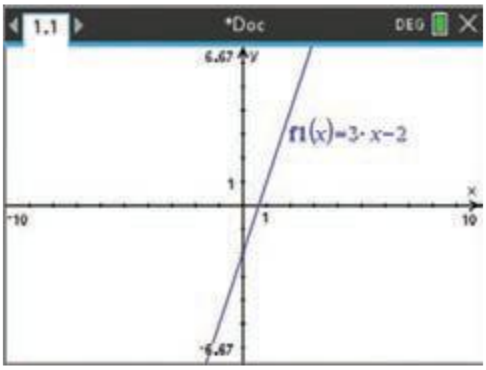
Simultaneous linear equations can be in the form $y = a + bx$ or $Ax + By = C$. We can solve simultaneous equations using CAS.



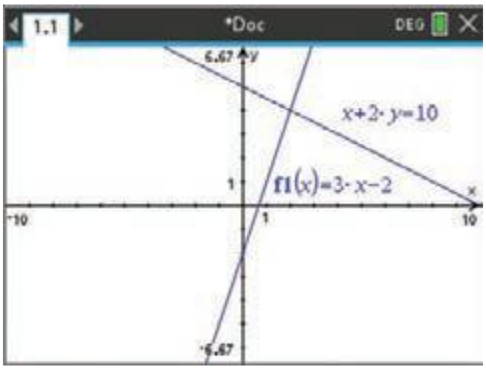
USING CAS 2 Graphing and solving simultaneous equations

Graph and solve the pair of simultaneous equations $y = 3x - 2$ and $x + 2y = 10$.

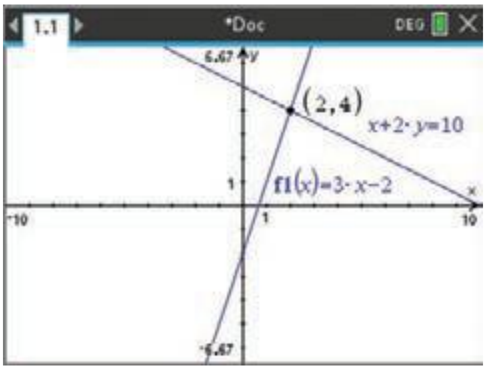
TI-Nspire



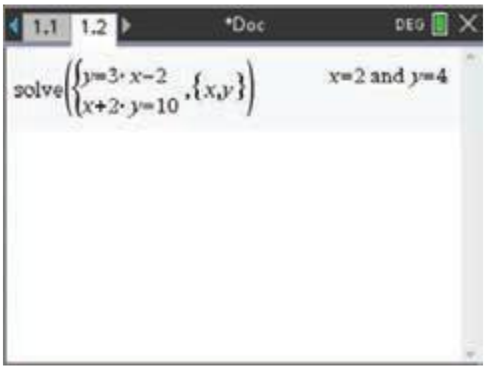
1 Start a new document and add a **Graphs** page.
2 In the **Graph Entry Line**, enter **3x–2**.
3 Press **enter** to graph the line.



4 Press **tab** or **ctrl** + **G** to display the **Graph Entry Line** again.
5 Press **menu** > **Graph Entry/Edit** > **Relation**.
6 Enter the relation **x+2y=10**.
7 Press **enter** to graph the relation.



8 Press **menu > Analyze > Intersection**.
9 When prompted for the **lower bound**, use the arrow keys to move to the left of the point of intersection and press **enter**.
10 When prompted for the **upper bound**, use the right arrow key to move to the right of the point of intersection and press **enter**.
11 The coordinates of the point of intersection **(2, 4)** will be displayed on the screen.



12 Insert a **Calculator** page.
13 Press **menu** > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.
14 On the next screen, keep the default values and select **OK**.
15 Enter the two equations into the template and press **enter**.
16 The coordinates of the point of intersection **(2, 4)** will be displayed.

The solution is $x = 2$ and $y = 4$.

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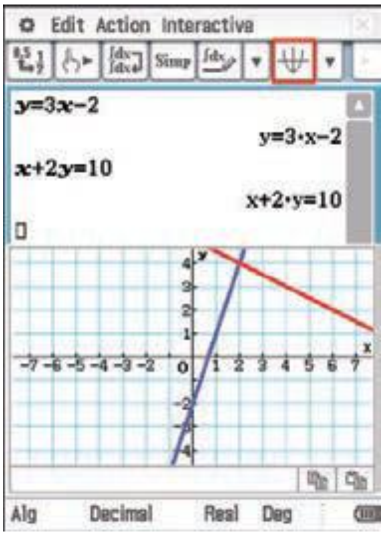
ClassPad



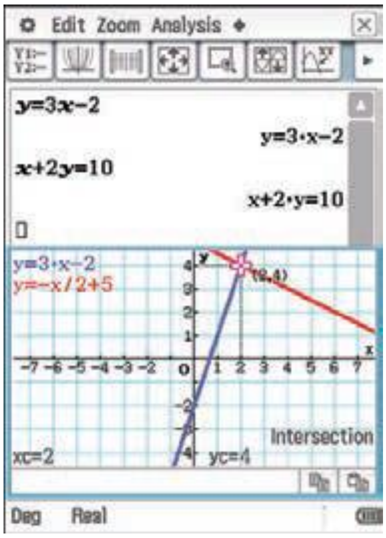
1 Tap **Main** and clear all equations.

2 Enter the equations **y=3x–2** and **x+2y=10**.

3 Tap the **Graph** tool to add a graph window.

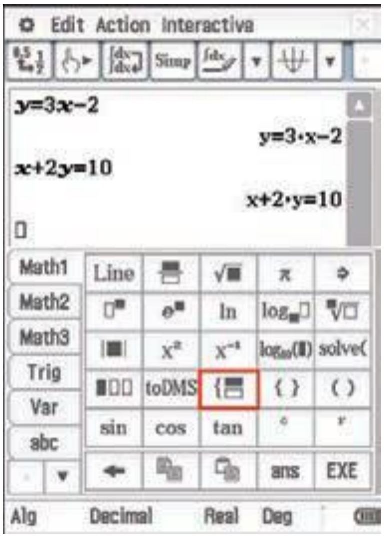
4 Drag the equations into the graph window.

Note: if the graphs do not display, tap the **View Window** tool and tap **Default** then **OK**.



5 With the graph window highlighted, tap **Analysis** > **G-Solve** > **Intersection**.

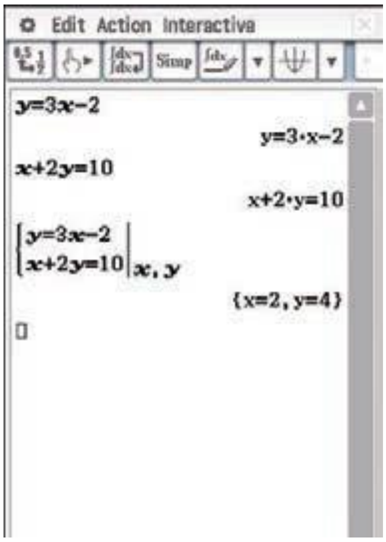
6 The coordinates of the point of intersection **(2, 4)** will be displayed.



7 Tap **Main**.

8 Open the **Keyboard** > **Math1**.

9 Select the **simultaneous equations** template.



10 Enter or copy the equations into the template.

11 Enter **x,y** in the lower right corner of the template.

12 Press **EXE**.

13 The coordinates of the point of intersection **(2, 4)** will be displayed.

The solution is $x = 2$ and $y = 4$.

Modelling with simultaneous equations

Simultaneous equations can be used to solve real-life problems where there is more than one unknown.

Steps for solving worded problems using simultaneous equations

To solve worded problems using simultaneous equations:

1 Identify the unknowns and assign a pronumeral for each of them.
2 Set up two equations by converting the given information into mathematical symbols.
3 Solve the simultaneous equations using CAS.
4 Answer the question in sentence form.

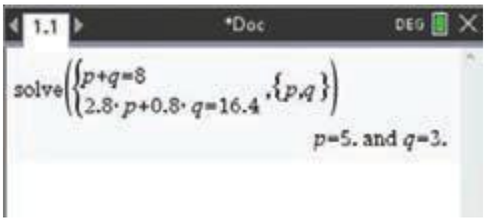
WB p. 72

WORKED EXAMPLE 10 Solving problems using simultaneous equations

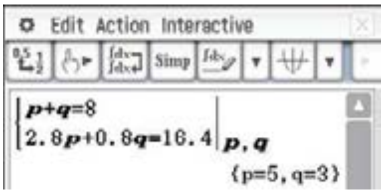
Eleanor spent \$16.40 on pens and pencils. She purchased a total of 8 items. If the pens cost \$2.80 each and the pencils cost 80 cents each, how many pens and pencils did Eleanor purchase?

Steps	Working
1 Identify the unknowns and assign a pronumeral for each of them.	Let p = number of pens q = number of pencils
2 Set up two equations by converting the given information into mathematical symbols.	There are 8 items, so $p + q = 8$ Pens cost \$2.80 each and pencils cost \$0.80 each. cost of pens = $2.8 \times p$ cost of pencils = $0.8 \times q$ total cost of pens and pencils = $2.8p + 0.8q$ Eleanor spent \$16.40.
Convert units where necessary.	$2.8p + 0.8q = 16.4$
3 Solve the simultaneous equations using CAS.	

TI-Nspire



ClassPad



4 Answer the question in sentence form.	Eleanor purchased 5 pens and 3 pencils.

EXERCISE 4.3 Simultaneous linear equations

ANSWERS p. 503

Recap

80–100% 60–79% 0–59%

1 © VCAA 2016 1GRQ2 94% A phone company charges a fixed, monthly line rental fee of \$28 and \$0.25 per call. Let n be the number of calls that are made in a month. Let C be the monthly phone bill, in dollars. The equation for the relationship between the monthly phone bill, in dollars, and the number of calls is

A $C = 28 + 0.25n$ **B** $C = 28n + 0.25$ **C** $C = n + 28.25$

D $C = 28(n + 0.25)$ **E** $C = 0.25(n + 28)$

2 © VCAA 2007 1GRQ2 86% A builder's fee, C dollars, can be determined from the rule $C = 60 + 55n$, where n represents the number of hours worked. According to this rule, the builder's fee will be

A \$60 for 1 hour of work **B** \$110 for 2 hours of work **C** \$500 for 8 hours of work

D \$550 for 10 hours of work **E** \$1150 for 10 hours of work

4.3

Mastery

3 WORKED EXAMPLE 9 For each of the following equations

a $x + 2y = 6$ **b** $x - 2y = -4$ **c** $2x - y = 6$ **d** $4x + y = 3$

- **i** find the x-intercept
- **ii** find the y-intercept
- **iii** and hence, sketch the graph by hand.

4 Using CAS 2 Graph and solve the following pairs of simultaneous equations.

a $y = x + 3$, $2x - y = -4$ **b** $y = -x + 2$, $y = x + 6$ **c** $2x + y = 3$, $4x - y = 3$ **d** $3x - y = 4$, $2x - y = 2$

5 WORKED EXAMPLE 10 Beachside Ice creams sells ice cream in single cones and double cones. The price of a single cone is \$3.80, whereas the price of a double cone is \$5.20. On a particular day, they sold a total of 450 cones and their total takings for the day were \$1990. How many of each type of cone did Beachside Ice creams sell on the day?

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2020 1GRQ2 95% At a school concert, the entry fee for adults was different from the entry fee for children. The entry fee for three adults and four children was \$67.00. The entry fee for two adults and five children was \$57.50. Let x be the entry fee for an adult. Let y be the entry fee for a child. A pair of simultaneous equations that could be used to represent the situation above is

A $3x + 2y = 57.5$, $4x + 5y = 67$ **B** $3x + 2y = 67$, $4x + 5y = 57.5$ **C** $3x + 4y = 57.5$, $2x + 5y = 67$

D $3x + 4y = 67$, $2x + 5y = 57.5$ **E** $4x + 3y = 67$, $5x + 2y = 57.5$

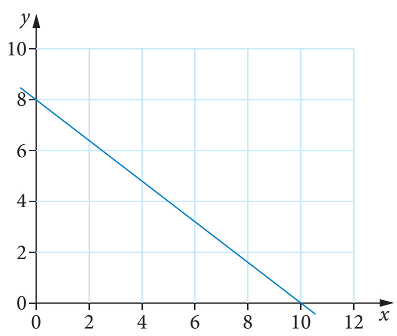
7 © VCAA 2020 1GRQ4 65% A straight line is graphed below.

An equation for this straight line is

A $10x + 8y = 40$ **B** $4x + 5y = 40$

C $4x - 5y = 40$ **D** $5x + 4y = 40$

E $8x + 10y = 40$



8 © VCAA 2008 1GRQ6 65% At the local bakery, the cost of four doughnuts and six buns is \$14.70. The cost of three doughnuts and five buns is \$11.90. At this bakery, the cost of one doughnut and two buns will be

A \$2.80 **B** \$3.80 **C** \$3.85 **D** \$4.55 **E** \$4.85

9 © VCAA 2011 1GRQ4 64% The fare, \$$F$, to travel a distance of n kilometres in a taxi is given by the rule $F = a + bn$. To travel a distance of 20 kilometres, the taxi fare is \$18.20. To travel a distance of 30 kilometres, the taxi fare is \$25.70. The charge per kilometre, b, is

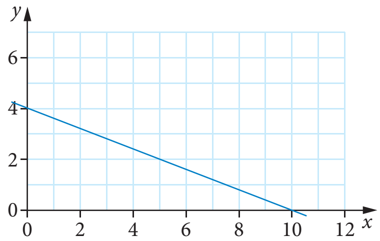
A \$0.75 **B** \$0.88 **C** \$0.91 **D** \$1.33 **E** \$3.20

10 © VCAA 2003 1GRQ6 62% For the pair of simultaneous equations $2x - 3y = 7$ and $3x = 5 - y$, the solution is

A $x = -2, y = -1$ **B** $x = -1, y = -3$ **C** $x = -1, y = 2$

D $x = 2, y = -3$ **E** $x = 2, y = -1$

11 © VCAA 2015 1GRQ4 62% The graph of a straight line is shown.



An equation for this straight line is

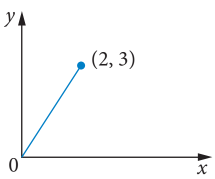
A $2x + 5y = 20$ **B** $10x + 4y = 20$

C $2x - 5y = 20$ **D** $5x + 2y = 20$

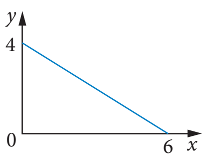
E $4x - 10y = 20$

12 © VCAA 2008 1GRQ5 48% A mixture contains two liquids, A and B. Liquid A costs \$2 per litre and liquid B costs \$3 per litre. Let x be the volume (in litres) of liquid A purchased. Let y be the volume (in litres) of liquid B purchased. Which graph shows all possible volumes of liquid A and liquid B that can be purchased for exactly \$12?

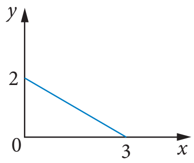
A



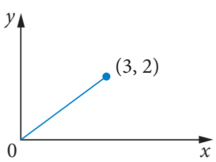
B



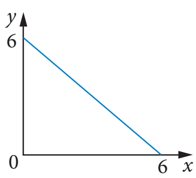
C



D



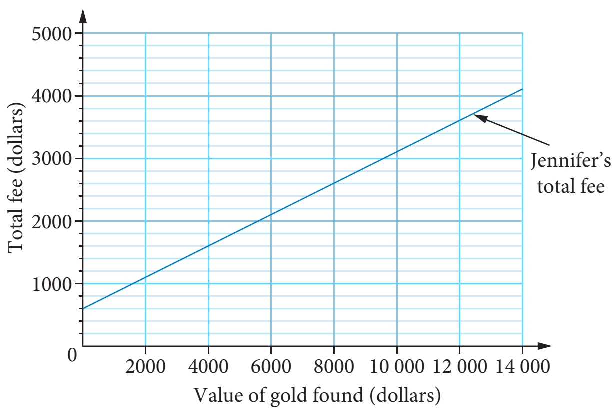
E



13 © VCAA 2004 1GRQ6 45% The cost, \$$C$, of hiring a boat for x hours is given by the equation $C = ax + b$ where a is the hourly rate and b is a fixed booking fee. When the boat is hired for 4 hours the cost is \$320. When the boat is hired for 6 hours the cost is \$450. When the boat is hired for one hour the cost is

A \$65 **B** \$75 **C** \$77 **D** \$80 **E** \$125

14 © VCAA 2018 2GRQ3ab (2 marks) Robert wants to hire a geologist to help him find potential gold locations. One geologist, Jennifer, charges a flat fee of \$600 plus 25% commission on the value of gold found. The following graph displays Jennifer's total fee in dollars.



Another geologist, Kevin, charges a total fee of \$3400 for the same task.

a 60% Copy the above graph and add a graph of the line representing Kevin's fee. 1 mark

b 41% For what value of gold found will Kevin and Jennifer charge the same amount for their work? 1 mark

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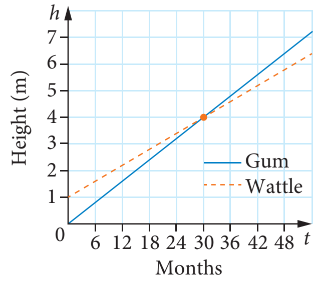
15 (6 marks) The graph shows the growth of two different types of trees, measured from the time they were seedlings.



a Which tree was taller as a seedling and what was the difference in height? 1 mark

b After how many months do the trees reach the same height? 1 mark

c What is that height? 1 mark

d Find an equation for the height of the

 i gum tree 1 mark

 ii wattle tree. 1 mark

e Are these equations good models for the height of the trees after 20 years? Why or why not? 1 mark

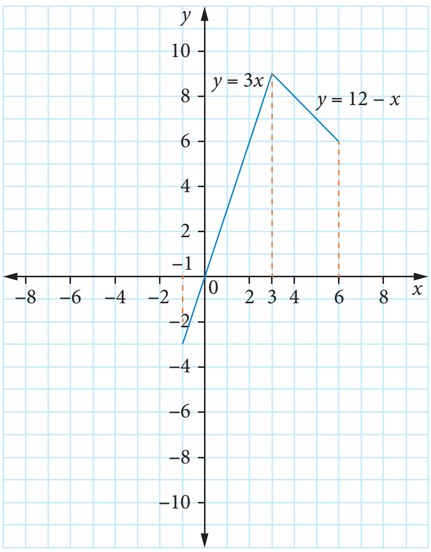
4.4 Piecewise linear graphs

Video playlist
Piecewise linear graphs

A **piecewise linear graph** is a graph that is made up of more than one straight line piece. There are two types of piecewise linear graphs: **line segment graphs** and **step graphs**.

Line segment graphs

A **line segment graph** is a piecewise linear graph that joins two or more straight line pieces. For example:



The equations for this graph are:

$y = 3x$ $\quad -1 \le x < 3$ ← This means for x values between −1 and 3, including −1.

$y = 12 - x$ $\quad 3 \le x < 6$ ← This means for x values between 3 and 6, including 3.

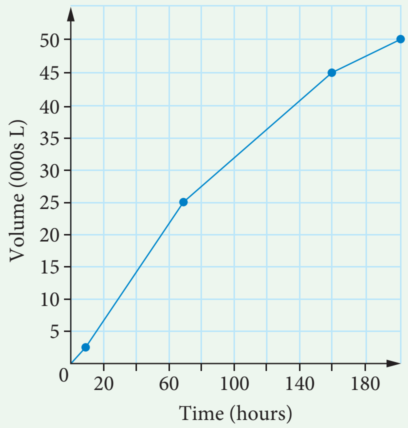
Line segment graphs are useful for displaying information that involves rates. The rate for each line segment is the slope of the line.

WB

p. 73

WORKED EXAMPLE 11 Interpreting line segment graphs

A rainwater tank holds 50 000 litres. The line segment linear graph shows the rate at which the rainwater tank fills with water from the start of winter.



a Explain how we know the rainwater tank fills at four different rates.

b Approximately how many litres does the rainwater tank hold after 160 hours?

c After how many hours from the start of winter is the rainwater tank filled to capacity?

d Approximately during which times does the tank fill the fastest?

e What is the rate, in litres per hour, that the tank is filling in the last 20 hours?

Steps	Working
a Refer to the line segments and their slopes.	The graph is made up of four different line segments and each line segment has a different slope. This means the rainwater tank fills at four different rates.
b Read from the graph and note how the vertical scale is written.	After 160 hours, the rainwater tank holds 45 000 litres.
c Read from the graph.	The rainwater tank filled to capacity after 200 hours.
d Identify which of the line segments has the greatest slope.	The second time period has the greatest slope, so the tank fills the fastest from approximately 10 hours to 70 hours after the start of winter.
e Find the slope of the last line segment. Convert the rate to litres per hour.	Slope of last line segment $= \frac{\text{rise} \uparrow}{\text{run} \rightarrow}$ $= \frac{5}{40}$ $= 0.125$ thousand litres per hour. The tank is filling at a rate of $0.125 \times 1000 = 125$ litres per hour in the last 20 hours.

Line segment graphs where the horizontal axis measures time and the vertical axis measures distance are called **distance-time graphs**. For distance-time graphs, the slope of the line segments measures speed, which is the rate at which distance is travelled.

4.4

WB

p. 74

WORKED EXAMPLE 12 Interpreting line segment distance-time graphs

The distance-time graph below shows how Celine walked to Nicky's house, stayed there for a while, then borrowed Nicky's bike and cycled home.

a The vertical intercept of this graph is 2 km. What does this mean?

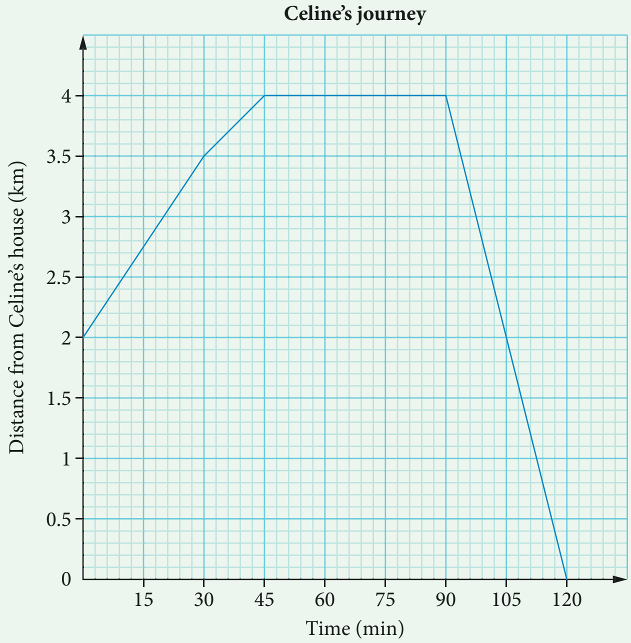
b Explain how we know Celine started walking slower after 30 minutes.

c Explain how we know the horizontal line segment represents the time Celine is at Nicky's house.

d How long did Celine stay at Nicky's house?

e What is the distance between Celine's house and Nicky's house?

f Calculate Celine's cycling speed in kilometres/hour during the last section of her journey.



Steps	Working
a The vertical intercept is the initial value. What is the vertical scale measuring?	Celine started her journey 2 km from her own house.
b Refer to the slopes of the line segments, which measure speed.	The slope of the line segment after 30 minutes is less than the slope of the line segment before 30 minutes, so Celine started walking slower after 30 minutes.
c Refer to the vertical axis value.	The horizontal line indicates that Celine is the same distance (4 km) from her house during this time, so she is at Nicky's house.
d Read from the graph.	Celine stayed at Nicky's house from the 45-minute mark to the 90-minute mark, so Celine stayed at Nicky's house for 45 minutes.
e Read the distance using the vertical scale of the graph.	The distance between Celine's house and Nicky's house is 4 km.
f Calculate the slope from the graph in kilometres/minute and convert to kilometres/hour.	cycling speed $= \frac{\text{rise} \rightarrow}{\text{run} \downarrow} = \frac{4\text{ km}}{30\text{ min}}$ $= \frac{4\text{ km}}{\frac{1}{2}\text{ h}}$ $= 8\text{ km/h}$

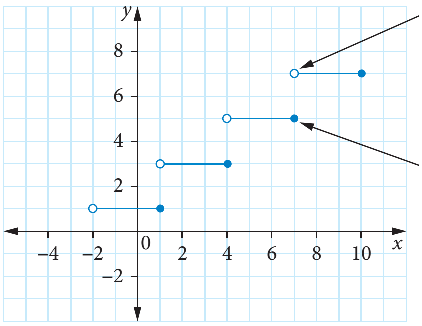
Exam hack

Always check to see if all the information is given in the same units. Convert units where necessary.

Worksheet
Step graphs

Step graphs

A **step graph** is a piecewise linear graph that has only horizontal straight line pieces. For example:



WB

p. 75

WORKED EXAMPLE 13 Interpreting step graphs

This step graph shows the daily parking charges at a car park.

a Find the charge for parking for

  i 55 minutes

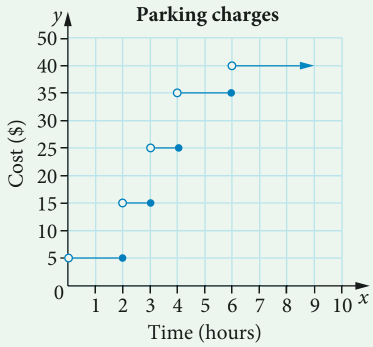
  ii 3 hours

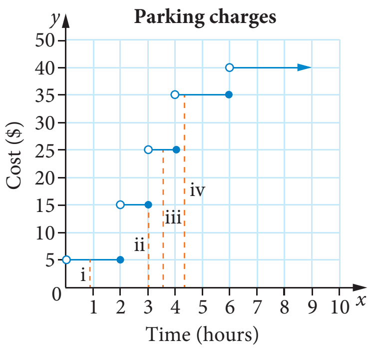
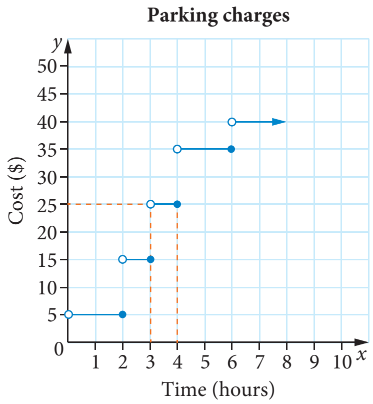
  iii $3\frac{1}{2}$ hours

  iv 4 hours 20 minutes.

b What range of times can a driver park for $25?

c What does the arrow on the $40 step mean?



Steps	Working
a Read from the graph. 	**i** 55 minutes parking will cost $5. **ii** 3 hours of parking will cost $15. **iii** $3\frac{1}{2}$ hours of parking will cost $25. **iv** 4 hours 20 minutes of parking will cost $35.
b **1** Find the relevant step on the graph. Remember, a closed circle means the value is included and an open circle means it's not included. Write the answer.	 For $25 the driver can park for a time that is more than 3 hours up to and including 4 hours.
c The arrow indicates the line continues.	After 6 hours, the parking charge remains constant at $40, so the maximum daily charge is $40.

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VCE QUESTION ANALYSIS

© VCAA 2009 2GRQ1 2009 Examination 2 Graphs and relations Question 1 (7 marks)

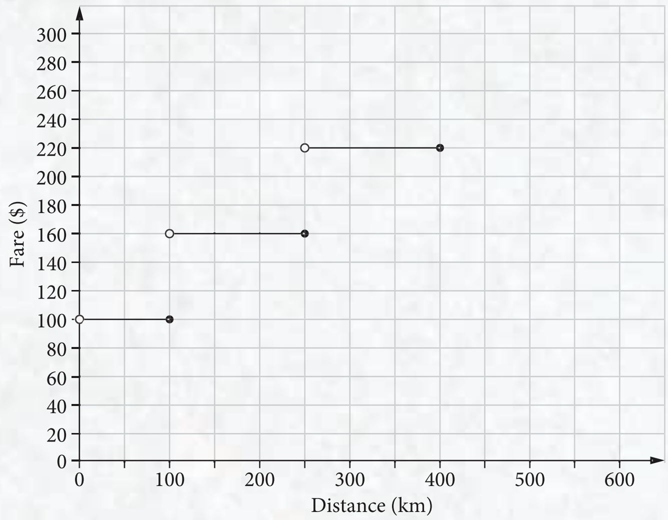
4.4

Fair Go Airlines offers air travel between destinations in regional Victoria. The table shows the fares for some distances travelled.

Distance (km)	Fare
$0 < distance \leq 100$	\$100
$100 < distance \leq 250$	\$160
$250 < distance \leq 400$	\$220

a What is the maximum distance a passenger could travel for \$160? 1 mark

The fares for the distances travelled in the table are graphed below.



b The fare for a distance longer than 400 km, but not longer than 550 km, is \$280. Copy the graph and draw this information on it. 1 mark

Fair Go Airlines is planning to change its fares. A new fare will include a service fee of \$40, plus 50 cents per kilometre travelled. An equation used to determine this new fare is given by

$$fare = 40 + 0.5 \times distance$$

c A passenger travels 300 km. How much will this passenger save on the fare calculated using the equation above compared to the fare shown in the table? 1 mark

d At a certain distance between 250 km and 400 km, the fare, when calculated using either the new equation or the table, is the same. What is this distance? 2 marks

e An equation connecting the maximum distance that may be travelled for each fare in the table can be written as

$$fare = a + b \times maximum\ distance$$

Determine a and b. 2 marks

Video playlist VCE question analysis: Linear functions, graphs, equations and models

Reading the question

- Note the words 'longer than' and 'not longer than' and the mathematical symbols they represent.
- Part **c** is asking how much the passenger 'saves' on the fare, not how much they pay.
- The maximum distances for each fare are indicated both in the table and on the graph.

Thinking about the question

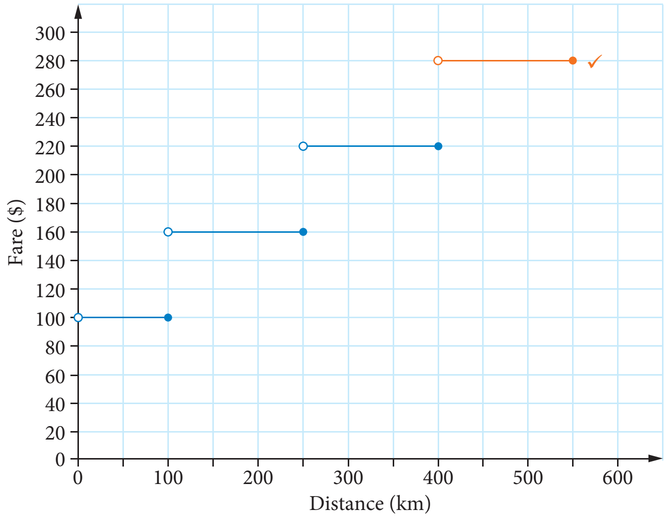
- How do the words 'longer than' and 'not longer than' appear on the graph?
- Part **d** involves calculating two different methods.
- In part **e**, what do a and b represent in the equation of a straight line?

Worked solution (✓ = 1 mark)

a Read from the vertical axis and then down to the horizontal axis, noting that the dot means the point is included.

The maximum distance a passenger could travel for \$160 is **250 km**. ✓

b



c fare = 40 + 0.5 × distance

If distance = 300 km, then fare = 40 + 0.5 × 300 = 190

So, the cost using this equation is \$190.

The fare shown in the table for travelling 300 km is \$220.

So, the passenger will save **\$30** ✓ on the fare calculated using the equation compared to the fare shown in the table.

d From the graph, the fare for the distance between 250 km and 400 km is \$220.

The equation is fare = 40 + 0.5 × distance

To find the distance when fare = 220, solve using CAS if necessary.

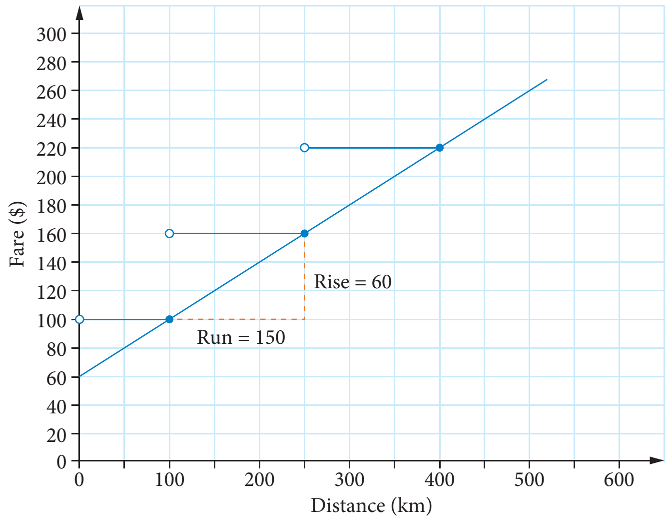
40 + 0.5 × distance = 220 ✓

$$\text{distance} = \frac{220 - 40}{0.5} = 360$$

The distance when the two fares are the same is **360 km**. ✓

e Draw a line connecting the maximum distance that may be travelled for each fare.

Find the slope using $\frac{\text{rise}}{\text{run}}$ and find the vertical axis intercept.



slope $= \frac{60}{150} = \frac{2}{5} = 0.4$ and vertical intercept from the graph is 60.

The equation of the line is fare = $a + b \times$ maximum distance, so a is the vertical axis intercept and b is the slope.

Therefore, $a = \mathbf{60}$ ✓ and $b = \mathbf{0.4}$. ✓

Student performance

80–100% 60–79% 0–59%

a 91% The most common incorrect answer was 249 km, suggesting confusion about what ≤ 250 means.

b 91% Answers without the correct open and closed circles for the endpoints were incorrect. The most common error showed the line ending at a distance of 525 km, which suggested a misreading of the horizontal scale.

Success percentages for parts **a** and **b** were averaged.

c 60%

d 60% Students must clearly identify their answer to a question. For this question, several answers just consisted of the line $220 = 40 + 0.5 \times 360$.

e 60% Some students found only the equation $160 = a + 250b$ and then made up a value for a (usually $a = 0$) to find a value of b.

Success percentages for parts **c**, **d** and **e** were averaged.

Areas of strength

- drawing in a missing section of a step graph
- substitution into a linear equation that does not require transposition
- finding the coordinates of the intersection of two lines

Areas of weakness

- using a straight edge to draw a straight line rather than presenting a freehand line
- setting up simultaneous equations

EXERCISE 4.4 Piecewise linear graphs

ANSWERS p. 505

Recap

80–100% 60–79% 0–59%

1 The solution to the simultaneous equations $7x - y = -8$ and $-5x - y = 16$ is

A $x = 4, y = 36$ **B** $x = -12, y = -44$ **C** $x = -2, y = 11$

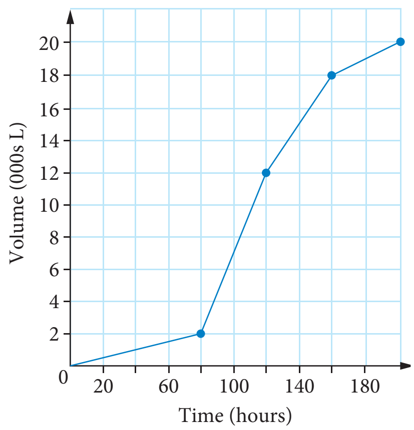
D $x = -4, y = -36$ **E** $x = -2, y = -6$

2 © VCAA 2005 1GRQ6 57% One afternoon at the beach Mr Smith bought four ice creams and three drinks for his family at a cost of \$21.40. Mrs Brown bought five of the same ice creams and two of the same drinks for \$20.80. Based on these prices, the cost of one drink is

A \$2.80 **B** \$2.90 **C** \$3.00 **D** \$3.30 **E** \$3.40

Mastery

3 WORKED EXAMPLE 11 A rainwater tank holds 20 000 litres. The line segment linear graph shows the rate at which the rainwater tank fills with water from the start of the rainy season.



a Explain how we know the rainwater tank fills at four different rates.

b Approximately how many litres does the rainwater tank hold after 160 hours?

c After how many hours from the start of the rainy season is the rainwater tank filled to capacity?

d Approximately during which times does the tank fill the fastest?

e What is the rate, in litres per hour, at which the tank is filling in the first 80 hours?

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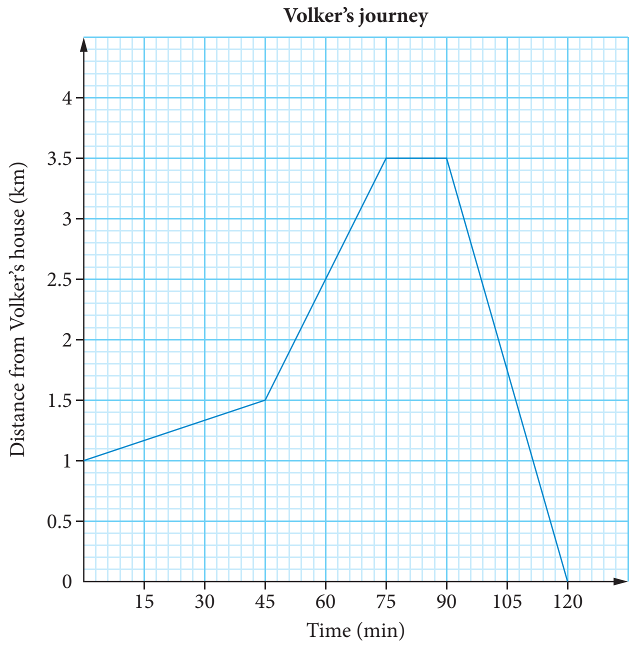
4 WORKED EXAMPLE 12 The distance-time graph below shows how Volker and Herb walked from the park to Herb's house. Volker stayed there a while and then jogged home.



a The vertical intercept of this graph is 1 km. What does this mean?

b Explain how we know Volker and Herb started walking faster after 45 minutes?

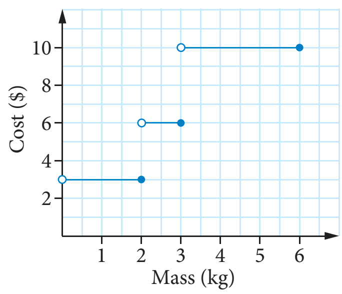
c Explain how we know the horizontal line segment represents the time Volker and Herb were at Herb's house.

d How long did Volker stay at Herb's house?

e What is the distance between Volker's house and Herb's house?

f Calculate Volker's jogging speed in kilometres/hour during the last section of his journey.

5 WORKED EXAMPLE 13 This step graph shows the cost of sending parcels of different weights by air freight.



a Find the cost of sending

i one parcel weighing 3.4 kg

ii one parcel weighing 1.7 kg

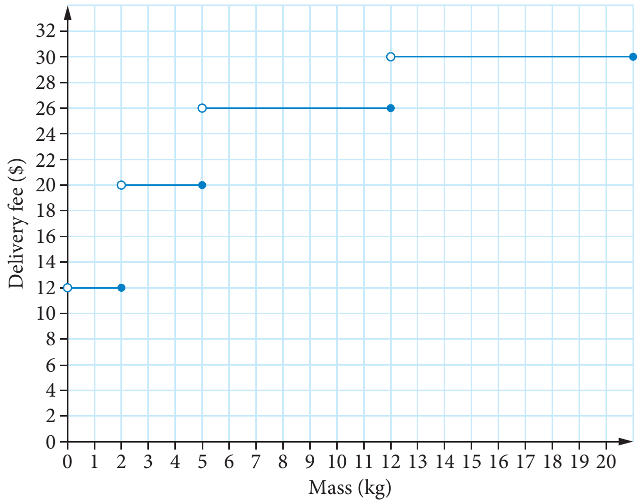
iii two parcels, weighing 2 kg and 3 kg.

b What range of weights for a single parcel can be sent for $6?

c Why do you think there is no arrow on the $10 step?

6 © VCAA 2020 1GRQ3 91% The *delivery fee* for a parcel, in dollars, charged by a courier company is based on the *weight* of the parcel, in kilograms.

This relationship is shown in the step graph below for parcels that weigh up to 20 kg.



Which one of the following statements is **not** true?

A The *delivery fee* for a 4 kg parcel is $20.

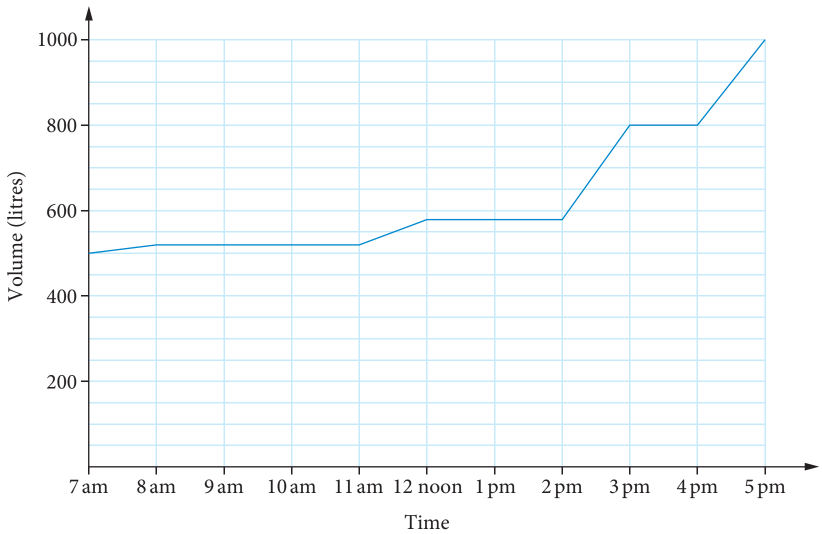
B The *delivery fee* for a 12 kg parcel is $26.

C The *delivery fee* for a 13 kg parcel is the same as the *delivery fee* for a 20 kg parcel.

D The *delivery fee* for a 10 kg parcel is $14 more than the *delivery fee* for a 2 kg parcel.

E The *delivery fee* for a 12 kg parcel is $18 more than the *delivery fee* for a 2 kg parcel.

7 © VCAA 2017 1GRQ2 89% The graph shows the volume of water in a water tank between 7 am and 5 pm on one day.



Which one of the following statements is **true**?

A The volume of water in the tank decreases between 8 am and 11 am.

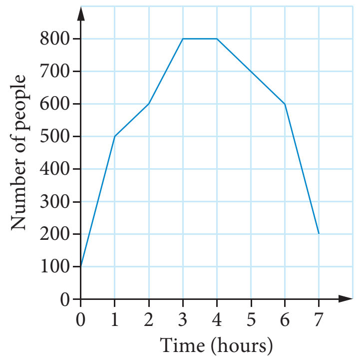
B The volume of water in the tank increases at the greatest rate between 4 pm and 5 pm.

C The volume of water in the tank is constant between 12 noon and 2 pm.

D The tank is filled with water at a constant rate of 100 L per hour.

E More water enters the tank during the first five hours than during the last five hours.

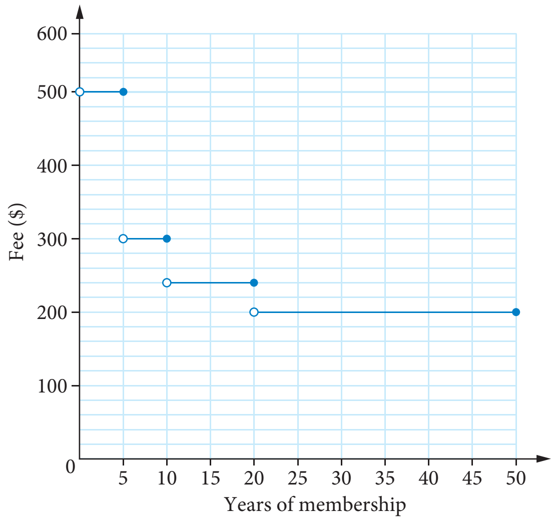
8 © VCAA 2020 1GRQ1 87% The graph shows the *number of people* who attended a market over a seven-hour time period.



For how many hours were there at least 600 people at the market?

A 2 **B** 3 **C** 4
D 6 **E** 7

9 © VCAA 2017 1GRQ4 72% The annual *fee* for membership of a car club, in dollars, based on *years of membership* of the club is shown in the step graph.



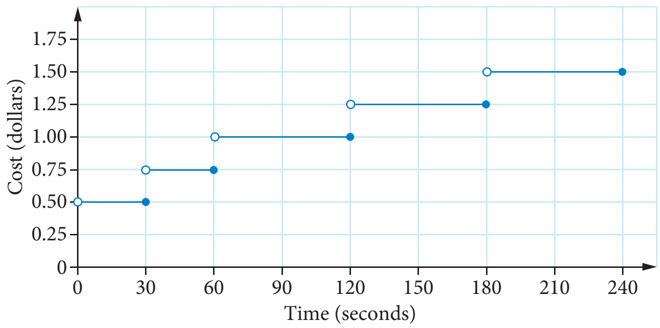
In the Martin family:

- Hayley has been a member of the club for four years
- John has been a member of the club for 20 years
- Sharon has been a member of the club for 25 years.

What is the total fee for membership of the car club for the Martin family?

A \$200 **B** \$600 **C** \$720
D \$900 **E** \$940

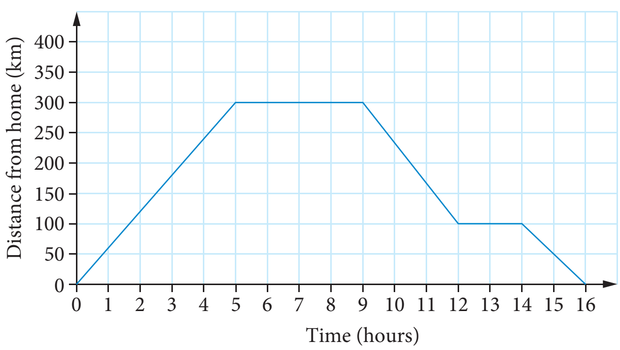
10 © VCAA 2004 1GRQ1 68% The graph shows the cost (dollars) of mobile telephone calls up to 240 seconds long.



The cost of making a 90-second call followed by a 30-second call is

A \$1.00 **B** \$1.20 **C** \$1.25 **D** \$1.50 **E** \$1.75

11 © VCAA 2004 1GRQ4 59% The graph shows a distance-time graph for a car travelling from home along a long straight road over a 16-hour period.



In which one of the time intervals is the speed of the car greatest?

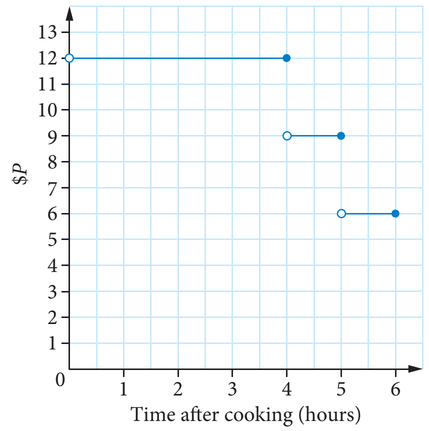
A 0 to 5 hours **B** 5 to 9 hours
C 9 to 12 hours **D** 12 to 14 hours
E 14 to 16 hours

12 © VCAA 2018N 1GRQ2 A supermarket sells roasted chickens. For the first four hours after cooking, the roasted chickens are sold at full price. After this time, the selling price of each roasted chicken is reduced. The price of a roasted chicken, $\$P$, at any time up to six hours after cooking is shown in the step graph.

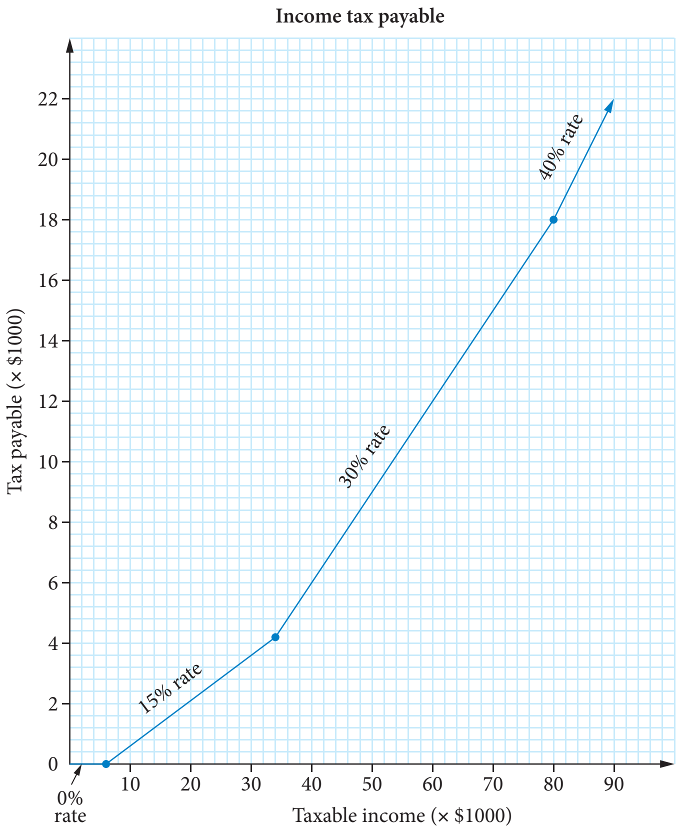
A roasted chicken is sold five hours after cooking. By how much has the full price of the roasted chicken been reduced?

A \$3 **B** \$4 **C** \$8

D \$9 **E** \$12



13 (4 marks) The line segment graph shows the income tax payable in a particular country.



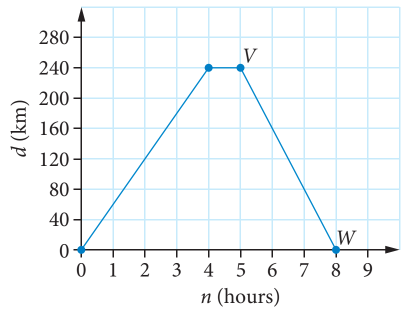
a What is the income range for the first two tax rates? 2 marks

b Find the tax payable on each of the following incomes from the graph.

i \$30 000 1 mark

ii \$84 000 1 mark

14 © VCAA 2020 2GRQ1 (3 marks) Kyla owns and manages a truck and car care business. After a major repair on a truck, one of the mechanics took it on a long test drive. The test drive started at 12 noon. After four hours, the mechanic stopped to rest for one hour and then returned to the workshop. The graph below shows the truck's distance from the workshop, d, in kilometres, and the number of hours of test driving, n, after 12 noon.

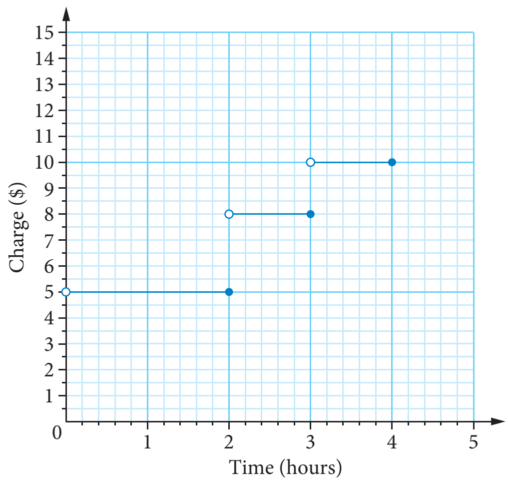


a 88% At what time of the day did the mechanic arrive back at the workshop? 1 mark

b 89% Find the average speed, in kilometres per hour, of the truck during the first four hours of the test drive. 1 mark

c 43% On the return trip, the truck travelled at an average speed of 80 km/h. The equation of the line segment VW that represents this part of the test drive is of the form $d = k - 80n$. Find the value of k. 1 mark

15 © VCAA 2019N 2GRQ2 (3 marks) Customers who visit Rumi's shop park their cars in an underground car park. The graph below shows the total charge for parking, in dollars, according to the number of hours parked in one visit.



a On Monday, Leon parked in the underground car park for two hours and Lucy parked for one-and-a-half hours.

What was the total combined charge for Leon and Lucy? 1 mark

b Customers who park for more than four hours are charged $11.50 if they do not stay longer than five hours. Copy the graph and add this information. 1 mark

c Another customer, Pam, parked in the underground car park on both Tuesday and Wednesday. Each day she parked for more than two hours but less than four hours. Pam was charged less than $20 in total for these two days.

Write down the two possible amounts that Pam could have been charged. 1 mark

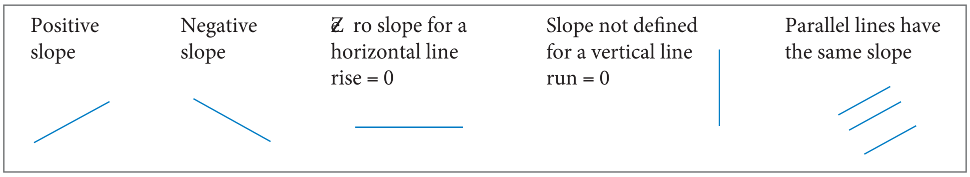
4 Chapter summary

Linear functions in the form $y = a + bx$

- A **linear function** is an equation that can be written in the form $y = a + bx$, where x and y are variables and a and b are constants.
- The graph of a linear function is a straight line.
- The **coordinates** of points on the line are found by substituting x values into the equation to calculate corresponding y values.
- Functions can be drawn using a table of values.
- To determine whether a point lies on a line, substitute the coordinates into the equation of the line and check if the result is true or false.
- For a straight line in the form $y = a + bx$:

  a = the y-**intercept** = the y value when $x = 0$

  b = the **slope** = $\dfrac{\text{rise} \uparrow}{\text{run} \rightarrow} = \dfrac{\text{vertical distance between two points}}{\text{horizontal distance between the same two points}}$



Interpreting linear functions

- When interpreting linear functions in the form $y = a + bx$:

  a = the initial value

  b = the constant rate of change
- We can use linear equations to make predictions about real-life situations, but not all calculated values make sense in the real world.
- Linear functions can be used to model the cost and revenue for a business.
- Profit of a business is also a linear function and can be calculated using

  profit = revenue − cost

Simultaneous equations

- To graph linear relations in the form $Ax + By = C$:
  1. find the x-intercept by substituting $y = 0$ and solving for x
  2. find the y-intercept by substituting $x = 0$ and solving for y
  3. draw a straight line through the two intercepts.
- When $B = 0$, the graph is a vertical line.
- If we graph two linear equations on the same axes, the **intersection** of the two lines is the point that lies on both lines.
- The point of intersection of two lines is the solution to the two **simultaneous equations**.
- Simultaneous linear equations can be in the form $y = a + bx$ or $Ax + By = C$.
- Both forms of linear equations can be solved using CAS.

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- To solve worded problems using simultaneous equations:
  1 Identify the unknowns and assign a pronumeral for each of them.
  2 Set up two equations by converting the given information into mathematical symbols.
  3 Solve the simultaneous equations using CAS.
  4 Answer the question in sentence form.

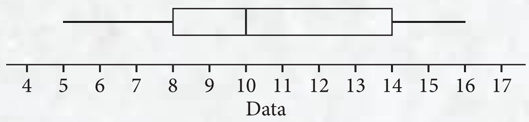
Piecewise linear graphs

- A **piecewise linear graph** is a graph that is made up of more than one straight line piece.
- A **line segment graph is a piecewise graph that joins** two or more straight line pieces.
- The rate for each line segment in a line segment graph is the slope of the line.
- **Distance-time graphs** are line segment graphs where the horizontal axis measures time and the vertical axis measures distance.
- The slope of the line segments in a distance-time graph measures speed, which is the rate at which distance is travelled.
- A **step graph** is a piecewise graph that has only horizontal straight line pieces.

Cumulative examination 1

Total number of marks: 12 Reading time: 5 minutes Writing time: 27 minutes

Use the following information to answer the next two questions.



1 The percentage of data below 10 is

A 25% **B** 50% **C** 75%

D 100% **E** none of the above

2 The value of Q_1 is

A 5 **B** 8 **C** 10 **D** 14 **E** 16

3 The sequence 4, 9, 24, 69 … is generated from the recurrence relation $t_{n+1} = Rt_n - 3$, where $t_0 = 4$.

The value of R is

A 2 **B** 3 **C** 5 **D** 7 **E** 15

4 © VCAA 2002 1BRMQ6 A machine is purchased for $36 000. Its value depreciates at a rate of $0.16 for each unit it produces. On average, the machine produces 24 000 units a year. Using the unit cost method of depreciation, the value of the machine after six years of use is closest to

A $3840 **B** $7200 **C** $12 960 **D** $23 040 **E** $32 160

5 © VCAA 2010 1BRMQ4 The price of a one-way airfare between two cities varies each day according to demand.

- On Monday the price is $160.
- The price on Tuesday is 25% greater than the price on Monday.
- The price on Wednesday is 10% less than the price on Tuesday.
- The price on Thursday is 25% less than the price on Wednesday.
- The price on Friday is 20% greater than the price on Thursday.
- The Saturday price is the same as the Friday price.

The price on Saturday is

A $150 **B** $162 **C** $176 **D** $288 **E** $330

6 © VCAA 2008 1BRMQ6 Sam and Charlie each invest $5000 for three years. Sam's investment earns simple interest at the rate of 7.5% per annum. Charlie's investment earns interest at the rate of 7.5% per annum compounding annually. At the conclusion of three years, correct to the nearest cent, Sam will have

A $86.48 less than Charlie. **B** $86.48 more than Charlie.

C $132.23 less than Charlie. **D** $132.23 more than Charlie.

E the same as Charlie.

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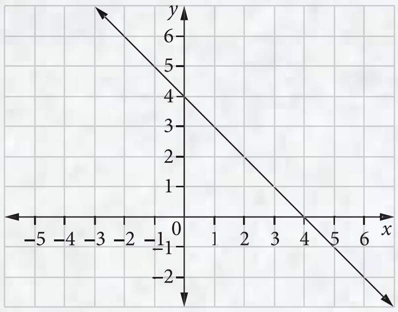
7 Which linear equation matches this graph?



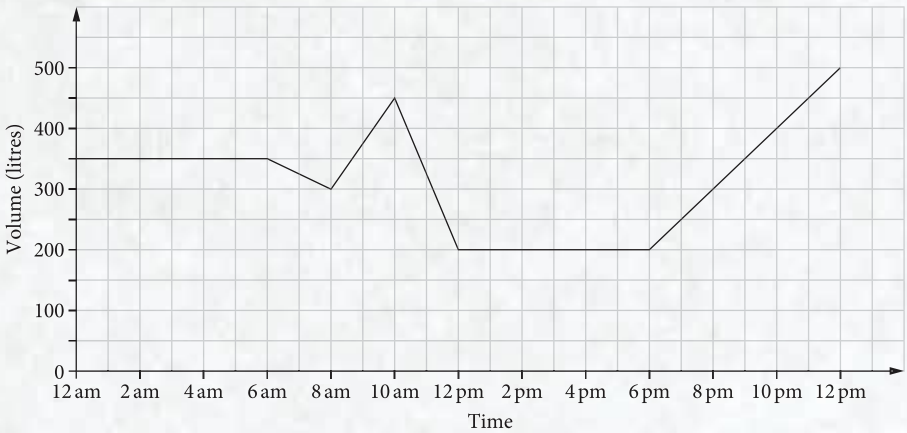
A $y = 4 - 4x$ **B** $y = 4 + 4x$

C $y = -4 + x$ **D** $y = -4 - 4x$

E $y = 4 - x$

Use the following information to answer the next two questions.

The volume of water that is stored in a tank over a 24-hour period is shown in the graph.



8 © VCAA 2010 1GRQ1 What is the difference in the volume of water (in litres) in the tank between 8 am and 6 pm?

A 50 **B** 100 **C** 120 **D** 200 **E** 400

9 © VCAA 2010 1GRQ2 The rate of increase in the volume of water in the tank (in litres/hour) between 8 am and 10 am is

A 37.5 **B** 50 **C** 75 **D** 125 **E** 150

10 © VCAA 2010 1GRQ5 The cost in dollars, C, of making n pottery mugs is given by the equation $C = 150 + 6n$. A loss will result from selling

A 60 mugs at \$9.00 each **B** 70 mugs at \$8.50 each **C** 80 mugs at \$7.50 each

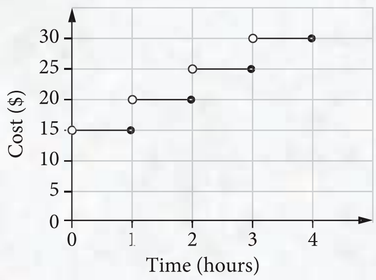
D 90 mugs at \$8.00 each **E** 100 mugs at \$9.50 each

11 © VCAA 2002 1GRQ4 For the pair of simultaneous equations $4x = 7 - y$ and $5x + 7y = 3$, the solution is

A $x = -2, y = -1$ **B** $x = -2, y = 1$ **C** $x = 1, y = 3$

D $x = -1, y = 2$ **E** $x = 2, y = -1$

12 © VCAA 2007 1GRQ5 The cost of hiring one motorbike for up to 4 hours is shown in the graph. Two motorbikes were hired. The **total** charge for hiring the two motorbikes was \$45.



The time for which each motorbike was hired could have been

A 1 hour and 2 hours **B** 1 hour and 3 hours

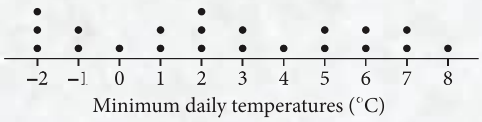
C 1.5 hours and 2 hours **D** 1.5 hours and 3 hours

E 2 hours and 3.5 hours

Cumulative examination 2

Total number of marks: 17 Reading time: 4 minutes Writing time: 26 minutes

1 (2 marks) This dot plot shows the minimum daily temperatures (°C) in Hobart over a three-week period.



a What is the mode? 1 mark

b Find the IQR. 1 mark

2 © VCAA 2017N 2CQ5a–c (3 marks) The snooker table at a community centre was purchased for $3000. After purchase, the value of the snooker table was depreciated using the flat rate method of depreciation. The value of the snooker table, V_n, after n years, can be determined using the recurrence relation

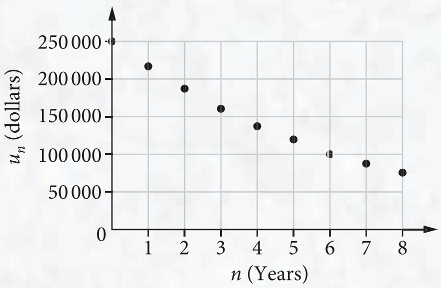
$$V_0 = 3000, \quad V_{n+1} = V_n - 180$$

a What is the annual depreciation in the value of the snooker table? 1 mark

b Use recursion to show that the value of the snooker table after two years, V_2, is $2640. 1 mark

c After how many years will the value of the snooker table first fall below $2000? 1 mark

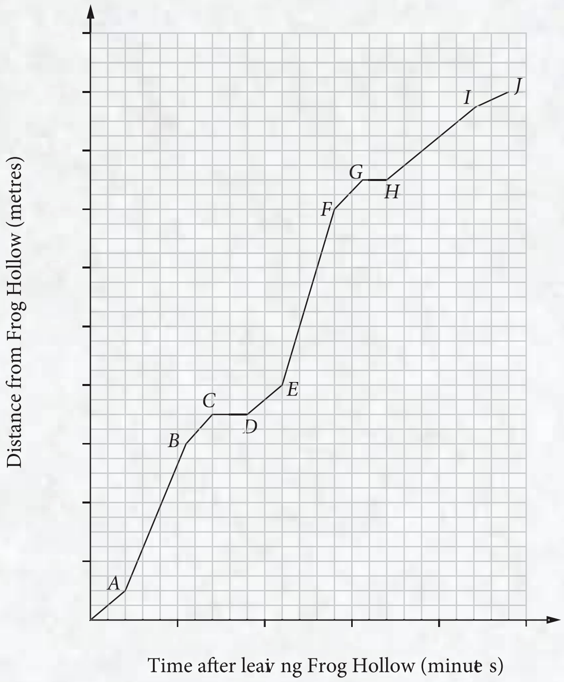
3 © VCAA 2012 1BRMQ7 MODIFIED (2 marks) The following graph shows the decreasing value of an asset over eight years.



a What type of depreciation is being used? 1 mark

b If u_n is the value of the asset over n years, and rate of depreciation is 14%, write the rule for calculating the value of the asset over n years. 1 mark

4 © VCAA 2002 2GRQ1 (4 marks) At the Gum Flat Fun Park there are many attractions. One that appeals especially to the younger visitors is the train Puffing Polly. The distance-time graph represents a train trip for Puffing Polly from Frog Hollow to Eagle Hill, stopping at two stations on the way.



a What is the total time for which Puffing Polly is stopped at the two stations on the way? 1 mark

b **i** Which line segment of the graph represents the section of the trip when Puffing Polly is travelling the fastest? 1 mark

ii Find its speed for this section of the trip, stating clearly the units used in your answer. 2 marks

5 (6 marks) Kami wants to hire a car for the three days of a long weekend and is choosing between two rental companies.

CarsRus charges $45 per day plus $1.85 per kilometre.

WeRent charges $80 per day plus $1.55 per kilometre.

a If d is the distance travelled in kilometres, and C is the cost in dollars, write an equation representing the cost of hiring a car for three days from

i CarsRus 1 mark

ii WeRent. 1 mark

b **i** After how many kilometres will the cost for hiring a car be the same for both companies? 1 mark

ii What is the cost at this time? 1 mark

c If Kami plans to drive an average of 100 km each day on the long weekend, which plan would be better for her and what would be her cost? Give reasons for your answer. 2 marks

CHAPTER

5 MATRICES

STUDY DESIGN COVERAGE

Study design coverage

UNIT 1, AREA OF STUDY 4: DISCRETE MATHEMATICS

Matrices

- use of matrices to store and display information that can be presented in a rectangular array of rows and columns such as databases and links in social networks and road networks
- types of matrices (row, column, square, zero and identity) and the order of a matrix
- matrix addition, subtraction, multiplication by a scalar, and matrix multiplication including determining the power of a square matrix using technology as applicable
- use of matrices, including matrix products and powers of matrices, to model and solve problems, for example costing or pricing problems, and squaring a matrix to determine the number of ways pairs of people in a network can communicate with each other via a third person
- inverse matrices and their applications including solving a system of simultaneous linear equations
- introduction to transition matrices (assuming the next state only relies on the current state), working with iterations of simple models linked to, for example, population growth or decay, including informal consideration of long run trends and steady state.

Video playlists (7):

5.1 Introducing matrices
5.2 Matrix addition, subtraction and scalar multiplication
5.3 Matrix multiplication
5.4 Inverse matrices
5.5 Matrix applications
5.6 Transition matrices
VCE question analysis Matrices

Worksheets (2):

5.2 Addition and subtraction of matrices
5.3 Multiplying matrices

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

5.1 Introducing matrices

Video playlist
Introducing matrices

What do databases, costing and pricing, quantum mechanics, and links in communication, social and road networks have in common? They can all be modelled by matrices.

The order of a matrix

Matrices are collections of numbers arranged into rows and columns inside square brackets. The **order of a matrix** tells us how many rows and columns it has. We always write the order in the form *number of rows* × *number of columns*, so Matrix Q is a 4 × 2 (pronounced *four by two*) matrix. The numbers in the **matrix** are called **elements**. Matrix Q has 4 × 2 = 8 elements.

Column 1   Column 2

$$Q = \begin{bmatrix} 5 & 12 \\ 9 & 0 \\ 17 & 22 \\ 1 & 19 \end{bmatrix} \begin{matrix} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \\ \leftarrow \text{Row 3} \\ \leftarrow \text{Row 4} \end{matrix}$$

Order of a matrix

Order of a matrix = *number of rows* × *number of columns*

A matrix with m rows and n columns has order $m \times n$.

WB
p. 77

WORKED EXAMPLE 1 Understanding the order of matrices

The table shows the quantity (in grams) of the main ingredients used to bake a batch of various types of muffins.

Type of muffin

Ingredient	Choc Chip	Apple	Savoury	Banana	Berry
Sugar	120	80	50	75	100
Flour	255	130	150	175	150
Butter	125	90	130	160	175

Find the

Steps | **Working**

a matrix M that could be used to show this information, stating its order and number of elements

Rewrite the information in the table as a matrix.

$$M = \begin{bmatrix} 120 & 80 & 50 & 75 & 100 \\ 225 & 130 & 150 & 175 & 150 \\ 125 & 90 & 130 & 160 & 175 \end{bmatrix}$$

The order of M is 3 × 5.

M has 15 elements.

Exam hack

The order of a matrix with m rows and n columns is $m \times n$ ('m **by** n'). The number of elements in a matrix is $m \times n$ ('m **times** n'). Even though they both have a ×, they represent two different things.

b matrix that could be used to show the quantity of flour used for apple muffins and state its order

Find the information in the table and write as a matrix.

$[130]$

The order is 1×1.

c 1×5 matrix that could be used to show the quantity of butter needed for each type of muffin

Find the information in the table and write as a matrix with the given order.

$$\begin{bmatrix} 125 & 90 & 130 & 160 & 175 \end{bmatrix}$$

d 5×1 matrix that could be used to show the quantity of sugar needed for each type of muffin

Find the information in the table and write as a matrix with the given order.

$$\begin{bmatrix} 120 \\ 80 \\ 50 \\ 75 \\ 100 \end{bmatrix}$$

e 1×3 matrix that could be used to show the quantity of each ingredient needed for a batch of berry muffins

Find the information in the table and write as a matrix with the given order.

$$\begin{bmatrix} 100 & 150 & 175 \end{bmatrix}$$

f 3×1 matrix that could be used to show the total quantities of each main ingredient needed if batches of all the five muffin types are made.

Find the information in the table and write as a matrix with the given order.

$$\begin{bmatrix} 120 + 80 + 50 + 75 + 100 \\ 255 + 130 + 150 + 175 + 150 \\ 125 + 90 + 130 + 160 + 175 \end{bmatrix}$$

$$= \begin{bmatrix} 425 \\ 860 \\ 680 \end{bmatrix}$$

g Copy the following labelled matrix showing the information from the table and fill in the missing numbers. The ingredients are shown by S = sugar, F = flour and B = butter.

	S	F	B
Choc Chip	120	255	☐
Apple	☐	130	90
Savoury	50	☐	130
Banana	☐	☐	160
Berry	☐	☐	☐

Find the information in the table and complete the matrix.

	S	F	B
Choc Chip	120	255	125
Apple	80	130	90
Savoury	50	150	130
Banana	75	175	160
Berry	100	150	175

5.1

Types of matrices

Type of matrix	Description	Examples	Order of examples
Row matrix	A matrix with just one row.	$\begin{bmatrix} 7 & -3 & 15 \end{bmatrix}$ $\begin{bmatrix} 9 & -3 & 0 & 0 \end{bmatrix}$	1 × 3 and 1 × 4
Column matrix	A matrix with just one column.	$\begin{bmatrix} 45 \\ -32 \end{bmatrix}$ $\begin{bmatrix} 31 \\ 25 \\ 50 \end{bmatrix}$	2 × 1 and 3 × 1
Square matrix	A matrix that has the same number of rows as columns.	$\begin{bmatrix} 7 & 12 & 0 & 8 \\ -3 & 9 & -2 & 5 \\ 6 & -1 & 0 & 0 \\ 10 & -6 & 13 & 4 \end{bmatrix}$ $\begin{bmatrix} 0 & -12 \\ -4 & 1 \end{bmatrix}$	4 × 4 and 2 × 2
Zero matrix	A matrix where all the numbers are 0.	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	3 × 2 and 1 × 6
Identity matrix (I) (also called the **unit matrix**)	A square matrix where all the elements in the diagonal from top left to bottom right are 1, and every other element is 0. We use I to indicate this matrix.	$I = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$ $I = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}$	4 × 4 and 2 × 2

WB
p. 79

WORKED EXAMPLE 2 Identifying types of matrices

For each of the following matrices state the order and whether it's a row, column, square, zero or identity matrix.

a $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 14 \\ 50 \\ -91 \\ 35 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Steps	Working
Does the matrix have • just one row (row matrix) or just one column (column matrix) • the same number of rows and columns (square matrix) • all zeros (zero matrix) • 1s it in the diagonal from top left to bottom right and zeros everywhere else (identity matrix)?	**a** 5 × 2, zero matrix **b** 4 × 1, column matrix **c** 3 × 3, square matrix, zero matrix **d** 6 × 6, square matrix, identity matrix **e** 1 × 5, row matrix

EXERCISE 5.1 Introducing matrices

ANSWERS p. 506

Mastery

1 WORKED EXAMPLE 1 The table shows how different types of tickets were sold to a school production of *Fame*.

	Sold by school office	Sold online	Sold at theatre
Student ticket	172	67	30
Adult ticket	3	139	10
Concession ticket	0	65	9
Teacher ticket	11	15	17

Find the

a matrix F that could be used to show this information, stating its order and number of elements

b matrix that could be used to show the number of adult tickets sold by the school office and state its order

c 4×1 matrix that could be used to show the number of tickets sold at the theatre

d 3×1 matrix that could be used to show the number of concession tickets sold in each of the different ways

e 1×4 matrix that could be used to show the number of each type of ticket sold online

f 4×1 matrix that could be used to show the totals sold for each of the four tickets types.

g Copy the following labelled matrix showing the information from the table and fill in the missing numbers. The ticket types are shown by S = student, A = adult, C = concession and T = teacher.

$$\begin{array}{l} \\ \text{Office} \\ \text{Online} \\ \text{Theatre} \end{array} \begin{array}{c} \begin{matrix} S & A & C & T \end{matrix} \\ \begin{bmatrix} 172 & \square & 0 & \square \\ 67 & 139 & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \end{array}$$

2 WORKED EXAMPLE 2 For each of the following matrices, state the order and whether it is a row, column, square, zero or identity matrix.

a $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

e $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

f $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3 Answer true or false to each of the following statements.

a A 6×10 matrix has 60 elements.

b A 5×7 matrix has 7 rows.

c A 4×1 matrix is a column matrix.

d If a matrix has 3 elements, it has to be either a row matrix or a column matrix.

e An identity matrix with five 1s has 25 elements.

4 © VCAA 2011 1MQ1 95% The matrix shows the airfares (in dollars) that are charged by Zeniff Airlines to fly between Adelaide (A), Melbourne (M) and Sydney (S).

$$\begin{array}{c} \text{From} \\ \begin{array}{ccc} A & M & S \end{array} \end{array}$$
$$\begin{bmatrix} 0 & 85 & 89 \\ 85 & 0 & 99 \\ 97 & 101 & 0 \end{bmatrix} \begin{matrix} A \\ M \\ S \end{matrix} \;\text{To}$$

The cost to fly from Melbourne to Sydney with Zeniff Airlines is

A \$85 **B** \$89 **C** \$97 **D** \$99 **E** \$101

5 © VCAA 2007 1MQ2 93% The number of tourists visiting three towns, Oldtown, Newtown and Twixtown, was recorded for three years. The data is summarised in the table.

	2004	2005	2006
Oldtown	975	1002	1390
Newtown	2105	1081	1228
Twixtown	610	1095	1380

The 3 × 1 matrix that could be used to show the number of tourists visiting the three towns in the year **2005** is

A $\begin{bmatrix} 975 & 1002 & 1390 \end{bmatrix}$ **B** $\begin{bmatrix} 1002 & 1081 & 1095 \end{bmatrix}$ **C** $\begin{bmatrix} 975 \\ 1002 \\ 1390 \end{bmatrix}$

D $\begin{bmatrix} 1002 \\ 1081 \\ 1095 \end{bmatrix}$ **E** $\begin{bmatrix} 975 & 1002 & 1390 \\ 2105 & 1081 & 1228 \\ 610 & 1095 & 1380 \end{bmatrix}$

6 © VCAA 2010 1MQ1 92% The order of the matrix $\begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$ is

A 2 × 2 **B** 2 × 3 **C** 3 × 2 **D** 4 **E** 6

7 © VCAA 2009 1MQ3 75% The number of people attending the morning, afternoon and evening sessions at a cinema is given in the table. The admission charges (in dollars) for each session are also shown in the table.

	Session		
	Morning	Afternoon	Evening
Number of people attending	25	56	124
Admission charge (\$)	12	15	20

A column matrix that can be used to list the number of people attending each of the three sessions is

A $\begin{bmatrix} 25 & 56 & 124 \end{bmatrix}$ **B** $\begin{bmatrix} 25 \\ 56 \\ 124 \end{bmatrix}$ **C** $\begin{bmatrix} 12 & 15 & 20 \end{bmatrix}$

D $\begin{bmatrix} 12 \\ 15 \\ 20 \end{bmatrix}$ **E** $\begin{bmatrix} 25 & 56 & 124 \\ 12 & 15 & 20 \end{bmatrix}$

8 What type of matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

A zero matrix **B** identity matrix **C** 3×2 matrix
D square matrix **E** 2×3 matrix

9 For the following matrices:

$J = \begin{bmatrix} 3 & 5 \\ 6 & 2 \end{bmatrix}$ $K = \begin{bmatrix} -2 \\ 5 \\ 1.1 \end{bmatrix}$ $L = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 1 & -2 \\ 5 & 0 & 8 \end{bmatrix}$ $M = \begin{bmatrix} 12 \end{bmatrix}$

which statement is **true**?

A All of these matrices are square matrices. **B** K is a 1×3 matrix.
C L has 10 elements. **D** J is an identity matrix.
E M is a 1×1 matrix.

10 Which of the following is **not** true about $M = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$?

A M is a 3×3 matrix **B** M is a column matrix **C** M is a square matrix
D M has 9 elements **E** M has seven zero elements

11 The orders of the matrices that could be used to represent the information in the table are

A 6×2 or 2×6
B 6×2 only
C 2×6 only
D 7×3 or 3×7
E 7×3 only

Senior students' favourite takeaway food

	Year 11	Year 12
Pizza	32	15
Chicken	38	27
Fish and chips	30	33
Hamburgers	9	32
Kebabs	5	2
Other	3	7

12 © VCAA 2012 1MQ7 MODIFIED (4 marks) A store has three outlets, A, B and C. These outlets sell dresses, jackets and skirts made by the fashion house Ocki. The table lists the number of Ocki dresses, jackets and skirts that are currently held at each outlet.

	Size 10	Size 12	Size 14	Size 16
Outlet A	2 dresses	3 jackets	1 skirt	4 jackets
Outlet B	1 skirt	1 jacket	3 jackets	1 dress
Outlet C	2 skirts	2 dresses	2 dresses	1 jacket

a Write the 1×4 matrix that gives the number of Ocki clothing items held in each size in Outlet B. 1 mark

b What is the name given to this type of matrix? 1 mark

c The incomplete matrix is designed to show the total number of Ocki dresses (D), jackets (J) and skirts (S) held in each size at the three outlets.

$$\begin{array}{r} \\ \text{Size 10} \\ \text{Size 12} \\ \text{Size 14} \\ \text{Size 16} \end{array} \begin{array}{c} \quad D \quad J \quad S \\ \begin{bmatrix} 2 & 0 & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \end{array}$$

i What is the order of the matrix? 1 mark

ii Copy and complete the matrix. 1 mark

Video playlist Matrix addition, subtraction and scalar multiplication

Worksheet Addition and subtraction of matrices

5.2 Matrix addition, subtraction and scalar multiplication

Equal matrices

For two matrices to be equal, they must have the same order *and* have all the same elements in the same places.

So if $\begin{bmatrix} 8 & 3 \\ 4 & x \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 4 & 7 \end{bmatrix}$ then $x = 7$.

But $\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \neq \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$.

Addition and subtraction of matrices

Only matrices that have the same order can be added or subtracted. To do this we add or subtract each pair of corresponding elements. The answer we get has the same order as the original two matrices.
For example:

$$\begin{bmatrix} 2 & -7 \\ 11 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 5 & 12 \end{bmatrix} = \begin{bmatrix} 2+1 & -7+3 \\ 11+5 & 3+12 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 16 & 15 \end{bmatrix}$$

	A	+	B	=	C
Order:	2×2	+	2×2	=	2×2

$$\begin{bmatrix} 2 & -7 \\ 11 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 5 & 12 \end{bmatrix} = \begin{bmatrix} 2-1 & -7-3 \\ 11-5 & 3-12 \end{bmatrix} = \begin{bmatrix} 1 & -10 \\ 6 & -9 \end{bmatrix}$$

	A	−	B	=	C
Order:	2×2	−	2×2	=	2×2

Adding and subtracting matrices

When adding or subtracting matrices, add or subtract pairs of corresponding elements.

For addition or subtraction of matrices to be defined, they must have the same order.

The answer has the same order as the matrices being added or subtracted.

Exam hack

Matrix questions often use the word 'defined'. If something is defined, it means it is possible. If it's not defined, then it is impossible.

Scalar multiplication

A **scalar** is a regular number that isn't in a matrix. When we multiply a matrix by a scalar, we multiply each element by the scalar. **Scalar multiplication** can be done to any matrix. The answer we get has the same order as the original matrix. For example:

$$4 \times \begin{bmatrix} 5 & 2 \\ 7 & -3 \\ 12 & 0 \end{bmatrix} = \begin{bmatrix} 4 \times 5 & 4 \times 2 \\ 4 \times 7 & 4 \times (-3) \\ 4 \times 12 & 4 \times 0 \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 28 & -12 \\ 48 & 0 \end{bmatrix}$$

	$4A$	=	C
Order:	3×2	=	3×2

Exam hack

We don't always write the multiplication sign between the scalar and the matrix.

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Multiplying a matrix by a scalar

When multiplying a matrix by a regular number called a scalar, multiply each element by the scalar.

Scalar multiplication is defined for any matrix.

The answer has the same order as the original matrix.

5.2

WORKED EXAMPLE 3 Adding, subtracting and multiplying matrices by a scalar

If $A = \begin{bmatrix} 0 \\ -2 \\ 10 \end{bmatrix}$, $B = \begin{bmatrix} 12 \\ -4 \\ 8 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -5 & 0 \\ -2 & 8 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 7 \\ 1 & 3 & 4 \end{bmatrix}$, calculate each of the following, giving a reason if the addition or subtraction is not defined.

a $A - B$ **b** $C + D$ **c** $11C$ **d** $\frac{1}{4}B$ **e** $A - 2B$ **f** $5A + D$

WB p. 80

Steps

1 Check that the matrices have the same order.

2 Add or subtract corresponding elements.

3 Multiply each element by the scalar.

Working

a $A - B = \begin{bmatrix} 0 \\ -2 \\ 10 \end{bmatrix} - \begin{bmatrix} 12 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 - 12 \\ -2 + 4 \\ 10 - 8 \end{bmatrix} = \begin{bmatrix} -12 \\ 2 \\ 2 \end{bmatrix}$

b $C + D = \begin{bmatrix} 2 & -5 & 0 \\ -2 & 8 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 7 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2+3 & -5+0 & 0+7 \\ -2+1 & 8+3 & 4+4 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 7 \\ -1 & 11 & 8 \end{bmatrix}$

c $11C = 11 \times \begin{bmatrix} 2 & -5 & 0 \\ -2 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 11 \times 2 & 11 \times -5 & 11 \times 0 \\ 11 \times -2 & 11 \times 8 & 11 \times 4 \end{bmatrix} = \begin{bmatrix} 22 & -55 & 0 \\ -22 & 88 & 44 \end{bmatrix}$

d $\frac{1}{4}B = \frac{1}{4} \times \begin{bmatrix} 12 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \times 12 \\ \frac{1}{4} \times -4 \\ \frac{1}{4} \times 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

e $A - 2B = \begin{bmatrix} 0 \\ -2 \\ 10 \end{bmatrix} - 2 \times \begin{bmatrix} 12 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 10 \end{bmatrix} - \begin{bmatrix} 24 \\ -8 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 - 24 \\ -2 + 8 \\ 10 - 16 \end{bmatrix} = \begin{bmatrix} -24 \\ 6 \\ -6 \end{bmatrix}$

f $5A + D$ isn't defined because the order of $5A$ is 3×1 and the order of D is 2×3. Matrices must have the same order for addition to be possible.

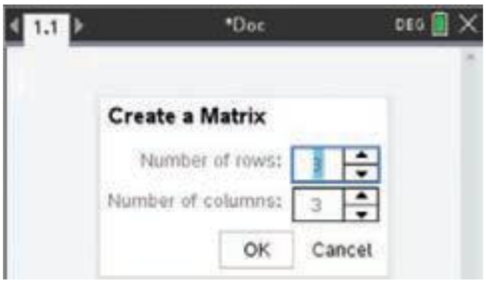
USING CAS 1 Addition, subtraction and scalar multiplication of matrices

Given that $Q = \begin{bmatrix} 3 & -5 & 10 \\ 11 & 6 & 17 \\ -2 & 15 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 12 & 2 & 1 \\ -3 & 4 & 7 \\ 22 & 14 & -7 \end{bmatrix}$, evaluate $4Q - 2R$ and $3Q + 7R$.

TI-Nspire

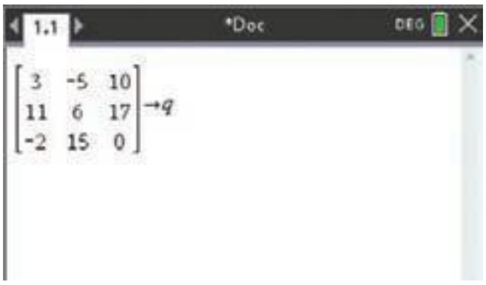
1 From a **Calculator** page, press the **template** key.

2 Select the **3×3** matrix template.



3 In the **Create a Matrix** screen, keep the default values of **3** rows and **3** columns.

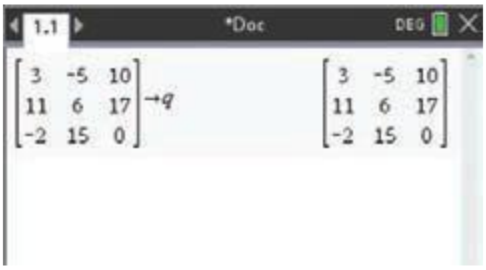
4 Select **OK**.



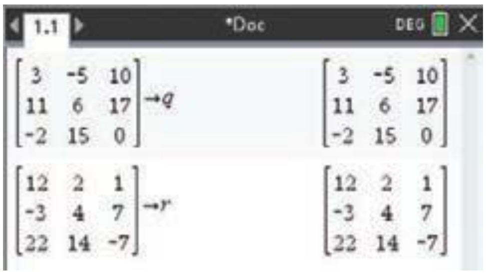
5 Enter the values for matrix Q.

6 Press **ctrl** > **var** to store the matrix as the letter **q**.

7 Press **enter**.

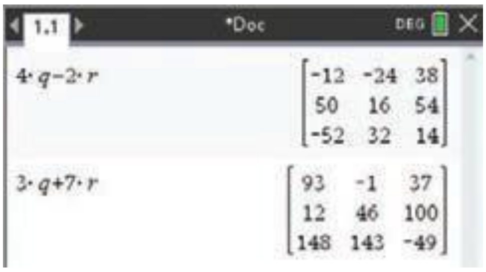


8 The matrix is now stored as the letter **q**.



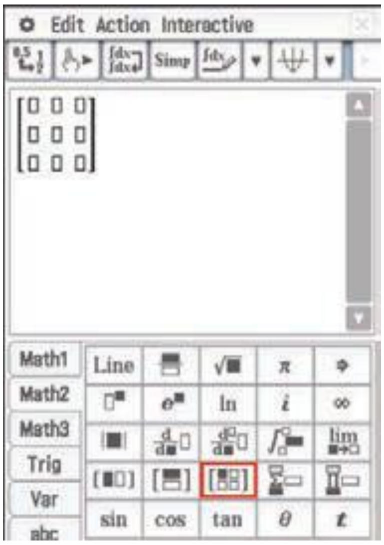
9 Create a new **3×3** matrix and enter the values for matrix R.

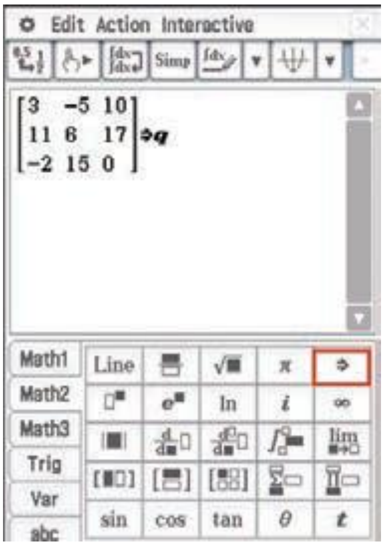
10 Press **ctrl** > **var** to store the matrix as the letter **r**.

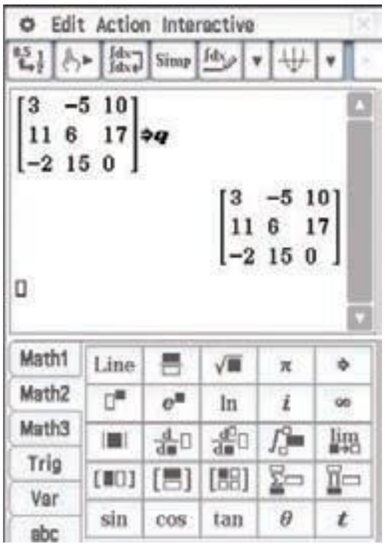


11 Use the matrices stored in **q** and **r** to complete the matrix operations.

ClassPad







1 Open the **Main** application and clear all values.

2 Open the **Keyboard** and tap **Math2**.

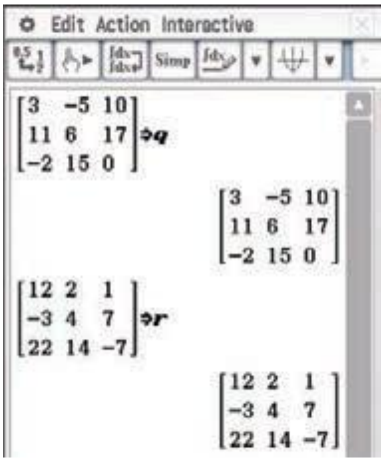
3 Tap on the **2×2** matrix template **twice** to create a **3×3** matrix.

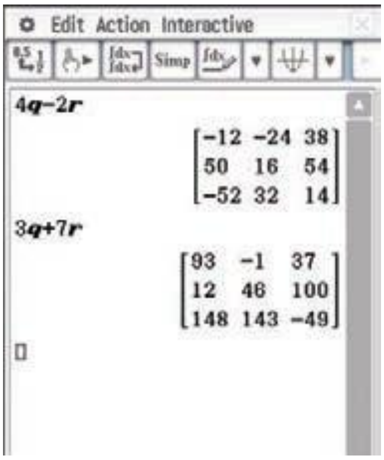
4 Enter the values for matrix *Q*.

5 Tap the **store** arrow to store the matrix as the variable **q**.

6 Press **EXE**.

7 The matrix is now stored as the variable **q**.





8 Create a new **3×3** matrix and enter the values for matrix *R*.

9 Tap the **store** arrow to store the matrix as the variable **r**.

10 Use the matrices stored in **q** and **r** to complete the matrix operations.

WB

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WORKED EXAMPLE 4 Finding missing elements in matrix equations

Find the value of x, y and z in each of the following.

a $\begin{bmatrix} 6 & 12 \\ x & 7 \end{bmatrix} - \begin{bmatrix} 6 & -5 \\ 1 & 2z \end{bmatrix} = \begin{bmatrix} 0 & y \\ 4 & -3 \end{bmatrix}$

b $3\begin{bmatrix} x & 2 & -1 \end{bmatrix} + 2\begin{bmatrix} 4 & 5 & y \end{bmatrix} = \begin{bmatrix} 14 & z & 11 \end{bmatrix}$

Steps	Working		
a Using the elements in the same row and column of each matrix, write down equations involving the unknowns. Solve, using CAS if necessary.	$x - 1 = 4$ $x = 5$	$12 - (-5) = y$ $y = 17$	$7 - 2z = -3$ $2z = 10$ $z = 5$
b Using the elements in the same row and column of each matrix, write down equations involving the unknowns. Solve, using CAS if necessary.	$3x + 8 = 14$ $3x = 6$ $x = 2$	$-3 + 2y = 11$ $2y = 14$ $y = 7$	$6 + 10 = z$ $z = 16$

WB

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WORKED EXAMPLE 5 Working with matrices using addition, subtraction and scalar multiplication

The cost prices of four different laptops in a store are \$890, \$999, \$1300 and \$1950. The selling price of each of these four laptops is 1.4 times the cost price.

Steps | **Working**

a Show a matrix calculation involving a column matrix that will give the selling price of each laptop.

Use scalar multiplication.

$$1.4 \times \begin{bmatrix} 890 \\ 999 \\ 1300 \\ 1950 \end{bmatrix}$$

b If the selling price also needs to allow for a \$60 commission for the salesman, show how this can be included in the matrix calculation.

Use matrix addition.

$$1.4 \times \begin{bmatrix} 890 \\ 999 \\ 1300 \\ 1950 \end{bmatrix} + \begin{bmatrix} 60 \\ 60 \\ 60 \\ 60 \end{bmatrix}$$

c The store has a sale where laptops with a cost price less than \$1000 have their selling price reduced by \$30, and laptops with a cost price greater than \$1000 have their selling price reduced by \$90. Show how this can be included in the matrix calculation of the selling price.

Use matrix subtraction.

$$1.4 \times \begin{bmatrix} 890 \\ 999 \\ 1300 \\ 1950 \end{bmatrix} + \begin{bmatrix} 60 \\ 60 \\ 60 \\ 60 \end{bmatrix} - \begin{bmatrix} 30 \\ 30 \\ 90 \\ 90 \end{bmatrix}$$

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EXERCISE 5.2 Matrix addition, subtraction and scalar multiplication

ANSWERS p. 506

5.2

Recap

80–100% 60–79% 0–59%

1 © VCAA 2017 1MQ1 98% Kai has a part-time job. Each week, he earns money and saves some of this money. The matrix shows the amounts earned (E) and saved (S), in dollars, in each of three weeks.

$$\begin{bmatrix} 300 & 100 \\ 270 & 90 \\ 240 & 80 \end{bmatrix}$$

How much did Kai save in week 2?

A $80 **B** $90 **C** $100 **D** $170 **E** $270

2 Matrix T has 4 rows and 3 columns. How many elements does it have?

A 3 **B** 4 **C** 7 **D** 12 **E** 16

Mastery

3 WORKED EXAMPLE 3 Given the matrices $A = \begin{bmatrix} -1 & 7 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -12 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 5 \\ 2 & 3 \end{bmatrix}$, calculate the following, giving a reason if the addition or subtraction is not defined.

a $C + D$ **b** $A - D$ **c** $4C$

d $\frac{1}{3}B$ **e** $C - 10B$ **f** $3A + D$

4 Using CAS 1 Given that $M = \begin{bmatrix} 5 & 9 \\ 3 & 8 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$, evaluate the following using CAS.

a $3M + 3N$ **b** $5M + 4N$ **c** $6M - 4N$ **d** $12N - 3M$

5 WORKED EXAMPLE 4 Find the value of x, y and z in each of the following.

a $\begin{bmatrix} 5 \\ y \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} x \\ 7 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 9 \\ z \end{bmatrix}$

b $\begin{bmatrix} 7 & x \\ 9 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ y & 3 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 10 & z \end{bmatrix}$

c $5\begin{bmatrix} x & 3 \\ 7 & -2 \end{bmatrix} + 2\begin{bmatrix} 5 & y \\ 11 & 8 \end{bmatrix} = \begin{bmatrix} 25 & 37 \\ 57 & z \end{bmatrix}$

d $3\begin{bmatrix} 10 & 5 \\ 4 & x \\ 8 & 7 \end{bmatrix} + 4\begin{bmatrix} 1 & 9 \\ 0 & 6 \\ y & 2 \end{bmatrix} = \begin{bmatrix} 34 & 51 \\ z & 45 \\ 19 & 29 \end{bmatrix}$

6 WORKED EXAMPLE 5 The cost prices of three different wireless earbuds in a store are $79, $199 and $399. The selling price of each pair of these wireless earbuds is 1.3 times the cost price.

a Show a matrix calculation involving a column matrix that will give the selling price of each pair of earbuds.

b If the selling price also needs to allow for a $15 commission for the salesman, show how this can be included in the matrix calculation.

c The store has a sale where wireless earbuds with a cost price less than $100 have their prices reduced by $20 and wireless earbuds with a cost price greater than $100 have their prices reduced by $45. Show how this can be included in the matrix calculation of the selling price.

7 $2\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 \\ 1 & 6 \end{bmatrix}$ is equal to

A $5\begin{bmatrix} 2 & 2 \\ 1 & 10 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 7 \\ 3 & 26 \end{bmatrix}$ **C** $6\begin{bmatrix} 2 & 2 \\ 1 & 10 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & 4 \\ 3 & 26 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 4 \\ 3 & 8 \end{bmatrix}$

8 © VCAA 2008 1MQ1 94% If $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 8 & d \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 11 \end{bmatrix}$, then d is equal to

A −11 **B** −10 **C** 7 **D** 10 **E** 11

9 © VCAA 2009 1MQ1 93% $3\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 \\ 2 & -7 \end{bmatrix}$ equals

A $\begin{bmatrix} 4 & 3 \\ 4 & -5 \end{bmatrix}$ **B** $6\begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 3 \\ 4 & 2 \end{bmatrix}$ **D** $5\begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & 6 \\ 7 & 4 \end{bmatrix}$

10 © VCAA 2019 1MQ1 87% Consider the following four matrix expressions.

$\begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 8 & 0 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 8 & 0 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

How many of these four matrix expressions are defined?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

11 © VCAA 2006 1MQ1 84% The matrix $\begin{bmatrix} 12 & 36 \\ 0 & 24 \end{bmatrix}$ is equal to

A $12\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$ **B** $12\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ **C** $12\begin{bmatrix} 0 & 24 \\ -12 & 12 \end{bmatrix}$ **D** $12\begin{bmatrix} 0 & 24 \\ 0 & 12 \end{bmatrix}$ **E** $12\begin{bmatrix} 1 & 3 \\ -12 & 2 \end{bmatrix}$

12 © VCAA 2017N 1MQ1 $\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} + 3\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$ is equal to

A $\begin{bmatrix} 8 & 4 \\ 2 & 10 \end{bmatrix}$ **B** $\begin{bmatrix} 11 & 1 \\ 5 & 13 \end{bmatrix}$ **C** $\begin{bmatrix} 16 & 8 \\ 4 & 20 \end{bmatrix}$ **D** $\begin{bmatrix} 24 & 12 \\ 6 & 30 \end{bmatrix}$ **E** $\begin{bmatrix} 38 & 34 \\ 31 & 40 \end{bmatrix}$

13 © VCAA 2018N 1MQ2 Consider the matrix equation $2 \times \begin{bmatrix} 3 & 0 \\ 4 & -1 \end{bmatrix} + W = \begin{bmatrix} 6 & 2 \\ 7 & 0 \end{bmatrix}$. Which one of the following is matrix W?

A $\begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} 12 & 2 \\ 15 & -2 \end{bmatrix}$ **E** $\begin{bmatrix} 12 & -2 \\ 15 & -2 \end{bmatrix}$

5.3 Matrix multiplication

Multiplying matrices

Matrix multiplication involves multiplying the elements of each row in the first matrix by the elements of each column in the second matrix, and then adding them. For example:

$(1 \times 7) + (2 \times 8) + (3 \times 9) = 50$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 50 & \\ & \end{bmatrix}$$

$(1 \times 10) + (2 \times 11) + (3 \times 12) = 68$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 50 & 68 \\ & \end{bmatrix}$$

	A	$\times$	B	$=$	C
Order:	2×3		3×2		2×2

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 50 & 68 \\ 122 & \end{bmatrix}$$

$(4 \times 7) + (5 \times 8) + (6 \times 9) = 122$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 50 & 68 \\ 122 & 167 \end{bmatrix}$$

$(4 \times 10) + (5 \times 11) + (6 \times 12) = 122$

Exam hack

We don't always write the multiplication sign between matrices.

Matrix multiplication order

Matrix multiplication $AB = C$ is only defined if the number of columns in A is the same as the number of rows in B. The product matrix C has the same number of rows as A and the same number of columns as B.

$$AB = C$$

Order: $(m \times n)\ (n \times q) = (m \times q)$

Product is defined

Exam hack

Order is important when multiplying matrices! Except in some special cases, $AB \neq BA$.

Multiplying by identity matrices

As we've seen, the identity matrix I is a square matrix where all the elements in the diagonal from top left to bottom right are 1, and every other element is 0. Multiplying the identity matrix with another matrix leaves the other matrix unchanged. It acts like a 1 does when multiplying regular numbers. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 10 & 13 \\ 12 & 18 & 11 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 10 & 13 \\ 12 & 18 & 11 \end{bmatrix}$$

Video playlist Matrix multiplication

Worksheet Multiplying matrices

Powers of matrices

Matrices can be raised to powers in a similar way to regular numbers. For a matrix A:

$A^2 = A \times A$

$A^3 = A \times A \times A$

and so on.

Only square matrices can be raised to a power. For non-square matrices the product is not defined.

$$AA = A^2$$

Order: $(m \times m)\ (m \times m) = (m \times m)$

Product is defined for square matrices

$$AA$$

Order: $(m \times n)\ (m \times n)$

Product is not defined for non-square matrices

Multiplying matrices

When multiplying matrices $AB = C$:

- multiply the elements of each row in A by the elements of each column in B, and then add them
- the product is defined if number of columns in A equal number of rows in B
- C has the same number of rows as A and the same number of columns as B.

When raising matrices to powers:

- only powers of square matrices are defined
- the power of a matrix will always have the same order as the original matrix.

WB p. 83

WORKED EXAMPLE 6 Multiplying matrices

If $A = \begin{bmatrix} 3 & 6 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & 1 \\ 0 & 4 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 6 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix}$, for each of the following

a AB **b** BA **c** BC **d** BD

e CD **f** B^3 **g** $A^2 + 5A$

i state whether or not the expression is defined, giving a reason.

For those that are defined

ii state the order of the answer before performing the calculation

iii do the calculation to find the answer.

Steps	Working
a **i** Do the number of columns in A equal the number of rows in B?	A has order 2×2, B has order 2×3 number of columns in A = number of rows in B So AB is defined.
ii How many rows does A have? How many columns does B have?	$(2 \times 2)\ (2 \times 3) = (2 \times 3)$ AB has order 2×3. Note: Giving the answer as (2×3) is incorrect. You can only use brackets in the working.

	iii Calculate AB.	$AB = \begin{bmatrix} 3 & 6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 5 & 1 \\ 0 & 4 & 7 \end{bmatrix}$ $= \begin{bmatrix} (3\times 2)+(6\times 0)=6 & (3\times 5)+(6\times 4)=39 & (3\times 1)+(6\times 7)=45 \\ (1\times 2)+(0\times 0)=2 & (1\times 5)+(0\times 4)=5 & (1\times 1)+(0\times 7)=1 \end{bmatrix}$ $= \begin{bmatrix} 6 & 39 & 45 \\ 2 & 5 & 1 \end{bmatrix}$
b	**i** Do the number of columns in B equal number of rows in A?	B has order 2×3, A has order 2×2. number of columns in $B \neq$ number of rows in A So BA is not defined.
c	**i** Do the number of columns in B equal the number of rows in C?	B has order 2×3, C has order 1×3. number of columns in $B \neq$ number of rows in C So BC is not defined.
d	**i** Do the number of columns in B equal the number of rows in D?	B has order 2×3, D has order 3×1. number of columns in $B =$ number of rows in D So BD is defined.
	ii How many rows does B have? How many columns does D have?	$(2 \times 3)\ (3 \times 1) = (2 \times 1)$ BD has order 2×1.
	iii Calculate BD.	$BD = \begin{bmatrix} 2 & 5 & 1 \\ 0 & 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} (2\times 5)+(5\times 9)+(1\times 7) \\ (0\times 5)+(4\times 9)+(7\times 7) \end{bmatrix} = \begin{bmatrix} 62 \\ 85 \end{bmatrix}$
e	**i** Do the number of columns in C equal number of rows in D?	C has order 1×3, D has order 3×1. number of columns in $C =$ number of rows in D So CD is defined.
	ii How many rows does C have? How many columns does D have?	$(1 \times 3)\ (3 \times 1) = (1 \times 1)$ CD has order 1×1.
	iii Calculate CD.	$CD = \begin{bmatrix} 1 & 6 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(6\times 9)+(2\times 7) \end{bmatrix} = \begin{bmatrix} 72 \end{bmatrix}$
f	**i** Is B a square matrix?	B is a not square matrix. Only powers of square matrices are defined, so B^3 is not defined.
g	**i** Is A a square matrix?	A is a square matrix. Powers of square matrices are always defined, so A^2 is defined.
	ii How many rows does A have? How many columns does A have?	$(2 \times 2)\ (2 \times 2) = (2 \times 2)$ A^2 has order 2×2.
	Is the sum possible?	$5A$ has order 2×2. Matrices must have the same order to be added, so $A^2 + 5A$ is defined.

iii Calculate $A^2 + 5A$.

$$A^2 = \begin{bmatrix} 3 & 6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (3\times 3)+(6\times 1) & (3\times 6)+(6\times 0) \\ (1\times 3)+(0\times 1) & (1\times 6)+(0\times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 18 \\ 3 & 6 \end{bmatrix}$$

$$A^2 + 5A = \begin{bmatrix} 15 & 18 \\ 3 & 6 \end{bmatrix} + 5\begin{bmatrix} 3 & 6 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 18 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 15 & 30 \\ 5 & 0 \end{bmatrix}$$

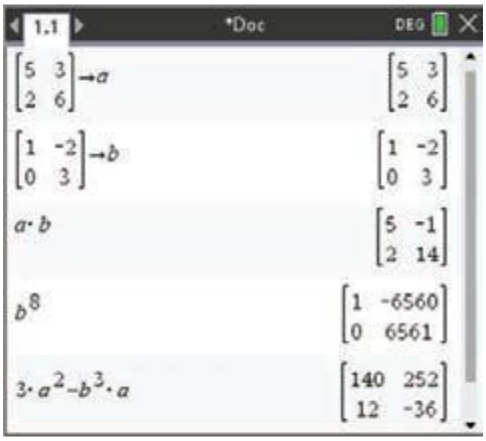
$$= \begin{bmatrix} 30 & 48 \\ 8 & 6 \end{bmatrix}$$

USING CAS 2 Multiplication and powers of matrices

If $A = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$, find

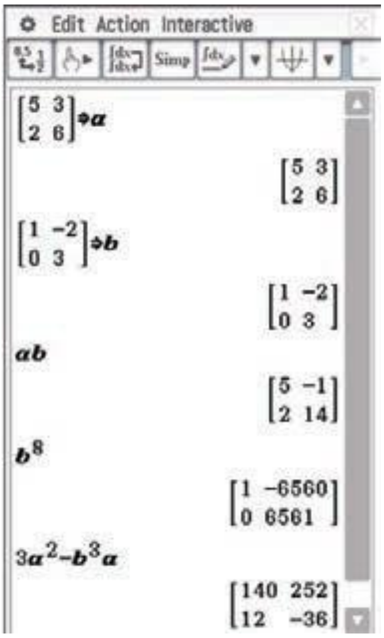
a AB **b** B^8 **c** $3A^2 - B^3A$

TI-Nspire



1. Create the two matrices and store them as variables **a** and **b**.
2. Perform the calculations as shown above.
3. For part **a**, remember to insert a multiplication sign between the **a** and **b**.

ClassPad



1. Create the two matrices and store them as variables **a** and **b**.
2. Perform the calculations as shown above. When you use variables instead of letters, there is no need to insert multiplication signs.

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EXERCISE 5.3 Matrix multiplication

ANSWERS p. 506

5.3

Recap

80–100% 60–79% 0–59%

1 © VCAA 2007 1MQ1 97% The matrix sum $\begin{bmatrix} 0 & -4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ -2 & 2 \end{bmatrix}$ is equal to

A $\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}$

B $\begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$

C $\begin{bmatrix} 5 & -4 \\ 0 & 7 \end{bmatrix}$

D $\begin{bmatrix} 0 & 5 & -4 & 4 \\ 2 & -2 & 5 & 2 \end{bmatrix}$

E $\begin{bmatrix} 0 & -4 & 5 & 4 \\ 2 & 5 & -2 & 2 \end{bmatrix}$

2 Consider the following four matrix expressions.

$\begin{bmatrix} 6 & -3 \end{bmatrix} - \begin{bmatrix} -7 & 9 \end{bmatrix}$

$3\begin{bmatrix} 10 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 & 8 \end{bmatrix}$

$2\begin{bmatrix} 6 & 0 \\ 12 & -7 \\ 3 & 5 \end{bmatrix} - 6\begin{bmatrix} 9 \\ 0 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 0 & -4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 15 \\ 12 \end{bmatrix}$

How many of these four matrix expressions are defined?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

Mastery

3 WORKED EXAMPLE 6 If $A = \begin{bmatrix} 3 & 4 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 3 \\ 0 & 2 \\ 4 & 1 \end{bmatrix}$, for each of the following

a AB **b** BA **c** BC **d** BD

e DC **f** $C^2 - 2C$ **g** D^4

i state whether or not the expression is defined, giving a reason.

For those that are defined

ii state the order of the answer before performing the calculation.

iii do the calculation to find the answer.

4 Using CAS 2 If $M = \begin{bmatrix} 2 & 6 \\ 8 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} 4.5 & 5.2 \\ 2.8 & 3.6 \end{bmatrix}$, find

a MN **b** M^4 **c** $3M - NM^2$

Exam practice

80–100% 60–79% 0–59%

5 © VCAA 2014 1MQ1 92% $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ is equal to

A $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

C $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 4 & 0 \\ 4 & 1 & 1 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$

E $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

6 © VCAA 2011 1MQ2 92% If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $AB + 2C$ equals

A $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ **B** $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ **E** $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

7 © VCAA 2013 1MQ1 91% $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2 \times \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}$ equals

A $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ **B** $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ **C** $\begin{bmatrix} 2 \\ -2 \\ -2 \\ 2 \end{bmatrix}$ **D** $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix}$ **E** $\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

8 © VCAA 2009 1MQ2 85% The matrix $\begin{bmatrix} 12 & 15 & 3 \\ -6 & 0 & 24 \end{bmatrix}$ can also be written as

A $\begin{bmatrix} 12 & 15 & 3 \end{bmatrix} + \begin{bmatrix} -6 & 0 & 24 \end{bmatrix}$

B $\begin{bmatrix} 12 \\ -6 \end{bmatrix} + \begin{bmatrix} 15 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 24 \end{bmatrix}$

C $\begin{bmatrix} 3 \\ 6 \end{bmatrix}\begin{bmatrix} 4 & 5 & 1 \\ -1 & 0 & 4 \end{bmatrix}$

D $\frac{1}{3} \times \begin{bmatrix} 4 & 5 & 1 \\ -2 & 0 & 8 \end{bmatrix}$

E $3 \times \begin{bmatrix} 4 & 5 & 1 \\ -2 & 0 & 8 \end{bmatrix}$

9 © VCAA 2013 1MQ2 80% Matrix A has three rows and two columns. Matrix B has four rows and three columns. Matrix $C = B \times A$ has

A two rows and three columns.

B three rows and two columns.

C three rows and three columns.

D four rows and two columns.

E four rows and three columns.

10 If M is a 3×3 matrix, then the matrix I that will give $MI = M$ is

A $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

11 If the matrix product AB exists, where $A = \begin{bmatrix} 6 & 12 \\ 8 & 19 \\ 21 & 4 \\ 17 & 32 \\ 5 & 29 \end{bmatrix}$, then which one of the following could be the matrix B?

A $\begin{bmatrix} 5 & 6 & 3 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 \\ 9 \\ 11 \\ 1 \end{bmatrix}$ **C** $\begin{bmatrix} 3 \\ 2 \\ 8 \\ 9 \\ 4 \end{bmatrix}$

D $\begin{bmatrix} 2 \\ 13 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 11 \end{bmatrix}$

12 Which one of the following products cannot be found?

A $\begin{bmatrix} 2 & 7 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 20 & 12 \end{bmatrix}$ **C** $\begin{bmatrix} 6 \\ 3 \\ 7 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 7 & 6 & 8 \\ 3 & 5 & 9 \end{bmatrix}$

D $\begin{bmatrix} 7 & 15 \\ 3 & 6 \\ 72 & 12 \\ 11 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 8 & 13 \end{bmatrix}$ **E** $\begin{bmatrix} 0.5 & 16 \\ 3.8 & 2 \\ 14 & 2.1 \end{bmatrix}\begin{bmatrix} 7 & 6 & 12 & 44 \\ 8 & 52 & 1 & 6 \end{bmatrix}$

13 © VCAA 2006 1MQ2 75% Let $A = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 9 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \end{bmatrix}$

Using these matrices, the matrix product that is **not** defined is

A AB **B** AC **C** BA **D** BC **E** CB

14 © VCAA 2012 1MQ2 72% If $A = \begin{bmatrix} 8 & 1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 12 \\ 6 & 0 \end{bmatrix}$, then matrix $AB = \begin{bmatrix} 30 & 96 \\ 24 & 48 \end{bmatrix}$.

The element '24' in the matrix AB is correctly obtained by calculating

A $4 \times 6 + 2 \times 0$ **B** $4 \times 3 + 2 \times 6$ **C** $3 \times 4 + 12 \times 1$

D $4 \times 2 + 8 \times 2$ **E** $8 \times 3 + 1 \times 0$

15 (8 marks) $A = \begin{bmatrix} 5 & 4 \\ 11 & 8 \\ 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 9 & 12 \\ 10 & 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 8 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 7 \\ 3 & 8 \\ 9 & 2 \end{bmatrix}$.

a Which pair of matrices can be added? 1 mark

b Give a reason why the matrix product AC exists. 1 mark

c Which two of the possible matrix products formed by pairs from the given matrices is of order 3×3? 2 marks

d Which two of the possible matrix products formed by pairs from the given matrices is of order 2×2? 2 marks

e Which matrix can be raised to a power? Give a reason for your answer. 2 marks

5.4 Inverse matrices

Video playlist
Inverse matrices

The inverse matrix

We have seen that multiplying the identity matrix I with another matrix A leaves A unchanged. It acts like a 1 when multiplying regular numbers.

So $AI = IA = A$ is like saying $7 \times 1 = 1 \times 7 = 7$.

The **inverse** matrix A^{-1} is the matrix which when multiplied by A results in the identity matrix I. It acts like the inverse of a regular number.

So $AA^{-1} = A^{-1}A = I$ is like saying $7 \times \frac{1}{7} = \frac{1}{7} \times 7 = 1$.

For example, where $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$:

$$AA^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 5\times 3+7\times(-2) & 5\times(-7)+7\times 5 \\ 2\times 3+3\times(-2) & 2\times(-7)+3\times 5 \end{bmatrix} = \begin{bmatrix} 15-14 & -35+35 \\ 6-6 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3\times 5+(-7)\times 2 & 3\times 7+(-7)\times 3 \\ -2\times 5+5\times 2 & -2\times 7+5\times 3 \end{bmatrix} = \begin{bmatrix} 15-14 & 21-21 \\ -10+10 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The inverse matrix

If the matrix A^{-1} is the inverse of the matrix A, then

$$AA^{-1} = A^{-1}A = I$$

Only square matrices have inverses.

Not *all* square matrices have inverses.

A and A^{-1} always have the same order.

Finding the determinant and the inverse of a matrix

To find the inverse of a matrix, we first need to find the **determinant**. We will start with 2 × 2 matrices.

The determinant

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A) = ad - bc$ and

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The determinant only exists for square matrices.

The inverse does not exist when $\det(A) = 0$.

WB

p. 85

WORKED EXAMPLE 7 Finding the determinant and inverse of a 2 × 2 matrix

For each of the following matrices, find

i the determinant (if it exists)

ii the inverse (if it exists).

a $A = \begin{bmatrix} 3 & 8 \\ 2 & 6 \end{bmatrix}$ **b** $B = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$ **c** $C = \begin{bmatrix} 4 & 20 \\ 2 & 10 \end{bmatrix}$ **d** $D = \begin{bmatrix} 3 & -1 \\ 4 & 8 \\ 7 & -2 \end{bmatrix}$

Steps	Working
a **i** Is it a square matrix?	yes
For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, use determinant = $ad - bc$.	$\det(A) = 3\times 6 - 8\times 2$ $= 18 - 16$ $= 2$
ii Is the determinant equal to 0? If not, use inverse = $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$	$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -8 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -1 & -1.5 \end{bmatrix}$

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b **i** Is it a square matrix?

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, use determinant = $ad - bc$.

yes

$\det(B) = 7 \times 2 - 3 \times 5$
$= 14 - 15$
$= -1$

ii Is the determinant equal to 0?

If not, use inverse = $\dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$B^{-1} = -1\begin{bmatrix} 2 & -3 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5 & 7 \end{bmatrix}$

c **i** Is it a square matrix?

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, use determinant = $ad - bc$.

yes

$\det(C) = 4 \times 10 - 20 \times 2$
$= 40 - 40$
$= 0$

ii Is the determinant equal to 0?

If not, use inverse = $\dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det(C) = 0$, so C^{-1} doesn't exist.

d **i** Is it a square matrix?

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, use determinant = $ad - bc$.

no

$\det(D)$ doesn't exist.

ii Is the determinant equal to 0?

If not, use inverse = $\dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Only square matrices have inverses, so D^{-1} doesn't exist.

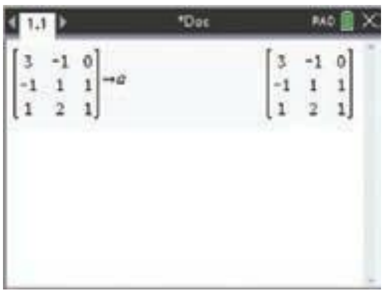
We can use CAS to find determinants and inverses. For a 2 × 2 matrix it may be quicker to do it by hand, but for higher order square matrices the calculation is more complex, and we will always use CAS.

USING CAS 3 Finding the determinant and inverse of a matrix

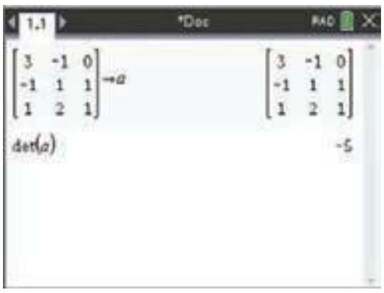
For $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, find

i $\det(A)$ **ii** A^{-1}

TI-Nspire

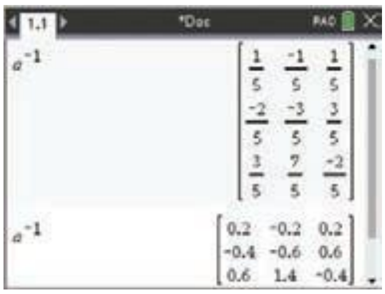


1 In a **Calculator** page, create a **3×3** matrix and store it as the variable **a**.



2 Press **menu** > **Matrix & Vector** > **Determinant**.

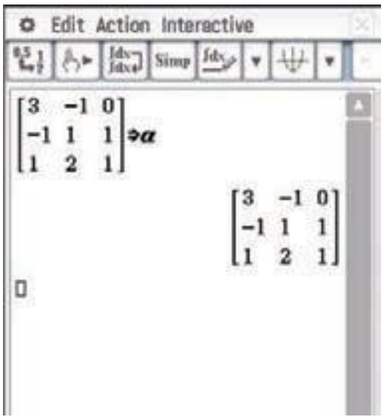
3 Enter **a** and press **enter** to find the determinant.



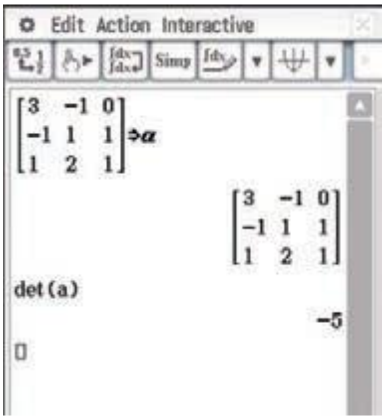
4 Find **a**$^{-1}$ to calculate the inverse matrix.

5 Press **ctrl + enter** for the approximate answer.

ClassPad



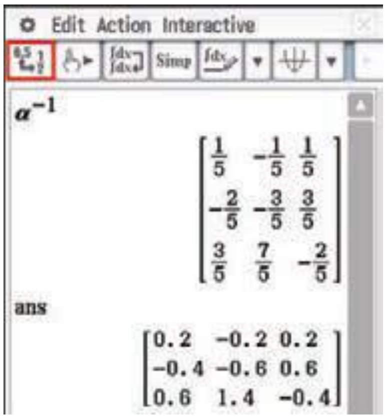
1 In the **Main** application, create the **3×3** matrix and store it in for the variable **a**.



2 Tap **Interactive** > **Matrix** > **Calculation** > **det**.

3 In the dialogue box, enter **a**.

4 Press **OK**.

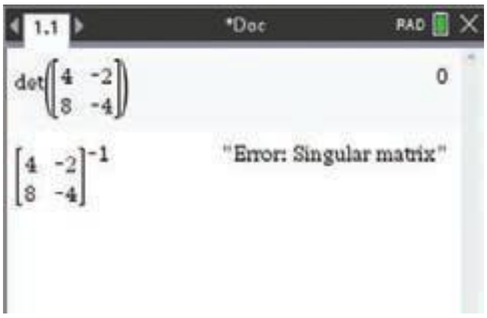


5 Find $\mathbf{a}^{-1}$ to calculate the inverse matrix.

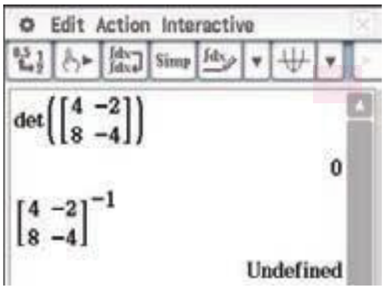
6 Tap the **Convert** tool to display the matrix in decimal form.

When the determinant of a matrix is equal to 0, it is known as a **singular matrix**. The inverse of a singular matrix does not exist. When attempting to find the inverse of a singular matrix, CAS will display an error message.

TI-Nspire



ClassPad



Solving simultaneous equations using matrices

We looked at simultaneous equations involving two unknowns in Chapter 4. Simultaneous equations can be written in matrix form. For example:

$$\begin{aligned} 4x + y &= 6 \\ 2x + 9y &= 20 \end{aligned} \quad \text{can be written as} \quad \begin{bmatrix} 4 & 1 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \end{bmatrix}$$

Matrices can be used to solve simultaneous equations with a large number of unknowns. Simultaneous equations with two unknowns involve a 2 × 2 matrix, simultaneous equations with three unknowns involve a 3 × 3 matrix and so on.

We will use CAS to solve the matrix equations but we will also look at how to solve examples with two unknowns by hand. The solution involves calculating the inverse of the square matrix. This means if there is no solution to the simultaneous equations, the determinant of the matrix will be zero.

5.4

WORKED EXAMPLE 8 Solving simultaneous equations with two unknowns using matrices

Solve the following simultaneous equations using matrices, showing all the steps.

$5x + 3y = 13$

$6x + 4y = 16$

Steps	Working
1 Write the simultaneous equations in matrix form.	$\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$
2 Find the determinant of the matrix.	$\det = 4 \times 5 - 3 \times 6$ $= 20 - 18$ $= 2$
3 Find the inverse of the square matrix in the matrix equation.	$\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}$
4 Multiply by the inverse on the left of both sides of the matrix equation and simplify.	$\begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}\begin{bmatrix} 13 \\ 16 \end{bmatrix}$
A matrix multiplied by its inverse is the identity matrix.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}\begin{bmatrix} 13 \\ 16 \end{bmatrix}$
A matrix multiplied by the identity matrix is unchanged.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}\begin{bmatrix} 13 \\ 16 \end{bmatrix}$
5 Multiply the matrices on the right of the equation.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 26 - 24 \\ -39 + 40 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
Equate the elements to solve the simultaneous equations.	$x = 2, y = 1$

WB p. 86

Exam hack

When solving simultaneous equations using matrices, always multiply by the inverse on the left of both sides of the matrix equation.

We can use CAS to solve simultaneous equations with two or more unknowns. When writing the simultaneous equations as matrix equations, insert a zero where a pronumeral is missing.

USING CAS 4 Solving simultaneous equations using matrices

Solve these simultaneous equations using matrices.

$2x + 3y - z = 7$

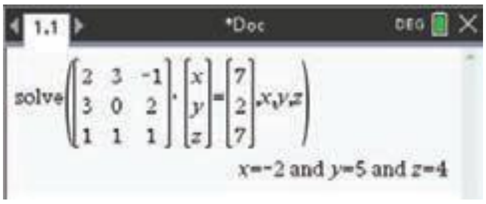
$3x + 2z = 2$

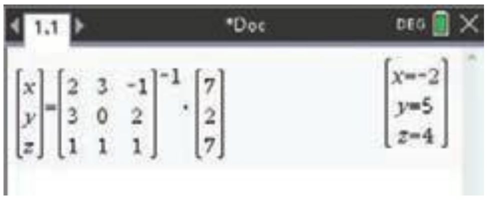
$x + y + z = 7$

TI-Nspire

1 Write the simultaneous equations as a matrix equation.

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix}$$





2 In a **Calculator** page, set up the matrix equation using **3×3** and **3×1** matrices.

3 To create a **3×1** matrix, use the **3×3** matrix template, then select **3** rows and **1** column.

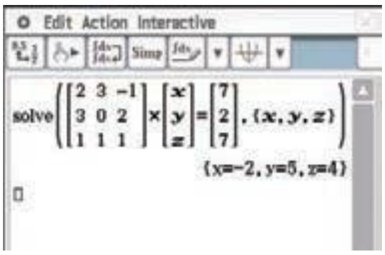
4 Solve the matrix equation for the variables **x**, **y** and **z**.

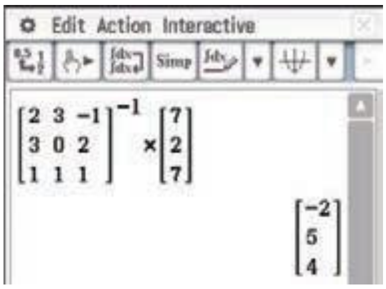
5 Alternatively, find the inverse of the **3×3** matrix and multiply it by the **3×1** matrix on the right-hand side of the equation.

ClassPad

1 Write the simultaneous equations as a matrix equation.

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix}$$





2 In the **Main** application, set up the matrix equation using **3×3** and **3×1** matrices.

3 To create a **3×1** matrix, tap **twice** on the **2×1** matrix template.

4 Solve the matrix equation for the variables **x**, **y** and **z**.

5 Alternatively, find the inverse of the **3×3** matrix and multiply it by the **3×1** matrix on the right-hand side of the equation.

The solution is $x = -2$, $y = 5$ and $z = 4$.

WORKED EXAMPLE 9 Solving problems using inverse matrices

A factory has three identical robots that assemble three different models of computers (A, B and C).

Robot 1 assembles three model As, two model Bs and five model Cs in 113 minutes.
Robot 2 assembles one model A and three model Bs in 56 minutes.
Robot 3 assembles two model Bs and one model C in 40 minutes.

Let a = the amount of time in minutes it takes for a robot to assemble one model A.
Let b = the amount of time in minutes it takes a robot to assemble one model B.
Let c = the amount of time in minutes it takes a robot to assemble one model C.

a Write three simultaneous equations in terms of a, b and c.

b Write the simultaneous equations in matrix form.

c Solve the matrix equation and hence, find how long it takes a robot to assemble each of the three computer models.

Steps	Working
a Use the information in the question to write three simultaneous equations. Write the pronumerals so they line up under each other.	$3a + 2b + 5c = 113$ $a + 3b = 56$ $2b + c = 40$
b Rewrite in matrix form, adding zeros where necessary.	$\begin{bmatrix} 3 & 2 & 5 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 113 \\ 56 \\ 40 \end{bmatrix}$
c **1** Solve with CAS by finding the inverse of the 3×3 matrix and multiplying by the inverse on the left of both sides of the matrix equation.	$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 10 \end{bmatrix}$
2 Write the answer.	It takes a robot 11 minutes to assemble model A, 15 minutes to assemble model B, and 10 minutes to assemble model C.

WB p. 87

EXERCISE 5.4 Inverse matrices

ANSWERS p. 507

Recap

80–100% 60–79% 0–59%

1 © VCAA 2020 1MQ2 MODIFIED 79% Matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 0 \\ 4 & 5 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 2 & 0 & 3 & 1 \\ 4 & 5 & 2 & 0 \end{bmatrix}$.

Matrix $Q = A \times B$. The element in row 4 and column 1 of matrix Q is determined by the calculation

A $0 \times 0 + 3 \times 5$ **B** $1 \times 1 + 2 \times 0$ **C** $1 \times 2 + 2 \times 4$

D $4 \times 1 + 5 \times 0$ **E** $4 \times 2 + 5 \times 4$

2 © VCAA 2006 1MQ3 84% Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Then $A^3(B - C)$ equals

A $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 6 \\ 6 & -3 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 10 \\ 10 & -5 \end{bmatrix}$

Mastery

3 WORKED EXAMPLE 7 For each of the following matrices, find

i the determinant (if it exists) **ii** the inverse (if it exists).

a $A = \begin{bmatrix} 6 & 2 \\ 8 & 1 \end{bmatrix}$ **b** $B = \begin{bmatrix} 10 & 5 \\ 4 & 2 \end{bmatrix}$ **c** $C = \begin{bmatrix} 9 & 4 & 3 \\ -7 & -1 & 2 \end{bmatrix}$

d $D = \begin{bmatrix} -10 & 5 \\ -1 & 1 \end{bmatrix}$ **e** $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **f** $F = \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix}$

4 Using CAS 3 For each of the following matrices, A, find

i $\det(A)$ **ii** A^{-1}

a $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 4 & -2 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & 1 & 4 \\ 0 & -2 & 1 \\ -1 & -4 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 4 & 3 & 6 \\ 2 & -10 & 7 \\ -1 & -8 & 1 \end{bmatrix}$

5 WORKED EXAMPLE 8 Solve the following simultaneous equations using matrices, showing all the steps.

$4x - y = -5$
$x - 3y = 7$

6 Using CAS 4 Solve each of the following simultaneous equations using matrices.

a $5x - y + 2z = 1$
$-x - 3y = -5$
$x + 5z = 19$

b $x + y + z = 12$
$2x + 2y - 3z = 9$
$-x + y - z = -4$

c $10x - 5y = -40$
$x + 3z = 7$
$4y - z = 38$

7 WORKED EXAMPLE 9 A worker assembles a particular model of laptop, printer, modem and router. They assemble

- two laptops, five printers, three modems and one router in 167 minutes
- four laptops and ten printers in 240 minutes
- six printers, twelve modems and three routers in 273 minutes
- five modems and two routers in 82 minutes

Let a = the amount of time in minutes it takes her to assemble one laptop.
Let b = the amount of time in minutes it takes her to assemble one printer.
Let c = the amount of time in minutes it takes her to assemble one modem.
Let d = the amount of time in minutes it takes her to assemble one router.

a Write four simultaneous equations in terms of a, b, c and d.

b Write the simultaneous equations in matrix form.

c Solve the matrix equation and hence, find how long it takes her to assemble each of the four items.

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2009 1MQ4 82% The matrix equation $\begin{bmatrix} 4 & 2 & 8 \\ 2 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 6 \end{bmatrix}$ can be used to solve the system of simultaneous linear equations

A $4x + 2y + 8z = 7$
$2x + 3y = 2$
$3x - y = 6$

B $4x + 2y + 8z = 7$
$2x + 3y = 2$
$3y - z = 6$

C $4x + 2y + 8z = 7$
$2y + 3z = 2$
$3x - z = 6$

D $4x + 2y + 8z = 7$
$2x + 3z = 2$
$3y - z = 6$

E $4x + 2y + 8z = 7$
$2x + 3z = 2$
$3x - z = 6$

5.4

9 © VCAA 2018 1MQ1 78% Which one of the following matrices has a determinant of zero?

A $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$

10 © VCAA 2008 1MQ6 75% The solution of the matrix equation $\begin{bmatrix} 0 & -3 & 2 \\ 1 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$ is

A $\begin{bmatrix} 1 \\ 24 \\ 2 \end{bmatrix}$ **B** $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ **D** $\begin{bmatrix} -11 \\ \frac{4}{3} \\ 8 \end{bmatrix}$ **E** $\begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$

11 © VCAA 2010 1MQ5 73% A system of three simultaneous linear equations is written in matrix form as follows.

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$$

One of the three linear equations is

A $x - 2y + z = 4$ **B** $x + y + 3z = 11$ **C** $2x - y = -5$

D $x + 3z = 11$ **E** $3y - z = -5$

12 © VCAA 2011 1MQ3 MODIFIED Each of the following four matrix equations represents a system of simultaneous linear equations.

$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ $\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

How many of these systems of simultaneous linear equations *don't* have a solution?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

13 © VCAA 2007 1MQ4 64% Consider the following system of three simultaneous linear equations.

$2x + z = 5$

$x - 2y = 0$

$y - z = -1$

This system of equations can be written in matrix form as

A $\begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

C $\begin{bmatrix} 2 & 1 & 5 \\ 1 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

E $\begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

14 © VCAA 2013 1MQ6 **41%** A worker can assemble 10 bookcases and four desks in 360 minutes, and eight bookcases and three desks in 280 minutes. If each bookcase takes b minutes to assemble and each desk takes d minutes to assemble, the matrix $\begin{bmatrix} b \\ d \end{bmatrix}$ will be given by

A $\begin{bmatrix} -1.5 & 2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$ **B** $\begin{bmatrix} 10 & 4 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & -4 \\ -8 & 10 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$

D $\begin{bmatrix} 5 & -2 \\ -4 & 1.5 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$ **E** $\begin{bmatrix} 10 \\ 4 \end{bmatrix}\begin{bmatrix} 360 \end{bmatrix} + \begin{bmatrix} 8 \\ 3 \end{bmatrix}\begin{bmatrix} 280 \end{bmatrix}$

15 © VCAA 2017N 1MQ5 Yvette needs to buy a total of nine pens and markers. Five pens and four markers will cost \$31.00. Four pens and five markers will cost \$32.00. Let p be the cost of a pen. Let m be the cost of a marker. Consider the following matrix equations.

$$\begin{bmatrix} p \\ m \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 31 \\ 32 \end{bmatrix} \qquad \begin{bmatrix} p \\ m \end{bmatrix} = \begin{bmatrix} 32 \\ 31 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} p \\ m \end{bmatrix} = -\frac{1}{9}\begin{bmatrix} 4 & -5 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 32 \\ 31 \end{bmatrix} \qquad \begin{bmatrix} p \\ m \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{bmatrix}\begin{bmatrix} 32 \\ 31 \end{bmatrix}$$

How many of the matrix equations above could Yvette solve to get a matrix that contains the price of a pen and the price of a marker?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

16 © VCAA 2009 2MQ2 **69%** (3 marks) Tickets for a school function are sold at the school office, the function hall and online. Different prices are charged for students, teachers and parents. The table shows the number of tickets sold at each place and the total value of sales.

	School office	Function hall	Online
Student tickets	283	35	84
Teacher tickets	28	4	3
Parent tickets	5	2	7
Total sales	\$8712	\$1143	\$2609

For this function

- student tickets cost \$$x$
- teacher tickets cost \$$y$
- parent tickets cost \$$z$.

a Use the information in the table to find the missing values of this matrix equation.

$$\begin{bmatrix} 283 & 28 & 5 \\ \square & 4 & \square \\ 84 & 3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8712 \\ 1143 \\ 2609 \end{bmatrix}$$

1 mark

b Use the matrix equation to find the cost of a teacher ticket to the school function. 2 marks

5.5 Matrix applications

Costing and pricing matrices

Matrices can be used to solve costing or pricing problems involving different product categories.

WORKED EXAMPLE 10 Solving costing and pricing problems using matrices

A health food store owner purchases small packets of peppermint tea for $2.50 each and large packets of peppermint tea for $3.50 each. In the last two weeks, she has purchased the number of tea packets shown.

	Small peppermint tea packets	Large peppermint tea packets
Week 1	86	70
Week 2	64	52

Steps | **Working**

a Find the two matrices that can be multiplied to give the total purchase cost of packets of peppermint tea in each of the two weeks and complete the multiplication. State the total peppermint tea costs for each week.

We need a matrix product that calculates

number of small peppermint tea packets × cost of small peppermint tea packets + number of large peppermint tea packets × cost of large peppermint tea packets.

$$\begin{matrix}\text{Week 1}\\ \text{Week 2}\end{matrix}\begin{bmatrix} 86 & 70 \\ 64 & 52 \end{bmatrix}\begin{bmatrix} 2.5 \\ 3.5 \end{bmatrix} = \begin{bmatrix} 460 \\ 342 \end{bmatrix}$$

The total peppermint tea cost for week 1 is $460.
The total peppermint tea cost for week 2 is $342.

b The health food store owner sells goods at 175% of the cost price. She recorded her purchase costs over the last two weeks for peppermint tea and three other teas in the following table.

	Week 1	Week 2
Peppermint tea		
Apple tea	$473	$542
Chamomile tea	$628	$745
Ginger tea	$263	$220

i Represent these costs in a 4 × 2 cost matrix, C.

ii Using scalar multiplication, represent the selling prices of these goods in a 4 × 2 matrix, S.

i The table already has 4 rows and 2 columns. Fill in the missing information from part **a**.

$$C = \begin{bmatrix} 460 & 342 \\ 473 & 542 \\ 628 & 745 \\ 263 & 220 \end{bmatrix}$$

ii Convert the percentage to a decimal and multiply the cost matrix by the decimal.

$$175\% = \frac{175}{100} = 1.75 \text{ as a decimal}$$

$$S = 1.75C = 1.75\begin{bmatrix} 460 & 342 \\ 473 & 542 \\ 628 & 745 \\ 263 & 220 \end{bmatrix}$$

$$= \begin{bmatrix} 805.00 & 598.50 \\ 827.75 & 948.50 \\ 1099.00 & 1303.75 \\ 460.25 & 385.00 \end{bmatrix}$$

Video playlist Matrix applications

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c **i** Create a profit matrix.

ii Calculate the total profit to be made if all goods purchased over these two weeks are sold.

i To create a profit matrix:

profit = selling price – cost price

$$\text{profit} = S - C$$

$$= \begin{bmatrix} 805.00 & 598.50 \\ 827.75 & 948.50 \\ 1099.00 & 1303.75 \\ 460.25 & 385.00 \end{bmatrix} - \begin{bmatrix} 460 & 342 \\ 473 & 542 \\ 628 & 745 \\ 263 & 220 \end{bmatrix}$$

$$= \begin{bmatrix} 345.00 & 256.50 \\ 354.75 & 406.50 \\ 471.00 & 558.75 \\ 197.25 & 165.00 \end{bmatrix}$$

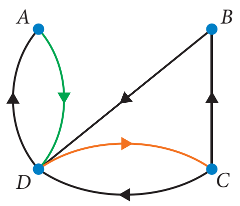
ii The total profit can be found by adding all of the elements in the profit matrix.

total profit = 345.00 + 256.50 + 354.75 + 406.50 + 471.00 + 558.75 + 197.25 + 165.00
= $2754.75

Communication diagrams and matrices

Matrices are useful when investigating how people and systems communicate. Communication links can be shown either by a diagram or a matrix. **Communication diagrams** show the direct links with arrows. **Communication matrices** are square matrices that show the direct links with '1's and '0's everywhere else. Links with the same sender and receiver are called **redundant links**. These don't count as communication, so the diagonal from the top left to the bottom right of communication matrices are always all '0's.

Communication diagram



Communication matrix

Receiver

$$M = \text{Sender} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

A communicates with *D*

D communicates with *C*

Direct links shown by '1's in a communication matrix are called **one-step communications**. **Two-step communications** are communication that occur via another link. For the matrix *M*, A has a two-step communication with *C* but not a one-step communication: *C*: $A \rightarrow D \rightarrow C$.

Exam hack

Communications can be one-way (*A* to *B* but not *B* to *A*) or two-way (*A* to *B and B* to *A*). Don't confuse this with one-step and two-step communications.

Communication matrices

A communication matrix

- is a square matrix where a '1' indicates direct one-step communication and a '0' indicates non-communication
- has all zeros in the diagonal from the top left to the bottom right, indicating redundant links where the sender and receiver are the same
- can be used to find two-step communications where communication occurs via another link.

WORKED EXAMPLE 11 Working with communication matrices and diagrams

5.5

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Steps	Working
a The communication matrix M shows how direct messages can be sent between four people: Arnie (A), Billie (B), Cathy (C) and Detlev (D).	

$$M = \text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}}$$

i List who each person can send direct messages to.

ii Explain why the diagonal from the top left to the bottom right is all zeros.

iii Draw a communication diagram showing the communication links given in the matrix.

iv How could Detlev get a message to Billie in two steps?

i Look at each row in order.

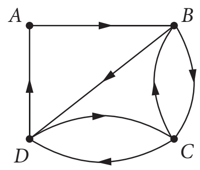
A '1' means the person can send a direct message.

Arnie can send direct messages to Billie.
Billie can send direct messages to Cathy and Detlev.
Cathy can send direct messages to Billie and Detlev.
Detlev can send direct messages to Arnie and Cathy.

ii Refer to redundant links.

The diagonal represents links where the sender and receiver are the same. This isn't considered communication, so they are redundant links.

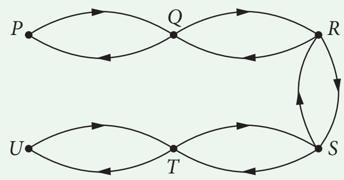
iii Draw a diagram with arrows that match the list of possible direct messages.



iv Find how the message could be passed on in two steps.

Detlev → Cathy → Billie
or
Detlev → Arnie → Billie

b Write the communication matrix for the following communication diagram.



Set up a 6 × 6 square matrix where a '1' indicates communication and a '0' indicates non-communication.

$$\text{Sender}\ \begin{matrix} P \\ Q \\ R \\ S \\ T \\ U \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} P & Q & R & S & T & U \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}}$$

Using two-step communication

Squaring a communication matrix gives us the number of two-step communications. The values that appear in the top left to bottom right diagonal of the squared matrix give the number of two-step communications where the sender and receiver are the same. These are redundant links that should be ignored.

M	M^2
Number of one-step communications	Number of two-step communications
$$\text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}}$$	$$\text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}}}$$
No one-step communication from A to C.	1 two-step communications from A to C.

Exam hack

While M^2 gives the *number* of two-step communications, we need to look at M to find what the two-step communications are.

Using two-step communication

For a communication matrix M

- M^2 gives the number of two-step communications
- the values in the diagonal from the top left to the bottom right of M^2 are redundant two-step links.

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WORKED EXAMPLE 12 Working with two-step communication

For the communication matrix representing the connections between four computers, find the following.

$$M = \text{From}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{To}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}}$$

a The number of ways C can connect with A by connecting directly to one other computer.

b The list of all the two-step connections from A to C.

c The total number of redundant two-step connections.

d The list of redundant two-step connections from D to D.

Steps	Working
a Find M^2 using CAS and read the number of two-step connections from the matrix.	$M^2 = \text{From}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{To}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}}}$ There are 2 ways C can connect with A by connecting directly to one other computer.
b Use M to find the two-step connections.	The two-step connections from A to C are $A \rightarrow B \rightarrow C$ and $A \rightarrow D \rightarrow C$

9780170448192

c Add the values in the top left to bottom right diagonal of M^2.	The total number of redundant two-step connections is $2 + 2 + 2 + 2 = 8$
d Use M^2 to find the number of redundant two-step connections. Use M to find the redundant two-step connections.	There are two redundant two-step connections from D to D: $D \rightarrow A \rightarrow D$ and $D \rightarrow C \rightarrow D$

5.5

EXERCISE 5.5 Matrix applications

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Recap

80–100% 60–79% 0–59%

1 How many of the following matrices have an inverse?

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 4 & 4 & 4 \\ 5 & 5 & 5 \\ 6 & 6 & 6 \end{bmatrix} \quad \begin{bmatrix} 20 & 2 \\ 30 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A 0 **B** 1 **C** 2 **D** 3 **E** 4

2 © VCAA 2012 1MQ3 41% The **solution** of the simultaneous equations

$$\begin{aligned} x + z &= 6 \\ 2y + z &= 8 \\ 2x + y + 2z &= 15 \end{aligned}$$

is given by

A $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & -1 & 2 \\ -2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

B $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

C $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

D $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

E $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

Mastery

3 WORKED EXAMPLE 10 The manager of a garden supply store purchases small solar garden lights for $3 each and regular solar garden lights for $5. In the last two weeks, he purchased the number of solar garden lights shown.

	Small solar garden lights	Regular solar garden lights
Week 1	133	98
Week 2	75	62

a Find the two matrices that can be multiplied to give the total purchase cost of solar garden lights in each of the two weeks and complete the multiplication. State the total of solar garden light costs for each week.

b The manager sells goods at 155% of the cost price. He recorded his purchase costs over the last two weeks for solar garden lights and three other items in the following table.

	Week 1	Week 2
Pots	\$1060	\$1555
Gardening tools	\$3029	\$1124
Solar lights		
Wheelbarrows	\$896	\$2130

i Represent these costs in a 4 × 2 cost matrix, *C*.

ii Using scalar multiplication, represent the selling prices of these goods in a 4 × 2 matrix, *S*.

c **i** Create a profit matrix.

ii Calculate the total profit to be made if all of the goods purchased over these two weeks are sold.

4 WORKED EXAMPLE 11

a The communication matrix *M* shows how direct messages can be sent between four people: Ahmed (*A*), Beth (*B*), Crystal (*C*) and Daniella (*D*).

$$M = \text{Sender} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\begin{matrix}\text{Receiver} \\ A \quad B \quad C \quad D\end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

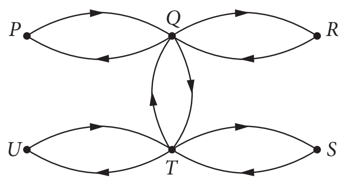
i List who each person can send direct messages to.

ii Explain why the diagonal from the top left to the bottom right is all zeros.

iii Draw a communication diagram showing the communication links given in the matrix.

iv How could Crystal get a message to Ahmed in two steps?

b Write the communication matrix for the following communication diagram.



5 WORKED EXAMPLE 12 For the communication matrix representing the connections between four computers, find the following.

$$M = \text{From} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\begin{matrix}\text{To} \\ A \quad B \quad C \quad D\end{matrix}}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$$

a The number of ways *B* can connect with *A* by connecting directly to one other computer.

b The list of all the two-step connections from *B* to *A*.

c The total number of redundant two-step connections.

d The list of redundant two step connections from *C* to *C*.

6 Which one of the following could be a one-step communication matrix?

A $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$

7 Given the following matrix M^2 is a two-step communication matrix, which of the statements is **false**?

$$M^2 = \text{From} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{To}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}}$$

A There is one two-step communication from C to D.

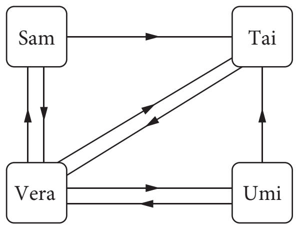
B There are two two-step communications from B to A.

C There are two two-step redundant links from C to C.

D There is one two-step communication from A to C.

E There are no two-step communications from B to D.

8 © VCAA 2020 1MQ5 86% The diagram shows the direct communication links that exist between Sam (S), Tai (T), Umi (U) and Vera (V). For example, the arrow from Umi to Vera indicates that Umi can communicate directly with Vera.



A communication matrix can be used to convey the same information. In this matrix:

- a '1' indicates that a direct communication link exists between a sender and a receiver
- a '0' indicates that a direct communication link does not exist between a sender and a receiver.

The communication matrix could be

A $\text{Sender} \begin{matrix} S \\ T \\ U \\ V \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} S & T & U & V \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}}$

B $\text{Sender} \begin{matrix} S \\ T \\ U \\ V \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} S & T & U & V \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}}$

C $\text{Sender} \begin{matrix} S \\ T \\ U \\ V \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} S & T & U & V \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}}}$

D $\text{Sender} \begin{matrix} S \\ T \\ U \\ V \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} S & T & U & V \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}}$

E $\text{Sender} \begin{matrix} S \\ T \\ U \\ V \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} S & T & U & V \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix}}}$

9 © VCAA 2006 1MQ5 78% A company makes Regular (R), Queen (Q) and King (K) size beds. Each bed comes in either the Classic style or the more expensive Deluxe style. The price of each style of bed, in dollars, is listed in a price matrix P, where

$$P = \begin{bmatrix} 145 & 210 & 350 \\ 185 & 270 & 410 \end{bmatrix} \begin{matrix} \text{Classic} \\ \text{Deluxe} \end{matrix}$$

(columns R, Q, K)

The company wants to increase the price of all beds. A new price matrix, listing the increased prices of the beds, can be generated from P by forming a **matrix product** with the matrix M, where

$$M = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.35 \end{bmatrix}$$

This new price matrix is

A $\begin{bmatrix} 145 & 210 & 350 \\ 185 & 270 & 410 \end{bmatrix}$

B $\begin{bmatrix} 234.90 & 340.20 & 567 \\ 299.70 & 437.40 & 664.20 \end{bmatrix}$

C $\begin{bmatrix} 174 & 252 & 420 \\ 222 & 324 & 492 \end{bmatrix}$

D $\begin{bmatrix} 174 & 252 & 420 \\ 249.75 & 364.50 & 553.50 \end{bmatrix}$

E $\begin{bmatrix} 195.75 & 283.50 & 472.50 \\ 249.75 & 364.50 & 553.50 \end{bmatrix}$

10 © VCAA 2010 1MQ2 70% Peter bought only apples and bananas from his local fruit shop.

The matrix $N = \begin{bmatrix} 3 & 4 \end{bmatrix}$ (columns A, B) lists the number of apples (A) and bananas (B) that Peter bought.

The matrix $C = \begin{bmatrix} 0.37 \\ 0.43 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$ lists the cost (in dollars) of one apple and one banana respectively.

The matrix product, NC, gives

A the total amount spent by Peter on the fruit that he bought.

B the total number of pieces of fruit that Peter bought.

C the individual amounts that Peter spent on apples and bananas respectively.

D the total number of pieces of fruit that Peter bought and the total amount that he spent.

E the individual number of apples and bananas that Peter bought and the individual amounts that Peter spent on these apples and bananas respectively.

11 © VCAA 2008 1MQ2 61% Apples cost \$3.50 per kg, bananas cost \$4.20 per kg and carrots cost \$1.89 per kg. Ashley buys 3 kg of apples, 2 kg of bananas and 1 kg of carrots. A matrix product to calculate the total cost of these items is

A $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 3.50 \\ 4.20 \\ 1.89 \end{bmatrix}$

B $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3.50 & 4.20 & 1.89 \end{bmatrix}$

C $\begin{bmatrix} 3.50 \times 2 & 4.20 \times 3 & 1.89 \times 1 \end{bmatrix}$

D $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 3.50 & 4.20 & 1.89 \end{bmatrix}$

E $\begin{bmatrix} 3.50 & 4.20 & 1.89 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

12 © VCAA 2019 1MQ7 54% The communication matrix below shows the direct paths by which messages can be sent between two people in a group of six people, U to Z.

$$\text{Sender}\begin{array}{c} \\ U \\ V \\ W \\ X \\ Y \\ Z \end{array}\overset{\text{Receiver}}{\begin{array}{c} \begin{matrix} U & V & W & X & Y & Z \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}}$$

A '1' in the matrix shows that the person named in that row can send a message directly to the person named in that column. For example, the '1' in row 4, column 2 shows that X can send a message directly to V. In how many ways can Y get a message to W by sending it directly to one other person?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

13 © VCAA 2009 2MQ1 88% (3 marks) Three types of cheese, Cheddar (C), Gouda (G) and Blue (B), will be bought for a school function. The cost matrix P lists the prices of these cheeses, in dollars, at two stores, Foodway and Safeworth.

$$P = \begin{bmatrix} 6.80 & 5.30 & 6.20 \\ 7.30 & 4.90 & 6.15 \end{bmatrix}\begin{array}{l} \text{Foodway} \\ \text{Safeworth} \end{array}$$

a What is the order of matrix P? 1 mark

The number of packets of each type of cheese needed is listed in the quantity matrix Q.

$$Q = \begin{bmatrix} 8 \\ 11 \\ 3 \end{bmatrix}\begin{array}{l} G \\ G \\ B \end{array}$$

b **i** Evaluate the matrix $W = PQ$. 1 mark

ii At which store will the total cost of the cheese be lower? 1 mark

14 © VCAA 2016 2MQ2 (2 marks) The travel company has five employees, Amara (A), Ben (B), Cheng (C), Dana (D) and Elka (E). The company allows each employee to send a direct message to another employee only as shown in the communication matrix G. The matrix G^2 is also shown below.

$$G = \text{Sender}\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\overset{\text{Receiver}}{\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}} \qquad G^2 = \text{Sender}\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\overset{\text{Receiver}}{\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}}$$

The '1' in row E, column D of matrix G indicates that Elka (*sender*) can send a direct message to Dana (*receiver*). The '0' in row E, column C of matrix G indicates that Elka cannot send a direct message to Cheng.

a 89% To whom can Dana send a direct message? 1 mark

b 71% Cheng needs to send a message to Elka, but cannot do this directly. Write down the names of the employees who can send the message from Cheng directly to Elka. 1 mark

15 © VCAA 2006 2MQ1 68% (4 marks) A manufacturer sells three products, A, B and C, through outlets at two shopping centres, Eastown (E) and Noxland (N). The number of units of each product sold per month through each shop is given by the matrix Q, where

$$\begin{matrix} & A & B & C & \\ Q = & \begin{bmatrix} 2500 & 3400 & 1890 \\ 1765 & 4588 & 2456 \end{bmatrix} & & & \begin{matrix} E \\ N \end{matrix} \end{matrix}$$

a Write down the order of matrix Q. 1 mark

The matrix P shown gives the selling price, in dollars, of products A, B, C.

$$P = \begin{bmatrix} 14.50 \\ 21.60 \\ 19.20 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$

b **i** Evaluate the matrix M, where $M = QP$. 1 mark

ii What information do the elements of matrix M provide? 1 mark

c Explain why the matrix PQ is **not** defined. 1 mark

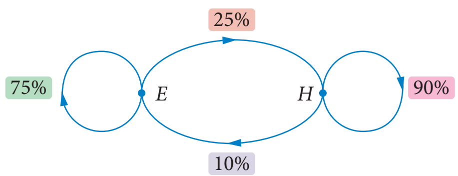
5.6 Transition matrices

Video playlist Transition matrices

Constructing transition diagrams and matrices

A **state** is a condition at a point in time. A **transition matrix** is a square matrix that shows a change from one **state** to another, where the change follows the same rules each time. The following **transition diagram** and matching transition matrix show how shoppers at two shopping centres, Eastfield and Highland, change the centre they visit from one week to the next.

Transition diagram



Transition matrix

$$\begin{matrix} & \text{This week} & \\ & \begin{matrix} E & H \end{matrix} & \\ T = & \begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix} & \begin{matrix} E \\ H \end{matrix} \text{ Next week} \end{matrix}$$

- 75% of shoppers who shop at Eastfield one week will shop at Eastfield the next week
- 25% of shoppers who shop at Eastfield one week will shop at Highland the next week
- 90% of shoppers who shop at Highland one week will shop at Highland the next week
- 10% of shoppers who shop at Highland one week will shop at Eastfield the next week

We will only be looking at situations where each of the columns in the transition matrix add up to 1, which means the total number involved in the **transition** stays the same.

Transition diagrams and matrices

Transition diagrams

- show transitions using percentages or decimals
- have all the arrow percentages *from* a single point add up to 100%
- don't show transitions that are 0%.

Transition matrices

- are square matrices
- show the transitions as decimals
- have each column adding up to 1
- show transitions that are 0.

9780170448192

WORKED EXAMPLE 13 Constructing transition matrices

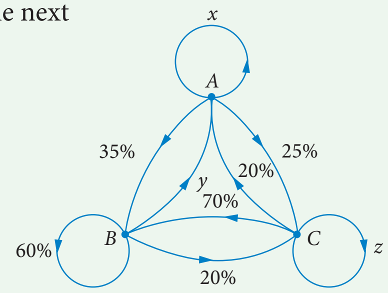
Construct transition matrices for each of the following.

Steps | **Working**

a For this transition diagram showing changes from one year to the next

i find x, y, z

ii construct the matching transition matrix.



i All the arrow percentages *from* a single point add up to 100%.

Solve for the unknowns, using CAS if necessary.

For arrows from A: $x + 35 + 25 = 100$

$x = 40\%$

For arrows from B: $60 + y + 20 = 100$

$y = 20\%$

For arrows from C: $70 + 20 + z = 100$

$z = 10\%$

ii Convert all the percentages to decimals and construct the matrix.

This year

$$\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0.4 & 0.2 & 0.2 \\ 0.35 & 0.6 & 0.7 \\ 0.25 & 0.2 & 0.1 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \end{matrix} \text{ Next year}$$

b A long-term study has established that in a particular town, if it rains on any day, there is a 75% chance of it raining the next day, and if it is dry on any day, there is an 80% chance it will be dry the next day.

1 Set up a 2×2 matrix using R for 'rain' and D for 'dry' with a transition from 'This day' to 'Next day'. Enter the percentages from the question as decimals.

This day

$$T = \begin{array}{c} \begin{matrix} R & D \end{matrix} \\ \begin{bmatrix} 0.75 & \\ & 0.8 \end{bmatrix} \end{array} \begin{matrix} R \\ D \end{matrix} \text{ Next day}$$

2 Enter the remaining elements of the matrix by using the fact that the columns of a transition matrix must add up to 1.

This day

$$T = \begin{array}{c} \begin{matrix} R & D \end{matrix} \\ \begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix} \end{array} \begin{matrix} R \\ D \end{matrix} \text{ Next day}$$

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Interpreting transition matrices

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WORKED EXAMPLE 14 Interpreting transition matrices

A polling company has established a pattern of voter behaviour in a particular electorate regarding the three major political parties, Labor (L), Coalition (C) and Greens (G). The pattern from one election to the next has been incorporated into the transition matrix T, where

$$T = \begin{matrix} & \text{This election} & \\ L & C & G \end{matrix}$$

$$T = \begin{bmatrix} 0.75 & 0.1 & 0.5 \\ 0 & 0.8 & 0 \\ 0.25 & 0.1 & 0.5 \end{bmatrix} \begin{matrix} L \\ C \\ G \end{matrix} \text{ Next election}$$

In a previous election, 43 536 people voted Labor, 32 750 voted Coalition and 11 206 voted Greens.

a How many people who voted Labor in this election are expected to vote Greens in the next election?

b How many people who voted Coalition in this election are expected to vote Coalition in the next election?

c How many people are expected to vote Labor in the next election?

d If 47% of the people voted Coalition one year, what percentage are expected to vote Greens in the next election?

Steps	Working
a Locate the relevant element in the transition matrix and multiply by the number of voters.	0.25 of people who voted Labor in this election are expected to vote Greens in the next election. $0.25 \times 43\,536 = 10\,884$ voters
b Locate the relevant element in the transition matrix and multiply by the number of voters.	0.8 of people who voted Coalition in this election are expected to vote Coalition in the next election. $0.8 \times 32\,750 = 26\,200$ voters
c Locate the relevant elements in the transition matrix, multiply by the number of voters in each case, and add.	L to L, C to L, G to L $0.75 \times 43\,536 + 0.1 \times 32\,750 + 0.5 \times 11\,206 = 41\,530$ 41 530 voters are expected to vote Labor in the next election.
d Locate the relevant element in the transition matrix and multiply by the percentage.	0.1 of Coalition voters are expected to vote Greens the next election. $0.1 \times 47\% = 4.7\%$

The state matrix

The power of transition matrices is that they can be used to make predictions. We can do this by multiplying the transition matrix by a **state matrix**, which shows the numbers in each category at a point in time. The **initial state matrix** contains the numbers at the start.

In the previous worked example, the initial state matrix is $\begin{bmatrix} 43\,536 \\ 32\,750 \\ 11\,206 \end{bmatrix} \begin{matrix} L \\ C \\ G \end{matrix}$

To predict how many people will vote for each party in the next election, we calculate

$$\begin{bmatrix} 0.75 & 0.1 & 0.5 \\ 0 & 0.8 & 0 \\ 0.25 & 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} 43\,536 \\ 32\,750 \\ 11\,206 \end{bmatrix} \begin{matrix} L \\ C \\ G \end{matrix} = \begin{bmatrix} 41\,530 \\ 26\,200 \\ 19\,762 \end{bmatrix} \begin{matrix} L \\ C \\ G \end{matrix}$$

So at the next election, we predict 41 530 will vote Labor, 26 200 will vote Coalition and 19 762 will vote Greens.

To predict how many people will vote for each party five elections from now, we use powers.

$$\begin{bmatrix} 0.75 & 0.1 & 0.5 \\ 0 & 0.8 & 0 \\ 0.25 & 0.1 & 0.5 \end{bmatrix}^5 \begin{bmatrix} 43\,536 \\ 32\,750 \\ 11\,206 \end{bmatrix} \begin{matrix} L \\ C \\ G \end{matrix} = \begin{bmatrix} 50\,532 \\ 10\,732 \\ 26\,228 \end{bmatrix} \begin{matrix} L \\ C \\ G \end{matrix}$$

So five elections from now, we predict 50 532 will vote Labor, 10 732 will vote Coalition and 26 228 will vote Greens.

The state matrix rule

The rule for finding the state matrix S_n after n transitions is

$$S_n = T^n S_0$$

where

T is a transition matrix

S_0 is the initial state matrix.

Exam hack

Remember the importance of order with matrices. Always multiply the transition matrix on the left.

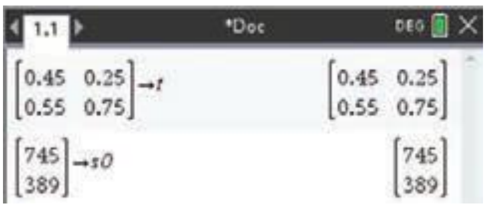
USING CAS 5 Finding state matrices using the rule

A market research company has analysed the petrol-buying patterns of motorists in two petrol stations in a small town, GasStop (G) and Oils (O). It has established that the movement between the two occurs according to this transition matrix.

$$\begin{matrix} & \text{This week} \\ & \begin{matrix} G & \quad O \end{matrix} \\ T = & \begin{bmatrix} 0.45 & 0.25 \\ 0.55 & 0.75 \end{bmatrix} \begin{matrix} G \\ O \end{matrix} \text{ Next week} \end{matrix}$$

This week, 745 motorists went to GasStop and 389 went to Oils. How many motorists go to each petrol station after six weeks?

TI-Nspire



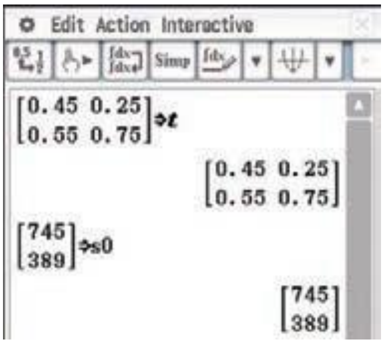
1 In a **Calculator** page, create a **2×2** transition matrix and store it in **t**.

2 Create a **2×1** initial state matrix for the number of motorists and store it in **s0**.

3 To find the number of motorists after six weeks, calculate **t**^{6}**×s0**.

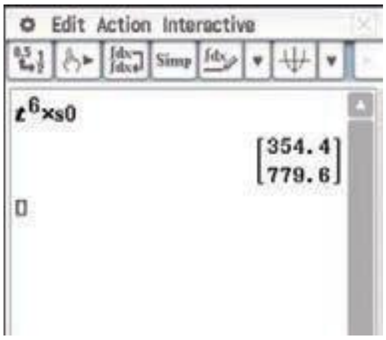
Rounding to the nearest whole number, 354 motorists went to GasStop and 780 went to Oils.

ClassPad



1 In the **Main** application, create a **2×2** transition matrix and store it in **t**.

2 Create a **2×1** initial state matrix for the number of motorists and store it in **s0**.



3 To find the number of motorists after six weeks, calculate $\mathbf{t^6 \times s0}$.

Rounding to the nearest whole number, 354 motorists went to GasStop and 780 went to Oils.

Exam hack

Round to the nearest whole number if it makes sense to do so, even if the question doesn't specifically ask you to.

Long-term trends

Although transition matrices involve constant change, in the long term the state matrix often gets to a point where it stops changing from one transition to the next. This is called the **steady-state matrix** (or **equilibrium state matrix**). The transition changes continue to occur, but they reach a point where they are cancelling themselves out, so the result ends up the same as the previous one.

To find the steady-state matrix we need to use the state matrix rule to show that the state matrix is the same for two consecutive high powers (e.g. 39 and 40).

For example, for the regular transition matrix above $T = \begin{bmatrix} 0.2 & 0.5 & 0.8 \\ 0 & 0.2 & 0.1 \\ 0.8 & 0.3 & 0.1 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 3400 \\ 2100 \\ 800 \end{bmatrix}$

$$S_{39} = \begin{bmatrix} 0.2 & 0.5 & 0.8 \\ 0 & 0.2 & 0.1 \\ 0.8 & 0.3 & 0.1 \end{bmatrix}^{39} \begin{bmatrix} 3400 \\ 2100 \\ 800 \end{bmatrix} = \begin{bmatrix} 0.4894 & 0.4894 & 0.4894 \\ 0.05674 & 0.05674 & 0.05674 \\ 0.4539 & 0.4539 & 0.4539 \end{bmatrix} \begin{bmatrix} 3400 \\ 2100 \\ 800 \end{bmatrix} = \begin{bmatrix} 3083 \\ 357 \\ 2860 \end{bmatrix}$$

$$S_{40} = \begin{bmatrix} 0.2 & 0.5 & 0.8 \\ 0 & 0.2 & 0.1 \\ 0.8 & 0.3 & 0.1 \end{bmatrix}^{40} \begin{bmatrix} 3400 \\ 2100 \\ 800 \end{bmatrix} = \begin{bmatrix} 0.4894 & 0.4894 & 0.4894 \\ 0.05674 & 0.05674 & 0.05674 \\ 0.4539 & 0.4539 & 0.4539 \end{bmatrix} \begin{bmatrix} 3400 \\ 2100 \\ 800 \end{bmatrix} = \begin{bmatrix} 3083 \\ 357 \\ 2860 \end{bmatrix}$$

So the equilibrium state matrix is $\begin{bmatrix} 3083 \\ 357 \\ 2860 \end{bmatrix}$.

Exam hack

For the questions you will be asked, 39 and 40 are high enough to find the steady-state matrix, though any other higher pair of consecutive powers can also be used.

WORKED EXAMPLE 15 Finding the steady-state matrix

A fleet of delivery vans starts each day at one of two depots, A or B. By the end of the day, the vans end up at either of the two depots according to the following transition matrix.

$$T = \begin{array}{c} \text{This day} \\ \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.75 & 0.45 \\ 0.25 & 0.55 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \end{array} \text{ Next day}$$

At the start of a particular day, there are 100 vans at depot A and 120 vans at depot B.

Steps	Working
a Find the steady-state matrix.	
1 Use CAS and the rule $S_n = T^n S_0$ for two large consecutive values of n.	$S_0 = \begin{bmatrix} 100 \\ 120 \end{bmatrix} \begin{array}{c} A \\ B \end{array}$, choose $n = 39$ and 40 $S_{39} = T^{39}S_0 = \begin{bmatrix} 0.75 & 0.45 \\ 0.25 & 0.55 \end{bmatrix}^{39} \begin{bmatrix} 100 \\ 120 \end{bmatrix} \approx \begin{bmatrix} 141 \\ 79 \end{bmatrix}$ $S_{40} = T^{40}S_0 = \begin{bmatrix} 0.75 & 0.45 \\ 0.25 & 0.55 \end{bmatrix}^{40} \begin{bmatrix} 100 \\ 120 \end{bmatrix} \approx \begin{bmatrix} 141 \\ 79 \end{bmatrix}$
2 Are the two state matrices the same?	$S_{39} = S^{40}$, so $\begin{bmatrix} 141 \\ 79 \end{bmatrix} \begin{array}{c} A \\ B \end{array}$ is the steady-state matrix.
b How many vans will be at each depot in the long term?	
Read from the steady-state matrix.	In the long term, 141 vans will be at depot A and 79 vans will be at depot B.
c What percentage of vans are at depot B in the long term? Round your answer to the nearest percentage.	
Calculate from the steady-state matrix.	In the long term, $\frac{79}{220}$ = 36% will be at depot B.

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VCE QUESTION ANALYSIS

 2006 2MQ2 2006 Examination 2 Matrices Question 2 (7 marks)

A new shopping centre called Shopper Heaven (S) is about to open. It will compete for customers with Eastown (E) and Noxland (N). Market research suggests that each shopping centre will have a regular customer base, but attract and lose customers on a weekly basis as follows.

80% of Shopper Heaven customers will return to Shopper Heaven next week
12% of Shopper Heaven customers will shop at Eastown next week
8% of Shopper Heaven customers will shop at Noxland next week

76% of Eastown customers will return to Eastown next week
9% of Eastown customers will shop at Shopper Heaven next week
15% of Eastown customers will shop at Noxland next week

85% of Noxland customers will return to Noxland next week
10% of Noxland customers will shop at Shopper Heaven next week
5% of Noxland customers will shop at Eastown next week

Video playlist VCE question analysis: Matrices

a Use this information to copy and complete the below transition matrix T (express percentages as proportions, for example, write 76% as 0.76).

$$T = \begin{array}{c} \text{This week} \\ \begin{matrix} S & E & N \end{matrix} \\ \left[\begin{matrix} & & \\ & & \\ & & \end{matrix}\right] \end{array} \begin{matrix} S \\ E \\ N \end{matrix} \text{ Next week}$$

2 marks

During the week that Shopper Heaven opened, it had 300 000 customers. In the same week, Eastown had 120 000 customers and Noxland had 180 000 customers.

b Use this information to copy and complete the below column matrix, K_0.

$$K_0 = \left[\begin{matrix} \\ \\ \end{matrix}\right] \begin{matrix} S \\ E \\ N \end{matrix}$$

1 mark

c Use T and K_0 to write and evaluate a matrix product that determines the number of customers expected at each of the shopping centres during the following week. 2 marks

d Show, by calculating at least two appropriate state matrices, that, in the long term, the number of customers expected at each centre each week is given by the

$$\text{matrix } K = \begin{bmatrix} 194\,983 \\ 150\,513 \\ 254\,504 \end{bmatrix}$$

2 marks

Reading the question

- The instructions clearly state to write percentages as decimals in the transition matrix.
- Note part **c** asks to write and evaluate.
- Part **d** states you need to show at least two matrix calculations.

Thinking about the question

- Both parts **a** and **b** show you what the matrices should look like.
- Part **c** is asking you to write the matrices you are multiplying, as well as to calculate the answer.
- How do you show the long-term trend for state matrices?

Worked solution (✓ = 1 mark)

a Convert all percentages to decimals. For example:

80% of Shopper Heaven customers will return to Shopper Heaven next week and

12% of Shopper Heaven customers will shop at Eastown next week

are entered in the transition matrix as

$$T = \begin{array}{c} \text{This week} \\ \begin{matrix} S & E & N \end{matrix} \\ \left[\begin{matrix} 0.80 & & \\ 0.12 & & \\ & & \end{matrix}\right] \end{array} \begin{matrix} S \\ E \\ N \end{matrix} \text{ Next week}$$

$$\boldsymbol{T = \begin{array}{c} \textbf{This week} \\ \begin{matrix} S & E & N \end{matrix} \\ \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.76 & 0.05 \\ 0.08 & 0.15 & 0.85 \end{bmatrix} \end{array} \begin{matrix} S \\ E \\ N \end{matrix} \textbf{ Next week}} \quad ✓ ✓$$

5.6

b $K_0 = \begin{bmatrix} 300\,000 \\ 120\,000 \\ 180\,000 \end{bmatrix} \begin{matrix} S \\ E \\ N \end{matrix}$ ✓

c $K_1 = TK_0 = \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.76 & 0.05 \\ 0.08 & 0.15 & 0.85 \end{bmatrix}\begin{bmatrix} 300\,000 \\ 120\,000 \\ 180\,000 \end{bmatrix} = \begin{bmatrix} 268\,800 \\ 136\,200 \\ 195\,000 \end{bmatrix}$ ✓

There are **268 800 at Shopper Heaven, 136 200 at Eastown and 195 000 at Noxland.** ✓

d Find two consecutive high powers of T that give the same result for T^nK_0.

Any two consecutive high powers which are 39 or higher give the result.

One correct answer is:

$$K_{39} = T^{39}K_0 = \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.76 & 0.05 \\ 0.08 & 0.15 & 0.85 \end{bmatrix}^{39}\begin{bmatrix} 300\,000 \\ 120\,000 \\ 180\,000 \end{bmatrix} = \begin{bmatrix} 194\,983 \\ 150\,513 \\ 254\,504 \end{bmatrix}$$

and

$$K_{40} = T^{40}K_0 = \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.76 & 0.05 \\ 0.08 & 0.15 & 0.85 \end{bmatrix}^{40}\begin{bmatrix} 300\,000 \\ 120\,000 \\ 180\,000 \end{bmatrix} = \begin{bmatrix} 194\,983 \\ 150\,513 \\ 254\,504 \end{bmatrix}$$ ✓

and this is the same as K^{39}. ✓

Student performance

80–100% 60–79% 0–59%

a 78% Some students wrote percentages despite being asked for proportions. Some simply copied the figures in order, as printed into the three columns. The percentage, 8%, was sometimes written as a proportion of 0.8.

b 78%

Success percentages for parts **a** and **b** were averaged.

c 46% The expression and the result were both required for full marks.

d 46% There was more than one correct answer. Calculations involving *any* two consecutive high powers that are 39 or higher were correct. The long-term trend is shown by stating the two state matrices are the same.

Success percentages for parts **c** and **d** were averaged.

EXERCISE 5.6 Transition matrices

ANSWERS p. 508

Recap

80–100% 60–79% 0–59%

1 © VCAA 2017 1MQ2 90% The matrix shows how five people, Alan (A), Bevan (B), Charlie (C), Drew (D) and Esther (E), can communicate with each other.

A ‘1’ in the matrix shows that the person named in that row can send a message directly to the person named in that column. For example, the ‘1’ in row 3 and column 4 shows that Charlie can send a message directly to Drew. Esther wants to send a message to Bevan. Which one of the following shows the order of people through which the message is sent?

$$\text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}\overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D & E \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}}$$

A Esther – Bevan

B Esther – Charlie – Bevan

C Esther – Charlie – Alan – Bevan

D Esther – Charlie – Drew – Bevan

E Esther – Charlie – Drew – Alan – Bevan

2 © VCAA 2008 1MQ3 72% The cost prices of three different electrical items in a store are \$230, \$290 and \$310 respectively. The selling price of each of these three electrical items is 1.3 times the cost price plus a commission of \$20 for the salesman. A matrix that lists the selling price of each of these three electrical items is determined by evaluating

A $1.3 \times \begin{bmatrix} 230 \\ 290 \\ 310 \end{bmatrix} + [20]$ **B** $1.3 \times \begin{bmatrix} 230 \\ 290 \\ 310 \end{bmatrix} + 1.3 \times 20$ **C** $1.3 \times \begin{bmatrix} 230 \\ 290 \\ 310 \end{bmatrix} + \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$

D $1.3 \times \begin{bmatrix} 230 \\ 290 \\ 310 \end{bmatrix} + 1.3 \times \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$ **E** $1.3 \times \begin{bmatrix} 230 + 20 \\ 290 + 20 \\ 310 + 20 \end{bmatrix}$

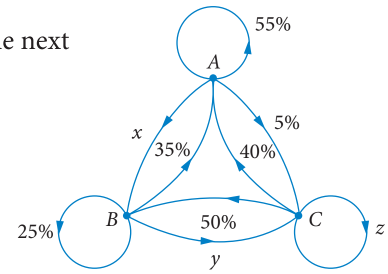
Mastery

3 WORKED EXAMPLE 13

a For this transition diagram showing changes from one year to the next

i find x, y, z

ii construct the matching transition matrix.



b Construct a transition matrix for the following situation.

If a train arrives at a certain station on time, the next train has a 95% chance of being on time. If a train arrives late, the next train has a 70% chance of being late.

4 WORKED EXAMPLE 14 A colony of birds migrates to three different locations A, B and C. The birds change their location each year according to the transition matrix T, where

$$T = \begin{array}{c} \text{This year} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.35 & 0.2 & 0.3 \\ 0.25 & 0.55 & 0 \\ 0.4 & 0.25 & 0.7 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \\ C \end{array} \text{Next year}$$

This year there are 440 birds in location A, 620 birds in location B and 280 birds in location C.

a How many birds in location B this year are expected to be in location A next year?

b How many birds in location C this year are expected to be in location C next year?

c How many birds are expected to be in location A next year?

d If 52% of the birds were in location A one year, what percentage are expected to be in location C the next year?

5 Using CAS 5 A market research company has analysed the takeaway buying patterns of people regarding two chains, Sammy's Seafood (S) and Karl's Kebabs (K). It has established that the movement between the two occurs according to this transition matrix.

$$\begin{array}{c} \text{This week} \\ \begin{array}{cc} S & K \end{array} \end{array}$$

$$T = \begin{bmatrix} 0.2 & 0.55 \\ 0.8 & 0.45 \end{bmatrix} \begin{matrix} S \\ K \end{matrix} \text{ Next week}$$

In the current week, 9840 people bought from Sammy's Seafood and 7680 bought from Karl's Kebabs. How many people go to each fast food outlet after

a 3 weeks **b** 5 weeks **c** 10 weeks?

6 WORKED EXAMPLE 15 A fleet of trams starts each day at one of two depots, A or B. By the end of the day, the trams end up at either of the two depots according to the following transition matrix.

$$\begin{array}{c} \text{This day} \\ \begin{array}{cc} A & B \end{array} \end{array}$$

$$T = \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix} \begin{matrix} A \\ B \end{matrix} \text{ Next day}$$

At the start of a particular day, there are 40 trams at depot A and 100 trams at depot B.

a Find the steady-state matrix.

b How many trams will be at each depot in the long term?

c What percentage of trams are at depot A in the long term? Round your answer to the nearest percentage.

7 State whether each matrix could be a transition matrix, giving a reason when it would not be possible.

a $\begin{bmatrix} 0.8 & 0.9 \\ 0.5 & 0.1 \end{bmatrix}$ **b** $\begin{bmatrix} 0.4 & 0.75 \\ 0.6 & 0.25 \end{bmatrix}$ **c** $\begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \\ 0.6 & 0.7 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.5 & 1 & 0 \end{bmatrix}$

Exam practice

80–100% 60–79% 0–59%

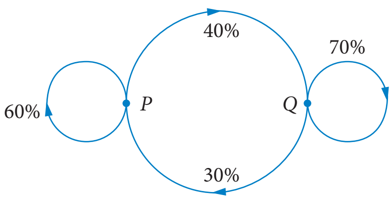
8 © VCAA 2009 1MQ6 87% T is a transition matrix, where

$$\begin{array}{c} \text{From} \\ \begin{array}{cc} P & Q \end{array} \end{array}$$

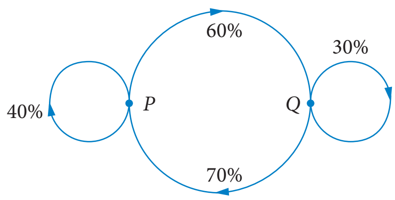
$$T = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{matrix} P \\ Q \end{matrix} \text{ To}.$$

An equivalent transition diagram, with proportions expressed as percentages, is

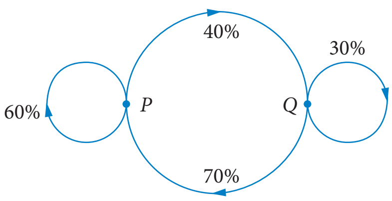
A



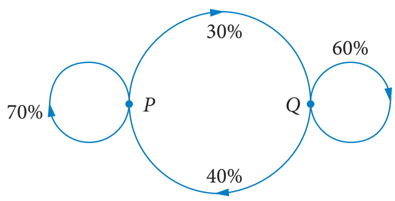
B



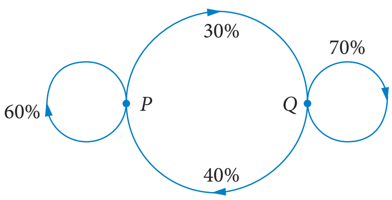
C



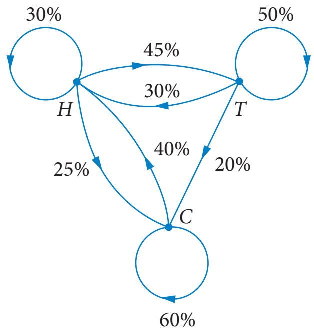
D



E



9 © VCAA 2016 1MQ6 86% Families in a country town were asked about their annual holidays. Every year, these families choose between staying at home (H), travelling (T) and camping (C). The transition diagram shows the way families in the town change their holiday preferences from year to year.



A transition matrix that provides the same information as the transition diagram is

A $T = \begin{array}{c} \text{From} \\ \begin{array}{ccc} H & T & C \end{array} \\ \begin{bmatrix} 0.30 & 0.75 & 0.65 \\ 0.75 & 0.50 & 0.20 \\ 0.65 & 0.20 & 0.60 \end{bmatrix} \end{array} \begin{array}{c} H \\ T \\ C \end{array} \text{To}$

B $T = \begin{array}{c} \text{From} \\ \begin{array}{ccc} H & T & C \end{array} \\ \begin{bmatrix} 0.30 & 0.30 & 0.40 \\ 0.45 & 0.50 & 0 \\ 0.25 & 0.20 & 0.60 \end{bmatrix} \end{array} \begin{array}{c} H \\ T \\ C \end{array} \text{To}$

C $T = \begin{array}{c} \text{From} \\ \begin{array}{ccc} H & T & C \end{array} \\ \begin{bmatrix} 0.30 & 0.30 & 0.40 \\ 0.45 & 0.50 & 0.20 \\ 0.25 & 0.20 & 0.60 \end{bmatrix} \end{array} \begin{array}{c} H \\ T \\ C \end{array} \text{To}$

D $T = \begin{array}{c} \text{From} \\ \begin{array}{ccc} H & T & C \end{array} \\ \begin{bmatrix} 0.30 & 0.30 & 0.40 \\ 0.45 & 0.50 & 0.20 \\ 0.25 & 0.20 & 0.40 \end{bmatrix} \end{array} \begin{array}{c} H \\ T \\ C \end{array} \text{To}$

E $T = \begin{array}{c} \text{From} \\ \begin{array}{ccc} H & T & C \end{array} \\ \begin{bmatrix} 0.30 & 0.45 & 0.25 \\ 0.30 & 0.50 & 0.20 \\ 0.40 & 0 & 0.60 \end{bmatrix} \end{array} \begin{array}{c} H \\ T \\ C \end{array} \text{To}$

10 © VCAA 2009 1MQ7 84% In a country town, people only have the choice of doing their food shopping at a store called Marks (M) or at a newly opened store called Foodies (F). In the first week that Foodies opened, only 300 of the town's 800 shoppers did their food shopping at Marks. The remainder did their food shopping at Foodies. A state matrix S_0 that can be used to represent this situation is

A $S_0 = \begin{bmatrix} 300 \\ 800 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$

B $S_0 = \begin{bmatrix} 500 \\ 300 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$

C $S_0 = \begin{bmatrix} 800 \\ 300 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$

D $S_0 = \begin{bmatrix} 300 \\ 500 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$

E $S_0 = \begin{bmatrix} 800 \\ 500 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$

11 © VCAA 2013 1MQ3 63% A coffee shop sells three types of coffee, Brazilian (B), Italian (I) and Kenyan (K). The regular customers buy one cup of coffee each per day and choose the type of coffee they buy according to the following transition matrix, T.

$$T = \begin{matrix} & \text{Choose today} & \\ & \begin{matrix} B & I & K \end{matrix} & \\ & \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.8 \end{bmatrix} & \begin{matrix} B \\ I \\ K \end{matrix} \text{ Choose tomorrow} \end{matrix}$$

On a particular day, 84 customers bought Brazilian coffee, 96 bought Italian coffee and 81 bought Kenyan coffee. If these same customers continue to buy one cup of coffee each per day, the number of these customers who are expected to buy each of the three types of coffee in the long term is

A Brazilian 85, Italian 85, Kenyan 91

B Brazilian 87, Italian 58, Kenyan 116

C Brazilian 88, Italian 86, Kenyan 87

D Brazilian 89, Italian 89, Kenyan 83

E Brazilian 116, Italian 89, Kenyan 58

12 © VCAA 2006 1MQ8 56% Australians go on holidays either within Australia or overseas. Market research shows that

- 95% of those who had their last holiday in Australia said that their next holiday would be in Australia
- 20% of those who had their last holiday overseas said that their next holiday would also be overseas.

A transition matrix that could be used to describe this situation is

A $\begin{bmatrix} 0.95 \\ 0.20 \end{bmatrix}$

B $\begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 0.20 \\ 0.80 \end{bmatrix}$

C $\begin{bmatrix} 0.95 & 0.95 \\ 0.20 & 0.20 \end{bmatrix}$

D $\begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix}$

E $\begin{bmatrix} 0.95 & 0.80 \\ 0.05 & 0.20 \end{bmatrix}$

13 © VCAA 2007 1MQ6 51% A colony of fruit bats feeds nightly at three different locations, A, B and C. Initially, the number of bats from the colony feeding at each of the locations was as follows.

$$T = \begin{matrix} & \text{This night} & \\ & \begin{matrix} A & B & C \end{matrix} & \\ & \begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.7 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \text{ Next night} \end{matrix}$$

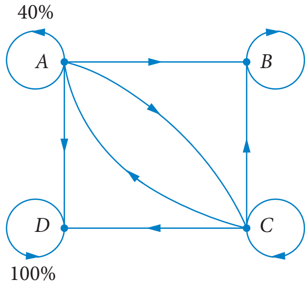
Location	Number of bats
A	1568
B	1105
C	894

The bats change feeding locations according to the transition matrix T. If this pattern of feeding continues, the number of bats feeding at location A in the long term will be closest to

A 1254 **B** 1543 **C** 1568 **D** 1605 **E** 1725

14 © VCAA 2008 1MQ7 MODIFIED (2 marks) A large population of mutton birds migrates each year to a remote island to nest and breed. There are four nesting sites on the island, A, B, C and D. Researchers suggest that the following transition matrix can be used to predict the number of mutton birds nesting at each of the four sites in subsequent years. An incomplete equivalent transition diagram is also given.

$$T = \begin{array}{c} \text{This year} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{cccc} 0.4 & 0 & 0.2 & 0 \\ 0.35 & 1 & 0.15 & 0 \\ 0.15 & 0 & 0.55 & 0 \\ 0.1 & 0 & 0.1 & 1 \end{array}\right] \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array} \text{ Next year}$$



a Copy and complete the transition diagram. 1 mark

b If 2800 mutton birds nest at site C in 2008, how many of these are predicted to nest at site A in 2009? 1 mark

15 © VCAA 2015 2MQ2 (3 marks) The ability level of students in a music school is assessed regularly and classified as beginner (B), intermediate (I) or advanced (A). After each assessment, students either stay at their current level or progress to a higher level. Students cannot be assessed at a level that is lower than their current level. The expected number of students at each level after each assessment can be determined using the transition matrix, T_1.

$$T_1 = \begin{array}{c} \text{Before assessment} \\ \begin{array}{ccc} B & I & A \end{array} \\ \left[\begin{array}{ccc} 0.50 & 0 & 0 \\ 0.48 & 0.80 & 0 \\ 0.02 & 0.20 & 1 \end{array}\right] \end{array} \begin{array}{c} B \\ I \\ A \end{array} \text{ After assessment}$$

a The element in the third row and third column of matrix T_1 is the number 1. Explain what this tells you about the advanced-level students. 1 mark

Let matrix S_n be a state matrix that lists the number of students at beginner, intermediate and advanced levels after n assessments. The number of students in the school, immediately before the first assessment of the year, is shown in matrix S_0.

$$S_0 = \left[\begin{array}{c} 20 \\ 60 \\ 40 \end{array}\right] \begin{array}{c} B \\ I \\ A \end{array}$$

b **i** Write down the matrix S_1 that contains the expected number of students at each level after one assessment. Write the elements of this matrix correct to the nearest whole number. 1 mark

ii How many intermediate-level students have become advanced-level students after one assessment? 1 mark

5 Chapter summary

Introducing matrices

- **Matrices** are collections of numbers arranged into rows and columns inside square brackets.
- **Order** of a matrix = *number of rows* × *number of columns*.
- A matrix with m rows and n columns has order $m \times n$.
- A matrix with m rows and n columns has mn elements (e.g. a 5×3 matrix has 15 elements).

Types of matrices

Type of matrix	Description	Examples	Order of examples
Row matrix	A matrix with just one row.	$\begin{bmatrix} 7 & -3 & 15 \end{bmatrix}$ $\begin{bmatrix} 9 & -3 & 0 & 0 \end{bmatrix}$	1 × 3 and 1 × 4
Column matrix	A matrix with just one column.	$\begin{bmatrix} 45 \\ -32 \end{bmatrix}$ $\begin{bmatrix} 31 \\ 25 \\ 50 \end{bmatrix}$	2 × 1 and 3 × 1
Square matrix	A matrix that has the same number of rows as columns.	$\begin{bmatrix} 7 & 12 & 0 & 8 \\ -3 & 9 & -2 & 5 \\ 6 & -1 & 0 & 0 \\ 10 & -6 & 13 & 4 \end{bmatrix}$ $\begin{bmatrix} 0 & -12 \\ -4 & 1 \end{bmatrix}$	4 × 4 and 2 × 2
Zero matrix	A matrix where all the numbers are '0'.	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	3 × 2 and 1 × 6
Identity matrix (I) (also called the **unit matrix**)	A square matrix where all the elements in the diagonal from top left to bottom right are '1', and every other element is '0'. We use I to indicate this matrix.	$I = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$ $I = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}$	4 × 4 and 2 × 2

Matrix addition and subtraction

- When adding or subtracting matrices, add or subtract pairs of corresponding elements.
- For addition or subtraction of matrices to be defined, they must have the same order.
- The answer has the same order as the matrices being added or subtracted.

Scalar multiplication

- A **scalar** is a regular number that isn't in a matrix.
- When multiplying a matrix by a scalar, multiply each element by the scalar.
- Scalar multiplication is defined for any matrix.
- The answer has the same order as the original matrix.

Matrix multiplication

When multiplying matrices $AB = C$:

- Multiply the elements of each row in A by the elements of each column in B and add them.

$$(1 \times 7) + (2 \times 8) + (3 \times 9) = 50$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 50 & \\ & \end{bmatrix}$$

	A	$\times$	B	$=$	C
Order:	2×3		3×2		2×2

- The product is defined if the number of columns in A = the number of rows in B.
- C has the same number of rows as A and the same number of columns as B.

$$AB = C$$

Order: $(m \times n)\ (n \times q) = (m \times q)$

Product is defined

- Multiplying the identity matrix with another matrix, leaves the other matrix unchanged.
- When raising matrices to powers:
- only powers of square matrices are defined
- the power of a matrix will always have the same order as the original matrix.

Inverse matrices

- If the matrix A^{-1} is the **inverse** of the matrix A then $AA^{-1} = A^{-1}A = I$.
- Only square matrices have inverses.
- Not all square matrices have inverses.
- A and A^{-1} always have the same order.
- For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A) = ad - bc$
- $A^{-1} = \dfrac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- The determinant only exists for square matrices.
- The inverse does not exist when $\det(A) = 0$.
- Use CAS to find determinants, inverses, and to solve simultaneous equations.
- Solving simultaneous equations using matrices involves finding the inverse.
- When the determinant of a matrix is equal to 0, it is known as a **singular matrix**. The inverse of a singular matrix does not exist.

Communication matrices

A **communication matrix** M is a square binary matrix where

- a '1' indicates direct **one-step communication** and a '0' indicates non-communication
- the top left to bottom right diagonal has all '0's, indicating **redundant links** where the sender and receiver are the same
- M^2 gives the number of **two-step communications** where communication occurs via another link.

Transition diagrams and matrices

Transition diagrams

- show transitions using percentages
- have all the arrow percentages *from* a single point add up to 100%
- don't show transitions that are 0%.

Transition matrices

- are square matrices
- show the transitions as decimals
- have each column adding up to 1
- show transitions that are '0'.

The state matrix

- The rule for finding the state matrix S_n after n transitions is

  $S_n = T^n S_0$

  where

  T is a transition matrix

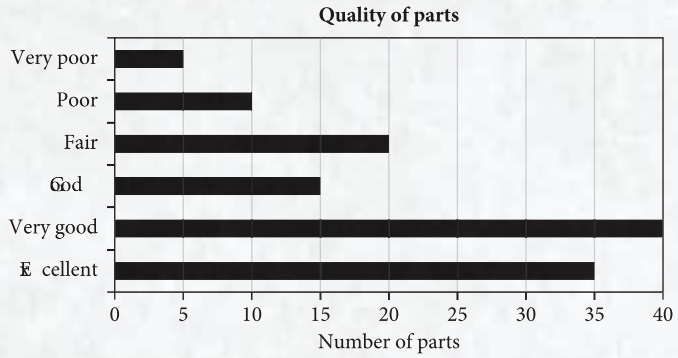
  S_0 is the initial state matrix.
- For large n, the state matrix S_n often stops changing and becomes the **steady-state matrix**.
- To find the steady-state matrix check that two large consecutive values of n give the same state matrix (e.g. 39 and 40).

Cumulative examination 1

Total number of marks: 14 Reading time: 5 minutes Writing time: 32 minutes

Use the following information to answer the next two questions.

The bar chart shows the results of a quality control survey of parts manufactured at a factory.



1 The type of data collected is best described as

A categorical and nominal data.
B numerical and ordinal data.
C categorical and ordinal data.
D numerical and discrete data.
E numerical and continuous data.

2 The total number of parts surveyed was

A 20 **B** 40 **C** 120 **D** 125 **E** 132

3 © VCAA 2019N 1CQ17 MODIFIED A sequence of numbers is generated by the recurrence relation

$u_0 = 2, \quad u_{n+1} = 3u_n - 1$

What is the value of u_3?

A 2 **B** 5 **C** 11 **D** 41 **E** 122

4 © VCAA 2013 1BRMQ8 MODIFIED A car is purchased for \$25 000. The value of the car is to be depreciated each year by 20% using the reducing balance method. By how much has the car been depreciated after four years?

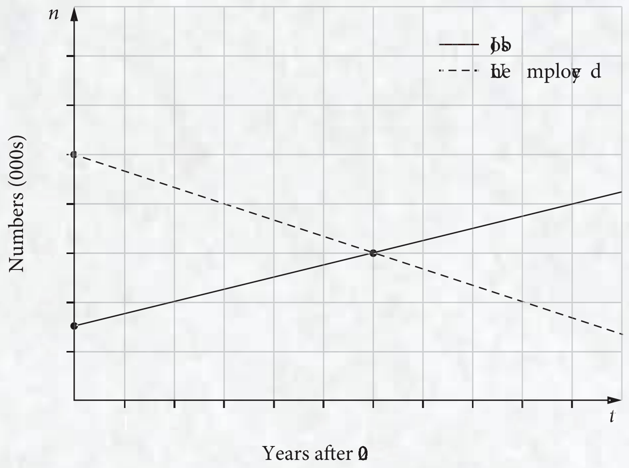
A \$5000 **B** \$10 240 **C** \$12 200 **D** \$12 800 **E** \$14 760

5 © VCAA 2015 1BRMQ2 An investment property was purchased for \$600 000. Over a 10-year period, its value increased to \$850 000. The increase in value, as a percentage of the purchase price, is closest to

A 4.2% **B** 25.0% **C** 29.4% **D** 41.7% **E** 70.6%

Use the following information to answer the next two questions.

The number of unemployed people since the year 2000 in a particular area is declining at a steady rate, while the number of jobs available is rising, according to the graph.



6 How many people were unemployed in the year 2000?

A 6 **B** 30 **C** 50 **D** 15 000 **E** 50 000

7 When the number of unemployed people equalled the number of jobs, approximately how many people were unemployed?

A 6 **B** 30 **C** 15 000 **D** 30 000 **E** 50 000

8 If $P = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}$, then the order of PQ is

A 1×1 **B** 3×1 **C** 1×3 **D** 3×3 **E** 2×3

9 © VCAA 2015 1MQ6 A carpenter can make four coffee tables and seven stools in a total of 33 hours. The carpenter can make two coffee tables and three pencil boxes in a total of 12 hours. The carpenter can make five stools and one pencil box in a total of 10 hours. The time, in hours, that it takes to make one coffee table is closest to

A 2 **B** 3 **C** 4 **D** 5 **E** 6

10 © VCAA 2019N 1MQ1 The number of individual points scored by Rhianna (*R*), Suzy (*S*), Tina (*T*), Ursula (*U*) and Vicki (*V*) in five basketball matches (*F*, *G*, *H*, *I*, *J*) is shown in matrix *P* below.

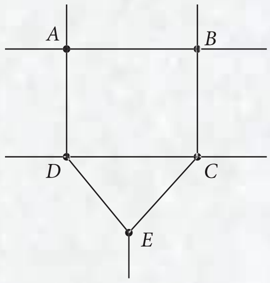
$$P = \begin{array}{c} \text{Match} \\ \begin{array}{ccccc} F & G & H & I & J \end{array} \\ \begin{bmatrix} 2 & 0 & 3 & 1 & 8 \\ 4 & 7 & 2 & 5 & 3 \\ 6 & 4 & 0 & 0 & 5 \\ 1 & 6 & 1 & 4 & 5 \\ 0 & 5 & 3 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{l} R \\ S \\ T \;\; \text{Player} \\ U \\ V \end{array}$$

Who scored the highest number of points and in which match?

A Suzy in match *I* **B** Tina in match *H* **C** Vicki in match *F*

D Ursula in match *G* **E** Rhianna in match *J*

11 © VCAA 2009 1MQ5 MODIFIED *A*, *B*, *C*, *D* and *E* are five intersections joined by roads as shown in the diagram. Some of these roads are one-way only.



The matrix indicates the direction that cars can travel along each of these roads.

In this matrix

- 1 in column *A* and row *B* indicates that cars can travel directly from *A* to *B*
- 0 in column *B* and row *A* indicates that cars cannot travel directly from *B* to *A* (either it is a one-way road or no road exists).

$$M = \text{From intersection} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \overset{\text{To intersection}}{\overset{\begin{matrix} A & B & C & D & E \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}}$$

Cars can travel in both directions between intersections

A *A* and *D* **B** *B* and *C* **C** *C* and *D* **D** *D* and *E* **E** *C* and *E*

12 © VCAA 2017N 1MQ2 The cost of fruit at a stall, in dollars per kilogram, is shown in the table.

Apples	\$2.50
Pears	\$3.20
Bananas	\$1.90

Sean wants to buy 2 kg of apples, 1 kg of pears and 3 kg of bananas. Which one of the following matrix products will result in a matrix that contains the total cost of Sean's fruit purchase, in dollars?

A $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2.50 \\ 3.20 \\ 1.90 \end{bmatrix}$

B $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2.50 & 3.20 & 1.90 \end{bmatrix}$

C $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2.50 \\ 3.20 \\ 1.90 \end{bmatrix}$

D $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2.50 \\ 3.20 \\ 1.90 \end{bmatrix}$

E $\begin{bmatrix} 2.50 \\ 3.20 \\ 1.90 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$

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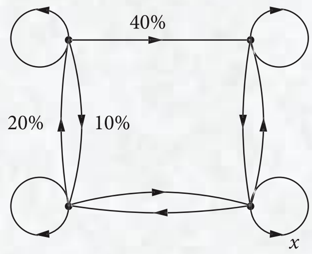
13 © VCAA 2018 1MQ6 MODIFIED A transition matrix, V, is shown.

$$\begin{array}{cc} & \text{This month} \\ & \begin{array}{cccc} L & T & F & M \end{array} \\ V = & \begin{bmatrix} 0.6 & 0.6 & 0.2 & 0.0 \\ 0.1 & 0.2 & 0.0 & 0.1 \\ 0.3 & 0.0 & 0.8 & 0.4 \\ 0.0 & 0.2 & 0.0 & 0.5 \end{bmatrix} \begin{array}{c} L \\ T \\ F \\ M \end{array} \text{ Next month} \end{array}$$

The following transition diagram has been constructed from the transition matrix V. The labelling in the transition diagram is not yet complete.



The proportion for one of the transitions is labelled x. The value of x is

A 20% **B** 50% **C** 60% **D** 70% **E** 80%

14 © VCAA 2016 1MQ7 MODIFIED Each week, the 300 students at a primary school choose art (A), music (M) or sport (S) as an afternoon activity. The transition matrix shows how the students' choices change from week to week. In the first week there are 100 students in each activity.

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{ccc} A & M & S \end{array} \\ T = & \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{bmatrix} \begin{array}{c} A \\ M \\ S \end{array} \text{ Next week} \end{array}$$

Based on the information, it can be concluded that, in the long term

A no student will choose sport.

B all students will choose to stay in the same activity each week.

C all students will have chosen to change their activity at least once.

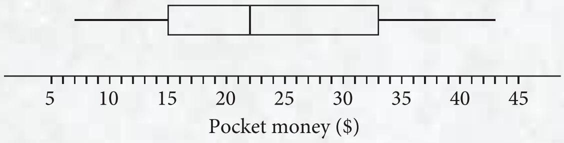
D more students will choose to do music than sport.

E the number of students choosing to do art and music will be the same.

Cumulative examination 2

Total number of marks: 19 Reading time: 5 minutes Writing time: 26 minutes

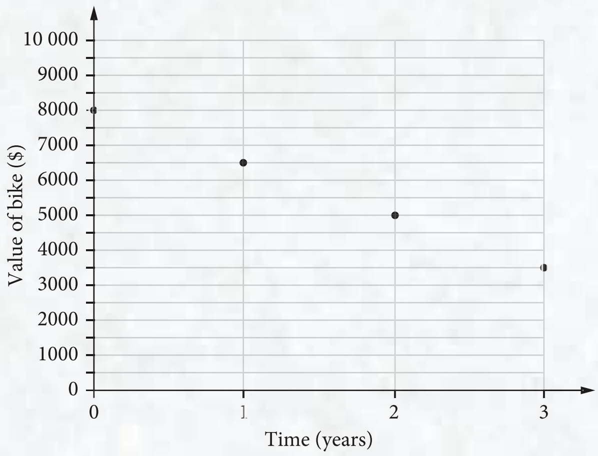
1 (2 marks) This boxplot represents the amount of pocket money, in dollars, earned by a sample of 48 children.



a What percentage of children earned less than $15? 1 mark

b How many children earned $22 or more of pocket money? 1 mark

2 © VCAA 2013 2BRMQ1 (4 marks) Hugo is a professional bike rider. The value of his bike will be depreciated over time using the flat rate method of depreciation. The graph shows his bike's initial purchase price and its value at the end of each year for a period of three years.



a What was the initial purchase price of the bike? 1 mark

b **i** Show that the bike depreciates in value by $1500 each year. 1 mark

ii Assume that the bike's value continues to depreciate by $1500 each year. Determine its value five years after it was purchased. 1 mark

The unit cost method of depreciation can also be used to depreciate the value of the bike. In a two-year period, the total depreciation calculated at $0.25 per kilometre travelled will equal the depreciation calculated using the flat rate method of depreciation as described above.

c Determine the number of kilometres the bike travels in the two-year period. 1 mark

3 (2 marks) N-mart sold a computer for $1600 in May. They reduced the price of all goods by 20% for their June sale, and then increased their June prices by 20% at the start of July.

a What was the overall percentage change to the price of the computer from May to July? 1 mark

b If N-mart had instead decided to decrease the $1600 computer to $1200 in their June sale, what would be the percentage price decrease? 1 mark

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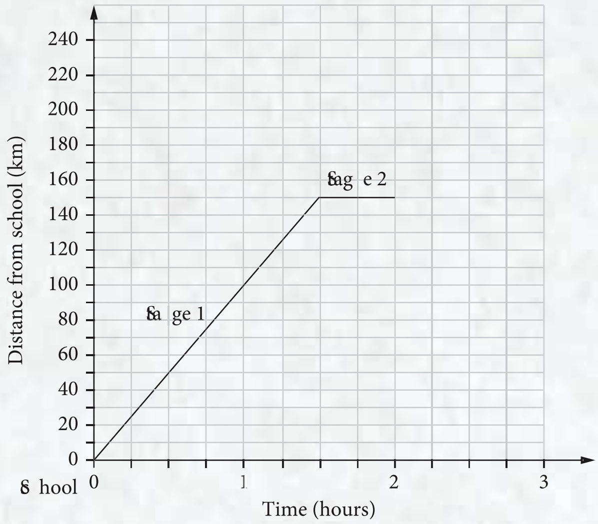
4 © VCAA 2013 2GRQ1ab (2 marks) The distance–time graph shows the first two stages of a bus journey from a school to a camp.



a At what constant speed, in kilometres per hour, did the bus travel during Stage 1 of the journey? 1 mark

b For how many minutes did the bus stop during Stage 2 of the journey? 1 mark

5 © VCAA 2021 2MQ1 (2 marks) Elena imports three brands of olive oil: Carmani (C), Linelli (L) and Ohana (O). The number of 1 litre bottles of these oils sold in January 2021 is shown in matrix J.

$$J = \begin{bmatrix} 2800 \\ 1700 \\ 2400 \end{bmatrix} \begin{matrix} C \\ L \\ O \end{matrix}$$

a What is the order of matrix J? 1 mark

b Elena expected that in February 2021 the sales of all three brands of olive oil would increase by 5%. She multiplied matrix J by a scalar value, k, to determine the expected volume of sales for February. Write down the value of the scalar k. 1 mark

6 © VCAA 2012 2MQ1 MODIFIED (4 marks) Matrix F shows the flight connections for an airline that serves four cities, Anvil (A), Berga (B), Cantor (C) and Dantel (D). In this matrix, the '1' in column C row B, for example, indicates that, using this airline, you can fly directly from Cantor to Berga. The '0' in column C row D, for example, indicates that you cannot fly directly from Cantor to Dantel.

$$F = \text{From} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{To}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}}$$

a Copy and complete the following sentence.

On this airline, you can fly directly from Berga to __________ and __________. 1 mark

b List the route that you must follow to fly from Anvil to Cantor. 1 mark

c Evaluate the matrix product $G = KF$, where $K = [1\ 1\ 1\ 1]$. 1 mark

d In the context of the problem, what information does matrix G contain? 1 mark

7 © VCAA 2008 2MQ4 MODIFIED (3 marks) At the end of each academic year, students at the university will have either passed (P), failed (F) or deferred (D) the year. Experience has shown that

- 88% of students who pass this year will also pass next year
- 10% of students who pass this year will fail next year
- 2% of students who pass this year will defer next year
- 52% of students who fail this year will pass next year
- 44% of students who fail this year will fail next year
- 4% of students who fail this year will defer next year
- 65% of students who defer this year will pass next year
- 10% of students who defer this year will fail next year
- 25% of students who defer this year will defer next year.

Twelve hundred and thirty students began a business degree in 2007. By the end of the 2007 academic year, 880 students had passed, 230 had failed, while 120 had deferred the year. No students have dropped out of the business degree permanently.

a Copy and complete the following transition matrix.

$$\begin{matrix} & \text{This year} & \\ P & F & D \end{matrix}$$
$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \begin{matrix} P \\ F \\ D \end{matrix} \text{ Next year}$$

1 mark

b Write a matrix product that calculates the state matrix for the end of the 2009 academic year. 1 mark

c Use the state matrix from part **b** to predict the number of business students who will **defer** the 2009 academic year. 1 mark

CHAPTER

6

RELATIONSHIPS BETWEEN NUMERICAL VARIABLES

Study Design coverage

UNIT 2, AREA OF STUDY 1: DATA ANALYSIS, PROBABILITY AND STATISTICS

Investigating relationships between two numerical variables

- response and explanatory variables
- scatterplots and their use in identifying and qualitatively describing the association between two numerical variables in terms of direction, form and strength
- informal interpretation of association and causation
- use of a line of good fit by eye to make predictions, including the issues of interpolation and extrapolation
- interpretation of a line of good fit, its intercept and slope in the context of the data.

Video playlists (4):

6.1 Explanatory and response variables
6.2 Scatterplots
6.3 Lines of good fit
VCE question analysis Relationships between numerical variables

Worksheets (6):

6.2 Scatterplots and association • A page of scatterplots • Correlations matching game
6.3 Significant figures • A page of scatterplots • Height vs shoe size

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

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6.1 Explanatory and response variables

6.1

Dealing with two numerical variables

Most of the statistical examples we've looked at so far involved only one variable. The examples involving back-to-back stem plots and parallel boxplots explored the association or relationship between a numerical variable and a categorical variable. In this chapter, we will be exploring associations between two numerical variables.

Here are some examples of questions that involve analysing the association between two numerical variables.

- Is there an association between age and the amount of time spent sleeping?
- Can the number of social media friends be used to predict the number of real-life friends?
- Can the length of common colds be explained by the amount of vitamin C a person takes?
- What is the relationship between the number of books in a home and NAPLAN literacy results?

Video playlist Explanatory and response variables

Identifying explanatory and response variables

It's important to decide which of the variables is the **explanatory variable** and which is the **response variable**. An explanatory variable is a variable that we expect to predict or explain the changes observed in another variable, which is called a response variable.

In some situations, it's clear which of the two is the explanatory variable. For example, if researchers wish to investigate the association between age and the amount of time spent sleeping, they look at how a person's age can affect the amount of time a person spends sleeping (not how the amount of time spent sleeping can affect a person's age). So, *age* is the explanatory variable and the *amount of time spent sleep* is the response variable.

Exam hack

When deciding which is the explanatory variable, look for the words 'explain changes' or 'predict' in the question. If they don't appear, think about which of the variables is the most likely to affect the other variable.

WORKED EXAMPLE 1 Identifying explanatory and response variables

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For each of the following, identify the explanatory variable, giving a reason for your answer.

a Can the number of social media friends be used to predict the number of real-life friends?

b Can the length of a common cold be explained by the amount of vitamin C a person takes?

c What is the relationship between the number of books in a home and NAPLAN literacy results?

Steps	Working
a **1** Identify the two variables.	*number of social media friends* and *number of real-life friends*
2 Do the words 'explain changes' or 'predict' appear in the question? If not, which variable is most likely to affect the other?	*Number of social media friends* is being used to predict the other variable, so it is the explanatory variable.
b **1** Identify the two variables.	*length of a common cold* and *amount of vitamin C*
2 Do the words 'explain changes' or 'predict' appear in the question? If not, which variable is most likely to affect the other?	*Amount of vitamin C* is being used to explain changes to the other variable, so it is the explanatory variable.

c	**1** Identify the two variables.	*number of books in a home* and *NAPLAN literacy results*
	2 Do the words 'explain changes' or 'predict' appear in the question? If not, which variable is most likely to affect the other?	NAPLAN literacy results are not likely to affect the number of books in a home, but the number of books in a home may affect NAPLAN literacy results. So, *number of books in a home* is the explanatory variable.

EXERCISE 6.1 Explanatory and response variables

ANSWERS p. 509

Mastery

1 WORKED EXAMPLE 1 For each of the following, identify the explanatory variable, giving a reason for your answer.

- **a** Can the length of time a person sleeps be predicted by the amount of coffee they drink?
- **b** Can an AFL team's position on the ladder explain changes in the crowd size at an AFL match?
- **c** Is there an association between the score in a mini-golf tournament and the age of the person?
- **d** What is the relationship between the price of a house and the number of bedrooms in the house?
- **e** Is there an association between the amount of alcohol consumed and the number of car accidents?

2 For each of the following, identify the response variable.

- **a** A criminologist analysed data on the number of crimes committed per month in a big city and the number of police officers patrolling the city.
- **b** A study was undertaken to see whether a particular species of Australian cricket chirps faster at night if the temperature is higher.

Exam practice

80–100% 60–79% 0–59%

3 A study is conducted to investigate the relationship between the monthly water cost of a household and the number of teenagers in the household. The response variable is

- **A** nominal.
- **B** categorical.
- **C** the *household cost.*
- **D** the *monthly water cost.*
- **E** the *number of teenagers in the household.*

4 Research is undertaken to see whether the amount of study time the day before a Year 11 mathematics test can predict test scores. Which of the following statements is **not** true?

- **A** The response variable is *test score*.
- **B** The explanatory variable is *amount of study the day before test*.
- **C** Both variables are ordinal.
- **D** The response variable is numerical.
- **E** The explanatory variable is numerical.

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5 **a** © VCAA 2008 2CQ4 MODIFIED The arm spans (in cm) and heights (in cm) for a group of 13 boys have been measured. The aim is to find whether arm span can be predicted from height. Name the explanatory variable.

b **i** © VCAA 2019 2CQ4a MODIFIED 84% The relative humidity (%) at 9 am and 3 pm on 14 days in November 2017 was measured with the aim of predicting the relative humidity at 3 pm (humidity 3 pm) from the relative humidity at 9 am (humidity 9 am). Name the explanatory variable.

ii Would it make sense to undertake a study where the other variable was the explanatory variable? Give a reason for your answer.

6.2 Scatterplots

Constructing scatterplots

A **scatterplot** can be used to investigate the association between two numerical variables. Scatterplots are constructed by plotting points on a Cartesian plane, where the horizontal or x-axis is used for the explanatory variable and the vertical or y-axis is used for the response variable.

Video playlist
Scatterplots

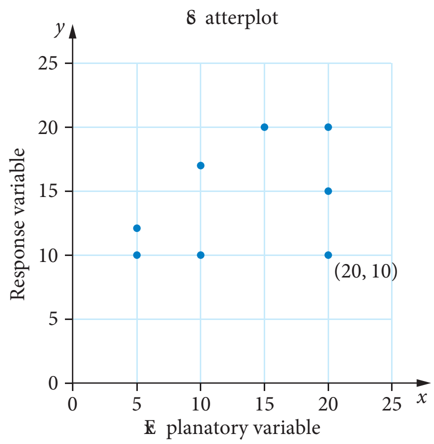
Explanatory variable (x)	5	5	10	10	15	20	20	20
Response variable (y)	10	12	10	17	20	10	15	20

To construct a scatterplot from a table:

1 decide which variable is the explanatory variable (x) and which is the response variable (y)

2 set up an appropriate scale for each axis and label them clearly with the variable names

3 plot each pair of data points from the table using a dot (•).



Exam hack

To remember which axis is used for which variable, use the fact that 'explanatory' starts with 'ex' for x-axis.

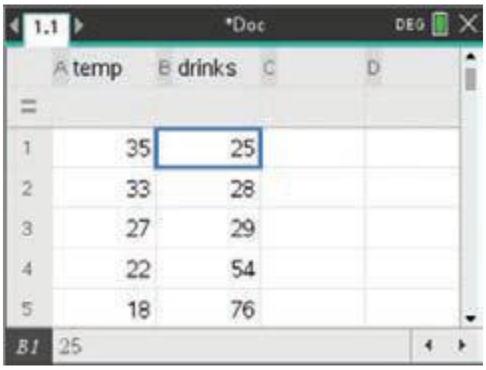
USING CAS 1 Constructing a scatterplot

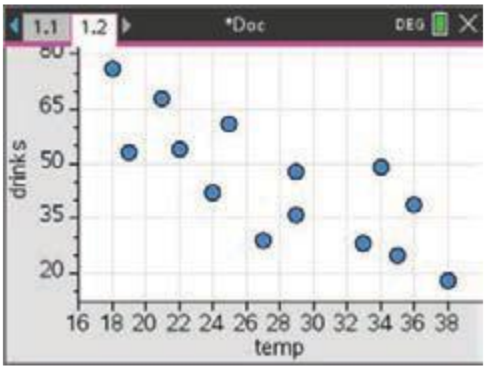
The number of hot drinks sold at a milk bar per day and the day's maximum temperature were recorded over a period of two weeks.

Temperature	35	33	27	22	18	29	38	36	24	25	29	34	21	19
No. of drinks sold	25	28	29	54	76	48	18	39	42	61	36	49	68	53

Construct a scatterplot for this information.

TI-Nspire





1 Start a new document and add a **Lists & Spreadsheet** page.

2 Label column **A** as **temp** and column **B** as **drinks**.

3 Enter the data from the table, as shown above.

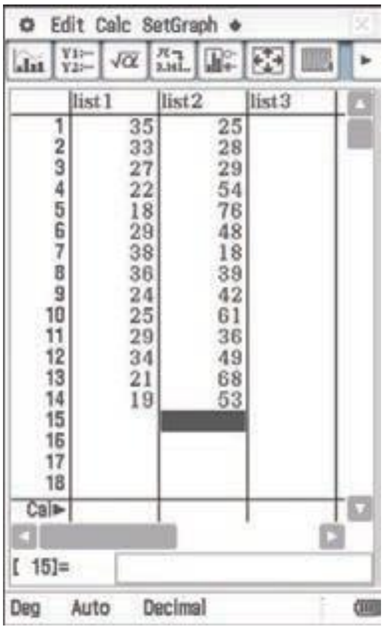
4 Insert a **Data & Statistics** page.

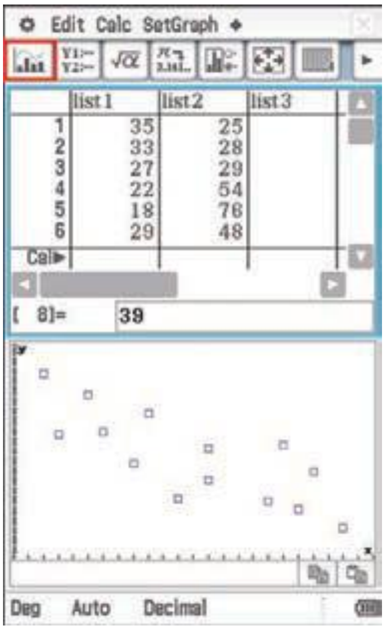
5 For the horizontal axis, select **temp**.

6 For the vertical axis, select **drinks**.

7 A scatterplot of the data will be displayed.

ClassPad





1 Tap **Menu** and open the **Statistics** application.

2 Clear all lists.

3 Enter the data from the table into **list1** and **list2**, as shown above.

4 Tap **Graph***. A scatterplot of the data will be displayed in the lower window.

*There is no need to change **SetGraph** as the default settings are for a **Scatterplot** of **list1** and **list2**.

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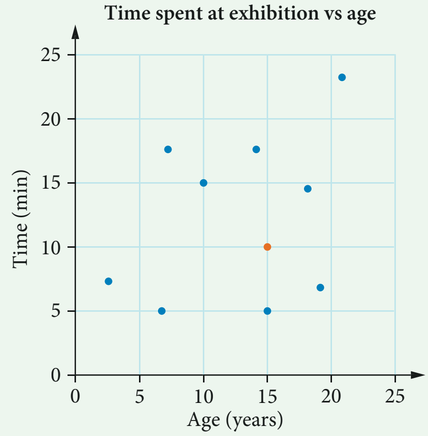
Interpreting scatterplots

To interpret a scatterplot, we need to understand what each of the variables represents and how it's being measured.

6.2

WORKED EXAMPLE 2 Interpreting scatterplots

A study was conducted on how long people of various ages spent at an exhibition at the Melbourne Aquarium and a scatterplot was plotted of the data.



Steps	Working
a What is the explanatory variable?	
The explanatory variable appears on the horizontal axis.	*age* (years)
b What is the response variable?	
The response variable appears on the vertical axis.	*time* (min)
c How many people were in the study?	
Count the number of dots.	10 people
d What does the orange dot represent?	
Read from both axes.	a 15-year-old who was at the exhibition for 10 minutes
e How many teenagers were in the study?	
Read from the horizontal axis and count the dots between 13 and 20.	5
f What was the shortest time spent at the exhibition?	
Read from the vertical axis.	5 minutes

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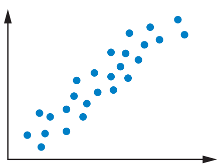
Scatterplots and association

There are three ways that scatterplots can be used to describe the association between two numerical variables.

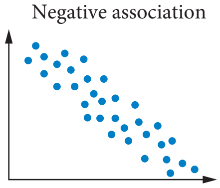
Worksheets
Scatterplots and association
A page of scatterplots
Correlations matching game

1 Direction

Positive association



Negative association

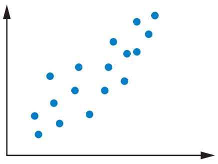


The direction of a positive association *rises* from left to right. This indicates that the y (response) variable tends to *increase* as the x (explanatory) variable increases.

The direction of a negative association *falls* from left to right. This indicates that the y (response) variable tends to *decrease* as the x (explanatory) variable increases.

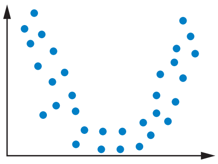
2 Form

Linear



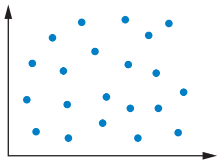
Data in general follows a straight line pattern.

Non-linear



Data does not occur in a straight line pattern but does follow a curved pattern.

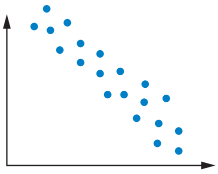
No association



Data is randomly scattered and shows no pattern at all.

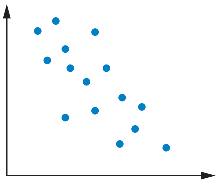
3 Strength

& ong association



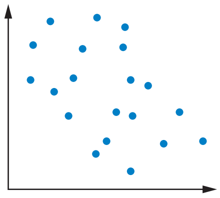
Data points are all reasonably close together.

Moderate association



Data points are more spread out than a strong association.

Weak association



Data points are widely spread out.

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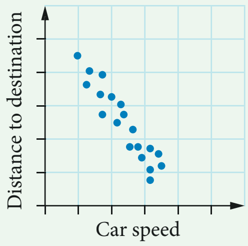
WORKED EXAMPLE 3 Scatterplots and association

For each of the following scatterplots

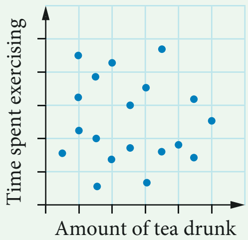
i describe the association between the two variables in terms of direction, form and strength

ii explain what this means in terms of the variables.

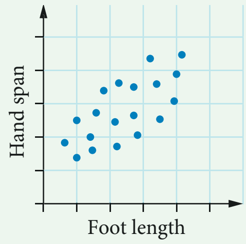
a



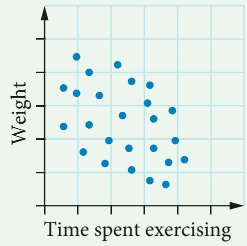
b



c



d



Steps	Working
a i Is the data sloping up or down? Does the data follow a straight line pattern? How spread out are the data points?	negative, linear and strong
ii Refer to the variables.	The distance to a destination decreases as car speed increases.
b i Is the data sloping up or down? Does the data follow a straight line pattern? How spread out are the data points?	no association
ii Refer to the variables.	There appears to be no association between amount of tea drunk and the time spent exercising.
c i Is the data sloping up or down? Does the data follow a straight line pattern? How spread out are the data points?	positive, linear and moderate
ii Refer to the variables.	Hand span tends to increase as foot length increases.

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d **i** Is the data sloping up or down? Does the data follow a straight line pattern? How spread out are the data points?	negative, linear and weak
ii Refer to the variables.	There is some evidence to suggest that weight decreases as time spent exercising increases.

6.2

Association and causation

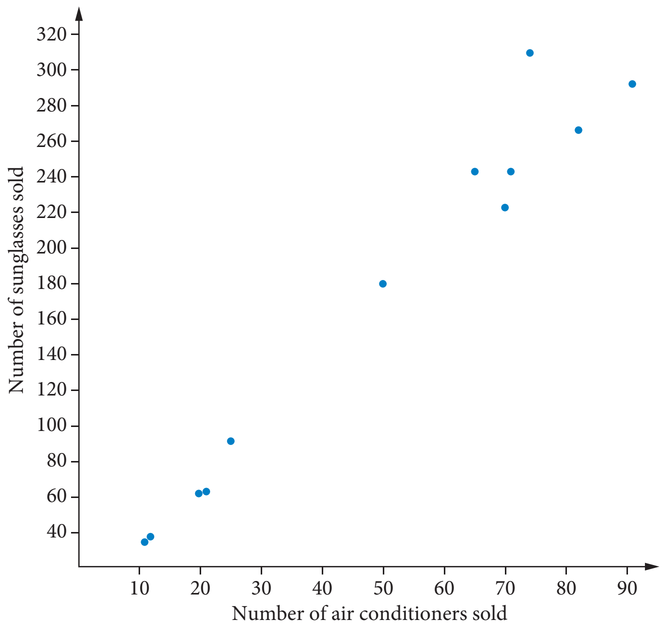
If two variables have an association, it doesn't necessarily mean that one *causes* the other. Association is not the same as **causation**. For example, a store recorded the following monthly sales of air conditioners and sunglasses over a year.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Number of air conditioners sold	74	65	50	25	21	20	11	12	70	71	82	91
Number of sunglasses sold	310	243	180	91	63	62	35	37	223	243	266	292

These sales numbers give the following scatterplot.

The scatterplot indicates that there is a strong positive association between the variables *number of air conditioners sold* and *number of sunglasses sold*. However, clearly neither one of these variables is causing the other. What's more likely is that another variable, *outdoor temperature*, is contributing to the changes in both of them.

It is relatively easy to show an association between two variables. It's much harder to show that one variable *causes* a change in another variable.



WORKED EXAMPLE 4 Exploring causation

WB p. 99

For each of the following associations between pairs of variables, suggest another variable that could be the underlying cause of the association between the two.

Steps	**Working**
a A positive association between the *number of mobile phones sold in Melbourne* and the *amount spent on Thai takeaway in Melbourne*	
Which variable might be causing changes in both?	*Population* changes could be the underlying cause of the association between the two.
b A negative association between *ski jackets sold* ($) and *fans sold* ($).	
Which variable might be causing changes in both?	*Outdoor temperature* changes could be the underlying cause of the association between the two.
c A positive association between the *height* of children and *mathematical ability*.	
Which variable might be causing changes in both?	*Age* could be the underlying cause of the association between the two.

EXERCISE 6.2 Scatterplots

ANSWERS p. 509

Recap

1 A study is undertaken to investigate the association between the amount of chocolate eaten before bed and the length of nightly sleep. Which of the following statements is true?

A *Amount of chocolate eaten* is the response variable.

B The study is investigating whether *length of nightly sleep* can explain changes in *amount of chocolate eaten*.

C *Length of nightly sleep* is the explanatory variable.

D *Amount of chocolate eaten* is the explanatory variable.

E The study is predicting the *amount of chocolate eaten* from the *length of nightly sleep*.

2 A consumer group conducts research to determine whether the price of a laptop is affected by its weight. The weight of 50 laptops is measured and each weight is recorded next to their price on a spreadsheet. The explanatory and response variables are, respectively

A the 50 *laptops* and their *prices*.

B the *price of the laptop* and the *weight of the laptop*.

C the *weight of the laptop* and the *price of the laptop*.

D the *prices of the laptops* and the *spreadsheet*.

E the *consumer group* and the *study*.

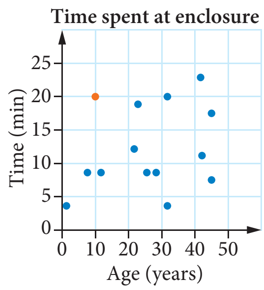
Mastery

3 Using CAS 1 The number of customers in a store over a two-week period was recorded and the results are shown below.

Time (days)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Number of customers	76	92	63	79	81	64	102	71	86	119	86	78	111	92

Construct a scatterplot for this information.

4 WORKED EXAMPLE 2 A study was made on how long people of various ages spent at an enclosure at the zoo and a scatterplot was plotted of the data.



a What is the explanatory variable?

b What is the response variable?

c How many people were in the study?

d What does the orange dot represent?

e How many people in their 20s were in the study?

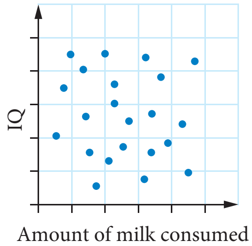
f Why could we reasonably conclude that the two people who stayed for the shortest time were together?

5 WORKED EXAMPLE 3 For each of the following scatterplots

i describe the association between the two variables in terms of direction, form and strength

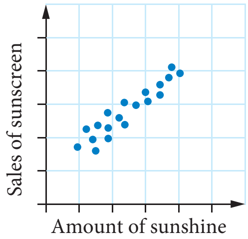
ii explain what this means in terms of the variables.

a

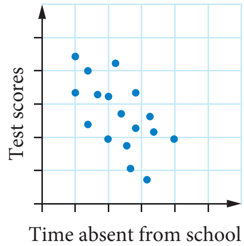


b

c



d



6 For each pair of variables, state whether they would generally have a positive association, a negative association or no association.

a number of chocolate bars eaten and how many pets a person owns

b kilojoules consumed and weight gained

c alcohol consumption and reaction time

d height and the number of hours sleep a person has per night

e temperature and number of people at a beach

f time spent travelling home and distance from home

7 WORKED EXAMPLE 4 For each of the following associations between pairs of variables, suggest another variable that could be the underlying cause of the association between the two.

a A negative association between the *number of people injured skiing* and the *number of swimsuits sold*.

b A positive association between the *total consumption of avocados in Victoria* (kg) and the *number of VCE students*.

c A positive association between the *amount of money spent going to the movies in a household* ($) and the *number of mobile phones in a household*.

d A positive association between the *number of ambulances in a state* and the *number of teachers*.

e A positive association between *hand-eye coordination in children* and *handspan*.

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2016 1CQ12 67% There is a strong positive association between a country's Human Development Index and its carbon dioxide emissions. From this information, it can be concluded that

A increasing a country's carbon dioxide emissions will increase the Human Development Index of the country.

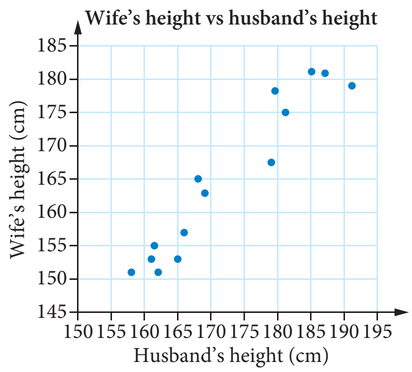
B decreasing a country's carbon dioxide emissions will increase the Human Development Index of the country.

C this association must be a chance occurrence and can be safely ignored.

D countries that have higher human development indices tend to have higher levels of carbon dioxide emissions.

E countries that have higher human development indices tend to have lower levels of carbon dioxide emissions.

9 The scatterplot shows that



A there is no association between a husband's height and a wife's height.

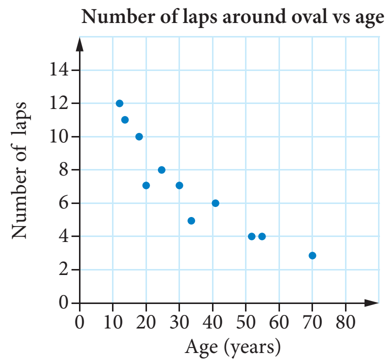
B there is a strong negative linear association between a husband's height and a wife's height.

C there is a non-linear association between a husband's height and a wife's height.

D there is a moderate positive linear association between a husband's height and a wife's height.

E there is a strong positive linear association between a husband's height and a wife's height.

10 Eleven people of different ages ran around an oval for 30 minutes. The number of laps they ran was recorded and the results are displayed in the scatterplot.



Which one of the following statements is **false**?

A The youngest person in the group ran the most laps.

B *Age* is the explanatory variable and *number of laps* is the response variable.

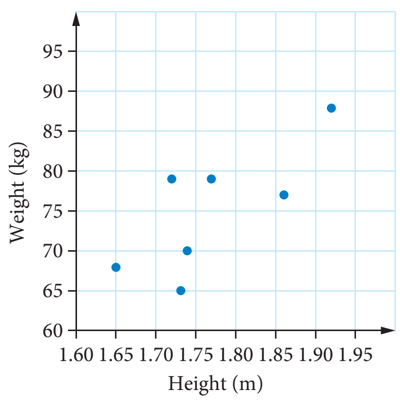
C The association between *age* and *number of laps* can be described as strong.

D The association between *age* and *number of laps* can be described as positive.

E The association between *age* and *number of laps* can be described as non-linear.

11 © VCAA 2004 2CQ1a MODIFIED (2 marks) The table gives the heights (m) and weight (kg) of a sample of nine people. On the scatterplot, the points representing the data for seven of these people has been plotted with *height* on the horizontal axis and *weight* on the vertical axis.

Height (m)	Weight (kg)
1.65	68
1.68	**63**
1.72	79
1.73	65
1.74	70
1.77	79
1.78	**81**
1.86	77
1.92	88



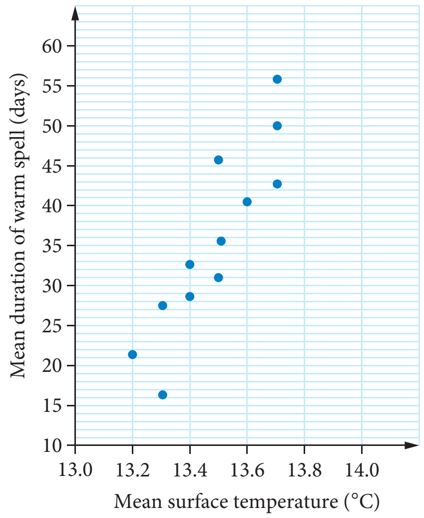
a 92% Copy the scatterplot and plot points representing the data for the remaining two people (shown in **bold** in the table) on it. 1 mark

b If these nine people were teenagers, what other variable could be the underlying cause of the association between *height* and *weight*? 1 mark

9780170448192

12 © VCAA 2007 2CQ3a MODIFIED (2 marks) The table displays the *mean surface temperature* (in °C) and the *mean duration of warm spell* (in days) in Australia for 13 years selected at random from the period 1960 to 2005. This data set has been used to construct the scatterplot.

Mean surface temperature (°C)	Mean duration of warm spell (days)
13.2	21.4
13.3	16.3
13.3	27.6
13.4	32.6
13.4	28.7
13.5	30.9
13.5	45.9
13.5	35.5
13.6	40.6
13.7	42.8
13.7	49.9
13.7	55.8
13.8	**53.1**



a 86% The scatterplot is incomplete. Copy and complete the scatterplot by plotting the **bold** data value given in the table. Mark the point with a cross (×). 1 mark

b What other variable could be the underlying cause of the association between *mean surface temperature* and *mean duration of warm spell*? 1 mark

13 (3 marks) The *annual rainfall* and *number of bushfires* in a region were recorded over 8 years and the results are shown.

Rainfall (cm)	16	14	19	28	22	24	23	20
No. of bushfires	20	32	26	10	18	11	9	21

a Sketch a scatterplot for this information. 1 mark

b Interpret the scatterplot in terms of direction, form and strength. 1 mark

c What other variable could be the underlying cause of the association between *annual rainfall* and *number of bushfires*? 1 mark

14 (3 marks) People were asked how many years of education they had (including primary and high school as well as university and TAFE etc.) and asked how many children they had. The results are shown below.

Years of education	15	18	13	13	12	17	17	18	16	10	19	19	18	22	16	16
Number of children	2	4	6	3	1	2	1	2	0	7	1	3	4	2	1	2

a Draw a scatterplot for this information. 1 mark

b Describe the association shown in terms of form, direction and strength. 1 mark

c Write a brief sentence to describe the association between the two variables. 1 mark

Video playlist
Lines of good fit

Worksheets
Significant figures

A page of scatterplots

Height vs shoe size

6.3 Lines of good fit

Rounding to decimal places

Dealing with data often involves giving approximate rather than exact answers, so **rounding** is important. Rounding involves replacing a number with an approximate value. We have already come across rounding to decimal places. For example:

28.3597 rounded to the nearest tens is 30
rounded to the nearest units (or whole number) is 28
rounded to one decimal place is 28.4
rounded to two decimal places is 28.36
rounded to three decimal places is 28.360
rounded to four decimal places is 28.3597
rounded to five decimal places is 28.359 70

Exam hack

Don't round off too early! If you are working with CAS, keep all the decimal places in the calculation and don't round off to the stated number of decimal places until the very end.

Rounding to decimal places

When rounding to a number of decimal places

- look at the next digit, and round down if it is 0–4 and round up if it is 5–9
- if 9s need to be rounded up, round the 9 to 0 and carry the rounding over to the next digit on the left
- include trailing zeros in decimals, if necessary.

Rounding to significant figures

When dealing with data, it often makes sense to round to **significant figures** rather than to decimal places. In the following examples, the significant figures are in red.

Example	No. of significant figures	What type of digits are significant?	What type of digits are not significant?
29.418	5	non-zero digits	
530.0027	7	non-zero digits and zeros between non-zero digits.	
4.650	4	non-zero digits and trailing zeros in decimals	
0.0065	2		leading zeros in decimals
800	1		trailing zeros in whole number

When rounding to significant figures, use the same rounding rules as for rounding to a number of decimal places:

- '0–4 round down' and '5–9 round up'

  e.g. 27 501 rounded to two significant figures is 28 000
- round the 9 to 0 and carry the rounding over to the next digit on the left

  e.g. 3.9722 rounded to two significant figures is 4.0
  497 rounded to two significant figures is 500
- include trailing zeros in decimals, if necessary

  e.g. 4 rounded to two significant figures is 4.0

Exam hack

Don't confuse 'rounding to significant figures' with the'number of significant figures'. For example, 497 rounded to two significant figures is 500, but 500 has one significant figure.

Significant figures

Significant figures:

- any non-zero digit
- zeros between non-zero digits
- trailing zeros in decimals.

Not significant figures:

- leading zeros in decimals
- trailing zeros in whole numbers.

When rounding to significant figures, use usual rounding rules:

- '0–4 round down' and '5–9 round up'
- round the 9 to 0 and carry the rounding over to the next digit on the left
- include trailing zeros in decimals, if necessary.

Exam hack

Note that the two methods of rounding can give very different results.
10 618.028 rounded to two decimal places is 10 618.03.
10 618.028 rounded to two significant figures is 11 000.

WORKED EXAMPLE 5 Rounding to decimal places versus significant figures

WB p. 100

Round each number to

i two decimal places **ii** two significant figures

a 2.414 **b** 7.368 **c** 0.5561 **d** 34 700
e 15.881 **f** 95 008.037 **g** 34.8976

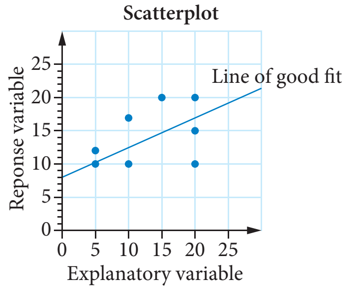
		Steps	Working
a	**i**	Focus on the first two decimal places.	2.414 rounded to two decimal places is 2.41.
	ii	Focus on the first two significant figures.	2.414 rounded to two significant figures is 2.4.
b	**i**	Focus on the first two decimal places.	7.368 rounded to two decimal places is 7.37.
	ii	Focus on the first two significant figures.	7.368 rounded to two significant figures is 7.4.
c	**i**	Focus on the first two decimal places.	0.5561 rounded to two decimal places is 0.56.
	ii	Focus on the first two significant figures.	0.5561 rounded to two significant figures is 0.56.
d	**i**	Focus on the first two decimal places.	34 700 rounded to two decimal places is 34 700.00.
	ii	Focus on the first two significant figures.	34 700 rounded to two significant figures is 35 000.
e	**i**	Focus on the first two decimal places.	15.881 rounded to two decimal places is 15.88.
	ii	Focus on the first two significant figures.	15.881 rounded to two significant figures is 16.
f	**i**	Focus on the first two decimal places.	95 008.037 rounded to two decimal places is 95 008.04.
	ii	Focus on the first two significant figures.	95 008.037 rounded to two significant figures is 95 000.
g	**i**	Focus on the first two decimal places.	34.8976 rounded to two decimal places is 34.90.
	ii	Focus on the first two significant figures.	34.8976 rounded to two significant figures is 35.

Finding a line of good fit

A **line of good fit** is a straight line that represents the data on a scatterplot and helps us to make predictions. We find a line of good fit 'by eye', which means finding the straight line on a scatterplot that

- is close to all the points
- has approximately as many points above the line as below
- may pass through some of the points, none of the points, or all of the points.

We can find the equation of the line of good fit by finding the intercept on the vertical axis and calculating the slope. Always select points that are easy to read from the graph.



Exam hack

The best way to draw a line of good fit is to use a clear plastic ruler.

Line of good fit equation

The equation of the line of good fit is

$$\textit{response variable} = a + b \times \textit{explanatory variable}$$

where

a = the intercept of the line on the vertical axis

$b = \frac{\textbf{rise}}{\textbf{run}}$ = the slope of the line

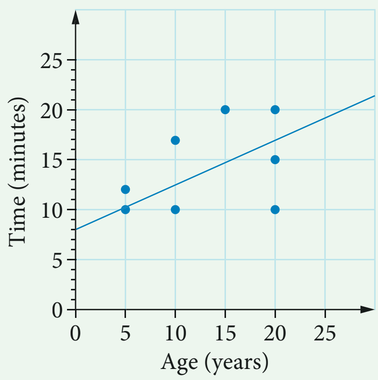
Making predictions

We can use the equation of the line of good fit to make predictions by substituting values into it and solving. The equation can be used to predict *within* the original data range, which is called **interpolation**, as well as *outside* the original data range, which is called **extrapolation**. Predictions based on extrapolation are not as reliable as those based on interpolation because we cannot be certain that the equation applies to values outside the range of the data values we have.

WB
p. 101

WORKED EXAMPLE 6 Working with the line of good fit equation

A line of good fit has been drawn on the following scatterplot showing the time spent by people of various ages at a particular carnival game.



Steps	Working
a Find the intercept of the line.	
Where does the line touch the vertical axis?	intercept = 8

b Calculate the slope of the line, correct to two significant figures.	
1 Choose two easy-to-identify points on the line.	(0, 8) and (25, 19)
2 Calculate the slope from these two points, giving the answer to the required rounding.	$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{19 - 8}{25 - 0}$ $= \frac{11}{25}$ $= 0.44$ rounded to two significant figures
c What is the equation of the line of good fit?	
Use *response variable* = *a* + *b* × *explanatory variable*, where *a* is the intercept and *b* is the slope.	$time = 8 + 0.44 \times age$
d **i** Use the line to predict the number of minutes a 16-year-old would spend at the carnival game, rounding to two significant figures.	
Substitute the value into the equation in place of *age* and solve for *time*, giving the answer to the required rounding.	$time = 8 + 0.44 \times age$ $= 8 + 0.44 \times 16$ $= 15.04$ A 16-year-old would spend 15 minutes at the carnival game, rounded to two significant figures.
ii State whether this prediction involves interpolation or extrapolation, giving a reason.	
Was the value used within or outside the original data range?	16 is within the original data range of 5 to 20 years, so this involves interpolation.
e **i** Use the line to predict the number of minutes a 23-year-old would spend at the carnival game, rounding to two significant figures.	
Substitute the value into the equation in place of *age* and solve for *time*, giving the answer to the required rounding.	$time = 8 + 0.44 \times age$ $= 8 + 0.44 \times 23$ $= 18.12$ A 23-year-old would spend 18 minutes at the carnival game, rounded to two significant figures.
ii State whether this prediction involves interpolation or extrapolation, giving a reason.	
Was the value used within or outside the original data range?	23 is outside the original data range of 5 to 20 years, so this involves extrapolation.
f Which of the predictions in parts **d** and **e** is more reliable? Justify your answer.	
Decide which of the two is the more reliable prediction and justify your decision.	The prediction for the 16-year-old involves interpolation, so it is more reliable than the prediction for a 23-year-old, which involves extrapolation.

Interpreting a line of good fit

The equation of a line of good fit can help us make statements about the two variables involved.

Interpreting a line of good fit equation

We interpret the equation of a line of good fit

response variable = $a + b \times$ *explanatory variable*

by saying

- the intercept is a
  This means the ***response variable*** is a **units** when the ***explanatory variable*** is zero **units**.
- the slope is b
  This means on average the ***response variable*** increases/decreases by b **units** for every 1-**unit** increase in the ***explanatory variable***.

Use the word 'increases' when b is positive and 'decreases' when b is negative.

Replace the words in **bold** with the appropriate variable names or units of measure for each variable.

WB

p. 103

WORKED EXAMPLE 7 Interpreting a line of good fit equation

A study of the association between the age and price of a particular car model has resulted in the line of good fit equation

price = 35 000 − 3670 × *age*

where the price is measured in dollars and the age in years.

a Identify and interpret the intercept.

b Identify and interpret the slope.

Steps	Working
For the equation of the line of good fit *response variable* = $a + b \times$ *explanatory variable* where a = the intercept of the line b = the slope of the line	**a** intercept = 35 000 This is the *price* when the *age* = 0. This means that the car cost \$35 000 when it was new.
	b slope = −3670 This means on average the *price* of the car decreases by \$3670 for every 1-year increase in the *age*.

Exam hack

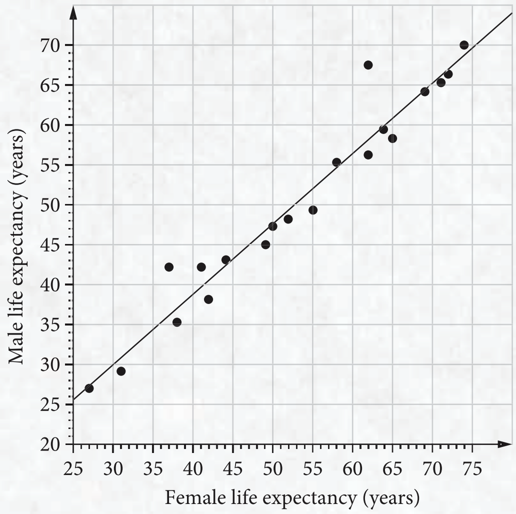
Make sure you always write the line of good fit equation using the variables names. You will lose a mark if you write it using x and y.

VCE QUESTION ANALYSIS

6.3

© VCAA 2015 2CQ3&4 MODIFIED 2015 Examination 2 Core Questions 3 & 4 (6 marks)

The scatterplot plots male life expectancy (*male*) against female life expectancy (*female*) in 1950 for a number of countries. A line of good fit has been fitted to the scatterplot as shown.



a Name the response variable. 1 mark

b Describe the association between *male* life expectancy and *female* life expectancy in terms of strength, direction and form. 1 mark

The slope of the line of good fit shown is 0.88.

c Interpret the slope in terms of the variables *male* life expectancy and *female* life expectancy. 1 mark

The equation of the line of good fit is

$$male = 3.6 + 0.88 \times female$$

In a particular country in 1950, *female* life expectancy was 35 years.

d Use the equation to predict *male* life expectancy for that country. 1 mark

> **Exam hack**
>
> If the question doesn't ask for rounding, don't round the answer.

e State whether this prediction involves interpolation or extrapolation, giving a reason that refers to a data range from the graph. 2 marks

Reading the question

- In part **b**, three things are required.
- Note that the slope of the line of good fit has been given so you don't need to find it from the graph.
- Part **e** is worth 2 marks, so you need to give a reason.

Thinking about the question

- Be clear on what 'interpreting' the slope means.
- You need to quote the data range in your answer for part **e** to get full marks.

Video playlist
VCE question analysis: Relationships between numerical variables

Worked solution (✓ = 1 mark)

a The label of the vertical axis gives the response variable. There is no need to include '(years)'.

***male* life expectancy** ✓

b Most of the data points on the scatterplot are close to the line of good fit, so it's a strong association.

The data is sloping up, so it's a positive association.

The line is straight not curved, so it's a linear association.

strong, positive, linear ✓

c When the slope b is positive, it means that the *response variable* on average increases by b units for every 1-unit increase in the *explanatory variable*.

Male life expectancy on average increased by 0.88 years for each 1-year increase in female life expectancy. ✓

d Substitute *female* = 35 into the equation *male* = 3.6 + 0.88 × *female*.

male = 3.6 + 0.88 × 35 = **34.4 years** ✓

e **35 is within the female life expectancy data range on the scatterplot from 27 to 74** ✓**, so this involves interpolation.** ✓

Student performance 80–100% 60–79% 0–59%

a Writing just *male* is not enough.

b 63% Many students did not state the form as linear.

c 40% This question was not well answered, with many students describing the scatterplot instead of interpreting the slope of the line as asked. The increase in *male* life expectancy needed to be related to 1-unit (year) increase in *female* life expectancy. Answers such as 'On average, *male* life expectancy increased by 0.88 years for every increase in *female* life expectancy' were not accepted.

d 78%

e Students need to include the data range values for full marks.

EXERCISE 6.3 Lines of good fit

ANSWERS p. 510

Recap

1 Consider the following scatterplot.

Which statement best describes the association between M and H?

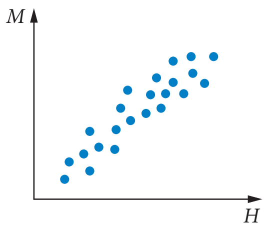
A no association

B non-linear association

C negative, linear and weak

D positive, linear and weak

E positive, linear and strong



2 © VCAA 2016S 1CQ12 A large study of secondary-school male students shows that there is a negative association between the time spent playing sport each week and the time spent playing computer games. From this information, it can be concluded that

A male students who spend a lot of time playing computer games do not play sport.

B encouraging male students to spend less time playing sport will increase the time they spend playing computer games.

C encouraging male students to spend more time playing sport will reduce the time they spend playing computer games.

D male students who spend more time playing sport tend to spend less time playing computer games.

E male students who tend to spend more time playing sport tend to spend more time playing computer games.

Mastery

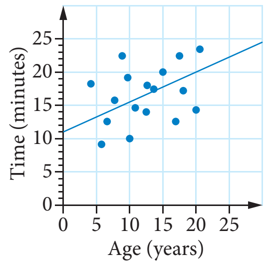
3 WORKED EXAMPLE 5 Round each number to

i two decimal places **ii** two significant figures.

a 8.532 **b** 43.948 **c** 0.251 **d** 46 000 **e** 82.687

f 14.0049 **g** 3.8974

4 WORKED EXAMPLE 6 A line of good fit has been drawn on the following scatterplot showing the time spent by people of various ages watching a busker.



a Find the intercept of the line.

b Calculate the slope of the line, correct to two significant figures.

c What is the equation of the line of good fit?

d **i** Use the line to predict the number of minutes a 14-year-old would spend watching the busker, rounding to two significant figures.

ii State whether this prediction involves interpolation or extrapolation, giving a reason.

e **i** Use the line to predict the number of minutes a 24-year-old would spend watching the busker, rounding to two significant figures.

ii State whether this prediction involves interpolation or extrapolation, giving a reason.

f Which of the predictions in parts **d** and **e** are more reliable? Justify your answer.

5 WORKED EXAMPLE 7 For each of the following lines of good fit

i identify and interpret the intercept. **ii** identify and interpret the slope.

a A study of the association between the age and value of a collectible figurine has resulted in the line of good fit equation

$$value = 240 + 90 \times age$$

where the value is measured in dollars and the age in years.

b A scatterplot showing the association between the age and price of a particular model of mobile phone has resulted in the line of good fit equation

$$price = 1850 - 300 \times age$$

where the price is measured in dollars and the age in years.

c A line of good fit fitted to a scatterplot showing the association between the age and value of a particular video game has the equation

$$value = 105 - 28 \times age$$

where the value is measured in dollars and the age in years.

Exam practice

80–100% 60–79% 0–59%

6 How many significant figures does 4060.0030 have?

A 2 **B** 3 **C** 4 **D** 7 **E** 8

7 A calculation gives an answer of 23.672 930 64. Which of the following is true?

A The answer rounded to three decimal places is 23.672.

B The answer rounded to three significant figures is 23.6.

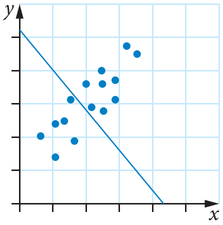
C The answer rounded to three significant figures is 23.0.

D The answer rounded to three decimal places is 23.6.

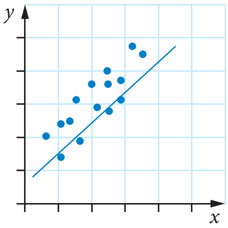
E The answer rounded to three decimal places is 23.673.

8 Which one of the following shows a line of good fit?

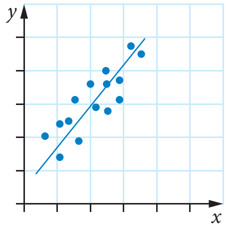
A



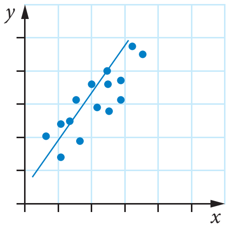
B



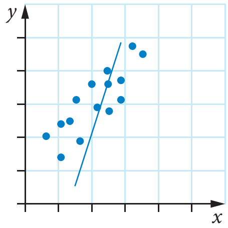
C



D



E



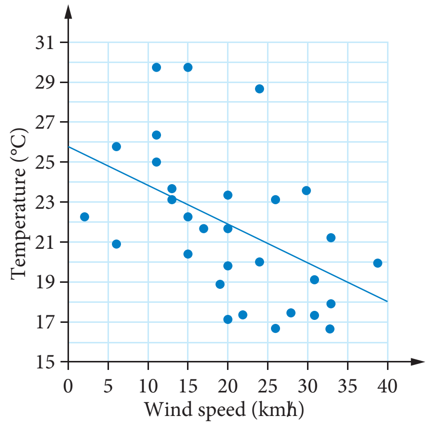
9 © VCAA 2003 1CQ8 MODIFIED 89% Eighteen students sat for a 15-question multiple-choice test. The number of errors made by each student on the test and the time they reported studying for the test was recorded. The equation for the line of good fit for the data is

number of errors = 8.8 – 0.120 × *study time*

Using the line of good fit, it can be estimated that, on average, a student reporting a study time of 35 minutes would make

A 4.3 errors. **B** 4.6 errors. **C** 4.8 errors. **D** 5.0 errors. **E** 13.0 errors.

10 © VCAA 2012 1CQ8 73% The maximum wind speed and maximum temperature were recorded each day for a month. The data is displayed in the scatterplot and a line of good fit has been fitted. The response variable is *temperature*. The explanatory variable is *wind speed*. The equation of the line of good fit is closest to



A *temperature* = 25.7 – 0.191 × *wind speed*

B *wind speed* = 25.7 – 0.191 × *temperature*

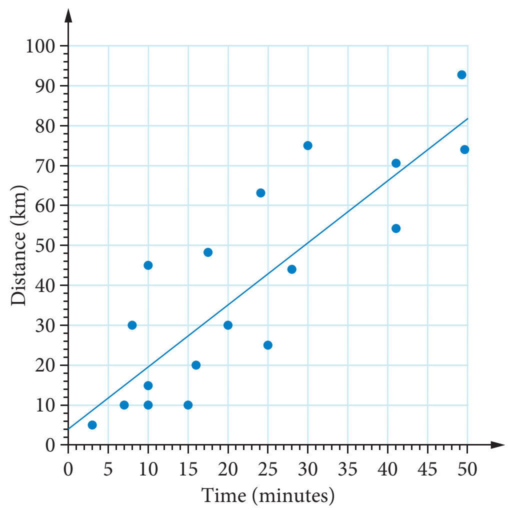
C *temperature* = 0.191 + 25.7 × *wind speed*

D *wind speed* = 25.7 + 0.191 × *temperature*

E *temperature* = 25.7 + 0.191 × *wind speed*

11 © VCAA 2015 1CQ9 62% A line of good fit has been fitted to the scatterplot to enable *distance*, in kilometres, to be predicted from *time*, in minutes.

The equation of this line is closest to



A *distance* = 3.5 + 1.6 × *time*

B *time* = 3.5 + 1.6 × *distance*

C *distance* = 1.6 + 3.5 × *time*

D *time* = 1.8 + 3.5 × *distance*

E *distance* = 3.5 + 1.8 × *time*

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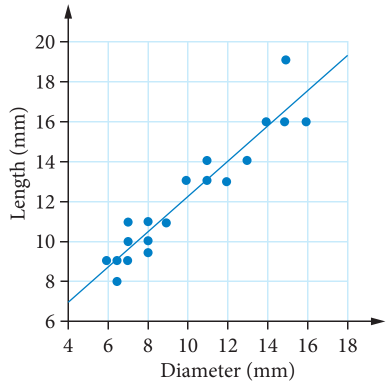
Use the following information to answer the next two questions.

The lengths and diameters (in mm) of a sample of jellyfish were recorded and displayed in the scatterplot. The line of good fit for this data is shown.

The equation of the line of good fit is

length = 3.5 + 0.87 × *diameter*



6.3

12 © VCAA 2007 1CQ8 MODIFIED **51%** From the equation of the line of good fit, it can be concluded that for these jellyfish, on average

A there is a 3.5 mm increase in *diameter* for each 1 mm increase in *length*.

B there is a 3.5 mm increase in *length* for each 1 mm increase in *diameter*.

C there is a 0.87 mm increase in *diameter* for each 1 mm increase in *length*.

D there is a 0.87 mm increase in *length* for each 1 mm increase in *diameter*.

E there is a 4.37 mm increase in *diameter* for each 1 mm increase in *length*.

13 The intercept of the line of good fit is

A 0.87 mm **B** 3.5 mm **C** 4 mm **D** 6 mm **E** 7 mm

14 © VCAA 2018 1CQ10 MODIFIED **51%** In a study of the association between a person's *height*, in centimetres, and *body surface area*, in square metres, the following line of good fit was obtained.

body surface area = −1.1 + 0.019 × *height*

Which one of the following is a conclusion that can be made from this line of good fit?

A An increase of 1 m^2 in *body surface area* is associated with an increase of 0.019 cm in *height*.

B An increase of 1 cm in *height* is associated with an increase of 0.019 m^2 in *body surface area*.

C There is no association between a person's *height* and *body surface area*.

D A person's *body surface area*, in square metres, can be determined by adding 1.1 cm to their *height*.

E A person's *height*, in centimetres, can be determined by subtracting 1.1 from their *body surface area*, in square metres.

15 © VCAA 2005 1CQ9 **47%** The equation of a line of good fit for the lengths (in metres) and wingspans (in metres) of eight commercial aeroplanes is

wingspan = −2.99 + 0.96 × *length*

From this equation it can be concluded that, on average, for these aeroplanes, wingspan

A decreases by 2.03 metres with each one metre increase in length.

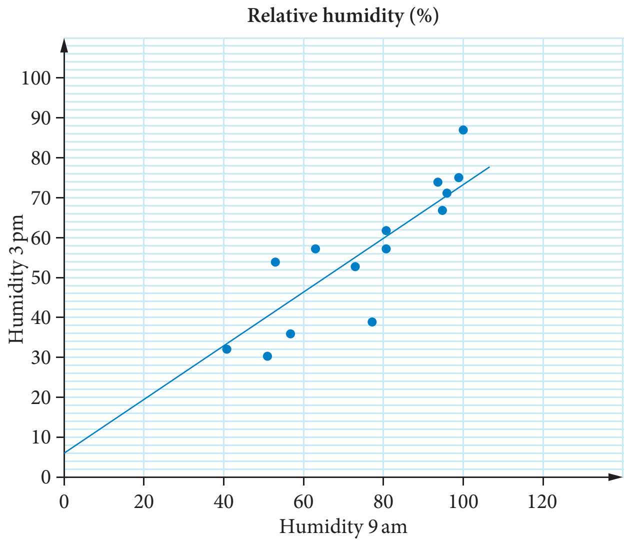
B increases by 0.96 metres with each one metre increase in length.

C decreases by 0.96 metres with each one metre increase in length.

D increases by 2.99 metres with each one metre increase in length.

E decreases by 2.99 metres with each one metre increase in length.

16 © VCAA 2019 2CQ4 MODIFIED (9 marks) The relative humidity (%) at 9 am and 3 pm on 14 days in November 2017 is shown in the scatterplot with a line of good fit.



Data: Australian Government, Bureau of Meteorology, www.bom.gov.au

a Name the explanatory variable. 1 mark

b Determine the value of the intercept of the line of good fit and interpret it in the context of the data. 2 marks

c Determine the value of the slope of the line of good fit and interpret it in the context of the data. 2 marks

d Determine the equation of the line of good fit, rounding the values to two significant figures. 2 marks

e Use the line of good fit equation to predict the humidity at 3 pm if the humidity at 9 am was 33% and 62%, respectively, to two significant figures. Explain which of these predictions is more reliable, giving a reason. 2 marks

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6 Chapter summary

Explanatory and response variables

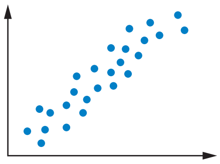
- An **explanatory variable** is a variable that we expect to predict or explain the changes observed in another variable, which is called a **response variable**.
- If the words 'explain changes' or 'predict' don't appear in the question, decide which of the two variables is the most likely to affect the other variable.

Scatterplots

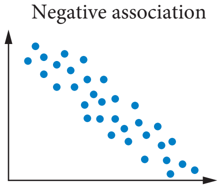
There are three ways **scatterplots** can be used to describe the association between two numerical variables:

1 Direction

Positive association

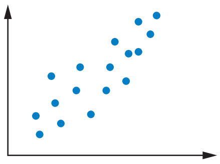


Negative association

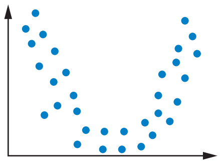


2 Form

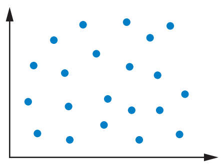
Linear



Non-linear

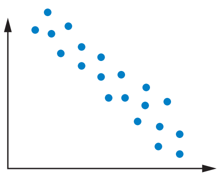


No association

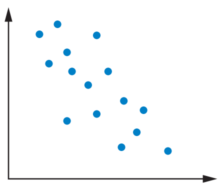


3 Strength

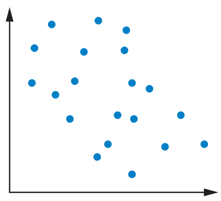
Strong association



Moderate association



Weak association



Association and causation

- Just because two variables have a strong association, it doesn't necessarily mean that one *causes* the other.
- It is relatively easy to show an association between two variables. It's much harder to show that one variable *causes* change in another variable.

Rounding

Rounding rules

- '0–4 round down' and '5–9 round up'
- round the 9 to 0 and carry the rounding over to the next digit on the left
- include trailing zeros in decimals, if necessary.

Significant figures

Significant figures:

- any non-zero digit
- zeros between non-zero digits
- trailing zeros in decimals.

Not significant figures:

- leading zeros in decimals
- trailing zeros in whole numbers.

Line of good fit

A **line of good fit** is a straight line that represents the data on a scatterplot and helps us make predictions. We find a line of good fit 'by eye' that

- is close to all the points
- has approximately as many points above the line as below
- may pass through some of the points, none of the points, or all of the points.

The equation of the line of good fit is

response variable = $a + b \times$ *explanatory variable*

where

a = the intercept of the line on the vertical axis

$b = \frac{\text{rise}}{\text{run}}$ = the slope of the line.

Making predictions

- We predict values by using the equation of the line of good fit and substituting values into it and solving.
- Predicting *within* the original data range is called **interpolation**.
- Predicting *outside* the original data range is called **extrapolation**.
- Predictions based on extrapolation are not as reliable as those based on interpolation.

Interpreting a line of good fit equation

We interpret the equation of a line of good fit *response variable* = $a + b \times$ *explanatory variable* by saying:

- the intercept is a
  This means the **response variable** is a **units** when the **explanatory variable** is zero **units**.
- the slope is b
  This means on average the **response variable** increases/decreases by b units for every 1-**unit** increase in the **explanatory variable**.

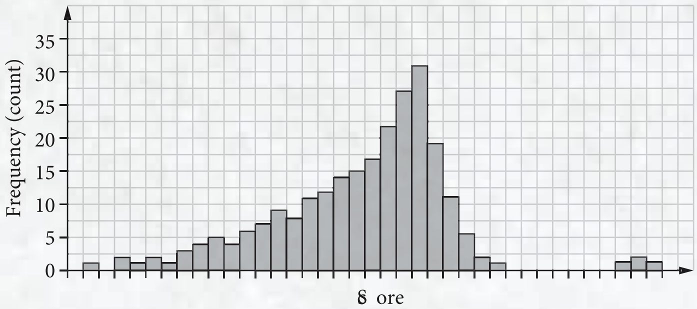
Use the word 'increases' when b is positive and 'decreases' when b is negative.

Replace the words in bold with the appropriate variable names or units of measure for each variable.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 23 minutes

1 © VCAA 2005 1CQ3 The histogram shown is best described as



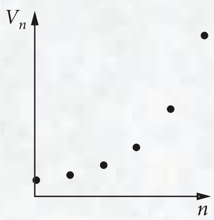
A negatively skewed.
B positively skewed.
C symmetric.
D negatively skewed with outliers.
E positively skewed with outliers.

2 The sequence generated by the recurrence relation $u_0 = 10$, $u_{n+1} = u_n - 3$ is

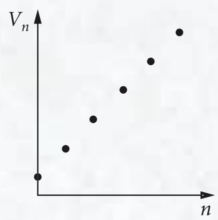
A a decreasing geometric sequence.
B an increasing geometric sequence.
C a decreasing arithmetic sequence.
D an increasing arithmetic sequence.
E neither arithmetic nor geometric.

3 A machine is purchased for \$50 000 and is depreciated on a reducing balance basis at a rate of 12% per annum. Which of the following would best match the shape of the graph of n against V_n, the value of the machine after n years?

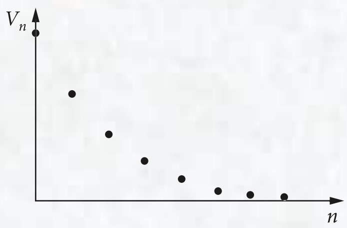
A



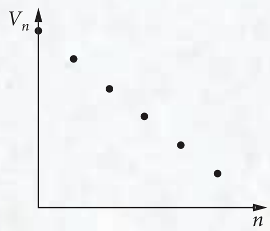
B



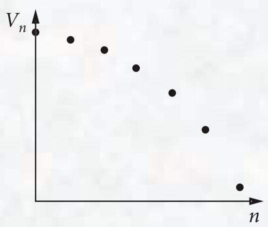
C



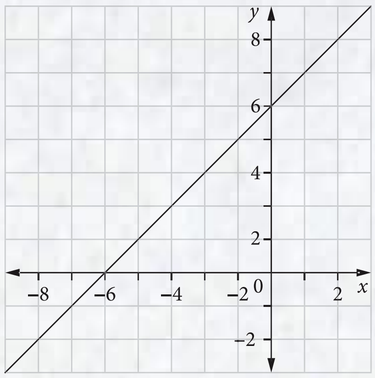
D



E



4 © VCAA 2013 1GRQ1 The equation of the line shown on the graph is



A $y = x - 6$ **B** $y = x + 6$

C $y = 6 - x$ **D** $y = -6$

E $y = 6$

5 © VCAA 2010 1MQ3 The total cost of one ice cream and three soft drinks at Catherine's shop is $9. The total cost of two ice creams and five soft drinks is $16. Let x be the cost of an ice cream and y be the cost of a soft drink.

The matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ is equal to

A $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

D $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$ **E** $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

6 Research is undertaken to see whether stress scores can predict scores for a particular test. Which of the following statements is **not** true?

A A line of good fit can be drawn for this data.

B A scatterplot can be drawn for this data.

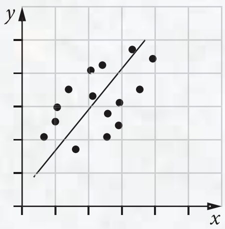
C The variable *test score* is ordinal.

D The explanatory variable is *stress score*.

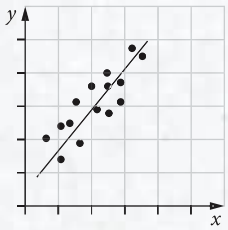
E Both variables are numerical.

7 Which one of the following shows a line of good fit to data that has a strong negative association?

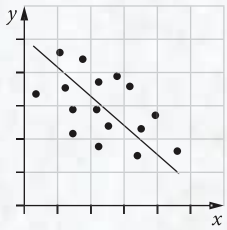
A



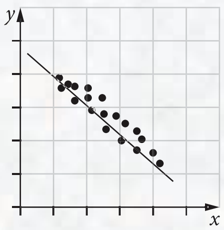
B



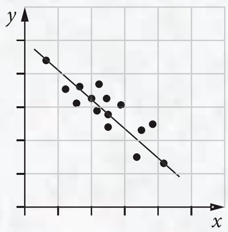
C



D



E



Use the following information to answer the next two questions.

The number of mice in a colony as weeks pass can be modelled by a line of good fit with the equation

number of mice = 12 + 10 × *number of weeks.*

8 Which one of the following statements is **true**?

A The number of mice on average increases by 12 for every one week that passes.

B The number of mice on average decreases by 12 for every one week that passes.

C The number of mice on average increases by 10 for every one week that passes.

D The number of mice on average decreases by 10 for every one week that passes.

E The number of weeks on average increases by 10 for every extra mouse in the colony.

9 Which one of the following statements is **false**?

A The model predicts that the number of mice in the colony is continually increasing.

B The number of mice in the colony after two weeks is 32.

C The initial population of mice in the colony was 12.

D The number of mice in the colony after 10 weeks is 112.

E It will take four weeks for the number of mice in the colony to reach 60.

10 © VCAA 2014 1CQ9 The equation of a line of good fit is used to predict the fuel consumption, in kilometres per litre of fuel, from a car's weight, in kilograms. The equation predicts that a car with a weight of 900 kg will travel 10.7 km per litre of fuel, while a car with a weight of 1700 kg will travel 6.7 km per litre of fuel. The slope of this line of good fit is closest to

A –250 **B** –0.005 **C** –0.004 **D** 0.005 **E** 200

Cumulative examination 2

Total number of marks: 18 Reading time: 5 minutes Writing time: 27 minutes

1 (2 marks) Bungee jumpers must be weighed before they jump. The weights, in kg, of 40 jumpers were recorded in the following table.

Weight (kg)	Frequency
30–<40	3
40–<50	8
50–<60	10
60–<70	10
70–<80	7
80–<90	2
Total	40

a Which statistical graph is the best choice to display this information? 1 mark

b How would you best describe the shape of the distribution? 1 mark

2 © VCAA 2016S 2CQ7 MODIFIED (2 marks) Hugo is a professional bike rider. The value of his bike will be depreciated over time using the flat rate method of depreciation. The value of Hugo's bike, in dollars, after *n* years, V_n, can be modelled using the recurrence relation

$$V_0 = 8400, \quad V_{n+1} = V_n - 1200$$

a Using the recurrence relation, write down calculations to show that the value of Hugo's bike after two years is \$6000. 1 mark

b Hugo will sell his bike when its value reduces to \$3600. After how many years will Hugo sell his bike? 1 mark

3 (3 marks) The Fantastic Fellows retail store sold a television for \$1200 in 2023. At the start of 2024, it increased all its prices by 10%. At the end of 2024, it has a '10% off' clearance sale.

a What are the terms for the percentage price increase at the start of 2024 and the percentage price decrease at the end of 2024? 1 mark

b What was the overall percentage change to the price of the television from 2023 to the end of 2024? 1 mark

c If the store had instead decided to increase the \$1200 television to \$1500 at the start of 2024, what would the percentage price increase have been? 1 mark

4 © VCAA 2016 2MQ1 (3 marks) A travel company arranges flight (F), hotel (H), performance (P) and tour (T) bookings. Matrix C contains the number of each type of booking for a month.

$$C = \begin{bmatrix} 85 \\ 38 \\ 24 \\ 43 \end{bmatrix} \begin{matrix} F \\ H \\ P \\ T \end{matrix}$$

a Write down the order of matrix C. 1 mark

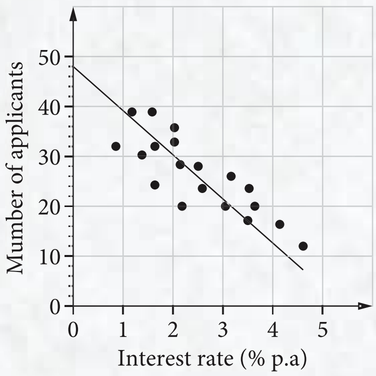
A booking fee, per person, is collected by the travel company for each type of booking. Matrix G contains the booking fees, in dollars, per booking.

$$\begin{matrix} & F & H & P & T \\ G = & [\,40 & 25 & 15 & 30\,] \end{matrix}$$

b **i** Calculate the matrix product $J = G \times C$. 1 mark

ii What does matrix J represent? 1 mark

5 (8 marks) Happy Home Loans recorded the data on the following scatterplot to see if there was an association between the loan interest rate and the number of home loan applications over a year. A line of good fit has been fitted to the data.



a State the explanatory variable and response variable. 1 mark

b For each variable, state whether it is discrete or continuous. 1 mark

c Determine the value of the slope of the line of good fit and interpret it in the context of the data. 2 marks

d Determine the equation of the line of good fit, rounding the values to the nearest whole number. 2 marks

e Use the equation of the line of good fit to predict the *number of applications* if the *interest rate* was 2.5% p.a. and 5% p.a., rounding to the nearest whole number. Explain which of these predictions is more reliable, giving a reason. 2 marks

CHAPTER

7 GRAPHS AND NETWORKS

Study Design coverage

UNIT 2, AREA OF STUDY 2: DISCRETE MATHEMATICS

Graphs and networks

- introduction to the notations, conventions and representations of types and properties of graphs, including edge, loop, vertex, the degree of a vertex, isomorphic and connected graphs and the adjacency matrix
- description of graphs in terms of faces (regions), vertices and edges and the application of Euler's formula for planar graphs
- connected graphs: walks, trails, paths, cycles and circuits with practical applications
- weighted graphs and networks, and an introduction to the shortest path problem (solution by inspection only) and its practical application
- trees and minimum spanning trees, greedy algorithms and their use to solve practical problems.

STUDY DESIGN COVERAGE

Video playlists (5):

7.1 Introducing graphs and networks
7.2 Types of graphs
7.3 Walks and weighted graphs
7.4 Minimum spanning trees
VCE question analysis Graphs and networks

Worksheets (5):

7.1 Adjacency matrices 1 • Adjacency matrices 2
7.2 Planar graphs
7.4 Minimum spanning trees • Shortest paths and trees

Nelson MindTap

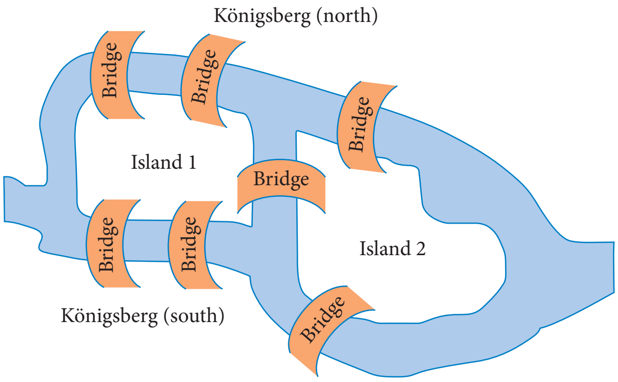
To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Introducing graphs and networks

7.1 Introducing graphs and networks

Network diagrams and graphs

One of the most famous mathematical problems in history involved seven bridges that spanned a river in the German town of Königsberg (now Kaliningrad, Russia). It became a challenge to see if someone could complete a walk in a way that crossed each bridge only once. No one had managed to do it, but no one could prove that it was impossible. The problem was finally solved in 1735 by the Swiss mathematician Leonhard Euler (pronounced 'Oiler'), who invented a new diagram called a **network diagram** and a new area of mathematics called graph theory.

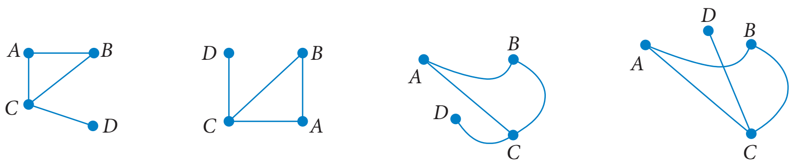


In modern times, graph theory is used to represent real-life situations such as road systems, maps and friendship networks. The Google search engine is based on graph theory.

A **network** is a group of interconnected elements such as people, places or things, and a network diagram shows these connections. Network diagrams are usually called **graphs**, so we will be using the term 'graph', although these are different to the graphs you've come across before.

Vertices, edges and isomorphic graphs

A graph consists of points called **vertices** that are connected by lines called **edges**. Two vertices that are connected by one or more edges are called **adjacent vertices**. A **vertex** is usually labelled by a single capital letter, but it can also appear without a label. Edges are indicated by the two vertices they connect. A graph with vertices *A*, *B*, *C* and *D* and edges *AB*, *AC*, *BC* and *DC* is shown below drawn in four different ways.



Although these four graphs look different, they show exactly the same network of connections, so they are considered the same graph. In all four of these graphs

A and *B* are connected

A and *C* are connected

B and *C* are connected

C and *D* are connected

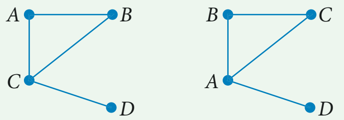
Graphs that show exactly the same connections are called **isomorphic graphs**.

> **Exam hack**
>
> When deciding whether graphs are isomorphic, look closely at the labels of the vertices and where the edges cross.

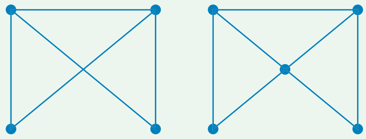
WORKED EXAMPLE 1 Identifying isomorphic graphs

For each of the following pairs of graphs, state whether or not they are isomorphic and give a reason for your answer.

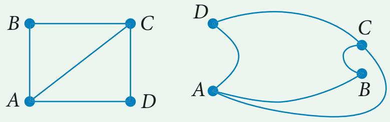
a



b



c



Steps	Working
Do the graphs have the same number of vertices? Do the graphs have the same number of edges? Do the graphs show *exactly* the same connections?	**a** These two graphs are *not* isomorphic because, although they have the same number of vertices and edges, not all the connections are the same. In the first graph, *C* and *D* are connected, but in the second graph they aren't.
	b These two graphs are *not* isomorphic because they have different numbers of vertices and edges.
	c These two graphs are isomorphic because they show exactly the same connections.

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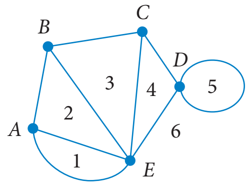
Features of a graph

Faces are the enclosed regions of a graph. Before we count faces, we need to check that no edges are crossing. If they are, we will need to redraw the graph with no crossings where possible.

This graph has six faces. Note that

- the **loop** at *D* where an edge starts and ends at the same vertex creates a face
- the two **multiple edges** between *A* and *E* create a face
- the region outside the graph counts as a face.



The **degree** of a vertex is the number of edges connected to that vertex. For the above graph: *B* has degree 3, *D* has degree 4, and *E* has degree 5. A loop adds 2 to the degree of a vertex because it's connected to the vertex twice.

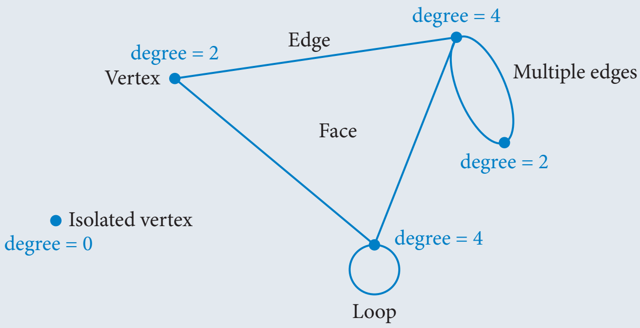
The **degree sum** of a graph is the sum of the degrees of all the vertices. For any graph

degree sum = 2 × number of edges

This means the degree sum is always even.

An **isolated vertex**, which is a vertex with no edges connected to it, has degree 0.

Features of a graph



degree sum = 2 + 4 + 2 + 4 + 0 = 12

degree sum = 2 × number of edges = 2 × 6 = 12

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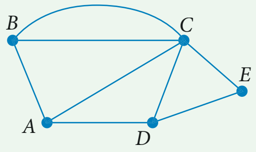
WORKED EXAMPLE 2 Identifying the features of graphs

For each of the following graphs

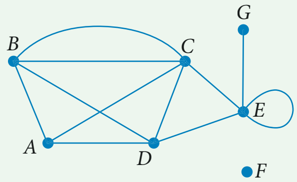
i count and list the vertices, edges and faces

ii show that the degree sum is twice the number of edges.

a



b

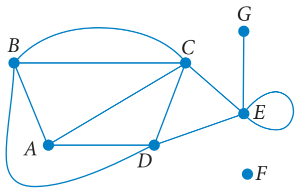


Steps	Working
a **i** Count and list the number of vertices and edges.	5 vertices: *A*, *B*, *C*, *D*, *E*
	8 edges: *AB*, *AC*, *AD*, *BC* × 2, *CD*, *CE*, *DE*
Count and list the number of enclosed regions plus the region outside the graph.	5 faces: 4 enclosed and 1 outside the graph
ii Find the degree of each vertex and add them. Show that multiplying the number of edges by 2 gives the same result.	(table and working below)

Vertex	*A*	*B*	*C*	*D*	*E*	Sum
Degree	3	3	5	3	2	16

degree sum = 2 × number of edges
= 2 × 8
= 16

b **i** **1** Redraw the graph to uncross the intersecting edges that have no vertex at the point of intersection.



2 Count and list the number of vertices and edges.

7 vertices: *A*, *B*, *C*, *D*, *E*, *F*, *G*

11 edges: *AB*, *AC*, *AD*, *BC* × 2, *BD*, *CD*, *CE*, *DE*, *EE*, *EG*

Count and list the number of enclosed regions plus the region outside the graph.

7 faces: 6 enclosed and 1 outside the graph

ii Find the degree of each vertex and add them.

Show that multiplying the number of edges by 2 gives the same result.

Vertex	*A*	*B*	*C*	*D*	*E*	*F*	*G*	Sum
Degree	3	4	5	4	5	0	1	22

degree sum = 2 × number of edges
= 2 × 11
= 22

Exam hack

Watch out for edges that cross without a vertex at the intersection. Your count of the number of faces in the graph will be wrong unless you redraw the graph by uncrossing the edges.

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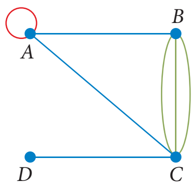
Adjacency matrices

Adjacency matrices can be used to describe graphs by showing the number of edges connecting each pair of vertices. For example:

Graph



Adjacency matrix

	A	B	C	D
A	1	1	1	0
B	1	0	3	0
C	1	3	0	1
D	0	0	1	0

Description

- 1 loop A to A
- 3 edges between B and C
- 1 edge between A and B, A and C, C and D
- 0 edges everywhere else

Adjacency matrices

Adjacency matrices

- show the number of connections between each pair of vertices of a graph
- are always square matrices
- have row 1 = column 1, row 2 = column 2 and so on.

7.1

Video playlist Introducing graphs and networks

Worksheets Adjacency matrices 1

Adjacency matrices 2

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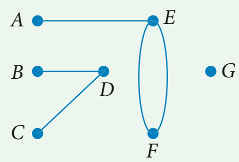
WORKED EXAMPLE 3 Finding adjacency matrices

Represent the following graph using an adjacency matrix.



Steps | **Working**

1 Set up the square matrix with the vertices as row and column labels.

Fill in the first row by counting the connections between *A* and each of the other vertices.

	A	B	C	D	E	F	G
A	0	0	0	0	1	0	0
B							
C							
D							
E							
F							
G							

2 Fill in the first column by copying the first row.

	A	B	C	D	E	F	G
A	0	0	0	0	1	0	0
B	0						
C	0						
D	0						
E	1						
F	0						
G	0						

3 Continue the steps for each row and column.

	A	B	C	D	E	F	G
A	0	0	0	0	1	0	0
B	0	0	0	1	0	0	0
C	0	0	0	1	0	0	0
D	0	1	1	0	0	0	0
E	1	0	0	0	0	2	0
F	0	0	0	0	2	0	0
G	0	0	0	0	0	0	0

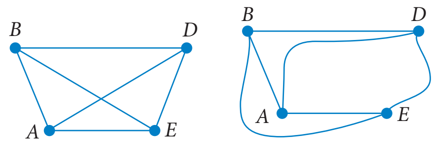
EXERCISE 7.1 Introducing graphs and networks

ANSWERS p. 511

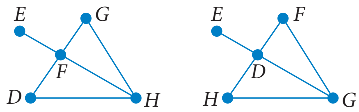
Mastery

1 WORKED EXAMPLE 1 For each of the following pairs of graphs, state whether or not they are isomorphic and give a reason for your answer.

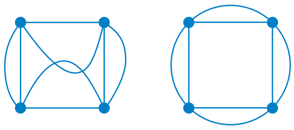
a



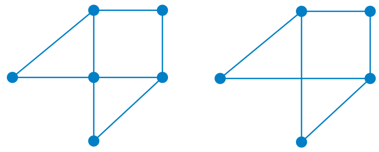
b



c



d

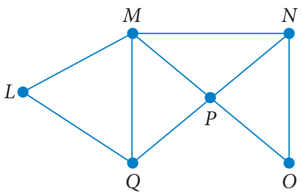


2 WORKED EXAMPLE 2 For each of the following graphs

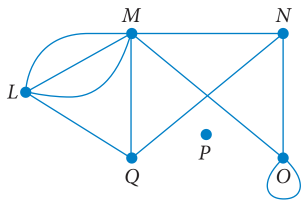
i count and list the vertices, edges and faces

ii show that the degree sum is twice the number of edges

a



b



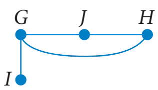
3 For each graph below, find the number of

i vertices **ii** isolated vertices **iii** edges

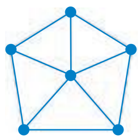
iv faces **v** loops **vi** vertices with even degrees

vii pairs of vertices with multiple edges and find the degree sum.

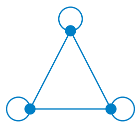
a



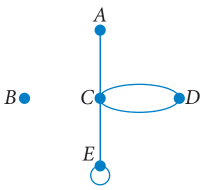
b



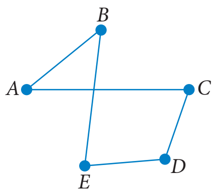
c



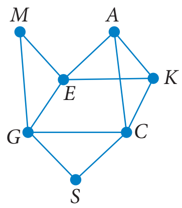
d



e

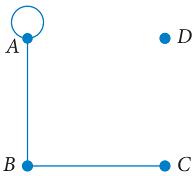


f

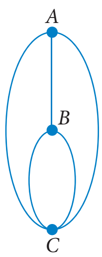


4 WORKED EXAMPLE 3 Represent each of the following graphs using an adjacency matrix.

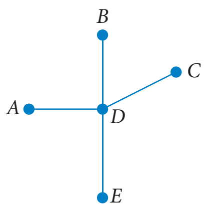
a



b



c

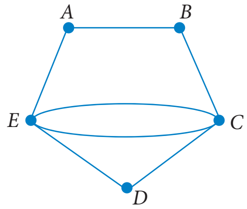


5 © VCAA 2010 1NQ3 95% A graph that can be drawn from the adjacency matrix

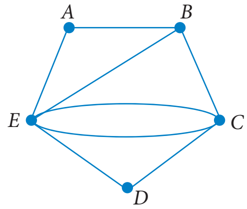
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

is

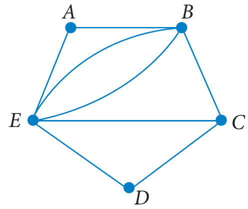
A



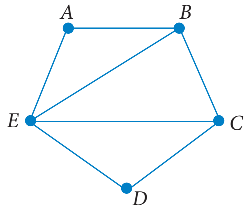
B



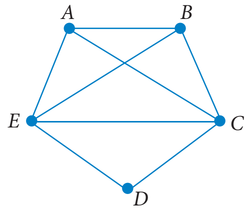
C



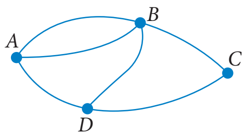
D



E



6 © VCAA 2007 1NQ3 93% Consider the following graph.



An adjacency matrix that could be used to represent this graph is

A $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

B $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

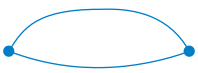
C $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

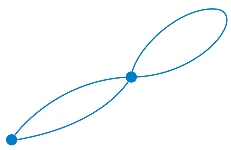
E $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

7 © VCAA 2017 1NQ1 92% Which one of the following graphs contains a loop?

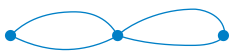
A



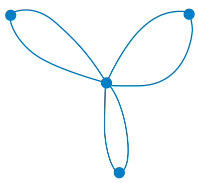
B



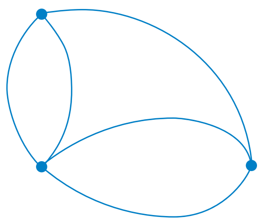
C



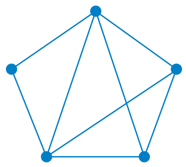
D



E

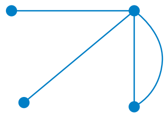


8 © VCAA 2010 1NQ2 89% The number of edges in the graph is



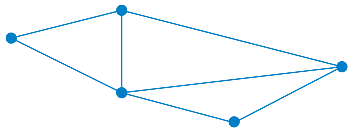
A 5 **B** 7 **C** 8
D 10 **E** 11

9 © VCAA 2005 1NQ1 87% The sum of the degrees of all the vertices in this network diagram is



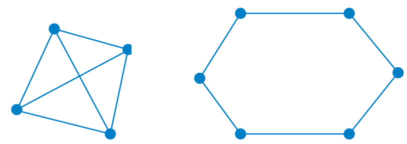
A 6 **B** 7 **C** 8
D 9 **E** 10

10 © VCAA 2006 1NQ1 85% The number of vertices with an odd degree in the network diagram is



A 1 **B** 2 **C** 3
D 4 **E** 5

11 © VCAA 2017 1NQ2 83% Two graphs, labelled Graph 1 and Graph 2, are shown.

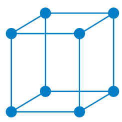


Graph 1 Graph 2

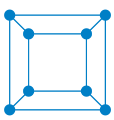
The sum of the degrees of the vertices of Graph 1 is

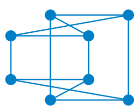
A two less than the sum of the degrees of the vertices of Graph 2.
B one less than the sum of the degrees of the vertices of Graph 2.
C equal to the sum of the degrees of the vertices of Graph 2.
D one more than the sum of the degrees of the vertices of Graph 2.
E two more than the sum of the degrees of the vertices of Graph 2.

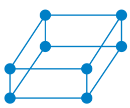
12 © VCAA 2011 1NQ5 81% A network is represented by the following graph.

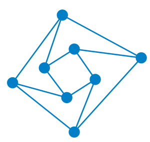


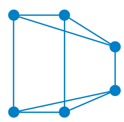
Which one of the following graphs could **not** be used to represent the same network?

A


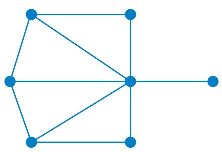
B


C


D


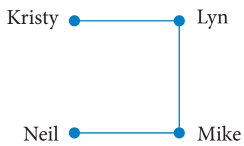
E


13 © VCAA 2011 1NQ1 61% In the network diagram shown, the number of vertices of even degree is



A 2 **B** 3 **C** 4
D 5 **E** 6

14 © VCAA 2010 2NQ1 (2 marks) In a competition, members of a team work together to complete a series of challenges. The members of one team are Kristy (*K*), Lyn (*L*), Mike (*M*) and Neil (*N*). In one of the challenges, these four team members are only allowed to communicate directly with each other as indicated by the edges of the following network diagram.



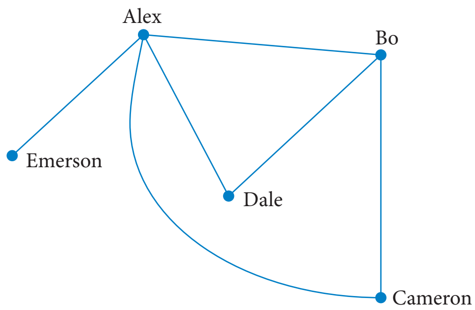
$$\begin{array}{c} \\ K \\ L \\ M \\ N \end{array} \begin{array}{c} \begin{array}{cccc} K & L & M & N \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & f & 0 & 1 \\ 0 & g & 1 & 0 \end{array}\right] \end{array}$$

The adjacency matrix also shows the allowed lines of communication.

a 79% Explain the meaning of a **zero** in the adjacency matrix. 1 mark

b 79% Write down the values of *f* and *g* in the adjacency matrix. 1 mark

15 © VCAA 2020 2NQ1 (3 marks) The Sunny Coast Cricket Club has five new players join its team: Alex, Bo, Cameron, Dale and Emerson. The graph below shows the players who have played cricket together before joining the team. For example, the edge between Alex and Bo shows that they have previously played cricket together.



a 95% How many of these players had Emerson played cricket with before joining the team? 1 mark

b 58% Who had played cricket with both Alex and Bo before joining the team? 1 mark

c 91% During the season, another new player, Finn, joined the team. Finn had not played cricket with any of these players before. Copy the graph and represent this information. 1 mark

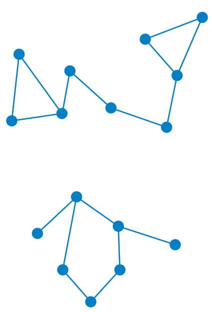
7.2 Types of graphs

Video playlist
Types of graphs

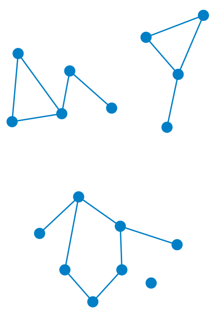
Connected graphs

A **connected graph** is a graph where there is a path from any vertex to any other vertex. For example:

Connected graphs



Not connected graphs

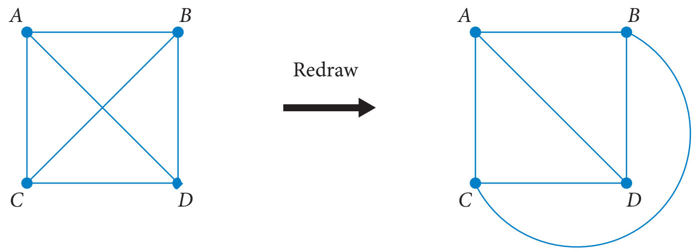


Worksheet
Planar graphs

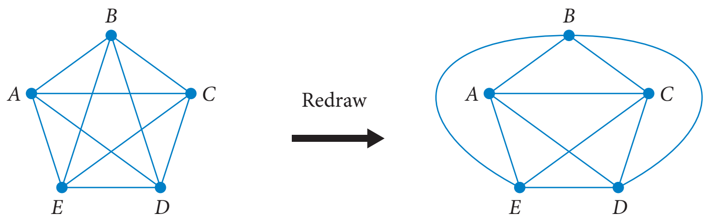
Planar graphs

Planar graphs are connected graphs that can be drawn so no edges are crossing. It doesn't matter whether it's actually drawn with crossed edges. What's important is how it *can* be drawn.

This is a planar graph. Although two edges are crossing, it can be redrawn so that no edges cross.



This is *not* a planar graph. It's possible to redraw it so that *some* of the edges aren't crossing, but it's impossible to redraw it with *no* edges at all crossing.



Trying to uncross the last two edges always results in a different cross.

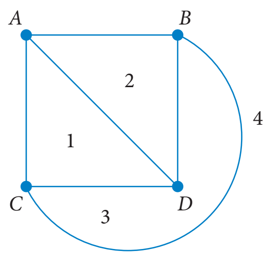
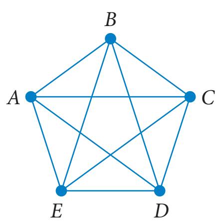
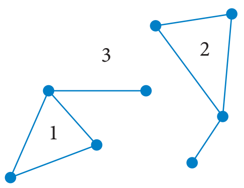
Euler's formula

Euler's formula applies to graphs that are *both* planar and connected:

number of vertices + number of faces − number of edges = 2

or

$v + f - e = 2$

Graph	Features	Euler's formula
	connected planar	$v = 4, f = 4, e = 6$ so $v + f - e = 4 + 4 - 6 = 2$ Euler's formula works for this graph.
	connected not planar	Euler's formula doesn't work because, although it's connected, it isn't planar. We can't identify all the faces if we can't uncross all the edges.
	not connected planar	Euler's formula doesn't work because although it's planar, it isn't connected. For this graph: $v = 8, f = 3, e = 8$ giving $v + f - e = 8 + 3 - 8 = 3$

Euler's formula

For connected planar graphs

$$v + f - e = 2$$

where

v = the number of vertices

f = the number of faces

e = the number of edges.

The formula is often rearranged to give **Euler's formula**:

$$v + f = e + 2$$

7.2

WB p. 107

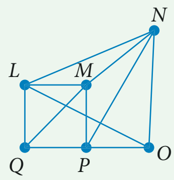
WORKED EXAMPLE 4 Verifying Euler's formula

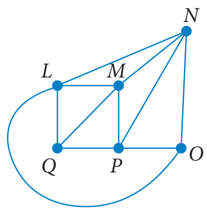
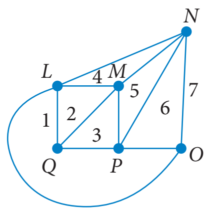
For the graph shown

a redraw it to show it is a planar graph

b state whether or not it is a connected graph, giving a reason

c verify that Euler's formula works or show that it doesn't.



Steps	Working
a To uncross the edges, move edge LO around the outside of the graph.	 The graph can be redrawn without any edges crossing, so it is a planar graph.
b Use the definition of a connected graph.	There is a path from each vertex to every other vertex, so it is a connected graph.
c Count the number of vertices, faces and edges, and substitute into Euler's formula to see if the result is 2.	 $v = 6, f = 7, e = 11$ $v + f - e = 6 + 7 - 11$ $= 2$ Euler's formula works for this graph.

WB
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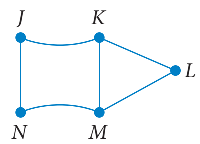
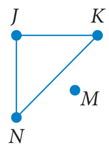
WORKED EXAMPLE 5	**Using Euler's formula**
a A connected planar graph has 13 edges and 6 faces. How many vertices does it have?	
b A connected planar graph has 11 vertices and 5 faces. How many edges does this graph have?	
Steps	**Working**
a Substitute the known values into Euler's formula and solve to find v.	Substitute $e = 13$ and $f = 6$ into $v + f - e = 2$. $v + 6 - 13 = 2$ $v = 2 - 6 + 13$ $= 9$ The number of vertices is 9.
b Substitute the known values into Euler's formula and solve to find e.	Substitute $v = 11$ and $f = 5$ into $v + f - e = 2$. $11 + 5 - e = 2$ $e = 11 + 5 - 2$ $= 14$ The number of edges is 14.

Subgraphs

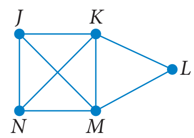
A **subgraph** is part of a larger graph. It can only have vertices and edges that are in the larger graph.



and



are both subgraphs of

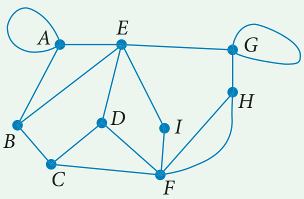


As with all graphs, it doesn't matter whether the edges are drawn curved or straight.

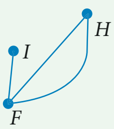
WB
p. 109

WORKED EXAMPLE 6 **Identifying subgraphs**

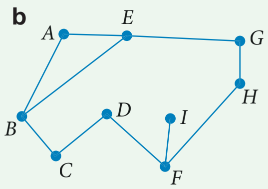
For each of the following graphs, state whether it is a subgraph of this graph, giving reasons for your answers.



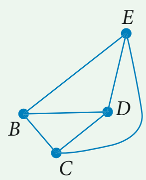
a



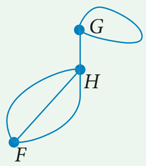
b



c



d



Steps	**Working**
Does the graph contain *only* vertices and edges from the original graph?	**a** It is a subgraph because it only has vertices and edges from the larger graph.
	b It is a subgraph because it only has vertices and edges from the larger graph.
	c It is not a subgraph because it has two edges (*BD* and *EC*) that are not in the larger graph.
	d It is not a subgraph because it has three edges connecting *F* and *H* and the larger graph has only two.

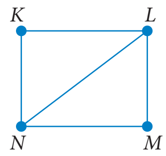
EXERCISE 7.2 Types of graphs

ANSWERS p. 511

7.2

Recap

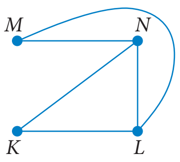
1 © VCAA 2015 1NQ5 92% The graph represents a friendship network.



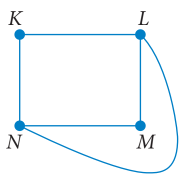
The vertices represent the four people in the friendship network: Kwan (K), Louise (L), Milly (M) and Narelle (N). An edge represents the presence of a friendship between a pair of these people. For example, the edge connecting K and L shows that Kwan and Louise are friends.

Which one of the following graphs does **not** contain the same information?

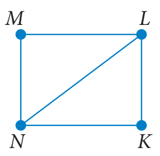
A



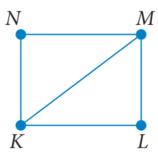
B



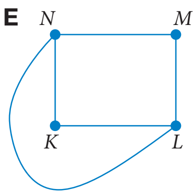
C



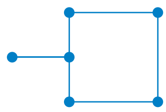
D



E



2 © VCAA 2012 1NQ1 76% The sum of the degrees of all the vertices in the graph is



A 6 **B** 8 **C** 9 **D** 11 **E** 12

Mastery

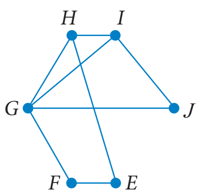
3 WORKED EXAMPLE 4 For each of the following graphs

i redraw it to show it is a planar graph

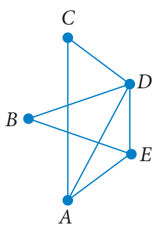
ii state whether or not it is a connected graph, giving a reason

iii verify that Euler's formula works or show that it doesn't.

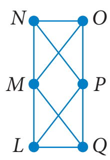
a



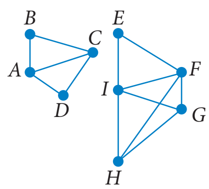
b



c



d



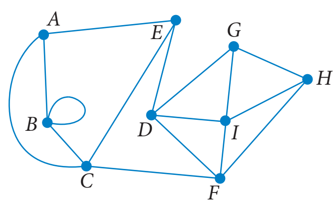
4 WORKED EXAMPLE 5

a A connected planar graph has 8 edges and 5 faces. How many vertices does this graph have?

b A connected planar graph has 10 vertices and 6 faces. How many edges does this graph have?

c A connected planar graph has 9 vertices and 10 edges. How many faces does this graph have?

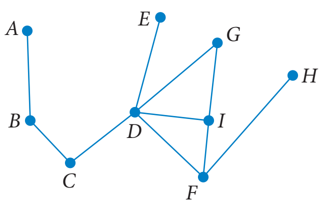
5 WORKED EXAMPLE 6 For each of the following graphs, state whether it is a subgraph of



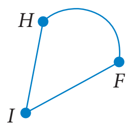
giving reasons for your answer.

a

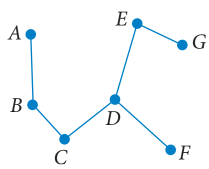
b



c



d



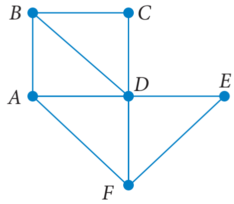
Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2020 1NQ1 **89%** A connected planar graph has seven vertices and nine edges. The number of faces that this graph will have is

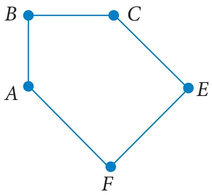
A 1 **B** 2 **C** 3 **D** 4 **E** 5

7 © VCAA 2004 1NQ1 MODIFIED **83%** A subgraph of the graph

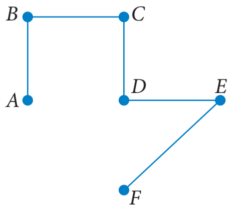


is

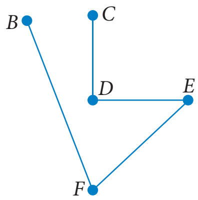
A



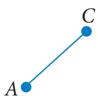
B



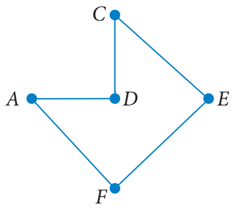
C



D



E

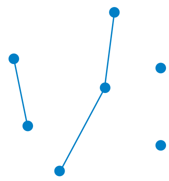


8 © VCAA 2007 1NQ2 **76%** A connected planar graph has 12 edges. This graph could have

A 5 vertices and 6 faces. **B** 5 vertices and 8 faces. **C** 6 vertices and 8 faces.

D 6 vertices and 9 faces. **E** 7 vertices and 9 faces.

9 © VCAA 2009 1NQ1 **74%** Consider the following graph. The smallest number of edges that need to be added to make this a connected graph is



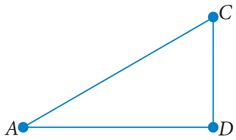
A 1 **B** 2 **C** 3

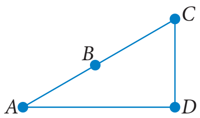
D 4 **E** 5

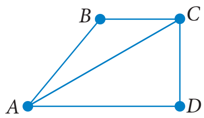
10 © VCAA 2009 1NQ4 **68%** A connected planar graph has 10 edges and 10 faces. The number of vertices for this graph is

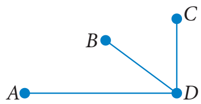
A 2 **B** 5 **C** 8 **D** 12 **E** 20

11 © VCAA 2008 1NQ2 67% The graph on the right is a subgraph of which of the following graphs?

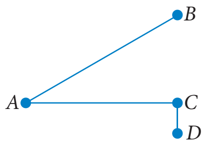


A


B


C


D

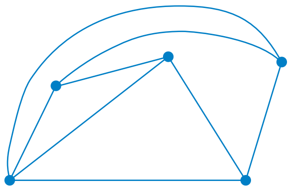

E


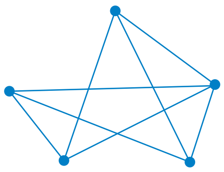
12 © VCAA 2005 1NQ5 66% A connected planar graph has 10 vertices and 15 edges. A number of edges are removed to leave a connected graph with 10 vertices and 3 faces. The number of edges that were removed is

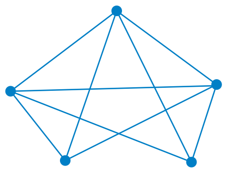
A 4 **B** 5 **C** 6 **D** 7 **E** 8

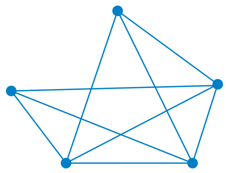
13 © VCAA 2016 1NQ5 58% Consider the planar graph shown.

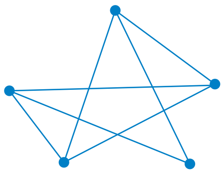
Which one of the following graphs can be redrawn as the planar graph on the right?

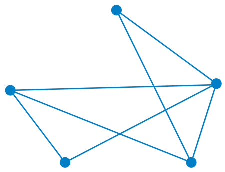


A


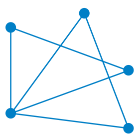
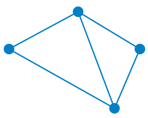
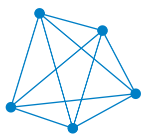
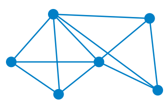
B


C


D


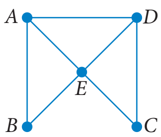
E


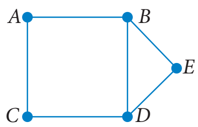
14 © VCAA 2019N 1NQ6 Four graphs are shown below. How many of these graphs are planar?

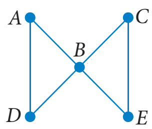





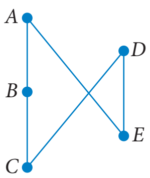
A 0 **B** 1 **C** 2 **D** 3 **E** 4

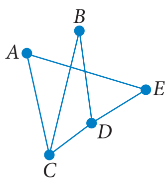
15 Which one of the following is a planar graph with five vertices and two faces?

A


B


C


D


E


Video playlist
Walks and weighted graphs

7.3 Walks and weighted graphs

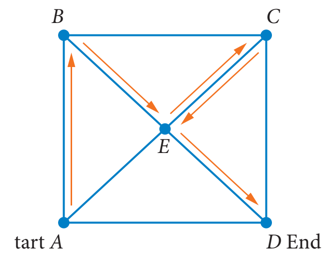
Types of walks

Graphs can be used to solve travelling and exploring problems where we want to find the shortest route between different locations. These problems involve moving from one vertex to another via an edge and are described by listing the vertices that are visited in order. You will need to know the following definitions when dealing with travelling and exploring problems.

A **walk** is a sequence of connected vertices.

A–B–E–C–E–D is an example of a walk in this graph:

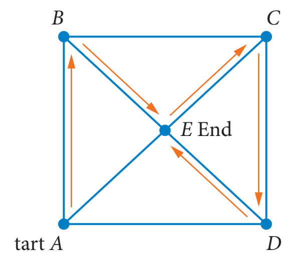
In a walk, edges and vertices can be repeated.
In this example, the edge *EC* is travelled along twice and the vertex *E* is visited twice.



A **trail** is a walk with no repeated edges.

A–B–E–C–D–E is an example of a trail in this graph:

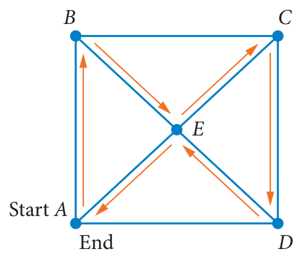
In a trail, vertices can be repeated. In this example, the vertex *E* is visited twice.



A **circuit** is a walk with no repeated edges that starts and finishes at the same vertex.

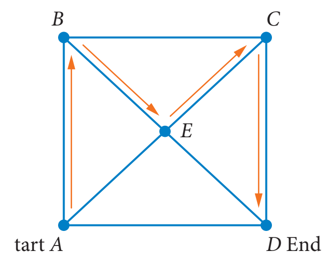
A–B–E–C–D–E–A is an example of a circuit in this graph:

In a circuit, vertices can be repeated. In this example, the vertices *A* and *E* are visited twice.



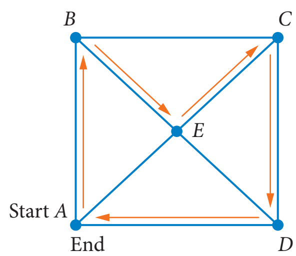
A **path** is a walk with no repeated vertices.

A–B–E–C–D is an example of a path in this graph:



A **cycle** is a walk with no repeated vertices that starts and finishes at the same vertex.

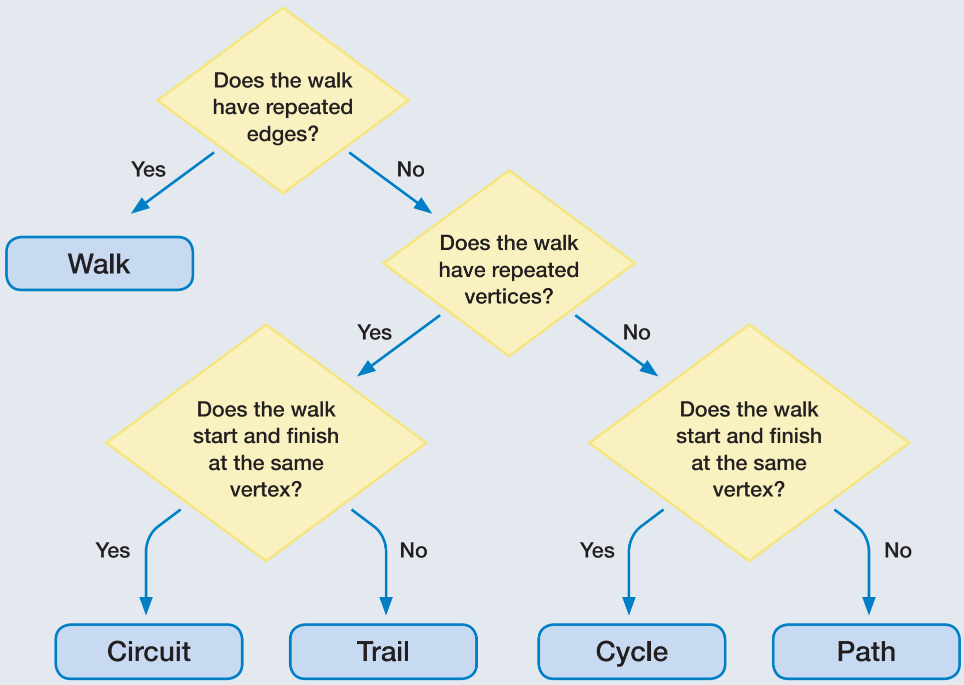
A–B–E–C–D–A is an example of a cycle in this graph:



Types of walks

Walks that:	Trail	Circuit	Path	Cycle
have no repeated edges	✓	✓	✓	✓
have no repeated vertices			✓	✓
start and finish at the same vertex		✓		✓

Walk classification chart

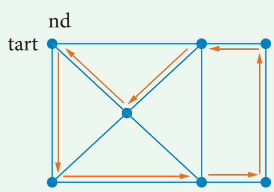


WORKED EXAMPLE 7 Classifying walks shown on a graph

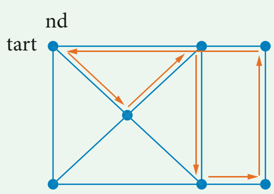
WB p. 110

For each of the following walks, state whether it is a trail, path, circuit, cycle or a walk only, and give a reason for your answer.

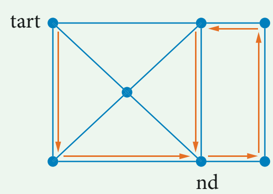
a



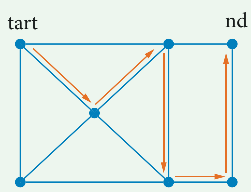
b



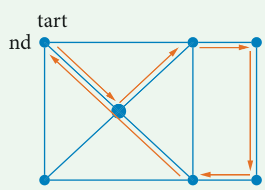
c



d



e



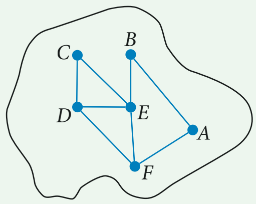
Steps	Working
Use the Walk classification chart on page 317 and ask the three questions, in this order, for each one: **1** Does the walk have repeated edges? **2** Does the walk have repeated vertices? **3** Does the walk start and finish at the same vertex?	**a** cycle: no repeated edges, no repeated vertices (except the first and last vertex), and starts and finishes at the same vertex **b** circuit: no repeated edges, a repeated vertex, and starts and finish at the same vertex **c** trail: no repeated edges, a repeated vertex, and doesn't start and finish at the same vertex **d** path: no repeated edges, no repeated vertices, and doesn't start and finish at the same vertex **e** walk only: repeated edge

WB

p. 111

WORKED EXAMPLE 8 Classifying walks from a list of vertices

Byron is trekking along paths on Wombat Island. For each of the following walks, state whether it is a trail, path, circuit, cycle or a walk only, and give a reason for your answer.



a $C–E–B–A–F–E–D$ **b** $D–C–E–B–A–F$

c $D–C–E–B–A–F–D$ **d** $D–C–E–B–A–F–E–D$

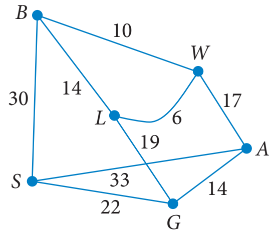
e $C–D–E–F–A–B–E–D$ **f** $F–E–B–A–F–E$

Steps	Working
Use the Walk classification chart to ask three questions, in this order, for each one: **1** Does the walk have repeated edges? **2** Does the walk have repeated vertices? **3** Does the walk start and finish at the same vertex?	**a** trail: no repeated edges, a repeated vertex *E*, and doesn't start and finish at the same vertex **b** path: no repeated edges, no repeated vertices, and doesn't start and finish at the same vertex **c** cycle: no repeated edges, no repeated vertices (except the first and last vertex), and starts and finishes at the same vertex **d** circuit: no repeated edges, a repeated vertex *E*, and starts and finish at the same vertex **e** walk only: repeated edge (*DE* is the same edge as *ED*) **f** walk only: repeated edge *FE*

Weighted graphs and shortest paths

A **weighted graph** (sometimes just called a network) is a graph where extra information such as distances, times or costs is labelled on the edges.

The weighted graph here shows the distances, in kilometres, by road between several towns. Note that the lengths of the edges don't need to match the weight. For example, the edge showing a distance of 30 km doesn't have to be drawn three times as long as the edge showing a distance of 10 km. As with all the graphs we've been dealing with, it doesn't matter what length the edges are drawn or whether they are curved or straight.

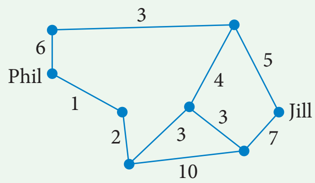


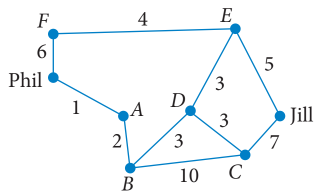
We use weighted graphs to solve **shortest path** problems. These are problems involving finding the shortest distance, shortest time or least cost from a starting point to an end point. For example, in the above graph we might want to find the route from *S* to *W* with the shortest distance.

WORKED EXAMPLE 9 Finding the shortest path

WB p. 112

The network shows the travel times, in minutes, along a series of roads. Find the shortest time, in minutes, that it takes Phil to travel from his house to Jill's house by listing all the options.



Steps	Working
1 Add labels to the vertices.	
2 List the path options and calculate the total time of each option.	Phil–*F*–*E*–Jill takes 6 + 4 + 5 = 15 min Phil–*A*–*B*–*C*–Jill takes 1 + 2 + 10 + 7 = 20 min Phil–*A*–*B*–*D*–*C*–Jill takes 1 + 2 + 3 + 3 + 7 = 16 min Phil–*A*–*B*–*D*–*E*–Jill takes 1 + 2 + 3 + 3 + 5 = 14 min
3 Write the answer.	The shortest time needed for Phil to travel to Jill's house is 14 minutes.

EXERCISE 7.3 Walks and weighted graphs

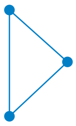
ANSWERS p. 512

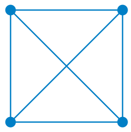
Recap

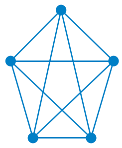
1 © VCAA 2002 1NQ1 **76%** A connected planar graph of 15 edges has 8 faces. The number of vertices in this graph is

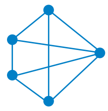
A 3 **B** 7 **C** 9 **D** 23 **E** 25

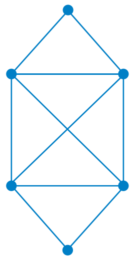
2 Which one of the following graphs is not planar?

A


B


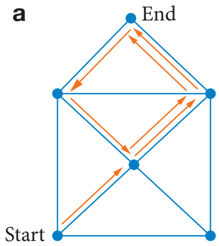
C


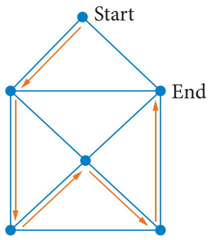
D


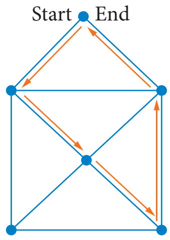
E


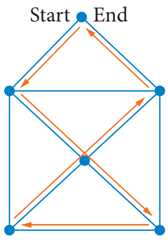
Mastery

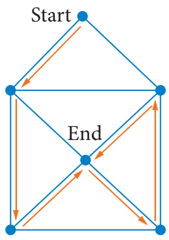
3 WORKED EXAMPLE 7 For each of the following walks, state whether it is a trail, path, circuit, cycle or a walk only, and give a reason for your answer.

a


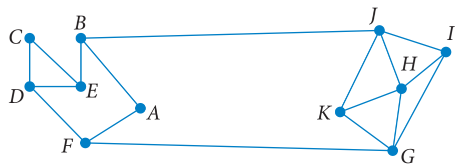
b


c


d


e


4 WORKED EXAMPLE 8 Elke is hiking along tracks in the Ranges National Park. For each of the following walks, state whether it is a trail, path, circuit, cycle or a walk only, and give a reason for your answer.



a $K–H–I–J–H–G–K$

b $C–E–B–A–F–D–C$

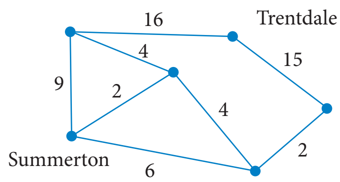
c $A–F–G–K–H$

d $G–K–H–I–J–K–H–G$

e $D–E–C–D–F–G$

f $G–K–J–I–H–K–G$

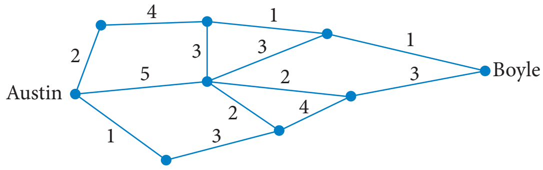
5 WORKED EXAMPLE 9 The network shows the travel times, in minutes, along a series of roads. Find the shortest time, in minutes, that it takes to travel from Summerton to Trentdale by listing all the options.



Exam practice

80–100% | 60–79% | 0–59%

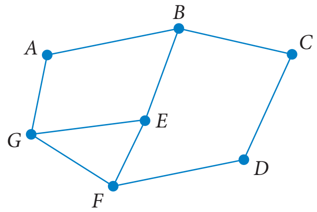
6 © VCAA 2009 1NQ2 **93%** The network shows the distances, in kilometres, along roads that connect the cities of Austin and Boyle. The shortest distance, in kilometres, from Austin to Boyle is



A 7 **B** 8 **C** 9
D 10 **E** 11

7 © VCAA 2020 1NQ2 MODIFIED **93%** Consider the graph.

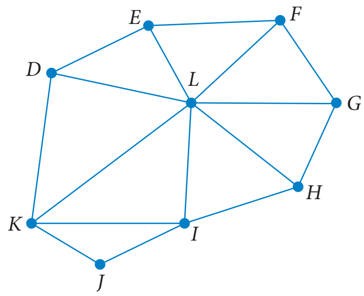
Which one of the following is **not** a cycle for this graph?



A *ABCDFEGA* **B** *BAGEFDCB*
C *CDFEGABC* **D** *DCBAGFED*
E *EGABCDFE*

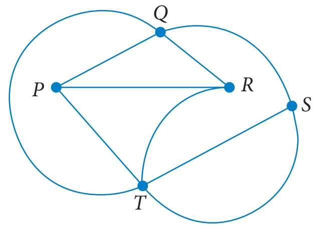
8 © VCAA 2008 1NQ3 MODIFIED **89%** A cycle for the graph is

A *KJIHGLFEDK* **B** *DKLIJHGFED*
C *DEFGHIJKD* **D** *JIKDLHGFE*
E *GHILKJILDEFG*



9 © VCAA 2018 1NQ4 **88%** Consider the following graph.

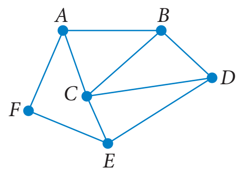
Which one of the following is **not** a path for this graph?



A *PRQTS* **B** *PQRTS* **C** *PRTSQ*
D *PTQSR* **E** *PTRQS*

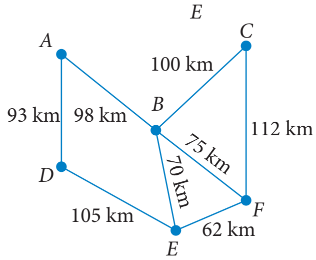
10 The total number of paths connecting *A* and *C* without returning to a vertex already used is

A 3 **B** 4 **C** 5 **D** 7 **E** 8



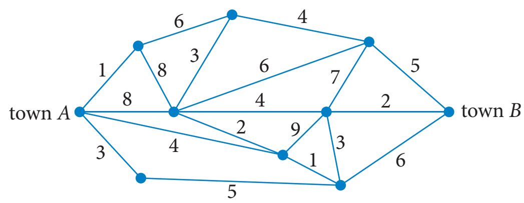
11 The graph represents six towns and the roads connecting them. The distance between the towns (via the roads) is written on the edges.

The difference between the longest route and the shortest route between town *A* and town *F* (without passing a town more than once) is



A 137 km **B** 173 km **C** 307 km
D 310 km **E** 480 km

12 © VCAA 2007 1NQ4 66% The network shows the distances, in kilometres, along a series of roads that connect town A to town B. The shortest distance, in kilometres, to travel from town A to town B is

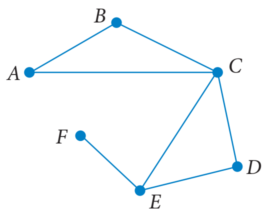


A 9 **B** 10 **C** 11

D 12 **E** 13

13 For the graph shown, how many of the following are possible?

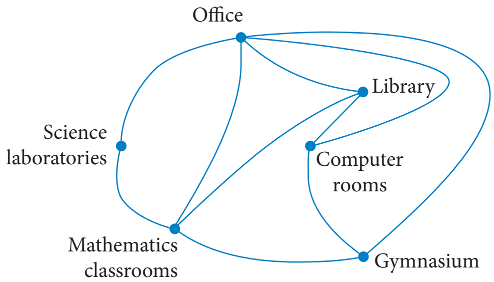
walk circuit trail cycle path



A 0 **B** 1 **C** 2

D 3 **E** 4

14 © VCAA 2019 2NQ1 MODIFIED (3 marks) Fencedale High School has six buildings. The network below shows these buildings represented by vertices. The edges of the network represent the paths between the buildings.

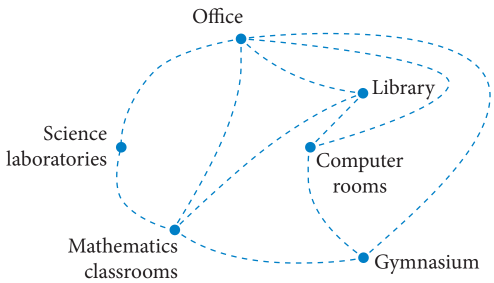


a 96% Which building in the school can be reached directly from all other buildings? 1 mark

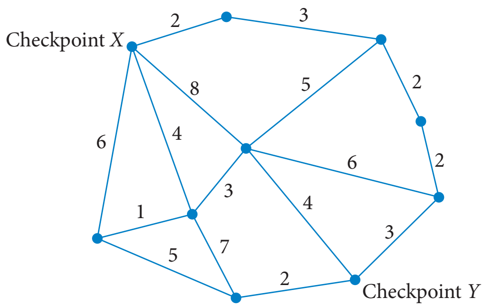
b A school tour is to start at the office, visiting each building only once.

i What type of walk is this route? 1 mark

ii 93% Copy the diagram below and draw in a possible route for this school tour that finishes at the office. 1 mark



15 © VCAA 2010 2NQ2 MODIFIED (3 marks) The diagram shows a network of tracks (represented by edges) between checkpoints (represented by vertices) in a short-distance running course. The numbers on the edges indicate the time, in minutes, a team would take to run along each track.



A challenge requires teams to run from checkpoint X to checkpoint Y using these tracks.

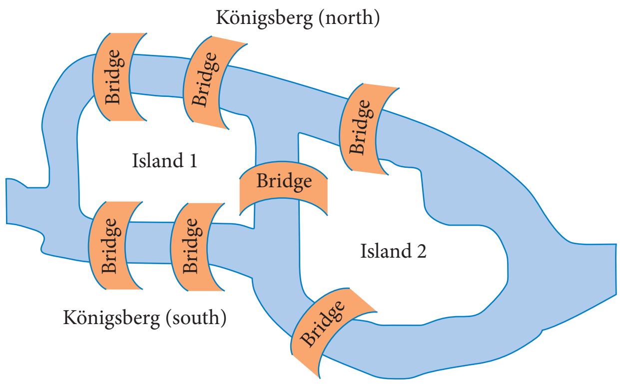
a 78% What would be the shortest possible time for a team to run from checkpoint X to checkpoint Y? 1 mark

b Teams are required to follow a route from checkpoint X to checkpoint Y that passes through every checkpoint once only.

i What type of walk is this route? 1 mark

ii 78% Copy the network diagram and on it draw in the route from checkpoint X to checkpoint Y that passes through every checkpoint once only. 1 mark

16 (8 marks) Leonhard Euler showed it was impossible to complete a walk in Königsberg that visited both sides of the river and the two islands by crossing each bridge only once.



a In a graph showing this, are the bridges vertices or edges? 1 mark

b Draw a graph of the Königsberg bridge problem that includes the names of the four locations. 1 mark

c What sort of walk was Euler investigating? 1 mark

The German mathematician Carl Hierholzer went one step further than Euler and proved that for a trail without repeated edges to be possible, a connected graph has to have *exactly* zero or two vertices of odd degree.

d List the degree of each of the four locations and use Hierholzer's proof to show that it is impossible to visit the locations by crossing each Königsberg bridge only once. 2 marks

e Redraw the graph deleting one edge connecting Königsberg (north) and Island 1 and explain how this now makes a trail without repeated edges possible. 2 marks

f Add numbered arrows to your graph to show a trail that has no repeated edges (called an **Eulerian trail**). 1 mark

Video playlist
Minimum spanning trees

Worksheets
Minimum spanning trees

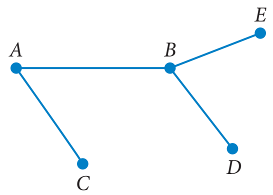
Shortest paths and trees

7.4 Minimum spanning trees

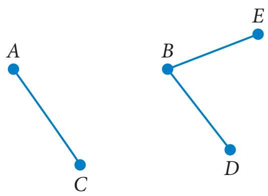
Trees and spanning trees

A **tree** is a connected graph with no loops, multiple edges, or cycles.

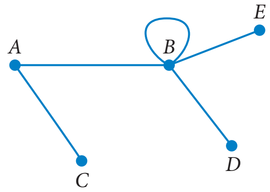
This graph is an example of a tree:



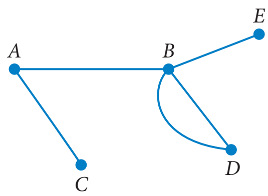
These are *not* trees:



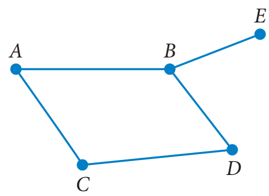
not a connected graph



contains a loop



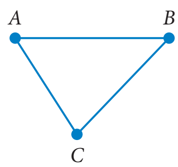
contains multiple edges



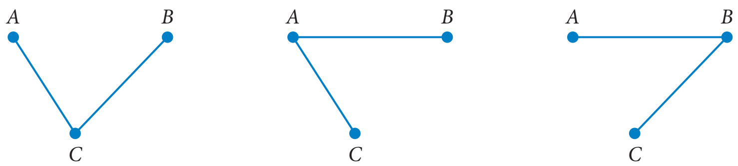
contains a cycle

A **spanning tree** is a tree subgraph that connects all the vertices of the original graph. Every connected graph has at least one spanning tree.

The three spanning trees for



are shown below:



Trees

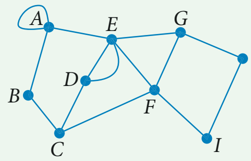
- A **tree** is a connected graph with no loops, multiple edges or cycles.
- The number of edges in a tree is always one less than the number of vertices.

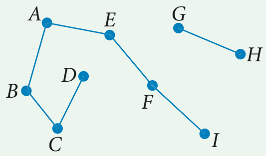
Spanning trees

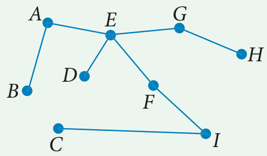
- A **spanning tree** is a tree subgraph that connects all the vertices of the original graph.
- Every connected graph has at least one spanning tree.

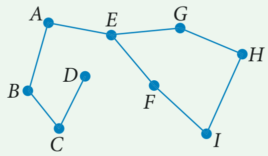
WORKED EXAMPLE 10 Identifying spanning trees

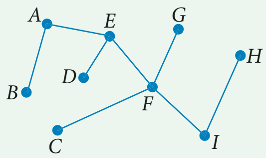
Which of the following graphs are a spanning tree of the graph shown? For those that are spanning trees, verify that the number of edges is one less than the number of vertices. For those that aren't spanning trees, give a reason.

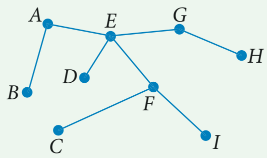


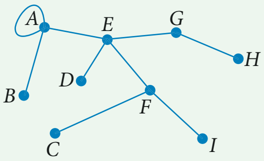
a


b


c


d


e


f


Steps	Working
Is it connected? Does it have all the vertices of the original graph? Does it have no loops? Does it have no multiple edges? Does it have no cycles?	**a** This graph is not a spanning tree. It is not connected.
	b This graph is not a spanning tree. It has an edge (*CI*) that isn't in the original graph.
	c This graph is not a spanning tree. It has a cycle.
	d This graph is a spanning tree. It has 9 vertices and 8 edges.
	e This graph is a spanning tree. It has 9 vertices and 8 edges.
	f This graph is not a spanning tree. It has a loop.

WB
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Minimum connector problems

Spanning trees are used to solve **minimum connector** problems. These problems, like the shortest path problems, involve weights on the edges of the graph that represent quantities like distance and cost. A minimum connector is the minimum weight path connecting *all* the vertices in a graph. This is different to the shortest path, which is the minimum weight path between *two* particular vertices and doesn't have to contain all the vertices in a graph.

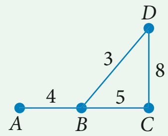
To solve a minimum connector problem, we need to find the **minimum spanning tree**, which is the spanning tree with the smallest total weight. Minimum spanning trees are used when connecting broadband networks like the NBN, water pipe systems and in real-time face recognition software.

WB

p. 114

WORKED EXAMPLE 11 Finding minimum spanning trees by inspection

Find all the spanning trees for the network shown, and hence, find the total weight of the minimum spanning tree.



Steps

1 Work out how many edges need to be removed from the graph to create a spanning tree:

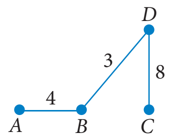
- The number of edges in a tree is always one less than the number of vertices.
- A spanning tree has the same number of vertices as the original graph.

2 Remove each edge in turn to see which options result in a spanning tree. Calculate the total weight of each spanning tree and find the one with the smallest total weight.

Working

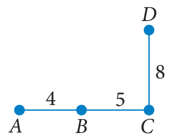
1. A spanning tree has 4 vertices and 3 edges. The original graph has 4 edges, so we need to remove one edge to create a spanning tree.

2.



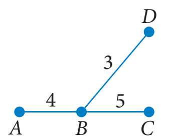
Remove edge *BC*.

Total weight of spanning tree = 15



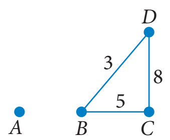
Remove edge *BD*.

Total weight of spanning tree = 17



Remove edge *CD*.

Total weight of spanning tree = 12



Remove edge *AB*.

This isn't a connected graph, so it's not a tree.

The total weight of the minimum spanning tree is 12.

Greedy algorithms

When graphs become more complex, it becomes too difficult to solve problems like shortest path and minimum connector by inspection. In these cases, we solve them by using a **greedy algorithm**. A greedy algorithm is a series of steps where at each step a choice is made that is the best at that moment. So, for a shortest path and minimum connector problem, it involves choosing the smallest weight at each time.

Prim's algorithm is a greedy algorithm that gives a series of steps to find a minimum spanning tree for a graph.

Prim's algorithm for finding a minimum spanning tree

1 Start at any vertex and choose the edge with the lowest weight connected to this vertex.
2 Look at *all* the edges connecting to the vertices you've chosen so far (*not just the last vertex connected*) and choose the edge with the lowest weight that doesn't connect to a vertex already in the tree. If there are edges with equal lowest weights, choose one of them.
3 Repeat step 2 until all the vertices in the graph are included in the tree.

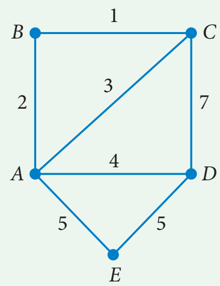
Exam hack

Questions don't always say to use Prim's algorithm, but you can use it for any minimum spanning tree problem.

WB p. 116

WORKED EXAMPLE 12 Finding minimum spanning trees using Prim's algorithm

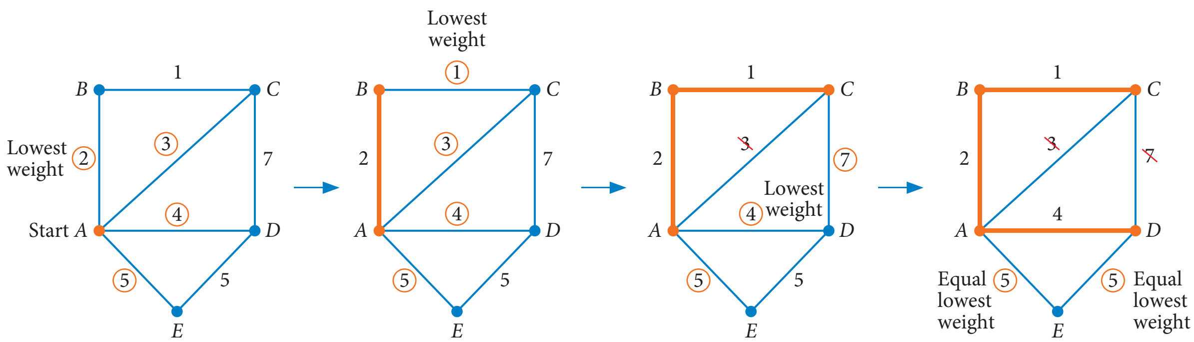
Use Prim's algorithm to find the minimum spanning tree for the weighted graph shown, and hence, find the total weight of the minimum spanning tree.



Steps

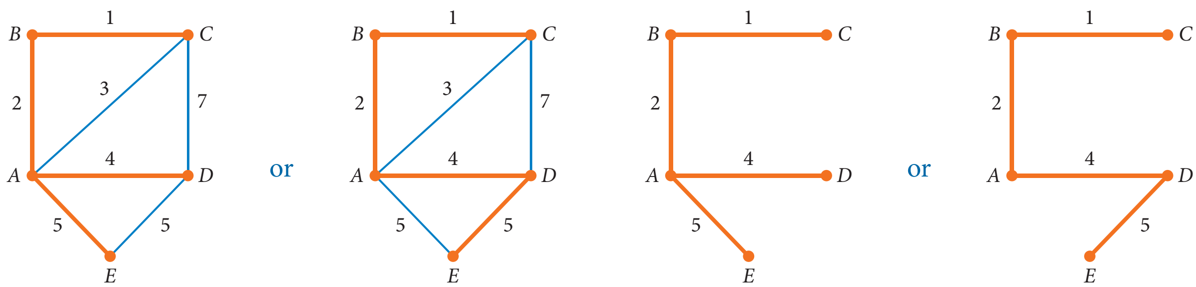
1 Start at any vertex and choose the edge with the lowest weight connected to this vertex.
2 Look at *all* the edges connecting to the vertices you've chosen so far (*not just the last vertex connected*) and choose the edge with the lowest weight that doesn't connect to a vertex already in the tree. If there are edges with equal lowest weights, choose one of them.
3 Repeat step **2** until all the vertices in the graph are included in the tree.

Working



With the next step all the vertices have been included:

This gives us two minimum spanning trees with the same total weight:



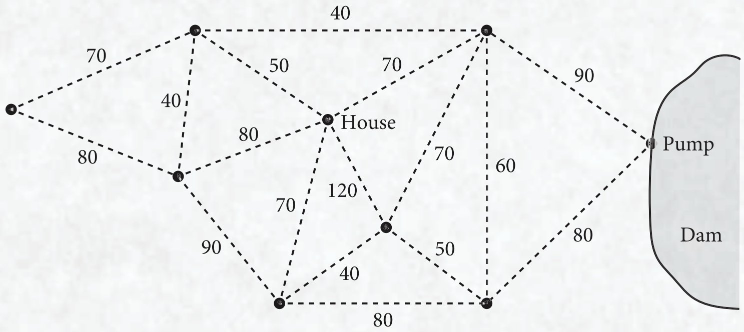
The total weight of the minimum spanning tree is $1 + 2 + 4 + 5 = 12$.

Video playlist
Introducing graphs and networks

VCE QUESTION ANALYSIS

© VCAA 2012 2NQ1 MODIFIED 2012 Examination 2 Networks and decision mathematics Question 1 (5 marks)

The total length of pipe that supplies water from a pump to the eight locations on a farm (including the house) is a minimum. This minimum length of pipe is laid along some of the edges in this network. (All distances are in metres.)



a Determine the shortest distance between the house and the pump. 1 mark

b How many vertices on the network diagram have an odd degree? 1 mark

c The farmer has found a way of travelling from his house to the pump and visiting all locations only once. What type of walk is this? 1 mark

d Copy the diagram and on it **draw** the minimum length of pipe that is needed to supply water to all locations on the farm. 1 mark

e What is the mathematical term that is used to describe this minimum length of pipe in part **d**? 1 mark

Reading the question

- What is the definition of a degree of a vertex?
- Be clear on the definitions of different types of walks.
- Part **d** asks you to draw the minimum length of pipe, not calculate its length.

Thinking about the question

- You need to know the difference between a shortest path problem and a minimum connector problem.
- Part **a** refers to the shortest distance between *two* vertices.
- Part **d** refers to the minimum distance between *all* the vertices.

Worked solution (✓ = 1 mark)

a By inspection the shortest distance is **160 metres**. ✓

b Counting the number of edges connecting to each vertex tells us there are **2** ✓ vertices that have odd degrees.

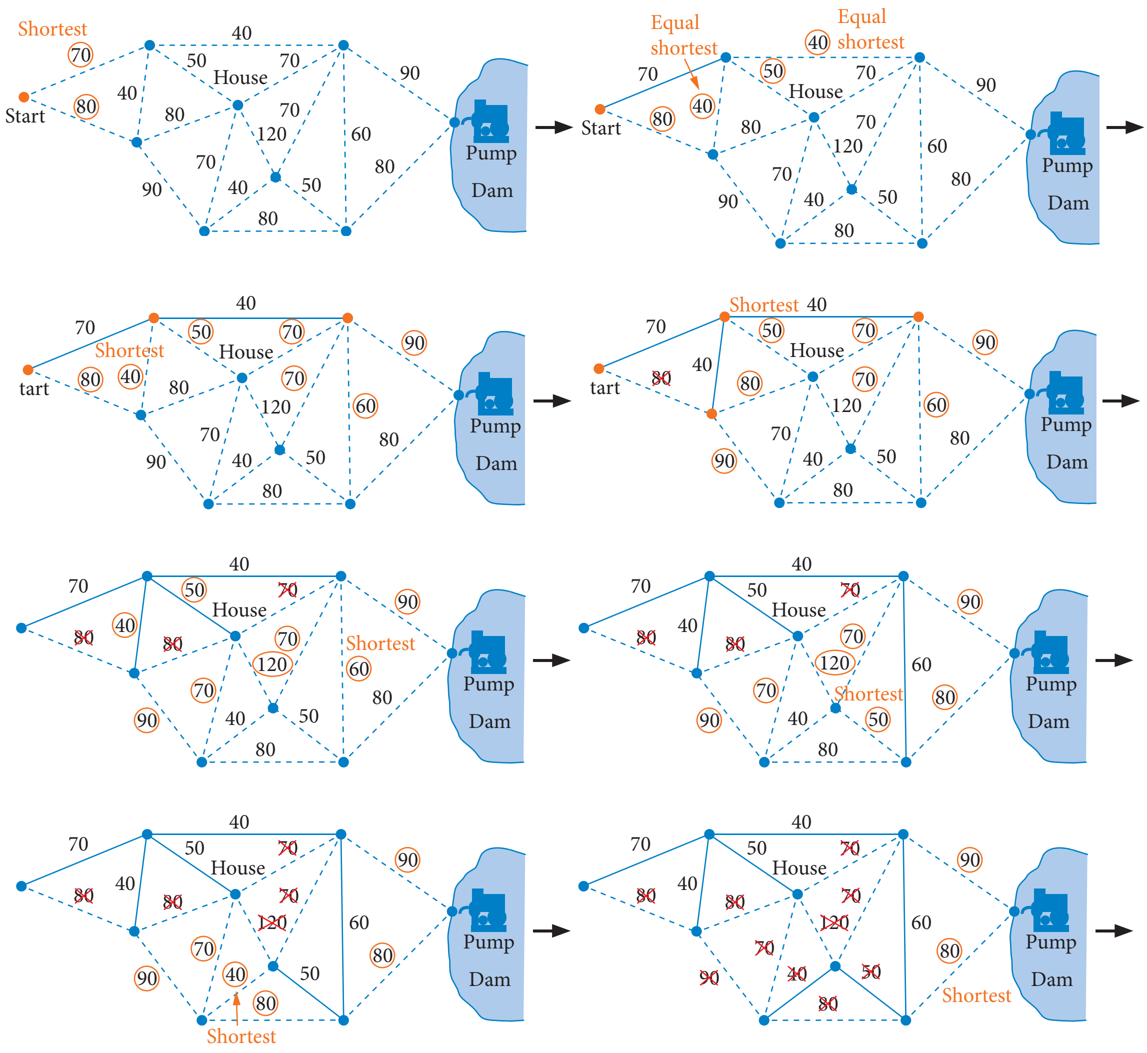
c Does the walk have repeated edges? No

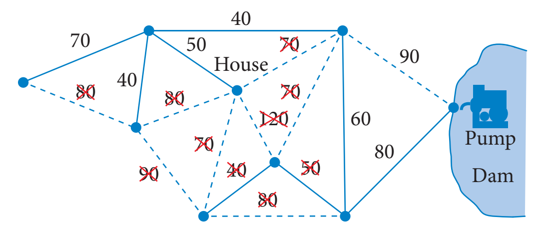
Does the walk have repeated vertices? No

Does the walk start and finish at the same vertex? No

The walk is a **path**. ✓

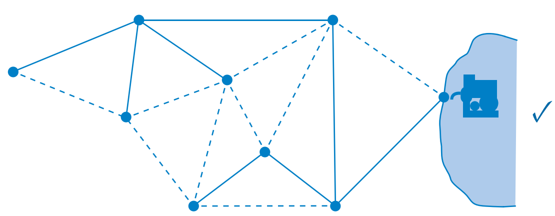
d Using Prim's algorithm:





All the vertices have been included.

Draw the minimum length of pipe that is needed to supply water to all locations on the farm.



e This is a **minimum spanning tree.** ✓

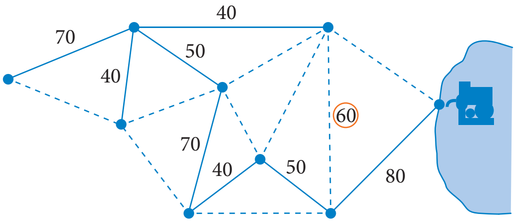
Student performance

a This was straightforward.

b A large number of incorrect responses ranged from 1 to 7.

c This relies on students being clear on the definitions of walks, trails, paths, cycles and circuits.

d This question was not well done by many students. The following was a common incorrect answer:



This answer may have been found by starting at the pump and then selecting the shortest edge from only the very last vertex connected, rather than any of the already-connected vertices. After choosing the third edge (from the pump) marked 40, the edge marked 60 should have been selected next, rather than just choosing the smallest edge that immediately followed the 40. Using Prim's algorithm is a weakness that needs to be addressed.

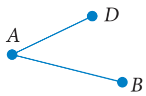
e Many students gave incorrect answers such as shortest path and others.

EXERCISE 7.4 Minimum spanning trees

ANSWERS p. 513

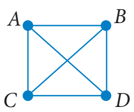
Recap

1 © VCAA 2010 1NQ1 MODIFIED The graph

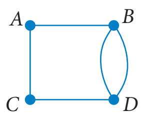


is a subgraph of which one of the following graphs?

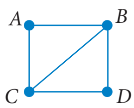
A



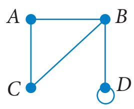
B



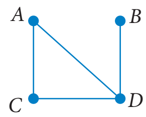
C



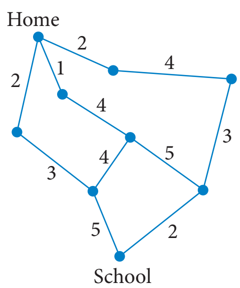
D



E



2 © VCAA 2017N 1NQ1 Hunter rides his bike to school each day. The edges of the network shown represent the roads that Hunter can use to ride to school. The numbers on the edges give the distance, in kilometres, along each road.

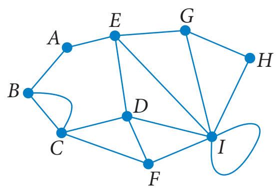


What is the shortest distance that Hunter can ride between home and school?

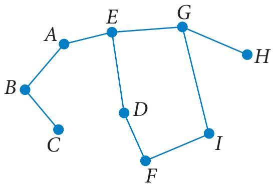
A 10 km **B** 11 km **C** 12 km **D** 14 km **E** 23 km

Mastery

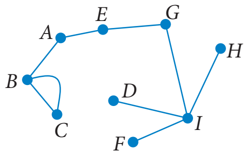
3 WORKED EXAMPLE 10 Which of the following graphs are a spanning tree of the graph on the right? For those that are spanning trees, verify that the number of edges is one less than the number of vertices. For those that aren't spanning trees, give a reason.



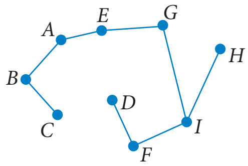
a



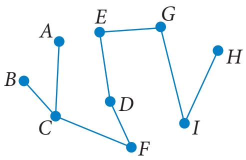
b



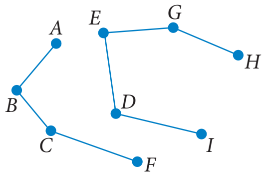
c



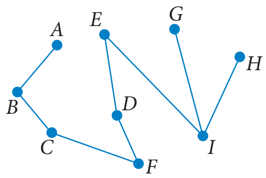
d



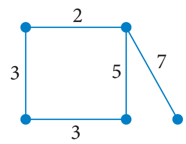
e



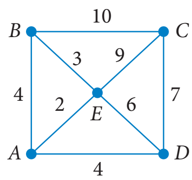
f



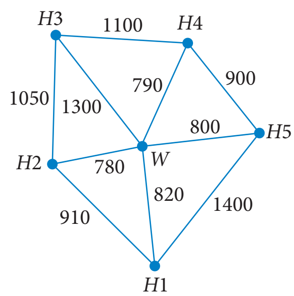
4 WORKED EXAMPLE 11 Find all the spanning trees for the network shown, and hence, find the total weight of the minimum spanning tree.



5 WORKED EXAMPLE 12 Use Prim's algorithm to find the minimum spanning tree for the weighted graph shown, and hence, find the total weight of the minimum spanning tree.



6 The Plumbtree plumbing company has been hired to connect five huts in a national park with running water. The weighted graph shows the distances, in metres, from the water source W to each of the five huts. Use Prim's algorithm to find the minimum spanning tree, and hence, find the minimum length of piping needed to connect to all five houses.



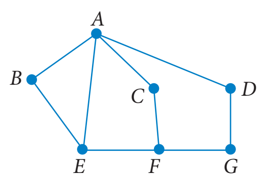
Exam practice

80–100% 60–79% 0–59%

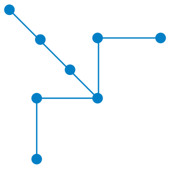
7 The minimum number of edges to be removed from this graph to create a spanning tree is

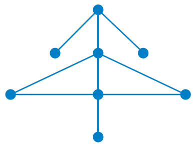
A 1 **B** 2 **C** 3

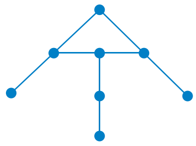
D 4 **E** 5

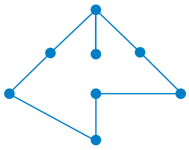


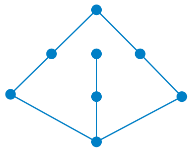
8 © VCAA 2013 1NQ1 **91%** Which of the following graphs is a tree?

A


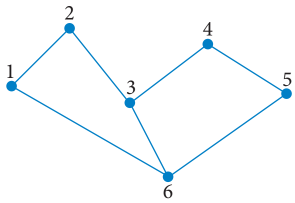
B


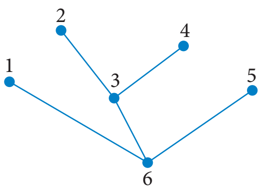
C


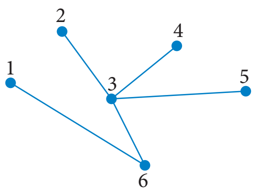
D


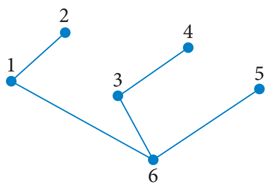
E


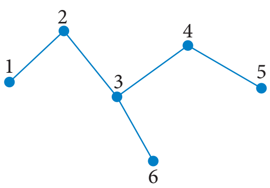
9 © VCAA 2020 1NQ3 **81%** Which one of the following is **not** a spanning tree for the network?

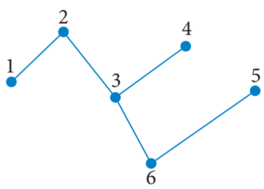


A


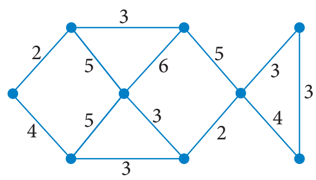
B


C


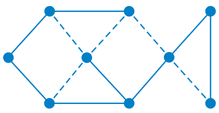
D


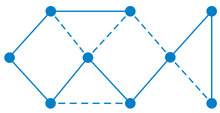
E


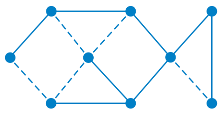
10 © VCAA 2013 1NQ3 **78%** The vertices of the graph represent nine computers in a building. The computers are to be connected with optical fibre cables, which are represented by edges. The numbers on the edges show the costs, in hundreds of dollars, of linking these computers with optical fibre cables.

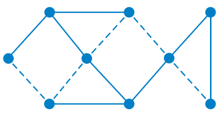


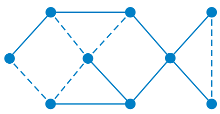
Based on the same set of vertices and edges, which one of the following graphs shows the cable layout (in bold) that would link all the computers with optical fibre cables for the minimum cost?

A


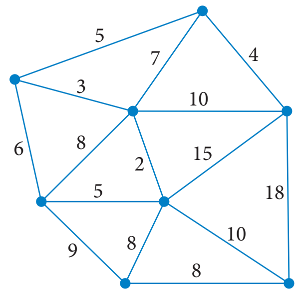
B


C


D


E


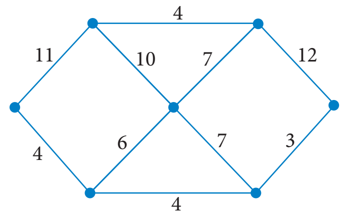
11 © VCAA 2010 1NQ5 76% For this network, the length of the minimum spanning tree is



A 30 **B** 31 **C** 35

D 39 **E** 45

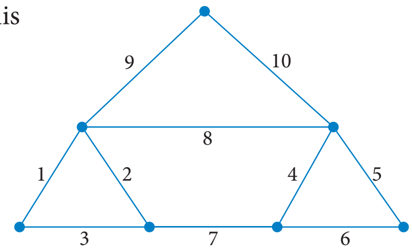
12 © VCAA 2004 1NQ3 75% The length of the minimum spanning tree for this network is



A 15 **B** 22 **C** 28

D 34 **E** 35

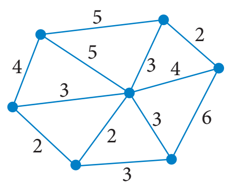
13 © VCAA 2006 1NQ4 70% The minimum spanning tree for this network will include the edge that has a weight of



A 3 **B** 6 **C** 8

D 9 **E** 10

14 © VCAA 2008 2NQ1 (3 marks) James, Dante, Tahlia and Chanel are four children playing a game. In this children's game, seven posts are placed in the ground. The network shows the distances, in metres, between the seven posts. The aim of the game is to connect the posts with ribbon using the shortest length of ribbon. This will be a minimum spanning tree.

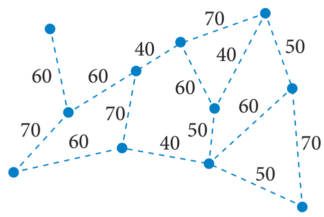


a Copy the diagram and draw a minimum spanning tree for this network on it. 1 mark

b Determine the length, in metres, of this minimum spanning tree. 1 mark

c How many different minimum spanning trees can be drawn for this network? 1 mark

15 © VCAA 2011 2NQ2 (2 marks) At the Farnham showgrounds, eleven locations require access to water. These locations are represented by vertices on the network diagram shown. The dashed lines on the network diagram represent possible water pipe connections between adjacent locations. The numbers on the dashed lines show the minimum length of pipe required to connect these locations in metres.



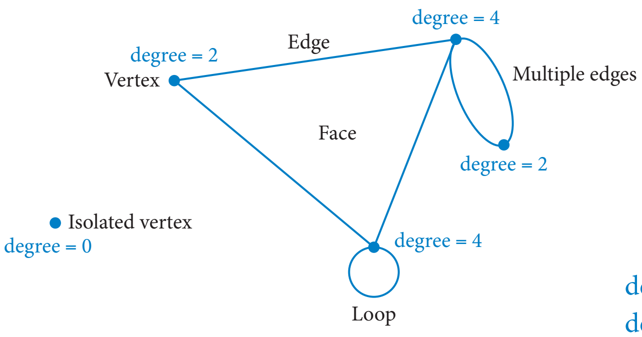
All locations are to be connected using the smallest total length of water pipe possible.

a Copy the diagram and on it show where these water pipes will be placed. 1 mark

b Calculate the total length, in metres, of water pipe that is required. 1 mark

7 Chapter summary

Features of a graph



degree sum = 2 + 4 + 2 + 4 + 0 = 12

degree sum = 2 × number of edges = 2 × 6 = 12

Adjacency matrices

Adjacency matrices

- show the number of connections between each pair of vertices of a graph
- are always square matrices
- have row 1 = column 1, row 2 = column 2 and so on.

Types of graphs

- **Isomorphic graphs** are graphs that show exactly the same connections.
- A **planar graph** is a connected graph that can be drawn so that it doesn't have any edges crossing.
- A **connected graph** is a graph where there is a path from any vertex to any other vertex.
- A **subgraph** is part of a larger graph.
- A **weighted graph** is a graph where extra information such as distances, times or costs is labelled on the edges.
- A **tree** is a connected graph with no loops, multiple edges, or cycles.
  The number of edges in a tree is always one less than the number of vertices.
- A **spanning tree** is a tree subgraph that connects all the vertices of the original graph.
  Every connected graph has at least one spanning tree.
- A **minimum spanning tree** is the spanning tree with the smallest total weight.

Euler's formula

- For connected planar graphs

  $v + f - e = 2$

  where

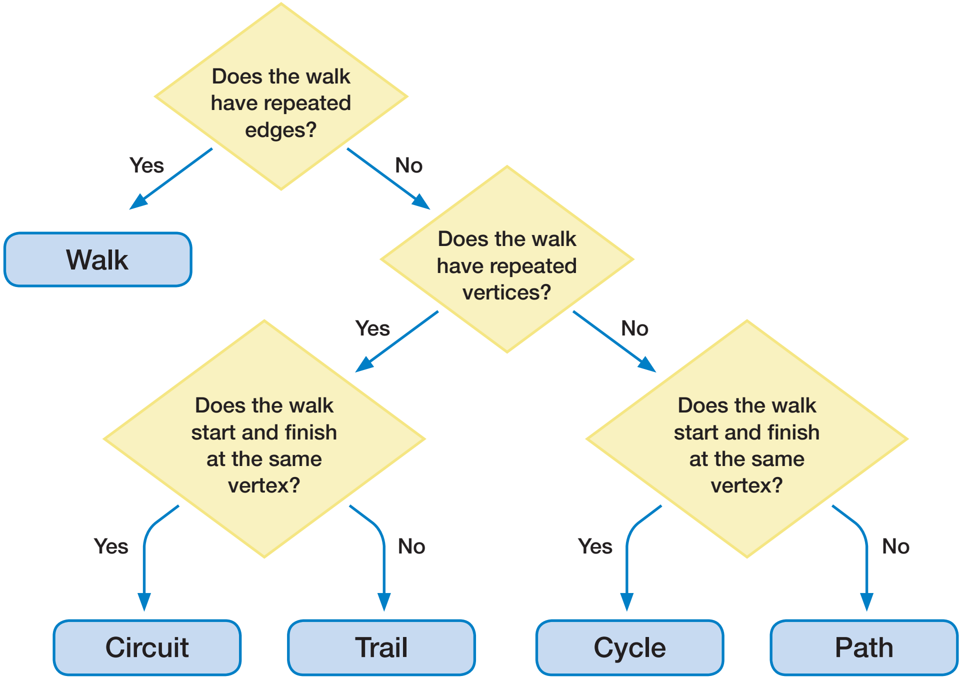
  v = the number of vertices

  f = the number of faces

  e = the number of edges.

Types of walks

- A **walk** is a sequence of connected vertices.
- A **trail** is a walk with no repeated edges.
- A **circuit** is a walk with no repeated edges that starts and finishes at the same vertex.
- A **path** is a walk with no repeated vertices.
- A **cycle** is a walk with no repeated vertices that starts and finishes at the same vertex.



Walk summary

Walks that:	Trail	Circuit	Path	Cycle
have no repeated edges	✓	✓	✓	✓
have no repeated vertices			✓	✓
start and finish at the same vertex		✓		✓

Finding the shortest path

- A **weighted graph** is a graph where extra information is labelled on the edges.
- The **shortest path** can be found by inspection if the graph isn't complex.

Finding a minimum spanning tree

- A **tree** is a connected graph with no loops, multiple edges, or cycles.
- The number of edges in a tree is always one less than the number of vertices.
- A **spanning tree** is a tree subgraph that connects all the vertices of the original graph.
- A **minimum spanning tree** is the spanning tree with the smallest total weight.
- Use **Prim's algorithm** for complex graphs:
  1 Start at any vertex and choose the edge with the lowest weight connected to this vertex.
  2 Look at all the edges connecting to the vertices you've chosen so far (not just the last vertex connected) and choose the edge with the lowest weight that doesn't connect to a vertex already in the tree. If there are edges with equal lowest weights, choose one of them.
  3 Repeat step 2 until all the vertices in the graph are included in the tree.
- Prim's algorithm is an example of a **greedy algorithm**, which is a series of steps where at each step a choice is made that is the best at that moment.

Cumulative examination 1

Total number of marks: 13 Reading time: 5 minutes Writing time: 30 minutes

1 © VCAA 2018N 1CQ5 The variables *height* (less than 1.83 m, 1.83 m and over) and *enthusiasm for playing basketball* (low, medium, high) are

A both ordinal variables.

B both nominal variables.

C a nominal and an ordinal variable respectively.

D an ordinal and a nominal variable respectively.

E a numerical and an ordinal variable respectively.

2 An asset purchased for \$10 000 is depreciated by a flat rate of 15% in the first year and a flat rate of 30% each year thereafter. Which calculation gives the value of the asset after three years?

A 10 000 – 1500 – 3000 – 3000 **B** 10 000 – 1500 – 3000

C 10 000 – 3000 – 3000 – 3000 **D** 10 000 – 1500 – 1500 – 3000

E 10 000 – 1500 – 1500 – 1500

3 © VCAA 2014 1BRMQ2 An internet car market site charges \$120 to advertise a car for sale. The car is sold for \$15 000. The \$120 charge as a percentage of the selling price of the car is

A 0.008% **B** 0.08% **C** 0.80% **D** 1.20% **E** 1.25%

4 © VCAA 2016S 1GRQ5 The Domestics Cleaning Company provides household cleaning services. For two hours of cleaning, the cost is \$55. For four hours of cleaning, the cost is \$94. The rule for the cost of cleaning services is *cost* = $a + b \times$ *hours* where a is a fixed charge, in dollars, and b is the charge per hour of cleaning, in dollars per hour. Using this rule, the cost for five hours of cleaning is

A \$19.50 **B** \$97.50 **C** \$99.50 **D** \$113.50 **E** \$121.50

5 © VCAA 2015 1MQ2 Four matrices are shown.

$$W = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \quad X = \begin{bmatrix} 4 & 1 & 5 \\ 2 & 0 & 6 \end{bmatrix} \quad Y = \begin{bmatrix} 7 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 8 & 5 & 0 \\ 1 & 9 & 3 \\ 4 & 2 & 7 \end{bmatrix}$$

Which one of the following matrix products is **not** defined?

A $W \times Y$ **B** $X \times W$ **C** $Y \times X$ **D** $Z \times W$ **E** $Z \times Y$

6 The best description the association between the two variables shown in the scatterplot is

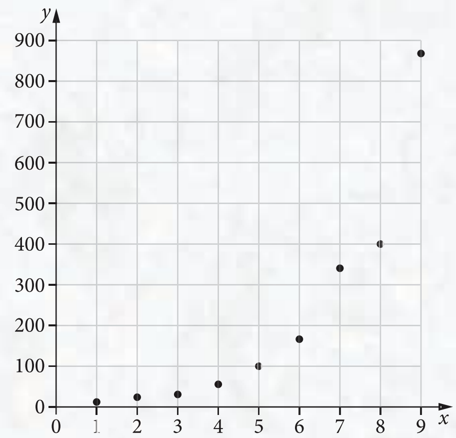
A no association

B weak

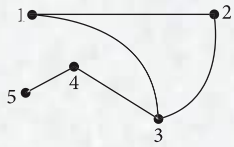
C non-linear

D moderate

E linear



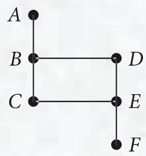
7 © VCAA 2002 1NQ3 Which one of the following adjacency matrices could be used to represent the graph?



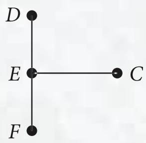
A $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

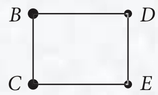
D $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

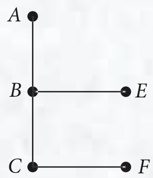
8 Which of the following graphs is not a subgraph of this graph?

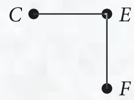


A 

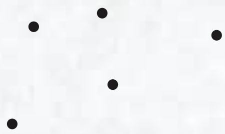
B 

C 

D 

E 

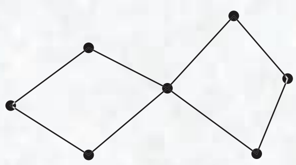
9 © VCAA 2018 1NQ1 Consider the graph with five isolated vertices shown.



To form a tree, the minimum number of edges that must be added to the graph is

A 1 **B** 4 **C** 5 **D** 6 **E** 10

10 © VCAA 2018N 1NQ1 Consider the following graph.



Which one of the following statements is **not** true for this graph?

A This graph has seven vertices.

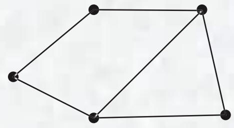
B There are no isolated vertices.

C All vertices have an even degree.

D Six of the vertices have the same degree.

E The sum of the degrees of the vertices is 14.

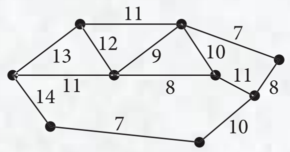
11 © VCAA 2019N 1NQ2 Consider the graph. Euler's formula will be verified for this graph. What values of e, v and f will be used in this verification?



A $e = 5, v = 5, f = 2$ **B** $e = 5, v = 5, f = 3$ **C** $e = 6, v = 5, f = 2$

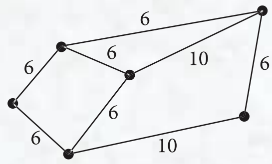
D $e = 6, v = 5, f = 3$ **E** $e = 6, v = 6, f = 3$

12 © VCAA 2019N 1NQ5 In the graph shown, the vertices represent electricity transformer substations. The numbers on the edges of the graph show the length, in kilometres, of cables that connect these substations. What is the minimum length of cable, in kilometres, that is necessary to make sure that each substation remains connected to the network?



A 65 **B** 71 **C** 73 **D** 74 **E** 77

13 © VCAA 2017N 1NQ7 Consider the weighted graph shown.



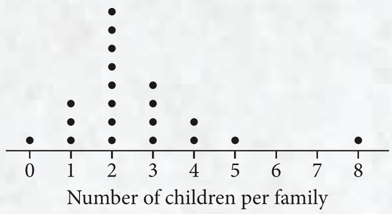
How many different minimum spanning trees are possible?

A 2 **B** 3 **C** 4 **D** 5 **E** 6

Cumulative examination 2

Total number of marks: 17 Reading time: 4 minutes Writing time: 26 minutes

1 (4 marks) This dot plot shows the number of children in each family living on Willard Crescent.



a What is the mode? 1 mark

b What is the median? 1 mark

c If the outlier 8 is removed from the data set, explain why there is no effect on

i the mode 1 mark

ii the median 1 mark

2 © VCAA 2019 2CQ7d MODIFIED (1 mark) Phil is a builder who has purchased a large set of tools for $60 000. He depreciates the value of the tools by a flat rate of 8% of the purchase price per annum. Let V_n be the value of the tools after n years, in dollars. Write down a recurrence relation that could be used to model the value of the tools using this flat rate depreciation.

3 © VCAA 2019N 2MQ2 (2 marks) Three television channels, C_1, C_2 and C_3, will broadcast the International Games in the town of Gillen. Gillen's 2000 residents are expected to change television channels from hour to hour as shown in the transition matrix T below. The option for residents not to watch television (*NoTV*) at that time is also indicated in the transition matrix.

$$\begin{array}{cc} & \text{This hour} \\ & \begin{array}{cccc} C_1 & C_2 & C_3 & NoTV \end{array} \\ T = & \begin{bmatrix} 0.50 & 0.05 & 0.10 & 0.20 \\ 0.10 & 0.60 & 0.20 & 0.20 \\ 0.25 & 0.10 & 0.50 & 0.10 \\ 0.15 & 0.25 & 0.20 & 0.50 \end{bmatrix} \begin{array}{l} C_1 \\ C_2 \\ C_3 \\ NoTV \end{array} \text{ Next hour} \end{array}$$

The state matrix G_0 below lists the number of Gillen residents who are expected to watch the games on each of the channels at the start of a particular day (9.00 am). Also shown is the number of Gillen residents who are not expected to watch television at that time.

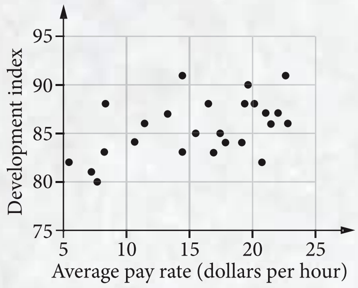
$$G_0 = \begin{bmatrix} 100 \\ 400 \\ 100 \\ 1400 \end{bmatrix} \begin{array}{l} C_1 \\ C_2 \\ C_3 \\ NoTV \end{array}$$

a Copy and complete the calculation below to show that 835 Gillen residents are not expected to watch television (*NoTV*) at 10.00 am that day. 1 mark

$$\square \times 100 + \square \times 400 + \square \times 100 + \square \times 1400 = 835$$

b Determine the number of residents expected to watch the games on C_3 at 11.00 am that day. 1 mark

4 © VCAA 2013 2CQ3 MODIFIED (3 marks) The development index and the average pay rate for workers, in dollars per hour, for a selection of 25 countries are displayed in the scatterplot. The table contains the values of some statistics that have been calculated for this data.



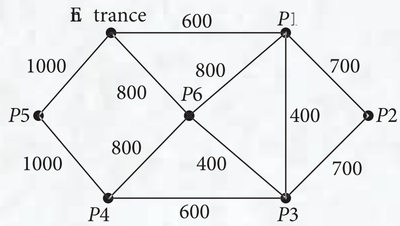
Statistic	Average pay rate (x)	Development index (y)
Mean	$\bar{x} = 15.7$	$\bar{y} = 85.6$
Standard deviation	$s_x = 5.37$	$s_y = 2.99$

a What is the response variable? 1 mark

b What is the strength and direction of the association? 1 mark

c Which of the variables has the greater spread? Quote two statistics to give a reason for your answer. 1 mark

5 © VCAA 2013 2NQ1 MODIFIED (4 marks) The vertices in the network diagram show the entrance to a wildlife park and six picnic areas in the park: $P1$, $P2$, $P3$, $P4$, $P5$ and $P6$. The numbers on the edges represent the lengths, in metres, of the roads joining these locations.



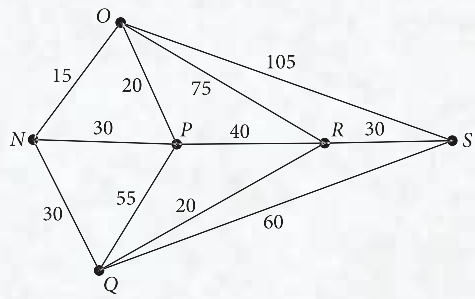
a In this graph, what is the degree of the vertex at the entrance to the wildlife park? 1 mark

b What is the shortest distance, in metres, from the entrance to picnic area $P3$? 1 mark

c A park cleaner follows a route that starts at the entrance and passes through each picnic area once, ending at picnic area $P1$. Write down the order in which the park cleaner will visit the six picnic areas. 1 mark

d If the cleaner then walks back to the entrance, what is the mathematical term for the walk he has taken? 1 mark

6 © VCAA 2017 2NQ1 (3 marks) Bus routes connect six towns. The towns are Northend (N), Opera (O), Palmer (P), Quigley (Q), Rosebush (R) and Seatown (S). The graph gives the cost, in dollars, of bus travel along these routes. Bai lives in Northend (N) and he will travel by bus to take a holiday in Seatown (S).



a Bai considers travelling by bus along the route Northend (N) – Opera (O) – Seatown (S). How much would Bai have to pay? 1 mark

b If Bai takes the cheapest route from Northend (N) to Seatown (S), which other town(s) will he pass through? 1 mark

c Euler's formula, $v + f = e + 2$, holds for this graph. Copy and complete the formula by writing the appropriate numbers in the boxes. 1 mark

$$\boxed{}_{v} + \boxed{}_{f} = \boxed{}_{e} + \boxed{2}$$

CHAPTER

8

VARIATION

Study Design coverage

UNIT 2, AREA OF STUDY 3: FUNCTIONS, RELATIONS AND GRAPHS

Variation

- numerical, graphical and algebraic approaches to direct and inverse variation
- transformation of data to linearity to establish relationships between variables, for example y and x^2, y and $\frac{1}{x}$, and y and $\log_{10}(x)$
- modelling of given non-linear data using the relationships $y = kx^2 + c$, $y = \frac{k}{x}$, where $k > 0$, and $y = k\log_{10}(x) + c$, where $k > 0$.

Video playlists (4):

8.1 Direct and inverse variation
8.2 Transforming non-linear data
8.3 Modelling non-linear data
VCE question analysis Variation

Worksheets (2):

8.1 Direct and inverse proportion • Variation problems

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

8.1 Direct and inverse variation

8.1

Direct variation

Variation describes how one variable changes in relation to another variable.

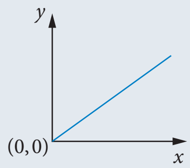
Direct variation occurs when the value of one variable is a constant multiple of the corresponding value of the other variable, so the percentage increase in one variable results in the *same percentage increase* in the other variable. We know there is direct variation between two variables if doubling one variable results in *doubling the other*. The equation for the relationship is $y = kx$, and its graph is a straight line passing through the origin at $(0, 0)$.

The k in variation equations is called the **constant of variation**. We will only be looking at examples where k is positive.

Direct variation ($k > 0$)

- y varies directly with x.
- $y = kx$, where k is a constant.
- As x increases, y *increases* by the same percentage.
- If x is doubled, y is *doubled*.

The graph looks like this:



Video playlist
Direct and inverse variation

Worksheets
Direct and inverse proportion

Variation problems

WB

p. 117

WORKED EXAMPLE 1 Working with direct variation

The cost of carrots, c (\$), varies directly with their total weight, w (kg), as shown in the table.

w (kg)	0.5	1	1.5	2
c (\$)				4.40

a Write an equation showing this variation including k, the constant of variation.

b Find the value of k.

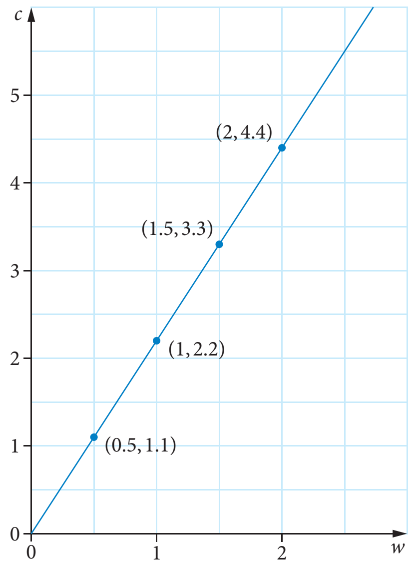
c Copy and complete the table.

d Draw a graph of the variation, labelling the values from the table.

e Show a calculation from the table of values that verifies that doubling the weight doubles the cost.

f Find the cost of 12 kg of carrots.

Steps	Working
a Write the equation in terms of the given variables and the constant of variation, k.	$c = kw$
b Substitute a pair of known values into the equation and solve for k, using CAS if necessary.	When $w = 2$, $c = 4.4$ $4.4 = k \times 2$ $k = \frac{4.4}{2}$ $= 2.2$

c	Write the equation using the value of k and substitute the values from the table into the equation to complete the table.	$c = 2.2w$ For $w = 0.5$, $c = 2.2 \times 0.5 = 1.1$ For $w = 1$, $c = 2.2 \times 1 = 2.2$ For $w = 1.5$, $c = 2.2 \times 1.5 = 3.3$
d	Plot the points from the table.	
e	Use the simplest example from the table for the calculation.	0.5 kg costs \$1.10. 1 kg costs $2 \times 1.1 = 2.2 =$ \$2.20.
f	Substitute the value into the variation equation and solve, using CAS if necessary.	Substitute $w = 12$: $c = 2.2w$ $= 2.2 \times 12$ $= 26.4$ The cost of 12 kg of carrots is \$26.40.

w (kg)	0.5	1	1.5	2
c (\$)	1.10	2.20	3.30	4.40

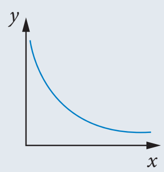
Inverse variation

Inverse variation occurs when the percentage increase in one variable results in the *same percentage decrease* in the other variable. We know there is inverse variation between two variables if doubling one variable results in *halving the other*. The equation for the relationship is $y = \frac{k}{x}$ and its graph is a curve that approaches but never reaches zero.

Inverse variation ($k > 0$)

- y varies inversely with x.
- $y = \frac{k}{x}$, where k is a constant.
- As x increases, y *decreases* by the same percentage.
- If x is doubled, y is *halved*.

The graph looks like this:



8.1

WORKED EXAMPLE 2 Working with inverse variation

For a team of school cleaners, the number of cleaners, n, varies inversely with the time, t (minutes), it takes to clean the school, as shown in the table.

n	1	2	3	4
t (minutes)			60	

a Write an equation showing this variation including k, the constant of variation.

b Find the value of k.

c Copy and complete the table.

d Draw a graph of the variation, labelling the values from the table.

e Show a calculation from the table of values that verifies that doubling the number of cleaners halves the amount of time it takes to clean the school.

f Find how many minutes it would take 10 cleaners to clean the school.

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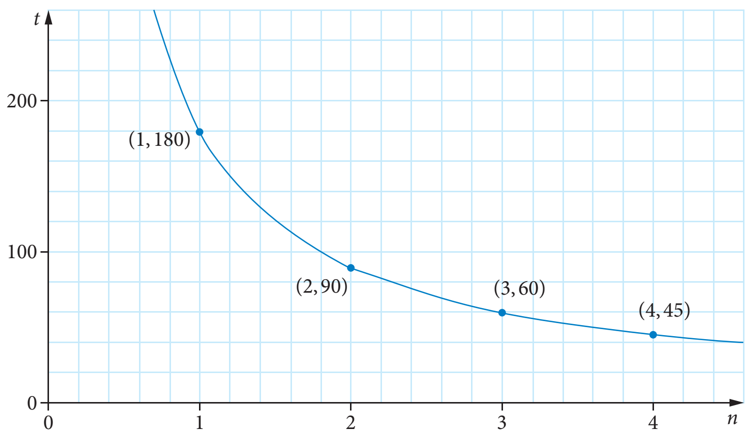
Steps	Working
a Write the equation in terms of the given variables and the constant of variation, k.	$t = \frac{k}{n}$
b Substitute a pair of known values into the equation and solve for k, using CAS if necessary.	When $n = 3$, $t = 60$ $60 = \frac{k}{3}$ $k = 60 \times 3$ $= 180$
c Write the equation using the value of k and substitute the values from the table into the equation to complete the table.	$t = \frac{180}{n}$ For $n = 1$, $t = \frac{180}{1} = 180$ For $n = 2$, $t = \frac{180}{2} = 90$ For $n = 4$, $t = \frac{180}{4} = 45$
d Plot and label the points from the table.	
e Use the simplest example from the table for the calculation.	Two cleaners take 90 minutes to clean the school. Four cleaners take $90 \div 2 = 45$ minutes to clean the school.

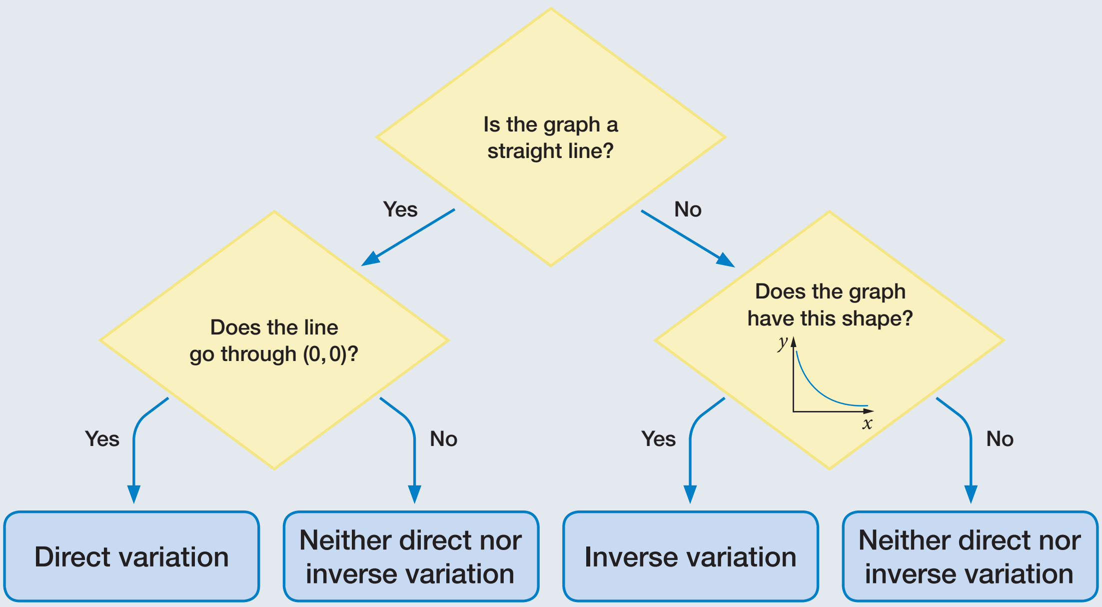
n	1	2	3	4
t (minutes)	180	90	60	45

f Substitute the value into the variation equation and solve, using CAS if necessary.	Substitute $n = 10$: $t = \dfrac{180}{n}$ $= \dfrac{180}{10}$ $= 18$ It would take 10 cleaners 18 minutes to clean the school.

Direct versus inverse variation

To determine whether two variables vary directly or inversely we can look at the features of the graph of the equation.

Identifying direct and inverse variation from a graph



WORKED EXAMPLE 3 Identifying direct and inverse variation from a graph

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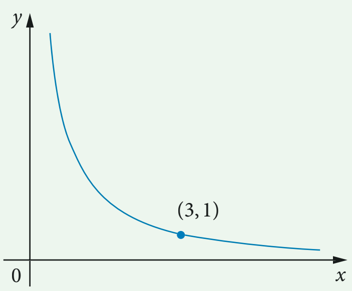
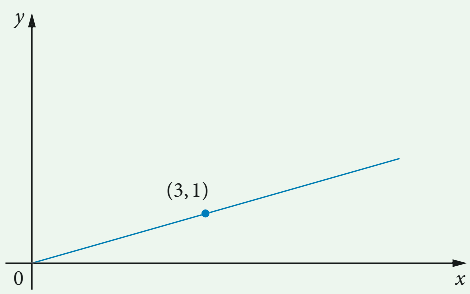
For each of the following, state whether direct, inverse or neither variation is involved, giving a reason for your answer. If the variation is direct or inverse, find the equation of variation.

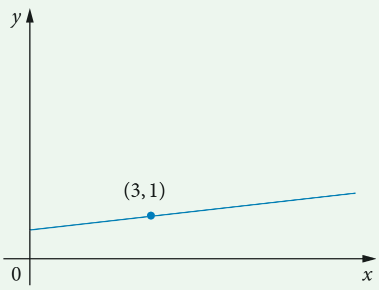
a



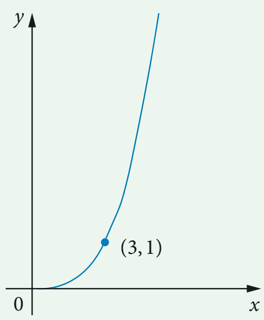
b

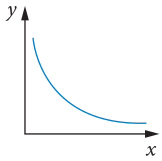
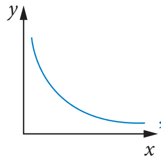
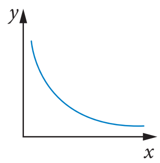
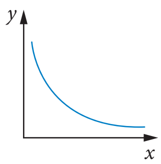
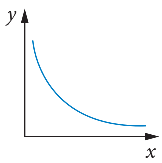
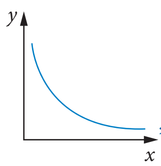


c



d



Steps	Working
a **1** Is the graph a straight line? Does the line go through $(0, 0)$? Does the graph have this shape? 	The graph has the shape  , so inverse variation is involved.
2 Using either $y = kx$ or $y = \frac{k}{x}$, substitute a point in and solve to find k, using CAS if necessary.	Using $y = \frac{k}{x}$ and $(3, 1)$, we get $1 = \frac{k}{3}$ $k = 1 \times 3$ $= 3$ The equation of variation is $y = \frac{3}{x}$.
b **1** Is the graph a straight line? Does the line go through $(0, 0)$? Does the graph have this shape? 	The graph is a straight line that goes through $(0, 0)$, so direct variation is involved.
2 Using either $y = kx$ or $y = \frac{k}{x}$, substitute a point in and solve to find k, using CAS if necessary.	Using $y = kx$ and $(3, 1)$, we get $1 = k \times 3$ $k = \frac{1}{3}$ The equation of variation is $y = \frac{1}{3}x$.
c Is the graph a straight line? Does the line go through $(0, 0)$? Does the graph have this shape? 	The graph is a straight line that doesn't go through $(0, 0)$, so neither direct nor inverse variation is involved.
d Is the graph a straight line? Does the line go through $(0, 0)$? Does the graph have this shape? 	The graph isn't a straight line and it doesn't have the shape  , so neither direct nor inverse variation is involved.

WB

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WORKED EXAMPLE 4 Identifying direct and inverse variation from a graph without (0, 0)

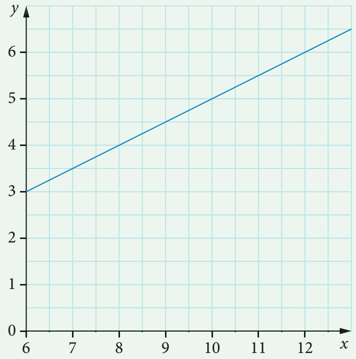
For the following variation graphs

i describe what is unusual about the horizontal axis

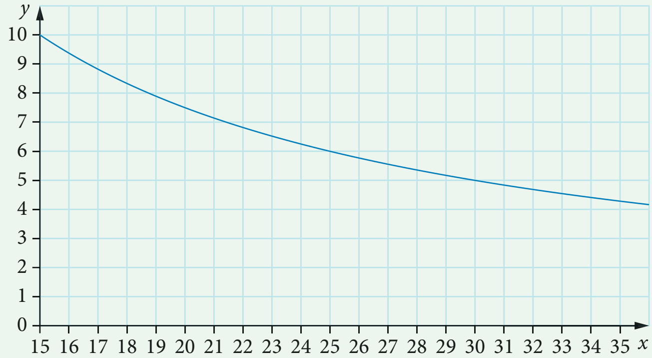
ii show a calculation that verifies the type of variation

iii find the variation equation.

a



b



Steps	Working
a **i** What is the first value on the horizontal axis?	The horizontal axis doesn't start at zero.
ii Find two points that can be read from the graph where one x value is double the other x value. Find what happens to the value.	Two points where one x value is double the other x value are (6, 3) and (12, 6). Double 6 is 12 and double 3 is 6. So, this is direct variation.
iii Using either $y = kx$ or $y = \frac{k}{x}$, substitute a point in and solve to find k, using CAS if necessary.	Using $y = kx$ and (6, 3), we get $3 = k \times 6$, $k = \frac{3}{6} = \frac{1}{2}$. The equation of variation is $y = \frac{1}{2}x$.
b **i** What is the first value on the horizontal axis?	The horizontal axis doesn't start at zero.
ii Find two points that can be read from the graph where one x value is double the other x value. Find what happens to the value.	Two points where one x value is double the other x value are (15, 10) and (30, 5). Double 15 is 30 and halving 10 is 5. So, this is inverse variation.
iii Using either $y = kx$ or $y = \frac{k}{x}$, substitute a point in and solve to find k, using CAS if necessary.	Using $y = \frac{k}{x}$ and (15, 10), we get $15 = \frac{k}{10}$, $k = 15 \times 10 = 150$. The equation of variation is $y = \frac{150}{x}$.

Exam hack

If the straight-line graph is drawn in a way that it's hard to tell if it goes through (0, 0), check to see if you can find values on the graph where doubling the x value results in doubling the y value to show that direct variation is involved.

EXERCISE 8.1 Direct and inverse variation

ANSWERS p. 513

Mastery

1 WORKED EXAMPLE 1 The cost of potatoes, c (\$), varies directly with their total weight, w (kg), as shown in the table.

w (kg)	0.5	1	1.5	2
c (\$)			3.90	

a Write an equation showing this variation including k, the constant of variation.

b Find the value of k.

c Copy and complete the table.

d Draw a graph of the variation, labelling the values from the table.

e Show a calculation from the table of values that verifies that doubling the weight doubles the cost.

f Find the cost of 15 kg of potatoes.

2 WORKED EXAMPLE 2 For a team of painters, the number of painters, n, varies inversely with the time, t (hours), it takes to paint an apartment block, as shown in the table.

n	1	2	3	4
t (hours)				33

a Write an equation showing this variation including k, the constant of variation.

b Find the value of k.

c Copy and complete the table.

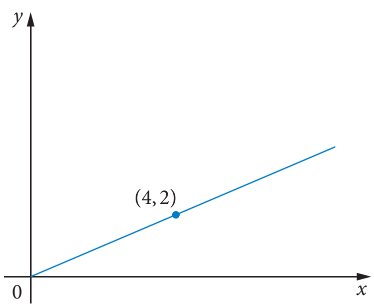
d Draw a graph of the variation, labelling the values from the table.

e Show a calculation from the table of values that verifies that doubling the number of painters halves the amount of time it takes to paint an apartment block.

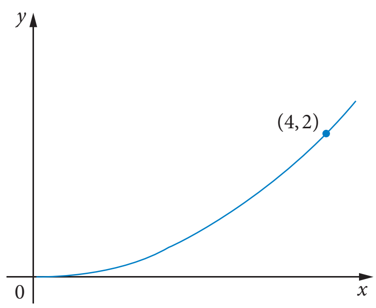
f Find how many hours it would take 12 painters to paint an apartment block.

3 WORKED EXAMPLE 3 For each of the following, state whether direct, inverse or neither variation is involved, giving a reason for your answer. If the variation is direct or inverse, find the equation of variation.

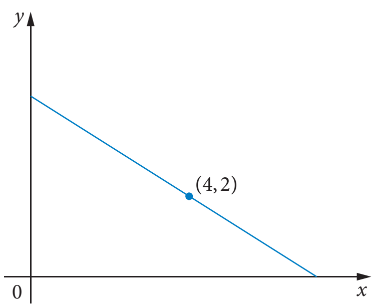
a



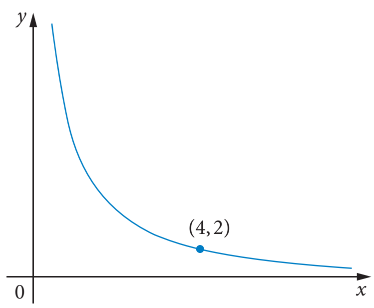
b



c



d



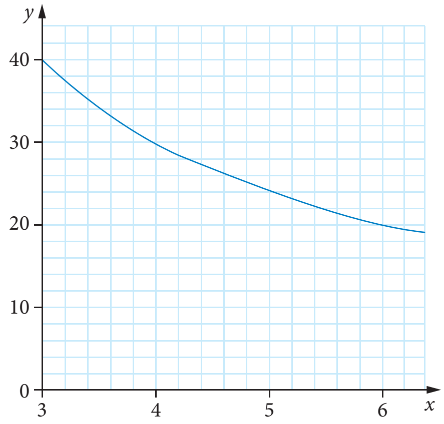
4 WORKED EXAMPLE 4 For the following variation graphs

i describe what is unusual about the horizontal axis

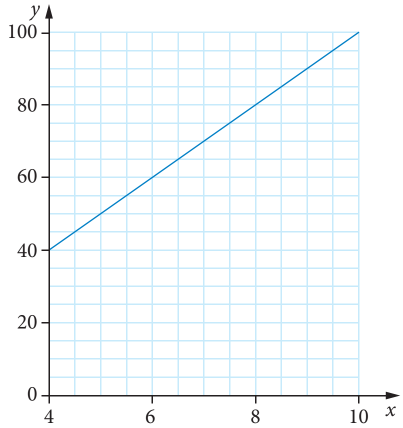
ii show a calculation that verifies the type of variation

iii find the variation equation.

a



b

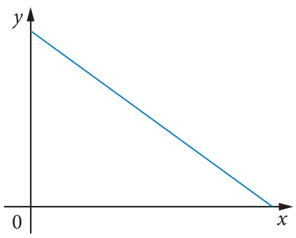


Exam practice

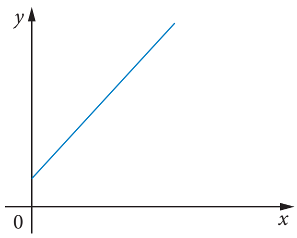
80–100% 60–79% 0–59%

5 Which one of the following graphs represents direct variation?

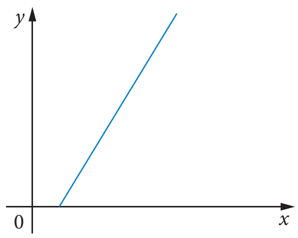
A



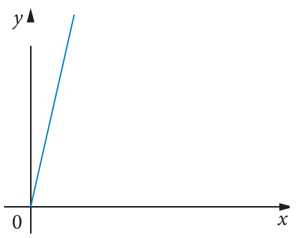
B



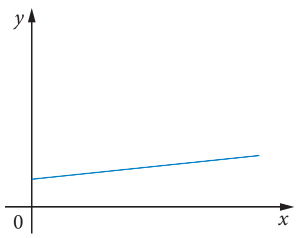
C



D



E



6 Which one of the following equations represents inverse variation?

A $y = 4x$ **B** $y = \frac{x}{4}$ **C** $y = x + 4$ **D** $y = 4 - x$ **E** $y = \frac{4}{x}$

7 How many of the following equations represent direct variation?

$y = \frac{2}{3}x$ $y = 2x + 3$ $y = \frac{x}{23}$ $y = \frac{23}{x}$

A 0 **B** 1 **C** 2 **D** 3 **E** 4

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8.1

8 In which of the following is y **not** varying directly with x?

A

x	1	2	3	4
y	5	10	15	20

B

x	10	20	30	40
y	2	4	6	8

C

x	0	1	2	3
y	0	4	8	12

D

x	1	2	3	4
y	4	7	10	13

E

x	6	9	12	15
y	3	4.5	6	7.5

9 Which of the following is **not** true about inverse variation?

A When one variable doubles, the other variable halves.

B When one variable increases, the other variable decreases by the same percentage.

C The graph of the variation equation is a curve that never reaches zero.

D If one variable increases by 30%, the other variable decreases by 30%.

E The equation is in the form $y = \frac{k}{x}$, where k is a variable.

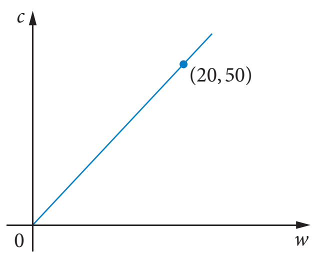
10 The cost per person, c (\$), of hiring a venue for a party varies inversely with the number of people, n, attending, as shown in the table.

n	30	50	100	120
c (\$)	120	72	36	30

The variation equation for this is

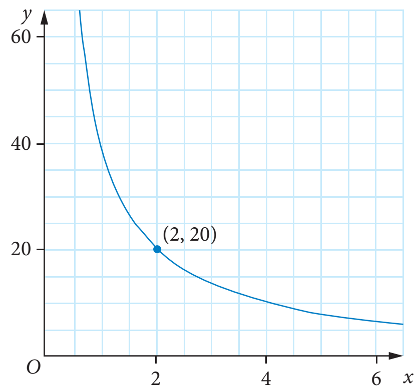
A $c = \frac{3600}{n}$ **B** $c = 4n$ **C** $c = \frac{n}{0.25}$ **D** $c = \frac{4}{n}n$ **E** $c = \frac{120}{n}$

11 The variation equation for the graph shown is



A $c = 30w$ **B** $c = 2.5w$ **C** $c = 4w$ **D** $c = 0.4w$ **E** $c = \frac{w}{2.5}$

12 © VCAA 2014 1GRQ3 **74%** The point (2, 20) lies on the graph of $y = \frac{k}{x}$, as shown.



The value of k is

A 5 **B** 10 **C** 20 **D** 40 **E** 80

13 A removalist company has a formula that measures the total time in minutes it takes for their workers to pack a household's belongings according to how many workers are packing. How many of the following four statements are true?

This is an example of direct variation.

The formula is in the form $t = kn$, where t is the total time in minutes and n is the number of workers.

As the number of workers increases, the total time decreases.

If the number of workers is doubled, the total time will double.

A 0 **B** 1 **C** 2 **D** 3 **E** 4

14 (7 marks) The time taken for a truck to travel a distance of 100 km along a stretch of road varies inversely with the speed (km/h) according to the table.

s (km/h)	40	75	80	100
t (minutes)				60

a Find the value of k and hence, write the equation of variation. 2 marks

b Copy and complete the table. 1 mark

c Draw a graph of the variation, labelling the values from the table. 1 mark

d Show a calculation from the table of values that verifies that doubling the speed halves the travelling time. 1 mark

e If the truck travelled the distance at a speed of 120 km/h, what would be the travelling time? 1 mark

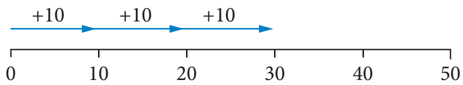
f What speed would the truck have to travel at to cover the distance in 96 minutes? 1 mark

8.2 Transforming non-linear data

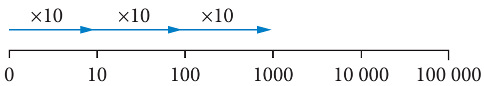
8.2

Linear and log scales

All the graphs we've used so far have had **linear scales**. On a linear scale, we *add* the same number to move from one scale mark to the next. In this example, the linear scale involves adding 10 each time.



There are some situations where it is better to use a **log scale** (or **logarithmic scale**). On a log scale, we *multiply* the same number to move from one scale mark to the next. In this example, the log scale involves multiplying by 10 each time.



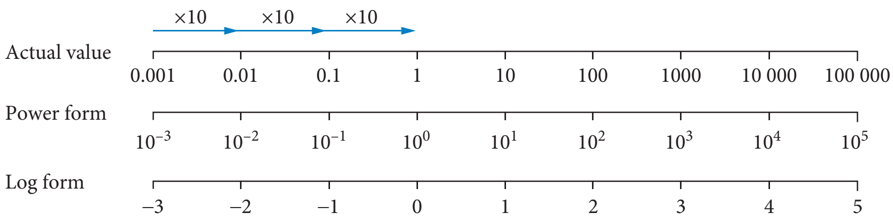
Video playlist
Transforming non-linear data

This is called a 'log base 10' or '$\log_{10}$' scale. Just as a linear scale can increase by numbers other than +10, a log scale can increase by numbers other than ×10. We will only be considering $\log_{10}$ scales, so when we refer to a log scale, we assume it's $\log_{10}$.

A log scale allows us to plot values such as 2, 17, 2567 and 98 654 on the *same* graph, whereas a linear base 10 scale would need to be very long to plot all of these values! Log scales are used in predicting the spread of corona viruses, trends in social media, and when measuring the brightness of stars and the magnitude of earthquakes.

Reading log scales

Log scales let us plot very small values and very large values together. The following shows three different ways that the $\log_{10}$ scale from 0.001 to 100 000 can be written:



We will use the log form to represent log scales. The log form is a rearrangement of the power form. For example:

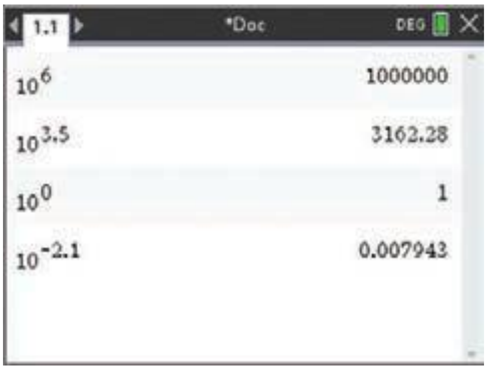
$10^4 = 10\,000$ is rearranged to $\log_{10} 10\,000 = 4$

Log form to actual values

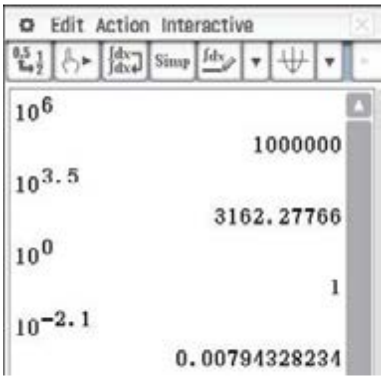
To work out the actual values involved in a log form scale, we will need to write the numbers on the scale as powers of 10. Unless it's a simple whole number, we will need to use CAS or a scientific calculator.
For example:

Number on log form scale	Actual value
6	$10^6 = 1\,000\,000$
3.5	$10^{3.5} = 3162.28$ (rounded to two decimal places)
0	$10^0 = 1$
–2.1	$10^{-2.1} = 0.008$ (rounded to three decimal places)

TI-Nspire



ClassPad

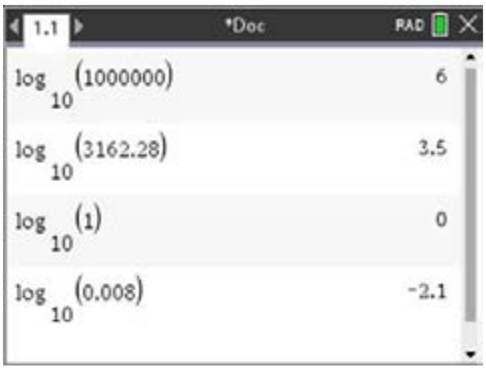


Actual values to log form

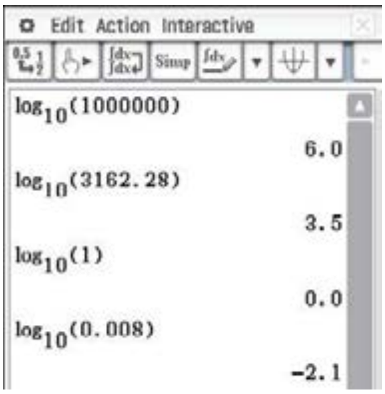
If the actual value is known and we want to find the number appearing on the log = scale, we use the $\log_{10}$ function on CAS or a scientific calculator.

Actual value	Number appearing on log form scale
1 000 000	$\log_{10} 1\,000\,000 = 6$
3162.28	$\log_{10} 3162.28 = 3.5$ (rounded to two significant figures)
1	$\log_{10} 1 = 0$
0.008	$\log_{10} 0.008 = -2.1$ (rounded to two significant figures)

TI-Nspire



ClassPad



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Linearisation

In Chapter 6, we looked at finding a line of good fit to represent data on a scatterplot. Not all data associations are straight lines, however. Some associations follow curves. These are called **non-linear associations**. The way that we deal with non-linear associations is to apply a **transformation** to one of the variables so that the association between the two variables becomes closer to a straight line. This is called **linearisation**.

We will be looking at three ways of linearising data for variables x and y:

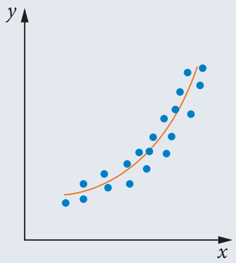
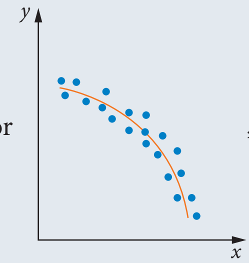
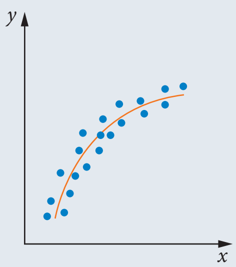
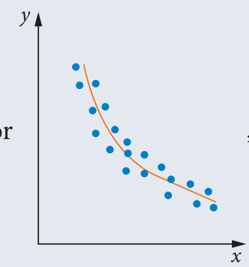
1 Plotting y values against x^2 values.

2 Plotting y values against $\frac{1}{x}$ values.

3 Plotting y values against $\log x$ values.

Linearising data

To linearise data:

- If the data has the shape  or  , use x^2.
- If the data has the shape  or  , use either $\log x$ or $\frac{1}{x}$.

WORKED EXAMPLE 5 Linearising data

WB p. 125

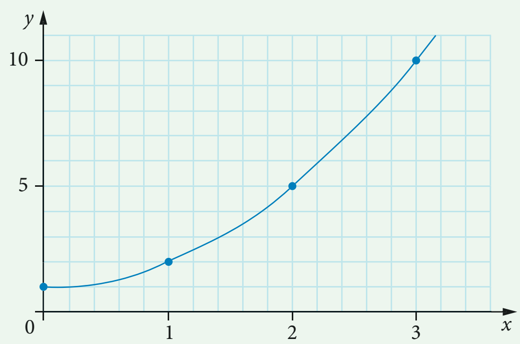
Linearise each of the following by transforming the variables as shown.

i Set up a table of values for the points marked on the graph.

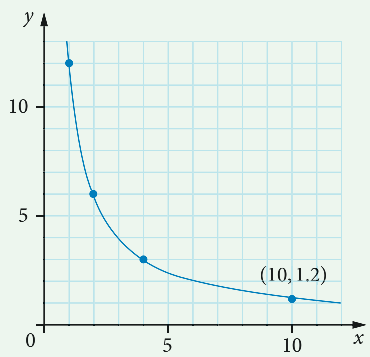
ii Add a row to the table and include the values for the transformed variable.

iii Draw the transformed graph with the transformed points to verify it is a straight line.

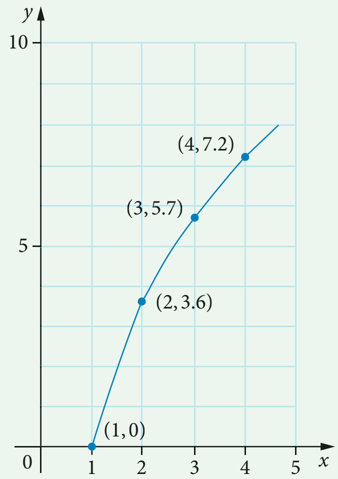
a Linearise by plotting y against x^2.



b Linearise by plotting y against $\frac{1}{x}$.



c Linearise by plotting y against $\log x$.



Steps	**Working**

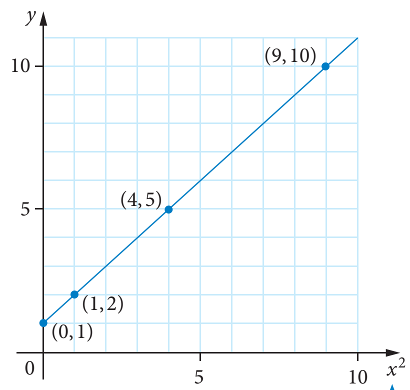
a **i** Set up a table of values for the points shown on the graph.

x	0	1	2	3
y	1	2	5	10

ii Add a row to the table of values and include the values for x^2.

x	0	1	2	3
x^2	0	1	4	9
y	1	2	5	10

iii Sketch the linearised graph by plotting the y values against the x^2 values and labelling the horizontal axis x^2. Show the points on the line.



The x-axis label always matches the transformed variable.

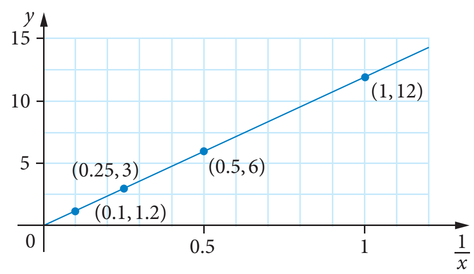
b **i** Set up a table of values for the points shown on the graph.

x	1	2	4	10
y	12	6	3	1.2

ii Add a row to the table of values and include the values for $\frac{1}{x}$.

x	1	2	4	10
$\frac{1}{x}$	1	$\frac{1}{2} = 0.5$	$\frac{1}{4} = 0.25$	$\frac{1}{10} = 0.1$
y	12	6	3	1.2

iii Sketch the linearised graph by plotting the y values against the $\frac{1}{x}$ values and labelling the horizontal axis $\frac{1}{x}$. Show the points on the line.



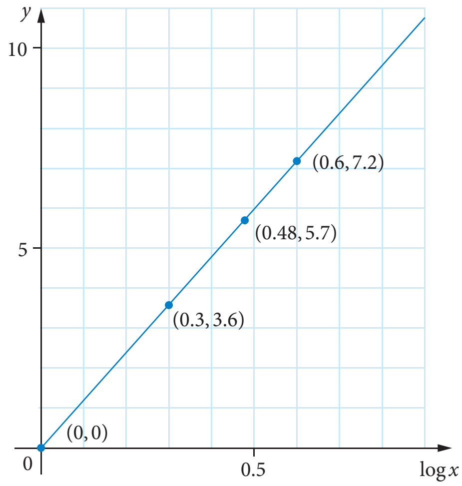
c **i** Set up a table of values for the points shown on the graph.

x	1	2	3	4
y	0	3.6	5.7	7.2

ii Add a row to the table of values and include the values for $\log x$.

x	1	2	3	4
$\log x$	0	0.30	0.48	0.60
y	0	3.6	5.7	7.2

iii Sketch the linearised graph by plotting the y values against the $\log x$ values and labelling the horizontal axis $\log x$. Show the points on the line.



Exam hack

Always remember to look at the x-axis label to see whether you are dealing with a transformed graph.

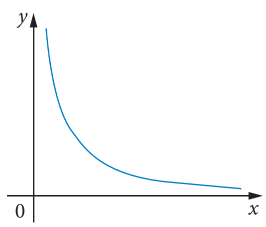
EXERCISE 8.2 Transforming non-linear data

ANSWERS p. 514

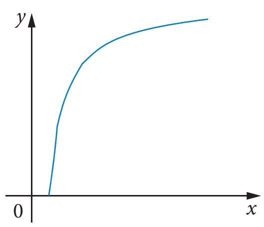
Recap

1 Which one of the following graphs represents inverse variation?

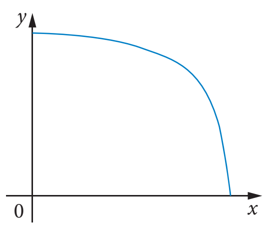
A



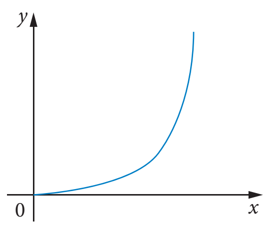
B



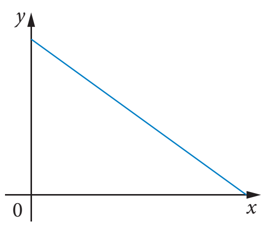
C



D



E



2 The cost of prawns, c (\$), varies directly with their weight, w (kg), as shown in the table.

w (kg)	0.5	1	1.5	2
c (\$)	3.75	7.50	11.25	15

The variation equation for this is

A $c = \dfrac{7.5}{w}$ **B** $w = 7.5c$ **C** $w = \dfrac{7.5}{c}$ **D** $c = 7.5w$ **E** $c = \dfrac{w}{7.5}$

Mastery

3 a For each of the following numbers appearing on a log scale, find their actual value, giving your answers rounded to three decimal places if necessary.

i 3 **ii** 2.4 **iii** 0 **iv** 0.8

v 1 **vi** −1 **vii** −1.5

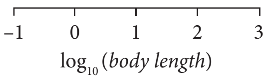
b For each of the following actual values, find the number that would appear on a log scale, giving your answers rounded to three significant figures.

i 10 000 **ii** 10 000 000 **iii** 73 **iv** 689

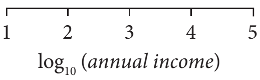
v 0.6 **vi** 30 000 **vii** 9

4 Rewrite each of these $\log_{10}$ scales to show the actual values involved.

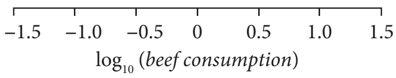
a *Body length* in mm of animals.



b *Annual income* in dollars of a group of people.

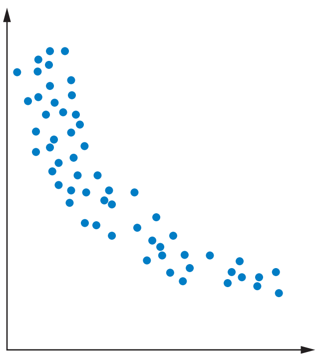


c Annual per capita *beef consumption* in kg of countries rounded to one decimal place.

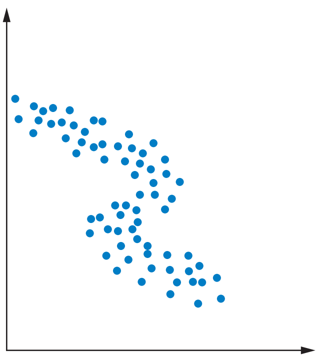


5 For each of the following scatterplots, state which of the transformations x^2, $\frac{1}{x}$ or $\log x$ (if any) can be used to linearise the association.

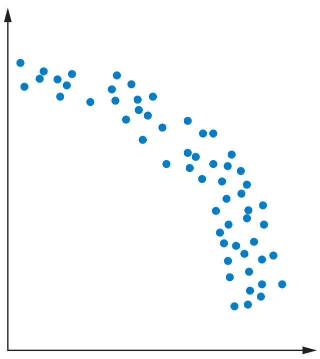
a



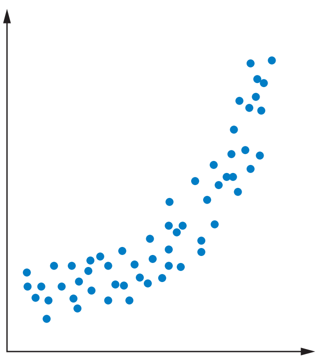
b



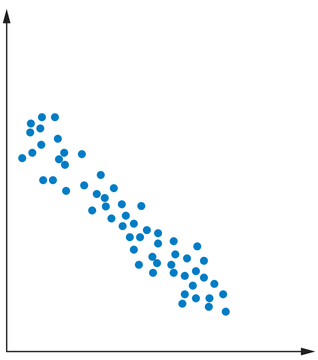
c



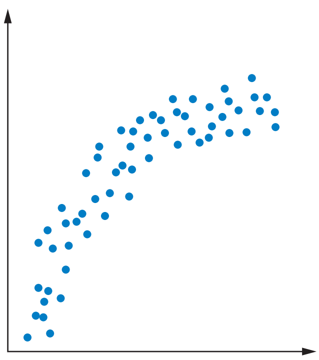
d



e



f



8.2

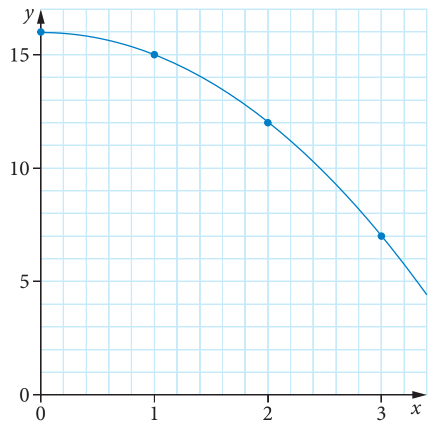
6 WORKED EXAMPLE 5 Linearise each of the following by transforming the variables as shown.

i Set up a table of values for the points marked on the graph.

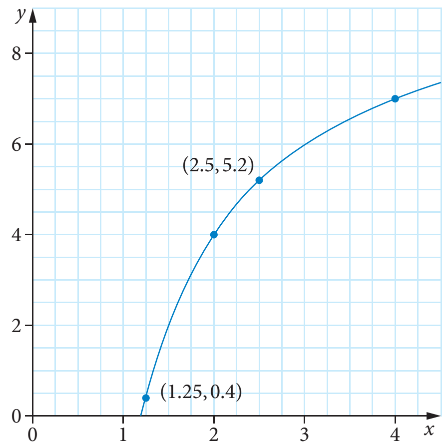
ii Add a row to the table and include the values for the transformed variable.

iii Draw the transformed graph with the transformed points to verify it is a straight line.

a Linearise by plotting y against x^2.



b Linearise by plotting y against $\frac{1}{x}$.



c Linearise by plotting y against $\log x$.

Exam practice

80–100% 60–79% 0–59%

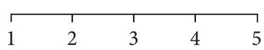
7 A log scale indicates a value of –2. This means the actual value is

A log (–2) **B** 0.01 **C** 0.3 **D** 10 **E** 100

8 The actual value 90 would appear on a log scale rounded to two significant figures as

A 1.9 **B** 1.95 **C** 2 **D** 2.0 **E** 1 000 000 000

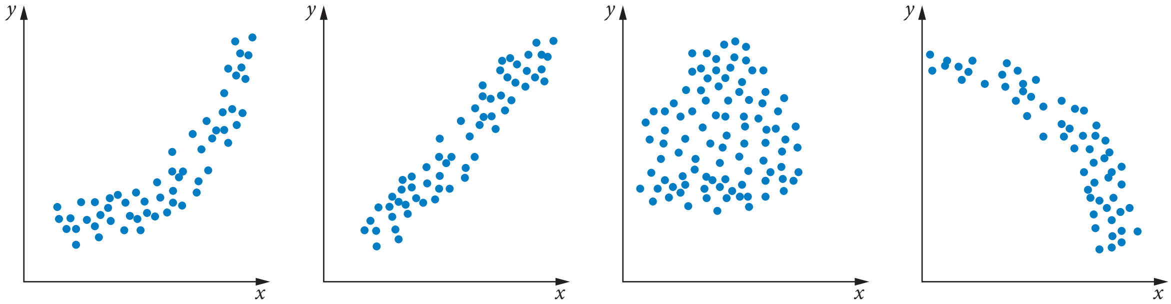
9 The annual income, in dollars, of people living in a town is shown on the following log scale.



The largest annual income in the town could be

A \$97 000 **B** \$108 000 **C** \$260 000 **D** \$4 050 000 **E** \$5 000 000

10 How many of the following scatterplots could be linearised by plotting y against $\log x$?



A 0 **B** 1 **C** 2 **D** 3 **E** 4

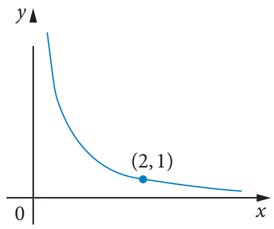
11 Given the table of values

x	0	1	2	3
y	4	5	8	13

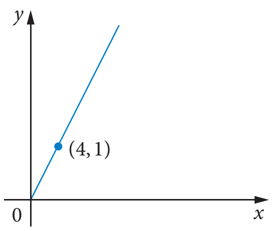
which of the following points would lie on the linearised graph where y is plotted against x^2?

A $(0, 16)$ **B** $(1, 25)$ **C** $(4, 2)$ **D** $(9, 13)$ **E** $(2, 8)$

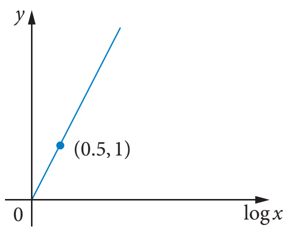
12 Which of the following show a linearised version of this graph?



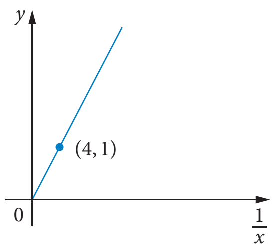
A



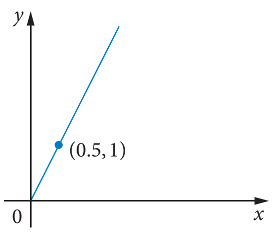
B



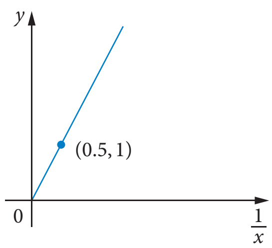
C



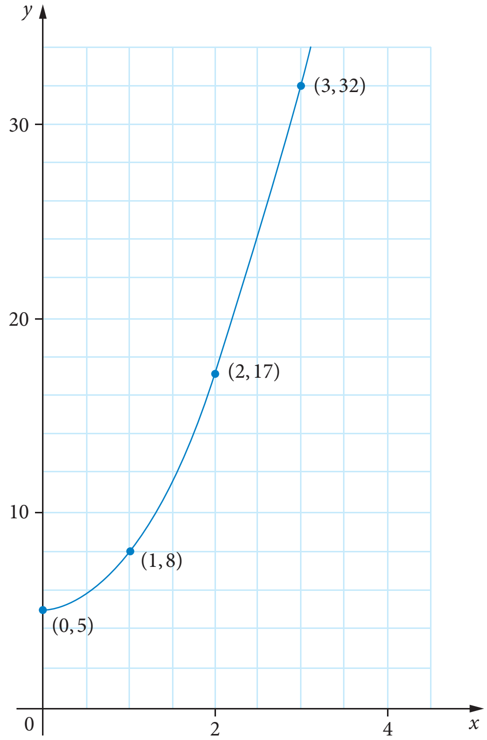
D



E



13 Which of the options lists four points on a linearised version of this graph?



A (0, 5), (0, 8), (0.3, 17), (0.5, 32)

B (0, 5), (1, 8), (4, 17), (9, 32)

C (0, 25), (1, 64), (2, 289), (3, 1024)

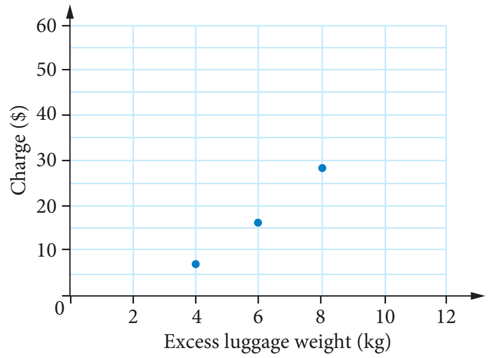
D (0, 5), (1, 8), (2, 17), (3, 32)

E (0, 5), (1, 8), (0.5, 17), (0.3, 32)

14 © VCAA 2009 2GRQ2ab MODIFIED (2 marks) Luggage over 20 kg in weight is called excess luggage. Fair Go Airlines charges for transporting excess luggage. The charges for some excess luggage weights are shown in the table.

Excess luggage weight (kg)	4	6	8	10
Charge ($)	$7.20	$16.20	$28.80	$45.00

a 96% Copy and complete this graph by plotting the charge for excess luggage weight of 10 kg. 1 mark



b A graph of the charge against (*excess luggage weight*)^2 is to be constructed. Find the missing (*excess luggage weight*)^2 value in this table, and then draw this graph including the missing point. 1 mark

Excess luggage weight (kg)	4	6	8	10
(excess luggage weight)^2 (kg^2)	16	36		100
Charge ($)	$7.20	$16.20	$28.80	$45.00

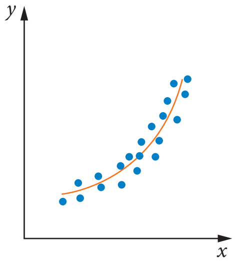
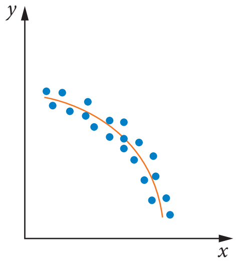
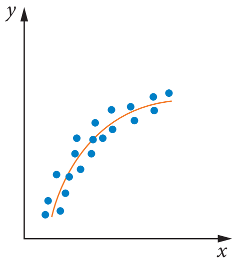
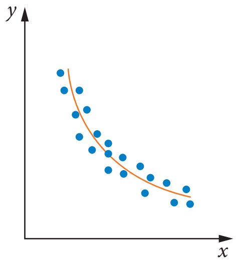
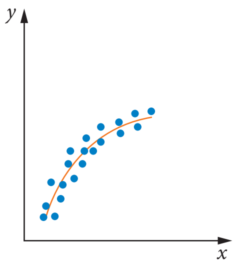
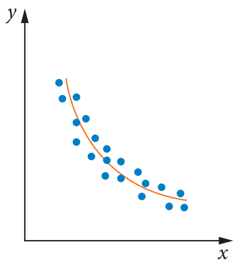
8.3 Modelling non-linear data

Video playlist
Modelling non-linear data

Modelling non-linear data with curves

Real-life data doesn't always fall neatly on a curve; however, we can still find curves of good fit to model the data.

Modelling non-linear data with curves

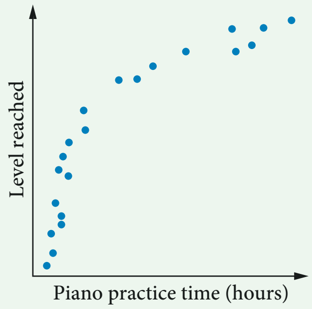
Type of relationship	Form of equation	Shape of curve		
		as x increases, y increases		as x increases, y decreases
squared	$y = kx^2 + c$		or	
log	$y = k\log x + c$		or	
inverse	$y = \frac{k}{x} + c$		or	

9780170448192

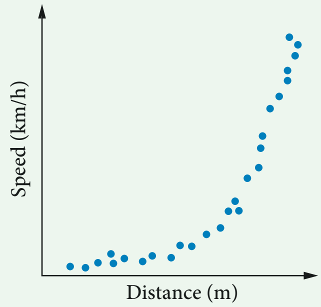
WORKED EXAMPLE 6 Modelling non-linear data with curves

For each of the following non-linear scatterplots, identify the shape of the curve (squared, log or inverse) and then write the equation of best fit that models the data.

a



b



Steps	Working
a **1** Identify the options for the type of relationship.	$\frac{1}{x}$, $\log x$
2 Choose the correct form of the equation from $y = kx^2 + c$, $y = k\log x + c$, $y = \frac{k}{x} + c$ using the variable names.	$\textit{level reached} = k\log(\textit{piano practice time}) + c$ $\textit{level reached} = \frac{k}{(\textit{piano practice time})} + c$
b **1** Identify the options for the type of relationship.	x^2
2 Choose the correct form of the equation from $y = kx^2 + c$, $y = k\log x + c$, $y = \frac{k}{x} + c$ using the variable names.	$\textit{speed} = k(\textit{distance})^2 + c$

8.3

WB
p. 127

Making predictions for non-linear data

If we can identify the shape of the curve (squared, log or inverse) and then write the equation of best fit, we can make generalisations about the trend in the data. This allows us to make predictions for the future. This type of modelling was used during the COVID pandemic to create models and develop strategies.

WB p. 128

WORKED EXAMPLE 7 Working with non-linear data

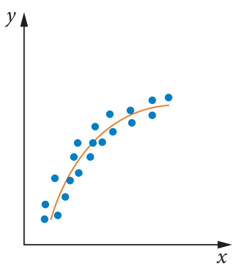
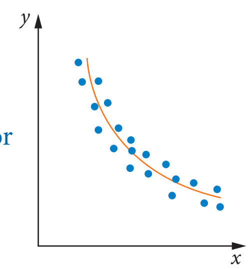
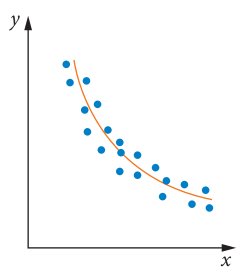
The data for the association between the maximum daily temperature (°C) and the daily number of boxes of hot pies sold at the school canteen has been modelled by the inverse relationship given by the following equation.

$$\textit{number of boxes of pies} = \frac{90}{\textit{temperature}} - 0.8$$

a Use the equation to predict the number of boxes of pies sold when the temperature is

i 17°C **ii** 25°C **iii** 30°C

b What shape is the original data?

Steps	**Working**
a Substitute the value into the equation. Round the answer if necessary.	**i** $\textit{number of boxes of pies} = \frac{90}{17} - 0.8 = 4.49$ When the *temperature* is 17°C, the equation predicts 4 boxes of pies will be sold. **ii** $\textit{number of boxes of pies} = \frac{90}{25} - 0.8 = 2.8$ When the *temperature* is 25°C, the equation predicts 3 boxes of pies will be sold. **iii** $\textit{number of boxes of pies} = \frac{90}{30} - 0.8 = 2.2$ When the *temperature* is 30°C, the equation predicts 2 boxes of pies will be sold.
b 1 Which relationship has been used?	$\frac{1}{x}$
2 Which data shape matches this relationship? Decide whether y increases or decreases as x increases.	$\frac{1}{x}$ relationships have the shape  or  As the *temperature* increases, the *number of boxes of hot pies* sold decreases, so the data shape is 

Exam hack

Sometimes we need to round to the nearest whole number, even though we are not specifically told to, because of the context of the question.

9780170448192

VCE QUESTION ANALYSIS

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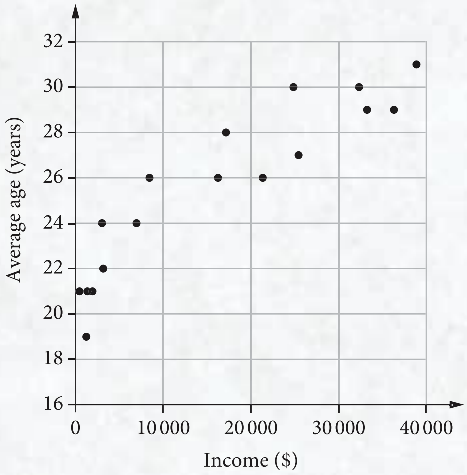
8.3

The average age of women at first marriage in years (*average age*) and average yearly income in dollars per person (*income*) were recorded for a group of 17 countries. The results are displayed in the table. A scatterplot of the data is also shown.

Income ($)	Average age (years)
1750	21
3200	22
8600	26
16 000	26
17 000	28
21 000	26
24 500	30
32 000	30
38 500	31
33 000	29
25 500	27
36 000	29
1300	19
600	21
3050	24
6900	24
1400	21



Video playlist
VCE question analysis: Variation

The association between *average age* and *income* is non-linear.

a Name two types of relationships that could be used to model the data. 1 mark

b Copy and complete the following to determine the equation that allows *average age* to be predicted from log (*income*).

$$\square = 2.39 + 5.89 \times \square$$ 1 mark

c Use this equation to predict the average age of women at first marriage in a country with an average yearly income of $20 000 per person. Write your answer correct to one decimal place. 1 mark

d According to the equation, what is the average yearly income in dollars for a woman whose first marriage was at age 26? 1 mark

e Find the difference between the mean of the incomes of all the 26-year-olds in the group and the value you calculated for part **d**. 1 mark

Reading the question

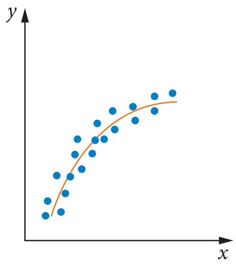
- Both a table of values and a scatterplot have been given in the question.
- The question tells you that there is a log relationship.
- Part **e** requires the answer for part **d**.

Thinking about the question

- Have you used both the table of values and scatterplot at some stage?
- Parts **c** and **d** require the answer for part **b**.
- Are you clear about what the mean is referring to in part **e**?

Worked solution (✓ = 1 mark)

a The shape of the graph is,



so the relationship can be either an **inverse or log.** ✓

b The variables are given as *average age* and log (*income*) and the horizontal axis label is Income ($), so the equation is

***average age* = 2.39 + 5.89 × log (*income*)** ✓

c $average\ age = 2.39 + 5.89 \times \log(20\,000) = 2.39 + 5.89 \times 4.301\ldots = 27.723\ldots$

So, the average age of women at first marriage in a country with an average yearly income of $20 000 per person correct to one decimal place is **27.7 years**. ✓

d Solve $26 = 2.39 + 5.89 \times \log(income)$ using the CAS solve function.

Otherwise:

$$26 = 2.39 + 5.89 \times \log(income)$$
$$5.89 \times \log(income) = 26 - 2.39$$
$$5.89 \times \log(income) = 23.61$$
$$\log(income) = \frac{23.61}{5.89}$$
$$\log(income) = 4.008\ldots$$
$$\text{income} = 10^{4.008}\ldots = 10\,197$$

The average yearly income in dollars for a woman whose first marriage was at age 26 is **$10 197.** ✓

e Use the table of values. There are three incomes for 26-year-olds. Find the mean by adding them together and dividing by 3: $\frac{8600 + 16\,000 + 21\,000}{3} = 15\,200$

Subtract the value from part **d**.

$15\,200 - 10\,197 = 5003$

The difference is **$5003**. ✓

Student performance

a Two answers are required. One of them is indicated in part **b**.

b This is relatively straight forward.

c Unrealistic answers for 'average age at first marriage' were given. Students are expected to recognise unrealistic answers.

d Incorrect answers will occur if students round before the end.

e Students should use the table of values rather than estimating from the scatterplot. When a question requires a previous answer, a mark is awarded if the correct working is shown, even if the answer is incorrect.

EXERCISE 8.3 Modelling non-linear data

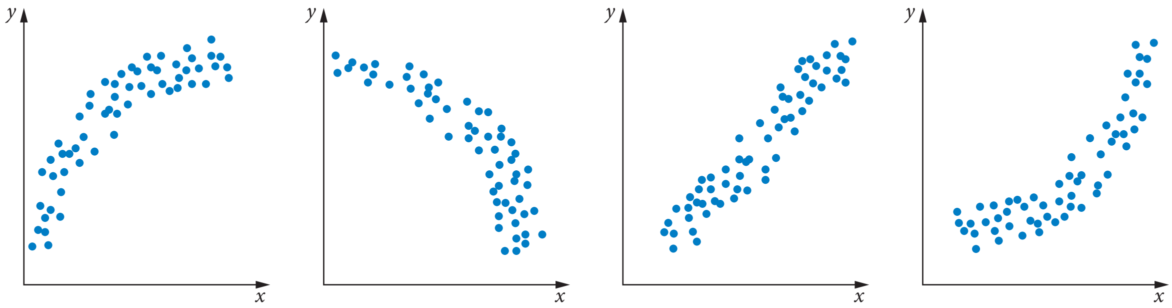
ANSWERS p. 515

Recap

1 The actual value 3000 would appear on a log scale rounded to three significant figures as

A 3.477 **B** 3.48 **C** 30.0 **D** 30 **E** 9000

2 How many of the following scatterplots could be linearised by plotting y against x^2?

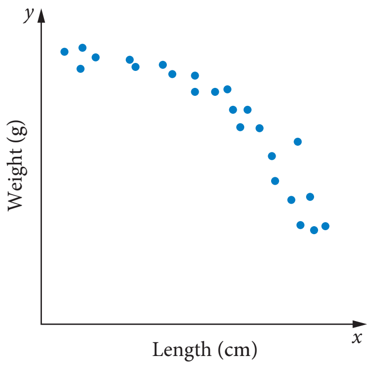


A 0 **B** 1 **C** 2 **D** 3 **E** 4

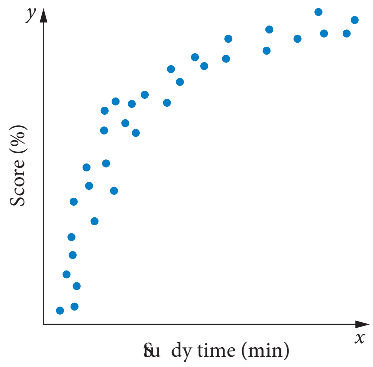
Mastery

3 WORKED EXAMPLE 6 For each of the following non-linear scatterplots, identify the shape of the curve (squared, log or inverse) and then write the equation of best fit that models the data.

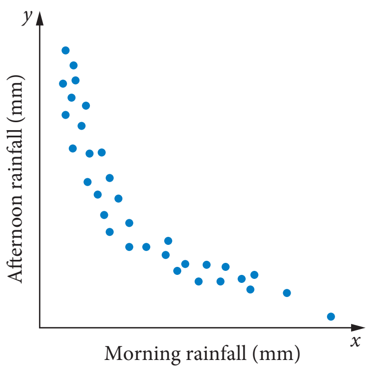
a



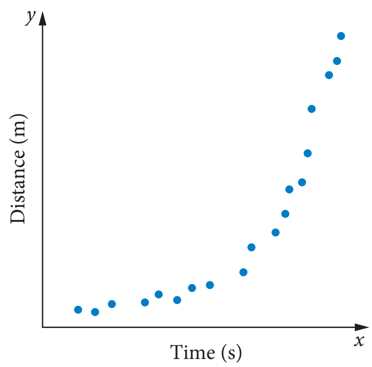
b



c



d



4 WORKED EXAMPLE 7 The data for the association between the diameter of a particular fruit (cm) and the number of seeds it contains has been modelled by the log relationship given by the following equation.

$$\textit{number of seeds} = 6.6\log(\textit{diameter}) + 4$$

a Use the equation to predict the number of seeds in a fruit of diameter

i 8 cm **ii** 4 cm **iii** 2 cm

b What shape is the original data?

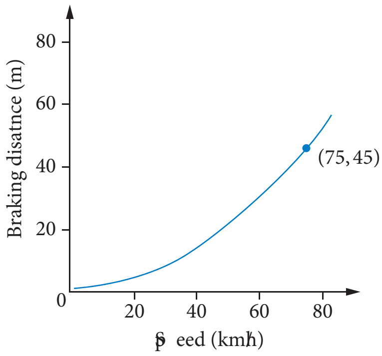
5 © VCAA 2015 1CQ11 69% The relationship between the *weight* of a mouse, in grams, and its *age*, in weeks, is modelled by the equation

$$weight = -7 + 30\log_{10}(age)$$

This equation predicts that a mouse aged five weeks has a weight, in grams, that is closest to

A 14 **B** 21 **C** 23 **D** 41 **E** 143

6 © VCAA 2015 1GRQ6 62% The graph below shows the braking distance, in metres, of a car at different speeds, in kilometres per hour. The coordinates of a point on the graph are also shown.



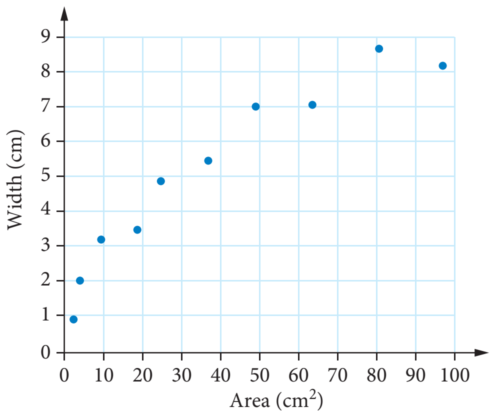
The relationship between *braking distance* and *speed* can be modelled by an equation of the form

$$braking\ distance = k \times (speed)^2$$

Using this model, the braking distance, in metres, when the speed is 60 km/h is

A 24.0 **B** 28.8 **C** 30.0 **D** 32.2 **E** 48.5

7 © VCAA 2013 1CQ10 MODIFIED The data in the scatterplot shows the *width*, in cm, and the *surface area*, in cm^2, of leaves sampled from 10 different trees. The scatterplot is non-linear.



The equation that models this relationship is

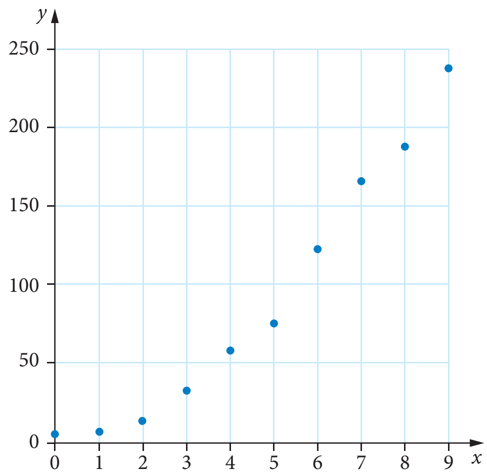
$$width = 4.3 \times \log(area) - 0.4$$

Using this equation, a leaf with a surface area of 120 cm^2 is predicted to have a width, in cm, closest to

A 8.5 **B** 9.2 **C** 10.6 **D** 72.5 **E** 97.8

8 © VCAA 2006 1CQ9 42% A student uses the following data to construct the scatterplot shown.

x	0	1	2	3	4	5	6	7	8	9
y	5	7	14	33	58	76	124	166	188	238



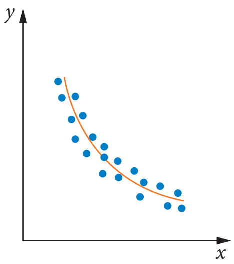
She then models the relationship with an equation. The equation which most closely models the data is

A $y = 7.1 + 2.9x^2$ **B** $y = -29.5 + 26.8x^2$ **C** $y = 26.8 - 29.5x^2$

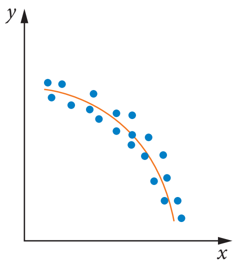
D $y = 1.3 + 0.04x^2$ **E** $y = -2.2 + 0.3x^2$

9 A student has collected data that shows there is an inverse relationship between the maximum daily temperature (°C), x, and the daily number of ice creams sold at an ice creamery, y. The shape of the scatterplot is

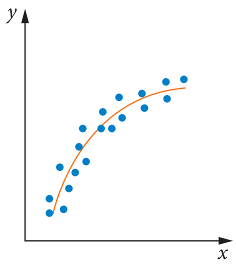
A



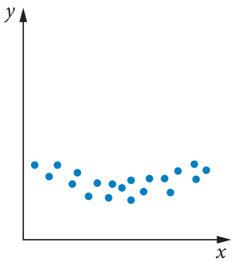
B



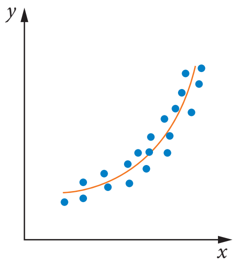
C



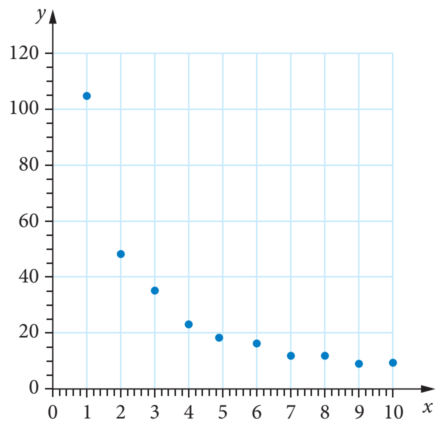
D



E



10 © VCAA 2018 1CQ11 MODIFIED Freya's data generates the following non-linear scatterplot.

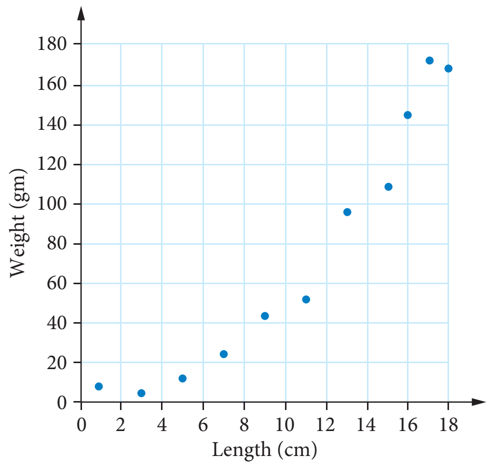


The equation that most closely models this data is

A $y = \frac{105.5}{x} - 0.5$ **B** $y = 0.012x - 0.0039$ **C** $y = -0.082x + 7.8$

D $y = \frac{59.7}{x} + 45.3$ **E** $y = \frac{45.3}{x} + 59.7$

11 © VCAA 2019 1CQ12 MODIFIED For the non-linear scatterplot shown, which of the following equations best models this data?



A $weight = -1.34 + \frac{0.546}{length}$ **B** $weight = -1.34 + 0.546 \times (length)^2$

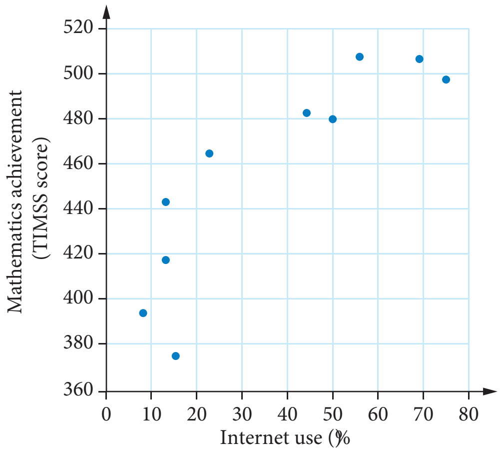
C $weight = 3.93 - 0.008\,64 \times \log(length)$ **D** $weight = 34.6 - 10.5 \times length$

E $weight = 34.6 - \frac{10.5}{length}$

12 © VCAA 2018 1CQ12 MODIFIED A relationship is modelled by the equation $y = 3.1 - 2.3\log_{10}(x)$. This equation is used to predict the value of y when $x = 1.1$. The value of y to two significant figures is

A 2.9 **B** 3 **C** 3.0 **D** 3.1 **E** 3.2

13 © VCAA 2009 1CQ12 MODIFIED The *mathematics achievement level* (TIMSS score) for Year 8 students and the general rate of *internet use* (%) for 10 countries are displayed in the scatterplot.



To model the data, it would be best to plot

A *mathematics achievement* against *internet use*.

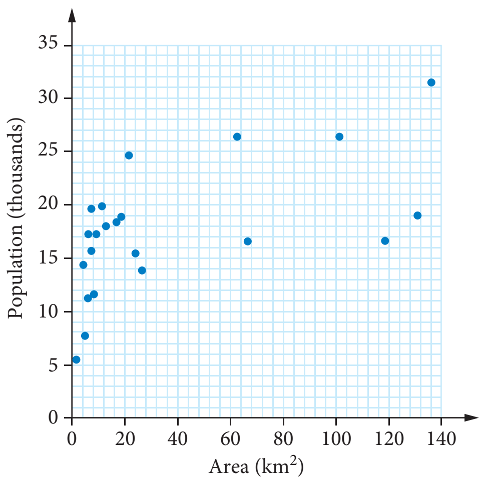
B log(*mathematics achievement*) against log(*internet use*).

C *mathematics achievement* against log(*internet use*).

D *mathematics achievement* against log(*internet use*)^2.

E $\dfrac{1}{\textit{mathematics achievement}}$ against *internet use*.

14 © VCAA 2016S 2CQ4 MODIFIED (2 marks) The scatterplot shows the *population*, in thousands, and the *area*, in square kilometres, for a sample of 21 outer suburbs of the same city.



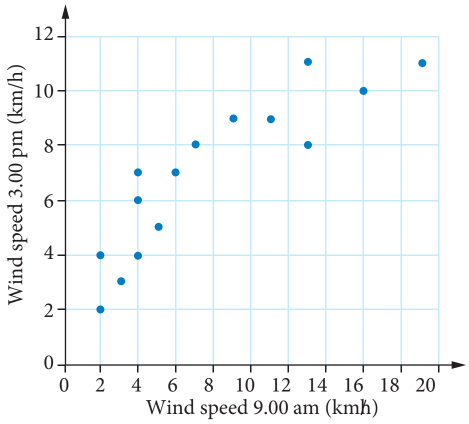
In the outer suburbs, the relationship between *population* and *area* is non-linear. The equation that allows the population of an outer suburb to be predicted from the logarithm of its area is

$$\textit{population} = 7700 + 7700\log(\textit{area})$$

a Use the equation to predict the population of an outer suburb with an area of 90 km^2. Round your answer to the nearest one thousand people. 1 mark

b What other type of relationship could be used to model the data? 1 mark

15 © VCAA 2012 2CQ4 MODIFIED (8 marks) The wind speeds (in km/h) at a weather station at 9.00 am and 3.00 pm respectively were recorded on 18 days in November, and a scatterplot has been constructed from this data set.



Let the wind speed at 9.00 am be represented by the variable *ws9.00 am* and the wind speed at 3.00 pm be represented by the variable *ws3.00 pm*. The association between *ws9.00 am* and *ws3.00 pm* shown in the scatterplot is non-linear.

a Name two types of relationships that could be used to model the data. 1 mark

b Copy and complete the following to determine the equation that allows *ws3.00 pm* to be predicted from $\frac{1}{ws9.00\,am}$.

$$\boxed{} = 12 - 20 \times \boxed{}$$ 1 mark

c Use the equation in part **b** to predict the wind speed at 3.00 pm on a day when the wind speed at 9.00 am is 24 km/h. Write your answer correct to the nearest whole number. 1 mark

d A second model has been applied to the data. Copy and complete the following to determine the equation that allows *ws3.00 pm* to be predicted from log (*ws9.00 am*).

$$\boxed{} = 0.77 + 9.2 \times \boxed{}$$ 1 mark

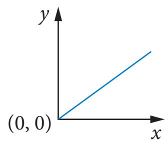
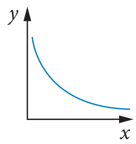
e Use the equation in part **d** to predict the wind speed at 3.00 pm on a day when the wind speed at 9.00 am is 24 km/h. Write your answer correct to the nearest whole number. 1 mark

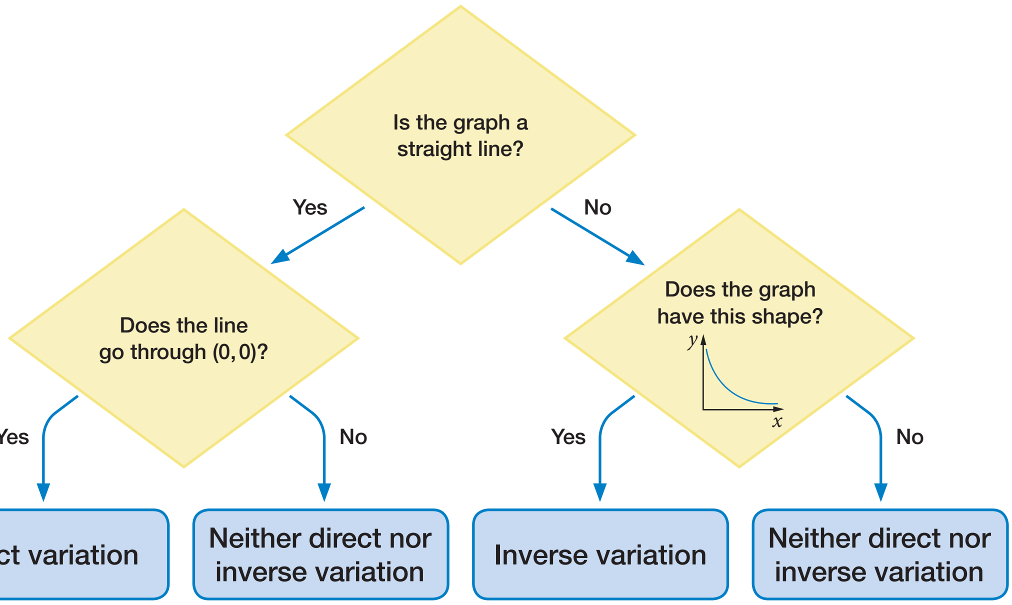
f Calculate the *ws3.00 pm* value, to two significant figures, that each of the two equations predict when *ws9.00 am* is 8, and hence, state which of the two is closer to the actual value recorded. 3 marks

 9780170448192

8 Chapter summary

Direct and inverse variation

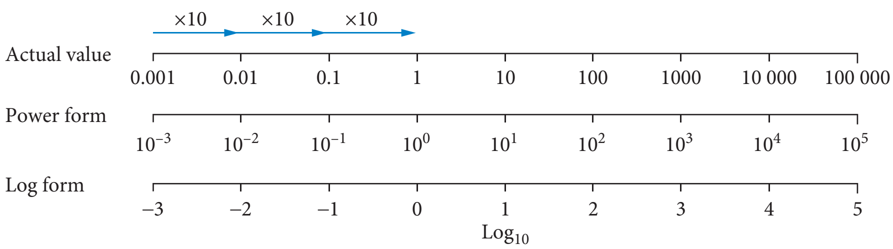
Direct variation ($k > 0$)	Inverse variation ($k > 0$)
• y varies directly with x.	• y varies inversely with x.
• $y = kx$, where k is a constant	• $y = \frac{k}{x}$, where k is a constant
• As x increases, y *increases* by the same percentage.	• As x increases, y *decreases* by the same percentage.
• If x is doubled, y is *doubled*.	• If x is doubled, y is *halved*.
• The graph looks like this: 	• The graph looks like this: 



Linear and log scales

- On a **linear scale**, we *add* the same number to move from one scale mark to the next.
- On a **log scale**, we *multiply* the same number to move from one scale mark to the next.

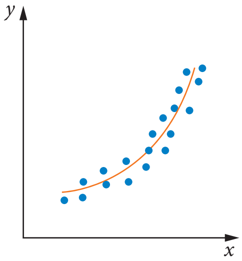
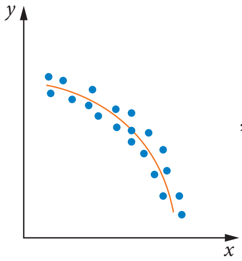
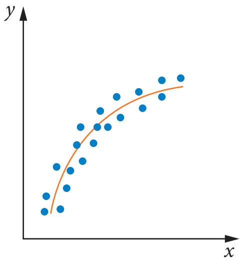
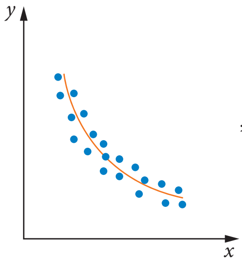
  Log scales can be written in three different ways:



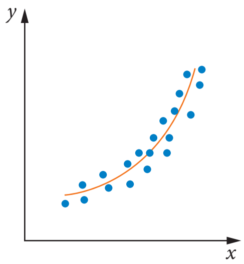
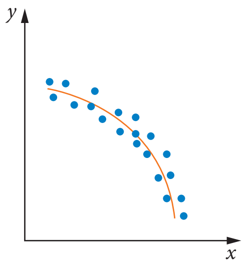
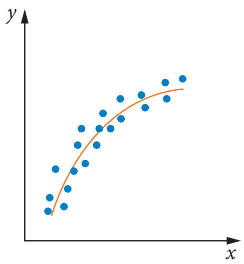
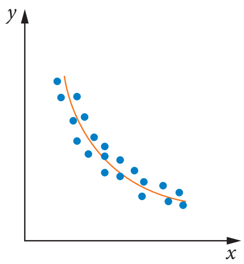
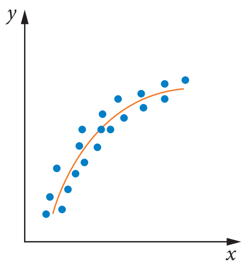
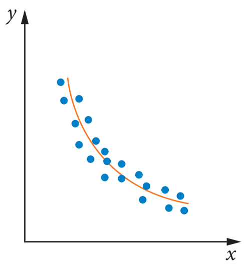
- The log form is a rearrangement of the power form, e.g. $10^4 = 10\,000$ is rearranged to $\log_{10} 10\,000 = 4$.
- Use the log key on CAS to find the value of logs that aren't multiples of 10. e.g. $\log 12 = 1.079\ldots$

Linearisation

To linearise data:

- If the data has the shape  or  , use x^2.
- If the data has the shape  or  , use either $\log x$ or $\frac{1}{x}$.

Modelling non-linear data with curves

Type of relationship	Form of equation	Shape of curve		
		as x increases, y increases		as x increases, y decreases
squared	$y = kx^2 + c$		or	
log	$y = k\log x + c$		or	
inverse	$y = \frac{k}{x} + c$		or	

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Cumulative examination 1

Total number of marks: 14 Reading time: 5 minutes Writing time: 32 minutes

1 © VCAA 2002 1CQ8 The following data was recorded from measurements made on 12 men. The sample of men has been drawn from a population whose mass has a normal distribution, with a mean of 81.1 kg and a standard deviation of 17.9 kg.

Age (years)	Mass (kg)	Waist (cm)
26	84	84
29	72	74
32	67	89
32	59	75
34	97	106
37	112	114
39	67	80
40	91	101
41	98	101
43	89	94
45	117	126
51	62	82

The percentage of men in this population with a mass greater than that of the heaviest man in this sample is closest to

A 0.05% **B** 2.5% **C** 5% **D** 50% **E** 95%

2 The rule $u_n = 8 + 11n$ could be used to find the 50th value of which of the following sequences?

A 8, 19, 27 … **B** 11, 19, 17 … **C** 8, 19, 30 … **D** 8, 18, 28 … **E** 11, 10, 8 …

3 © VCAA 2010 1BRMQ3 Peter received a quote from the Artificial Grass Company for his new front lawn. The quote is for \$1880 plus a Goods and Services Tax (GST) of 10%. The final amount that Peter pays for the new front lawn is

A \$188 **B** \$1880 **C** \$1890 **D** \$1899 **E** \$2068

4 © VCAA 2005 1GRQ5 An electrician charges a fixed call-out fee of \$50 and then charges \$65 per hour for each hour worked. For n hours worked, the total charge in dollars is

A 115 **B** $n + 115$ **C** $50n + 65$ **D** $65n + 50$ **E** $115n$

5 If $A = \begin{bmatrix} 8 & 7 \\ 3 & 2 \end{bmatrix}$, which of the following is true?

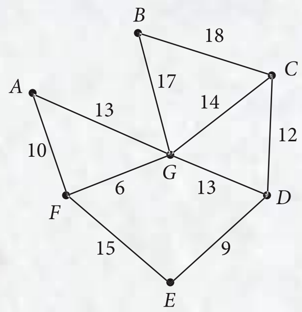
A $\det(A) = \det(A^{-1})$ **B** $\det(A) = \det(A^2)$ **C** $\det(A) = \det(A^T)$

D $\det(A) = \det(A^3)$ **E** $\det(A) = \det(A^{A-1})$

6 What is 0.000 40 rounded to two significant figures?

A 0.0 **B** 0.00 **C** 0.000 **D** 0.0004 **E** 0.000 40

7 The shortest path from A to E is



A 6 **B** 10 **C** 16 **D** 25 **E** 34

8 Which of the following is true about direct variation?

A As one variable increases, the other variable decreases.

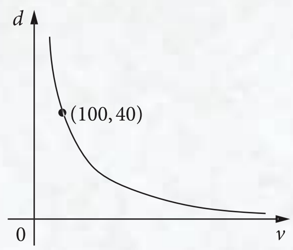
B As one variable increases, the other variable halves.

C If one variable is increased by 10%, the other variable is increased by 10%.

D If one variable is increased by 10%, the other variable is decreased by 10%.

E If one variable is increased by 10%, the other variable is doubled.

9 The inverse variation equation for the graph shown is

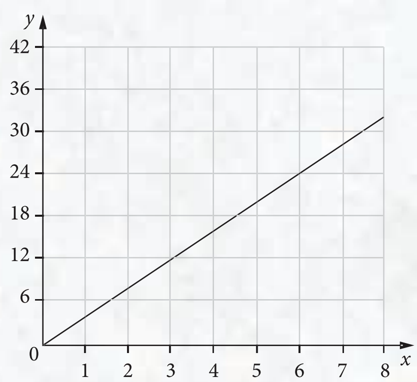


A $d = 0.4v$ **B** $d = \frac{v}{4000}$ **C** $d = \frac{0.4}{v}$ **D** $d = \frac{v}{0.4}$ **E** $d = \frac{4000}{v}$

10 The point (2.5, 6) lies on the direct variation graph $y = kx$. The value of k rounded to one decimal place is

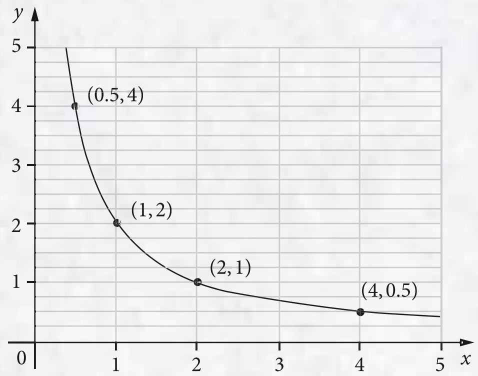
A 0.4 **B** 2.4 **C** 2.5 **D** 8.5 **E** 15

11 What is the variation equation for the graph shown?



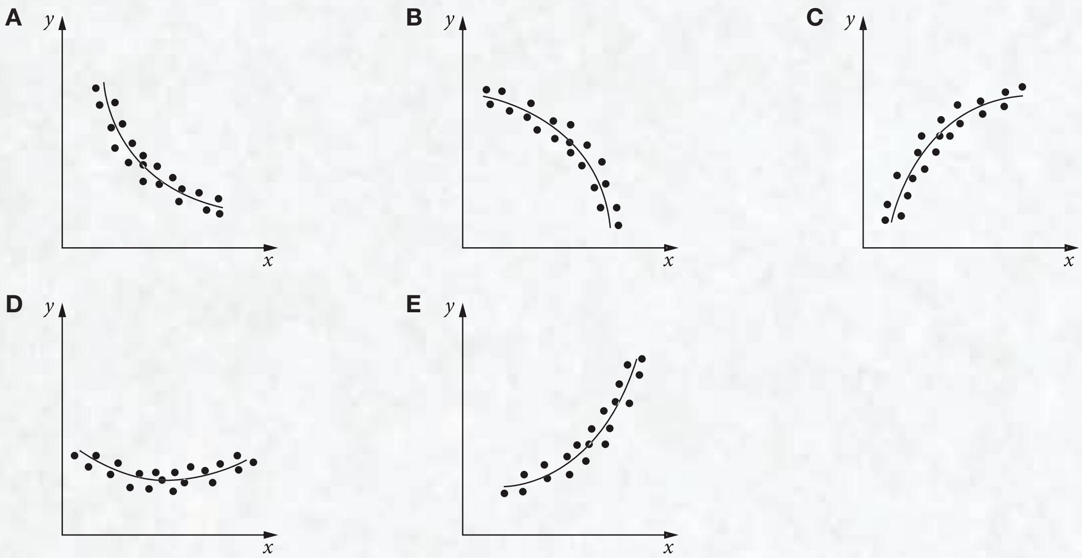
A $y = 3x$ **B** $y = 4x$ **C** $y = 6x$ **D** $y = 36x$ **E** $y = 0.25x$

12 Which of the options lists four points on a linearised version of this graph which plots y against $\frac{1}{x}$?

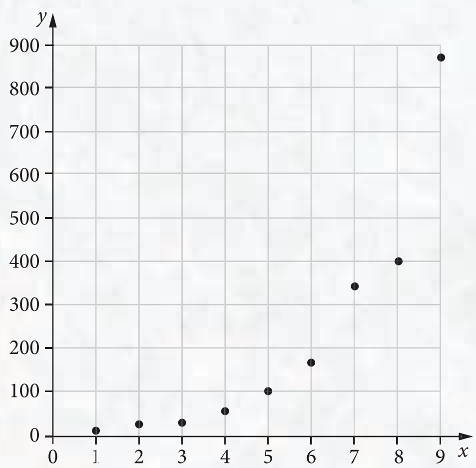


A (0.25, 4), (1, 2), (4, 1), (16, 0.5)
B (4, 0.5), (2, 1), (1, 2), (0.5, 4)
C (0.25, 0.5), (0.5, 1), (1, 2), (2, 4)
D (0.5, 0.25), (1, 0.5), (2, 0.5), (4, 2)
E (4, 0.25), (2, 1), (1, 4), (0.5, 16)

13 A student has collected data that shows there is a log relationship between the maximum daily temperature (°C), x, and the daily number of heaters sold, y. The shape of the scatterplot is



14 The data for the scatterplot shown can best be modelled by an equation in the form



A $y = k\log x + c$ **B** $y = \frac{k}{x} + c$ **C** $y = \frac{k}{x^2} + c$ **D** $y = kx^2 + c$ **E** $y = kx + c$

Cumulative examination 2

Total number of marks: 20 Reading time: 6 minutes Writing time: 30 minutes

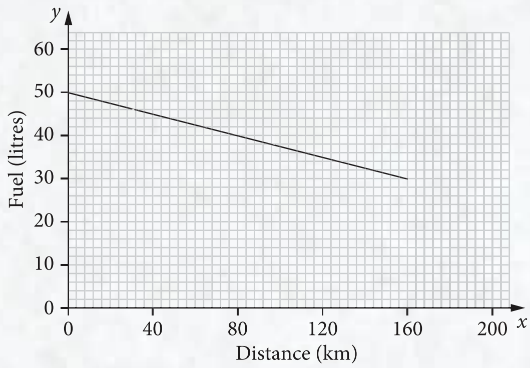
1 © VCAA 2016 2CQ6ab MODIFIED (3 marks) Ken's caravan had a purchase price of \$38 000. The value of the caravan has been depreciated using the flat rate method of depreciation at a rate of 7.2% each year.

a What is the annual amount of depreciation? 1 mark

b Let C_n be the value of the caravan after n years after it was purchased. Write down a recurrence relation, in terms of C_{n+1} and C_n, that models the value of the caravan. 1 mark

c List the value of the caravan after each of the first four years. 1 mark

2 © VCAA 2006 2GRQ2 (4 marks) In one particular week, Harry began with 50 litres of fuel in the tank of his van. After he had travelled 160 km, there were 30 litres of fuel left in the tank of his van. The amount of fuel remaining in the tank of Harry's van followed a linear trend as shown in the graph.



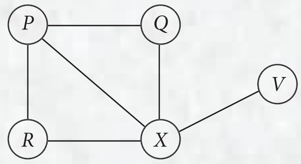
a Determine the equation of the line shown in the graph. 2 marks

b Assume that this linear trend continues and that Harry does not add fuel to the tank of his van. How much **further** will he be able to travel before the tank is empty? 1 mark

c Harry stopped to refuel his van when there were 12 litres of fuel left in the tank. He completely filled the tank in $3\frac{1}{2}$ minutes when fuel was flowing from the pump at a rate of 18 litres per minute. How much fuel does the tank hold when it is completely full? Write your answer in litres. 1 mark

9780170448192

3 © VCAA 2013 2MQ1 (3 marks) Five trout-breeding ponds, P, Q, R, X and V, are connected by pipes, as shown in the diagram.



The matrix W is used to represent the information in this diagram.

$$W = \begin{array}{c} \\ P \\ Q \\ R \\ X \\ V \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & X & V \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

In matrix W

- the 1 in column 1, row 2, for example, indicates that a pipe directly connects pond P and pond Q.
- the 0 in column 1, row 5, for example, indicates that pond P and pond V are not directly connected by a pipe.

a Find the sum of the elements in row 3 of matrix W. 1 mark

b In terms of the breeding ponds described, what does the sum of the elements in row 3 of matrix W represent? 1 mark

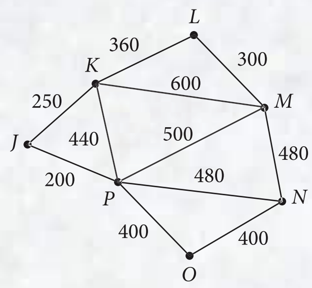
The pipes connecting pond P to pond R and pond P to pond X are removed. Matrix N will be used to show this situation. However, it has missing elements.

c Copy and complete matrix N by filling in the missing elements in row 1 and column 1. 1 mark

$$N = \begin{array}{c} \\ P \\ Q \\ R \\ X \\ V \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & X & V \end{array} \\ \begin{bmatrix} 0 & \_ & \_ & \_ & \_ \\ \_ & 0 & 0 & 1 & 0 \\ \_ & 0 & 0 & 1 & 0 \\ \_ & 1 & 1 & 0 & 1 \\ \_ & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

9780170448192

4 © VCAA 2015 2NQ1abd i (3 marks) A factory requires seven computer servers to communicate with each other through a connected network of cables. The servers, J, K, L, M, N, O and P, are shown as vertices on the graph below.



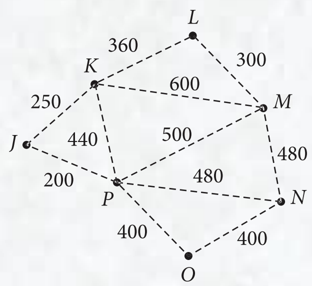
The edges on the graph represent the cables that could connect adjacent computer servers. The numbers on the edges show the cost, in dollars, of installing each cable.

a What is the cost, in dollars, of installing the cable between server L and server M? 1 mark

b What is the cheapest cost, in dollars, of installing cables between server K and server N? 1 mark

The computer servers will be able to communicate with all the other servers as long as each server is connected by cable to at least one other server. The cheapest installation that will join the seven computer servers by cable in a connected network follows a minimum spanning tree.

c Copy the plan shown and draw the minimum spanning tree on it. 1 mark



5 © VCAA 2005 2CQ1 MODIFIED (7 marks) Cars depreciate in value over time. **Table 1** gives the average value of a car (of the same brand and model) at different ages.

Table 1

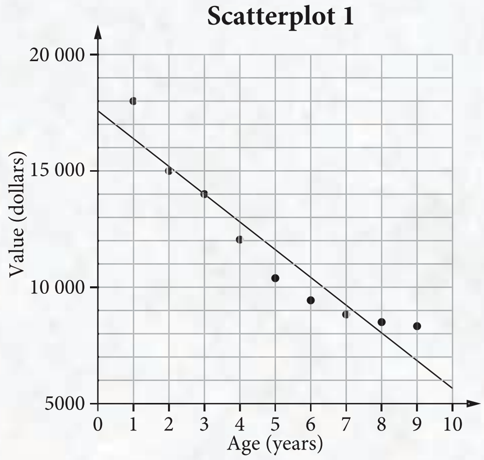
Age (years)	1	2	3	4	5	6	7	8	9
Value (dollars)	18 100	15 050	13 900	11 900	10 400	9600	8900	8500	8400

The data is to be used to build a mathematical association that will enable the average value of this brand and model of car to be predicted from its age.

a In this situation, what is the response variable? 1 mark

Scatterplot 1 is constructed from the data and a line of good fit is fitted as shown.

Scatterplot 1



b Give a reason why a straight line doesn't appear to be a good fit for the data. 1 mark

The scatterplot indicates that a logarithmic transformation of the horizontal (*age*) axis may linearise the data. The original data has been reproduced in **Table 2**. An extra row has been added for the transformed variable, log (*age*). The table is incomplete.

Table 2

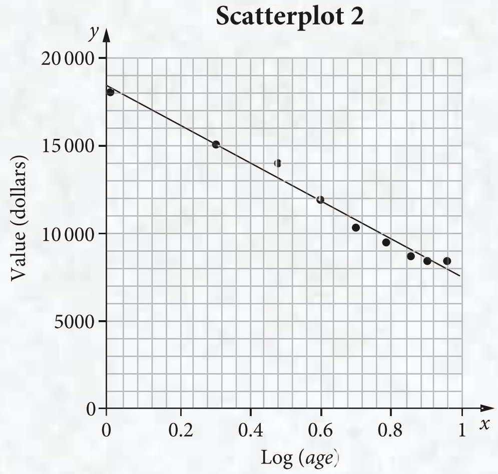
Age (years)	1	2	3	4	5	6	7	8	9
Log (age)	0	0.30	0.48	0.60	0.70	0.78	0.85	0.90	
Value (dollars)	18 100	15 050	13 900	11 900	10 400	9 600	8 900	8 500	8 400

c What is the missing *value*? Write your answer correct to two decimal places. 1 mark

In **Scatterplot 2**, *value* is plotted against log (*age*). A line of good fit fitted to the transformed data is also drawn.

d Use the information in **Scatterplot 2** to describe the association between value and log (*age*) in terms of **direction** and **strength**. 2 marks

Scatterplot 2



e The equation of this line of good fit is

$$value = 18\,300 - 10\,800 \times \log(age)$$

Use this equation to predict the value of a car that is three years old. Write your answer correct to the nearest hundred dollars. 1 mark

f What is the difference between the predicted value when the car is three years old in part **e** and the actual value given in **Table 1**? 1 mark

CUMULATIVE EXAMINATION 2

CHAPTER

9 MEASUREMENT, SCALE AND SIMILARITY

STUDY DESIGN COVERAGE

Study Design coverage

UNIT 2, AREA OF STUDY 4: SPACE AND MEASUREMENT

Space, measurement and applications of trigonometry

- units of measurement of length, angle, area, volume and capacity
- exact and approximate answers, scientific notation, significant figures and rounding
- similar shapes including the conditions for similarity
- perimeter and areas of triangles, quadrilaterals, circles including arcs and sectors and composite shapes, and practical applications
- volumes and surface areas of solids (spheres, cylinders, pyramids and prisms and composite objects) and practical applications, including simple applications of Pythagoras' theorem in three dimensions
- similar objects and the application of linear scale factor $k > 0$ to scale lengths, surface areas and volumes with practical applications.

Video playlists (7):

9.1 Measurement
9.2 Pythagoras' theorem
9.3 Perimeter and area
9.4 Volume
9.5 Surface area
9.6 Similarity and scale
VCE question analysis Measurement, scale and similarity

Skillsheets (3):

9.2 Pythagoras' theorem
9.3, 9.5 Solid shapes
9.6 Finding sides in similar triangles

Worksheets (22):

9.1 Length, area and volume conversions
9.2 Pythagoras' theorem time trial • Pythagorean two-step problems • Applications of Pythagoras' theorem • Pythagoras' problems
9.3 Units of length and perimeter • Area ID • Areas of composite shapes • Composite areas • A page of circular shapes • Applications of area
9.4 A page of solid shapes • Volumes of solids • Measurement in the home • Volumes of water
9.5 Nets of solids • A page of solid shapes • Surface area of solids • Surface area • Formula matching game
9.6 Finding sides in similar figures • Areas and volumes of similar figures

Puzzles (2):

9.1 Scientific notation puzzle
9.5 Surface area riddle

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Measurement

Worksheet
Length, area and volume conversions

9.1 Measurement

Units of measurement

The following units of measurement for length, area, volume, capacity and angles will be covered in this chapter. These diagrams show how to convert between the units of measurement.

Length units

× 1000 × 100 × 10

km kilometre → m metre → cm centimetre → mm millimetre

÷ 1000 ÷ 100 ÷ 10

Area units (square units)

$\times 1000^2$ $\times 100^2$ $\times 10^2$

km^2 square kilometre → m^2 square metre → cm^2 square centimetre → mm^2 square millimetre

$\div 1000^2$ $\div 100^2$ $\div 10^2$

Volume units (cubic units)

$\times 1000^3$ $\times 100^3$ $\times 10^3$

km^3 cubic kilometre → m^3 cubic metre → cm^3 cubic centimetre → mm^3 cubic millimetre

$\div 1000^3$ $\div 100^3$ $\div 10^3$

Capacity units

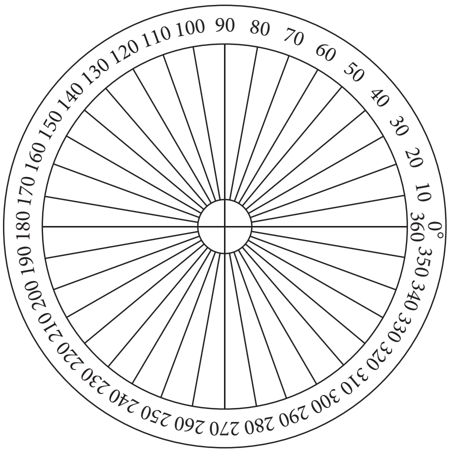
× 1000 × 1000 × 1000

ML megalitre → kL kilolitre → L litre → mL millilitre

÷ 1000 ÷ 1000 ÷ 1000

Angle unit

Degrees (°)



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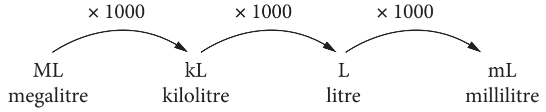
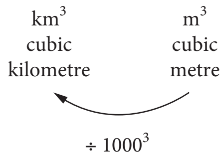
WORKED EXAMPLE 1	Converting units of measurement
Convert each of the following units of measurement.	
a 18 645 square centimetres to square metres	**b** 5 metres to millimetres
c 9.62 ML to mL	**d** $4200\,m^3$ to km^3
Steps	**Working**
a m^2 square metre ← cm^2 square centimetre ($\div 100^2$)	$18\,645\,cm^2 = 18\,645 \div 100^2\,m^2$ $= 1.8645\,m^2$
b m metre → cm centimetre → mm millimetre (× 100, × 10)	$5\,m = 5 \times 100 \times 10\,mm$ $= 5000\,mm$

c		$9.62\,\text{ML} = 9.62 \times 1000 \times 1000 \times 1000\,\text{mL}$ $= 9\,620\,000\,000\,\text{mL}$
d		$4200\,\text{m}^3 = 4200 \div 1000^3\,\text{km}^3$ $= 0.000\,004\,2\,\text{km}^3$

Scientific notation

When converting units of measurement, the numbers can quickly become very large or very small. This is why we often use **scientific notation** when working with measurement problems. A number written in scientific notation takes the form:

(a number between 1 and 10) × (a power of 10)

Puzzle
Scientific notation puzzle

Scientific notation for numbers greater than 1

Number	Scientific notation
476.3	4.763×10^2
5200	5.2×10^3
56 000	5.6×10^4
674 359.86	$6.743\,598\,6 \times 10^5$
4 942 300 000	4.9423×10^9

Scientific notation for numbers less than 1

Number	Scientific notation
0.0456	4.56×10^{-2}
0.0023	2.3×10^{-3}
0.0005	5×10^{-4}
0.000 067 403	6.7403×10^{-5}
0.000 000 004 942 3	4.9423×10^{-9}

Converting *to* scientific notation

Move the decimal place n times until a number between 1 and 10 is attained.

- Moving the decimal point to the left, the power of 10 is n.
- Moving the decimal point to the right, the power of 10 is $-n$.

or use CAS.

WORKED EXAMPLE 2 Converting to scientific notation

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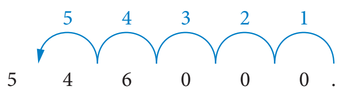
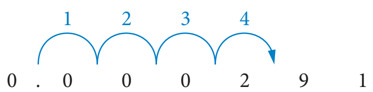
Convert the following to scientific notation.

a 546 000 **b** 0.000 291

Steps	Working
a **1** Count the number of times needed to move the decimal point to attain a number between 1 and 10. Has the decimal point moved left or right?	 Moving the decimal point five places to the left gives 5.46.
2 Write the answer.	$546\,000 = 5.46 \times 10^5$
b **1** Count the number of times needed to move the decimal point to attain a number between 1 and 10. Has the decimal point moved left or right?	 Moving the decimal point four places to the right gives 2.91.
2 Write the answer.	$0.000\,291 = 2.91 \times 10^{-4}$

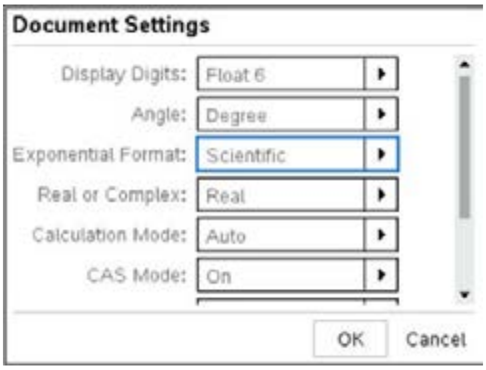
USING CAS 1 Converting to scientific notation

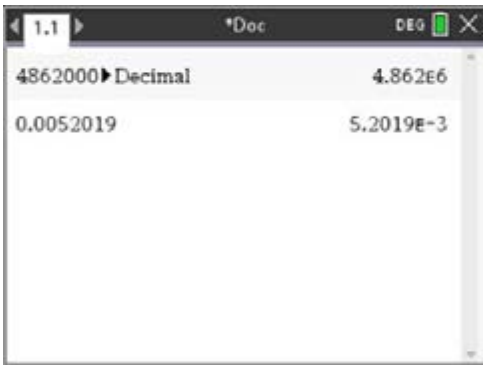
Convert the following to scientific notation.

a 4 862 000 **b** 0.005 201 9

[Note that when CAS gives 'E' in the answer, this means 'multiplied by 10 to the power of'.]

TI-Nspire





1 Press **home** > **Settings** > **Document Settings**.

2 Change the **Exponential Format:** field to **Scientific***.

3 Press **OK**.

*When finished, remember to reset this field to **Normal**.

4 Enter **4 862 000.**

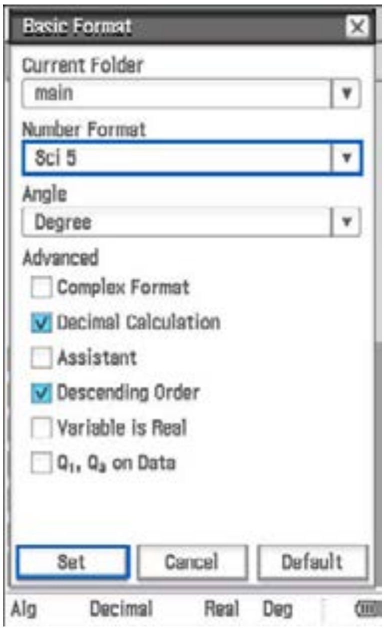
5 Press **menu** > **Number** > **Convert to Decimal**.

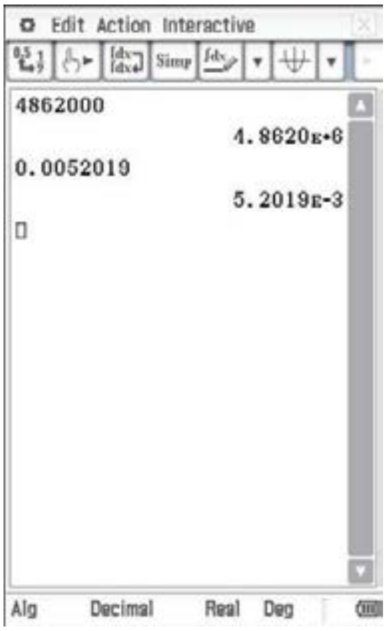
6 Press **enter**.

7 Enter **0.005 201 9**.

8 Press **enter**.

ClassPad





1 Tap ⚙ > **Basic Format**.

2 Change the **Number Format** field to one of the **Sci 0** to **Sci 9** settings. The setting above is **Sci 5***.

3 Tap **Set**.

*When finished, remember to reset this field to **Normal 1** or **Normal 2**.

4 Enter **4 862 000**.

5 Tap **EXE**.

6 Enter **0.005 201 9**.

7 Tap **EXE**.

a 4 862 000 = 4.862×10^{6} **b** 0.005 201 9 = 5.2019×10^{-3}

Converting *from* scientific notation

If n, the power of 10,

- is positive, move the decimal power to the right n times, inserting zeros if necessary
- n is negative, move the decimal power to the left n times, inserting zeros if necessary

or do the multiplication using CAS.

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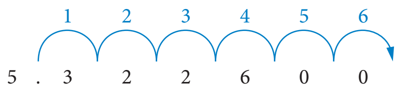
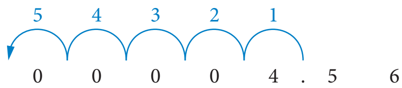
WORKED EXAMPLE 3 Converting from scientific notation

Convert the following from scientific notation.

a 5.3226×10^6 **b** 4.56×10^{-5}

Steps	Working
a **1** What is the power of 10? Does the decimal point need to move left or right? Do zeros need to be inserted? Or do the multiplication using CAS, if necessary.	10^6 means to move the decimal point 6 places to the right. 
2 Write the answer.	$5.3226 \times 10^6 = 5\,322\,600$
b **1** What is the power of 10? Does the decimal point need to move left or right? Do zeros need to be inserted? Or do the multiplication using CAS, if necessary.	10^{-5} means to move the decimal point 5 places to the left. 
2 Write the answer.	$4.56 \times 10^{-5} = 0.000\,045\,6$

Rounding measurements to significant figures

When working with measurement problems, we often give approximate values rather than exact answers, so rounding is important. When small and large numbers are involved, it often makes sense to round to significant figures rather than to decimal places. We looked at rounding to significant figures in Chapter 6.

Significant figures

Significant figures:

- any non-zero digit
- zeros between non-zero digits
- trailing zeros in decimals.

Not significant figures:

- leading zeros in decimals
- trailing zeros in whole numbers.

When rounding to significant figures, use usual rounding rules:

- '0–4 round down' and '5–9 round up'
- round the 9 to 0 and carry the rounding over to the next digit on the left
- include trailing zeros in decimals, if necessary.

EXERCISE 9.1 Measurement

ANSWERS p. 516

Mastery

1 WORKED EXAMPLE 1 Convert each of the following units of measurement.

a 43 681 centimetres to kilometres
b 7 cubic metres to cubic centimetres
c 5 square metres to square centimetres
d 600 mL to L
e 50 m^3 to mm^3
f 4.2 kL to L
g 250 000 mm^2 to cm^2
h 6800 cm^2 to m^2
i 1.5 km to mm

2 WORKED EXAMPLE 2 Convert the following to scientific notation.

a 564
b 45 000
c 0.0055
d 0.0008
e 63 000 000
f 0.000 029 7

3 Using CAS 1 Convert the following to scientific notation.

a 72 310 290
b 0.046 03

4 WORKED EXAMPLE 3 Convert the following from scientific notation.

a 3.2×10^4
b 9×10^6
c 4.537×10^5
d 5.03×10^{-4}
e 8×10^{-3}
f 6.14×10^{-4}

Exam practice

5 152 m^2 is equivalent to

A 0.0152 cm^2
B 1.52 cm^2
C 15 200 cm^2
D 1 520 000 cm^2
E 15 200 000 cm^2

6 How many litres are in a megalitre?

A 10^2
B 10^3
C 10^5
D 10^6
E 10^7

7 Which of the following is the number of square centimetres in 300 m^2?

A 3
B 300
C 3000
D 3 000 000
E 30 000 000

8 The distance from Earth to Mars is 225 300 000 km. What is the distance from Earth to Mars in scientific notation?

A 22.53×10^8 km
B 22.53×10^9 km
C 2.253×10^9 km
D 2253×10^8 km
E 2.253×10^8 km

Use the following information to answer the next two questions.

A 2013 UN study reported that out of the world's estimated 7 billion (7 000 000 000) people, 6 billion had access to mobile phones, whereas only 4.5 billion had access to working toilets.

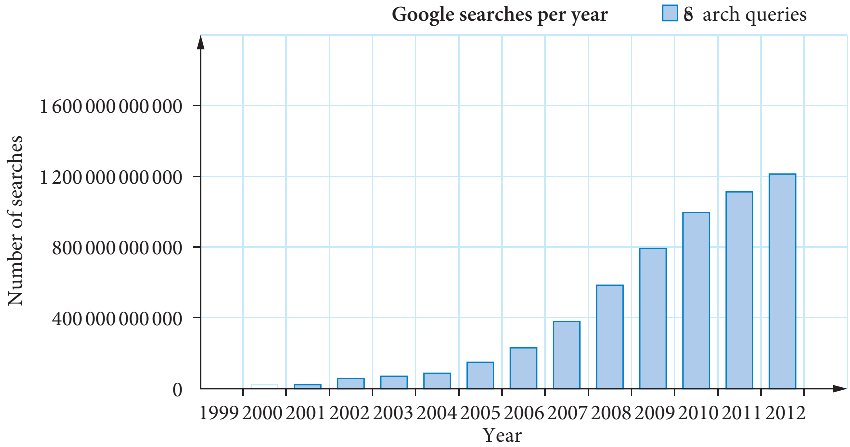
9 In scientific notation, how many people didn't have access to working toilets?

A 2.5×10^8
B 2.5 billion
C 4 500 000 000
D 2.5×10^9
E 4.5×10^9

10 Which of the following *doesn't* indicate the number of people who had access to working toilets?

A 4 500 000 000
B 4.5×10^9
C 4.5×10^{10}
D 4.5 billion
E four thousand five hundred million

11 The following graph shows the number of Google queries made from 1999 to 2012.



Which one of the following gives the best approximation to the number of searches made in 2012 in scientific notation?

A 1 200 000 000 000 **B** 1.2 trillion **C** 1.2×10^{12}

D 1.2×10^{11} **E** 1.2×10^{13}

9.2 Pythagoras' theorem

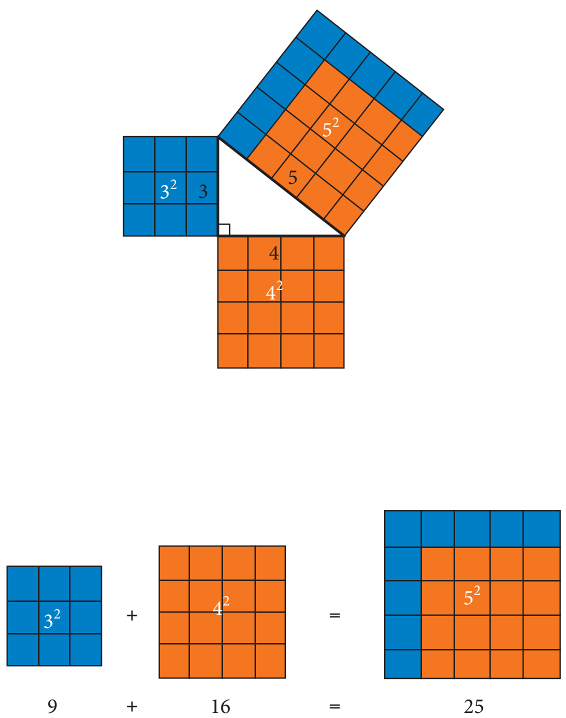
Pythagoras' theorem in two dimensions

A **right-angled triangle** is a triangle where one of the angles is 90°. **Pythagoras' theorem** is a rule for calculating the length of the longest side of a right-angled triangle given the length of the other two sides. For any right-angled triangle, we can write an equation in the form:

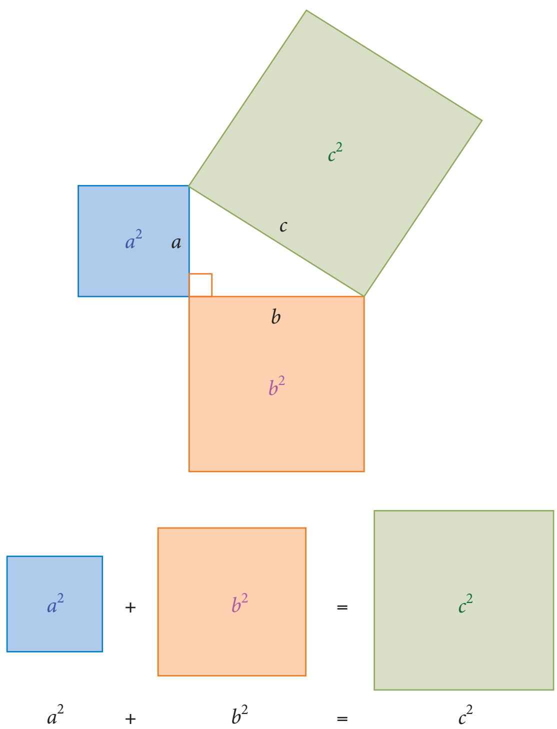
longest side squared = one short side squared + the other short side squared

The longest side is called the **hypotenuse**. It is always opposite the right angle.

Example of Pythagoras' theorem



General form of Pythagoras' theorem



Video playlist Pythagoras' theorem

Skillsheet Pythagoras' theorem

Worksheets Pythagoras' theorem time trial

Pythagorean two-step problems

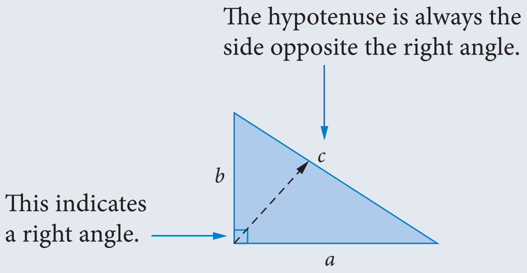
Applications of Pythagoras' theorem

Pythagoras' problems

Pythagoras' theorem

In a right-angled triangle with side lengths a, b and c, where c is the hypotenuse:

$$c^2 = a^2 + b^2$$



Exam hack

The following side lengths often appear in Pythagoras' theorem questions:

3, 4, 5 $(3^2 + 4^2 = 5^2)$
6, 8, 10 $(6^2 + 8^2 = 10^2)$
5, 12, 13 $(5^2 + 12^2 = 13^2)$
9, 12, 15 $(9^2 + 12^2 = 15^2)$

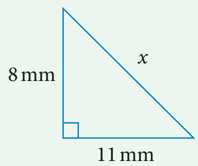
WB
p. 132

WORKED EXAMPLE 4 Using Pythagoras' theorem to find unknown sides

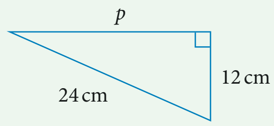
For each of the following right-angled triangles, find the unknown values

- **i** correct to two decimal places
- **ii** correct to two significant figures.

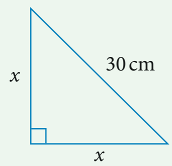
a



b



c



Steps	Working
a i 1 Identify the hypotenuse, c. The other two sides are a and b. Use Pythagoras' theorem to find the unknown side, using CAS if necessary.	$c = x$, $a = 8$, $b = 11$ $c^2 = a^2 + b^2$ $x^2 = 8^2 + 11^2$ $= 185$ $x = \sqrt{185}$ $= 13.601\ldots$
2 Write your answer in the required units and round to the required level of accuracy.	$x = 13.601\ldots$ $x \approx 13.60$ mm
ii Write your answer in the required units and round to the required level of accuracy.	$x = 13.601\ldots$ $x \approx 14$ mm
b i 1 Identify the hypotenuse, c. The other two sides are a and b. Use Pythagoras' theorem to find the unknown side, using CAS if necessary.	$c = 24$, $a = 12$, $b = p$ $c^2 = a^2 + b^2$ $24^2 = 12^2 + p^2$ $p^2 = 24^2 - 12^2$ $= 432$ $p = \sqrt{432}$ $= 20.784\ldots$
2 Write your answer in the required units and round to the required level of accuracy.	$p = 20.784\ldots$ $p \approx 20.78$ cm
ii Write your answer in the required units and round to the required level of accuracy.	$p = 20.784\ldots$ $p \approx 21$ cm

9780170448192

c **i** **1** Identify the hypotenuse, c. The other two sides are a and b. Use Pythagoras' theorem to find the unknown side, using CAS if necessary.

$c = 30,\ a = x,\ b = x$

$c^2 = a^2 + b^2$

$30^2 = x^2 + x^2$

$2x^2 = 30^2$

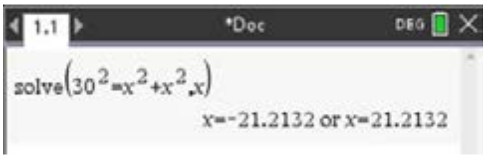
$= 900$

$x^2 = 450$

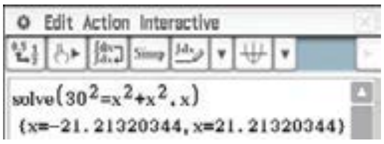
$x = \sqrt{450}$

$= 21.213\ldots$

TI-Nspire



ClassPad



2 Write your answer in the required units and round to the required level of accuracy.

$x = 21.213\ldots$

$x \approx 21.21$ cm

ii Write your answer in the required units and round to the required level of accuracy.

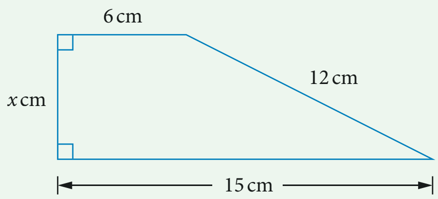
$x = 21.213\ldots$

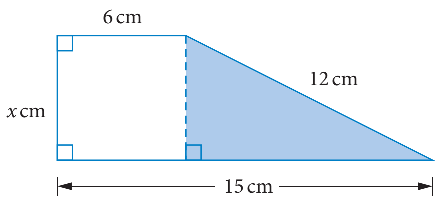
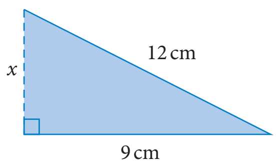
$x \approx 21$ cm

WORKED EXAMPLE 5 Using Pythagoras' theorem with shapes that contain right-angled triangles

WB p. 134

Calculate the value of x, correct to one decimal place.

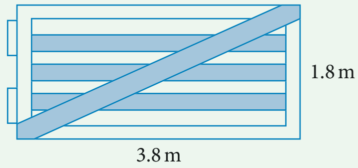


Steps	**Working**
1 Identify the right-angled triangle.	
2 Label the sides of the triangle using the information given.	$15 - 6 = 9$ cm 
3 Identify the hypotenuse, c. The other two sides are a and b. Use Pythagoras' theorem to find the unknown side, using CAS if necessary.	$c = 12,\ a = x,\ b = 9$ $c^2 = a^2 + b^2$ $12^2 = x^2 + 9^2$ $144 = x^2 + 81$ $144 - 81 = x^2$ $x^2 = 63$ $x = \sqrt{63}$ $= 7.937\ldots$
4 Write your answer in the required units and round to the required level of accuracy.	$x = 7.9$ cm

WB
p. 135

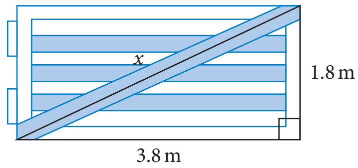
WORKED EXAMPLE 6 Solving problems using Pythagoras' theorem in two dimensions

A rectangular wooden gate has dimensions 3.8 m by 1.8 m. What is the length of its diagonal plank, to the nearest centimetre?



Steps | **Working**

1 Identify the right-angled triangle and label the unknown side x.



2 Use Pythagoras' theorem to find the unknown side, using CAS if necessary.

$$a^2 + b^2 = c^2$$
$$(1.8)^2 + (3.8)^2 = x^2$$
$$x^2 = 17.68$$
$$x = 4.204\ldots$$

3 Convert to the required units, round to the required accuracy, and write the answer.

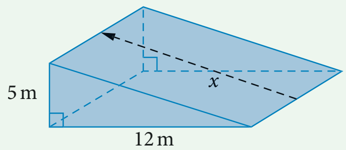
4.204… m
= 4.204… × 100
= 420.4… cm
The diagonal plank has length 420 cm.

Pythagoras' theorem in three dimensions

WB
p. 136

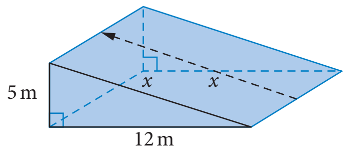
WORKED EXAMPLE 7 Solving problems using Pythagoras' theorem in three dimensions

For this ski jump ramp, what is the distance (in metres) a skier would travel if they skied directly up the centre of the ramp?

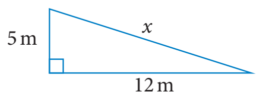


Steps | **Working**

1 Picture the diagram in three dimensions and find the relevant right-angled triangle with one unknown side.



2 Redraw the right-angled triangle separately, labelling the two known sides and one unknown side.



3 Use Pythagoras' theorem to find the unknown side, using CAS if necessary.

$$c^2 = a^2 + b^2$$
$$x^2 = 5^2 + 12^2$$
$$5^2 + 12^2 = x^2$$
$$x^2 = 169$$
$$x = 13$$

4 Write your answer in the required units and round to the required level of accuracy.

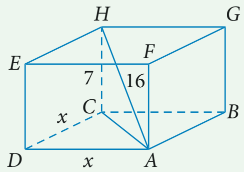
A skier would travel 13 m if they skied directly up the centre of the ramp.

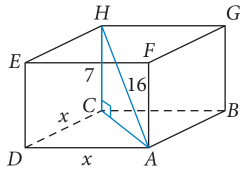
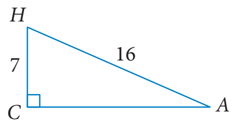
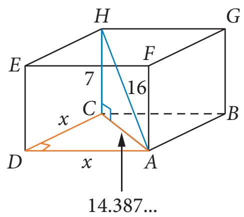
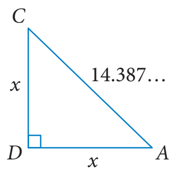
WORKED EXAMPLE 8 Solving two-step problems using Pythagoras' theorem in three dimensions

This rectangular box has a square base.
The lengths of AH and CH are 16 cm and 7 cm respectively.

a Find the length of AC correct to three significant figures.

b Use your answer to part **a** to find x, the length of AD, correct to three significant figures.



Steps	**Working**
a **1** Picture the diagram in three dimensions and find the relevant right-angled triangle with one unknown side.	
2 Redraw the right-angled triangle separately, labelling the two known sides and one unknown side.	
3 Use Pythagoras' theorem to find the unknown side, using CAS if necessary.	$a^2 + b^2 = c^2$ $(AC)^2 + 7^2 = 16^2$ $(AC)^2 = 16^2 - 7^2$ $(AC)^2 = 207$ $AC = 14.387\ldots$
4 Write your answer in the required units and round to the required level of accuracy.	$AC \approx 14.4$ cm
b **1** Picture the diagram in three dimensions and find the relevant right-angled triangle that includes the side whose length we found in part **a**.	
2 Redraw the right-angled triangle separately, labelling the sides. Include the unrounded answer from part **a**.	
3 Use Pythagoras' theorem and the unrounded answer to part **a** to find x, using CAS if necessary.	$a^2 + b^2 = c^2$ $x^2 + x^2 = (AC)^2$ $2x^2 = (14.387\ldots)^2$ $2x^2 = 207$ $x^2 = 103.5$ $x = \sqrt{103.5}$ $= 10.173\ldots$
4 Write your answer in the required units and round to the required level of accuracy.	$AD = 10.2$ cm

9.2

WB
p. 137

Recap

1 $2.49\text{ m}^2 =$ ________ cm^2?

A 2.5 **B** 249 **C** 2490 **D** 24 900 **E** 249 000

2 The speed of light is approximately 3×10^8 m/s. This can also be written as

A 300 000 000 m/s **B** 30 000 000 m/s **C** 3108 m/s

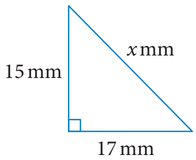
D 3 000 000 000 m/s **E** 240 m/s

Mastery

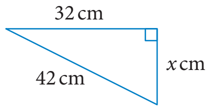
3 WORKED EXAMPLE 4 For each of the following right-angled triangles, find the unknown value

i correct to two decimal places **ii** correct to two significant figures.

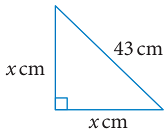
a



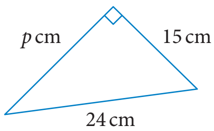
b



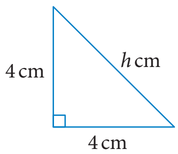
c



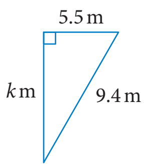
d



e

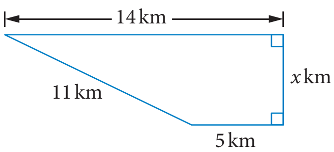


f

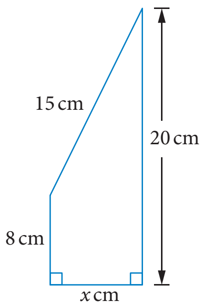


4 WORKED EXAMPLE 5 Calculate the value of x for each of the following, correct to one decimal place.

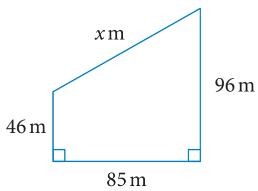
a



b

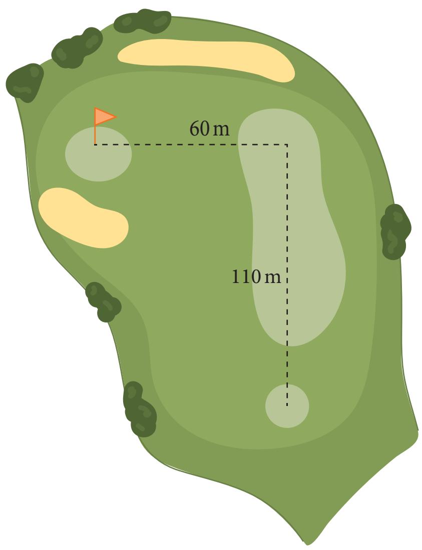


c

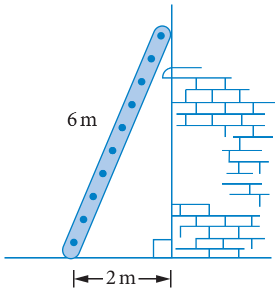


5 WORKED EXAMPLE 6

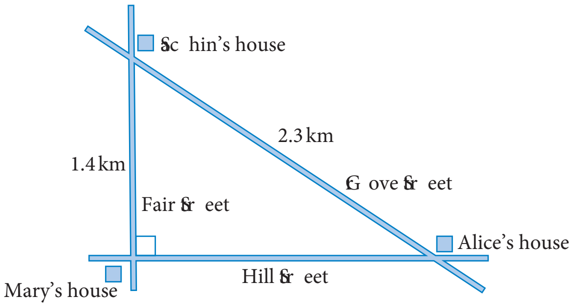
a Jordan scored a hole-in-one on the hole shown on a golf course. How far did he hit the ball? Give your answer to the nearest metre.



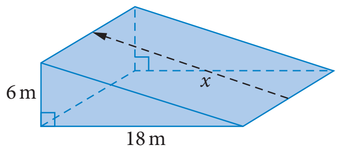
b A 6 m ladder leans against a house so that its base is 2 m from the bottom of the house. How far up the wall of the house does the ladder reach, to the nearest centimetre?



c How many metres does Alice have to walk along Hill Street from her house to get to Mary's house? Give your answer to the nearest metre.



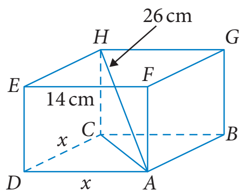
6 WORKED EXAMPLE 7 For this snowboard jump, what is the distance (in metres) a snowboarder would travel if they snowboarded directly up the centre of the jump?



7 WORKED EXAMPLE 8 This rectangular box has a square base. The lengths of AH and CH are 26 cm and 14 cm respectively.

a Find the length of AC correct to three significant figures.

b Use your answer to part **a** to find x, the length of AD, correct to three significant figures.



Exam practice

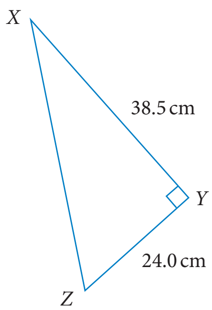
80–100% 60–79% 0–59%

8 © VCAA 2017 1GMQ2 92% A right-angled triangle, XYZ, has side lengths $XY = 38.5$ cm and $YZ = 24.0$ cm, as shown in the diagram.

The length of XZ, in centimetres, is closest to

A 24.8 **B** 30.1 **C** 38.8

D 45.4 **E** 62.5

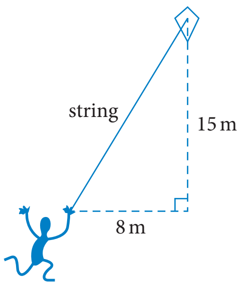


9 © VCAA 2018 1GMQ1 92% Henry flies a kite attached to a long string, as shown in the diagram.

The horizontal distance of the kite to Henry's hand is 8 m. The vertical distance of the kite above Henry's hand is 15 m. The length of the string, in metres, is

A 13 **B** 17 **C** 23

D 161 **E** 289

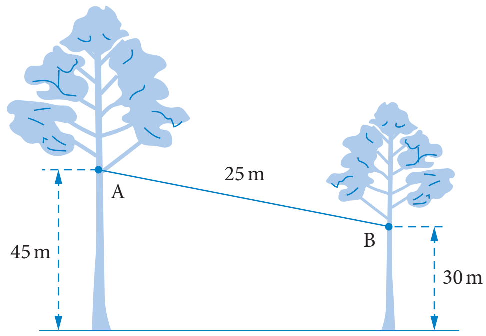


10 © VCAA 2020 1GMQ3 83% Two trees stand on horizontal ground. A 25 m cable connects the two trees at point A and point B, as shown in the diagram.

Point A is 45 m above the ground and point B is 30 m above the ground. The horizontal distance, in metres, between point A and point B is

A 10 **B** 15 **C** 20

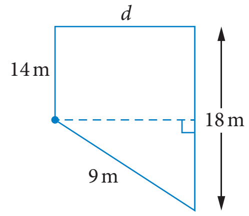
D 30 **E** 35



11 What is the value of *d*, correct to three significant figures?

A 8.06 m **B** 9.85 m **C** 10.7 m

D 11.3 m **E** 15.6 m

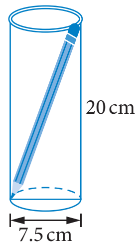


12 Rachel wants to use an old tennis ball container as a pencil case. If the container is a cylinder with a diameter of 7.5 cm and a height of 20 cm, and she has five pencils of the following lengths: 19.5 cm, 20 cm, 21 cm, 21.5 cm and 22 cm, how many of them are too long to fit inside the container?

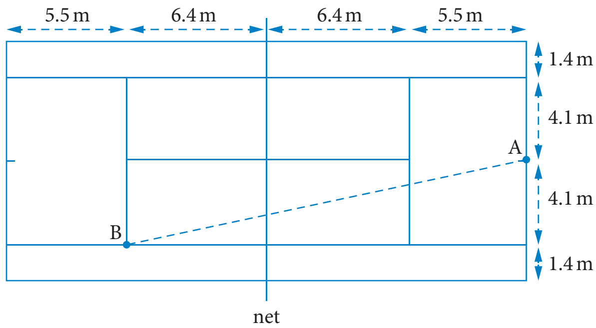
A 0 **B** 1

C 2 **D** 3

E 4



13 © VCAA 2018 2GMQ3ab (2 marks) Frank owns a tennis court. A diagram of his tennis court is shown. Assume that all intersecting lines meet at right angles. Frank stands at point A. Another point on the court is labelled point B.



a 73% What is the straight-line distance, in metres, between point A and point B? Round your answer to one decimal place. 1 mark

b 60% Frank hits a ball when it is at a height of 2.5 m directly above point A. Assume that the ball travels in a straight line to the ground at point B. What is the straight-line distance, in metres, that the ball travels? Round your answer to the nearest whole number. 1 mark

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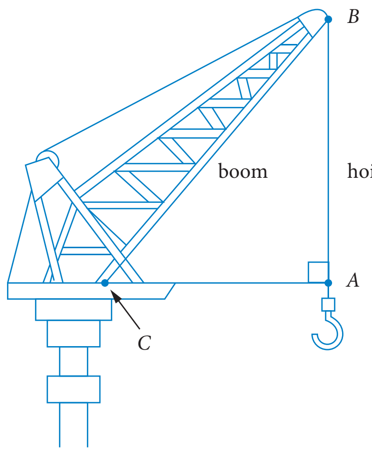
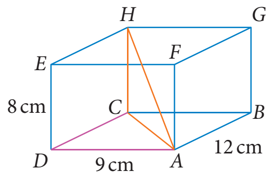
14 © VCAA 2019 2GMQ3a i **60%** (1 mark) The diagram shows a crane that is used to transfer shipping containers between a port and a cargo ship.

The length of the boom, BC, is 25 m. The length of the hoist, AB, is 15 m. Write a calculation to show that the distance AC is 20 m.



9.3

15 (6 marks) The rectangular box has the length, width and height shown. Two right-angled triangles have been drawn.



a Copy the diagram and mark in the right angle in each of the two triangles. — 2 marks

b Redraw ΔACD separately from the three-dimensional figure and label the known side lengths. — 1 mark

c What is the length of AC, correct to the nearest centimetre? — 1 mark

d Redraw ΔACH separately from the three-dimensional figure and label the known sides. — 1 mark

e What is the length of AH, correct to the nearest centimetre? — 1 mark

9.3 Perimeter and area

Video playlist Perimeter and area

Worksheets Units of length and perimeter

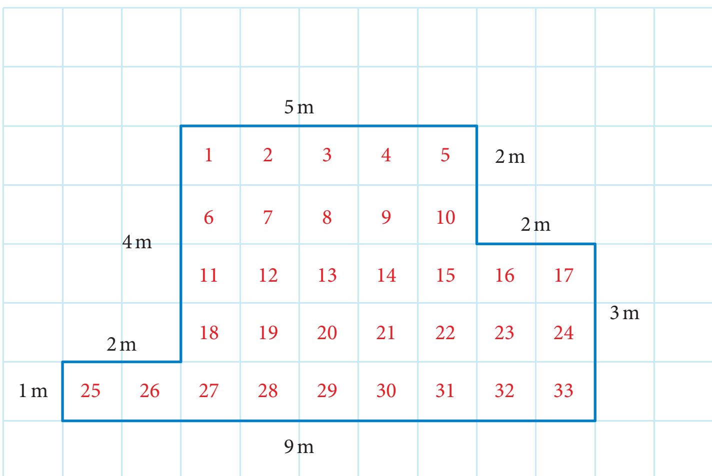
Area ID

Perimeter and area of quadrilaterals, triangles and circles

The **perimeter** of a **shape** is a measure of the total distance around the outside of the shape. The **area** of a shape is a measure of the amount of space inside the shape. For example:

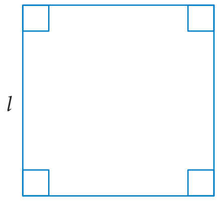
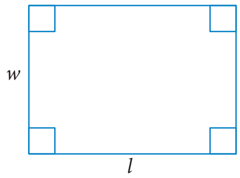
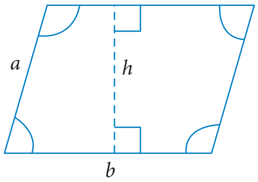
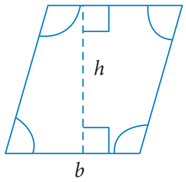
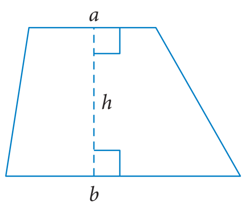
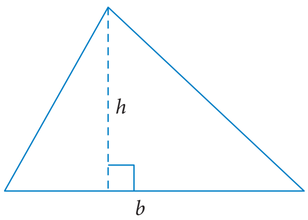
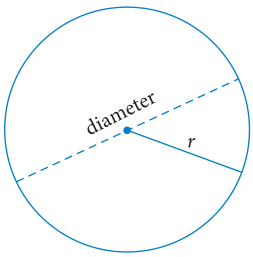
The perimeter of this shape is 1 + 2 + 4 + 5 + 2 + 2 + 3 + 9 = 28 m

The area of this shape is 33 m^2.



We will be looking at the perimeter and area of **quadrilaterals** (shapes with four straight sides), triangles and circles. When dealing with circles we also use the word **circumference** for perimeter. The **radius** of a circle is the distance from its centre to the circumference, and the **diameter** of a circle is the distance from one side to the other through the centre.

We usually use formulas to calculate the perimeter and area of these shapes:

Shape		Perimeter	Area
Square Four sides All sides are equal All angles are 90°		$P = 4l$	$A = l^2$
Rectangle Four sides Opposite sides are equal in length All angles are 90°		$P = 2l + 2w$	$A = lw$
Parallelogram Four sides Opposite sides are equal in length and parallel		$P = 2a + 2b$	$A = bh$
Rhombus Four sides All sides are equal in length Opposite sides are parallel		$P = 4b$	$A = bh$
Trapezium Four sides Two sides are parallel		Add the lengths of the four sides	$A = \frac{1}{2}(a + b)h$
Triangle Three sides		Add the lengths of the three sides	$A = \frac{1}{2}bh$
Circle Every point is the same distance from the centre		$C = 2\pi r$	$A = \pi r^2$

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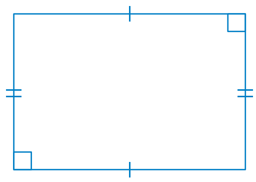
Exam hack

Sides equal in length are often indicated by dashes.

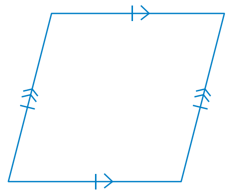
For example,



is a rectangle.

Parallel sides are often indicated by arrows.

For example,



is a rhombus.

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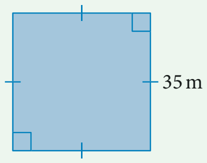
WORKED EXAMPLE 9 Calculating the perimeter and area of quadrilaterals, triangles and circles

For each of the following shapes, calculate

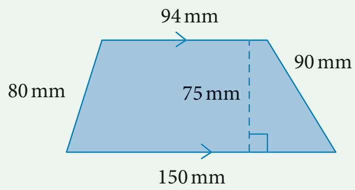
i the perimeter to the nearest centimetre

ii the area to the nearest square centimetre.

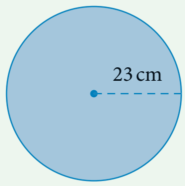
a



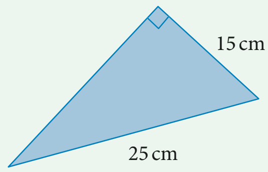
b



c



d



Steps	Working
a i 1 Identify the shape.	square
2 State the values from the diagram.	$l = 35$
Use the perimeter formula for that shape.	$P = 4l$ $= 4 \times 35$ $= 140$
3 Write your answer in the required units and round to the required level of accuracy.	$P = 140$ m $= 140 \times 100$ cm $= 14\,000$ cm
ii 1 Use the area formula for that shape.	$A = l^2$ $= 35^2$ $= 1225$
2 Write your answer in the required units and round to the required level of accuracy.	$A = 1225\text{ m}^2$ $= 1225 \times 100^2\text{ cm}^2$ $= 12\,250\,000\text{ cm}^2$
b i 1 Identify the shape.	trapezium
2 State the values from the diagram.	$a = 94$, $b = 150$, other sides = 80 and 90, $h = 75$
Use the perimeter formula for that shape.	$P = 94 + 150 + 80 + 90$ $= 414$
3 Write your answer in the required units and round to the required level of accuracy.	$P = 414$ mm $= 414 \div 10$ cm $= 41.4$ cm $P = 41$ cm

ii **1** Use the area formula for that shape.

$A = \frac{1}{2}(a + b)h$

$= \frac{1}{2}(94 + 150) \times 75$

$= 9150$

2 Write your answer in the required units and round to the required level of accuracy.

$A = 9150\ \text{mm}^2$

$= 9150 \div 10^2\ \text{cm}^2$

$= 91.50\ \text{cm}^2$

$A = 92\ \text{cm}^2$

c **i** **1** Identify the shape.

circle

2 State the values from the diagram.

$r = 23$

Use the perimeter formula for that shape.

$C = 2\pi r$

$= 2\pi \times 23$

$= 144.513\ldots$

3 Write your answer in the required units and round to the required level of accuracy.

$C = 145\ \text{cm}$

ii **1** Use the area formula for that shape.

$A = \pi r^2$

$= \pi \times 23^2$

$= 1661.902\ldots$

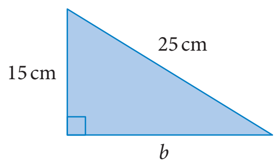
2 Write your answer in the required units and round to the required level of accuracy.

$A = 1662\ \text{cm}^2$

d **i** **1** Identify the shape.

triangle

2 Redraw the diagram to identify the values.
State the values from the diagram.
Calculate other values needed.
Use the perimeter formula for that shape.



$h = 15,\ b = ?$

Using Pythagoras' theorem:

$a^2 + b^2 = c^2$

$15^2 + b^2 = 25^2$

$b^2 = 25^2 - 15^2$

$= 400$

$b = 20$

$P = 20 + 15 + 25 = 60$

3 Write your answer in the required units and round to the required level of accuracy.

$P = 60\ \text{cm}$

ii **1** Use the area formula for that shape.

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 20 \times 15$

$= 150$

2 Write your answer in the required units and round to the required level of accuracy.

$A = 150\ \text{cm}^2$

Exam hack

Always use the π value on CAS when doing circle-related calculations.

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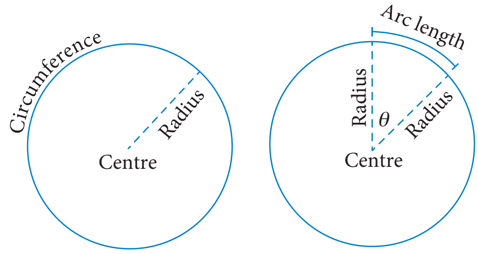
Arcs and sectors

An **arc** is a part of the circumference of a circle formed by two radiuses. If the angle between two radiuses is known, we can calculate the **arc length**.

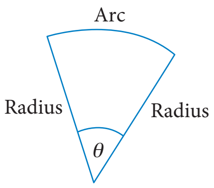


circumference of circle = $2\pi r$

$$\text{arc length} = \frac{\theta}{360} \text{ of circle}$$

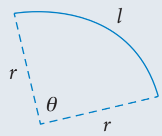
$$= \frac{\theta}{360} \times 2\pi r = \frac{\pi r\theta}{180}$$

A **sector** is the part of a circle formed by two radiuses and the arc between them.



Arcs and sectors

For a sector of a circle of radius r, angle θ, and arc length l



$$\text{arc length } l = \frac{\pi r\theta}{180}$$

$$\text{sector perimeter} = 2r + l$$

$$\text{sector area} = \frac{1}{2}rl$$

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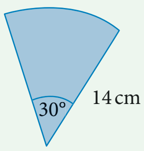
WORKED EXAMPLE 10 Calculating the perimeter and area of sectors

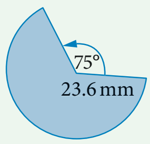
Calculate the following correct to three significant figures for each of the sectors shown.

i arc length **ii** perimeter **iii** area

a



b



Steps	Working
a i 1 State the values from the diagram.	$r = 14,\ \theta = 30$
Use the arc length formula.	$l = \frac{\pi r\theta}{180}$ $= \frac{\pi \times 14 \times 30}{180}$ $= 7.330\ldots$
2 Write your answer in the required units and round to the required level of accuracy.	arc length = 7.33 cm
ii 1 State the values from the diagram.	$r = 14,\ \theta = 30,\ l = 7.330\ldots$
Use the sector perimeter formula, using unrounded values where necessary.	perimeter = $2r + l$ $= 2 \times 14 + 7.330\ldots$ $= 35.330\ldots$
2 Write your answer in the required units and round to the required level of accuracy.	perimeter = 35.3 cm
iii 1 State the values from the diagram.	$r = 14,\ \theta = 30,\ l = 7.330\ldots$
Use the sector area formula, using unrounded values where necessary.	area = $\frac{1}{2}rl$ $= \frac{1}{2} \times 14 \times 7.330\ldots$ $= 51.312\ldots$
2 Write your answer in the required units and round to the required level of accuracy.	area = 51.3 cm^2

b **i** **1** State the values from the diagram.

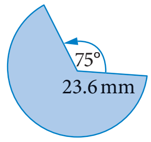
$r = 23.6,\ \theta = 360 - 75 = 285$

Use the arc length formula.

$l = \dfrac{\pi r \theta}{180}$

$= \dfrac{\pi \times 23.6 \times 285}{180}$

$= 117.390\ldots$



2 Write your answer in the required units and round to the required level of accuracy.

arc length = 117 mm

ii **1** State the values from the diagram.

$r = 23.6,\ \theta = 285,\ l = 117.390\ldots$

Use the sector perimeter formula, using unrounded values where necessary.

perimeter $= 2r + l$

$= 2 \times 23.6 + 117.390\ldots$

$= 164.590\ldots$

2 Write your answer in the required units and round to the required level of accuracy.

perimeter = 165 mm

iii **1** State the values from the diagram.

$r = 23.6,\ \theta = 285,\ l = 117.390\ldots$

Use the sector area formula, using unrounded values where necessary.

area $= \frac{1}{2}rl$

$= \frac{1}{2} \times 23.6 \times 117.390\ldots$

$= 1385.212\ldots$

2 Write your answer in the required units and round to the required level of accuracy.

area = 1390 mm^2

Perimeter and area of composite shapes

Skillsheet Solid shapes

Worksheets Areas of composite shapes; Composite areas; A page of circular shapes; Applications of area

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A **composite shape** is formed by combining two or more shapes. Use the formulas for quadrilaterals, triangles and circles to calculate the perimeter and area of composite shapes.

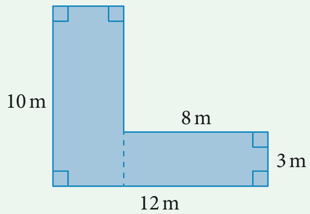
WORKED EXAMPLE 11 Calculating the perimeter and area of composite shapes

For each of the following shapes, calculate

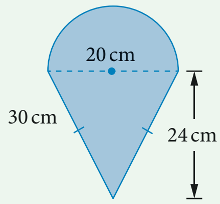
i the perimeter **ii** the area.

Give your answers using the units stated in each diagram to two significant figures.

a



b

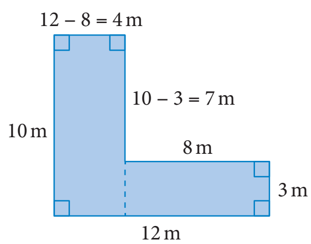


Steps / **Working**

a **i** **1** Identify the shapes that make up the composite shape.

2 rectangles

2 Calculate the missing lengths.



3 Add all the lengths.

$P = 10 + 4 + 7 + 8 + 3 + 12$

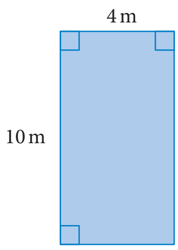
$= 44$

4 Write your answer in the required units and round to the required level of accuracy.

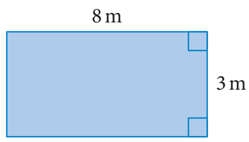
$P = 44$ m

ii **1** Separate the shapes that make up the composite shape.

rectangle



rectangle



2 Use the area formulas for those shapes.

$l = 10$, $w = 4$

$A = lw$
$= 4 \times 10$
$= 40$

$l = 8$, $w = 3$

$A = lw$
$= 8 \times 3$
$= 24$

3 Add the areas. Write your answer in the required units and round to the required level of accuracy.

$A = 40 + 24$
$= 64\text{ m}^2$

b **i** **1** Identify the shapes that make up the composite shape.

semi-circle and triangle

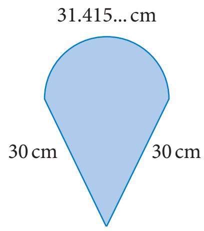
2 Calculate the missing lengths.

The curve of a semi-circle is half a circumference.

$r = 10$

$C = 2\pi r$
$= 62.831\ldots$

$\frac{1}{2}C = \frac{1}{2} \times 62.831\ldots$
$= 31.415\ldots$



3 Add all the lengths.

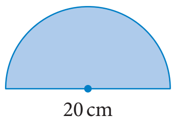
$P = 30 + 31.415\ldots + 30$
$= 91.415\ldots$

4 Write your answer in the required units and round to the required level of accuracy.

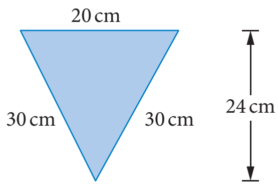
$P = 91\text{ cm}$

ii **1** Separate the shapes that make up the composite shape.

semi-circle



triangle



2 Use the area formulas for those shapes.

The area of a semi-circle is half the area of a circle.

$r = 10$

$A = \pi r^2$
$= \pi \times 100$
$= 314.159\ldots$

$\frac{1}{2}A = 157.079\ldots$

$b = 20$, $h = 24$

$A = \frac{1}{2}bh$
$= \frac{1}{2} \times 20 \times 24$
$= 240$

3 Add the areas. Write your answer in the required units and round to the required level of accuracy.

$A = 157.079\ldots + 240$
$= 400\text{ cm}^2$

Some composite shapes are made by taking a piece away from a shape.

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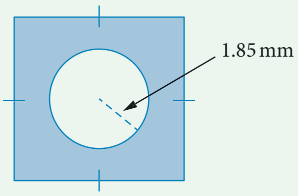
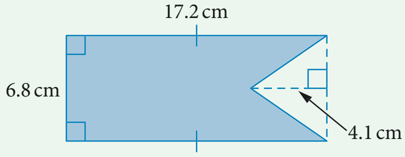
WORKED EXAMPLE 12 Calculating the area of composite shapes where a shape is removed

Find the following shaded areas. Give your answers using the units stated in each diagram to three significant figures.

a



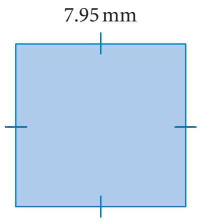
b



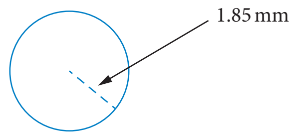
Steps | **Working**

a **1** Separate the shapes that make up the composite shape.

square



circle



2 Use the area formulas for those shapes.

$l = 7.95$

$A = l^2$

$= (7.95)^2$

$= 63.2025$

$r = 1.85$

$A = \pi r^2$

$= \pi \times (1.85)^2$

$= 10.752\ldots$

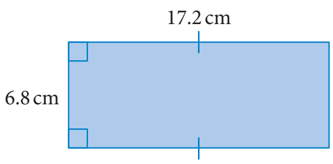
3 Subtract the empty space area from the other area. Write your answer in the required units and round to the required level of accuracy.

$A = 63.2025 - 10.752$

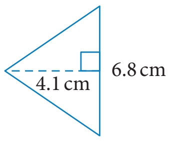
$= 52.5\text{ mm}^2$

b **1** Separate the shapes that make up the composite shape.

rectangle



triangle



2 Use the area formulas for those shapes.

$l = 17.2,\ w = 6.8$

$A = lw$

$= 17.2 \times 6.8$

$= 116.96$

$b = 6.8,\ h = 4.1$

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 6.8 \times 4.1$

$= 13.94$

3 Subtract the empty space area from the other area. Write your answer in the required units and round to the required level of accuracy.

$A = 116.96 - 13.94$

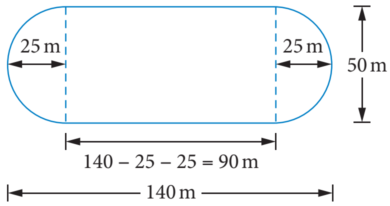
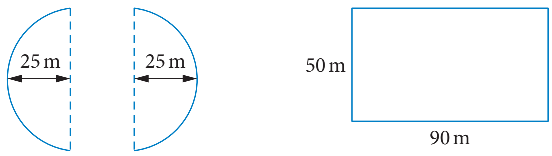
$= 103\text{ cm}^2$

WORKED EXAMPLE 13 Applying perimeter and area formulas

A sportsground 140 m long and 50 m wide has semi-circular ends.

a A fence is to be built all the way around, with four gates each 2 m wide. The gates cost $150 each and the fencing costs $45 per metre. Calculate the cost of fencing the sportsground, correct to the nearest dollar.

b The sportsground needs to be covered in grass that costs $90 per square metre. Calculate the cost of the grass, correct to the nearest dollar.

Steps	Working
a 1 Decide if it is a perimeter or area problem.	perimeter
2 Sketch the shape, showing the measurements. Calculate the missing lengths.	
3 Add all the lengths. Add or delete any other given amounts. Write your unrounded answer in the required units.	The two semi-circles together make a full circle circumference. $r = 25$ $\quad C = 2\pi r$ $= 2\pi \times 25$ $= 157.079\ldots$ $P = 157.079\ldots + 90 + 90$ $= 337.079\ldots$ m four 2 m gates = 4×2 = 8 m fence required = $337.079\ldots - 8$ $= 329.079\ldots$ m
4 Calculate the cost and write the answer to the required level of accuracy.	total cost = fence cost + gate cost $= 329.079\ldots \times 45 + 4 \times \$150$ $= \$15\,408.58\ldots$ The cost of fencing is $15 409.
b 1 Decide if it is a perimeter or area problem. Separate the shapes that make up the composite shape.	area 2 semi-circles = 1 circle rectangle 
2 Use the area formulas for those shapes.	$r = 25$ $A = \pi r^2$ $= \pi \times 25^2$ $= 1963.495\ldots \text{ m}^2$ $l = 90,\ w = 50$ $A = lw$ $= 90 \times 50$ $= 4500 \text{ m}^2$
3 Add all the areas. Write your unrounded answer in the required units.	total area = $1963.495\ldots + 4500$ $= 6463.495\ldots \text{ m}^2$
4 Calculate the cost and write the answer to the required level of accuracy.	cost = $6463.495\ldots \times \$90$ $= \$581\,714.58\ldots$ The cost of grass is $581 715.

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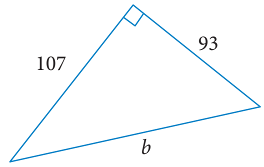
EXERCISE 9.3 Perimeter and area

ANSWERS p. 516

Recap

1 Which of the following is a true statement for the triangle shown?



A $b = 53$ (correct to two significant figures)

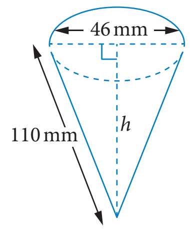
B $(107)^2 + b^2 = (93)^2$

C $b = 142$ (correct to three significant figures)

D $(107)^2 - b^2 = (93)^2$

E The length of the hypotenuse $= (107)^2 + (93)^2$

2 Find the height, h, of this ice-cream cone, correct to the nearest millimetre.



A 98 mm **B** 100 mm **C** 107 mm

D 108 mm **E** 112 mm

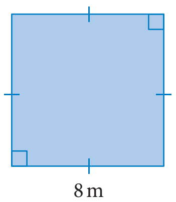
Mastery

3 WORKED EXAMPLE 9 For each of the following shapes, calculate

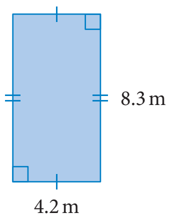
i the perimeter to the nearest centimetre

ii the area to the nearest square centimetre.

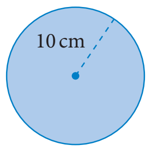
a



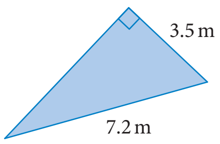
b



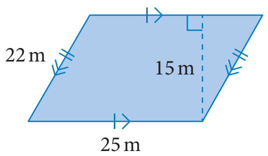
c



d



e



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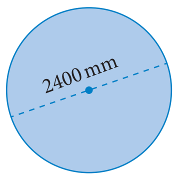
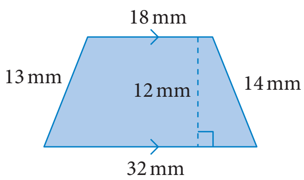


g



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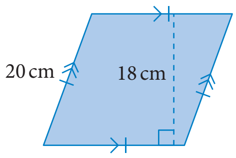
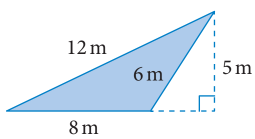


i



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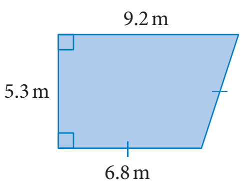
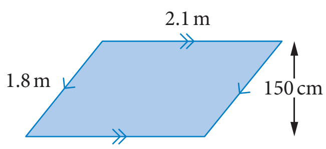
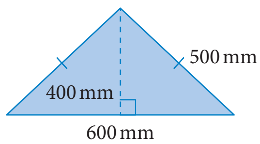


k



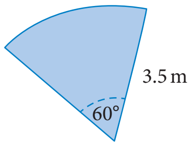
l



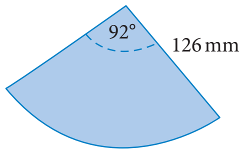
4 WORKED EXAMPLE 10 Calculate the following, correct to three significant figures, for each of the sectors shown.

i arc length **ii** perimeter **iii** area.

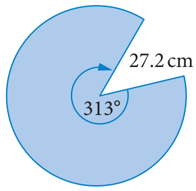
a



b



c

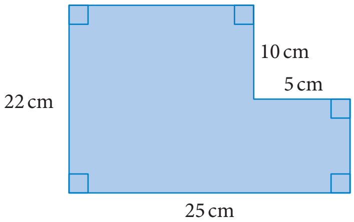


5 WORKED EXAMPLE 11 For each of the following shapes, calculate

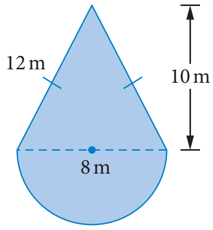
i the perimeter **ii** the area.

Give your answers using the units stated in each diagram to two significant figures.

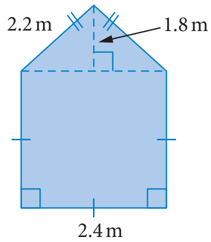
a



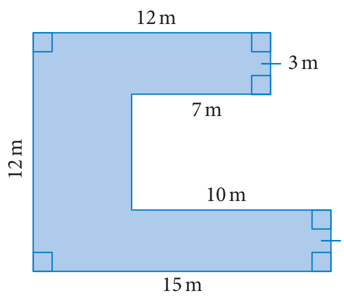
b



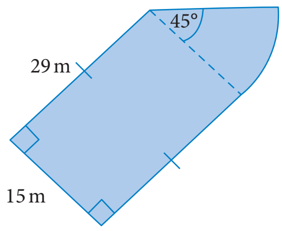
c



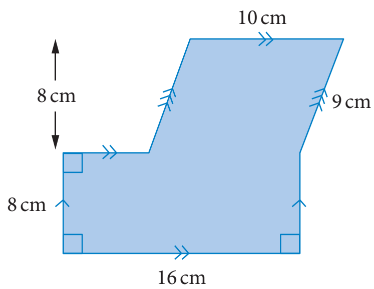
d



e

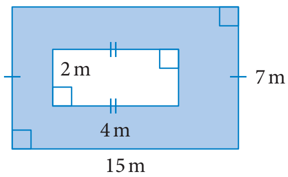


f

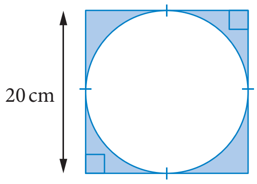


6 WORKED EXAMPLE 12 Find the following shaded areas. Give your answers using the units stated in each diagram to three significant figures.

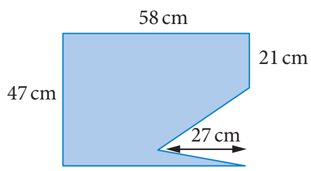
a



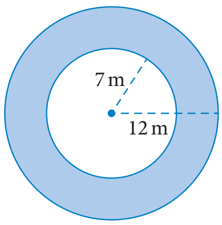
b



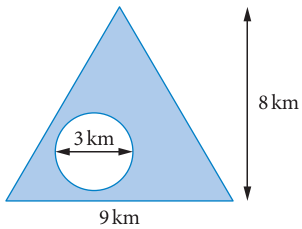
c



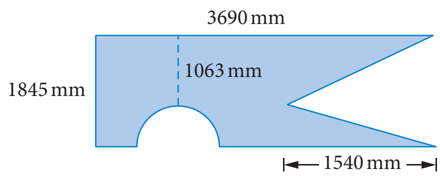
d



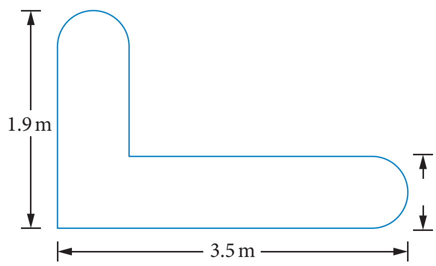
e



f



7 WORKED EXAMPLE 13 The L-shaped island kitchen bench shown is 600 mm wide and has semi-circular ends.



a Copy the diagram and add in the missing measurement in metres.

b The stone surface costs $1000 per square metre. Calculate the cost of the stone, correct to the nearest dollar.

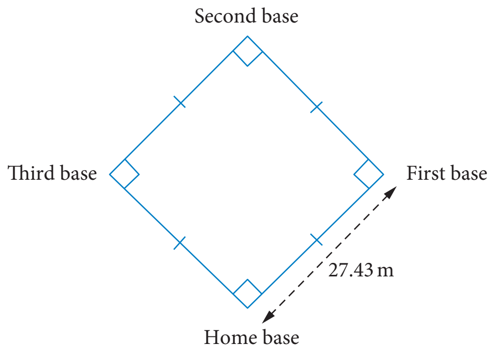
c The edges around the outside of the bench will require polishing at a cost of $55 per metre. Calculate the cost of polishing the edges of the bench, correct to the nearest dollar.

80–100% 60–79% 0–59%

8 © VCAA 2019 1GMQ1 91% The four bases of a baseball field form four corners of a square of side length 27.43 m, as shown in the diagram.

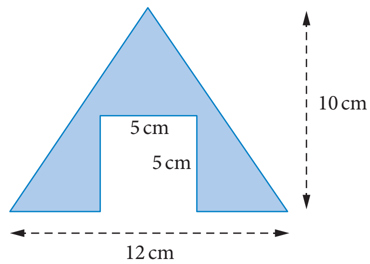
A player ran from home base to first base, then to second base, then to third base and finally back to home base. The minimum distance, in metres, that the player ran is

A 27.43 **B** 54.86 **C** 82.29
D 109.72 **E** 164.58



9 © VCAA 2016 1GMQ1 87% The shaded area in the diagram, in square centimetres, is

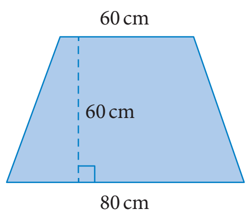
A 35 **B** 45 **C** 60
D 85 **E** 95



10 © VCAA 2015 1GTQ1 86% The top of a table is in the shape of a trapezium, as shown.

The area of the tabletop, in square centimetres, is

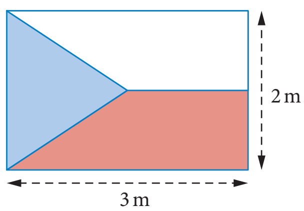
A 200 **B** 260 **C** 4200
D 4800 **E** 288 000



11 © VCAA 2020 1GMQ2 82% A flag consists of three different coloured sections: red, white and blue. The flag is 3 m long and 2 m wide, as shown in the diagram.

The blue section is an isosceles triangle that extends to half the length of the flag. The area of the blue section, in square metres, is

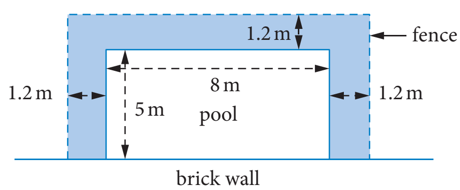
A 0.75 **B** 1.5 **C** 2
D 3 **E** 6



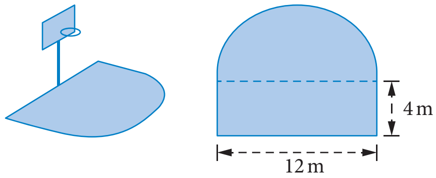
12 © VCAA 2021N 1GMQ3 A rectangular swimming pool is enclosed by a brick wall on one side and a fence along the other three sides. The pool is 8 m long and 5 m wide. The fence is 1.2 m from the pool on each side. The area between the pool and the fence will be concreted, as shown shaded in the diagram.

The total area that will be concreted, in square metres, is

A 21.60 **B** 24.48 **C** 28.08
D 36.96 **E** 40.00



13 © VCAA 2015 1GTQ2 **69%** A one-on-one basketball court is a composite shape made up of a rectangle and a semicircle, as shown.

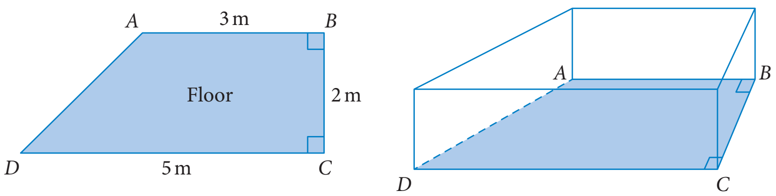


A boundary line is painted around the perimeter of the shape. The total length of the boundary line, in metres, is closest to

A 38.8 **B** 57.7 **C** 66.8

D 76.5 **E** 85.7

14 © VCAA 2014 2GTQ1 **75%** (2 marks) The floor of a chicken coop is in the shape of a trapezium. The floor, *ABCD*, and the chicken coop are shown.

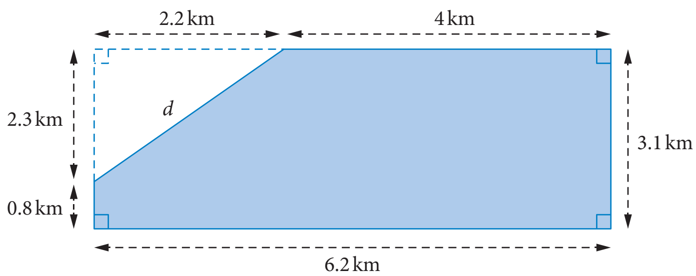


$AB = 3$ m, $BC = 2$ m and $CD = 5$ m

a What is the area of the floor of the chicken coop? Write your answer in square metres. 1 mark

b What is the perimeter of the floor of the chicken coop? Write your answer in metres, correct to one decimal place. 1 mark

15 © VCAA 2017N 2GMQ1 (3 marks) A dairy farm is situated on a large block of land. The shaded area in the diagram represents the block of land.

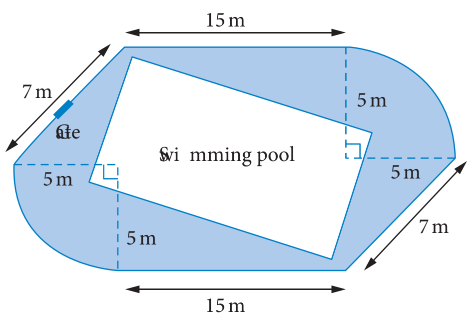


a Show that the length d is 3.2 km, rounded to one decimal place. 1 mark

b Using $d = 3.2$, calculate the perimeter, in kilometres, of this block of land. 1 mark

c Calculate the area of this block of land. Round your answer to the nearest square kilometre. 1 mark

16 A swimming pool is set in a wooden deck, as shown, with quarter circle sections. A pool fence is to be erected around the border, with a 1200 mm wide gate. The cost of materials for a 1200 mm high fence is \$42/m, and the gate costs \$120. Installation costs are \$10/m for the fence and gate.



a Calculate the perimeter of the deck in metres, correct to three significant figures.

b Calculate the cost of materials to complete the job to the nearest dollar.

c Calculate the cost of installing the fence and gate to the nearest dollar.

Video playlist
Volume

Worksheet
A page of solid shapes

Volumes of solids

Measurement in the home

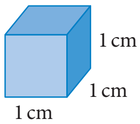
Volumes of water

9.4 Volume

Volume and capacity

The **volume** of a three-dimensional object or **solid** is the amount of space it takes up. The **capacity** of a three-dimensional object is the amount of liquid it can hold. We measure volume in cubic units based on metres, and we measure capacity in units based on litres. We can convert between the two.

1 cubic centimetre (cm^3) holds 1 millilitre (mL)



Volume and capacity conversion

When calculating capacity, calculate volume first and then convert.

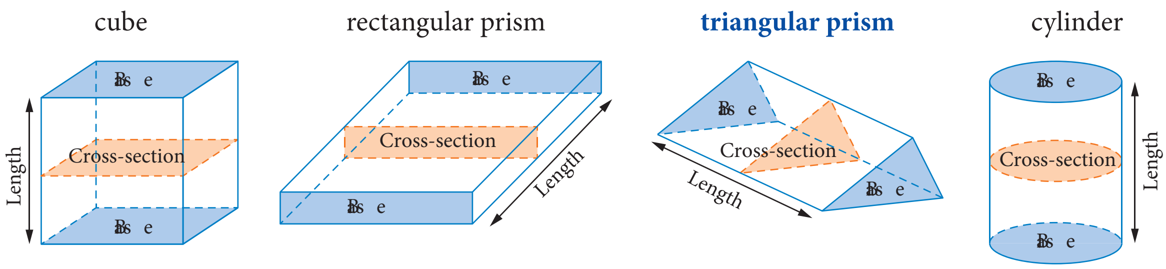
$1\ cm^3 = 1$ millilitre (mL)

$1000\ cm^3 = 1$ litre (L)

$1\ m^3 = 1000$ litres (L)

Volumes of prisms and cylinders

A **prism** is a three-dimensional object with straight edges that has the same **cross-section** along its full length. A **cylinder** isn't a prism because it has curves, but it does have the same cross-section along its full length. For cylinders, the length is often called the height.



Volume of prisms and cylinders

Prisms and cylinders have the same cross-section from one **base** to the other.

volume of prisms and cylinders = area of the base × length

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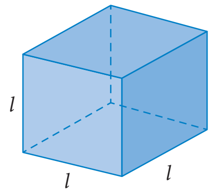
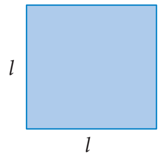
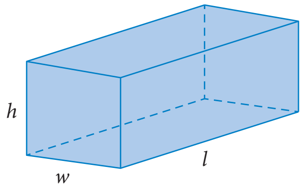
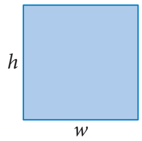
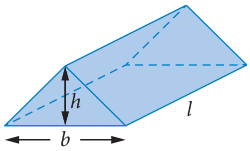
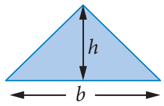
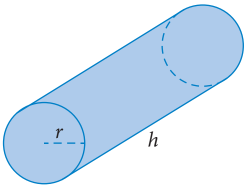
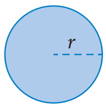
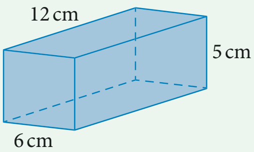
Object	Diagram	Area of base	Volume formula
cube		 $A = l^2$	$V = l^3$
rectangular prism		 $A = wh$	$V = whl$
triangular prism		 $A = \frac{1}{2}bh$	$V = \frac{1}{2}bhl$
cylinder		 $A = \pi r^2$	$V = \pi r^2 h$

WORKED EXAMPLE 14 Calculating the volume and capacity of prisms and cylinders

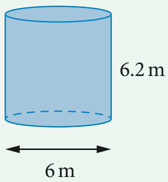
WB p. 146

Calculate the volume (V) and capacity (C) of each of the following, rounding your answer to three significant figures.

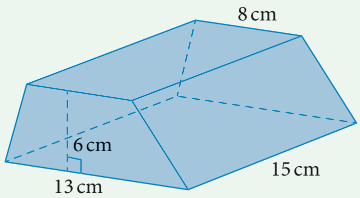
a



b



c

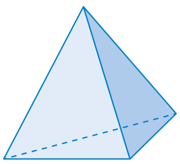


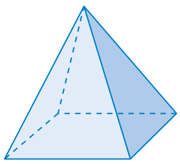
Steps	Working
a **1** Identify the object.	rectangular prism
Use a formula, or	$V = whl$
volume = base area × length	$w = 6, h = 5, l = 12$
if it is a prism we have no formula for.	$V = 6 \times 5 \times 12$
	$= 360$
2 Write the volume including units and round to the required level of accuracy.	$V = 360\ \text{cm}^3$
3 Convert volume to capacity including units and round to the required level of accuracy.	$1\ \text{cm}^3 = 1$ millilitre (mL)
	$C = 360\ \text{mL}$

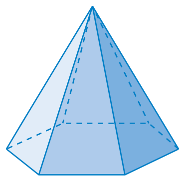
b	**1** Identify the object. Use a formula, or volume = base area × length if it is a prism we have no formula for.	cylinder $V = \pi r^2 h$ $r = 3, h = 6.2$ $V = \pi \times 3^2 \times 6.2$ $= 175.300\ldots$
	2 Write the volume including units and round to the required level of accuracy.	$V = 175\ \text{m}^3$
	3 Convert volume to capacity including units and round to the required level of accuracy.	$1\ \text{m}^3 = 1000$ litres (L) $C = 175\,000$ L
c	**1** Identify the object. Use a formula, or volume = base area × length if it is a prism we have no formula for.	prism with a trapezium base $V = \text{base area} \times \text{length} = \frac{1}{2}(a + b)h \times l$ $a = 8, b = 13, h = 6, l = 15$ $V = \frac{1}{2}(8 + 13) \times 6 \times 15$ $= 945$
	2 Write the volume including units and round to the required level of accuracy.	$V = 945\ \text{cm}^3$
	3 Convert volume to capacity including units and round to the required level of accuracy.	$1\ \text{cm}^3 = 1$ millilitre (mL) $C = 945$ mL

Volumes of pyramids, cones and spheres

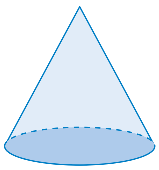
A **pyramid** has a base and triangular faces meeting at a point called the **apex**. Pyramids take their names from the shape of their base. For example:


triangular-based pyramid

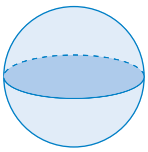

square-based pyramid


hexagonal-based pyramid

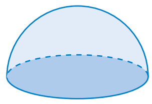
A **cone** is similar to a pyramid except its base is a circle.



A **sphere** looks like ball.



A **hemisphere** is half a sphere.

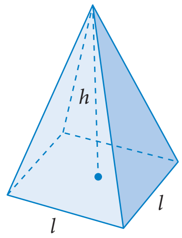
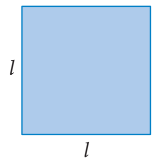
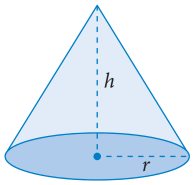
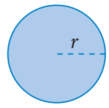
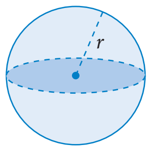


We can calculate the volume of pyramids and cones if we know the area of the base and the height. To calculate the volume of a sphere or hemisphere, we only need to know the radius.

Volume of pyramids, cones and spheres

$$\text{Volume of pyramids and cones} = \frac{1}{3} \times \text{area of base} \times \text{height}$$

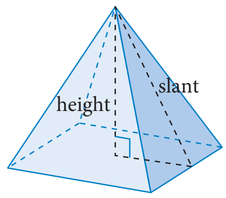
$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

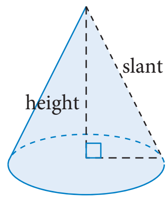
Object	Diagram	Area of base	Volume formula
square-based pyramid		 $A = l^2$	$V = \frac{1}{3}l^2h$
cone		 $A = \pi r^2$	$V = \frac{1}{3}\pi r^2 h$
sphere		no base	$V = \frac{4}{3}\pi r^3$

9.4

Exam hack

If you have been given the **slant length** of a pyramid or cone, use Pythagoras' theorem to find the height.





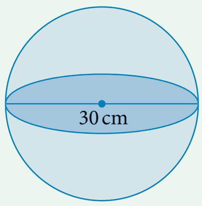
WORKED EXAMPLE 15 Calculating the volume and capacity of pyramids, cones and spheres

WB

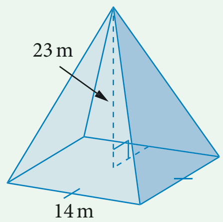
p. 147

Calculate the volume (V) and capacity (C) of each of the following, rounding your answer to two significant figures.

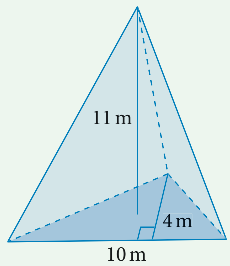
a



b



c



Steps	Working
a **1** Identify the object. Use a formula, or volume = $\frac{1}{3}$ × area of base × height if it is a pyramid we have no formula for.	sphere $V = \frac{4}{3}\pi r^3$ $r = 15$ $V = \frac{4}{3} \times \pi \times 15^3$ $= 14\,137.16\ldots$
2 Write the volume including units and round to the required level of accuracy.	$V = 14\,000\ \text{cm}^3$
3 Convert volume to capacity including units and round to the required level of accuracy.	$1\ \text{cm}^3 = 1$ millilitre (mL) $C = 14\,000$ mL

b **1** Identify the object. Use a formula, or volume = $\frac{1}{3}$ × area of base × height if it is a pyramid we have no formula for.

pyramid

$V = \frac{1}{3}l^2h$

$w = 14,\ l = 14,\ h = 23$

$V = \frac{1}{3} \times 14 \times 14 \times 23$

$= 1502.666\ldots$

2 Write the volume including units and round to the required level of accuracy.

$V = 1500\ \text{m}^3$

3 Convert volume to capacity including units and round to the required level of accuracy.

$1\ \text{m}^3 = 1000$ litres (L)

$C = 1\,500\,000\ \text{L}$

c **1** Identify the object. Use a formula, or Volume = $\frac{1}{3}$ × area of base × height if it is a pyramid we have no formula for.

pyramid with a triangular base

$b = 10,\ h = 4$

area of base $= \frac{1}{2}bh$

$= \frac{1}{2} \times 10 \times 4$

$= 20$

height of pyramid = 11

$V = \frac{1}{3}$ × area of base × height of pyramid

$= \frac{1}{3} \times 20 \times 11$

$= 73.333\ldots$

2 Write the volume including units and round to the required level of accuracy.

$V = 73\ \text{m}^3$

3 Convert volume to capacity including units and round to the required level of accuracy.

$1\ \text{m}^3 = 1000$ litres (L)

$C = 73\,000\ \text{mL}$

WB

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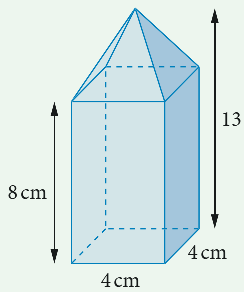
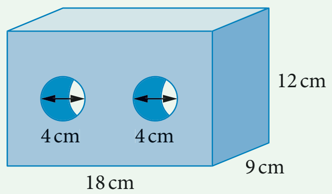
WORKED EXAMPLE 16 Calculating the volume of composite objects

Calculate the volume of the following, giving your answers to the nearest whole unit.

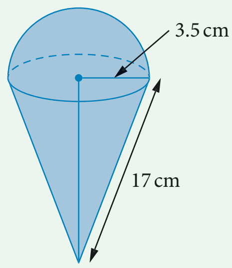
a



b



c

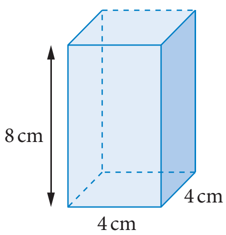


Steps | **Working**

a **1** Identify the objects that make up the composite object

Use the formulas needed, calculating any missing values.

rectangular prism



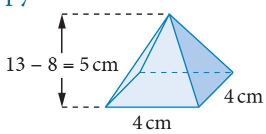
$w = 4,\ h = 8,\ l = 4$

$V = whl$

$= 4 \times 8 \times 4$

$= 128\ \text{cm}^3$

pyramid



$w = 4,\ l = 4,\ h = 5$

$V = \frac{1}{3}l^2h$

$V = \frac{1}{3} \times 4 \times 4 \times 5$

$= 26.666\ldots\ \text{cm}^3$

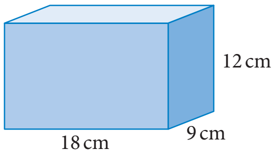
2 Calculate the total volume by adding or subtracting the volumes. Write your answer in the required units and round to the required level of accuracy.

total volume = rectangular prism volume + pyramid volume
$= 128 + 26.66\ldots$
$= 154.666\ldots$
$= 155\text{ cm}^3$

b **1** Identify the objects that make up the composite object

Use the formulas needed, calculating any missing values.

rectangular prism

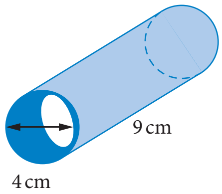


$w = 18,\ h = 12,\ l = 9$

$V = whl$

$V = 18 \times 12 \times 9$
$= 1944\text{ cm}^3$

two cylinders



$r = 2,\ h = 9$

$V = \pi r^2 h$
$= \pi \times 2^2 \times 9$
$= 113.09\ldots\text{ cm}^3$

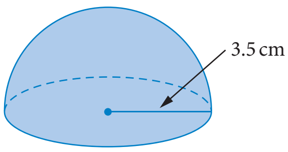
2 Calculate the total volume by adding or subtracting the volumes. Write your answer in the required units and round to the required level of accuracy.

total volume = rectangular prism volume − 2 × cylinder volume
$= 1944 - 2 \times 113.09\ldots$
$= 1717.80\ldots$
$= 1718\text{ cm}^3$

c **1** Identify the objects that make up the composite object

Use the formulas needed, calculating any missing values.

hemisphere (half sphere)

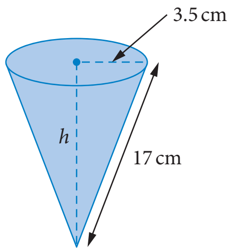


$V = \frac{1}{2} \times \frac{4}{3}\pi r^3$

$r = 3.5$

$V = \frac{1}{2} \times \frac{4}{3} \times \pi \times 3.5^3$
$= 89.797\ldots\text{ cm}^3$

cone



Using Pythagoras' theorem:

$17^2 = 3.5^2 + h^2$
$h = \sqrt{17^2 - 3.5^2}$
$= 16.63\ldots\text{ cm}$

$V = \frac{1}{3}\pi r^2 h$

$r = 3.5,\ h = 16.63\ldots$

$V = \frac{1}{3}\pi \times 3.5^2 \times 16.636\ldots$
$= 213.407\ldots\text{ cm}^3$

2 Calculate the total volume by adding or subtracting the volumes. Write your answer in the required units and round to the required level of accuracy.

total volume = hemisphere volume + cone volume
$= 89.797\ldots + 213.407\ldots$
$= 303.204\ldots$
$= 303\text{ cm}^3$

EXERCISE 9.4 Volume

ANSWERS p. 517

Recap

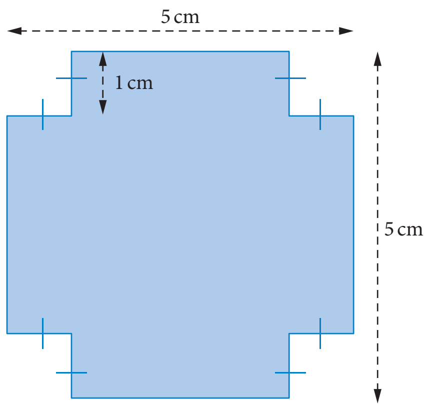
1 © VCAA 2021N 1GMQ1 A stove has one circular burner. This burner has a radius of 11 cm. The area of the top of this burner, in square centimetres, is closest to

A 35 **B** 69 **C** 95 **D** 380 **E** 1521

2 © VCAA 2019N 1GMQ1 A piece of cardboard is shown in the diagram.

The area of the cardboard, in square centimetres, is

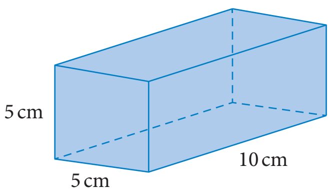
A 4 **B** 5 **C** 21
D 25 **E** 29



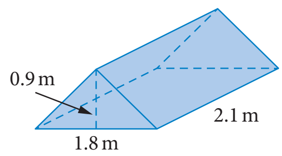
Mastery

3 WORKED EXAMPLE 14 Calculate the volume (V) and capacity (C) of each of the following, rounding your answer to two significant figures.

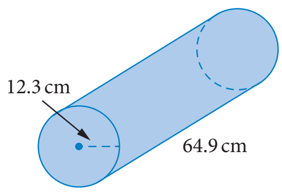
a



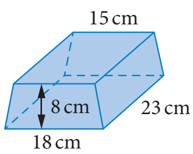
b



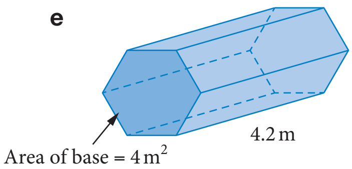
c



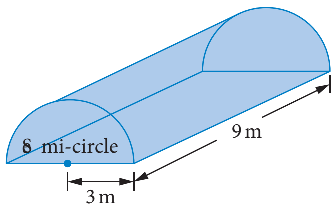
d



e

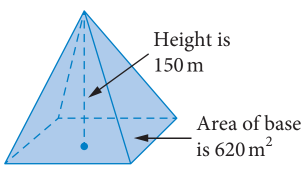


f

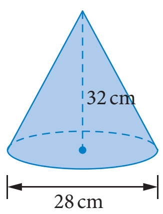


4 WORKED EXAMPLE 15 Calculate the volume (V) and capacity (C) of each of the following, rounding your answer to two significant figures.

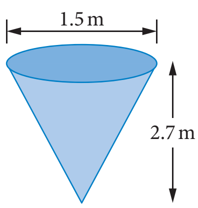
a



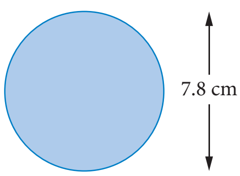
b



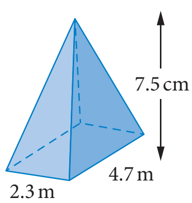
c



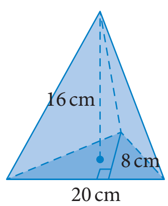
d



e

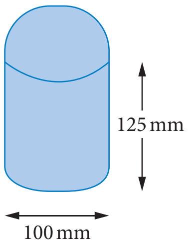


f

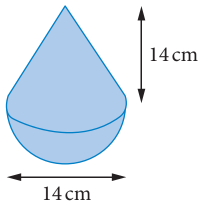


5 WORKED EXAMPLE 16 Calculate the volume of the following, giving your answers to the nearest whole unit.

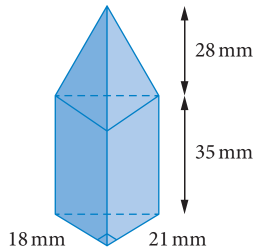
a



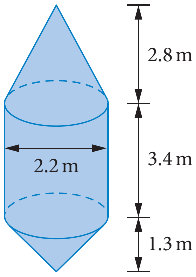
b



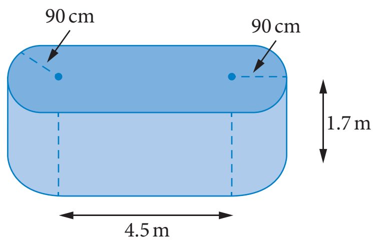
c



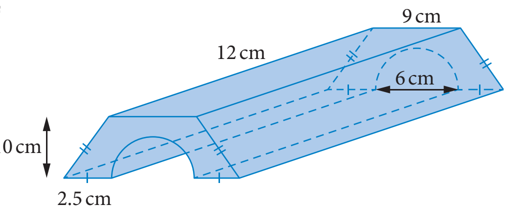
d



e



f

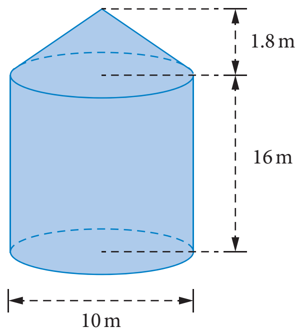


9.4

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2017 1GMQ4 80% A grain storage silo in the shape of a cylinder with a conical top is shown in the diagram.

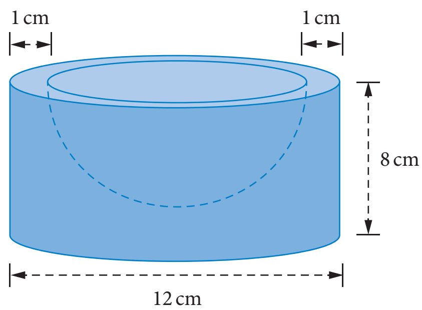


The volume of this silo, in cubic metres, is closest to

A 550 **B** 1304 **C** 1327

D 1398 **E** 2560

7 © VCAA 2015 1GTQ6 74% A cylindrical block of wood has a diameter of 12 cm and a height of 8 cm. A hemisphere is removed from the top of the cylinder, 1 cm from the edge, as shown.

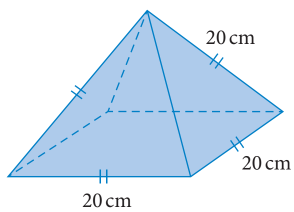


The volume of the block of wood, in cubic centimetres, after the hemisphere has been removed is closest to

A 452 **B** 606 **C** 643

D 1167 **E** 1357

8 © VCAA 2018N 1GMQ5 MODIFIED A square-based pyramid is shown in the diagram.



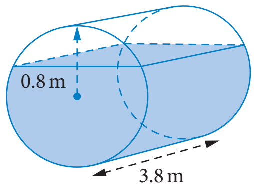
The base lengths of this pyramid are 20 cm. The slant edges of this pyramid are 20 cm. Which one of the following calculations gives the volume of this pyramid in cubic centimetres?

A $\frac{1}{3} \times 20 \times 20 \times 20$ **B** $\frac{1}{3} \times 10 \times 10 \times \sqrt{200}$

C $\frac{1}{3} \times 10 \times 20 \times \sqrt{200}$ **D** $\frac{1}{3} \times 20 \times 20 \times \sqrt{200}$

E $\frac{1}{3} \times 20 \times 20 \times 20 \times \sqrt{200}$

9 © VCAA 2018N 1GMQ7 A cylindrical fuel tank is shown in the diagram.

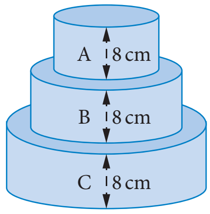


The radius of the fuel tank is 0.8 m. The length of the fuel tank is 3.8 m. The depth of fuel in the tank is 1.2 m.

One thousand litres of fuel has a volume of 1 m^3. The amount of fuel in this tank is closest to

A 5094 litres. **B** 5730 litres. **C** 6147 litres.
D 6587 litres. **E** 7420 litres.

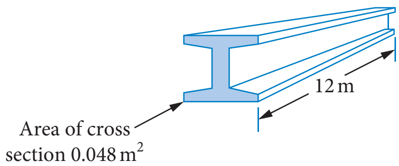
10 © VCAA 2019N 1GMQ6 A cake in the shape of three cylindrical sections is shown in the diagram.



Each section of the cake has a height of 8 cm, as shown in the diagram. The middle section of the cake, B, has twice the volume of the top section of the cake, A. The bottom section of the cake, C, has twice the volume of the middle section of the cake, B. The volume of the top section of the cake, A, is 900 cm^3. The diameter of the bottom section of the cake, C, in centimetres, is closest to

A 12 **B** 18 **C** 24
D 36 **E** 48

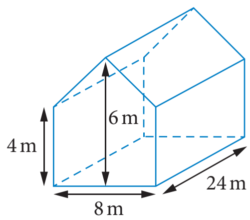
11 © VCAA 2007 1GTQ4 66% A steel beam used for constructing a building has a cross-sectional area of 0.048 m^2 as shown. The beam is 12 m long.



In cubic metres, the volume of this steel beam is closest to

A 0.576 **B** 2.5 **C** 2.63
D 57.6 **E** 2500

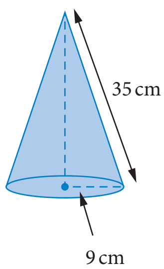
12 © VCAA 2006 1GTQ4 65% The building shown in the diagram is 8 m wide and 24 m long. The side walls are 4 m high. The peak of the roof is 6 m vertically above the ground.



In cubic metres, the volume of this building is

A 384 **B** 576 **C** 960
D 1152 **E** 4608

13 The volume, in cubic centimetres, of the witches' hat shown is closest to



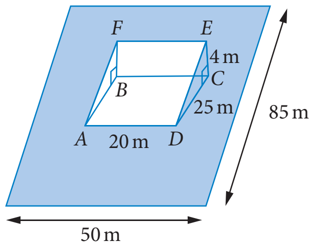
A 34 **B** 85 **C** 319
D 2868 **E** 2969

9.4

14 © VCAA 2012 2GTQ1 69% (3 marks) A rectangular block of land has width 50 metres and length 85 metres.

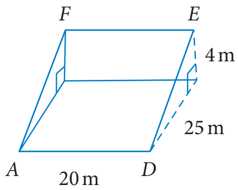
a Calculate the area of this block of land. Write your answer in m^2. 1 mark

In order to build a house, the builders dig a hole in the block of land. The hole has the shape of a triangular prism, *ABCDEF*. The width $AD = 20$ m, length $DC = 25$ m and height $EC = 4$ m are shown in the diagram.



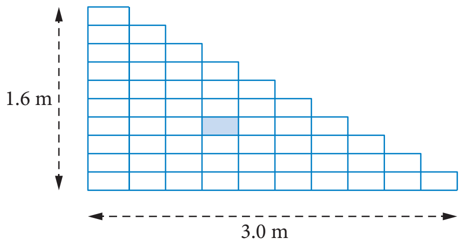
b Calculate the volume of the triangular prism, *ABCDEF*. Write your answer in m^3. 1 mark

Once the triangular prism shape has been dug, a fence will be placed along the two sloping edges, *AF* and *DE*, and along the edges *AD* and *FE*.



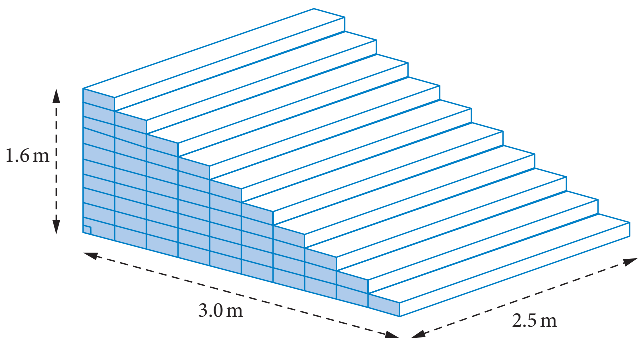
c Calculate the total length of fencing that will be required. Write your answer, in metres, correct to one decimal place. 1 mark

15 © VCAA 2013 2GTQ2 53% (3 marks) A concrete staircase leading up to a grandstand has 10 steps. The staircase is 1.6 m high and 3.0 m deep. Its cross-section comprises identical rectangles. One of these rectangles is shaded in the diagram.



a Find the area of the shaded rectangle in square metres. 1 mark

The concrete staircase is 2.5 m wide.



b Find the volume of the solid concrete staircase in cubic metres. 2 marks

Video playlist
Surface area

Skillsheet
Solid shapes

Worksheets
Nets of solids

A page of solid shapes

Surface area of solids

Surface area

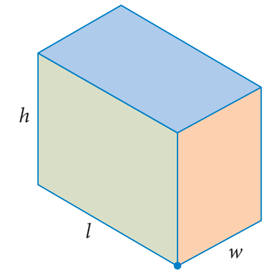
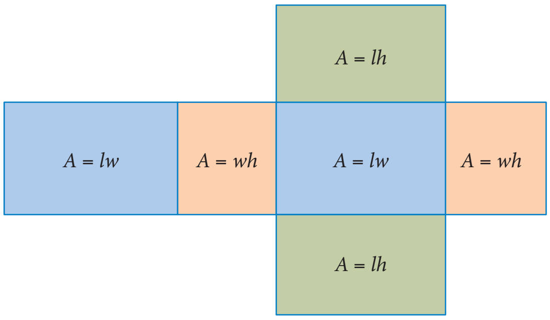
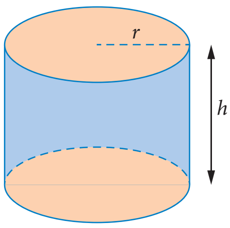
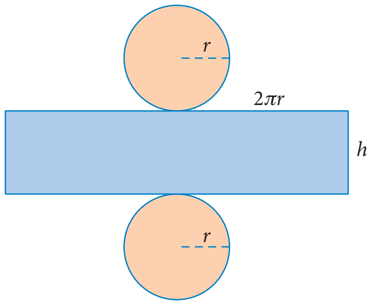
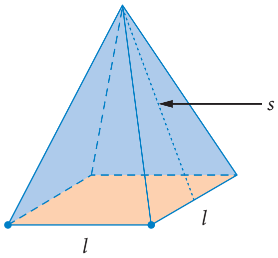
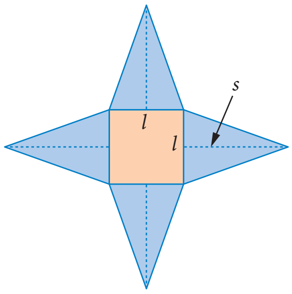
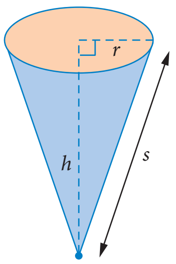
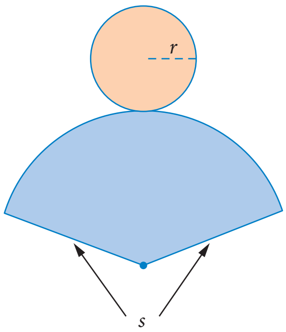
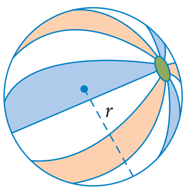
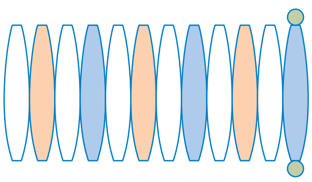
Formula matching game

Puzzle
Surface area riddle

9.5 Surface area

Surface area and nets

The **surface area** of a three-dimensional object is the area of all of its faces added together. Drawing or visualising the **net** of a three-dimensional object helps us to identify the shapes of all the faces. A net is a two-dimensional shape that can be folded up to form the three-dimensional object.

Three-dimensional object	Net	Surface area (SA) formula
rectangular prism 		3 pairs of rectangles $SA = 2(lw + wh + lh)$
cylinder 		2 circles and 1 rectangle $SA = 2\pi r^2 + 2\pi rh$ $= 2\pi r(r + h)$
square-based pyramid  s = slant		1 square and 4 triangles $SA = l^2 + 4 \times \left(\frac{1}{2}ls\right)$ $= l^2 + 2ls$
cone  s = slant		1 circle and a sector of a circle $SA = \pi r^2 + \pi rs$ $= \pi r(r + s)$
sphere 	 The net of a sphere doesn't help us to calculate the surface area.	$SA = 4\pi r^2$

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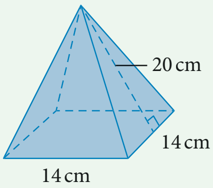
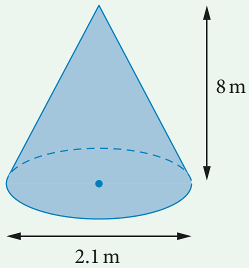
WORKED EXAMPLE 17 Using surface area formulas

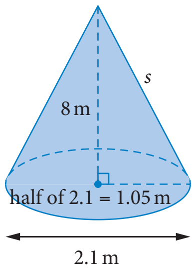
Calculate the surface area (SA) of each of the following, rounding your answer to two significant figures.

a



b



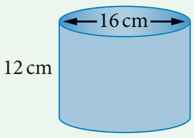
Steps	Working
a **1** Identify the object and the shapes that make up the net.	square-based pyramid 1 square and 4 triangles
Use the formula needed, calculating any missing values.	$SA = l^2 + 2ls$ $l = 14,\ s = 20$ $SA = (14)^2 + 2 \times 14 \times 20$ $= 756\ \text{cm}^2$
2 Write your answer in the required units and round to the required level of accuracy.	$SA = 760\ \text{cm}^2$
b **1** Identify the object and the shapes that make up the net.	cone 1 circle and a sector of a circle
Use the formula needed, calculating any missing values.	$SA = \pi r(r + s)$ $r = \frac{1}{2} \times 2.1 = 1.05,\ s = ?$  $s^2 = 1.05^2 + 8^2 = 65.1025$ $s = \sqrt{65.1025} = 8.068\ldots$ $SA = \pi r(r + s)$ $= \pi \times 1.05 \times (1.05 + 8.068\ldots)$ $= 30.079\ldots$
2 Write your answer in the required units and round to the required level of accuracy.	$SA = 30\ \text{m}^2$

WB p. 150

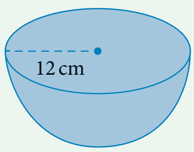
WORKED EXAMPLE 18 Applying surface area formulas

Calculate the surface area (SA) of each of the following, rounding your answer to two significant figures.

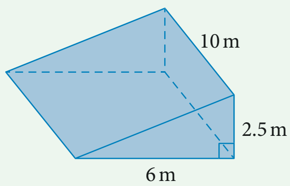
a An open-top cylindrical tin



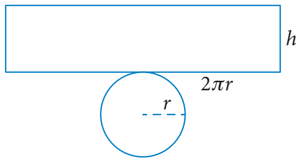
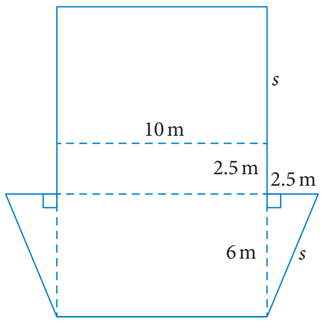
b A hemisphere salad bowl with a flat lid



c A solid wooden ramp



WB p. 151

Steps	Working
a **1** Identify the object and the shapes that make up the net, drawing the net if necessary. Adapt the longer version of the formulas needed, calculating any missing values.	cylinder  1 circle and 1 rectangle $SA = \pi r^2 + 2\pi rh$ $r = 8, h = 12$ $SA = \pi(8^2) + 2\pi \times 8 \times 12$ $= 804.247\ldots$
2 Write your answer in the required units and round to the required level of accuracy.	$SA = 800\,\text{cm}^2$
b **1** Identify the object and the shapes that make up the net, drawing the net if necessary. Adapt the longer version of the formulas needed, calculating any missing values.	half a sphere circle and curved surface of half a sphere $SA = \pi r^2 + \frac{1}{2} \times 4\pi r^2$ $= \pi r^2 + 2\pi r^2$ $= 3\pi r^2$ $r = 12$ $SA = 3 \times \pi \times 12^2$ $= 1357.168\ldots$
2 Write your answer in the required units and round to the required level of accuracy.	$SA = 1400\,\text{cm}^2$
c **1** Identify the object and the shapes that make up the net, drawing the net if necessary. Adapt the longer version of the formulas needed, calculating any missing values.	3 rectangles and 2 triangles  area of rectangles = lw area of triangles = $\frac{1}{2}bh$ $s = ?$ $s^2 = 2.5^2 + 6^2 = 42.25$ $s = \sqrt{42.25} = 6.5$ $SA = 6.5 \times 10 + 2.5 \times 10 + 6 \times 10 + 2 \times \left(\frac{1}{2} \times 2.5 \times 6\right)$ $= 165$
2 Write your answer in the required units and round to the required level of accuracy.	$SA = 170\,\text{m}^2$

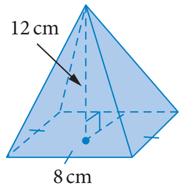
EXERCISE 9.5 Surface area

ANSWERS p. 517

Recap

80–100% 60–79% 0–59%

1 Which of the following calculations would give the volume for the pyramid shown?



A $V = \frac{1}{3} \times 8^2 \times 12$

B $V = \frac{1}{3} \times 8 \times 12^2$

C $V = \frac{1}{3} \times 8 \times 12$

D $V = \frac{1}{3} \times 4^2 \times 12$

E $V = \frac{1}{3} \times 4^2 \times 12.6$

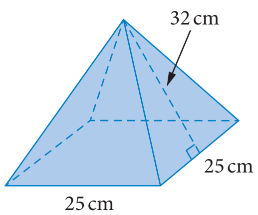
2 © VCAA 2008 1GTQ7 70% Sand is poured out of a truck and forms a pile in the shape of a circular cone. The diameter of the base of the pile of sand is 2.6 m. The height is 1.2 m. The volume (in m^3) of sand in the pile is closest to

A 2.1 **B** 3.1 **C** 6.4 **D** 8.5 **E** 25.5

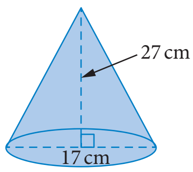
Mastery

3 WORKED EXAMPLE 17 Calculate the surface area (*SA*) of each of the following, rounding your answer to three significant figures.

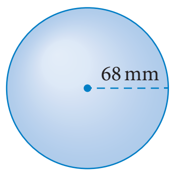
a



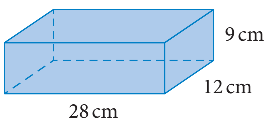
b



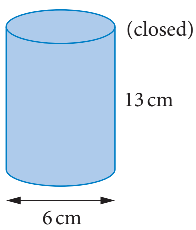
c



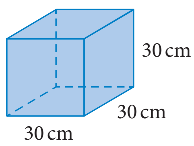
d



e

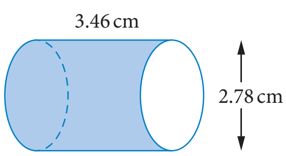


f

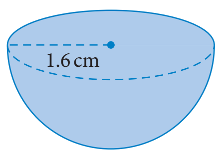


4 WORKED EXAMPLE 18 Calculate the surface area (*SA*) of each of the following, rounding your answer to two significant figures.

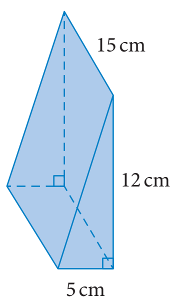
a cylindrical pipe open at both ends



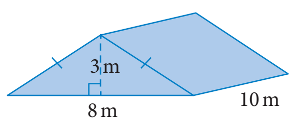
b spherical marble cut in half



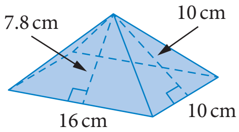
c bookend



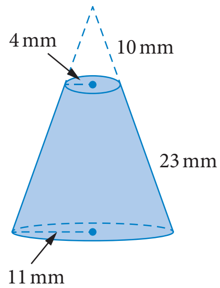
d wooden ramp



e souvenir Egyptian pyramid



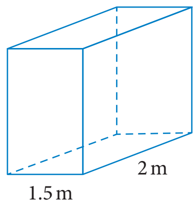
f chocolate formed by cutting a small cone off the end of a large cone



Exam practice

80–100% 60–79% 0–59%

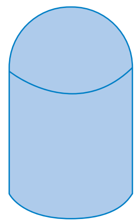
5 © VCAA 2006 1GTQ6 68% The rectangular box shown in this diagram is closed at the top and at the bottom. It has a volume of 6 m^3. The base dimensions are 1.5 m × 2 m.



The surface area of this box is

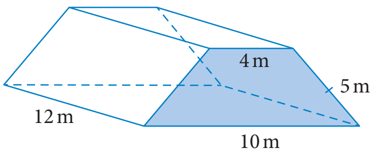
A 10 m^2 **B** 13 m^2 **C** 13.5 m^2
D 20 m^2 **E** 27 m^2

6 For how many surfaces would we need to calculate the area to find the surface area of this closed solid?



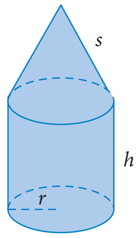
A 2 **B** 3 **C** 4
D 6 **E** 10

7 © VCAA 2013 1GTQ7 58% A greenhouse is built in the shape of a trapezoidal prism, as shown in the diagram. The cross-section of the greenhouse (shaded) is an isosceles trapezium. The parallel sides of this trapezium are 4 m and 10 m respectively. The two equal sides are each 5 m. The length of the greenhouse is 12 m. The five exterior surfaces of the greenhouse, **not** including the base, are made of glass. The total area, in m^2, of the glass surfaces of the greenhouse is



A 196 **B** 212 **C** 224
D 344 **E** 672

8 Which formula would correctly calculate the surface area of the following?



A $2\pi r^2 + 2\pi rh + 2\pi rs$ **B** $\pi r^2 + 2\pi rh + \pi rs$
C $\pi r^2 + 2\pi r^2 h + \pi rs$ **D** $\pi r^2 + 2\pi h + \pi rs$
E $\pi r^2 + 2\pi rh + \frac{1}{3}\pi rs$

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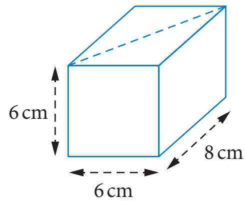
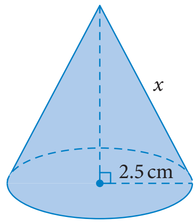
9 © VCAA 2020 1GMQ8 **57%** A cake is in the shape of a rectangular prism, as shown in the diagram.



The cake is cut in half to create two equal portions. The cut is made along the diagonal, as represented by the dotted line. The surface area, in square centimetres, of one portion of the cake is

A 132 **B** 180 **C** 192
D 212 **E** 264

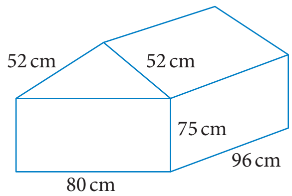
10 © VCAA 2018 1GMQ8 **51%** A cone with a radius of 2.5 cm is shown in the diagram.



The slant edge, x, of this cone is also shown. The volume of this cone is 36 cm^3. The surface area of this cone, including the base, can be found using the rule surface area = $\pi r(r + x)$. The surface area of this cone, including the base, in square centimetres, is closest to

A 20 **B** 42 **C** 63
D 67 **E** 90

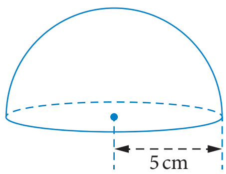
11 © VCAA 2017N 1GMQ4 Paula has built a model house using a triangular prism on top of a rectangular box. The dimensions of the model house are shown on the diagram.



Paula will paint the outside walls and the roof of the model house. The area that will be painted, in square centimetres, is closest to

A 12 600 **B** 26 400 **C** 36 400
D 37 700 **E** 39 000

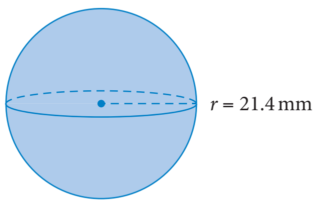
12 © VCAA 2019 1GMQ3 **48%** An ice-cream dessert is in the shape of a hemisphere. The dessert has a radius of 5 cm.



The top and the base of the dessert are covered in chocolate. The surface area, in square centimetres, that is covered in chocolate is closest to

A 52 **B** 157 **C** 236
D 314 **E** 942

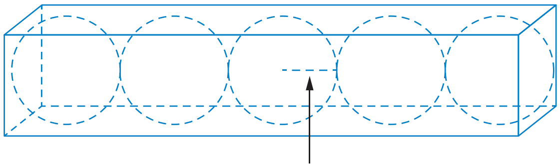
13 © VCAA 2016 2GMQ1 (2 marks) A golf ball is spherical in shape and has a radius of 21.4 mm, as shown in the diagram.



Assume that the surface of the golf ball is smooth.

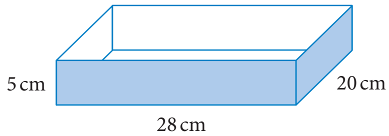
a **74%** What is the surface area of the golf ball shown? Round your answer to the nearest square millimetre. 1 mark

b 82% Golf balls are sold in a rectangular box that contains five identical golf balls, as shown in the diagram.



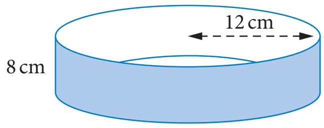
What is the minimum length, in millimetres, of the box? 1 mark

14 © VCAA 2018N 2GMQ1ab (3 marks) Shannon is a baker. One of her baking tins has a rectangular base of length 28 cm and width 20 cm. The height of this baking tin is 5 cm, as shown in the diagram.



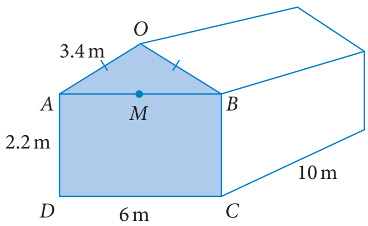
a What is the volume of this tin, in cubic centimetres? 1 mark

Another baking tin has a circular base with a radius of 12 cm. The height of this baking tin is 8 cm, as shown in the following diagram.



b Shannon needs to cover the inside of both the base and side of this tin with baking paper. What is the area of baking paper required, in square centimetres? Round your answer to one decimal place. 2 marks

15 © VCAA 2008 2GTQ2 61% (5 marks) The shed has the shape of a prism. Its front face, *AOBCD*, is shaded in the diagram. *ABCD* is a rectangle and *M* is the midpoint of *AB*.



a Show that the length of *OM* is 1.6 m. 1 mark

b Show that the area of the front face of the shed, *AOBCD*, is 18 m^2. 1 mark

c Find the volume of the shed in m^3. 1 mark

d All inside surfaces of the shed, including the floor, will be painted.

i Find the total area that will be painted in m^2. 1 mark

One litre of paint will cover an area of 16 m^2.

ii Determine the number of litres of paint that is required. 1 mark

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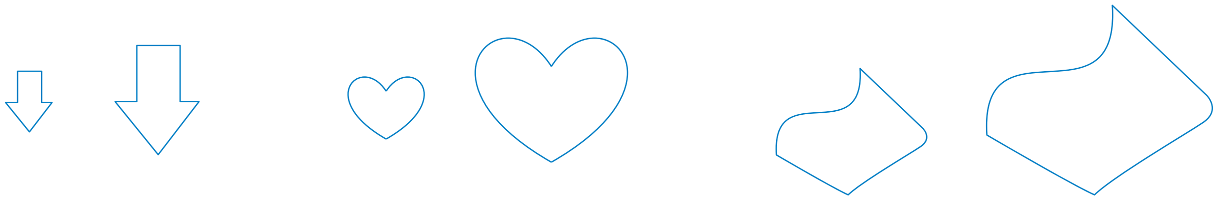
9.6 Similarity and scale

9.6

Similar shapes and scale factors

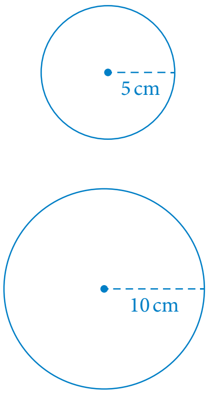
Two shapes are **similar** if they have exactly the same shape but are different sizes. These pairs of shapes are similar:



A **scale factor** measures how much a shape needs to be enlarged or reduced to produce the similar shape. For example:

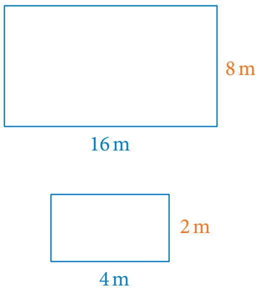
Video playlist Similarity and scale

Circles



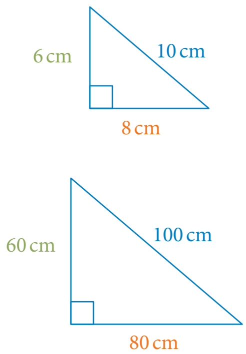
scale factor = $\frac{10}{5} = 2$

Rectangles



scale factor = $\frac{4}{16} = \frac{2}{8} = \frac{1}{4} = 0.25$

Triangles



scale factor = $\frac{100}{10} = \frac{80}{8} = \frac{60}{6} = 10$

Similar shapes and scale factors

k = scale factor for similar shapes = $\dfrac{\text{any length of second shape}}{\text{matching length of first shape}}$

$k > 1$ gives a larger second shape

$k < 1$ gives a smaller second shape

WB

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Skillsheet Finding sides in similar triangles

Worksheet Finding sides in similar figures

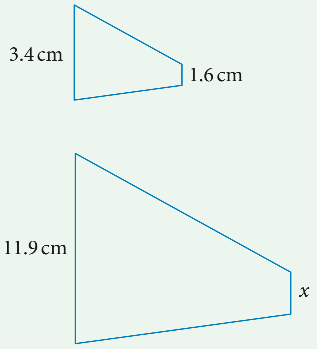
WORKED EXAMPLE 19 Working with scale factors

For each of these pairs of similar shapes, find

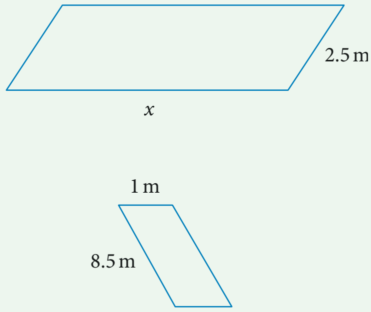
i the scale factor k

ii the value of x, rounded to one decimal place.

a



b



Steps	Working
a **i** Find a pair of matching lengths and use $k = \dfrac{\text{any length of second shape}}{\text{matching length of first shape}}$	$k = \dfrac{11.9}{3.4} = 3.5$
ii Use the scale factor to find and solve an equation for the unknown, using CAS if necessary. Write your answer to the required accuracy.	$\dfrac{x}{1.6} = 3.5$ $x = 3.5 \times 1.6$ $= 5.6\text{ cm}$
b **i** Find a pair of matching lengths and use scale factor $= \dfrac{\text{any length of second shape}}{\text{matching length of first shape}}$	$k = \dfrac{1}{2.5} = 0.4$
ii Use the scale factor to find and solve an equation for the unknown, using CAS if necessary. Write your answer to the required accuracy.	$\dfrac{8.5}{x} = 0.4$ $0.4x = 8.5$ $x = \dfrac{8.5}{0.4}$ $= 21.3\text{ cm}$

Exam hack

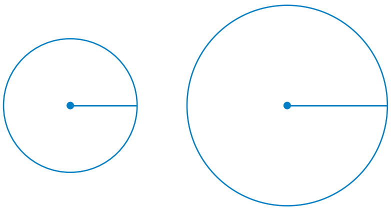
Don't round your answer if you're not asked to round it.

Identifying similar shapes

We can't always tell if two shapes are similar just by looking at them. It depends on the shape and often the angles within the shape.

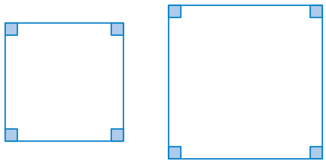
9.6

Circles



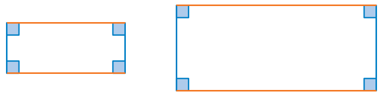
All circles are similar. The radius is the only length measurement needed.

Squares



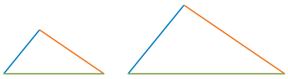
All squares are similar. The scale factors for each matching pair of sides will be the same because all sides in each shape are of equal length.

Rectangles



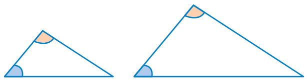
Rectangles are similar if the two pairs of matching sides have the same scale factor.

Triangles



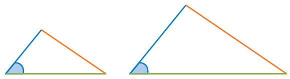
Side-Side-Side (SSS) or

Triangles are similar if the three pairs of matching sides have the same scale factor.



Angle-Angle (AA) or

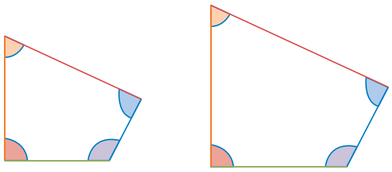
Triangles are similar if two pairs of matching angles are equal.

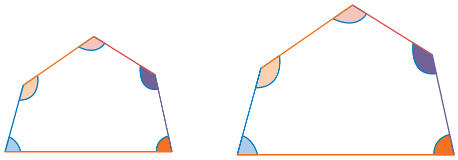


Side-Angle-Side (SAS)

Triangles are similar if two pairs of matching sides have the same scale factor and the angles between them are equal.

All shapes with straight sides (polygons)



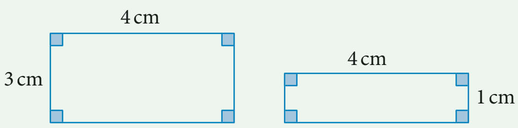


Polygons are similar if every pair of matching sides have the same scale factor *and* all pairs of matching angles are equal.

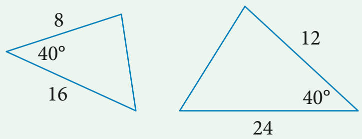
WORKED EXAMPLE 20 Identifying similar shapes

State whether the following pairs of shapes are similar, giving a reason.

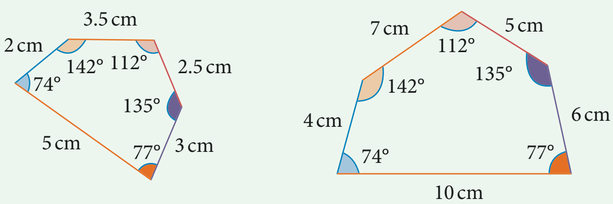
a



b



c



WB

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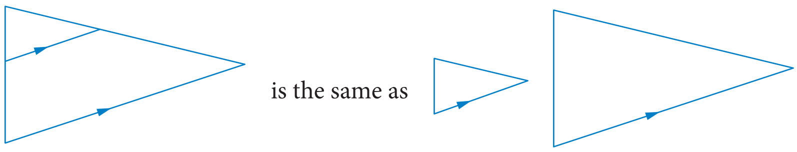
Steps	Working
a **1** Identify the shape.	rectangle
2 Calculate the scale factors of the matching sides in order, starting with the largest values in each shape. If necessary, look at the angles.	$\frac{4}{4} = 1, \frac{1}{3} = 0.333...$ The scale factors are not the same, so the shapes aren't similar.
b **1** Identify the shape.	triangle
2 Calculate the scale factors of the matching sides in order, starting with the largest values in each shape. If necessary, look at the angles.	$\frac{24}{16} = 1.5, \frac{12}{8} = 1.5$ The scale factors are the same. Use SAS. The angle between the two sides is 40° for both triangles, so the triangles are similar.
c **1** Identify the shape.	polygon
2 Calculate the scale factors of the matching sides in order, starting with the largest values in each shape. If necessary, consider the angles.	$\frac{10}{5} = \frac{7}{3.5} = \frac{6}{3} = \frac{5}{2.5} = \frac{4}{2} = 2$ The scale factors are the same. The pairs of matching angles are equal in each shape, so the shapes are similar.

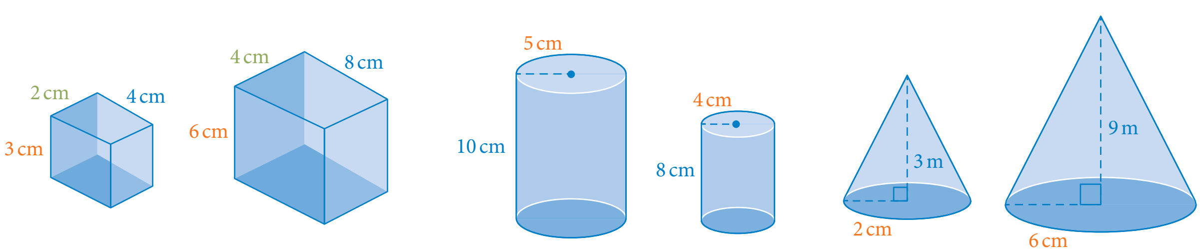
Exam hack

Questions sometimes present two similar shapes like this where the smaller and larger shape are in the one diagram.



Similar three-dimensional objects

Similarity works the same way for three dimensional objects. Three dimensional objects are similar if they have exactly the same shape and the scale factor of pairs of matching lengths are the same.
For example, these pairs are all similar:



scale factor $= \frac{8}{4} = \frac{6}{3} = \frac{4}{2} = 2$

scale factor $= \frac{8}{10} = \frac{4}{5} = 0.8$

scale factor $= \frac{9}{3} = \frac{6}{2} = 3$

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Scaling areas and volumes

When a shape is scaled by a factor of k:

area of second shape
$= k^2 \times$ area of first shape

When a three-dimensional object is scaled by a factor of k:

surface area of second object
$= k^2 \times$ surface area of first object

volume of second object
$= k^3 \times$ volume of first object

where

$$k = \frac{\text{any length of second shape/object}}{\text{matching length of first shape/object}}$$

9.6

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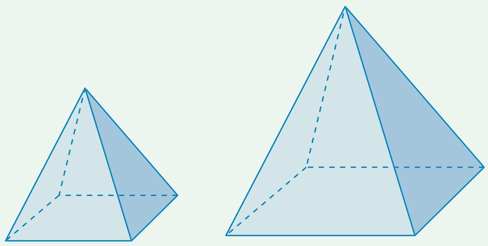
Worksheet
Areas and volumes of similar figures

WORKED EXAMPLE 21 Scaling areas and volumes

A square-based pyramid with a base of length of 4 cm is enlarged to produce a similar pyramid with a base length of 10 cm.



Find

a the scale factor k

b the surface area of the larger pyramid if the surface area of the smaller pyramid is 64 cm^2

c the volume of the smaller pyramid if the volume of the larger pyramid is 32 cm^3.

Round your answers to three significant figures.

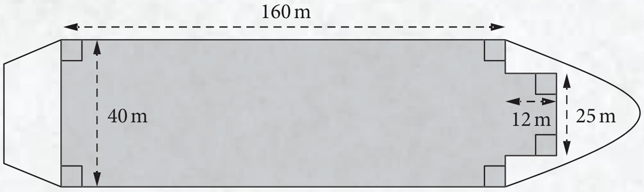
Steps	Working
a Use $k = \frac{\text{any length of second object}}{\text{matching length of first object}}$	$k = \frac{10}{4} = 2.5$
b 1 Use surface area of second object $= k^2 \times$ surface area of first object	surface area of larger pyramid $= k^2 \times$ surface area of smaller pyramid $= (2.5)^2 \times 64$ $= 400$
2 Write your answer in the required units and round to the required level of accuracy.	The surface area of the larger pyramid is 400 cm^2.
c 1 Use volume of second object $= k^3 \times$ volume of first object	volume of larger pyramid $= k^3 \times$ volume of smaller pyramid; $32 = (2.5)^3 \times$ volume of smaller pyramid; volume of smaller pyramid $= \frac{32}{(2.5)^3} = 2.05$
2 Write your answer in the required units and round to the required level of accuracy.	The volume of the smaller pyramid is 2.05 cm^3.

Video playlist VCE question analysis: Measurement, scale and similarity

VCE QUESTION ANALYSIS

© VCAA 2019 2GMQ1 2019 Examination 2 Geometry and measurement Question 1 (4 marks)

The following diagram shows a cargo ship viewed from above.

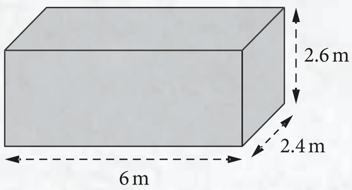


The shaded region illustrates the part of the deck on which shipping containers are stored.

a What is the area, in square metres, of the shaded region? 1 mark

Each shipping container is in the shape of a rectangular prism.

Each shipping container has a height of 2.6 m, a width of 2.4 m and a length of 6 m, as shown in the diagram.

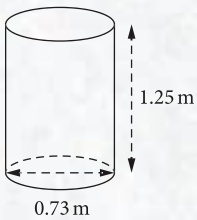


b What is the volume, in cubic metres, of one shipping container? 1 mark

c What is the surface area, in square metres, of the outside of one shipping container? 1 mark

d One shipping container is used to carry barrels. Each barrel is in the shape of a cylinder.

Each barrel is 1.25 m high and has a diameter of 0.73 m, as shown in the diagram. Each barrel must remain upright in the shipping container.



What is the maximum number of barrels that can fit in one shipping container? 1 mark

Reading the question

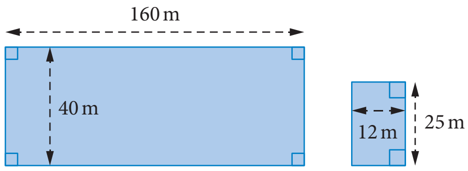
- Identify the shapes and three-dimensional objects involved.
- Be clear on whether you are using an area, volume or surface area formula.

Thinking about the question

- For part **d**, picture how upright barrels will be stacked inside the container.
- What happens in part **d** if you don't get neat whole barrels in your calculation?

Worked solution (✓ = 1 mark)

a Separate the shapes into the two rectangles that make up the composite shape.



Area = 160 × 40 + 12 × 25

= **6700 m^2** ✓

b Use the formula for the volume of a rectangular prism.

$V = whl$

= 2.4 × 2.6 × 6

= **37.44 m^3** ✓

c Use the formula for the surface area of a rectangular prism.

$SA = 2(lh + wh + lw)$

= 2(6 × 2.6 + 2.4 × 2.6 + 6 × 2.4)

= **72.48 m^2** ✓

d Calculate the number of upright barrels that can fit the width, height and length of the shipping container.

Width of the shipping container = 2.4 m, width of a barrel = 0.73 m

So, the number of barrels that can be placed across the width = 2.4 ÷ 0.73 = 3.287... which is 3 barrels.

Length of the shipping container = 6 m, length of a barrel = 0.73 m

So, the number of barrels that can be placed along the length = 6 ÷ 0.73 = 8.219... which is 8 barrels.

Height of the shipping container = 2.6 m, height of a barrel = 1.25 m

So, the number of barrels that can be placed on top of each other = 2.6 ÷ 1.25 = 2.08 which is 2 barrels.

This means the maximum number of barrels that can fit in one shipping container = 3 × 8 × 2 = **48** ✓

Student performance 80–100% 60–79% 0–59%

a 86% This question was well done.

b 88% No rounding was asked for, so the exact value had to be given.

c 70% Again, no rounding was asked for, so the exact value had to be given. The question was mostly done well but some students incorrectly answered 74.88, by calculating 4(6 × 2.6) + 2(2.6 × 2.4).

d 22% There were many incorrect answers. The most common incorrect answer was 71, which was calculated using $\frac{\text{volume of container}}{\text{volume of one barrel}}$. This method didn't work because whole barrels don't fit perfectly into the container space.

EXERCISE 9.6 Similarity and scale

ANSWERS p. 517

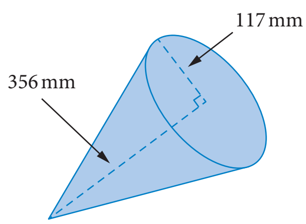
Recap

1 The surface area of this closed cone is

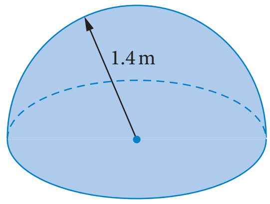
A 84 657.3 mm^2 **B** 170 087.6 mm^2

C 173 858.9 mm^2 **D** 180 744.6 mm^2

E 347 717.8 mm^2



2 To calculate the surface area of this hemisphere, we would use the calculation



A $\frac{1}{2} \times 4\pi \times 1.4^2$

B $4\pi \times 1.4^2$

C $\frac{1}{2} \times 4\pi \times 2.8^2$

D $\frac{1}{2} \times 4\pi \times 1.4^2 + \pi \times 1.4^2$

E $4\pi \times 1.4^2 + \pi \times 1.4^2$

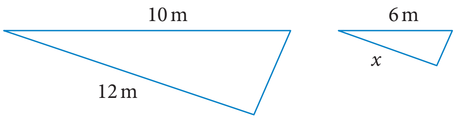
Mastery

3 WORKED EXAMPLE 19 For each of these pairs of similar shapes, find

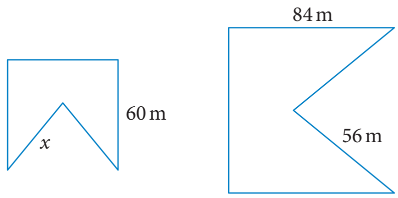
i the scale factor k

ii the value of x, rounded to one decimal place.

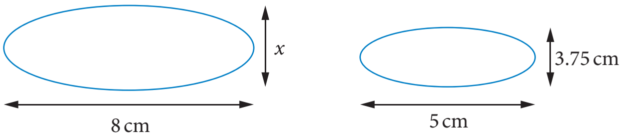
a



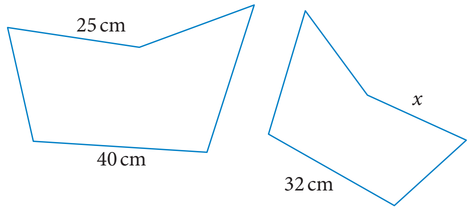
b



c

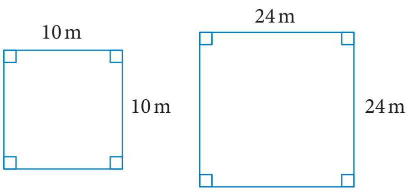


d

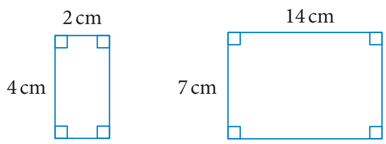


4 WORKED EXAMPLE 20 State whether the following pairs of shapes are similar, giving a reason.

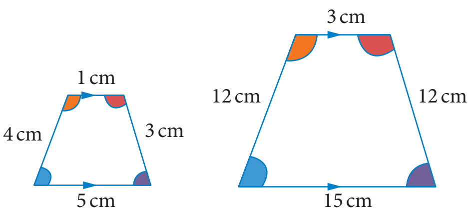
a



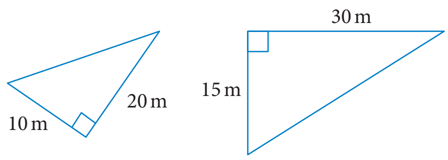
b



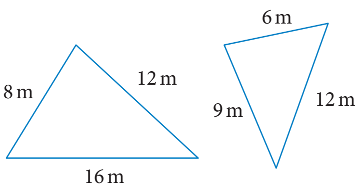
c



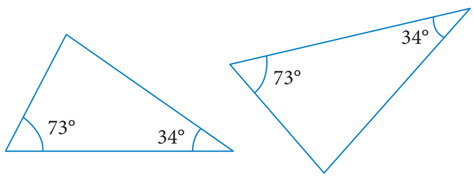
d



e



f



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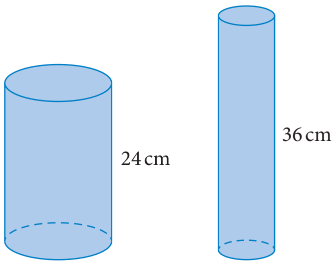
5 WORKED EXAMPLE 21 For each of the following pairs of similar three-dimensional objects, find

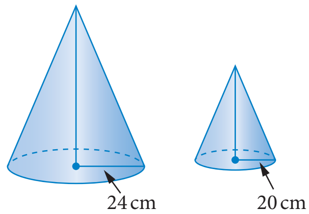
- **i** the scale factor k
- **ii** the surface area of the smaller object if the surface area of the larger object is $48\ cm^2$
- **iii** the volume of the larger object if the volume of the smaller object is $937.5\ cm^3$.

Round your answers to three significant figures.

a Two similar cylinders with heights 24 cm and 36 cm respectively.



b Two similar cones with bases of radius 24 cm and 20 cm respectively.



9.6

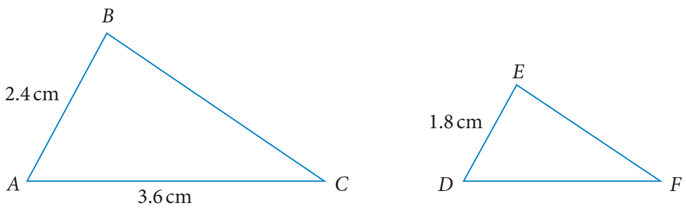
Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2021N 1GMQ2 The length and width of a rectangular photograph were increased by a scale factor of $k = 4$. The area of this photograph has increased by a factor of

A 2 **B** 4 **C** 8 **D** 16 **E** 64

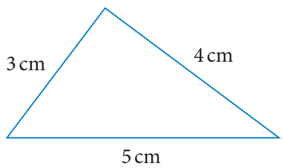
7 © VCAA 2016 1GMQ2 92% Triangle *ABC* is similar to triangle *DEF*.



The length of *DF*, in centimetres, is

A 0.9 **B** 1.2 **C** 1.8 **D** 2.7 **E** 3.6

8 © VCAA 2019 1GMQ4 87% Triangle *M*, shown here, has side lengths of 3 cm, 4 cm and 5 cm.



Four other triangles have the following side lengths:

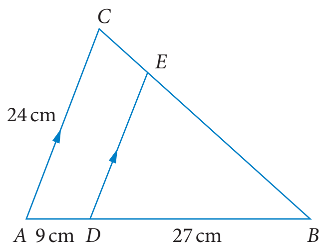
- Triangle *N* has side lengths of 3 cm, 6 cm and 8 cm.
- Triangle *O* has side lengths of 4 cm, 8 cm and 12 cm.
- Triangle *P* has side lengths of 6 cm, 8 cm and 10 cm.
- Triangle *Q* has side lengths of 9 cm, 12 cm and 15 cm.

The triangles that are similar to triangle *M* are

A triangle *N* and triangle *O*.
B triangle *N*, triangle *O* and triangle *P*.
C triangle *O* and triangle *P*.
D triangle *O* and triangle *Q*.
E triangle *P* and triangle *Q*.

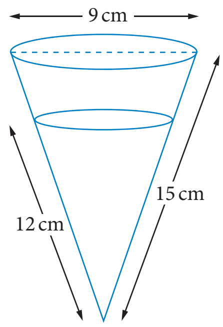
9 © VCAA 2006 1GTQ7 69% In the diagram, $AD = 9$ cm, $AC = 24$ cm and $DB = 27$ cm. Line segments AC and DE are parallel. The length of DE is

A 6 cm **B** 8 cm **C** 12 cm
D 16 cm **E** 18 cm



10 © VCAA 2003 1GTQ9 65% Two cones, as shown, have the same angle at the base. The larger cone has a slant length of 15 cm and the smaller cone has a slant length of 12 cm. The diameter of the larger cone is 9 cm. The diameter of the smaller cone is

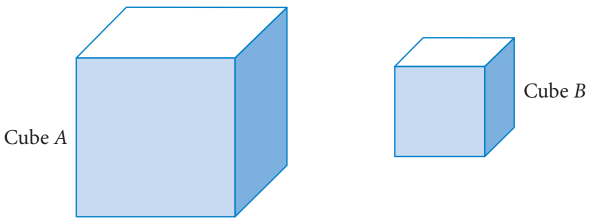
A 2.0 cm **B** 3.6 cm **C** 4.5 cm
D 6.0 cm **E** 7.2 cm



11 © VCAA 2004 1GTQ4 64% A triangle has sides of length 20 cm, 48 cm and 52 cm. A second triangle which is similar to the first triangle has a longest side of 65 cm. The perimeter of the second triangle is

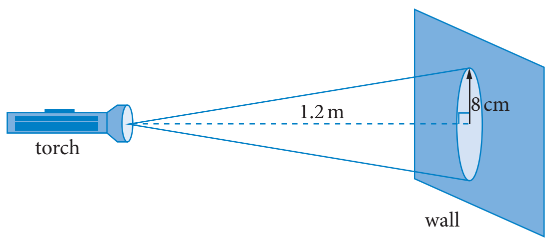
A 96 cm **B** 120 cm **C** 125 cm **D** 133 cm **E** 150 cm

12 © VCAA 2010 1GTQ4 46% Cube A and cube B are shown. The side length of cube A is 1.5 times the side length of cube B. The **surface** area of cube B is 256 cm^2. The **surface** area of cube A is



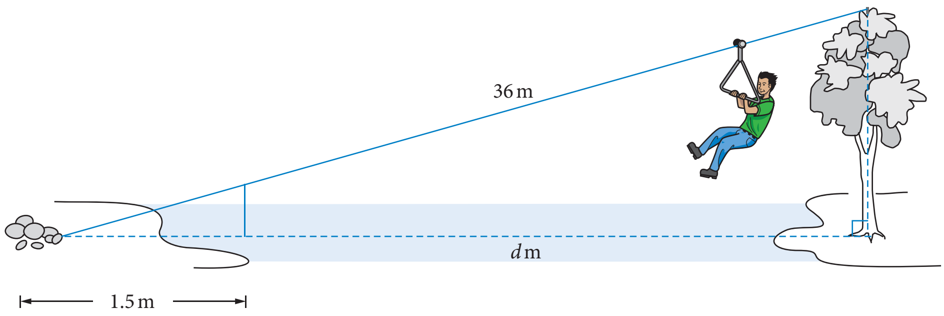
A 114 cm^2 **B** 256 cm^2 **C** 384 cm^2 **D** 576 cm^2 **E** 864 cm^2

13 © VCAA 2002 1GTQ4 46% A torch, which is held horizontally, is shone on to a wall from a distance of 1.2 metres as shown. The circular area of light it creates on the wall has a radius of 8 centimetres. The torch is now moved an additional 2 metres away from the wall. The radius of the circular area of light on the wall is now closest to



A 3 cm **B** 10 cm **C** 13 cm **D** 16 cm **E** 21 cm

14 (1 mark) A flying fox is constructed between the top of a tree and a pile of stones. It has a length of 36 m. A post is put into the ground 1.5 m from the pile of stones. The length of cable from the pile of stones to the top of the post is 2 m. What is the (horizontal) distance between the tree and the pile of stones?



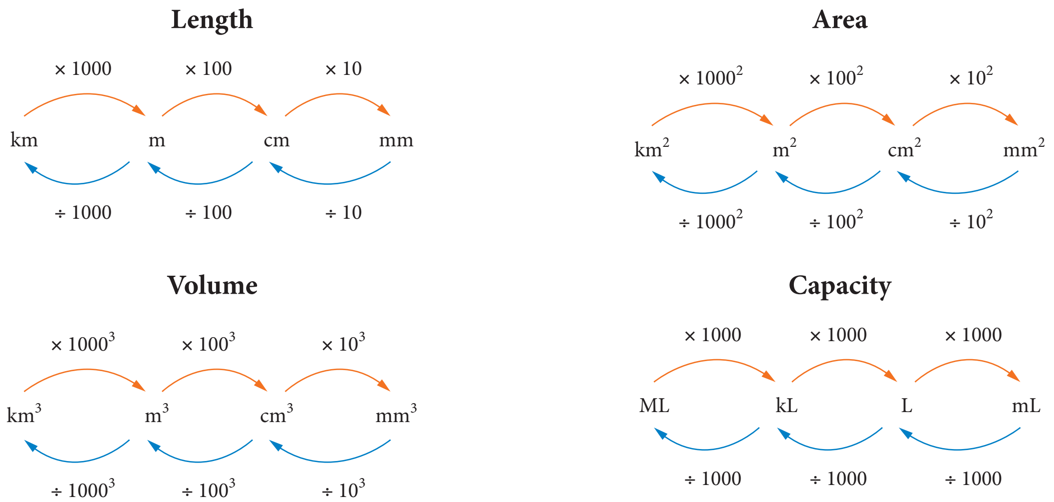
15 (2 marks) A model car is a scaled down version of a real car. The real car is 20 times the length of the model car.

a If it takes 15 mL of paint to cover the model, how much paint is needed to paint the real car? Give your answer in litres. 1 mark

b The fuel tank of the real car holds 60 L. What would be the capacity of the fuel tank on the model? Give your answer in millilitres. 1 mark

9 Chapter summary

Converting units of measurement



Scientific notation

- A number written in **scientific notation** takes the form (a number between 1 and 10) × (a power of 10).
- To convert *to* scientific notation, move the decimal place n places until a number between 1 and 10 is attained.

  Moving the decimal point to the left, the power of 10 is n.

  Moving the decimal point to the right, the power of 10 is $-n$.
- To convert *from* scientific notation, if n, the power of 10

  is positive, move the decimal power to the right n places, inserting zeros if necessary

  is negative, move the decimal power to the left n places, inserting zeros if necessary

  or use CAS.

Rounding measurements to significant figures

Significant figures:

- any non-zero digit
- zeros between non-zero digits
- trailing zeros in decimals.

Not significant figures:

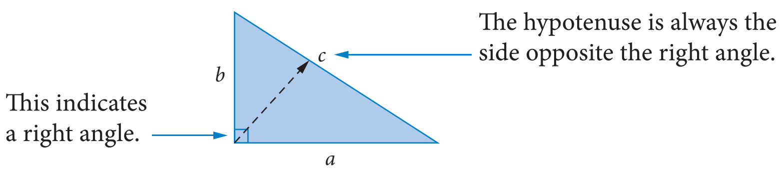
- leading zeros in decimals
- trailing zeros in whole numbers.

When rounding to significant figures, use usual rounding rules:

- '0–4 round down' and '5–9 round up'
- round the 9 to 0 and carry the rounding over to the next digit on the left
- include trailing zeros in decimals, if necessary.

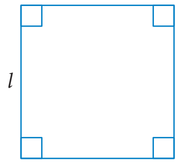
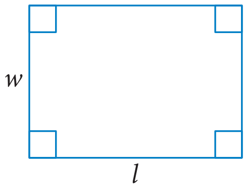
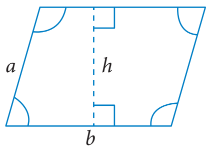
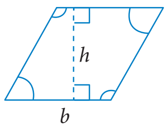
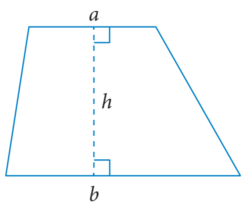
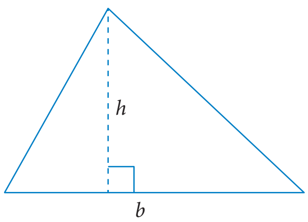
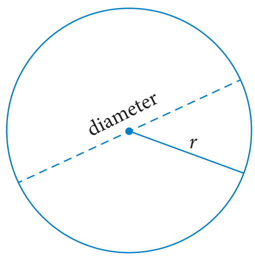
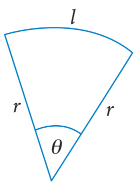
Pythagoras' theorem

- $c^2 = a^2 + b^2$, where c is the **hypotenuse** (longest side).



Perimeter and area

- The **perimeter of a shape** is a measure of the total distance around the outside of the **shape**.
- The **area** of a shape is a measure of the amount of space inside the shape.

Shape		Perimeter	Area
Square Four sides All sides are equal in length All angles are 90°		$P = 4l$	$A = l^2$
Rectangle Four sides Opposite sides are equal in length All angles are 90°		$P = 2l + 2w$	$A = lw$
Parallelogram Four sides Opposite sides are equal in length and parallel		$P = 2a + 2b$	$A = bh$
Rhombus Four sides All sides are equal in length Opposite sides are parallel		$P = 4b$	$A = bh$
Trapezium Four sides Two sides are parallel		Add the lengths of the four sides	$A = \frac{1}{2}(a + b)h$
Triangle Three sides		Add the lengths of the three sides	$A = \frac{1}{2}bh$
Circle Every point is the same distance from the centre		$C = 2\pi r$	$A = \pi r^2$
Sectors Part of a circle formed by two radiuses and the arc between them		$P = 2r + l$ where $l = \frac{\pi r\theta}{180}$	$A = \frac{1}{2}rl$ where $l = \frac{\pi r\theta}{180}$

Volume and capacity

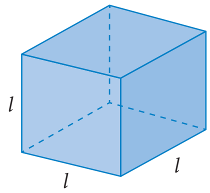
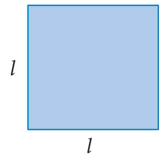
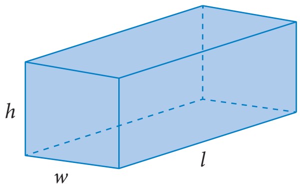
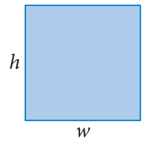
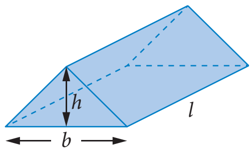
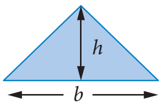
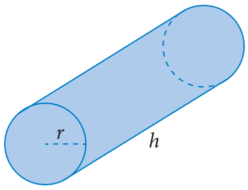
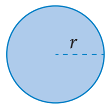
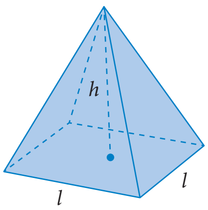
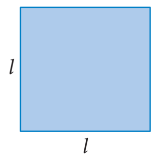
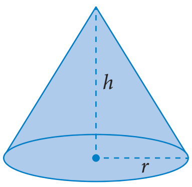
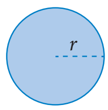
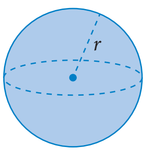
- The **volume** of a three-dimensional object is the amount of space it takes up.
- The **capacity** of a three-dimensional object is the amount of liquid it can hold.
- When calculating capacity, calculate volume first and then convert:

  $1\text{ cm}^3 = 1$ millilitre (mL)

  $1000\text{ cm}^3 = 1$ litre (L)

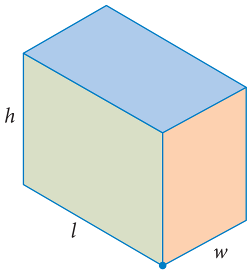
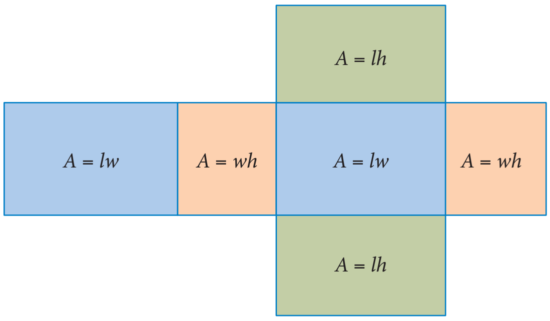
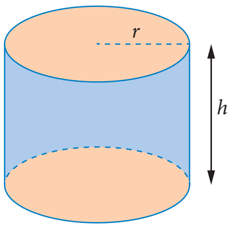
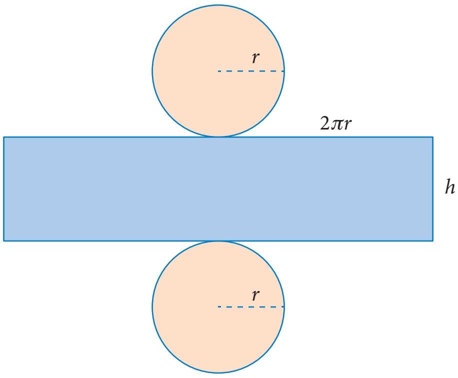
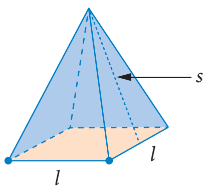
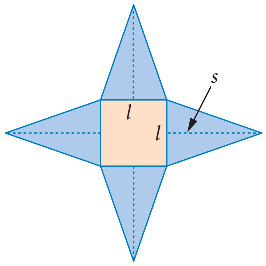
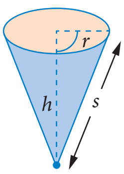
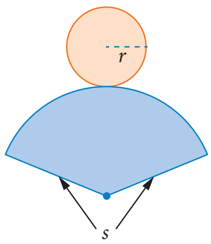
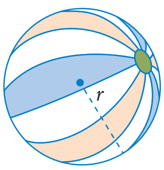
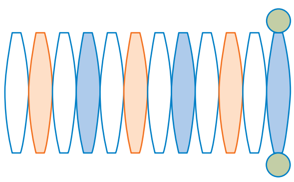
  $1\text{ m}^3 = 1000$ litres (L)
- **Prisms** and **cylinders** have the same cross-section from one base to the other.
- Volume of prisms and cylinders = area of the base × length
- Volume of pyramids and cones = $\frac{1}{3}$ × area of the base × height

Volume formulas

Object	Diagram	Area of base	Volume formula
cube		 $A = l^2$	$V = l^3$
rectangular prism		 $A = wh$	$V = whl$
triangular prism		 $A = \frac{1}{2}bh$	$V = \frac{1}{2}bhl$
cylinder		 $A = \pi r^2$	$V = \pi r^2 h$
square-based pyramid		 $A = l^2$	$V = \frac{1}{3}l^2 h$
cone		 $A = \pi r^2$	$V = \frac{1}{3}\pi r^2 h$
sphere		No base	$V = \frac{4}{3}\pi r^3$

Surface area

- The **surface area** of a three-dimensional object is the area of all of its faces added together.
- Drawing or visualising the **net** of a three-dimensional object helps to identify the shapes of all the faces. A net is a two-dimensional shape that can be folded up to form the three-dimensional object.

Three-dimensional object	Net	Surface area (SA) formula
rectangular prism 		3 pairs of rectangles $SA = 2(lw + wh + lh)$
cylinder 		2 circles and 1 rectangle $SA = 2\pi r^2 + 2\pi rh$ $= 2\pi r(r + h)$
square-based pyramid  s = slant		1 square and 4 triangles $SA = l^2 + 4 \times \left(\frac{1}{2}ls\right)$ $= l^2 + 2ls$
cone  s = slant		1 circle and a sector of a circle $SA = \pi r^2 + \pi rs$ $= \pi r(r + s)$
sphere 	 The net of a sphere doesn't help us calculate the surface area.	$SA = 4\pi r^2$

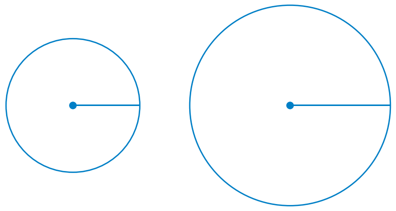
Similarity and scale

- Two shapes are **similar** if they have exactly the same shape but are different sizes.
- k = **scale factor** for similar shapes = $\dfrac{\text{any length of second shape}}{\text{matching length of first shape}}$

  $k > 1$ gives a larger second shape

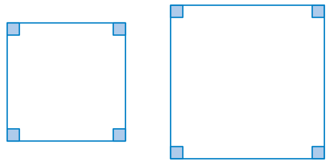
  $k < 1$ gives a smaller second shape

Circles



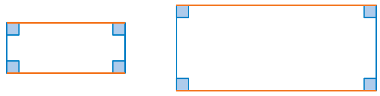
All circles are similar. The radius is the only length measurement needed.

Squares



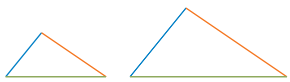
All squares are similar. The scale factors for each matching pair of sides will be the same because all sides in each shape are of equal length.

Rectangles



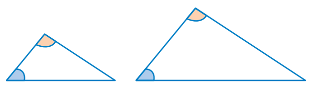
Rectangles are similar if the two pairs of matching sides have the same scale factor.

Triangles



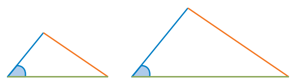
Side-Side-Side (SSS) or

Triangles are similar if the three pairs of matching sides have the same scale factor.



Angle-Angle (AA) or

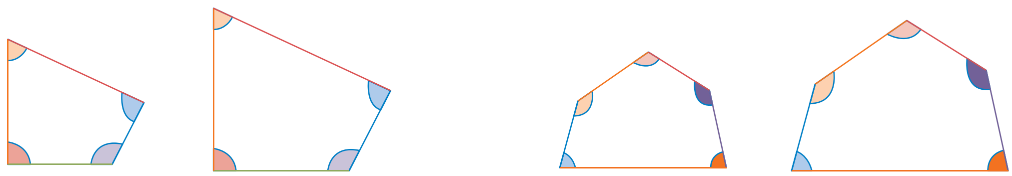
Triangles are similar if two pairs of matching angles are equal.



Side-Angle-Side (SAS)

Triangles are similar if two pairs of matching sides have the same scale factor and the angles between them are equal.

All shapes with straight sides (polygons)



Polygons are similar if every pair of matching sides have the same scale factor *and* all pairs of matching angles are equal.

- When a shape is scaled by a factor of k:

  area of second shape = k^2 × area of first shape
- When a three-dimensional object is scaled by a factor of k:

  surface area of second object = k^2 × surface area of first object

  volume of second object = k^3 × volume of first object

  where

  $$k = \frac{\text{any length of second shape/object}}{\text{matching length of first shape/object}}$$

Cumulative examination 1

Total number of marks: 16 Reading time: 7 minutes Writing time: 36 minutes

1 © VCAA 2008 1CQ7 The pulse rates of a large group of 18-year-old students are approximately normally distributed with a mean of 75 beats/minute and a standard deviation of 11 beats/minute. The percentage of 18-year-old students with pulse rates less than 53 beats/minute or more than 86 beats/minute is closest to

A 2.5% **B** 5% **C** 16% **D** 18.5% **E** 21%

2 © VCAA 2003 1NPQ9 MODIFIED The first five values of a sequence of numbers are

20, 10, 20, 10, 20 …

A recurrence relation that generates this sequence is

A $u_0 = 20, \quad u_{n+1} = 20 - u_n$ **B** $u_0 = 20, \quad u_{n+1} = u_n - 20$ **C** $u_0 = 20, \quad u_{n+1} = 0.5u_n$

D $u_0 = 20, \quad u_{n+1} = u_n - 10$ **E** $u_0 = 20, \quad u_{n+1} = 30 - u_n$

3 © VCAA 2013 1BRMQ3 \$10 000 is invested for five years. Interest is earned at a rate of 8% per annum, compounding quarterly. Which one of the following calculations will give the total interest earned, in dollars, by this investment?

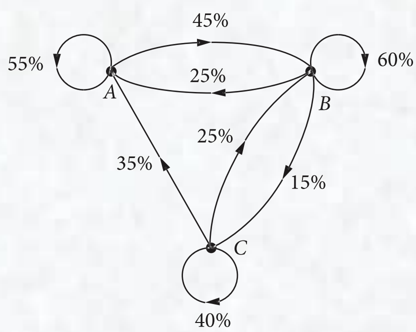
A $10\,000 \times 1.02^5 - 10\,000$ **B** $10\,000 \times 1.02^{20} - 10\,000$ **C** $10\,000 \times 1.08^5 - 10\,000$

D $10\,000 \times 1.08^{20} - 10\,000$ **E** $10\,000 \times 1.02^{20}$

4 © VCAA 2011 1GRQ5 The cost, \$$C$, of making x kilograms of chocolate fudge is given by $C = 60 + 5x$. The revenue, \$$R$, from selling x kilograms of chocolate fudge is given by $R = 15x$. A particular quantity of chocolate fudge was made and sold. It resulted in a loss of \$20. The number of kilograms of chocolate fudge made and sold was

A 2 **B** 4 **C** 8 **D** 12 **E** 16

5 © VCAA 2020 1MQ4 In a particular supermarket, the three top-selling magazines are *Angel* (A), *Bella* (B) and *Crystal* (C). The transition diagram shows the way shoppers at this supermarket change their magazine choice from week to week.



A transition matrix that provides the same information as the transition diagram is

A

$$\begin{array}{c} \text{This week} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.55 & 0.70 & 0.35 \\ 0.70 & 0.60 & 0.40 \\ 0.35 & 0.40 & 0.40 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \text{ Next week} \\ C \end{array}$$

B

$$\begin{array}{c} \text{This week} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.55 & 0.60 & 0.25 \\ 0.45 & 0.15 & 0.35 \\ 0 & 0.25 & 0.40 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \text{ Next week} \\ C \end{array}$$

C

$$\begin{array}{c} \text{This week} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.55 & 0.25 & 0.35 \\ 0.45 & 0.60 & 0.25 \\ 0 & 0.15 & 0.40 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \text{ Next week} \\ C \end{array}$$

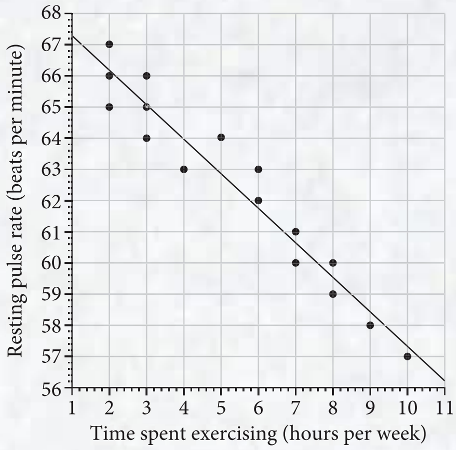
D

$$\begin{array}{c} \text{This week} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.55 & 0.25 & 0.35 \\ 0.45 & 0.60 & 0.25 \\ 0.35 & 0.15 & 0.40 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \text{ Next week} \\ C \end{array}$$

E

$$\begin{array}{c} \text{This week} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.55 & 0.25 & 0 \\ 0.45 & 0.60 & 0.25 \\ 0 & 0.15 & 0.75 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \text{ Next week} \\ C \end{array}$$

6 © VCAA 2018 1CQ8 MODIFIED The scatterplot displays the *resting pulse rate*, in beats per minute, and the *time spent exercising*, in hours per week, of 16 students. A line of good fit has been fitted to the data.



The equation of this line of good fit is closest to

A *resting pulse rate* = 67.2 – 0.91 × *time spent exercising*

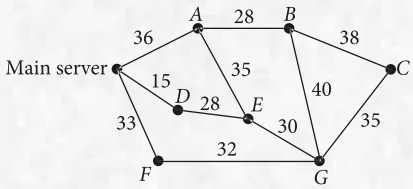
B *resting pulse rate* = 67.2 – 1.10 × *time spent exercising*

C *resting pulse rate* = 68.3 – 0.91 × *time spent exercising*

D *resting pulse rate* = 68.3 – 1.10 × *time spent exercising*

E *resting pulse rate* = 67.2 + 1.10 × *time spent exercising*

7 © VCAA 2019 1NQ5 The following diagram shows the distances, in metres, along a series of cables connecting a main server to seven points, *A* to *G*, in a computer network.



The minimum length of cable, in metres, required to ensure that each of the seven points is connected to the main server directly or via another point is

A 175 **B** 203 **C** 208 **D** 221 **E** 236

8 How many of the following equations represent inverse variation?

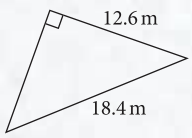
$y = \frac{4}{5}x$ $\quad y = \frac{5}{x}$ $\quad y = \frac{4}{x}$ $\quad y = \frac{4}{5x}$

A 0 **B** 1 **C** 2 **D** 3 **E** 4

9 Which of the following is the calculation needed to convert 2 cubic metres to cubic centimetres?

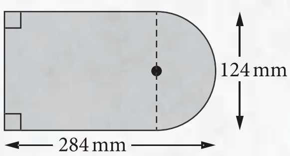
A 2 ÷ 1000 **B** 2 × 1000 **C** 2 × 1 000 000 **D** 2 ÷ 1 000 000 **E** 2 × 100

10 What is the missing side length for the triangle shown, correct to one decimal place?



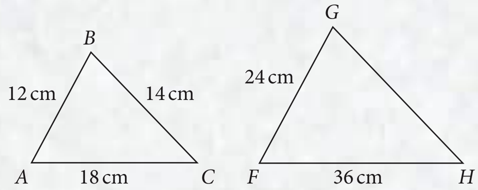
A 13.4 m **B** 13.9 m **C** 18.4 m **D** 22.3 m **E** 22.4 m

11 The perimeter of the following shape rounded to two significant figures is



A 408 mm **B** 760 mm **C** 763 mm **D** 34 000 mm^2 **E** 35 216 mm^2

12 © VCAA 2009 1GTQ1 The two triangles, *ABC* and *FGH*, are similar. The length of *GH* is

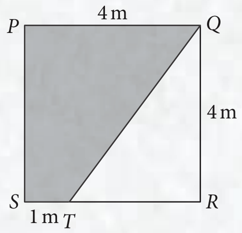


A 14 cm **B** 24 cm **C** 26 cm **D** 28 cm **E** 32 cm

13 During the COVID-19 pandemic people were restricted to travelling a limit of 5 km from their houses. How many square kilometres is this closest to?

A 25 **B** 31 **C** 32 **D** 78 **E** 79

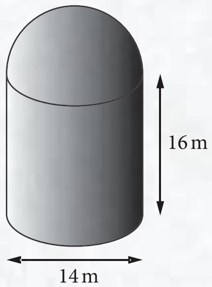
14 © VCAA 2012 1GTQ2 *PQRS* is a square of side length 4 m, as shown in the diagram. The distance *ST* is 1 m.



The shaded area *PQTS* shown in the diagram, in m^2, is closest to

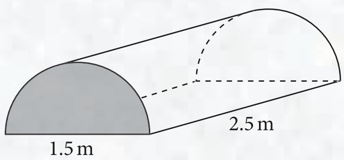
A 6 **B** 8 **C** 9 **D** 10 **E** 12

15 Which of the following is closest to the volume for the solid shown?



A 3191 cm^3 **B** 2808 m^3 **C** 3181 m^3 **D** 3191 m^3 **E** 3192 m^3

16 © VCAA 2008 1GTQ6 A tent with semi-circular ends is in the shape of a prism. The diameter of the ends is 1.5 m. The tent is 2.5 m long. The surface area (in m^2) of the tent, including the base, is closest to



A 5.5 **B** 7.7 **C** 8.8 **D** 11.4 **E** 15.3

Cumulative examination 2

Total number of marks: 22 Reading time: 6 minutes Writing time: 33 minutes

1 © VCAA 2007 2BRMQ3b (2 marks) Khan will depreciate his \$900 fax machine for taxation purposes. He considers two methods of depreciation.

Flat rate depreciation

Under flat rate depreciation the fax machine will be valued at \$300 after five years.

a Calculate the annual depreciation in dollars. 1 mark

Unit cost depreciation

Suppose Khan sends 250 faxes a year. The \$900 fax machine is depreciated by 46 cents for each fax it sends.

b Determine the value of the fax machine after five years. 1 mark

2 © VCAA 2018 2MQ1 (3 marks) A toll road is divided into three sections, E, F and G. The cost, in dollars, to drive one journey on each section is shown in matrix C.

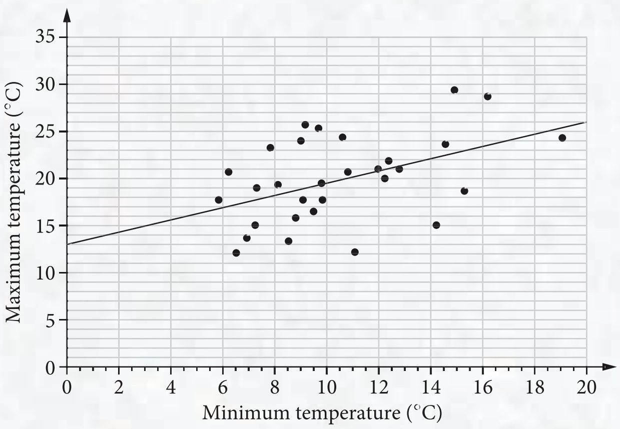
$$C = \begin{bmatrix} 3.58 \\ 2.22 \\ 2.87 \end{bmatrix} \begin{matrix} E \\ F \\ G \end{matrix}$$

a What is the cost of one journey on section G? 1 mark

b Write down the order of matrix C. 1 mark

c One day Kim travels once on section E and twice on section G. His total toll cost for this day can be found by the matrix product $M \times C$. Write down the matrix M. 1 mark

3 © VCAA 2012 2CQ2 MODIFIED (5 marks) The maximum temperature and the minimum temperature at a weather station on each of the 30 days in November 2011 are displayed in the scatterplot. A line of good fit has been fitted to the data.



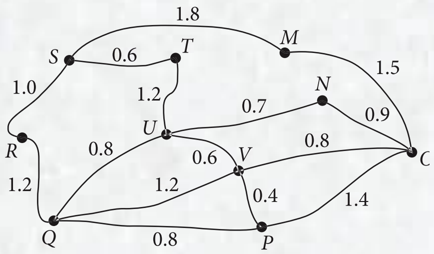
a Describe the association between the maximum temperature and the minimum temperature in terms of strength and direction. 1 mark

b What does the vertical intercept of the line of good fit predict in terms of maximum temperature and minimum temperature. 1 mark

c What maximum temperature, to the nearest degree, does the line of good fit predict when the minimum temperature is 14°C. 1 mark

d Interpret the slope of the line of good fit in terms of maximum temperature and minimum temperature. 2 marks

4 © VCAA 2020 2NQ3abi (2 marks) A local fitness park has 10 exercise stations: *M* to *V*.
The edges on the graph below represent the tracks between the exercise stations.
The number on each edge represents the length, in kilometres, of each track.



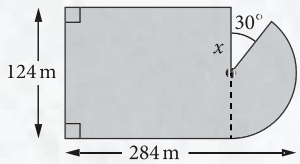
The Sunny Coast cricket coach designs three different training programs, **all starting at exercise station *S*.**

Training program number	Training details
1	The team must run to exercise station *O*.
2	The team must run along all tracks just once.
3	The team must visit each exercise station and return to exercise station *S*.

a What is the shortest distance, in kilometres, covered in training program 1? 1 mark

b What type of walk is used to describe training program 2? 1 mark

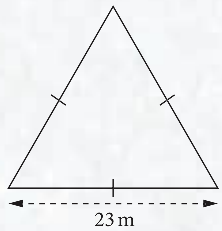
5 (2 marks) A fence has been constructed around the grassed area in the following shape.



a What is the length x shown? 1 mark

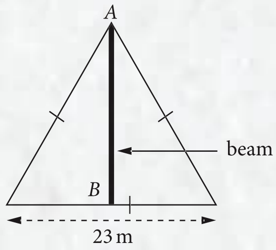
b What is the total length of the fence to the nearest metre? 1 mark

6 © VCAA 2021N 2GMQ1abcd 2ab MODIFIED (6 marks) Shona is in charge of decorations for an event.
She wants to hang the decorations from the ceiling. The ceiling is triangular in shape, as shown in the diagram. All three sides of the ceiling are 23 m long.



a What is the perimeter, in metres, of the ceiling? 1 mark

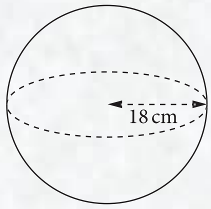
The decorations are to be hung from a beam *AB* that runs across the centre of the ceiling, as shown in the diagram.



b Write a calculation that shows that the length of this beam *AB*, rounded to one decimal place, is 19.9 m. 1 mark

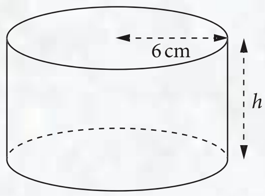
c What is the area, in square metres, of the ceiling? Round your answer to the nearest whole number. 1 mark

Shona wants to hang spheres from the beam. Each sphere has a radius of 18 cm, as shown in the diagram below.



d What is the volume, in cubic centimetres, of one sphere? Round your answer to the nearest whole number. 1 mark

Shona will place cylindrical bowls on each table at the event as a centrepiece.
Each cylindrical bowl has a radius of 6 cm, as shown in the diagram.



Each bowl has a volume of 1244 cm^3.

e Write a calculation that shows that the height, *h*, of one cylindrical bowl, rounded to the nearest whole number, is 11 cm. 1 mark

f A candle, also in the shape of a cylinder, is to be placed upright inside each bowl so that it touches the base of the bowl.

The candle has a radius of 3 cm and a height of 18 cm. Once the candle has been placed inside the bowl, the remaining volume of the bowl will be filled with sand. What volume of sand, in cubic centimetres, is required to fill the cylindrical bowl once the candle is placed inside it? Round your answer to the nearest whole number. 1 mark

7 (2 marks) A chocolate bar is in the shape of a rectangular prism. The manufacturer wants to increase profits by reducing the size of the bar.

a If the dimensions of the new bar are reduced by one quarter, what is the scale factor? 1 mark

b If the original bar was 320 g, and weight is directly proportional to volume, how much does the new bar weigh? 1 mark

CHAPTER 10

APPLICATIONS OF TRIGONOMETRY

Study Design coverage

UNIT 2, AREA OF STUDY 4: SPACE AND MEASUREMENT

Space, measurement and applications of trigonometry

- the use of trigonometric ratios and Pythagoras' theorem to solve practical problems involving a right-angled triangle in two dimensions, including the use of angles of elevation and depression
- the use of the sine rule, including the ambiguous case, the cosine rule, as a generalisation of Pythagoras' theorem, and their application to solving practical problems involving non-right-angled triangles, including three-figure (true) bearings in navigation.

Video playlists (4):

10.1 Trigonometric ratios

10.2 Applying trigonometry

10.3 The sine and cosine rules

VCE question analysis Applications of trigonometry

Worksheets (25):

10.1 Trigonometric ratios • Identifying the correct trigonometric ratio • Calculating lengths and angles • Trigonometry review • Mixed trig questions

10.2 Angles of elevation and depression • Trigonometry problems • Bearings 1 • Bearings 2 • Identifying bearings • A page of bearings • NSW map bearings • Bearings match-up • 16 points of the compass • Elevations and bearings

10.3 Trigonometric calculations • Discovering the sine rule • The sine rule – Finding lengths of sides • The sine rule – Finding angles • The cosine rule – Angles and sides • Applying trigonometry • Finding an unknown side • Finding an unknown angle • Sine rule problems • Cosine rule problems

Puzzles (4):

10.1 Trigonometry match-up

10.2 Angles of elevation and depression • Trigonometry crossword • Every which way

Nelson MindTap

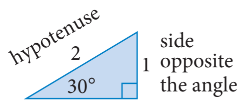
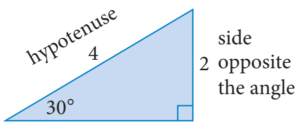
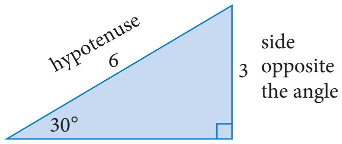
To access resources above, visit **cengage.com.au/nelsonmindtap**

10.1 Trigonometric ratios

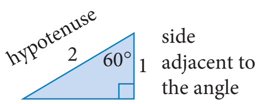
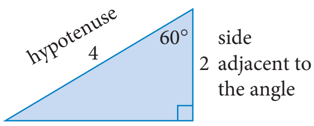
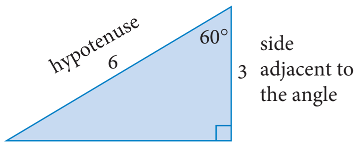
What is a trigonometric ratio?

Trigonometry makes it possible to calculate the height of Mount Everest without climbing it. It is used to work out how far it is to shore when we are at sea, and the distance from the Earth to the Sun. Trigonometry allows us to calculate lengths and angles that we can't measure directly. How does it do this? It does it by using ratios in right-angled triangles.

Consider the three similar right-angled triangles shown in the table.

Triangle	Triangle ratio	Trigonometric ratio
	$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} = 0.5$	$\sin(30°) = 0.5$
	$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{2}{4} = 0.5$	$\sin(30°) = 0.5$
	$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{6} = 0.5$	$\sin(30°) = 0.5$

Every right-angled triangle with an angle of 30° will have the same **ratio** between the sides, so no matter what the length of the sides, $\sin(30°) = 0.5$.

Triangle	Triangle ratio	Trigonometric ratio
	$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} = 0.5$	$\cos(60°) = 0.5$
	$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{4} = 0.5$	$\cos(60°) = 0.5$
	$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{6} = 0.5$	$\cos(60°) = 0.5$

Every right-angled triangle with an angle of 60° will have the same ratio between the sides, so no matter what the length of the sides, $\cos(60°) = 0.5$.

10.1

Video playlist Trigonometric ratios

Worksheets Trigonometric ratios

Identifying the correct trigonometric ratio

Calculating lengths and angles

Trigonometry review

Mixed trig questions

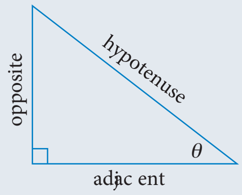
Puzzle Trigonometry match-up

Finding an unknown side of a right-angled triangle

A **trigonometric ratio** can be used to calculate the length of an unknown side in a right-angled triangle if one angle and one side are known.

Finding an unknown side of a right-angled triangle

There is a trigonometric ratio for each pair of sides in a right-angled triangle. The opposite and adjacent sides of the right-angled triangle are named in relation to a given angle, θ (theta).



Function	Abbreviation	Ratio	Initials
sine	sin	$\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$	SOH
cosine	cos	$\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	CAH
tangent	tan	$\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}}$	TOA

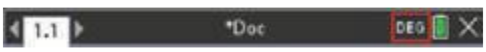
Exam hack

The three ratios can be remembered simply by their initials, SOH-CAH-TOA (pronounced 'soh car toe-ah'), but there are many other ways to remember it. For example, '**S**tudying **O**ur **H**omework **C**an **A**lways **H**elp **T**o **O**btain **A**chievement'.

Finding trigonometric functions with CAS

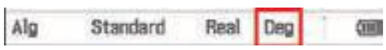
Before we begin trigonometric calculations, we must have our calculator set to degree mode. So when we enter angles in our calculator, all calculations will be based on angles in degrees.

TI-Nspire



In the top right-hand corner of the screen, click on the angle setting to the left of the battery icon to toggle between degrees and radians.

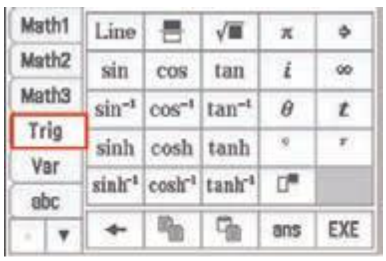
ClassPad



In the bottom right-hand corner of the screen, click on the angle setting to the left of the battery icon to toggle between degrees and radians.

The trigonometric functions can be accessed as follows.

Press the **trig** key.



Open the **Keyboard** and select **Trig**.

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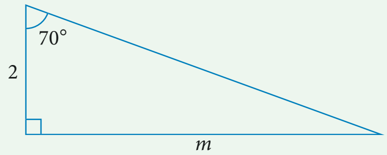
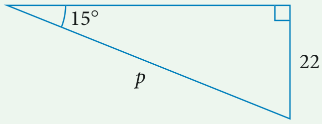
WORKED EXAMPLE 1 Finding an unknown side of a right-angled triangle

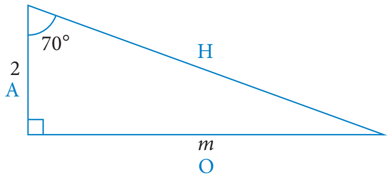
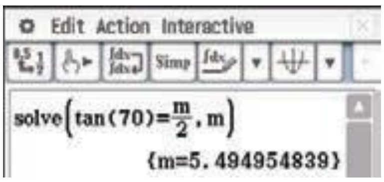
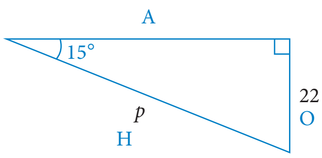
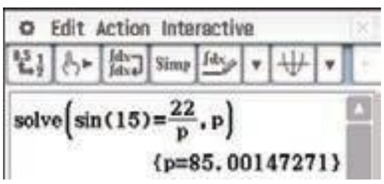
Find the value of the pronumeral, correct to two decimal places, in each right-angled triangle.

a



b



Steps	Working
a 1 Indicate the sides of the right-angled triangle with the letters O, A and H to show opposite, adjacent and hypotenuse.	
2 Identify whether the labelled sides are opposite, adjacent or hypotenuse and select the matching trigonometric ratio.	m is the opposite side. 2 is the adjacent side. Use $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
3 Substitute the known values and solve the equation for the unknown to the required level of accuracy, using the **solve** function on CAS if necessary.	$\tan(70°) = \dfrac{m}{2}$ $m = 2 \times \tan(70°)$ $= 5.495\ldots$ $m \approx 5.49$
TI-Nspire 	**ClassPad** 
b 1 Indicate the sides of the right-angled triangle with the letters O, A and H to show opposite, adjacent and hypotenuse.	
2 Identify whether the labelled sides are opposite, adjacent or hypotenuse and select the matching trigonometric ratio.	22 is the opposite side. p is the hypotenuse. Use $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$
3 Substitute the known values and solve the equation for the unknown to the required level of accuracy, using the **solve** function on CAS if necessary.	$\sin(15°) = \dfrac{22}{p}$ $p = 22 \div \sin(15°)$ $= 85.001\ldots$ $p \approx 85.00$
	

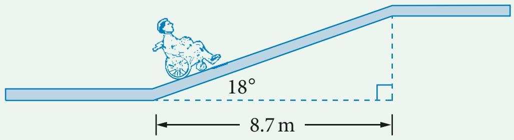
WB p. 155

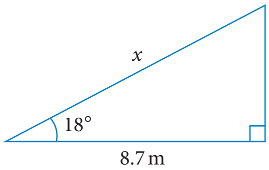
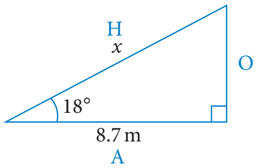
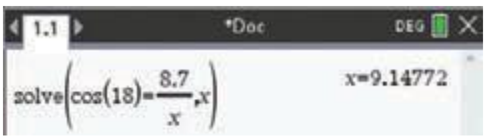
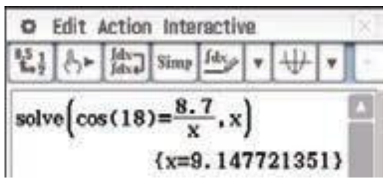
WB

p. 157

WORKED EXAMPLE 2 Solving problems involving an unknown side of a right-angled triangle

A wheelchair ramp is inclined at 18° to the horizontal. How long is the ramp if it links two levels 8.7 m apart horizontally? Round your answer correct to one decimal place.



Steps	Working
1 Redraw the triangle using the given values. Represent the length of the ramp by *x*.	
2 Write the letters O, A and H on the right-angled triangle to show the opposite and adjacent sides and the hypotenuse.	
3 Identify whether the labelled sides are opposite, adjacent or hypotenuse and select the matching trigonometric ratio.	8.7 is the adjacent side. x is the hypotenuse. Use $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$
4 Substitute the known values and solve the equation for the unknown, using the **solve** function on CAS if necessary.	$\cos(18°) = \frac{8.7}{x}$ $x = \frac{8.7}{\cos(18°)} = 9.1477\ldots$
TI-Nspire 	**ClassPad** 
5 Write your answer in the required units and round to the required level of accuracy.	The length of the ramp is 9.1 m.

Finding an unknown angle in a right-angled triangle

Trigonometric ratios can also be used to find angles in right-angled triangles when two sides are known. This can be done using the **inverse trigonometric function** on CAS.

If we know the angle in a right-angled triangle, a trigonometric function gives us the ratio of the sides. An inverse trigonometric function does the reverse. If we know the ratio of the sides, the inverse trigonometric function gives us the angle.

For example, we know $\sin(30°) = 0.5$.

Using the inverse of sine, we know that $\sin^{-1}(0.5) = 30°$.

Inverse trigonometric functions

Inverse trigonometric functions can be used to find unknown angles in right-angled triangles.

If $\sin\theta = a$, then $\sin^{-1}(a) = \theta$.

If $\cos\theta = b$, then $\cos^{-1}(b) = \theta$.

If $\tan\theta = c$, then $\tan^{-1}(c) = \theta$.

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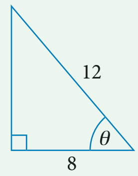
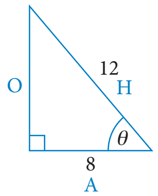
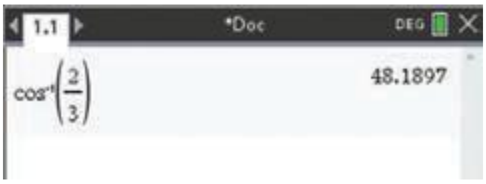
WORKED EXAMPLE 3 Finding an unknown angle in a right-angled triangle

Find the value of θ, correct to the nearest degree.



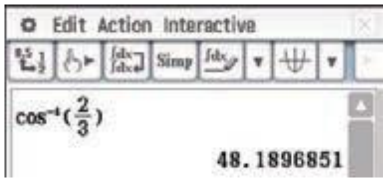
Steps	Working
1 Indicate the sides of the right-angled triangle with the letters O, A and H to show opposite, adjacent and hypotenuse.	
2 Identify whether the labelled sides are opposite, adjacent or hypotenuse and select the matching trigonometric ratio.	8 is the adjacent side. 12 is the hypotenuse. Use $\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
3 Substitute the known values and solve for θ with CAS, using the inverse trigonometric function and rounding to the nearest degree.	$\cos(\theta) = \dfrac{8}{12} = \dfrac{2}{3}$ $\theta = \cos^{-1}\left(\dfrac{2}{3}\right)$ $\theta \approx 48°$

TI-Nspire



Press **trig** to open the mini-palette to access the inverse trigonometry functions.

ClassPad



Open the **Keyboard** and tap **Trig** to access the inverse trigonometry functions.

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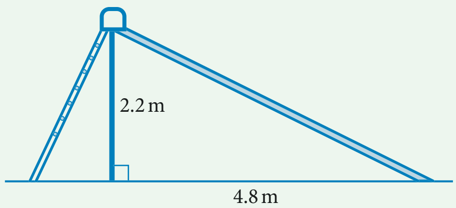
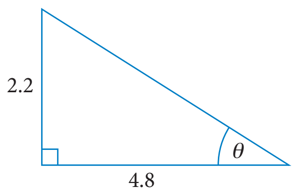
WORKED EXAMPLE 4 Solving problems involving an unknown angle of a right-angled triangle

A playground slide is made up of a ladder and a metal slide. The top of the metal slide is 2.2 m from the ground, and the bottom of the slide is 4.8 m from a point directly under the top of the slide. What angle does the slide make with the ground, correct to the nearest degree?



Steps	Working
1 Redraw the triangle using the given values. Represent the required angle by θ.	

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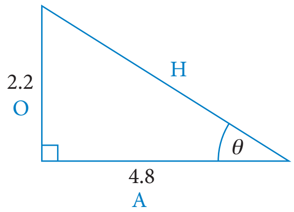
2 Write the letters O, A and H on the right-angled triangle to show the opposite and adjacent sides and the hypotenuse.



3 Identify whether the labelled sides are opposite, adjacent or hypotenuse and select the matching trigonometric ratio.

2.2 is the opposite side.

4.8 is the adjacent side.

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

4 Substitute the known values and solve for θ with CAS, using the inverse trigonometric function and rounding to the nearest degree.

$$\tan(\theta) = \frac{2.2}{4.8}$$

$$\theta = \tan^{-1}\left(\frac{2.2}{4.8}\right)$$

$$\theta = 24.624\ldots$$

5 Write your answer in the required units and round to the required level of accuracy.

The slide makes an angle of 25° with the ground.

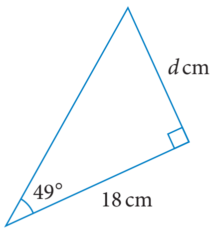
EXERCISE 10.1 Trigonometric ratios

ANSWERS p. 518

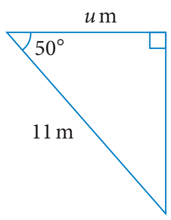
Mastery

1 WORKED EXAMPLE 1 Find the value of the pronumeral, correct to two decimal places, in each right-angled triangle.

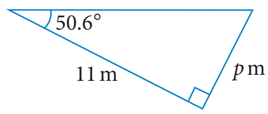
a



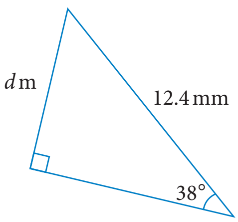
b



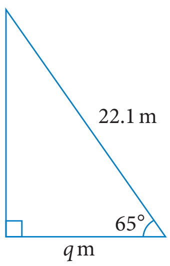
c



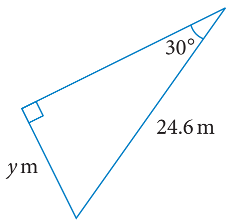
d



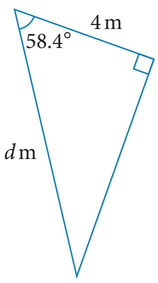
e



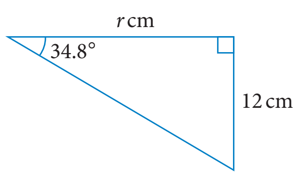
f



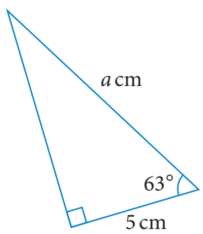
g



h

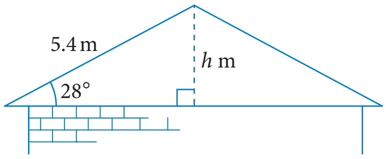


i

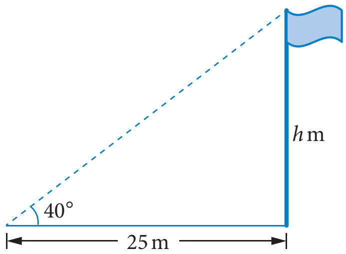


2 WORKED EXAMPLE 2 Find the required length in each of the following, giving your answer correct to two decimal places.

a The pitch of a roof is 28°. If the peak of the roof is 5.4 m from the gutter, how much higher is the peak than the gutter?

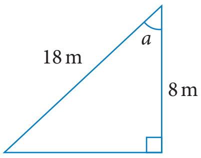


b Find the height, h m, of this flagpole.

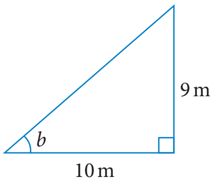


3 WORKED EXAMPLE 3 Find the value of the pronumeral in each triangle, correct to the nearest degree.

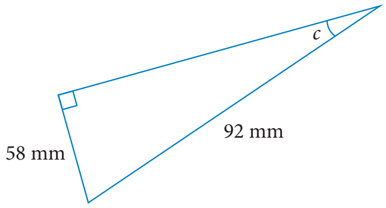
a



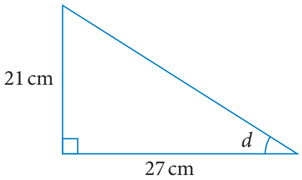
b



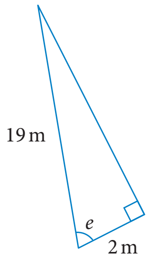
c



d



e



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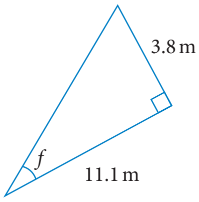

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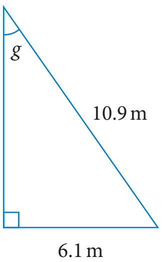

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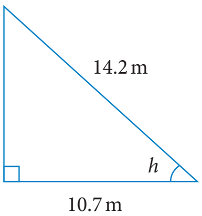
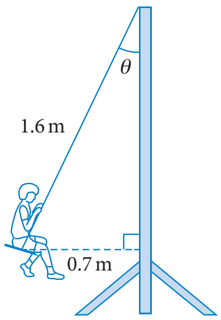
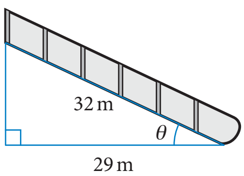


4 WORKED EXAMPLE 4 Find the required angle for each of the following to the nearest degree.

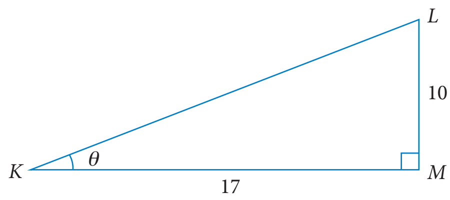
a Selwa is on a swing of length 1.6 m. When she is 0.7 m away from the swing's frame, what angle does the swing make with the vertical?



b An escalator ramp has a length of 32 m and covers a horizontal distance of 29 m. What is its angle of inclination, θ?



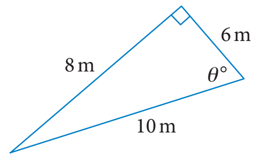
5 © VCAA 2002 1GTQ1 76% For triangle KLM, the size of the angle is closest to



A 27° **B** 30° **C** 36°

D 54° **E** 60°

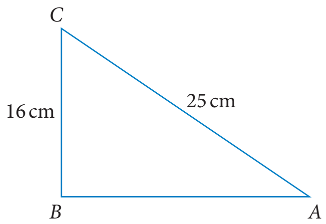
6 © VCAA 2007 1GTQ1 75% For the triangle shown, the value of $\cos(\theta°)$ is equal to



A $\frac{6}{10}$ **B** $\frac{6}{8}$ **C** $\frac{8}{10}$

D $\frac{10}{8}$ **E** $\frac{8}{6}$

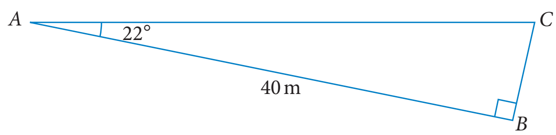
7 © VCAA 2004 1GTQ1 75% For the right-angled triangle ABC, with $BC = 16$ cm and $AC = 25$ cm, the size of angle BAC is closest to



A 7° **B** 25° **C** 33°

D 38° **E** 40°

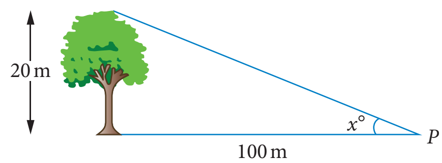
8 © VCAA 2002 1GTQ3 67% Tom and Matt are having a competition to see who can kick a football the longest distance. They both kick from point A, and the length of the kick is measured from point A to the point where the ball first makes contact with the ground. Tom's kick lands 40 metres away at point B. Matt kicks the ball at an angle of 22° to the direction of Tom's kick and it lands at point C, such that BC makes a right angle with AB as shown in the diagram below.



The length of Matt's kick, AC, is closest to

A 16 m **B** 37 m **C** 43 m **D** 46 m **E** 50 m

9 © VCAA 2017N 1GMQ2 A vertical tree of height 20 m stands on horizontal ground. The tree is 100 m away from point P, as shown in the diagram.

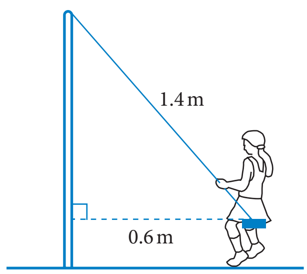


The value of the angle x, shown in the diagram above, is closest to

A 3° **B** 11° **C** 12°

D 78° **E** 79°

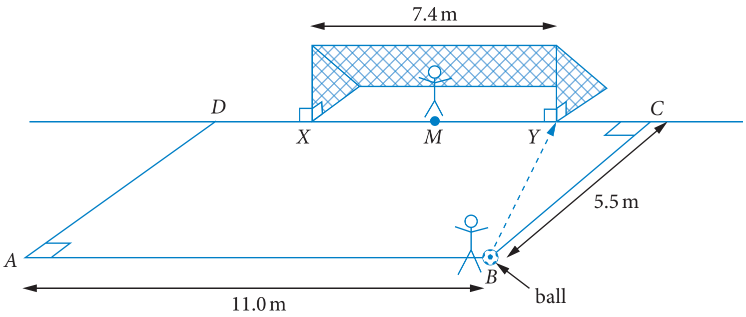
10 Alexis is on a swing of length 1.4 m. She pulls back 0.6 m from the frame to commence swinging. Which of the following is the expression that will give the angle that the swing makes with the vertical?



A $\cos^{-1}\left(\frac{0.6}{1.4}\right)$ **B** $\sin^{-1}\left(\frac{1.4}{0.6}\right)$ **C** $\tan^{-1}\left(\frac{0.6}{1.4}\right)$

D $\cos^{-1}\left(\frac{1.4}{0.6}\right)$ **E** $\sin^{-1}\left(\frac{0.6}{1.4}\right)$

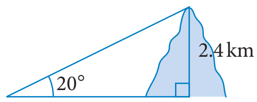
11 © VCAA 2010 1GTQ5 66% A soccer goal is 7.4 metres wide. A rectangular region *ABCD* is marked out directly in front of the goal. In this rectangular region, $AB = DC = 11.0$ metres and $AD = BC = 5.5$ metres. The goal line *XY* lies on *DC* and *M* is the midpoint of both *DC* and *XY*.



Ben kicks the ball from point *B*. It travels in a straight line to the base of the goal post at point *Y* on the goal line. Angle *CBY*, the angle that the path of the ball makes with the line *BC*, is closest to

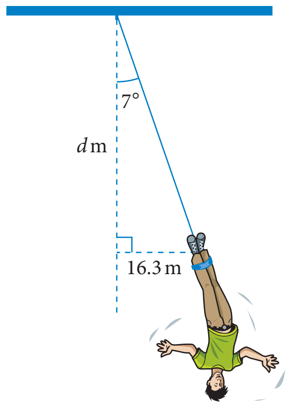
A 18° **B** 33° **C** 45° **D** 67° **E** 72°

12 Danielle flew in a helicopter from the ground at sea level to the summit of a mountain. The helicopter flew at an incline to ground level of 20° and the summit was 2.4 km above sea level. What is the total distance that the helicopter flew (to the nearest kilometre)?

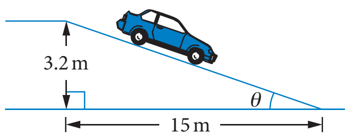


A 2 km **B** 3 km **C** 5 km **D** 6 km **E** 7 km

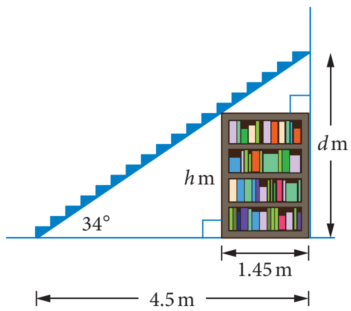
13 (1 mark) A bungee jumper leaps off a bridge at an angle of 7° to the vertical. If he is swinging 16.3 m off-centre as shown, calculate the vertical distance that he has dropped, measured to where his feet are tied, correct to one decimal place.



14 (1 mark) Find the angle of inclination, θ, of this ramp in a multi-storey car park, correct to the nearest degree.



15 (3 marks) A staircase is inclined at an angle of 34° and has a horizontal length of 4.5 m. A bookcase, 1.45 m long, is placed under the stairs. Find, correct to the nearest metre,



a the vertical height, *h*, of the tallest bookcase that will fit under the stairs — 1 mark

b the distance, *d*, of the top step of the staircase from the ground — 1 mark

c the length from the bottom of the staircase to the top. — 1 mark

Video playlist
Applying trigonometry

Worksheets
Angles of elevation and depression

Trigonometry problems

Puzzles
Angles of elevation and depression

Trigonometry crossword

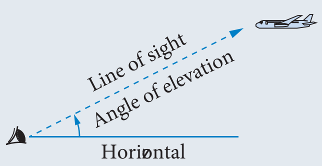
10.2 Applying trigonometry

Angles of elevation and depression

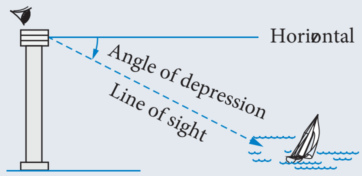
When we *look up* at an object above us, the angle that our line of sight makes with the horizontal is called the **angle of elevation**. When we *look down* at an object below us, the angle that our line of sight makes with the horizontal is called the **angle of depression**.

Angles of elevation and depression

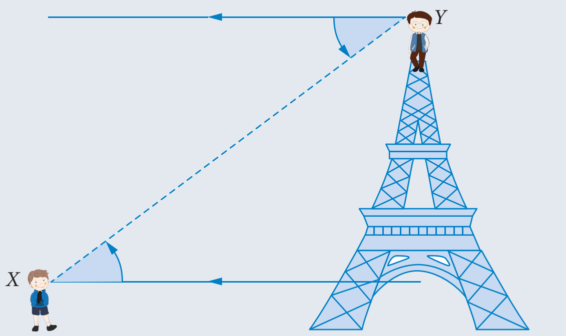
Angle of elevation – looking up



Angle of depression – looking down



angle of elevation X to Y = angle of depression Y to X



Two angles drawn inside a Z shape are called **alternate angles** and are always equal.

Exam hack

If a question refers to angles of elevation or depression, make sure you label the angle from the horizontal and not from the vertical.

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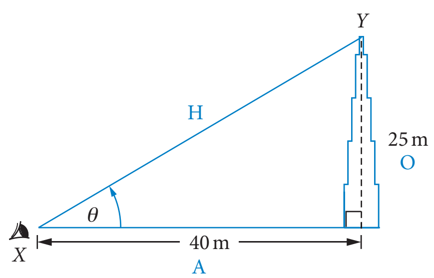
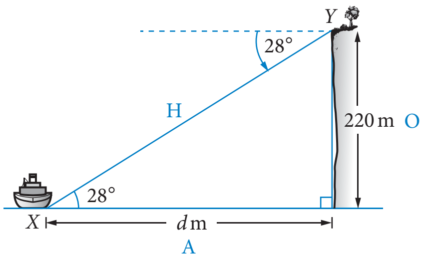
WORKED EXAMPLE 5 Solving problems involving angles of elevation and depression

Find each of the following, to the nearest whole unit, by drawing a diagram.

a Amala stands 40 m from the base of a 25 m tower and looks up at the top of the tower. Find the angle of elevation θ of the top of the tower from Amal.

b Sergio is standing on a 220 m high vertical cliff looking down at a ship. The angle of depression of the ship is 28°. Find the distance d from the ship to the base of the cliff.

Steps	Working
a **1** Draw a diagram with all the measurements including the angle. If necessary, use: angle of elevation X to Y = angle of depression Y to X Write the letters O, A and H on the right-angled triangle to show the opposite and adjacent sides and the hypotenuse.	

Step	Working
2 Identify whether the labelled sides are opposite, adjacent or hypotenuse and select the matching trigonometric ratio.	40 is the adjacent side. 25 is the opposite side. Use $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$
3 Substitute the known values and solve with CAS, rounding to the required level of accuracy.	$\tan(\theta) = \frac{25}{40}$ $\theta = \tan^{-1}\left(\frac{25}{40}\right)$ $\approx 32°$
b **1** Draw a diagram with all the measurements including the angle. If necessary, use: angle of elevation X to Y = angle of depression Y to X Write the letters O, A and H on the right-angled triangle to show the opposite and adjacent sides and the hypotenuse.	
2 Identify whether the labelled sides are opposite, adjacent or hypotenuse and select the matching trigonometric ratio.	d is the adjacent side. 220 is the opposite side. Use $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$
3 Substitute the known values and solve with CAS. Write your answer in the required units and round to the required level of accuracy.	$\tan(28°) = \frac{220}{d}$ $d = \frac{220}{\tan(28°)}$ ≈ 414 m

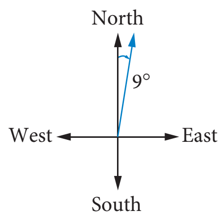
Exam hack

If you're asked to draw a diagram for elevation/depression problems, always draw the triangle you need to solve in the same position – for example, with the right angle at the bottom right.

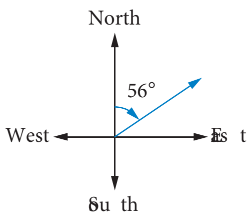
Three-figure bearings

A **bearing** is an angle that shows the direction of one point from another point. A **three-figure bearing** (also called a **true bearing**) is the angle measured in the clockwise direction from north. A three-figure bearing always has three figures. For example:

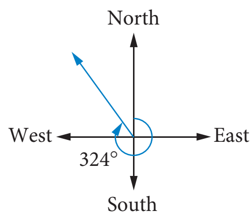
Three-figure bearing = 009°



Three-figure bearing = 056°



Three-figure bearing = 324°



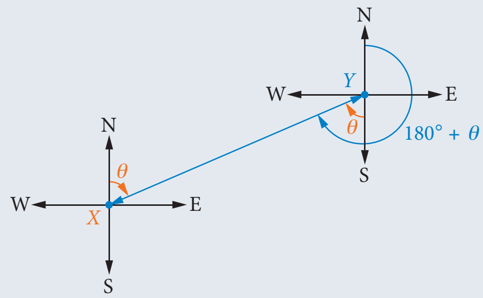
Worksheets
Bearings 1
Bearings 2
Identifying bearings
A page of bearings
NSW map bearings
Bearings match-up
16 points of the compass
Elevations and bearings

Puzzle
Every which way

Three-figure bearings

A three-figure bearing is the angle of Y from X measured in a clockwise direction from north.

If the bearing of Y from X is θ, then the bearing of X from Y is $180° + \theta$, where $\theta < 180°$.



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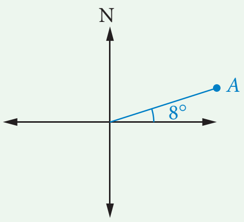
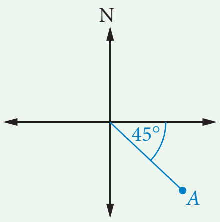
WORKED EXAMPLE 6 Calculating three-figure bearings

For each of the following, calculate the three-figure bearing of the point A and draw a diagram showing the angle.

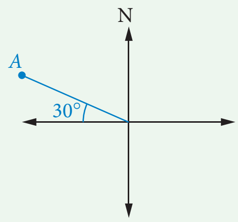
a



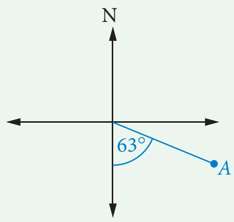
b

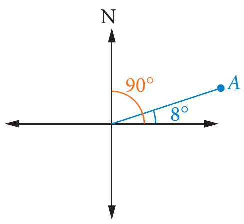
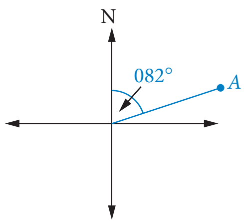
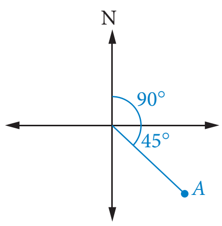
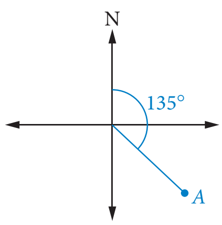


c

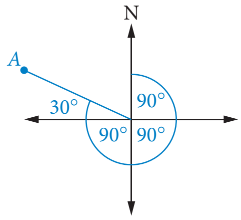


d



Steps	Working
a **1** Find the angle clockwise from north, adding or subtracting 90° angles if necessary.	 $90° - 8° = 82°$
2 Write your answer, making sure the answer has three figures. Draw a diagram showing the angle.	The three-figure bearing is 082°. 
b **1** Find the angle clockwise from north, adding or subtracting 90° angles if necessary.	 $90° + 45° = 135°$
2 Write your answer, making sure the answer has three figures.	The three-figure bearing is 135°. 

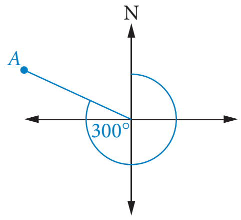
c **1** Find the angle clockwise from north, adding or subtracting 90° angles if necessary.



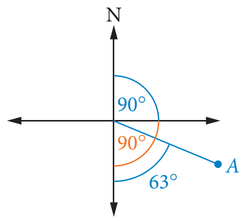
$90° + 90° + 90° + 30° = 300°$

2 Write your answer, making sure the answer has three figures.

The three-figure bearing is 300°.



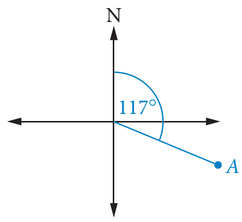
d **1** Find the angle clockwise from north, adding or subtracting 90° angles if necessary.



$90° + (90° - 63°) = 117°$

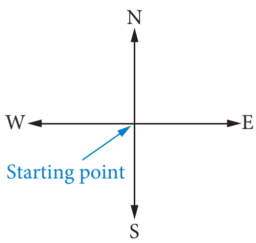
2 Write your answer, making sure the answer has three figures.

The three-figure bearing is 117°.



Exam hack

Watch for the word 'from' in a bearing question. This tells you where your starting point is.



WB

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WORKED EXAMPLE 7 Applying three-figure bearings

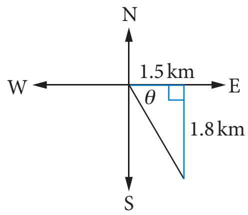
A hot air balloon travelled 1.5 km due east and then 1.8 km south.

a What is its three-figure bearing from its starting point, to the nearest degree?

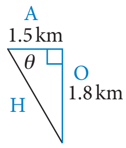
b How far is it from its starting point? Round your answer to one decimal place.

Steps | **Working**

a 1 Draw a diagram showing the information as a right-angled triangle. Include θ as an unknown angle in the triangle.



2 Use the triangle to find θ. Identify whether the labelled sides are opposite, adjacent or hypotenuse and select the matching trigonometric ratio.



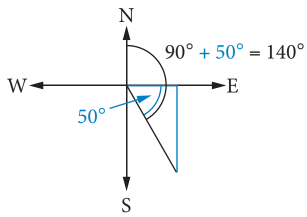
Use $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$

3 Substitute the known values and solve with CAS, rounding to the nearest whole unit.

$$\tan(\theta) = \frac{1.8}{1.5}$$

$$\theta = \tan^{-1}\left(\frac{1.8}{1.5}\right) \approx 50°$$

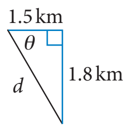
4 Mark the three-figure bearing angle on the diagram. Find the angle clockwise from north, adding or subtracting 90° angles if necessary. Write the answer to the required level of accuracy.



$90° + 50° = 140°$

The balloon is at a bearing of 140° from its starting point.

b 1 Use Pythagoras' theorem to find d, the distance from the starting point.



$$d^2 = 1.5^2 + 1.8^2$$
$$= 5.49$$
$$d = 2.343\ldots$$

2 Write your answer in the required units and round to the required level of accuracy.

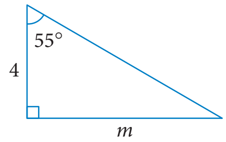
The balloon is 2.3 km from its starting point.

EXERCISE 10.2 Applying trigonometry

ANSWERS p. 518

Recap

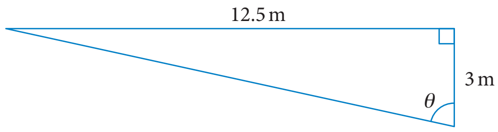
1 The value of the pronumeral, correct to two decimal places, in the diagram is



A 2.29 **B** 2.80 **C** 3.28

D 5.71 **E** 6.97

2 What calculation will give the value θ in the following triangle?



A $\theta = \tan^{-1}\left(\frac{12.5}{3}\right)$ **B** $\theta = \sin^{-1}\left(\frac{12.5}{3}\right)$ **C** $\theta = \cos^{-1}\left(\frac{3}{12.5}\right)$

D $\theta = \tan^{-1}\left(\frac{3}{12.5}\right)$ **E** $\theta = \sin^{-1}\left(\frac{3}{12.5}\right)$

Mastery

3 WORKED EXAMPLE 5 Find each of the following, to the nearest whole unit, by drawing a diagram.

a Lisbet stands 156 m from the base of a 180 m tower and looks up at the top of the tower. Find the angle of elevation, θ, of the top of the tower from Lisbet.

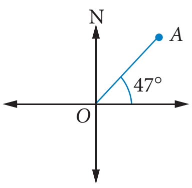
b From her apartment, 130 m above ground level, Zahiya sights the park at an angle of depression of 50°. What is the distance, d, to the park from the base of Zahiya's building?

c Jess is standing on her balcony, which is 60 m above street level. She spots her friend, Albie, on his balcony, which is 20 m above street level. If the two buildings are 35 m apart, what is the angle of depression, θ, of Albie from Jess?

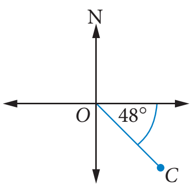
d A zipline has been built with a 30° angle of elevation from the ground to the top of a tower. The height of the tower is 35 m. How long is the zipline?

4 WORKED EXAMPLE 6 For each of the following, calculate the three-figure bearing of the point A and draw a diagram showing the angle.

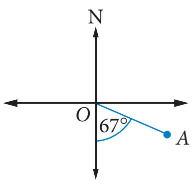
a



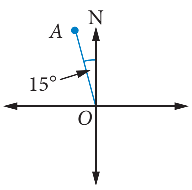
b



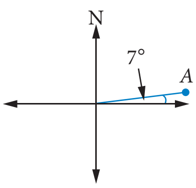
c



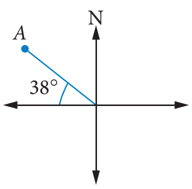
d



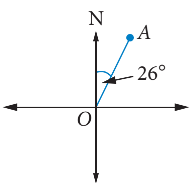
e



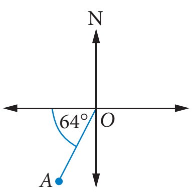
f



g



h



5 WORKED EXAMPLE 7 For each of the following, find

i the ship's three-figure bearing from its starting point to the nearest degree

ii how far the ship is from its starting point, rounding your answer to the nearest kilometre.

a A ship sails 22 km due east and then 10 km south.

b A ship sails due south for 20 km and due east for a further 15 km.

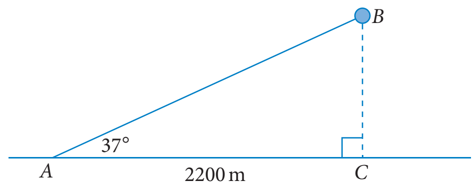
c A ship sails due west for 7 km and due north for 5 km.

d A ship sails due west for 50 km and due south for 12 km.

Exam practice

80–100% 60–79% 0–59%

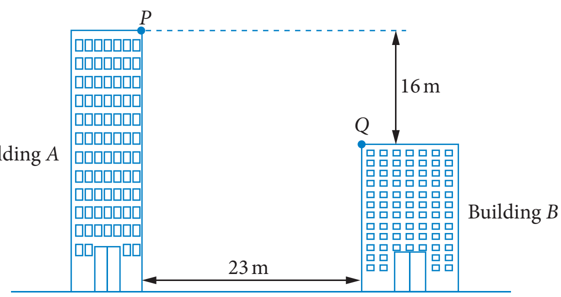
6 © VCAA 2007 1GTQ2 79% For an observer on the ground at A, the angle of elevation of a weather balloon at B is 37°. C is a point on the ground directly under the balloon. The distance AC is 2200 m. To the nearest metre, the height of the weather balloon above the ground is



A 1324 m **B** 1658 m **C** 1757 m

D 2919 m **E** 3655 m

7 © VCAA 2011 1GTQ2 71% The point Q on building B is visible from the point P on building A, as shown in the diagram. Building A is 16 metres taller than building B. The horizontal distance between point P and point Q is 23 metres. The angle of depression of point Q from point P is closest to



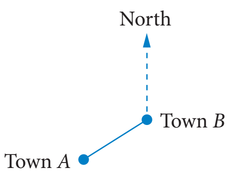
A 35° **B** 41° **C** 44° **D** 46° **E** 55°

8 © VCAA 2019N 1GMQ3 A waterfall in a national park is 4 km east of a camp site. A lookout tower is 4 km south of the waterfall. The bearing of the camp site from the lookout tower is

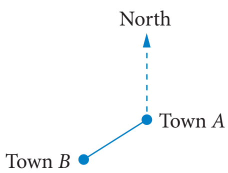
A 045° **B** 090° **C** 135° **D** 300° **E** 315°

9 © VCAA 2019 1GMQ2 68% Town B is located on a bearing of 060° from Town A. The diagram that could illustrate this is

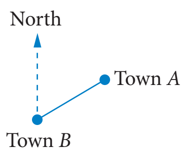
A



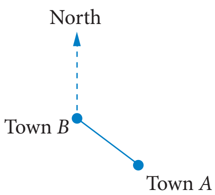
B



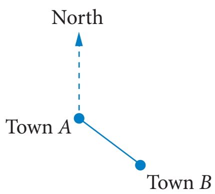
C



D

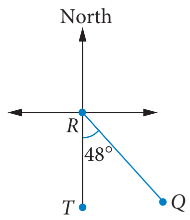


E



9780170448192

10 © VCAA 2009 1GTQ3 54% The locations of three towns, Q, R and T, are shown in the diagram. Town T is due south of town R. The angle TRQ is 48°. The bearing of town R from town Q is

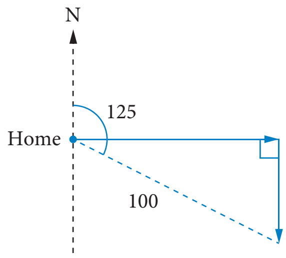


A 048° **B** 132° **C** 138°
D 228° **E** 312°

11 © VCAA 2015 1GTQ4 51% Town A is due west of town B. Town C is due south of town B. The bearing of town A from town C is

A between 000° and 090° **B** between 090° and 180° **C** exactly 135°
D between 180° and 270° **E** between 270° and 360°

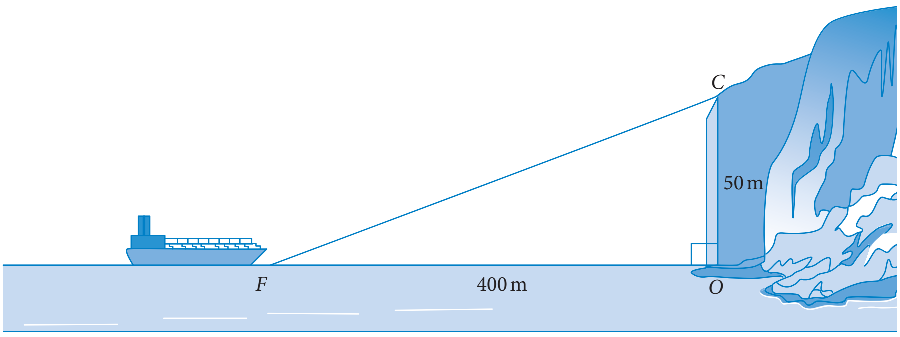
12 © VCAA 2018N 1GMQ2 Alan walked directly east from his home and then directly south. His final position was 100 m from his home on a bearing of 125°, as shown in the diagram.



The distance, in metres, that Alan walked directly east is closest to

A 42.8 **B** 57.3 **C** 81.9
D 142.8 **E** 181.9

13 © VCAA 2009 2GTQ1 78% (3 marks) A ferry, F, is 400 metres from point O at the base of a 50 metre high cliff, OC.

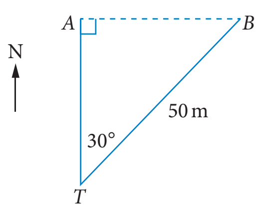


a Show that the slope of the line FC in the diagram is 0.125. 1 mark

b Calculate the angle of elevation of point C from F. Write your answer in degrees, correct to one decimal place. 1 mark

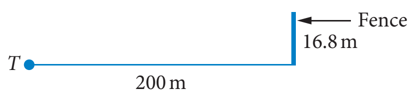
c Calculate the distance FC, in metres, correct to one decimal place. 1 mark

14 © VCAA 2016 2GMQ2 (2 marks) Salena practises golf at a driving range by hitting golf balls from point T. The first ball that Salena hits travels directly north, landing at point A. The second ball that Salena hits travels 50 m on a bearing of 030°, landing at point B. The diagram shows the positions of the two balls after they have landed.



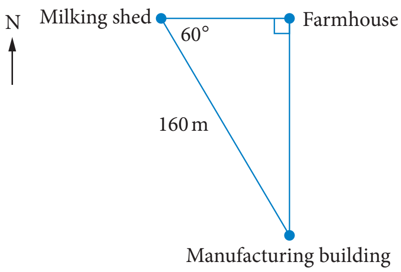
a 78% How far apart, in metres, are the two golf balls? 1 mark

b 66% A fence is positioned at the end of the driving range. The fence is 16.8 m high and is 200 m from the point T.



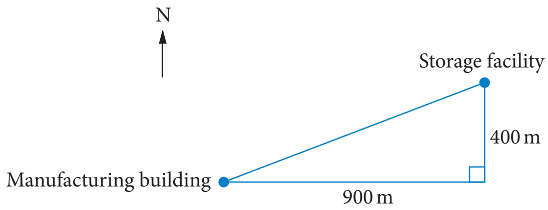
What is the angle of elevation from T to the top of the fence? Round your answer to the nearest degree. 1 mark

15 © VCAA 2017N 2GMQ2 (2 marks) The dairy farm has a farmhouse, a milking shed and a manufacturing building. The farmhouse is located due east of the milking shed. The manufacturing building is located due south of the farmhouse. The manufacturing building is 160 m from the milking shed, as shown.



a How far east of the milking shed is the manufacturing building located? 1 mark

b A storage facility is located 900 m east and 400 m north of the manufacturing building, as shown.



What is the bearing of the storage facility from the manufacturing building? Round your answer to the nearest degree. 1 mark

10.3 The sine and cosine rules

Video playlist The sine and cosine rules

Worksheets Trigonometric calculations

Discovering the sine rule

The sine rule – Finding lengths of sides

The sine rule – Finding angles

The cosine rule – Angles and sides

Applying trigonometry

Finding an unknown side

Finding an unknown angle

Sine rule problems

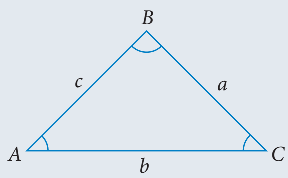
Cosine rule problems

The sine rule

So far, we have been using trigonometry to solve problems involving right-angled triangles. The **sine rule** can be used to find unknown side lengths and angles of non-right-angled triangles.

The sine rule

For the triangle

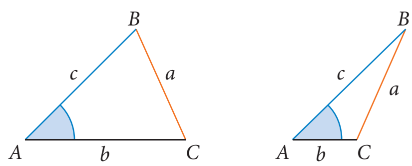


where A, B and C are the angles and a, b and c are the sides opposite each respective angle

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

The ambiguous case for the sine rule

When the sine rule is used to find an angle, there are sometimes two possible answers. This is called the **ambiguous case**. For example, these two triangles have two equal sides and one equal angle, but angle B and angle C are different in each triangle:

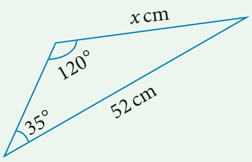


If this happens, we choose the correct answer by deciding from the diagram if the angle we are finding is less than 90° or greater than 90°.

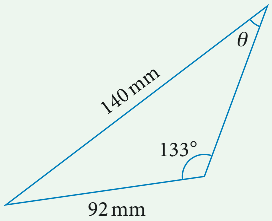
WORKED EXAMPLE 8 Using the sine rule for non-right-angled triangles

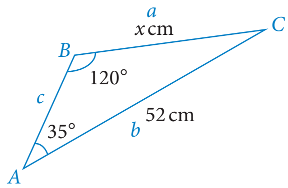
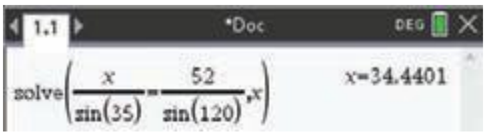
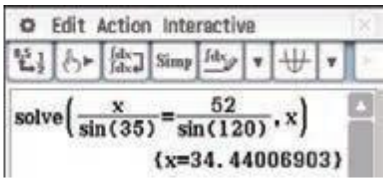
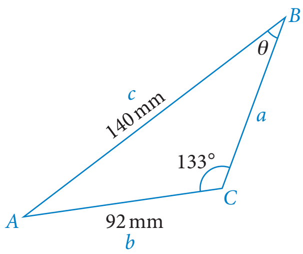
Find the unknown value for each of the following to the nearest whole unit.

a



b



Steps	Working
a **1** Label the angles A, B, C and label the sides opposite the angles a, b, c.	
2 List the known values and the value that needs to be found. Select the equality needed from the sine rule: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$	$A = 35°$, $B = 120°$, $a = x$, $b = 52$ $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$
3 Substitute the values into the sine rule and solve using CAS.	$\frac{x}{\sin(35)} = \frac{52}{\sin(120)}$
TI-Nspire 	**ClassPad** 
4 Write your answer in the required units and round to the required level of accuracy.	$x = 34$ cm
b **1** Label the angles A, B, C and label the sides opposite the angles a, b, c.	
2 List the known values and the value that needs to be found. Select the equality needed from the sine rule: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$	$B = \theta$, $C = 133°$, $b = 92$, $c = 140$ $\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$

10.3

WB

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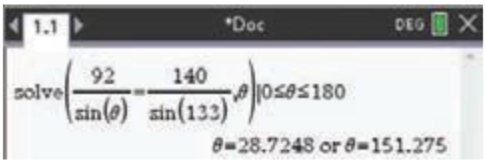
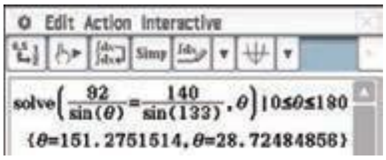
3 Substitute the values into the sine rule and solve using CAS.

$$\frac{92}{\sin(\theta)} = \frac{140}{\sin(133)}$$

TI-Nspire



ClassPad



4 Choose the correct answer for the ambiguous case by deciding from the diagram if θ is less than 90° or greater than 90°.

Write your answer in the required units and round to the required level of accuracy.

Two values are possible, to the nearest degree:

$\theta = 29°$ and $\theta = 151°$

θ is less than 90°.

$\theta = 29°$

WB
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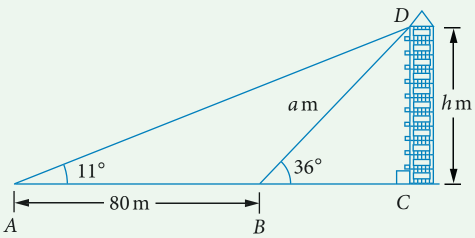
WORKED EXAMPLE 9 Solving problems using the sine rule

Ryan observes a tower at an 11° angle of elevation. Walking 80 m towards the tower, he finds that the angle of elevation increases to 36°.

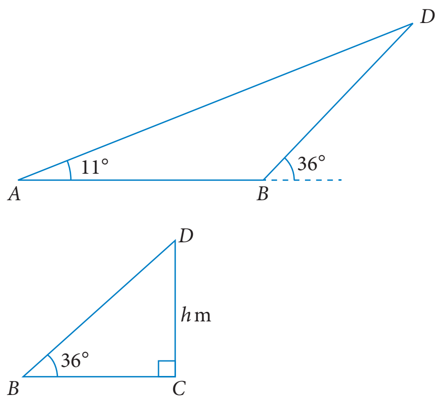
a Draw the two triangles involved and show the sizes of all the angles.

b Hence evaluate the height, h metres, of the tower correct to two decimal places.

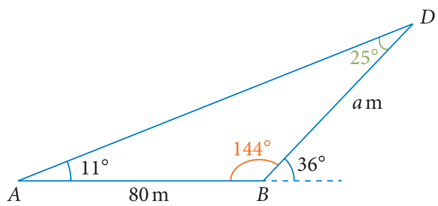


Steps | **Working**

a **1** Draw the two triangles separately showing all the information given in the diagram.

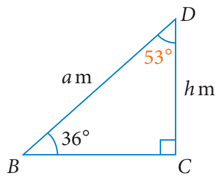


2 Use the fact that a straight angle is 180° and the angles of a triangle sum to 180°.



$180° - 36° = 144°$

$180° - 11° - 144° = 25°$



$180° - 36° - 90° = 53°$

b **1** Select the equality needed from the sine rule to find the unknown in the first triangle: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ Solve using CAS.	$\frac{a}{\sin(11)} = \frac{80}{\sin(25)}$ $a = 36.119\ldots$
2 Use the unknown length found and the sine rule to calculate the required value from the second triangle. Solve using CAS.	$\frac{36.119\ldots}{\sin(90)} = \frac{h}{\sin(36)}$ $h = 21.230\ldots$
3 Write your answer in the required units and round to the required level of accuracy.	The height of the tower is 21.23 m.

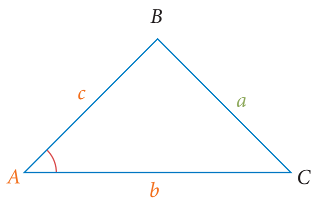
Exam hack

The sine rule also works for right-angled triangles. On CAS, it can be quicker to use the sine rule for a right-angled triangle than to use SOH-CAH-TOA.

The cosine rule

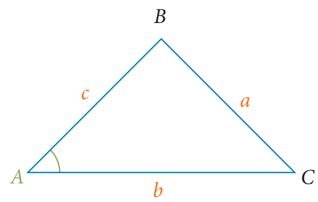
The **cosine rule** is another way to find unknown sides and angles in non-right-angled triangles.

If two sides and the angle between them are known, use the cosine rule to find the unknown side:



$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

If three sides are known, use the rearranged cosine rule to find any angle:



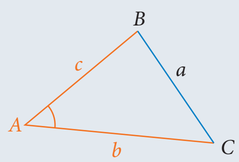
$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

The cosine rule

The cosine rule for triangle *ABC* can be used in two forms.

- When finding an unknown side $a^2 = b^2 + c^2 - 2bc\cos(A)$
- When finding an unknown angle $\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$

where angle *A* is the angle between sides *b* and *c*.



Exam hack

If $A = 90°$, then $\cos(A) = \cos(90°) = 0$, and so the cosine rule becomes $a^2 = b^2 + c^2$, which is Pythagoras' theorem.

WB

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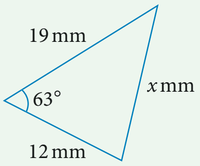
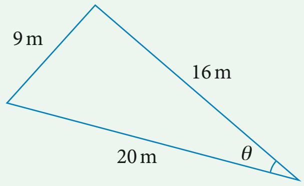
WORKED EXAMPLE 10 Using the cosine rule for non-right-angled triangles

Find the unknown value for each of the following to the nearest whole unit.

a



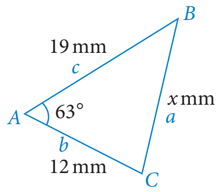
b



Steps | **Working**

a

1 Label the shown angle A, then label the other angles B and C and the sides opposite the angles a, b, c.



2 List the known values and the value that needs to be found.

Select the required version of the cosine rule.

$a^2 = b^2 + c^2 - 2bc\cos(A)$

or

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

Solve using CAS.

$A = 63°$, $a = x$, $b = 12$, $c = 19$

To find an unknown side use

$a^2 = b^2 + c^2 - 2bc\cos(A)$

$x^2 = 12^2 + 19^2 - 2 \times 12 \times 19 \times \cos(63°)$

$= 297.98\ldots$

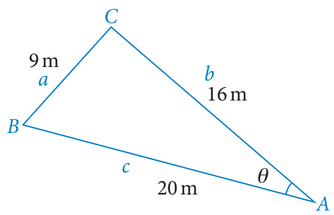
$x = 17.3\ldots$

3 Write your answer in the required units and round to the required level of accuracy.

$x = 17$ mm

b

1 Label the shown angle A, then label the other angles B and C, and the sides opposite the angles a, b, c.



2 List the known values and the value that needs to be found.

Select the required version of the cosine rule.

$a^2 = b^2 + c^2 - 2bc\cos(A)$

or

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

Solve using CAS.

$A = \theta$, $a = 9$, $b = 16$, $c = 20$

To find an unknown angle, use

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos(\theta) = \frac{16^2 + 20^2 - 9^2}{2 \times 16 \times 20} = 0.898\ldots$$

$\theta = \cos^{-1}(0.898\ldots)$

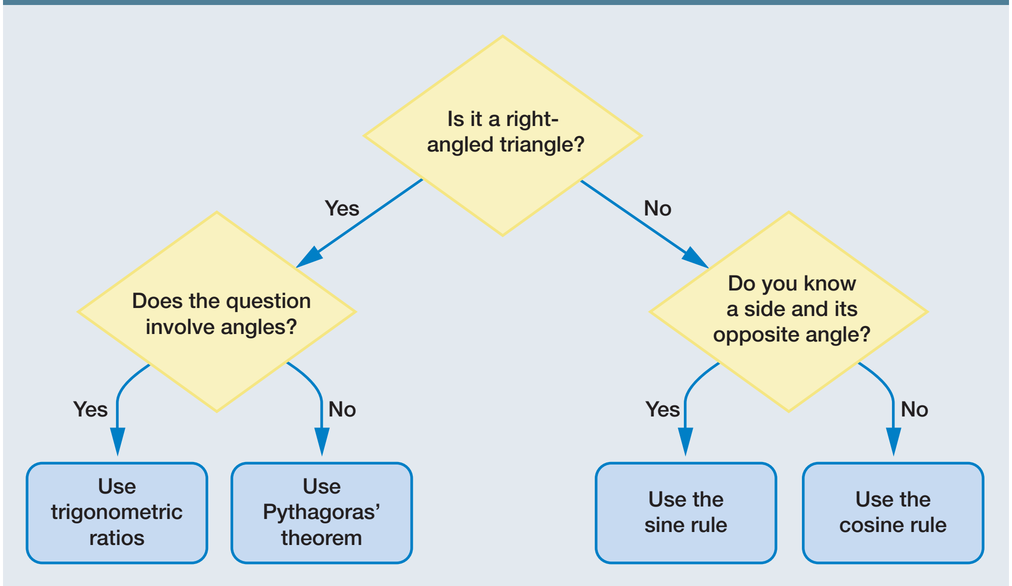
$\theta = 26.046\ldots$

3 Write your answer in the required units and round to the required level of accuracy.

$\theta = 26°$

Solving problems involving triangles

Selecting the method to find unknowns in a triangle



WORKED EXAMPLE 11 Solving problems involving non-right-angled triangles

WB p. 169

Putri is going on a three-day bushwalking adventure. She starts at camp site X and walks for 15 km on a bearing of 150° to camp site Y. On the second day, she walks 20 km due north to camp site Z. On the third day, Putri plans to return to camp site X.

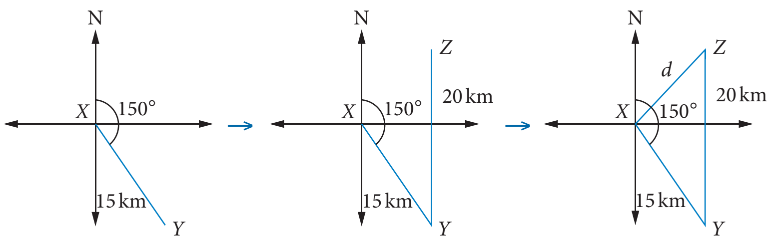
a Find the distance, to the nearest kilometre, that Putri needs to travel to return to camp site X.

b What three-figure bearing does Putri need to travel on in order to return to camp site X? Express your answer correct to the nearest degree.

Steps | **Working**

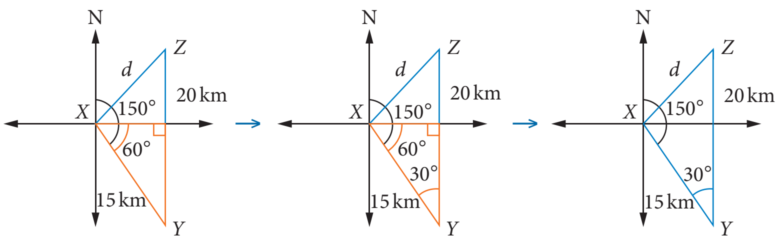
a 1 Draw a diagram showing the information as a triangle, including the unknown we are asked to find.

Day 1 | Day 2 | Day 3



2 Calculate an angle in the triangle.

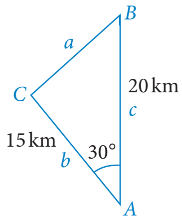
Use the bearing | Triangle angles sum to 180° | 30° is an angle in the triangle



3 Is it a right-angled triangle? — no

Is a side and its opposite angle known? — no → Use the cosine rule

4 Redraw the triangle. Label the shown angle A, then label the other angles B and C, and the sides opposite the angles a, b, c.



5 List the known values and the value that needs to be found.

Select the required version of the cosine rule.

$a^2 = b^2 + c^2 - 2bc\cos(A)$

or

$\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$

Solve using CAS.

$A = 30°$, $a = d$, $b = 15$, $c = 20$

To find an unknown side, use

$a^2 = b^2 + c^2 - 2bc\cos(A)$

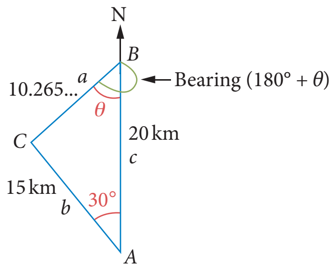
$d^2 = 15^2 + 20^2 - 2 \times 15 \times 20 \times \cos(30°)$

$d = 10.265\ldots$

6 Write your answer in the required units and round to the required level of accuracy.

The distance that Putri needs to travel to return to camp site X is 10 km.

b **1** Add north to the bearing starting point on the redrawn triangle, include the unrounded distance found in part **a**, and show the angle to find.



2 Is it a right-angled triangle?

no

Is a side and its opposite angle known?

yes → Use the sine rule

3 Select the equality needed from the sine rule to find the unknown in the first triangle:

$\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$

Solve using CAS, rounding to the nearest degree.

$a = 10.265\ldots$, $A = 30°$, $b = 15$, $B = \theta$

$\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)}$

$\dfrac{10.265}{\sin(30)} = \dfrac{15}{\sin(\theta)}$

$\theta \approx 47°$

4 Answer the question by calculating the three-figure bearing.

$180° + 47° = 227°$

The three-figure bearing is 227°.

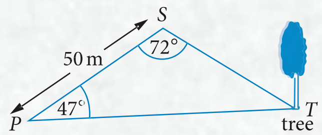
Exam hack

When you're solving problems involving non-right-angled triangles, you will usually need to remember that the angles inside a triangle add to 180° and that a straight-line angle is 180°.

VCE QUESTION ANALYSIS

© VCAA 2012 2GTQ3 2012 Examination 2 Geometry and trigonometry Question 3 (2 marks)

A tree is growing near a block of land. The base of the tree, T, is at the same level as the corners, P and S, of the block of land.



a Show that, correct to two decimal places, distance ST is 41.81 metres. 1 mark

b From point S, the angle of elevation to the top of the tree is 22°. Calculate the height of the tree. Write your answer, in metres, correct to one decimal place. 1 mark

Reading the question

- Part **a** is a 'show that' question.
- Part **b** involves an angle of elevation.

Thinking about the question

- Is the triangle in part **a** a right-angled triangle? If not, what are your options?
- Part **b** involves an angle above ground level, so draw the triangle from scratch rather than on top of the diagram in the question.
- Do you need to use your answer for part **a**?

Exam hack

If you need a part **a** answer for part **b** and you can't find the answer to part **a**, refer to it as x and show the steps you would take for part **b** using x in your working. You will get a mark for part **b** if your steps are correct.

Worked solution (✓ = 1 mark)

a This isn't a right-angled triangle. When only one side is given, use the sine rule.

First find the angle at the tree: 180° – 47° – 72° = 61°.

Use the sine rule to find the distance ST.

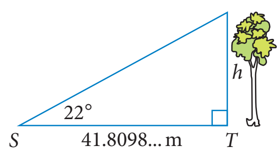
$$\frac{ST}{\sin(47°)} = \frac{50}{\sin(61°)}$$

Solve for ST.

$$ST = \frac{50 \times \sin(47°)}{\sin(61°)} \approx 41.8098\ldots \approx 41.81\text{ m} \ ✓$$

Save the calculator value for part **b**.

b Draw a triangle showing the angle of elevation from S to the top of the tree. Include the known side ST from part **b** and the side you need to find, h.



h is opposite and 41.8098… m (ST) is adjacent, so use tangent.

$$\tan(22°) = \frac{h}{41.8098...}$$

Solve using CAS, where necessary. Round your answer to one decimal place.

$$h = 41.8098\ldots \times \tan(22°)$$
$$= 16.8922\ldots\text{ m}$$
$$\approx 16.9\text{ m} \ ✓$$

10.3

Video playlist VCE question analysis: Applications of trigonometry

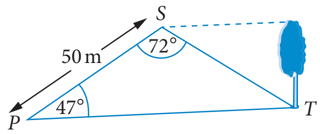
Student performance

80–100% | 60–79% | 0–59%

a 51% This ‘show that’ question required the end result of an appropriate calculation to be 41.81.

b 51% This question was very poorly answered by many students, who appeared to be unable to interpret the diagram and written description. The most common incorrect answer came from students who connected point S to the top of the tree, as shown in the diagram, then incorrectly assumed a right triangle had been formed with ST as the hypotenuse and then used

$\sin(22°) = \frac{x}{41.81} = 15.7.$

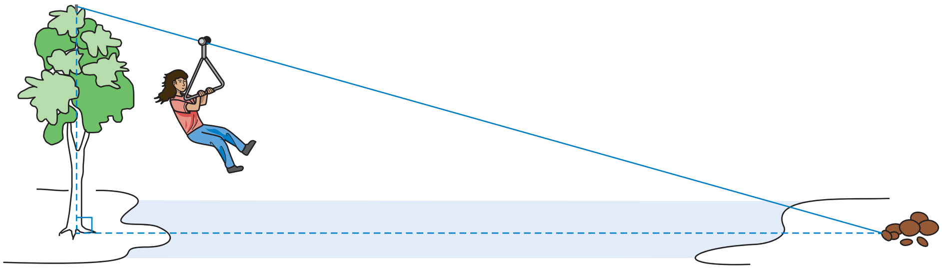


EXERCISE 10.3 The sine and cosine rules

ANSWERS p. 518

Recap

1 A flying fox is constructed between the top of a tree and a pile of rocks. It has a length of 34 m and a 38° angle of elevation. Which of the following is the (horizontal) distance between the tree and the pile of rocks, correct to one decimal place?



A 20.9 m **B** 26.0 m **C** 26.6 m **D** 26.7 m **E** 26.8 m

2 © VCAA 2015 1GTQ4 MODIFIED Town A is due south of town B. Town C is due east of town A. The bearing of town A from town C is

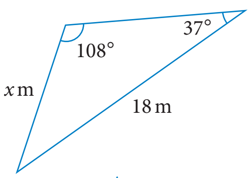
A between 000° and 090° **B** between 090° and 180° **C** exactly 270°

D between 180° and 270° **E** between 270° and 360°

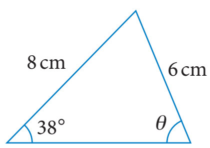
Mastery

3 WORKED EXAMPLE 8 Find the unknown value for each of the following to the nearest whole unit.

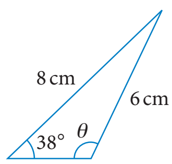
a



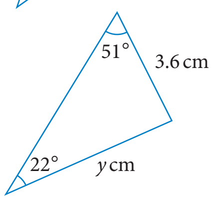
b



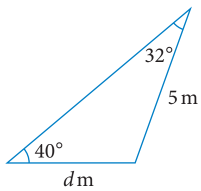
c



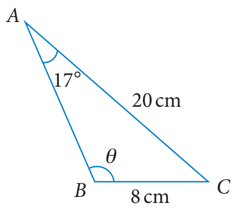
d



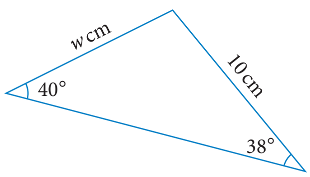
e



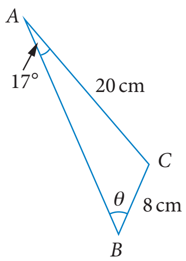
f



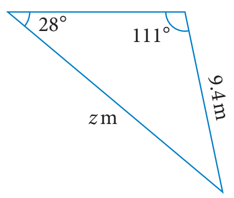
g



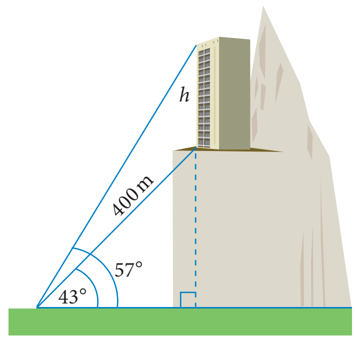
h



i



4 WORKED EXAMPLE 9 Reshan observes the top and bottom of a mountain resort to be at angles of elevation of 57° and 43° respectively. He is 400 m from the bottom of the resort.

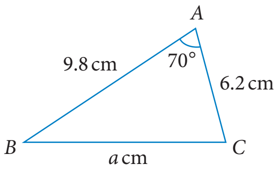


a Draw the two triangles involved and show the sizes of all the angles.

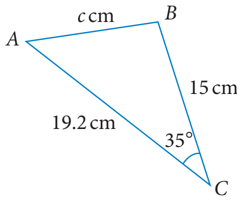
b Hence evaluate the height, h metres, of the resort correct to two decimal places.

5 WORKED EXAMPLE 10 Find the unknown value for each of the following to the nearest whole unit.

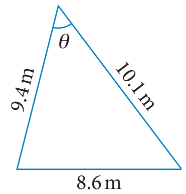
a



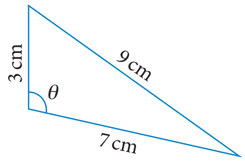
b



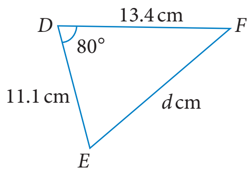
c



d



e



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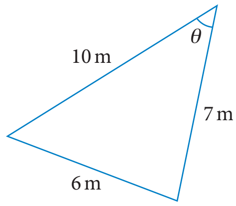

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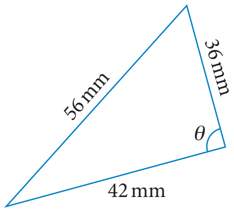

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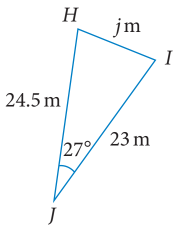

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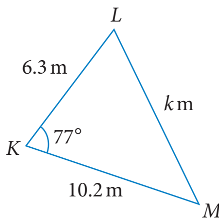


6 WORKED EXAMPLE 11 Abibi is going on a long distance run. He starts at point X and runs for 12 km on a bearing of 140° to point Y. Then he runs 16 km due north to point Z. He then decides to head back to his starting point, X.

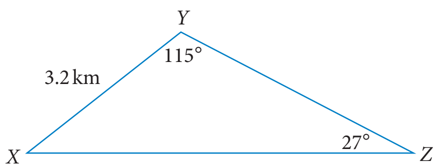
a Find the distance, to the nearest kilometre, that Abibi needs to travel to return to camp site X.

b What three-figure bearing does Abibi need to travel on in order to return to camp site X? Express your answer correct to the nearest degree.

Exam practice

80–100% 60–79% 0–59%

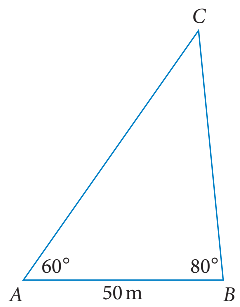
7 © VCAA 2004 1GTQ3 83% The diagram shows the route of a cross-country race. Point X lies due west of point Z. Given that the length XY is 3.2 km, the length XZ is closest to



A 1.5 km **B** 1.6 km **C** 6.4 km

D 7.0 km **E** 7.6 km

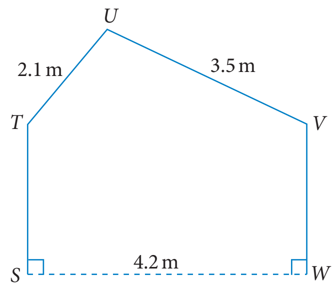
8 © VCAA 2011 1GTQ4 75% In triangle ACB, $\angle CAB = 60°$ and $\angle ABC = 80°$. The length of side $AB = 50$ m. The length of side AC is closest to



A 57 m **B** 67 m **C** 77 m

D 81 m **E** 100 m

9 © VCAA 2003 1GTQ8 64% A cross-section of a glass greenhouse is shown in the diagram. The sides of the glass panels TU and UV are 2.1 metres and 3.5 metres long respectively. The greenhouse is 4.2 metres wide. The walls ST and WV are vertical and equal in height.

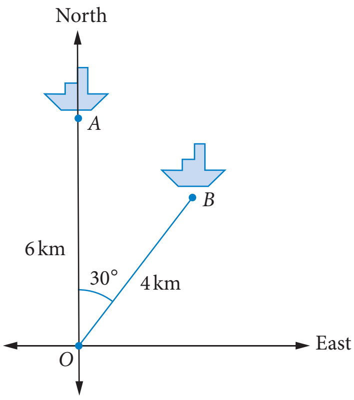


The size of $\angle TUV$ is

A 44.4° **B** 45.6° **C** 86.2°

D 93.8° **E** 109.6

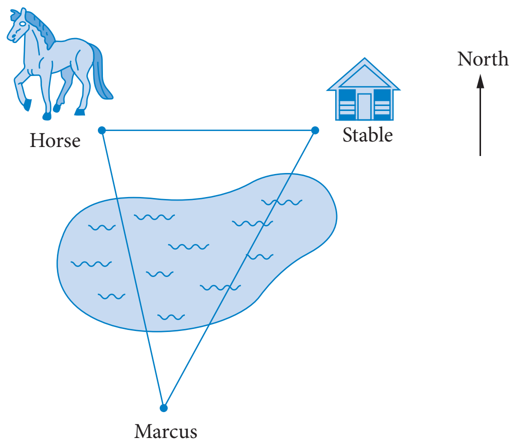
10 © VCAA 2005 1GTQ4 60% Two ships are observed from point O. At a particular time their positions A and B are as shown. The distance between the ships at this time is



A 3.0 km **B** 3.2 km **C** 4.5 km

D 9.7 km **E** 10.4 km

11 © VCAA 2016 1GMQ6 51% Marcus is on the opposite side of a large lake from a horse and its stable. The stable is 150 m directly east of the horse. Marcus is on a bearing of 170° from the horse and on a bearing of 205° from the stable.

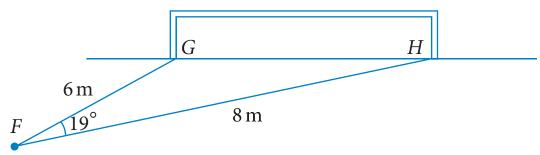


The straight-line distance, in metres, between Marcus and the horse is closest to

A 45 **B** 61 **C** 95

D 192 **E** 237

12 © VCAA 2002 1GTQ6 51% In a game of beach soccer, a player at point F is attempting to kick a goal. She is 6 metres from one goal post at G and 8 metres from the other goal post at H. From where she is standing, the angle through which she can shoot at the goal and still score a goal is 19°, as shown.



The distance between the goal posts, GH, is closest to

A 2.3 m **B** 2.6 m **C** 3.0 m

D 5.3 m **E** 9.2 m

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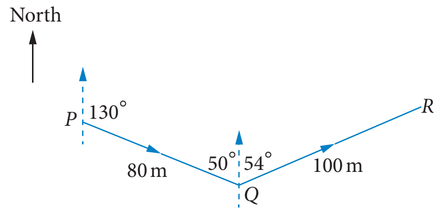
13 © VCAA 2016 2GMQ4 (3 marks) During a game of golf, Salena hits a ball twice, from P to Q and then from Q to R. The path of the ball after each hit is shown in the diagram below.

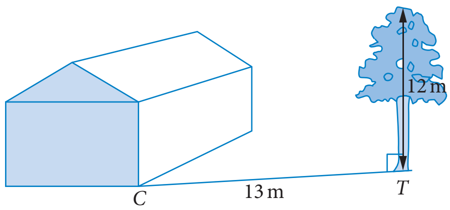


After Salena's first hit, the ball travelled 80 m on a bearing of 130° from point P to point Q.
After Salena's second hit, the ball travelled 100 m on a bearing of 054° from point Q to point R.

a 61% Another ball is hit and travels directly from P to R. Use the cosine rule to find the distance travelled by this ball. Round your answer to the nearest metre. 2 marks

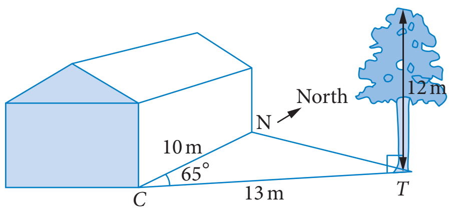
b 21% What is the bearing of R from P? Round your answer to the nearest degree 1 mark

14 © VCAA 2008 2GTQ3abc (3 marks) A tree, 12 m tall, is growing at point T near the shed. The distance, CT, from corner C of the shed to the centre base of the tree is 13 m.



a 58% Calculate the angle of elevation of the top of the tree from point C. Write your answer, in degrees, correct to one decimal place. 1 mark

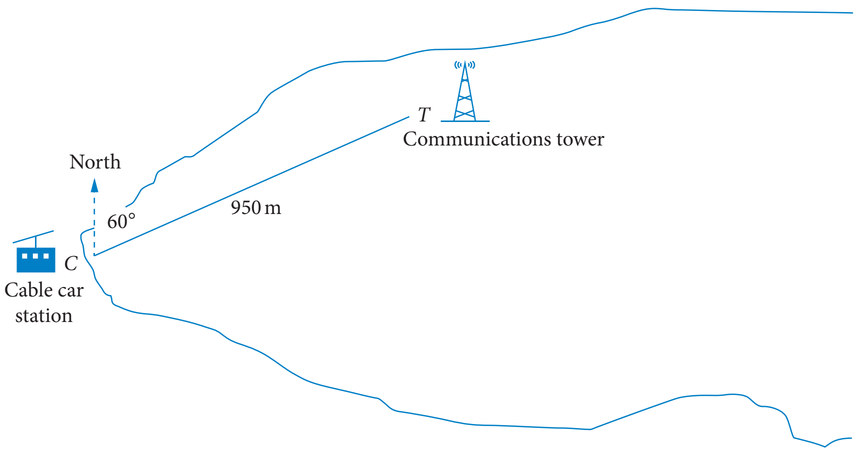
N and C are two corners at the base of the shed. N is due north of C. The angle, TCN, is 65°.



b 58% Show that, correct to one decimal place, the distance, NT, is 12.6 m. 1 mark

c Calculate the angle, CNT, correct to the nearest degree. 1 mark

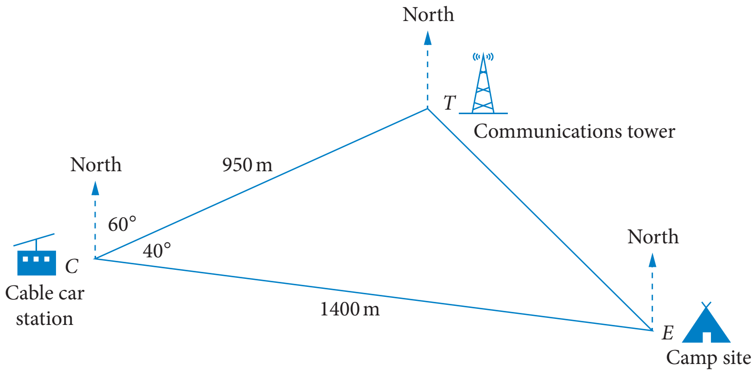
15 © VCAA 2015 2GTQ2 (4 marks) There are plans to construct a series of straight paths on the flat top of the mountain. A straight path will connect the cable car station at C to a communications tower at T, as shown in the diagram.



The bearing of the communications tower from the cable car station is 060°. The length of the straight path between the communications tower and the cable car station is 950 m.

a How far north of the cable car station is the communications tower? 1 mark

Paths will also connect the cable car station and the communications tower to a camp site at E, as shown.



The length of the straight path between the cable car station and the camp site is 1400 m. The angle TCE is 40°.

b **i** What will be the length of the straight path between the communications tower and the camp site? Write your answer correct to the nearest metre. 1 mark

ii Use the cosine rule to find the bearing of the camp site from the communications tower. Write your answer correct to the nearest degree. 2 marks

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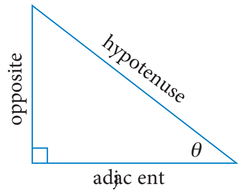
10 Chapter summary

Finding an unknown side of a right-angled triangle



Function	Abbreviation	Ratio	Initials
sine	sin	$\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$	SOH
cosine	cos	$\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	CAH
tangent	tan	$\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}}$	TOA

Inverse trigonometric functions

- Inverse trigonometric functions can be used to find unknown angles in right-angled triangles.

  If $\sin\theta = a$, then $\sin^{-1}(a) = \theta$.

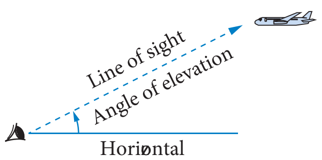
  If $\cos\theta = b$, then $\cos^{-1}(b) = \theta$.

  If $\tan\theta = c$, then $\tan^{-1}(c) = \theta$.

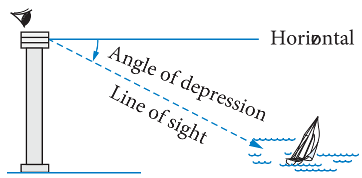
- When using CAS, make sure to set your calculator to degree mode.

Angles of elevation and depression

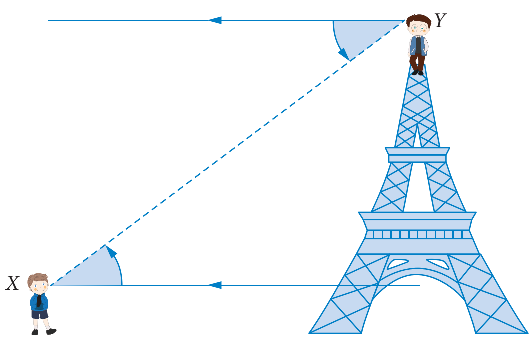
Angle of elevation – looking up



Angle of depression – looking down

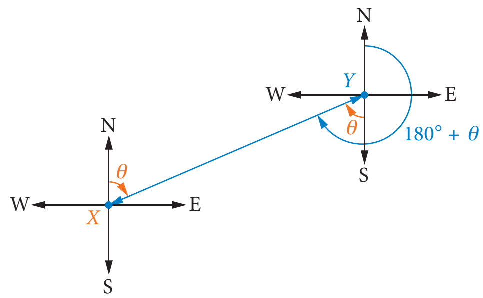


- angle of elevation X to Y = angle of depression Y to X



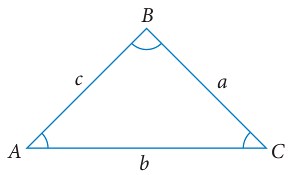
Three-figure bearings

- A **three-figure bearing** (also called a **true bearing**) is the angle measured in the clockwise direction from north.
- A three-figure bearing always has three figures (e.g. 5° is written as 005°).
- If the bearing of Y from X is θ, then the bearing of X from Y is $180° + \theta$, where $\theta < 180°$.



The sine rule

For the triangle



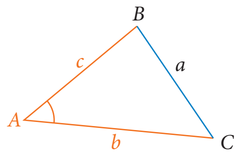
where A, B and C are the angles and a, b and c are the sides opposite each respective angle

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

The cosine rule

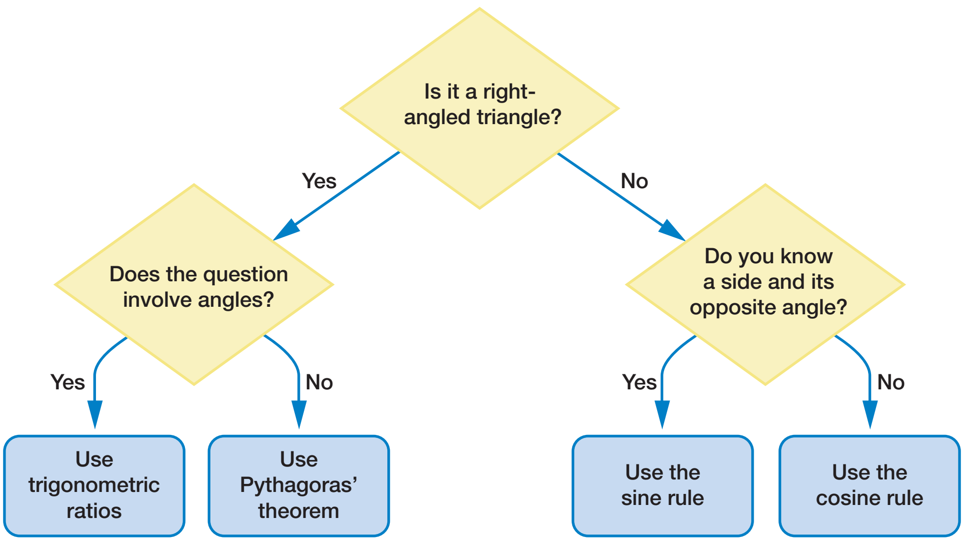
The **cosine rule** for triangle ABC can be used in two forms.

- When finding an unknown side $a^2 = b^2 + c^2 - 2bc\cos(A)$
- When finding an unknown angle $\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$



where angle A is the angle between sides b and c.

Selecting the method to find unknowns in a triangle



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Cumulative examination 1

Total number of marks: 16 Reading time: 7 minutes Writing time: 36 minutes

1 © VCAA 2003 1CQ3 The test scores obtained when 2500 students sit for an examination has a normal distribution with a mean of 64 and a standard deviation of 12. From this information we can conclude that the number of these students who obtained marks between 52 and 76 is closest to

A 68 **B** 95 **C** 850 **D** 1700 **E** 2375

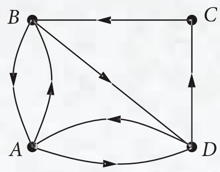
2 A sequence is defined by $u_0 = 4$, $u_{n+1} = -5u^n$. The rule for the nth value is

A $u_n = 4(-5)^n$ **B** $u_n = 4(5)^n$ **C** $u_n = -5(4)^n$ **D** $u_n = 5(4)^n$ **E** $u_n = 4\left(\frac{1}{5}\right)^n$

3 © VCAA 2017N 1GRQ2 Players at a football club pay a fee of $130 each year. They also pay a fee of $12 for every game they play in that year. Last year, Jenny paid a total of $262 in fees at this football club. How many games did Jenny play last year?

A 10 **B** 11 **C** 12 **D** 13 **E** 14

4 Which of the matrices represents the communication diagram?



A
$$\text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}}$$

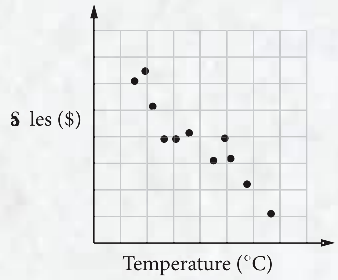
B
$$\text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}}$$

C
$$\text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}}}$$

D
$$\text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}}$$

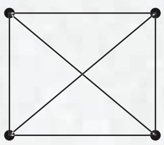
E
$$\text{Sender}\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \overset{\text{Receiver}}{\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}}$$

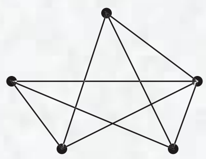
5 Which best describes the association between the variables *temperature* and *sales* in the following scatterplot in terms of strength and direction.

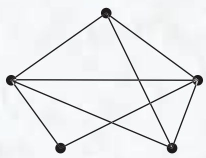


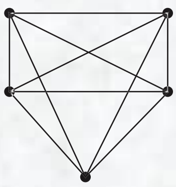
A strong and positive **B** moderate and positive **C** weak and negative
D moderate and negative **E** strong and negative

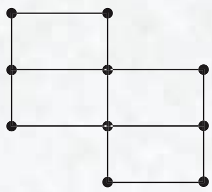
6 © VCAA 2018 1NQ6 Which one of the following graphs is **not** a planar graph?

A


B


C


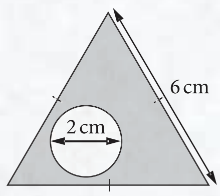
D


E


7 Which one of the following equations represents direct variation?

A $y = 11x + 6$ **B** $y = 11x - 6$ **C** $y = 6x + 11$ **D** $y = \dfrac{11x}{6}$ **E** $y = \dfrac{11}{6x}$

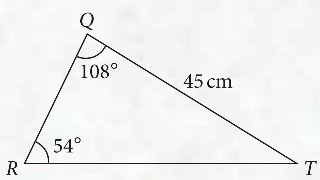
8 © VCAA 2010 1GTQ3 An equilateral triangle of side length 6 cm is cut from a sheet of cardboard. A circle is then cut out of the triangle, leaving a hole of diameter 2 cm as shown.



The area of cardboard **remaining**, as shown by the shaded region in the diagram, is closest to

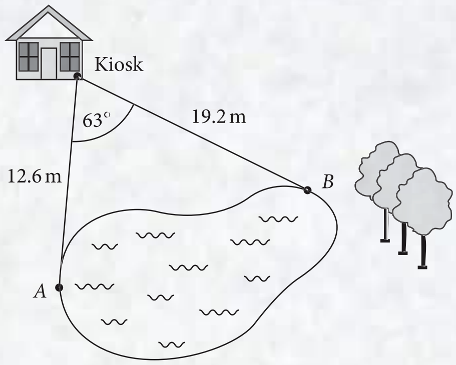
A 3 cm^2 **B** 9 cm^2 **C** 12 cm^2 **D** 15 cm^2 **E** 16 cm^2

9 © VCAA 2006 1GTQ2 The length of *RT* in the triangle shown is closest to



A 17 cm **B** 33 cm **C** 45 cm **D** 53 cm **E** 57 cm

10 © VCAA 2013 1GTQ2 The distances from a kiosk to points A and B on opposite sides of a pond are found to be 12.6 m and 19.2 m respectively. The angle between the lines joining these points to the kiosk is 63°. The distance, in m, across the pond between points A and B can be found by evaluating



A $\frac{1}{2} \times 12.6 \times 19.2 \times \sin(63°)$

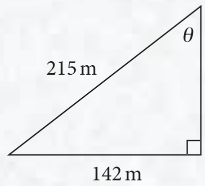
B $\frac{19.2 \times \sin(63°)}{12.6}$

C $\sqrt{12.6^2 + 19.2^2}$

D $\sqrt{12.6^2 + 19.2^2 - 2 \times 12.6 \times 19.2 \times \cos(63°)}$

E $\sqrt{s(s-12.6)(s-19.2)(s-63)}$, where $s = \frac{1}{2}(12.6 + 10.2 + 63)$

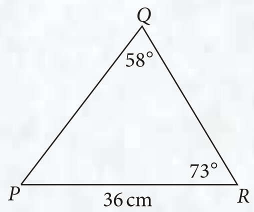
11 © VCAA 2006 1GTQ1 For the triangle shown, the size of angle θ is closest to



A 33° **B** 41° **C** 45°

D 49° **E** 57°

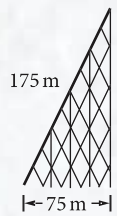
12 © VCAA 2003 1GTQ2 MODIFIED The length of PQ in the triangle is



A 31.9 cm **B** 34.4 cm **C** 40.6 cm

D 42.5 cm **E** 43.7 cm

13 A section of a rollercoaster is shown in the diagram. What is the angle of elevation of this section to the nearest degree?



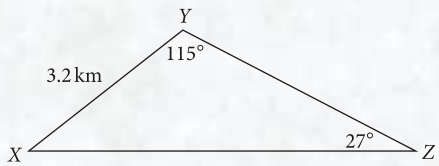
A 1° **B** 23° **C** 25°

D 65° **E** 69°

14 © VCAA 2013S 1GT1 The angle of depression of a car sighted from the top of a tower is 23°. The tower is 34.6 metres tall and the base of the tower and the car are at the same level. The horizontal distance of the car from the base of the tower is closest to

A 14.7 m **B** 15.7 m **C** 34.7 m **D** 81.5 m **E** 88.6 m

15 © VCAA 2004 1GTQ2 The diagram shows the route of a cross-country race. Point X lies due west of point Z. The bearing of point Y from point X is



A 038° **B** 052° **C** 063°

D 218° **E** 232°

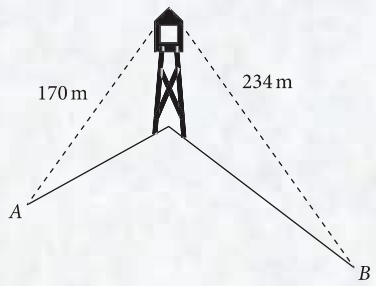
16 © VCAA 2020 1GMQ10 An 80 m high lookout tower stands in the centre of town. Two landmarks, on the same horizontal plane, are visible from the top of the lookout tower.

The direct distance from the top of the lookout tower to the base of Landmark A is 170 m.

The direct distance from the top of the lookout tower to the base of Landmark B is 234 m.

The bearing of Landmark B from Landmark A is 105°.

The bearing of Landmark B from the lookout tower is 142°.



The direct distance along the ground, in metres, between Landmark A and Landmark B is closest to

A 127 **B** 135 **C** 246 **D** 297 **E** 320

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Cumulative examination 2

Total number of marks: 18 Reading time: 5 minutes Writing time: 27 minutes

1 © VCAA 2008 2BRMQ4 MODIFIED (2 marks) Michelle intends to own a $17 000 car for 15 years.

If the car depreciates by $900 each year

a what will be the value of the car after 15 years? 1 mark

b determine the annual flat rate of depreciation, correct to one decimal place. 1 mark

2 (3 marks) The Technology Superstore sold a laptop for $1600 in August. They reduced all their prices by 20% for their September sale, and then increased all their September prices by 20% at the start of October.

a What is the price of the computer in September? 1 mark

b What is the price of the computer in October? 1 mark

c Explain why the 20% decrease and 20% increase didn't cancel each other out? 1 mark

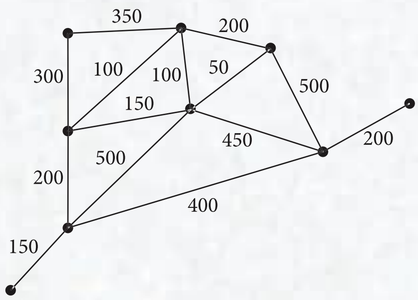
3 © VCAA 2019N 1MQ5 MODIFIED (1 mark) A population of birds feeds at two different locations, *A* and *B*, on an island. The change in the percentage of the birds at each location from year to year can be determined from the transition matrix *T* shown.

$$\begin{array}{c} \text{This year} \\ \begin{array}{cc} A & B \end{array} \end{array}$$

$$T = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \begin{matrix} A \\ B \end{matrix} \ \text{Next year}$$

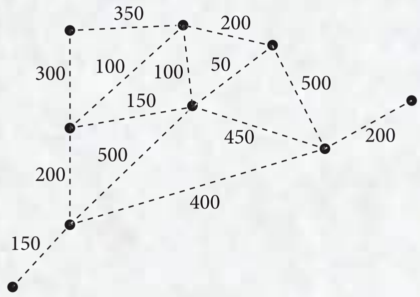
In 2018, 55% of the birds fed at location *B*. What percentage of the birds are expected to feed at location *A* in 2019?

4 © VCAA 2017 2NQ3 (2 marks) While on holiday, four friends visit a theme park where there are nine rides. On the graph shown, the positions of the rides are indicated by the vertices. The numbers on the edges represent the distances, in metres, between rides.



a Electrical cables are required to power the rides. These cables will form a connected graph. The shortest total length of cable will be used. Give a mathematical term to describe a graph that represents these cables. 1 mark

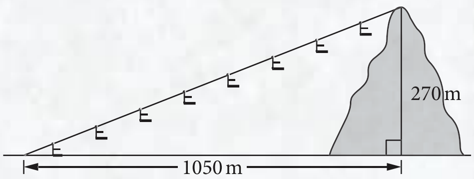
b Copy the following and use it to draw the graph that represents these cables. 1 mark



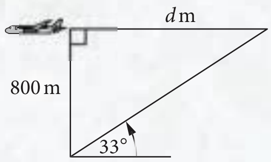
5 (2 marks) A chairlift takes skiers up to the top of a hill 270 m high and a horizontal distance of 1050 m away.

a How long is the chairlift's cable, correct to two decimal places, assuming the cable must go to the top and back? 1 mark

b Another chairlift with half the cable length comes up to the same peak from the other side of the hill. What is the horizontal distance from the start of this chairlift to the peak, correct to two decimal places? 1 mark



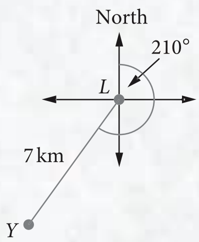
6 (2 marks) Milos observed a plane at a 33° angle of elevation. It flew at a constant height of 800 m until it was directly above him. Find each of the following values, correct to two decimal places.



a How far was the plane from Milos when he first saw it? 1 mark

b What was the distance, *d*, travelled by the plane? 1 mark

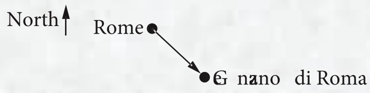
7 © VCAA 2009 2GTQ2ab (2 marks) A yacht, *Y*, is 7 km from a lighthouse, *L*, on a bearing of 210° as shown in the diagram.



a A ferry can also be seen from the lighthouse. The ferry is 3 km from *L* on a bearing of 135°. Copy the diagram and label the position of the ferry, *F*, and show an angle to indicate its bearing. 1 mark

b Determine the angle between *LY* and *LF*. 1 mark

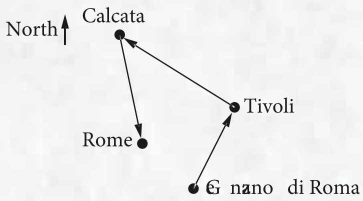
8 © VCAA 2019N 2GMQ3 (4 marks) A helicopter trip started in Rome and travelled directly to Genzano di Roma, as shown in the diagram.



Genzano di Roma is 37 km from Rome on a bearing of 153°.

a How far south of Rome is Genzano di Roma? Round your answer to the nearest kilometre. 1 mark

From Genzano di Roma, the helicopter then followed a route to Tivoli, Calcata and then back to Rome, as shown in the diagram.



Tivoli is 25 km from Genzano di Roma on a bearing of 016°. Calcata is 42 km from Tivoli on a bearing of 309°.

b Calculate the distance between Genzano di Roma and Calcata. Round your answer to the nearest kilometre. 2 marks

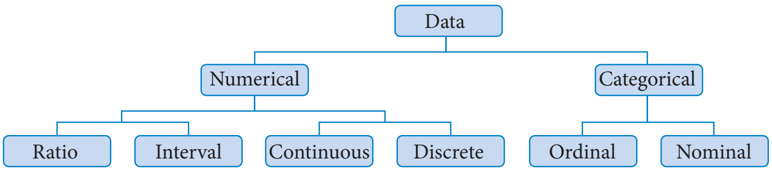
c Calcata is 40.3 km north and 11.8 km west of Rome. Calculate the bearing of Rome from Calcata. Round your answer to the nearest degree. 1 mark

Answers

CHAPTER 1

EXERCISE 1.1

1



2 **a** **i** numerical **ii** discrete, ratio
b **i** numerical **ii** continuous, ratio
c **i** numerical **ii** discrete, interval
d **i** numerical **ii** continuous, ratio
e **i** numerical **ii** continuous, ratio
f **i** categorical **ii** ordinal
g **i** categorical **ii** nominal
h **i** categorical **ii** ordinal
i **i** numerical **ii** discrete, ratio
j **i** categorical **ii** ordinal
k **i** numerical **ii** discrete, ratio
l **i** categorical **ii** nominal
m **i** numerical **ii** continuous, ratio
n **i** categorical **ii** ordinal
o **i** numerical **ii** continuous, interval
p **i** categorical **ii** nominal

3 **a** **i** 6.5 screens **ii** 4 and 5 screens
iii 13 screens
b **i** 15 marks **ii** 15 and 18 marks
iii 11 marks
c **i** 8 min **ii** 3 min
iii 17 min
d **i** 6 cm **ii** 5 cm and 6 cm
iii 2 cm

4 E **5** D **6** D **7** E
8 D **9** E **10** A **11** A
12 B

13 **a** **i** 23.6°C
ii No mode because every data value occurs once.
iii 9.9°C
b **i** 24.0°C **ii** 24.0°C **iii** 10.0°C
c Rounding early in a calculation makes a difference to your answers. 23.6°C in part **a** is the more accurate of the two medians because you lose accuracy if you round before the last step of a calculation.
d The 24°C in part **b** gives the most helpful information about the mode. Data rounded to a large number of decimal places will often give no information about the mode because the level of accuracy is so great, data values don't repeat.

EXERCISE 1.2

1 E **2** C

3 **a**

	Frequency	
Fish and chip packs	Number	Percentage
Kiddie pack	2	8%
Snack pack	9	36%
Family pack	7	28%
Hawaiian pack	6	24%
Potato cake pack	1	4%
Dim sim pack	0	0%
Total	25	100%

b

	Frequency	
Kebab types	Number	Percentage
lamb	5	31.25%
chicken	4	25%
mix	3	18.75%
falafel	2	12.5%
vegan	2	12.5%
Total	16	100%

4 **a**

Score	Frequency
0–<10	1
10–<20	3
20–<30	3
30–<40	4
40–50	5
Total	16

The modal interval is 40–<50.

b

Score	Frequency
0–<5	5
5–<10	0
10–<15	4
15–<20	3
20–25	3
Total	15

The modal interval is 0–<5.

c

Score	Frequency
0–<10	2
10–<20	2
20–<30	1
30–<40	5
40–<50	2
50–60	2
Total	14

The modal interval is 30–<40.

5 **a** 50 **b** Labor **c** 50 **d** 480

e 31.25%

6 E **7** B **8** D **9** C

10 B **11** C **12** C **13** B

14 **a** 10, 7, 8 **b** 32%

15 **a** 31 **b** 61%, $\frac{49+45}{153} \times 100 = 61.4379\ldots$

EXERCISE 1.3

1 D **2** D

3 **a** true **b** false **c** true **d** false

e false **f** true **g** true

4 **a** 8 **b** 75 **c** 15.4%

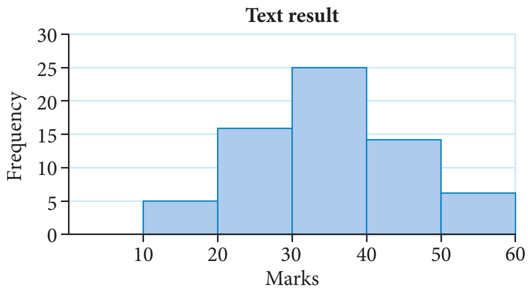
d positively skewed with a possible outlier

e 60–<120 minutes

f

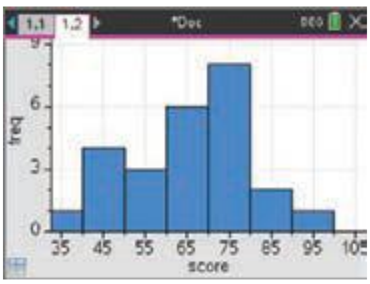
Amount of time studying (min)	Frequency
0–<60	35
60–<120	40
120–<180	27
180–<240	13
240–<300	16
300–<360	0
360–<420	0
420–<480	5
Total	**136**

5 **a**

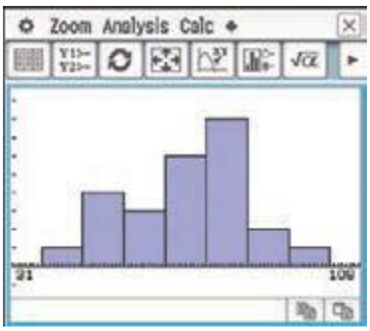


b The histogram is approximately symmetric.

6 TI-Nspire



ClassPad



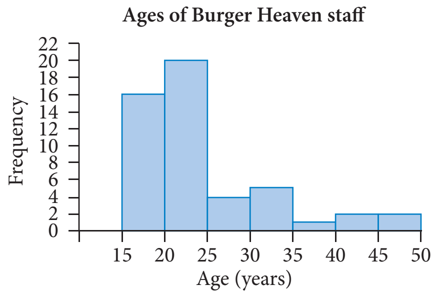
7 C **8** E **9** D **10** C

11 B

12 **a** 14 countries **b** 16.4%

13 **a** 50

b



c 4 **d** 8%

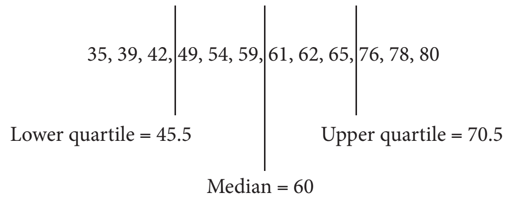
e Burger Heaven employs a greater proportion of younger people, particularly 15- to 25-year-olds.

EXERCISE 1.4

1 D **2** E

3 **a** min = 35, Q_1 = 45.5, median = 60, Q_3 = 70.5, max = 80

b



i $\frac{3}{12} = 25\%$ **ii** $\frac{6}{12} = 50\%$ **iii** $\frac{9}{12} = 75\%$

4 **a** **i** lower fence:

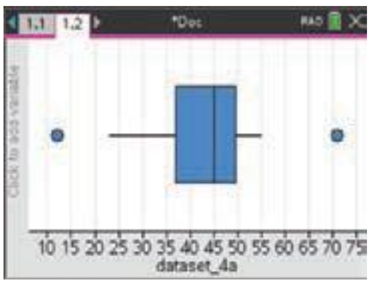
$Q_1 - 1.5 \times \text{IQR} = 37 - 1.5 \times 12.5 = 18.25$

upper fence:

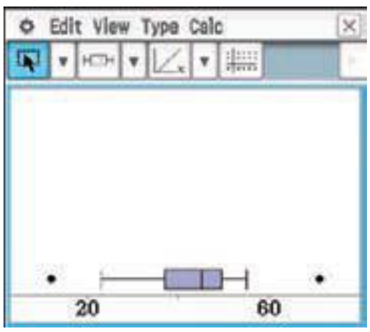
$Q_3 + 1.5 \times \text{IQR} = 49.5 + 1.5 \times 12.5 = 68.25$

12 is less than 18.25, so it's a possible outlier.
23 isn't less than 18.25, so it's *not* an outlier.
71 is greater than 68.25, so it's a possible outlier.

ii TI-Nspire



ClassPad



b **i** lower fence:

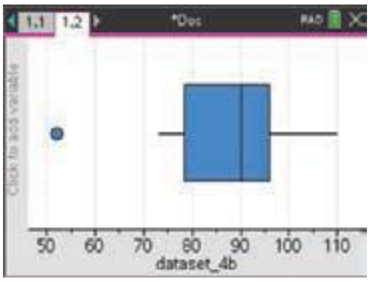
$Q_1 - 1.5 \times IQR = 78.5 - 1.5 \times 17.5 = 52.25$

upper fence:

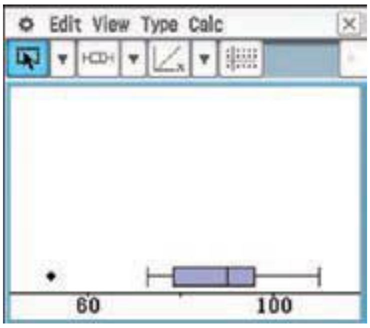
$Q_3 + 1.5 \times IQR = 96 + 1.5 \times 17.5 = 122.25$

52 is less than 52.25, so it's a possible outlier.
105 isn't greater than 122.25, so it's *not* an outlier.
110 isn't greater than 122.25, so it's *not* an outlier.

ii TI-Nspire



ClassPad



c **i** lower fence:

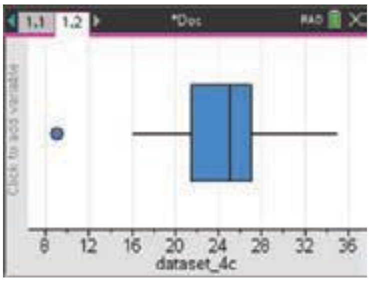
$Q_1 - 1.5 \times IQR = 21.5 - 1.5 \times 5.5 = 13.25$

upper fence:

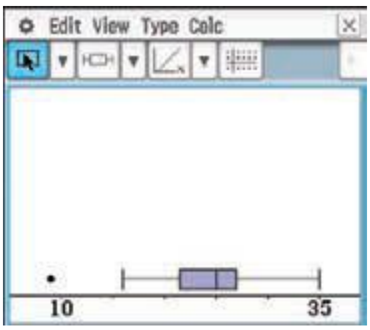
$Q_3 + 1.5 \times IQR = 27 + 1.5 \times 5.5 = 35.25$

9 is less than 13.25, so it's a possible outlier.
33 isn't greater than 35.25, so it's *not* an outlier.
35 isn't greater than 35.25, so it's *not* an outlier.

ii TI-Nspire



ClassPad



5 **a** **i** min = 4, $Q_1 = 5$, median = 6, $Q_3 = 7$, max = 9

ii 25% **iii** 100%

iv 25% **v** 15

vi scores less than 2 **vii** scores greater than 10

b **i** min = 5, $Q_1 = 7$, median = 9, $Q_3 = 10$, max = 14

ii 75% **iii** 75%

iv 25% **v** 0

vi scores less than 2.5 **vii** scores greater than 14.5

c **i** min = 3, $Q_1 = 5$, median = 7, $Q_3 = 9$, max = 10

ii 50% **iii** 100%

iv 25% **v** 15

vi No scores at the lower end would be considered outliers.

vii No scores at the upper end would be considered outliers.

6 **a** positively skewed; the box and whisker in the positive direction are longer than the box and whisker in the negative direction.; no outliers

b approximately symmetric; median approximately in the middle of box and whiskers about the same length; one outlier shown by dot

c negatively skewed; the box and whisker in the negative direction are longer than the box and whisker in the positive direction; two outliers shown by dots

7 C **8** D **9** B **10** E

11 D **12** B **13** D

14 **a** 12 **b** 15 **c** positively skewed

d 50% **e** 90

15 **a** $Q_3 = 52$ **b** IQR = 24 **c** 76

d The upper fence is $Q_3 + 1.5 \times IQR = 52 + 1.5 \times 24 = 88$. An outlier is a value greater than the upper fence. The dot shown is 76 which is less than 88, meaning it should not be drawn as an outlier.

EXERCISE 1.5

1 D **2** C

3 **a** **i** 2 **ii** 5 **iii** 2

iv 1 **v** 4 **vi** 3

b approximately symmetric

4 **a** **i** 31 **ii** 54 **iii** 34

iv 29 **v** 42.5 **vi** 13.5

b 63 and 77 are possible outliers

5 **a** **i** 25

ii upper fence $= Q_3 + 1.5 \times IQR$
$= 40 + 1.5 \times 25$
$= 77.5$

60 < 77.5, so 60 is **not** an outlier.

b **i** 10

ii lower fence $= Q_1 - 1.5 \times IQR$
$= 90 - 1.5 \times 10$
$= 75$

60 < 75, so 60 is an outlier.

c **i** 11

ii upper fence = $Q_3 + 1.5 \times \text{IQR}$
$= 37 + 1.5 \times 11$
$= 53.5$

60 > 53.5, so 60 is an outlier.

d **i** 12

ii lower fence = $Q_1 - 1.5 \times \text{IQR}$
$= 84.5 - 1.5 \times 12$
$= 66.5$

60 < 66.5, so 60 is an outlier.

6 **a**

Stem	Leaf
3	5 4 7 8
4	0 1 4 9
5	1 3 3 3 4 6 7
6	0 2 2 8
7	3 5 9

Key: 6 | 8 = 68

b 22 matches **c** 79 **d** 59%

e approximately symmetric

7 C **8** B **9** A **10** B

11 B **12** B

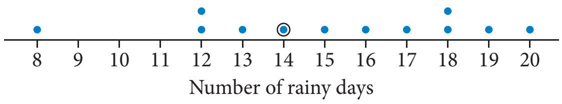
13 **a** mode = 78, range = 9

b $Q_1 = 75$, $Q_3 = 78$, IQR = 78 − 75 = 3

$Q_1 - 1.5 \times \text{IQR} = 75 - 1.5 \times 3 = 70.5$

Therefore, 70 is an outlier because it is less than 70.5.

14 **a**



b **i** 15.5 **ii** 92%

15 **a** **i** 25.0 **ii** 28.2 years

b 1.1 years

c $Q_1 - 1.5 \times \text{IQR} = 29.9 - 1.5 \times 1.1 = 28.25$

26.0 < 28.25, so the age of 26.0 would be shown on a boxplot as an outlier.

EXERCISE 1.6

1 D **2** E

3 **a**

Camera 1		Camera 2
9 8 8 4	6	
6 5 5 4 3 2	7	
9 8 4 4 3 2 2	8	4 8
5 1	9	2 6 6 7
5	10	0 0 3 9 9 9
	11	0 0 1 4 5 6 8 9
	12	
	13	
Key: 4 | 8 = 84 km/h	14	Key: 8 | 4 = 84 km/h

Other numbers can be used as keys.

b negatively skewed

c Camera 1: median = 79 km/h, range = 41 km/h, IQR = 13.5 km/h

Camera 2: median = 109 km/h, range = 35 km/h, IQR = 16 km/h

d The second road has the **greater speeding problem**. The median speed for the first road is 79 km/h, which is 1 km/h below the 80 km/h speed limit. The median speed for the second road is 109 km/h, which is 9 km/h above the 100 km/h speed limit.

4 **a**

Irina		Steven
	1	
	1	6
4 2	2	2
9 9 8 7	2	5 6
3 3 2 2	3	0 2 2 3
7 6	3	8 8 9 9

Key: 2 | 3 = 32 pamphlets Key: 3 | 2 = 32 pamphlets

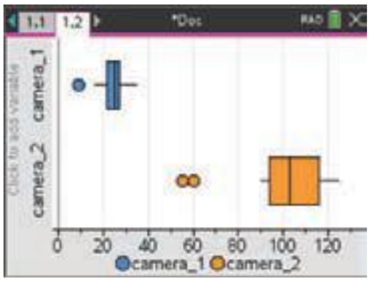
Other numbers can be used as keys.

b negatively skewed

c Irina: median = 30.5 range = 15 IQR = 5.5
Steven: median = 32 range = 23 IQR = 12.5

d Although the medians (30.5 and 32) are similar, Irina's range (15) and IQR (5.5) are considerably lower than Steven's range (23) and IQR (12.5). This means Steven's deliveries have more variability and are less consistent than Irina's, which indicates that Irina is the better delivery person.

5 TI-Nspire



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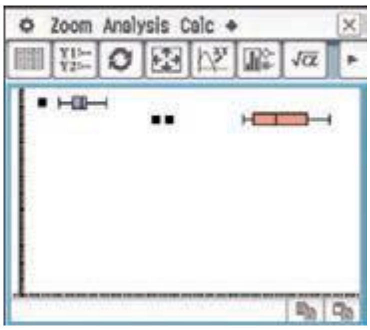


6 **a** Colebrook **b** Ashville

c Ballinga **d** Ballinga

e Colebrook has noticeably higher average July temperatures than Ashville and Ballinga. Colebrook's median (26°C) is much higher than Ashville's (10°C) and Ballinga's (12°C).

7 D **8** B **9** A **10** C

11 C **12** C **13** D

14 **a** The wind direction with the lowest recorded wind speed was **south-east**.

The wind direction with the largest range of recorded wind speeds was **north-east**.

b north-east

c north-west, south-west and west

d north-west and south-west

EXERCISE 1.7

1 B **2** E

3 **a** **i** 13 **ii** 3.5

b **i** 50 **ii** 2.9

4 $\bar{x} = 38.07, s = 4.91$ **5** $\bar{x} = 7.31, s = 1.89$

6 $\bar{x} = 26.1$, median = 26, mode = 24, range = 11, IQR = 4.5 and $s_x = 3.1$

7 **a** **i** $\bar{x} = 19.27, s = 2.05$ **ii** 7 data values

b **i** $\bar{x} = 27.31, s = 9.78$ **ii** 9 data values

8 **a** 34%% **b** 0.15% **c** 49.85% **d** 50%

9 E **10** B **11** D **12** C

13 C **14** C

15 **a** 6.8 **b** 1.91 **c** 15

CUMULATIVE EXAMINATION 1

1 C 91% **2** B 88% **3** D 75%

4 B 74% **5** E 57% **6** B

7 C 57% **8** E 54% **9** D

10 C

CUMULATIVE EXAMINATION 2

1 **a** positively skewed

b **i** 26 **ii** $\frac{20}{103} = 19.4\%$ 70%

2 **a** 30.8 mm

b upper fence = 31.6 + 1.5 × 1.3 = 33.55

Since 33.5 < 33.55, 33.5 mm is not an outlier.

3 **a** estimates: 10 cm; actual: 26 cm

b estimates: 34 cm; actual: 11 cm

c estimates: 21 cm; actual: 4 cm

d estimates: positively skewed; actual: approximately symmetrical

e The estimates median of 10 cm is considerably less than the actual median of 26 cm. This supports Edgar's theory that many people tend to underestimate the length of a piece of string.

4 **a** **i** $\bar{x} = 39.43, s = 9.83$ **ii** 10 data values

b 68% **c** 71%

CHAPTER 2

EXERCISE 2.1

1 **a** **i** 4, 9, 14, 19, 24

ii Add 5 to generate each new value.

b **i** 1.5, 2, 2.5, 3, 3.5

ii Add 0.5 to generate each new value.

c **i** 2, 1, 0, −1, −2

ii Subtract 1 to generate each new value.

d **i** 1, 3, 9, 27, 81

ii Multiply by 3 to generate each new value.

e **i** 20, 15, 10, 5, 0

ii Subtract 5 to generate each new value.

f **i** 1, −3, 9, −27, 81

ii Multiply by −3 to generate each new value.

2 **a** constant **b** increasing

c decreasing **d** oscillating

e constant **f** increasing

g decreasing, limiting value **h** oscillating

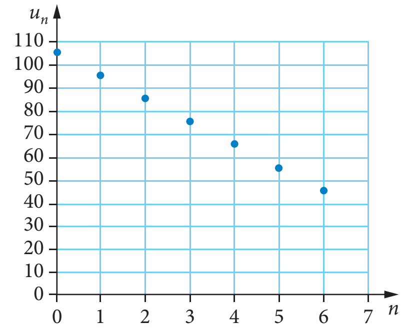
3 **a** **i** Each new value is generated by subtracting 10 to the previous value.

ii $u_2 = 85, u_5 = 55$

iii

n	0	1	2	3	4	5	6
u_n	105	95	85	75	65	55	45

iv



v slope = −10, which is the amount being added to generate each new value.

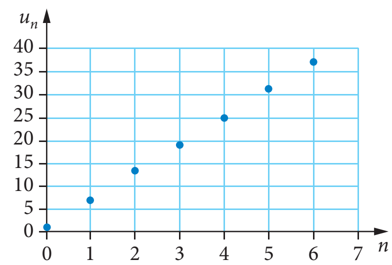
b **i** Each new value is generated by adding 6 to the previous value.

ii $u_2 = 13, u_5 = 31$

iii

n	0	1	2	3	4	5	6
u_n	1	7	13	19	25	31	37

iv



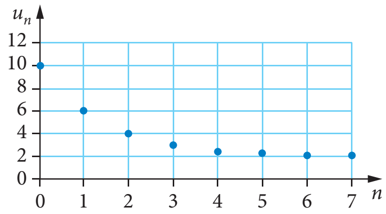
v slope = 6, which is the amount being added to generate each new value.

4 A **5** D **6** B **7** E

8 B **9** A **10** E

11 **a** There is no fixed amount being added or subtracted each time.

b 2.031 25

c 

d 2

e decreasing, limiting value

EXERCISE 2.2

1 E **2** C

3 a i $a = 8, d = 2$

ii $u_0 = 8$,
$u_1 = u_0 + 2 = 8 + 2 = 10$,
$u_2 = u_1 + 2 = 10 + 2 = 12$,
$u_3 = u_2 + 2 = 12 + 2 = 14$
The first four values in the sequence are 8, 10, 12, 14.

b i $a = 1, d = -3$

ii $u_0 = 1$,
$u_1 = u_0 - 3 = 1 - 3 = -2$,
$u_2 = u_1 - 3 = -2 - 3 = -5$,
$u_3 = u_2 - 3 = -5 - 3 = -8$
The first four values in the sequence are 1, −2, −5, −8.

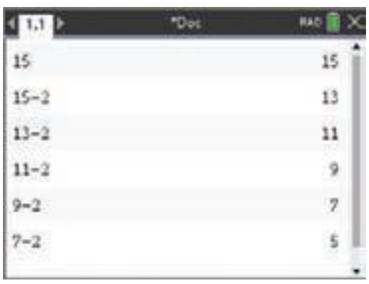
c i $a = 20, d = 5$

ii $u_0 = 20$
$u_1 = u_0 + 5 = 20 + 5 = 25$
$u_2 = u_1 + 5 = 25 + 5 = 30$
$u_3 = u_2 + 5 = 30 + 5 = 35$
The first four values in the sequence are 20, 25, 30, 35.

d i $a = -3, d = -4$

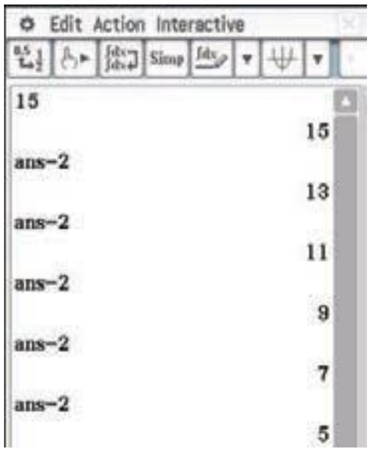
ii $u_0 = -3$
$u_1 = u_0 - 4 = -3 - 4 = -7$
$u_2 = u_1 - 4 = -7 - 4 = -11$
$u_3 = u_2 - 4 = -11 - 4 = -15$
The first four values in the sequence are −3, −7, −11, −15.

4 a TI-Nspire



It is an arithmetic sequence.

ClassPad

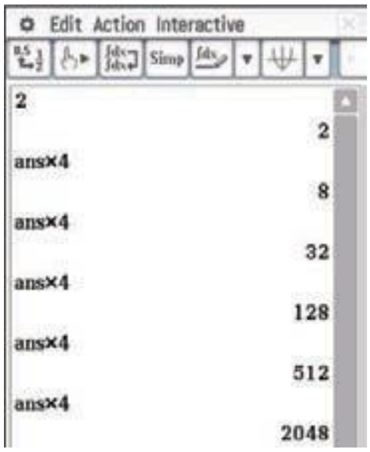


It is an arithmetic sequence.

b TI-Nspire

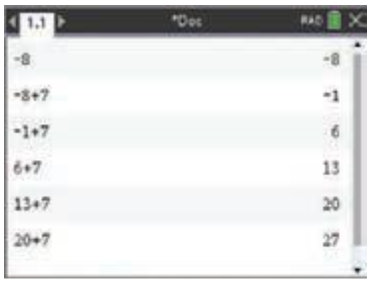
It is not an arithmetic sequence.

ClassPad



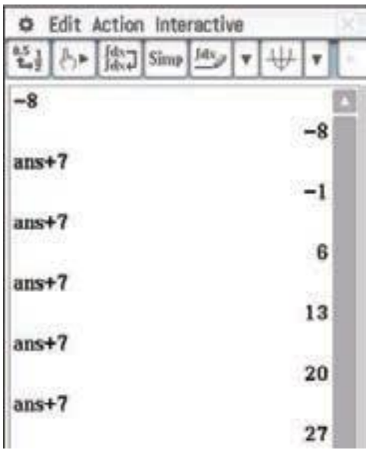
It is not an arithmetic sequence.

c TI-Nspire



It is an arithmetic sequence.

ClassPad

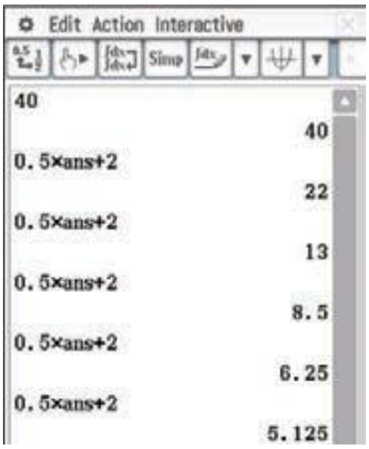


It is an arithmetic sequence.

d TI-Nspire

It is not an arithmetic sequence.

ClassPad



It is not an arithmetic sequence.

5 a i We are adding 2 to generate each new value, so it is an arithmetic sequence.

ii $u_n = 5 + 2n$ **iii** $u_{10} = 25$, $u_{50} = 105$

b i We are subtracting 7 to generate each new value, so it is an arithmetic sequence.

ii $u_n = 100 - 7n$ **iii** $u_{10} = 30$, $u_{50} = -250$

c i We are subtracting 4 to generate each new value, so it is an arithmetic sequence.

ii $u_n = 20 - 4n$ **iii** $u_{10} = -20$, $u_{50} = -180$

d i We are adding 10 to generate each new value, so it is an arithmetic sequence.

ii $u_n = 45 + 10n$ **iii** $u_{10} = 145$, $u_{50} = 545$

6 a $a = 11, d = 8$ **b** $a = -10, d = 6$

c $a = 31, d = -2$ **d** $a = -9, d = -4$

7 a 1, 8, 15, 22, 29 **b** 10, 4, −2, −8, −14

c 9, 12, 15, 18, 21 **d** −2, −6, −10, −14, −18

8 a $u_{10} = -30$, $u_{20} = -110$ **b** $u_{10} = -106$, $u_{20} = -196$

c $u_{10} = 114$, $u_{20} = 224$ **d** $u_{10} = 50$, $u_{20} = 120$

9 a $u_0 = -4$, $u_{n+1} = u_n + 3$ **b** $u_0 = 30$, $u_{n+1} = u_n - 9$

c $u_0 = 1$, $u_{n+1} = u_n + 2.5$ **d** $u_0 = 3.8$, $u_{n+1} = u_n - 1.1$

10 A **11** D **12** A **13** B

14 D **15** E

16 a arithmetic; we are subtracting 40 to generate each new value

b $d = -40$

c

n	0	1	2	3
u_n	100	60	20	−20

d $u_n = 100 - 40n$

e $u_{48} = 100 - 40 \times 48 = 100 - 1920 = -1820$

EXERCISE 2.3

1 B **2** B

3 a $120

b i

n	Account balance after n years ($)
0	3000
1	3000 + 120 = 3120
2	3120 + 120 = 3240
3	3240 + 120 = 3360
4	3360 + 120 = 3480

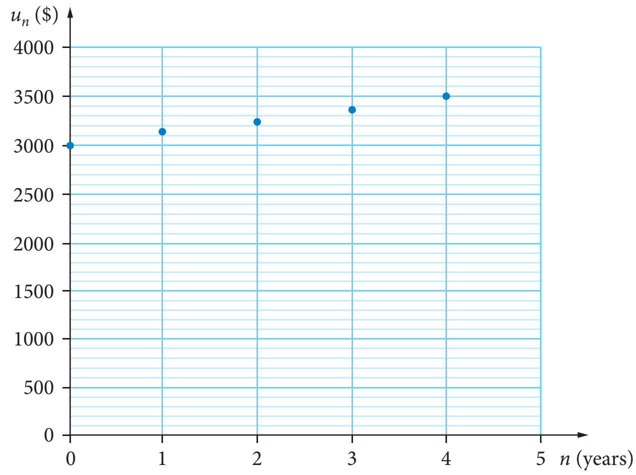
The total amount earned after four years is $3480.

ii after three years **iii** $480

c Let u_n = the account balance after n years.

$u_0 = 3000$, $u_{n+1} = u_n + 120$

d

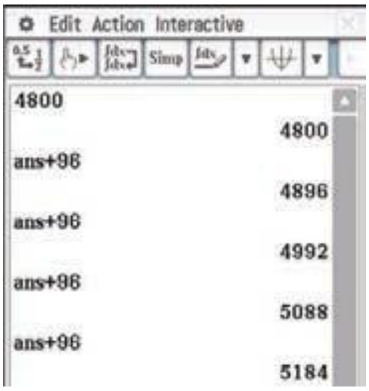


The points form an increasing straight line.

4 a \$5184

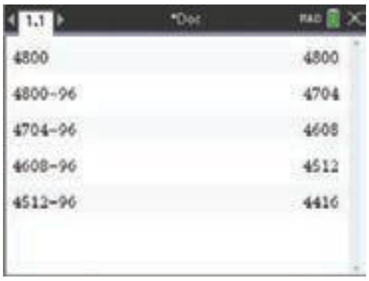
TI-Nspire

ClassPad



b \$4416

TI-Nspire



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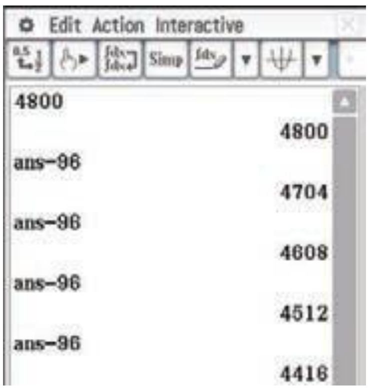


5 a \$330 **b** $u_n = 11\,000 + 330n$ **c** \$13 640

6 a i $u_0 = 9000, u_{n+1} = u_n + 1080$
ii $u_n = 9000 + 1080n$ **iii** \$18 720

b i $u_0 = 6500, u_{n+1} = u_n + 260$
ii $u_n = 6500 + 260n$ **iii** \$8840

c i $u_0 = 8400, u_{n+1} = u_n + 252$
ii $u_n = 8400 + 252n$ **iii** \$10 668

d i $u_0 = 7000, u_{n+1} = u_n + 770$
ii $u_n = 7000 + 770n$ **iii** \$13 930

7 C **8** E **9** B **10** C

11 B

12 a $u_0 = 10\,000, u_{n+1} = u_n + 400$
b a straight line with a negative slope
c $u_n = 10\,000 + 400n$
d \$11 600

13 a 5000, 8000, 11 000
b arithmetic sequence
c $A_0 = 5000, A_{n+1} = A_n + 3000$
d $A_n = 5000 + 3000n$
e \$20 000

EXERCISE 2.4

1 B **2** B

3 a \$2125

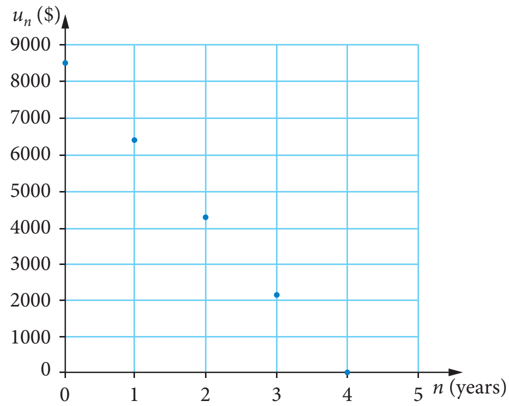
b

n	Value after n years (\$)
0	8500
1	8500 – 2125 = 6375
2	6375 – 2125 = 4250
3	4250 – 2125 = 2125
4	2125 – 2125 = 0

i \$2125 **ii** after two years
iii after four years

c $u_0 = 8500, u_{n+1} = u_n - 2125$

d



4 a \$22 000
b $u_1 = 100\,000 - 22\,000 = 78\,000$
$u_2 = 78\,000 - 22\,000 = 56\,000$
c 22%

5 a i \$72 000 **ii** $u_n = \$1\,800\,000 - 72\,000n$
iii \$1 368 000 **iv** 25 years

b i \$7500 **ii** $u_n = 50\,000 - 7500n$
iii \$5000 **iv** 7 years

c i \$2025 **ii** $u_n = 22\,500 - 2025n$
iii \$10 350 **iv** 12 years

6 a The amount of depreciation is determined by applying a rate per unit of use: \$750 every time the record is played.

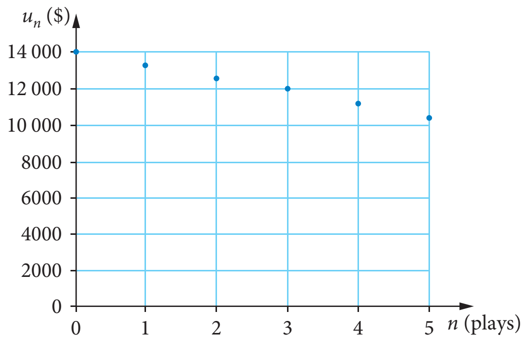
b

n	Value after n uses ($)
0	14 000
1	14 000 – 750 = 13 250
2	13 250 – 750 = 12 500
3	12 500 – 750 = 11 750
4	11 750 – 750 = 11 000
5	11 000 – 750 = 10 250

i $11 000 **ii** after three plays

c Let u_n = value of the record after n plays.
$u_0 = 14\,000$, $u_{n+1} = u_n - 750$

d



7 a $u_n = 60\,000 - 0.54n$ **b** $38 400

8 a Let u_n = value of the concert violin after n concerts.
$u_0 = 8540$, $u_{n+1} = u_n - 6$

b Let u_n = value of the truck after n kilometres.
$u_0 = 170\,000$, $u_{n+1} = u_n - 2$

c Let u_n = value of the van after n kilometres.
$u_0 = 90\,000$, $u_{n+1} = u_n - 0.84$

d Let u_n = value of the photocopier after n copies.
$u_0 = 18\,000$, $u_{n+1} = u_n - 0.01$

9 a i unit cost

ii $u_0 = 10\,000$, $u_{n+1} = u_n - 1500$,
n = number of times the piece of jewellery is worn

iii $u_n = 10\,000 - 1500n$

b i flat rate

ii $u_0 = 10\,000$, $u_{n+1} = u_n - 500$,
n = the number of years

iii $u_n = 10\,000 - 500n$

10 A **11** D **12** A

13 a $u_0 = 75\,000$
$u_1 = 75\,000 - 3375 = 71\,625$
$u_2 = 71\,625 - 3375 = 68\,250$

b i $3375 **ii** 4.5%

14 a $\frac{200}{25} = \$8$ **b** $G_n = 3264 - 8 \times n$ **c** $3024

CUMULATIVE EXAMINATION 1

1 D **2** E **3** A
4 C **5** E **6** D
7 D **8** E 76% **9** A
10 C

CUMULATIVE EXAMINATION 2

1 a 0.5 79%

b median = 28; range = 56; IQR = 17 79%

c 1 | 2 4 **6** 69%

d The distribution is approximately symmetric. 69%

2 a The vertical scale has some values between 0 and 2000 missing.

b The values are in an increasing straight line.

c $2000 **d** $2600

e $3200 **f** after nine years

3 a $S_0 = 8500$, $S_{n+1} = S_n - 867$

b $5032 **c** 10.2%

CHAPTER 3

EXERCISE 3.1

1 a i We are multiplying by 4 to generate each new value.

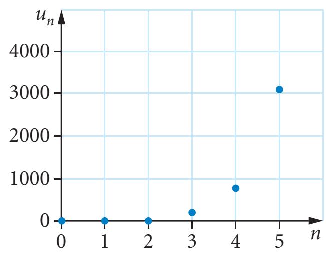
ii $R = \frac{12}{3} = 4$, $R = \frac{48}{12} = 4$, $R = \frac{192}{48} = 4$

iii $u_2 = 48$, $u_4 = 768$

iv

n	0	1	2	3	4	5
u_n	3	12	48	192	768	3072

v



b i We are dividing by 3 to generate each new value, so we are multiplying by $\frac{1}{3}$ or 0.33…

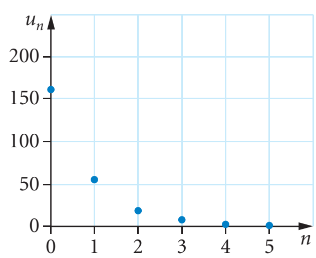
ii $R = \frac{54}{162} = 0.33\ldots$, $R = \frac{18}{54} = 0.33\ldots$,
$R = \frac{6}{18} = 0.33\ldots$

iii $u_2 = 18$, $u_4 = 2$

iv

n	0	1	2	3	4	5
u_n	162	54	18	6	2	0.66…

v



2 a 5 **b** 0.2 **c** 1.6 **d** 0.9

3 a i $a = 4$, $R = 10$

ii $u_0 = 4$
$u_1 = 10u_0 = 10 \times 4 = 40$
$u_2 = 10u_1 = 10 \times 40 = 400$
$u_3 = 10u_2 = 10 \times 400 = 4000$
Sequence is 4, 40, 400, 4000.

b **i** $a = -1, R = 6$

ii $u_0 = -1$

$u_1 = 6u_0 = 6 \times -1 = -6$

$u_2 = 6u_1 = 6 \times -6 = -36$

$u_3 = 6u_2 = 6 \times -36 = -216$

Sequence is –1, –6, –36, –216.

c **i** $a = 80, R = 0.25$

ii $u_0 = 80$

$u_1 = 0.25u_0 = 0.25 \times 80 = 20$

$u_2 = 0.25u_1 = 0.25 \times 20 = 5$

$u_3 = 0.25u_2 = 0.25 \times 5 = 1.25$

Sequence is 80, 20, 5, 1.25.

d **i** $a = 12, R = 1.5$

ii $u_0 = 12$

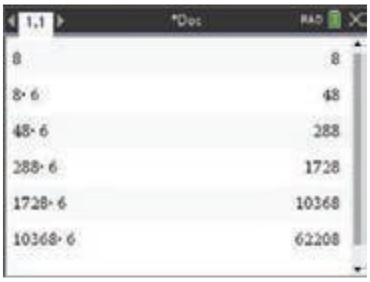
$u_1 = 1.5u_0 = 1.5 \times 12 = 18$

$u_2 = 1.5u_1 = 1.5 \times 18 = 27$

$u_3 = 1.5u_2 = 1.5 \times 27 = 40.5$

Sequence is 12, 18, 27, 40.5.

4 TI-Nspire



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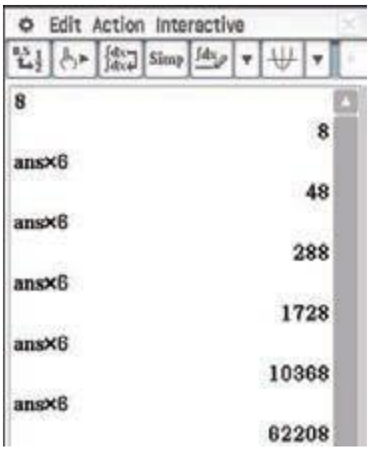


5 **a** **i** We are multiplying by 3 to generate each new value, so this is a geometric sequence.

ii $u_n = 8 \times 3^n$

iii $u_6 = 5832, u_9 = 157\,464$

b **i** We are multiplying by 0.125 to generate each new value, so this is a geometric sequence.

ii $u_n = 16\,000\,000 \times (0.125)^n$

iii $u_6 = 61.035, u_9 = 0.119$

c **i** We are multiplying by 0.9 to generate each new value, so this is a geometric sequence.

ii $u_n = 2240 \times (0.9)^n$

iii $u_6 = 1190.428, u_9 = 867.822$

d **i** We are multiplying by 1.5 to generate each new value, so this is a geometric sequence.

ii $u_n = 4 \times (1.5)^n$

iii $u_6 = 45.563, u_9 = 153.773$

6 **a** $u_0 = 2, u_{n+1} = 2u_n$ **b** $u_0 = 3, u_{n+1} = 5u_n$

c $u_0 = 5, u_{n+1} = 6u_n$ **d** $u_0 = 48, u_{n+1} = (0.5)u_n$

e $u_0 = 2, u_{n+1} = 2.8u_n$

7 A **8** E **9** A **10** C

11 D **12** E **13** D

14 **a** geometric; we are multiplying by 2 to generate each new value.

b $R = 2$

c

n	0	1	2	3
u_n	3	6	12	24

d $u_n = 3 \times 2^n$

e $u_{10} = 3 \times 2^{10} = 3 \times 1024 = 3072$

15 **a** $h_0 = 96, h_{n+1} = 0.25h_n$

b $h_1 = 0.25 \times 96 = 24, h_2 = 0.25 \times 24 = 6$

c 1.5 m

EXERCISE 3.2

1 C **2** D

3 **a**

n	Compound interest ($)	Value of investment ($)	Simple interest ($)	Value of investment ($)
0	–	3000	–	3000
1	$\frac{5}{100} \times 3000$ = 150	3000 + 150 = 3150	$\frac{5}{100} \times 3000$ = 150	3000 + 150 = 3150
2	$\frac{5}{100} \times 3150$ = 157.50	3150 + 157.50 = 3307.50	$\frac{5}{100} \times 3000$ = 150	3150 + 150 = 3300
3	$\frac{5}{100} \times 3307.50$ = 165.38	3307.50 + 165.38 = 3472.88	$\frac{5}{100} \times 3000$ = 150	3300 + 150 = 3450
4	$\frac{5}{100} \times 3472.88$ = 173.64	3472.88 + 173.64 = 3646.52	$\frac{5}{100} \times 3000$ = 150	3450 + 150 = 3600

b $3646.52 **c** $46.52

4 **a** **i** 52 **ii** 468 **iii** $\frac{3}{52}\%$ **iv** $35.77

b **i** 365 **ii** 3285 **iii** $\frac{7}{365}\%$ **iv** $21.10

c **i** 12 **ii** 108 **iii** $\frac{5}{12}\%$ **iv** $41.67

d **i** 4 **ii** 36

iii $\frac{6}{4}\% = \frac{3}{2}\%$ **iv** $210.00

e **i** 26 **ii** 234

iii $\frac{8}{26}\% = \frac{4}{13}\%$ **iv** $67.69

5 **a** $u_0 = 20\,000, u_{n+1} = 1.031u_n$

b $u_0 = 16\,000, u_{n+1} = 1.002u_n$

c $u_0 = 21\,000, u_{n+1} = 1.012u_n$

6 a i 0.5% ii $u_n = 1.005^n \times 75\,000$
iii $82 454.89 iv $7454.89
b i 1% ii $u_n = 1.01^n \times 45\,000$
iii $54 364.90 iv $9364.90
c i 8.5% ii $u_n = 1.085^n \times 6000$
iii $28 269.38 iv $22 269.38
d i 2.5% ii $u_n = 1.025^n \times 34\,000$
iii 54 354.11 iv $20 354.11

7 E **8** E **9** D **10** B
11 B **12** C **13** D **14** D

15 a $\frac{4.4}{4}\% = 1.1\%$ **b** $3876.97 **c** $1020.86

EXERCISE 3.3

1 E **2** D

3 a $3707

n	Depreciation after n years ($)	Value after n years ($)
0	–	10 000
1	$\frac{18}{100} \times 10000 = 1800$	10 000 – 1800 = 8200
2	$\frac{18}{100} \times 8200 = 1476$	8200 – 1476 = 6724
3	$\frac{18}{100} \times 6724 = 1210$	6724 – 1210 = 5514
4	$\frac{18}{100} \times 5514 = 993$	5514 – 993 = 4521
5	$\frac{18}{100} \times 4521 = 814$	4521 – 814 = 3707

b $993
c $u_0 = 10\,000,\ u_{n+1} = 0.82u_n$

4 a 21% **b** after five years

5 a The truck is being depreciated by 20% of its *value* each year. If it was flat rate depreciation, it would be depreciated by 20% of its *initial value* each year.
b $u_n = 0.8^n \times 250\,000$ **c** $33 554

6 a i reducing balance
ii $u_0 = 20\,000,\ u_{n+1} = 0.85u_n$, n = the number of years
iii $u_n = (0.85)^n \times 20\,000$
b i flat rate
ii $u_0 = 20\,000,\ u_{n+1} = u_n - 3000$, n = the number of years
iii $u_n = 20\,000 - 3000n$
c i unit cost
ii $u_0 = 20\,000,\ u_{n+1} = u_n - 0.1$, n = number of pages
iii $u_n = 20\,000 - 0.1n$

7 A **8** A **9** C **10** A
11 B

12 a $u_1 = 0.9 \times 60\,000 = 54\,000,\ u_2 = 0.9 \times 54\,000 = 48\,600$
b 10% **c** 11 years

13 5.7%

14 a $11 440 **b** $10 953.17
c Reducing balance method will give the greater depreciation of $11 046.83.

EXERCISE 3.4

1 D **2** A

3 a i $12 ii –$7.20 iii –$5.40 iv $21
b i $75 ii $51 iii $63.60 iv $55.20

4 a i 40% ii 70% iii 20%
b i 20% ii 10% iii 50%

5 a $125.00 **b** $112.50 **c** $132.00 **d** $148.15

6 a i $2591 ii $259
b i $1975 ii $180
c i $98 ii $9
d i $144 642 ii $14 464

7 a i $1.22 ii $122
b i 65 grams ii 1950 grams
c i 120 widgets ii 3600 widgets
d i $87 ii $52 200

8 a i 1100 students ii 385 students
b i 400 megabytes ii 288 megabytes

9 a i $0.0157 and $0.0148
ii 1 L
b i $0.5556 and $0.5333
ii 15 cans
c i $0.0168 and $0.0175
ii 500 g

10 B **11** D **12** E
13 C **14** E **15** A

16 a 20% **b** $330

17 a $8.00 **b** $352 **c** $99

EXERCISE 3.5

1 C **2** B

3 a 189.1% **b** 578.9% **c** 471.4% **d** 410.8%

4 a i $u_1 = \left(1 + \frac{r}{100}\right) u_n = \left(1 + \frac{2.7}{100}\right) \times 1.29 = 1.324\,83$
The price of the litre of milk at the end of 2023 is $1.32.
ii $u_2 = \left(1 + \frac{r}{100}\right) u_n = \left(1 + \frac{3.5}{100}\right) \times 1.324\,83 = 1.3711\ldots$
The price of the litre of milk at the end of 2024 is $1.37.
iii $u_3 = \left(1 + \frac{r}{100}\right) u_n = \left(1 + \frac{3.3}{100}\right) \times 1.3711\ldots = 1.4164\ldots$
The price of the litre of milk at the end of 2025 is $1.42.
b 10.1%
c 2.7% + 3.5 + 3.3% = 9.5%
Each year's inflation compounds with the previous year's inflation.

5 a $41.31 **b** $832.50 **c** $15 198.37

6 a 0.75%

b **i** \$10 000 + (0.0075 × \$10 000) – \$456.85
= \$9618.15

ii \$9618.15 + (0.0075 × \$9618.15) – \$456.85
= \$9233.44

iii \$9233.44 + (0.0075 × \$9233.44) – \$456.85
= \$8845.84

c Total amount of money paid = 24 × \$456.85
= \$10 964.40

d Total interest = \$10 964.40 – \$10 000 = \$964.40

7 **a** **i** \$2125 **ii** \$2610 **iii** \$2568

iv \$2546 **v** \$2837

b wait 6 months and pay cash

8 D **9** D **10** B **11** B

12 A **13** C

14 **a** 0.0548% **b** \$6200.50 **c** \$200.50 **d** 0.667%

e $6000 + \left(\frac{0.667}{100} \times 6000\right) - 271.36$

f \$512.64 **g** Laura

CUMULATIVE EXAMINATION 1

1 C **2** D **3** C **4** C

5 C **6** C **7** B **8** B

9 D **10** A

CUMULATIVE EXAMINATION 2

1 **a** 2.5 hours **b** 2.625 hours

2 **a** $u_0 = 16\,000$, $u_{n+1} = u_n + 1200$

b \$19 600 **c** five years

3 **a** \$1254 **b** \$7106 **c** \$7600

4 **a** \$330 **b** \$462 **c** 8%

5 **a** **i** \$1904 **ii** after five years

b **i** $u_n = 3100 - 0.03n$ **ii** \$1750

CHAPTER 4

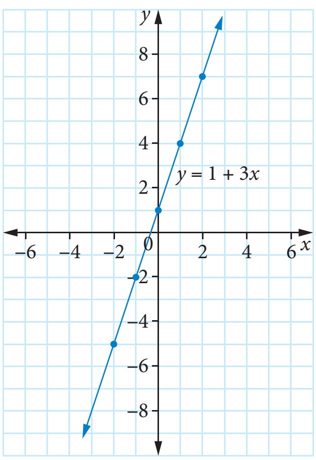
EXERCISE 4.1

1 **a** **i** $y = -5, y = -2, y = 1, y = 4, y = 7$

ii

x	–2	–1	0	1	2
y	–5	–2	1	4	7

iii

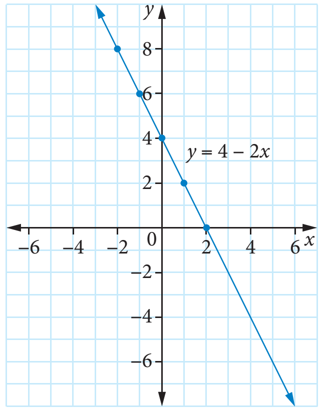


b **i** $y = 8, y = 6, y = 4, y = 2, y = 0$

ii

x	–2	–1	0	1	2
y	8	6	4	2	0

iii

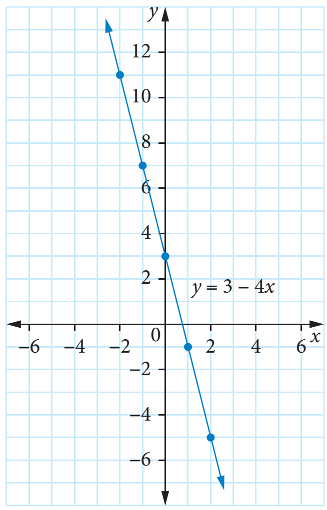


c **i** $y = 11, y = 7, y = 3, y = -1, y = -5$

ii

x	–2	–1	0	1	2
y	11	7	3	–1	–5

iii

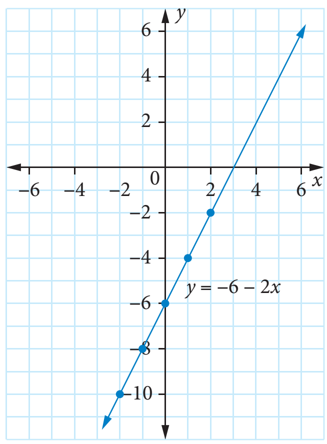


d **i** $y = -10, y = -8, y = -6, y = -4, y = -2$

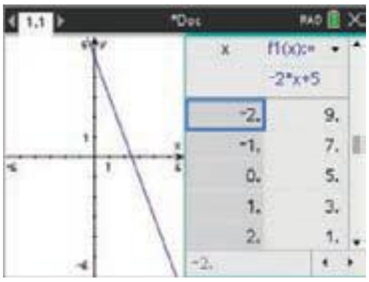
ii

x	–2	–1	0	1	2
y	–10	–8	–6	–4	–2

iii



2 a TI-Nspire



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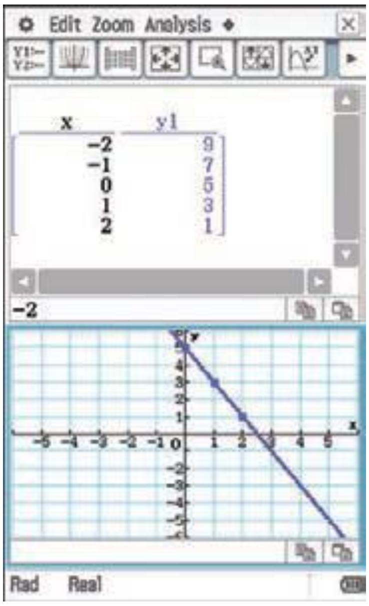
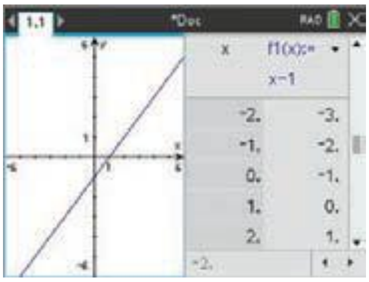
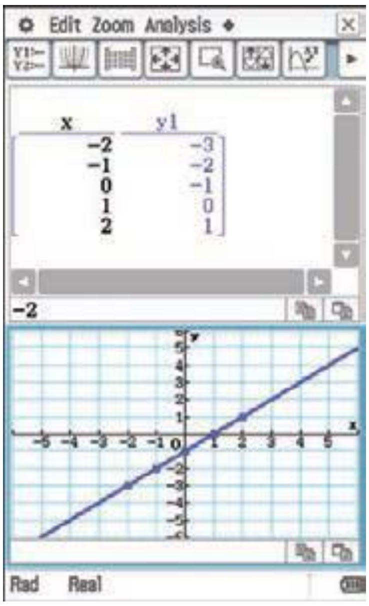


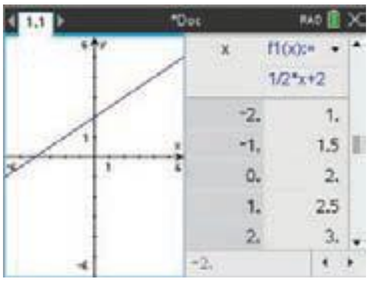
b TI-Nspire



ClassPad



c TI-Nspire



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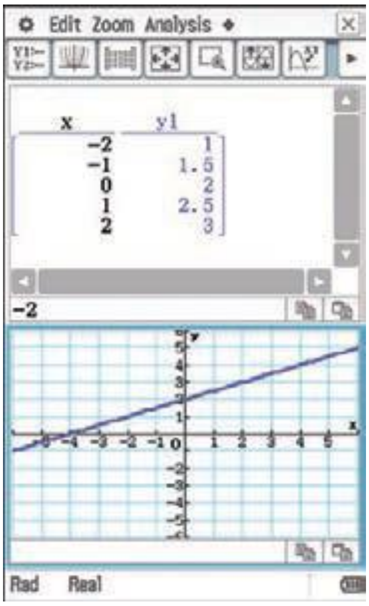
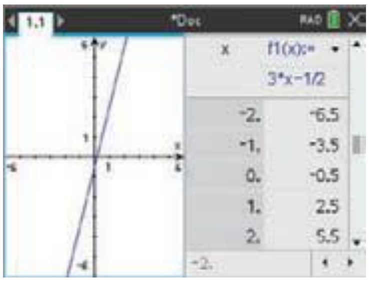


d TI-Nspire



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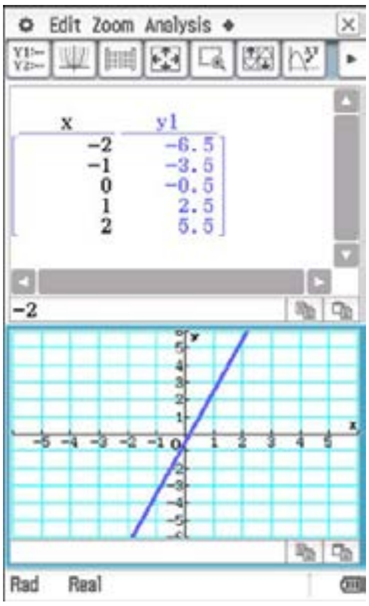

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3 **a** $(-1, 5)$ doesn't lie on the line $y = 15x - 5$ as $5 \neq -15 - 5$.

b $(-1, 5)$ doesn't lie on the line $y = -5x - 1$ as $5 \neq 5 - 1$.

c $(-1, 5)$ lies on the line $y = -10x - 5$ as $5 = 10 - 5$.

4 **a** **i** negative **ii** $-\frac{1}{2}$

b **i** negative **ii** $-\frac{5}{2}$

c **i** negative **ii** $-\frac{1}{4}$

d **i** positive **ii** 2

e **i** zero

f **i** not defined

5 **a** −2 **b** −1 **c** 3

6 **a** $y = 3 + 2x$ **b** $y = -1 + 3x$ **c** $y = 4x$

d $y = 2 - 2x$ **e** $y = 1$

7 B **8** C **9** D **10** D

11 B **12** D **13** B

EXERCISE 4.2

1 B **2** D

3 **a** 1000 **b** 3 **c** $C = 1000 + 3k$

d \$1000 **e** \$3 **f** \$13 600

g 3667 kilolitres

4 **a** 184 chirps/min **b** 8 chirps/min

c 4°C is an unrealistic temperature for an Australian summer.

5 **a** revenue = $5.5n$ **b** profit = $1.5n - 110$

c \$190 **d** loss of \$35

e 474; The profit needs to be *at least* \$600 and selling 473 key rings won't quite make that profit. So, the number of key rings that need to be sold is 474.

6 D **7** A **8** C **9** A

10 E **11** E

12 **a** 18 000 yen **b** 90 yen

13 **a** Substituting (20, 567) into the equation gives $567 = M \times 20$. Solving this equation gives $M = 28.35$.

b 5.67 g **c** 8 ounces

14 **a** profit = $24n - (6n + 260) = 18n - 260$

b 43 students

15 **a** $C = 22 + 80t$ **b** C **c** \$14.62

d 22, starting cost of call (at 0 min)

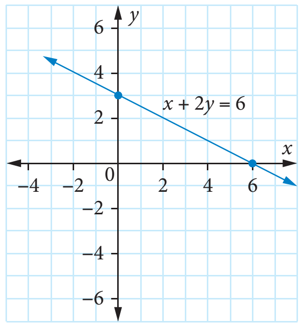
e \$2.40 **f** 7 min

EXERCISE 4.3

1 A **2** C

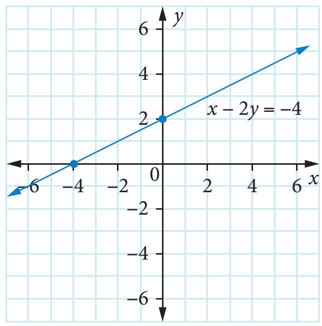
3 **a** **i** (6, 0) **ii** (0, 3)

iii



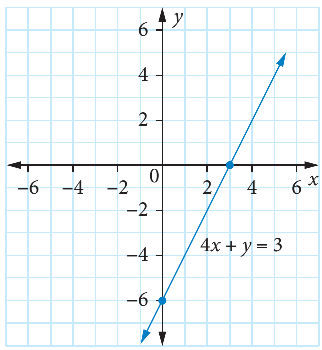
b **i** (−4, 0) **ii** (0, 2)

iii



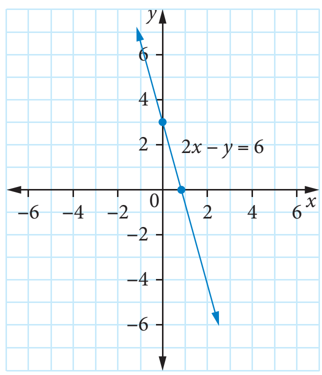
c **i** (3, 0) **ii** (0, −6)

iii



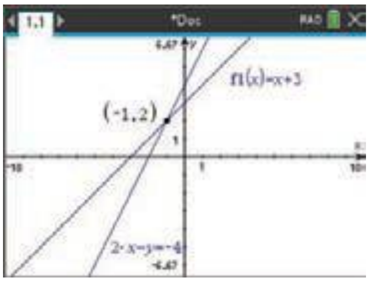
d **i** $\left(\frac{3}{4}, 0\right)$ **ii** (0, 3)

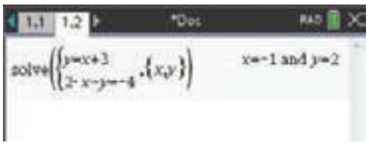
iii



4 **a** $x = -1, y = 2$

TI-Nspire





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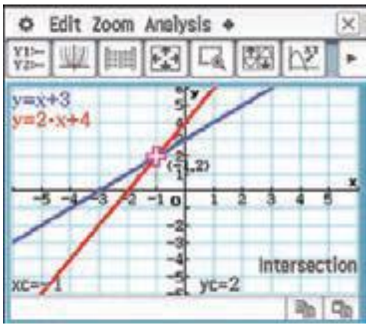
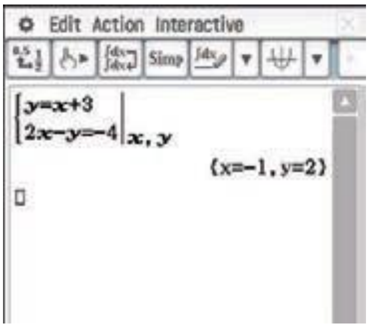
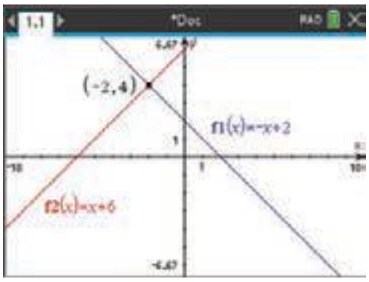
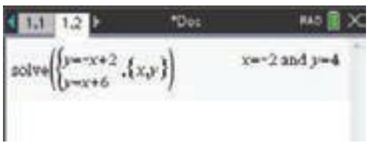




b $x = -2, y = 4$

TI-Nspire





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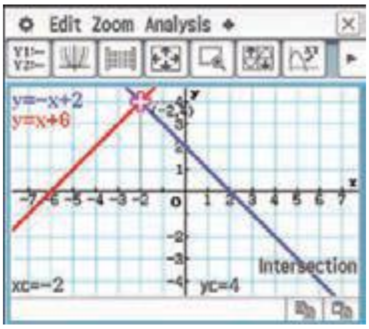
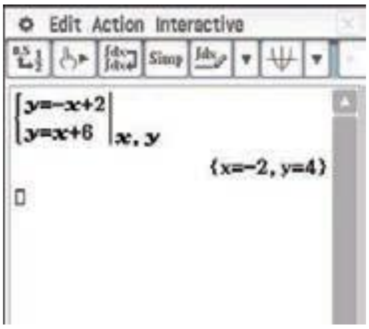
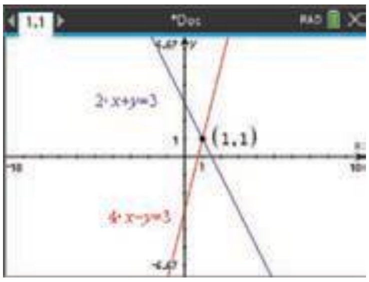
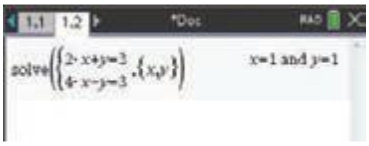




c $x = 1, y = 1$

TI-Nspire





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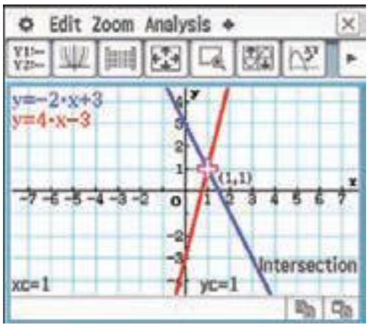

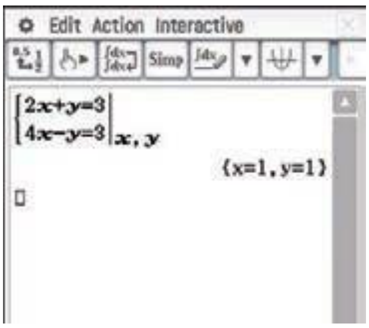

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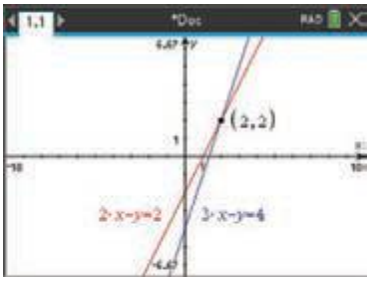
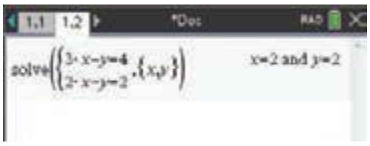
d $x = 2, y = 2$

TI-Nspire





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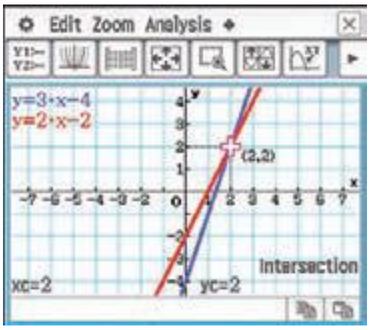
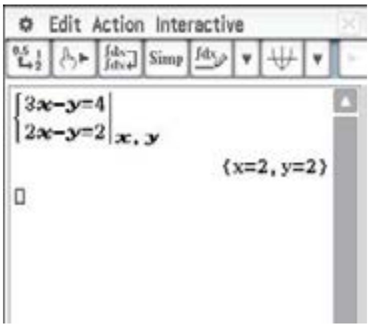


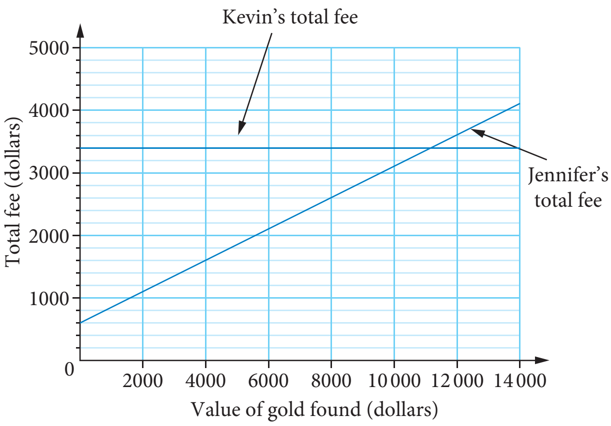


5 250 single cones and 200 double cones

6 D **7** B **8** D **9** A

10 E **11** A **12** B **13** E

14 **a**



b $11 200

15 **a** wattle tree; 1 metre **b** 30 months

c 4 metres

d **i** $h = \frac{2}{15}t$ **ii** $h = 1 + \frac{1}{10}t$

e No, trees don't always grow at the same rate and the graph may only apply for the time period given.

EXERCISE 4.4

1 E **2** E

3 **a** The graph is made up of four different line segments and each line segment has a different slope. This means the rainwater tank fills at four different rates.

b 18 000 litres

c 200 hours

d approximately 80 hours to 120 hours

e 25 litres per hour

4 **a** They started their journey at the park 1 km from Volker's house.

b The slope of the line segment after 45 minutes is greater than the slope of the line segment before, so Volker and Herb started walking faster.

c The horizontal line indicates that Volker and Herb were the same distance (3.5 km) from Volker's house during this time, so they are at Herb's house.

d 15 minutes **e** 3.5 km **f** 7 km/h

5 **a** **i** $10 **ii** $3 **iii** $6

b more than 2 kg up to 3 kg

c 6 kg is the heaviest item that can be sent in this type of parcel delivery.

6 E **7** C **8** C **9** E

10 D **11** C **12** A

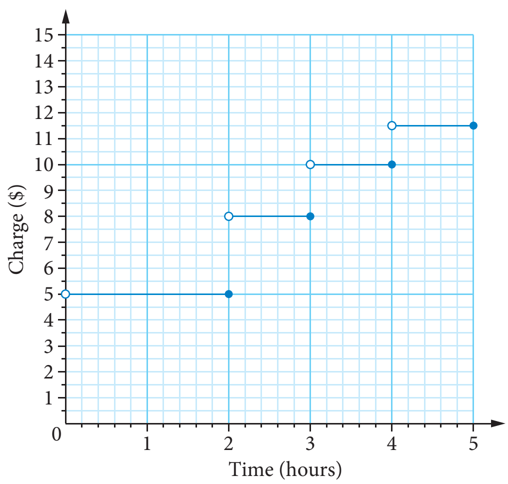
13 **a** 0% for up to $6000 and 15% for over $6000 up to $34 000.

b **i** $3600 **ii** $19 600

14 **a** 8 pm **b** 60 km/h **c** $k = 640$ km/h

15 **a** $10

b



c $16 and $18

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CUMULATIVE EXAMINATION 1

1 B	**2** B	**3** B
4 C 61%	**5** B 82%	**6** A 51%
7 E	**8** B 88%	**9** C 67%
10 C 68%	**11** E 60%	**12** D 60%

CUMULATIVE EXAMINATION 2

1 a −2°C and 2°C **b** 6°C

2 a $180

b $V_0 = 3000$, $V_1 = 3000 - 180 = 2820$
So $V_2 = 2820 - 180 = 2640$.

c 6 years

3 a reducing balance depreciation

b $u_n = 0.86^n \times 250\,000$

4 a 3.5 minutes

b i Section E–F **ii** 200 m/min, or 12 km/h

5 a i $C = 135 + 1.85d$ **ii** $C = 240 + 1.55d$

b i 350 km **ii** $782.50

c CarsRus, as it is cheaper; CarsRus charges $690 and WeRent charges $705.

CHAPTER 5

EXERCISE 5.1

1 a $F = \begin{bmatrix} 172 & 67 & 30 \\ 3 & 139 & 10 \\ 0 & 65 & 9 \\ 11 & 15 & 17 \end{bmatrix}$

The order of F is 4 × 3. F has 12 elements.

b [3]; The order is 1 × 1.

c $\begin{bmatrix} 30 \\ 10 \\ 9 \\ 17 \end{bmatrix}$ **d** $\begin{bmatrix} 0 \\ 65 \\ 9 \end{bmatrix}$

e $\begin{bmatrix} 67 & 139 & 65 & 15 \end{bmatrix}$ **f** $\begin{bmatrix} 269 \\ 152 \\ 74 \\ 43 \end{bmatrix}$

g

	S	A	C	T
Office	172	3	0	11
Online	67	139	65	15
Theatre	30	10	9	17

2 a 2 × 1, column matrix

b 3 × 3, square matrix, identity matrix

c 1 × 4, row matrix, zero matrix

d 3 × 3, square matrix

e 3 × 1, column matrix

f 4 × 4, square matrix, identity matrix

3 a true **b** false **c** true **d** true

e true

4 E **5** D **6** C **7** B

8 E **9** E **10** B **11** A

12 a $\begin{bmatrix} 1 & 1 & 3 & 1 \end{bmatrix}$ **b** row matrix

c i 4 × 3

ii

	D	J	S
Size 10	2	0	3
Size 12	2	4	0
Size 14	2	3	1
Size 16	1	5	0

EXERCISE 5.2

1 B **2** D

3 a Addition is not defined because the matrices have a different order.

b $\begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} -4 & 16 \end{bmatrix}$

d $\begin{bmatrix} 1 & -4 \end{bmatrix}$ **e** $\begin{bmatrix} -31 & 124 \end{bmatrix}$

f $\begin{bmatrix} -3 & 26 \\ 8 & 3 \end{bmatrix}$

4 a $\begin{bmatrix} 24 & 21 \\ 24 & 36 \end{bmatrix}$ **b** $\begin{bmatrix} 37 & 37 \\ 35 & 56 \end{bmatrix}$

c $\begin{bmatrix} 18 & 62 \\ -2 & 32 \end{bmatrix}$ **d** $\begin{bmatrix} 21 & -51 \\ 51 & 24 \end{bmatrix}$

5 a $x = 4, y = -4, z = 7$ **b** $x = 9, y = -1, z = 1$

c $x = 3, y = 11, z = 6$ **d** $x = 7, y = -\frac{5}{4}, z = 12$

6 a $1.3 \times \begin{bmatrix} 79 \\ 199 \\ 399 \end{bmatrix}$ **b** $1.3 \times \begin{bmatrix} 79 \\ 199 \\ 399 \end{bmatrix} + \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$

c $1.3 \times \begin{bmatrix} 79 \\ 199 \\ 399 \end{bmatrix} + \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} - \begin{bmatrix} 20 \\ 45 \\ 45 \end{bmatrix}$

7 D **8** D **9** A **10** C

11 B **12** C **13** B

EXERCISE 5.3

1 A **2** B

3 a i AB has order (1 × 3) (3 × 1), number of columns in A = number of rows in B, so AB is defined.

ii 1 × 1 **iii** [54]

b i BA has order (3 × 1) (1 × 3), number of columns in B = number of rows in A, so BA is defined.

ii 3 × 3 **iii** $\begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 10 \\ 15 & 20 & 50 \end{bmatrix}$

c i BC has order (3 × 1) (2 ×2), number of columns in B ≠ number of rows in C, so BC is not defined.

d i BD has order (3 × 1) (3 ×2), number of columns in B ≠ number of rows in D, so BD is not defined.

e i DC has order (3 × 2) (2 ×2), number of columns in D = number of rows in C, so DC is defined.

ii 3 × 2 **iii** $\begin{bmatrix} 4 & 3 \\ 6 & 2 \\ -1 & 1 \end{bmatrix}$

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f i C is a square matrix and powers of square matrices are always defined, so C^2 is defined. C^2 has order 2×2 and $2C$ has order 2×2. Matrices must have the same order to be subtracted, so $C^2 - 2C$ is defined.

ii 2×2 iii $\begin{bmatrix} 3 & 0 \\ -6 & -1 \end{bmatrix}$

g D has order 3×1. Only powers of square matrices are defined. D is not a square matrix so D^4 is not defined.

4 a $\begin{bmatrix} 25.8 & 32 \\ 44.4 & 52.4 \end{bmatrix}$ b $\begin{bmatrix} 3904 & 3270 \\ 4360 & 4449 \end{bmatrix}$

c $\begin{bmatrix} -436 & -413.4 \\ -265.6 & -280.2 \end{bmatrix}$

5 A **6** A **7** B **8** E
9 D **10** C **11** D **12** C
13 D **14** B

15 a A and D
b C has the same number of rows as A has columns
c AB and DB
d BA and BD
e C as only square matrices can be raised to a power.

EXERCISE 5.4

1 E **2** A

3 a i $\det(A) = -10$ ii $A^{-1} = \begin{bmatrix} -0.1 & 0.2 \\ 0.8 & -0.6 \end{bmatrix}$

b i $\det(B) = 0$ ii B^{-1} doesn't exist.

c i $\det(C)$ doesn't exist. ii C^{-1} doesn't exist.

d i $\det(D) = -5$ ii $D^{-1} = \begin{bmatrix} -0.2 & 1 \\ -0.2 & 2 \end{bmatrix}$

e i $\det(E) = 1$ ii $E^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

f i $\det(F) = 0$ ii F^{-1} doesn't exist.

4 a i $\det(A) = -4$

ii $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & -0.5 \\ 0.5 & 0 & -0.25 \end{bmatrix}$

b i $\det(A) = 1$

ii $A^{-1} = \begin{bmatrix} 2 & -17 & 9 \\ -1 & 9 & -5 \\ -2 & 19 & -10 \end{bmatrix}$

c i $\det(A) = 1$

ii $A^{-1} = \begin{bmatrix} 46 & -51 & 81 \\ -9 & 10 & -16 \\ -26 & 29 & -46 \end{bmatrix}$

5 $x = -2, y = -3$

6 a $x = -1, y = 2, z = 4$
b $x = 5, y = 4, z = 3$
c $x = 1, y = 10, z = 2$

7 a
$$\begin{aligned} 2a + 5b + 3c + d &= 167 \\ 4a + 10b &= 240 \\ 6b + 12c + 3d &= 273 \\ 5c + 2d &= 82 \end{aligned}$$

b $\begin{bmatrix} 2 & 5 & 3 & 1 \\ 4 & 10 & 0 & 0 \\ 0 & 6 & 12 & 3 \\ 0 & 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 167 \\ 240 \\ 273 \\ 82 \end{bmatrix}$

c It takes the worker 20 minutes to assemble a laptop, 16 minutes to assemble a printer, 12 minutes to assemble a modem, and 11 minutes to assemble a router.

8 D **9** D **10** B **11** D
12 C **13** B **14** A **15** C

16 a 35 and 2
b A teacher ticket costs $32.

EXERCISE 5.5

1 A **2** A

3 a $\begin{matrix} \text{Week 1} \\ \text{Week 2} \end{matrix} \begin{bmatrix} 133 & 98 \\ 75 & 62 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 889 \\ 535 \end{bmatrix}$

The total solar light cost for week 1 is $889.
The total solar light cost for week 2 is $535.

b i $C = \begin{bmatrix} 1060 & 1555 \\ 3029 & 1124 \\ 898 & 535 \\ 896 & 2130 \end{bmatrix}$

ii $S = 1.55C = 1.55 \begin{bmatrix} 1060 & 1555 \\ 3029 & 1124 \\ 898 & 535 \\ 896 & 2130 \end{bmatrix}$

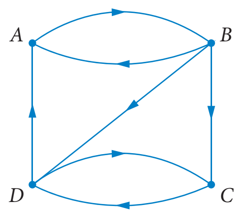
$= \begin{bmatrix} 1643.00 & 2410.25 \\ 4694.95 & 1742.20 \\ 1391.90 & 829.25 \\ 1388.80 & 3301.50 \end{bmatrix}$

c i $\text{profit} = S - C = \begin{bmatrix} 583.00 & 855.25 \\ 1665.95 & 618.20 \\ 493.90 & 294.25 \\ 492.80 & 1171.50 \end{bmatrix}$

ii $6174.85

4 a i Ahmed can send direct messages to Beth.
Beth can send direct messages to Ahmed, Crystal and Daniella.
Crystal can send direct messages to Daniella.
Daniella can send direct messages to Ahmed and Crystal.

ii The diagonal represents links where the sender and receiver are the same. This isn't considered communication, so they are redundant links.

iii


iv Crystal → Daniella → Ahmed

b

Receiver

$$\text{Sender}\ \begin{matrix} \\ P \\ Q \\ R \\ S \\ T \\ U \end{matrix}\begin{matrix} P & Q & R & S & T & U \\ \end{matrix}$$

$$\begin{matrix} P \\ Q \\ R \\ S \\ T \\ U \end{matrix}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(columns: $P\ Q\ R\ S\ T\ U$; rows: Sender)

5 **a** 2 ways **b** $B \to C \to A$ and $B \to D \to A$

c 4 **d** $C \to A \to C$ and $C \to B \to C$

6 B **7** D **8** D **9** D

10 A **11** E **12** A

13 **a** 2×3

b **i** $\begin{bmatrix} 131.30 \\ 130.75 \end{bmatrix}$ **ii** Safeworth

14 **a** Ben and Elka **b** Amara and Dana

15 **a** 2×3

b **i** $M = \begin{bmatrix} 145\,978.00 \\ 171\,848.50 \end{bmatrix}$

ii Total revenue from selling products *A*, *B* and *C* at Eastown and Noxland.

c number of columns in $P \neq$ number of rows in Q

EXERCISE 5.6

1 E **2** C

3 **a** **i** $x = 40\%, y = 40\%, z = 10\%$

ii This year

$$\begin{matrix} & A & B & C & \\ & \begin{bmatrix} 0.55 & 0.35 & 0.4 \\ 0.4 & 0.25 & 0.5 \\ 0.05 & 0.4 & 0.1 \end{bmatrix} & & & \begin{matrix} A \\ B \\ C \end{matrix}\ \text{Next year} \end{matrix}$$

b This train

$$T = \begin{matrix} O & L \end{matrix}\ \begin{bmatrix} 0.95 & 0.3 \\ 0.05 & 0.7 \end{bmatrix}\begin{matrix} O \\ L \end{matrix}\ \text{Next train}$$

4 **a** 124 **b** 196 **c** 362 **d** 20.8%

5 **a** 7022 at Sammy's Seafood and 10 498 at Karl's Kebabs

b 7124 at Sammy's Seafood and 10 396 at Karl's Kebabs

c 7138 at Sammy's Seafood and 10 382 at Karl's Kebabs

6 **a** $\begin{bmatrix} 105 \\ 35 \end{bmatrix}\begin{matrix} A \\ B \end{matrix}$

b 105 trams will be at depot A and 35 trams will be at depot B.

c 75%

7 **a** no; columns don't all add to 1

b yes

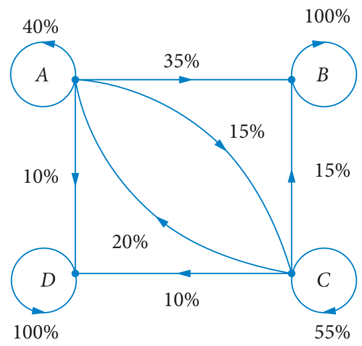
c no; not a square matrix

d no; columns don't all add to 1

8 C **9** B **10** D **11** B

12 E **13** D

14 **a**



b 560

15 **a** All of the advanced-level students stay as advanced-level students.

b **i** $S_1 = \begin{bmatrix} 10 \\ 58 \\ 52 \end{bmatrix}$

ii 12 intermediate-level students

CUMULATIVE EXAMINATION 1

1 C **2** D **3** D

4 E **5** D 64% **6** E

7 D **8** D **9** D 56%

10 E **11** E 75% **12** A

13 C 81% **14** D 52%

CUMULATIVE EXAMINATION 2

1 **a** 25% **b** 24

2 **a** \$8000 72%

b **i** 8000 – 6500 = 1500 72%

ii \$500 72%

c 12 000 km 72%

3 **a** a decrease of 4% **b** 25%

4 **a** 100 km/h **b** 30 minutes

5 **a** 3×1 93% **b** 1.05 48%

6 **a** Anvil and Dantel 79%

b Anvil – Berga – Dantel – Cantor 79%

c $\begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix}$ 79%

d The matrix *G* lists, for each city, the total number of direct flight connections from that city to another city in the network. 79%

7 **a** This year

$$\begin{matrix} P & F & D \end{matrix}$$
$$\begin{bmatrix} 0.88 & 0.52 & 0.65 \\ 0.10 & 0.44 & 0.10 \\ 0.02 & 0.04 & 0.25 \end{bmatrix}\begin{matrix} P \\ F \\ D \end{matrix}\ \text{Next year}$$

b $\begin{bmatrix} 0.88 & 0.52 & 0.65 \\ 0.10 & 0.44 & 0.10 \\ 0.02 & 0.04 & 0.25 \end{bmatrix}^2\begin{bmatrix} 880 \\ 230 \\ 120 \end{bmatrix} = \begin{bmatrix} 996.9 \\ 191.4 \\ 41.7 \end{bmatrix}\begin{matrix} P \\ F \\ D \end{matrix}$

c 42 students will defer the 2009 academic year.

CHAPTER 6

EXERCISE 6.1

1 a *length of time sleeping* and *amount of coffee drunk*; *amount of coffee drunk* is being used to predict the other variable, so it is the explanatory variable.

b *AFL team's position on the ladder* and *crowd size at an AFL match*; *AFL team's position on the ladder* is being used to explain changes to the other variable, so it is the explanatory variable.

c *score in a mini-golf tournament* and *age*; the score in a mini-golf tournament is not likely to affect the age of the person, but the age of the person may affect their mini-golf score. So, *age* is the explanatory variable.

d *price of a house and number of bedrooms in the house*; the price of a house is not likely to affect the number of bedrooms in the house, but the number of bedrooms in the house may affect the price of the house. So, *number of bedrooms in the house* is the explanatory variable.

e *amount of alcohol consumed* and *number of car accidents*; the number of car accidents is not likely to affect the amount of alcohol consumed, but the amount of alcohol consumed may affect the number of car accidents. So, *amount of alcohol consumed* is the explanatory variable.

2 a *number of crimes per month*

b *chirp rate*

3 D **4** C

5 a *height*

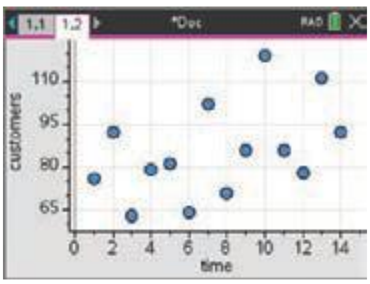
b i *humidity 9 am*

ii No, a study with the aim of predicting *humidity 9 am* from *humidity 3 pm* does not make sense because 9 am occurs before 3 pm so we already know what the humidity was at 9 am.

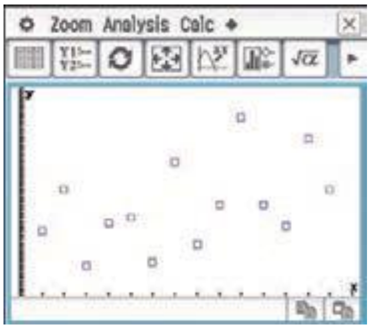
EXERCISE 6.2

1 D **2** C

3 TI-Nspire



ClassPad



4 a *age* (years) **b** *time* (min) **c** 14 people

d a 10-year-old who was at the enclosure for 20 minutes

e 4 **f** parent/child or guardian/child

5 a i no association

ii There appears to be no association between the amount of milk consumed and IQ.

d i positive, linear and weak

ii There is limited evidence to suggest that the size of the kitchen increases as the income per household increases

c i positive, linear and strong

ii The sales of sunscreen increase as the amount of sunshine increases.

b i negative, linear and moderate

ii Test scores tend to decrease as the time absent from school increases.

6 a no association **b** positive association

c positive association **d** no association

e positive association **f** positive association

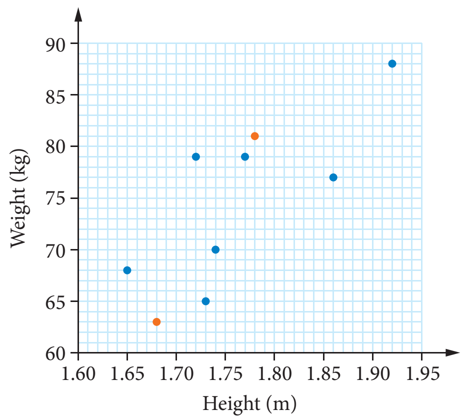
7 a *outdoor temperature* **b** *population*

c *household income* **d** *population*

e *age*

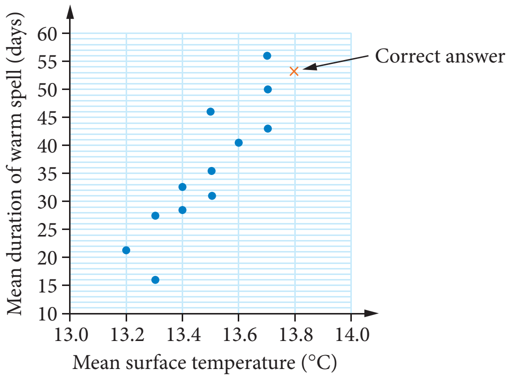
8 D **9** E **10** D

11 a



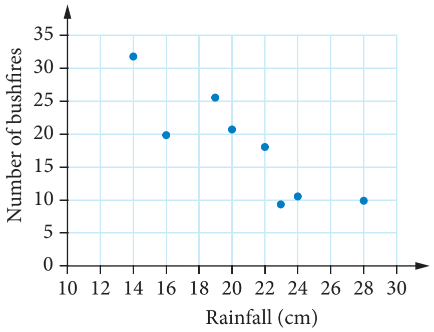
b *age*

12 **a**



b *climate change*

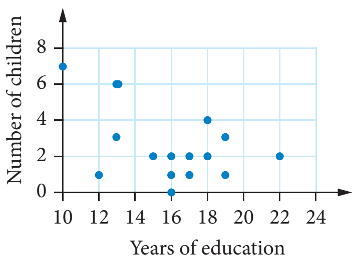
13 **a** Number of bushfires vs rainfall



b The association is negative, linear and strong.

c *climate change*

14 **a** Number of children vs years of education



b linear, negative and weak

c There is limited evidence to suggest that the number of children a person will have should decrease as the number of years of education increases.

EXERCISE 6.3

1 E **2** D

3 **a** **i** 8.53 **ii** 8.5

b **i** 43.95 **ii** 44

c **i** 0.25 **ii** 0.25

d **i** 46 000.00 **ii** 46 000

e **i** 82.69 **ii** 83

f **i** 14.00 **ii** 14

g **i** 3.90 **ii** 3.9

4 **a** 11 **b** 0.45 **c** *time* = 11 + 0.45 × *age*

d **i** 17 minutes

ii 14 is within the original data range of 4 to 21 years, so this involves interpolation.

e **i** 22 minutes

ii 24 is outside the original data range of 4 to 21 years, so this involves extrapolation.

f The prediction for the 14-year-old involves interpolation, so it is more reliable than the prediction for a 24-year-old, which involves extrapolation.

5 **a** **i** 240. This means the figurine cost \$240 when it was new.

ii 90. This means on average the *value* of the figurine increases by \$90 for every 1-year increase in the *age*.

b **i** 1850. This means that the mobile phone cost \$1850 when it was new.

ii 300. This means on average the *price* of the mobile phone decreases by \$300 for every one-year increase in the *age*.

c **i** 105. This means that the video game cost \$105 when it was new.

ii 28. This means on average the *value* of the video game decreases by \$28 for every one-year increase in the *age*.

6 E **7** E **8** C **9** B

10 A **11** A **12** D **13** E

14 B **15** B

16 **a** *humidity 9 am*

b 6. When the *humidity 9 am* is 0%, the *humidity 3pm* is 6%

c 0.675. The *humidity 3 pm* on average increases by 0.675% for every 1% increase in *humidity 9 am*.

d *humidity 3 pm* = 6.0 + 0.68 × *humidity 9 am*

e 28% and 48%. The second prediction is more reliable because it involves interpolation, whereas the first prediction involves extrapolation.

CUMULATIVE EXAMINATION 1

1 D 74% **2** C **3** C

4 B 69% **5** E 35% **6** C

7 E **8** C **9** E

10 B 58%

CUMULATIVE EXAMINATION 2

1 **a** histogram **b** approximately symmetric

2 **a** $V_1 = 8400 - 1200 = 7200$, $V_2 = 7200 - 1200 = \$6000$

b after four years

3 **a** mark-up and discount

b decrease of 1%

c 25%

4 **a** 4 × 1 91%

b **i** [6000] 79%

ii the total booking fees collected for the month 42%

5 **a** explanatory variable = *interest rate*
response variable = *number of applications*

b *interest rate* is continuous, *number of applications* is discrete

c –9. The number of applications on average decreases by 9 for every 1% increase in the interest rate.

d *number of applications* = 48 – 9 × *interest rate*

e 26 and 3. The first prediction is more reliable because it involves interpolation, whereas the second prediction involves extrapolation.

CHAPTER 7

EXERCISE 7.1

1 a Isomorphic because they show exactly the same connections.

b Not isomorphic because they don't have exactly the same connections. For example, in the first graph E connects to F but in the second graph it doesn't.

c Isomorphic because they show exactly the same connections.

d Not isomorphic because they have different numbers of vertices and edges.

2 a i 6 vertices: L, M, N, O, P, Q
9 edges: $LM, LQ, MQ, MP, MN, NP, NO, PO, PQ$
5 faces: 4 enclosed and 1 outside the graph

ii

Vertex	L	M	N	O	P	Q	Sum
Degree	2	4	3	2	4	3	18

degree sum = 2 × number of edges = 2 × 9 = 18

b i 6 vertices: L, M, N, O, P, Q
10 edges: LM × 3, $LQ, MQ, MN, MO, NO, NQ, OO$
8 faces: 7 enclosed and 1 outside the graph

ii

Vertex	L	M	N	O	P	Q	Sum
Degree	4	6	3	4	0	3	20

degree sum = 2 × number of edges = 2 × 10 = 20

3 a i 4 **ii** 0 **iii** 4 **iv** 2
v 0 **vi** 2 **vii** 0; degree sum = 8

b i 6 **ii** 0 **iii** 10 **iv** 6
v 0 **vi** 0 **vii** 0; degree sum = 20

c i 3 **ii** 0 **iii** 6 **iv** 5
v 3 **vi** 3 **vii** 0; degree sum = 12

d i 5 **ii** 1 **iii** 5 **iv** 3
v 1 **vi** 3 **vii** 1; degree sum = 10

e i 5 **ii** 0 **iii** 5 **iv** 2
v 0 **vi** 5 **vii** 0; degree sum = 10

f i 7 **ii** 0 **iii** 11 **iv** 6
v 0 **vi** 5 **vii** 0; degree sum = 22

4 a
$$\begin{array}{c|cccc|} & A & B & C & D \\ A & 1 & 1 & 0 & 0 \\ B & 1 & 0 & 1 & 0 \\ C & 0 & 1 & 0 & 0 \\ D & 0 & 0 & 0 & 0 \end{array}$$

b
$$\begin{array}{c|ccc|} & A & B & C \\ A & 0 & 1 & 2 \\ B & 1 & 0 & 2 \\ C & 2 & 2 & 0 \end{array}$$

c
$$\begin{array}{c|ccccc|} & A & B & C & D & E \\ A & 0 & 0 & 0 & 1 & 0 \\ B & 0 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 1 & 0 \\ D & 1 & 1 & 1 & 0 & 1 \\ E & 0 & 0 & 0 & 1 & 0 \end{array}$$

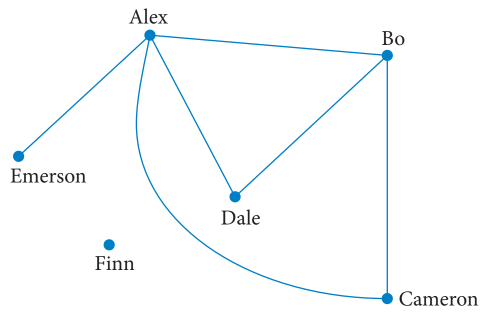
5 B **6** A **7** B **8** C
9 C **10** B **11** C **12** E
13 B

14 a A zero means the team members are not allowed to communicate directly with each other.

b $f = 1, g = 0$

15 a 1 player **b** Cameron and Dale

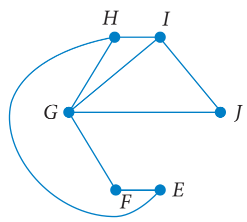
c



EXERCISE 7.2

1 D **2** E

3 a i

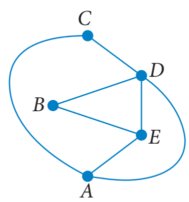


Other answers are possible. Teacher to check.

ii It is a connected graph because there is a path from each vertex to every other vertex.

iii $v = 6, f = 4, e = 8; v + f - e = 6 + 4 - 8 = 2$

b i

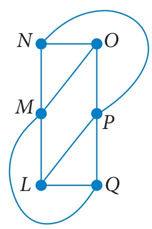


Other answers are possible. Teacher to check.

ii It is a connected graph because there is a path from each vertex to every other vertex.

iii $v = 5, f = 4, e = 7; v + f - e = 5 + 4 - 7 = 2$

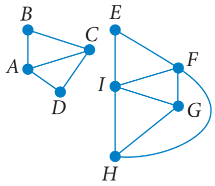
c i



Other answers are possible. Teacher to check.

ii It is a connected graph because there is a path from each vertex to every other vertex.

iii $v = 6, f = 6, e = 10; v + f - e = 6 + 6 - 10 = 2$

d **i** 

Other answers are possible. Teacher to check.

ii It is not a connected graph because there isn't a path from each vertex to every other vertex (e.g. C to H).

iii Euler's formula doesn't work. $v = 9, f = 7, e = 13$; $v + f - e = 9 + 7 - 13 = 3$

4 **a** $v = 5$ **b** $e = 14$ **c** $f = 3$

5 **a** It is not a subgraph because there is no loop at A in the larger graph.

b It is not a subgraph because it has an edge (CD) that's not in the larger graph.

c It is a subgraph because it only has vertices and edges from the larger graph.

d It is not a subgraph because it has two edges (CD and EG) that are not in the larger graph.

6 D **7** B **8** C **9** C

10 A **11** B **12** A **13** A

14 D **15** D

EXERCISE 7.3

1 C **2** C

3 **a** This walk has two repeated edges, so it's a walk only.

b This walk has no repeated edges, no repeated vertices, and doesn't start and finish at the same vertex, so it's a path.

c This walk has no repeated edges, no repeated vertices (except the first and last vertex), and starts and finishes at the same vertex, so it's a cycle.

d This walk has no repeated edges, a repeated vertex, and starts and finish at the same vertex, so it's a circuit.

e This walk has no repeated edges, a repeated vertex, and doesn't start and finish at the same vertex, so it's a trail.

4 **a** This walk has no repeated edges, a repeated vertex H, and starts and finish at the same vertex, so it's a circuit.

b This walk has no repeated edges, no repeated vertices (except the first and last vertex), and starts and finishes at the same vertex, so it's a cycle.

c This walk has no repeated edges, no repeated vertices, and doesn't start and finish at the same vertex, so it's a path.

d This walk has a repeated edge KH, so it's a walk only.

e This walk has no repeated edges, a repeated vertex D, and doesn't start and finish at the same vertex, so it's a trail.

f This walk has a repeated edge (GK is the same edge as KG), so it's a walk only.

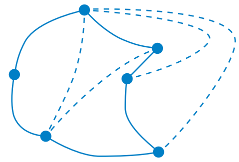
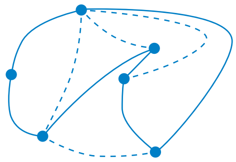
5 22 minutes

6 B **7** D **8** A **9** D

10 D **11** C **12** B **13** E

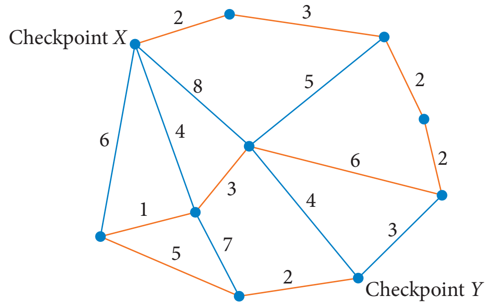
14 **a** office

b **i** cycle

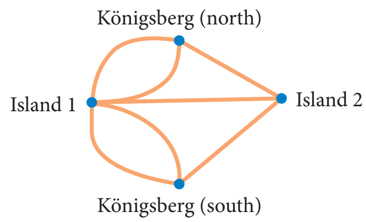
ii  or 

15 **a** 11 minutes

b **i** path

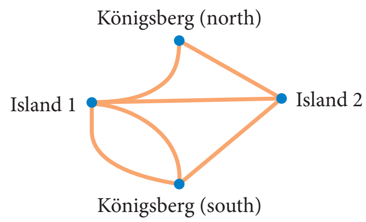
ii 

16 **a** edges

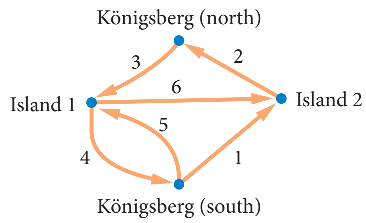
b 

c trail

d Island 1 degree = 5, Königsberg (north) degree = 3, Island 2 degree = 3, Königsberg (south) degree = 3. All degrees are odd, so a trail without repeated edges is impossible.

e 

Island 1 degree = 4, Königsberg (north) degree = 2, Island 2 degree = 3, Königsberg (south) degree = 3.
There are now exactly two vertices of odd degree so this makes a trail without repeated edges possible.

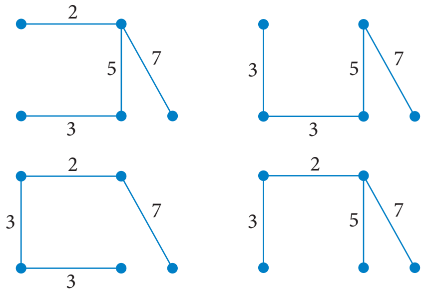
f 

Other answers are possible.

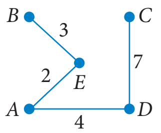
EXERCISE 7.4

1 A 2 A

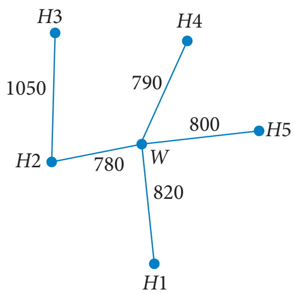
3 **a** This graph is not a spanning tree. It has a cycle.
b This graph is not a spanning tree. It has a multiple edge.
c This graph is a spanning tree. It has 9 vertices and 8 edges.
d This graph is not a spanning tree. It has an edge (AC) that isn't in the original graph.
e This graph is not a spanning tree. It is not connected.
f This graph is a spanning tree. It has 9 vertices and 8 edges.

4


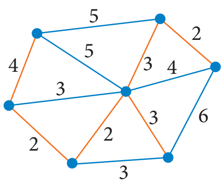
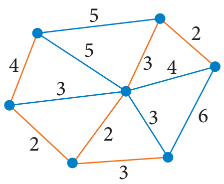
Total weight of minimum spanning tree = 15

5


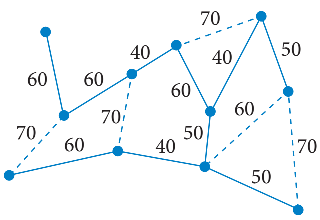
Total weight = 16

6 4240 m


7 C 8 A 9 B 10 A
11 C 12 C 13 D

14 **a**

or


b 16
c Two, as shown in part **a** above.

15 **a**


b 510 m

CUMULATIVE EXAMINATION 1

1 A 2 A 3 C 59%
4 D 5 E 88% 6 C
7 A 88% 8 D 9 B 92%
10 E 11 D 12 B
13 C

CUMULATIVE EXAMINATION 2

1 **a** 2 **b** 2
c **i** Removing the outlier does not change the most frequently occurring value, so the mode remains 2.
ii There would only be 19 values if the outlier was removed, so the 10th value is in the position of the median, which is still 2.

2 $V_0 = 60\,000,\ V_{n+1} = V_n - 4800$ 52%

3 **a** $0.15 \times 100 + 0.25 \times 400 + 0.20 \times 100 + 0.50 \times 1400$
$= 835$
b 356

4 **a** development index **b** weak and positive
c Average pay rate has the greater spread because $s_x = 5.37$ and $s_y = 2.99$, so $s_x > s_y$.

5 **a** 3 **b** 1000 m
c *E-P5-P4-P6-P3-P2-P1* **d** cycle

6 **a** \$120 94%
b Quigley and Rosebush 79%
c $6 + 7 = 11 + 2$ 83%

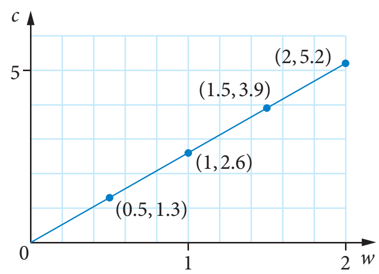
CHAPTER 8

EXERCISE 8.1

1 **a** $c = kw$ **b** $k = 2.6$

c

w (kg)	0.5	1	1.5	2
c (\$)	1.30	2.60	3.90	5.20

d


e 1 kg costs \$2.60, 2 kg costs $2 \times 2.6 = 5.2 =$ \$5.20
f \$39

2 **a** $t = \frac{k}{n}$ **b** $k = 132$

c

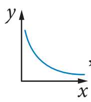
n	1	2	3	4
t (hours)	132	66	44	33

d

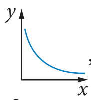
e Two painters take 66 hours to paint an apartment block. Four painters take 66 ÷ 2 = 33 hours to paint an apartment block.

f 11 hours

3 a The graph is a straight line that goes through (0, 0), so direct variation is involved. $y = \frac{1}{2}x$

b The graph isn't a straight line and it doesn't have the shape , so neither direct nor inverse variation is involved.

c The graph is a straight line that doesn't go through (0, 0), so neither direct nor inverse variation is involved.

d The graph has the shape , so inverse variation is involved. $y = \frac{8}{x}$.

4 a **i** The horizontal axis doesn't start at zero.

ii Two points where one x value is double the other x value are (3, 40) and (6, 20). Double 3 is 6 and halving 40 is 20. So, this is inverse variation.

iii $y = \frac{120}{x}$

b **i** The horizontal axis doesn't start at zero.

ii Two points where one x value is double the other x value are (4, 40) and (8, 80). Double 4 is 8 and doubling 40 is 80. So, this is direct variation.

iii $y = 10x$

5 D **6** E **7** C **8** D

9 E **10** A **11** B **12** D

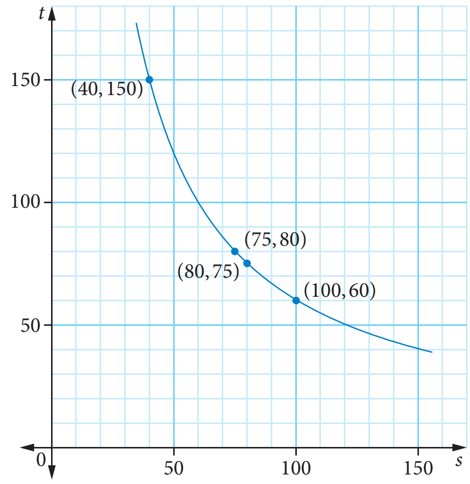
13 B

14 a $k = 6000$, $t = \frac{6000}{s}$

b

s (km/h)	40	75	80	100
t (minutes)	150	80	75	60

c



d Travelling at a speed of 40 km/h gives a travelling time of 150 minutes. Doubling the speed to 80 km/h halves the travelling time to 75 minutes.

e 50 min **f** 62.5 km/h

EXERCISE 8.2

1 A **2** D

3 a **i** 1000 **ii** 251.189 **iii** 1

iv 6.310 **v** 10 **vi** 0.1

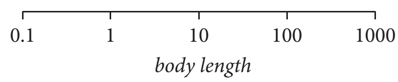
vii 0.032

b **i** 4.00 **ii** 7.00 **iii** 1.86

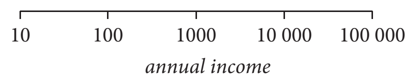
iv 2.84 **v** −0.222 **vi** 4.48

vii 0.954

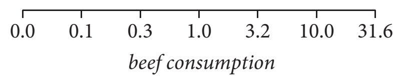
4 a



b



c



5 a $\frac{1}{x}$ or $\log x$ **b** none **c** x^2

d x^2 **e** none **f** $\frac{1}{x}$ or $\log x$

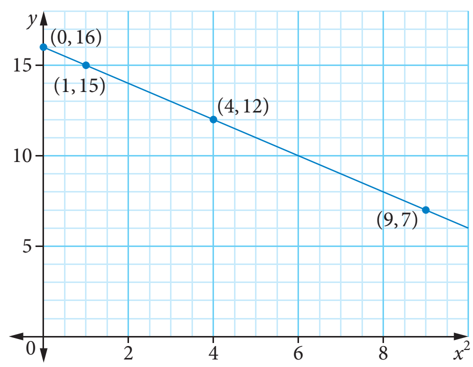
6 a **i**

x	0	1	2	3
y	16	15	12	7

ii

x	0	1	2	3
x^2	0	1	4	9
y	16	15	12	7

iii



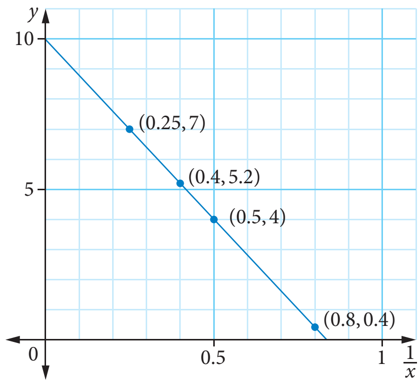
b **i**

x	1.25	2	2.5	4
y	0.4	4	5.2	7

ii

x	1.25	2	2.5	4
$\frac{1}{x}$	$\frac{1}{1.25} = 0.8$	$\frac{1}{2} = 0.5$	$\frac{1}{2.5} = 0.4$	$\frac{1}{4} = 0.25$
y	0.4	4	5.2	7

iii



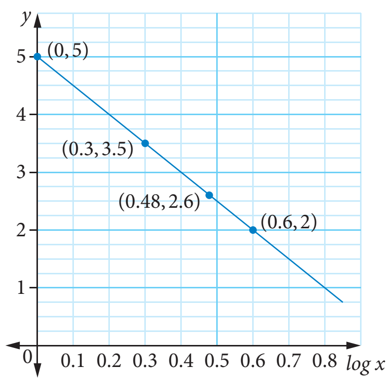
c **i**

x	1	2	3	4
y	5	3.5	2.6	2

ii

x	1	2	3	4
$\log x$	0	0.30	0.48	0.60
y	5	3.5	2.6	2

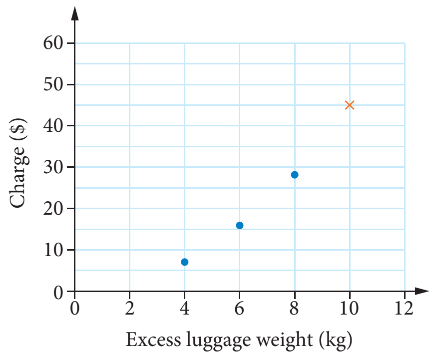
iii



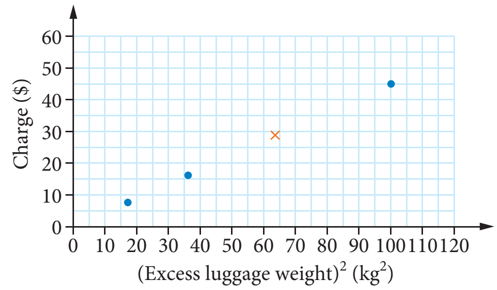
7 B **8** D **9** A **10** A

11 D **12** E **13** B

14 **a** Correct point marked at (10, 45).



b 64 (on the table) and (64, 28.8) on the graph



EXERCISE 8.3

1 B **2** C

3 **a** $weight = k(length)^2 + c$

b $score = \dfrac{k}{study\ time} + c$; $score = k\log(study\ time) + c$

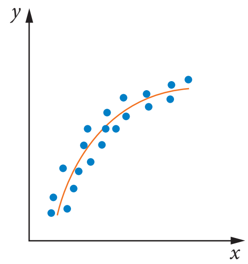
c $afternoon\ rainfall = \dfrac{k}{morning\ rainfall} + c$,

$afternoon\ rainfall = k\log(morning\ rainfall) + c$

d $distance = k(time)^2 + c$

4 **a** **i** 10 seeds **ii** 8 seeds **iii** 6 seeds

b



5 A **6** B **7** A **8** A

9 A **10** A **11** B **12** C

13 C

14 **a** 23 000 **b** inverse

15 **a** log and inverse

b $ws3.00\,pm = 12 - 20 \times \dfrac{1}{ws9.00\,am}$

c 11 km/h

d $ws3.00\,pm = 0.77 + 9.2 \times \log(ws9.00\,am)$

e 13 km/h

f For the inverse model: $ws3.00\,pm$ = 9.5 km/h.
For the log model: $ws3.00\,pm$ = 9.1 km/h.

The log model was closer to the actual value from the scatterplot of 8 km/h.

CUMULATIVE EXAMINATION 1

1 B 41% **2** C **3** E 87%

4 D 91% **5** C **6** E

7 D **8** C **9** E

10 B **11** B **12** C

13 A **14** D

CUMULATIVE EXAMINATION 2

1 **a** \$2736

b $C_0 = 38\,000$, $C_{n+1} = C_n - 2736$

c \$35 264, \$32 528, \$29 792, \$27 056

2 **a** $f = -\frac{1}{8}d + 50$ **b** 240 km **c** 75 litre

54%

3 **a** 2

b Pond *R* connects directly by pipe to two other ponds.

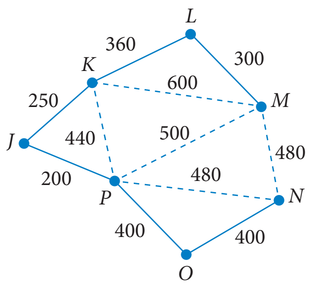
c

	P	Q	R	X	V
P	0	1	0	0	0
Q	1	0	0	1	0
R	0	0	0	1	0
X	0	1	1	0	1
V	0	0	0	1	0

73%

4 **a** \$300 **b** \$920

c



5 **a** *value* 76% **b** The data is non-linear.

c 0.95 **d** negative and strong 68%

e \$13 100 68% **f** \$800

CHAPTER 9

EXERCISE 9.1

1 **a** 0.436 81 km **b** 7 000 000 cm^3

c 50 000 cm^2 **d** 0.6 L

e 50 000 000 000 mm^3 **f** 4200 L

g 2500 cm^2 **h** 0.68 m^2

i 0.000 0015 mm

2 **a** 5.64×10^2 **b** 4.5×10^4

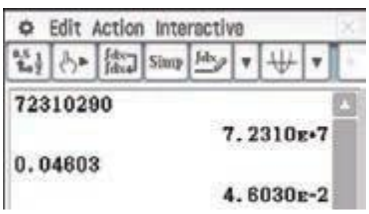
c 5.5×10^{-3} **d** 8×10^{-4}

e 6.3×10^7 **f** 2.97×10^{-5}

3 **a** $7.231\,029 \times 10^7$ **b** 4.603×10^{-2}

TI-Nspire

ClassPad



4 **a** 32 000 **b** 9 000 000 **c** 453 700

d 0.000 503 **e** 0.008 **f** 0.000 614

5 D **6** D **7** D **8** E

9 D **10** C **11** C

EXERCISE 9.2

1 D **2** A

3 **a** **i** 22.67 mm **ii** 23 mm

b **i** 27.20 cm **ii** 27 cm

c **i** 30.41 cm **ii** 30 cm

d **i** 18.73 cm **ii** 19 cm

e **i** 5.66 cm **ii** 5.7 cm

f **i** 7.62 m **ii** 7.6 m

4 **a** 6.3 km **b** 9.0 cm **c** 98.6 m

5 **a** 125 m **b** 566 cm **c** 1825 m

6 19 m

7 **a** 21.9 cm **b** 15.5 cm

8 D **9** B **10** C

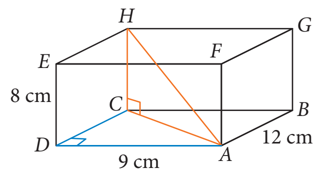
11 A **12** C

13 **a** 18.8 m **b** 19 m

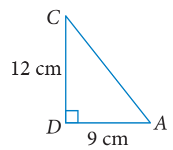
14 $\sqrt{25^2 - 15^2} = 20$ or $25^2 - 15^2 = 400$, $\sqrt{400} = 20$.

Incorrect: $25^2 - 15^2 = \sqrt{400} = 20$.

15 **a**

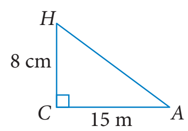


b



c 15 cm

d



e 17 cm

EXERCISE 9.3

1 C **2** B

3 **a** **i** 3200 cm **ii** 640 000 cm^2

b **i** 2500 cm **ii** 348 600 cm^2

c **i** 63 cm **ii** 314 cm^2

d **i** 1699 cm **ii** 110 111 cm^2

e **i** 9400 cm **ii** 3 750 000 cm^2

f **i** 754 cm **ii** 45 239 cm^2

g **i** 8 cm **ii** 3 cm^2

h **i** 80 cm **ii** 360 cm^2

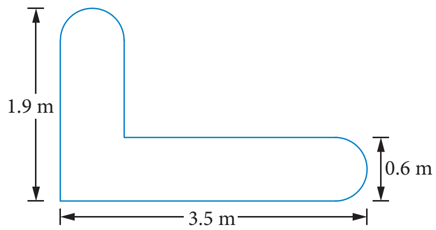
i **i** 2600 cm **ii** 200 000 cm^2

j **i** 2810 cm **ii** 424 000 cm^2

k **i** 780 cm **ii** 31 500 cm^2

l **i** 160 cm **ii** 1200 cm^2

4 a i 3.67 m   ii 10.7 m   iii 6.41 m^2
b i 202 mm   ii 454 mm   iii 12 700 mm^2
c i 149 cm   ii 203 cm   iii 2020 cm^2
5 a i 94 cm   ii 500 cm^2
b i 37 m   ii 65 m^2
c i 12 m   ii 7.9 m^2
d i 68 m   ii 110 m^2
e i 100 m   ii 520 m^2
f i 66 cm   ii 210 cm^2
6 a 97.0 m^2 **b** 85.8 cm^2 **c** 2380 cm^2
d 298 m^2 **e** 28.9 km^2 **f** 4 430 000 mm^2
7 a



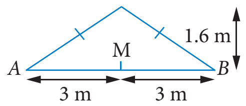
b \$2803   **c** \$566
8 D   **9** A   **10** C
11 B   **12** B   **13** A
14 a 8 m^2 **b** 12.8 m
15 a $\sqrt{2.2^2 + 2.3^2} = 3.182\ldots \approx 3.2$
b 17.3 km   **c** 17 km^2
16 a 59.7 m   **b** \$2577   **c** \$597 (including gate)

EXERCISE 9.4

1 D   **2** C
3 a $V = 250\text{ cm}^3$, $C = 250$ mL
b $V = 1.7\text{ m}^3$, $C = 1700$ L
c $V = 31\,000\text{ cm}^3$, $C = 31\,000$ mL
d $V = 3000\text{ cm}^3$, $C = 3000$ mL
e $V = 17\text{ m}^3$, $C = 17\,000$ L
f $V = 130\text{ m}^3$, $C = 130\,000$ L
4 a $V = 31\,000\text{ m}^3$, $C = 31\,000\,000$ L
b $V = 6600\text{ cm}^3$, $C = 6600$ mL
c $V = 1.6\text{ m}^3$, $C = 1600$ L
d $V = 250\text{ cm}^3$, $C = 250$ mL
e $V = 27\text{ m}^3$, $C = 27\,000$ L
f $V = 430\text{ cm}^3$, $C = 430$ mL
5 a 1 243 547 mm^3 **b** 1437 cm^3
c 8379 mm^3 **d** 18 m^3
e 18 m^3 **f** 1030 cm^3
6 B   **7** C   **8** D   **9** C
10 C   **11** A   **12** C   **13** D
14 a 4250 m^2 **b** 1000 m^3 **c** 90.6 m
15 a 0.048 m^2 **b** 6.6 m^3

EXERCISE 9.5

1 A   **2** A
3 a 2230 cm^2 **b** 983 cm^2 **c** 58 100 mm^2
d 1390 cm^2 **e** 302 cm^2 **f** 5400 cm^2
4 a 30 cm^2 **b** 24 cm^2 **c** 510 cm^2
d 200 m^2 **e** 380 cm^2 **f** 1400 cm^2
5 D   **6** B   **7** C   **8** B
9 C   **10** D   **11** E   **12** C
13 a 5755 mm^2 **b** 214 mm
14 a 2800 cm^3 **b** 1055.6 cm^2
15 a $OM = \sqrt{3.4^2 - 3^2} = 1.6$ m
b An example for the triangle shown is:
area of the triangle $= \frac{1}{2}bh = \frac{1}{2} \times 6 \times 1.6 = 4.8\text{ m}^2$



c 180 m^3
d i 208 m^2 ii 13 litres

EXERCISE 9.6

1 D   **2** D
3 a i 0.6   ii 7.2 m
b i 1.4   ii 40.0 m
c i 0.625   ii 6.0 cm
d i 0.8   ii 20.0 cm
4 a Similar because the shapes are squares so the scale factors will always be the same.
b Similar because the shapes are rectangles and $\frac{14}{4} = \frac{7}{2} = 3.5$
c Not similar because not all the scale factors are the same. $\frac{12}{4} = 3$ and $\frac{12}{3} = 4$
d Similar because $\frac{30}{20} = \frac{15}{10} = 1.5$ and the angle between the two sides is 90° for both triangles (SAS).
e Similar because $\frac{12}{16} = \frac{9}{12} = \frac{6}{8} = 0.75$ (SSS)
f Similar because two matching angles are equal. (AA)
5 a i 1.5   ii 21.3 cm^2 iii 3160 cm^3
b i 0.833   ii 33.3 cm^2 iii 1620 cm^3
6 D   **7** D   **8** E   **9** E
10 E   **11** E   **12** D   **13** E
14 27 m
15 a 6 L   **b** 7.5 mL

CUMULATIVE EXAMINATION 1

1 D 44%   **2** E 43%   **3** B 40%
4 B 65%   **5** C 90%   **6** D 46%
7 B 75%   **8** D   **9** C
10 A   **11** B   **12** D 95%
13 E   **14** D 73%   **15** C
16 D 39%

CUMULATIVE EXAMINATION 2

1 **a** \$120 **b** \$325

2 **a** \$2.87 99% **b** 3×1 94%

c $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$ 66%

3 **a** moderate, positive

b It predicts that, when the minimum temperature is 0°C, the maximum temperature will be 13°C.

c 22°C

d On average, it is predicted that the maximum temperature increases by 0.65°C for each 1°C increase in the minimum temperature.

4 **a** 3.2 km 58% **b** trail 66%

5 **a** 62 m **b** 854 m

6 **a** 69 m

b length $AB = \sqrt{23^2 - 11.5^2} = 19.9$ m

c 229 m^2 **d** 24 429 cm^3

e $h = \dfrac{1244}{36\pi} = 11$ cm **f** 993 cm^3

7 **a** 0.75 **b** 135 g

CHAPTER 10

EXERCISE 10.1

1 **a** 20.71 cm **b** 7.07 m **c** 13.39 m

d 7.63 mm **e** 9.34 m **f** 12.30 m

g 7.63 m **h** 17.27 cm **i** 11.01 cm

2 **a** 2.54 m **b** 20.98 m

3 **a** 64° **b** 42° **c** 39° **d** 38°

e 84° **f** 19° **g** 34° **h** 41°

4 **a** 26° **b** 25°

5 B **6** A **7** E **8** C

9 B **10** E **11** A **12** E

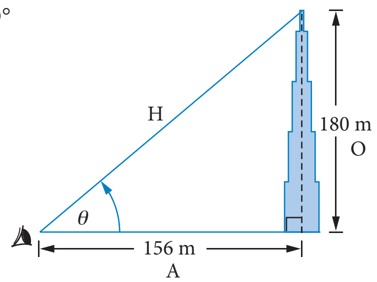
13 132.8 m **14** 12°

15 **a** 2 m **b** 3 m **c** 5 m

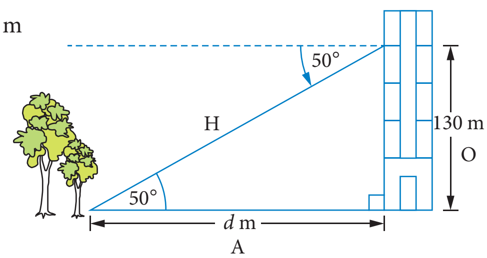
EXERCISE 10.2

1 D **2** A

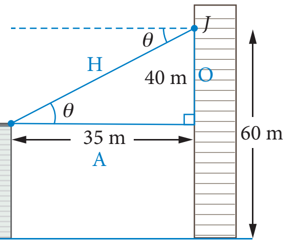
3 **a** 49°



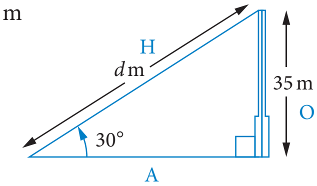
b 109 m



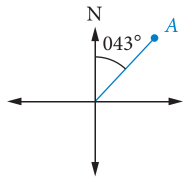
c 49°



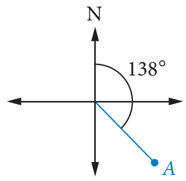
d 70 m



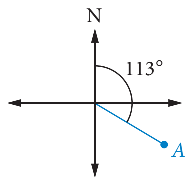
4 **a** 043°



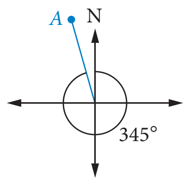
b 138°



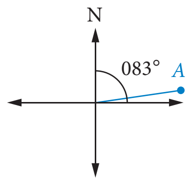
c 113°



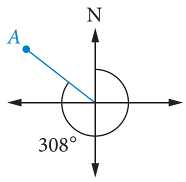
d 345°



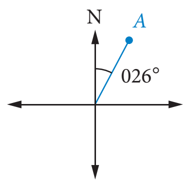
e 083°



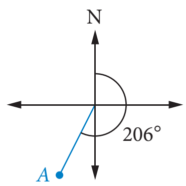
f 308°



g 026°



h 206°



5 **a** **i** 114° **ii** 24 km

b **i** 143° **ii** 25 km

c **i** 306° **ii** 9 km

d **i** 257° **ii** 51 km

6 B **7** A **8** E **9** A

10 E **11** E **12** C

13 **a** $\dfrac{50}{400} = 0.125$ **b** 7.1° **c** 403.1 m

14 **a** 25 m **b** 5°

15 **a** 80 m **b** 066°

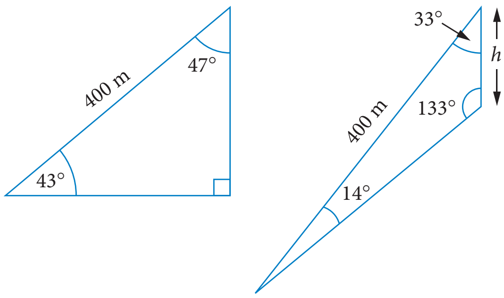
9780170448192

EXERCISE 10.3

1 E **2** C

3 **a** 11 m **b** 55° **c** 125° **d** 5 cm
e 4 m **f** 133° **g** 10 cm **h** 47°
i 19 m

4 **a**



b 132.31 m

5 **a** 10 cm **b** 11 cm **c** 52°
d 123° **e** 16 cm **f** 36°
g 91° **h** 11 m **i** 11 m

6 **a** 10 km **b** 229°

7 C **8** C **9** D

10 B **11** E **12** C

13 **a** 142 m **b** 087°

14 **a** 42.7°

b $NT^2 = 10^2 + 13^2 - 2 \times 10 \times 13 \times \cos(65°)$
$NT^2 = 159.119\ldots$
$NT = \sqrt{159.119\ldots}$
$NT = 12.614\ldots \approx 12.6$

c 69°

15 **a** 475 m

b **i** 908 m

ii 142°; must show cosine rule use for full marks:
either $1400^2 = 950^2 + 908^2 - 2 \times 950 \times 908 \times \cos(T)$,
$T = 97.76\ldots$
or $950^2 = 1400^2 + 908^2 - 2 \times 1400 \times 908 \times \cos(E)$,
$E = 42.24\ldots$

CUMULATIVE EXAMINATION 1

1 D 47% **2** A **3** B

4 C **5** E **6** D 41%

7 D **8** C 61% **9** D 82%

10 D 81% **11** B 77% **12** C

13 D **14** D **15** B 51%

16 C 45%

CUMULATIVE EXAMINATION 2

1 **a** $3500 **b** 5.3%

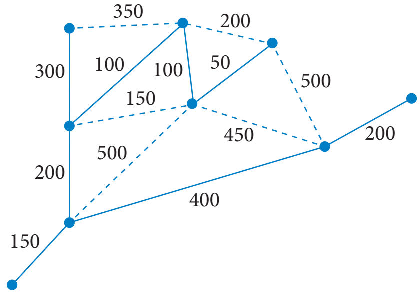
2 **a** $1280 **b** $1536

c 20% of $1600 ≠ 20% of $1280

3 58%

4 **a** Minimum spanning tree 63%

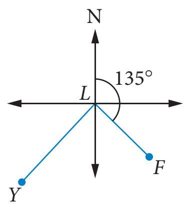
b 53%



5 **a** 2168.32 m **b** 470.05 m

6 **a** 1468.86 m **b** 1231.89 m

7 **a**



b 75°

8 **a** 33 km **b** 57 km **c** 164°

Glossary and index

adjacency matrix A representation of connections between each pair of vertices of a graph. (p. 305)

adjacent vertices Two vertices that are connected by one or more edges. (p. 302)

alternate angles Two equal angles drawn inside a *Z* shape. (p. 460)

ambiguous case (in trigonometry) A situation where it is possible to draw two different triangles that both match the information given. (p. 468)

angle of depression The angle made between the horizontal and a direction below the horizontal. (p. 460)

angle of elevation The angle made between the horizontal and a direction above the horizontal. (p. 460)

apex The point at one end of a cone and the point where all of the triangular faces of a pyramid meet. (p. 412)

arc A part of the circumference of a circle formed by two radiuses. (p. 401)

arc length The length of part of a circle's circumference. (p. 401)

area The amount of space inside a shape. (p. 397)

arithmetic sequence A sequence of numbers that is formed by adding or subtracting a constant number to each preceding value. (p. 81)

asset Items purchased by businesses to help them function. (p. 97)

asymmetric distribution *See* **skewed distribution**.

back-to-back stem plot A statistical graph used when dealing with two sets of data values for the same variable where the original data values are visible. (p. 46)

balance The value of an investment or loan at any time. (p. 91)

bar chart A graphical display used for categorical data, where the frequency of each different category is shown using a vertical column or a horizontal bar. (p. 14)

base The surface that a three-dimensional object stands on. (p. 410)

bearing An angle that shows the direction of one point from another point. (p. 461)

bi-modal Data with two modes. (p. 7)

BNPL A platform that allows the buyer to pay a percentage of the purchase price at the time of purchase and pay the balance in regular instalments. (p. 147)

box-and-whisker plot *See* **boxplot**.

boxplot (box-and-whisker plot) A graphical display of numerical data based on the five-number summary, IQR and outliers. (p. 31)

Buy now pay later scheme *See* **BNPL**.

capacity The amount of liquid a three-dimensional object can hold. (p. 410)

cash Notes and coins. (p. 147)

categorical data Data involving categories. (p. 5)

causation A relationship between two variables where one variable is known to cause the other. (p. 277)

centre of a distribution The single value that best represents the distribution. (p. 7)

circuit A walk with no repeated edges that starts and finishes at the same vertex. (p. 316)

circumference The perimeter of a circle. (p. 398)

column matrix A matrix that has just one column. (p. 210)

common difference The fixed amount that is being added to generate each new value of an arithmetic sequence. (p.85)

common ratio The constant number being multiplied from one value to the next in a geometric sequence. (p. 117)

communication diagram A diagram showing one-way or two-way arrows between points indicating when communication occurs. (p. 238)

communication matrix A square matrix where communication is indicated by a '1' and non-communication is indicated by a '0'. (p. 238)

composite shape A shape formed by combining two or more shapes. (p. 402)

compound interest Interest that is added to the principal, where the interest for the next time period is calculated using this new balance. (p. 124)

compounding period The length of the time period before interest compounds. (p. 124)

cone A three-dimensional object with a circular base and a curved surface that joins the base to an apex. (p. 412)

connected graph A graph where there is a path from any vertex to any other vertex. (p. 309)

constant of variation The k value in variation equations such as $y = kx^2$ and $y = \frac{k}{x}$. (p. 343)

constant rate of change A measure of the change in y as x changes for a linear function. (p. 173)

constant sequence A sequence whose values are all the same. (p. 80)

continuous data Numerical data that can be measured to an increasing level of accuracy. (p. 5)

coordinates A set of values showing an exact position. (p. 163)

cosine rule A formula used to find unknown sides and angles in non-right-angled triangles. (p. 471)

cost (of a business) The amount of money going out of a business. (p. 176)

credit card A card that allows the user to delay making payments for up to a month. (p. 147)

cross-section The two-dimensional shape that is visible when a solid object is cut parallel to its base. (p. 410)

cubic unit A cube of length 1 used to measure volume. (p. 384)

cycle A walk with no repeated vertices that starts and finishes at the same vertex. (p. 316)

cylinder A three-dimensional figure that has equal circles at each end, joined by a curved surface. (p. 410)

data Information collected through observation that can be used to make informed decisions. (p. 4)

debit card A card that allows the user to make payments from a bank account. (p. 147)

decreasing sequence A sequence whose values keep getting smaller. (p. 80)

degree (of a vertex) The number of edges connected to a vertex. (p. 303)

degree sum (of a graph) The sum of the degrees of all the vertices. (p. 303)

depreciation The decrease in value of assets bought by a business over time. (p. 97)

determinant A number that plays an important role in finding the inverse of a matrix. (p. 228)

diameter The distance from one side of a circle to the other through the centre. (p. 398)

direct variation Variation where a percentage increase in one variable results in the same percentage increase in the other variable. (p. 343)

discount A percentage price reduction. (p. 137)

discrete data Numerical data can't be measured to an increasing level of accuracy. (p. 5)

distance-time graph A line segment graph where the horizontal axis measures time and the vertical axis measures distance. (p. 189)

domain of interpretation The range of values for which a model is valid. (p. 175)

dot plot Graphical display for categorical or numerical discrete data. (p. 38)

edge The lines connecting the vertices of a graph. (p. 302)

element A value in a matrix. (p. 208)

equilibrium state matrix *See* **steady-state matrix**.

Euler's formula A formula that applies to graphs that are *both* planar and connected: number of vertices + number of faces – number of edges = 2. (p. 310)

Eulerian trail A walk with no repeated edges that includes every *edge* in a graph. (p. 323)

explanatory variable A variable that we expect to predict or explain the changes observed in another variable. (p. 271)

extrapolation A prediction made outside the original data range. (p. 284)

face A region of a graph that is enclosed by three or more edges, or the region outside the graph. (p. 303)

five-number summary Five key points in a data distribution, consisting of the minimum value, the lower quartile, the median, the upper quartile and the maximum value. (p. 28)

flat rate depreciation Depreciation where the future value of an asset is reduced by a fixed amount every year, expressed either in dollars or as a fixed percentage of the purchase price. (p. 97)

frequency table A table used to organise large amounts of data, with data values in one column and the corresponding frequencies in another. (p. 12)

future value The new reduced value of an asset being depreciated at any point in time or the balance of loans and investments at any point in time. (p. 97)

geometric sequence A sequence of numbers that is formed by multiplying the preceding value by a constant. (p. 117)

Goods and Services Tax *See* **GST**.

graph (network diagram) A diagram consisting of vertices that are connected by edges. (p. 302)

greedy algorithm A series of steps where at each step a choice is made that is the best at that moment. (p. 326)

grouped frequency table A frequency table where numerical data has been grouped into regular intervals. (p. 13)

GST Currently a 10% tax on most sales in Australia. (p. 140)

hemisphere Half of a sphere. (p. 412)

histogram Graphical display for numerical data (discrete or continuous) with vertical, joined columns. (p. 18)

hypotenuse The longest side of a right-angled triangle. (p. 389)

identity matrix A square matrix where all the elements in the leading diagonal are '1' and the other elements are '0'. (p. 210)

increasing sequence A sequence whose values keep getting larger. (p. 79)

inflation Price rises over time. (p. 145)

initial state matrix A column matrix representing the starting state of a transition. (p. 248)

initial value The first value. (p. 173)

interest The fee for using someone else's money. (p. 91)

interpolation A prediction made within the original data range. (p. 284)

interquartile range (IQR) The measure of the spread of the middle 50% of the data values. (p. 30)

intersection (of two lines) The point that lies on both lines. (p. 182)

interval data A numerical data that has no fixed beginning. (p. 6)

inverse (of a matrix) The matrix that, when multiplied by another matrix, results in the identity matrix. (p. 227)

inverse trigonometric function A function that calculates an angle of a right-angled triangle given the ratio of two of the sides. (p. 454)

inverse variation Variation where a percentage increase in one variable results in the same percentage decrease in the other variable. (p. 344)

isolated vertex A vertex that is not connected to any other vertex in a graph. (p. 303)

isomorphic graphs Graphs that show exactly the same connections. (p. 302)

limiting value sequence A sequence whose values tend towards a value but never reach it. (p. 80)

line of good fit A straight line that represents the data on a scatterplot. (p. 284)

line segment graph A piecewise linear graph that joins two or more straight line pieces. (p. 187)

linear equation *See* **linear function**.

linear function An equation that can be written in the form $y = a + bx$, where x and y are variables and a and b are constants. (p. 163)

linear scale A scale used on a graph where the same number is *added* to move from one scale mark to the next. (p. 353)

linearisation The process of changing non-linear relationships into straight-line form. (p. 355)

log scale A scale used on a plot or graph where the same number is *multiplied* to move from one scale mark to the next. (p. 353)

logarithmic scale *See* **log scale**.

loop An edge that starts and finishes on the same vertex. (p. 303)

lower fence The value below which a data point may be considered an outlier. (p. 30)

lower quartile The data point that has 25% of the data below it. (p. 28)

mark-up A percentage price increase. (p. 137)

matrices Plural of **matrix**.

matrix A rectangular arrangement of numbers organised into rows and columns, usually presented in square brackets. (p. 208)

matrix multiplication The multiplication of a matrix by another matrix. (p. 221)

mean The value often referred to in everyday life as the average, calculated using $\bar{x} = \frac{\Sigma x}{n}$, and considered to be a measure of centre. (p. 56)

median The midpoint that divides a distribution into two equal halves. (p. 8)

megalitre (ML) A measure of capacity that is equal to 1 million litres. (p. 384)

minimum connector The path connecting all the vertices in a weighted graph with the smallest total weight. (p. 325)

minimum spanning tree The spanning tree with the smallest total weight. (p. 325)

modal category The category with the highest frequency. (p. 7)

modal interval The interval with the highest frequency. (p. 13)

mode The most frequently occurring category or value. (p. 7)

multiple edges More than one edge connecting two vertices. (p. 303)

negatively skewed distribution A distribution that has a tail at the lower end. (p. 19)

net A two-dimensional figure that shows all of the faces of a solid and can be folded up to form that solid. (p. 420)

network A group of interconnected elements such as people, places or things. (p. 302)

network diagram *See* **graph**.

nominal data Categorical data that doesn't have a natural order, even when numbers are involved. (p. 5)

non-linear association A relationship between two variables that is not a straight line. (p. 355)

normal distribution A distribution with a bell shape that is symmetric about the mean, peaks in the centre and tails off towards zero on both sides. (p. 61)

***n*th value** (of a sequence) An algebraic representation of any value in a sequence. (p. 87)

***n*th value rule** A rule to find any value of a sequence without listing all the values up to the *n*th value. (p. 87)

numerical data Data involving numbers that have a mathematical meaning such as counting or measuring. (p. 4)

one-step communication A direct communication between *A* and *B*. (p. 238)

order of a matrix The number of rows and columns in a matrix. (p. 208)

ordinal data Categorical data that has a natural order, but doesn't involve counting anything and can't be measured to an increasing level of accuracy. (p. 5)

oscillating sequence A sequence whose values switch between positive and negative. (p. 80)

outlier An extreme high or low value in the data. (p. 20)

parallel boxplots A graph where two or more boxplots are shown on the same axis. (p. 48)

parallelogram A four-sided shape where opposite sides are the same length and parallel. (p. 398)

path A walk with no repeated vertices. (p. 316)

per annum (p.a.) Per year. (p. 91)

percentage A number written as a proportion of one hundred and indicated by the symbol %. (p. 12)

percentage change The amount of increase or decrease of a quantity written as a percentage of the quantity. (p. 138)

perimeter The total distance around the outside of a shape. (p. 397)

personal loan A loan involving a signed contract and regular payments based on a quoted percentage rate of interest per annum. (p. 147)

piecewise linear graph A graph made up of more than one straight line piece. (p. 187)

planar graph A connected graph that can be drawn so that no edges cross. (p. 310)

polygon A closed shape with straight sides. (p. 429)

positively skewed distribution A distribution that has a tail at the upper end. (p. 19)

Prim's algorithm A series of steps to find a minimum spanning tree for a graph. (p. 326)

principal The amount of money invested or borrowed. (p. 91)

prism A three-dimensional object with straight edges that has the same cross-section along its full length. (p. 410)

pronumeral A letter or symbol that takes the place of an unknown or generalised number. (p. 175)

purchasing power The change in the value of money over time due to the change in prices. (p. 147)

pyramid A three-dimensional object with a base and triangular faces meeting at a point called the apex. (p. 412)

Pythagoras' theorem A rule for calculating the third side of a right-angled triangle given the length of the other two sides. (p. 389)

quadrilateral A shape with four straight sides. (p. 398)

quartiles The three points that divide a set of data into quarters. (p. 28)

radius The distance from the centre of a circle to the circumference. (p. 398)

range A measure of the spread of the data; the difference between the largest and smallest observations. (p. 8)

rate *See* **constant rate of change**.

ratio A comparison of two like quantities such as lengths. (p. 451)

ratio data Numerical data that has a fixed beginning. (p. 6)

recurrence relation A relation that tells us how a particular value in a sequence can be found from the previous value in the same sequence. (p. 85)

recursion *See* **recursive computation**.

recursive computation Calculations that continually use the previous answer to generate the next answer. (p. 85)

reducing balance depreciation Depreciation where the future value of an asset is reduced every year by a fixed percentage of its value in the preceding year. (p. 131)

redundant link A communication link where the sender and receiver are the same. (p. 238)

response variable A variable whose changes we expect to be predicted or explained by another variable. (p. 271)

revenue (of a business) The amount of money going into a business. (p. 176)

rhombus A four-sided shape where all sides are equal in length and opposite sides are parallel. (p. 398)

right-angled triangle A triangle where one of the angles is 90°. (p. 389)

rise The vertical distance between two points. (p. 284)

rounding Replacing a number with an approximate value. (p. 282)

row matrix A matrix that has just one row. (p. 210)

run The horizontal distance between two points. (p. 284)

scalar A number that is not in a matrix. (p. 214)

scalar multiplication The multiplication of every element in a matrix by the same number. (p. 214)

scale factor A measurement of how much a shape needs to be enlarged or reduced to produce a similar shape. (p. 427)

scatterplot A graph used to compare two numerical variables, where the explanatory variable is plotted on the x-axis and the response variable is plotted on the y-axis. (p. 273)

scientific notation A number in the form (a number between 1 and 10) × (a power of 10). (p. 385)

sector The part of a circle formed by two radiuses and the arc between them. (p. 401)

sequence A list of numbers called values. (p. 78)

shape A two-dimensional figure. (p. 397)

shape of a distribution A description of data as symmetric, positively skewed or negatively skewed. (p. 19)

shortest path The path between two vertices of a graph with the smallest weight. (p. 319)

significant figures (rounding to a number of) A method of rounding involving all the non-zero digits of a number plus the zeros that are included between them, or that are final zeros and signify accuracy. (p. 282)

similar shapes Two-dimensional figures that have exactly the same shape but are different sizes. (p. 427)

simple interest The fixed amount of interest paid at regular time periods calculated as a percentage of the amount of money invested or borrowed. (p. 91)

simultaneous equations Two or more linear equations, each with two or more variables, which are being solved to find values that are common solutions to all the equations. (p. 182)

sine rule A formula that is used to find sides and angles in non-right-angled triangles. (p. 468)

singular matrix A matrix that doesn't have an inverse. (p. 230)

skewed distribution A distribution that is non-symmetric and has a tail. (p. 20)

slant length The distance from a point on the perimeter of the base to the apex of a pyramid or cone. For a pyramid, the distance is measured along the centre of a triangular face. (p. 412)

slope (gradient) The measure of the steepness of a line. (p. 166)

solid A three-dimensional object. (p. 410)

spanning tree A tree subgraph that includes all the vertices of the original graph. (p. 324)

sphere A round, three-dimensional figure where all the points on the surface are the same distance from its centre. (p. 412)

spread of a distribution How much data varies around the centre of a distribution. (p. 8)

square matrix A matrix that has the same number of rows as columns. (p. 210)

square unit A square of side length 1 unit used to measure area. (p. 384)

standard deviation A measurement of the spread of data about the mean. (p. 57)

state A condition or a location at a point in time. (p. 246)

state matrix A column matrix representing a state at a point in time of a transition. (p. 248)

steady-state matrix A column matrix representing the final state of a transition. (p. 250)

stem plot (stem-and-leaf plot) A graphical display for numerical data that is either discrete or continuous. (p. 39)

stem-and-leaf plot *See* **stem plot**.

step graph A piecewise linear graph that has only horizontal straight line pieces. (p. 187)

subgraph A graph that is part of a larger graph. (p. 312)

subject (of an equation or formula) The single variable on the left-hand side of an equation or formula. (p. 181)

substitute To replace a pronumeral by a numerical value in order to evaluate an expression. (p. 165)

surface area The sum of all the areas of the faces of a three-dimensional object. (p. 420)

symmetric distribution A distribution that has the same shape on both sides of the median. (p. 19)

three-figure bearings (true bearings) Three-digit angular directions measured in degrees in a clockwise direction from north. (p. 461)

trail A walk with no repeated edges. (p. 316)

transformation A change made from one type of variable to another. (p. 355)

transition A change from one state to another. (p. 246)

transition diagram A diagram that shows transitions from one state to another using arrows with corresponding percentages. (p. 246)

transition matrix A square matrix that shows a change from one state to another, where the change follows the same rules each time. (p. 246)

trapezium A four-sided shape where two sides are parallel. (p. 398)

tree A connected graph with no loops, multiple edges, or cycles. (p. 324)

triangular prism A prism with a triangular base. (p. 410)

trigonometric ratios The three ratios in right-angled triangles: $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$ (p. 452)

trigonometry A branch of mathematics that studies the relationship between the lengths of sides and the size of angles in a triangle. (p. 451)

true bearings *See* **three-figure bearings**.

two-step communication A communication between *A* and *B*, where *A* can communicate with *C*, which can then communicate with *B*. (p. 238)

unit cost depreciation A reduction in the value of an asset according to the amount of use it has had. (p. 101)

unitary method A way of solving problems by calculating the value of one unit. (p. 140)

upper fence The value above which a data point may be considered an outlier. (p. 30)

upper quartile The data point that has 75% of the data below it. (p. 28)

value (of a sequence) The name for a number in a sequence. (p. 78)

variable (in algebra) A letter or symbol used to represent a quantity that can have many different values in a particular situation. (p. 163)

variable (in statistics) Something measurable or observable that changes between individual observations or over time. (p. 4)

variation The way one variable changes in relation to another variable. (p. 33)

vertex A point on a graph or network diagram. (p. 302)

vertices Plural of **vertex**.

volume The amount of space a three-dimensional object takes up. (p. 410)

walk A sequence of connected vertices. (p. 316)

weighted graph A type of graph where extra information such as distances, times or costs is labelled on the edges. (p. 319)

whiskers Parts of a boxplot that show the minimum and maximum values if there are no outliers. (p. 31)

***x*-intercept** The x value where a line crosses the x-axis. (p. 181)

yearly inflation rate An average based on the individual inflation rates of many items. (p. 145)

***y*-intercept** The y value where the line crosses the y-axis. (p. 169)

zero matrix A matrix where all the elements are '0'. (p. 210)

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